ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

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Dynamical Evolution of Quantum States: a Geometrical Approach.

Relatore:
Chiar.mo Prof.
Elisa Ercolessi

Presentata da:
Simone Pasquini

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“. . . TO VERONICA
OF COURSE . . . ”

Abstract

Questo lavoro di tesi si inserisce nel recente filone di ricerca che ha lo scopo di studiare le strutture della Meccanica quantistica facendo impiego della geometria differenziale. In particolare, lo scopo della tesi é analizzare la geometria dello spazio degli stati quantistici puri e misti. Dopo aver riportato i risultati noti relativi a questo argomento, vengono calcolati esplicitamente il tensore metrico e la forma simplettica come parte reale e parte immaginaria del tensore di Fisher per le matrici densitá 2×2 e 3×3. Quest'ultimo altro non é che la generalizzazione di uno strumento molto usato in Teoria dell'Informazione: l'Informazione di Fisher. Dal tensore di Fisher si puó ottenere un tensore metrico non solo sulle orbite generate dall'azione del gruppo unitario ma anche su percorsi generati da trasformazioni non unitarie. Questo fatto apre la strada allo studio di tutti i percorsi possibili all'interno dello spazio delle matrici densitá, che in questa tesi viene esplicitato per le matrici 2×2 e affrontato utilizzando il formalismo degli operatori di Kraus. Proprio grazie a questo formalismo viene introdotto il concetto di semigruppo dinamico che riflette la non invertibilitá di evoluzioni non unitarie causate dall'interazione tra il sistema sotto esame e l'ambiente. Viene infine presentato uno schema per intraprendere la stessa analisi sulle matrici densitá 3×3, e messe in evidenza le differenze con il caso 2×2.

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Introduction

This work belongs to the recent attempts to describe Quantum Mechanics using differential geometry. During the last decades the interest in geometrical aspects of Quantum Mechanics arose higher and higher for many reasons. First of all this subject opens fundamental questions and leads to the heart of Quantum Mechanics. For example, one requires the space of quantum states to be a linear space in order to incorporate the superposition principle that characterizes the behaviour of quantum "objects". In fact, this is expressed in one of the Quantum Mechanics postulates. Alternatively, you can consider the postulate on composite systems, whose content has also a geometrical nature: one requires that the states space of the composite system is the tensorial product of the states space of each sub-system.
Second, the most successful quantum theories have a solid geometrical background. This is so in Quantum Field Theory, in gauge field theory in particular. One can simply consider the very first definitions in Quantum Field Theory: we are used to defining fields according to their transformations under the action of the Lorentz group. These kind of definitions are purely geometrical.
Furthermore one of the most fascinating theory of the 20^{th} century, General Relativity by Einstein, is in essence a geometrical theory. Even if, until now, nobody has been able to unify Quantum Theory with General Relativity, several attempts have been made to rephrase Quantum Theory as a Geometrical Theory. Hence, the study of geometrical aspects of Quantum Mechanics could bring new results in this context.
It is also interesting to notice that Quantum Theory reveals its geometrical nature when considering systems with a finite number of (internal) degrees of freedom, as q-bits or more generally q-dits or d-level systems. This is also connected to another important theory of our days: Quantum Information Theory. In fact, in this contest, the q-bit (a two levels Quantum system) plays the main role. Quantum Information Theory has a wide range of applications, from quantum computing to quantum metrology, that is the study of making high-resolution and highly sensitive measurements of physical parameters using Quantum theory to describe the physical systems.

Thus, in this work our first issue will be a geometrical description of the space of quantum states. In fact, it is well known that the Hilbert space is not the most natural space to describe a Quantum system: due to the probabilistic interpretation, a Quantum state is not specified by a vector in the Hilbert space but by a normalized vector. Also one has to eliminate a redundant phase. This leads to the projective Hilbert space that is the space of equivalence classes of vectors that represent the same physical state. Unluckily, the projective Hilbert space is not a linear space and the easiest way to face up with this problem is using tensorial techniques. This problem is presented in Chapter 1, where, following [5], we analyse the space of pure states of a d-level Quantum system. Nevertheless, for practical and theoretical application, the whole space of states that contains both pure and mixed quantum states is more interesting. The whole space of states has a more complex and richer geometrical structure than the space of pure states. To characterize this space we will find a metric tensor and a symplectic form. These can be found, essentially, in two ways.
The first approach, that is presented in Chapter 2, was pointed out in [10]. It consists in seeing the whole space of states as the set of density matrices, and, in turn, the set of density matrices is seen as a convex cone in the space of the Hermitian matrices. These latter form the Lie algebra of the unitary group. Hence we study the action of the Unitary group on the Hermitian matrices analysing the orbits of Hermitian matrices under the co-adjoint action of the unitary group. With this method it is possible to define explicitly a metric tensor and a symplectic form on each unitary orbit. In particular, we evaluate these tensors for a two-level quantum system (i.e q-bit) and for a three-level quantum systems (q-trit).

On the other hand, as was pointed out in [8, 6, 7], the space of density matrices can be studied considering it as a stratified space where each stratum is an orbit obtained by the co-adjoint action of the quotiented unitary group. In this context it is possible to define the so called Fisher Tensor on each unitary orbit, and one can obtain from this a metric tensor and a symplectic form. Quantum Fisher Information is the Quantum counterpart of the classical Fisher information, a tool which is widely used in the field of information and optimization theory. Both classical and Quantum Fisher Information are used to evaluate how much information an observable random variable carries about a parameter on which the variable depends on. Actually, in this work we will study the tensorial generalization of the quantum Fisher Information. In particular in Chapter 3 we have review the theoretical framework while in Chapter 4 we explicitly evaluate the symplectic form and the metric tensor for q-bits and q-trits. We will show that these tensors and the previously calculated ones are essentially the same, up to normalisation constants. Moreover we discuss how the Fisher Tensor allows to define a metric on general paths on the whole space of density matrices and not only on unitary orbits. In particular in Chapter 4 we also evaluate the metric tensor, for the q-bit and q-trits case, for the transversal paths that connect different unitary orbits.

Finally, in the last part of the thesis, we will try to give an interpretation of such transverse direction in term of dynamical evolution. Indeed, it is well known that an unitary transformation describes the dynamical evolution of a closed Quantum system. Thus the unitary orbits have a precise physical meaning: they are the path that a state "draws" in the space of density states when it evolves without interacting with other systems or with the environment. But what is the physical meaning of the transversal path? In Chapter 5 we answer to this question by introducing the superoperator formalism. It turns out that the transversal paths are " drawn" by a state that evolves interacting with other systems or with the environment. Hence, this kind of paths describes the evolution of an open quantum system. In particular, using the Kraus Representation of a superoperator, we are able to show the semi-group structure underlying these "transversal" evolutions. This means that, in general, when an open system evolves there is an arrow on time so that the system cannot be brought back to the initial state. In Chapter 5 we explicitly present all these features for a q-bit, while in Chapter 6 we show how we would like to proceed the study for q-trits. This latter case is more complex than the q-bit one and we start our analysis the space of diagonal density matrices finding out the existence of a precise and regular pattern that could be useful to continue the research.

Chapter 1

The Geometry of Quantum Mechanics

In this chapter we will first recall some basic notions in order to fix the language and the notation that will be employed. We will show that the Hilbert space can be seen as the total space of a principal bundle with base space the projective Hilbert Space and fiber $\mathbb{C}_0$. Moreover, we will point out the isomorphism between the projective Hilbert Space and the rank-one projectors. Finally we will show that the space of rank one-projectors is a Khäler manifold. In this chapter we will follow the articles [5, 10], where all the proofs can be found.

1.1 An alternative description

Dirac's approach to Quantum Mechanics uses Hilbert spaces as fundamental object to start with, as a consequence of the fact that one needs a superposition rule, and hence a linear structure, in order to gain an appropriate description of the interference phenomena that are at the heart of Quantum Mechanics. For what follows it should be noticed that a Complex Hilbert space carries a natural complex structure (naively, multiplication of vectors by the imaginary unit).

Nevertheless, it is well known that a complete measurement in Quantum Mechanics does not provide us with a uniquely defined vector in some Hilbert space, but with a "ray". A ray is an equivalence class of vectors that differ by multiplication through a non-zero complex number. Even if it is possible to fix a normalisation condition, an overall phase remains unobservable. Considering a finite-dimensional Hilbert space $\mathcal{H}$ with $dim\mathcal{H}_{\mathbb{C}} = n$ and quotienting it with respect to both multiplication operation, i.e. multiplication by the norm of a complex number and multiplication by a phase, one obtains a double fibration:

$$\begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathcal{H}_0 \\ & & \downarrow \\ U(1) & \longrightarrow & S^{2n-1} \\ & & \downarrow \\ & & \mathbb{P}(\mathcal{H}) \end{array} \tag{1.1}$$

where $\mathcal{H}_0 = \mathcal{H} - \mathbf{0}$, S^{2n-1} is the 2n-1-dimensional sphere and $\mathbb{P}(\mathcal{H})$ is the projective Hilbert space.

Definition 1.1. The Projective Hilbert Space $\mathbb{P}(\mathcal{H})$ is defined as:

$$\mathbb{P}(\mathcal{H}) = \{[|\psi\rangle] : |\psi\rangle, |\varphi\rangle \in [|\psi\rangle] \Leftrightarrow |\psi\rangle = \lambda |\varphi\rangle, |\psi\rangle, |\varphi\rangle \in \mathcal{H}_0, \lambda \in \mathbb{C}_0\} \tag{1.2}$$

where $[|\psi\rangle]$ denotes the equivalence class to which $|\psi\rangle \in \mathcal{H}$ belongs, and $\mathbb{C}_0 = \mathbb{C} - \{0\}$.

Remark 1. By definition $P(\mathcal{H}) \equiv \mathbb{C}P^{n-1}$ also known as complex projective space.

Remark 2. Notice that in this way $\mathcal{H}$ acquires the structure of a principal fiber bundle, with base space $\mathbb{P}(\mathcal{H})$ and typical fiber $\mathbb{C}_0 = U(1) \times \mathbb{R}_+$.

Definition 1.2. In general a principal fiber bundle is composed by

1. a t-dimensional manifold T, called total space;

2. a m-dimensional manifold M, called base space;

3. a map $\pi : T \mapsto M$, called projection, such that π is surjective and continuous function;

4. a topological space $\mathfrak{F}$, called typical fiber, such that $\mathfrak{F}$ is omeomorphic to all the fibers $\pi^{-1}(m)$ with $m \in M$)

In symbols:

$$\begin{array}{ccc} \mathfrak{F} & \longrightarrow & T \\ & & \downarrow \pi \\ & & M \end{array} \tag{1.3}$$

If the total space is the direct product of two manifolds $T = M \times N$, the projection is simply given by $\pi : T \to M, \pi(m,n) = m$ where $m \in M$ and we have that, for all $n \in N$, $\pi^{-1}(m_0 \in M) = \{(m_0, n), n \in N\}$ is homeomorphic to N, that is the fiber of the point m_0, while M is the base of the product. This structure is said trivial bundle and it is usually represented with this notation:

$$\begin{array}{ccc} N & \longrightarrow & T = M \times N \\ & & \Big\downarrow \pi \\ & & M \end{array} \tag{1.4}$$

The Hermitian structure of $\mathcal{H}$ allows the association of the equivalence class $[|\psi\rangle]$ with the rank-one projector

$$\rho_\psi = \frac{|\psi\rangle \langle\psi|}{\langle\psi|\psi\rangle} \tag{1.5}$$

with the properties:

$$\rho_\psi^\dagger = \rho_\psi; \tag{1.6}$$

$$Tr\rho_\psi = 1; \tag{1.7}$$

$$\rho_\psi^2 = \rho_\psi. \tag{1.8}$$

The space of rank one-projectors is usually denoted as $D_1^1(\mathcal{H})$ and it is easy to understand that we can identify it with $P(\mathcal{H})$ and conclude that a complete measurement will yield a rank-one projector also called *pure state*. Moreover transition probabilities and expectation values of self-adjoint linear operators (associated to the dynamical variables) depend only on the projectors associated with the states. In particular:
if $A = A^\dagger$ the expectation value in the state $|\psi\rangle$ is

$$\langle A\rangle_\psi = \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle} \equiv Tr\{\rho_\psi A\} \tag{1.9}$$

if $|\psi\rangle$ and $|\phi\rangle$ are two states, the normalized transition probability from $|\psi\rangle$ to $|\phi\rangle$ is

$$\frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle} \equiv Tr\{\rho_\psi\rho_\phi\} \tag{1.10}$$

Hence it is clear that the most natural setting for Quantum Mechanics is not the Hilbert space itself but $\mathbb{P}(\mathcal{H})$, or, equivalently, the space $D_1^1(\mathcal{H})$. Nevertheless, the superposition rule remains a fundamental quantum feature. It is possible to show [5] that a superposition of rank-one projectors which yields another rank-one projector is possible, but requires the arbitrary choice of a fiducial projector.
In conclusion, the whole Quantum Mechanics can be formulated using objects

that belong to the Projective Hilbert space. The latter is no more a linear space but it carries a rich manifold structure. In this context, the notion of linear transformation loses meaning and we are led to consider the tensor counterpart of all the Quantum Mechanics features.

1.2 Kähler structure on the space of pure states

1.2.1 $\mathcal{H}_{\mathbb{R}}$ as a Kähler manifold

Let $\mathcal{H}$ an n-dimensional vector space on $\mathbb{C}$. The Geometric approach to Quantum Mechanics is based on considering the "realification" $\mathcal{H}_{\mathbb{R}}$ di $\mathcal{H}$. $\mathcal{H}_{\mathbb{R}}$ coincides with $\mathcal{H}$ as a group (Abelian under addition) but in which only multiplication by real scalars is allowed. Now we are going to build $\mathcal{H}_{\mathbb{R}}$. Let $(e_1, ..., e_n)$ be a basis for $\mathcal{H}$, once a basis has been chosen $\mathcal{H} \equiv \mathbb{C}^n$. Then a basis for $\mathcal{H}_{\mathbb{R}}$ is $(e_1, ...e_n, ie_1, ..., ie_n)$ in this way $\mathcal{H}_{\mathbb{R}} \equiv \mathbb{R}^{2n}$.
If $\mathbf{x} \in \mathbb{C}^n$ then $\mathbf{x} = x^k e_k$ with $x^k = u^k + iv^k$ where $x^k \in \mathbb{C}$ and $u^k, v^k \in \mathbb{R}$. In short $\mathbf{x}$ correspond to the vector in $\mathcal{H}_{\mathbb{R}}$: $\mathbf{u}$+i$\mathbf{v}$ or $(\mathbf{u}, \mathbf{v})$ where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.
Let $A : H \to H$ be a linear operator, we could build its *realified* counterpart: $A_{\mathbb{R}} : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$.
We are looking for $A_{\mathbb{R}}$ that coincides pointwise with A: if $A\mathbf{x} = \mathbf{x'}$
with $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ and $\mathbf{x'} = \mathbf{u'} + i\mathbf{v'}$, then $A_{\mathbb{R}}(\mathbf{u}, \mathbf{v}) = (\mathbf{u'}, \mathbf{v'})$.
So we can represent A with a matrix of the form: $A = \alpha + i\beta$ where α and β are $n \times n$ real matrices and $A_{\mathbb{R}}$ with a $2n \times 2n$ real matrix:

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \tag{1.11}$$

The multiplication for the imaginary unit in $\mathcal{H}$ will be reproduced in $\mathcal{H}_{\mathbb{R}}$ defining a linear operator J. If $-i\mathbf{x} = -i(\mathbf{u} + i\mathbf{v}) = -i\mathbf{u} + \mathbf{v}$
then in short:

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{v} \\ \mathbf{-u} \end{bmatrix} \tag{1.12}$$

moreover

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \alpha\mathbf{u} - \beta\mathbf{v} \\ \beta\mathbf{u} \ + \alpha\mathbf{v} \end{bmatrix} \tag{1.13}$$

It is trivial understand that $\alpha = 0$ and $\beta = -1$; hence:

$$J = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix} \tag{1.14}$$

with the property $J^2 = -\mathbb{I}_{2n\times 2n}$ In order to prove that the projective Hilbert Space is a Kähler manifold we have to introduce some definitions.

