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Non-Gaussianities from a sudden change in the inflaton speed of sound

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At the end of this work, I think some thanks are due.

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Abstract

Il concetto di inflazione è stato introdotto nei primi anni '80 per risolvere alcuni problemi del modello cosmologico standard, quali quello dell'orizzonte e quello della piattezza.
Le predizioni dei più semplici modelli inflazionari sono in buon accordo con le osservazioni cosmologiche più recenti, che confermano sezioni spaziali piatte e uno spettro di fluttuazioni primordiali con statistica vicina a quella gaussiana.
I più recenti dati di *Planck* [1], pur in ottimo accordo con una semplice legge di potenza per lo spettro a scale $k > 0.08\,\mathrm{Mpc}^{-1}$, sembrano indicare possibili deviazioni a scale maggiori, seppur non a un livello statisticamente significativo a causa della varianza cosmica.
Queste deviazioni nello spettro possono essere spiegate da modelli inflazionari che includono una violazione della condizione di lento rotolamento (*slow-roll*) e che hanno precise predizioni per lo spettro.
Per uno dei primi modelli, caratterizzato da una discontinuità nella derivata prima del potenziale proposto da Starobinsky [2], lo spettro ed il bispettro delle fluttuazioni primordiali sono noti analiticamente [3].

In questa tesi estenderemo tale modello a termini cinetici non standard, calcolandone analiticamente il bispettro e confrontando i risultati ottenuti con quanto presente in letteratura.
In particolare, l'introduzione di un termine cinetico non standard permetterà di ottenere una velocità del suono per l'inflatone non banale, che consentirà di estendere i risultati noti, riguardanti il bispettro, per questo modello.
Innanzitutto studieremo le correzioni al bispettro noto in letteratura dovute al fatto che in questo caso la velocità del suono è una funzione dipendente dal tempo; successivamente, cercheremo di calcolare analiticamente un ulteriore contributo al bispettro proporzionale alla derivata prima della velocità del suono (che per il modello originale è nullo).

Introduction

General Relativity (GR) and Quantum Field Theory (QFT) are two of the most remarkable developments of the last century.
From the beginning, scientists tried to unify these two theories, but they quickly realized that it is not possible to treat gravity like the other forces, because this theory is not renormalizable: for this reason, gravity is often considered as a background, a curvature of space-time where the other forces act on. Such a semiclassical approach is called *Quantum field theory in curved space-time.*
One of the first subjects this theory was applied to has been cosmology, particularly for the study of the early stages of the Universe. The Standard Cosmological Model presents some problems which are not explainable within the standard theory of Big Bang (e.g. the flatness and the horizon problem). In order to solve this issues, in late $70s$/early $80s$ the idea of *inflation* was introduced: it consists in a fast, quasi-exponential expansion in space starting about 10^{-37}s after a possible initial singularity. In terms of a QFT, the simplest models involve a real scalar field, called *inflaton*, as the responsible for this quasi-exponential expansion.
As already stated, the idea of inflation was able to explain some important results of observations, as the high level of isotropy of the cosmic microwave background (CMB), the spatial flatness, the nearly scale-invariant spectrum of nearly Gaussian perturbations, in agreement with the large-scales structure (LSS).
With the results available from the *Planck* satellite (which can be found, e.g., in [1, 4], and [5, 6]) the tilt of the primordial power spectrum has been accurately measured and the level of non-Gaussianity has been tightly constrained, confirming that the simplest models of slow-roll inflation are a good fit to CMB observations, whereas non-standard models producing large non-Gaussianities are not.
On the other side, few puzzles in the CMB pattern of temperature anisotropies on large scales appear, from unexpected features in the temperature power spectrum. For instance, a low amplitude at $\ell < 40$ and a dip in the temperature power spectrum lower than that expected in a simple ΛCDM model.
Recently, many models which predict a temporary violation of the slow-roll condition have been proposed to explain the low amplitude cited above; some of these models can also produce a feature at $\ell \sim 20$.
Inflationary models which aim at explaining this feature at $\ell < 40$ generally predict non-Gaussian signals which can be compared with CMB data. Since the anomaly in temperature power spectrum is not statistically significant because of the cosmic variance, it is important to search for additional, possible counterparts at the bispectrum level.

This thesis is devoted to study inflationary models which can explain these features. In particular, this work aims to study non-Gaussianities which are originated by a sudden jump in the speed of sound. To obtain such a jump, we consider theories for which the inflaton Lagrangian contains a non-standard kinetic term and a discontinuity in the first derivative of the potential (this model, with a standard kinetic term, was proposed by Starobinsky in 1992, [2]). On the one hand, in this way we are able to recover the results found in [3] in a suitable limit, while on the other hand we can provide its generalization which comes from the terms proportional to the derivative of c_s. In addition, we can also compute a second contribution to the bispectrum which comes from a vertex proportional to the logarithmic derivative of the speed of sound, which for the model with a standard kinetic term vanishes.
In details, the following work is structured as follows:

i) In chapter 1 we briefly review some basic concepts of cosmology, such as the Friedman equations and the conceptual issues which led to inflation. We then discuss more advanced topics, as inflation driven by a field with a general Lagrangian.

ii) In chapter 2 we review the relativistic theory of cosmological perturbations; we then introduce the Mukhanov-Sasaki equation for the gauge-invariant scalar field fluctuations and the curvature perturbation.

iii) In chapter 3 we discuss in general power spectrum and bispectrum of curvature perturbations, giving details of their calculations.

iv) In chapter 4 we present the model with a discontinuity in the first derivative of the potential, originally introduced by Starobinsky, describing the motivations for which it is interesting in the comparison with observations. We compute both the power spectrum and the bispectrum (the latter given in [3]).

v) In chapter 5 we provide two different extensions to the model originally introduced in [2], obtained by generalizing the Lagrangian with higher order powers in the kinetic term. We first compute the correction to the bispectrum found in the previous chapter, and then we consider another contribution to the bispectrum which comes from a different vertex, which vanishes for the model with standard kinetic term.

vi) In the two appendixes we briefly review the concepts of classical Gaussian and non-Gaussian random fields. In the second one, instead, we give the full expressions of the original result for the bispectra for the models studied in chapter 5.

Throughout this work, we consider natural units for simplicity, i.e. units for which $c = \hbar = k_{\rm B} = 1$. In addition, we assume the metric signature $(-,+,+,+)$, while for tensors we use Greek letters for space-time indexes ($\mu = 0, \dots, 3$) and Latin letters for spatial indexes ($i = 1, \dots, 3$).

Contents

Chapter 1

The Standard Big-Bang Cosmological Model

Modern cosmology aims to explain the origin and the evolution of our universe. The first models were developed just after the publication of the Theory of General Relativity (1916).
On the one hand, modern cosmology is based on three assumptions: General Relativity is the theory which describes the evolution of our universe; there is homogeneity and isotropy; we can model the universe as a perfect fluid. On the other hand, there are three observational evidences which are predicted by the model: the expansion of the universe, the existence of the cosmic microwave background (CMB) and the abundance of light elements from the primordial nucleosynthesis.
Applying General Relativity to our universe, in which the matter energy-momentum tensor is that of a perfect fluid, leads to the *Standard Big-Bang Cosmological Model.* This model is usually referred to as ΛCDM, since it assumes the existence of both *cold dark matter* (CDM in the acronym) and *dark energy* (denoted by Λ): indeed "visible" matter constitutes only about 5% of the total energy [7]; about 26% is matter which only acts by means of the gravitational attraction (for this reason called 'dark'), while the leftover 69% is called 'dark' energy, and it is thought to be the responsible for the accelerate expansion of our universe.
However, as we shall see, this model exhibits some conceptual issues, which are solved by introducing the concept of inflation.
This chapter is divided in two sections:

i) in the first one, we briefly present a summary of how General Relativity applies to our universe, from the Friedman equations, to some considerations about its geometry, up to the definition of the concepts of horizon which are significant for our purposes.

ii) in the second section, instead, we derive the concept of inflation in a straightforward way, i.e. from the problems of the standard model to their solution with the concept of a field which permeates the universe. Eventually, we generalize this concept to the so called k-inflations, which will be widely used later.

1.1 Friedman-Lemaître-Robertson-Walker universe

The starting point is the assumption of the *cosmological principle*, which states that there are neither preferred observers nor directions: this is equivalent to assume homogeneity and spatial isotropy.

1.1.1 Friedman equations

These assumptions strongly reduce the number of degrees of freedom: for instance, the cosmological principle requires that the line element can be rewritten as [8, 9]

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-k\,r^2} + r^2\,d\Omega^2\right], \tag{1.1}$$

which defines the *Friedman-Lemaître-Robertson-Walker metric* (FLRW).
Here $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$ is the angular part of the spherical line element, while k is a parameter called *curvature* which can take just three values

$$k = \begin{cases} +1 \\ 0 \\ -1 \end{cases}, \tag{1.2}$$

which correspond to *closed*, *flat* and *open* spatial sections, respectively.
The coordinates r is dimensionless, while the *scale factor* $a(t)$ has the dimension of a length.

It will be useful to deal with a new time coordinate, the *conformal time*, defined as

$$d\tau \equiv \frac{dt}{a(t)}, \tag{1.3}$$

so that eq. (1.1) becomes

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \frac{dr^2}{1-k\,r^2} + r^2\,d\Omega^2\right]. \tag{1.4}$$

The cosmological principle also applies to Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.5}$$

with $T_{\mu\nu}$ is the energy-momentum tensor of the fluid

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)u_\mu u_\nu. \tag{1.6}$$

A first equation which comes from (1.5) is [10]

$$3\ddot{a} = -4\pi G(\rho + 3p)a, \tag{1.7}$$

where $\dot{}$ denotes the derivative with respect to t.
A second one is

$$a\ddot{a} + 2\dot{a}^2 + 2k = 4\pi G(\rho - p)a^2. \tag{1.8}$$

Combining the two, and defining the *Hubble parameter*

$$H \equiv \frac{\dot{a}}{a}\,, \tag{1.9}$$

we get the system of two equation

$$H^2 = \frac{8}{3}\pi G\rho - \frac{k}{a^2} \tag{1.10a}$$

$$\ddot{a} = -\frac{4}{3}\pi G(\rho + 3p)a\,, \tag{1.10b}$$

which constitute the *Friedman equations*.
Together with these equations, we also have a *continuity equation*

$$\dot{\rho} = -3H(\rho + p)\,. \tag{1.11}$$

It is important to note that eqs. (1.10a), (1.10b) and (1.11) are not independent, but one of these comes from the combination of the other two.
The continuity equation (1.11) can be recast as

$$\rho' = -3\mathcal{H}(\rho + p)\,, \tag{1.12}$$

where $'$ which denotes the derivative with respect to τ and $\mathcal{H}$ is the Hubble parameter in conformal time

$$\mathcal{H} \equiv \frac{a'}{a}\,. \tag{1.13}$$

Taking the derivative of (1.10a) with respect to t, and using (1.11), we also have

$$\dot{H} = -4\pi G(\rho + p)\,. \tag{1.14}$$

Since they will be widely used in the following, it is worthwhile to write the Friedman equation in conformal time

$$\mathcal{H}^2 = \frac{8}{3}\pi G\rho a^2\,, \tag{1.15}$$

which can combined together to obtain some other useful relations

$$\mathcal{H}^2 - \mathcal{H}' = 4\pi G a^2(\rho + p) \tag{1.16a}$$

$$2\mathcal{H}' + \mathcal{H}^2 = -8\pi G p a^2\,. \tag{1.16b}$$

It is useful to rewrite the Friedman equation in a slightly different form; by defining a *critical density* ρ_c and a *density parameter* Ω as

$$\rho_c \equiv \frac{3H^2}{8\pi G} \tag{1.17a}$$

$$\Omega \equiv \frac{\rho}{\rho_c}\,, \tag{1.17b}$$

the preceding relation becomes

$$H^2(\Omega - 1) = \frac{k}{a^2}\,. \tag{1.18}$$

It is then clear that the hierarchy between Ω and 1 has a one-to-one correspondence with the spatial geometry of the universe.

On the one hand, for fluids with a state equation parameter $w \equiv \frac{p}{\rho} > -\frac{1}{3}$, the r.h.s. of eq. (1.10b) is always negative; on the other hand, observations suggest that $H > 0$, i.e. the universe is expanding. This implies that the scale factor is a concave function of time, and therefore that there always exists a time $t = 0$ for which $a(t = 0) = 0$: this is what in the standard model of universe is known as Big-Bang.

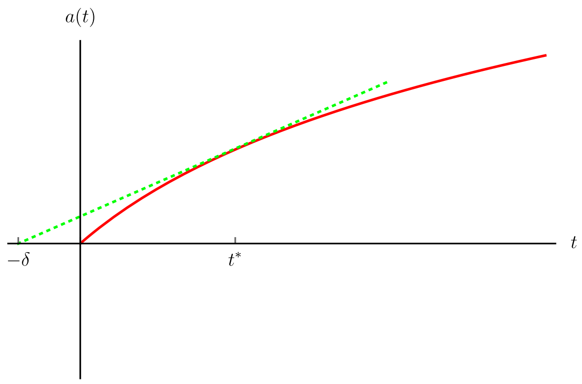

Figure 1.1: Evolution of the scale factor as a function of time.

This fact is independent from the curvature: the behaviour at late times, instead, depends on the value of k.
Furthermore, it is possible to get a rough estimate of the age of the universe by considering the inverse of the Hubble parameter, i.e. H^{-1}: in fact, from Fig. 1.1 we see that at a given instant t^* the scale factor can be approximated with a straight line of equation $y(t)$ such that $a(t^*) = y(t^*)$. The slope of this line is clearly $a'(t^*)$, so that its equation is

$$y(t) = a'(t^*)(t + \delta)\,. \tag{1.19}$$

By evaluating this at $t = t^*$ we have

$$a(t^*) = y(t^*) = a'(t^*)(t^* + \delta), \qquad \Rightarrow \qquad t^* = \frac{1}{H(t^*)} - \delta\,. \tag{1.20}$$

Therefore, at a given instant t^*, the age of the universe can be approximated by excess as $H(t^*)^{-1}$: for example, at the present time $H_0 = 67.3(10)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ [7], and consequently

$$t_0 = 4.59 \times 10^{17}\,\mathrm{s} \approx 1.45 \times 10^{10}\,\mathrm{yr} \tag{1.21}$$

Another very important quantity to describe the evolution of our universe is the *redshift* z, which relates the scale factor at a certain time with the scale factor today: it is defined as

$$1 + z \equiv \frac{a_0}{a}\,, \tag{1.22}$$

where $a_0 \equiv a(t_{\text{today}})$, and usually it is assumed $a_0 = 1$ (since now on we make this assumption), so that z varies from 0 to $+\infty$.

As previously noted, for each value of k there is a time in which $a = 0$: nevertheless, for big z (early times), we can see that the curvature is negligible, i.e. $k \approx 0$. To prove this, let's consider a single-component fluid with equation of state

$$p = w\,\rho\,. \tag{1.23}$$

From eq. (1.11), the density depends on the scale factor as

$$\rho \sim a^{-3(1+w)}\,, \tag{1.24}$$

so that the Hubble parameter can be expressed as a function of the redshift as

$$H^2(z) = H_0^2\,(1+z)^2\left[1-\Omega_0+\Omega_0(1+z)^{1+3w}\right]\,. \tag{1.25}$$

Now, dividing eq. (1.18) by itself evaluated at the present time t_0, thanks to (1.25), we get

$$\Omega(z) = \frac{\Omega_0(1+z)^{1+3w}}{1-\Omega_0+\Omega_0(1+z)^{1+3w}} = 1 - \frac{1-\Omega_0}{1-\Omega_0+\Omega_0(1+z)^{1+3w}} \xrightarrow[z\to\infty]{} 1 \tag{1.26}$$

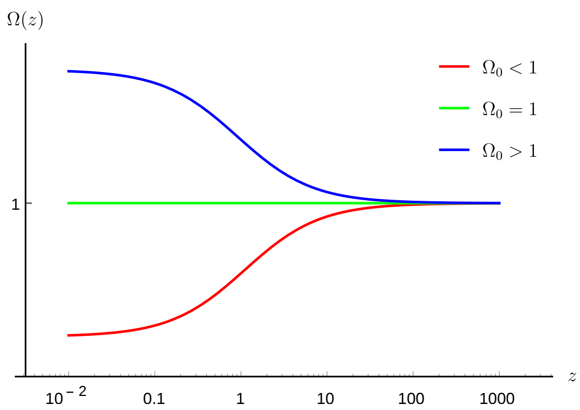

Figure 1.2: Evolution of the density parameter as a function of the redshift.

This aspect is very important since if we go sufficiently back in time we can approximate the universe as spatially flat, considering the so called Einstein-de Sitter (EdS) model of universe.
If $k = 0$, the second equation of (1.10) can be integrated to get

$$a \sim t^{\frac{2}{3(1+w)}} \qquad \Rightarrow \qquad H = \frac{2}{3(1+w)}\,\frac{1}{t}\,, \tag{1.27}$$

and from the previous discussion we know that we can use these results every time we study the very early universe.

1.1.2 Particle horizon and Hubble radius

Let's consider the line element (1.4) for a photon which propagates radially in a spatially flat universe [11]: it is characterized by

$$d\tau = \pm dr\,. \tag{1.28}$$

Therefore, the maximum comoving distance from which an observer will be able to receive signals at time t is

$$\chi_P(t) \equiv \int_{t_i}^{t} \frac{dt}{a(t)} = \int_{\tau_i}^{\tau} d\tau = \tau - \tau_i\,, \tag{1.29}$$

where t_i is the time corresponding to the initial singularity (we do not denote it with 0 for a reason which will be clear in a moment). The quantity $\chi_P(t)$ is called *comoving particle horizon.* The corresponding physical particle horizon is simply

$$R_P(t) = a(t) \int_{t_i}^{t} \frac{dt}{a(t)}\,. \tag{1.30}$$

Another very important quantity is the *comoving Hubble radius*, defined as

$$\chi_H \equiv \frac{1}{aH}\,, \tag{1.31}$$

together with its physical counterpart

$$R_H = \frac{1}{H}\,. \tag{1.32}$$

Thanks to (1.27), the comoving Hubble radius can also be expressed by

$$\chi_H = \frac{a^{\frac{1+3w}{2}}}{H_0}\,, \tag{1.33}$$

from which we note that the Hubble radius *grows with time* if $w > -\frac{1}{3}$; in addition

$$\chi_P = \int_{\ln a_i}^{\ln a} \frac{d\ln a}{aH} = \frac{1}{H_0} \int_{\ln a_i}^{\ln a} d\ln a\, a^{\frac{1+3w}{2}} = \frac{2}{3(1+w)H_0} \left(a^{\frac{1+3w}{2}} - a_i^{\frac{1+3w}{2}} \right) = \\ = \tau - \tau_i\,. \tag{1.34}$$

For standard fluids, the comoving horizon gets a negligible contribution from early times since

$$\tau_i \sim a_i^{\frac{1+3w}{2}} \xrightarrow[a_i \to 0]{w > -\frac{1}{3}} 0\,. \tag{1.35}$$

Thus, the particle horizon at time t is given by

$$\chi_P(t) = \frac{2}{3(1+w)H_0}\, a(t)^{\frac{1+3w}{2}} = \frac{2}{3(1+w)} \frac{1}{aH} = \frac{2}{3(1+w)} \chi_H(t)\,, \tag{1.36}$$

which ensures that the particle horizon and the Hubble radius are of the same order. Anyway, even if they are of the same order, their meaning is quite different: while the particle horizon χ_P is the maximum (comoving) distance photon have travelled from the Big-Bang, the Hubble radius is the distance over which photon can travel in a Hubble time, i.e. roughly the time in which the scale factor doubles.

1.2 Inflation

In this subsection, we present some of the conceptual problems of the standard cosmological Big-Bang scenario. Secondly, we show how these can be solved by inflation, and we discuss its simplest formulation in terms of a real scalar field. Eventually, we generalize this formulation to the so called k-inflations, in which the Lagrangian has a non-trivial kinetic term.

1.2.1 Problems of the Standard Big-Bang Cosmological Model

The Standard Big-Bang Cosmological Model we introduced in the previous section exhibits some problems, both from theoretical and observational points of view. Among these, we could cite the flatness problem, the horizon problem, the entropy and the relics problem. Here, we discuss in detail the first two issues.
These problems were solved in early 80s by introducing the idea of *inflation* as a phase of the universe characterized by a fast, quasi-exponential, expansion with a non-standard equation of state.

Horizon problem

We have seen from eq. (1.35) that if the universe is filled by a standard fluid, the initial singularity takes place at $\tau_i = 0$. Moreover, from eq. (1.33) we know that if we go sufficiently back in time two points are always separated by a distance which is grater than the Hubble sphere: this means that it is always possible to find an instant in which two points are not in causal contact.
This is represented in figure 1.3

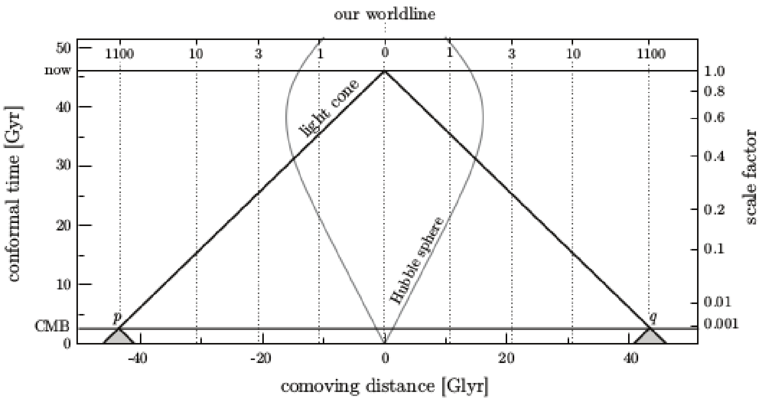

Figure 1.3: Representation of the horizon problem (figure taken from [12]).

At this point an issue arises: observations suggest that the CMB is characterized by an almost isotropic radiation. This seems to violate the concept of causal contact: distant points in the sky should not have influenced each other in remote past times since their past light cones did not overlap at the time of CMB formation (about 380 000 years after Big-Bang). This is known as the *horizon problem*.

Flatness problem

Another typical issue related to the Cosmological Standard Model is the flatness problem: it can be viewed as a fine-tuning problem.
Let's consider the evolution of the density parameter Ω [13]: from eqs. (1.18) and (1.27) we have

$$\Omega - 1 \sim \frac{k}{a^2\, a^{-3(1+w)}} \sim a^{1+3w}\,, \tag{1.37}$$

which indicates that for standard state parameters the density parameter increases in time.
So, if we consider the ratio between the density parameter today and at a given past time, e.g. at the Planck time (assuming that our physics theory is valid till to those scales), we have

$$\frac{|\Omega - 1|_{t_0}}{|\Omega - 1|_{t_{PL}}} \sim \frac{a_0^2}{a_{PL}^2} \sim \frac{T_{PL}^2}{T_0^2} \sim 10^{64}\,, \tag{1.38}$$

where, again, we have neglected the matter-dominated phase for simplicity and assumed that radiation-dominated phase began with the Big-Bang.
In any case, since we know that today $\Omega \sim \mathcal{O}(1)$, we see that in the past Ω had to be extremely close to 1: as we do not know any reason why this should be happen, we consider this as a fine tuning problem.

1.2.2 Inflationary solution to Standard Big-Bang Cosmological Model problems

Both the horizon and the flatness problem can be naturally solved with the same idea, which naturally leads to the concept of inflation.

Solution to the horizon problem

Since we have seen that, ultimately, the horizon problem arises from the fact that past light cones of two distant points do not overlap in the remote past, we could search for a scenario in which it is always possible to have a region of overlapping between the past light cones of any two points. We see that this request is satisfied if we accept a phase of a *shrinking Hubble radius* [12]: in other words, we want

$$\frac{d}{dt}\chi_H = \frac{d}{dt}\frac{1}{aH} < 0\,. \tag{1.39}$$

From eq. (1.10b) we have that

$$\frac{d}{dt}\chi_H = -\frac{1}{aH^2}(H^2 + \dot{H}) = \frac{4\pi G\rho}{3aH^2}(1+3w) < 0 \qquad \Leftrightarrow \qquad w < -\frac{1}{3}\,. \tag{1.40}$$

Hence, to have a Hubble radius which decreases in time, it is necessary to have a violation of the standard equation of state for fluids.
This request has also another consequence: similarly to eq. (1.35), we have

$$\tau_i \sim a_i^{\frac{1+3w}{2}} \xrightarrow[a_i \to 0]{w<-\frac{1}{3}} -\infty\,. \tag{1.41}$$

Thus, a shrinking Hubble sphere implies that the Big-Bang singularity happens at a negative infinite conformal time; in such a way the horizon problem is naturally solved. In fact, now, it is no more possible to approach the initial singularity enough to get two points separated by a region greater than the Hubble radius: conversely, there always exists an instant in past in which the past light cones overlap, as shown in figure 1.4

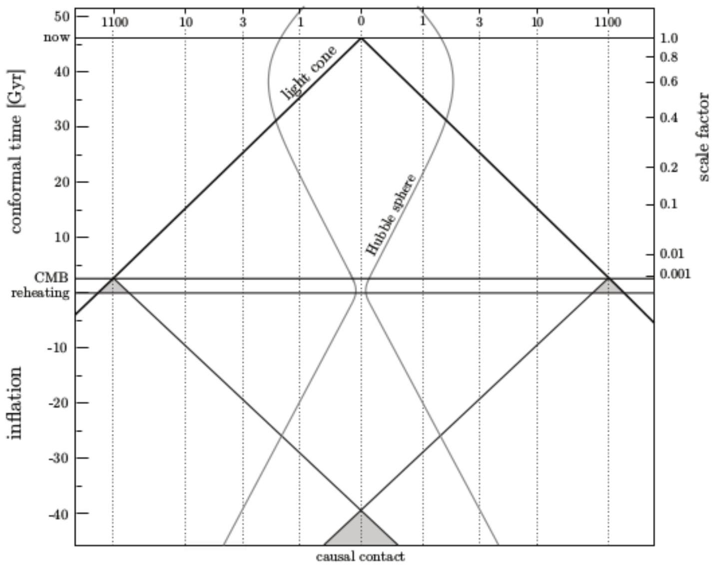

Figure 1.4: Representation of the solution to horizon problem (figure taken from [12]).

A particular case, i.e. $w \approx -1$, is particularly simple. From the continuity equation (1.11), in fact, we find that in this case $\rho \approx$ const, which implies that the solution to the Friedman equation (1.10a) is

$$a(t) \approx e^{\sqrt{\frac{\rho}{3M_{Pl}^2}}\, t} \,. \tag{1.42}$$

This stage, characterized by a scale factor which grows quasi-exponentially in time, is known as *inflation*. Since the equation of state parameter $w = -1$ is typical of the cosmological constant Λ, this is usually written as

$$a(t) \approx e^{\sqrt{\frac{\Lambda}{3}}\, t} \,, \tag{1.43}$$

which represents a quasi-de Sitter expansion. If this holds true, we get from eq. (1.3)

$$\tau = -\frac{1}{aH} \,. \tag{1.44}$$

In this way we have the asymptotic behaviours at early and late times, i.e.

$$\tau \xrightarrow{a \to 0} -\infty \tag{1.45a}$$

$$\tau \xrightarrow{a \to \infty} 0 \,. \tag{1.45b}$$

This expressions will be useful later on.

To determine how much inflation has to last, we can require that the Hubble radius at the beginning of inflation t_I was greater than today. Assuming that from the end of inflation until today the universe has been dominated by radiation ($w = \frac{1}{3}$), we have that

$$\frac{\chi_H(t_E)}{\chi_H(t_0)} = \frac{a_0 H_0}{a_E H_E} \sim \frac{a_E}{a_0} \sim \frac{T_0}{T_E} \sim 10^{-28}\,, \tag{1.46}$$

where E indicates the end of inflation and we have approximated $T_E \sim 10^{15}\,\mathrm{GeV}$, $T_0 \sim 10^{-3}\,\mathrm{eV}$.
Imposing now that the Hubble radius at the beginning of inflation was greater than today we get

$$\chi_H(t_I) > \chi_H(t_0) \sim 10^{28} \chi_H(t_E)\,. \tag{1.47}$$

Since $H \approx$ const. it therefore implies

$$\frac{a_E}{a_I} \gtrsim 10^{28} \qquad \Rightarrow \qquad N \equiv \ln \frac{a_E}{a_I} \gtrsim 64\,. \tag{1.48}$$

N is the number of e-folds of inflation.

Solution to the flatness problem

This issue too can be solved with an inflationary phase: in particular, by requiring that at the beginning of inflation $\Omega|_{t_I} \sim \mathcal{O}(1)$ and that after the end of inflation the radiation-dominated era began, eq. (1.38) imposes

$$e^{2N} = \frac{a_E^2}{a_I^2} \sim \frac{|\Omega - 1|_{t_E}}{|\Omega - 1|_{t_I}} \sim \frac{|\Omega - 1|_{t_0}}{|\Omega - 1|_{t_I}} \frac{T_E^2}{T_0^2} \sim 10^{56} \qquad \Rightarrow \qquad N \sim 64\,. \tag{1.49}$$

Again, with a number of e-folds of about 60-70, the flatness problem is naturally solved.

1.2.3 Standard single-field inflation

We have seen that with inflation it is possible to solve some problems that otherwise would seem unexplainable. In terms of modern theoretical physics, the simplest model is to consider inflation as a stage driven by a single scalar field, called *inflaton*, in a flat FLRW space-time described by the Lagrangian [14, 15]

$$\mathcal{L} = -\frac{1}{2}\, g^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi - V(\phi)\,. \tag{1.50}$$

The corresponding energy-momentum tensor is

$$T_{\mu\nu} = -2\,\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} + g_{\mu\nu}\,\mathcal{L} = \partial_\mu \phi\, \partial_\nu \phi + g_{\mu\nu} \left(-\frac{1}{2}\, g^{\rho\sigma} \partial_\rho \phi\, \partial_\sigma \phi - V(\phi) \right)\,. \tag{1.51}$$

With the hypothesis of the flat space, this energy-momentum tensor is that of a perfect fluid, so that the energy density and the pressure are

$$\rho = T_{00} = \frac{\dot{\phi}^2}{2} + V(\phi) \tag{1.52a}$$

$$p = T_{ii} = \frac{\dot{\phi}^2}{2} - V(\phi)\,. \tag{1.52b}$$

From the Lagrangian (1.50), the equations of motion are

$$0 = \partial^\mu \left(\frac{\delta \mathcal{L}}{\delta \partial^\mu \phi} \sqrt{-g} \right) - \sqrt{-g}\, \frac{\delta \mathcal{L}}{\delta \phi} = -\partial_t \left(a^3 \dot{\phi} \right) - a^3 \frac{dV}{d\phi} \,, \tag{1.53}$$

and consequently the Klein-Gordon equation reads

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0 \,, \tag{1.54}$$

or in conformal time

$$\phi'' + 2\mathcal{H}\phi' + V_\phi a^2 = 0 \,. \tag{1.55}$$

where prime and $_\phi$ denote the derivative with respect to τ and ϕ, respectively.
The Friedman equations, instead, become

$$\ddot{a} = -\frac{8\pi G}{3} \left(\dot{\phi}^2 - V(\phi) \right) \tag{1.56a}$$

$$H^2 = \frac{8\pi G}{3} \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right) \,. \tag{1.56b}$$

In conformal time, the density and the pressure in eqs. (1.52) become

$$\rho = \frac{\phi'^2}{2a^2} + V \tag{1.57a}$$

$$p = \frac{\phi'^2}{2a^2} - V \,. \tag{1.57b}$$

As a consequence, the state parameter takes the form

$$w = \frac{p}{\rho} = \frac{\phi'^2 - 2a^2 V}{\phi'^2 + 2a^2 V} \,. \tag{1.58}$$

Slow-roll inflation

Taking the derivative of relation (1.56) with respect to time we get

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_{Pl}^2} \,. \tag{1.59}$$

We now introduce a quantity which will be fundamental for all future considerations.
It is defined as

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2} = -\frac{d \ln H}{dN} = \frac{\dot{\phi}^2}{2H^2 M_{Pl}^2} \,. \tag{1.60}$$

We have seen that inflation is characterized by $H \approx$ const, and therefore $\epsilon_1 \ll 1$.
This condition implies that the kinetic term has a small contribution in the energy density.
From this, we can define a second parameter ϵ_2 as

$$\epsilon_2 \equiv \frac{d \ln \epsilon_1}{dN} = \frac{\dot{\epsilon}_1}{H\epsilon_1} \,. \tag{1.61}$$

These parameters are called *Hubble flow functions* (HFF), or *slow-roll parameters*. It is customary to make another assumption on the second HFF, i.e. $\epsilon_2 \ll 1$. Therefore we have

$$\epsilon_1 \ll 1 \qquad \longrightarrow \qquad \dot{\phi}^2 \ll V(\phi) \tag{1.62a}$$

$$\epsilon_2 \ll 1 \qquad \longrightarrow \qquad \ddot{\phi} \ll H\dot{\phi}\,. \tag{1.62b}$$

These conditions determine the so called *slow-roll inflation*.

Iteratively we can define the n-th slow roll parameter as

$$\epsilon_n \equiv \frac{d\ln\epsilon_{n-1}}{dN} = \frac{\dot{\epsilon}_{n-1}}{H\epsilon_{n-1}}\,. \tag{1.63}$$

Another interesting quantity is

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}\,. \tag{1.64}$$

Combining eqs. (1.60) and (1.64), the HFF ϵ_2 can be written as

$$\epsilon_2 = \frac{H}{\dot{\phi}^2}\left(\frac{2\ddot{\phi}\dot{\phi}}{H^2} - \frac{2\dot{\phi}^2\dot{H}}{H^3}\right) = 2\left(\frac{\ddot{\phi}}{H\dot{\phi}} + \epsilon_1\right) = 2(\epsilon_1 - \delta)\,. \tag{1.65}$$

Another parametrization which is sometimes used deals with the flow parameters [16]

$${}^l\lambda_H \equiv (2M_{Pl}^2)^l\,\frac{(H')^{l-1}}{H^l}\,\frac{d^{l+1}H}{d\phi^{l+1}}, \qquad , \qquad l > 1\,. \tag{1.66}$$

This parametrization is connected to the previous one via

$$\frac{d}{dN}\,{}^l\lambda_H = {}^l\lambda_H\left(\frac{l-1}{2}\,\epsilon_2 + \epsilon_1\right) - {}^{l+1}\lambda_H\,. \tag{1.67}$$

Slow-roll inflation is usually thought as driven by a potential which is almost constant in a certain region, as showed in Fig. 1.5

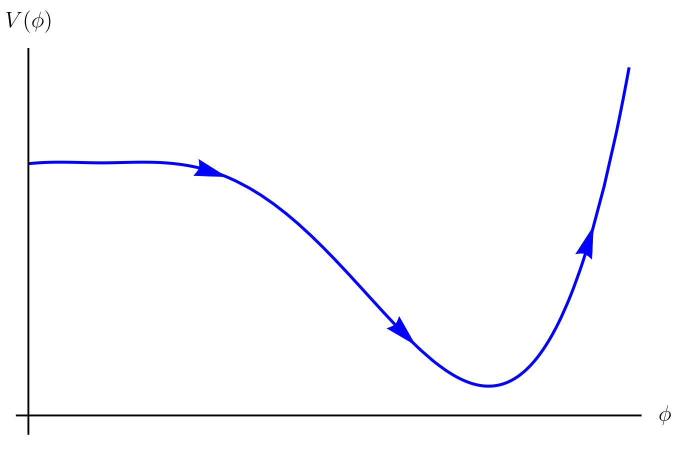

Figure 1.5: Potential for a slow-roll inflation.

After the phase of slow-roll, the field starts to roll down the potential, eventually oscillating around the minimum of the potential. This regime of coherent oscillations can be thought to be responsible for the emission of energy, the production of particles and the thermalisation of the universe.

Slow-roll approximation allows to significantly simplify the equations: for instance, the Friedman equation (1.56) reduces to

$$H^2 \approx \frac{V}{3M_{Pl}^2}\,, \tag{1.68}$$

while the Klein-Gordon equation (1.54)

$$3H\dot{\phi} + V_\phi \approx 0\,. \tag{1.69}$$

In addition, it is possible to relate the HFF with another class of parameters, defined in terms of the derivative of the potential. The first two are

$$\epsilon_V \equiv \frac{M_{Pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2\,, \tag{1.70}$$

and

$$\eta_V \equiv 2\epsilon_V - M_{Pl}^2\left[\left(\frac{V_\phi}{V}\right)^2 - \frac{V_{\phi\phi}}{V}\right]\,. \tag{1.71}$$

To determine the hierarchy between the HFF and ϵ_V and η_V, we firstly calculate the derivatives of the potential.

