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Real forms of Lie algebras and Lie superalgebras

Tesi di Laurea Magistrale
in Algebra

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Do or do not.
There is no try.

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Introduzione

In questa tesi vogliamo studiare le forme reali di algebre e superalgebre di Lie semisemplici complesse. L'elaborato è diviso in tre capitoli, i primi due riguardano il caso classico, in altre parole le algebre di Lie, mentre nel terzo capitolo esaminiamo il caso delle superalgebre di Lie.

Il primo capitolo è sostanzialmente usato per definire gli strumenti base e la notazione, a partire dalla definizione di algebra di Lie, passando per la definizione della *forma di Cartan-Killing* e di *spazi di radice*, sino ad arrivare alla classificazione delle algebre si Lie semisemplici nelle famiglie standard A_m, B_m, C_m, D_m e le algebre eccezionali, con relativi sistemi di radici e *diagrammi di Dynkin.*

Nel secondo capitolo invece entriamo più nel dettaglio del nostro studio, definiamo la nozione di algebra compatta e, piú importante, di sottoalgebra compatta massimale $\mathfrak{u}$, che come vedremo è unica a meno di automorfismi interni. La sottoalgebra compatta massimale gioca un ruolo fondamentale nella determinazione di tutte le forme reali $\mathfrak{g}_0$ di una stessa $\mathfrak{g}$ complessa semisemplice. Seguono poi le definizioni di *decomposizione di Cartan*, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, ed *involuzione di Cartan*, che esiste per ogni algebra di Lie reale ed è unica a meno di coniugazione attraverso elementi di $\mathrm{Int}\mathfrak{g}_0$, il gruppo degli automorfismi interni di $\mathfrak{g}$. Dopo questo importante risultante, si dimostra che, data $\mathfrak{g}$ algebra semisemplice complessa, ogni coppia di forme compatte è coniugata attraverso un elemento di $\mathrm{Int}(\mathfrak{g})$. Da ció segue quasi immediatamente il risultato su cui poggia il nostro lavoro: le involuzioni di Cartan di $\mathfrak{g}^{\mathbb{R}}$, l'algebra di Lie $\mathfrak{g}$ complessa vista come algebra reale, sono coniugate rispetto alla forme reali compatte di $\mathfrak{g}$. Introduciamo poi i diagrammi di *Vogan di una tripla* $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$, che per le algebre reali giocano lo stesso ruolo dei diagrammi di Dynkin nel problema di classificazione e ci limitiamo a studiare i *diagrammi di Vogan astratti senza frecce*, in quanto assumiamo che

$$\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$$

dove $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ e $\mathfrak{h}$ è la sottoalgebra di Cartan di $\mathfrak{g}$ che è la complessificazione di $\mathfrak{g}_0$, con fine ultimo la dimostrazione del teorema di *Borel de Siebenthal*.

Nella parte finale del capitolo viene presentato l'algoritmo "push the button", un interessante metodo in cui si agisce direttamente sui diagrammi di Vogan e che ci permette di ottenere una dimostrazione alternativa del teorema di *Borel de Siebenthal*, nonchè di trovare un algoritmo per dire direttamente se due diagrammi di Vogan sono equivalenti, cioè corrispondono alla stessa algebra di Lie.

Il terzo capitolo invece riguarda il caso delle superalgebre e ricalca la struttura del capitolo del caso classico. Inizialmente vengono fornite definizione di base riguardanti le superalgebre di Lie, tra le quali anche le definizioni di *superalgebre di Lie classiche* e *di Cartan*. Poi studiamo i *sistemi di radice* e in particolare descriveremo quelli delle algebre classiche. Una volta forniti i sistemi di radici, diamo la definizione di *forma reale* e studiamo in particolare le forme reali di $A(m|n)$. In ultimo introduciamo i *diagrammi di Vogan*, i *diagrammi di Vogan astratti* e l'algoritmo "push the button" nel caso delle superalgebre.

Introduction

In this work we want to study the real forms of semisimple complex Lie alegbras and Lie superalgebras. We have three different chapters, the first two are about the classical case, in other terms about Lie algebras, while, in the third one, we examinate the case of the superalgebras.
In the first chapter we define the basic instruments and the notation for the Lie algebras. We start defining what a Lie algebra is, then we give the definition of *Killing form* and of the *root spaces*. It ends with the classification of the Lie algebra in the *standard families* A_m , B_m , C_m , D_m and the exceptional ones with their root sistems and *Dynkin diagrams*.

In the second chapter we go deeper in our study, we define the notion of compact Lie algebra, and especially of *maximally compact* Lie subalgebra $\mathfrak{u}$ which is unique up to inner automorphism.
The maximally compact subalgebra plays a foundamental role in the classification of all the real forms $\mathfrak{g}_0$ of the same semisimple complex Lie algebra $\mathfrak{g}$. Then we define what a *Cartan decomposition* $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is, and what a *Cartan involution* is, that it exists for all the real Lie algebra and it is unique up to conjugation through element of $\text{Int}\mathfrak{g}_0$. After this important result, we prove that, given a semisimple complex Lie algebra $\mathfrak{g}$, any teo compact forms of $\mathfrak{g}$ are conjugate by an element of $\text{Int}\mathfrak{g}$. Then it follows the main result on which our work is based: the Cartan involution of $\mathfrak{g}^{\mathbb{R}}$, which is the complex Lie algebra $\mathfrak{g}$ seen as a real one, are conjugate with respect to the compact real forms of $\mathfrak{g}$. Then we define what a *Vogan diagram of the triple* $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ and the *abstract Vogan diagram with no arrows*, since we study also the case in which

$$\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$$

where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan subalgebra of the complexification g og $\mathfrak{g}_0$ and we reach the *Borel de Siebenthal* theorem. In the last part of the chapter we develop the "push the button" algorithm, which helps us to achieve an alternative proof of the *Borel de Siebentahl theorem* and a direct algorithm to determine if two Vogan diagram are equivalent, which means that they correspond to the same real form.

The third chapter is about the Lie superalgebras and their real forms and it is similar in structure to the first chapter treating the classical case. In the first part we give the basic definitions for the Lie superalgebras as the definition of *classical Lie superalgebra* and of *Lie superlagebra of Cartan type*. Then we investigate the *root systems* and we give a presentation of the root system for the classical families. After that we give the definition of *real form* and we study, in particular, the real forms of $A(m|n)$. At the end of the chapter we introduce the notion of *Vogan diagram* and the *abstract Vogan diagram* and the "push the button" algorithm in the super case.

Chapter 1

Lie algebras

This chapter does not want to be a detailed description of classical Lie algebras but wants to establish the notation and recall some important definitions in order to help us with our work. Due to this purpose, we do not give all the proofs of the theorems we will state, but we give always the references where such proofs can be found.
At the end of the chapter, we can also find some examples of Lie algebras, roots system and Dynkin diagrams.

1.1 Preliminar definitions

First of all, we have to define what a Lie algebra is, then we define what semisimple and simple Lie algebra are.
At the end of the section we also introduce the notion of Cartan subalgebra. We work in an arbitrary commutative field F with characteristic 0.

Definition 1.1. Let $\mathfrak{g}$ be a vector space $\mathfrak{g}$ over a field F, with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted $(x, y) \to [x, y]$ and called the bracket or commutator of x and y. $\mathfrak{g}$ is called a *Lie algebra* over F, if the following axioms are satisfied:

1. The bracket is bilinear;
2. $[x, x] = 0, \quad \forall x \in \mathfrak{g};$
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall x, y, z \in \mathfrak{g}.$

It is useful for us to introduce the concept of morphism of Lie algebras and $\mathfrak{gl}(V)$.

Definition 1.2. A linear transformation $\phi : \mathfrak{g} \to \mathfrak{g}'$, where $\mathfrak{g}, \mathfrak{g}'$ are Lie algebras, is called *morphism* if $\phi([x, y]) = [\phi(x), \phi(y)],\ \forall x, y \in \mathfrak{g}$.

Example 1.3. Let V be a finite dimensional vector space over F and denote with $\mathrm{End}(V)$ the set of the linear transformations from V to V. If we define the bracket as $[x, y] = xy - yx$ with $x, y \in \mathrm{End}(V)$ we have that $(\mathrm{End}(V), [,])$ is a Lie algebra and we call it $\mathfrak{gl}(V)$.

In order to study the structure of a Lie algebra, we have to define the notion of ideal.

Definition 1.4. A subspace I of a Lie algebra $\mathfrak{g}$ is called an *ideal* of $\mathfrak{g}$ if $x \in \mathfrak{g}, y \in I$ implies $[x, y] \in I$.

Thanks to this definition, we are already able to define what a simple Lie algebra is.

Definition 1.5. Let $\mathfrak{g}$ be a Lie algebra. We call $\mathfrak{g}$ *simple* if $\mathfrak{g}$ has no ideals except itself and 0, and if moreover $[\mathfrak{g}, \mathfrak{g}] \neq 0$ (i.e. $\mathfrak{g}$ is not abelian).

Now we want to define what a semisimple Lie algebra is. To reach our purpose we need to go through the definition of solvable Lie algebra.

Definition 1.6. Define a sequence of ideals of $\mathfrak{g}$, called *derived series*, by $\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \ldots, \mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$. $\mathfrak{g}$ is called *solvable* if $\mathfrak{g}^{(n)} = 0$ for some n.

Next we assemble a few simple observations about solvability, for the proof see [1], chapter 1.

Theorem 1.7. *Let $\mathfrak{g}$ be a Lie algebra,*

a) If $\mathfrak{g}$ is solvable, then so are all subalgebras and homomorphic images of $\mathfrak{g}$.

b) If I is a solvable ideal of $\mathfrak{g}$ such that $\mathfrak{g}/I$ is solvable, then $\mathfrak{g}$ itself is solvable.

c) If I, J are solvable ideals of $\mathfrak{g}$, then so is $I + J$.

As a first application of this proposition we can prove the existence of a unique and maximal solvable ideal called the *radical* of $\mathfrak{g}$ and denoted by Rad $\mathfrak{g}$.

Definition 1.8. Let be $\mathfrak{g}$ a Lie algebra. We call $\mathfrak{g}$ a *semisimple* Lie algebra if $\mathrm{Rad}\mathfrak{g} = 0$.

Theorem 1.9 (Cartan's Criterion). *Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$, V finite dimensional. Suppose that* $\mathrm{Tr}(xy) = 0 \ \forall x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$. *Then $\mathfrak{g}$ is solvable.*

For the proof of this theorem see on [1], chapter 2.

1.2 Root space decomposition

In this section $\mathfrak{g}$ denotes a nonzero semisimple Lie algebra.
Our purpose is to study in detail the structure of $\mathfrak{g}$, via its adjoint representation. The Killing form will play a crucial role.
First of all, we have to introduce what a representation of a Lie algebra is and in particular the adjoint one.

Definition 1.10. A *representation* of a Lie algebra $\mathfrak{g}$ is an homomorphism of Lie algebras

$$\phi : \mathfrak{g} \to \mathfrak{gl}(V)$$

If we define the morphism $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_x y = [x, y]$. We have the following definitions.

Definition 1.11. Let be $\mathfrak{g}$ a Lie algebra, given an element x of a Lie algebra $\mathfrak{g}$, the *adjoint representation* is the morphism:

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathfrak{gl}(V) \\ x &\to \mathrm{ad}_x \end{aligned}$$

for all $x \in \mathfrak{g}$

Now we are ready to introduce the Killing form.

Definition 1.12. Let $\mathfrak{g}$ any Lie algebra, if $x, y \in \mathfrak{g}$ we define the *Killing form* the follow bilinear form on $\mathfrak{g}$:

$$\kappa(x, y) = Tr(\mathrm{ad}_x \mathrm{ad}_y)$$

With this new tools, now we can define a toral subalgebra, show some properties of its subalgebras and define what a root system is.

Definition 1.13. We call a subalgebra *toral* if it consists of semisimple elements.

It is easy to see that a toral subalgebra is abelian too, which is verified in [1].
Now we fix a *maximal toral subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$, i.e. a toral subalgebra not properly included in any other.
Since $\mathfrak{h}$ is abelian, $\mathrm{ad}_{\mathfrak{g}}\mathfrak{h}$ is a commuting family of semisimple endomorphism so it is *simultaneously diagonalizable*.
So we find that

$$\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$, where α ranges over $\mathfrak{h}^*$.
Now we finally give the definitions of *root system* and *root space decomposition.*

Definition 1.14. The set of all nonzero $\alpha \in \mathfrak{h}^*$ for which $\mathfrak{g}_\alpha \neq 0$ is denoted by Δ and its elements are called *roots* of $\mathfrak{g}$ relative to $\mathfrak{h}$.

With the above notation we have what we call the *root space decomposition* also called *root decomposition*:

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

For more details see in [1].
We can now continue with a simple observation about the root space decomposition.

Theorem 1.15. *If $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ is a root decomposition, we have:*

- *For all $\alpha, \beta \in \mathfrak{h}^*, [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.*
- *If $x \in \mathfrak{g}_\alpha, \alpha \neq 0$, then ad_x is nilpotent.*
- *If $\alpha, \beta \in \mathfrak{h}^*$, $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$, relative to the Killing form κ of $\mathfrak{g}$.*

Let's now see some properties, which are all verified in [1].

Theorem 1.16. *Let Δ a root system:*

1. *Δ spans $\mathfrak{h}^*$;*
2. *If $\alpha \in \Delta$ then $-\alpha \in \Delta$;*
3. *Let $\alpha \in \Delta, x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$. Then $[x, y] = \kappa(x, y)t_\alpha$ where $t_\alpha \in \mathfrak{h}$ is defined by $\kappa(t_\alpha, h) = \alpha(h), \ \forall h \in \mathfrak{h}$;*
4. *If $\alpha \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional with basis t_α;*
5. *$\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha)$, for $\alpha \in \Delta$;*
6. *If $\alpha \in \Delta$ and x_α is any nonzero element of $\mathfrak{g}_\alpha$, then there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ span a three dimensional simple subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, F)$ via $x_\alpha \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y_\alpha \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h_\alpha \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;*

7. $h_\alpha = \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$; $h_\alpha = -h_{-\alpha}$;

8. *If* $\alpha \in \Delta$ *the only scalar multiples of* α *which are roots are* $\alpha, -\alpha$;

9. *If* $\alpha, \beta \in \Delta$, *then* $\beta(h_\alpha) \in \mathbb{Z}$, *and* $\beta - \beta(h_\alpha)\alpha \in \Delta$, *the numbers* $\beta(h_\alpha)$ *are called Cartan Integers;*

10. *If* $\alpha, \beta, \alpha + \beta \in \Delta$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$;

11. *Let* $\alpha, \beta \in \Delta$, $\beta \neq \pm\alpha$. *Let* r, q *b (respectively) the largest integers for which* $\beta - r\alpha, \beta + q\alpha$ *are roots. Then all* $\beta + i\alpha \in \Delta$ $(-r \leq i \leq q)$, *and* $\beta(h_\alpha) = r - q$;

12. $\mathfrak{g}$ *is generated (as Lie algebra) by all the root spaces* $\mathfrak{g}_\alpha$.

Now, we are at this point: we have $\mathfrak{g}$, which is a semisimple Lie algebra, $\mathfrak{h}$ a maximal toral subalgebra, $\Delta \subset \mathfrak{h}^*$ the set of roots and $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ our root space decomposition.
Since the restriction to $\mathfrak{h}$ of the Killing form is nondegenerate we can transfer the form to $\mathfrak{h}^*$, letting $(\gamma, \delta) = \kappa(t_\gamma t_\delta)$, $\forall \gamma, \delta \in \mathfrak{h}^*$. Since we know that Δ spans $\mathfrak{h}^*$ we can choose a basis $\alpha_1, \ldots, \alpha_l$ and write $\beta \in \Delta$ as $\beta = \sum_{i=1}^{l} c_i \alpha_i$, with $c_i \in F$. It is possible to show that all the $c_i \in \mathbb{Q}$.
Now we can show the result of this section, proved in [1]

Theorem 1.17. *Let be* $\mathfrak{g}$ *a Lie algebra,* $\mathfrak{h}$ *its maximal toral subalgebra,* Δ *a root system and* $E = \mathbb{R} \oplus_{\mathbb{Q}} E_{\mathbb{Q}}$ *where* $E_{\mathbb{Q}}$ *is the* $\mathbb{Q}$*-subspace of* $\mathfrak{h}^*$ *spanned by all the roots. Then:*

1. Δ *spans* E, *and* 0 *does not belong to* Δ;

2. *If* $\alpha \in \Delta$ *then* $-\alpha \in \Delta$, *but no other scalar multiple of* α *is a root;*

3. *If* $\alpha, \beta \in \Delta$, *then* $\beta - \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Delta$;

4. *If* $\alpha, \beta \in \Delta$, *then* $\beta - \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.

1.3 Root Systems, Weyl Group, Cartan Matrix, Dynkin Diagram

In this section we define the notion of a root system by a few axioms, what a reflection and the Weyl group are. After that, we define the Cartan matrix, that is a matrix strictly connected to the Lie algebra and last, but not least, the Dynkin Diagram, which is a sort of graphic representation of the Cartan Matrix which will be useful for the classification.

1.4 Root systems and Weyl group

Let be E a fixed real euclidean space, i.e. a finite dimensional vector space over $\mathbb{R}$ endowed with a positive definite symmetric bilinear form (α, β). Geometrically a reflection in E is an invertible linear transformation leaving pointwise fixed some *hyperplane* (subspace of dimension one) and sending any vector orthogonal to that hyperplane into its negative. Evidently a reflection is orthogonal. Any nonzero vector α determines a reflection σ_α, with reflecting hyperplane $P_\alpha = \{\beta \in E | (\beta, \alpha) = 0\}$. Of course, nonzero vectors proportional to α yield the same reflection. It is easy to write down an explicit formula: $\sigma_\beta = \beta - \dfrac{2(\beta, \alpha)}{\alpha, \alpha} = \langle \beta, \alpha \rangle$.

Definition 1.18. A subset of Δ of the euclidean space E is called a *root system* in E if the following axioms are satisfied:

1. Δ is finite, spans E, and does not contain 0;
2. If $\alpha \in \Delta$, the only multiples of α in Δ are $\pm\alpha$;
3. If $\alpha \in \Delta$, the reflection σ_α leaves Δ invariant;
4. If $\alpha, \beta \in \Delta$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$

Definition 1.19. Let be Δ a root system in E. We denote by $\mathcal{W}$ the subgroup of $\mathrm{GL}(E)$ generated by the reflection $\sigma_\alpha (\alpha \in \Delta)$ and we call it the *Weyl Group* of Δ.

By the third and the first axioms respectively we can see that $\mathcal{W}$ permutes the set Δ and it is finite. The following lemma shows how a certain automorphism of E acts on $\mathcal{W}$ by conjugation, for the proof see in [1],chapter 3.

Lemma 1.20. *Let Δ be a root system in E, with Weyl group $\mathcal{W}$. If $\sigma \in \mathrm{GL}(E)$ leaves Δ invariant, then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma_\alpha}$ for all $\alpha \in \Delta$, and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Delta$.*

The fourth axiom limits severely the possible angles occurring between pairs of roots. Recall that the cosine of the angle θ between vectors $\alpha, \beta \in E$ is given by the formula $\|\alpha\|\|\beta\| \cos\theta = (\alpha, \beta)$.
Therefore, $\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = 2\frac{\|\beta\|}{\|\alpha\|} \cos\theta$ and $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\cos^2\theta$. This last number is a nonnegative integer; but $0 \leq \cos^2\theta \leq 1$, and $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ have like sign, so the following possibilities are the only ones when $\alpha \neq \pm\beta$ and $\|\beta\| \geq \|\alpha\|$. Since $4\cos\theta = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ we could have only a few values for $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$, values that are reported in the following table.

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\|\|\beta\|\|^2/\|\|\alpha\|\|^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Table 1.1: Values of $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$

Lemma 1.21. *Let α, β be nonproportional roots. If $(\alpha, \beta) > 0$(i.e. if the angle between α and β is strictly acute), then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$, then $\alpha + \beta$ is a root.*

This last lemma is proved in [1].
Let us introduce some important notions about root systems:

Definition 1.22. A subset $S \in \Delta$ is called a *base* if:

1. S is a base of E;
2. each root β can be written as $\beta = \sum k_\alpha \alpha$ $(\alpha \in S)$ with integral coefficients k_α all nonnegative or all nonpositive.

Remark 1.23. We can denote with $\Pi = \text{Span}\{S\} \cap \Delta$

The roots in S are called *simple*. As a consequence of the first property Card $S = \dim E = l$ and the expression for β in the second one tell us that this expression is unique. Thanks to this fact we can define the *height* of a root by $ht\beta = \sum_{\alpha \in S} k_\alpha$. If all the coefficients are nonnegative we call β a *positive root* and we denote it writing $\beta \in \Delta^+$, otherwise we will write $\beta \in \Delta^-$.

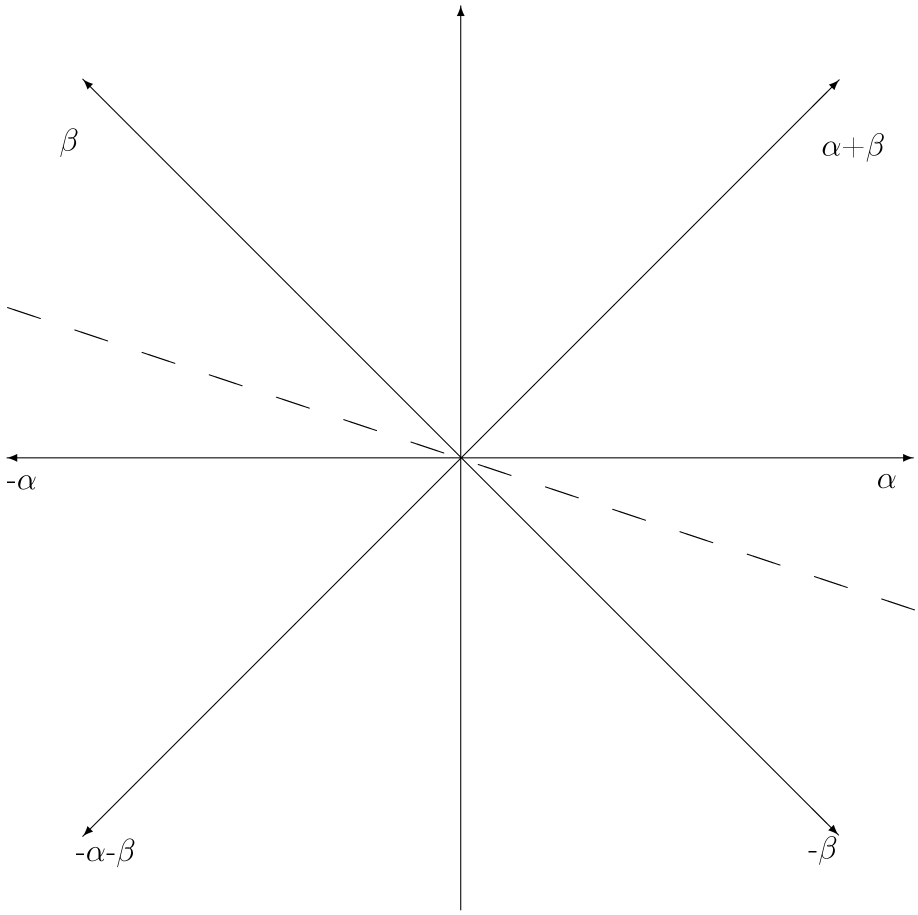


Figure 1.1: Root system of A_2

Lemma 1.24. *If S is a base of Δ, then $(\alpha, \beta) \leq 0$ for $\alpha \neq \beta$ in S, and $\alpha - \beta$ is not a root.*

Also this lemma is proved in [1].
It is possible to show that Δ has a base, to reach this results we have to introduce some new tools:

Definition 1.25. Let be $\gamma \in E$, we call it *regular* if $\gamma \in E - \cup_{\alpha \in \Delta} P_\alpha$, where $P_\alpha = \{\beta \in E | (\beta, \alpha) = 0\}$

Then if γ is regular, it is clear that $\Delta = \Delta^+(\gamma) \cup \Delta^-(\gamma)$; we can call now $\alpha \in \Delta^+(\gamma)$ *decomposable* if $\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in \Delta$, *indecomposable* otherwise. With the following theorem, and it is possible to show that this Δ is a base, see in [1], chapter 3. Now we will see some behavior of simple

roots:
All the proofs are on [1]:

Lemma 1.26. *If α is positive but not simple, then $\alpha - \beta$ is a root (necessarily positive) for some $\beta \in S$*

Corollary 1.27. *Each $\beta \in \Delta^+$ can be written in the form $\alpha_1 + \cdots + \alpha_k$ ($\alpha_i \in S$ not necessarily distinct) in a such way that each partial sum is a root.*

Lemma 1.28. *Let α be simple. Then σ_α permutes the positive roots other than α.*

Corollary 1.29. *Set $\delta = \dfrac{1}{2}\sum_{\beta \in \Delta^+} \beta$. Then $\sigma_\alpha(\delta) = \delta - \alpha$ for all $\alpha \in S$.*

Corollary 1.30. *If $\sigma = \sigma_1 \dots \sigma_t$ is an expression for $\sigma \in \mathcal{W}$ in terms of reflections corresponding to simple roots, with t as small as possible, then $\sigma(\alpha) \in \Delta^+$.*

Now we can finally discuss some properties of the Weyl group, see [1], chapter 3.