Definition 1.3 (Complex Manifold)**.** A manifold Z that can be mapped on $\mathbb{C}^n$ and with analytic diffeomorphism as compatibility condition between charts is called to be a Complex Manifold. Then, on the tangent bundle TZ we can define the *complex structure* $J_0 : TZ \to TZ$ such that:

$$\forall v \in TZ : J_0(v) = -iv \tag{1.15}$$

and $J_0^2 = -\mathbb{I}$

Definition 1.4 (Kähler Manifold)**.** Let K be a real and even-dimensional manifold with:

- a complex structure J such that $J^2 = -\mathbb{I}$
- a closed, non-degenerate two-form satisfying:

  $$\omega(x, Jy) + \omega(Jx, y) = 0 \tag{1.16}$$

  with $x, y \in TK$. In other words ω is a symplectic structure.
- a positive (0,2)-tensor $g(.,.)$ such that:

  $$g(.,.) =: \omega(., J(.)) \qquad g(x, y) =: \omega(x, Jy) \tag{1.17}$$

  Note that equation (1.16) implies that g is symmetric and non-degenerate iff ω is non-degenerate. In this latter case g is a metric.

In this case K is said Kähler Manifold.

Remark 3.
The property $J^2 = -\mathbb{I}$ implies:

$$\omega(Jx, Jy) = \omega(x, y); \qquad g(Jx, Jy) = g(x, y) \tag{1.18}$$

while the equation (1.17) implies:

$$g(x, Jy) + g(Jx, y) = 0 \tag{1.19}$$

Using equation (1.17) and substituting $y \to Jy$ we obtain:

$$\omega = -g(., J(.)) \tag{1.20}$$

This equation permits to define a Kähler manifold starting from a metric. Moreover, using the same tricks, we could start from g and ω and require that $J = g^{-1} \circ \omega$ is such that $J^2 = -1$. In literature (g, ω, J) is said to be a compatible triple.

Coming back to the complex vector space, if it is endowed with the Hermitian product $\langle x, y \rangle_{\mathcal{H}}$ (being by convention $\mathbb{C}$-linear in y and anti-linear in x), then $\mathcal{H}$ is a complex Hilbert space. Using the Hermitian product we can build up a metric and a symplectic form. Firstly we can separate the real part of the Hermitian product from the imaginary one:

$$h(x, y) = g(x, y) + i\omega(x, y) \tag{1.21}$$

with

$$g(x, y) = Re\, h(x, y) \qquad \text{and} \qquad \omega = Im\, h(x, y)$$

Note that in this case x, y are vectors in $\mathcal{H}$. Being $h(.,.)$ a positive-definite sesquilinear form and non-degenerate, it is easy to show that g is symmetric, positive and non-degenerate and ω is antisymmetric and non-degenerate. Let us consider $\mathcal{H}_{\mathbb{R}}$ with is tangent bundle $T\mathcal{H}_{\mathbb{R}} = \mathcal{H}_{\mathbb{R}} \times \mathcal{H}_{\mathbb{R}}$.

Remark 4. We will make use of the following notation.
Points in $\mathcal{H}_{\mathbb{R}}$ will be denoted with Latin letters, while points in the tangent space $T_x\mathcal{H}_{\mathbb{R}}$ will be denoted with Greek letters. For example $(x, \phi) \in T\mathcal{H}_{\mathbb{R}}$ denotes a point $x \in \mathcal{H}_{\mathbb{R}}$ and a tangent vector in x: $\phi \in T_x\mathcal{H}_{\mathbb{R}} \approx \mathcal{H}_{\mathbb{R}}$.

In order to promote g and ω to (0,2)-tensor fields, we associate with every point $x \in \mathcal{H}$ the constant vector field:

$$X_\psi =: (x, \psi) \tag{1.22}$$

Hence we consider g and ω as (0,2)-tensor fields, defining:

$$g(x)(X_\psi, X_\phi) =: g(\psi, \phi) \quad \text{and} \quad \omega(x)(X_\psi, X_\phi) =: \omega(\psi, \phi) \tag{1.23}$$

in this way g becomes a Riemannian metric and ω a symplectic form. In the same way we can define:

$$J(x)(X_\psi) = (x, J\psi) \tag{1.24}$$

where $J\psi = -i\psi$. Then J has been promoted to a (1,1) tensor field. It is easy to show that g, ω and J satisfy the compatibility conditions encoded in the equations (1.16),(1.17) and (1.18), hence $\mathcal{H}_{\mathbb{R}}$ is a Kähler Manifold, with J as the complex structure. Explicitly, if $(e_1, ...e_n)$ is an orthonormal basis for $\mathcal{H}$, and $x = (u, v)$ and $y = (u', v')$ then

$$g(x, y) = u \cdot u' + v \cdot v' \qquad \text{and} \qquad \omega(x, y) = u \cdot v' - v \cdot u' \tag{1.25}$$

Introducing real coordinates $x^1, ...x^{2n}$ on $H_{\mathbb{R}} \approx \mathbb{R}^{2n}$ we can write:

$$g = g_{ij} dx^i \otimes dx^j \tag{1.26}$$

$$\omega = \omega_{ij} dx^i \wedge dx^j \tag{1.27}$$

$$J = J^i_j dx^j \otimes \frac{\partial}{\partial x^i} \tag{1.28}$$

Hence

$$J^2 = -\mathbb{I} \Leftrightarrow J^j{}_k J^k{}_i = -\delta^i{}_j \tag{1.29}$$

1.2.2 The tensorial approach

We have just proved that $\mathcal{H}_\mathbb{R}$ is a linear Kähler manifold. Because of the non-linearity of $P(\mathcal{H})$ (remeber the definition 1.2) we will use a tensorial description of these structures.
Being g and ω non-degenerate, seen as (0,2)-tensor field on $T\mathcal{H}_\mathbb{R}$, they have their inverses: two controvariant (2,0)-tensors. In particular G, that is a metric tensor, and Ω, that is a Poisson tensor, both map $T^*\mathcal{H}_\mathbb{R}$ to $T\mathcal{H}_\mathbb{R}$, such that:

$$G \circ g = \Omega \circ \omega = \mathbb{I}_{T\mathcal{H}_\mathbb{R}} \tag{1.30}$$

These two controvariant tensors can be used to define a Hermitian product on the cotangent bundle $T^*\mathcal{H}_\mathbb{R} \approx \mathcal{H}^*_\mathbb{R}$, i.e. the dual Hilbert space. Hence: given two one-forms α and β in $\mathcal{H}^*_\mathbb{R}$, we have:

$$\langle \alpha, \beta \rangle_{\mathcal{H}^*_\mathbb{R}} = G(\alpha, \beta) + i\Omega(\alpha, \beta) \tag{1.31}$$

These tensor fields induces two real brackets on smooth, real-valued functions on $H_\mathbb{R}$:

1. the symmetric Jordan bracket $\{f, h\}_g := G(df, dh)$
2. the anti-symmetric Poisson bracket $\{f, h\}_\omega := \Omega(df, dh)$

We can extend these brackets to complex functions by complex linearity and set:

$$\{f, h\}_H = \langle df, dh \rangle_{\mathcal{H}^*_\mathbb{R}} = \{f, h\}_g + i\{f, h\}_\omega \tag{1.32}$$

To make all these structures more explicit we can choose an orthonormal basis$\{e_k\}_{k=1,\dots n}$ in $\mathcal{H}$. This induces global coordinates (q_k, p_k) with $k = 1, ..., n$ on $\mathcal{H}_\mathbb{R}$ by:

$$\langle e_k, x \rangle = (q^k + ip^k)(x), \qquad \forall x \in \mathcal{H} \tag{1.33}$$

Then

$$J = dp^k \otimes \frac{\partial}{\partial q^k} - dq^k \otimes \frac{\partial}{\partial p^k} \tag{1.34}$$

$$g = dq^k \otimes dq^k + dp^k \otimes dp^k \tag{1.35}$$

$$\omega = dq^k \otimes dp^k - dp^k \otimes dq^k \tag{1.36}$$

It is easy to show that:

$$G = \frac{\partial}{\partial q_k} \otimes \frac{\partial}{\partial q_k} + \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial p_k} \tag{1.37}$$

and

$$\Omega = \frac{\partial}{\partial q_k} \otimes \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial q_k} \tag{1.38}$$

In this way the brackets just defined become:

$$\{f, h\}_g = \frac{\partial f}{\partial q_k}\frac{\partial h}{\partial q_k} + \frac{\partial f}{\partial p_k}\frac{\partial h}{\partial p_k} \tag{1.39}$$

$$\{f, h\}_\omega = \frac{\partial f}{\partial q_k}\frac{\partial h}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial h}{\partial q_k} \tag{1.40}$$

Moreover

$$G + i\Omega = \left(\frac{\partial}{\partial q_k} \otimes \frac{\partial}{\partial q_k} + \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial p_k}\right) + i\left(\frac{\partial}{\partial q_k} \otimes \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial q_k}\right) \tag{1.41}$$

Using complex coordinates $z^k = q^k + ip^k$

$$G + i\Omega = 4\frac{\partial}{\partial z^k} \otimes \frac{\partial}{\partial \bar{z}^k} \tag{1.42}$$

and

$$\begin{aligned}\{f, h\}_{\mathcal{H}} =& 2\left(\frac{\partial f}{\partial z^k}\frac{\partial h}{\partial \bar{z}^k} + \frac{\partial h}{\partial z^k}\frac{\partial f}{\partial \bar{z}^k}\right) + i\left[\frac{2}{i}\left(\frac{\partial f}{\partial z^k}\frac{\partial h}{\partial \bar{z}^k} - \frac{\partial h}{\partial z^k}\frac{\partial f}{\partial \bar{z}^k}\right)\right] = \\ =& 4\frac{\partial f}{\partial z^k}\frac{\partial h}{\partial \bar{z}^k}\end{aligned} \tag{1.43}$$

Every complex linear operator $A \in gl(\mathcal{H})$ on $\mathcal{H}$ induces the quadratic function

$$f_A(x) = \frac{1}{2}\langle x|Ax\rangle_{\mathcal{H}} = \frac{1}{2}z^\dagger A z \tag{1.44}$$

Remark 5. Note that f_A is real if and only if A is Hermitian, $A = A^\dagger$

Using the definitions of Jordan and Poisson brackets it is easy to show that

$$\{f_A, f_B\}_g = f_{AB+BA} \quad \text{and} \quad \{f_A, f_B\}_\omega = f_{\frac{AB-BA}{i}} \tag{1.45}$$

where, $A, B \in gl(\mathcal{H})$. So it is clear that the Jordan bracket of two quadratic functions f_A and f_B is related to the Jordan bracket between the operators A and B, that is:

$$[A, B]_+ = AB + BA \tag{1.46}$$

The same thing is true for the Poisson bracket between the quadratic functions that is related to the Poisson bracket between operators:

$$[A, B]_- = \frac{AB - BA}{i} \tag{1.47}$$

Moreover from these relations, it is easy to check that

$$\{f_A, f_B\}_{\mathcal{H}} = 2f_{AB} \tag{1.48}$$

Let us consider real and smooth functions on $\mathcal{H}_{\mathbb{R}}$; we can define $\forall\ f$ on $\mathcal{H}_{\mathbb{R}}$ two vector fields

Definition 1.5 (Gradient vector field)**.** $\forall\ f$, smooth function on $H_{\mathbb{R}}$, the gradient vector field ($grad_f$) associated with f is defined as

$$g(grad_f, .) = df \qquad \text{or} \qquad grad_f = G(., df) \tag{1.49}$$

Definition 1.6 (Hamiltonian vector field)**.** $\forall\ f$, smooth function on $\mathcal{H}_{\mathbb{R}}$, the hamiltonian vector field (Ham_f) associated with f is defined as

$$\omega(Ham_f, .) = df \qquad \text{or} \qquad Ham_f = \Omega(., df) \tag{1.50}$$

So that

$$grad_f = \frac{\partial f}{\partial q^k}\frac{\partial}{\partial q^k} + \frac{\partial f}{\partial p^k}\frac{\partial}{\partial p^k} \quad Ham_f = \frac{\partial f}{\partial p^k}\frac{\partial}{\partial q^k} - \frac{\partial f}{\partial q^k}\frac{\partial}{\partial p^k} \tag{1.51}$$

and then we can rewrite:

$$\{f, h\}_g = g(grad_f, grad_h) \qquad \{f, h\}_\omega = \omega(Ham_f, Ham_h) \tag{1.52}$$

Remark 6.

$$J(grad_f) = Ham_f \tag{1.53}$$

in fact,using equations (1.51), (1.34) we have:

$$J(\frac{\partial f}{\partial q^k}\frac{\partial}{\partial q^k} + \frac{\partial f}{\partial p^k}\frac{\partial}{\partial p^k}) = (dp^t \otimes \frac{\partial}{\partial q^t} - dq^t \otimes \frac{\partial}{\partial p^t})(\frac{\partial f}{\partial q^k}\frac{\partial}{\partial q^k} + \frac{\partial f}{\partial p^k}\frac{\partial}{\partial p^k})$$

Moreover, using the orthonormality conditions between the basis of the coordinate system:

$$dp^k\frac{\partial}{\partial p^t} = dq^k\frac{\partial}{\partial q^t} = \delta^k_t \quad dq^k\frac{\partial}{\partial p^t} = dp^k\frac{\partial}{\partial q^t} = 0 \tag{1.54}$$

we obtain:

$$\frac{\partial f}{\partial p^k}\frac{\partial}{\partial q^k} - \frac{\partial f}{\partial q^k}\frac{\partial}{\partial p^k} = Ham_f$$

□

Turning to linear operators, $\forall A \in gl(\mathcal{H})$ such that $A : \mathcal{H} \to \mathcal{H}$, we can associate:

1. a quadratic function, as explained before

$$f_A(x) = \frac{1}{2}\langle x|Ax\rangle_{\mathcal{H}} = \frac{1}{2}z^\dagger Az \tag{1.55}$$

2. a vector field $\tilde{A}$ such that $\mathcal{H} \to T\mathcal{H}$ via $x \to (x, \psi = Ax)$
3. a (1,1)-tensor field T_A: $T_x\mathcal{H} \ni (x, \psi) \to (x, A\psi)$

Theorem 1.2.1.

$$grad_{f_A} = \tilde{A} \tag{1.56}$$

Proof. From one side we have

$$g(\tilde{A}(z), \psi) = g(Az, \psi) = Reh(Az, \psi) = \frac{1}{2}(\langle Az|\psi\rangle_{\mathcal{H}} + \langle\psi|Az\rangle_{\mathcal{H}}) \tag{1.57}$$

from the other one, using $(.,.)$ as the pairing between vectors and covectors, we have:

$$\begin{aligned}(\psi, df_A) &= \left((\psi^i\frac{\partial}{\partial z^i} + \bar{\psi}^i\frac{\partial}{\partial \bar{z}^i}), (\frac{\partial f_A}{\partial z^k}dz^k + \frac{\partial f_A}{\partial \bar{z}^k}d\bar{z}^k)\right) \\ &= \left((\psi\frac{\partial}{\partial z} + \psi^\dagger\frac{\partial}{\partial z^\dagger}), (\frac{\partial f_A}{\partial z}dz + \frac{\partial f_A}{\partial z^\dagger}dz^\dagger)\right)\end{aligned}$$

Moreover

$$f_A = \frac{1}{2}z^\dagger Az \quad \frac{\partial f_A}{\partial \bar{z}^\dagger} = \frac{1}{2}Az \quad \frac{\partial f_A}{\partial z} = \frac{1}{2}\bar{z}A$$

Then

$$\left((\frac{1}{2}dz^\dagger Az + \frac{1}{2}z^\dagger Adz), (\psi\frac{\partial}{\partial z} + \psi^\dagger\frac{\partial}{\partial z^\dagger})\right) =$$

$$= (\frac{1}{2}\psi^\dagger Az + \frac{1}{2}z^\dagger A\psi) = \frac{1}{2}(\langle Az|\psi\rangle_{\mathcal{H}} + \langle\psi|Az\rangle_{\mathcal{H}})$$

In conclusion we have just proved that

$$g(\tilde{A}(z), \psi) = (\psi, df_A) \qquad \forall\psi \tag{1.58}$$

Looking at the very first definition of $grad_f$ is clear that this latter equality proves the theorem. □

Theorem 1.2.2.

$$Ham_{f_A} = iA \tag{1.59}$$

Proof. Using the compatibilty conditions encoded in (1.18),(1.17) ,(1.16), and the previous result we have:

$$(\psi, df_A) = g(\tilde{A}(z), \psi) = \omega(Az, J\psi) = -\omega(JAz, \psi) = \omega(iAz, \psi) \quad \forall\psi \tag{1.60}$$