- Zeroth derivative
  If we can combine eqs. (1.60) and (1.68), we get [17, 1]

$$V = 3M_{Pl}^2 H^2\left(1 - \frac{\epsilon_1}{3}\right)\,. \tag{1.72}$$

- First derivative
  Furthermore, from the equation of motion (1.54), together with HFF (1.60) and (1.65), we find that the first derivative of the potential can be cast as

$$\begin{aligned} V_\phi = -\ddot{\phi} - 3H\dot{\phi} = H\dot{\phi}\left(-\frac{\epsilon_2}{2} + \epsilon_1 - 3\right) = \\ = -3\sqrt{2\epsilon_1}H^2 M_{Pl}\left(1 - \frac{\epsilon_1}{3} + \frac{\epsilon_2}{6}\right)\,. \end{aligned} \tag{1.73}$$

- Second derivative
  Determining the second derivative is a little more tedious: first of all, if we take the time derivative of the equation of motion (1.54), we get

$$\dddot{\phi} + 3\dot{H}\dot{\phi} + 3H\ddot{\phi} + V_{\phi\phi}\dot{\phi} = 0\,. \tag{1.74}$$

  From eq. (1.65) we can find out an expression for $\ddot{\phi}$ in terms of ϵ_1 and ϵ_2; substituting it, together with its derivative

$$\dddot{\phi} = (\dot{H}\dot{\phi} + H\ddot{\phi})\left(\frac{\epsilon_2}{2} - \epsilon_1\right) + H\dot{\phi}\left(\frac{\dot{\epsilon}_2}{2} - \dot{\epsilon}_1\right)\,, \tag{1.75}$$

into (1.74), we obtain

$$\begin{aligned}
0 = &\left(\frac{\epsilon_2}{2}-\epsilon_1\right)\left[\dot{H}\dot{\phi}+H^2\dot{\phi}\left(\frac{\epsilon_2}{2}-\epsilon_1\right)\right]+H^2\dot{\phi}\left(\frac{\epsilon_2\epsilon_3}{2}-\epsilon_1\epsilon_2\right)-3H^2\dot{\phi}\epsilon_1+\\
&+3H^2\dot{\phi}\left(\frac{\epsilon_2}{2}-\epsilon_1\right)+V_{\phi\phi}\dot{\phi}=\\
=&\,\dot{\phi}\Big\{-H^2\epsilon_1\left(\frac{\epsilon_2}{2}-\epsilon_1\right)+H^2\left(\frac{\epsilon_2}{2}-\epsilon_1\right)^2+H^2\left(\frac{\epsilon_2\epsilon_3}{2}-\epsilon_1\epsilon_2\right)-3H^2\epsilon_1+\\
&+3H^2\left(\frac{\epsilon_2}{2}-\epsilon_1\right)+V_{\phi\phi}\Big\}, \qquad (1.76)
\end{aligned}$$

where in the first line we have used the general relation (1.63) to express $\dot{\epsilon}_1$ and $\dot{\epsilon}_2$, and in the second line eq. (1.60) for $\dot{H}$.
From the above equation we have

$$V_{\phi\phi}=H^2\left[\frac{5\epsilon_1\epsilon_2}{2}-2{\epsilon_1}^2-\frac{{\epsilon_2}^2}{4}+6\epsilon_1-\frac{3\epsilon_2}{2}-\frac{\epsilon_2\epsilon_3}{2}\right]. \qquad (1.77)$$

Now, we just have to put these expression for V and its derivatives into eqs. (1.70) and (1.71): so doing, we find

$$\epsilon_V=\frac{M_{Pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2=\epsilon_1\frac{\left(1-\frac{\epsilon_1}{3}+\frac{\epsilon_2}{6}\right)^2}{\left(1-\frac{\epsilon_1}{3}\right)^2}, \qquad (1.78)$$

and

$$\eta_V=M_{Pl}^2\frac{V_{\phi\phi}}{V}=\frac{2\epsilon_1-\frac{\epsilon_2}{2}+\frac{5\epsilon_1\epsilon_2}{6}-\frac{2{\epsilon_1}^2}{3}-\frac{{\epsilon_2}^2}{12}-\frac{\epsilon_2\epsilon_3}{6}}{1-\frac{\epsilon_1}{3}}. \qquad (1.79)$$

In the same way, after some lengthy calculations, one finds

$$\begin{aligned}
\xi_V^2\equiv M_{Pl}^2\frac{V_{\phi\phi\phi}V_\phi}{V^2}=&\frac{1-\frac{\epsilon_1}{3}+\frac{\epsilon_2}{6}}{(1-\frac{\epsilon_1}{3})^2}\Big[4{\epsilon_1}^2-3\epsilon_1\epsilon_2+\frac{\epsilon_2\epsilon_3}{2}-\epsilon_1{\epsilon_2}^2+3{\epsilon_1}^2\epsilon_2-\\
&-\frac{4}{3}{\epsilon_1}^3-\frac{7}{6}\epsilon_1\epsilon_2\epsilon_3+\frac{{\epsilon_2}^2\epsilon_3}{6}+\frac{\epsilon_2{\epsilon_3}^2}{6}+\frac{\epsilon_2\epsilon_3\epsilon_4}{3}\Big]. \qquad (1.80)
\end{aligned}$$

The crucial fact is that expressions (1.78)-(1.80) are exact, so that the use of the HFF parametrization allows to reconstruct the derivatives of the potential without any additional approximation.

As a final consideration, we note that in the slow-roll regime, the number of e-folds can be expressed through eq. (1.69) as

$$N=\int_{t_I}^{t_E}H\,dt=\int_{\phi_I}^{\phi_E}d\phi\,\frac{H}{\dot{\phi}}\approx-\frac{1}{M_{Pl}^2}\int_{\phi_I}^{\phi_E}d\phi\,\frac{V}{V_\phi}. \qquad (1.81)$$

1.2.4 Non-standard kinetic term

As we have seen, an inflation as driven by a single scalar field constitutes the simplest model we can think of; many extensions of this model have been developed, and one of the most interesting is called k-inflation [18, 19], which represents a more general class of models in which the Lagrangian density is a generic function

$$\mathcal{L} = P(\chi, \phi)\,, \tag{1.82}$$

where χ is the kinetic term of the theory

$$\chi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi\,. \tag{1.83}$$

In the following, we consider homogeneous fields, for which $\phi = \phi(t)$, so that

$$\chi = \frac{\dot{\phi}^2}{2}\,. \tag{1.84}$$

For the moment, anyway, let's keep χ unexpressed, so that the considerations we make are valid in general.

The Lagrangian is indicated with P because it can be thought as a "pressure": we can convince of it by calculating the energy-momentum tensor from eq. (1.51). Denoting with $_\chi$ the derivative with respect to χ we get

$$T_{\mu\nu} = -2\,\frac{\delta P}{\delta g^{\mu\nu}} + g_{\mu\nu}\,P = P\,g_{\mu\nu} - 2\,P_\chi \frac{\delta \chi}{\delta g^{\mu\nu}} = P\,g_{\mu\nu} + P_\chi\,\partial_\mu\phi\,\partial_\nu\phi\,. \tag{1.85}$$

The first term of r.h.s. is the same than (1.6) if $P = p$; by defining

$$u_\mu = \frac{\partial_\mu \phi}{\sqrt{2\chi}}\,, \tag{1.86}$$

eq. (1.85) is

$$T_{\mu\nu} = P\,g_{\mu\nu} + 2\chi P_\chi u_\mu u_\nu\,, \tag{1.87}$$

from which we can identify the energy density with

$$\rho = 2\chi P_\chi - P\,, \tag{1.88}$$

and consequently write the energy-momentum tensor as

$$T_{\mu\nu} = P\,g_{\mu\nu} + (P+\rho) u_\mu u_\nu, \qquad , \qquad \begin{cases} \rho = 2\chi P_\chi - P \\ u_\mu = \dfrac{\partial_\mu \phi}{\sqrt{2\chi}} \end{cases} . \tag{1.89}$$

We see from this expression that the Lagrangian (1.82) can be used to describe the potential motions of hydrodynamical fluid as well as to draw a useful analogy with hydrodynamics in the case of arbitrary Lagrangian for a scalar field. Indeed, if P depends only on χ, then $\rho = \rho(\chi)$.

An important quantity for the following analysis is the *speed of sound* c_s, which is defined as [18]

$$c_s^2 \equiv \frac{P_\chi}{\rho_\chi} = \frac{P_\chi}{P_\chi + 2\chi P_{\chi\chi}}\,. \tag{1.90}$$

Like for the slow-roll parameters, it is useful to consider the logarithmic derivative of c_s with respect to the number of e-fold: this quantity is denoted by s

$$s \equiv \frac{d\ln c_s}{dN}\,. \tag{1.91}$$

It is convenient to compute the derivative of the speed of sound firstly: this is

$$\begin{aligned}\frac{dc_s}{dN} &= \frac{1}{2c_s}\,\frac{P_{\chi N}(P_\chi + 2\chi P_{\chi\chi}) - P_\chi(P_{\chi N} + 2\chi_N P_{\chi\chi} + 2\chi P_{\chi\chi N})}{(P_\chi + 2\chi P_{\chi\chi})^2} = \\ &= \frac{1}{2c_s}\left[c_s^2\,\frac{P_{\chi N}}{P_\chi} - c_s^2\,\frac{P_{\chi N} + 2\chi_N P_{\chi\chi} + 2\chi P_{\chi\chi N}}{P_\chi + 2\chi P_{\chi\chi}}\right]\,.\end{aligned} \tag{1.92}$$

Hence, the explicit expression for s is

$$s = \frac{1}{2}\left[\frac{P_{\chi N}}{P_\chi} - \frac{P_{\chi N} + 2\chi_N P_{\chi\chi} + 2\chi P_{\chi\chi N}}{P_\chi + 2\chi P_{\chi\chi}}\right] \tag{1.93}$$

These two quantities could obviously have been defined in the previous section, but in that case they would have been trivial, since we would have found $c_s = 1$, $s = 0$. Now we determine the expressions for the slow-roll parameters for the lagrangian $P(\chi, \phi)$.

Combining the Friedman equation (1.10a) with the continuity equation (1.11) we find

$$2H\dot{H} = -\frac{H}{M_{Pl}^2}(P+\rho) = -\frac{2H\chi P_\chi}{M_{Pl}^2} \qquad \Rightarrow \qquad \dot{H} = -\frac{\chi P_\chi}{M_{Pl}^2}\,. \tag{1.94}$$

The first HFF, i.e. ϵ_1, is then given by

$$\epsilon_1 = \frac{\chi P_\chi}{H^2 M_{Pl}^2}\,. \tag{1.95}$$

The time derivative of ϵ_1 is therefore

$$\begin{aligned}\dot{\epsilon}_1 &= \frac{\chi\dot{\chi}P_{\chi\chi} + \chi\dot{\phi}P_{\chi\phi} + \dot{\chi}P_\chi}{H^2 M_{Pl}^2} - \frac{2\chi P_\chi \dot{H}}{H^3 M_{Pl}^2} = \\ &= \frac{\dot{\chi}}{H^2 M_{Pl}^2}\left(\chi P_{\chi\chi} + P_\chi\right) + \frac{\chi\dot{\phi}P_{\chi\phi}}{H^2 M_{Pl}^2} + 2H\epsilon_1^2\,.\end{aligned} \tag{1.96}$$

Thus, the second HFF ϵ_2 reads

$$\epsilon_2 = \frac{\dot{\epsilon}_1}{H\epsilon_1} = \frac{1}{H}\left(\frac{\dot{\chi}P_{\chi\chi}}{P_\chi} + \frac{\dot{\chi}}{\chi} + \frac{\dot{\phi}P_{\chi\phi}}{P_\chi}\right) + 2\epsilon_1\,. \tag{1.97}$$

For Lagrangian (1.82), the equations of motion are

$$\begin{aligned}
0 &= \partial^\mu \left(\frac{\delta P}{\delta \partial^\mu \phi} \sqrt{-g} \right) - \sqrt{-g}\, \frac{\delta P}{\delta \phi} = \partial^\mu \left(P_\chi \, \frac{\delta \chi}{\delta \partial^\mu \phi} a^3 \right) - a^3 \, P_\phi = \\
&= \partial_t \left(a^3 P_\chi \, \dot{\phi} \right) - a^3 \, \nabla \cdot (P_\chi \, \nabla \phi) - a^3 P_\phi = \\
&= a^3 \, \partial_t \left(P_\chi \, \dot{\phi} \right) - 3a^2 \dot{a} \, P_\chi \, \dot{\phi} - a^3 \, \nabla \cdot (P_\chi \, \nabla \phi) - a^3 P_\phi \, .
\end{aligned} \tag{1.98}$$

If we consider homogeneous fields we find the generalized Klein-Gordon equation

$$\frac{d}{dt} \, (P_\chi \, \dot{\phi}) + 3H P_\chi \, \dot{\phi} - P_\phi = 0 \, . \tag{1.99}$$

This equation can be also written in a different way. In fact, from eqs. (1.84) and (1.90), we have

$$\begin{aligned}
0 &= P_{\chi\chi} \dot{\chi} \, \dot{\phi} + P_{\chi\phi} \, \dot{\phi}^2 + P_\chi \, \ddot{\phi} + 3H P_\chi \, \dot{\phi} - P_\phi = \\
&= P_\chi \left[\ddot{\phi} \left(1 + 2\chi \, \frac{P_{\chi\chi}}{P_\chi} \right) + 3H \dot{\phi} + \frac{P_{\chi\phi}}{P_\chi} \, \dot{\phi}^2 - \frac{P_\phi}{P_\chi} \right] = \\
&= \frac{P_\chi}{c_s^2} \left[\ddot{\phi} + 3H c_s^2 \dot{\phi} + \frac{P_{\chi\phi}}{\rho_\chi} \, \dot{\phi}^2 - \frac{P_\phi}{\rho_\chi} \right] \, ,
\end{aligned} \tag{1.100}$$

and therefore [20]

$$\ddot{\phi} + 3H c_s^2 \dot{\phi} + \frac{P_{\chi\phi}}{\rho_\chi} \, \dot{\phi}^2 - \frac{P_\phi}{\rho_\chi} = 0 \, . \tag{1.101}$$

We can also relate HFF with the speed of sound of the inflaton [21]. First of all we note that from (1.97) we can write

$$\epsilon_2 = 2(\epsilon_1 - \delta) + p \, , \tag{1.102}$$

where δ is given by (1.64) and

$$p \equiv \frac{\dot{P}_\chi}{H P_\chi} \, . \tag{1.103}$$

The quantity p is

$$p = \frac{P_{\chi\chi} \, \dot{\chi}}{H P_\chi} + \frac{P_{\chi\phi} \, \dot{\phi}}{H P_\chi} = \frac{P_{\chi\phi} \, \dot{\phi}}{H P_\chi} - \delta \left(\frac{1}{c_s^2} - 1 \right) \, . \tag{1.104}$$

However, thanks to the Klein-Gordon equation (1.99), we also have

$$P_\phi = \dot{P}_\chi \dot{\phi} + P_\chi \ddot{\phi} + 3H P_\chi \dot{\phi} = H P_\chi \dot{\phi} \, (p - \delta + 3) \, , \tag{1.105}$$

from which

$$p = \frac{P_\phi}{H P_\chi \dot{\phi}} + \delta - 3 \, . \tag{1.106}$$

Combining the two expressions for p we get

$$p = \frac{2q(3 - \delta) - \delta(c_s^{-2} - 1)}{1 - 2q} \, , \tag{1.107}$$

where we have defined

$$q \equiv \frac{\chi P_{\chi\phi}}{P_\phi} \, . \tag{1.108}$$

If we substitute (1.107) into (1.102) we are able to express ϵ_2 as a function of ϵ_1, c_s^2 and the derivatives of the Lagrangian.
For Lagrangians for which $P_{\chi\phi} = 0$, we simply have

$$\epsilon_2 = 2\epsilon_1 - \frac{\delta}{c_s^2}\left(c_s^2 + 1\right) \, . \tag{1.109}$$

As a final comment, it is worthwhile to stress that the Lagrangian (1.82) is not the most general we can consider. In fact, we can add to it a term [22]

$$\mathcal{G}(\chi, \phi)\Box\phi \, . \tag{1.110}$$

This latter modification is sometimes called *Galileon model*.

Chapter 2

Relativistic perturbation theory

The Universe we observe is almost perfectly homogeneous and isotropic, as the cosmological principle states. However, in order to understand the formation of the structures we observe (such as galaxies, clusters and so on), a small amount of inhomogeneities must be accepted [14, 15, 23]. This is not a failure of the standard theory of Big-Bang we briefly discussed in the previous chapter: on the contrary, the *cosmological perturbation theory* is one of the greatest successes of modern cosmology.

Since observed inhomogeneities are very small, of the order of 1 part in 10^5, the problem can be faced with a perturbative approach [24].

This chapter is divided in three sections:

i) in the first one, we develop the theory of cosmological perturbations in terms of general relativity; this approach is doubly convenient: on the one hand, it is valid on all scales, also when the Newtonian treatment of gravity is inadequate; on the other hand, it can be applied up to relativistic energies;

ii) in the second section, instead, we define a fundamental quantity for later considerations: the concept of comoving curvature perturbation. In particular, we note how it is related to the gravitational potentials and the inflaton field;

iii) eventually, in the last section, we derive the Mukhanov-Sasaki equation, which represents the fundamental equation of motion for the comoving curvature perturbation, and whose solution will be widely exploited in the next chapters to compute the spectrum and the bispectrum of such a quantity.

2.1 Einstein equations

We consider all the quantities, i.e. the metric tensor, the energy-momentum tensor etc., to first order in perturbation theory around a time-dependent, isotropic and homogeneous cosmology.
In the following we denote with an overline the unperturbed quantities so that, for instance, a generic quantity A is

$$A = \overline{A} + \delta A \,. \tag{2.1}$$

2.1.1 Metric fluctuations

From these considerations, the metric tensor is

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \delta g_{\mu\nu} \,. \tag{2.2}$$

This has the consequence that, for a flat space-time, the background line element

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \delta_{ij}\, dx^i dx^j\right] \tag{2.3}$$

takes the most general form to first order in perturbations

$$ds^2 = a^2(\tau)\left\{-(1+2A)d\tau^2 + 2B_i\, d\tau\, dx^i + \left[(1+2C)\delta_{ij} + h_{ij}\right]\, dx^i dx^j\right\}\,, \tag{2.4}$$

where h_{ij} is symmetric by construction.
Thus, up to second order corrections

$$\begin{aligned} g_{\mu\nu} &= a^2 \begin{pmatrix} -(1+2A) & B_i \\ B_i & (1+2C)\delta_{ij} + h_{ij} \end{pmatrix} \\ g^{\mu\nu} &= \frac{1}{a^2} \begin{pmatrix} -(1-2A) & B^i \\ B^i & (1-2C)\delta^{ij} - h^{ij} \end{pmatrix} \end{aligned} \,. \tag{2.5}$$

The metric tensor has 10 independent components: 1 comes from A, 3 from B_i, and 6 from the symmetric tensor field δ_{ij}.

SVT decomposition

In order to keep track more efficiently of the equations we will obtain, it is worthwhile to rewrite eq. (2.4) in a more useful way.
For linear perturbations, we can split the metric fluctuations in their scalar, vector and tensor components, where this distinction depends on their transformation properties on spatial hypersurfaces. For this reason, this kind of decomposition is called *SVT decomposition* ("SVT" stands for "scalar-vector-tensor").
In the following, we analyse in details only scalar and tensor perturbations, since vector modes are not predicted in single-field inflation models.
Quantities in (2.5) are eigenvectors of the spatial Laplace operator, and at the linear order the scalar, vector and tensor parts do not mix with each other. For this reason, we can decompose both the vector and the tensor part as follows.

- **Vectors**.
  Under very general hypothesises, a vector field v_i can always be decomposed in a scalar part and a vector, divergenceless part, i.e.

  $$\begin{cases} v_i = \partial_i v + \hat{v}_i \\ \partial^i \hat{v}_i = 0 \end{cases} \,. \tag{2.6}$$

  The number of components of v_i, i.e. 3, is maintained, since v contributes with 1 component, and $\hat{v}_i$ with 2, being divergenceless.

- **Tensors**.
  A similar procedure applies for a symmetric tensor field t_{ij}, which can be decomposed in a scalar part and a vector and a traceless tensor part, both divergenceless:

$$\begin{cases} t_{ij} = 2s\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}p + 2\partial_{(i}\hat{u}_{j)} + 2\hat{u}_{ij} \\ \partial^i\hat{u}_i = 0 = \partial^i\hat{u}_{ij} \\ \hat{u}^i{}_i = 0 \end{cases} , \tag{2.7}$$

  where

$$\begin{cases} \partial_{\langle i}\partial_{j\rangle}p \equiv \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\nabla^2\right)p \\ \partial_{(i}\hat{u}_{j)} \equiv \frac{1}{2}\left(\partial_i\hat{u}_j + \partial_j\hat{u}_i\right) \end{cases} , \tag{2.8}$$

  and the factors 2 are for convenience.
  This kind of decomposition preserves the number of independent components, as it has to be. From eq. (2.7) we have

$$\begin{array}{lll} s, p \ \text{scalars} & \Rightarrow & 1+1=2 \ \text{components} \\ \hat{u}_i \ : \ \partial^i\hat{u}_i = 0 & \Rightarrow & 3-1=2 \ \text{components} \\ \hat{u}_{ij} \ : \ \partial^i\hat{u}_{ij} = 0 = \hat{u}^i{}_i & \Rightarrow & 6-3-1=2 \ \text{components} \end{array} \tag{2.9}$$

  Thus, we have 2+2+2=6 total components, and this is consistent with the fact that t_{ij} is a rank 3 symmetric tensor.

Overall, the total number of independent components is preserved: in fact, with this decomposition we have $1+1+2+2+2+2 = 10$ degrees of freedom, as in eq. (2.5).
With such a kind of decomposition, we can rearrange the line element (2.4) as

$$ds^2 = a(\tau)\Big\{ -(1+2A)d\tau^2 + \\ + 2(\partial_i B + \hat{B}_i)\, d\tau\, dx^i + \Big[(1+2C)\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E + 2\partial_{(i}\hat{E}_{j)} + 2\hat{E}_{ij}\Big]\, dx^i dx^j \Big\} . \tag{2.10}$$

Since in general relativity every system is equivalent to the others, we are always allowed to make coordinates transformation such as

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu \quad , \quad \xi^\mu = (T, \boldsymbol{L}) \quad |\xi^\mu| \ll 1 \quad . \tag{2.11}$$

Such a kind of transformation is called *gauge transformation.*
Both T and L^i are functions of time and space, i.e. $T = T(\tau, x^i)$, $L^i = L^i(\tau, x^i)$.
Again, the vector part of ξ^μ can be decomposed according to (2.6) as

$$\begin{cases} L^i = \hat{L}^i + \partial^i L \\ \partial_i\hat{L}^i = 0 \end{cases} \Rightarrow \begin{cases} \xi^0 = T \\ \xi^i = \hat{L}^i + \partial^i L \end{cases} . \tag{2.12}$$

Since the metric tensor transforms as

$$\tilde{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}\, g_{\alpha\beta} \, , \tag{2.13}$$

it is quite straightforward to determine the transformation laws for the parameters in (2.10).
Taking into account that

$$a^2(\tau) \to [a(\tau+T)]^2 = \left[a(\tau) + T\, a'(\tau) + \mathcal{O}(\xi^2)\right]^2 = a^2(\tau)\,[1 + 2\mathcal{H}\,T] + \mathcal{O}(\xi^2)\,, \tag{2.14}$$

where $'$ indicates a derivative with respect to τ, and $\mathcal{H}$ is the Hubble parameter in conformal time, one has, for example

$$\begin{aligned} -a^2(\tau)(1+2A) = g_{00} &= \frac{\partial\tilde{x}^\alpha}{\partial x^0}\frac{\partial\tilde{x}^\beta}{\partial x^0}\,\tilde{g}_{\alpha\beta} = \frac{\partial\tilde{x}^0}{\partial x^0}\frac{\partial\tilde{x}^0}{\partial x^0}\,\tilde{g}_{00} + \mathcal{O}(\xi^2) = \\ &= \left(\delta^0{}_0 + T'\right)^2 \left[-a^2(\tilde{\tau})(1+2\tilde{A})\right] + \mathcal{O}(\xi^2) = \\ &= -(1+2T')a^2(\tau)(1+2\mathcal{H}\,T)(1+2\tilde{A}) + \mathcal{O}(\xi^2) = \\ &= -a^2(\tau)(1+2\tilde{A}+2\mathcal{H}T+2T') + \mathcal{O}(\xi^2)\,. \end{aligned} \tag{2.15}$$

By comparison it is easy to get the expression for $\tilde{A}$, which reads

$$\tilde{A} = A - T' - \mathcal{H}T\,. \tag{2.16}$$

In a similar way all the others coefficients are computed

$$\begin{array}{lll} \tilde{A} = A - T' - \mathcal{H}T & \tilde{E} = E - L & \\ \tilde{B} = B + T - L' & \tilde{\hat{E}}_i = \hat{E}_i - \hat{L}_i & \tilde{C} = C - \mathcal{H}T - \dfrac{1}{3}\nabla^2 L \quad . \\ \tilde{\hat{B}}_i = \hat{B}_i - \hat{L}'_i & \tilde{\hat{E}}_{ij} = \hat{E}_{ij} & \end{array} \tag{2.17}$$

From these relations we can build up a set of quantities which do not change under gauge transformations, which for this reason are called *gauge-invariant*.
For instance, if we consider

$$\Psi \equiv A + \mathcal{H}(B - E') + (B - E')'\,, \tag{2.18}$$

this is gauge-invariant by construction.
The same property is fulfilled by

$$\Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E\,. \tag{2.19}$$

These two quantities are known as *Bardeen potentials*.
Other invariant quantities, although not as interesting as (2.18) and (2.19), are

$$\hat{\Psi}_i \equiv \hat{E}'_i - \hat{B}_i \qquad , \qquad \hat{E}_{ij}\,. \tag{2.20}$$

These gauge-invariant variables can be thought as the "real" perturbations, since they cannot be removed by means of coordinates transformations [11].

Since, as we mentioned above, it is always possible to make a change of coordinates, we can *fix the gauge* in order to simplify equations: some popular gauge choices are

- **Newtonian gauge**.
  It is defined by
$$B = 0 = E\,. \tag{2.21}$$
  From relations (2.18) and (2.19), the line element (2.10) becomes
$$ds^2 = a(\tau)\left[-(1+2\Psi)d\tau^2 + (1-2\Phi)\delta_{ij}\,dx^i dx^j\right]\,. \tag{2.22}$$
  This choice is called "Newtonian" since the line element is that of a small deviation from the flat space, from which we can identify Ψ with the gravitational potential.

- **Spatially flat gauge**.
  It is defined by
$$C = 0 = E\,. \tag{2.23}$$

2.1.2 Perturbations of the energy-momentum tensor

In an unperturbed Universe the energy-momentum tensor is given by eq. (1.6) with $g_{\mu\nu} \to \eta_{\mu\nu}$; for a comoving observer $\overline{x}^{\,i} = \text{const}$, so that $\overline{u}^{\,i} = 0 = \overline{u}_i$, while

$$ds^2 = -d\tau^2 \qquad \Rightarrow \qquad -1 = -a^2(\overline{u}^0)^2 \qquad \Rightarrow \qquad \overline{u}^0 = \frac{1}{a} \quad , \quad \overline{u}_0 = -a\,, \tag{2.24}$$

where $\overline{u}^0 = \dfrac{d\tau}{ds}$.
Perturbing the relation
$$\overline{g}_{\mu\nu}\overline{u}^\mu\overline{u}^\nu = -1 \tag{2.25}$$
one finds
$$0 = \overline{g}_{\mu\nu}(\overline{u}^\mu\delta u^\nu + \overline{u}^\nu\delta u^\mu) + \overline{u}^\mu\overline{u}^\nu\delta g_{\mu\nu} = \overline{u}^\mu\overline{u}^\nu\delta g_{\mu\nu} + 2\,\overline{u}_\mu\delta u^\mu\,. \tag{2.26}$$

Taking into account that $\overline{u}^\mu = \frac{\delta^\mu{}_0}{a}$ and $\delta g_{00} = -2a^2 A$, eq. (2.26) becomes

$$-2A - 2a\delta u^0 = 0 \qquad \Rightarrow \qquad \delta u^0 = -\frac{A}{a}\,. \tag{2.27}$$

Defining $\delta u^i = \frac{v^i}{a}$ we obtain

$$u^\mu = \frac{1}{a}\left(1 - A, v^i\right)\,. \tag{2.28}$$

The expression for u_μ can be easily found by contracting it with $g_{\mu\nu}$, so that at the leading order
$$u_\mu = a\left[-(1+A), v_i + B_i\right]\,. \tag{2.29}$$
As for the metric tensor, the first order perturbation in $T_{\mu\nu}$ is

$$\delta T^\mu{}_\nu = \delta^\mu{}_\nu\delta p + (\delta p + \delta\rho)\overline{u}^\mu\overline{u}_\nu + (\overline{p} + \overline{\rho})(\overline{u}^\mu\delta u_\nu + \overline{u}_\nu\delta u^\mu)\,, \tag{2.30}$$

where we have not taken into account a possible anisotropic term.
With relations (2.28), (2.29) it is easy to get the explicit components of (2.30). In particular, one finds

$$\begin{aligned} \delta T^0{}_0 &= -\delta\rho \\ \delta T^0{}_i &= (\overline{\rho}+\overline{p})(v_i + B_i) \\ \delta T^i{}_0 &= -(\overline{\rho}+\overline{p})v^i \equiv -q^i \\ \delta T^i{}_j &= \delta^i{}_j\,\delta p \end{aligned} \; . \tag{2.31}$$

In a second reference frame there will be a different energy-momentum tensor $\tilde{T}^\mu{}_\nu$, related to this by

$$\tilde{T}^\mu{}_\nu = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\,\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\,T^\alpha{}_\beta\,, \tag{2.32}$$

while the two frames are related by (2.11).
Combining (2.11) with (2.32), it can be shown that the following relation holds true

$$\delta\tilde{T}^\mu{}_\nu = \delta T^\mu{}_\nu - \overline{T}^\mu{}_{\nu,\gamma}\,\xi^\gamma + \overline{T}^\gamma{}_\nu\,\xi^\mu{}_{,\gamma} - \overline{T}^\mu{}_\gamma\,\xi^\gamma{}_{,\nu}\,. \tag{2.33}$$

By substituting the explicit components, we readily obtain

$$\begin{aligned} \delta\tilde{\rho} &= \delta\rho - \overline{\rho}'\,T \\ \delta\tilde{p} &= \delta p - \overline{p}'\,T \\ \tilde{q}^i &= q^i + (\overline{\rho}+\overline{p})L^{i\,\prime} \\ \tilde{v}^i &= v^i + L^{i\,\prime} \end{aligned} \; , \tag{2.34}$$

where, again, thanks to (2.6) v_i can be decomposed as $v_i = \hat{v}_i + \partial_i v$.

At this point it is worthwhile to introduce a dimensionless quantity, called *density contrast*, defined as

$$\delta \equiv \frac{\delta\rho}{\overline{\rho}}\,. \tag{2.35}$$

Similarly to eqs. (2.18) and (2.19), we can define the following quantity

$$\Delta \equiv \delta + \frac{\overline{\rho}'}{\overline{\rho}}(B+v)\,, \tag{2.36}$$

which is gauge-invariant by construction.

Similarly to what we have done in the case of the perturbations of the metric, we can impose gauge conditions. In particular we can consider

- **Uniform density gauge**.
  It is defined by
  $$\delta = 0 = \delta\rho\,. \tag{2.37}$$

- **Comoving gauge**.
  It is defined by
  $$q^i = 0 = B^i\,. \tag{2.38}$$

Adiabatic fluctuations

A fluctuation is called *adiabatic* if its value at given point in the perturbed space-time is its value in the unperturbed one at the same spatial point, but at a different time, for every species.
In other words, it means that, for a quantity A, we have

$$\delta A(\tau, \boldsymbol{x}) = \overline{A}(\tau + \delta\tau(\boldsymbol{x}), \boldsymbol{x}) \approx \overline{A}(\tau, \boldsymbol{x}) + \overline{A}'(\tau, \boldsymbol{x})\, \delta\tau(\boldsymbol{x})\,. \tag{2.39}$$

However, from the general definition of fluctuations

$$\delta A(\tau, \boldsymbol{x}) \equiv \overline{A}(\tau + \delta\tau) - \overline{A}(\tau)\,, \tag{2.40}$$

so that

$$\delta A(\tau, \boldsymbol{x}) \approx \overline{A}'(\tau)\, \delta\tau(\boldsymbol{x})\,. \tag{2.41}$$

This is the case, for example, for pressure and density.
Since eq. (2.41) is valid for every species, we obtain

$$\delta\tau = \frac{\delta\rho_i}{\overline{\rho}_i{}'} = \frac{\delta\rho_j}{\overline{\rho}_j{}'}\,. \tag{2.42}$$

From relation (1.12) one finds

$$\mathcal{H} = -\frac{\overline{\rho}'}{3\overline{\rho}(1+w)}\,, \tag{2.43}$$

in such a manner that (2.42) becomes

$$\frac{\delta_i}{1+w_i} = \frac{\delta_j}{1+w_j}\,. \tag{2.44}$$

Because of relation (2.42), the speed of sound c_s^2 can be computed as

$$c_s^2 \equiv \frac{\delta p}{\delta\rho} = \frac{p'}{\rho'}\,. \tag{2.45}$$

This is an important consequence, as we will note.

2.1.3 Perturbations of Einstein equations

In the following we assume the Newtonian gauge defined in (2.21), so that

$$\begin{aligned} g_{\mu\nu} &= a^2 \begin{pmatrix} -(1+2\Psi) & 0 \\ 0 & (1-2\Phi)\delta_{ij} \end{pmatrix} \\ g^{\mu\nu} &= \frac{1}{a^2} \begin{pmatrix} -(1-2\Psi) & 0 \\ 0 & (1+2\Phi)\delta^{ij} \end{pmatrix} \end{aligned}\,. \tag{2.46}$$

The Christoffel symbols then read

$$\begin{aligned} \Gamma^0_{00} &= \mathcal{H} + \Psi' \\ \Gamma^0_{0i} &= \partial_i\Psi \\ \Gamma^i_{00} &= \partial^i\Psi \\ \Gamma^0_{ij} &= \delta_{ij}\left[\mathcal{H} - 2\mathcal{H}(\Psi+\Phi) - \Phi'\right] \\ \Gamma^i_{j0} &= \delta^i{}_j(\mathcal{H} - \Phi') \\ \Gamma^i_{jk} &= -(\delta^i{}_j\partial_k\Phi + \delta^i_k\partial_j\Phi) + \delta_{jk}\partial^i\Phi \end{aligned}\,. \tag{2.47}$$

Continuity equation

The energy momentum tensor satisfies the continuity equation

$$0 = \nabla_\mu T^\mu{}_\nu \,, \tag{2.48}$$

which in fact is a system of four differential equations.

i) $\nu = 0$

From eq. (2.31) and (2.47) we get

$$\begin{aligned} 0 &= T^\mu{}_{0,\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha{}_0 - \Gamma^\alpha_{\mu 0} T^\mu{}_\alpha = \\ &= -\left[\overline{\rho}\,' + \delta\rho' + \partial_i q^i + 3(\mathcal{H} - \Phi')(\overline{\rho} + \overline{p}) + 3\mathcal{H}(\delta\rho + \delta p)\right] + \mathcal{O}(2)\,, \end{aligned}$$

where $\mathcal{O}(2)$ means that second order perturbations have been neglected. Separating the leading order from the first order in perturbations we get two equations, which combined together give

$$\delta' + 3\mathcal{H}\delta\left(\frac{\delta P}{\delta\rho} - \frac{\overline{p}}{\overline{\rho}}\right) + \left(1 + \frac{\overline{p}}{\overline{\rho}}\right)\left(\nabla\cdot\boldsymbol{v} - 3\Phi'\right) = 0\,, \tag{2.49}$$

which represents the relativistic version of the continuity equation.

ii) $\nu = i$

From eq. (2.31) and (2.47) we get

$$\begin{aligned} 0 &= T^\mu{}_{i,\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha{}_i - \Gamma^\alpha_{\mu i} T^\mu{}_\alpha = \\ &= q_i' + \partial_i\delta p + 4q_i\mathcal{H} + (\overline{\rho} + \overline{p})\partial_i\Psi + \mathcal{O}(2)\,. \end{aligned}$$

Taking into account eq. (1.12), and the definition of q^i in (2.31), one finds

$$\boldsymbol{v}' + \mathcal{H}\boldsymbol{v} - 3\mathcal{H}\frac{\overline{p}\,'}{\overline{\rho}\,'}\boldsymbol{v} + \frac{\nabla\delta p}{\overline{\rho} + \overline{p}} + \nabla\Psi = 0\,. \tag{2.50}$$

Einstein equations

The next step is to compute the Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,. \tag{2.51}$$

In order to do this we firstly have to calculate the Riemann tensor, and from it to determine the Riemann symbol and the Riemann scalar.
The computation is quite easy, even if a bit tedious, so we simply write the main stages

$$R_{00} = -3\mathcal{H}' + \nabla^2\Psi + 3\mathcal{H}(\Phi' + \Psi') + 3\Phi'' + \mathcal{O}(2)\,. \tag{2.52}$$

$$R_{0i} = 2\partial_i\Phi' + 2\mathcal{H}\partial_i\Psi + \mathcal{O}(2)\,. \tag{2.53}$$

$$R_{ij} = \delta_{ij}\left[\mathcal{H}' - 2\mathcal{H}'(\Phi+\Psi) - \mathcal{H}\Psi' - 5\mathcal{H}\Phi' - \Phi'' + \nabla^2\Phi + 2\mathcal{H}^2 - 4\mathcal{H}^2(\Phi+\Psi)\right] + \\ + \partial_i\partial_j(\Phi-\Psi) + \mathcal{O}(2)\,. \tag{2.54}$$

$$R = g^{00}R_{00} + g^{ij}R_{ij} = \\ = \frac{1}{a^2}\left[6(\mathcal{H}'+\mathcal{H}^2) - 2\nabla^2\Psi + 4\nabla^2\Phi - 12\Psi(\mathcal{H}'+\mathcal{H}^2) - 6\Phi'' - 6\mathcal{H}(\Psi'+3\Phi')\right]\,. \tag{2.55}$$

Due to these equations, the components of the Einstein tensor $G_{\mu\nu}$ are

$$G_{00} = 3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi'\,. \tag{2.56}$$

$$G_{0i} = 2\partial_i(\Phi' + \mathcal{H}\Psi)\,. \tag{2.57}$$

$$G_{ij} = \delta_{ij}\left[\nabla^2(\Psi-\Phi) + 2\Phi'' - (2\mathcal{H}'+\mathcal{H}^2) + 2(2\mathcal{H}'+\mathcal{H}^2)(\Phi+\Psi) + 2\mathcal{H}\Psi' + 4\mathcal{H}\Phi'\right] + \\ + \partial_i\partial_j(\Phi-\Psi)\,. \tag{2.58}$$

Now that we have the Einstein tensor, and the energy-momentum tensor, we can write the Einstein equations explicitly.
Separating the components we obtain:

i) Component $i \neq j$
In this case $T_{ij} = 0$, from which it follows

$$\partial_i\partial_j(\Phi-\Psi) = 0 \qquad \Rightarrow \qquad \Phi = \Psi\,, \tag{2.59}$$

where we have assumed that the potential are regular at infinity, so that the integration constant vanishes.

ii) Component $0i$
In this case

$$T_{0i} = g_{00}T^0{}_i = -a^2\, q_i\,, \tag{2.60}$$

and consequently

$$\partial_i(\Phi' + \mathcal{H}\Phi) = -4\pi G a^2 q_i\,. \tag{2.61}$$

By assuming again the decaying at infinity it is equivalent to

$$\Phi' + \mathcal{H}\Phi + 4\pi G a^2 v(\overline{\rho} + \overline{p}) = 0\,. \tag{2.62}$$

iii) Component 00
In this case

$$T_{00} = g_{00}\,T^0{}_0 = a^2(1+2\Phi)\overline{\rho}(1+\delta) = a^2\overline{\rho}(1+\delta+2\Phi) + \mathcal{O}(2)\,, \tag{2.63}$$

from which

$$3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi' = 8\pi G a^2\overline{\rho}(1+\delta+2\Phi)\,. \tag{2.64}$$

It is now worthwhile to consider the unperturbed order and the perturbed one separately: so doing we get

- zeroth order
  At the unperturbed order we recover the Friedman equation (1.15)

$$\mathcal{H}^2 = \frac{8\pi G}{3}\,\overline{\rho} a^2 \,. \tag{2.65}$$

- first order
  The lowest order in perturbations gives

$$\nabla^2\Phi = 4\pi G a^2\delta + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi)\,. \tag{2.66}$$

  Substituting in it eq. (2.62) and then (2.36), we can recast this equation into

$$\nabla^2\Phi = 4\pi G a^2\overline{\rho}\Delta\,. \tag{2.67}$$

iv) Component ii
In this case

$$T_{ii} = g_{ij}\,T^j{}_i = a^2\delta_{ij}(1-2\Phi)\delta^j{}_i(\overline{p}+\delta p) = a^2(\overline{p}+\delta p - 2\Phi) + \mathcal{O}(2)\,, \tag{2.68}$$

and then

$$2\Phi'' - (2\mathcal{H}' + \mathcal{H}^2) + 4(2\mathcal{H}' + \mathcal{H}^2)\Phi + 6\mathcal{H}\Phi' = 8\pi G a^2(\overline{p} + \delta p - 2\overline{p}\,\Phi)\,. \tag{2.69}$$

Again, let's separate the orders in perturbations:

- zeroth order
  At the unperturbed order we find

$$2\mathcal{H}' + \mathcal{H}^2 = -8\pi G a^2\overline{p}\,, \tag{2.70}$$

  which is nothing but the equation (1.16b).