Theorem 1.31. *Let S be a base of Δ, then:*

1. *If $\gamma \in E$ regular, there exists $\sigma \in \mathcal{W}$ such that $(\sigma(\alpha), \alpha) > 0, \forall \alpha \in S$;*
2. *If S' is another base of Δ, then $\sigma(S') = S$ for some $\sigma \in \mathcal{W}$ (so $\mathcal{W}$ acts simply transitively on the set of bases);*
3. *If α is any root, there exist $\sigma \in \mathcal{W}$ such that $\sigma(\alpha) \in S$;*
4. *$\mathcal{W}$ is generated by the $\sigma_\alpha, \alpha \in S$;*

1.5 Cartan Matrix and Dynkin diagram

Definition 1.32. Fix an ordering $(\alpha_1, \dots, \alpha_l)$ of simple roots. The matrix $(\langle \alpha_i, \alpha_j \rangle)$ is then called the *Cartan matrix* of Δ.

It is obvious that the matrix depends on the order we choose for the simple roots, i.e. if we choose a different order we will have a permutation of the columns and the rows of that matrix, but it is not a problem because we have already seen that the Weyl group acts transitively on the collection of bases. Notice that the matrix is not singular since S is a base for E.

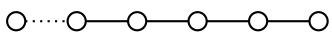
Theorem 1.33. *Let $\Delta' \subset E'$ be another root system, with base $S' = (\alpha'_1, \dots, \alpha'_l)$. If $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ for $1 \leq i, j \leq l$, then the bijection $\alpha_i \mapsto \alpha'_i$ extends (uniquely) to an isomorphism $\Phi : E \to E'$ mapping Δ onto Δ' and satisfying $\langle \Phi(\alpha), \Phi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$. Therefore, the Cartan matrix of Δ determines Δ up to isomorphism.*

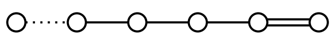
The proof of this last theorem is in [1]. If α, β are distinct positive roots, we know that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0, 1, 2, 3$. Now we can define the *Coxeter graph* of Δ which is a graph having l vertices, where l is the dimesion of our root space, the $i-$th joined the to the $j-$th $(i \neq j)$ by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges. Whenever a double or triple edge occurs in the Coxeter graph of Δ, we can add an arrow pointing to the shorter of the two roots. This additional information allow us to recover the Cartan integers; we call the resulting figure *Dynkin diagram* of Δ.
If we recall that Δ is called irreducible if and only if Δ cannot be partioned into two proper, orthogonal subset, we can easily understand that Δ is irreducible if and only if the Dynkin diagram is connected in the usual sense. In general, we could have as many connected components of the Dynkin diagram as the partition of S into mutually orthogonal subsets, we will call them S_i. If E_i is the span of S_i it is clear that $E = E_1 \oplus E_2 \oplus \cdots \oplus E_t$. It is possible to show that all the E_i are $\mathcal{W}$-invariant .

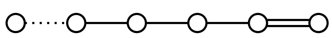
Theorem 1.34. *Δ decomposes (uniquely) as the union of irreducible root system Δ_i (in subspaces E_i of E) such that $E = E_1 \oplus E_2 \oplus \cdots \oplus E_t$ (orthogonal direct sum).*

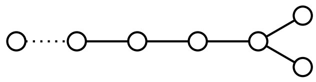
The discussion we have already have show that it is sufficient to classify the irridicibile root systems or the Dynkin diagrams, for the proof see in [1], chapter 3. The possibilities of different Dynkin diagram are restricted by the angles that the edge of the graph can have, angles that are expressed in table 1.1.

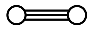
Theorem 1.35. *If Δ is an irreducible root system of rank l, its Dynkin diagram is one of the following:*

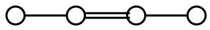
$A_l, l \geq 1$ 

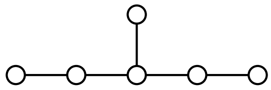
$B_l, l \geq 2$ 

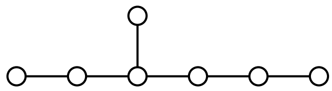
$C_l, l \geq 3$ 

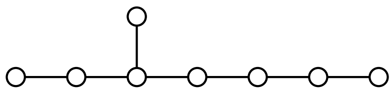
$D_l, l \geq 4$ 

G_2 

F_4 

E_6 

E_7 

E_8 

Proposition 1.36. *Let $\mathfrak{g}$ be a classical Lie algebra, its Dynkin diagram is one of the Dynkin of the previous Theorem.*

Remark 1.37. As a consequence of the last two theorem and of Theorem 1.33, we have that if we have two classical Lie $\mathfrak{g}$, $\mathfrak{g}'$ algebras with the same Dynkin diagram, $\mathfrak{g} \cong \mathfrak{g}'$.

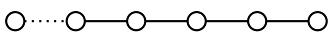
Example 1.38. We can introduce the four kind of classical Lie algebra with their root systems.
The first one, which is called $A_m = \mathfrak{sl}(m+1, F) = \{x \in \mathfrak{gl}(m+1, F) : tr(x) = 0\}$. We have that its dimension is $(m+1)^2 - 1$ and a base is $\{E_{i,i} - E_{i+1,i+1}, i = 1, \ldots, m\} \bigcup \{E_{i,j} : i \neq j, i, j = 1, \ldots, m-1\}$.
If we define $\varepsilon_i \in \mathfrak{h}^* = \varepsilon_i(\text{diag}(a_1, \ldots, a_n)) = a_i$ we can give a basis of its root system as follows:

$$\Delta_{A_{m-1}} = \{\varepsilon_i - \varepsilon_j, i \neq j, i, j = 1, \ldots, m\}$$

so if we define $\alpha_i = \varepsilon_i - \varepsilon_{i+1},\ i = 1, \ldots, m-1$ we have that

$$S_{m-1} = \{\alpha_1, \ldots, \alpha_{m-1}\}$$

and its Dynkin diagram is:

$A_m, m \geq 1$ 

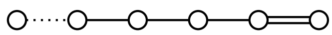
For the following ones we have to introduce the concept of a Lie algebra associated to a bilinear form. Let f be a nondegenarate bilinear form, if s is the matrix associated to f, our Lie algebra is:

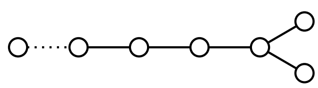
$$L_f = \{x \in \mathfrak{gl}(n, f) : x^t s + sx = 0\}$$

If we take $s = J$ where $J^2 = \mathrm{Id}$ we obtain two different kind of Lie algebra. We will have $B_m = \mathfrak{so}(2m+1, F)$ and a base is given by $E_{i,j} - E_{j^*,i^*}$ where $i^* = n+1-i$ and $j^* = n+1-j$ and $n = 2m+1$ and $D_m = \mathfrak{so}(2m, F)$. A root systems for B_m and D_m is given by:

$$\Delta_{B_m} = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j : i \neq j, i, j = 1, \dots, m\}$$
$$\Delta_{D_m} = \{\pm\varepsilon_i \pm \varepsilon_j : i \neq j, i, j = 1, \dots, m\}$$

If we define $\alpha : i = \varepsilon_i \varepsilon_{i+1}$ and $\alpha_m = \varepsilon_m$ we have the following base for B_m: $S_{B_m} = \{\alpha_1, \dots, \alpha_m\}$ and if we define $\alpha : i = \varepsilon_i \varepsilon_{i+1}$ for $i = 1, \dots, m-1$ and $\alpha_m = \varepsilon_{m-1} - \varepsilon_m$ we have $S_{D_m} = \{\alpha_1, \dots, \alpha_m\}$ and the Dynkin diagram are the following:

$B_m, m \geq 2$ 

$D_m, m \geq 4$ 

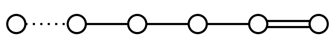
Last but not least if we consider a bilinear form with the following matrix we will obtain $C_m = \mathfrak{sp}(2m, F)$:

$$s = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$$

The root system is:

$$\Delta_{C_m} = \{\pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j : i \neq j, i, j = 1, \dots, m\}$$

and if we define $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, m-1$ and $\alpha_m = \varepsilon_m$ we obtain $S_{C_m} = \{\alpha_1, \dots, \alpha_m\}$ as a basis and the following Dynkin diagram:

$C_m, m \geq 3$ 

Chapter 2

Real Forms and Cartan Decomposition

In this chapter we see what a compact Lie subalgebra and a Cartan decomposition are and their relation to the root space decomposition.
To reach this purpose we need some properties of the Killing form and finally we introduce Vogan diagrams, which are strictly related with the Dynkin diagrams explained in the previous chapter.

2.1 Compact Lie Algebras

Let us recall Lie second and third theorem. For its proof see[2] page 662.

Theorem 2.1. *Every finite-dimensional Lie algebra over* $\mathbb{R}$ *is isomorphic to the Lie algebra of a simply connected analytic group.*

Thanks to this theorem, we can now define what a compact Lie subalgebra is.

Definition 2.2. Let $\mathfrak{g}$ be a semisimple complex Lie algebra, we define as $\mathrm{Int}(\mathfrak{g})$ the analytic subgroup of $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$. Thus $\mathrm{Int}(\mathfrak{g})$ is the identity component of $\mathrm{Ad}(G)$ and equals to $\mathrm{Ad}(G)$ if G is connected.

Definition 2.3. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ and K^* th analytic subgroup of $\mathrm{Int}(\mathfrak{g})$ which corresponds (according to the second Lie theorem) to the subalgebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$. The subalgebra $\mathfrak{k}$ is called *compactly imbedded subalgebra of* $\mathfrak{g}$ if K^* is compact. The lie algebra $\mathfrak{g}$ is said to be *compact* if it is compactly imbedded in itself or equivalently if $\mathrm{Int}(\mathfrak{g})$ is compact.

Lets see some properties and properties of the compact Lie algebras.

Theorem 2.4. *If G is a Lie group with Lie algebra $\mathfrak{g}$ and if K is a compact subgroup with corresponding Lie subalgebra $\mathfrak{k}$, then $\mathfrak{k}$ is a compactly embedded subalgebra of $\mathfrak{g}$. In particular, the Lie algebra of a compact Lie group is always a compact Lie algebra.*

Proof. Since K is compact, so is the identity component K^0. Then $\mathrm{Ad}_{\mathfrak{g}}(K^0)$ must be compact, being the continuous image of a compact group. The groups $\mathrm{Ad}_{\mathfrak{g}}(K^0)$ and $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})$ are both analytic connected subgroups of $\mathrm{GL}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ and hence are isomorphic as Lie groups. Therefore $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})$ is compact. □

The next proposition and its corollary give properties of a compact Lie algebra.

Remark 2.5. Let G be a compact Lie group, we have that G admits an invariant measure under Ad which is the *Haar measure*, see on [2] page 239. So when $\mathrm{Ad}(G)$ invariant product is defined as follows:

$$(x, y) = \frac{1}{|G|}\int_G \langle \mathrm{Ad}(g)x, \mathrm{Ad}(g)y\rangle \mathrm{dg}$$

where $\langle , \rangle$ is any product on $\mathfrak{g}$.

Definition 2.6. A Lie algebra $\mathfrak{g}$ is called *reductive* if its radical coincides with its center.

Proposition 2.7. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$, G its adjoint group. Then the following statements are equivalent:*

(i) $\mathfrak{g}$ *is reductive and* $[\mathfrak{g}, \mathfrak{g}]$ *is of compact type;*

(ii) G *is compact;*

(iii) *If* $X \in \mathfrak{g}$, $\mathrm{ad}X$ *is semisimple and has only pure imaginary eigenvalues.*

Sketch of proof. (See on [4], chapter 4, for a complete proof). (i) ⇒ (ii) Let C be the center of $\mathfrak{g}$, $\mathfrak{g}_1 = [L, L]$. Then $Y^y = Y$ for $Y \in C, y \in G$. So if $Int(G)_1$ is the adjoint group of $[\mathfrak{g}, \mathfrak{g}]$, $y \to y|_{[\mathfrak{g},\mathfrak{g}]}$ is an isomorphism of G onto G_1. So G is compact.
(ii) ⇒ (iii) Let G be compact, so it follows as we said in the previous remark, that there is an inner product for $\mathfrak{g}$ which is positive deinite and invariant under G. It is easy to see that the $\mathrm{ad}X (X \in \mathfrak{g})$ are skew-symmetric with respect to this inner product. A standard result in linear algebra implies that all eigenvalues of $\mathrm{ad}X$ are pure imaginary. If $X \in \mathfrak{g}$ and $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ invariant under $\mathrm{ad}X$, then X leaves the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{g}$

invariant. Hence ad$X$ is semisimple for all $X \in \mathfrak{g}$.
(iii) $\Rightarrow$ (i) The adjoint representation of $\mathfrak{g}$ is semisimple, so $\mathfrak{g}$ is reductive. Let $[\mathfrak{g}, \mathfrak{g}]$ the derived algebra of $\mathfrak{g}$ and ω the Casimir polynomial of the derived algebra. So, according to the previous observation, we have that all eiegenvalues of $(\text{ad}X)^2$ are ≤ 0, so $\omega(X) = (tr(X))^2 \leq 0$. Moreover, if $X \in \mathfrak{g}$ and $\omega(X) = 0$ we have that all the eigenvalues are equal to zero. So ad$X$ is nilpotent, showing that ad$X = 0$. Thus $X = 0$. $-\omega$ is therefore a positive definite quadratic form on the derived algebra. If G_1 is the adjoint group of the derivate we have that it is a closed subgroup of $GL([\mathfrak{g}, \mathfrak{g}])$. On the other hand, G_1 is contained in the orthogonal group of the derived algebra, with respect to ω, which is compact. So G_1 is compact and we have proven (i). □

The next proposition is a kind of converse of the previous corollary.

Proposition 2.8. *If the Killing form of a real Lie algebra* $\mathfrak{g}$ *is negative definite, then* $\mathfrak{g}$ *is a compact Lie algebra.*

Proof. By Cartan's criterion for semisimplicity, Theorem 1.9, $\mathfrak{g}$ is semisimple. We also have that $\text{Int}(\mathfrak{g}) = (\text{Aut}_{\mathbb{R}}\mathfrak{g})_0$ (see [2], Proposition 1.97 and 1.98). Consequently $\text{Int}(\mathfrak{g})$ is a closed subgroup of $\text{GL}(\mathfrak{g})$. On the other hand, the negativity of the Killing form is an inner product on $\mathfrak{g}$ in which every member of ad$\mathfrak{g}$ is skew symmetric. Therefore the corresponding analytic group Int$\mathfrak{g}$ acts by orthogonal transformations. Since Int$\mathfrak{g}$ is then exhibited as a closed subgroup of the orthogonal group, it is compact. □

2.2 Real Forms

In this section we introduce what a real form is and what a Cartan decomposition and a Cartan involution are.

Definition 2.9. Let V be a vector space over $\mathbb{R}$ of finite dimension. A *complex structure on* V is an $\mathbb{R}$-linear endomorphism J of V such that $J^2 = -\text{Id}$, where Id is the identity mapping of V.

A vector space V over $\mathbb{R}$ with a complex structure J can be turned into a vector space $\widetilde{V}$ over $\mathbb{C}$, by putting

$$(a + ib)X = aX + bJX, \quad X \in V, a, b \in \mathbb{R}$$

In fact $J^2 = -\text{Id}$ implies $\alpha(\beta X) = (\alpha\beta X)$ for $\alpha, \beta \in \mathbb{C}$, so it is obvious that $\dim_{\mathbb{C}}\widetilde{V} = \frac{1}{2}\dim_{\mathbb{R}}V$. We call $\widetilde{V}$ the *complex vector space associated to* V.

On the other hand, if we have the vector space E over $\mathbb{C}$ we can consider the vector space $E^{\mathbb{R}}$ over $\mathbb{R}$ where the multiplication by i is given by the complex structure J and it is clear that $E = \widetilde{E^{\mathbb{R}}}$.
A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is said to have a *complex structure* J, if J is a complex structure over the vector space $\mathfrak{g}$ and in addition

$$[X, JY] = J[X, Y], \quad \forall X, Y \in \mathfrak{g}$$

From which simply follows:

$$[JX, JY] = -[X, Y], \quad \forall X, Y \in \mathfrak{g}$$

So the complex vector space $\tilde{\mathfrak{g}}$ becomes a Lie algebra over $\mathbb{C}$ with the following bracket operation:

$$\begin{aligned}[(a+ib)X, (c+id)Y] &= [aX + bJX, cY + idY] = \\ &= ac[X, Y] + bcJ[X, Y] + adJ[X, Y] - bd[X, Y]\end{aligned}$$

In a similar way, we can introduce a complex structure J instead the multiplication for i of a complex Lie algebra to reach a real one.
Now suppose that V is an arbitrary finite dimensional vector space over $\mathbb{R}$, so the product $V \times V$ is a vector space too and we can choose the endomorphism $J : V \times V : (X, Y) \to (-Y, X)$ which is a complex structure on $V \times V$. The complex space $\widetilde{(V \times V)}$ is called *complexification of* V and will be denoted $V^{\mathbb{C}}$. In the same way we can define $\mathfrak{g}$ the *complexification of a Lie algebra* $\mathfrak{g}_0$, where $\mathfrak{g}_0$ is a real Lie algebra, writing $X + JY$ with $X, Y \in \mathfrak{g}_0$ and the following bracket:

$$[X + JY, Z + JT] = [X, Z] - [Y, T] + J([Y, Z] + [X, T])$$

So we will denote with $\mathfrak{g}_0$ a real Lie algebra, $\mathfrak{g}$ its complexification and with $\mathfrak{g}^{\mathbb{R}}$ a Lie algebra over $\mathbb{R}$ with a complex structure J derived from multiplication by i on $\mathfrak{g}$.

Lemma 2.10. *Let κ_0, κ and $\kappa^{\mathbb{R}}$ denote the Killing forms of the Lie algebras $\mathfrak{g}_0, \mathfrak{g}$ and $\mathfrak{g}^{\mathbb{R}}$. Then*

$$\begin{aligned}\kappa_0(X, Y) &= \kappa(X, Y) \quad for\ X, Y \in \mathfrak{g}_0 \\ \kappa^{\mathbb{R}}(X, Y) &= 2Re(\kappa(X, Y)) \quad for\ X, Y \in \mathfrak{g}^{\mathbb{R}}\end{aligned}$$

Proof. The first relation is obvious, for the second suppose X_i $(1 \leq i \leq n)$ is any basis of $\mathfrak{g}$; let $B + iC$ denote the matrix of $\mathrm{ad}X\mathrm{ad}Y$ with respect to this basis, B and C being real. Then $X_1, \ldots, X_n, JX_1, \ldots, JX_n$ is a basis of $\mathfrak{g}^{\mathbb{R}}$

and since the linear transformation $\mathrm{ad}X\mathrm{ad}Y$ of $\mathfrak{g}^{\mathbb{R}}$ commutes with J, it has the matrix expression

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

from which the second relation above follows. □

Due to this lemma we have the following.

Proposition 2.11. *Let $\mathfrak{g}$ a Lie algebra, $\mathfrak{g}_0, \mathfrak{g}, \mathfrak{g}^{\mathbb{R}}$, are all semisimple if and only if one of them is.*

Definition 2.12. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. A *real form* of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0$ of the real algebra $\mathfrak{g}^{\mathbb{R}}$ such that

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus J\mathfrak{g}_0$$

In this case, each $Z \in \mathfrak{g}$ can be uniquely written as

$$Z = X + iY, \quad X, Y \in \mathfrak{g}_0$$

Thus $\mathfrak{g}$ is isomorphic to the complexification of $\mathfrak{g}_0$. The mapping σ of $\mathfrak{g}$ onto itself is given by $\sigma : X + iY \to X - iY (X, Y \in \mathfrak{g}_0)$ is called the *conjugation* of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. The mapping σ has the properties

$$\sigma(\sigma(X)) = X, \quad \sigma(X+Y) = \sigma(x) + \sigma(Y)$$
$$\sigma(\alpha x) = \bar{\alpha}\sigma(X), \quad \sigma[X, Y] = [\sigma X, \sigma Y]$$

for $X, Y \in \mathfrak{g}, \alpha \in \mathbb{C}$. Thus σ is not an automorphism of $\mathfrak{g}$, but it is an automorphism of $\mathfrak{g}^{\mathbb{R}}$. We can now observe that the set $\mathfrak{g}_0$ of fixed point of σ is a real form of $\mathfrak{g}$ and σ is the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. We have that $J\mathfrak{g}_0$ is the eigenspace of σ for the eigenvalue -1 and $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 + J\mathfrak{g}_0$.