This proves the Theorem. □

1.2.3 The momentum map

We consider now the action of the unitary group $\mathrm{U}(\mathcal{H})$ on $\mathcal{H}$. Notice that $\mathrm{U}(\mathcal{H})$ is the group that preserves both the metric g and the symplectic form ω,hence the triple. In what follows we will denote with $u(\mathcal{H})$ the Lie algebra of $\mathrm{U}(\mathcal{H})$ of anti-Hermitian operators and $u^*(\mathcal{H})$ the dual space of $u(\mathcal{H})$. In particular we use the following convention: we will identify the space of Hermitian operators $(A = A^\dagger)$ with the dual $u^*(\mathcal{H})$ of the Lie algebra $u(\mathcal{H})$ of the Unitary group $\mathrm{U}(\mathcal{H})$ on $\mathcal{H}$, according to the pairing between Hermitian operators $A \in u^*(\mathcal{H})$ and anti-Hermitian operators $T \in u(\mathcal{H})$:

$$\langle A, T\rangle = \frac{i}{2}Tr(AT) \tag{1.61}$$

In this way the multiplication by i establishes an isomorphism between $u(\mathcal{H})$ and his dual: $u(\mathcal{H}) \ni T \mapsto iT \in u^*(\mathcal{H})$. This is fundamental since this isomorphism allows us to identify the adjoint action of the group $\mathrm{U}(\mathcal{H})$ with the co-adjoint one. The notions of adjoint and co-adjoint action are very deep and have a lot of applications. An interested reader can find a thorough work on these subjects in [11]. Nevertheless, for our purpose, the following naive and intuitive explanations that can be found in Appendix D, are enough.
The adjoint action of the group $\mathrm{U}(\mathcal{H})$ can be defined as:

$$Ad_U(T) = UTU^\dagger \tag{1.62}$$

where $T \in u(\mathcal{H})$ and $U \in \mathrm{U}(\mathcal{H})$. On the other hand the co-adjoint action of $\mathrm{U}(\mathcal{H})$ is the action on $u^*(\mathcal{H})$; because of the just noticed isomorphism, it is possible to define it like the adjoint action: $Ad_U(A) = UAU^\dagger$ where $A \in u^*(\mathcal{H})$ and $U \in \mathrm{U}(\mathcal{H})$.
Moreover, it is easy to check that $u^*(\mathcal{H})$ is a Lie Algebra endowed with:

$$[A, B]_- = \frac{AB - BA}{i} \tag{1.63}$$

and considering the Jordan Bracket:

$$[A, B]_+ = AB + BA \tag{1.64}$$

we have that: if $A, B \in u^*(\mathcal{H})$ then $[A, B]_+ \in u^*(\mathcal{H})$ as well.
It is possible to equip $u^*(\mathcal{H})$ with an inner product:

$$\langle A, B\rangle_{u^*} = \frac{1}{2} Tr(AB) \tag{1.65}$$

Remark 7. The following relations hold:

$$\langle [A, C]_-, B\rangle_{u^*} = \langle A, [C, B]_-\rangle_{u^*} \quad \langle [A, C]_+, B\rangle_{u^*} = \langle A, [C, B]_+\rangle_{u^*} \tag{1.66}$$

Proof.

$$\begin{aligned}\langle [A, C]_-, B\rangle_{u^*} = \frac{1}{2} Tr([A, C]_- B) = \frac{1}{2i}(Tr(ACB) - Tr(CAB)) = \\ = \frac{1}{2i}(Tr(A(CB - CB))) = \langle A, [C, B]_-\rangle_{u^*}\end{aligned}$$

In the same way it is easy to show the second equality. □

In what follows we use the definitions and properties recalled in Appendix D. With any $A \in u^*(\mathcal{H})$, we can associate the so-called fundamental vector field that corresponds to the element $\frac{1}{i}A \in u(\mathcal{H})$ defined by the formula

$$\frac{d}{dt} e^{-\frac{t}{i}A}(x)|_{t=0} = iA(x) = \phi_{\frac{A}{i}}(x) \qquad \forall x \in \mathcal{H} \tag{1.67}$$

Now, we would like to define the momentum map.

Definition 1.7 (Momentum Map)**.** Let $\mu : \mathcal{H}_{\mathbb{R}} \mapsto u^*(\mathcal{H})$ be a smooth map. For every element in $u(\mathcal{H})$, i.e. $\frac{A}{i}$, we denote with:

$$\mu^{\frac{A}{i}} : \mathcal{H}_{\mathbb{R}} \mapsto \mathbb{R} \qquad \forall A \in u^*(\mathcal{H}) \tag{1.68}$$

the smooth map such that

$$\mu^{\frac{A}{i}}(x) = \langle \mu(x), \frac{A}{i}\rangle \qquad \forall x \in \mathcal{H}_{\mathbb{R}} \tag{1.69}$$

μ is said a momentum map if:

1. The map μ is U-equivariant (Appendix D)

2. $\forall \xi \in u(\mathcal{H})$ (without loss of generality we can assume $\xi = \frac{A}{i}$) the fundamental vector field of $\frac{A}{i}$ is the Hamiltonian vector field of $\mu^{\frac{A}{i}}$, using the definitions we have:
$$d\mu^{\frac{A}{i}} = \omega(\phi_{\frac{A}{i}}, .) \tag{1.70}$$

We already know that the fundamental vector field of $\frac{A}{i}$ is $\phi_{\frac{A}{i}} = iA$, but we also know from (1.59) that $iA = Ham_{f_A}$. Hence:

$$d\mu^{\frac{A}{i}} = d\,\langle \mu, \frac{A}{i} \rangle = \omega(\phi_{\frac{A}{i}}, .) = \omega(Ham_{f_A}, .) = df_A.$$

Finally :
$$\mu^{\frac{A}{i}}(x) = \langle \mu(x), \frac{A}{i} \rangle = f_A(x) = \frac{1}{2}\,\langle x, Ax \rangle_{\mathcal{H}} \tag{1.71}$$

In such a way we have obtained explicitly the *momentum map*; in particular:

$$\langle \mu(x), \frac{A}{i} \rangle = \frac{i}{2} Tr(\mu(x)\frac{A}{i}) = \frac{1}{2} Tr(\mu(x)A) \tag{1.72}$$

Hence $Tr(\mu(x)A) = \langle x, Ax \rangle_H$ and then we can write μ in Dirac notation:

$$\mu(x) = |x\rangle \langle x|\,. \tag{1.73}$$

Moreover with every $A \in u^*(\mathcal{H})$ we can associate the linear function $\hat{A} : u^*(\mathcal{H}) \mapsto \mathbb{R}$ defined as
$$\hat{A} =: \langle A, . \rangle_{u^*} \tag{1.74}$$

After the identification between the tangent space at every point of $u^*(\mathcal{H})$ with $u^*(x)$ itself we can consider $\hat{A}$ as a one-form.
Then, we can define two controvariant tensors, a symmetric tensor:

$$R(\hat{A}, \hat{B})(C) =: \langle C, [A, B]_+ \rangle_{u^*} \tag{1.75}$$

and a Poisson tensor:

$$I(\hat{A}, \hat{B})(C) =: \langle C, [A, B]_- \rangle_{u^*} \tag{1.76}$$

where $A, B, C \in u^*(\mathcal{H})$

Remark 8. We notice that the quadratic function f_A is the pull-back of $\hat{A}$ via the momentum map. In fact:

$$\mu^*(\hat{A})(x) = \hat{A} \circ \mu(x) = \langle A, \mu(x) \rangle_{u^*} = \frac{1}{2} Tr(A\,|x\rangle \langle x|) = \frac{1}{2}\,\langle x, Ax \rangle_H = f_A(x)$$

Moreover if $\xi = \mu(x)$

$$(\mu_* G)(\hat{A}, \hat{B})(\xi) = G(df_A, df_B)(x) = \{f_A, f_B\}_g(x) = f_{[A,B]_+} = R(\hat{A}, \hat{B})(\xi) \tag{1.77}$$

where the last inequality follow from $\{f_A, f_B\}_g = f_{AB+BA}$.
We can do the same with Ω and I. In short we have proved that

$$\mu_* G = R \qquad \mu_* \Omega = I \tag{1.78}$$

Summing up: we have found that there exists a map $\mu : H_{\mathbb{R}} \mapsto u^*(\mathcal{H})$; then we have prove that the push-forward via μ of contravariant tensors, G and Ω defined in $H_{\mathbb{R}}$ correspond to the contravariant tensors, R and I, defined in $u^*(\mathcal{H})$. We also notice that R and I together form a complex tensor related to the Hermitian product on $u^*(\mathcal{H})$

$$(R + iI)(\hat{A}, \hat{B})(\xi) = 2 \langle \xi, AB \rangle_{u^*} \tag{1.79}$$

In conclusion we would like to define two (1,1)-tensors that will be used in the following paragraph. In particular, $\tilde{R}$ and $\tilde{J}$: $Tu^*(H) \mapsto Tu^*(H)$ such that:

$$\tilde{R}_\xi(A) =: [\xi, A]_+ = R(\hat{A}, .)(\xi) \qquad \tilde{j}_\xi(A) =: [\xi, A]_- = I(\hat{A}, .)(\xi) \tag{1.80}$$

Where the last equalities follow from the relations shown in Remark 7.

Remark 9. Actually

$$R(\hat{A}, .)(\xi) = \langle \xi, [A, .]_+ \rangle_{u^*} = \langle [\xi, A]_+, . \rangle_{u^*} \tag{1.81}$$

ant then

$$\tilde{R}_\xi(A) =: \langle [\xi, A]_+, . \rangle_{u^*} \tag{1.82}$$

In short, we write

$$\tilde{R}_\xi(A) =: [\xi, A]_+ \tag{1.83}$$

1.2.4 The space of pure states

We have seen that the image of $\mathcal{H}$ under the momentum map consists of the set of all non-negative Hermitian operators of rank one that we will denote as

$$P^1(\mathcal{H}) = \mu(x) = |x\rangle \langle x| \, ; |x\rangle \in \mathcal{H} - \{0\} \tag{1.84}$$

Notice that $|x\rangle$ is not necessarily normalized. Moreover the co-adjoint action of the unitary group $\mathrm{U}(\mathcal{H})$, as defined before, $(U, \rho) \mapsto U\rho U^\dagger$ where $U \in \mathrm{U}(\mathcal{H})$ and $\rho \in P^1(\mathcal{H})$, foliates $P^1(\mathcal{H})$ into the space

$$D^1_r = |x\rangle \langle x| : \langle x, x \rangle_{\mathcal{H}} = r \tag{1.85}$$

In fact unitary transformations do not change the norm of $|x\rangle$. In particular we have already defined $D_1^1(\mathcal{H})$, the space of pure states, and we have argued that this space, rather than the whole Hilbert space, is enough to develop a quantum description of the system. Now, $D_1^1(\mathcal{H})$ can be seen, in a more formal way, as the image via the momentum map of the unit sphere:

$$S_{\mathcal{H}} = \{x \in \mathcal{H}, \langle x, x\rangle = 1\} \tag{1.86}$$

In what follows we will, finally, show that $D_1^1(\mathcal{H})$ is a Kähler Manifold.
Let $\xi \in u^*(\mathcal{H})$ be the image via momentum map of a unit vector $x \in S_{\mathcal{H}}$, then $\xi = |x\rangle\langle x|$ with $\langle x|x\rangle = 1$ and $\xi^2 = \xi$. The tangent space of the coadjoint $\mathrm{U}(\mathcal{H})$-orbit at ξ is generated by vectors of the form $[A, \xi]_- = \frac{1}{i}(A\xi - \xi A)$.
The Poisson tensor field as defined before is: $I(\hat{A}, \hat{B})(\xi) = \langle \xi, [A, B]_-\rangle_{u^*} = \langle [\xi, A]_-, B\rangle$ and it can be used to define a new invertible map $\tilde{I}$ that associates to any one-form $\hat{A}$ the tangent vector at ξ:

$$\tilde{I}_\xi(\hat{A}) := I(\hat{A}, .)(\xi) = [\xi, A]_- \tag{1.87}$$

Denoting $\tilde{\eta}_\xi$ its inverse such that

$$\tilde{\eta}_\xi([\xi, A]_-) = \hat{A} \tag{1.88}$$

We can define a canonical two-form on $u^*(H)$

$$\eta_\xi([A, \xi]_-, [B, \xi]_-) := (\tilde{\eta}[A, \xi]_-, [B, \xi]_-) = \left(-\hat{A}, [B, \xi]_-\right) = -\langle A, [B, \xi]_-\rangle . \tag{1.89}$$

Using the invariance of the u^*-inner product, it is easy to check that the following equalities hold:

$$\begin{aligned} \eta_\xi([A, \xi]_-, [B, \xi]_-) = \left(-\hat{A}, [B, \xi]_-\right) = -\langle A, [B, \xi]_-\rangle = \\ = \frac{1}{2i} Tr(AB\xi - a\xi B) = \frac{1}{2i} Tr(\xi(AB - BA)) = -\langle \xi, [A, B]_-\rangle_{u^*} = \\ = +\langle A, [\xi, B]_-\rangle_{u^*} = \langle [A, \xi]_-, B\rangle_{u^*} \end{aligned} \tag{1.90}$$

We have proved the following theorem:

Theorem 1.2.3. *The restriction of the two-form η_ξ to the $U(\mathcal{H})$-orbit on $D_1^1(\mathcal{H})$ defines a canonical symplectic form η characterized by the property*

$$\eta_\xi([A, \xi]_-, [B, \xi]_-) = \langle [A, \xi]_-, B\rangle_{u^*} = -\langle \xi, [A, B]_-\rangle_{u^*} \tag{1.91}$$

In the same way we could star from the Jordan tensor $R(\hat{A},\hat{B})(\xi) =: \langle \xi, [A,B]_+ \rangle_{u^*}$ and define a tensor $\tilde{R}(\hat{A}) := R(\hat{A},.) = [\xi, A]_+$ and consider its inverse $\tilde{\sigma}_\xi([\xi,A]_+) = \hat{A}$.
We obtain a covariant tensor:

$$\sigma_\xi = (([\xi,A]_+),[\xi,B]_+) = \langle [\xi,A]_+, B\rangle_{u^*} = \langle \xi, [A,B]_+\rangle_{u^*} \tag{1.92}$$

Nevertheless, at this step σ_ξ is only a "partial" tensor; in fact it is defined only on vectors of the form $[A,\xi]_+$, that are not the most general vectors in the distribution tangent to the U($\mathcal{H}$)−orbit, but they are the vectors that belong to the image of the map $\tilde{R}$. On the other hand we are working on $D_1^1(\mathcal{H})$ where we have some remarkable properties

$$\begin{aligned}[A,\xi]_- = [A,\xi^2]_- = [A,\xi]_- + \xi[A,\xi]_- = \frac{1}{i}(A\xi\xi - \xi A\xi + \xi A\xi - \xi\xi A) = \\ = ([A,\xi]_-\xi + \xi[A,\xi]_-) = [[A,\xi]_-,\xi]_+\end{aligned}$$

So that

$$\sigma_\xi([A,\xi]_-,[B,\xi]_-) = \sigma_\xi\left([[A,\xi]_-,\xi]_+, [[B,\xi]_-,\xi]_+\right) \tag{1.93}$$

And for the definition of σ we have

$$\sigma_\xi([A,\xi]_-,[B,\xi]_-) = \langle \xi, \left[[[A,\xi]_-,\xi]_+, [[B,\xi]_-,\xi]_+\right]_+\rangle$$

With some algebra, using the commutator properties, it is easy but a little tedious to show that

$$\left[[[A,\xi]_-,\xi]_+, [[B,\xi]_-,\xi]_+\right]_+ = [[A,\xi]_-,[B,\xi]]_+$$

Then

$$\sigma_\xi([A,\xi]_-,[B,\xi]_-) = \langle \xi, [[A,\xi]_-,[B,\xi]]_+\rangle$$

Finally using the trace properties

$$\begin{aligned}\langle \xi, [[A,\xi]_-,[B,\xi]]_+\rangle_{u^*} = \frac{1}{2}Tr(\xi[[A,\xi]_-,[B,\xi]]_+) = \frac{1}{2}Tr(\xi[[A,\xi]_-[B,\xi]]_+) = \\ = \langle [A,\xi]_-,[B,\xi]_-\rangle_{u^*}\end{aligned}$$

Hence we have proved:

Theorem 1.2.4. *On the $U(\mathcal{H})$-orbit $D_1^1(\mathcal{H})$ we can define a symmetric covariant tensor σ such that:*

$$\sigma([A,\xi]_-,[B,\xi]_-) = \langle [A,\xi]_-,[B,\xi]_-\rangle_{u^*} \tag{1.94}$$

In order to define a Kähler structure we need also a complex structure. In [10] the detailed proof of the following theorem can be found:

Theorem 1.2.5. *When restricted to $D_1^1(\mathcal{H})$ the (1,1)-tensor $\tilde{I}$, satisfies:*

$$\tilde{I}^3 = -\tilde{I} \tag{1.95}$$

and become invertible.Then: $\tilde{I}^2 = -\mathbb{I}$, and therefore it can be used to define a complex structure j such that the compatibility relations hold:

$$\eta_\xi([A,\xi]_-, j_\xi([B,\xi]_-)) = \sigma_\xi([A,\xi]_-, [B,\xi]_-) \tag{1.96}$$

$$\eta_\xi(j_\xi([A,\xi]_-), j_\xi([B,\xi]_-)) = \sigma_\xi([A,\xi]_-, [B,\xi]_-) \tag{1.97}$$

All together the three previous theorems establish the proof of the following theorem, which is the main result of this paragraph:

Theorem 1.2.6. (D_1^1, j, σ, η)*is a Kähler manifold.*

In conclusion we would like to enunciate as a result the following theorem which has an intuitive meaning:

Theorem 1.2.7. *For any $y, y' \in \mathcal{H}$ the vectors $(\mu_*)_x(y), (\mu_*)_x(y')$ are tangent to the $U(\mathcal{H})$-orbit in $u^*(x)$ at $\xi = \mu(x)$ and*

$$\sigma_\xi((\mu_*)_x(y), (\mu_*)_x(y')) = g(y, y') \tag{1.98}$$

$$\eta_\xi((\mu_*)_x(y), (\mu_*)_x(y')) = -\omega(y, y') \tag{1.99}$$

$$j_\xi(\mu_*)_x(y) = (\mu_*)_x(Jy) \tag{1.100}$$

This last theorem says that the Kähler structure of $D_1^1(\mathcal{H})$ come from the original Kähler manifold structure of $\mathcal{H}_\mathbb{R}$.