- first order
  At the first order we have remained with

$$\Phi'' + 2(2\mathcal{H}' + \mathcal{H}^2)\Phi + 3\mathcal{H}\Phi' = 4\pi G a^2(\delta p - 2\overline{p}\,\Phi)\,. \tag{2.71}$$

  Using again eq. (1.16b) for the term proportional to $\overline{p}$ in the rhs, we have

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2\delta p\,. \tag{2.72}$$

The Einstein equations we have just found are usually solved trough the expansion in Fourier modes, so that a generic quantity $A(\tau, \boldsymbol{x})$ is substituted by

$$A(\tau,\boldsymbol{x}) = \int \frac{d\boldsymbol{k}}{(2\pi)^3}\,A(\tau,\boldsymbol{k})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \equiv \int \frac{d\boldsymbol{k}}{(2\pi)^3}\,A_{\boldsymbol{k}}\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{2.73}$$

By imposing that $A(\tau, x)$ has to be real, we get the reality condition

$$A^*_{\boldsymbol{k}} = A_{-\boldsymbol{k}}\,. \tag{2.74}$$

2.2 Curvature perturbation

Let's go back to the most general expression for the perturbed metric (2.5): the induced metric on a constant time hypersurface is

$$\gamma_{ij} = a^2 \left[(1+2C)\delta_{ij} + 2E_{ij}\right] \, , \tag{2.75}$$

where $2E_{ij} = h_{ij}$.
As a consequence, the inverse induced metric is

$$\gamma^{ij} = \frac{1}{a^2} \left[(1-2C)\delta_{ij} - 2E_{ij}\right] \, . \tag{2.76}$$

The corresponding Christoffel symbol, considering scalar perturbations only, is

$${}^{(3)}\Gamma^i_{jk} = 2\delta^i{}_{(j}\partial_{k)}C + 2\partial_{(j}E_{k)}{}^i - \delta_{jk}C_,{}^i - E_{jk,}{}^i \, . \tag{2.77}$$

Since it is already a first order quantity in perturbations, it is quite straightforward to find the expression for the Ricci symbol, which reads

$${}^{(3)}R = \frac{1}{a^2}\left[\partial_a(\delta^{ij}\,{}^{(3)}\Gamma^a_{ij}) - \partial^i\,{}^{(3)}\Gamma^a_{ia}\right] \, . \tag{2.78}$$

By assumption, E_{ij} is traceless, so that the first term is

$${}^{(3)}\Gamma^a_{ij} = 2\partial_i E^{ia} - \partial^a C \, , \tag{2.79}$$

while the second one is

$${}^{(3)}\Gamma^a_{ia} = \delta^a{}_a \partial_i C = 3\partial_i C \, . \tag{2.80}$$

Thus eq. (2.78) is

$${}^{(3)}R = \frac{1}{a^2}\left[\partial_a(2\partial_i E^{ia} - \partial^a C) - 3\partial^i\partial_i C\right] = \frac{1}{a^2}\left[2\partial_i\partial_j E^{ij} - 4\nabla^2 C\right] \, . \tag{2.81}$$

As we mentioned, we are just considering scalar perturbations: for this reason, from (2.10) we know that

$$E_{ij} = \partial_{\langle i}\partial_{j\rangle}E \equiv \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\nabla^2\right)E \, , \tag{2.82}$$

from which

$${}^{(3)}R = \frac{1}{a^2}\left[\frac{4}{3}\nabla^4 E - 4\nabla^2 C\right] \, , \tag{2.83}$$

and finally

$$a^2\,{}^{(3)}R = -4\nabla^2\left(C - \frac{\nabla^2 E}{3}\right) \equiv 4\nabla^2\mathcal{K} \, . \tag{2.84}$$

$\mathcal{K}$ is called *curvature perturbation*: from this we define a *comoving curvature perturbation* $\mathcal{R}$, defined as the curvature perturbation in the comoving gauge (2.38), characterized by $B^i = 0 = q^i$.
However, it is convenient to have a gauge-invariant expression for $\mathcal{R}$, so that we will be able to calculate it in any gauges: since in the comoving gauge B and v vanish,

we are free to add a linear combination of these to eq. (2.84).
All these things considered, we define

$$\mathcal{R} \equiv -C + \frac{\nabla^2 E}{3} - \mathcal{H}(B+v)\,, \tag{2.85}$$

which is manifestly gauge-invariant if we take into account expressions (2.17) and (2.34).

Since we will make all our considerations in the Newtonian gauge, it is worthwhile to determine some properties of the comoving curvature perturbation in such a gauge.
Newtonian gauge is defined by condition (2.21), so that (2.85) simplifies to

$$\mathcal{R} = \Psi - \mathcal{H}v \tag{2.86}$$

From the Einstein equation (2.62) we can eliminate the velocity v to obtain

$$\mathcal{R} = \Psi + \mathcal{H}\,\frac{\Psi' + \mathcal{H}\Psi}{4\pi G a^2(\overline{\rho}+\overline{p})}\,. \tag{2.87}$$

Thanks to the Friedman equation (1.15) we can recast this equation into

$$\mathcal{R} = \Psi + \mathcal{H}\,\frac{2(\Psi' + \mathcal{H}\Psi)}{3\mathcal{H}^2(1+w)} = \Psi + \frac{2}{3}\,\frac{\Psi' + \mathcal{H}\Psi}{\mathcal{H}(1+w)} \tag{2.88}$$

2.3 Mukhanov-Sasaki equation

At very high energy matter is well described in term of fields [25]; for this reason, we should substitute in the expression for the energy-momentum tensor that for a scalar field.
In this section, we consider the general context of k-inflation, in such a manner that the considerations we do are valid also for the case of inflation with a standard kinetic term. We consider the general Lagrangian (1.82) [14]

$$\mathcal{L} = P(\chi, \phi)\,. \tag{2.89}$$

In this context, we perturb the scalar field as

$$\phi(\tau, \boldsymbol{x}) = \overline{\phi}(\tau) + \delta\phi(\tau, \boldsymbol{x})\,, \tag{2.90}$$

so that the kinetic terms becomes

$$\begin{aligned} \chi &= -\frac{1}{2} g^{\mu\nu}\partial_\mu\overline{\phi}\,\partial_\nu\overline{\phi} - g^{\mu\nu}\partial_\mu\overline{\phi}\,\partial_\nu\delta\phi = -\frac{1}{2}g^{00}\overline{\phi}'^2 - g^{00}\overline{\phi}'\delta\phi' = \\ &= \frac{1}{a^2}\left(\frac{\overline{\phi}'^2}{2} + \overline{\phi}'\delta\phi' - \overline{\phi}'^2\Psi\right) \equiv \overline{\chi} + \delta\chi\,, \end{aligned} \tag{2.91}$$

where as usual we have chosen Newtonian gauge.
From this relation we find that the first order in perturbation of the kinetic term is

$$\delta\chi = \frac{1}{a^2}\left(\overline{\phi}'\delta\phi' - \overline{\phi}'^2\Psi\right) = 2\overline{\chi}\left(\frac{\delta\phi'}{\overline{\phi}'} - \Psi\right)\,. \tag{2.92}$$

For the energy-momentum tensor we have to refer to eq. (1.89): in particular, since it is the tensor of perfect fluid by hypothesis, it is

$$\delta T^0{}_0 = -\delta\rho\,. \tag{2.93}$$

In order to obtain an expression for $\delta\rho$, let's consider the continuity equation before: since $\rho = \rho(\phi, \chi)$, we have

$$-3\mathcal{H}(P+\rho) = \rho' = \rho_\phi \phi' + \rho_\chi \chi' \qquad \Rightarrow \qquad \rho_\phi = -3\,\frac{\mathcal{H}}{\phi'}(P+\rho) - \rho_\chi\,\frac{\chi'}{\phi'}\,. \tag{2.94}$$

Then, using definition (1.90) and eq. (2.92), we find

$$\begin{aligned}\delta\rho = \rho_\chi\delta\chi + \rho_\phi\delta\phi &= \rho_\chi\left(\delta\chi - \overline{\chi}'\,\frac{\delta\phi}{\overline{\phi}'}\right) - 3\,\mathcal{H}(P+\rho)\frac{\delta\phi}{\overline{\phi}'} = \\ &= \frac{P+\rho}{c_s^2}\left(\frac{\delta\phi'}{\overline{\phi}'} - \Psi - \frac{\overline{\phi}''}{\overline{\phi}'^2}\,\delta\phi + \mathcal{H}\,\frac{\delta\phi}{\overline{\phi}'}\right) - 3\,\mathcal{H}(P+\rho)\frac{\delta\phi}{\overline{\phi}'} = \\ &= \frac{P+\rho}{c_s^2}\left[\frac{d}{d\tau}\left(\frac{\delta\phi}{\overline{\phi}'}\right) + \mathcal{H}\,\frac{\delta\phi}{\overline{\phi}'} - \Psi\right] - 3\,\mathcal{H}(P+\rho)\frac{\delta\phi}{\overline{\phi}'}\,,\end{aligned} \tag{2.95}$$

from which

$$\delta T^0{}_0 = -\frac{P+\rho}{c_s^2}\left[\frac{d}{d\tau}\left(\frac{\delta\phi}{\overline{\phi}'}\right) + \mathcal{H}\,\frac{\delta\phi}{\overline{\phi}'} - \Psi\right] + 3\,\mathcal{H}(P+\rho)\frac{\delta\phi}{\overline{\phi}'}\,. \tag{2.96}$$

Determining the expression for $\delta T^0{}_i$, instead, is simpler, since

$$\begin{aligned}T^0{}_i = g^{00}T_{0i} &= -\frac{1}{a^2}(1-2\Psi)(P+\rho)u_0 u_i = -\frac{1}{a^2}(1-2\Psi)(P+\rho)\frac{\partial_0\phi\,\partial_i\phi}{2\chi} = \\ &= -\frac{1}{a^2}(P+\rho)\frac{\overline{\phi}'\,\partial_i\delta\phi}{2\overline{\chi}} = -(P+\rho)\frac{\partial_i\delta\phi}{\overline{\phi}'} = \delta T^0{}_i\end{aligned} \tag{2.97}$$

since it is first order in perturbations.
By comparison with eq. (2.31) we note that also in this case

$$v = -\frac{\delta\phi}{\overline{\phi}'}\,, \tag{2.98}$$

and then from (2.86)

$$\mathcal{R} = \Psi + \mathcal{H}\frac{\delta\phi}{\overline{\phi}'}\,. \tag{2.99}$$

Furthermore, from eqs. (2.56)-(2.58), the Einstein tensor is

$$\begin{aligned}G^0{}_0 = g^{00}G_{00} &= -\frac{1}{a^2}(1-2\Psi)(3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi') = \\ &= -\frac{3\mathcal{H}^2}{a^2} - \frac{2}{a^2}[\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Psi)]\,.\end{aligned} \tag{2.100}$$

$$G^0{}_i = g^{0i}G_{00} = -\frac{2}{a^2}\partial_i(\Phi' + \mathcal{H}\Psi)\,. \tag{2.101}$$

$$\begin{aligned}G_{ij} = \delta_{ij}&\left[\nabla^2(\Psi-\Phi) + 2\Phi'' - (2\mathcal{H}' + \mathcal{H}^2) + 2(2\mathcal{H}' + \mathcal{H}^2)(\Phi+\Psi) + 2\mathcal{H}\Psi' + 4\mathcal{H}\Phi'\right] + \\ &+ \partial_i\partial_j(\Phi - \Psi)\,.\end{aligned} \tag{2.102}$$

As in the previous considerations, we just have to compute the Einstein equations.

i) Component $i \neq j$
As usual, $T_{ij} = 0$, from which it follows

$$\partial_i \partial_j (\Phi - \Psi) = 0 \qquad \Rightarrow \qquad \Phi = \Psi\,, \tag{2.103}$$

where we have assumed that the potential are regular at infinity, so that the integration constant vanishes.

ii) Component $0i$
If the potential are sufficiently regular we can drop the spatial derivative to find

$$\Psi' + \mathcal{H}\Psi = 4\pi G a^2 (P+\rho)\frac{\delta\phi}{\bar{\phi}'}\,, \tag{2.104}$$

which can be written in the equivalent form

$$\frac{d}{d\tau}\left(\frac{a^2\Psi}{\mathcal{H}}\right) = \frac{4\pi G a^4 (P+\rho)}{\mathcal{H}^2}\left(\mathcal{H}\frac{\delta\phi}{\bar{\phi}'} + \Psi\right). \tag{2.105}$$

iii) Component 00
We directly consider the first order in perturbation: thanks to (2.104) we get

$$\begin{aligned} \nabla^2\Psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Psi) &= \nabla^2\Psi - \cancel{12\mathcal{H}\pi G a^2 (P+\rho)\frac{\delta\phi}{\bar{\phi}'}} = \\ &= 4\pi G a^2 \left\{\frac{P+\rho}{c_s^2}\left[\frac{d}{d\tau}\left(\frac{\delta\phi}{\bar{\phi}'}\right) + \mathcal{H}\frac{\delta\phi}{\bar{\phi}'} - \Psi\right] - \cancel{3\,\mathcal{H}(P+\rho)\frac{\delta\phi}{\bar{\phi}'}}\right\}, \end{aligned} \tag{2.106}$$

and using again (2.104) together with (1.16a)

$$\begin{aligned} \nabla^2\Psi &= \frac{4\pi G a^2 (P+\rho)}{c_s^2}\left[\frac{d}{d\tau}\left(\frac{\delta\phi}{\bar{\phi}'}\right) + \mathcal{H}\frac{\delta\phi}{\bar{\phi}'} - \Psi\right] = \\ &= \frac{4\pi G a^2 (P+\rho)}{\mathcal{H}c_s^2}\left[\mathcal{H}\frac{d}{d\tau}\left(\frac{\delta\phi}{\bar{\phi}'}\right) + \Psi' + \mathcal{H}'\frac{\delta\phi}{\bar{\phi}'}\right] = \\ &= \frac{4\pi G a^2 (P+\rho)}{\mathcal{H}c_s^2}\frac{d}{d\tau}\left[\mathcal{H}\frac{\delta\phi}{\bar{\phi}'} + \Psi\right] \end{aligned} \tag{2.107}$$

The quantity in the square brackets is nothing but $\mathcal{R}$, so that this relation can be recast, thanks to the Friedman equations, as

$$\nabla^2\Psi = \frac{3\mathcal{H}}{2c_s^2}(1+w)\mathcal{R}'\,, \tag{2.108}$$

or in Fourier space as

$$\frac{3}{2}(1+w)\mathcal{R}'_{\boldsymbol{k}} = -\frac{c_s^2 k^2}{\mathcal{H}}\,\Psi_{\boldsymbol{k}}\,. \tag{2.109}$$

This equation is fundamental, since it ensures that on super-horizon scales (i.e. with $k \ll \mathcal{H}$) the comoving curvature perturbation is almost constant. It is customary to say that on scales much larger than the horizon, the curvature perturbation is frozen.

Equations (2.105) and (2.107) can be written in a different way by introducing the variables

$$z \equiv \frac{a^2\sqrt{P+\rho}}{\mathcal{H}c_s} \tag{2.110a}$$

$$\theta \equiv \frac{1}{zc_s} = \frac{\mathcal{H}}{a^2\sqrt{P+\rho}}\,, \tag{2.110b}$$

and

$$v \equiv z\mathcal{R} = \frac{a^2\sqrt{P+\rho}}{\overline{\phi}'c_s}\left(\delta\phi + \frac{\overline{\phi}'}{\mathcal{H}}\,\Psi\right) \tag{2.111a}$$

$$u \equiv \frac{\Psi}{4\pi\sqrt{P+\rho}}\,. \tag{2.111b}$$

With these quantities, they become

$$\nabla^2 u = \frac{z}{c_s}\left(\frac{v}{z}\right)' \qquad , \qquad v = \frac{\theta}{c_s}\left(\frac{u}{\theta}\right)'\,. \tag{2.112}$$

Applying the laplacian to the second equation, and taking into account the first one, we get

$$\nabla^2 v = \frac{1}{zc_s^2}\frac{d}{d\tau}\left(zc_s\,\frac{z}{c_s}\left(\frac{v}{z}\right)'\right) = \frac{1}{zc_s^2}\left(zv'' - vz''\right)\,. \tag{2.113}$$

If we now go to Fourier space, the latter reads

$$v_{\boldsymbol{k}}'' + \left(c_s^2k^2 - \frac{z''}{z}\right)v_{\boldsymbol{k}} = 0\,, \tag{2.114}$$

which is the *generalized Mukhanov-Sasaki equation.*
Equation (2.114) for the curvature perturbation $\mathcal{R}$ is

$$\mathcal{R}_{\boldsymbol{k}}'' + 2\,\frac{z'}{z}\,R_{\boldsymbol{k}}' + c_s^2k^2\mathcal{R}_{\boldsymbol{k}} = 0\,. \tag{2.115}$$

Chapter 3

Spectrum and bispectrum

3.1 Spectrum of curvature perturbation

In this section we determine the power spectrum for the curvature perturbation. Firstly, we briefly remind some concepts of the harmonic oscillator, since the analogy between this simple system and our treatment of $\mathcal{R}$ is quite strong and evident. We then define quantitatively the power spectrum, and we study how this is related to observational parameters.

3.1.1 Brief review of harmonic oscillator

In this subsection we briefly summarize some results for the harmonic oscillator which will be useful for our future considerations.
Let's start from the action [26]

$$S = \int dt \left(\frac{\dot{x}}{2} - \frac{\omega^2 \, x^2}{2} \right) = \int dt \, L \,, \tag{3.1}$$

where both x and ω are time-dependent, and the mass is $m = 1$.
From this action, the equations of motion can be found as

$$\delta S = 0 \qquad \Rightarrow \qquad \ddot{x} + \omega^2 x = 0 \,. \tag{3.2}$$

To quantize the system, the first step is to define the conjugate momentum p as

$$p = \frac{\delta L}{\delta \, \dot{x}} = \dot{x} \,. \tag{3.3}$$

Secondly, we promote both x and p to operators, and impose the commutation rule

$$[\hat{x}, \hat{p}] \equiv i \,. \tag{3.4}$$

Now, we can decompose the position operator $\hat{x}$ as

$$\hat{x} \equiv v(t) a + \text{h.c.} \,, \tag{3.5}$$

where v is a complex number and a is a time-independent operator.
Inserting this decomposition into (3.4), we find

$$i = (v\dot{v}^* - v^*\dot{v})[a, a^\dagger] \equiv iW[v, v][a, a^\dagger]\,, \tag{3.6}$$

where W is the Wronskian [27, 28].
Since we can choose, without loss of generality, v in such a manner that $W[v, v]$ is positive. Furthermore, by rescaling v so that $W[v, v] = 1$, we obtain the commutation rule

$$[a, a^\dagger] = 1\,, \tag{3.7}$$

which is the usual commutation relation between ladder operators.
The vacuum of the system is a vector $|0\rangle$, and it is defined through the action of the destruction operator a on it, i.e.

$$a\,|0\rangle \equiv 0\,. \tag{3.8}$$

Conversely, the excited states are defined as

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|0\rangle\ . \tag{3.9}$$

The Hamiltonian for the system is

$$\begin{aligned} \hat{H} &= \frac{\hat{p}^2}{2} + \frac{\omega^2\hat{x}^2}{2} \\ &= \frac{1}{2}\left[(\dot{v}^2 + \omega^2 v^2)aa + \text{h.c.}\right] + \frac{1}{2}\left[(|\dot{v}|^2 + \omega^2|v|^2)(aa^\dagger + a^\dagger a)\right]\ . \end{aligned} \tag{3.10}$$

By demanding that the vacuum state $|0\rangle$ is a state with zero energy, we find

$$\langle 0|\hat{H}|0\rangle = 0 \qquad \Rightarrow \qquad \dot{v} = \pm i\omega v\,. \tag{3.11}$$

The positivity of $W[v, v]$ selects the minus sign, and if we choose $W[v, v] = 1$ we find the normalized solution

$$v(t) = \sqrt{\frac{1}{2\omega}}\,e^{-i\omega t}\,. \tag{3.12}$$

3.1.2 Curvature perturbation power spectrum

We now apply the considerations for the harmonic oscillator to the variables v and $\mathcal{R}$ introduced in sections 2.2 and 2.3.
First of all, we note that the Mukhanov-Sasaki equation (2.114) in real space comes from the action [29, 30]

$$S = \frac{1}{2}\int d\tau\, d\boldsymbol{x}\left[(v')^2 - c_s^2(\partial_i v)^2 + \frac{z''}{z}v^2\right] \equiv \int d\tau\mathcal{L}\,. \tag{3.13}$$

The next step is to get the conjugate momentum of v, which is simply

$$\pi \equiv \frac{\delta\mathcal{L}}{\delta v'} = v'\,. \tag{3.14}$$

Then, we follow the standard procedure to quantize a system: we promote v and π to the operators $\hat{v}$ and $\hat{\pi}$, and impose the usual commutation relation (remember $\hbar = 1$)

$$[\hat{v}(\tau, \boldsymbol{x}), \hat{\pi}(\tau, \boldsymbol{y})] \equiv i\delta(\boldsymbol{x} - \boldsymbol{y}), \qquad \text{other vanishing}\,. \tag{3.15}$$

The operator $\hat{v}$ follows the same equation as the corresponding classic variable [14]

$$\hat{v}'' - \left(c_s^2 \nabla^2 + \frac{z''}{z}\right)\hat{v} = 0\,, \tag{3.16}$$

whose general solution can be written in Fourier space as

$$\hat{v}(\tau, \boldsymbol{x}) = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\left[\hat{a}(\boldsymbol{k})v(\tau, \boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{b}(\boldsymbol{k})v^*(\tau, \boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]\,. \tag{3.17}$$

Here, $\hat{a}$ and $\hat{b}$ are operators, while the mode functions fulfil the same Mukhanov-Sasaki equation (2.114)

$$v_{\boldsymbol{k}}'' + \left(c_s^2 k^2 - \frac{z''}{z}\right)v_{\boldsymbol{k}} = 0\,, \tag{3.18}$$

with $v_{\boldsymbol{k}} \equiv v(\tau, \boldsymbol{k})$.
The commutations rules (3.15) can be realized if we consider $\hat{a}$ and $\hat{b}$ satisfying the usual commutation relations for creator/annihilation operators

$$\left[\hat{a}(\boldsymbol{k}), \hat{b}(\boldsymbol{p})\right] = (2\pi)^3\delta(\boldsymbol{k} - \boldsymbol{p}) \quad , \quad \text{other vanishing}\,, \tag{3.19}$$

together with the condition on the mode functions

$$v_{\boldsymbol{k}} v_{\boldsymbol{k}}'^* - v_{\boldsymbol{k}}^* v_{\boldsymbol{k}}' = i\,. \tag{3.20}$$

For this reason, we will indicate $\hat{a}(\boldsymbol{k})$ with $a_{\boldsymbol{k}}$ and $\hat{b}(\boldsymbol{k})$ with $a_{\boldsymbol{k}}^\dagger$.

For our future considerations, it is more useful to deal with the curvature perturbation $\hat{\mathcal{R}}_{\boldsymbol{k}}$ rather than $\hat{v}_{\boldsymbol{k}}$.
Therefore we quantize $\mathcal{R}_{\boldsymbol{k}}$ as [3, 31, 32]

$$\hat{\mathcal{R}}_{\boldsymbol{k}}(\tau) = \mathcal{R}_{\boldsymbol{k}}(\tau)a_{\boldsymbol{k}} + \mathcal{R}_{-\boldsymbol{k}}^*(\tau)a_{-\boldsymbol{k}}^\dagger \tag{3.21a}$$

$$\left[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}^\dagger\right] = (2\pi)^3\delta(\boldsymbol{k} - \boldsymbol{k}')\,, \tag{3.21b}$$

so that

$$\hat{\mathcal{R}}(\tau, \boldsymbol{x}) = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\left[\mathcal{R}_{\boldsymbol{k}}(\tau)a_{\boldsymbol{k}} + \mathcal{R}_{-\boldsymbol{k}}^*(\tau)a_{-\boldsymbol{k}}^\dagger\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{3.22}$$

We have seen that the Mukhanov-Sasaki variable $v_{\boldsymbol{k}}$ satisfies the equation

$$v_{\boldsymbol{k}}'' + \left(c_s^2 k^2 - \frac{z''}{z}\right)v_{\boldsymbol{k}} = 0\,, \tag{3.23}$$

with

$$z = \frac{a^2\sqrt{P+\rho}}{\mathcal{H}c_s} = \frac{a\sqrt{P+\rho}}{Hc_s}\,. \tag{3.24}$$

Taking into account relations (1.14) and (1.60), we can recast z as

$$z = \frac{aM_{Pl}}{c_s}\sqrt{2\epsilon_1}\,. \tag{3.25}$$

Eq. (3.23) is the equation for a harmonic oscillator with a time-dependent frequency

$$\omega_k^2(\tau) = c_s^2 k^2 - \frac{z''}{z} \tag{3.26}$$

From eq. (3.25) we are able to express the ratio z''/z in terms of the HFF [20]. Let's firstly note that

$$\frac{z''}{z} = \frac{a^2}{z}(\ddot{z} + H\dot{z})\,. \tag{3.27}$$

Thus, we have to calculate the first and the second derivative of z:

- First derivative
  The first derivative is

$$\begin{aligned}\dot{z} &= M_{Pl}\sqrt{2}\left[\left(\frac{\dot{a}}{c_s} - \frac{a\,\dot{c}_s}{{c_s}^2}\right)\sqrt{\epsilon_1} + \frac{a}{2c_s}\frac{\dot{\epsilon}_1}{\sqrt{\epsilon_1}}\right] = z\left[H - \frac{\dot{c}_s}{c_s} + \frac{1}{2}\frac{\dot{\epsilon}_1}{\epsilon_1}\right] = \\ &= Hz\left(1 - s + \frac{\epsilon_2}{2}\right)\,. \end{aligned} \tag{3.28}$$

- Second derivative
  Taking a further derivative we find

$$\begin{aligned}\ddot{z} &= (\dot{H}z + H\dot{z})\left(1 - s + \frac{\epsilon_2}{2}\right) + Hz\left(\frac{\dot{\epsilon}_2}{2} - \dot{s}\right) = \\ &= \left[-H^2 z\epsilon_1 + H^2 z\left(1 - s + \frac{\epsilon_2}{2}\right)\right]\left(1 - s + \frac{\epsilon_2}{2}\right) + H^2 z\left(\frac{\epsilon_2\epsilon_3}{2} - \frac{\dot{s}}{H}\right) = \\ &= H^2 z\left[1 - \epsilon_1 + \epsilon_2 - 2s - \frac{\epsilon_1\epsilon_2}{2} + \epsilon_1 s - \epsilon_2 s + \frac{{\epsilon_2}^2}{4} + s^2 + \frac{\epsilon_2\epsilon_3}{2} - \frac{\dot{s}}{H}\right]\,. \end{aligned} \tag{3.29}$$

It is now straightforward to get the expression for z''/z, and we have [33]

$$\frac{z''}{z} = a^2H^2\left[2 - \epsilon_1 + \frac{3\epsilon_2}{2} - 3s - \frac{\epsilon_1\epsilon_2}{2} + \epsilon_1 s - \epsilon_2 s + \frac{{\epsilon_2}^2}{4} + s^2 + \frac{\epsilon_2\epsilon_3}{2} - \frac{\dot{s}}{H}\right] \tag{3.30}$$

At sufficiently early times, all the modes where deep inside the horizon, which means that $k \gg aH = \mathcal{H}$, or equivalently $|k\tau| \gg 1$. We know from eq. (1.45a) that in the inflationary slow-roll scenario, early times correspond to infinite negative conformal time. Thus, at zeroth order in HFF, in the remote past

$$\omega_k^2 \approx c_s^2 k^2 - \frac{2}{\tau^2} \xrightarrow{\tau\to-\infty} c_s^2 k^2\,, \tag{3.31}$$

where eq. (3.30) for slowly varying slow-roll parameters has been used.
In this limit, the Mukhanov-Sasaki equation (3.23) has two independent solutions $v_{\boldsymbol{k}} \propto e^{\pm i c_s k\tau}$; however, in a similar way to section 3.1.1, one can shows that the only

acceptable mode is $v_{\boldsymbol{k}} \propto e^{-ic_s k\tau}$.
To get an explicit expression for such a mode, we exploit the normalization condition on the wronskian, i.e. $W[v,v]=1$. Expressing the mode function as $v_{\boldsymbol{k}} = Ce^{-ic_s k\tau}$, we have

$$W[v,v]=1 \qquad \Leftrightarrow \qquad C = \frac{1}{\sqrt{2c_s k}}\,. \tag{3.32}$$

As a consequence, we find out that at very early times

$$v_{\boldsymbol{k}}(\tau) = \frac{e^{-ic_s k\tau}}{\sqrt{2c_s k}}\,. \tag{3.33}$$

This defines a preferable set of mode function and a unique vacuum, called *Bunch-Davies vacuum.* Eq. (3.33) is nothing but a plane wave which propagates in Minkowski space-time.

We now presume a phase of slow-roll with a constant speed of sound [28]. At first order in HFF, we have

$$\frac{z''}{z} \approx \mathcal{H}^2\left[2-\epsilon_1+\frac{3\epsilon_2}{2}\right]\,. \tag{3.34}$$

In addition, the slow-roll parameter ϵ_1 can be written as

$$\epsilon_1 = -\frac{\dot{H}}{H^2} = 1-\frac{\mathcal{H}'}{\mathcal{H}^2} \qquad \Rightarrow \qquad \frac{d}{d\tau}\left(\frac{1}{\mathcal{H}}\right) = \epsilon_1 - 1\,. \tag{3.35}$$

If we consider ϵ_1 as almost constant, we can integrate this equation to get

$$\mathcal{H} = \frac{1}{\tau(\epsilon_1-1)} \approx -\frac{1}{\tau}(1+\epsilon_1)\,. \tag{3.36}$$

Equation (3.30) then reduces to

$$\frac{z''}{z} \approx \frac{1}{\tau^2}\left[2+3\epsilon_1+\frac{3\epsilon_2}{2}\right] = \frac{\nu^2-\frac{1}{4}}{\tau^2}\,, \tag{3.37}$$

where we have defined

$$\nu \equiv \frac{3}{2}+\epsilon_1+\frac{\epsilon_2}{2}\,. \tag{3.38}$$

The Mukhanov-Sasaki (3.23) then becomes

$$v''_{\boldsymbol{k}} + \left(c_s^2k^2 - \frac{4\nu^2-1}{4\tau^2}\right)v_{\boldsymbol{k}} = 0\,. \tag{3.39}$$

Let's introduce now the quantity $x \equiv -c_s k\tau$, so that the equation above is

$$\frac{d^2 v_{\boldsymbol{k}}}{dx^2} + \left(1-\frac{4\nu^2-1}{4x^2}\right)v_{\boldsymbol{k}} = 0\,. \tag{3.40}$$

By defining

$$v_{\boldsymbol{k}}(x) \equiv \sqrt{\frac{x}{c_s k}}\,y(x)\,, \tag{3.41}$$

we finally recast it as

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - \nu^2\right) y = 0 \,. \tag{3.42}$$

This is a Bessel equation in its canonical form; therefore, the solution can be written as a combination of the cylindrical harmonics $J_\nu(x)$ and $Y_\nu(x)$. However, in this case, it is more convenient to write the solution as a combination of the Hankel functions $H_\nu^{(1,2)}(x) \equiv J_\nu(x) \pm i Y_\nu(x)$.
Being $x = -c_s k\tau$, let's write the solution as

$$v_{\boldsymbol{k}}(\tau) \equiv -\frac{\sqrt{-\pi\tau}}{2} \left[\alpha_k H_\nu^{(1)}(-c_s k\tau) + \beta_k H_\nu^{(2)}(-c_s k\tau)\right] \,, \tag{3.43}$$

with α_k and β_k which are called *Bogoliubov coefficients*.
From the property of the Hankel functions

$$\lim_{x \to \infty} H_\nu^{(1,2)}(x) = \sqrt{\frac{1}{x\pi}} \,(1 \mp i)\, e^{\pm i\left(x - \frac{\pi\nu}{2}\right)} \,, \tag{3.44}$$

we can write an expression for the mode function valid asymptotically in the past. This is

$$\lim_{c_s k\tau \to -\infty} v_{\boldsymbol{k}}(\tau) = -\frac{(1-i)\,\alpha_k\, e^{-i\left(c_s k\tau + \frac{\pi\nu}{2}\right)} + (1+i)\,\beta_k\, e^{i\left(c_s k\tau + \frac{\pi\nu}{2}\right)}}{2\sqrt{c_s k}} \,. \tag{3.45}$$

Evaluating this at zeroth order in HFF ($\nu = \frac{3}{2}$), if we want it to be equal to (3.33), we constraint the Bogoliubov coefficients to be

$$a_k = 1 \qquad , \qquad \beta_k = 0 \,. \tag{3.46}$$

Thus, the general solution of the Mukhanov-Sasaki equation (3.23), at first order in HFF, for an almost constant speed of sound, is

$$v_{\boldsymbol{k}}(\tau) = -\frac{\sqrt{-\pi\tau}}{2}\, H_\nu^{(1)}(-c_s k\tau) \,. \tag{3.47}$$

At the leading order in slow-roll parameters, $\nu = \frac{3}{2}$; then, the Hankel function takes the simple form

$$H_{3/2}^{(1)}(-c_s k\tau) = \sqrt{\frac{2}{\pi(-c_s k\tau)^3}}\,(-i + c_s k\tau) e^{-i c_s k\tau} \,, \tag{3.48}$$

and then

$$v_{\boldsymbol{k}}(\tau) = \frac{-i}{\sqrt{2k^3 c_s^3}}\, \frac{1 + i c_s k\tau}{\tau}\, e^{-i c_s k\tau} = \frac{e^{-i c_s k\tau}}{\sqrt{2 c_s k}} \left(1 - \frac{i}{c_s k\tau}\right) \,, \tag{3.49}$$

which reduces to (3.33) in the asymptotic past, as it should.

Now, from definition of z in (3.25), we find the expression for $\mathcal{R}_{\boldsymbol{k}}$[1]

$$\mathcal{R}_{\boldsymbol{k}}(\tau) = \frac{iH}{2M_{Pl}\sqrt{\epsilon_1 c_s k^3}}(1 + i c_s k\tau) e^{-i c_s k\tau} \,, \tag{3.50}$$

[1]So far, we have treated c_s as a constant. In fact, if this is not true, all the exponential factors become $e^{i\int c_s k d\tau}$, in accordance with [33].

where we have taken into account that if $\nu = \frac{3}{2}$ we have a de Sitter Universe with $a = -(H\tau)^{-1}$.