Definition 2.13. A real form of $\mathfrak{g}$ that contains $\mathfrak{h}_0$ for some Cartan subalgebra $\mathfrak{h}$ is called *split real form.*

Example 2.14. Let us see an example of real form and of a compact subalgebra.
Let $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C} = \mathrm{span}\{\mathfrak{h}, X_\alpha, X_{-\alpha}\}$ where

$$H_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; X_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; X_{-\alpha} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

We have the real form $\mathfrak{g}_0 = \mathfrak{sl}_2\mathbb{R}$, which is split, and also another real form $\mathfrak{u}_0 = \mathfrak{su}(2) = \text{span}\{X, Y, Z\} \subset \mathfrak{sl}_2\mathbb{C}$ where

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \; Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \; Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This is also a real form because $\mathfrak{span}_{\mathbb{C}}(X, Y, Z) = \mathfrak{sl}_2\mathbb{C}$, but it is not split.
We have only to verify that $\mathfrak{su}(2)$ is compact, that is a real form as $\mathfrak{sl}_2\mathbb{R}$ is obvious.
To see that $\mathfrak{u}_0$ is compact we give as isomorphism Φ to $\mathfrak{so}_{\mathbb{R}}(3) = \{x \in \mathfrak{gl}_3(\mathbb{R}) | X = -X^t\}$ which is compact:

$$\begin{aligned} \Phi : &\mathfrak{su}(2) \to \mathfrak{so}_{\mathbb{R}}(3) \\ &X \to \widetilde{X} \\ &Y \to \widetilde{Y} \\ &Z \to \widetilde{Z} \end{aligned}$$

where

$$\widetilde{X} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; \widetilde{Y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \; \widetilde{Z} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

which generate $\mathfrak{so}_{\mathbb{R}}(3)$, and Φ is well defined on the bracket.

Remark 2.15. Following the notation of the first chapter, for each $\alpha \in \Delta$ a vector $X_\alpha \in \mathfrak{g}_\alpha$ can be chosen such that for all $\alpha, \beta \in \Delta$:

$$\begin{aligned} [X_\alpha, X_{-\alpha}] &= H_\alpha, \; [H, X_\alpha] = \alpha(H) X_\alpha \text{ for } H \in \mathfrak{h}; \\ [X_\alpha, X_\beta] &= 0 \quad \text{if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \notin \Delta; \\ [X_\alpha, X_\beta] &= N_{\alpha,\beta} X_{\alpha,\beta}, \quad \text{if } \alpha + \beta \in \Delta \end{aligned}$$

where the constant $N_{\alpha,\beta}$ is integral and satisfies

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}$$

The $\{H_\alpha, \; X_\alpha, \; X_{-\alpha},\}_{\alpha\in\Delta}$ form the Chevalley-Weyl basis.
To see that every semisimple Lie algebra has a Chevalley-Weyl basis see on [3] page 176.

Theorem 2.16. *Every semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has a real form which is compact.*

Sketch of proof. Let κ denote the Killing form of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and Δ the root system. For each $\alpha \in \Delta$ we select $X_\alpha \in \mathfrak{g}_\alpha$. Since $[X_\alpha, X_{-\alpha}] = H_\alpha$ implies $\kappa(X_\alpha, X_{-\alpha}) = 1$, see Theorem 1.15 and consequently

$$\kappa(X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha}) = -2$$

$$\kappa(i(X_\alpha + X_{-\alpha}), i(X_\alpha + X_{-\alpha})) = -2$$

$$\kappa(X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha})) = 0$$

$$\kappa(iH_\alpha, iH_\alpha) = -\alpha(H_\alpha) < 0$$

Since $\kappa(X_\alpha, X_\beta) = 0$ if $\alpha + \beta \neq 0$, it follows that κ is strictly negative definite on the $\mathbb{R}$-linear subspace

$$\mathfrak{u}_0 = \sum_{\alpha \in \Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha \in \Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha \in \Delta} \mathbb{R}(i(X_\alpha + X_{-\alpha}))$$

Moreover $\mathfrak{g} = \mathfrak{u}_0 \oplus i\mathfrak{u}_0$. Using $N_{\alpha,\beta} = -N_{\alpha,-\beta}$, where $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$, see Remark 2.15 which implies that $N_{\alpha,\beta}$ is real, we see that $X, Y \in \mathfrak{u}_0$ implies $[X, Y] \in \mathfrak{u}_0$, so $\mathfrak{u}_0$ is a real form of $\mathfrak{g}$, and due to the fact that the Killing form is strictly negative, we have that it is compact. □

2.3 Cartan Decomposition

In this section we will see what a Cartan decomposition and a Cartan involution are, the connection between them and the connection to the maximal compact subalgebra of a given real Lie algebra.

Definition 2.17. Let $\mathfrak{g}_0$ be a semisimple Lie algebra over $\mathbb{R}$, $\mathfrak{g}$ its complexification, σ the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. A direct decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$ into a subalgebra $\mathfrak{k}_0$ and a vector subspace $\mathfrak{p}_0$ is called a *Cartan decomposition* if there exists a compact real form $\mathfrak{u}_0$ of $\mathfrak{g}$ such that

$$\sigma \cdot \mathfrak{u} \subset \mathfrak{u}, \quad \mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{u}_o, \quad \mathfrak{p}_0 = \mathfrak{g}_0 \cap (i\mathfrak{u}_o)$$

Since every semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has a real form which is compact, we will see, in this section, that every semisimple Lie algebra $\mathfrak{g}_0$ over $\mathbb{R}$ has a Cartan decomposition.

Definition 2.18. An involutive automorphism θ of a semisimple Lie algebra $\mathfrak{g}_0$ is called a *Cartan involution* if the bilinear form $\kappa(X, \theta Y)$ is strictly positive definite.

Let be $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$ and let $R(\mathfrak{g})$ the set of its real forms.
Lets us define also $\mathrm{AInv}(\mathfrak{g}) = \{\sigma : \mathfrak{g} \to \mathfrak{g} \mid \sigma^2 = \mathrm{Id}, \sigma \text{ antilinear}\}$.
It is obvious that there exists a bijection between this two sets. Our goal is to show what kind of relationship exists between $R(\mathfrak{g})$ and the Cartan involutions. If we consider $u \subset \mathfrak{g}$, (we can consider it bacaues of Theorem 2.16), we can define $\tau_u \in AInv(\mathfrak{g})$ as the conjugation with respect to $\mathfrak{u}$ and $\xi_\sigma = \sigma \circ \tau_u$ where $\sigma \in AInv(\mathfrak{g})$.
Last but not least also define the following set:

$$A(\mathfrak{g}) = \{\eta \in \mathrm{Int}(\mathfrak{g}) | \eta^{-1} = \tau_u \circ \eta \circ \tau_u\}$$

Theorem 2.19. *The map:*

$$\begin{aligned} AInv(\mathfrak{g}) &\to A(\mathfrak{g}) \\ \sigma &\to \xi_\sigma = \sigma \circ \tau_u \end{aligned}$$

is a bijection and $\xi_\sigma \in AInv(\mathfrak{g})$ *if and only if* $\sigma \circ \tau_u = \tau_u \circ \sigma$.

Proof. To prove that $\xi_\sigma \in A(\mathfrak{g})$ we have to check if $\xi_\sigma^{-1} = \tau_u \circ \xi_\sigma \circ \tau_u$, but $\xi_\sigma^{-1} = \tau_u \circ \sigma \circ \tau_u \circ \tau_u$, so the inverse map is $\xi \circ \tau_u = \sigma_\xi$.
We have now to see the second point of our statement, it is easy to see because we have only to see what ξ_σ^2 is:

$$\begin{aligned} \xi_\sigma^2 &= Id \\ \sigma \circ \tau_u \circ \sigma \circ \tau_u &= Id \\ \sigma \circ \tau_u &= \tau_u \circ \sigma \end{aligned}$$

□

We have now to see how the group $\mathrm{Int}(\mathfrak{g})$ acts on the sets.

Proposition 2.20. *Let be* $\alpha \in \mathrm{Int}(\mathfrak{g})$*, and* $\mathrm{R}(\mathfrak{g})$, $\mathrm{AInv}(\mathfrak{g})$ and $\mathrm{A}(\mathfrak{g})$ *as in the previous definition. Then we have the well defined actions:*

i. $\alpha \circ g_0 \overset{def}{=} \alpha(g_0),\ g_0 \in R(\mathfrak{g})$

ii. $\alpha \circ \sigma \overset{def}{=} \alpha \circ \sigma \circ \alpha^{-1},\ \sigma \in AInv(\mathfrak{g})$

iii. $\alpha \circ \xi \overset{def}{=} \alpha \circ \xi \circ (\tau_u \circ \alpha^{-1} \circ \tau_u),\ \xi \in A(\mathfrak{g})$

Proof. The first point is obvious that is well defined so let us check the second and the third one and also if the actions respect the bijections. ii)

$$\begin{aligned}
&(\alpha \circ \sigma)^2 \stackrel{?}{=} Id \\
&(\alpha \circ \sigma)^2 = \alpha \circ \sigma \circ \alpha^{-1} \circ \alpha \circ \sigma \circ \alpha^{-1} \\
&(\alpha \circ \sigma)^2 = \alpha \circ \sigma \circ Id \circ \sigma \circ \alpha^{-1} \\
&(\alpha \circ \sigma)^2 = \alpha \circ Id \circ \alpha^{-1} \\
&(\alpha \circ \sigma)^2 = Id
\end{aligned}$$

iii)

$$\begin{aligned}
(\alpha \circ \xi \circ (\tau_u \circ \alpha^{-1} \circ \tau_u))^{-1} &= (\tau_u \circ \alpha^{-1} \circ \tau_u)^{-1} \circ \xi^{-1} \circ \alpha^{-1} \\
&= \tau_u \circ \alpha \circ \tau_u \circ \xi^{-1} \circ \alpha^{-1} \\
&= \tau_u \circ \alpha \circ \tau_u \circ \tau_u \circ \xi \circ \tau_u \circ \alpha^{-1} \\
&= \tau_u \circ \alpha \circ \xi \circ \tau_u \circ \alpha^{-1} \\
&= \tau_u \circ (\alpha \circ \xi) \circ \tau_u
\end{aligned}$$

And now we will check that the actions respect the bijections: Consider the map:

$$\begin{aligned}
AInv(\mathfrak{g}) &\to A(\mathfrak{g}) \\
\sigma &\to \sigma \circ \tau_u = \xi_\sigma \\
\alpha \circ \sigma &\stackrel{?}{\to} \alpha \circ \xi_\sigma
\end{aligned}$$

If we remember that $\alpha \circ \sigma = \alpha \circ \sigma \circ \alpha^{-1}$ we have:

$$\begin{aligned}
\alpha \circ \sigma \circ \alpha^{-1} \circ \tau_u &= \alpha \circ \sigma \circ \tau_u \circ \tau_u \circ \alpha^{-1} \circ \tau_u \\
&= \alpha \circ \xi_\sigma \circ \tau_u \circ \alpha^{-1} \circ \tau_u
\end{aligned}$$

Meanwhile if we consider the inverse map:

$$\begin{aligned}
A(\mathfrak{g}) &\to AInv(\mathfrak{g}) \\
\xi &\to \xi \circ \tau_u \\
\alpha \circ \xi &\stackrel{?}{\to} \alpha \circ \xi \circ \tau_u
\end{aligned}$$

This is easy to verify because: $\alpha \circ \xi \circ \tau_u \circ \alpha^{-1} \circ \tau_u \circ \tau_u = \alpha \circ \xi \circ \tau_u \circ \alpha^{-1}$ □

Thanks to these verification we can now give the following theorem.

Theorem 2.21.

$$\frac{AInv(\mathfrak{g})}{\mathrm{Int}(\mathfrak{g})} \cong \frac{A(\mathfrak{g})}{\mathrm{Int}(\mathfrak{g})} \cong \frac{R(\mathfrak{g})}{\mathrm{Int}(\mathfrak{g})}$$

Thanks to this Cartan's result we know that if we have $\sigma \in AInv(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ exists an alement ξ_σ which is a Cartan involution for every equivalence class $[\sigma]$, we post pone the proof later because we have to see some preliminar results:

Lemma 2.22. *Let $\mathfrak{g}_0$ be in $R(\mathfrak{g})$, so there is an $\alpha \in \mathrm{Int}(\mathfrak{g})$ such that:*

1. $\sigma_{\alpha(g_0)} \circ \tau_u = \tau_u \circ \sigma_{\alpha(g_0)}$;
2. $\sigma_{\alpha(g_0)} \circ \tau_u | \alpha(g_0)$ *is a Cartan involution of* $\alpha(g_0)$

Proof. The Hermitian form κ_{τ_u} on $\mathfrak{g} \times \mathfrak{g}$ given by

$$\kappa_{\tau_u}(X, Y) = -\kappa(X, \tau_u Y), \quad X, Y \in \mathfrak{g}$$

is strictly positive definite since $\mathfrak{u}$ is compact. The linear transformation $N = \sigma\tau_u$ where σ is the conjugation of $\mathfrak{g}$ with respect to $R(\mathfrak{g})$ is an automorphism of the complex algebra $\mathfrak{g}$ and hence leaves the Killing form invariant. Using $\sigma^2 = \tau_u^2 = Id$ we obtain:

$$\kappa(NX, \tau Y) = \kappa(X, N^{-1}\tau Y) = \kappa(X, \tau NY)$$

or

$$\kappa_{\tau_u}(NX, Y) = \kappa_{\tau_u}(X, NY)$$

This shows that N is self-adjoint with respect to κ_{τ_u}. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ with respect to which N is represented by a diagonal matrix. Then the endomorphism $P = N^2$ is represented by a diagonal matrix with positive diagonal elements $\lambda_1, \ldots, \lambda_n$. For each $t \in \mathbb{R}$, let P^t denote the linear transformation of $\mathfrak{g}$ represented by the diagonal matrix with diagonal elements $(\lambda_i)^t > 0$. Then each P^t commutes with N. Let c_{ij}^k denote the constants determined by

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$$

for $1 \leq i, j \leq n$. Since P is an automorphism, we have

$$\lambda_i \lambda_j c_{ij}^k = (\lambda_k) c_{ij}^k, \quad (1 \leq i, j \leq n)$$

This equation implies

$$(\lambda_i)^t(\lambda_j)^t c_{ij}^k = (\lambda_k)^t c_{ij}^k, \quad (t \in \mathbb{R}),$$

which shows that each P^t is an automorphism of $\mathfrak{g}$.
Consider now the mapping $\tau_1 = P^t \tau_u P^{-t}$ of $\mathfrak{g}$ into itself. The subspace $P^t u$ is a compact real form of $\mathfrak{g}$ and τ_1 is the conjugation of $\mathfrak{g}$ with respect to this form. Moreover we have $\tau_u N \tau_u^{-1} = N^{-1}$ so $\tau_u N \tau_u^{-1} = P^{-1}$. By a simple matrix computation the relation $\tau_u P = P^{-1}\tau_u$ implies $\tau_u P^t = P^{-t}\tau_u$ for all $t \in \mathbb{R}$. Consequently,

$$\begin{aligned}\sigma\tau_1 &= \sigma P^t \tau_u P^{-t} = \sigma\tau_u P^{-2t} = NP^{-2t}\\ \tau_1\sigma &= (\sigma\tau_1)^{-1} = P^{2t}N^{-1} = N^{-1}P^{2t}.\end{aligned}$$

If $t = \frac{1}{4}$ then $\sigma\tau_1 = \tau_1\sigma$. Thus the automorphism $\alpha = P^{\frac{1}{4}}$ has the desired properties. $\mathfrak{k}_0$ is compactly imbedded in $\mathfrak{g}_0$, and it is maximal. If $\mathfrak{k}_0$ were not maximal, let $\mathfrak{k}_1$ be a compactly imbedded subalgebra of $\mathfrak{g}_0$, properly containing $\mathfrak{k}_0$. Then there exists an element $X \neq 0$ in $\mathfrak{k}_1 \cap \mathfrak{p}_0$. Then $\tau_u \mathfrak{g}_0 \subset \mathfrak{g}_0$ and the bilinear form, as we have already seen, is symmetric and strictly positive definite. Since:

$$\kappa([X,Y], \tau_u Z) = -\kappa(Y, [X, \tau_u Z]) = \kappa(Y, [\tau_u X, \tau_u Z])$$

we have

$$\kappa_{\tau_u}(\mathrm{ad}X(Y), Z) = \kappa_{\tau_u}(Y, \mathrm{ad}X(Z).$$

Thus $\mathrm{ad}X$ has all its eigenvalues real, and not all zero. But then the power $e^{\mathrm{ad}X}$ can not lie in a compact matrix group. This contraddicts the fact that $\mathfrak{k}_1$ is a compactly embedded subalgebra of $\mathfrak{g}_0$. □

Theorem 2.23. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{u}_0$ be a compact real form of $\mathfrak{g}$, and let τ be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}_0$. If $\mathfrak{g}$ is regarded as a real Lie algebra $\mathfrak{g}^{\mathbb{R}}$, then τ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$.*

Proof. It is clear that τ is an involution. The Killing forms $\kappa_{\mathfrak{g}}$ of $\mathfrak{g}$ and $\kappa_{\mathfrak{g}^{\mathbb{R}}}$ are related by

$$\kappa_{\mathfrak{g}^{\mathbb{R}}}(Z_1, Z_2) = 2\mathrm{Re}\kappa_{\mathfrak{g}}(Z_1, Z_2),$$

see Lemma 2.10. Write $Z \in \mathfrak{g}$ as $X + iY$ with $X, Y \in \mathfrak{u}_0$. Then

$$\begin{aligned}\kappa_{\mathfrak{g}}(Z, \tau Z) &= \kappa_{\mathfrak{g}}(X + iY, x - iY)\\ &= \kappa_{\mathfrak{g}}(X, X) + \kappa_{\mathfrak{g}}(Y, Y)\\ &= \kappa_{\mathfrak{u}_0}(X, X) + \kappa_{\mathfrak{u}_0}(Y, Y)\end{aligned}$$

and the right side is < 0 unless $Z = 0$. It follows that

$$(\kappa_{\mathfrak{g}^{\mathbb{R}}})_\tau(Z_1, Z_2) = -\kappa_{\mathfrak{g}^{\mathbb{R}}}(Z_1, \tau Z_2) = -2Re\kappa_{\mathfrak{g}}(Z_1, \tau Z_2)$$

is positive definite on $\mathfrak{g}^{\mathbb{R}}$, and therefore τ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$. □

Corollary 2.24. *If $\mathfrak{g}_0$ is a real Lie algebra, then $\mathfrak{g}_0$ has a Cartan involution*

Proof. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$, and choose a compact real form $\mathfrak{u}$ of $\mathfrak{g}$. Let σ and τ be the conjugations of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and $\mathfrak{u}_0$. If we regard $\mathfrak{g}$ as a real Lie algebra $\mathfrak{g}^{\mathbb{R}}$, then σ and τ are involutions of $\mathfrak{g}^{\mathbb{R}}$ and the previous theorem shows that τ is a Cartan involution, so we can find $\varphi \in \mathrm{Int}(\mathfrak{g}^{\mathbb{R}}) = \mathrm{Int}\mathfrak{g}$ such that $\varphi\tau\varphi^{-1}$ commutes with σ, see Theorem 2.21
Here $\varphi\tau\varphi^{-1}$ is the conjugation of $\mathfrak{g}$ with respect to $\varphi(\mathfrak{u}_0)$, which is another real form of $\mathfrak{g}$. Thus

$$(\kappa_{\mathfrak{g}^{\mathbb{R}}})_{\varphi\tau\varphi^{-1}}(Z_1, Z_2) = -2Re\kappa_{\mathfrak{g}}(Z_1, \varphi\tau\varphi^{-1}Z_2)$$

is positive definite on $\mathfrak{g}^{\mathbb{R}}$.
The Lie algebra $\mathfrak{g}_0$ is characterized as the fixed set of σ. If $\sigma X = X$, then

$$\sigma(\varphi\tau\varphi^{-1}X) = \varphi\tau\varphi^{-1}\sigma X = \varphi\tau\varphi^{-1}X.$$

Hence $\varphi\tau\varphi^{-1}$ restricts to an involution θ of $\mathfrak{g}_0$. We have

$$-\kappa_{\mathfrak{g}_0}(X, \theta Y) = -\kappa_{\mathfrak{g}}(X, \varphi\tau\varphi^{-1}Y) = \frac{1}{2}(\kappa_{\mathfrak{g}^{\mathbb{R}}})_{\varphi\tau\varphi^{-1}}(X, Y).$$

Thus B_θ is positive definite on $\mathfrak{g}_0$, and θ is a Cartan involution. □

Corollary 2.25. *If $\mathfrak{g}_0$ is a real semisimple Lie algebra, then any two Cartan involutions of $\mathfrak{g}_0$ are conjugate via* $\mathrm{Int}\mathfrak{g}_0$.

Proof. Let θ and θ' be two Cartan involutions. Taking $\sigma = \theta'$ in Theorem 2.22, we can find $\varphi \in \mathrm{Int}(\mathfrak{g}_0)$ such that $\varphi\theta\varphi^{-1}$ commutes with θ'. Here $\varphi\theta\varphi^{-1}$ is another Cartan involution of $\mathfrak{g}_0$. So we may as well assume that θ and θ' commute from the outset. We shall prove that $\theta = \theta'$.
Since θ and θ' commute, they have compatible eigenspace decomposition into $+1$ and -1 eigenspaces. By simmetry it is enough to show that no nonzero $X \in \mathfrak{g}_0$ is in the $+1$ eigenspace for θ and the -1 eigenspace for θ'. Assuming the contrary, suppose that $\theta X = X$ and $\theta' X = -X$. Then we have

$$0 < -\kappa(X, \theta X) = -\kappa(X, X)$$
$$0 < -\kappa(X, \theta' X) = +\kappa(X, X),$$

contradiction. We conclude that $\theta = \theta'$. □

Corollary 2.26. *If $\mathfrak{g}$ is a complex semisimple Lie algebra, then any two compact real forms of $\mathfrak{g}$ are conjugate via* $\mathrm{Int}\mathfrak{g}$.