Chapter 2

The Space of Mixed States

In this chapter, at first, we will discuss the main features of the space of density states ([13, 14]). Then, we will recollect a geometrical description of density states space given by [10]. In particular we will prove that each unitary-orbit, through the density states space, is a Kähler Manifold. Finally we will explicitly evaluate the metric and symplectic form for the space of density states related to a two and three levels quantum system.

2.1 Density states

In the previous chapter we argued that,in order to describe the state of a quantum system, one should use the projector associated with the vector lying in the Hilbert Space. In fact we can make the following association

$$|\psi\rangle \Longleftrightarrow \rho_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \qquad \forall\psi \in \mathcal{H} \tag{2.1}$$

to avoid phase ambiguities and normalization.
Moreover we have already proved that ρ_ψ is a projector, that is:

$$\rho_\psi = \rho_\psi^2 \qquad \rho_\psi = \rho_\psi^\dagger \tag{2.2}$$

and that:

$$Tr[\rho_\psi] = 1 \tag{2.3}$$

One can easily see that the following properties are true as well:

$$\|\rho_\psi\| = 1 \qquad \langle\xi|\,\rho_\psi\,|\xi\rangle \geq 0 \quad \forall\,|\xi\rangle \in \mathcal{H} \tag{2.4}$$

An operator with the previous five properties is said *pure density operator* or *pure density matrix*. Hence we could show that any **pure** state is in ono-to-one correspondence with a density matrix. We have talked about pure states to distinguish them from another kind of states called **mixed** states. Sometimes it is not possible to know precisely the state of a quantum system a priori but it is only known that the system can be in a given state among a set of states compatible with the boundary conditions. For example you could think about a beam of unpolarized electrons; taken by chance an electron from the beam you only know that this electron has a probability equal to 50% to be spin up ($|+\rangle$) or to be spin down ($|-\rangle$). Naively speaking, while a pure state has an uncertainty because of its quantum nature, a mixed state has in addition a classic uncertainty. For example we can think about two pure states of an electron such as:

$$|e_1\rangle = \frac{\sqrt{3}}{2}|+\rangle + \frac{1}{2}|-\rangle \qquad |e_2\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

In the first case, evaluating the spin of the electron, it will be found a spin up electron with probability $3/4$ and a spin down electron with probability $1/4$, while in the second one it will be found a spin up electron with probability $1/2$ and a spin down electron with probability $1/2$. On the other hand to build a mixed state we have to mix classically some pure states, adding a classic contribution to the uncertainty; for example we can consider a polarized electron beam in which, taking by chance an electron it will be in the state $|e_1\rangle$ with probability $2/5$ and in the state $|e_2\rangle$ with probability $3/5$. How can we describe suitably this situation? Moreover, it is clear that this situation actually occurs very frequently in the experimental environment. Thus it is fundamental to give a rigorous description of mixed states. Let denote the compatible states with $|\psi_k\rangle$ with $k \in I$. We will only deal with a discrete I. We suppose that the system is in the state $|\psi_k\rangle$ with a probability p_k, then obviously:

$$0 \leqslant p_k \leqslant 1 \qquad \sum_{k\in I} p_k = 1 \tag{2.5}$$

Specifying the couple $(|\psi_k\rangle, k)$ for each $k \in I$, we define a *statistical ensemble*. We can use the density matrices formalism and define the density matrices of a mixed state:

$$\rho = \sum_{k\in I} p_k |\psi_k\rangle\langle\psi_k| = \sum_{k\in I} p_k \rho_k \tag{2.6}$$

Clearly a pure state is a mixed state with only one $p_{\bar{k}} \neq 0 \Rightarrow p_{\bar{k}} = 1$. Coming back to the example above we see that the pure states are described by rank-one projectors:

$$\rho_{e_1} = |e_1\rangle\langle e_1| \qquad \rho_{e_2} = |e_2\rangle\langle e_2| \tag{2.7}$$

while the mixed one is given by the density matrix:

$$\rho = \frac{2}{5}\rho_{e_1} + \frac{3}{5}\rho_{e_2}$$

The mixed state density matrix inherits some properties from the pure state density matrix; in particular:

1. ρ is bounded because it is a combination of unitary norm operators
2. $\rho^\dagger = \sum p_k \rho_k^\dagger = \sum p_k \rho_k = \rho$
3. ρ is positive because it is a linear combination, with positive coefficients, of positive operators
4. $Tr[\rho] = \sum p_k Tr[\rho_k] = \sum p_k = 1$

We have lost the idempotence property with respect to the density matrix associated with a pure state. Thanks to this we could prove that:

Theorem 2.1.1. *Given a ρ operator with the properties from 1 to 4, then ρ is associated with a pure state if and only if $\rho^2 = \rho$.*

Moreover

Theorem 2.1.2. *ρ is associated with a pure state iff $Tr[\rho^2] = 1$,otherwise $Tr[\rho^2] \leq 1$.*

It is possible to find other properties and the redefinitions of quantum postulates for the density matrices in a variety of texts, for example in [14].

2.2 Geometry of the space of density states

We are now ready to give a geometrical description of the Density states space. First of all we will define the space of non-negatively operator from $gl(\mathcal{H})$,the space of non-negatively operator from $gl(\mathcal{H})$ with rank k and the set of density states

Definition 2.1. The space of non-negatively defined operators is

$$P(H) = \{\rho | \rho = T^\dagger T \qquad \forall T \in gl(\mathcal{H})\}$$

Note that $\rho \in P(\mathcal{H}) \Rightarrow \rho \in u^*(\mathcal{H})$.

Definition 2.2. The space of non-negatively defined operators with rank k is

$$P^k(\mathcal{H}) = \{\rho|\rho \in P(\mathcal{H}), rank(\rho) = k\}$$

Definition 2.3. The set of density states is

$$D(\mathcal{H}) = \{\rho|\rho \in P(\mathcal{H}), Tr(\rho) = 1\} \tag{2.8}$$

From these definitions we can also define:

$$D^k(\mathcal{H}) = D(\mathcal{H}) \cap P^k(\mathcal{H})$$

Remark 10. It is well known, [2], that the state of density matrices $D(\mathcal{H})$ is a convex cone in $u^*(\mathcal{H})$. In particular, every matrix in $D(\mathcal{H})$ can be written as a convex combination of pure states, then the pure states are the extreme points of $D(\mathcal{H})$. This result will be used later.

2.2.1 Density matrices and unitary orbits

We would like to exploit more deeply the connection between the space of density states $D(\mathcal{H})$ and the co-adjoint orbit of the unitary group. Later in this chapter we will study unitary orbits in the space $u^*(\mathcal{H})$ of hermitian matrices, bigger than $D(\mathcal{H})$, this will allow us to built and calculate explicitly a metric tensor, a symplectic form, and a complex structure. We want now to focus only on the set of density matrices, in fact the co-adjoint action of the unitary group on a density matrix, leaving the trace invariant and, generates orbit that "contains" only density matrices. Nevertheless working only on $D(\mathcal{H})$ we can use other results[8, 17] that we will recollect in what follows. First of all we have the following proposition from linear algebra:

Proposition 2.2.1. *Let ρ_1 and ρ_2 two density matrices on $\mathcal{H} \simeq \mathbb{C}^n$ then: ρ_1 and ρ_2 are unitarily equivalent (i.e. $\rho_2 = U\rho_1 U^\dagger$ for some unitary matrix U) if and only if ρ_1 and ρ_2 have the same spectrum; that is the same eigenvalues including multiplicity. Moreover we have $Tr[(\rho_2)^r] = Tr[(\rho_1)^r]$ for all $r = 1, 2, \ldots, n$*

This has an important consequence: the orbit $O_\rho = \{U\rho U^\dagger | U \in \mathrm{U}(n)\}$ is uniquely determined by the spectrum of ρ. In particular two density matrices belong to the same orbit iff they have the same eigenvalues λ_i with the same multiplicities n_i. Moreover each orbit can be represented by a diagonal density matrix:

$$\rho = diag\{\lambda_1 \mathbb{I}_{n_1}, \ldots, \lambda_r \mathbb{I}_{n_r}\} \tag{2.9}$$

where we can ordinate the λ_i such that $\lambda_i > \lambda_j$ if $i < j$ in order to have an unique representation. Recalling that each $\lambda_i \in [0,1]$ such that $\sum_{i=1}^{r} \lambda_i n_i = 1$, we see that there infinitely many distinct orbits that correspond to the infinite choices for λ_i. We can conclude that $\mathrm{U}(n)$ partitions the set $D(\mathcal{H})$ into a infinite (uncountably) family of orbits or strata. To understand the nature of these strata we have to use the notion of subgroup of isotropy or stabilizer that can be found in the Appendix C, and recall that $\mathrm{U}(n)$, being a compact Lie group, is a compact topological group.

Proposition 2.2.2. *If G is a compact topological group acting on a Hausdorff space X and G_x is the stabilizer at x then the map$\phi : G/G_x \mapsto O_x$ is a homeomorphism.*

Using this preposition is easy to prove [17]

Theorem 2.2.3. *Let $U(n)$ act on $D(\mathcal{H})$ by the co-adjoint action and ρ a density matrix with $r \geq 1$ eigenvalues λ_i with multiplicity n_i, then the orbit of ρ is homeomorphic to the manifold*

$$U(n)/[U(n_1) \times U(n_2) \times \cdots \times U(n_r)] \tag{2.10}$$

of real dimension $n^2 - \sum_{n=1}^{r} n_i^2$.

As consequence of this last theorem we have for a pure state of a n-level quantum system:

$$\mathrm{U}(n)/[\mathrm{U}(1) \times \mathrm{U}(n-1)] = \mathbb{P}(\mathcal{H}) \tag{2.11}$$

We will illustrate these results with explicit applications in the following part of this work.

2.3 Orbits on $u^*(\mathcal{H})$

As anticipated before, we are ready to study the geometry of $u^*(\mathcal{H})$

2.3.1 Smooth manifold on $P^k(\mathcal{H})$

In [10] it is shown that P^k is a smooth manifold in two way:

1. choosing explicitly a coordinates system
2. studying the geometry of $u^*(\mathcal{H})$

For our purpose is enough to focus on the second one only.

Definition 2.4 ($u^*_{k_+,k_-}(\mathcal{H})$)**.** Let us denote with $u^*_{k_+,k_-}(\mathcal{H})$ the set of Hermitian operators ξ whose spectrum contains k_+ positive and k_- negative eigenvalues counted with multiplicities, respectively. Then $rank(\xi) = k = k_- + k_+$.

Note that comparing the definitions of $P^k(\mathcal{H})$ and $u^*_{k_+,k_-}(\mathcal{H})$ we have that $P^k(H) = u^*_{k,0}(\mathcal{H})$. Fixing an orthogonal basis in $\mathcal{H}$ we can identify $u^*_{k_+,k_-}(\mathcal{H})$ with $u^*_{k_+,k_-}(n)$ i.e. $n \times n$ Hermitian matrices of rank k with the corresponding spectrum.
Let $D^{k_+}_{k_-}$ the diagonal matrix $diag(1, ..., 1, -1, ..., -1, 0, ...0)$ with k_+-times 1 and k_--times -1. We can build with $D^{k_+}_{k_-}$ a "semiHermitian" product in $\mathbb{C}^n$, that we denote with $\langle ., . \rangle_{k_+,k_-}$ defining for $a, b \in \mathbb{C}^n$:

$$\langle a, b\rangle = \sum_{i,j=0}^{n} \bar{a}_i [D^{k_+}_{k_-}]^{i,j} b_j = \sum_{j=0}^{k_+} \bar{a}_j b_j - \sum_{j=k_++1}^{k_++k_-} \bar{a}_j b_j \tag{2.12}$$

We can prove the following

Theorem 2.3.1. *Any $\xi = (a_{ij} \in u^*_{k_+,k_-}(n)$ can be written in the form $\xi = T^\dagger D^{k_+}_{k_-} T$ with $T \in GL(n, \mathbb{C})$.*

In other words this theorem means that the matrix element a_{ij} can be obtained using $\langle ., . \rangle_{k_+,k_-}$; in particular $a_{ij} = \langle \alpha_i, \alpha_j \rangle_{k_+,k_-}$ where α_i denotes the $i-$th column of T.
Proof
ξ is hermitian so it can be diagonalized using an unitary matrix U:

$$U\xi U^\dagger = diag(\lambda_1, ..., \lambda_n)$$

where $\lambda_1, ..., \lambda_{k_+} > 0$,$\lambda_{k_++1}, ..., \lambda_{k_++k_-} < 0$ and $\lambda_{k_+k_-+1}, ..., \lambda_n = 0$. Then $\xi = T^\dagger D^{k_+}_{k_-} T$ where $T = CU$ and $C = diag(\sqrt{|\lambda_1|}, ..., \sqrt{|\lambda_{k_++k_-}|}, 1, ..., 1)$,in fact:

$$\xi = U^\dagger C^\dagger D^{k_+}_{k_-} CU \Rightarrow diag(\lambda_1, ..., \lambda_n) = U\xi U^\dagger = C^\dagger D^{k_+}_{k_-} C$$

□

We are now ready to show an important result that allow us to consider $P^k(\mathcal{H})$ a smooth manifold.

Theorem 2.3.2. *The family*

$$\{u^*_{k_+,k_-}(\mathcal{H}) : k_+, k_- > 0, k = k_+ + k_- \leq n\}$$

of subsets of $u^*(\mathcal{H})$ is exactly the family of orbits of the smooth action of the group $GL(H)$ given by:

$$GL(H) \times u^*(\mathcal{H}) \ni (T, \xi) \mapsto T\xi T^\dagger \in u^*(H). \tag{2.13}$$

In particular, every $u^*_{k_+,k_-}(\mathcal{H})$ *is a connected submanifold of* $u^*(H)$ *and the tangent space to* $u^*_{k_+,k_-}(H)$ *at* ξ *is characterized by:*

$$B \in T_\xi u^*_{k_+,k_-}(\mathcal{H}) \Leftrightarrow [\langle Bx, y\rangle_H = 0]\forall x, y \in Ker(\xi).$$

Moreover for $P^k(\mathcal{H})$ we have another result:

Theorem 2.3.3. *the following statements are equivalent:*

1. $u^*_{k_+,k_-}(\mathcal{H})$ *intersect* $P(H)$
2. $u^*_{k_+,k_-}(\mathcal{H})$ *is contained in* $P(H)$
3. $k_- = 0$
4. $u^*_{k_+,k_-}(\mathcal{H}) = P^k(\mathcal{H})$ *with* $k = k_+ + k_-$

Proof. If $u^*_{k_+,k_-}(\mathcal{H})$ intersect $P(\mathcal{H})$, then it contains an element with non-negative spectrum; but $u^*_{k_+,k_-}(\mathcal{H})$ is a $GL(\mathcal{H})$-orbit and we have just proved that the number of negative eigenvalues cannot change along the orbit, then $k_- = 0$. From definition we have $P^k = u^*_{k_+,0}$ when $k_- = 0$ and obviously $P^k(\mathcal{H}) \subseteq P(\mathcal{H})$. □

With this theorem we have proved that P^k is a smooth manifold. Moreover in [10] the authors shows that:

Theorem 2.3.4. *Let* $\gamma : \mathbb{R} \mapsto u^*(n)$ *be a smooth curve in the space of Hermitian matrices which lies entirely in* $P(n)$ *we have:*

$$\gamma(t) \in P^k(n) \qquad \Longrightarrow \qquad \dot{\gamma(t)} \in T_{\gamma(t)}P^k(n) \tag{2.14}$$

This theorem means that smooth curves in $u^*(n)$ which lies entirely in $P(n)$ cannot cross $P^k(n)$ transversally. Moreover recalling that $D(n) = P(n) \cap \{T \in u^*(n) \ : \ Tr[T] = 1\}$, we can see $D(n)$ as the level set of the function $Tr : P(\mathcal{H}) \mapsto \mathbb{R}$ corresponding to the value 1. Since $Tr[t\rho] = tTr[\rho]$ and P^k is invariant with respect to homoteties with $t > 0$, we have that the function Tr is regular on each P^k. Hence also D^k is a smooth manifold. Because we have that topologically $P^k \simeq D^k \times \mathbb{R}$ the manifolds D^k are also connected. Then:

Theorem 2.3.5. *The space* $D^k(\mathcal{H})$ *of density states of rank* $k \leq n$ *are smooth and connected submanifold in* $u^*(\mathcal{H})$*. Moreover the stratification into submanifolds* $D^k(\mathcal{H})$ *is maximal; i.e. every smooth curve in the space of Hermitian matrices which lies entirely in* $D(\mathcal{H})$ *is such that:*

$$\gamma(t) \in D^k(n) \qquad \Longrightarrow \qquad \dot{\gamma(t)} \in T_{\gamma(t)}D^k(n) \tag{2.15}$$

We conclude this section with another result from [10].