Now that we have an expression for $\hat{\mathcal{R}}_{\boldsymbol{k}}$, we are able to define a quantity which will become fundamental later: the primordial power spectrum of curvature perturbation.
The two-point correlation function of $\hat{\mathcal{R}}_{\boldsymbol{k}}$ is defined in terms of the power spectrum as

$$\begin{aligned}
\langle \hat{\mathcal{R}}_{\boldsymbol{k}} \hat{\mathcal{R}}_{\boldsymbol{k}'} \rangle &= \langle 0 | \left(\mathcal{R}_{\boldsymbol{k}}(\tau) a_{\boldsymbol{k}} + \mathcal{R}^*_{-\boldsymbol{k}}(\tau) a^\dagger_{-\boldsymbol{k}} \right) \left(\mathcal{R}_{\boldsymbol{k}'}(\tau) a_{\boldsymbol{k}'} + \mathcal{R}^*_{-\boldsymbol{k}'}(\tau) a^\dagger_{-\boldsymbol{k}'} \right) | 0 \rangle = \\
&= \mathcal{R}_{\boldsymbol{k}} \mathcal{R}^*_{-\boldsymbol{k}'} \langle 0 | a_{\boldsymbol{k}} a^\dagger_{-\boldsymbol{k}} | 0 \rangle = \mathcal{R}_{\boldsymbol{k}} \mathcal{R}^*_{-\boldsymbol{k}'} \langle 0 | \left[a_{\boldsymbol{k}}, a^\dagger_{-\boldsymbol{k}} \right] | 0 \rangle = \\
&= (2\pi)^3 \delta(\boldsymbol{k} + \boldsymbol{k}') |\mathcal{R}_{\boldsymbol{k}}|^2 \equiv P_{\mathcal{R}}(k) \, (2\pi)^3 \delta(\boldsymbol{k} + \boldsymbol{k}') \,,
\end{aligned} \tag{3.51}$$

with the δ function which comes from commutation relation and insures the invariance under translation for the background [34].
$P_{\mathcal{R}} = |\mathcal{R}(\tau, \boldsymbol{k})|^2$ is the *dimensional power spectrum*; however, it is more useful to define a *dimensionless power spectrum* $\mathcal{P}_{\mathcal{R}}(k)$ as

$$\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} |\mathcal{R}_{\boldsymbol{k}}|^2 \,. \tag{3.52}$$

From eq. (3.50) we readily find

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{H^2}{8\pi^2 M_{Pl}^2 \epsilon_1 c_s} \left[1 + \left(\frac{c_s k}{aH} \right)^2 \right] \xrightarrow{\text{superhorizon}} \frac{H^2}{8\pi^2 M_{Pl}^2 \epsilon_1 c_s} \,. \tag{3.53}$$

We have seen that during inflation the Hubble radius decreases (cf. eq. (1.40)); thus, if we suppose that inflation lasts a sufficient lapse of time, every scales crosses the horizon sooner or later. However, from eq. (2.109) we know that on super-horizon scales the comoving curvature perturbation $\mathcal{R}$, and consequently the power spectrum $\mathcal{P}_{\mathcal{R}}$, are almost constant.
For this reason, we will approximate the curvature power spectrum at the horizon crossing as

$$\mathcal{P}_{\mathcal{R}}(k) = \left. \frac{H^2}{8\pi^2 M_{Pl}^2 \epsilon_1 c_s} \right|_{k=aH} \,. \tag{3.54}$$

Since the r.h.s. of the above equation is evaluated at the horizon crossing, the power spectrum $\mathcal{P}_{\mathcal{R}}$ is purely a function of k. In particular, if it is k-independent, i.e. $\propto k^0$, we say that the power spectrum is *scale-invariant*.
However, since H and ϵ_1 are functions of time, even if slowly varying, we assume a slight deviation from scale-invariance, and parametrize the power spectrum as

$$\mathcal{P}_{\mathcal{R}}(k) \equiv A_s \left(\frac{k}{k_*} \right)^{n_s - 1} \,, \tag{3.55}$$

where k_* is a reference (or pivot) scale, e.g. $k_* = 0.05\,\mathrm{Mpc}^{-1}$, and n_s is called *scalar spectral index*, while the amplitude is given by

$$A_s = \frac{H^2}{8\pi^2 M_{Pl}^2 \epsilon_1 c_s} \,. \tag{3.56}$$

The measured amplitude A_s for the spectrum at this scale k_* is [7]

$$A_s \quad : \qquad \ln(10^{10}A_s) = 3.094 \pm 0.034\,. \tag{3.57}$$

From the definition of scale-invariant power spectrum, it is clear that such a condition corresponds to $n_s = 1$; in this case, $\mathcal{P}_{\mathcal{R}}$ is known as Harrison-Zel'dovich spectrum.

If n_s has no wavelength dependence, it can be determined as

$$n_s - 1 = \frac{d\ln\mathcal{P}_{\mathcal{R}}}{d\ln k}\,. \tag{3.58}$$

Taking into account definitions (1.60), (1.61) and (1.91) we obtain from eq. (3.54)

$$\begin{aligned} n_s - 1 &= \frac{d\ln\mathcal{P}_{\mathcal{R}}}{dN}\frac{dN}{d\ln k} = \left(2\frac{d\ln H}{dN} - \frac{d\ln\epsilon_1}{dN} - \frac{d\ln c_s}{dN}\right)\frac{dN}{d\ln k} = \\ &= (-2\epsilon_1 - \epsilon_2 - s)\frac{dN}{d\ln k} = (-2\epsilon_1 - \epsilon_2 - s)\left[\frac{d}{dN}(N + \ln H)\right]^{-1} \approx \\ &\approx (-2\epsilon_1 - \epsilon_2 - s)(1+\epsilon_1) \approx (-2\epsilon_1 - \epsilon_2 - s)\,. \end{aligned} \tag{3.59}$$

Thus, the scalar spectral index is related to the HFF by

$$n_s = 1 - 2\epsilon_1 - \epsilon_2 - s\,, \tag{3.60}$$

from which it is clear that the slow-roll parameters are the responsible for the deviation from scale invariance.
Observations suggest that there is a slight deviation from $n_s = 1$; in particular, for the pivot scale $k_* = 0.05\,\mathrm{Mpc}^{-1}$, the value of the spectral index at 68% CL is [7]

$$n_s = 0.9652 \pm 0.0062\,. \tag{3.61}$$

It is important to stress that the expression for the power spectrum (3.51) is analogous to that for the two-points correlation function for a Gaussian field (see eq. (A.13) in appendix). Therefore, we could wonder if it is correct to identify $\mathcal{R}_{\boldsymbol{k}}$ with a Gaussian random variable, with variance [35]

$$\sigma^2_{\mathcal{R}} = 4\pi^3|\mathcal{R}_{\boldsymbol{k}}|^2\,. \tag{3.62}$$

Indeed, this identification is correct, since when we have quantized the system, we have treated $\hat{\mathcal{R}}_{\boldsymbol{k}}$ as a quantum free field made up of a collection of harmonic oscillators with mode functions $\mathcal{R}_{\boldsymbol{k}}$. In addition, by construction, each mode started out in its own ground state, which for a free harmonic oscillator is a stationary state represented by a Gaussian wavepacket with variance $\sigma^2_{\mathcal{R}} \propto |\mathcal{R}_{\boldsymbol{k}}|^2$. Secondly, in expression (3.54), which is evaluated at horizon crossing, the phase of $\mathcal{R}_{\boldsymbol{k}}$ are random, expect for the reality condition, and this is another characteristic of Gaussian fields [15].
Furthermore, linear theory does not mix up different modes, so that if two modes start as independent, they will remain as long as the linear approximation is adequate.
For it reason, it is customary to say that inflation generates primordial Gaussian fluctuations.

3.2 Tensor modes

In the previous section we have analysed the scalar curvature perturbation and its power spectrum, which indeed come from scalar perturbations of the metric. In a similar way, we can consider tensor perturbation, and treat them separately. In fact, this is possible since at the linear order perturbations in the metric (2.5) do not mix with each others.
Thus, the line element which contains tensor perturbations is

$$ds^2 = a^2 \left[-d\tau^2 + (\delta_{ij} + h_{ij}) dx^i dx^j \right] . \tag{3.63}$$

h_{ij} is a rank 3, symmetric tensor, so that it would have 6 independent components: however, since it is traceless and divergenceless, it has only two degrees of freedom, which are called *gravitational waves polarizations*.
It is customary to write

$$h_{ij} = h_+ e^+_{ij} + h_\times e^\times_{ij} \equiv \sum_s h_s \, e^s_{ij} \, , \tag{3.64}$$

where e^s_{ij} $(s = +, \times)$ are called *polarization tensors*, and have the properties

$$\begin{cases} e^s_{ij} = e^s_{ji} \\ k^i e^s_{ij} = 0 \\ e^s_{ii} = 0 \\ e^s_{ij}{}^*(\boldsymbol{k}) = e^s_{ij}(-\boldsymbol{k}) \\ \sum_s e^s_{ij}{}^*(\boldsymbol{k}) e^{s'}_{ij}(\boldsymbol{k}) = 4\delta_{ss'} \end{cases} . \tag{3.65}$$

Without loss of generality, we can write

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{3.66}$$

which represents a perturbation in the xy-plane. This means that the wave vector $\boldsymbol{k}$ propagates along the z-axis.

From the Einstein equations, it can be shown that the amplitudes h_s satisfy the equations

$$\ddot{h}_{s,\boldsymbol{k}} + 3H\dot{h}_{s,\boldsymbol{k}} + \frac{k^2}{a^2} \, h_{s,\boldsymbol{k}} = 0 \, . \tag{3.67}$$

Introducing the new function

$$\tilde{h}_{s,\boldsymbol{k}} \equiv \frac{aM_{Pl}}{\sqrt{2}} \, h_{s,\boldsymbol{k}} \, , \tag{3.68}$$

the previous equation corresponds to

$$\tilde{h}''_{s,\boldsymbol{k}} + \left(k^2 - \frac{a''}{a} \right) \tilde{h}_{s,\boldsymbol{k}} = 0 \, , \tag{3.69}$$

which is the same as the Mukhanov-Sasaki equation with $z''/z \to a''/a$ and $c_s = 1$. For this reason, we do not repeat here the full calculation, but we only present the main results.
In analogy with what we have done in the previous section, the quantity a''/a can be recast as

$$\frac{a''}{a} = \frac{1}{\tau^2}(2 + 3\epsilon_1) = \frac{4\nu_t - 1}{4\tau^2}\,, \tag{3.70}$$

where

$$\nu_t \equiv \frac{3}{2} + \epsilon_1\,. \tag{3.71}$$

Therefore, the solution $\tilde{h}_{\boldsymbol{k}}$ is the same for both the polarizations, and it is equal to (3.47), with the substitutions $c_s \to 1$ and $\nu \to \nu_t$.
If we consider the leading order in HFF, the power spectrum $\mathcal{P}_h(k)$ on superhorizon scales is

$$\mathcal{P}_h(k) \equiv \frac{k^3}{2\pi^2}|h_{ij}(\boldsymbol{k})|^2 = \frac{k^3}{2\pi^2}\sum_{s,p} e^s_{ij} e^{p\,*}_{ij} h_{s,\boldsymbol{k}} h^*_{p,\boldsymbol{k}} = \frac{k^3}{2\pi^2}\,4|\tilde{h}_{\boldsymbol{k}}|^2\,\frac{2}{a^2 M_{Pl}^2} \approx \frac{2H^2}{\pi^2 M_{Pl}^2}\,. \tag{3.72}$$

If we assume that this spectrum is almost scale-invariant, we can rewrite it as

$$\mathcal{P}_h(k) \equiv A_t \left(\frac{k}{k_*}\right)^{n_t}\,, \tag{3.73}$$

where the amplitude is given by

$$A_t = \frac{2H^2}{\pi^2 M_{Pl}^2}\,, \tag{3.74}$$

and the spectral index for tensor modes by

$$n_t \equiv \frac{d\ln\mathcal{P}_h}{d\ln k} = 2\frac{\dot{H}}{H^2}(1 + \epsilon_1) \approx -2\epsilon_1 = 3 - 2\nu_t\,. \tag{3.75}$$

In the case of tensor perturbations, the scale invariance is characterized by $n_t = 0$.

For single-field inflationary models, an important is the so called *tensor-to-scalar ratio*, which is defined as

$$r \equiv \frac{A_t}{A_s}\,. \tag{3.76}$$

Combining eqs. (3.56) and (3.74), the so called consistency condition between the tensor-to-scalar ratio and the tensor spectral index holds

$$r \approx 16\epsilon_1\,. \tag{3.77}$$

The above relation holds to first order in the slow-roll parameters, and a modified one is needed to second order. When the inflaton has non-trivial speed of sound, the first order consistency condition in (3.77) is modified as

$$r \approx 16 c_s \epsilon_1\,. \tag{3.78}$$

With the pivot scale $k_* = 0.05\,\mathrm{Mpc}^{-1}$, the the Planck alone 2015 95% CL upper limit is [1]

$$r_{0.05} < 0.11\,. \tag{3.79}$$

Another pivot scale which is often used is $k_* = 0.002\,\mathrm{Mpc}^{-1}$. For this different scale the Planck 2015 upper limit is

$$r_{0.002} < 0.10\,. \tag{3.80}$$

At this scale, the constraints in the plane (n_s, r) with some representative inflationary models are shown in Fig. 3.1

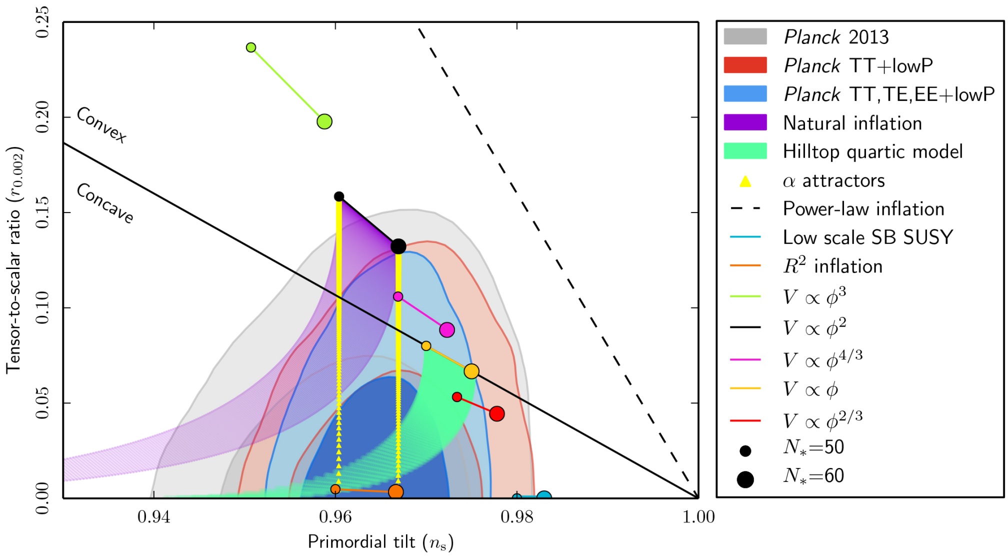

Figure 3.1: Marginalized joint 68% and 95% CL regions for n s and $r_{0.002}$ from *Planck* in combination with other data sets, compared to the theoretical predictions of selected inflationary models (figure taken from [1]).

3.3 Bispectrum of curvature perturbation

In the previous section we have defined the power spectrum for primordial curvature perturbation, and we have seen that inflation gives rise to Gaussian fluctuations.
However, this is true if we calculate the correlation functions in the vacuum state, and if we assume to be valid the approximation of a linear theory. Thus, if we introduce a perturbation, it is quite natural that modes start to mix with each other, and that they can evolve from their vacuum state.
This is the idea behind the so called *in-in formalism*: one tries to study how different modes which are initially in their vacuum state evolve in the presence of a non-linear interaction term.

Our next goal is then to calculate the bispectrum, which is the correspondent quantity for the three-points correlation function.

As a first step, we present a summary on the key points of the in-in formalism. Then, we show how the bispectrum can be calculated from a general third order Lagrangian.

3.3.1 The 'in-in' formalism

To connect fundamental theory with observations, a central object to compute is the time correlation function of a certain operator $A(t)$ [28]. What one wants to compute is therefore the quantity

$$\langle A(t)\rangle \,, \tag{3.81}$$

where the expectation value is calculated with respect to the initial state $|0_{in}\rangle$ [36]: such an initial state is usually assumed to coincide with the Bunch-Davies vacuum.

This is usually realized by working in the interaction picture, in which the Hamiltonian of the system is written as [37, 38]

$$H = H_0 + H_I \,, \tag{3.82}$$

where H_0 is the free Hamiltonian.
In this picture, operators are assumed to evolve with the free Hamiltonian H_0 as

$$A_{int}(t) \equiv e^{\frac{i}{\hbar}H_0 t} A(0) e^{-\frac{i}{\hbar}H_0 t} \,. \tag{3.83}$$

States, on the other hand, are assumed to evolve as

$$|\psi_{int}(t)\rangle \equiv e^{\frac{i}{\hbar}H_0 t}|\psi(t)\rangle = e^{\frac{i}{\hbar}H_0 t} e^{-\frac{i}{\hbar}H t}|\psi(0)\rangle \,. \tag{3.84}$$

Taking a time derivative of this gives

$$\begin{aligned} i\hbar\frac{\partial|\psi_{int}(t)\rangle}{\partial t} &= -\left(H_0 e^{\frac{i}{\hbar}H_0 t} e^{-\frac{i}{\hbar}H t} - e^{\frac{i}{\hbar}H_0 t} H e^{-\frac{i}{\hbar}H t}\right)|\psi(0)\rangle = e^{\frac{i}{\hbar}H_0 t} V e^{-\frac{i}{\hbar}H t}|\psi(0)\rangle = \\ &= e^{\frac{i}{\hbar}H_0 t} V e^{-\frac{i}{\hbar}H_0 t} e^{\frac{i}{\hbar}H_0 t} e^{-\frac{i}{\hbar}H t}|\psi(0)\rangle = H_{I,int}|\psi_{int}(t)\rangle \,, \end{aligned} \tag{3.85}$$

whose solution is

$$|\psi_{int}(t)\rangle = T\exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, H_{int}(t')\right\}|\psi_{int}(t_0)\rangle \,, \tag{3.86}$$

where we have omitted the subscript $_I$, and T is the time-ordering operator.
However, given the evolution operator

$$U(t,t_0) \equiv e^{-\frac{i}{\hbar}H(t-t_0)} \,, \tag{3.87}$$

it is also true that

$$\begin{aligned} |\psi_{int}(t)\rangle &= e^{\frac{i}{\hbar}H_0 t} e^{-\frac{i}{\hbar}H t}|\psi(0)\rangle = e^{\frac{i}{\hbar}H_0 t} U(t,t_0) e^{-\frac{i}{\hbar}H t}|\psi(0)\rangle = \\ &= e^{\frac{i}{\hbar}H_0 t} U(t,t_0) e^{-\frac{i}{\hbar}H_0 t} e^{\frac{i}{\hbar}H_0 t} e^{-\frac{i}{\hbar}H t}|\psi(0)\rangle = U_{int}(t,t_0)|\psi_{int}(t_0)\rangle \,. \end{aligned} \tag{3.88}$$

By comparison, then, we have an explicit expression for the evolution operator in the interaction picture, i.e.

$$U_{int}(t,t_0) = T\exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, H_{int}(t')\right\}\,. \tag{3.89}$$

From eq. (3.84) it is immediate to find an expression for the time correlation $\langle A(t)\rangle$: in particular, this is [28]

$$\langle A(t)\rangle = \langle 0_{in}|\,\overline{T}\exp\left\{i\int_{t_0}^{t} dt'\, H_{int}(t')\right\}\, A(t)\, T\exp\left\{-i\int_{t_0}^{t} dt'\, H_{int}(t')\right\}|0_{in}\rangle\ , \tag{3.90}$$

where we have switched to natural units, as usual, and $\overline{T}$ is the anti time-ordering operator.

However, a more useful way to compute this time averaged was found by Weinberg [39], and it is

$$\langle A(t)\rangle = \sum_{n=0}^{\infty} i^n \int_{t_0}^{t} dt_n \int_{t_0}^{t_n} dt_{n-1}\ldots\int_{t_0}^{t_2} dt_1 \langle [H_{int}(t_1),[H_{int}(t_2),\ldots[H_{int}(t_n),A(t)]\ldots]]\rangle\,. \tag{3.91}$$

The two formulations (3.90) and (3.91) can be shown to be equivalent by recurrence. If one assumes that their time derivative are equivalent up to order N, then they are equal also at order $N+1$.

For our future computations, we will use formulation (3.91).

3.3.2 Curvature perturbation bispectrum

As for the power spectrum $\mathcal{P}_{\mathcal{R}}$, we define the bispectrum $\mathcal{B}_{\mathcal{R}}$ as

$$\langle \hat{\mathcal{R}}_{\boldsymbol{k}_1}\hat{\mathcal{R}}_{\boldsymbol{k}_2}\hat{\mathcal{R}}_{\boldsymbol{k}_3}\rangle \equiv \mathcal{B}_{\mathcal{R}}(k_1,k_2,k_3)\,(2\pi)^3\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\,, \tag{3.92}$$

where again, the δ function is a consequence of invariance under translation.

Another definition, which will be used in our future calculations, is given through a quantity $\mathcal{G}$ defined as [3]

$$\langle \hat{\mathcal{R}}_{\boldsymbol{k}_1}\hat{\mathcal{R}}_{\boldsymbol{k}_2}\hat{\mathcal{R}}_{\boldsymbol{k}_3}\rangle \equiv \frac{\mathcal{G}(k_1,k_2,k_3)}{k_1k_2k_3}\,\frac{\mathcal{P}^2_{\mathcal{R}}(k)}{k_1^2k_2^2k_3^2}\,(2\pi)^7\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\,. \tag{3.93}$$

By comparison of (3.92) with (3.93) it is immediate to get

$$\frac{\mathcal{G}(k_1,k_2,k_3)}{k_1k_2k_3} = \frac{\mathcal{B}_{\mathcal{R}}(k_1,k_2,k_3)}{\mathcal{P}^2_{\mathcal{R}}(k)}\,\frac{k_1^2k_2^2k_3^2}{(2\pi)^4}\,. \tag{3.94}$$

In literature, the l.h.s. of the above expression is sometimes indicated as $\mathcal{S}(k_1,k_2,k_3)$. Other parametrizations are given in terms of the functions $\mathcal{A}(k_1,k_2,k_3) \equiv k_1k_2k_3\,\mathcal{S}(k_1,k_2,k_3) = \mathcal{G}(k_1,k_2,k_3)$ and $\mathcal{F} \equiv (k_1k_2k_3)^{-2}\mathcal{S}(k_1,k_2,k_3)$ [40].

The dependence of the bispectrum (i.e. $\mathcal{B}_{\mathcal{R}}$ or the other quantities defined above) on k_1, k_2 and k_3 are usually split into two kinds: the *shape* and the *running*.

The shape refers to the dependence of $\mathcal{B}_{\mathcal{R}}$ on the ratios k_2/k_1 and k_3/k_1 while keeping fixed the overall momentum $K \equiv k_1 + k_2 + k_3$. Particularly important are the *equilateral shape* ($k_2/k_1 = 1 = k_3/k_1$) and the *squeezed shape* ($k_2/k_1, k_3/k_1 \gg 1$). On the other hand, the running refers to the dependence of $\mathcal{B}_{\mathcal{R}}$ on the overall momentum $K = k_1 + k_2 + k_3$ while keeping fixed the ratios k_2/k_1 and k_3/k_1.

Together with these definitions, non-Gaussianities are often parametrized in terms of a dimensionless parameter, indicated as f_{NL} [31, 41].
At the end of the previous section, we have noted that at the leading order, we can consider $\hat{\mathcal{R}}$ as Gaussian. Therefore we can parametrize non-Gaussianities in terms of the power spectrum as [42, 11]

$$\mathcal{B}_{\mathcal{R}}(k_1,k_2,k_3) = \frac{6}{5}\, f_{NL}\left[\mathcal{P}_{\mathcal{R}}(k_1)\mathcal{P}_{\mathcal{R}}(k_2) + \mathcal{P}_{\mathcal{R}}(k_1)\mathcal{P}_{\mathcal{R}}(k_3) + \mathcal{P}_{\mathcal{R}}(k_2)\mathcal{P}_{\mathcal{R}}(k_3)\right] . \tag{3.95}$$

See appendix A for a brief review on Gaussian and non-Gaussian classical random fields.
In the previous expression, the factor $\frac{6}{5}$ is conventional and follows from the fact that this treatise has been originally adopted for the gravitational potential Φ [43, 44], which during matter domination is related to the curvature perturbation by $\Phi = \frac{3}{5}\,\mathcal{R}$ (another factor 2 comes from (A.25)).
If we assume a scale-invariant power spectrum $\mathcal{P}_{\mathcal{R}}$, thanks to eq.(3.52) we can rewrite eq. (3.95) as

$$\begin{aligned}\mathcal{B}_{\mathcal{R}}(k_1,k_2,k_3) &= \frac{6}{5}\, f_{NL}\frac{(2\pi)^4}{4}\mathcal{P}^2_{\mathcal{R}}\left[\frac{1}{k_1^3k_2^3} + \frac{1}{k_1^3k_3^3} + \frac{1}{k_2^3k_3^3}\right] = \\ &= \frac{3}{10}\, f_{NL}(2\pi)^4\mathcal{P}^2_{\mathcal{R}}\frac{k_1^3+k_2^3+k_3^2}{k_1^3\,k_2^3\,k_3^2}\,,\end{aligned} \tag{3.96}$$

and find the usual expression for f_{NL} [41, 28]

$$f_{NL} = \frac{10}{3(2\pi)^4}\frac{k_1^3\,k_2^3\,k_3^2}{k_1^3+k_2^3+k_3^3}\frac{\mathcal{B}_{\mathcal{R}}(k_1,k_2,k_3)}{\mathcal{P}^2_{\mathcal{R}}}\,. \tag{3.97}$$

Eventually, exploiting eq. (3.94) which relates $\mathcal{B}_{\mathcal{R}}$ with $\mathcal{G}$ we can write it as

$$f_{NL} = \frac{10}{3}\frac{1}{k_1^3+k_2^3+k_3^3}\,\mathcal{G}(k_1,k_2,k_3)\,. \tag{3.98}$$

3.3.3 Second and third order action

The three-point function is calculated from a third order action S_3. This action derives directly from the general action for the single inflaton field

$$S = \int dx\,\sqrt{-g}\left[\frac{M_{Pl}}{2}R + P(\chi,\phi)\right]\,. \tag{3.99}$$

In fact, in the ADM formalism [45, 46], this action can be expanded up to the cubic order in $\mathcal{R}$.
Doing so, ones firstly find the quartic action [3, 28]

$$S_2 = \int dx\,a^3\epsilon_1 M_{Pl}\left(\frac{\dot{\mathcal{R}}^2}{c_s^2} - \frac{(\partial\mathcal{R})^2}{a^2}\right)\,. \tag{3.100}$$

An explicit expression for the third order action will be given later on.

To compute the three-point correlation function, we exploit eq. (3.91) to find [3, 31]

$$\langle \hat{\mathcal{R}}_{\boldsymbol{k}_1}(t)\hat{\mathcal{R}}_{\boldsymbol{k}_2}(t)\hat{\mathcal{R}}_{\boldsymbol{k}_3}(t)\rangle = -i\int_{t_0}^{t} dt' \, \langle \Big[\hat{\mathcal{R}}_{\boldsymbol{k}_1}(t)\hat{\mathcal{R}}_{\boldsymbol{k}_2}(t)\hat{\mathcal{R}}_{\boldsymbol{k}_3}(t), \hat{H}_{int}(t')\Big]\rangle \, . \tag{3.101}$$

Here we have taken into account just the first term of the sum in (3.91) since the third order action is suppressed by one more order in HFF relative to the quadratic action [28].
The Hamiltonian in the interaction picture can be obtained directly from the third order action as [39]

$$H_{int} = -L_3 \qquad \text{where} \qquad L_3 \; : \; S_3 = \int dt \, L_3 \, . \tag{3.102}$$

Thus, if we have an explicit expression for S_3, we are immediately able to compute the integrand of eq. (3.101).
The third order action for the k-inflation Lagrangian in (3.99) is [47, 48]

$$\begin{aligned} S_3 = \int dx \, \Bigg\{ & a^3 \mathcal{C}_1 M_{Pl}^2 \mathcal{R}\dot{\mathcal{R}}^2 + a\mathcal{C}_2 M_{Pl}^2 \mathcal{R}(\partial\mathcal{R})^2 + a^3\mathcal{C}_3 M_{Pl}\dot{\mathcal{R}}^3 + a^3\mathcal{C}_4\dot{\mathcal{R}}(\partial_i\mathcal{R})(\partial_i\kappa) + \\ & + \frac{a^3}{M_{Pl}^2}\mathcal{C}_5\partial^2\mathcal{R}(\partial\kappa)^2 + \mathcal{F}_1 \left.\frac{\delta L_2}{\delta\mathcal{R}}\right|_1 \Bigg\} , \end{aligned} \tag{3.103}$$

where the coefficients $\mathcal{C}_i$, $(i = 1, \ldots, 5)$ and $\mathcal{F}_1 \left.\frac{\delta L_2}{\delta\mathcal{R}}\right|_1$ are given by

$$\mathcal{C}_1 = \frac{\epsilon_1}{c_s^4}\left(\epsilon_1 - \epsilon_2 - 3 + 3c_s^2\right) \tag{3.104a}$$

$$\mathcal{C}_2 = \frac{\epsilon_1}{c_s^2}\left(\epsilon_1 + \epsilon_2 - c_s^2 - 2s + 1\right) \tag{3.104b}$$

$$\mathcal{C}_3 = \frac{\Sigma - c_s^2(2\lambda + \Sigma)}{H^3 M_{Pl} c_s^2} \tag{3.104c}$$

$$\mathcal{C}_4 = \frac{\epsilon_1}{2c_s^2}\left(\epsilon_1 - 4\right) \tag{3.104d}$$

$$\mathcal{C}_5 = \frac{\epsilon_1}{4} \tag{3.104e}$$

$$\begin{aligned} \mathcal{F}_1 = & -\frac{1}{2HM_{Pl}^2}\left[\partial_k\mathcal{R}\partial_k\kappa - \partial^{-2}\partial_i\partial_j(\partial_i\mathcal{R}\partial_j\kappa)\right] - \frac{1}{Hc_s^2}\mathcal{R}\dot{\mathcal{R}} + \\ & + \frac{1}{4M_{Pl}^2 a^2}\Big[(\partial\mathcal{R})^2 - \partial^{-2}\partial_i\partial_j(\partial_i\mathcal{R}\partial_j\mathcal{R})\Big] \end{aligned} \tag{3.104f}$$

$$\left.\frac{\delta L_2}{\delta\mathcal{R}}\right|_1 = -2M_{Pl}^2\epsilon_1\left[\frac{a^3}{c_s^2}\Big(3H\dot{\mathcal{R}} + H\epsilon_2\dot{\mathcal{R}} - 2Hs\dot{\mathcal{R}} + \ddot{\mathcal{R}}\Big) - a\partial^2\mathcal{R}\right] \tag{3.104g}$$

$$\kappa = \frac{M_{Pl}^2\epsilon_1}{c_s^2}\,\partial^{-2}\dot{\mathcal{R}} \tag{3.104h}$$

$$\Sigma = \chi P_\chi + 2\chi^2 P_{\chi\chi} \tag{3.104i}$$

$$\lambda = \chi^2 P_{\chi\chi} + \frac{2}{3}\chi^3 P_{\chi\chi\chi} \tag{3.104j}$$

with ∂^{-2} the inverse of Laplacian.
In the cubic action (3.103), the last term of r.h.s. survives only at second order in $\mathcal{R}$. Thus, when we will evaluate the three-point function, we will not take it into account [47].

From the expression of the coefficients $\mathcal{C}_i$, $(i = 1, \ldots, 5)$, it is clear that in standard slow-roll inflation, the bispectrum is always negligible with respect to the power spectrum, since it is at least proportional to HFF.
In order to get a significant bispectrum, there are several possibilities: first of all, one can consider a temporary violation of the slow-roll regime; secondly, Lagrangian with a non-trivial speed of sound can be taken into account; eventually, one can study a system described by multiple field.
In the following, we will widely study the case of a violation of slow-roll, together with non-standard Lagrangians which give rise to a variable speed of sound.

Chapter 4

Discontinuity in the first derivative of the inflaton potential

4.1 Primordial perturbations and CMB power spectrum

We have seen that the inflationary scenario predicts an almost scale-invariant power spectrum. The observable quantities which allow to reconstruct such a power spectrum are the CMB anisotropies and the large-scale structures (LSS). However, the inflationary paradigm cannot predict exact initial conditions, but just their statistical properties: for this reason, a statistical analysis of data is required. Since we have just one realization of our Universe, the mean values are inevitably affected by an error, which is called *cosmic variance*.
As far as the CMB is concerned, the statistical analysis has to be done on a spherical surface, since the cosmic microwave background originated about 300 000 yr after the Big-Bang ($z \approx 1100$), and propagated from a spherical surface which is called *last scattering surface*. In particular, we are able to measure the fluctuations in temperature of the CMB as a function of the angle, and from these to constrain the cosmological parameters.
The most natural set of functions to describe these fluctuations is constituted by the spherical harmonics $Y_{\ell m}$, so that

$$\frac{\Delta T}{T}(\theta, \phi) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi) \,. \tag{4.1}$$

From the coefficients $a_{\ell m}$ we define the *angular power spectrum* C_ℓ as

$$C_\ell \equiv \langle |a_{\ell m}|^2 \rangle \,. \tag{4.2}$$

It is customary to define an angular power spectrum as

$$\mathcal{D}_\ell^{TT} \equiv \ell(\ell+1)\frac{C_\ell}{2\pi} \,. \tag{4.3}$$

From *Planck* 2015 data, we have that this spectrum is [7]

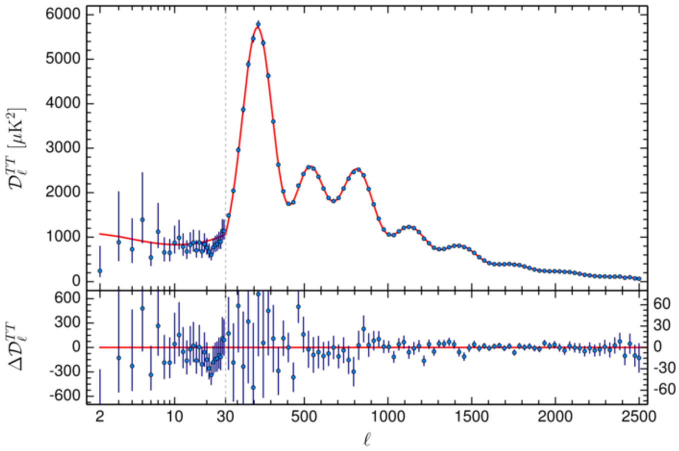

Figure 4.1: The *Planck* 2015 temperature power spectrum (figure taken from [7]).

The multipole ℓ is related to the angle by the formula

$$\theta \approx \frac{\pi}{\ell}\,, \tag{4.4}$$

from which we see that the biggest uncertainty in the above spectrum corresponds to large angular scales (and therefore to large scales). In addition, we see that in the power spectrum there is a feature at $\ell \approx 20$, a dip which is lower than the prediction of the simple ΛCDM model. For this reason, many models have been studied in order to improve the fit at low ℓ. One possibility is to consider a scenario in which the slow-roll regime is not valid at all scales, but it is temporarily broken. Observations seem to suggest such a scenario; furthermore, this breaking in the slow-roll condition should have taken place at large scales, so that this could be a good way to further constrain the temperature power spectrum at large angular scales.

4.2 The original model with a discontinuity in the first derivative of the potential

In this section we introduce and discuss a model firstly introduced by Starobinsky characterized by a discontinuity in the first derivative of the inflaton potential [2]. This model is included in the context of standard single-field inflation, i.e. with the standard Klein-Gordon equation (1.54) and Friedman equation (1.56b).
The Starobinsky model consists of a linear potential with a sharp change in its slope at a given point, which we define with ϕ_0. Let the slope of the potential, before and after such a transition, be A_+ and A_- (both assumed to be positive), respectively. In other words, the potential is

$$V(\phi) = \begin{cases} V_0 + A_+(\phi-\phi_0) & , \quad \phi > \phi_0 \\ V_0 + A_-(\phi-\phi_0) & , \quad \phi < \phi_0 \end{cases}\,, \tag{4.5}$$

which can also be written in a single line as

$$V(\phi) = V_0 + \left[A_- + (A_+ - A_-)\,\theta(\phi-\phi_0)\right](\phi-\phi_0)\,, \tag{4.6}$$

with θ Heaviside step function.

4.2.1 Evolution of the background

In this subsection we study the background for this model, i.e. we determine the dependence of the inflaton field and the Hubble flow function on time.

For our future numerical estimations, we use the following values for the parameters [32]

Parameter	**Value**
ϕ_0	$0.707\, M_{Pl}$
V_0	$2.37 \times 10^{-12} M_{Pl}^4$
A_+	$3.35 \times 10^{-14} M_{Pl}^3$
A_-	$7.26 \times 10^{-15} M_{Pl}^3$

Table 4.1: Numerical values for the potential in (4.6).