Proof. Each compact real form has an associated conjugation of $\mathfrak{g}$ that determines it, and this conjugation is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$, by Theorem 2.23. Applying Corollary 2.25 to $\mathfrak{g}^{\mathbb{R}}$, we see that the two conjugations are conjugated by a member of $\mathrm{Int}(\mathfrak{g}^{\mathbb{R}})$. Since $\mathrm{Int}(\mathfrak{g}^{\mathbb{R}}) = \mathrm{Int}(\mathfrak{g})$ the corollary follows. □

Corollary 2.27. *If $\mathfrak{g}$ is a complex semisimple Lie algebra, then the only Cartan involutions of $\mathfrak{g}^{\mathbb{R}}$ are the conjugations with respect to the compact real forms of $\mathfrak{g}$.*

Proof. Theorem 2.16 and Theorem 2.23 produce a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$ that is conjugations with respect to some compact real form of $\mathfrak{g}$. Any other Cartan involution is conjugate to this one, according to Corollary 2.25, and hence is also the conjugation with respect to a compact real form of $\mathfrak{g}$. □

A Cartan involution θ of $\mathfrak{g}_0$ yields the eigenspace decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

of $\mathfrak{g}_0$ into ± 1 eigenspaces, and these must bracket according to the rules

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, \quad [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0 \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0$$

Since θ is an involution,

$$\mathfrak{k}_0 \text{ and } \mathfrak{p}_0 \text{ are orthogonal under } \kappa_{\mathfrak{g}_0} \text{ and under } \kappa_\theta.$$

In fact, if X is in $\mathfrak{k}_0$ and Y is in $\mathfrak{p}_0$, then $\mathrm{ad}X\mathrm{ad}Y$ carries $\mathfrak{k}_0$ to $\mathfrak{p}_0$ and viceversa. Thus it has trace 0, and $\kappa_{\mathfrak{g}_0}(X, Y) = 0$; since $\theta Y = -Y, \kappa_\theta(X, Y) = 0$ also. Since κ_θ is positive definite, the eigenspaces $\mathfrak{k}_0$ nd $\mathfrak{p}_0$ have the property that

$$\kappa_{\mathfrak{g}_0}, \text{ is} \begin{cases} \text{negative definite on } \mathfrak{k}_0 \\ \text{positive definite on } \mathfrak{p}_0 \end{cases}$$

so we have a Cartan decomposition. Conversely a Cartan decomposition determines a Cartan involution θ by the formula

$$\theta = \begin{cases} +1 \text{ on } \mathfrak{k}_0 \\ -1 \text{ on } \mathfrak{p}_0 \end{cases}$$

Proposition 2.28. *Let $\mathfrak{g}_0$ a real semisimple Lie algebra and $\mathfrak{g}$ its complexification, θ is a Cartan involution of $\mathfrak{g}_0$ if and only if:*

1)

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0,$$

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, \quad [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0 \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0;$$

2) $\kappa_{\mathfrak{g}_0}$ is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$

Proof. Let θ be a Cartan involution and $\mathfrak{k}_0$, $\mathfrak{p}_0$ the ± 1 eigenspaces, then 1) and 2) are clear. We now have to show that an involution satisfying 1) and 2) is a Cartan decomposition.
This implication is also easy because we have only to give a compact real form which has the desired properties, this compact, see the proof of Theorem 2.16 where we show why this is compact, real form is $\mathfrak{u} = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ and we consider the conjugation σ of $\mathfrak{g}$ respect to $\mathfrak{g}_0$. □

Lemma 2.29. *If $\mathfrak{g}_0$ is a real semisimple Lie algebra and θ is a Cartan involution, then*

$$(\mathrm{ad}X)^* = -\mathrm{ad}\theta X, \quad \forall X \in \mathfrak{g}_0$$

where adjoint $(\cdot)^*$ is defined relative to the inner product κ_θ.

Proof. We have

$$\begin{aligned} B_\theta((\mathrm{ad}\theta X)Y, Z) &= -\kappa([\theta X, Y], \theta Z) \\ &= \kappa(Y, [\theta X, \theta Z]) = \kappa(Y, \theta[X, Z]) \\ &= -\kappa_\theta(Y, (\mathrm{ad}X)Z) = -\kappa_\theta((\mathrm{ad}X)^* Y, Z). \end{aligned}$$

□

Proposition 2.30. *If $\mathfrak{g}_0$ is a real semisimple Lie algebra, then $\mathfrak{g}_0$ is isomorphic to a Lie algebra of a real matrices that is close under transpose. If a Cartan involution θ of $\mathfrak{g}_0$ has been specified, then the isomorphism may be chosen so that θ is carried to negative transpose.*

Proof. Let θ be a Cartan involution of $\mathfrak{g}_0$ and define the inner product B_θ on $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is semisimple, $\mathfrak{g}_0 \cong \mathrm{ad}\mathfrak{g}_0$. The matrices of $\mathrm{ad}\mathfrak{g}_0$ in an orthonormal basis relative to κ_θ will be the required Lie algebra of matrices. We have only to show that $\mathrm{ad}\mathfrak{g}_0$ is close under adjoint. But this follows from Lemma 2.29 and the fact that $\mathfrak{g}_0$ is closed under θ. □

We have already seen that every real semisimple Lie algebra has a Cartan subalgebra, now we want to investigate the conjugacy class of Cartan subalgebras and some their relationship to each other.

Proposition 2.31. *Any Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is conjugate via* $\mathrm{Int}\mathfrak{g}_0$ *to a θ stable Cartan subalgebra.*

Proof. Let $\mathfrak{h}$ be the complexification of $\mathfrak{h}_0$, and let σ be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Let $\mathfrak{u}_0$ be the compact real form constructed from $\mathfrak{h}$ and let τ be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}_0$. The construction of $\mathfrak{u}_0$ has the property that $\tau(\mathfrak{h}) = \mathfrak{h}$. The conjugations σ and τ are involutions of $\mathfrak{g}^{\mathbb{R}}$, and τ is a Cartan involution by Theorem 2.23. The conjugations σ and τ are involutions of $\mathfrak{g}^{\mathbb{R}}$, and τ is a Cartan involution. Lemma 2.22 shows that the element φ of $\mathrm{Int}\mathfrak{g}^{\mathbb{R}} = \mathrm{Int}\mathfrak{g}$ given by $\varphi = ((\sigma\tau)^2)^{\frac{1}{4}}$ has the property that the Cartan involution $\tilde{\eta} = \varphi\tau\varphi^{-1}$ of $\mathfrak{g}^{\mathbb{R}}$ commutes with σ. Since $\sigma(\mathfrak{h}) = \mathfrak{h}$ and $\tau(\mathfrak{h}) = \mathfrak{h}$, it follows that $\varphi(\mathfrak{h}) = \mathfrak{h}$. Therefore $\tilde{\eta}(\mathfrak{h}) = \mathfrak{h}$.
Since $\tilde{\eta}$ and σ commute, it follows that $\tilde{\eta}(\mathfrak{g}_0) = \mathfrak{g}_0$. Since $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$, we obtain $\tilde{\eta}(\mathfrak{h}_0) = \mathfrak{h}_0$.
Put $\eta = \tilde{\eta}|_{\mathfrak{g}_0}$, so that $\eta(\mathfrak{h}_0) = \mathfrak{h}_0$. Since $\tilde{\eta}$ is the conjugation of $\mathfrak{g}$ with respect to the compac real form $\varphi(\mathfrak{u}_0)$, the proof of Corollary 2.24 shows that η is a Cartan involution of $\mathfrak{g}_0$. Corollary 2.25 shows that η and θ are conjugate cia $\mathrm{Int}\mathfrak{g}_0$, say $\theta = \psi\eta\psi^{-1}$ with $\psi \in \mathrm{Int}\mathfrak{g}_0$. Then $\psi(\mathfrak{h}_0)$ is a Cartan subalgebra of $\mathfrak{g}_0$, and

$$\theta(\psi(\mathfrak{h}_0)) = \psi\eta\psi^{-1}\psi(\mathfrak{h}_0) = \psi(\eta\mathfrak{h}_0) = \psi(\mathfrak{h}_0).$$

shows that it is θ stable. □

Thus it is sufficient to study θ stable Cartan subalgebras. When $\mathfrak{h}_0$ is θ stable, we can write $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ with $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$. We can define the *compact dimension* as $\dim\mathfrak{t}_0$ and the *noncompact dimension* as $\dim\mathfrak{a}_0$ which are unchanged when $\mathfrak{h}_0$ is conjugated via $\mathrm{Int}\mathfrak{g}_0$ to another θ stable Cartan subalgebra.
We say that a θ stable subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ is *maximally compact* if its compact dimension is as large as possible, *maximally noncompact* if its noncompact dimension is as large as possible.

Theorem 2.32. *Let $\mathfrak{t}_0$ be a maximal abelian subspace of $\mathfrak{k}_0$. Then $\mathfrak{h}_0 = Z_{\mathfrak{g}_0}(\mathfrak{t}_0)$ is a stable Cartan subalgebra of $\mathfrak{g}_0$ of the form $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ with $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$.*

Proof. The subalgebra $\mathfrak{h}_0$ is θ stable and hence is a vector space direct sum $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, where $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0$. Since $\mathfrak{h}_0$ is θ stable, it is reductive and $[\mathfrak{h}_0, \mathfrak{h}_0]$

is semisimple. We have $[\mathfrak{h}_0, \mathfrak{h}_0] = [\mathfrak{a}_0, \mathfrak{a}_0]$, and $[\mathfrak{a}_0, \mathfrak{a}_0] \subseteq \mathfrak{t}_0$ since $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$ and $\mathfrak{h}_0 \cap \mathfrak{k}_0 = \mathfrak{t}_0$. Thus the semisimple Lie algebra $[\mathfrak{h}_0, \mathfrak{h}_0]$ is abelian and must be 0. Consequently $\mathfrak{h}_0$ is abelian.

It is clear that $\mathfrak{h} = (\mathfrak{h}_0)^{\mathbb{C}}$ is maximal abelian in $\mathfrak{g}$, and $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{t}_0)$ are skrew adjoint, the members of $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{a}_0)$ are selfadjoint, and $\mathfrak{t}_0$ commutes with $\mathfrak{a}_0$. Finally, we have that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and hence $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. □

With any θ stable subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, $\mathfrak{t}_0$ is an abelian subspace of $\mathfrak{k}_0$, so $\mathfrak{h}_0$ is maximally compact if and only if $\mathfrak{t}_0$ is a maximal abelian subspace of $\mathfrak{k}_0$.

Proposition 2.33. *Among θ stable Cartan subalgebras $\mathfrak{h}_0$ of $\mathfrak{g}_0$, the maximally noncompact ones are all conjugate via K, and the maximally compact ones are all conjugate via K, where $K = \mathrm{Int}_{\mathfrak{g}_0}(\mathfrak{k}_0)$.*

Proof. Let $\mathfrak{h}_0$ and $\mathfrak{h}_0'$ be given Cartan subalgebras. In the first case, as we observed above, $\mathfrak{h}_0 \cap \mathfrak{p}_0$ and $\mathfrak{h}_0' \cap \mathfrak{p}_0$ are maximal abelian in $\mathfrak{p}_0$ and there is no loss of generality in assuming that $\mathfrak{h}_0 \cap \mathfrak{p}_0 = \mathfrak{h}_0' \cap \mathfrak{p}_0$. Thus $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ and $\mathfrak{h}_0' = \mathfrak{t}_0' \oplus \mathfrak{a}_0$ where $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}_0$. Define $\mathfrak{m}_0 = Z_{\mathfrak{k}_0}(\mathfrak{a}_0$. Then $\mathfrak{t}_0$ and $\mathfrak{t}_0'$ are in $\mathfrak{m}_0$ and are maximal abelian there. Let $M = Z_K(\mathfrak{a}_0)$. This is a compact subgroup of K with Lie algebra $\mathfrak{m}_0$, and we let M_0 be its identity component. Now we have that $\mathfrak{t}_0$ and $\mathfrak{t}_0'$ are conjugate via M_0, and this conjugacy clearly fixes $\mathfrak{a}_0$. Hence $\mathfrak{h}_0$ and $\mathfrak{h}_0'$ are conjugate via K.

In the second case, $\mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{h}_0' \cap \mathfrak{k}_0$ are maximal abelian in $\mathfrak{k}_0$, so we can assume that $\mathfrak{h}_0 \cap \mathfrak{k}_0 = \mathfrak{h}_0' \cap \mathfrak{k}_0$. So Theorem 2.32 shows that $\mathfrak{h}_0 = \mathfrak{h}_0'$ and the proof is complete. □

If we examine the proof of the first part of this last proposition, we find that we can adjust it to obtain root data that determine a Cartan subalgebra up to conjugacy. As a consequence there are only finitely many conjugacy classes of Cartan subalgebras.

Lemma 2.34. *Let $\mathfrak{h}_0$ and $\mathfrak{h}_0'$ be θ stable Cartan subalgebras of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \cap \mathfrak{p}_0 = \mathfrak{h}_0' \cap \mathfrak{p}_0$. Then $\mathfrak{h}_0$ and $\mathfrak{h}_0'$ are conjugate via K.*

Proof. Since the $\mathfrak{p}_0$ parts of two Cartan subalgebras are the same and since Cartan subalgebras are abelian, the $\mathfrak{k}_0$ parts $\mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{h}_0' \cap \mathfrak{k}_0$ are both contained in $\tilde{\mathfrak{m}} = Z_{\mathfrak{k}_0}(\mathfrak{h}_0 \cap \mathfrak{p}_0)$. The Cartan subalgebras are maximal abelian in $\mathfrak{g}_0$. Let $\widetilde{M} = Z_K(\mathfrak{h}_0 \cap \mathfrak{p}_0)$. This is a compact Lie group with Lie algebra $\tilde{m}_0$, and we let $\widetilde{M_0}$ be its identity component, so we have that $\mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{h}_0' \cap \mathfrak{k}_0$ are conjugate via $\widetilde{M_0}$, and this conjugacy clearly fixes $\mathfrak{h}_0 \cap \mathfrak{p}_0$. Hence $\mathfrak{h}_0$ and $\mathfrak{h}_0'$ are conjugate via K. □

Lemma 2.35. *Let* $\mathfrak{a}_0$ *be a maximal abelian subspace of* $\mathfrak{p}_0$*, and let* Σ *be the set of restricted-roots of* $(\mathfrak{g}_0, \mathfrak{a}_0)$*. Suppose that* $\mathfrak{h}_0$ *is a* θ *stable Cartan subalgebra such that* $\mathfrak{h}_0 \cap \mathfrak{p}_0 \subseteq \mathfrak{a}_0$*. Let* $\Sigma' = \{\lambda(\mathfrak{h}_0 \cap \mathfrak{p}_0) = 0\}$*. Then* $\mathfrak{h}_0 \cap \mathfrak{p}_0$ *is the common kernel of all* $\lambda \in \Sigma'$*.*

Proof. Let $\mathfrak{a}_0'$ be the common kernel of all $\lambda \in \Sigma'$. Then $\mathfrak{h}_0 \cap \mathfrak{p}_0 \subseteq \mathfrak{a}_0'$, and we are to prove that equality holds. Since $\mathfrak{h}_0$ is a maximal abelian in $\mathfrak{g}_0$, it is enough to prove that $\mathfrak{h}_0 + \mathfrak{a}_0'$ is abelian.
Let $\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0 \oplus \bigoplus_{\lambda\in\Sigma}(\mathfrak{g}_0)_\lambda$ be the restricted-root space decomposition of $\mathfrak{g}_0$, and let $X = H_0 + X_0 + \sum_{\lambda\in\Sigma} X_\lambda$ be an element of $\mathfrak{g}_0$ that centralize $\mathfrak{h}_0 \cap \mathfrak{p}_0$. Bracketing the formula for X with $H \in \mathfrak{h}_0 \cap \mathfrak{p}_0$, we obtain $0 = \sum_{\lambda\in\Sigma-\Sigma'} \lambda(H)X_\lambda$, from which we conclude that $\lambda(H)X_\lambda = 0$ for all $H \in \mathfrak{h}_0 \cap \mathfrak{p}_0$ and all $\lambda \in \Sigma - \Sigma'$. Since that λ's in $\Sigma - \Sigma'$ have $\lambda(\mathfrak{h}_0 \cap \mathfrak{p}_0)$ not identically 0, we see that $X_\lambda = 0$ for all $\lambda \in \Sigma - \Sigma'$. Thus any X that centralize $\mathfrak{h}_0 \cap \mathfrak{p}_0$ is of the form

$$X = H_0 + X_0 + \sum_{\lambda\in\Sigma'} X_\lambda$$

Since $\mathfrak{h}_0$ is abelian, the elements $X \in \mathfrak{h}_0$ are of his form, and $\mathfrak{a}_0'$ commutes with any X of this form. Hence $\mathfrak{h}_0 + \mathfrak{a}_0'$ is abelian, and the proof is complete. □

Proposition 2.36. *Up to conjugacy by* Int$\mathfrak{g}_0$*, there are only finitely many Cartan subalgebras of* $\mathfrak{g}_0$*.*

Proof. Fix a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}_0$. Let $\mathfrak{h}_0$ be a Cartan subalgebra. Without loss of generality we can assume that $\mathfrak{h}_0$ is θ stable and that $\mathfrak{h}_0 \cap \mathfrak{p}_0$ is contained in $\mathfrak{a}_0$. Lemma 2.35 associates to a $\mathfrak{h}_0$ a subset of the set Σ of restricted roots that determines $\mathfrak{h}_0 \cap \mathfrak{p}_0$, and Lemma 2.34 shows that $\mathfrak{h}_0 \cap \mathfrak{p}_0$ determines $\mathfrak{h}_0$ up to conjugacy. Hence the number of conjugacy classes of Cartan subagebras is bounded by the number of subset of Σ. □

2.4 Vogan Diagrams

We want to associate to a real Lie algebra $\mathfrak{g}_0$ a diagram consisting of the Dynkin diagram of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ with some additional information superimposed. This diagram will be called *Vogan diagram.*
Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, let $\mathfrak{g}$ be its complexification, let θ be a Cartan involution, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition, and let κ the associated Killing form. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we assume that

$$\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}.$$

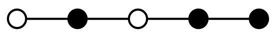
This makes $\mathfrak{h}$ compact and θ stabilizes all of root spaces, fact that it cannot always do. We have that $\Delta = \Delta_{\mathfrak{k}} \oplus \Delta_{\mathfrak{p}}$ where $\Delta_{\mathfrak{k}}$ are the compact roots and $\Delta_{\mathfrak{p}}$ the noncompact ones and that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and

$$\theta \text{ is equal to } 1 \text{ on } \mathfrak{k} \text{ and to } -1 \text{ on } \mathfrak{p}.$$

For the general case see on [2], chapter VI section 8.

Definition 2.37. Let $\mathfrak{g}_0$ a real Lie algebra and $\mathfrak{g}$ its complexification. We define the *Vogan diagram of the triple* $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ as the Dynkin diagram of Δ^+ with painted or not painted vertices, according as the corresponing simple root is noncompact or compact.

Example 2.38. If $\mathfrak{g}_0 = \mathfrak{su}(3,3)$, let us take θ to be negative conjugate transpose, $\mathfrak{h}_0$ to be the diagonal subalgebra. We have that $\Delta = \{\varepsilon_i - \varepsilon_j | \ 1 \leq i \neq j \leq 6\}$ and Δ^+ to be determined by the conditions $\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_4 \geq \varepsilon_5 \geq \varepsilon_3 \geq \varepsilon_6$, so we have that $S = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_4, \varepsilon_4 - \varepsilon_5, \varepsilon_5 - \varepsilon_3, \varepsilon_3 - \varepsilon_6$. The Dynkin diagram is of type A_5. In particular, θ acts as the identity in the Dynkin diagram. The compact roots $\varepsilon_i - \varepsilon_j$ are those with i and j in the same set $\{1,2,3\}$ or $\{4,5,6\}$, while the noncompact roots are those with i and j in opposite sets. Then among the simple roots, $\varepsilon_1 - \varepsilon_2$ is compact, $\varepsilon_2 - \varepsilon_4$ is noncompact etc. Hence the Vogan diagram is



If we use the standard ordering on the ε_i, with $1 \leq i \leq 6$ and that a root $\varepsilon_i - \varepsilon_j$ is compact if i and j are in the same set $\{1,2,3\}$ or $\{4,5,6\}$, noncompact if i and j are in opposite sets we have that the only noncompact root is $\varepsilon_3 - \varepsilon_4$.

Remark 2.39. Note that if we choose a real form we can have more associated Vogan diagram, but if we choose a Vogan diagram we have only one real form associated to it.

Theorem 2.40. *Let $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ be real semisimple Lie algebras. If two triples $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ and $(\mathfrak{g}_0', \mathfrak{h}_0', (\Delta')^+)$ have the same Vogan diagram, then $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ are isomorphic.*

Remark 2.41. This theorem is an analog for real semisimple Lie algebras of the Isomorphism Theorem 1.33 for complex semisimple Lie algebras.

Proof. Since the Dynkin diagrams are the same, the Isomorphism Theorem 1.33 shows that there is no loss of generality in assuming $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ have the same complexification $\mathfrak{g}$. Let $\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ and $\mathfrak{u}_0' = \mathfrak{k}_0' \oplus i\mathfrak{p}_0'$ be the associated compact real forms of $\mathfrak{g}$. By Corollary 2.25, there exists $x \in \mathrm{Int}\mathfrak{g}$ such that $x\mathfrak{u}_0' = \mathfrak{u}_0$. The real form $x\mathfrak{g}_0'$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_0'$ and has Cartan decomposition $x\mathfrak{g}_0' = x\mathfrak{k}_0' \oplus x\mathfrak{p}_0'$. Since $x\mathfrak{k}_0' \oplus ix\mathfrak{p}_0' = x\mathfrak{u}_0' = \mathfrak{u}_0$, there is no loss of generality in assuming that $\mathfrak{u}_0' = \mathfrak{u}_0$ from the outset. Then

$$\theta(\mathfrak{u}_0) = \mathfrak{u}_0 \quad \text{and} \quad \theta'(\mathfrak{u}_0) = \mathfrak{u}_0$$

Let $\mathfrak{h}_0$ and $\mathfrak{h}_0'$ be the Cartan subalgebras. We have that there exists a $k \in \mathrm{Int}(\mathfrak{u}_0)$ with $k(\mathfrak{h}_0') = \mathfrak{h}_0$. Therefore $\mathfrak{h}_0$ and $\mathfrak{h}_0'$ have the same complexification, which we denote $\mathfrak{h}$.

Now that the complexification $\mathfrak{g}$ and $\mathfrak{h}$ have been aligned, the root systems are the same. Let the positive systems given in the respective triples be Δ^+ and Δ'^+. Now we have that there exists $k' \in \mathrm{Int}\mathfrak{u}_0$ normalizing $\mathfrak{u}_0 \cap \mathfrak{h}$ with $k'\Delta'^+ = \Delta^+$. replacing $\mathfrak{g}_0'$ by $k'\mathfrak{g}_0'$ and arguing as above, we may assume that $\Delta'^+ = \Delta^+$ from the outset. The next step is to choose normalizations of root vectors relative to $\mathfrak{h}$. For this proof let κ be the Killing form of $\mathfrak{g}$. We start with root vectors X_α produced from $\mathfrak{h}$, and then we construct a compact real form $\tilde{\mathfrak{u}}_0$ of $\mathfrak{g}$. The subalgebra $\tilde{\mathfrak{u}}_0$ is just $\mathfrak{u}_0 \cap \mathfrak{h}$. By Corollary 2.25, there exists $g \in \mathrm{Int}\mathfrak{g}$ such that $g\tilde{\mathfrak{u}}_0 = \mathfrak{u}_0$. Then $g\tilde{\mathfrak{u}}_0 = \mathfrak{u}_0$ is built from $g(\mathfrak{u}_0 \cap \mathfrak{h})$ and the root vectors gX_α. Since $\mathfrak{u}_0 \cap \mathfrak{h}$ and $g(\mathfrak{u}_0 \cap \mathfrak{h})$ are maximal abelian in $\mathfrak{u}_0$, there exists $u \in \mathrm{Int}\mathfrak{u}_0$ with $ug(\mathfrak{u}_0 \cap \mathfrak{h}) = \mathfrak{u}_0 \cap \mathfrak{h}$. Then $\mathfrak{u}_0$ is built from $ug(\mathfrak{u}_0 \cap \mathfrak{h})$ and the root vectors ugX_α. For $\alpha \in \Delta$, put $Y_\alpha = ugX_\alpha$. Then we have established that

$$\mathfrak{u}_0 = \sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha\in\Delta} \mathbb{R}(Y_\alpha - Y_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}i(Y_\alpha + Y_{-\alpha}).$$

We have not yet used the information that is superimposed on the Dynkin diagram of Δ^+. Since the automorphism of Δ^+ defined by θ and θ' are the same, θ and θ' have the same effect on $\mathfrak{h}^*$. Thus

$$\theta(H) = \theta'(H), \quad \forall H \in \mathfrak{h}$$

Then

$$\begin{aligned} \theta(Y_\alpha) = Y_\alpha = \theta'(Y_\alpha), &\quad \text{if } \alpha \text{ is unpainted} \\ \theta(Y_\alpha) = -Y_\alpha = \theta'(Y_\alpha), &\quad \text{if } \alpha \text{ is painted} \end{aligned}$$

This completes the proof. □

Let us now show all the real forms of A_m with $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ and the corresponding Vogan diagrams.