Proposition 2.3.6. *The boundary $\partial D((\mathcal{H})) = \cup_{k<n} D^k(\mathcal{H})$ of the density states is not a submanifold of $u^*(\mathcal{H})$ if $n = dim\mathcal{H} > 2$*

We will see example if this result when we will study the space of states for a q-trit.

2.4 Metric and symplectic tensor on U($\mathcal{H}$)-orbits

Now we would like to obtain an explicit formula to evaluate a metric tensor and a symplectic form on the orbits of the co-adjoint action of the unitary group. Let us consider the fundamental vector field in $\xi \in u^*_{k_+,k_-}$ generated by $a \in gl(H)$ with respect to the smooth action defined in the theorem 2.13, that is:

$$\tilde{a}(\xi) = \frac{d}{dt}\left(\tau_{e^{-ta}}\xi\right)|_{t=0} = \frac{d}{dt}\left(e^{-ta}\xi e^{-ta^\dagger}\right)|_{t=0} = -a\xi - \xi a^\dagger$$

This satisfies $\left[\tilde{a}, \tilde{b}\right] = -\widetilde{[a,b]}$ in fact:

$$\left[\tilde{a}, \tilde{b}\right](\xi) = \tilde{a}(\tilde{b}(\xi)) - \tilde{b}(\tilde{a}(\xi)) = \tilde{a}(-b\xi - \xi b^\dagger) - \tilde{b}(-a\xi - \xi a^\dagger) =$$

$$= +ab\xi + a\xi b^\dagger + b\xi a^\dagger + \xi b^\dagger a^\dagger - a \leftrightarrow b = [a,b]\xi + \xi[a,b]^\dagger =$$

$$= -\widetilde{[a,b]}(\xi)$$

In the next step we will prove that the foliations into submanifolds $u^*_{k_+,k_-}(\mathcal{H})$ can be obtained directly from the tensors I and R. We have already shown that the distribution D_I (see Appendix B) induced by the tensor I, defined in the previous chapter, is generated by vector fields of the form $I_A(\xi) = [A, \xi]$. Note that we are talking about distribution because we are considering only the space tangent to an U($\mathcal{H}$)-orbit. We have also proved that the distribution D_R induced by R is generated by vectors of the form $R_A = [A, \xi]_+$. After some simple calculations it is easy to see that:

Theorem 2.4.1. *The family $\{I_A, R_A | A \in u^*(\mathcal{H})\}$ of vector fields on $u^*(\mathcal{H})$ is the family of fundamental vector fields of the $GL(\mathcal{H})$-action:*

$$I_A(\xi) = \frac{1}{i}(A\xi - \xi A) = -(iA)\xi - \xi(iA)^\dagger = \widetilde{iA}(\xi)$$

$$R_A(\xi) = (A\xi + \xi A) = A\xi + \xi A^\dagger = \widetilde{-A}(\xi)$$

In particular,

$$[I_A, I_B] = I_{[A,B]_-} \qquad [R_A, R_B] = I_{[B,A]_-} \qquad [R_A, I_B] = R_{[A,B]_-}$$

so the distribution induced jointly by the tensors I and R is completely integrable and $u^*_{k_+,k_-}(H)$ are the maximal integrate submanifolds.

A corollary descends from this theorem:

Corollary 2.4.2. *The distributions $D_{gl} = D_R + D_I$ and D_I on $u^*(\mathcal{H})$ are involutive and can be integrated in foliations F_{gl} and F_I respectively. The leaves of the foliations F_{gl} are the orbits of the $GL(\mathcal{H})$-action $\xi \mapsto T\xi T^\dagger$, the leaves of F_I are the orbits of the $U(\mathcal{H})$-action.*

Recall that we have defined, in the previous chapter, two $(1,1)$-tensor fields: $\tilde{R}$, $\tilde{J}$:$Tu^*(\mathcal{H}) \mapsto Tu^*(\mathcal{H})$ such that:

$$\tilde{R}_\xi(A) =: [\xi, A]_+ = R_\xi(A) \qquad \tilde{J}_\xi(A) =: [\xi, A]_- = I_\xi(A) \tag{2.16}$$

where $A \in u^*(H) \simeq T_\xi(\mathcal{H})$. Obviously the image of $\tilde{J}$ is D_I and the image of $\tilde{R}$ is D_R.

Remark 11. It should be noticed that:

$$\tilde{J}_\xi \circ \tilde{R}_\xi(A) = [[A,\xi]_+, \xi]_- = [A, \xi^2]_- = [[A,\xi]_- \xi]_+ = \tilde{R}_\xi \circ \tilde{J}_\xi(A)$$

Recall that the $U(\mathcal{H})$-orbits O, as we have shown in the previous chapter, carry a canonical symplectic structure η, this structure is $U(\mathcal{H})$-invariant and then (O, η^O) ia a symplectic manifold. We have already shown that this symplectic structure is a part of a Kähler structure (D^1_1, σ, η, J) and are now ready to generalize this result.

Theorem 2.4.3.
1. *The image of $\tilde{J}_\xi$ is $T_\xi O$, where O is a $U(H)$-orbit, and $Ker(\tilde{J}_\xi)$ is the orthogonal complement of $T_\xi O$*

2. *The $\tilde{J}^2_\xi$ is a self-adjoint (with respect to $\langle .,. \rangle_{u^*}$) and negatively defined operator on $T_\xi O$*

3. *The $(1,1)$-tensor J on $u^*(H)$ defined by*

$$J_\xi(A) = (-(\tilde{J}_\xi)^2|_{T_\xi O})^{-1/2} \tilde{J}_\xi(A) \tag{2.17}$$

induces an U($\mathcal{H}$)-invariant complex structure J on every orbit O

4. The tensor

$$\gamma^O_\xi(A,B) = \eta^O_\xi(A, J_\xi(B)) \tag{2.18}$$

is an $U(H)$-invariant Riemannian metric on O and

$$\gamma^O_\xi(J_\xi(A), B) = \eta^O_\xi(A,B). \tag{2.19}$$

In particular, (O,J,η^O,γ^O) is a Kähler manifold. Moreover, if $\xi \in u^*(\mathcal{H})$ is a projector and $\xi \in O$, then $J_\xi = \tilde{J}_\xi$ and $\gamma^O(A,B) = \langle A, B\rangle_{u^*}$

Proof. 1. Recalling that $\tilde{J} : Tu^*(\mathcal{H}) \mapsto Tu^*(\mathcal{H})$ is such that $\tilde{J}_\xi(A) =: [\xi, A]_- = I_\xi(A)$ where $I_\xi(A)$ are the fundamental vector field of the $U(\mathcal{H})$-action and then they generate the space tangent to the O-orbit, it is easy to see that $T_\xi O$ is the image of $\tilde{J}_\xi$. Moreover:

$$\langle \tilde{J}_\xi(A), B\rangle_{u^*} = \langle [A,\xi]_-, B\rangle_{u^*} = \langle A, -[B,\xi]_-\rangle_{u^*} = -\langle A, \tilde{J}_\xi(B)\rangle_{u^*}$$

then

$$B \in Ker(\tilde{J}_\xi) \Longleftrightarrow B \perp \tilde{J}_\xi(A)$$

$\forall A \in u^*(\mathcal{H})$.
This implies that B is orthogonal to the image of $\tilde{J}_\xi$ that is $T_\xi O$.

2. From the equality $\langle \tilde{J}_\xi(A), B\rangle_{u^*} = -\langle A, \tilde{J}_\xi(B)\rangle_{u^*}$ we have: $\tilde{J}^\dagger_\xi = -\tilde{J}_\xi$. Then:

$$\langle \tilde{J}^2_\xi(A), B\rangle_{u^*} = -\langle \tilde{J}_\xi(A), \tilde{J}_\xi B\rangle_{u^*} = \langle A, \tilde{J}^2_\xi B\rangle_{u^*}$$

This implies that $(\tilde{J}^2_\xi)^\dagger = \tilde{J}^2_\xi$ Moreover:

$$\langle \tilde{J}^2_\xi(A), A\rangle_{u^*} = \langle [[A,\xi],\xi], A\rangle_{u^*} = -\langle [A,\xi],[A,\xi]\rangle_{u^*} < 0$$

for $[A,\xi] \in T_\xi O$ and $[A,\xi] \neq 0$

3. $\tilde{J}$ is $U(H)$-invariant:

$$\begin{aligned}\tilde{J}_{U\xi U^\dagger}(UAU^\dagger) = [UAU^\dagger, U\xi U^\dagger] = UA\xi U^\dagger - U\xi AU^\dagger = \\ = U[A,\xi]U^\dagger = U(\tilde{J}_\xi(A))U^\dagger\end{aligned}$$

Hence, $(-\tilde{J}^2_\xi)^{-\frac{1}{2}}$ is $U(\mathcal{H})$-invariant and its composition J.
The tensor J defines an almost complex structure (cfr. the Appendix A) on every orbit O:

$$J^2 = [(-(\tilde{J})^2)^{-1/2}\tilde{J}]^2 = (-(\tilde{J}^2))^{-1}\tilde{J}^2 = -\mathbb{I}$$

This means that each orbit is an almost complex manifold. Moreover if we show that J is also integrable then each O is a complex manifold. To show that J is integrable we have to consider the distribution N in the complexified tangent bundle $TO \otimes \mathbb{C}$ made of vectors such that:$X \in N \iff J(X) = iX$ and we have to prove that the Lie bracket of two vectors in N is still in N. Because of the $U(\mathcal{H})$-invariance it is sufficient to check it at one point $\xi \in O$. Let $-k_1^2, ... - k_m^2$, where $k_1, ..., k_m > 0$, be the eigenvalues of $(\tilde{J}_\xi^2)$. $\tilde{J}_\xi$ has eigenvalues $\pm ik_k$ with eigenvectors $a_k^\pm$, $k = 1, ..., m$. Hence:

$$J_\xi(a_k^\pm) = (-(\tilde{J}_\xi)^2)^{-1/2}\tilde{J}(a_k^\pm) = \pm ik_k(-(\tilde{J}_\xi(a_k^\pm))^2)^{-1/2} = \pm ia_k^\pm$$

This means that N_ξ is spanned by the vectors a_k^+ that are also eigenvectors of $\tilde{J}_\xi$, that is $\tilde{J}_\xi(a_k^+) = ik_k a_k^+$. Now it is easy to prove that the Lie bracket of two vectors in N is still in N.

$$\begin{aligned}\tilde{J}_\xi([a_k^+, a_l^+]_-) = [[a_k^+, a_l^+]_-, \xi]_- = \\ = \frac{1}{i^2}(a_k^+ a_l^+ \xi - \xi a_k^+ a_l^+ - a_l^+ a_k^+ \xi + \xi a_l^+ a_k^+) = \\ = \frac{1}{i^2}(a_k^+ a_l^+ \xi - \xi a_k^+ a_l^+ - a_l^+ a_k^+ \xi + \xi a_l^+ a_k^+ \\ + a_k^+ \xi a_l^+ - a_k^+ \xi a_l^+ + a_l^+ \xi a_k^+ - a_l^+ \xi a_k^+) = \\ = [[a_k^+, \xi]_-, a_l^+]_- + [a_k^+, [a_l^+, \xi]_-]_- = \\ = [ik_k a_k^+, a_l^+]_- + [a_k^+, ik_l a_l^+] = \\ = i(k_k + k_l)[a_k^+, a_l^+]_-\end{aligned}$$

Then $[a_k^+, a_l^+]_-$ is still in N.

4. The tensor $\gamma_\xi^O(A, B) = \eta_\xi^O(A, J_\xi(B))$ is $U(H)$-invariant because it is a composition of $U(\mathcal{H})$-invariant objects.From $\tilde{J}^\dagger = -\tilde{J}$ and $(\tilde{J}^2)^\dagger = \tilde{J}^2$, we have $J_\xi^\dagger = -J_\xi$. Moreover since J and $\tilde{J}$ commute we have:

$$J_\xi([A, \xi]) = J_\xi \circ \tilde{J}_\xi(A) = \tilde{J}_\xi \circ J_\xi(A) = [J_\xi(A), \xi]$$

Then,recalling the theorem (1.2.3) that is:

$$\eta_\xi([A, \xi]_-, [B, \xi]_-) = \langle [A, \xi]_-, B \rangle_{u^*} = - \langle \xi, [A, B]_- \rangle_{u^*}$$

, we have:

$$\begin{aligned}\eta_\xi^O([A, \xi]_-, J_\xi([B, \xi]_-)) = \eta_\xi^O([A, \xi]_-, [J_\xi(B), \xi]_-) = \\ = \langle [A, \xi]_-, J_\xi(B) \rangle_{u^*} = - \langle J_\xi([A, \xi]_-), B \rangle_{u^*} = - \langle [J_\xi(A), \xi]_-, B \rangle_{u^*} = \\ = -\eta_\xi^O([J_\xi(A), \xi]_-, [B, \xi]_-)\end{aligned}$$

Knowing this it is easy to show that γ is symmetric and proves 2.19.
From theorem 1.2.3 we have also:

$$\gamma^O_\xi([A,\xi]_-,[A,\xi]_-) = \langle [A,\xi]_-, J_\xi(A)\rangle_{u^*} = \langle A, -\tilde{J}_\xi J_\xi(A)\rangle_{u^*}$$

but $-\tilde{J}_\xi J_\xi = (-\tilde{J}^2)^{1/2}$ that is a positive operator so
$\gamma^O_\xi([A,\xi]_-,[A,\xi]_-) > 0$ for $[A,\xi] \neq 0$.
Finally if ξ is a projector,$\xi^2 = \xi$ then, $\tilde{J}^2_\xi = -[a,\xi]$ so $J_\xi = \tilde{J}_x i$ and:

$$\gamma^O_\xi([A,\xi]_-,[B,\xi]_-) = \langle [A,\xi]_-, J_\xi(B)\rangle_{u^*} = \langle [A,\xi]_-,[B,\xi]_-\rangle_{u^*}$$

□

2.5 An explicit choice of local coordinates

Now we will make the above theorem more explicit. Let us consider the case of matrices and suppose that $\xi = diag(\lambda_1, ..., \lambda_n) \in u^*(n)$. For simplicity we can start with $gl(n) = u^*(n)\otimes\mathbb{C}$ equipped with the bracket $[a,b]_- = \frac{ab-ba}{i}$ and with the hermitian product $\langle a,b\rangle_{gl} = \frac{1}{2}Tr(a^\dagger b)$. In this way we can consider $u^*(n)$ a Lie (real) subalgebra in $gl(n)$ with an induced hermitian product $\langle a,b\rangle_{u^*} = \frac{1}{2}Tr(ab)$.
Let E^k_l be the matrix with the element in the kth row and in the lth column equal to 1 and all the other entries equal to 0. We will now list some equalities, but we will give only one explicit calculations because to show the following identities one always uses the same technique.

$$\langle E^k_l, E^r_s\rangle_{gl} = \frac{1}{2}(\delta^l_s\delta^r_k) \tag{2.20}$$