With these values, the shape of the potential is

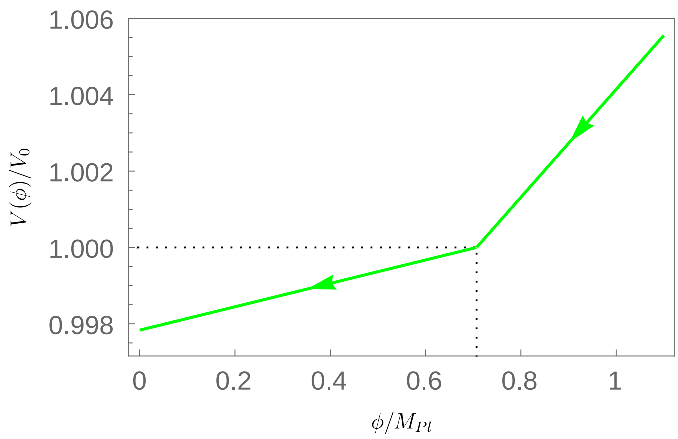

Figure 4.2: Potential for the Starobinsky model.

The simple structure of the potential in (4.6) allows us to study separately the evolution before and after the transition, and then to match these two regimes with appropriate conditions.

- **Before the transition**.
  Consider the case where the field is rolling down the potential from an initial value $\phi_i > \phi_0$. In slow-roll approximation, we have seen that the number of

e-folds can be expressed as (cf. eq. (1.81))

$$N = -\frac{1}{M_{Pl}^2}\int_{\phi_i}^{\phi_E} d\phi\,\frac{V}{V_\phi} = -\frac{1}{M_{Pl}^2}\int_{\phi_i}^{\phi_E} d\phi\left(\frac{V_0}{A_+}+\phi-\phi_0\right) =$$
$$= \frac{1}{2}\left[\phi^2+2\phi\left(\frac{V_0}{A_+}-\phi_0\right)-2\frac{V_0\phi_i}{A_+}-\phi_i^2+2\phi_0\phi_i\right]. \tag{4.7}$$

By solving for the inflaton field, we get

$$\phi_+ = -\left(\frac{V_0}{A_+}-\phi_0\right)+\left[\left(\phi_i-\phi_0+\frac{V_0}{A_+}\right)^2-2M_{Pl}^2N\right]^{\frac{1}{2}}, \tag{4.8}$$

where the subscript $_+$ indicates that we are considering quantities before the transition.
The number of e-folds at which the field reaches the values ϕ_0 is therefore

$$N_0 = \frac{\phi_i-\phi_0}{2M_{Pl}^2}\left[\phi_i-\phi_0+\frac{2V_0}{A_+}\right]. \tag{4.9}$$

Taking the derivative with respect to the number of e-folds gives

$$\frac{d\phi_+}{dN} = -M_{Pl}^2\left[\left(\phi_i-\phi_0+\frac{V_0}{A_+}\right)^2-2M_{Pl}^2N\right]^{-\frac{1}{2}}. \tag{4.10}$$

Actually, the Starobinsky model assumes that for a range of ϕ around ϕ_0, the constant V_0 is the dominant term in the potential, i.e. $|V-V_0| \ll V_0$, which means that the potential is vacuum dominated [3].
With this assumption, the expression (4.8) for ϕ_+ simplifies, and we get

$$\phi_+ \approx -\frac{V_0}{A_+}+\left[\frac{V_0^2}{A_+^2}+2\phi_i\frac{V_0}{A_+}-2M_{Pl}^2N\right]^{\frac{1}{2}} \approx$$
$$\approx -\frac{V_0}{A_+}\left[1-\left(1+2\phi_i\frac{A_+}{V_0}-2M_{Pl}^2N\frac{A_+^2}{V_0^2}\right)^{\frac{1}{2}}\right] \approx$$
$$\approx -\frac{V_0}{A_+}\left[-\phi_i\frac{A_+}{V_0}+M_{Pl}^2N\frac{A_+^2}{V_0^2}\right] = \phi_i-\frac{A_+M_{Pl}^2}{V_0}N\,, \tag{4.11}$$

from which

$$\frac{d\phi_+}{dN} \approx -\frac{A_+M_{Pl}^2}{V_0}. \tag{4.12}$$

Note that these equation can also be obtained from the Klein-Gordon equation in which the Hubble parameter is assumed to be

$$H^2 \approx \frac{V_0}{3M_{Pl}^2} \equiv H_0^2. \tag{4.13}$$

The approximation for the trajectory as a straight line with a negative slope is very accurate before the transition, as it is shown in fig.1 of [32].

- **After the transition**.
  After the transition, however, the slow-roll approximation ceases to be valid, and we have to consider the exact Klein-Gordon equation

$$\ddot{\phi}_- + 3H\dot{\phi}_- + V_\phi = 0\,, \tag{4.14}$$

  where the subscript $_-$ indicates that we are considering quantities after the transition.
  Switching to derivatives with respect to the number of e-folds, indicated with $_N$, the above equation is

$$H^2\phi_{NN,-} + (3-\epsilon_1)H^2\phi_{N,-} + V_\phi = 0\,. \tag{4.15}$$

  Since we have assumed the potential is dominated by the constant term V_0, at the leading order we can approximate the Hubble parameter as a constant, which implies $\epsilon_1 \ll 1$.
  Thus the Klein-Gordon equation simplifies to

$$H_0^2\phi_{NN,-} + 3H_0^2\phi_{N,-} + A_- \approx 0\,, \tag{4.16}$$

  whose solution is

$$\phi_- = -\frac{A_-}{3H_0^2}N - \frac{\alpha\, e^{-3N}}{3} + \beta\,. \tag{4.17}$$

The integration constants α and β are determined by demanding that the field and its first derivative are continuous at the transition, i.e.

$$-\frac{A_-}{3H_0^2}N_0 - \frac{\alpha\, e^{-3N_0}}{3} + \beta = \phi_0 \tag{4.18a}$$

$$\frac{A_-}{3H_0^2} + \alpha e^{-3N_0} = \frac{A_+ M_{Pl}^2}{V_0}\,, \tag{4.18b}$$

from which

$$\phi_- \approx \phi_0 + \frac{\Delta A}{9H_0^2}\left[1 - e^{-3(N-N_0)}\right] - \frac{A_-}{3H_0^2}(N-N_0)\,, \tag{4.19}$$

where $\Delta A \equiv A_- - A_+$.
The first derivative is

$$\frac{d\phi_-}{dN} \approx -\frac{A_-}{3H_0^2} + \frac{\Delta A e^{-3(N-N_0)}}{3H_0^2}\,. \tag{4.20}$$

The assumption for the Hubble parameter (4.13) gives the simple solution $a(t) = \exp\{H_0 t\}$. Furthermore, the condition $\epsilon_1 \ll 1$ is equivalent to say that inflation never ends, even if the slow-roll regime is temporarily broken.
For our future considerations, it will be useful to express the quantities in conformal time: for the previous considerations the conformal time and the scale factor are simply

$$\tau = -\frac{e^{-H_0 t}}{H_0} \tag{4.21a}$$

$$a(\tau) = -\frac{1}{H_0\tau}\,. \tag{4.21b}$$

In addition, since $H \approx H_0$, the relation between τ and N is

$$\tau = -\frac{e^{-N}}{H_0} \,. \tag{4.22}$$

Eqs. (4.11),(4.12),(4.19) and (4.20) become [3]

$$\phi_+ \approx \phi_i + \frac{A_+}{3H_0^2} \ln(-H_0\tau) \tag{4.23a}$$

$$\frac{d\phi_+}{d\tau} \approx \frac{A_+}{3H_0^2\tau} \tag{4.23b}$$

$$\phi_- \approx \phi_0 + \frac{\Delta A}{9H_0^2}\left(1 - \frac{\tau^3}{\tau_0^3}\right) + \frac{A_-}{3H_0^2} \ln\left(\frac{\tau}{\tau_0}\right) \tag{4.23c}$$

$$\frac{d\phi_-}{d\tau} \approx \frac{1}{3H_0^2\tau}\left(A_- - \Delta A\,\frac{\tau^3}{\tau_0^3}\right) \equiv \frac{A_-}{3H_0^2\tau}\left(1-\rho^3\tau^3\right) \,. \tag{4.23d}$$

where we have defined $\rho^3 \equiv \dfrac{\Delta A}{A_-}\dfrac{1}{\tau_0^3}$.
The first HFF is given by eq. (1.60), and it is

$$\epsilon_1 = \frac{\dot{\phi}^2}{2H^2M_{Pl}^2} = \frac{\phi'^2}{2H^2a^2M_{Pl}^2} \approx \frac{\phi'^2\tau^2}{2M_{Pl}^2} \,. \tag{4.24}$$

For this model, we therefore find

$$\epsilon_{1,+} \approx \frac{A_+^2}{18M_{Pl}^2H_0^4} \tag{4.25a}$$

$$\epsilon_{1,-} \approx \frac{A_-^2}{18M_{Pl}^2H_0^4}\left(1-\rho^3\tau^3\right)^2 , \tag{4.25b}$$

The second slow-roll parameter is

$$\epsilon_2 = 2\left(\frac{\ddot{\phi}}{H\dot{\phi}} + \epsilon_1\right) \,. \tag{4.26}$$

Before the transition we have, at the first non-vanishing order

$$\ddot{\phi}_+ = \frac{d}{dt}\frac{\phi_+'}{a} \approx -\frac{d}{dt}\frac{A_+}{3H} = -\frac{A_+\epsilon_{1,+}}{3} \,, \tag{4.27}$$

so that

$$\epsilon_{2,+} = 2\left(\frac{\ddot{\phi}_+}{H\dot{\phi}_+} + \epsilon_{1,+}\right) \approx 2(\epsilon_{1,+} + \epsilon_{1,+}) = 4\epsilon_{1,+} \,. \tag{4.28}$$

After the transition, the calculation is a bit more elaborated. First of all, let's write eq. (4.23c) in terms of the cosmic time, i.e.

$$\phi_- \approx \phi_0 + \frac{\Delta A}{9H_0^2}\left(1 - e^{-3H(t-t_0)}\right) - \frac{A_- t}{3H} \,. \tag{4.29}$$

Taking the first and the second derivative of this with respect to t we find

$$\dot{\phi}_- \approx \frac{\Delta A}{3H} e^{-3H(t-t_0)} - \frac{A_-}{3H} = -\frac{A_-}{3H}\left(1-\rho^3\tau^3\right) , \tag{4.30}$$

and

$$\ddot{\phi}_- \approx -\Delta A e^{-3H(t-t_0)} - \frac{A_-\epsilon_1}{3} = -\frac{A_-}{3}\left(\epsilon_1 + 3\rho^3\tau^3\right) , \tag{4.31}$$

where the second term comes from considering the Hubble parameter as a slowly varying quantity.
Thus, taking into account eq. (4.25b)

$$\epsilon_{2,-} = 2\left(\frac{\ddot{\phi}_-}{H\dot{\phi}_-} + \epsilon_{1,-}\right) \approx \frac{6\rho^3\tau^3}{1-\rho^3\tau^3} + 2\epsilon_1 \frac{2-\rho^3\tau^3}{1-\rho^3\tau^3} . \tag{4.32}$$

From this expression, we know that much after the transition we get again the slow-roll approximation $\epsilon_{2,-} \approx 4\epsilon_{1,-}$, but just after it we have a transient term which is dominant (note that the second term in (4.32) is sub-dominant because proportional to ϵ_1, which is always small).
Furthermore, eq. (4.32) is more accurate than eq. (2.17) in [32], where the corrective term is simply $4\epsilon_1$.

The two HFF have the following shapes (with values in table 4.1)

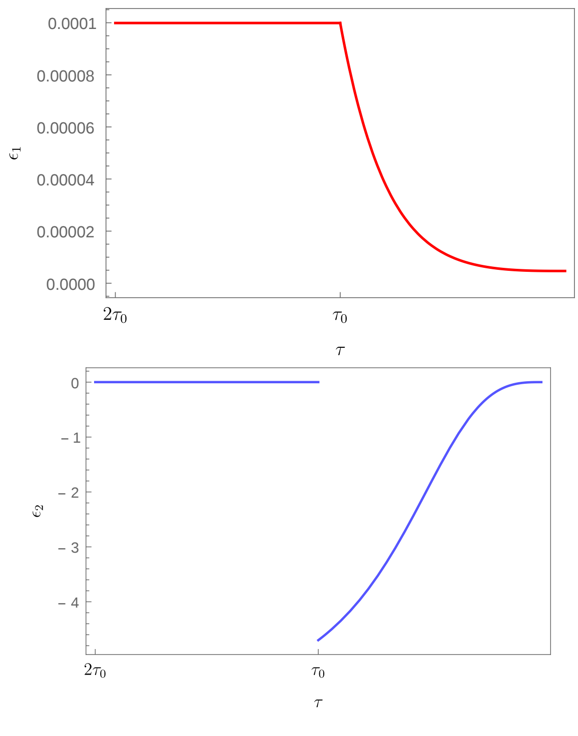

Figure 4.3: First and second HFF as a function of τ.

In addition, we can easily convince that the second term in (4.32) is sub-dominant with respect to the other by directly comparing the two

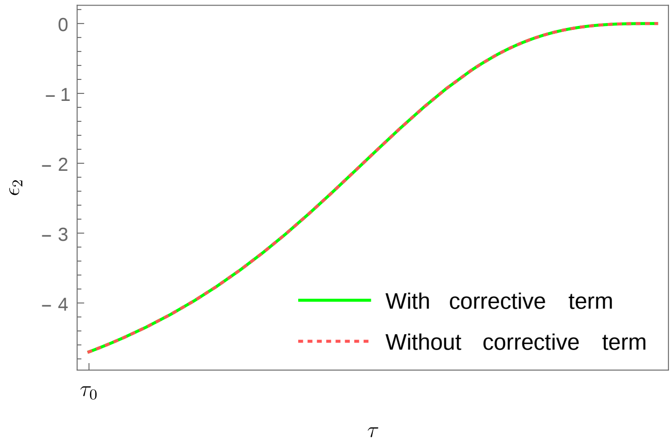

Figure 4.4: Comparison between the complete expression (4.32) and its dominant term.

As we can see, the correction due to the term proportional to ϵ_1 is negligible.

4.2.2 Power spectrum

In this section we calculate the spectrum of primordial curvature perturbation for the Starobinsky model. In particular, we follow the considerations made in section 3.1, and specialize them for this model.

First of all, since we are considering a model in which the Lagrangian has a standard kinetic term, the speed of sound is trivial, i.e. $c_s = 1$.
Before the transition, since we are in slow-roll regime, the solution to the Mukhanov-Sasaki equation (2.114) is given simply by (3.49), and it is

$$v_{\boldsymbol{k},+}(\tau) = \frac{-i}{\sqrt{2k^3}\tau}\,(1 + ik\tau)\, e^{-ik\tau} \tag{4.33}$$

or equivalently

$$\mathcal{R}_{\boldsymbol{k},+}(\tau) = \frac{iH}{2M_{Pl}\sqrt{\epsilon_1 k^3}}(1 + ik\tau)e^{-ik\tau}\,. \tag{4.34}$$

After the transition, however, the slow-roll approximation is no more valid, at least for a transitory period.
Therefore we could expect that the time-dependent frequency in the Mukhanov-Sasaki equation (2.114) is different from $2\mathcal{H}^2$.
Since the first slow-roll parameter is always small, relation (3.30) simplifies to

$$\frac{z''}{z} = a^2H^2\left[2 + \frac{3\epsilon_2}{2} + \frac{{\epsilon_2}^2}{4} + \frac{\epsilon_2\epsilon_3}{2}\right]. \tag{4.35}$$

However, being $V_{\phi\phi} = 0$, from eq. (1.77) we have

$$\frac{3\epsilon_2}{2} + \frac{{\epsilon_2}^2}{4} + \frac{\epsilon_2\epsilon_3}{2} \approx 0\,, \tag{4.36}$$

which allows us to consider even after the transition $z''/z \approx 2\mathcal{H}^2$.
In this case we already know the solution; we just have to take into account the general solution (3.43) for $\nu = \frac{3}{2}$ (since this is the case for $z''/z \approx 2\mathcal{H}^2$)

$$v_{\boldsymbol{k},-}(\tau) \approx \frac{-i}{\sqrt{2k^3}\tau}\left[\alpha_k(1+ik\tau)e^{-ik\tau} - \beta_k(1-ik\tau)e^{ik\tau}\right]. \tag{4.37}$$

Therefore, the corresponding curvature perturbation $\mathcal{R}_{\boldsymbol{k},-}$ is [33]

$$\mathcal{R}_{\boldsymbol{k},-}(\tau) \approx \frac{iH_0}{2M_{Pl}\sqrt{\epsilon_{1,-}\,k^3}}\left[\alpha_k(1+ik\tau)e^{-ik\tau} - \beta_k(1-ik\tau)e^{ik\tau}\right]. \tag{4.38}$$

It is then clear that we have to determine the two Bogoliubov coefficients α_k and β_k to achieve a complete solution of the problem. In order to do this, we impose matching conditions on $v_{\boldsymbol{k}}$ and $v'_{\boldsymbol{k}}$ at the transition.

First of all, from eqs. (4.25a) and (4.25b), together with $z = aM_{Pl}\sqrt{2\epsilon_1}$, we find

$$z_+ \approx -\frac{A_+}{3H_0^3\tau} \tag{4.39a}$$

$$z_- \approx -\frac{A_-}{3H_0^3\tau}(1-\rho^3\tau^3)\,, \tag{4.39b}$$

and then

$$z'_+ \approx \frac{A_+}{3H_0^3\tau^2} \tag{4.40a}$$

$$z'_- \approx \frac{A_-}{3H_0^3\tau^2} + \frac{2A_-\rho^3\tau}{3H_0^3}\,. \tag{4.40b}$$

At the time when the transition occurs, i.e. τ_0, we have

$$[z(\tau_0)]_\pm = 0 \tag{4.41a}$$

$$\left[z'(\tau_0)\right]_\pm = -\frac{\Delta A}{H_0^3\tau_0^2}\,, \tag{4.41b}$$

where $[X]_\pm \equiv X_+ - X_-$.
At the transition (or in a small interval around it, i.e. $\tau \approx \tau_0$), we can express z' as

$$z'(\tau\approx\tau_0) \approx z'_+(\tau\approx\tau_0) - \left[z'(\tau\approx\tau_0)\right]_\pm\,\theta(\tau-\tau_0) \approx \frac{A_+}{3H_0^3\tau_0^2} + \frac{\Delta A}{H_0^3\tau_0^2}\,\theta(\tau-\tau_0)\,. \tag{4.42}$$

Therefore, we can take one more derivative and write it in terms of a δ function, so that [2, 32]

$$\frac{z''}{z} = -\frac{3\Delta A}{A_+\tau_0}\,\delta(\tau-\tau_0)\,. \tag{4.43}$$

With this expression, we are able to obtain the two matching conditions on $v_{\boldsymbol{k}}$.

i) Combining eqs. (2.114) and (4.43) we know that $v_{\boldsymbol{k}}$ has to be continuous at the transition, i.e.

$$[v_{\boldsymbol{k}}(\tau_0)]_\pm = 0\,. \tag{4.44}$$

ii) By integrating the Mukhanov-Sasaki equation in a small interval around the transition τ_0, the last term of r.h.s. of (2.114) is negligible since we can treat $v_{\boldsymbol{k}}$ as a constant. Therefore, given $\varepsilon \ll 1$, taking into account the condition on $v_{\boldsymbol{k}}$

$$\begin{aligned}
0 &\approx \lim_{\varepsilon \to 0} \int_{\tau_0-\varepsilon}^{\tau_0+\varepsilon} d\tau \left(v''_{\boldsymbol{k}} + \frac{3\Delta A}{A_+\tau_0}\, \delta(\tau-\tau_0)\, v_{\boldsymbol{k}} \right) \approx \\
&\approx v'_{\boldsymbol{k}}(\tau_0+\varepsilon) - v'_{\boldsymbol{k}}(\tau_0-\varepsilon) + \frac{3\Delta A}{A_+\tau_0}\, v_{\boldsymbol{k}}(\tau_0)\, [\theta(\varepsilon) - \theta(-\varepsilon)] \approx \\
&\approx -[v_{\boldsymbol{k}}(\tau_0)]_\pm + \frac{3\Delta A}{A_+\tau_0}\, v_{\boldsymbol{k}}(\tau_0)\,. \qquad (4.45)
\end{aligned}$$

The second matching condition at the transition is, in this way

$$[v'_{\boldsymbol{k}}(\tau_0)]_\pm \approx \frac{3\Delta A}{A_+\tau_0}\, v_{\boldsymbol{k}}(\tau_0)\,. \qquad (4.46)$$

At this point we can match relations (4.33) and (4.37) requiring that at the transition the conditions (4.44) and (4.46) are satisfied.
Doing so, we obtain [3]

$$\alpha_k = 1 + \frac{3i\Delta A}{2A_+}\frac{k_0}{k}\left(1 + \frac{k_0^2}{k^2}\right) \qquad (4.47a)$$

$$\beta_k = -\frac{3i\Delta A}{2A_+}\frac{k_0}{k}\left(1 + \frac{ik_0}{k}\right)^2 e^{\frac{2ik}{k_0}}\,, \qquad (4.47b)$$

where we have indicated with k_0 the characteristic wave number of modes which leave the horizon at the transition. Since we are in a quasi-de Sitter universe, we have that

$$k_0 = a_0 H_0 = -\frac{1}{\tau_0}\,. \qquad (4.48)$$

The curvature power spectrum at late times (i.e. $\tau \to 0$) is simply

$$\begin{aligned}
\mathcal{P}_\mathcal{R}(\tau \to 0) &\approx \frac{k^3}{2\pi^2}\frac{H_0^2}{4M_{Pl}^2\epsilon_1 k^3}|\alpha_k - \beta_k|^2 = \\
&= \frac{9H_0^6}{4\pi^2 A_-^2}\left\{1 - \frac{3\Delta A}{A_+}\frac{k_0}{k}\sin\left(\frac{2k}{k_0}\right) + \right. \\
&\quad + \frac{3\Delta A}{2A_+^2}\left(\frac{k_0}{k}\right)^2\left[3\Delta A\left[1+\cos\left(\frac{2k}{k_0}\right)\right] - 4A_+\cos\left(\frac{2k}{k_0}\right)\right] + \\
&\quad + \frac{3\Delta A}{A_+^2}\left(\frac{k_0}{k}\right)^3 (A_+ - 3\Delta A)\sin\left(\frac{2k}{k_0}\right) + \frac{9\Delta A^2}{A_+^2}\left(\frac{k_0}{k}\right)^4 - \\
&\quad - \frac{9\Delta A^2}{A_+^2}\left(\frac{k_0}{k}\right)^5 \sin\left(\frac{2k}{k_0}\right) + \\
&\quad \left. + \frac{9\Delta A^2}{2A_+^2}\left(\frac{k_0}{k}\right)^6\left[1 - \cos\left(\frac{2k}{k_0}\right)\right]\right\}. \qquad (4.49)
\end{aligned}$$

This expression is the same as the original result, cf. eq. (10) in [2], simply written in a different way.

We can easily find two asymptotic values, in the limit of small and large scales. In the limit of $k \ll k_0$ we have

$$\mathcal{P}_{\mathcal{R}}(\tau \to 0) \xrightarrow{k \ll k_0} \frac{9H_0^6}{4\pi^2 A_+^2}\,. \tag{4.50}$$

Conversely, when $k \gg k_0$

$$\mathcal{P}_{\mathcal{R}}(\tau \to 0) \xrightarrow{k \gg k_0} \frac{9H_0^6}{4\pi^2 A_-^2}\,. \tag{4.51}$$

Overall, the spectrum of curvature perturbation is

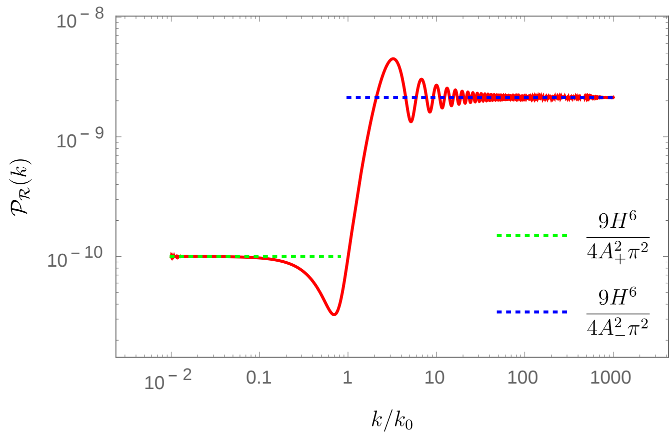

Figure 4.5: Spectrum of curvature perturbation at late times for the Starobinsky model.

The two asymptotic behaviours can be understood as follows: since on super-horizon scales the curvature perturbation is frozen, we can think that the power spectrum at late times is determined by the moment in which a certain wavelength crosses the Hubble radius. Then, large scales (which correspond to small k) cross the horizon before the transition, when the slow-roll regime is again valid, and their spectrum will be simply

$$\mathcal{P}_{\mathcal{R}} \xrightarrow{k \ll k_0} \frac{H^2}{8\pi^2 M_{Pl}^2 \epsilon_{1,+}} \approx \frac{9H^6}{4\pi^2 A_+^2} \tag{4.52}$$

On the other hand, small scales (big k) exit the horizon much after the transition, we the slow-roll regime is restored. Thus, their power spectrum will be

$$\mathcal{P}_{\mathcal{R}} \xrightarrow{k \gg k_0} \frac{H^2}{8\pi^2 M_{Pl}^2 \epsilon_{1,-}} \approx \frac{9H^6}{4\pi^2 A_-^2} \tag{4.53}$$

Between these two limits there are all the modes that cross the horizon in proximity of the transition: for such modes, the above considerations are no more valid, and one has to take into account the whole expression for $\mathcal{P}_{\mathcal{R}}$, which is a function of the wave number k.

4.2.3 Bispectrum

In the previous section we have computed the curvature power spectrum for the Starobinsky model. Now, we are ready to specialize the results of section 3.3 for this model and to calculate the bispectrum.

First of all, let's note that only the coefficients $\mathcal{C}_1$ and $\mathcal{C}_2$ of eq. (3.103) are dominant for this model, and they are simply

$$\mathcal{C}_1 \approx -\epsilon_1\epsilon_2 \tag{4.54a}$$
$$\mathcal{C}_2 \approx \epsilon_1\epsilon_2\,. \tag{4.54b}$$

As a consequence, the third order action is

$$\begin{aligned} S_3 &\approx -M_{Pl}^2 \int dt \int d\boldsymbol{x}\, \left[a^3\epsilon_1\epsilon_2\mathcal{R}\dot{\mathcal{R}}^2 - a\epsilon_1\epsilon_2\mathcal{R}(\partial\mathcal{R})^2\right] = \\ &= -M_{Pl}^2 \int dt \int d\boldsymbol{x}\, a\epsilon_1\epsilon_2 \left[\mathcal{R}\mathcal{R}'^2 - \mathcal{R}(\partial\mathcal{R})^2\right]\,. \end{aligned} \tag{4.55}$$

After an integration by parts, neglecting the boundary term, we come thanks to (3.102) to the Hamiltonian in the interaction picture [3, 32]

$$H_{int}(\tau) = M_{Pl}^2 \int d\boldsymbol{x}\, a\epsilon_1\epsilon_2 \left[\mathcal{R}\mathcal{R}'^2 + \frac{1}{2}\mathcal{R}^2\partial^2\mathcal{R}\right]\,. \tag{4.56}$$

In literature [40, 49] it is possible to find the previous third order action written as

$$S_3 \approx M_{Pl}^2 \int d\tau \int d\boldsymbol{x}\, \frac{a^2}{2}\,\epsilon_1\epsilon_2'\mathcal{R}^2\mathcal{R}'\,. \tag{4.57}$$

We can easily show that the formulations (4.55) and (4.57) are equivalent: by integrating by parts and neglecting the boundary term, we obtain

$$\begin{aligned} S_3 &\approx -M_{Pl}^2 \int d\tau \int d\boldsymbol{x}\, \frac{a^2}{2}\epsilon_1\epsilon_2 \left[2\frac{a'}{a}\mathcal{R}^2\mathcal{R}' + \frac{\epsilon_1'}{\epsilon_1}\mathcal{R}^2\mathcal{R}' + 2\mathcal{R}\mathcal{R}'^2 + \mathcal{R}^2\mathcal{R}''\right] = \\ &= -M_{Pl}^2 \int d\tau \int d\boldsymbol{x}\, \frac{a^2}{2}\epsilon_1\epsilon_2 \left\{2\mathcal{R}\mathcal{R}'^2 + \mathcal{R}^2\left[\mathcal{R}'' + 2\mathcal{R}'\left(\frac{a'}{a} + \frac{\epsilon_1'}{2\epsilon_1}\right)\right]\right\} = \\ &= -M_{Pl}^2 \int d\tau \int d\boldsymbol{x}\, \frac{a^2}{2}\epsilon_1\epsilon_2 \left\{2\mathcal{R}\mathcal{R}'^2 + \mathcal{R}^2\left[\mathcal{R}'' + 2\frac{z'}{z}\mathcal{R}'\right]\right\} = \\ &= -M_{Pl}^2 \int dt \int d\boldsymbol{x}\, a\epsilon_1\epsilon_2 \left[\mathcal{R}\mathcal{R}'^2 + \frac{1}{2}\mathcal{R}^2\partial^2\mathcal{R}\right]\,, \end{aligned} \tag{4.58}$$

where we have used the Mukhanov-Sasaki (2.115) in real space, and switched to cosmic time.

Once we have the Hamiltonian in the interaction picture, we can calculate the three-points correlation function via eq. (3.101) as

$$\langle\hat{\mathcal{R}}_{\boldsymbol{k}_1}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_2}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_3}(0)\rangle \approx -i\int_{-\infty}^{0} d\tau'\, a\,\langle\left[\hat{\mathcal{R}}_{\boldsymbol{k}_1}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_2}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_3}(0), \hat{H}_{int}(\tau')\right]\rangle\,. \tag{4.59}$$

Taking a time derivative of (4.38) we have

$$\begin{aligned}
\mathcal{R}'_{\boldsymbol{k},-}(\tau) \approx & \frac{iH_0}{2M_{Pl}\sqrt{\epsilon_{1,-}\,k^3}}\left\{\alpha_k\left[\frac{\epsilon'_{1,-}}{2\epsilon_{1,-}}(1+ik\tau)+k^2\tau\right]e^{-ik\tau}-\right.\\
& \left.-\,\beta_k\left[\frac{\epsilon'_{1,-}}{2\epsilon_{1,-}}(1+ik\tau)+k^2\tau\right]e^{ik\tau}\right\}=\\
= & \frac{iH_0}{2M_{Pl}\sqrt{\epsilon_{1,-}\,k^3}}\left\{\alpha_k\left[k^2\tau-\frac{\epsilon_{2,-}}{2\tau}(1+ik\tau)\right]e^{-ik\tau}-\right.\\
& \left.-\,\beta_k\left[k^2\tau-\frac{\epsilon_{2,-}}{2\tau}(1+ik\tau)\right]e^{ik\tau}\right\}\,. \qquad (4.60)
\end{aligned}$$

From the definition (3.21a) of $\hat{\mathcal{R}}_{\boldsymbol{k}}$ and the commutation rules (3.21b) for the creation/annihilation operators, we find [3]

$$\begin{aligned}
&\int d\boldsymbol{x}\left[\hat{\mathcal{R}}_{\boldsymbol{k}_1}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_2}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_3}(0),\hat{\mathcal{R}}(\tau,\boldsymbol{x})\hat{\mathcal{R}}'^{\,2}(\tau,\boldsymbol{x})\right]=\\
&\quad=(2\pi)^3\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)2\left[\mathcal{R}_{\boldsymbol{k}_1}(0)\mathcal{R}_{\boldsymbol{k}_2}(0)\mathcal{R}_{\boldsymbol{k}_3}(0)\mathcal{R}^*_{\boldsymbol{k}_1}(\tau)\mathcal{R}'^*_{\boldsymbol{k}_2}(\tau)\mathcal{R}'^*_{\boldsymbol{k}_3}(\tau)-\right.\\
&\qquad\left.-\,\mathcal{R}^*_{-\boldsymbol{k}_1}(0)\mathcal{R}^*_{-\boldsymbol{k}_2}(0)\mathcal{R}^*_{-\boldsymbol{k}_3}(0)\mathcal{R}_{-\boldsymbol{k}_1}(\tau)\mathcal{R}'_{-\boldsymbol{k}_2}(\tau)\mathcal{R}'_{-\boldsymbol{k}_3}(\tau)+1\leftrightarrow 2+1\leftrightarrow 3\right]\\
&\quad=(2\pi)^3\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)4i\Im\left[\mathcal{R}_{\boldsymbol{k}_1}(0)\mathcal{R}_{\boldsymbol{k}_2}(0)\mathcal{R}_{\boldsymbol{k}_3}(0)\mathcal{R}^*_{\boldsymbol{k}_1}(\tau)\mathcal{R}'^*_{\boldsymbol{k}_2}(\tau)\mathcal{R}'^*_{\boldsymbol{k}_3}(\tau)\right]+\\
&\qquad+1\leftrightarrow 2+1\leftrightarrow 3\,, \qquad (4.61)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\int d\boldsymbol{x}\left[\hat{\mathcal{R}}_{\boldsymbol{k}_1}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_2}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_3}(0),\hat{\mathcal{R}}^2(\tau,\boldsymbol{x})\partial^2\hat{\mathcal{R}}(\tau,\boldsymbol{x})\right]=\\
&\quad=-(2\pi)^3\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\left[k_1^2\left(\mathcal{R}_{\boldsymbol{k}_1}(0)\mathcal{R}_{\boldsymbol{k}_2}(0)\mathcal{R}_{\boldsymbol{k}_3}(0)\mathcal{R}^*_{\boldsymbol{k}_1}(\tau)\mathcal{R}^*_{\boldsymbol{k}_2}(\tau)\mathcal{R}^*_{\boldsymbol{k}_3}(\tau)\,-\right.\right.\\
&\qquad\left.\left.-\,\mathcal{R}^*_{-\boldsymbol{k}_1}(0)\mathcal{R}^*_{-\boldsymbol{k}_2}(0)\mathcal{R}^*_{-\boldsymbol{k}_3}(0)\mathcal{R}_{-\boldsymbol{k}_1}(\tau)\mathcal{R}_{-\boldsymbol{k}_2}(\tau)\mathcal{R}_{-\boldsymbol{k}_3}(\tau)\right)+1\leftrightarrow 2+1\leftrightarrow 3\right]=\\
&\quad=(2\pi)^3\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)2i\Im\left[k_1^2\mathcal{R}_{\boldsymbol{k}_1}(0)\mathcal{R}_{\boldsymbol{k}_2}(0)\mathcal{R}_{\boldsymbol{k}_3}(0)\mathcal{R}^*_{\boldsymbol{k}_1}(\tau)\mathcal{R}^*_{\boldsymbol{k}_2}(\tau)\mathcal{R}^*_{\boldsymbol{k}_3}(\tau)\right]+\\
&\qquad+1\leftrightarrow 2+1\leftrightarrow 3\,, \qquad (4.62)
\end{aligned}$$

where we have taken into account that $\mathcal{R}_{\boldsymbol{k}}$ depends on the wave number just through its amplitude, i.e. $\mathcal{R}_{\boldsymbol{k}}=\mathcal{R}_{-\boldsymbol{k}}$.
Putting these two results together, we find [3]

$$\begin{aligned}
\langle\hat{\mathcal{R}}_{\boldsymbol{k}_1}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_2}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_3}(0)\rangle \approx\ & (2\pi)^3\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\,2M_{Pl}^2\Im\Bigg[\mathcal{R}_{\boldsymbol{k}_1}(0)\mathcal{R}_{\boldsymbol{k}_2}(0)\mathcal{R}_{\boldsymbol{k}_3}(0)\cdot\\
&\cdot\int_{-\infty}^{0}d\tau\,a^2\epsilon_1\epsilon_2\mathcal{R}^*_{\boldsymbol{k}_1}(\tau)\Big(2\mathcal{R}'^*_{\boldsymbol{k}_2}(\tau)\mathcal{R}'^*_{\boldsymbol{k}_3}(\tau)-k_1^2\mathcal{R}^*_{\boldsymbol{k}_2}(\tau)\mathcal{R}^*_{\boldsymbol{k}_3}(\tau)\Big)\Bigg]+\\
&+1\leftrightarrow 2+1\leftrightarrow 3\,. \qquad (4.63)
\end{aligned}$$

The above integral can be split in two contributions, before and after the transition, respectively. Here, we focus on the computation after the transition, since it is the more important one for the comparison with observations.
In addition, we consider the equilateral limit (defined as that configuration in which

$k_1 = k_2 = k_3 \equiv k$) for simplicity.
In other words, we want to calculate the quantity

$$\mathcal{B}_{\mathcal{R}}(k) \approx 6M_{Pl}^2 \Im\left[\mathcal{R}_{\boldsymbol{k}}^3(0)\int_{\tau_0}^0 d\tau\, a^2\epsilon_1\epsilon_2\mathcal{R}_{\boldsymbol{k}}^*(\tau)\Big(2\mathcal{R}_{\boldsymbol{k}}^{\prime *2}(\tau) - k^2\mathcal{R}_{\boldsymbol{k}}^{*\,2}(\tau)\Big)\right]. \tag{4.64}$$

This calculation, even if a bit elaborated, can be done analytically with the assumptions we made. The result is

$$\begin{aligned}
\mathcal{B}_{\mathcal{R}}(k) \approx &-\frac{729\pi^3\Delta A H_0^{12}k_0^8}{2A_-^3A_+^4k^{15}}\\
&\Bigg\{9\Delta A\left(\frac{k^2}{k_0^2}+1\right)^2 k\cos\left(\frac{k}{k_0}\right)\left[9\Delta A + (A_- - 3A_+)\left(\frac{k}{k_0}\right)^2 + 2A_+\left(\frac{k}{k_0}\right)^4\right] -\\
&- 9k_0\left(\frac{k^2}{k_0^2}+1\right)^2\sin\left(\frac{k}{k_0}\right)\left[9\Delta A^2 - \left(A_-^2 - 4A_+A_- + 3A_+^2\right)\left(\frac{k}{k_0}\right)^4\right] -\\
&- k\cos\left(\frac{3k}{k_0}\right)\Bigg[81\Delta A^2 - 9\Delta A\,(9A_+ - 7A_-)\left(\frac{k}{k_0}\right)^2 - 9\Delta A\,(5A_- - 11A_+)\left(\frac{k}{k_0}\right)^4\\
&\quad - 3\left(9A_-^2 - 32A_+A_- + 27A_+^2\right)\left(\frac{k}{k_0}\right)^6 - 2A_+\,(7A_- - 13A_+)\left(\frac{k}{k_0}\right)^8\Bigg] +\\
&+ k_0\sin\left(\frac{3k}{k_0}\right)\Bigg[27\Delta A^2 - 54\Delta A^2\left(\frac{k}{k_0}\right)^2 - 54\Delta A\,(2A_- - 3A_+)\left(\frac{k}{k_0}\right)^4\\
&\quad - 2\left(9A_-^2 - 16A_+A_- + 9A_+^2\right)\left(\frac{k}{k_0}\right)^6 - \left(-9A_-^2 + 60A_+A_- - 67A_+^2\right)\left(\frac{k}{k_0}\right)^8 -\\
&\quad - 4A_+^2\left(\frac{k}{k_0}\right)^{10}\Bigg]\Bigg\}.
\end{aligned} \tag{4.65}$$

This expression is an original result of this work, but it is easy to check that its asymptotic behaviours coincide with the results obtained in [3]. In particular, in the limit of large scales ($k \ll k_0$) we have

$$\begin{aligned}
\mathcal{B}_{\mathcal{R}}(k) \xrightarrow{k\ll k_0} \approx &-\frac{3^7H_0^{12}\Delta A}{20A_+^4A_-}\frac{1}{k^6}\Bigg[\left(\frac{k}{k_0}\right)^2 +\\
&+ \frac{10228680A_-^2 - 13970880A_-A_+}{17463600A_-^2}\left(\frac{k}{k_0}\right)^4 +\\
&+ \frac{2794176A_+^2 - 5983054A_-^2 - 5231072A_-A_+}{17463600A_-^2}\left(\frac{k}{k_0}\right)^6 +\\
&+ \frac{1127045A_-^2 + 2508440A_+A_- - 190960A_+^2}{17463600A_-^2}\left(\frac{k}{k_0}\right)^8\Bigg].
\end{aligned} \tag{4.66}$$

On the other hand, in the limit of small scales ($k \gg k_0$) we find

$$\mathcal{B}_{\mathcal{R}}(k) \xrightarrow{k \gg k_0} \approx \frac{729 H_0^{12} \Delta A}{4 A_-^3 A_+^3} \frac{1}{k^6} \left[\frac{k}{k_0} \sin\left(\frac{3k}{k_0}\right) - \right.$$
$$\left. - \frac{9 \Delta A}{2 A_+} \cos\left(\frac{k}{k_0}\right) + \frac{13 A_+ - 7 A_-}{2 A_+} \cos\left(\frac{3k}{k_0}\right) \right] . \tag{4.67}$$

These two expressions coincide with eqs. (3.16)-(3.17) in [3].