Example 2.42. Let $\mathfrak{g}$ be $\mathfrak{sl}(m+1,\mathbb{C})$, and let be $\mathcal{B} = \{E_{ij} | 1 \leq i,j \leq m+1\}$ our basis of $\mathfrak{gl}(m+1,\mathbb{C})$.
With this notation we take as our Cartan subalgebra:
$\mathfrak{h} = \{diag(x_1,\ldots,x_{m+1}) | \sum_{i=0}^{m+1} x_i = 0\}$. So we have that $\varepsilon_i \in \mathfrak{h}^*$ and if we denote with $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ our root system is $\Delta = \langle \alpha_{ij} \rangle$ and that $\Delta^+ = \{\alpha_{ij} | i < j\}$ and $\Delta^- = \{\alpha_{ij} | j > i\}$ and $S = \{\alpha_{i,i+1} | 1 \leq i \leq m\}$.
Let us now consider the following basis $\mathcal{H} = \{H_i = E_{ii} - E_{i+1,i+1} | 1 \leq i \leq m\} \cap \{E_{ij}\}$. We can define the following scalar product:

$$\langle \Delta \rangle_{\mathbb{R}} = (\alpha_{i,i+1}, \alpha_{j,j+i}) = \alpha_{i,i+1}(H_j) \overset{def}{=} \begin{cases} 2 & j = i \\ -1 & |j-i| = 1 \\ 0 & \text{else} \end{cases}$$

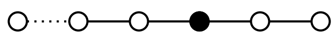
Let us consider

$$\begin{aligned} \mathfrak{su}(p,q) &= \{X \in \mathfrak{sl}(m+1,\mathbb{C}) |\ X^* \mathrm{Id}_{p,q} + \mathrm{Id}_{p,q} X = 0\} \\ &= \left\{ X = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} | A = -A^*, D = -D^* \right\} \end{aligned}$$

where $p+q = m+1, 1 \leq p,q \leq m$. It is easy to show that its complexification is $\mathfrak{sl}(m+1,\mathbb{C})$. The Cartan decomposition of $\mathfrak{su}(p,q)$ corresponding to the Cartan involution $\theta(X) = -X^*$ is

$$\mathfrak{k}_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \mathfrak{p}_0 = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$$

Now we can see that all the diagonal matrix of $\mathfrak{su}(p,q)$ are in $\mathfrak{k}_0$, so it is a maximally compact Cartan subalgebra, θ acts trivially on S, so we have only one simple root whose root space is not in $\mathfrak{k}_0$ which is $\varepsilon_p - \varepsilon_{p+1}$, so we have only one black vertex which is the p-th.



Thanks to Theorem 2.46 we will know that these are all the real forms of A_m.

Now we will investigate the question of existence.

Definition 2.43. We define an *abstract Vogan diagram with no arrows* to be an abstract Dynkin diagram with a subset of the roots, which is to be indicated by painting the vertices corresponding to the members of the subset. Every Vogan diagram, restricted to the case $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$, is of course an abstract Vogan diagram with no arrows.

To have a full description about *abstract Vogan diagram* see on [2], chapter VI.

Theorem 2.44. *If an abstract Vogan diagram is given, then there exist a real semisimple Lie algebra $\mathfrak{g}_0$, a Cartan involution θ, a maximally compact θ stable Cartan subalgebra $\mathfrak{h}_0$, and a positive system Δ^+ for $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ such that the given diagram is the Vogan diagram of $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$.*

Proof. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with the given abstract Dynkin diagram as its Dynkin diagram, and let $\mathfrak{h}$ be a Cartan subalgebra. Put $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, and let Δ^+ be the positive system determined by the given data. Introduce root vectors X_α normalized and define a compact real form $\mathfrak{u}_0$ of $\mathfrak{g}$ in terms of $\mathfrak{h}$ and the X_α. The formula for $\mathfrak{u}_0$ is

$$\mathfrak{u}_0 = \sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha) + \sum_{\alpha\in\Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}i(X_\alpha + X_{-\alpha}).$$

The given data determine an automorphism θ of the Dynkin diagram, which extends linearly to $\mathfrak{h}^*$ and is isometric. Thus $\theta(\Delta) = \Delta$. Thanks to this result we can transfer θ to $\mathfrak{h}$, retaining the same name. Define θ on the root vectors X_α for simple roots by

$$\theta X_\alpha = \begin{cases} X_\alpha & \text{if } \alpha \text{ is unpainted} \\ -X_\alpha & \text{if } \alpha \text{ is painted} \end{cases}$$

We have that θ extends to an automorphism of $\mathfrak{g}$ consistently with these definitions on $\mathfrak{h}$ and on the X_α's for α simple.
The main step is to prove that $\theta\mathfrak{u}_0 = \mathfrak{u}_0$. Let κ be the Killing form of $\mathfrak{g}$. For $\alpha \in \Delta$, define a constant a_α by $\theta X_\alpha = a_\alpha X_{\theta\alpha}$. Then $a_\alpha a_{-\alpha} = \kappa(a_\alpha X_{\theta\alpha}, a_{-\alpha} X_{-\theta\alpha}) = \kappa(\theta X_\alpha, \theta X_{-\alpha}) = 1$ shows that

$$a_\alpha a_{-\alpha} = 1.$$

We shall prove that

$$a_\alpha = \pm 1, \quad \forall \alpha \in \Delta$$

To prove this, it is enough to prove the result dor $\alpha \in \Delta^+$. We do so by induction on the level of α. If the level is 1, then $a_\alpha = \pm 1$ by definition. Thus it is enough to prove that if it holds for positive roots α and β and if $\alpha + \beta$ is a root, then it holds for $\alpha + \beta$. In the notation already used, we have:

$$\begin{aligned} \theta X_{\alpha+\beta} &= N_{\alpha,\beta}^{-1} \theta [X_\alpha X_\beta] = N_{\alpha.\beta}^{-1} [\theta X_\alpha, \theta X_\beta] \\ &= N_{\alpha,\beta}^{-1} a_\alpha a_\beta [X_{\theta\alpha}, X_{\theta\beta}] = N_{\alpha,\beta}^{-1} N_{\theta\alpha,\theta\beta} a_\alpha a_\beta X_{\theta\alpha+\theta\beta}]. \end{aligned}$$

Therefore

$$a_{\alpha+\beta} = N_{\alpha,\beta}^{-1} N_{\theta\alpha,\theta\beta} a_\alpha a_\beta$$

Here $a_\alpha a_\beta = \pm 1$ by assumption, but we know that θ is an automorphism of Δ and that the $N_{\alpha,\beta}$ and $N_{\theta\alpha,\theta\beta}$ are real with

$$N_{\alpha,\beta}^2 = \frac{1}{2} q(1+p) = |\alpha|^2 = \frac{1}{2} q(1+p)|\theta\alpha|^2 = N_{\theta\alpha,\theta\beta}^2.$$

Hence $a_{\alpha+\beta} = \pm 1$.
Let us see that

$$\theta(\mathbb{R}(X_\alpha - X_{-\alpha} + \mathbb{R}i(X_\alpha + X_{-\alpha})) \subseteq \mathbb{R}(X_{\theta\alpha} - X_{-\theta\alpha}) + \mathbb{R}i(X_{\theta\alpha} + X_{-\theta\alpha}).$$

If x and y are real and if $z = x + yi$, then we have

$$x(X_\alpha - X_{-\alpha}) + yi(X_\alpha + X_{-\alpha}) = zX_\alpha - \bar{z}X_{-\alpha}.$$

is of the form $wX_{\theta\alpha} - \bar{w}X_{-\theta\alpha}$, and this follows from the observations above. Since θ carries roots to roots,

$$\theta\left(\sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha)\right) = \sum_{\alpha\in\Delta} \mathbb{R}(iH_\alpha)$$

So we see that $\theta\mathfrak{u}_0 = \mathfrak{u}_0$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigenspaces for θ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since $\theta\mathfrak{u}_0 = \mathfrak{u}_0$, we have

$$\mathfrak{u}_0 = (\mathfrak{u}_0 \cap \mathfrak{k}) \oplus (\mathfrak{u}_0 \cap \mathfrak{p}).$$

Define $\mathfrak{k}_0 = (\mathfrak{u}_0 \cap \mathfrak{k})$ and $\mathfrak{p}_0 = (\mathfrak{u}_0 \cap \mathfrak{p})$, so that

$$\mathfrak{u}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0.$$

Since $\mathfrak{u}_0$ is a real form of $\mathfrak{g}$ as a vector space, so is

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

Since $\theta\mathfrak{u}_0 = \mathfrak{u}_0$ and since θ is an involution, we have the bracket relations

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0.$$

Therefore $\mathfrak{g}_0$ is closed under brackets and is a real form of $\mathfrak{g}$ as a Lie algebra. The involution θ is $+1$ on $\mathfrak{k}_0$ and is -1 on $\mathfrak{p}_0$; it is a Cartan involution of $\mathfrak{g}_0$,

since $\mathfrak{k}_0 \oplus i\mathfrak{p}_0 = \mathfrak{u}_0$ is compact.
So we have shown that θ maps $\mathfrak{u}_0 \cap \mathfrak{h}$ to itself, and therefore

$$\begin{aligned}\mathfrak{u}_0 &= (\mathfrak{u}_0 \cap \mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{u}_0 \cap \mathfrak{p}\mathfrak{h}) \\ &= (\mathfrak{k}_0 \cap \mathfrak{h}) \oplus (i\mathfrak{p}_0 \cap \mathfrak{h}) \\ &= (\mathfrak{k}_0 \cap \mathfrak{h}) \oplus i(\mathfrak{p}_0\mathfrak{h}).\end{aligned}$$

The abelian subspace $\mathfrak{u}_0 \cap \mathfrak{h}$ is a real form of $\mathfrak{h}$, and hence so is

$$\mathfrak{h}_0 = (\mathfrak{k}_0 \cap \mathfrak{h}) \oplus (\mathfrak{p}_0 \cap \mathfrak{h}).$$

The subspace $\mathfrak{h}_0$ is contained in $\mathfrak{g}_0$, and it is therefore a θ stable Cartan subalgebra of $\mathfrak{g}_0$. A real root α relative to $\mathfrak{h}_0$ has the property that $\theta\alpha = -\alpha$. Since θ preserves positivity relative to Δ^+, there are no real roots, and so $\mathfrak{h}_0$ is maximally compact. Let us verify that Δ^+ results from a lexicographic ordering that takes $i(\mathfrak{k}_0 \cap \mathfrak{h})$ before $\mathfrak{p}_0 \cap \mathfrak{h}$. Let $\{\beta_i\}_{i=1}^{l}$ be the set of simple roots of Δ^+ in 1-element orbits under θ. Relative to a basis $\{\alpha_i\}_{i=1}^{l+2m}$ consisting of all simple roots, let $\{\omega_i\}$ be the dual basis defined by $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$. We shall write ω_{β_j} in place of ω_i in what follows. We define a lexicographic ordering by using inner products with the ordered basis

$$\omega_{\beta_{j_1}}, \ldots, \omega_{\beta_{j_l}}$$

which takes $i(\mathfrak{k}_0 \cap \mathfrak{h})$ before $\mathfrak{p}_0 \cap \mathfrak{h}$. Let α be in Δ^+, and write

$$\alpha = \sum_{i=1}^{l} n_i \beta_i.$$

Then

$$\langle \alpha, \omega_{\beta_j} \rangle = n_j \geq 0$$

If all these inner products are 0, then all coefficients of α are 0, contradiction. Thus α has positive inner product with the first member of our ordered basis for which the inner product is nonzero, and the lexicographic ordering yields Δ^+ as positive system. Consequently $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ is a triple.
Our definitions of θ on $\mathfrak{h}^*$ and on the X_α for α simple make it clear that the Vogan diagram of $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ coincides with the given data. □

Now we want to show that we can always choose the simple roots so that we have one root painted in the Vogan diagram. Before doing this we show another property of real Lie algebras.

Theorem 2.45. *Let $\mathfrak{g}_0$ be a simple Lie algebra over $\mathbb{R}$, and let $\mathfrak{g}$ be its complexification. Then there are just two possibilities:*

1. *$\mathfrak{g}_0$ is complex, i.e. $\mathfrak{g}_0$ is of the form $\mathfrak{s}^{\mathbb{R}}$ for some complex $\mathfrak{s}$, and then $\mathfrak{g}$ is in $\mathbb{C}$ isomrphic to $\mathfrak{s} \oplus \mathfrak{s}$.*

2. *$\mathfrak{g}_0$ is not complex, and then $\mathfrak{g}$ is simple over $\mathbb{C}$.*

Proof. 1. Let J be multiplication by $\sqrt{-1}$ in $\mathfrak{g}_0$, and define an $\mathbb{R}$ linear map $L : \mathfrak{g} \to \mathfrak{s} \oplus \mathfrak{s}$ by $L(X + iY) = (X + JY, X - JY)$ for X and Y in $\mathfrak{g}_0$. We readily check that L is one-one and respects brackets. Since the domain and range have real dimension, L is an $\mathbb{R}$ isomorphism. Moreover L satisfies

$$\begin{aligned} L(i(X + iY)) &= L(-Y + iX) \\ &= (-Y + JX, -Y - JX) \\ &= (J(X + JY), -J(X - JY)). \end{aligned}$$

This equation exhibits L as a $\mathbb{C}$ isomorphism of $\mathfrak{g}$ with $\mathfrak{s} \oplus \bar{\mathfrak{s}}$, where $\bar{\mathfrak{s}}$ is the same real Lie algebra as $\mathfrak{g}_0$ but where the multiplication by $\sqrt{-1}$ is defined as multiplication by $-i$.
Now we have to show that $\bar{\mathfrak{s}}$ is $\mathbb{C}$ isomorphic to $\mathfrak{s}$. We already know that $\mathfrak{s}$ has a compact real form $\mathfrak{u}_0$. The conjugation τ of $\mathfrak{s}$ with respect to $\mathfrak{u}_0$ is $\mathbb{R}$ linear and respects brackets, and the claim is that τ is a $\mathbb{C}$ isomorphism of $\mathfrak{s}$ with $\bar{\mathfrak{s}}$. In fact, if U and V are in $\mathfrak{u}_0$, then

$$\begin{aligned} \tau(J(U + JV)) &= \tau(-V + JU) = -V - JU \\ &= -J(U - JV) = -J\tau(U + JV) \end{aligned}$$

and 1. follows.

2. Let bar denote conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. If $\mathfrak{a}$ is a simple ideal in $\mathfrak{g}$, then $\mathfrak{a} \cap \bar{\mathfrak{a}}$ and $\mathfrak{a} + \bar{\mathfrak{a}}$ are ideals in $\mathfrak{g}$ invariant under conjugation and hence are complexifications of ideals in $\mathfrak{g}_0$. Thus they are 0 or $\mathfrak{g}$. Since $\mathfrak{a} \neq 0, \mathfrak{a} + \bar{\mathfrak{a}} = \mathfrak{g}$.
If $\mathfrak{a} \cap \bar{\mathfrak{a}} = 0$, then $\mathfrak{g} = \mathfrak{a} \oplus \bar{\mathfrak{a}}$. The inclusion of $\mathfrak{g}_0$ into $\mathfrak{g}$, followed by projection to $\mathfrak{a}$, is an $\mathbb{R}$ homomorphism φ of Lie algebras. If $\ker\varphi$ is nonzero, then $\ker\varphi$ must be $\mathfrak{g}_0$. In this case $\mathfrak{g}_0$ is contained in $\bar{\mathfrak{a}}$. But conjugation fixes $\mathfrak{g}_0$, and thus $\mathfrak{g}_0 \subseteq \mathfrak{a} \cap \bar{\mathfrak{a}} = 0$, contradiction. We conclude that φ is one-one. A dimensional count shows that φ is an $\mathbb{R}$ isomorphism of $\mathfrak{g}_0$ onto $\mathfrak{a}$. But then $\mathfrak{g}_0$ is complex, contradiction.

We conclude that $\mathfrak{a} \cap \bar{\mathfrak{a}} = \mathfrak{g}$ and hence $\mathfrak{a} = \mathfrak{g}$. Therefore $\mathfrak{g}$ is simple, as asserted.

□

Now we want to reduce the redundancy of the Vogan diagrams that come out by having many choices for the positive system Δ^+. The idea is that we can always change Δ^+ so that at most one simple root is painted.

Theorem 2.46 (Borel and de Siebenthal Theorem)**.** *Let $\mathfrak{g}_0$ be a noncomplex simple real Lie algebra, and let the Vogan diagram with no arrows of $\mathfrak{g}_0$ be given so that corresponds to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$. Then there exists a simple system S' for $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, with the corresponding positive system Δ'^+ such that $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta'^+)$ is a triple and there is at most one painted simple root in its Vogan diagram. Furthermore suppose that the automorphism associated with the Vogan diagram is the identity, that $S' = \{\alpha_1, \dots, \alpha_l\}$, and that $\{\omega_1, \dots, \omega_l\}$ is the dual basis given by $\langle \omega_j, \alpha_k \rangle = \delta_{jk}$. Then the single painted simple root α_i may be chosen so that there is no i' with $\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0$.*

We start with two lemmas.

Lemma 2.47. *Let Δ be an irreducible abstract reduced root system in a real vector space V, let S be a simple system, and let ω and ω' be nonzero members of V that is dominant relative to S. Then $\langle \omega, \omega' \rangle > 0$.*

Proof. The first step is to show that in the expansion $\omega = \sum_{\alpha \in S} a_\alpha \alpha$, all the a_α are ≥ 0. Let us enumerate S as $\alpha_1, \dots, \alpha_l$ so that

$$\omega = \sum_{i=1}^{f} a_i \alpha_i - \sum_{i=r+1}^{s} b_i \alpha_i = \omega^+ - \omega^-$$

with all $a_i \geq 0$ and all $b_i > 0$. We shall show that $\omega^- = 0$. Since $\omega^- = \omega^+ - \omega$, we have

$$0 \leq |\omega^-|^2 = \langle \omega^+, \omega^- \rangle - \langle \omega^-, \omega \rangle = \sum_{i=1}^{r} \sum_{j=r+1}^{s} a_i b_j \langle \alpha_i, \alpha_j \rangle - \sum_{j=r+1}^{l} b_j \langle \omega, \alpha_j \rangle.$$

The first term on the right side is ≤ 0 and the second term on the right side is term-by-term ≤ 0 by hypothesis. Therefore the right side is ≤ 0, and we conclude that $\omega^- = 0$. Thus we can write $\omega = \sum_{j=1}^{l} a_j \alpha_j$ whit all $a_j \geq 0$. The next step is to show from the irreducibility of Δ that $a_j > 0$ for all j. Assuming the contrary, suppose that $a_i = 0$. Then

$$0 \leq \langle \omega, \alpha_i \rangle = \sum_{j \neq i} a_j \langle \alpha_j, \alpha_i \rangle$$

and every term on the right side is ≤ 0. Thus $a_j = 0$ for every α_j such that $\langle\alpha_j, \alpha_i\rangle < 0$. Since the Dynkin diagram is conneted, iteration of this argument shows that all coefficients are 0 once of them is 0.
Now we can complete the proof. For at least one index i, $\langle\alpha_i, \omega'\rangle > 0$ since $\omega' \neq 0$. Then

$$\langle\omega, \omega'\rangle = \sum_j a_j\langle\alpha_j, \omega'\rangle \geq a_i\langle\alpha_i, \omega'\rangle,$$

and the right side is > 0 since $a_i > 0$. This proves the lemma. □

Lemma 2.48. *Let $\mathfrak{g}_0$ be a noncomplex simple real Lie algebra, and let the Vogan diagram of $\mathfrak{g}_0$ be given that corresponds to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$. Let V be the span of the simple roots that are imaginary, let Δ_0 be the root system $\Delta \cap V$, let $\mathcal{H}$ be the subset of $i\mathfrak{t}_0$ paired with V, and let Λ be the subset of $\mathcal{H}$ where all roots of Δ_0 take integer values and all noncompact roots of Δ_0 take odd-integer values. Then Λ is nonempty. In fact, if $\alpha_1, \ldots, \alpha_m$ is any simple system for Λ_0 and if $\omega_1, \ldots, \omega_m$ in V are defined by $\langle\omega_j, \alpha_k\rangle = \delta_{jk}$, then the element*

$$\omega = \sum_{i \text{ with } \alpha_\text{i} \text{ noncompact}} \omega_i.$$

is in Λ.

Proof. Fix a simple system $\alpha_1, \ldots, \alpha_m$ for Δ_0, and let Δ_0^+ be the set of positive roots of Δ_0. Defin $\omega, \ldots, \omega_m$ by $\langle\omega_j, \alpha_k\rangle = \delta_{jk}$. If $\alpha = \sum_{i=1}^m n_i\alpha_i$ is a positive root of Δ_0, then $\langle\omega, \alpha\rangle$ is the sum of the n_i for which α_i is noncompact. This is certainly an integer.
We shall prove by induction on the level $\sum_{i=1}^m n_i$ that $\langle\omega, \alpha\rangle$ is even if α is compact, odd if α is noncompact. When the level is 1, this assertion is true by definition. In the general case, let α and β be in Δ_0^+ with $\alpha + \beta \in \Delta$, and suppose that the assertion is true for α and β. Since the sum of the n_i for which α_i is noncompact is additive, we are to prove that imaginary roots satisfy

compact+compact=compact
compact+noncompact=noncompact
noncompact+noncompact=compact.

But this is immediate from Corollary 1.15 and the previous observation about the behaviour of a θ Cartan involution with a Cartan decomposition. □

Proof of Theorem 2.46. Observe that the Dynkin diagram of Δ_0 is connected, i.e., that the roots in the Dynkin diagram of Δ fixed by the given automorphism form a connected set. There is no problem when the automorphism is the identity, and we observe the connectedness in the other cases one at a time by inspection.
Let $\Delta_0^+ = \Delta^+ \cap V$. The set Λ is discrete, being a subset of a lattice, and the previous lemma has just shown that it is nonempty. Let H_0 be a member of Λ with norm as small as possible. We know that we can choose a new positive system $\Delta_0^{'+}$ for Δ_0 that makes H_0 dominant. The main step is to show that

at most one simple root of $\Delta_0^{'+}$ is painted.