Proof.

$$\langle E^k_l, E^r_s\rangle_{gl} = \frac{1}{2}Tr((E^k_l)^\dagger E^r_s) = \frac{1}{2}Tr(E^l_k E^r_s)$$

Introducing the notation $(E^k_l)^a_b$ to indicate the element of the ath row and bth column of the matrix E^k_l so that:

$$(E^k_l)^a_b = \begin{cases} 0 & \forall a,b \quad with \quad a \neq k \quad or \quad b \neq l \\ 1 & a = k, b = l \end{cases}$$

we can compute:

$$(E^k_l E^r_s)^a_b = \sum_c ((E^k_l)^a_c (E^r_s)^c_b)$$

It is clear that if $a \neq k$ or $b \neq s$ we obtain zero. Then we only have to find the element $(E^k_l E^r_s)^k_s$. Moreover with this notation it is obvious that only if

$c = l = s$ this element is equal to 1 otherwise it is 0. In this way we have proved that the matrix product $(E^k_l E^r_s)$ is a matrix with 1 at the $k$the and sth column if and only if $l = r$. In conclusion:

$$E^k_l E^r_s = E^k_s \delta^r_l$$

Now we can easily compute the trace:

$$\frac{1}{2} Tr(E^l_k E^r_s) = \frac{1}{2} Tr(E^l_s \delta^r_k) = \frac{1}{2} \delta^r_k \delta^l_s$$

□

Following the previous proof it is easy to see that:

$$[E^k_l, E^r_s]_- = \frac{\delta^r_l E^k_s - \delta^k_s E^r_l}{i} \tag{2.21}$$

After noticing that:

$$E^k_l \xi = E^k_l \lambda l \qquad \text{and} \qquad \xi E^k_l = E^k_l \lambda_k$$

we have:

$$\tilde{J}_\xi(E^k_l) = [E^k_l, \xi]_- = i(\lambda_k - \lambda_l) E^k_l \qquad \tilde{J}^2_\xi(E^k_l) = -(\lambda_k - \lambda_l)^2 E^k_l \tag{2.22}$$

and the complex structure reads

$$J_\xi(E^k_l) = i \cdot sgn(\lambda_k - \lambda_l) E^k_l. \tag{2.23}$$

Then the complexified tangent space $T_\xi O \otimes \mathbb{C}$ is spanned by those E^k_l with $(\lambda_k - \lambda_l) \neq 0$, while the distribution N mentioned in the previous proof is spanned by E^k_l for which $(\lambda_k - \lambda_l) > 0$ We are ready to evaluate explicitly the symplectic form (defined only on $T_\xi O \otimes \mathbb{C}$):

$$\eta^O_\xi([E^k_l, \xi]_-, [E^r_s, \xi]_-) = \langle [E^k_l, \xi]_-, E^r_s \rangle_{gl} = \langle i(\lambda_k - \lambda_l) E^k_l, E^r_s \rangle_{gl}$$

Hence:

$$\eta^O_\xi([E^k_l, \xi]_-, [E^r_s, \xi]_-) = \frac{1}{2}(-i)(\lambda_k - \lambda_l)(\delta^l_s \delta^r_k) \tag{2.24}$$

On the other hand:

$$\begin{aligned}\eta^O_\xi([E^k_l, \xi]_-, [E^r_s, \xi]_-) &= \eta^O_\xi(i(\lambda_k - \lambda_l) E^k_l, i(\lambda_r - \lambda_s) E^r_s) = \\ &= -i(\lambda_k - \lambda_l) i(\lambda_r - \lambda_s) \eta^O_\xi(E^k_l, E^r_s)\end{aligned}$$

Comparing this last equation with (2.24) we have the following useful expression :

$$\eta^O_\xi(E^k_l, E^r_s) = \frac{1}{2i(\lambda_r - \lambda_s)}(\delta^l_s \delta^r_k) \tag{2.25}$$

From this we can compute also the metric:

$$\gamma^O_\xi(E^k_l, E^r_s) = \eta^O_\xi(E^k_l, J_\xi(E^r_s)) = \frac{1}{2i|\lambda_r - \lambda_s|}(\delta^l_s \delta^r_k) \tag{2.26}$$

Let us focus on $u^*(n)$, in which we can choose the following basis:

$$A^k_l = E^k_l + E^l_k, \quad k \leq l \qquad B^k_l = iE^k_l - iE^l_k, \quad k < l \tag{2.27}$$

Using the definition 2.27 and the previous identities it very easy to see that

$$J_\xi(A^k_l) = sgn(\lambda_k - \lambda_l)B^k_l \qquad J_\xi(B^k_l) = sgn(\lambda_l - \lambda_k)A^k_l \tag{2.28}$$

Moreover

$$\eta^O_\xi(B^k_l, A^r_s) = -\frac{\delta^k_r \delta^l_s}{(\lambda_k - \lambda_l)}, \qquad \eta^O_\xi(B^k_l, B^r_s) = \eta^O_\xi(A^k_l, A^r_s) = 0 \tag{2.29}$$

with $\lambda_k - \lambda_l$,$\lambda_r - \lambda_s \neq 0$. In a similar way we obtain

$$\gamma^O_\xi(B^k_l, A^r_s) = 0, \qquad \gamma^O_\xi(B^k_l, B^r_s) = \gamma^O_\xi(A^k_l, A^r_s) = \frac{\delta^k_r \delta^l_s}{|\lambda_k - \lambda_l|} \tag{2.30}$$

with $\lambda_k - \lambda_l$,$\lambda_r - \lambda_s \neq 0$. We can also rewrite the simplectic form and the metric:

$$\eta^O_\xi = \sum_{\lambda_k - \lambda_l \neq 0} \frac{1}{\lambda_k - \lambda_l} da^k_l \wedge db^k_l \qquad \gamma^O_\xi = \sum_{\lambda_k - \lambda_l \neq 0} \frac{1}{|\lambda_k - \lambda_l|}(db^k_l \otimes db^k_l + da^k_l \otimes da^k_l) \tag{2.31}$$

where $a \wedge b = ab - ba$ and:

$$b^k_l = \langle B^k_l, .\rangle_{u^*} \qquad A^k_l = \langle A^k_l, .\rangle_{u^*}$$

are coordinates on $u^*(N)$ such that $B^k_l = \partial_{b^k_l}$ and $A^k_l = \partial_{a^k_l}$.

2.5.1 The case U(2)

To make everything more explicit we will perform the calculation for 2×2 density matrices focussing our attention on Hermitian matrices. Every 2×2 Hermitian matrix can be written, in a suitable basis, as:

$$\xi = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{2.32}$$

We start from evaluating E^k_l:

$$E^1_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E^1_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E^1_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E^2_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.33}$$

Then we can evaluate $A^k_l = E^k_l + E^l_k, \quad k \leq l \qquad B^k_l = iE^k_l - iE^l_k, \quad k < l$,that are the basis of the tangent space:

$$A^1_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, A^2_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} A^1_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B^1_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{2.34}$$

We can also evaluate:

$$\eta^O_\xi = \frac{da^1_2 \wedge db^1_2}{\lambda_1 - \lambda_2} = \frac{da^1_2 \otimes db^1_2 - db^1_2 \otimes da^1_2}{\lambda_1 - \lambda_2} \tag{2.35}$$

Finally

$$\gamma^O_\xi = \frac{db^1_2 \otimes db^1_2 + da^1_2 \otimes da^1_2}{|\lambda_1 - \lambda_2|} \tag{2.36}$$

We can also evaluate the complex structure J using (2.28); in particular

$$J(A^1_2) = sgn(\lambda_1 - \lambda_2)B^1_2 \qquad \text{and} \qquad J(B^1_2) = sgn(\lambda_2 - \lambda_1)A^1_2 \tag{2.37}$$

and then we can write:

$$J = sgn(\lambda_1 - \lambda_2)(da^1_2 \otimes \partial_{b^1_2} - db^1_2 \otimes \partial_{a^1_2}) \tag{2.38}$$

2.5.2 The case U(3)

We can repeat the same calculation for 3×3 Hermitian matrices, that we can write as:

$$\xi = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2.39}$$

Also in this case we star by evaluating the basis for the 3×3 Hermitian matrices:

$$\begin{gathered} A^1_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A^2_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A^3_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ A^1_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A^1_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A^2_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \\ B^1_2 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B^1_3 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, B^2_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix} \end{gathered} \tag{2.40}$$

We can also calculate the simplectic form and the metric:

$$\eta_\xi^O = \frac{da_2^1 \wedge db_2^1}{\lambda_1 - \lambda_2} + \frac{da_3^1 \wedge db_3^1}{\lambda_1 - \lambda_3} + \frac{da_3^2 \wedge db_3^2}{\lambda_3 - \lambda_2} \tag{2.41}$$

That is:

$$\eta_\xi^O = \frac{da_2^1 \otimes db_2^1 - db_2^1 \otimes da_2^1}{\lambda_1 - \lambda_2} + \frac{da_3^1 \otimes db_3^1 - db_3^1 \otimes da_3^1}{\lambda_1 - \lambda_3} + \frac{da_3^2 \otimes db_3^2 - db_3^2 \otimes da_3^2}{\lambda_2 - \lambda_3} \tag{2.42}$$

Finally

$$\gamma_\xi^O = \frac{db_2^1 \otimes db_2^1 + da_2^1 \otimes da_2^1}{|\lambda_1 - \lambda_2|} + \frac{db_3^1 \otimes db_3^1 + da_3^1 \otimes da_3^1}{|\lambda_1 - \lambda_3|} + \frac{db_3^2 \otimes db_3^2 + da_3^2 \otimes da_3^2}{|\lambda_2 - \lambda_3|} \tag{2.43}$$

We can also evaluate the complex structure J using (2.28); in particular

$$\begin{array}{lcl} J(A_2^1) = sgn(\lambda_1 - \lambda_2)B_2^1 & \text{and} & J(B_2^1) = sgn(\lambda_2 - \lambda_1)A_2^1 \\ J(A_3^1) = sgn(\lambda_1 - \lambda_3)B_3^1 & \text{and} & J(B_3^1) = sgn(\lambda_3 - \lambda_1)A_3^1 \\ J(A_3^2) = sgn(\lambda_2 - \lambda_3)B_3^2 & \text{and} & J(B_3^2) = sgn(\lambda_3 - \lambda_2)A_3^2 \end{array} \tag{2.44}$$

and then we can write:

$$\begin{aligned} J = sgn(\lambda_1 - \lambda_2)(da_2^1 \otimes \partial_{b_2^1} - db_2^1 \otimes \partial_{a_2^1}) + \\ + sgn(\lambda_1 - \lambda_2)(da_3^1 \otimes \partial_{b_3^1} - db_3^1 \otimes \partial_{a_3^1}) + \\ + sgn(\lambda_2 - \lambda_3)(da_3^2 \otimes \partial_{b_3^2} - db_3^2 \otimes \partial_{a_3^2}) \end{aligned} \tag{2.45}$$

Chapter 3

The Fisher Tensor

*In this chapter we introduce the Fisher tensor starting from a well known instrument from the Quantum Information Theory: **the Fisher Information**. Motivated by the result, shown in [8], that the symmetric and the anti-symmetric part of the Fisher tensor are respectively a metric and a Symplectic form on the Co-adjoint orbit of the unitary group, we will calculate explicitly the Fisher Tensor for 2×2 and 3×3 density matrices. To reach this aim we will face up with the calculation of the Symmetric logarithmic derivative, introducing an "Algebraic method"([7]) to calculate it.*

3.1 Fisher Information

We start considering a generic quantum state $\rho(\theta)$, that could be pure or mixed, and in general depend on a real parameter θ. Let χ be a measurable space with measure dx, which represents the space of all possible outcomes of a measure. We can define a collection of non negative and self-adjoint operators $M(x)$ such that:

$$\int_{\chi} M(x)dx = \mathbb{I}. \tag{3.1}$$

This set of operators defines a Positive-Operator Valued Measure (for a clear introduction on POVM see [16]). Assuming that the outcomes of a measure on ρ is a random variable X taking value on χ, we have that for any measurable subset $B \in \chi$:

$$Pr(X \in B) = tr[\rho M(B)], \qquad \text{where} \qquad M(B) = \int_{B} M(x)dx \tag{3.2}$$

Hence the outcome X of a measure on $\rho(\theta)$ is described by a probability density:

$$p(x, \theta) = Tr[\rho(\theta)m(x)] \tag{3.3}$$

Now we can give some definitions:

Definition 3.1. The Score Function is

$$l_\theta = \log p(x,\theta) \tag{3.4}$$

Definition 3.2. Classical Fisher information is the expected value of the square derivative of the score function, so:

$$i(\theta, M) = E((dl_\theta)^2) = \int_\chi (dl_\theta)^2 p(x,\theta)dx \tag{3.5}$$

where we indicate with d the differential along the curve parametrized by θ

Using the definition above is quite simple to prove that:

$$i(\theta, M) = \int_\chi \frac{(Tr[d\rho(\theta)M(x)])^2}{Tr[\rho(\theta)M(x)]}dx \tag{3.6}$$

In fact:

$$dl_\theta = \frac{d}{d\theta}\log p(x,\theta) = \frac{1}{p(x,\theta)}\frac{dp(x,\theta)}{d\theta} = \frac{Tr[d\rho M]}{Tr[\rho M]}$$

The result follows immediately.

Definition 3.3. The Symmetric logarithmic derivative $d_l\rho$ of ρ is

$$d\rho = \frac{\rho d_l\rho + d_l\rho\rho}{2} \tag{3.7}$$

We can rewrite the classical Fisher information as:

$$i(\theta, M) = \int_{\chi_+} \frac{(ReTr[d\rho(\theta)M(x)])^2}{Tr[\rho(\theta)M(x)]}dx \tag{3.8}$$

where $\chi_+ = \chi - \chi_0$ and $\chi_0 = \{x : p(x,\theta) = 0\}$.

Proof.

$$Tr[d\rho(\theta)M(x)] = \frac{Tr[\rho d_l\rho M(x)] + Tr[d_l\rho\rho M(x)]}{2}$$

Because $Tr[A^T] = Tr[A]$ and then $Tr[A^\dagger] = Tr[A]^*$ we could rewrite $Tr[A^\dagger]^* = Tr[A]$. Hence:

$$Tr[d_l\rho\rho M] = (Tr[(d_l\rho\rho M)^\dagger])^* = Tr[M\rho d_l\rho]^* = Tr[\rho d_l\rho M]^*$$

$$Tr[d\rho(\theta)M(x)] = \frac{Tr[\rho d_l\rho M(x)] + Tr[\rho d_l\rho M(x)]^*}{2} = Re(Tr[\rho d_l\rho M(x)])$$

substituting this result in the integrand numerator, we conclude the proof.