Now that we have the explicit expressions for the three-point function and the power spectrum, we can calculate and plot the function $\mathcal{G}(k_1, k_2, k_3)$, which is defined in eq. (3.93).
In particular, if we are interested in the limit of large and small scales, we just have to take into account relations (4.50) and (4.51), together with (4.66) and (4.67).
Denoting with $_<$ the quantities for which $k \ll k_0$ we have

$$\frac{\mathcal{G}_<(k)}{k^3} \approx \frac{\mathcal{B}_{\mathcal{R}}(k)}{(2\pi)^4} \frac{k^6}{\mathcal{P}^2_{\mathcal{R},<}(k)} , \tag{4.68}$$

which is

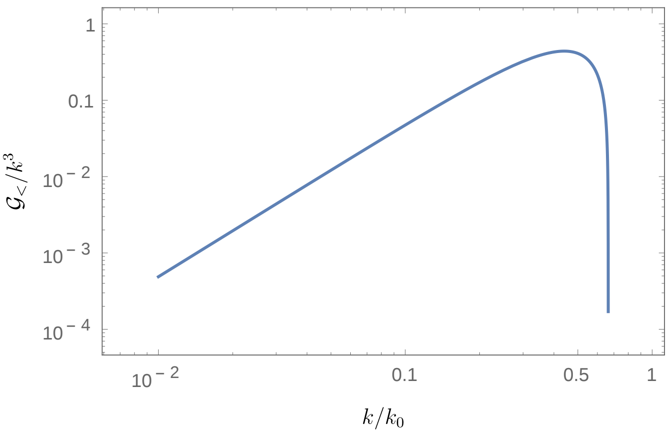

Figure 4.6: Function $\frac{\mathcal{G}(k)}{k^3}$ for large scales.

Conversely, if we indicate with $_>$ the quantities for which $k \gg k_0$ we have

$$\frac{\mathcal{G}_>(k)}{k^3} \approx \frac{\mathcal{B}_{\mathcal{R}}(k)}{(2\pi)^4} \frac{k^6}{\mathcal{P}^2_{\mathcal{R},>}(k)} , \tag{4.69}$$

which is

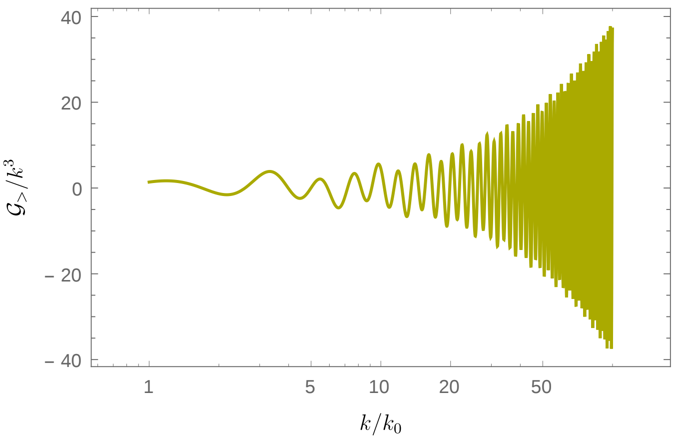

Figure 4.7: Function $\frac{\mathcal{G}(k)}{k^3}$ for small scales.

However, we can also consider the whole bispectrum (4.65), together with the formula for the power spectrum (4.49) which is valid on all scales: this time we have

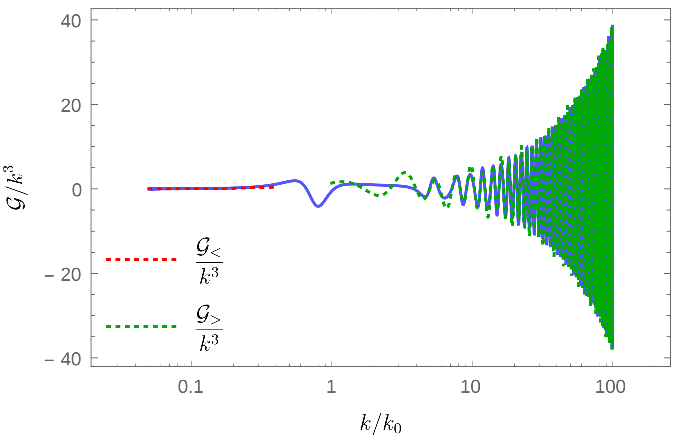

Figure 4.8: General behaviour of the function $\frac{\mathcal{G}(k)}{k^3}$.

As we can see, the bispectrum for this model is characterized by a growth for large k which never ends. This is due to the fact that we are an instantaneous transition in the potential (it is parametrized by a step function in eq. (4.6)). If one assumes that such a transition takes place in a finite lapse of time, for instance smoothing the step function with a hyperbolic tangent, the corresponding bispectrum is characterized by a growth followed by a suppression at large k, as discussed in [42, 50].

Moreover, if we want to calculate the parameter f_{NL}, from (3.98), we see that in the equilateral limit it is trivially

$$f_{NL} \approx \frac{10}{9}\frac{\mathcal{G}(k)}{k^3}\,. \tag{4.70}$$

Eventually, we can also evaluate the function $\mathcal{B}_{\mathcal{R}}(k)$ trough eq. (3.94)

$$\mathcal{B}_{\mathcal{R}}(k) \approx \mathcal{P}^2_{\mathcal{R}}(k)\,(2\pi)^4\,\frac{\mathcal{G}(k)}{k^9}\,. \tag{4.71}$$

Chapter 5

The effect of a non trivial speed of sound

In this chapter we provide an extension of the model originally studied by Starobinsky. In particular, we generalize the kinetic term in the Lagrangian.
We present two different non-standard models, characterized by different powers in higher orders of the kinetic term χ.
Under the same approximations, we discuss the contributions to the curvature spectrum and bispectrum for a model which is characterized at the transition by a jump in both ϵ_2 and s, but with continuous ϵ_1 and c_s. It is important to calculate the bispectrum for this model because, as it was showed in [33], for this model the power spectrum is degenerate with respect to a jump in ϵ_2 or s (i.e., it is possible to obtain the same power spectrum considering the a jump in s rather than in ϵ_2).

5.1 Overview on the models

In this section we present two different generalizations to the model with a discontinuity in the first derivative of the potential which are characterized by a non-trivial speed of sound. In addition, we assume a cosmological constant dominating the potential term in the Lagrangian, so that we can solve exactly the background dynamics after the transition.

5.1.1 First model

We consider the following Lagrangian

$$P(\chi, \phi) = \chi + \frac{\chi^2}{\Lambda^4} - V_0 - [A_- + (A_+ - A_-)\,\theta(\phi - \phi_0)]\,(\phi - \phi_0)\,, \qquad (5.1)$$

where Λ is a reference scale with the dimension of a mass and again, the potential is dominated by the constant term V_0.
It is easy to see that in the limit $\chi \ll \Lambda^4$, we recover a standard kinetic term, and the model falls into the case studied in section 4.2

Evolution of the background

Proceeding like in section 4.2.1, we first study the background for this model. The expressions of the first two HFF ϵ_1 and ϵ_2 are given in eqs. (1.95) and (1.97), while those for c_s^2 and s in (1.90) and (1.93). Then, we just have to specialize P in eq. (5.1) and its derivatives to this model:

$$P_\chi = 1 + \frac{\dot\phi^2}{\Lambda^4} \quad , \quad P_{\chi\chi} = \frac{2}{\Lambda^4} \quad , \quad \chi_N = \frac{\ddot\phi\dot\phi}{H} \quad , \quad P_{\chi N} = \frac{2}{H\Lambda^4} \,. \tag{5.2}$$

With these expressions, we determine the dependence of the HFF, c_s^2 and s on $\dot\phi$. The background quantities are

$$\epsilon_1 = \frac{\dot\phi^2}{2H^2M_{Pl}^2}\left(1+\frac{\dot\phi^2}{\Lambda^4}\right) \tag{5.3a}$$

$$\epsilon_2 = 2\left(\frac{\ddot\phi}{H\dot\phi}+\epsilon_1\right)+\frac{2\ddot\phi\dot\phi}{H(\Lambda^4+\dot\phi^2)} \tag{5.3b}$$

$$c_s^2 = \frac{\Lambda^4+\dot\phi^2}{\Lambda^4+3\dot\phi^2} \tag{5.3c}$$

$$s = \frac{\ddot\phi\dot\phi}{H}\left[\frac{1}{\Lambda^4+\dot\phi^2}-\frac{3}{\Lambda^4+3\dot\phi^2}\right] \,. \tag{5.3d}$$

The key point is that the introduction of a non-standard kinetic term in the Lagrangian makes possible to have a speed of sound which is different from 1. In particular, for the Lagrangian (5.1), c_s^2 varies between 1/3 and 1.

Following the same procedure as in 4.2.1, we study the Klein-Gordon equation (1.99) before and after the transition, separately.

- **Before the transition**.
  Before the transition we assume that the slow-roll approximation holds. Since

$$\frac{d}{dt}\left(P_\chi\,\dot\phi_+\right) \propto \ddot\phi_+ + \dot\phi_+^2 P_{\chi\chi}\ddot\phi_+ \propto \ddot\phi_+ \,, \tag{5.4}$$

  we can neglect the first term in the l.h.s. of the Klein-Gordon equation. Therefore, before the transition, the latter reduces to

$$3H_0 P_\chi\,\dot\phi_+ \approx P_\phi \,. \tag{5.5}$$

  Also in this case, we approximate the Hubble parameter with H_0, which means that the energy density is vacuum-dominated.
  Thus, the above equation can be written as

$$\frac{\dot\phi_+^3}{\Lambda^4}+\dot\phi_+ + \frac{A_+}{3H_0} \approx 0 \,. \tag{5.6}$$

This equation has three solutions, but only one of these is real: in particular, it is

$$\dot{\phi}_{+} \approx \frac{-2\sqrt[3]{3}H_0^2\Lambda^4 + \sqrt[3]{2}\Big[H_0^2\Lambda^4\left(\sqrt{9A_+^2+12H_0^2\Lambda^4}-3A_+\right)\Big]^{\frac{2}{3}}}{6^{2/3}H_0^{5/3}\Lambda^{4/3}\left(\sqrt{9A_+^2+12H_0^2\Lambda^4}-3A_+\right)^{\frac{1}{3}}} \,. \tag{5.7}$$

From this expression, we see that there is a critical value of Λ (be $\Lambda_{\chi^2,+}$) which distinguishes two different regimes. In particular, this value is

$$\Lambda^2_{\chi^2,+} = \frac{\sqrt{3}A_+}{2H_0} \,. \tag{5.8}$$

If the condition $\Lambda \ll \Lambda_{\chi^2,+}$ is satisfied, we can consider a Taylor expansion of this equation, and get an expression for $\dot{\phi}_+$ in the opposite limit with respect to the Starobinsky model: in this case, we have

$$\dot{\phi}_{+} \xrightarrow{\Lambda \ll \Lambda_{\chi^2,+}} -\left(\frac{A_+\Lambda^4}{3H_0}\right)^{\frac{1}{3}} \,. \tag{5.9}$$

In the opposite limit, i.e. $\Lambda \gg \Lambda_{\chi^2,+}$, we obviously recover the Starobinsky model, and therefore eq. (4.23b).
For the values of the parameters given in table 4.1, we have that

$$\Lambda_{\chi^2,+} \approx 2 \times 10^{-4} M_{Pl} \,. \tag{5.10}$$

- **After the transition**.
After the transition, we have to consider the complete Klein-Gordon equation

$$\frac{d}{dt}\left(P_\chi\,\dot{\phi}\right) + 3H_0 P_\chi\,\dot{\phi} - P_\phi \approx 0 \,. \tag{5.11}$$

Taking into account the expressions in (5.2) this is

$$\frac{\dot{\phi}_-^3}{\Lambda^4} + \dot{\phi}_- + \frac{A_-}{3H_0} - Be^{-3H(t-t_0)} \approx 0 \,. \tag{5.12}$$

In this case too only one solution is real, and it reads

$$\begin{aligned}\dot{\phi}_- \approx \frac{e^{-H_0(t-t_0)}\Lambda^{4/3}}{9\sqrt[3]{2}H_0^{1/3}}&\Bigg[81\sqrt{3}\sqrt{\Big[3\big(A_-e^{3H_0(t-t_0)}-3BH_0\big)^2+4H_0^2\Lambda^4e^{6H_0(t-t_0)}\Big]}-\\ &-243A_-e^{3H_0(t-t_0)}+729BH_0\Bigg]^{\frac{1}{3}}-\\ &-3\sqrt[3]{2}H_0^{1/3}\Lambda^{8/3}e^{H_0(t-t_0)}\times\\ &\times\Bigg[81\sqrt{3}\sqrt{\Big[3\big(A_-e^{3H_0(t-t_0)}-3BH_0\big)^2+4H_0^2\Lambda^4e^{6H_0(t-t_0)}\Big]}-\\ &-243A_-e^{3H_0(t-t_0)}+729BH_0\Bigg]^{-\frac{1}{3}}\end{aligned} \tag{5.13}$$

If we impose the condition that $\dot{\phi}$ is continuous at the transition, we find that B has to be

$$B = \frac{1}{3H_0}(A_- - A_+)\,, \tag{5.14}$$

so that

$$\begin{aligned}\dot{\phi}_- \approx \frac{e^{-H_0(t-t_0)}\Lambda^{4/3}}{9\sqrt[3]{2}H_0^{1/3}}\Bigg[&81\sqrt{3}\sqrt{\left[3\big(A_- e^{3H_0(t-t_0)} - \Delta A\big)^2 + 4H_0^2\Lambda^4 e^{6H_0(t-t_0)}\right]} - \\ &- 243\left(A_- e^{3H_0(t-t_0)} - \Delta A\right)\Bigg]^{\frac{1}{3}} - \\ - 3\sqrt[3]{2}H_0^{1/3}\Lambda^{8/3}e^{H_0(t-t_0)}\times& \\ \times\Bigg[81\sqrt{3}&\sqrt{\left[3\big(A_- e^{3H_0(t-t_0)} - \Delta A\big)^2 + 4H_0^2\Lambda^4 e^{6H_0(t-t_0)}\right]} - \\ &- 243\left(A_- e^{3H_0(t-t_0)} - \Delta A\right)\Bigg]^{-\frac{1}{3}}\,.\end{aligned} \tag{5.15}$$

In this case too, we can determine a critical value of the scale Λ which distinguish two regimes. In this case, this value is evidently

$$\Lambda^2_{\chi^2,-} = \frac{\sqrt{3}\left(A_- - \Delta A e^{-3H_0(t-t_0)}\right)}{2H_0} < \Lambda^2_{\chi^2,+}\,. \tag{5.16}$$

In the limit $\Lambda \ll \Lambda_{\chi^2,-}$, after the transition, the expression for $\dot{\phi}_-$ simplifies to

$$\dot{\phi}_- \xrightarrow{\Lambda\ll\Lambda_{\chi^2,-}} -\left[\frac{\Lambda^4\left(A_- - \Delta A e^{-3H_0(t-t_0)}\right)}{3H_0}\right]^{\frac{1}{3}}\,. \tag{5.17}$$

Again, in the other limit $\Lambda \gg \Lambda_{\chi^2,-}$ (after the transition), we recover eq. (4.23d).

Now that we have determined the explicit dependence on time for $\dot{\phi}$, we can investigate the values of Λ for which the non-standard kinetic term in (5.1) is dominant. In order to do so, we consider the ratio

$$\frac{\dfrac{\chi^2}{\Lambda^4}}{\chi^2 + \dfrac{\chi^2}{\Lambda^4}} = \frac{\dot{\phi}^2}{\dot{\phi}^2 + 2\Lambda^4}\,. \tag{5.18}$$

With the numerical values of table 4.1, we have

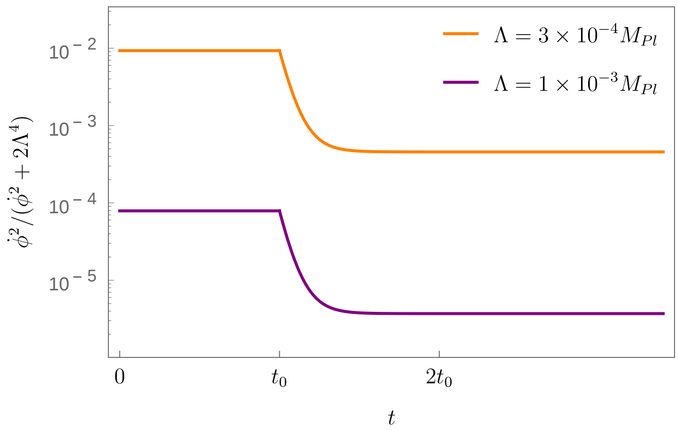

Figure 5.1: Ratio $\dot\phi^2/(\dot\phi^2+2\Lambda^4)$ as a function of time. The characteristic scale for this system is $V_0{}^{\frac{1}{4}} = 1.24\times 10^{-3} M_{Pl}$.

From this figure we note that for $\Lambda \gtrsim 3\times 10^{-4} M_{Pl}$ the non-standard kinetic term in Lagrangian (5.1) is small. In addition, we see that this value is consistent with the critical scale Λ_{χ^2} given in (5.8), which implies that when $\Lambda \gg \Lambda_{\chi^2}$ we can treat the non-standard kinetic term as a small perturbation.

5.1.2 Second model

A second generalization can be realized with a different kinetic term, i.e.

$$P(\chi,\phi) = \chi + \frac{\chi^{\frac{3}{2}}}{\Lambda^2} - V_0 - \left[A_- + (A_+ - A_-)\,\theta(\phi-\phi_0)\right](\phi-\phi_0)\,, \tag{5.19}$$

Evolution of the background

Proceeding like in section 4.2.1, we first study the background for this model. The expressions of the first two HFF ϵ_1 and ϵ_2 are given in eqs. (1.95) and (1.97), while those for c_s^2 and s in (1.90) and (1.93). Then, we just have to specialize P and its derivatives to this model. Here, we just list their values.

$$\begin{array}{lll} P_\chi = 1 + \dfrac{3\sqrt{2}}{4\Lambda^2}\,|\dot\phi| & P_{\chi\chi} = \dfrac{3\sqrt{2}}{4\Lambda^2}\,\dfrac{1}{|\dot\phi|} & \chi_N = \dfrac{\ddot\phi\dot\phi}{H} \\ P_{\chi N} = \dfrac{3\sqrt{2}}{4H\Lambda^2}\,\ddot\phi\,\dfrac{\dot\phi}{|\dot\phi|} & P_{\chi\chi N} = -\dfrac{3\sqrt{2}}{4H\Lambda^2}\,\dfrac{\ddot\phi}{\dot\phi^2}\,\dfrac{\dot\phi}{|\dot\phi|} & \end{array}, \tag{5.20}$$

where we have taken into account that $\chi^{\frac{1}{2}} = |\dot\phi|/2$ if we do not know the sign of $\dot\phi$. With these expressions, it is quite simple to determine the dependence of the HFF,

c_s^2 and s on $\dot{\phi}$. They are

$$\epsilon_1 = \frac{\dot{\phi}^2}{2H^2 M_{Pl}^2}\left(1+\frac{3\sqrt{2}|\dot{\phi}|}{4\Lambda^2}\right) \tag{5.21a}$$

$$\epsilon_2 = 2\left(\frac{\ddot{\phi}}{H\dot{\phi}}+\epsilon_1\right)+\frac{\frac{3\sqrt{2}\ddot{\phi}}{4H\Lambda^2}}{1+\frac{3\sqrt{2}|\dot{\phi}|}{4\Lambda^2}}\frac{\dot{\phi}}{|\dot{\phi}|} \tag{5.21b}$$

$$c_s^2 = \frac{1+\frac{3\sqrt{2}|\dot{\phi}|}{4\Lambda^2}}{1+\frac{3\sqrt{2}|\dot{\phi}|}{2\Lambda^2}} \tag{5.21c}$$

$$s = -\frac{\frac{3\sqrt{2}\ddot{\phi}}{8H\Lambda^2}}{\left(1+\frac{3\sqrt{2}|\dot{\phi}|}{4\Lambda^2}\right)\left(1+\frac{3\sqrt{2}|\dot{\phi}|}{2\Lambda^2}\right)}\frac{\dot{\phi}}{|\dot{\phi}|}\,. \tag{5.21d}$$

While in the previous model the speed of sound varies from 1/3 to 1, in this case the minimum value it can take is 1/2.

Following the same procedure as in 4.2.1, we study the Klein-Gordon equation (1.99) before and after the transition, separately.

- **Before the transition**.
  Before the transition the slow-roll approximation is always valid. Since

$$\frac{d}{dt}\left(P_\chi\,\dot{\phi}_+\right) \propto \ddot{\phi}_+ + \dot{\phi}_+^2 P_{\chi\chi}\ddot{\phi}_+ \propto \ddot{\phi}_+\,, \tag{5.22}$$

  we can neglect the first term in the l.h.s. of the Klein-Gordon equation. Therefore, before the transition, the latter reduces to

$$3H_0 P_\chi\,\dot{\phi}_+ \approx P_\phi\,. \tag{5.23}$$

  Also in this case, we approximate the Hubble parameter with H_0, which means that the energy density is vacuum-dominated.
  Thus, the above equation can be written as

$$\frac{3\sqrt{2}}{4\Lambda^2}\dot{\phi}_+\,|\dot{\phi}_+| + \dot{\phi}_+ + \frac{A_+}{3H_0} \approx 0\,. \tag{5.24}$$

  For $\chi \ll \Lambda^4$ we recover eq. (4.12). Then, since $A_+ > 0$, we have that $\dot{\phi}_+$ is always negative, so that we have solved the ambiguity upon its sign. Thus, eq. (5.24) becomes a quadratic equation for $\dot{\phi}$, which once solved gives

$$\dot{\phi}_+ = \frac{\sqrt{2}\Lambda^2}{3}\left(1 \pm \sqrt{1+\frac{A_+\sqrt{2}}{H_0\Lambda^2}}\right)\,. \tag{5.25}$$

Again, to establish the sign in the bracket, we consider the asymptotic behaviour

$$\dot\phi_+\Big|_{\chi\ll\Lambda^4} \approx -\frac{A_+}{3H_0}\,. \tag{5.26}$$

Expanding eq. (5.25) we have

$$\dot\phi_+ \approx \frac{\sqrt{2}\Lambda^2}{3}(1\pm 1) \pm \frac{A_+}{3H_0}\,. \tag{5.27}$$

This result is consistent with the previous equation only if we take the $-$ sign. So doing, we find

$$\dot\phi_+ = \frac{\sqrt{2}\Lambda^2}{3}\left(1-\sqrt{1+\frac{A_+\sqrt{2}}{H_0\Lambda^2}}\right) \equiv \frac{\sqrt{2}\Lambda^2}{3}(1-r_+)\,. \tag{5.28}$$

For this model, the critical value of Λ (denoted by $\Lambda_{\chi^{3/2}}$) is given by

$$\Lambda^2_{\chi^{3/2},+} = \frac{\sqrt{2}A_+}{H_0}\,. \tag{5.29}$$

With the parameters of table 4.1, its numerical value is

$$\Lambda_{\chi^{3/2},+} \approx 2.3\times 10^{-4} M_{Pl}\,. \tag{5.30}$$

- **After the transition**.
After the transition, we have to consider the complete Klein-Gordon equation

$$\frac{d}{dt}\left(P_\chi\,\dot\phi\right) + 3H_0 P_\chi\,\dot\phi - P_\phi \approx 0\,. \tag{5.31}$$

Taking into account the expression in (5.20), its solution is

$$\frac{3\sqrt{2}}{4\Lambda^2}\dot\phi_-\,|\dot\phi_-| + \dot\phi_- - \alpha \approx 0\,, \tag{5.32}$$

where

$$\alpha \equiv Be^{-3H_0(t-t_0)} - \frac{A_-}{3H_0}\,, \tag{5.33}$$

and B is a constant of integration.
To determine B, we can consider the limit of big Λ and impose the continuity with $\dot\phi_+$ at the transition: this gives

$$B = \frac{1}{3H}(A_- - A_+)\,. \tag{5.34}$$

Since for our parametrization $A_+ > A_-$, even $\dot\phi_-$ is always negative. Solving eq. (5.32) gives

$$\dot\phi_- \approx \frac{\sqrt{2}\Lambda^2}{3}\left[1\pm\sqrt{1+\frac{\sqrt{2}}{H\Lambda^2}\left[A_- - (A_- - A_+)e^{-3H_0(t-t_0)}\right]}\right] \tag{5.35}$$

Once more, we consider the asymptotic behaviour $\chi \ll \Lambda^4$ at late times; in this configuration we know from eq. (4.23d) that

$$\dot{\phi}_-\Big|_{\chi\ll\Lambda^4, t\gg t_0} \approx -\frac{A_-}{3H_0}\,, \tag{5.36}$$

from which we know that in (5.35) we have to take the $-$ sign, in such a way that

$$\dot{\phi}_- \approx \frac{\sqrt{2}\Lambda^2}{3}\left[1-\sqrt{1+\frac{\sqrt{2}}{H\Lambda^2}\left[A_- - \Delta A e^{-3H_0(t-t_0)}\right]}\right] \equiv \frac{\sqrt{2}\Lambda^2}{3}(1-r_-)\,. \tag{5.37}$$

After the transition, the critical value of Λ is

$$\Lambda^2_{\chi^{3/2},-} = \frac{\sqrt{2}}{H\Lambda^2}\left[A_- - \Delta A e^{-3H_0(t-t_0)}\right] < \Lambda^2_{\chi^{3/2},+}\,. \tag{5.38}$$

Like for the first model, we can investigate the values of Λ for which the non-standard kinetic term in (5.19) is dominant. In this case, the ratio to be considered is

$$\frac{\chi\left(\frac{\chi}{\Lambda^4}\right)^{\frac{1}{2}}}{\chi+\chi\left(\frac{\chi}{\Lambda^4}\right)^{\frac{1}{2}}} = \frac{\dot{\phi}}{\dot{\phi}-\sqrt{2}\Lambda^2}\,. \tag{5.39}$$

With the numerical values of table 4.1, we find

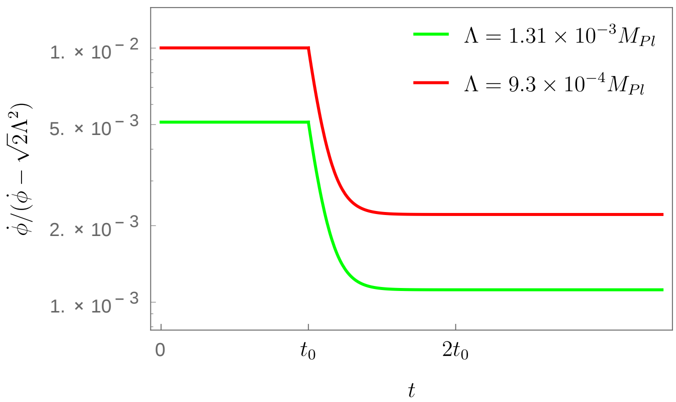

Figure 5.2: Ratio $\dot{\phi}/(\dot{\phi}-\sqrt{2}\Lambda^2)$ as a function of time. The characteristic scale for this system is $V_0^{\frac{1}{4}} = 1.24 \times 10^{-3} M_{Pl}$.

From the figure we note that the limit value for Λ, above which we can make a perturbative expansion, is different for the first model; in particular, while for the first model the perturbative approach was good for $\Lambda \gtrsim \Lambda_{\chi^2}$, for this model we have to choose $\Lambda \gtrsim 4\Lambda_{\chi^{3/2}}$ (here, we indicate with 'good' a ratio between the two kinetic terms smaller than 10^{-2}).

5.1.3 Comparison between the models

Now that we have the explicit expression of the HFF, c_s^2 and s for both the models, we can make a direct comparison between them.
In the following, we consider the time evolution of these quantities for two different values of the scale Λ: the first value of Λ is chosen so that the non-standard kinetic term is always much smaller that the standard one. This value is chosen to be $\Lambda = 5 \times 10^{-2} M_{Pl}$, so that

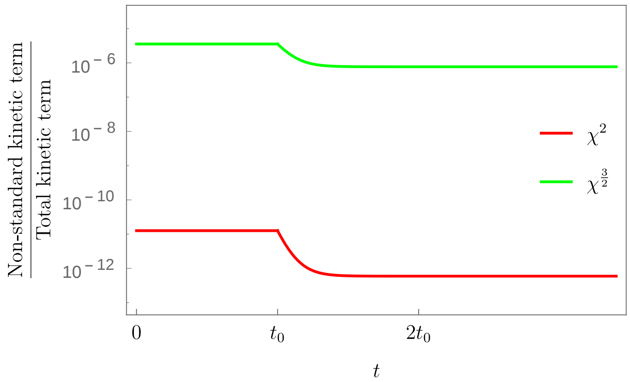

Figure 5.3: Ratio between the non-standard and the standard kinetic term for $\Lambda = 5 \times 10^{-2} M_{Pl}$. The characteristic scale for this system is $V_0^{\frac{1}{4}} = 1.24 \times 10^{-3} M_{Pl}$.

On the other hand, the second value we consider is $\Lambda = 5 \times 10^{-5} M_{Pl}$, for which the non-standard kinetic term begins to become important

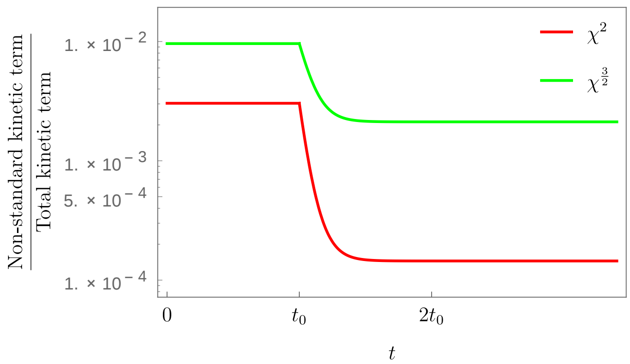

Figure 5.4: Ratio between the non-standard and the standard kinetic term for $\Lambda = 5 \times 10^{-5} M_{Pl}$. The characteristic scale for this system is $V_0^{\frac{1}{4}} = 1.24 \times 10^{-3} M_{Pl}$.

If we now consider the first two HFF, we find that their time evolution is

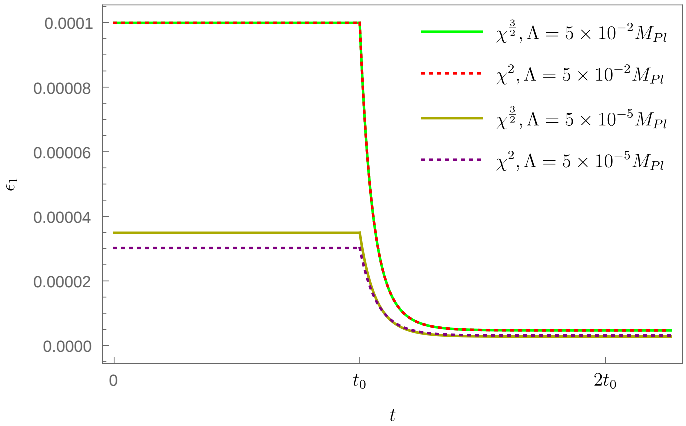

Figure 5.5: Evolution of ϵ_1 as a function of t for different Λ. The characteristic scale for this system is $V_0^{\frac{1}{4}} = 1.24 \times 10^{-3} M_{Pl}$.

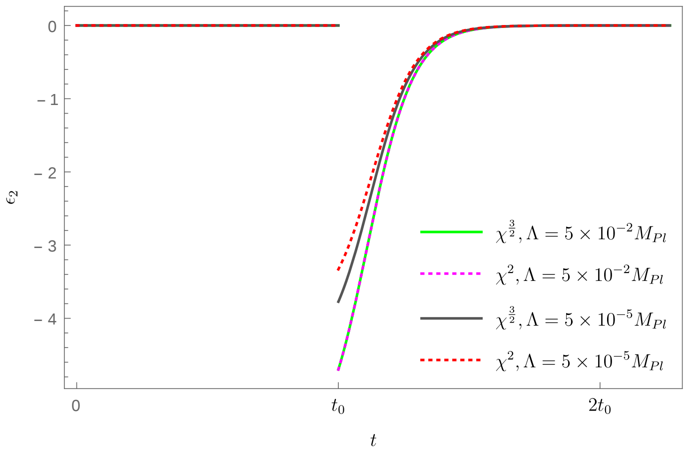

Figure 5.6: Evolution of ϵ_2 as a function of t for different Λ. The characteristic scale for this system is $V_0^{\frac{1}{4}} = 1.24 \times 10^{-3} M_{Pl}$.

As we can see from the figure, the first slow-roll parameter ϵ_1 remains always much smaller than 1, as required from the Starobinsky model which assumes that inflation never ends.
On the other hand, ϵ_2 is always discontinuous, and the jump do to the model with χ^2 is smaller than the other one.
Furthermore, as we could expect, these two model are almost indistinguishable for $\Lambda = 5 \times 10^{-2} M_{Pl}$, which confirms that the non-standard kinetic term is negligible at this scale Λ.

If we now consider the speed of sound and its logarithmic derivative, i.e. c_s^2 and s, we find

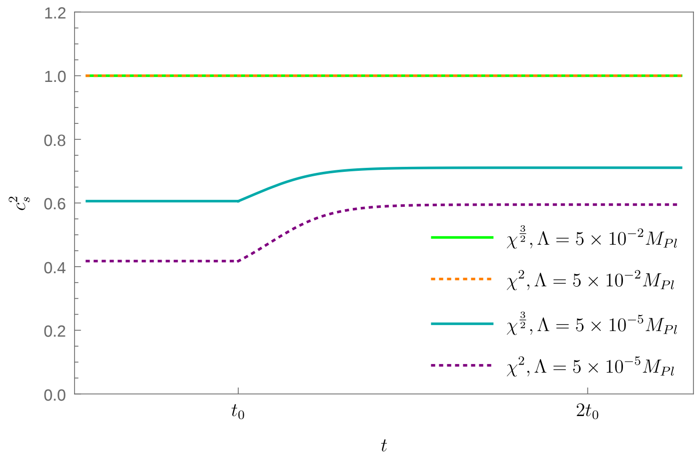

Figure 5.7: Evolution of c_s^2 as a function of t for different Λ. The characteristic scale for this system is $V_0^{\frac{1}{4}} = 1.24 \times 10^{-3} M_{Pl}$.