Suppose $H_0 = 0$. If α is in Δ_0, then $\langle H_0, \alpha\rangle$ is 0 and is not an odd integer. By definition of Λ, α is compact. Thus all roots of Δ_0 are compact, and the assert is true.
Now suppose $H_0 \neq 0$. Let $\alpha_1, \ldots, \alpha_m$ be the simple roots of Δ_0 relative to $\Delta_0^{'+}$ abd define $\omega_1, \ldots, \omega_m$ by $\langle \omega_j, \alpha_k\rangle = \delta_{jk}$. We can write $H_0 = \sum_{j=1}^m n_j\omega_j$ with $n_j = \langle H_0, \alpha_j\rangle$. The number n_j is an integer since H_0 is in Λ, and it is ≥ 0 since H_0 is dominant relative to $\Delta_0^{'+}$.
Since $H_0 \neq 0$, we have $n_i > 0$ for some i. Then $H_0 - \omega_i$ is dominant relative to $\Delta_0^{'+}$, and Lemma 2.47 shows that $\langle H_0 - \omega_i, \omega_i\rangle \geq 0$ with equality only if $H_0 = \omega_i$. If strict inequality holds, then the element $H_0 - 2\omega_i$ is in Λ and satisfies

$$|H_0 - 2\omega_i|^2 = |H_0|^2 - 4\langle H_0 - \omega_i, \omega_i\rangle < |H_0|^2$$

in contradiction with the minimal-norm condition on H_0. Hence equality holds, and $H_0 = \omega_i$.
Since H_0 is in Λ, a simple root α_j in $\Delta_0^{'+}$ is noncompact only if $\langle H_0, \alpha_j\rangle$ is an odd integer. Since $\langle H_0, \alpha_j\rangle = 0$ for $j \neq i$, the only possible noncompact simple root in $\Delta_0^{'+}$ is α_i.
If the automorphism associated with the Vogan diagram is the identity, we have proved the first conclusion of the theorem. For the second one we are assuming that $H_0 = \omega_i$; then an inequality $\langle \omega_i - \omega_{i'}, \omega_{i'}\rangle > 0$ would imply that

$$|H_0 - 2\omega_{i'}|^2 = |H_0|^2 - 4\langle \omega_i - \omega_{i'}, \omega_{i'}\rangle < |H_0|^2,$$

in contradiction with the minimal-norm condition on H_0.
To complete the proof of the theorem, we have to prove the first conclusion when the automorphism associated with the Vogan diagram is not the identity. Choose an element $s \in \mathcal{W}(\Delta_0)$ with $\Delta_0^+ = s\Delta_0^+$, and define $\Delta^{'+} = s\Delta^+$.

Let the simple roots of Δ^+ be $\beta_1, \dots, \beta_l$ with $\beta_1, \dots, \beta_l$ in Δ_0. Then the simple roots of $\Delta^{+'}$ are $s\beta_1, \dots, s\beta_l$. Thus $\Delta^{+'}$ has at most one simple root that is noncompact imaginary. □

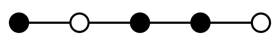
2.5 Graph Paintings

In this section we want to give a "graphic" algorithm to reduce the black vertices of a Vogan diagram with no arrows and to see if two Vogan diagrams are equivalent, which means that they correspond to the same real form.
By an abuse of terminology we identify the vertices of a Vogan diagram with the roots of a simple system of $\mathfrak{g}$. We encode the information contained in a Vogan diagram, by the pair consisting of a Dynkin diagram D and the $k-uple$ $(i_1, \dots, i_k)$, where the $i_1 < \cdots < i_k$ are the black vertices.
We introduce an operation $F[i]$ on the Vogan diagram which corresponds to the action on the root system of the reflections s_i of the noncompact root i.

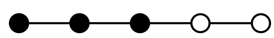
Definition 2.49. Let notation be as above, we define the operation $F[i]$ on the Vogan diagram $(D, (i_1, \dots, i_r))$ as follows:

- The colors of the vertex i of D and all vertices not adjacent to i remain unchanged.
- If the vertex j is joined to i by a double edge and j is long, the color of j remains unchanged.
- Apart from the above exceptions, $F[i]$ reverses the colors of all vertices adjacent to i.

Example 2.50. Let $(A_5, (1, 3, 4))$ be a Vogan diagram.



If we apply $F[3]$ we have $(1, 2, 3)$.



Proposition 2.51. *Let $(D, (i_1, \dots, i_r))$ be the Vogan diagram corresponding to the real form $\mathfrak{g}_0$ with the choice of simple system S. The operation $F[i]$ on $(D, (i_1, \dots, i_r))$ gives a new Vogan diagram $(D', (i'_1, \dots, i'_r))$ corresponding to the choice of simple system $s_i(S)$ where s_i is the reflection in $\mathcal{W}$, the Weyl group associated with the root $i \in \{i_1, \dots, i_r\}$.*

Sketch of proof. Let α be a black vertex and β a simple root.
We have that

$$s_\alpha\beta = \beta - 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha$$

Since both α, β are simple, $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} < 0$. As we have seen in the first chapter, Table 1.1, we have

$$2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = -1, -2, -3$$

We consider only the case $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = -1$, leaving the other 2 cases as an exercise to the reader. So we have that

$$S_\alpha\beta = \begin{cases} \beta+\alpha & \text{if the vertices } \alpha,\beta \text{ are connected,} \\ \beta & \text{if the vertices } \alpha,\beta \text{ are not connected.} \end{cases}$$

Hence s_α brings the simple system $\{\alpha, \beta\}$ to the simple system $\{-\alpha, \alpha+\beta\}$. If β is white we have that $\alpha+\beta$ is black, so from a Vogan diagram in which we had a black vertex α and a white vertex β we obtain a Vogan diagram with two black vertices $\alpha, \alpha+\beta$. Viceversa if β is black, $\alpha+\beta$ is white, from a Vogan diagram with two black vertices we obtain a Vogan with one black vertex and one white vertex. □

From now on we restrict ourself to consider $\mathfrak{g}$ as one of the Lie algebras belonging to the classical families A_n, B_n, C_n, D_n. Now we want to show that applying $F[i]$ a pair of painted vertices can be shifted leftward or rightward.

Lemma 2.52. *Let the notation be as above. If $i_1 < \cdots < i_k$:*

1. $(D, (i_1, \ldots, i_k)) \sim (D, (i_1, \ldots, i_{r-1}, i_r-c, i_{r+1}-c, i_{r+2}, \ldots, i_k))$ *whenever* $i_{r-1} < i_r - c$,
2. $(D, (i_1, \ldots, i_k)) \sim (D, (i_1, \ldots, i_{r-1}, i_r+c, i_{r+1}+c, i_{r+2}, \ldots, i_k))$ *whenever* $i_{r+1} + c < i_{r+2}$. *We require* $i_{r+1} + c \leq n+1$ *in* C_n *and* $i_{r+1} + c \leq n+2$ *in* D_n.

Proof. 1. Suppose we want to move $i_r.i_{r+1}$ leftward c steps, where $i_{r-1} < i_r - c$. It is equivalent to moving them 1 steps for c times, namely it suffices to show that

$$(D, (i_1, \ldots, i_k)) \sim (D, (i_1, \ldots, i_{r-1}, i_r - 1, i_{r+1} - 1, i_{r+2}, \ldots, i_k)). \tag{2.1}$$

By applying $F[i_r+1], F[i_r+2], \ldots, F]i_{r+1}-1]$ consecutively to $(i_1, \ldots, i_k)$ we obtain 2.1 and the point follows.

2. It is similar to the first point and the restriction on C_n, D_n follows from the properties of $F[i]$.

□

Example 2.53. In $(A_9, (1, 5, 7, 9))$, we can move the pair $5, 7$ leftward three steps and get $(A_9, (1, 5, 7, 9)) \sim (A_9, (1, 2, 4, 9))$ with $F[7],\ F[9],\ F[5],\ F[4],\ F[6],\ F[3],\ F[4]$.

Now we see a way to reduce the number of painted vertices, that can be used to find an alternative proof of Theorem 2.46. We show that, given a Vogan diagram, using operations $F[i]$ it is possible, by shifting leftward or rightward the pairs of noncompact roots, to reduce the number of black vertices.

Lemma 2.54. *Let the notation be as above:*

- *In A_n, B_n, $(D, (i_1, \ldots, i_k)) \sim (D, (i_2 - i_1, i_3, \ldots, i_k))$.*
- *In C_n, If $i_2 \leq n - 1$, $(D, (i_1, \ldots, i_k)) \sim (D, (i_2 - i_1, i_3, \ldots, i_k))$.*
- *In D_n, If $i_2 \leq n - 2$, $(D, (i_1, \ldots, i_k)) \sim (D, (i_2 - i_1, i_3, \ldots, i_k))$.*

Proof. We divide the arguments for $(i_1, \ldots, i_k)$ into two cases.

$i_1 = 1$ If $i_2 = 2$ then $F[1](D, (1, 2, i_3, \ldots, i_k)) \sim (D, (1, i_3, \ldots, i_k))$ and we are done. So suppose that $i_2 > 2$. Apply $F[1], F[2], \ldots, F[i_2 - 1]$ to $(D, (1, i_2, \ldots, i_k))$, we get $(D, (1, i_2, \ldots, i_k)) \sim (D, (i_2 - i_1, i_3, \ldots, i_k))$.

$i_1 > 1$ By Lemma 2.52, $(D, (i_1, \ldots, i_k)) \sim (D, (1, i_2 i_1 + 1, i_3, \ldots, i_k))$ and this is reduced to the first case. The extra conditions on C_n, D_n depends on how $F[i]$ acts, as in Lemma 2.52.

□

We now describe our algorithm based in the operation $F[i]$:

1. Using Lemma 2.1 we can shift pairs of noncompact roots to the left or to the right;

2. Using Lemma 2.54 we can reduce the number of noncompact roots, one by one.

Our algorithm $F[i]$ has another powerful property: if we consider only the Vogan diagram with no arrows, we can use this algorithm to find when two Vogan diagram are equivalent.

Theorem 2.55. *Two Vogan diagrams with no arrows are equivalent if and only if one can be transformed into the other by a sequence of $F[i]$ operations.*

Proof. The "if" part is obviuos since $F[i]$ preserves equivalence classes, in fact by Proposition 2.51 it correspond to a different choice of simple system of the same $\mathfrak{g}$, now we consider the converse. We recall that two equivalent Vogan diagrams correspond to the same Lie algebra under different choices of simple systems, see Example 2.38. The Weyl group $\mathcal{W} = \langle S_{\alpha_i} \rangle$ acts transitively on the simple systems, and so it acts transitively on each equivalence class of Vogan diagrams. Recall that $F[i]$ acts as a reflecition about the noncompact simple root α_i. let $\mathcal{W}_c$ and $\mathcal{W}_n$ denote the subgrous generated by reflections about the compact and noncompact simple roots, respectively. Clearly, $\mathcal{W}$ is generated by $\mathcal{W}_c$ and $\mathcal{W}_n$. Further, since $\mathcal{W}_c$ acts trivially on painting of the Vogan diagrams, it follows that $\mathcal{W}_n$ acts transitively on each equivalence class of Vogan diagrams. This proves the theorem. □

We now can state the *Borel de Siebenthal* theorem, which follows from our prevoius discussion.

Theorem 2.56 (Borel and de Siebenthal Theorem)**.** *Every Vogan diagram with no arrows are equivalent to a Vogan diagram with only one vertex painted.*

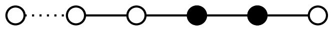
Now we want to generalize the result of Theorem 2.55 and see another connection to Theorem 2.46.

Corollary 2.57. *If a connected graph Γ is a Dynkin dagram, then*

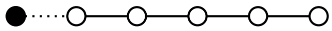
1. *every painting on Γ can be semplified via a sequence of $F[i]$ to a painting with single painted vertex;*
2. *every connected subgraph of Γ satisfies the first property.*

Proof. To prove the first point, let Γ be a Dynkin diagram. Suppose that p is a painting on Γ. By Theorem 2.46, $(\Gamma, p) \sim (\Gamma, s)$, where s paints just a single vertex of Γ. By Theorem 2.55 (Γ, p) can be transformed to (Γ, s) with some $F[i]$ operations. This proves the first properety. Since connected subgraph of a Dynkin diagram correspond to simple subalgebras, the second condition is trivial. □

Example 2.58. If we consider the following Vogan diagram

A_n 

and denote the first black vertex with number 4, if you apply $F[4] \circ F[3] \circ F[2] \circ F[1]$ you have only one black vertex:



Chapter 3

Lie Superalgebras

At the beginning of this chapter we define what a Lie superalgebra is and some of its properties in order to have some preliminary notions to discuss in the following chapter about their real forms.
Due to the this fact it is written only to give a few notions about this argument, we not give the proof of any proposition but we always give a reference.

3.1 Preliminary definitions

Let $\mathbb{K}$ be our ground field algebrically closed and of characteristic zero.

Definition 3.1. A *super vector space* is a $\mathbb{Z}/\mathbb{Z}_2$-graded vector space

$$V = V_0 \oplus V_1$$

where the elements of V_0 are called *even* and elements of V_1 are called *odd.*

Definition 3.2. The parity of $v \in V$, denoted by $p(v)$ or $|v|$, is defined only on non-zero *homogeneus* elements, that is elements of either V_0 or V_1:

$$p(v) = |v| = \begin{cases} 0 \text{ if } v \in V_0 \\ 1 \text{ if } v \in V_1 \end{cases}$$

We have that every element can be expressed as the sum of homogeneus elements, so we can give all the definitions, theorems and proofs considering only these elements.

Definition 3.3. The *superdimension* of a super vector space V is the pair (p, q) where $\dim(V_0) = p$ and $\dim(V_1) = q$ as ordinary vector spaces, we can also write $\dim(V) = p|q$.

Thanks to this definition we can define also what a basis is: if $\dim V = p|q$ we can find a basis $\{e_1, \dots, e_p\}$ of V_0 and a basis $\{\epsilon_1, \dots, \epsilon_q\}$ of V_1 so that V is canonically isomorphic to the $\mathbb{K}$-vector space generated by the $\{e_1, \dots, e_p, \epsilon_1, \dots, \epsilon_q\}$. We can denote this $\mathbb{K}$-vector space by $\mathbb{K}^{p|q}$ and we will call $(e_1, \dots, e_p, \epsilon_1, \dots, \epsilon_q)$ the *canonical basis* of $\mathbb{K}^{p|q}$. The (e_i) form a basis for $\mathbb{K}^p = \mathbb{K}_0^{p|q}$ and the (ϵ_j) form a basis for $\mathbb{K}^q = \mathbb{K}_1^{p|q}$.

Definition 3.4. A *morphism* from a super vector space V to a super vector space W is a linear map from V to W preserving $\mathbb{Z}/\mathbb{Z}_2$-grading. Let $\mathrm{Hom}(V, W)$ denote the vector space of morphisms $V \to W$.

Now we can define what a Lie superalgebra is.

Definition 3.5. A *super Lie algebra* is a super vector space $\mathfrak{g}$ with a morphism $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ called superbraket, or simply bracket, which satisfies the following condition:

1. Anti-Simmetry

$$[x, y] + (-1)^{|x||y|}[y, x] = 0$$

for $x, y \in \mathfrak{g}$ homogeneous.

2. The Jacobi-identity

$$[x, [y, z]] + (-1)^{|x||y|+|x||z|}[y, [z, x]] + (-1)^{|y||x|+|x||z|}[z, [x, y]] = 0$$

for $x, y, z \in \mathfrak{g}$ homogeneous.

The most important case of Lie superalgebra is the algebra of endomorphism, as in the classical case, called $\mathfrak{gl}(V)$.
If we have that $V = \mathbb{K}^{m|n}$ we can denote $\mathfrak{gl}(V)$ as $\mathfrak{gl}(m|n)$. The even part $\mathfrak{gl}(m|n)_0$ consists of the matrices with entries in $\mathbb{K}$ corresponding to endomorphisms preserving the parity, while the odd one consists of matrices that reverse the parity:

$$\mathfrak{gl}(m|n) = \mathfrak{gl}(m|n)_0 \oplus \mathfrak{gl}(m|n)_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}$$

where A and D are $(m \times m)$-matrices and $(n \times n)$-matrices, and B and C are $(n \times m)$-matrices.
Then $\mathfrak{gl}(m|n)$ is a Lie superalgebra with the following bracket:

$$[X, Y] = XY - (-1)^{|X||Y|}YX$$

Now we can define the *special linear Lie superalgebra*, denoted by $\mathfrak{sl}(m|n)$ and the *projective special linear Lie superalgebra* $\mathfrak{psl}(m|n)$ as

$$\mathfrak{sl}(m|n) = \{X \in \mathfrak{gl}(m|n) | \mathrm{str}(X) = 0\}$$

where str is the supertrace, defined as follow:

$$\mathrm{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}A - \mathrm{tr}B$$

and $\mathfrak{psl}(m|m) := \mathfrak{sl}(m|m)/\mathbb{K}\mathrm{I}_{2\mathrm{m}}$.

Definition 3.6. We say that a bilinear form f on a super vector space $V = V_0 \oplus V_1$ is *super symmetric* if

$$f(u, v) = (-1)^{|u||v|} f(v, u)$$

for every homogeneus elements $u, v \in V$. We say also that it is *consistent* if $f(u, v) = 0$ for $u \in V_0, v \in V_1$.

Now, we are ready to introduce the *orthosymplectic Lie superalgebra.*

Definition 3.7. Let f be a non-degenerate consistent super symmetric bilinear form on V, $\mathrm{dim}V = m+n$. We define the *orthosymplectic Lie superalgebra* as

$$\mathfrak{osp}(V) := \{X \in \mathfrak{gl}(m|n) | f(X, u, v) = -(-1)^{|X||u|} f(u, Xv)\}$$

Notice that n has to be even since f defines a non-degenerate skew-symmetric form on V_1.

Definition 3.8. We define the *strange series* $P(n)$ as

$$P(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right\} \subset \mathfrak{gl}(n+1|m+1)$$

where $A \in \mathfrak{sl}(n+1)$, B is symmetric and C skew-symmetric.
The *strange series* $Q(n)$ is defined as follows:

$$q(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}$$

$sq(n)$ are the matrices in $q(n)$ with $\mathrm{tr}(B) = 0$ and $Q(n-1) = \mathrm{psq}(n) = \mathrm{sq}(n)/\mathbb{K}I_{2n}$:

$$Q(n-1) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} | B \in \mathfrak{sl}_n \right\} / \mathbb{K}I_{2n}$$

3.2 Simple Lie Superalgebras

Simple Lie superalgebras have been classified by Kac and play a key role in many applications.

Definition 3.9. Let $\mathfrak{g}$ be a Lie superalgebra (always finite-dimensional). We say that $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is not abelian and it admits no non-trivial ideals. $\mathfrak{g}$ is of classical type if it is simple and $\mathfrak{g}_1$ is completely reducible as a g_0-module, where the action is given by the bracket. $\mathfrak{g}$ is basic if it is classical and it admits a consistent, non-degenerate, invariant bilinear form, that is to say, there exists a consistent, non-degenerate, bilinear form $\langle , \rangle : \mathfrak{g} \times \mathfrak{g}$ such that $\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$.

The simple Lie superalgebras divide into two main types: the classical type and the Cartan type. We make a list of such Lie superalgebras, for the proof see on [7].

Classical type. The classical type subdivides further into type 1 and type 2. Type 1 classical superalgebras are those for which $\mathfrak{g}_1$ is not irreducible as $\mathfrak{g}_0$-module and type 2 are those for which $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module.

Classical type 1. These superalgebras are:

$$\begin{aligned} A(m|m) &:= \mathfrak{sl}(m+1|n+1), \quad m \neq n \\ A(m|m) &:= \mathfrak{psl}(m+1|m+1), \\ C(n) &:= \mathfrak{osp}(2|2n-2), \quad P(n). \end{aligned}$$

and $\mathfrak{g}_1$ decomposes into two components as a $\mathfrak{g}_0$ module.

Classical type 2. The type 2 superalgebras are those for which $\mathfrak{g}_1$ is irreducible, so that there is no compatible $\mathbb{Z}$-grading. These Lie superalgebras are:

$$\begin{gathered} B(m|n) := \mathfrak{osp}(2m+1|2n) \\ D(m|n) := \mathfrak{osp}(2m|2n) \\ D(2,1;\alpha), \quad F(4), \quad G(3), \quad Q(n). \end{gathered}$$

where $D(2,1;\alpha)$ is a family with continuous parameter $\alpha \in \mathbb{K} \setminus \{0,1\}$. Two elements $D(2,1;\alpha), D(2,1;\beta)$ of this family are isomorphic if and only if α and β lie in the same orbit under the action of the group generated by $\alpha \to -1-\alpha, \alpha \to 1/\alpha$, see on [7].

Cartan Type. Let Sym(V) denote the symmetric algebra over the super vector space V. If our super vector space has dimension $m|n$ we can create an isomorphism between this symmetric algebra with the following polonomial algebra with m even indeterminates and n odds: $\mathrm{Sym}(V) \cong \mathbb{K}[x_1, \dots, x_m, \xi_1, \dots, \xi_n] =: A$. We define $W(m|n) := \mathrm{Der}(A)$ as the superalgebra of derivations of A, which is in general infinite-dimensional, however when $m = 0$ it is finite-dimensional. To semplify the notation we will write $W(n)$ instead of $W(0|n)$. Define $\Theta(n)$ as the associative superalgebra over A generated by $\theta\xi_1, \dots, \theta\xi_n$ with relations $\theta\xi_i \wedge \theta\xi_j = -\theta\xi_j \wedge \theta\xi_i, (i \neq j)$. This is a superalgebra with grading induced by $\deg(\theta\xi_i) = 1$. Now we can introduce the following superalgebras:

$$S(n) \overset{def}{=} \{D \in W(n) | D(\theta\xi_1 \wedge \dots \wedge \theta\xi_n) = 0\}$$
$$\tilde{S}(n) \overset{def}{=} \{D \in |D((1 + \xi_1\xi_2 \dots \xi_n)\theta\xi_1 \wedge \dots \wedge \theta\xi_n) = 0\} \text{ for even } n$$

which are subalgebras of $W(n)$ where some elements of $\Theta(n)$ are annihilated, those elements are called *volume forms*.
Now we can introduce our last superalgebra called $H(n)$ which is the commutator of $W(n)$ preserving a certain metric:

$$H(n) \overset{def}{=} [\tilde{H}(n), \tilde{H}(n)] \quad \text{where } \tilde{H}(n) \overset{def}{=} \{D \in W(n) | D(d\xi_1^2 + \dots + d\xi_n^2) = 0\}$$

Finally, we can enunciate the following theorem.

Theorem 3.10. *Every simple finite-dimensional Lie superalgebra over $\mathbb{K}$ is isomorphic to one of the following:*

1. *the classical Lie superalgebras, either isomorphic to a simple Lie algebra or to one of the following classical Lie superalgebras:*

   $$A(m|n), \quad B(m|n), \quad C(n), \quad D(m|n), \quad P(n), Q(n),$$

   for appropriate ranges of m and n,

   $$F(4), \quad G(3), \quad D(2,1;\alpha), \ \textit{for } \alpha \in \mathbb{K} \setminus \{0; -1\}$$

2. *the Lie superalgebras of Cartan type:*

   $$W(n), \quad S(n), \quad \tilde{S}(n) \textit{ for even } n, \quad H(n)$$

3.3 Root Systems, Cartan Matrix, Dynkin Diagram

Similarly to the ordinary setting, for Lie superalgebras we have the notion of Cartan subalgebras and the corresponding root decomposition.