□

One can also define:

Definition 3.4. Quantum Fisher information is the expectation value of the square of Symmetric logarithmic derivative on ρ

$$I_\theta \equiv E((d_l\rho)^2) = Tr[\rho(\theta)(d_l\rho(\theta))^2] \tag{3.9}$$

To be clear we specify that in this case, we implicitly assume a symmetric tensor product, that is:

$$(d_l\rho(\theta))^2 = \frac{d_l\rho(\theta) \otimes d_l\rho(\theta) + d_l\rho(\theta) \otimes d_l\rho(\theta)}{2} = d_l\rho(\theta) \otimes_S d_l\rho(\theta) \tag{3.10}$$

3.1.1 Geometric interpretation of Fisher information

We have already said that a pure state of a quantum system is not simply a vector in the Hilbert space, it would be more correct to think of it as a ray in the Hilbert Space. This led us to identified a $d-$dimensional Hilbert space with the base space of the following double fibration:

$$\begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathcal{H}_0 \\ & & \downarrow \\ U(1) & \longrightarrow & S^{2n-1} \\ & & \downarrow \\ & & \mathbb{P}(\mathcal{H}) \end{array} \tag{3.11}$$

We have done this defining a momentum map $\pi : |\psi\rangle \in \mathcal{H} = \mathbb{C}^d \mapsto \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$. It is a well known result (see for example [2] and [9]) that in the projective Hilbert Space $\mathbb{P}(\mathcal{H})$ one could define a metric g called Fubini Study metric and a compatible simplectic form ω. Together they form an hermitian structure $h = g + i\omega$. The pull back to $\mathcal{H}$ of these tensor via π acquires the form:

$$h = \frac{\langle d\psi|d\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle d\psi|\psi\rangle\,\langle\psi|d\psi\rangle}{\langle\psi|\psi\rangle^2} \tag{3.12}$$

Actually to use a correct notation we should write:

$$h = \frac{\langle d\psi| \otimes |d\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle d\psi|\psi\rangle \otimes \langle\psi|d\psi\rangle}{\langle\psi|\psi\rangle^2} \tag{3.13}$$

In this contest it has been shown ([9]) that the quantum Fisher information, seen as a tensor, can be identified with the symmetric part of the Hermitian form (3.12) that is:

$$h = \frac{\langle d\psi| \otimes_S |d\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle d\psi|\psi\rangle \otimes_S \langle\psi|d\psi\rangle}{\langle\psi|\psi\rangle^2} \tag{3.14}$$

We can prove this result heuristically following [6]. We suppose $\mathcal{H} = \mathbb{C}^d$ and we choose an orthonormal basis $\{|i\rangle\}_{i=1}^d$. To simplify the notation, we will omit the θ dependence, but in the following the symbol d denotes the derivative with respect to it. We can choose the basis such that:

$$\rho = |\psi\rangle\langle\psi| \qquad \text{with} \qquad |\psi\rangle = |1\rangle \tag{3.15}$$

$$|d\psi\rangle = \sum_{i=1}^{d} a_i |i\rangle\,, \tag{3.16}$$

and considering $M(x)$ operators as rank-one projector we can also write

$$M = |\xi\rangle\langle\xi| \qquad \text{with} \qquad |\xi\rangle = \sum_{i=1}^{d} \xi_i |i\rangle \tag{3.17}$$

where the coefficients must satisfy $\int_\chi \xi_i(x)^* \xi_j(x) = \delta_{ij}$. If we note that $\langle\psi|\psi\rangle = 1$ implies that the coefficient $a_1 = \langle d\psi|\psi\rangle$ is pure imaginary and so can be rewrite as $a_1 = ia$ with $a \in \mathbb{R}$. It is not difficult to check that

$$d_l\rho = 2d\rho \tag{3.18}$$

We can proceed in evaluating the quantum Fisher information using the equation (3.18) :

$$I = Tr[\rho(d_l\rho)^2] = 4Tr[\rho(d\rho)^2] = 4[\langle d\psi| \otimes_S |d\psi\rangle - \langle d\psi|\psi\rangle \otimes_S \langle\psi|d\psi\rangle] \tag{3.19}$$

Note that in this formula $d\rho$ indicates a matrix of one-forms. This result can be obtained from a different point of view that will be useful for the following discussion when we will generalize these concepts to mixed states. Let $|\psi_0\rangle \in \mathcal{H}_0 \equiv \mathbb{C}^d - \mathbf{0}$ be a normalized vector that we consider as a reference vector. If one acts on it with the unitary group $\mathrm{U}(d)$, it is possible to reach any point on the unit sphere S^{2d-1}. We also notice that the stabilizing subgroup is isomorphic to $\mathrm{U}(d-1)$. Because of a result from group theory we can write:

$$S^{2d-1} = \frac{U(d)}{U(d-1)} \tag{3.20}$$

If we want obtain the ray space, we have to eliminate the phase freedom, so:

$$\mathbb{P}(\mathcal{H}) = \frac{(\frac{U(d)}{U(d-1)})}{U(1)} \sim \frac{U(d)}{U(d-1) \times U(1)} \tag{3.21}$$

This result can be seen as an explicit application of the theorem 2.2.3: a pure state can be represented by a diagonal matrix with 2 eigenvalues with multiplicity $n_1 = 1$ and $n_2 = d-1$ As we have seen previously, we can consider as the projective space a subset of the dual $u^*(d)$ of the Lie algebra of the unitary group $U(d)$. More specifically, we have seen that the base space of this double fibration is one of the orbit of the co-adjoint action of $U(d)$ on the dual of its algebra $u^*(d)$. In fact we can take $\rho_0 = |\psi_0\rangle \langle\psi_0| \in u^*(d)$ and act upon it with the co-adjoint action of $U(d)$:

$$\rho = U\rho_0 U^\dagger, \qquad U \in U(d) \tag{3.22}$$

Obviously ρ_0 is invariant under the action of the stabilizer subgroup of $|\psi_0\rangle$ and under the action of $U(1)$. We have also shown that a generic tangent vector to the co-adjoint orbit is such that:

$$X \in T_\rho u^* \Rightarrow X = [K, \rho]_- = -i[K, \rho] \tag{3.23}$$

where $K \in u^*(d)$. Moreover if $\rho = |\psi\rangle \langle\psi|$ we have:

$$|\psi\rangle = U|\psi_0\rangle = e^{At}|\psi_0\rangle = |\psi_0\rangle + At|\psi_0\rangle + O(t^2) = |\psi_0\rangle + t|\chi\rangle + O(t^2)$$

but,

$$1 = \langle\psi_0|\psi_0\rangle = \langle\psi_0|U^\dagger U|\psi_0\rangle = \langle\psi|\psi\rangle = \langle\psi_0|\psi_0\rangle = 1 + t\langle\psi_0|\chi\rangle + c.c + O(t^2)$$

hence $\langle\psi_0|\chi\rangle = 0$. Considering the previous result it is easy to calculate:

$$X = |\chi\rangle\langle\psi| + |\psi\rangle\langle\chi| \qquad \text{with} \qquad \langle\psi_0|\chi\rangle = 0 \tag{3.24}$$

Using (3.23) it is easy to check that:

$$K = i(|\chi\rangle\langle\psi| - |\psi\rangle\langle\chi|) \tag{3.25}$$

Given two vectors $X, \tilde{X} \in T_\rho u^*$, identified by the operators $K, \tilde{K}$ determined respectively by the vectors $|\chi\rangle, |\tilde{\chi}\rangle$, it has been proved ([2]) that the metric tensor (the Fubini Study tensor) and the compatible Kostan Kirillov Souriau symplectic form (that is the canonical symplectic form on a co-adjoint orbit), that together form the Hermitian structure $H = g_{fs} + i\omega_{kks}$ on $\mathbb{P}(\mathcal{H})$, are:

$$g_{fs}(X, \tilde{X}) = \frac{1}{2} Tr[\rho[K, \tilde{K}]_+] = Re\langle\chi|\tilde{\chi}\rangle, \tag{3.26}$$

$$\omega_{kks}(X\tilde{X}) = -\frac{1}{2}Tr[\rho[K,\tilde{K}]_-] = Im\,\langle\chi|\tilde{\chi}\rangle \tag{3.27}$$

Noticing that putting $|d\psi\rangle = ia\,|\psi\rangle + |\chi\rangle$ where $\langle\chi|\psi\rangle = 0$ we have $d\rho = X = [K,\rho]_-$ where K is the same as in (3.25). So it is easy to associate the Quantum Fisher information with the real part of the hermitian structure on the Projective Hilbert space. In fact we have:

$$I = Tr[\rho(d\rho)^2] = 2Tr[\rho[K,K]_+]. \tag{3.28}$$

3.1.2 The Fisher tensor

Let us consider again the definition of the quantum Fisher information:

$$I(\theta) = Tr[\rho(d_l\rho)^2] \tag{3.29}$$

where, until now we meant, implicitly, to take the square as a symmetrized tensor product.
Now we would like to generalize this object and relax this restriction in order to compute the full Fisher Tensor:

Definition 3.5. The Fisher Tensor is defined as

$$\mathfrak{F} = Tr[\rho(d_l\rho \otimes d_l\rho)] \tag{3.30}$$

In the next pages we will focus mainly on this tensor or in some strategy to calculate it; for this reason it is important to understand why we are interested in Fisher tensor. The answer is quite simple to understand naively and it is enough for our scope, but more mathematical details can found on [8]. In this last work, the authors proved that the symmetric and the antisymmetric part of the Fisher Tensor are respectively a metric and a symplectic tensor. This will led us to show a quite easy way to evaluate the metric and the symplectic form on the space of density states, and allow us to explore "new directions" of evolution of a state with respect to what has been done in the previous chapter. Specifically, if in the previous chapter we have considered only unitary evolution that corresponds to an evolution of the basis states, keeping the eigenvalues of the density matrix fixed, now we can also study the case of density matrices whose eigenvalues are variable. We will call these directions **traverse directions**.

3.2 The Symmetric Logarithmic One-Form

Since the Fisher tensor contains the Symmetric Logarithm derivative and the latter is defined implicitly by the "ordinary" derivative, we need an explicit

formula to evaluate it. Now we will present a general method (following [7]) that resolves the problem any time the structure costants of the Lie Group are known. Note that, in this section, $d\rho$ denotes a section of the cotangent bundle that is a one form.
Let ρ be a rank-m matrix that represents a mixed state of a quantum n-level system. As we have discussed before we can write this $n \times n$ matrix as:

$$\rho = \sum_{i=1}^{m} k_i P_i \tag{3.31}$$

where:

$$P_i^2 = P_i, \qquad P_i^\dagger = P_i, \qquad \sum_i k_i = 1 \quad \text{with} \quad k_i \geq 0 \quad \forall i$$

As we have shown in Chapter 2, each ρ lies on a orbit $\mathcal{O}_n^{(m)}$, passing from a reference point ρ_0, of the co-adjoint action of the unitary group where:

$$\mathcal{O}_n^{(m)} = \{\rho \in u(n) | \rho = U\rho_0 U^\dagger, \quad U \in U(n)\} \tag{3.32}$$

and ρ_0 is diagonal:

$$\rho_0 = diag_n\{k_1, ..., k_m, 0, ...0\} \tag{3.33}$$

Moreover we can identify each ρ_0 with its expansion on the Lie algebra generators of $\mathfrak{su}(n)$ plus the identity matrix, in fact these form a basis for the Hermitian matrices vector space.

$$\rho = \rho_{\mathbb{I}} \mathbb{I} + \sum_{k=1}^{n^2-1} \rho_k t_k \tag{3.34}$$

where the t_k's are the Lie algebra generators in the n-dimensional fundamental representation, normalized with the usual scalar product between Hermitian matrices, i.e. $\langle t_i, t_j \rangle = \frac{Tr[t_i t_j]}{2} = \delta_{ij}$. Note that ρ_0 has an expansion restricted to the diagonal generator $t_{\hat{k}}$ which are $n-1$, which we suppose to be included in the set of all generators, that is:

$$\rho_0 = \rho_{\mathbb{I}} \mathbb{I} + \sum_{\hat{k}=1}^{n-1} \rho_{\hat{k}} t_{\hat{k}} = \rho_{\mathbb{I}} \mathbb{I} + \sum_{k=1}^{n^2-1} \rho_k t_k \tag{3.35}$$

Because of the identification between $u(n)$ and $u^*(n)$ we have similar expansions for:

$$d\rho = D_{\mathbb{I}} \mathbb{I} + \sum_{k=1}^{n^2-1} D_k t_k \tag{3.36}$$

and

$$d_l\rho = L_{\mathbb{I}}\mathbb{I} + \sum_{k=1}^{n^2-1} L_k t_k \tag{3.37}$$

In which L_i and D_i are one forms.

Remark 12. In the following we will consider only derivatives at ρ_0 because the expressions of $\rho, d\rho, d_l\rho$ can be found ,by adjoint acting on the base point expression given above. We can see this feature studying the tangents vectors that are in 1-1 correspondence with the 1-forms. In particular, considering k_i fixed, as we always assume unless we specify differently, if $\rho_0(\theta) = \sum k_i P_i$ we have:

$$X_{\rho_0} = \partial_\theta \rho_0(\theta) = \sum k_i \partial_\theta P_i(\theta)$$

and then because of the independence of U from θ:

$$X_\rho = \partial_\theta(U\rho(\theta)U^\dagger) = UX_{\rho_0}U^\dagger,$$

where X_{ρ_0} and X_ρ are seen as tangent vector to the curve $\rho(\theta)$.
This argument is valid also for θ-dependent parameters $k_i(\theta)$, in fact

$$X_{\rho_0} = \partial_\theta \rho_0(\theta) = \sum k_i(\theta)\partial_\theta P_i(\theta) + \sum \partial_\theta k_i(\theta) P_i(\theta)$$

Notice that the second term contains only derivative of the scalar coefficients and commute with ρ_0 because they both are diagonal matrices. We will consider this latter case later on, after having developed an explicit way to implement this kind of transformations.

Coming back to the main point and recalling that:

$$d\rho = \frac{1}{2}\{\rho, d_l\rho\} \tag{3.38}$$

we can put in this last equation the expansions (3.34),(3.36) and (3.37). We obtain:

$$d\rho = D_{\mathbb{I}}\mathbb{I} + \sum_{i=1}^{n^2-1} D_i t_i = \frac{1}{2}\{\rho_{\mathbb{I}}\mathbb{I} + \sum_{k=1}^{n^2-1} \rho_k t_k, L_{\mathbb{I}}\mathbb{I} + \sum_{j=1}^{n^2-1} L_j t_j\}.$$

Using the relation $\{t_i, t_j\} = \frac{4}{n}\delta ij + 2\sum_l f_{jkl} t_l$ we have:

$$D_{\mathbb{I}}\mathbb{I} + \sum_{l=1}^{n^2-1} D_l t_l = (L_{\mathbb{I}}\rho_{\mathbb{I}} + \frac{2}{n}\sum_{j=1}^{n^2-1} L_j\rho_j)\mathbb{I} + \sum_{l=1}^{n^2-1}(\rho_{\mathbb{I}}L_l + L_{\mathbb{I}}\rho_l + \sum_{j,k=1}^{n^2-1} L_j\rho_k f_{jkl})t_l. \tag{3.39}$$

This expression yields a set of equations that relate the coefficients of the ordinary and logarithmic derivatives.

$$
\begin{aligned}
D_{\mathbb{I}} &= L_{\mathbb{I}}\rho_{\mathbb{I}} + \frac{2}{n}\sum_{j=1}^{n^2-1} L_j\rho_j \\
D_l &= \rho_{\mathbb{I}}L_l + L_{\mathbb{I}}\rho_l + \sum_{j,k=1}^{n^2-1} L_j\rho_k f_{jkl}
\end{aligned}
\tag{3.40}
$$

Recalling that tangent vectors to co-adjoint orbits are of the form $[K_0,\rho]_- = \frac{K_0\rho-\rho K_0}{i} \in T_\rho u^*(n) \simeq u^*(n)$ with K_0 hermitian and that if $A \in u^*(n)$ then $iA \in u(n) \simeq T^*_\rho u(n)^*$, we can conclude that a one form in general has the form:

$$
d\rho_0 = [K_0,\rho_0], \qquad \text{where} \qquad K_0 = K_{\mathbb{I}}\mathbb{I} + \sum_{l=0}^{n^2-1} K_l t_l. \tag{3.41}
$$

If now we define $[t_i,t_j] = 2i\sum_k c_{ijk}t_k$ where $c_{ijk} = 0$ if t_i, t_j are diagonal and then commuting each other, we can recast $d\rho_0$ as:

$$
d\rho_0 = [\sum_{k=1}^{n^2-1} K_k t_k, \sum_{i=1}^{n^2-1} \rho_i t_i] = \sum_{k,i=1}^{n^2-1} K_k\rho_i[t_k,t_i] = 2i\sum_{k,i=1}^{n^2-1} K_k\rho_i \sum_{\bar{l}} c_{ki\hat{l}}t_{\bar{l}}
$$

where $\bar{l}$ are the indices of the non-diagonal generators. In conclusion we find

$$
d\rho_0 = 2i\sum_{\bar{l},k,i}^{n^2-1} K_k\rho_i c_{ki\bar{l}}t_l \tag{3.42}
$$

i.e. $d\rho_0$ has non vanishing components only on non diagonal generators $t_{\bar{l}}$. This last result means that some equations of the system (3.40) are homogeneous, in particular the equations for the components along the diagonal generators, that read:

$$
\begin{aligned}
D_{\mathbb{I}} &= L_{\mathbb{I}}\rho_{\mathbb{I}} + \frac{2}{n}\sum_{j=1}^{n^2-1} L_j\rho_j = 0 \\
D_{\hat{l}} &= \rho_{\mathbb{I}}L_{\hat{l}} + L_{\mathbb{I}}\rho_{\hat{l}} + \sum_{j,k=1}^{n^2-1} L_j\rho_k f_{jk\hat{l}} = 0
\end{aligned}
\tag{3.43}
$$

We have turn the problem of evaluating the symmetric logarithmic derivative into a linear algebra problem. In order to find the expression of the symmetric logarithmic derivative we have to solve the system 3.40 for n^2 unknown L_j while the homogeneous system 3.43 expresses the freedom in the definition of the symmetric derivative. This result is not surprising, in fact from the definition it is clear that $d_l\rho$ is defined up to matrices that anti-commute with ρ_0. However it is easy to prove that if ρ_0 is full rank there is non trivial Hermitian matrix that anti-commute with it. This means that in the latter case we have an unique solution for $d_l\rho_0$. Moreover the homogeneous system (3.43) contains equations only for the diagonal components $L_{\hat{i}}$ of $d_l\rho_0$. In fact $\rho_j L_j = 0$ if j refers to a non diagonal generator,and $\rho_j = 0$ because ρ_0 is diagonal. Moreover because $f_{\hat{k}j\hat{l}} \propto Tr[\{t_{\hat{k}}, t_j\}t_{\hat{l}}] = 0$ also the term $\rho_{\hat{k}} L_j f_{\hat{k}j\hat{l}} = 0$ whenever L_j is a non diagonal component. So the unique solution of (3.43) is the trivial one and then $d_l\rho_0$ has only non-diagonal components.