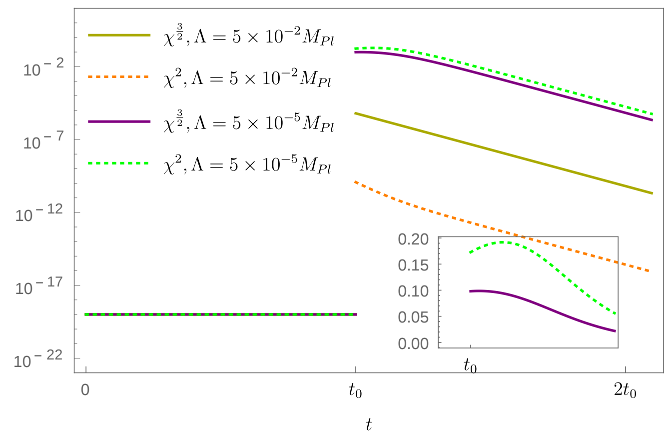

Figure 5.8: Evolution of s as a function of t for different Λ. The characteristic scale for this system is $V_0^{\frac{1}{4}} = 1.24 \times 10^{-3} M_{Pl}$.

Again, for the first value of Λ the two models reproduce the Starobinsky model almost perfectly. For the second value of Λ, instead, we note that this time the jump in s is smaller for the model with $\chi^{\frac{3}{2}}$.

5.2 Bispectrum of curvature perturbation

If we consider a model in which not only ϵ_2, but also s jumps, we can compute the bispectrum of curvature perturbation following the procedure indicated in section 3.3.3. First of all, we note that the vertex from which we have got the bispectrum in the previous chapter has to be corrected, since in this case we deal with a non-trivial speed of sound.

From the expressions of the coefficients $\mathcal{C}_1$ and $\mathcal{C}_2$ in eq. (3.103), we have that such a vertex reads

$$S_{3,\epsilon_2} = -M_{Pl}^2 \int dt \int d\boldsymbol{x}\, a\epsilon_1\epsilon_2 \left[\frac{\mathcal{R}\mathcal{R}'^2}{c_s^4} + \frac{\mathcal{R}^2\partial^2\mathcal{R}}{2c_s^2}\right], \tag{5.40}$$

which implies that the bispectrum takes the form

$$\mathcal{B}_{\mathcal{R}}(k)_{\epsilon_2} \approx 6M_{Pl}^2 \Im\left[\mathcal{R}_{\boldsymbol{k}}^3(0)\int_{\tau_0}^0 d\tau\, a^2\epsilon_1\epsilon_2\mathcal{R}_{\boldsymbol{k}}^*(\tau)\Big(\frac{2\mathcal{R}_{\boldsymbol{k}}'^{*2}(\tau)}{c_s^4} - k^2\frac{\mathcal{R}_{\boldsymbol{k}}^{*2}(\tau)}{c_s^2}\Big)\right]. \tag{5.41}$$

However, we stated that we there is a jump in s: therefore, we also have to take into account this second contribution. Again, from the considerations in sec. 3.3.3, the latter is

$$S_{3,s} = -2M_{Pl}^2 \int dt \int d\boldsymbol{x}\, a\frac{\epsilon_1 s}{c_s^2}\mathcal{R}^2\partial^2\mathcal{R}. \tag{5.42}$$

Taking into account this expression, the three-point function can be computed as

$$\langle\hat{\mathcal{R}}_{\boldsymbol{k}_1}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_2}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_3}(0)\rangle \approx -i\int_{-\infty}^0 d\tau'\, a\,\langle\Big[\hat{\mathcal{R}}_{\boldsymbol{k}_1}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_2}(0)\hat{\mathcal{R}}_{\boldsymbol{k}_3}(0), \hat{H}_{int}(\tau')\Big]\rangle, \tag{5.43}$$

with

$$\hat{H}_{int} = 2M_{Pl}^2 \int d\boldsymbol{x}\, a\frac{\epsilon_1 s}{c_s^2}\hat{\mathcal{R}}^2\partial^2\hat{\mathcal{R}}. \tag{5.44}$$

From the definition (3.21a) of $\hat{\mathcal{R}}_{\boldsymbol{k}}$ and the commutation rules (3.21b) for the creation/annihilation operators, we find that the commutator in (5.43) is of the same kind as in eq. (4.62).
The correspondent bispectrum in the equilateral limit is therefore

$$\mathcal{B}_{\mathcal{R}}(k)_s \approx 12M_{Pl}^2k^2\Im\left[\mathcal{R}_{\boldsymbol{k}}^3(0)\int_{\tau_0}^0 d\tau\, a^2\frac{\epsilon_1 s}{c_s^2}\mathcal{R}_{\boldsymbol{k}}^{*3}(\tau)\right]. \tag{5.45}$$

In the following, we treat the contributions coming from the two vertexes separately, in order to make evident the relative hierarchy.

The expression of the comoving curvature was found in [33], and it is

$$\mathcal{R}_{\boldsymbol{k}}(\tau) = \frac{iH}{2M_{Pl}\sqrt{\epsilon_1 c_s k^3}}\left[\alpha_k(1+ic_sk\tau)e^{-i\int c_sk\tau\, d\tau} - \beta_k(1-ic_sk\tau)e^{i\int c_sk\tau\, d\tau}\right]. \tag{5.46}$$

Its first derivative is therefore

$$\begin{aligned}\mathcal{R}_{\boldsymbol{k}}'(\tau) = {} & \frac{iH}{2M_{Pl}\sqrt{\epsilon_1 c_s k^3}}\times \\ & \times\left\{\alpha_k\left[c_s^2k^2\tau - \frac{1}{2}\left(\frac{\epsilon_1'}{\epsilon_1}+\frac{c_s'}{c_s}\right)(1+ic_sk\tau)e^{-i\int c_sk\tau\, d\tau}\right] - \right. \\ & \left. - \beta_k\left[c_s^2k^2\tau - \frac{1}{2}\left(\frac{\epsilon_1'}{\epsilon_1}+\frac{c_s'}{c_s}\right)(1-ic_sk\tau)e^{i\int c_sk\tau\, d\tau}\right]\right\}.\end{aligned} \tag{5.47}$$

In addition, the expressions for the Bogoliubov coefficients are

$$\alpha_k = 1 + \frac{i(\Delta s - \Delta\epsilon_2)(1 + c_s^2 k^2 \tau^2)}{4(c_s k\tau)^3} \tag{5.48a}$$

$$\beta_k = \frac{i(\Delta s - \Delta\epsilon_2)(1 + i c_s k\tau)^2 e^{-2ic_s k\tau}}{4(c_s k\tau)^3}\,, \tag{5.48b}$$

where we have defined Δ as the difference of a quantity evaluated after and before the transition respectively, i.e. $\Delta A \equiv A_- - A_+$.

5.3 Computation of power spectrum and bispectrum

After having analysed the two models, in this section we proceed to calculate the bispectrum of curvature perturbation.
We first compute the bispectrum for the vertex with ϵ_2 for the two models; we expect this calculation to reproduce the results of sec. 4.2.3 for $\Lambda^4 \gg \chi$, while we provide a first order correction in terms of powers in $(\chi/\Lambda^4)^n$ (with $n = 1$ for the first model and $n = 1/2$ for the second one) for $\Lambda^4 \gtrsim \chi$.
Secondly, we concentrate on the vertex with s. In this case, however, we limit to the computation of the bispectrum for the first model, since for the second one the calculation cannot be done analytically, unless with drastic approximations.

The only approximation we make is to consider the speed of sound c_s^2 as a constant when we compute the integral (5.41), and then we substitute its value at the transition. As a consequence, we can neglect the integrals and the term proportional to c_s' in (5.46) and (5.47), together with the term Δs in (5.48a) and (5.48b).

In order to plot the function $\mathcal{G}(k)/k^3$, we also have to consider the first order corrections to the power spectrum, which can be calculated analytically.

5.3.1 Curvature power spectrum

The non-standard kinetic term affects not only the curvature bispectrum, but also the power spectrum.
If we consider a scale for which $\chi \ll \Lambda^4$ (e.g. $\Lambda \sim 10^{-2} M_{Pl}$), as we could expect the power spectrum is almost the same as in the Starobinsky model

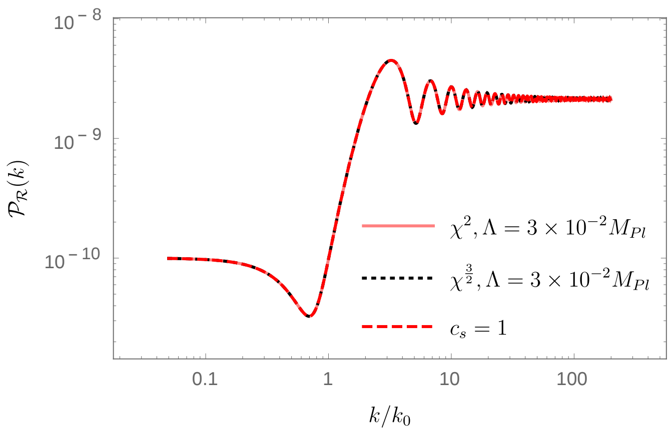

Figure 5.9: Curvature power spectrum for $\Lambda = 3 \times 10^{-2} M_{Pl}$.

In principle, we should expect that for smaller values of Λ, also the power spectrum would change. However, as we will see in a moment, the power spectrum is degenerate for a jump in s rather than in ϵ_2.

5.3.2 Contribution to the bispectrum from the ϵ_2-vertex

We have stated that we consider a Taylor expansion in terms of powers in χ/Λ^4 for all the quantities. It is then clear that the more we consider values of Λ for which the non-standard kinetic term is small compared to the standard one, the more this procedure is accurate.
In order to keep track of the approximations we make more efficiently, we can investigate the accuracy of such a power expansion. In fact, we make two approximations in the following: the first one, as already said, is to perform a Taylor expansion; the second one, is to consider the speed of sound c_s as a constant during the integration, and then to substitute its value at the transition to the result of the integral.
However, if we consider the integrands instead of the integrals, we can take into account also the exact expressions. In particular, in the following we make a comparison between the different expressions for the two integrands in (5.41). In order to do so, we plot both the exact, unexpanded integrands, respectively with the full expression for c_s and the integral in the exponential (called "exact, oscillating"), and with a constant, evaluated at the transition, c_s (called "exact, non-oscillating"); together with these, we also plot the expanded form of the integrands with c_s evaluated at the transition (called "approximated"). Note that for the computation of the integrals, we have used the "approximated" expression for the speed of sound. Since the integrands are function of τ, k and Λ, we can plot them by keeping one of these quantities fixed. For the first integrand in (5.41), if we fix the scale k, we plot the exact and approximated expression for two different values of Λ, respectively (almost) equal and slightly greater than its critical value (5.8). So doing, we find

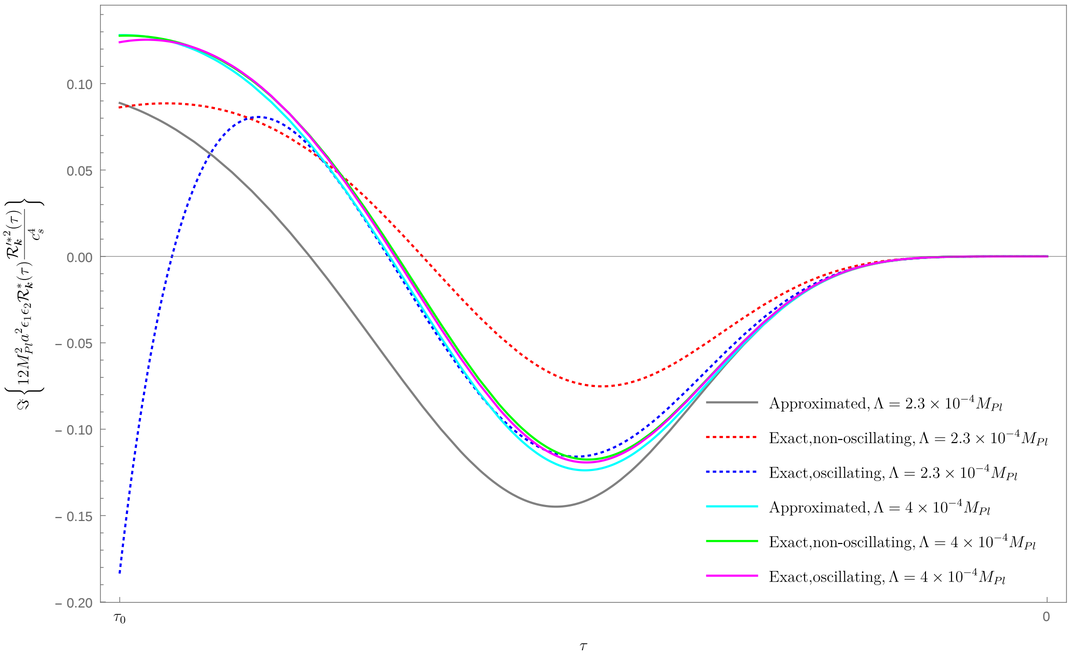

Figure 5.10: First integrand of (5.41) at fixed k for different values of Λ.

As we can see, just above the critical value (5.8), the approximated and the exact forms are rather different; however, this is clear, since in this regime we consider a Taylor expansion of a function which is not small at the point where we perform such an expansion. Conversely, for slightly bigger scales, we note that the power expansion is a good approximation.
By repeating the same considerations for the second integrand in (5.41), with the same values of k_* and Λ, we find

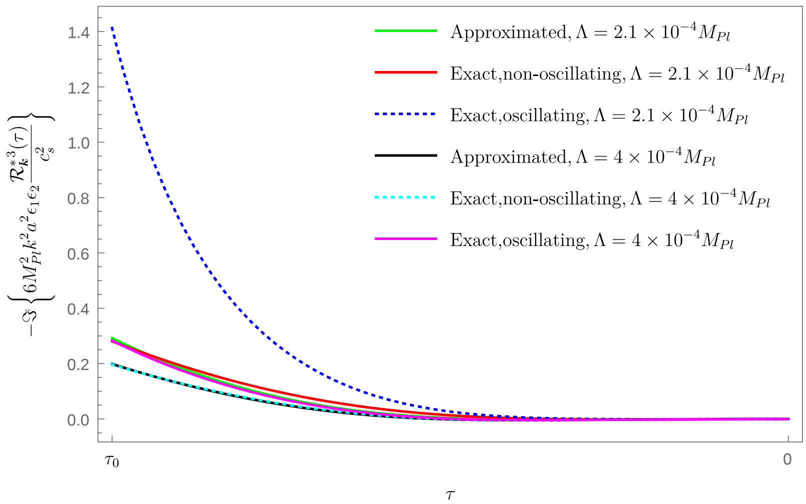

Figure 5.11: Second integrand of (5.41) at fixed k for different values of Λ.

In this case, we see that the approximation for value $\Lambda \gtrsim \Lambda_{\chi^2,+}$ is even better than in the previous one.
By performing the power expansion, we can analytically compute the correction to the bispectrum of the Starobinsky model calculated in [3]. In appendix B, we give the complete expressions for this quantity, for both the χ^2- and the $\chi^{\frac{3}{2}}$-models.

As we have stated before, one could think that also the power spectrum $\mathcal{P}_{\mathcal{R}}(k)$ in the definition (3.94) of $\mathcal{G}(k)$ should be expanded at first order in χ/Λ^4. However, if we plot the function $\mathcal{G}(k)/k^3$ in the limit $k \gg k_0$, we see that the result is almost the same whether we take the full, Λ-dependent form of $\mathcal{P}_{\mathcal{R}}$ or just the Λ-independent one.
For the first model, in fact, for $\Lambda = 3.7 \times 10^{-4} M_{Pl}$, we get

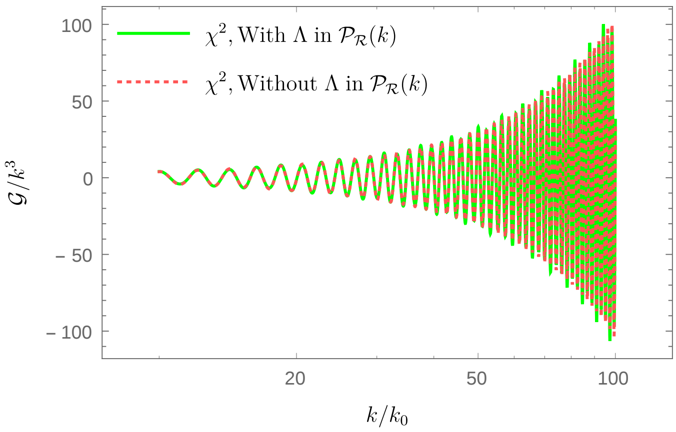

Figure 5.12: Function $\mathcal{G}(k)/k^3$ for small scales for the first model.

On the other hand, for the second model with $\Lambda = 9.5 \times 10^{-4} M_{Pl}$ we have

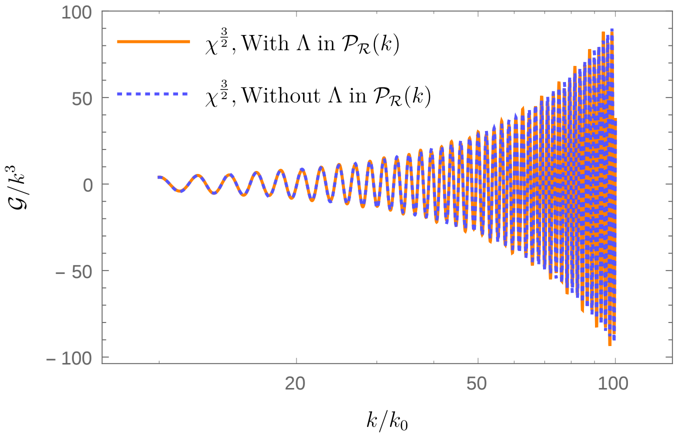

Figure 5.13: Function $\mathcal{G}(k)/k^3$ for small scales for the second model.

Since these plots have been obtained for the values of Λ we have used to define "good" the Taylor expansion, we can claim that these considerations are valid for all the range of values of Λ in which the perturbative approach is accurate. This is a confirm that the power spectrum is degenerate for a jump in s rather than in ϵ_2.

In this case too, when we consider values of Λ for which the non-standard kinetic term is negligible, the two models are undistinguishable, and in particular their bispectrum coincides with the one of the Starobinsky model.
For smaller values of Λ, instead, the bispectra are quite different for a fixed scale, as we can see from the following figure

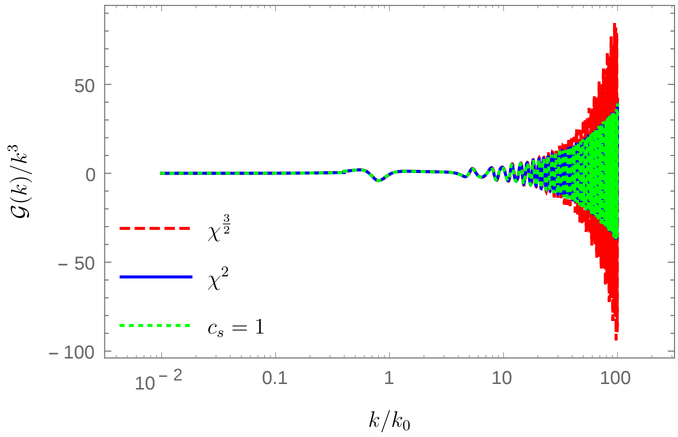

Figure 5.14: Curvature bispectrum for $\Lambda = 9.5 \times 10^{-4} M_{Pl}$.

As we can see, while for the χ^2-model the bispectrum is again very similar to that of the Starobinsky model, for the $\chi^{\frac{3}{2}}$-model the amplitude is significantly bigger. This difference between the two models can be better understood if we consider the asymptotic behaviours for the three-point functions for the two models. For the first model, in the limit $k \gg k_0$, we have

$$\mathcal{B}_{\mathcal{R}}(k)_{\chi^2} \xrightarrow{k\gg k_0} -\frac{729\Delta A H_0^{12}}{4A_-^3 A_+^2 k^5 k_0} \sin\left(\frac{3k}{k_0}\right) + \frac{243\Delta A H_0^{10}}{4A_-^3 k^4 k_0^2 \Lambda^4} \cos\left(\frac{3k}{k_0}\right) . \tag{5.49}$$

On the other hand, for the second model we find

$$\mathcal{B}_{\mathcal{R}}(k)_{\chi^{\frac{3}{2}}} \xrightarrow{k\gg k_0} -\frac{729\Delta A H_0^{12}}{4A_-^3 A_+^2 k^5 k_0} \sin\left(\frac{3k}{k_0}\right) + \frac{2187\Delta A H_0^{11}}{8\sqrt{2} A_-^3 A_+ k^4 k_0^2 \Lambda^2} \cos\left(\frac{3k}{k_0}\right) . \tag{5.50}$$

A crucial point is that the Λ-dependent term has the same shape for both the models, even if with different amplitudes. In particular, it grows more rapidly than the standard term found in [3]: while the Λ-independent term in the function $k^6 \mathcal{B}_{\mathcal{R}}$ grows like $\sim k \sin\left(\frac{3k}{k_0}\right)$, the original term found in this work grows like $\sim k^2 \cos\left(\frac{3k}{k_0}\right)$ both for the χ^2- and the $\chi^{\frac{3}{2}}$-models.

From these two expressions, it is also immediate to see that the Λ-dependent terms have different amplitudes for the two models. In fact, we find

$$\frac{\mathcal{B}_{\mathcal{R}}(k)_{\chi^2}}{\mathcal{B}_{\mathcal{R}}(k)_{\chi^{3/2}}} \xrightarrow{k\gg k_0} \frac{4\sqrt{2}A_+}{9H_0\Lambda^2}\,. \tag{5.51}$$

Therefore, since the bispectra scale with different powers of Λ, it is clear that when the non-standard term becomes relevant, those bispectra evolve in different ways for the same value of the scale Λ.

For smaller values of Λ, also the bispectrum for the first model becomes larger than the standard one. In fact, we have

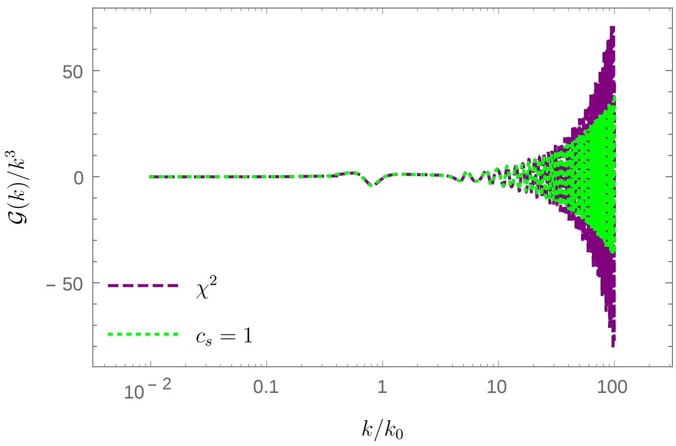

Figure 5.15: Curvature bispectrum for $\Lambda = 4 \times 10^{-4} M_{Pl}$.

Differently, we can get the same amplitude in the bispectrum if we consider different scales

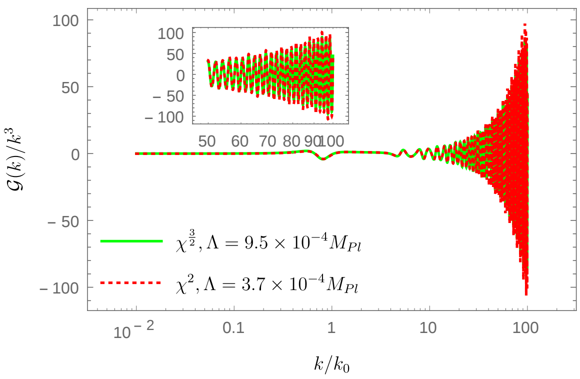

Figure 5.16: Curvature bispectrum for different values of Λ.

5.3.3 Contribution to the bispectrum from the s-vertex

The introduction of the new term in the Lagrangian allows to have another contribution to the bispectrum, due to the jump in the derivative of the speed of sound, i.e. s.
Such a contribution is given by eq. (5.45). While the preceding vertex was made by a Λ-independent term and a first order correction, in this case we just have a corrective term in powers of χ/Λ^4.

However, only for the χ^2-model such a bispectrum can be computed analytically, so that we consider just this case. The full expression for the bispectrum calculated from (5.45) is given in appendix B. Its shape is

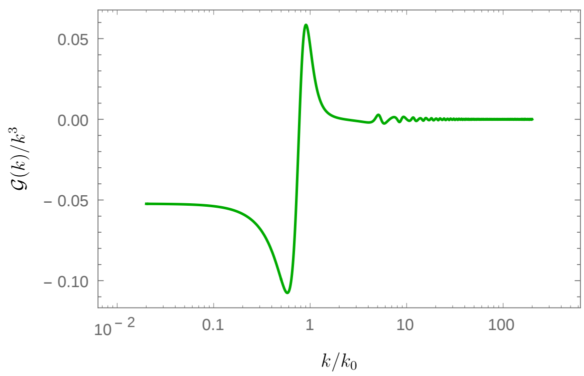

Figure 5.17: Bispectrum from the vertex proportional to s for $\Lambda = 3 \times 10^{-4} M_{Pl}$.

We note that this shape is completely different to the one coming from the vertex proportional to ϵ_2. In particular, this bispectrum does not grow indefinitely with the wave number, but it shows a feature near the transition, and two asymptotic values for small and large scales.
In addition, we see that the amplitude of such a bispectrum is much smaller than that coming from the other vertex.

Appendix A

Gaussian and non-Gaussian random fields

In this appendix we briefly review some features of Gaussian and non-Gaussian random fields, their principal properties and some relevant aspects for our treatise.

A.1 Gaussian random fields

Let's consider a generic real field $f(\boldsymbol{x})$, which can be Fourier expanded as

$$f(\boldsymbol{x}) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} . \tag{A.1}$$

Since $f(\boldsymbol{x})$ is real, $f(\boldsymbol{k})$ is subjected to the condition

$$f^*(\boldsymbol{k}) = f(-\boldsymbol{k}) . \tag{A.2}$$

Without loss of generality, we can parametrize $f(\boldsymbol{k})$ as $f(\boldsymbol{k}) \equiv f_{\boldsymbol{k}} = a_{\boldsymbol{k}} + i b_{\boldsymbol{k}}$, with a and b real and amplitude [35]

$$|f_{\boldsymbol{k}}| = \sqrt{a_{\boldsymbol{k}}^2 + b_{\boldsymbol{k}}^2} . \tag{A.3}$$

For these parameters the constraint for the reality of $f_{\boldsymbol{k}}$ is therefore

$$a_{-\boldsymbol{k}} = a_{\boldsymbol{k}} \quad , \quad b_{-\boldsymbol{k}} = -b_{\boldsymbol{k}} . \tag{A.4}$$

Different configurations of $f_{\boldsymbol{k}}$ are described by different sets of numbers $(a_{\boldsymbol{k}}, b_{\boldsymbol{k}})$. For this reason, if we want to randomly generate a field configuration of $f_{\boldsymbol{k}}$, we have to specify a probability distribution function (PDF) for the coefficients $(a_{\boldsymbol{k}}, b_{\boldsymbol{k}})$; in particular, if we require $f_{\boldsymbol{k}}$ to be a random Gaussian field, the PDF for the coefficients will be tightly constrained.
We define a *Gaussian distribution* for a single mode $\boldsymbol{k}$ the configuration in which the coefficients $(a_{\boldsymbol{k}}, b_{\boldsymbol{k}})$ are drawn from the distribution

$$P(a_{\boldsymbol{k}}, b_{\boldsymbol{k}}) = \frac{1}{2\pi\sigma_{\boldsymbol{k}}^2} \exp\left\{ -\frac{a_{\boldsymbol{k}}^2 + b_{\boldsymbol{k}}^2}{2\sigma_{\boldsymbol{k}}^2} \right\} , \tag{A.5}$$

which is normalized to unity

$$\int_{-\infty}^{\infty} da_{\boldsymbol{k}} \int_{-\infty}^{\infty} db_{\boldsymbol{k}}\, P(a_{\boldsymbol{k}}, b_{\boldsymbol{k}}) = 1\,. \tag{A.6}$$

If we assume statistical isotropy, the variance becomes a function only of the amplitude $k = |\boldsymbol{k}|$, i.e. σ_k.
Since we can identify $a_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}$, respectively, with the real and imaginary part of $f_{\boldsymbol{k}}$, we can rewrite eq. (A.5) in terms of amplitude and phase of $f_{\boldsymbol{k}}$ as

$$P(A,\theta) = \frac{1}{2\pi\sigma_k^2}\, \exp\left\{-\frac{A^2}{2\sigma_k^2}\right\}\,, \tag{A.7}$$

where

$$A \equiv a_{\boldsymbol{k}}^2 + b_{\boldsymbol{k}}^2 \tag{A.8a}$$

$$\tan\theta \equiv \frac{b_{\boldsymbol{k}}}{a_{\boldsymbol{k}}}\,. \tag{A.8b}$$

From eq. (A.7) we see that while the amplitude of $f_{\boldsymbol{k}}$ is drawn from a Gaussian PDF, its phase is random.

Now, let's consider a generic functional $Q\,[f_{\boldsymbol{k}}]$: we can define its mean value as

$$\langle Q\,[f_{\boldsymbol{k}}]\rangle = \prod_{\boldsymbol{k}} \int da_{\boldsymbol{k}} \int db_{\boldsymbol{k}}\, Q\,[f_{\boldsymbol{k}}]\, \frac{1}{2\pi\sigma_k^2}\, \exp\left\{-\frac{a_{\boldsymbol{k}}^2 + b_{\boldsymbol{k}}^2}{2\sigma_k^2}\right\}\,. \tag{A.9}$$

Consider the simple case in which $Q\,[f_{\boldsymbol{k}}] = b_{\boldsymbol{k}} b_{-\boldsymbol{k}'}$: then

$$\langle b_{\boldsymbol{k}} b_{-\boldsymbol{k}'}\rangle = \prod_{\boldsymbol{p}} \int da_{\boldsymbol{p}} \int db_{\boldsymbol{p}}\, b_{\boldsymbol{k}} b_{-\boldsymbol{k}'}\, \frac{1}{2\pi\sigma_p^2}\, \exp\left\{-\frac{a_{\boldsymbol{p}}^2 + b_{\boldsymbol{p}}^2}{2\sigma_p^2}\right\}\,. \tag{A.10}$$

The above integral is non-vanishing iff $\boldsymbol{k} = -\boldsymbol{k}'$. The integrals with $\boldsymbol{q} \neq \boldsymbol{k}$ give 1, so that we can write

$$\langle b_{\boldsymbol{k}} b_{-\boldsymbol{k}'}\rangle = -\langle b_{\boldsymbol{k}} b_{\boldsymbol{k}'}\rangle = \sigma_k^2 \delta(\boldsymbol{k} + \boldsymbol{k}')\,. \tag{A.11}$$

Similarly (recall that $a_{-\boldsymbol{k}} = a_{\boldsymbol{k}}$)

$$\langle a_{\boldsymbol{k}} a_{\boldsymbol{k}'}\rangle = \sigma_k^2 \delta(\boldsymbol{k} + \boldsymbol{k}')\,. \tag{A.12}$$

By combining these two results we have

$$\langle f_{\boldsymbol{k}} f_{\boldsymbol{k}'}\rangle = \langle a_{\boldsymbol{k}} a_{\boldsymbol{k}'}\rangle - \langle b_{\boldsymbol{k}} b_{\boldsymbol{k}'}\rangle = 2\sigma_k^2 \delta(\boldsymbol{k} + \boldsymbol{k}')\,. \tag{A.13}$$

Then, we note that Gaussian modes are uncorrelated, i.e. modes with different $\boldsymbol{k}$ have a vanishing correlation function.

In addition, all odd-points correlation functions vanish, while the even ones can be expressed in term of $\langle f(\boldsymbol{k}) f(\boldsymbol{k}')\rangle$. For instance

$$\langle a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2} a_{\boldsymbol{k}_3} a_{\boldsymbol{k}_4}\rangle = \prod_{\boldsymbol{p}} \int da_{\boldsymbol{p}} \int db_{\boldsymbol{p}}\, a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2} a_{\boldsymbol{k}_3} a_{\boldsymbol{k}_4}\, \frac{1}{2\pi\sigma_p^2}\, \exp\left\{-\frac{a_{\boldsymbol{p}}^2 + b_{\boldsymbol{p}}^2}{2\sigma_p^2}\right\}\,. \tag{A.14}$$

In this case the integral does not vanish only for pairs of equal momenta, i.e. $(k_1 = k_2, k_3 = k_4), (k_1 = k_3, k_2 = k_4), (k_1 = k_4, k_2 = k_3)$.
Therefore

$$\langle a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2} a_{\boldsymbol{k}_3} a_{\boldsymbol{k}_4} \rangle = \sigma_{k_1}^2 \sigma_{k_3}^2 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2)\delta(\boldsymbol{k}_3 + \boldsymbol{k}_4) + (1 \leftrightarrow 3) + (1 \leftrightarrow 4)\,, \tag{A.15}$$

and consequently

$$\begin{aligned}
\langle f_{\boldsymbol{k}_1} f_{\boldsymbol{k}_2} f_{\boldsymbol{k}_3} f_{\boldsymbol{k}_4} \rangle &= \langle a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2} a_{\boldsymbol{k}_3} a_{\boldsymbol{k}_4} \rangle + \langle b_{\boldsymbol{k}_1} b_{\boldsymbol{k}_2} b_{\boldsymbol{k}_3} b_{\boldsymbol{k}_4} \rangle + \langle a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2} b_{\boldsymbol{k}_3} b_{-\boldsymbol{k}_4} \rangle + \\
&\quad + \langle a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_3} b_{\boldsymbol{k}_2} b_{-\boldsymbol{k}_4} \rangle + \langle a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_4} b_{\boldsymbol{k}_2} b_{-\boldsymbol{k}_3} \rangle + \langle a_{\boldsymbol{k}_2} a_{\boldsymbol{k}_3} b_{\boldsymbol{k}_1} b_{-\boldsymbol{k}_4} \rangle + \\
&\quad + \langle a_{\boldsymbol{k}_2} a_{\boldsymbol{k}_4} b_{\boldsymbol{k}_1} b_{-\boldsymbol{k}_3} \rangle + \langle a_{\boldsymbol{k}_3} a_{\boldsymbol{k}_4} b_{\boldsymbol{k}_2} b_{-\boldsymbol{k}_3} \rangle = \\
&= 4\sigma_{k_1}^2 \sigma_{k_3}^2 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2)\delta(\boldsymbol{k}_3 + \boldsymbol{k}_4) + (1 \leftrightarrow 3) + (1 \leftrightarrow 4) = \\
&= \langle f_{\boldsymbol{k}_1} f_{\boldsymbol{k}_2} \rangle \langle f_{\boldsymbol{k}_3} f_{\boldsymbol{k}_4} \rangle + \langle f_{\boldsymbol{k}_1} f_{\boldsymbol{k}_3} \rangle \langle f_{\boldsymbol{k}_2} f_{\boldsymbol{k}_4} \rangle + \langle f_{\boldsymbol{k}_1} f_{\boldsymbol{k}_4} \rangle \langle f_{\boldsymbol{k}_2} f_{\boldsymbol{k}_3} \rangle\,.
\end{aligned} \tag{A.16}$$

It is then clear that for Gaussian random fields, all the information is encoded in the two-points correlation function.