Definition 3.11. Let $\mathfrak{g}$ be a simple Lie superalgebra. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{h}$ is nilpotent, self-normalizing Lie subalgebra of $\mathfrak{g}$. If $\alpha \in \mathfrak{h}_0^*$, we define the super vector space

$$\mathfrak{g}_\alpha \overset{def}{=} \{X \in \mathfrak{g} | [h, X] = \alpha(h)X \text{ for all } h \in \mathfrak{h}_0\}$$

If $\mathfrak{g}_\alpha \neq \{0\}$ for $\alpha \in \mathfrak{h}_0^* \setminus \{0\}$ we say α is a *root* and $\mathfrak{g}_\alpha$ a *root space*. A root is even if $\mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq \{0\}$, odd if $\mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq \{0\}$. Notice that $\dim(\mathfrak{g}_\alpha) = 1|0$ or $\dim(\mathfrak{g}_\alpha) = 0|1$ but in Q where a root can be both even and odd, see on [12] for more details. As in the ordinary case, if we denote $\Delta = \Delta_0 \cup \Delta_1$ as the set of all roots we have:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

For the proof of the root space decomposition, see on [7]

Definition 3.12. Let $\mathfrak{g}$ be a Lie classical superalgebra, we denote with κ the *Cartan-Killing form* defined as follows:

$$\kappa(x, y) = \mathrm{str}(\mathrm{ad}(x), \mathrm{ad}(y)),$$

where $x, y \in \mathfrak{g}$.

As one can easily check, this form is symmetric and consistent. However, quite differently from what happens in the classical setting, it is not always non-degenerate. In particular its restriction to a Cartan subalgebra of $\mathfrak{g}$ may be degenerate.
The fact that the Cartan–Killing form of a classical Lie superalgebra may be degenerate prompts the definition of *basic classical* Lie superalgebras.

Definition 3.13. A Lie superalgebra $\mathfrak{g}$ is a *basic classical* Lie superalgebra if $\mathfrak{g}$ is simple, $\mathfrak{g}_0$ is reductive, and $\mathfrak{g}$ admits a non-degenerate invariant symmetric consistent bilinear form.

The following table summarizes the classification of simple Lie superalgebras together with information about the existence of an invariant non-degenerate symmetric bilinear form.

Super Lie Algebra		
Classical Lie Alegbra		Cartan Type
Basic	Strange	
$A(m\vert n)$, $B(m\vert n)$, $C(n)$, and $D(m\vert n)$ $D(1,2;\alpha)$, $G(3)$, $F(4)$	$P(n)$, $Q(n)$	$W(n)$, $S(n)$, $\tilde{S}(n)$, $H(n)$

As in the classical case we can introduce the *Cartan matrix*:

Definition 3.14. The *Cartan matrix* A associated to the simple Lie superalgebra $\mathfrak{g}$ and the simple root system Π is defined as:

$$A = (a_{ij}) = (\alpha_i(h_j))$$

As it happens in the classical theory, we can associate a Dynkin diagram following the rules:

- Put as many nodes as simple roots.
- Connect the i-th node with the j-th node with $|a_{ij}a_{ji}|$ links.
- The i-th node is *white* if α_i is even, the j-th node is *black* if α_j is odd and $a_{ij} \neq 0$ and it is *grey* if α_i is odd and $\alpha_{ij} = 0$.
- The arrow goes from the long to the short root.

Unluckily, in the super case we do not have a bijection between Dynkin diagrams and Lie superalgebras, so we have to define what a *distinguished root system*, a *distinguished Cartan matrix* and a *distinguished Dynkin diagram* are. This fact happens beacuse a basic Lie superalgebra possesses many equivalent simple root system, which correspond to many inequivalent Dynkin diagrams. For a detailed discussion of this fact see on [12].

Definition 3.15. For each basic Lie superalgebra, there exists a simple root system for which the number of odd roots is the smallest one. Such a simple root system is called the *distinguished simple root system.* The associated Cartan matrix is called the distinguished Cartan matrix.

Definition 3.16. The *distinguished Dynkin diagram* is the Dynkin diagram associated to the distinguished simple root system to which corresponds the distinguished Cartan matrix. It is constructed as follows: the even dots are given by the Dynkin diagram of the even part $\mathfrak{g}_0$ (it may be not connected) and the odd dot corresponds to the lowest weight of the representation $\mathfrak{g}_1$ of $\mathfrak{g}_0$.

Remark 3.17. All the Dynkin diagram we use in this discussion are the distinguished ones.

3.4 The classical families: $A(m|n), B(m|n), C(n), D(m|n)$

$A(m|n)$. First we discuss about $A(m|n) = \mathfrak{sl}(m+1|n+1)$ for $m \neq n$. Let $\varepsilon_i, \delta_j \in \mathfrak{h}^*$, $1 \leq i \leq m+1$, $1 \leq j \leq n+1$, defined as $\varepsilon_i(\mathrm{diag}(a_1, \dots, a_{m+n+2})) = a_i,\ i = 1, \dots, m+1$, and $\delta_j(\mathrm{diag}(a_1, \dots, a_{m+n+2})) = a_{m+1+j},\ j = 1, \dots, n+1$. Its root system is:

$$\begin{aligned}\Delta &= \{\varepsilon_i - \varepsilon_j, \delta_k - \delta_l, \pm(\varepsilon_i - \delta_k)\},\\ \Delta_0 &= \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \pm\delta_k \pm \delta_l, \pm 2\delta_k\}\\ \Delta_1 &= \{\pm\varepsilon_i \pm \delta_k, \pm\delta_k\}, \quad 1 \leq k \neq l \leq n.\end{aligned}$$

And its simple root system:

$$\begin{aligned}\Pi = \{&\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{m+1} = \varepsilon_{m+1} - \delta_1,\\ &\alpha_{m+2} = \delta_1 - \delta_2, \dots, \alpha_{m+n-1} = \delta_n - \delta_{n+1}\}\end{aligned}$$

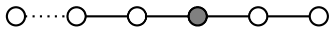
For $A(n|n), n > 1$, the root system and the simple root system are the same, the difference between them is that in this last one we have two relations between ε_i and δ_k, $\varepsilon_1 + \dots + \varepsilon_{m+1} = \delta_1 + \dots + \delta_{n+1} = 0$ instead of ε_i and δ_k, $\varepsilon_1 + \dots + \varepsilon_{m+1} = \delta_1 + \dots + \delta_{n+1}$ in the first case.
Let us now turn to the construction of the Cartan matrix and the Dynkin diagram associated to a classical Lie superalgebra $\mathfrak{g}$ with a simple root system $\Pi = \{\alpha_i\}_{i \in I}$. For each simple root $\alpha_i \in \Pi$, fix elements $e_i \in \mathfrak{g}_\alpha, f_i \in \mathfrak{g}_{-\alpha}$ and set $h_i = [e_i, f_i] \in \mathfrak{g}_0$ which is defined up to a costant. If $\alpha_i(h_i) \neq 0$, we fix it by imposing that $\alpha_i(h_i) = 2$.

In the case of $A(m|n)$ we can choose $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$, so we have that $h_i = E_{ii} - E_{i+1,i+1}$ for $i \neq m+1$, while $h_{m+1} = [e_{m+1}, f_{m+1}] = e_{m+1}f_{m+1} + f_{m+1}e_{m+1} = E_{m+1,m+1} + Em + 2, m + 2$.
The Cartan matrix has the form

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & \dots & 0 & \dots 0 & \\ -1 & 2 & -1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 2 & -1 & 0 & & & \\ 0 & \dots & & -1 & 0 & +1 & \dots & \dots & \dots \\ 0 & \dots & & 0 & -1 & 2 & -1 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -1 & 2 \end{pmatrix}$$

where the zero appears in row $(m+1)$ because $\alpha_{m+1}(h_{m+1}) = 0$.

So we have that the Dynkin diagram of $A(m|n)$ is:

$A(m|n)$ 

$B(m|n)$. $B(m|n) = \mathfrak{osp}(2m+1|2n)$, and we have that

$$\mathfrak{h} = \{h = \text{diag}(a_1, \ldots, a_m, -a_1, \ldots, -a_m, 0, b_1, \ldots, b_n, -b_1, \ldots, -b_n)\}.$$

Let $\varepsilon_i, \delta_j \in \mathfrak{h}^*$ be: for $h \in \mathfrak{h}$, let $\varepsilon_i(h) = a_1,\ i = 1, \ldots, m$ and $\delta_j(h) = b_j,\ j = 1, \ldots, m$.
Its root system for $m \neq 0$ is

$$\begin{aligned} \Delta_0 &= \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \pm\delta_k \pm \delta_l, \pm 2\delta_k\}, \\ \Delta_1 &= \{\pm\varepsilon_i \pm \delta_k, \pm\delta_k\} \quad 1 \leq i \neq j \leq m,\ 1 \leq k \neq l \leq n. \end{aligned}$$

and for $m = 0$ is

$$\Delta_0 = \{\pm\delta_k \pm \delta_l, \pm 2\delta_k\}, \quad \Delta_1 = \{\pm\delta_k\}, \quad 1 \leq k \neq l \leq n$$

The simple root systems for $B(m|n),\ B(0|n)$ are respectively

$$\begin{aligned} \Pi = \{&\alpha_1 = \delta_1 - \delta_2, \ldots, \alpha_{n-1} = \delta_{n-1} - \delta_n, \alpha_n = \delta_n - \varepsilon_1, \\ &\alpha_{n+1} = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{m+n+1} = \varepsilon_{m-1} - \varepsilon_m, \alpha_{m+n} = \varepsilon_m\} \end{aligned}$$

and

$$\Pi = \{\alpha_1 = \delta_1 - \delta_2, \ldots, \alpha_{n-1} = \delta_{n-1} - \delta_n, \alpha_n = \delta_n\}.$$

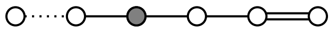
Then we have that the Cartan matrix for $B(m|n)$ with $m \neq 0$ is

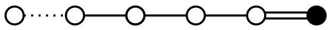
$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & \ldots & 0 & & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 & \ldots & 0 & & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 2 & -1 & 0 & & & & \\ 0 & \ldots & & -1 & 0 & +1 & \ldots & \ldots & & \ldots \\ 0 & \ldots & & 0 & -1 & 2 & -1 & \ldots & & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & & & & & & -1 & 2 & -1 \\ 0 & \ldots & & & & & & 0 & 2 & 2 \end{pmatrix}$$

and for $B(0|n)$ it is

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & \ldots & 0 & \ldots \\ -1 & 2 & -1 & \ldots & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & & & & -1 & 2 & -1 \\ 0 & \ldots & & & & 0 & -2 & 2 \end{pmatrix}$$

The Dynkin diagram is

$B(m|n)$ 

$B(0|n)$ 

$C(n) = \mathfrak{osp}(2|2n-2)$. We have that

$$\mathfrak{h} = \{h = \mathrm{diag}(a_1, -a_1, b_1, \ldots, b_{n-1}, -b_1, \ldots, -b_{n-1})\}$$

Define $\varepsilon_1, \delta_1, \ldots, \delta_{n-1} \in \mathfrak{h}^*$ as follows: for $h \in \mathfrak{h}$, let $\varepsilon_1(h) = a_1, \ldots, \varepsilon_m(h) = a_m, \delta_1(h) = b_1, \ldots, \delta_n(h) = b_n$.
The root system is:

$$\Delta_0 = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\delta_k, \pm\delta_k \pm \delta_l\},$$
$$\Delta_1 = \{\pm\varepsilon_i \pm \delta_k\}, \quad 1 \leq i \neq j \leq m,\ 1 \leq k \neq l \leq n$$

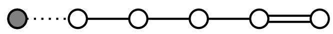
Instead, the root system is:

$$\Pi = \{\alpha_1 = \delta_1 - \delta_2, \ldots, \alpha_{n-1} = \delta_{n-1} - \delta_n, \alpha_n = \delta_n - \varepsilon_1,$$
$$\alpha_{n+1} = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{m+n-1} = \varepsilon_{m-1} - \varepsilon_m, \alpha_{m+n} = \varepsilon_{m-1} + \varepsilon_m\}$$

We have that the Cartan matrix is

$$A = \begin{pmatrix} 0 & +1 & 0 & \ldots & 0 & \ldots & 0 & \ldots \\ -1 & 2 & -1 & \ldots & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & & & & -1 & 2 & -2 \\ 0 & \ldots & & & & 0 & -1 & 2 \end{pmatrix}$$

The Dynkin diagram is

$C(n),\ n > 2$ 

$D(m|n)$**.** $D(m|n) = \mathfrak{osp}(2m|2n)$. We have that

$$\mathfrak{h} = \{h = \mathrm{diag}(a_1, \ldots, a_m, -a_1, \ldots, -a_m, b_1, \ldots, b_n, -b_1, \ldots, -b_n)\}.$$

The root system is:

$$\Delta_0 = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\delta_k \pm \delta_l\},$$
$$\Delta_1 = \{\pm\varepsilon_i \pm \delta_k\}, \quad 1 \leq i \neq j \leq m,\ 1 \leq k \neq l \leq n$$

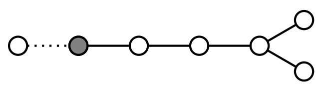
And the simple root system:

$$\Pi = \{\alpha_1 = \delta_1 - \delta_2, \ldots, \alpha_{n-1} = \delta_{n-1} - \delta_n, \alpha_n = \delta_n - \varepsilon_1,$$
$$\alpha_{n+1} = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{m+n-1} = \varepsilon_{m-1} - \varepsilon_m, \alpha_{m+n} = \varepsilon_{m-1} + \varepsilon_m\}.$$

The Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & & 0 \\ -1 & 2 & -1 & \ldots & 0 & \ldots & 0 & \ldots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & & -1 & 0 & +1 & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & & 0 & -1 & 2 & -1 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & & & & & & -1 & 2 & 0 \\ 0 & \ldots & & & & & & -1 & 0 & 2 \end{pmatrix}$$

The Dynkin diagram is

$D(m|n)$ 

3.5 Real Forms of Lie Superalgebras

In this section we show the parallelism and the differences between Lie superalgebras and Lie algebras and we discuss the real forms of $A(m,n), m \neq n$ using two different methods: the algebraic one and the graph painting. Before introducing what a real form of a classical semisimple complex Lie superalgebra is, we have to introduce some definitions and theorems starting from the definition of classical Lie superalgebra of the previous chapter.

Definition 3.18. Let $\mathfrak{g}$ be a classical Lie superalgebra over $\mathbb{C}$. A *semimorphism* C of $\mathfrak{g}$ is a semilinear transformation of $\mathfrak{g}$ which preserves the gradation, that is such that

$$\begin{aligned} &C(X+Y) = C(X) + C(Y) \\ &C(\alpha X) = \bar{\alpha} C(X) \\ &[C(X), C(Y)] = C([X,Y]) \end{aligned}$$

for all $X, Y \in \mathfrak{g}$ and $\alpha \in \mathbb{C}$.

All homomorphisms and semimorphisms of Lie superalgebras will be assumed to preserve $\mathfrak{g}_0, \mathfrak{g}_1$.

Proposition 3.19. *Let* $\mathfrak{g}$ *be a complex Lie superalgebra and let* C *be an involutive semimorphism of* $\mathfrak{g}$*. Then* $\mathfrak{g}_C = \{x + Cx | x \in \mathfrak{g}\}$ *is a real classical Lie superalgebra.*

Proof. From the definition of $\mathfrak{g}_C$ it is obvious that $\mathfrak{g}_C$ is a real simple Lie superalgebra and that its complexification is $\mathfrak{g}$: it is simple because $\mathfrak{g}$ is classical and so simple itself, and it is real because C is an involutive semimorphism. The only thing we have to check is that the representation $\mathfrak{g}_{0C} = \{x + Cx | x \in \mathfrak{g}_0\}$ on $\mathfrak{g}_{1C} = \{x + Cx | x \in \mathfrak{g}_1\}$ is completely reducible. Let $\tilde{V} = V \otimes \mathbb{C}$ be invariant with respect to $\mathfrak{g}_0$; hence, there exists a subspace W' supplementary to $\tilde{V}$ in $\mathfrak{g}_1$ and invariant by $\mathfrak{g}_0$. The subspace $W = (I + C)\{w' \in W' | (I - C)W' \in iV\}$ of $\mathfrak{g}_{1C}$ is supplementary to V and invariant by $\mathfrak{g}_{0C}$. Since $C(I + C) = C + I$, $W \subset \mathfrak{g}_{1C}$, moreover, if $g \in \mathfrak{g}_{0C}, [g, w] = (I + C)[g, w] = (I - C)[g, w'] = [g, (I + C)w']$, so W is invariant by $\mathfrak{g}_{0C}$. Last but not least W is supplementary to V, if we have $w = (I + C)w' \in V$, $2w' = (I + C)w' + (I - C)w' \in V + iV = \tilde{V}$; hence $w' = 0$ and thus $w' \in W'$ and $W \cap V = 0$. On the other hand, if $x \in \mathfrak{g}_{1C}$, one has $x = w' + v'$ where $w' \in W'$ and $v' \in \tilde{V}$. However, $(I - C)w' \in W$; hence, $2x = (I + C)x = (I + X)w' + (I + C)v' \in W + V$. □

Proposition 3.20. *If* $\mathfrak{g}$ *is a real classical Lie superalgebra, its complexification* $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$ *is a Lie superalgebra which is either classical or the direct sum of two isomorphic ideals which are classical.*

Proof. Let C be the conjugation in $\tilde{\mathfrak{g}}$ with respect to $\mathfrak{g}$. We note that the representation of $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \otimes \mathbb{C}$ on $\tilde{\mathfrak{g}}_1 = \mathfrak{g}_1 \otimes \mathbb{C}$ is completely reducible. Let indeed V be a complex subspace of $\tilde{\mathfrak{g}}_1$ which is invariant by $\tilde{\mathfrak{g}}_0$. Hence, V is invariant by $\mathfrak{g}_0$. Thus, there exists a subspace W' supplementary to V' in $\mathfrak{g}_1$ and invariant by $\mathfrak{g}_0$. Then $\tilde{\mathfrak{g}}_1 = \tilde{V}' \oplus \tilde{W}'$ is invariant by $\tilde{\mathfrak{g}}_0$, which proves the first point. If $\tilde{\mathfrak{g}}$ is not simple, it contains a simple graded ideal S. Then $(I + C)S = \mathfrak{g}$ is a graded ideal of $\mathfrak{g}$ so either $(I + C)S = 0$ or $(I + C)S = \mathfrak{g}$. However, $(I + C)S = 0$ is impossible since $S + CS = 0$ implies $iS + C(iS) \neq 0$. Hence, $(I + C)S = \mathfrak{g}$ and $\tilde{\mathfrak{g}} = \mathfrak{g} + i\mathfrak{g} = (I + C)S + (I - C)S = S + CS$. Since $S \cup CS$ is an ideal of S, we have $S \cup CS = 0$, which shows that $\tilde{\mathfrak{g}}$ is the direct sum of the two ideals S and CS. □

Proposition 3.21. *Let* $\mathfrak{g}$ *be a complex Lie superalgebra and let* C *and* C' *be two involutive semimorphism of* $\mathfrak{g}$*. The real forms* $\mathfrak{g}_C$ *and* $\mathfrak{g}_{C'}$ *are isomorphic if and only if there exists an automorphism* φ *of* $\mathfrak{g}$ *such that* $C' = \varphi C \varphi^{-1}$*.*

Proof. If $C' = \varphi C \varphi^{-1}$, it is clear that $\mathfrak{g}_C = \varphi \mathfrak{g}_{C'}$. Conversely, assume there exists an isomorphism ψ from $\mathfrak{g}_C$ into $\mathfrak{g}_{C'}$. The linear extension φ of ψ to $\mathfrak{g} = \mathfrak{g}_C + i\mathfrak{g}_C$ is defined by $\varphi(g + ig) = \psi g + i\psi g$, with $g \in \mathfrak{g}$, is an automorphism of $\mathfrak{g}$. Moreover, if $g' = \psi g \in \mathfrak{g}_{C'}$ we have $C'g' = g' = \psi g = \varphi g = \phi C g = \varphi C \varphi^{-1} g'$; hence, $C' \varphi C \varphi^{-1}$ is the identity on $\mathfrak{g}_{C'}$ and thus also on $\mathfrak{g}$. □

As a consequence of the previous two propositions, now, we can say:

Proposition 3.22. *Let $\mathfrak{g}$ be a real classical Lie algebra, $\mathfrak{g}$. Then there are only two possibilities:*

- *If the complexification of $\mathfrak{g}$ is not simple, $\mathfrak{g}$ is a complex classical Lie superalgebra considered as a real algebra;*
- *If the complexification $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ is simple, $\mathfrak{g}$ is the subalgebra of fixed points of an involutive semimorphism of $\tilde{\mathfrak{g}}$.*

Now we see how to classify the involutive semimorphism in order to classify, after that, the real forms of $A(m|n)$. Before the main proposition we have to show a small Lemma:

Lemma 3.23. *If $\mathfrak{g}$ is not $D(n)$ or $B(n)$ (defined in paragraph 3.2) and φ_0 is an inner automorphism of $\mathfrak{g}_0$, there exists an automorphism $\varphi = \varphi_0 + \varphi_1$ of $\mathfrak{g}$.*

Proof. Let ρ denote the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ and let B be any non-degenerate invariant bilinear form of $\mathfrak{g}$. If $\varphi_0 = e^{\mathrm{ad}(n)}, \varphi_1 = e^{\rho(n)}$ satisfy $\rho(\varphi_0 g)\varphi_1 = \varphi_1 \rho(g)$ for all simple ideals of $\mathfrak{g}_0$ and $n \in \mathfrak{g}_0$. On the center of $\mathfrak{g}_0 \varphi_0$ is the identity. On each of the simple ideals of $\mathfrak{g}_0$, the bilinear form B is a multiple of the Killing form: hence φ_0 is an isometry for B. On the other hand, $B(\rho(n)x, y) + B(x, \rho(n)y) = 0$ for all $x, y \in \mathfrak{g}_1$ implies that φ_1 is also an isometry for B. Then we have for all $g \in \mathfrak{g}_0, B([\varphi_1 x, y], \varphi_0 g) = B(\varphi_1 x, [\varphi_1 y, \varphi_0 g]) = B(\varphi_1 x, \varphi_a[y, g]) = B(x, [y, g]) = B([x, y], g) = B(\varphi_0[x, y], \varphi_0 g)$ and thus $[\varphi_1 x, \varphi_1 y] = \varphi_0[x, y]$. □

Proposition 3.24. *Le $\mathfrak{g}$ be a complex classical Lie superalgebra and let $C = C_0 + C_1$ be an involutive semimorphism of $\mathfrak{g}$. Assume C_0' is an involutive semimorphism of $\mathfrak{g}_0$ conjugate to C_0 in* $\mathrm{Aut}(\mathfrak{g}_0)$. *Then there exists an involutive semimorphism $C = C_0' + C_1'$ of $\mathfrak{g}$ which is conjugate to C in* $\mathrm{Aut}(\mathfrak{g})$.

Proof. Assume $C_0' = \varphi_0 C_0 \varphi_0^{-1}$, where $\varphi_0 \in \mathrm{Aut}(\mathfrak{g}_0)$. If there exists $\varphi = \varphi_0 + \varphi_1 \in \mathrm{Aut}(\mathfrak{g})$, then $C' = \varphi C \varphi^{-1}$ is an involutive semimorphism of $\mathfrak{g}$ and the proposition is proved. The existence of φ will follow from the previous Lemma when φ_0 is inner and $\mathfrak{g}$ is not $D(n)$ or $B(n)$. □

Now we see that for a given real form $\mathfrak{g}_{0C}$ of the Lie subalgebra $\mathfrak{g}_0$ there exists, up to isomorphism, at most two real forms $\mathfrak{g}_C$ which contains $\mathfrak{g}_{0C}$, we will see that these two forms are isomorphic. Let $C = C_0 + C_1$ and $C' = C'_0 + C'_1$ be the two involutive semimorphism which have the same restriction $C'_0 = C_0$ to $\mathfrak{g}_0$.

Lemma 3.25. *If the representation ρ of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is irreducible, then $C'_1 = \pm C_1$.*

Proof. The linear transformation $C_1 C'_1$ of $\mathfrak{g}$ commutes with $\rho(\mathfrak{g}_0)$; hence $C'_1 = \lambda C_1$ and $C_1^2 C_1'^2 = \mathrm{Id}$ implies $\lambda\bar{\lambda} = 1$. If $x, y \in \mathfrak{g}$, we have $C_0[x,y] = [C'_1 x, C'_1 y] = \lambda^2 [C_1 x, C_1 y] = \lambda^2 C_0[x,y]$, so $\lambda = \pm 1$. If the representation ρ of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is reducible, we write $\mathfrak{g}_1 = Y' \oplus Y''$ for the sum of the invariant subspaces and if $\mathfrak{g}_0$ is not semisimple, we denote by k_0 the element of the center of $\mathfrak{g}_0$ such that $\rho(k_0)|_{Y'} = \mathrm{Id}$ and $\rho(k_0)|_{Y''} = -\mathrm{Id}$. □

Lemma 3.26. *Let us use the same notation and hypothesis of the previous Lemma:*

1. *If C_1 preserves Y' and Y'', then $C_0 k_0 = k_0$ and if C_1 permutes Y' and Y'', then $C_0 k_0 = -k_0$;*

2. *If $C = C_0 + C_1$ and $C' = C_0 + C'_1$ preserve Y' and Y'', then they are conjugate in* $\mathrm{Aut}(\mathfrak{g})$*;*

3. *If they permute Y' and Y'', then C' is conjugate in* $\mathrm{Aut}(\mathfrak{g})$ *to $C_0 + C_1$ or to $C_0 - C_1$.*

where $\mathfrak{g}_1$ is the direct sum of the two subspaces Y', Y'' and $[Y', Y'] = [Y'', Y''] = 0$ and $[Y', Y''] = \mathfrak{g}_0$.

Proof. 1. Since the decomposition of $\mathfrak{g}_1$ into Y' and Y'' is unique, any semimorphism of $\mathfrak{g}$ preserves or permutes Y' and Y''. If $\mathfrak{g}_0$ is not semisimple, we set $C_0 k_0 = a k_0$, where $a\bar{a} = 1$. Then $[C_0 k_0, C_1 y] = C_1[k_0, y] = C_1 y$ with $y \in Y'$, which implies $a = 1$ if $C_1 Y' = Y'$ and $a = -1$ if $C_1 Y' = Y''$.

2. If we set $C_1 = C' + C''$, where $C' = C_1|_{Y'}$ and $C'' = C_1|_{Y''}$, by the same argument of the previous Lemma, we know that C' and C'' are unique up to a factor of modulus 1, so we may write $C'_1 = \lambda C' + \mu C''$. If $x \in Y'$ and $y \in Y''$, we have $C_0[x,y] = [\lambda C' x, \mu C'' y] = \lambda\mu [C' x, C'' y] = \lambda\mu C_0[x,y]$ and thus $\lambda\mu = 1$. However, the linear transformation ψ defined by $\psi y' = \lambda^{1/2} y'$, $\psi y'' = \lambda^{-1/2} y''$, and $\psi g_0 = g_0$ is an automorphism og $\mathfrak{g}$ and $C_0 + C'_1 = \psi(C_0 + C_1)\psi^{-1}$.

3. Since C_1C_1' preserves Y' and Y'' and commutes with $\rho(\mathfrak{g}_0)$, we may write $C_1'x = \lambda C_1 x$ and $C_1'y = \mu C_1 y$ for all $x \in Y'$ and $y \in Y''$. From $C_1^2 = C_1'^2 = \mathrm{Id}$, we deduce $\bar{\lambda}\mu = 1$ and $C_0[x,y] = [C_1'x, C_1'y]$ implies as above $\lambda\mu = 1$. We define an automorphism ψ of $\mathfrak{g}$ by $\psi y' = \lambda^{-1/2}y', \psi y'' = \lambda^{1/2}y''$ and $\psi g_0 = g_0$. If the real number λ is positive, we have $C_0 + C_1' = \psi(C_0 + C_1)\psi^{-1}$ and if λ is negative, we have $C_0 + C_1' = \psi(C_0 - C_1)\psi^{-1}$. □

3.6 Real Forms of $A(m|n)$

Now we consider $m \neq n$,as we have said in the previous chapter, $A(m|n) \cong \mathfrak{sl}(m|n)$.
If we write an element of $A(m|n)$ as in the previous chapter:

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sl}(m+1|n+1)$$

we can now say that $\mathfrak{g}_0$ is the direct sum of its one dimensional center K_0 and of the two simple ideals K_1 and K_2 of the respective type A_m and A_n where:

- $K_0 = \{X \in A(m|n) | C = D = 0,\ A = na\mathrm{Id}_m, B = ma\mathrm{Id}_n,\ a \in \mathbb{C}\}$;
- $K_1 = \{X \in A(m|n) | A = B = D = 0\}$;
- $K_2 = \{X \in A(m|n) | A = B = C = 0\}$.

The subspace $\mathfrak{g}_1$ is the direct sum of the two invariant subspaces $Y' = \{X \in A(m|n) | A = B = D = 0\}$ and $Y'' = \{X \in A(m|n) | A = B = C = 0\}$. The representation ρ' of $K_1 \oplus K_2$ on Y' is the tensor product of the natural representation π_1 of K_1 and the contragradient representation π_{n-1} of K_2, which we will abbreviate $\rho' = \pi_1(K_1) \otimes \pi_{n-1}(K_2)$. Similarly, $\rho'' = \pi_{m-1}(K_1) \otimes \pi_1(K_2)$.
Both the natural representation π_1 of A_m and its contragradient π_m are real for the real form $\mathfrak{sl}(m+1,\mathbb{R})$, they are unreal for $\mathfrak{su}^*(m+1)$ and real for $\mathfrak{su}(p, m+1-p)$. Hence the only real forms of $K_1 \oplus K_2$, for which the irreducible representations ρ' and ρ'' are real, are $\mathfrak{sl}(m,\mathbb{R}) \oplus \mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{su}^*(m) \oplus su^*(n)$ if m and n are even.
For the real form $\mathfrak{su}(p, m-p) \oplus \mathfrak{su}(q, n-q)$ of $K_1 \oplus K_2$, we will see that exists an extension to $\mathfrak{g}$ of the semimorphism C_0 which permutes Y' and Y''. We now consider the real forms of $K_1 \oplus K_2$ in which K_{1C} and K_{2C} are of different types. Since the representations ρ' and ρ'' are never real, any extension of C_0

to $\mathfrak{g}$ must permute Y' and Y''. However, the existence of a semilinear involution C_1 permuting Y' and Y'' such that $\rho'(C_0X)C_1 = C_1\rho''(X)$, implies that the weights of ρ'' are conjugate to those of $\rho' \circ C$. But this never happens and we check it in the following way:

If $K_{1C} \oplus K_{2C}$ is of the form	The weights of $\rho' \circ C$ are conjugate to those of
$\mathfrak{su}(p, m-o) \oplus \mathfrak{sl}(n, \mathbb{R})$	$\pi_{m-1}(K_1) \otimes \pi_{n-1}(K_2)$
$\mathfrak{su}(p, m-o) \oplus \mathfrak{su}^*(n)$	$\pi_{m-1}(K_1) \otimes \pi_{n-1}(K_2)$
$\mathfrak{sl}(m, \mathbb{R}) \oplus \mathfrak{su}^*(n)$	$\pi_1(K_1) \otimes \pi_{n-1}(K_2)$
$\mathfrak{sl}(m, \mathbb{R}) \oplus \mathfrak{su}^*(q, n-q)$	$\pi_1(K_1) \otimes \pi_1(K_2)$
$\mathfrak{su}^*(m) \oplus \mathfrak{sl}(n, \mathbb{R})$	$\pi_1(K_1) \otimes \pi_{n-1}(K_2)$
$\mathfrak{su}^*(m) \oplus \mathfrak{su}^*(q, n-q)$	$\pi_1(K_1) \otimes \pi_1(K_2)$

while $\rho'' = \pi_{m-1}(K_1) \otimes \pi(K_2)$.
The possible real forms are the following:

1. $\mathfrak{g}_{0C} = \mathfrak{sl}(m, \mathbb{R}) \oplus \mathbb{R}$, the involutive semimorphism is $CX = \bar{X}$ and it preserves Y' and Y''. With the notations of Lemma 3.26, we have $Ck_0 = k_0$ and hence $K_{0C} = \mathbb{R}$.

2. $\mathfrak{g}_{0C} = \mathfrak{su}^*(m) \oplus \mathfrak{su}^*(n) \oplus \mathbb{R}$ if m and n are even. The involutive semimorpshim is $CX = M\bar{X}M^{-1}$ where
$$M = \begin{pmatrix} \text{antidiag}(-\text{Id}_r, \text{Id}_r) & \\ & \text{antidiag}(-\text{Id}_s, \text{Id}_s) \end{pmatrix}.$$
Again C preserves Y' and Y'', $K_{0C} = \mathbb{R}$.

3. $\mathfrak{g}_{0C} = \mathfrak{su}(p, m-p) \oplus \mathfrak{su}(q, n-q) \oplus i\mathbb{R}$. The involutive semimorphism is $C_0X = -N\bar{X}^tN$ if $X \in \mathfrak{g}_0$ and $C_1X = iN\bar{X}^tN$ if $X \in \mathfrak{g}_1$, where $N = \text{diag}(-\text{Id}_p, \text{Id}_{m-p}, -\text{Id}_q, \text{Id}_{n-q})$. Since C permutes Y' and Y'', it follows that $Ck_0 = -k_0$ and $K_{0C} = i\mathbb{R}$.

We now prove that C_0 may be chosen up to conjugacy in $\text{Aut}(\mathfrak{g}_0)$. Every $X \in \text{Aut}(\mathfrak{g}_0)$ preserves the two ideals K_1 and K_2 and on each ideal is of the form $\psi\theta$ where θ is inner and $\psi(X) = -X^t$. All the three C_0, we have chosen to define the real forms, commutes with ψ, so any C_0' conjugate to C_0 is of the form $\theta C\theta^{-1}$. Finally because of Lemma 3.26, all semimorphism extending C_0 are conjugate in the first two cases. In the third, we define $\varphi \in \text{Aut}(\mathfrak{g})$ as $\varphi = \varphi_0 + \varphi_1$, where $\varphi_0X = -NX^tN$ if $X \in \mathfrak{g}_0$ and $\varphi_1C_1 = -C_1\varphi_1$, thus proving that $C_0 + C_1$ and $C_0 - C_1$ are conjugate by $\text{Aut}(\mathfrak{g})$.
Now we will investigate the case where $m = n$; the biggest difference between $A(m|n), m \neq n$ and $A(m|m)$ is that $\mathfrak{sl}(m|m)$ is not semisimple, but it has a one dimensional center. As we have said in the previous chapter, we will consider $\mathfrak{sl}(m|m)/\mathbb{K}\text{Id}_{2m}$. For the semimorphism C_0 which preserve

K_1 and K_2, we can apply the same reason of $A(m|n)$ and we obtain the real forms containing:

1. $\mathfrak{g}_{0C} = \mathfrak{sl}(m, \mathbb{R})$
2. $\mathfrak{g}_{0C} = \mathfrak{su}^*(m) \oplus \mathfrak{su}^*(m)$ if m is even
3. $\mathfrak{g}_{0C} = \mathfrak{su}(p, m-p) \oplus \mathfrak{su}(q, m-q)$
4. There is also the real form where $\mathfrak{g}_{0C} = \mathfrak{sl}(m, \mathbb{C})$ is the real algebra of dimension $2(m^2-1)$, defined by the semimorphism $CX = P\bar{C}XP$, with $P = \text{antidiag}(\text{Id}_m, \text{Id}_m)$, which permutes the two ideals K_1 and K_2 of $\mathfrak{g}_0$ and the subspaces Y', Y'' of $\mathfrak{g}_1$.

The proof of the first three cases are the same of the previous ones, where we have shown that all semimorphism extending C_0 are conjugate in $\text{Aut}(G)$. The fourth is a little bit different, the other possible semimorphism $C_0 - C_1$ is conjugate to $C_0 + C_1$ by the automorphism $\varphi = \varphi_0 + \varphi_1$, where $\varphi_0 X = -X^t$ if $X \in \mathfrak{g}_0$ and $\varphi_1 X = iX^t$ if $X \in \mathfrak{g}_1$.
To prove that C_1 may be chosen up to conjugacy by $\text{Aut}(\mathfrak{g}_0)$, it is sufficient to show that any semimorphism of $\mathfrak{g}_0$ extends to an automorphism of $\mathfrak{g}$. Any element of $\text{Aut}(\mathfrak{g}_0)$ may be written as a product $\psi_0\theta_0\eta_0$ or $\psi_0\theta_0$ or θ_0 where θ_0 is inner, $\psi_0 X = -X^t$, and $\eta_0 X = PXP$. By Lemma 3.25, we know that θ_0 extends to $\mathfrak{g}$. Automorphisms of $\mathfrak{g}$ extending ψ_0 and η_0 are defined, respectively, by $\psi_1 X = iX^t$ and $\eta_1 X = PXP$ if $X \in \mathfrak{g}$.

3.7 Vogan Diagrams

We have seen in section 2.4 what a Vogan diagram is, so now we want to define these diagrams in the super case. They are essential to classify the real forms of the Lie superalgebras.
Our discussion is only about *basic Lie superalgebras*, where we have that the Cartan subalgebra is totally even, hence this semplifies our discussion. We also are under the assumption that

$$\mathfrak{h} \subset \mathfrak{k}_0 \subset \mathfrak{g}_0$$

when $\mathfrak{h}$ is a fixed Cartan subalgebra. Hence, let $\mathfrak{g}$ denote a basic Lie superalgebra

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

and let

$$\mathfrak{g}_{\bar{0}} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{p}_{\bar{0}}$$

semisimple be the complexification of a CArtan decomposition of the semisimple part of the complex $<$lie algebra $\mathfrak{g}_{\bar{0}}$. We call $\mathfrak{g} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{p}, \mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{b}_{\bar{1}}$ a complex Cartan decomposition. It correspond to a unique real form $\mathfrak{g}_C$ of $\mathfrak{g}$, see on [8]. This is not the general, but for clarity of exposition we restrict ourselves to this. In this way we only obtain Vogan diagrams with no arrows.

Definition 3.27. Let D be a distinguished Dynkin diagram. The *Vogan diagram* of Lie superalgebras is the Vogan diagram of the even part of Lie superalgebras. In addition to that:

1. The vertices fixed by the Cartan involution of the even part is painted (or unpainted) depending whether the root is noncompact (or compact).
2. The odd root remains unchanged.

Definition 3.28. An *abstract Vogan diagram with no arrow* is an abstract Dynkin diagram with the subset of noncompact roots which is indicated by painting the vertices. Every Vogan diagram is of course an abstract Vogan diagram of a Lie superalgebra.

Theorem 3.29. *If an abstract Vogan diagram with no arrows is given, then, there exists a real Lie superalgebra $\mathfrak{g}_C$, a Cartan involution θ, a Cartan subalgebra and a positive system Δ_0^+ for $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ such that the given diagram is the Vogan diagram of $(\mathfrak{g}_C, \mathfrak{h}_0, \Delta_0^+)$.*

For the proof see on [9].
Now we present a modified version of Borel-de Siebenthal theorem for Lie superalgebras.

Theorem 3.30. *Let $\mathfrak{g}_C$ be a non complex real Lie superalgebra and let the Vogan diagram of $\mathfrak{g}_C$ be given that corresponding to the triple $(\mathfrak{g}_C, \mathfrak{h}_0, \Delta^+)$. Then exists a simple system Π' for $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, with corresponding positive system Δ^+, such that $(\mathfrak{g}_C, \mathfrak{h}_0, \Delta^+)$ is a triple and there is at most two painted simple root in its Vogan diagrams of $A(m|n), D(m|n)$ and at most three painted vertices in $D(2,1;\alpha)$. Furthermore suppose the automorphism associated with the Vogan diagram is the identity that $\Pi' = \alpha_1, \dots, \alpha_l$ and that $\omega_1, \dots, \omega_l$ is the dual basis for each even part such that $\langle \omega_j, \alpha_k \rangle = \delta_{jk}/\epsilon_{kk}$ where ϵ_{kk} is the diagonal entries to make Cartan matrix symmetric. The double painted simple root of even parts may be chosen so that there is no i' with $\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0$ for each even part.*

To see the proof of Theorem 3.30 see on [9] and [8]. As in the classical case we have the following properties for the roots:

compact+compact=compact
compact+noncompact=noncompact
noncompact+noncompact=compact.

3.8 Graph Paintings

In this section we study a procedure to obtain froam a given Vogan diagram another equivalent one, with fewer noncompact vertices.
Recall that in the Dynkin diagram of Lie superalgebras, we have that vertices can be white, grey, black or grey and white.

Definition 3.31. Two Dynkin diagrams Γ, Γ' are said to be *related* if they represent the same basic Lie superalgebra, and we denote this by $\Gamma \sim \Gamma'$.

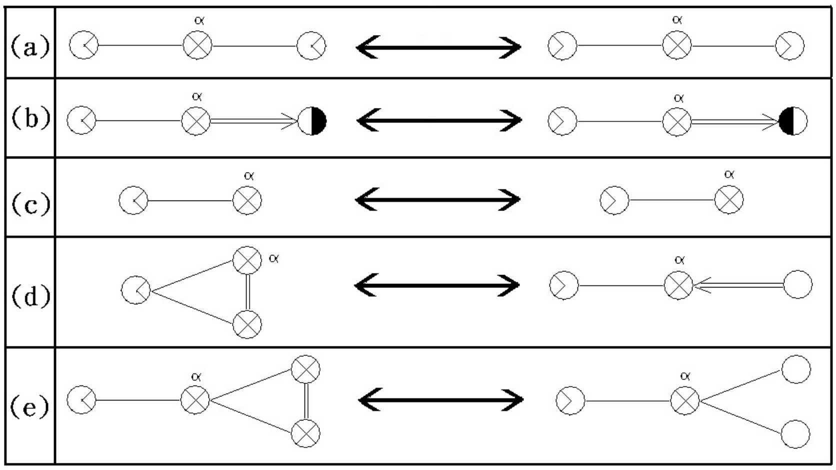
As in the classical case, we can indicate the black and grey verices of a Vogan diagram with the $k-uple$ $(i_1, i_1, \ldots, i_r)$ where $0 \leq i_1, \leq \cdots \leq i_r \leq n$ where n is the number of vertices of our diagram and the grey or black vertex appears odd number of times and a white vertex appears even number of times.
Let $\mathfrak{g}$ be a basic Lie superalgebra, with Cartan subalgebra $\mathfrak{h}$ and root system $\Delta \subset \mathfrak{h}^*$. Fix a simple system $\Pi \subset \Delta$. Its Dynkin diagram has Π as vertices, and its edges depend on the pairing of roots under an invariant supersymmetric form, since we do not ask for a positive definite form, roots may have zero length. Write $\Delta = \Delta_0 \cup \Delta_1$ for the even and odd roots. Let $\alpha \in \Pi$, if α is even, the its Weyl reflection s_α is an automorphism on Δ_0 and Δ_1, so Π and $s_\alpha \Pi$ produce the same Dynkin diagram. If α is odd, we define s_α as followes. Given $\beta \in \Pi$, we let $s_\alpha(\beta)$ be

$$s_\alpha(\beta) \begin{cases} \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha & \text{if } (\alpha,\alpha) \neq 0, \\ \beta + \alpha & \text{if } (\alpha,\alpha) = 0 \text{ and } (\alpha,\beta) \neq 0, \\ \beta & \text{if } (\alpha,\alpha) = (\alpha,\beta) = 0, \\ -\alpha & \text{if } \beta = \alpha \end{cases}$$

Theorem 3.32. *For any $\alpha \in \Pi$, $s_\alpha \Pi$ is again a simple system. For fixed Π, its orbit under such s_α, exhausts all simple systems of $(\mathfrak{g}, \mathfrak{h})$. Namely if $\Pi' \subset \Delta$ is another simple system, then there is a sequence $\Pi = \Pi_1 \to \Pi_2 \to \cdots \to \Pi_a = \Pi'$ such that each $\Pi_i \to \Pi_{i+1}$ is given by s_α for some odd root $\alpha \in \Pi_i$.*

The above method gives a practical way to find the Dynkin diagram related to a given one.
Now we look what happen when α is grey. If $\beta \neq \alpha$ is perpendicular to α, then β is not moved by s_α, otherwise β changes by a multiple of α. Hence the colors of the vertices that are not adjacent to α remain the same, as do the edges attached to these vertices. Therefore, it suffices to study the vertices which are adjacent to α, and this is described by the next proposition.

Proposition 3.33. *Let α be an odd root. Then the following table, in which the grey vertices are indicated with a crossed one, reveals the effect of the odd root reflection s_α.*



For the proof see on [13]. Also in this case we have some combinatorial rules.

Lemma 3.34. *Let $(\Gamma, (i_1, i_2, \dots, i_r))$ be a diagram as above.*

a) $(\Gamma, (i_1, i_2, \dots, i_r)) \sim (\Gamma, (i_1 - 1, i_2 - 1, i_3, \dots, i_r))$*;*

b) $(\Gamma, (i_1, i_2, \dots, i_r)) \sim (\Gamma, (i_2 - i_1, i_3, \dots, i_r))$*;*

c) $(\Gamma, (i)) \sim (\Gamma, (n + 1 - i))$

Proof. For part a), apply $s_{i_1}, s_{i_1+1}, \dots, s_{i_2-1}$ consecutively to $(\Gamma, (i_1, i_2, \dots, i_r))$. Namely,

$$
\begin{aligned}
(\Gamma, (i_1, i_2, \dots, i_r)) &\to (\Gamma, (i_1 - 1, i_1, i_1 + 1, i_2, \dots, i_r)) && \text{by } s_i \\
&\to \dots \to (\Gamma, (i_1 - 1, i_2 - 2, i_2 - 1, i_2, i_3, \dots, i_r)) && \text{by } s_{i_1+,}, \dots, s_{i_2-2} \\
&\to (\Gamma, (i_1 - 1, i_2 - 1, i_3, \dots, i_r)). && \text{by} s_{i_2-1}
\end{aligned}
$$

By applying a) inductively, we keep shifting the pair leftward and obtain

$$(\Gamma, (i_1, i_2, \ldots, i_r)) \sim (\Gamma, (\, i_2 - i_1 + 1, i_3, \ldots, i_r))$$

Then apply $s_1, s_2, \ldots, s_{i_2 - i_1}$ we obtain b). For c) we have only to apply s_j, $\forall j \in \{i, \ldots, n\}$ to (i). □

As in the classical case we can apply consecutively Lemma 3.34 to reach a diagram with the minimum number of black and grey vertices.

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