A.2 From Gaussian to non-Gaussian fields

From a general Gaussian curvature perturbation, i.e. $f_G(\boldsymbol{x})$, we can construct the correspondent non-Gaussian quantity $f(\boldsymbol{x})$ as

$$f(\boldsymbol{x}) = f_G(\boldsymbol{x}) + \, f_{NL}\left(f_G^2(\boldsymbol{x}) - \langle f_G^2(\boldsymbol{x})\rangle\right) \equiv f_G(\boldsymbol{x}) + f_{NG}(\boldsymbol{x})\,, \tag{A.17}$$

This quantity is defined “non-Gaussian” because, as we are going to demonstrate, it has a non-vanishing three-point function.
First of all, let’s compute the Fourier transform of $f_{NG}(\boldsymbol{x})$: this reads

$$\begin{aligned}
f_{NG}(\boldsymbol{k}) &= \, f_{NL} \int d\boldsymbol{x}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left[f_G^2(\boldsymbol{x}) - \langle f_G^2(\boldsymbol{x})\rangle\right] = \\
&= \, f_{NL}\left[\int d\boldsymbol{x} \int \frac{d\boldsymbol{p}}{(2\pi)^3} \int \frac{d\boldsymbol{q}}{(2\pi)^3} f_G(\boldsymbol{p}) f_G(\boldsymbol{q})\, e^{i(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}} - (2\pi)^3 \delta(\boldsymbol{k}) \langle f_G^2(\boldsymbol{x})\rangle\right] = \\
&= \, f_{NL}\left[\int \frac{d\boldsymbol{p}}{(2\pi)^3} f_G(\boldsymbol{p}) f_G(\boldsymbol{k}-\boldsymbol{p}) - (2\pi)^3\delta(\boldsymbol{k})\langle f_G^2(\boldsymbol{x})\rangle\right] = \\
&= \, f_{NL}\left[\int \frac{d\boldsymbol{p}}{(2\pi)^3} f_G^*(\boldsymbol{p}) f_G(\boldsymbol{k}+\boldsymbol{p}) - (2\pi)^3\delta(\boldsymbol{k})\langle f_G^2(\boldsymbol{x})\rangle\right]\,,
\end{aligned} \tag{A.18}$$

where the reality condition (2.74) has been used.
Thanks to (3.51), we can write $\langle f_G^2(\boldsymbol{x})\rangle$ as

$$\langle f_G^2(\boldsymbol{x})\rangle = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \int \frac{d\boldsymbol{p}}{(2\pi)^3} \langle f_G(\boldsymbol{k}) f_G(\boldsymbol{p})\rangle\, e^{i(\boldsymbol{k}+\boldsymbol{p})\cdot\boldsymbol{x}} = \int \frac{d\boldsymbol{k}}{(2\pi)^3} P_f(k)\,, \tag{A.19}$$

and consequently

$$f_{NG}(\boldsymbol{k}) = \, f_{NL} \int \frac{d\boldsymbol{p}}{(2\pi)^3} \left[f_G^*(\boldsymbol{p}) f_G(\boldsymbol{k}+\boldsymbol{p}) - (2\pi)^3 \delta(\boldsymbol{k}) P_f(k)\right] \tag{A.20}$$

Now, since

$$\langle f_G(\boldsymbol{k}) f_G(\boldsymbol{k}')\rangle = P_f(k)\,(2\pi)^3 \delta(\boldsymbol{k}+\boldsymbol{k}')\,, \tag{A.21}$$

we have, for the reality condition (2.74)

$$\langle f^*_G(\boldsymbol{p}) f_G(\boldsymbol{k}+\boldsymbol{p})\rangle = P_f(p)\,(2\pi)^3 \delta(\boldsymbol{k})\,. \tag{A.22}$$

With these relation, we are able to recast eq. (A.20) as

$$f_{NG}(\boldsymbol{k}) = f_{NL} \int \frac{d\boldsymbol{p}}{(2\pi)^3} \left[f^*_G(\boldsymbol{p}) f_G(\boldsymbol{k}+\boldsymbol{p}) - \langle f^*_G(\boldsymbol{k}) f_G(\boldsymbol{k}+\boldsymbol{p})\rangle \right]\,. \tag{A.23}$$

Since we have expressed the non-Gaussian part of the curvature perturbation uniquely in terms of Gaussian fields in Fourier space, can now calculate the three-point function and exploit relation (A.16) for Gaussian fields.
So doing we have

$$\begin{aligned}
\langle f_G(\boldsymbol{k}_1) f_G(\boldsymbol{k}_2) f_{NG}(\boldsymbol{k}_3)\rangle = {} & f_{NL} \int \frac{d\boldsymbol{p}}{(2\pi)^3} \Big[\langle f_G(\boldsymbol{k}_1) f_G(\boldsymbol{k}_2) f^*_G(\boldsymbol{p}) f_G(\boldsymbol{k}_3+\boldsymbol{p})\rangle - \\
& - \langle f_G(\boldsymbol{k}_1) f_G(\boldsymbol{k}_2)\rangle \langle f^*_G(\boldsymbol{k}) f_G(\boldsymbol{k}_3+\boldsymbol{p})\rangle \Big] = \\
= {} & f_{NL} \int \frac{d\boldsymbol{p}}{(2\pi)^3} \Big[\cancel{\langle f_G(\boldsymbol{k}_1) f_G(\boldsymbol{k}_2)\rangle \langle f^*_G(\boldsymbol{p}) f_G(\boldsymbol{k}_3+\boldsymbol{p})\rangle} + \\
& + \langle f_G(\boldsymbol{k}_1) f^*_G(\boldsymbol{p})\rangle \langle f_G(\boldsymbol{k}_2) f_G(\boldsymbol{k}_3+\boldsymbol{p})\rangle + \\
& + \langle f_G(\boldsymbol{k}_1) f_G(\boldsymbol{k}_3+\boldsymbol{p})\rangle \langle f_G(\boldsymbol{k}_2) f^*_G(\boldsymbol{p})\rangle - \\
& - \cancel{\langle f_G(\boldsymbol{k}_1) f_G(\boldsymbol{k}_2)\rangle \langle f^*_G(\boldsymbol{k}) f_G(\boldsymbol{k}_3+\boldsymbol{p})\rangle} \Big] = \\
= {} & f_{NL} (2\pi)^3 \int d\boldsymbol{p} \Big[P_f(k_1) P_f(k_2) \delta(\boldsymbol{k}_1 - \boldsymbol{p}) \delta(\boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{p}) + \\
& + P_f(k_1) P_f(k_2) \delta(\boldsymbol{k}_1 + \boldsymbol{k}_3 + \boldsymbol{p}) \delta(\boldsymbol{k}_2 - \boldsymbol{p}) \Big] = \\
= {} & 2\, f_{NL} (2\pi)^3 P_f(k_1) P_f(k_2) \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\,.
\end{aligned} \tag{A.24}$$

Thus, we have demonstrated that the definition of f in (A.17) gives non-vanishing three-point function.
In general, we find [35]

$$\langle f(\boldsymbol{k}_1) f(\boldsymbol{k}_2) f(\boldsymbol{k}_3)\rangle = 2\, f_{NL} (2\pi)^3 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3) \left[P_f(k_1) P_f(k_2) + \text{symm.} \right]\,. \tag{A.25}$$

Appendix B

Analytic expressions for the bispectrum

In this appendix we give the analytical expressions of the bispectrum for the models studied in chapter 5, up to the first, non-vanishing order in χ/Λ^4.

B.1 Contribution to the bispectrum from the ϵ_2-vertex

Here we give an explicit expression for the bispectrum deriving from the vertex proportional to $\epsilon_1\epsilon_2$, for both the two extensions of the Starobinsky model

B.1.1 First model

For the first model, i.e. the one for which the non-standard kinetic term is proportional to χ^2, the three-point correlation function is

$$\langle \hat{\mathcal{R}}_{\boldsymbol{k}}(0)\hat{\mathcal{R}}_{\boldsymbol{k}}(0)\hat{\mathcal{R}}_{\boldsymbol{k}}(0)\rangle_{\epsilon_2,\chi^2} \approx (2\pi)^3\delta(3\boldsymbol{k})\left[\mathcal{A}_1(k_0F_1 + k_0^2F_2 + k_0^3F_3 + k_0^4F_4) + \right. \\ \left. + \mathcal{A}_2(F_5 + k_0F_6 + k_0^2F_7 + k_0^3F_8 + k_0^4F_9)\right] , \qquad \text{(B.1)}$$

where

$$\mathcal{A}_1 = -\frac{81\pi^3\Delta A H_0^{10}}{4A_-^3A_+^4k^{15}k_0^2}\,, \qquad \text{(B.2)}$$

$$\mathcal{A}_2 = -\frac{81\pi^3\Delta A H_0^{10}}{4k^{15}\Lambda^4A_-^3A_+^4k_0^2}\,, \qquad \text{(B.3)}$$

$$F_1 = -36A_+^2H_0^2\sin\left(\frac{k}{k_0}\right)\left[(2k^{10} - 47k_0^2k^8 - 18k_0^4k^6 + 108k_0^6k^4 + 108k_0^8k^2 + 27k_0^{10}) + \right. \\ \left. + (4k^{10} - 67k_0^2k^8 + 18k_0^4k^6 + 162k_0^6k^4 + 54k_0^8k^2 - 27k_0^{10})\cos\left(\frac{2k}{k_0}\right)\right], \qquad \text{(B.4)}$$

$$
\begin{aligned}
F_2 = {} & 162A_+\Delta A H_0^2 k(k^2+k_0^2)^2(2k^4-3k_0^2k^2-9k_0^4)\cos\left(\frac{k}{k_0}\right)+\\
& +36A_-A_+H_0^2k(7k^8-48k_0^2k^6-72k_0^4k^4+72k_0^6k^2+81k_0^8)\cos\left(\frac{3k}{k_0}\right)-\\
& -18A_+^2H_0^2k(26k^8-81k_0^2k^6-99k_0^4k^4+81k_0^6k^2+81k_0^8)\cos\left(\frac{3k}{k_0}\right), \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
F_3 = {} & A_-H_0^2\sin\left(\frac{k}{k_0}\right)\Bigg[324A_-(k^8-10k_0^4k^4-12k_0^6k^2-3k_0^8)+\\
& +72A_+(-24k^8-10k_0^2k^6+99k_0^4k^4+108k_0^6k^2+27k_0^8)+\\
& +324A_-(k^8-2k_0^2k^6-12k_0^4k^4-6k_0^6k^2+3k_0^8)\cos\left(\frac{2k}{k_0}\right)-\\
& -72A_+(30k^8-16k_0^2k^6-135k_0^4k^4-54k_0^6k^2+27k_0^8)\cos\left(\frac{2k}{k_0}\right)\Bigg], \qquad \text{(B.6)}
\end{aligned}
$$

$$
\begin{aligned}
F_4 = {} & 162A_-\Delta A H_0^2k(k^2+k_0^2)(k^4+10k_0^2k^2+9k_0^4)\cos\left(\frac{k}{k_0}\right)+\\
& +162A_-^2H_0^2k(3k^6+5k_0^2k^4-7k_0^4k^2-9k_0^6)\cos\left(\frac{3k}{k_0}\right), \qquad \text{(B.7)}
\end{aligned}
$$

$$
F_5 = 2k\cos\left(\frac{3k}{k_0}\right)A_+^4(12k^{10}-299k_0^2k^8+630k_0^4k^6+1152k_0^6k^4-810k_0^8k^2-1053k_0^{10}), \qquad \text{(B.8)}
$$

$$
\begin{aligned}
F_6 = {} & -2\sin\left(\frac{k}{k_0}\right)A_+^2\\
& \Bigg[3(2k^{10}-119k_0^2k^8+228k_0^4k^6+1314k_0^6k^4+1350k_0^8k^2+333k_0^{10})A_-^2+\\
& +3\cos\left(\frac{2k}{k_0}\right)(4k^{10}-163k_0^2k^8+360k_0^4k^6+1404k_0^6k^4+648k_0^8k^2-333k_0^{10})A_-^2-\\
& -4\cos\left(\frac{2k}{k_0}\right)A_+(21k^{10}-303k_0^2k^8+181k_0^4k^6+1701k_0^6k^4+810k_0^8k^2-378k_0^{10})A_--\\
& -4A_+(15k^{10}-201k_0^2k^8+23k_0^4k^6+1323k_0^6k^4+1458k_0^8k^2+378k_0^{10})A_-+\\
& +2\cos\left(\frac{2k}{k_0}\right)A_+^2(90k^{10}-578k_0^2k^8+39k_0^4k^6+1809k_0^6k^4783k_0^8k^2-351k_0^{10})+\\
& +2A_+^2(54k^{10}-352k_0^2k^8-147k_0^4k^6+1215k_0^6k^4+1323k_0^8k^2+351k_0^{10})\Bigg], \qquad \text{(B.9)}
\end{aligned}
$$

$$F_7 = kA_+\Bigg[6\cos\left(\frac{3k}{k_0}\right)(7k^8 - 66k_0^2k^6 - 54k_0^4k^4 + 162k_0^6k^2 + 135k_0^8)A_-^3 + \\
+ 27\cos\left(\frac{k}{k_0}\right)\Delta A(k^2+k_0^2)^2(2k^4 - 15k_0^2k^2 - 21k_0^4)A_-^2 - \\
- 18\cos\left(\frac{k}{k_0}\right)\Delta A A_+(k^2+k_0^2)(k^6 - 43k_0^2k^4 - 147k_0^4k^2 - 135k_0^6)A_- + \\
+ 4\cos\left(\frac{3k}{k_0}\right)A_+^2(115k^8 - 462k_0^2k^6 - 855k_0^4k^4 + 864k_0^6k^2 + 1134k_0^8)A_- - \\
+ 18\cos\left(\frac{k}{k_0}\right)A_+^2\Delta A(k^2+k_0^2)(9k^6 - 19k_0^2k^4 - 129k_0^4k^2 - 117k_0^6)\Bigg], \tag{B.10}$$

$$F_8 = 6\sin\left(\frac{k}{k_0}\right)\Bigg[9(k^8 - 10k_0^4k^4 - 12k_0^6k^2 - 3k_0^8)A_-^4 + \\
+ 9\cos\left(\frac{2k}{k_0}\right)(k^8 - 2k_0^2k^6 - 12k_0^4k^4 - 6k_0^6k^2 + 3k_0^8)A_-^4 - \\
- 2\cos\left(\frac{2k}{k_0}\right)A_+(30k^8 - 70k_0^2k^6 - 225k_0^4k^4 - 72k_0^6k^2 + 45k_0^8)A_-^3 + \\
+ 2A_+(-24k^8 + 44k_0^2k^6 + 225k_0^4k^4 + 198k_0^6k^2\Bigg], \tag{B.11}$$

$$F_9 = 27k\cos\left(\frac{3k}{k_0}\right)(3k^6 + 5k_0^2k^4 - 7k_0^4k^2 - 9k_0^6)A_-^4 + \\
+ 27k\cos\left(\frac{k}{k_0}\right)\Delta A(k^2+k_0^2)^2(k^2+9k_0^2)A_-^3\,. \tag{B.12}$$

B.1.2 Second model

For the other model, i.e. the one for which the non-standard kinetic term is proportional to $\chi^{\frac{3}{2}}$, the three-point correlation function is

$$\langle\hat{\mathcal{R}}_{\boldsymbol{k}}(0)\hat{\mathcal{R}}_{\boldsymbol{k}}(0)\hat{\mathcal{R}}_{\boldsymbol{k}}(0)\rangle_{\epsilon_2,\chi^{3/2}} \approx (2\pi)^3\delta(3\boldsymbol{k})\Big[\mathcal{D}_1(G_1 + k_0G_2 + k_0^2G_3) + \\
+ \mathcal{D}_2(G_4 + k_0G_5 + k_0^2G_6 + k_0^3G_6)\Big], \tag{B.13}$$

where

$$\mathcal{D}_1 = \frac{729\pi^3\Delta A H_0^{12}}{2A_-^3A_+^4k^{15}k_0}, \tag{B.14}$$

$$\mathcal{D}_2 = -\frac{729\pi^3\Delta A H_0^{11}}{8\sqrt{2}k^{15}\Lambda^2A_-^3A_+^3k_0^2}, \tag{B.15}$$

$$
\begin{aligned}
G_1 = A_+^2 \Bigg\{ & 4k^{10} \sin\left(\frac{3k}{k_0}\right) + 2k_0 k^9 \left[9\cos\left(\frac{k}{k_0}\right) + 13\cos\left(\frac{3k}{k_0}\right)\right] - \\
& - k_0^2 k^8 \left[27\sin\left(\frac{k}{k_0}\right) + 67\sin\left(\frac{3k}{k_0}\right)\right] + 9k_0^3 k^7 \left[\cos\left(\frac{k}{k_0}\right) - 9\cos\left(\frac{3k}{k_0}\right)\right] - \\
& - 72k_0^4 k^6 \sin^3\left(\frac{k}{k_0}\right) - 9k_0^5 k^5 \left[13\cos\left(\frac{k}{k_0}\right) + 11\cos\left(\frac{3k}{k_0}\right)\right] + \\
& + 54k_0^6 k^4 \left[\sin\left(\frac{k}{k_0}\right) + 3\sin\left(\frac{3k}{k_0}\right)\right] + 27k_0^7 k^3 \left[3\cos\left(\frac{3k}{k_0}\right) - 7\cos\left(\frac{k}{k_0}\right)\right] + \\
& + 54k_0^8 k^2 \left[3\sin\left(\frac{k}{k_0}\right) + \sin\left(\frac{3k}{k_0}\right)\right] + 108k_0^{10} \sin^3\left(\frac{k}{k_0}\right) - \\
& - 162k_0^9 k \sin\left(\frac{k}{k_0}\right) \sin\left(\frac{2k}{k_0}\right) \Bigg\}, \qquad \text{(B.16)}
\end{aligned}
$$

$$
\begin{aligned}
G_2 = 2A_- A_+ \Bigg\{ & - k^9 \left[9\cos\left(\frac{k}{k_0}\right) + 7\cos\left(\frac{3k}{k_0}\right)\right] + \\
& + 6k_0 k^8 \left[3\sin\left(\frac{k}{k_0}\right) + 5\sin\left(\frac{3k}{k_0}\right)\right] + 48k_0^2 k^7 \cos\left(\frac{3k}{k_0}\right) + \\
& + 4k_0^3 k^6 \left[9\sin\left(\frac{k}{k_0}\right) - 4\sin\left(\frac{3k}{k_0}\right)\right] + 36k_0^4 k^5 \left[3\cos\left(\frac{k}{k_0}\right) + 2\cos\left(\frac{3k}{k_0}\right)\right] - \\
& - 9k_0^5 k^4 \left[7\sin\left(\frac{k}{k_0}\right) + 15\sin\left(\frac{3k}{k_0}\right)\right] + 36k_0^6 k^3 \left[5\cos\left(\frac{k}{k_0}\right) - 2\cos\left(\frac{3k}{k_0}\right)\right] - \\
& - 54k_0^7 k^2 \left[3\sin\left(\frac{k}{k_0}\right) + \sin\left(\frac{3k}{k_0}\right)\right] - 108k_0^9 \sin^3\left(\frac{k}{k_0}\right) + \\
& + 162k_0^8 k \sin\left(\frac{k}{k_0}\right) \sin\left(\frac{2k}{k_0}\right) \Bigg\}, \qquad \text{(B.17)}
\end{aligned}
$$

$$
\begin{aligned}
G_3 = -36A_-^2 k_0^2 \left(k^2 + k_0^2\right) & \left[k\cos\left(\frac{k}{k_0}\right) - k_0 \sin\left(\frac{k}{k_0}\right)\right]^2 \left[k^4 \sin\left(\frac{k}{k_0}\right) + \right. \qquad \text{(B.18)} \\
& \left. + k_0 k^3 \cos\left(\frac{k}{k_0}\right) - 3k_0^4 \sin\left(\frac{k}{k_0}\right) + 3k_0^3 k \cos\left(\frac{k}{k_0}\right)\right],
\end{aligned}
$$

$$
G_4 = k\cos\left(\frac{3k}{k_0}\right) A_+^2 \left(12k^{10} - 305k_0^2 k^8 + 540k_0^4 k^6 + 1278k_0^6 k^4 - 648k_0^8 k^2 - 1053k_0^{10}\right), \qquad \text{(B.19)}
$$

$$
\begin{aligned}
G_5 = 2\sin\left(\frac{k}{k_0}\right) A_+ & \\
\Bigg[& 2A_- \left(15k^{10} - 192k_0^2k^8 - 232k_0^4k^6 + 909k_0^6k^4 + 1269k_0^8k^2 + 351k_0^{10}\right) + \\
& + \cos\left(\frac{2k}{k_0}\right) A_+ \left(-90k^{10} + 578k_0^2k^8 + 171k_0^4k^6 - 1701k_0^6k^4 - 837k_0^8k^2 + 351k_0^{10}\right) + \\
& + 2\cos\left(\frac{2k}{k_0}\right) A_- \left(21k^{10} - 294k_0^2k^8 - 104k_0^4k^6 + 1431k_0^6k^4 + 837k_0^8k^2 - 351k_0^{10}\right) - \\
& - A_+ \left(54k^{10} - 352k_0^2k^8 - 333k_0^4k^6 + 999k_0^6k^4 + 1269k_0^8k^2 + 351k_0^{10}\right) \Bigg],
\end{aligned}
\tag{B.20}
$$

$$
\begin{aligned}
G_6 = k\cos\left(\frac{k}{k_0}\right) \Bigg[& 9A_+\Delta A\left(k^2 + k_0^2\right)\left(11k^6 + k_0^2k^4 - 111k_0^4k^2 - 117k_0^6\right) - \\
& - 9A_-\Delta A\left(k^2 + k_0^2\right)\left(k^6 - 5k_0^2k^4 - 91k_0^4k^2 - 117k_0^6\right) - \\
& - 9A_-^2\left(3k^8 - 24k_0^2k^6 - 76k_0^4k^4 + 52k_0^6k^2 + 117k_0^8\right) + \\
& + 2A_-A_+\left(118k^8 - 336k_0^2k^6 - 981k_0^4k^4 + 558k_0^6k^2 + 1053k_0^8\right) \Bigg],
\end{aligned}
\tag{B.21}
$$

$$
\begin{aligned}
G_7 = 18\sin\left(\frac{k}{k_0}\right) A_-^2 \Bigg[& \left(10k^8 + 13k_0^2k^6 - 91k_0^4k^4 - 141k_0^6k^2 - 39k_0^8\right) + \\
& + \cos\left(\frac{2k}{k_0}\right)\left(14k^8 + k_0^2k^6 - 129k_0^4k^4 - 93k_0^6k^2 + 39k_0^8\right) \Bigg].
\end{aligned}
\tag{B.22}
$$

B.2 Contribution to the bispectrum from the s-vertex

Here we give an explicit expression for the bispectrum deriving from the vertex proportional to s for the model with the χ^2 non-standard kinetic term.

The three-point correlation function for this model, at the lowest order χ/Λ^4, is

$$
\begin{aligned}
\langle \hat{\mathcal{R}}_{\boldsymbol{k}}(0)\hat{\mathcal{R}}_{\boldsymbol{k}}(0)\hat{\mathcal{R}}_{\boldsymbol{k}}(0)\rangle_{s,\chi^2} \approx (2\pi)^3\delta(3\boldsymbol{k})\, \frac{\pi^3\Delta A H_0^{10}}{A_-^3A_+^4k^{23}k_0\Lambda^4} \Bigg[& 18kk_0\mathcal{F}_1\cos\left(\frac{k}{k_0}\right) - \\
& - 40320kk_0^5\mathcal{F}_2\cos\left(\frac{2k}{k_0}\right) - 2\mathcal{F}_3\cos\left(\frac{3k}{k_0}\right) - 2\mathcal{F}_4\cos\left(\frac{4k}{k_0}\right) - \\
& - 2\mathcal{F}_5\sin\left(\frac{k}{k_0}\right) - 2\mathcal{F}_6\sin\left(\frac{2k}{k_0}\right) - 2\mathcal{F}_7\sin\left(\frac{3k}{k_0}\right) - \\
& - 2\mathcal{F}_8\sin\left(\frac{4k}{k_0}\right) \Bigg],
\end{aligned}
\tag{B.23}
$$

where

$$\begin{aligned}
\mathcal{F}_1 = {} & 54A_-^4 k_0^4 \big(k^2+k_0^2\big)^2 \left(-135k^6+567k_0^2k^4+1540k_0^4k^2+840k_0^6\right)+ \\
& + 3A_-^2A_+^2k_0^2\big(-675k^{12}-8523k_0^2k^{10}+46339k_0^4k^8+281727k_0^6k^6+484596k_0^8k^4+ \\
& + 347760k_0^{10}k^2+90720k_0^{12}\big)-18A_-^3A_+k_0^2\big(27k^{12}-1521k_0^2k^{10}+3689k_0^4k^8+30521k_0^6k^6+ \\
& + 53844k_0^8k^4+38640k_0^{10}k^2+10080k_0^{12}\big)+A_+^4\big(54k^{14}-2979k_0^2k^{12}+7183k_0^4k^{10}+ \\
& + 54221k_0^6k^8+157743k_0^8k^6+242298k_0^{10}k^4+173880k_0^{12}k^2+45360k_0^{14}\big)- \\
& - 2A_-A_+^3\big(27k^{14}-2745k_0^2k^{12}+851k_0^4k^{10}+71437k_0^6k^8+ \\
& + 295326k_0^8k^6+484596k_0^{10}k^4+347760k_0^{12}k^2+90720k_0^{14}\big)\,,
\end{aligned} \tag{B.24}$$

$$\begin{aligned}
\mathcal{F}_2 = {} & -27A_-^4k_0^4\big(k^2+k_0^2\big)^3+9A_-^3A_+k_0^2\big(k^2+k_0^2\big)^2\big(-k^4+13k_0^2k^2+12k_0^4\big)- \\
& - 3A_-^2A_+^2k_0^2\big(-7k^8+47k_0^2k^6+171k_0^4k^4+171k_0^6k^2+54k_0^8\big)+A_+^4\big(2k^{10}+k_0^2k^8- \\
& - 24k_0^4k^6-90k_0^6k^4-90k_0^8k^2-27k_0^{10}\big)+A_-A_+^3\big(-2k^{10}-13k_0^2k^8+ \\
& + 93k_0^4k^6+351k_0^6k^4+351k_0^8k^2+108k_0^{10}\big)\,,
\end{aligned} \tag{B.25}$$

$$\begin{aligned}
\mathcal{F}_3 = {} & -1926A_+^4k_0k^{15}+1458A_-A_+^3k_0k^{15}+38301A_+^4k_0^3k^{13}-25758A_-A_+^3k_0^3k^{13}- \\
& - 25029A_-^2A_+^2k_0^3k^{13}+13122A_-^3A_+k_0^3k^{13}+56862A_-^4k_0^5k^{11}-48681A_+^4k_0^5k^{11}- \\
& - 299214A_-A_+^3k_0^5k^{11}+662823A_-^2A_+^2k_0^5k^{11}-371790A_-^3A_+k_0^5k^{11}-427194A_-^4k_0^7k^9- \\
& - 526743A_+^4k_0^7k^9+2144070A_-A_+^3k_0^7k^9-3095739A_-^2A_+^2k_0^7k^9+1905606A_-^3A_+k_0^7k^9- \\
& - 820854A_-^4k_0^9k^7-816561A_+^4k_0^9k^7+3629124A_-A_+^3k_0^9k^7-5629419A_-^2A_+^2k_0^9k^7+ \\
& + 3637710A_-^3A_+k_0^9k^7+275562A_-^4k_0^{11}k^5+275562A_+^4k_0^{11}k^5-1102248A_-A_+^3k_0^{11}k^5+ \\
& + 1653372A_-^2A_+^2k_0^{11}k^5-1102248A_-^3A_+k_0^{11}k^5+1020600A_-^4k_0^{13}k^3+1020600A_+^4k_0^{13}k^3- \\
& - 4082400A_-A_+^3k_0^{13}k^3+6123600A_-^2A_+^2k_0^{13}k^3-4082400A_-^3A_+k_0^{13}k^3+408240A_-^4k_0^{15}k+ \\
& + 408240A_+^4k_0^{15}k-1632960A_-A_+^3k_0^{15}k+2449440A_-^2A_+^2k_0^{15}k-1632960A_-^3A_+k_0^{15}k\,,
\end{aligned} \tag{B.26}$$

$$\begin{aligned}
\mathcal{F}_4 = {} & 423360A_+^4k^9k_0^7-1028160A_-A_+^3k^9k_0^7+786240A_-^2A_+^2k^9k_0^7-181440A_-^3A_+k^9k_0^7- \\
& - 544320A_-^4k^7k_0^9-1693440A_+^4k^7k_0^9+5382720A_-A_+^3k^7k_0^9-6229440A_-^2A_+^2k^7k_0^9+ \\
& + 3084480A_-^3A_+k^7k_0^9-544320A_-^4k^5k_0^{11}-1451520A_+^4k^5k_0^{11}+4898880A_-A_+^3k^5k_0^{11}- \\
& - 5987520A_-^2A_+^2k^5k_0^{11}+3084480A_-^3A_+k^5k_0^{11}+544320A_-^4k^3k_0^{13}+725760A_+^4k^3k_0^{13}- \\
& - 2721600A_-A_+^3k^3k_0^{13}+3810240A_-^2A_+^2k^3k_0^{13}-2358720A_-^3A_+k^3k_0^{13}+ \\
& + 544320A_-^4kk_0^{15}+544320A_+^4kk_0^{15}-2177280A_-A_+^3kk_0^{15}+3265920A_-^2A_+^2kk_0^{15}- \\
& - 2177280A_-^3A_+kk_0^{15}\,,
\end{aligned} \tag{B.27}$$

$$
\begin{aligned}
\mathcal{F}_5 = {} & 6237A_+^4k^{14}k_0^2 - 11340A_-A_+^3k^{14}k_0^2 + 5103A_-^2A_+^2k^{14}k_0^2 + 13122A_-^4k^{12}k_0^4 - \\
& - 62964A_+^4k^{12}k_0^4 + 97470A_-A_+^3k^{12}k_0^4 + 9234A_-^2A_+^2k^{12}k_0^4 - 56862A_-^3A_+k^{12}k_0^4 - \\
& - 148716A_-^4k^{10}k_0^6 - 134586A_+^4k^{10}k_0^6 + 551520A_-A_+^3k^{10}k_0^6 - 847044A_-^2A_+^2k^{10}k_0^6 + \\
& + 578826A_-^3A_+k^{10}k_0^6 - 319788A_-^4k^8k_0^8 - 37098A_+^4k^8k_0^8 + 693522A_-A_+^3k^8k_0^8 - \\
& - 1595538A_-^2A_+^2k^8k_0^8 + 1258902A_-^3A_+k^8k_0^8 + 471420A_-^4k^6k_0^{10} + 657153A_+^4k^6k_0^{10} - \\
& - 2265732A_-A_+^3k^6k_0^{10} + 3031425A_-^2A_+^2k^6k_0^{10} - 1894266A_-^3A_+k^6k_0^{10} + \\
& + 1649970A_-^4k^4k_0^{12} + 1649970A_+^4k^4k_0^{12} - 6599880A_-A_+^3k^4k_0^{12} + 9899820A_-^2A_+^2k^4k_0^{12} - \\
& - 6599880A_-^3A_+k^4k_0^{12} + 1428840A_-^4k^2k_0^{14} + 1428840A_+^4k^2k_0^{14} - 5715360A_-A_+^3k^2k_0^{14} + \\
& + 8573040A_-^2A_+^2k^2k_0^{14} - 5715360A_-^3A_+k^2k_0^{14} + 408240A_-^4k_0^{16} + 408240A_+^4k_0^{16} - \\
& - 1632960A_-A_+^3k_0^{16} + 2449440A_-^2A_+^2k_0^{16} - 1632960A_-^3A_+k_0^{16}\,,
\end{aligned} \tag{B.28}
$$

$$
\begin{aligned}
\mathcal{F}_6 = {} & -141120A_+^4k^{10}k_0^6 + 201600A_-A_+^3k^{10}k_0^6 - 60480A_-^2A_+^2k^{10}k_0^6 - 272160A_-^4k^8k_0^8 - \\
& - 635040A_+^4k^8k_0^8 + 2177280A_-A_+^3k^8k_0^8 - 2721600A_-^2A_+^2k^8k_0^8 + 1451520A_-^3A_+k^8k_0^8 - \\
& - 544320A_-^4k^6k_0^{10} - 1209600A_+^4k^6k_0^{10} + 4233600A_-A_+^3k^6k_0^{10} - 5382720A_-^2A_+^2k^6k_0^{10} + \\
& + 2903040A_-^3A_+k^6k_0^{10} - 362880A_+^4k^4k_0^{12} + 1088640A_-A_+^3k^4k_0^{12} - 1088640A_-^2A_+^2k^4k_0^{12} + \\
& + 362880A_-^3A_+k^4k_0^{12} + 544320A_-^4k^2k_0^{14} + 544320A_+^4k^2k_0^{14} - 2177280A_-A_+^3k^2k_0^{14} + \\
& + 3265920A_-^2A_+^2k^2k_0^{14} - 2177280A_-^3A_+k^2k_0^{14} + 272160A_-^4k_0^{16} + 272160A_+^4k_0^{16} - \\
& - 1088640A_-A_+^3k_0^{16} + 1632960A_-^2A_+^2k_0^{16} - 1088640A_-^3A_+k_0^{16}\,,
\end{aligned} \tag{B.29}
$$

$$
\begin{aligned}
\mathcal{F}_7 = {} & -108A_+^4k^{16} + 10269A_+^4k_0^2k^{14} - 8748A_-A_+^3k_0^2k^{14} - 729A_-^2A_+^2k_0^2k^{14} + 13122A_-^4k_0^4k^{12} - \\
& - 83804A_+^4k_0^4k^{12} + 14094A_-A_+^3k_0^4k^{12} + 148230A_-^2A_+^2k_0^4k^{12} - 91854A_-^3A_+k_0^4k^{12} - \\
& - 201204A_-^4k_0^6k^{10} - 214542A_+^4k_0^6k^{10} + 1277640A_-A_+^3k_0^6k^{10} - 1994760A_-^2A_+^2k_0^6k^{10} + \\
& + 1132866A_-^3A_+k_0^6k^{10} + 187596A_-^4k_0^8k^8 + 238194A_+^4k_0^8k^8 - 602262A_-A_+^3k_0^8k^8 + \\
& + 677538A_-^2A_+^2k_0^8k^8 - 501066A_-^3A_+k_0^8k^8 + 1371492A_-^4k_0^{10}k^6 + 1370061A_+^4k_0^{10}k^6 - \\
& - 5601204A_-A_+^3k_0^{10}k^6 + 8463717A_-^2A_+^2k_0^{10}k^6 - 5604066A_-^3A_+k_0^{10}k^6 + 1173690A_-^4k_0^{12}k^4 + \\
& + 1173690A_+^4k_0^{12}k^4 - 4694760A_-A_+^3k_0^{12}k^4 + 7042140A_-^2A_+^2k_0^{12}k^4 - \\
& - 4694760A_-^3A_+k_0^{12}k^4 + 68040A_-^4k_0^{14}k^2 + 68040A_+^4k_0^{14}k^2 - 272160A_-A_+^3k_0^{14}k^2 + \\
& + 408240A_-^2A_+^2k_0^{14}k^2 - 272160A_-^3A_+k_0^{14}k^2 - 136080A_-^4k_0^{16} - 136080A_+^4k_0^{16} + \\
& + 544320A_-A_+^3k_0^{16} - 816480A_-^2A_+^2k_0^{16} + 544320A_-^3A_+k_0^{16}\,,
\end{aligned} \tag{B.30}
$$

$$\begin{aligned}\mathcal{F}_8 = {} & 60480A_+^4k^{10}k_0^6 - 120960A_-A_+^3k^{10}k_0^6 + 60480A_-^2A_+^2k^{10}k_0^6 - 136080A_-^4k^8k_0^8 - \\ & - 1224720A_+^4k^8k_0^8 + 3447360A_-A_+^3k^8k_0^8 - 3356640A_-^2A_+^2k^8k_0^8 + 1270080A_-^3A_+k^8k_0^8 + \\ & + 544320A_-^4k^6k_0^{10} + 604800A_+^4k^6k_0^{10} - 2298240A_-A_+^3k^6k_0^{10} + 3326400A_-^2A_+^2k^6k_0^{10} - \\ & - 2177280A_-^3A_+k^6k_0^{10} + 1360800A_-^4k^4k_0^{12} + 2086560A_+^4k^4k_0^{12} - 7620480A_-A_+^3k^4k_0^{12} + \\ & + 10342080A_-^2A_+^2k^4k_0^{12} - 6168960A_-^3A_+k^4k_0^{12} + 544320A_-^4k^2k_0^{14} + 544320A_+^4k^2k_0^{14} - \\ & - 2177280A_-A_+^3k^2k_0^{14} + 3265920A_-^2A_+^2k^2k_0^{14} - 2177280A_-^3A_+k^2k_0^{14} - 136080A_-^4k_0^{16} - \\ & - 136080A_+^4k_0^{16} + 544320A_-A_+^3k_0^{16} - 816480A_-^2A_+^2k_0^{16} + 544320A_-^3A_+k_0^{16}\,.\end{aligned} \tag{B.31}$$

Conclusion

In this thesis we have studied the bispectrum of curvature perturbations in an archetypal model of inflation with violation of the slow-roll condition [2], and its extensions to non-standard kinetic terms, which are fully original results of this thesis.

The study of the bispectrum in the original model [2] which includes a discontinuity in the first derivative of a linear potential has been chosen in view of many considerations and we mention here the two most important ones:

1. This model allows fully analytical calculation of the spectrum and bispectrum of curvature perturbations, whose accuracy in reproducing the exact (numerical) results is remarkable.

2. Although not preferred at a statistical significant level, this model (and others which also include features in the primordial curvature perturbations) provides a fit to the most recent Planck data for the CMB temperature and polarization anisotropies power spectrum [1] which is better than the one obtained with the simplest slow-roll inflationary models. This better fit is due to a combined effect of a lower amplitude for the spectrum for scales larger the one corresponding to the change in the derivative of the linear potential, followed by oscillations.

It is known that features in the primordial power spectrum are generated either by sudden changes in the potential and by sudden changes in the speed of sound, which is not trivial in models with a non-standard kinetic term, and that these two effects are degenerate in the power spectrum. The main motivation of this work is to understand how these sudden changes in the potential and the speed of sound of inflaton fluctuations can be disantangled in the bispectrum. As a class of models potentially amenable of analytical calculations, we have therefore considered extensions of the original model introduced by Starobinsky.
It is worthwhile to mention that features in the power spectrum are also generated by a regime of fast roll before slow-roll inflation [51]. However, this latter model produces a modification in the power spectrum which is less preferred by CMB data with respect to the model studied here [1].

In this work, we have first re-computed the bispectrum of curvature perturbations in the original model following [3], verifying that the results in the equilateral limit in the literature are obtained by considering only the varying part of the second slow-roll parameter, i.e. ϵ_2.

We then have generalized this model by considering a Lagrangian with a non-standard kinetic term. In particular, we have considered two different generalizations obtained by the addiction of a non-standard kinetic term with a power larger than 1.
The introduction of this new kinetic term has the important consequence to give a non-trivial speed of sound, which in turn leads to a discontinuity in its logarithmic derivative, s. As already said, these generalized models are characterized by degenerate power spectra, in the sense that the same feature for the power spectrum can be obtained both from a jump in ϵ_2 or s, with a suitable choice of the parameters. In order to break this degeneracy, then, we have calculated the curvature bispectrum, even if we have been able to get an analytic expression just for one of the models considered.
By comparing the results obtained for the model in which the non-standard kinetic term is proportional to $(\partial_\mu\phi\partial^\mu\phi)^2$, we have seen that the contribution to the bispectrum due to the vertex proportional to s is always much smaller than that coming from the vertex proportional to ϵ_2. This is due to the fact that for this model the jump in s is always much smaller than the jump in ϵ_2. Thus, in order to obtain comparable curvature bispectra, we should find a model for which the discontinuity in s and ϵ_2 are of the same order. A good candidate to realize this seems to be a model with the Born-Infeld Lagrangian, for which the background evolution seems to suggest that there exists a regime in which we approximately have $\Delta s \approx -\Delta\epsilon_2$. This is a promising direction which we plan to investigate in the future.

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