3.2.1 Trasverse direction

We note that *a priori* $d\rho$ can be defined in the whole (dual) Lie Algebra rather then on the orbit only, allowing the weights k_i to vary. So we have:

$$d^{tot}\rho = d^T\rho + d\rho \tag{3.44}$$

where we have used $d^T\rho$ to indicate $\sum dk_i P_i$ that is the differential along transverse direction. One may wonder why we are calling these directions transversal. The answer is quite intuitive if we recall the results reported in the previous chapter. There we have explained that the space of the (mixed) density matrix foliates into co-adjoint orbits. Each orbit passes through one and only one diagonal matrix ρ_0. So if change the diagonal elements of ρ we are moving from an orbit to another one. So according to [7] we consider:

$$d^{tot}\rho = d^T\rho + d\rho = \frac{1}{2}\{d_l^T\rho, \rho\} + \frac{1}{2}\{d_l\rho, \rho\} \tag{3.45}$$

where we have introduced the transversal symmetric logarithmic derivative $d_l^T\rho$.

Chapter 4

Fisher Tensor for Q-bits & Q-trits

In the following sections we will work out explicit calculations to evaluate the symmetric logarithmic derivatives and then the Fisher Tensor for two and three levels quantum systems. Moreover we will explore the " new directions" that we have introduced before and we compare our results with the result obtained in the previous chapter.

4.1 Two-level system

For a two-level system or q-bit the most general mixed state can always be expressed as:

$$\rho_0 = k_1 |\psi_1\rangle \langle\psi_1| + k_2 |\psi_2\rangle \langle\psi_2| \qquad \text{or} \qquad \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \tag{4.1}$$

where $|\psi_i\rangle$'s form an ortonormal basis, $k_1 + k_2 = 1$ and $k_i \geq 0 \; \forall i$. The generators of $U(2)$ are the Pauli matrices and the identity matrix, so we can decompose ρ_0 with respect to the basis:

$$\begin{gathered} t_{\mathbb{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, t_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ t_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, t_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{gathered} \tag{4.2}$$

Using the scalar product defined for Hermitian matrices $\langle A, B\rangle_{u^*} = \frac{1}{2}Tr[AB]$ it is easy to find the decomposition of ρ_0 in this basis of matrices; in particular we evaluate the coefficients:

$$\rho_i = \langle t_i, \rho_0 \rangle_{u^*} \tag{4.3}$$

and we obtain

$$\rho_{\mathbb{I}} = \frac{k_1 + k_2}{2}, \quad \rho_1 = 0, \quad \rho_2 = 0, \quad \rho_3 = \frac{k_1 - k_2}{2} \tag{4.4}$$

and then:

$$\rho_0 = \frac{k_1 + k_2}{2} t_{\mathbb{I}} + \frac{k_1 - k_2}{2} t_3. \tag{4.5}$$

Note that, as we have said before, ρ_0 has components only along the "directions" with diagonal generators.

4.1.1 Symmetric logarithmic derivative

The homogeneous system (3.43) is quite easy to solve because the structure constants f_{ijk} vanish and we are left with:

$$\begin{cases} D_{\mathbb{I}} = \rho_{\mathbb{I}} L_{\mathbb{I}} + \rho_3 L_3 = 0 \\ D_3 = \rho_3 L_{\mathbb{I}} + \rho_{\mathbb{I}} L_3 = 0 \end{cases} \tag{4.6}$$

If ρ_0 is a real mixed state that is $k_i \neq 0 \ \forall i$ the system has a unique solution $L_{\mathbb{I}} = L_3 = 0$ and, as discussed above, the symmetric logarithmic form is uniquely defined.
The system 3.40 becomes:

$$D_l = \rho_{\mathbb{I}} L_l + \rho_l L_{\mathbb{I}} = \rho_{\mathbb{I}} L_l \qquad \text{with} \qquad l = 1, 2 \tag{4.7}$$

$$\begin{cases} D_1 = \rho_{\mathbb{I}} L_1 \\ D_2 = \rho_{\mathbb{I}} L_2 \end{cases} \Leftrightarrow \begin{cases} L_1 = \frac{D_1}{\rho_{\mathbb{I}}} \\ L_2 = \frac{D_2}{\rho_{\mathbb{I}}} \end{cases} \tag{4.8}$$

We have just found the expression for the symmetric logarithmic derivative as a function of the coefficients of the standard derivative:

$$d_l \rho_0 = \frac{2}{k_1 + k_2} (D_1 t_1 + D_2 t_2) \tag{4.9}$$

Remark 13. The matrix associated to the homogeneous system is such that:

$$det \begin{bmatrix} \rho_{\mathbb{I}} & \rho_3 \\ \rho_3 & \rho_{\mathbb{I}} \end{bmatrix} = k_1 k_2 \tag{4.10}$$

In this form it is obvious that:

1. if k_1 and k_2 are not null, the system has as solution the trivial one;
2. if ρ_0 is a pure state, for example $k_1 = 1$ and $k_2 = 0$ the system has infinite solutions that depend on one parameter $L_3 = -L_{\mathbb{I}}$.

Before calculating the coefficients D_i we evaluate the Fisher Tensor as function of these.

4.1.2 Fisher tensor

Recall that the coefficients D_i and L_i are one-forms, then:

$$\mathfrak{F} = L_1 \otimes L_1 Tr[\rho_0 t_1 t_1] + L_1 \otimes L_2 Tr[\rho_0 t_1 t_2] + L_2 \otimes L_1 Tr[\rho_0 t_2 t_1] + L_2 \otimes L_2 Tr[\rho_0 t_2 t_2] \quad (4.11)$$

and then:

$$\mathfrak{F} = (k_1 + k_2) L_1 \otimes L_1 + (k_1 + k_2) L_2 \otimes L_2 + i(k_1 - k_2) L_1 \otimes L_2 - i(k_1 - k_2) L_2 \otimes L_1 \quad (4.12)$$

We introduce now $A \odot B = \frac{A \otimes B + B \otimes A}{2}$ and $A \wedge B = \frac{A \otimes B - B \otimes A}{2i}$. Note that in the previous chapter we used $A \wedge B = A \otimes B - B \otimes A$, so we have to remind this notation when we will compare the results. Note also that $A \odot A = A \otimes A$. In this notations we can rewrite $\mathfrak{F}$ as:

$$\mathfrak{F} = (k_1 + k_2)[L_1 \odot L_1 + L_2 \odot L_2] - 2(k_1 - k_2)(L_1 \wedge L_2) \quad (4.13)$$

It is manifest that the first term is symmetric and second one is anti-symmetric in the indices 1,2.

4.1.3 Evaluation of the ordinary differential

To compute the standard differential we have to consider an orbit generated by the co-adjoint action of the unitary group acting on ρ_0, since at the moment we are considering unitary transformations. In general

$$\rho^{'} = U \rho_0 U^{\dagger} \qquad \text{where} \qquad U = exp(ix_{\mathbb{I}} t_{\mathbb{I}} + i \sum_{k=1}^{3} x_k t_k) \quad (4.14)$$

In this way we have introduced a set of coordinates x_i so we calculate the differential using this particular choice of coordinates system. Before starting the evaluation we should note that:

$$d\rho^{'} = d(U(x_{\mathbb{I}}, x_1, x_2, x_3) \rho_0 U^{\dagger}(x_{\mathbb{I}}, x_1, x_2, 0)) = d(U(0, x_1, x_2, 0) \rho_0 U^{\dagger}(0, x_1, x_2, 0))$$

Writing the co-adjoint action in infinitesimal form (expanding the exponentials up to the first order in the x_i's) it is easy to see that the transformations along the $t_{\mathbb{I}}, t_3$ direction disappear, because these matrices are diagonal and commute with ρ_0. So we can put $x_{\mathbb{I}} = 0$ and $x_3 = 0$ and consider only unitary transformations that do not stabilize ρ_0 that is:

$$U = exp(+ix_1 t_1 + ix_2 t_2) \quad (4.15)$$

Then

$$\rho' = exp(+ix_1t_1 + ix_2t_2)\rho_0 exp(-ix_1t_1 - ix_2t_2)$$

and

$$d\rho_0 = \frac{\partial\rho'}{\partial x_1}\bigg|_{x_1=0} dx_1 + \frac{\partial\rho'}{\partial x_2}\bigg|_{x_2=0} dx_2 \tag{4.16}$$

An easy calculation yields:

$$d\rho_0 = \Big[\begin{bmatrix} 0 & ik_2 \\ ik_1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & ik_1 \\ ik_2 & 0 \end{bmatrix}\Big]dx_1 + \Big[\begin{bmatrix} 0 & k_2 \\ -k_1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & k_1 \\ k_2 & 0 \end{bmatrix}\Big]dx_2 \tag{4.17}$$

that is

$$\begin{bmatrix} 0 & i(k_2-k_1)dx_1 + (k_2-k_1)dx_2 \\ i(k_1-k_2)dx_1 + (k_2-k_1)dx_2 & 0 \end{bmatrix}. \tag{4.18}$$

If we use $r = k_2 - k_1$ and change the coordinate such that $z_1^* = x_2 + ix_1$ and $z_1 = x_2 - ix_1$ we obtain:

$$d\rho_0 = \begin{bmatrix} 0 & rdz_1^* \\ rdz_1 & 0 \end{bmatrix}. \tag{4.19}$$

Using one more time the scalar product we can decompose the differential on the basis of hermitian matrix as we did with ρ_0 and we have:

$$D_{\mathbb{I}} = D_3 = 0, \qquad D_1 = r\frac{dz_1 + dz_1^*}{2} \qquad D_2 = r\frac{dz_1 - dz_1^*}{2i} \tag{4.20}$$

Putting these results in (4.8) we obtain:

$$\begin{aligned} L_1 &= \frac{2r}{k_1+k_2}\left(\frac{dz_1 + dz_1^*}{2}\right) = \frac{2r}{k_1+k_2}dx_2, \\ L_2 &= \frac{2r}{k_1+k_2}\left(\frac{dz_1 - dz_1^*}{2i}\right) = -\frac{2r}{k_1+k_2}dx_1 \end{aligned} \tag{4.21}$$

Finally we can write $\mathfrak{F} = \mathfrak{F}^{\odot} + \mathfrak{F}^{\wedge}$

$$\mathfrak{F}^{\odot} = 4\frac{(k_1-k_2)^2}{k_1+k_2}(dx_1 \odot dx_1 + dx_2 \odot dx_2) = 4\frac{(k_1-k_2)^2}{k_1+k_2}(dz_1 \odot dz_1^*) \tag{4.22}$$

and

$$\begin{aligned}\mathfrak{F}^{\wedge} = 8\frac{(k_1-k_2)^3}{(k_1+k_2)^2}(dx_2 \wedge dx_1) = \\ = 8\frac{(k_1-k_2)^3}{(k_1+k_2)^2}(\frac{dx_2\otimes dx_1 - dx_1 \otimes dx_2}{2i}) = \\ = 4i\frac{(k_1-k_2)^3}{(k_1+k_2)^2}(dx_1 \otimes dx_2 - dx_2 \otimes dx_1)\end{aligned} \quad (4.23)$$

Or using z-coordinates we have:

$$\mathfrak{F}^{\wedge} = -4i\frac{(k_1-k_2)^3}{(k_1+k_2)^2}(dz_1 \wedge dz_1^*) = -2\frac{(k_1-k_2)^3}{(k_1+k_2)^2}(dz_1 \otimes dz_1^* - dz_1^* \otimes dz_1) \quad (4.24)$$

4.1.4 Comparison with previous results

Following the results shown in [8, 6], we will consider separately the real part of $\mathfrak{F}$, that is symmetric, and the imaginary part of $\mathfrak{F}$ and we will compare these with what we have found in the previous chapter,changing the eigenvalue from λ_i to k_i, that is:

$$\gamma_\xi^O = \frac{db_2^1 \otimes db_2^1 + da_2^1 \otimes da_2^1}{|k_1-k_2|} \quad (4.25)$$

and

$$\eta_\xi^O = \frac{da_2^1 \wedge db_2^1}{k_1-k_2} = \frac{da_2^1 \otimes db_2^1 - db_2^1 \otimes da_2^1}{k_1-k_2} \quad (4.26)$$

Recall that in these expression we have written the metric tensor and the symplectic tensor using as coordinates the

$$b_l^k = \langle B_l^k, . \rangle_{u^*} \qquad a_l^k = \langle A_l^k, . \rangle_{u^*}$$

where B_l^k and A_l^k were the matrices used as basis for the space of the hermitian matrix $T_\rho u(2)^* \simeq u(2)^*$. On the other hand in this paragraph we have used as basis for the space of hermitian matrix t_i. If we note that

$$A_2^1 = t_1 = \sigma_1, \qquad \text{and} \qquad B_2^1 = -t_2 = -\sigma_2, \quad (4.27)$$

we conclude that

$$a_2^1 = x_1 \qquad \text{and} \qquad b_2^1 = -x_2. \quad (4.28)$$

Then changing the coordinates we can compare:

$$G_{FS} = 4\tfrac{(k_1-k_2)^2}{k_1+k_2}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2) \longrightarrow \gamma_\xi^O = \tfrac{dx_1 \otimes dx_1 + dx_2 \otimes dx_2}{|k_1-k_2|} \quad (4.29)$$

and

$$\Omega_{FS} = 4\frac{(k_1-k_2)^3}{(k_1+k_2)^2}(dx_2 \otimes dx_1 - dx_1 \otimes dx_2) \longrightarrow \eta^O_\xi = \frac{dx_2\otimes dx_1 - dx_1 \otimes dx_2}{k_1-k_2} \quad (4.30)$$

where we have used the notation $-Im(\mathfrak{F}) = \Omega_{FS}$ and $Re(\mathfrak{F}) = G_{FS}$. It is evident that apart from the normalization coefficient, both tensors have the same tensorial structure. Note that the normalization coefficient is constant on every co-adjoint orbit; in fact we have already proved that each orbit passes through one and only one diagonal ρ_0 that fixes the values of k_1 and k_2. We have explained in Remark 3 that one can built a complex structure J starting from a metric and a symplectic tensor $J = g^{-1} \circ \omega$ requiring $J^2 = -\mathbb{I}$. If we evaluate $\tilde{J}(v) = G^{-1}_{FS} \circ \Omega_{FS}(v, .)$ we have:

$$\tilde{J} = (k_1 - k_2)(\partial_{x_2} \otimes \partial_{x_2} + \partial_{x_1} \otimes \partial_{x_1})(dx_2 \otimes dx_1 - dx_1 \otimes dx_2) = (k_1 - k_2)(dx_2 \otimes \partial_{x_1} - dx_1 \otimes \partial_{x_2}) \quad (4.31)$$

We have used $\tilde{J}$ because we have to find a normalization constant N such that:

$$J_{FS} = N\tilde{J} \qquad \text{and} \qquad J^2_{FS} = -\mathbb{I} \quad (4.32)$$

It is very easy to show that $N = |(k_1 - k_2)|$ and then:

$$J_{FS} = sgn(k_1 - k_2)(dx_2 \otimes \partial_{x_1} - dx_1 \otimes \partial_{x_2}) \quad (4.33)$$

In this way we have found the three components of a Kähler structure on the co-adjoint orbit. We can also compare J_{FS} with J founded previously that is:

$$J = sgn(\lambda_1 - \lambda_2)da^1_2 \otimes \partial_{b^1_2} + sgn(\lambda_2 - \lambda_1)db^1_2 \otimes \partial_{a^1_2}. \quad (4.34)$$

Performing the usual change of coordinates, we have:

$$J = sgn(k_1 - k_2)(dx_2 \otimes \partial_{x_1} - dx_1 \otimes \partial_{x_2}) = J_{FS} \quad (4.35)$$

We have a perfect agreement between the complex structures yielded by the two different method. In particular in the previous chapter we have built a metric tensor starting from a symplectic form and a complex structure, while in this chapter we have shown that starting from a metric and a symplectic form we are able to obtain the same complex structure J than before.

4.1.5 Transverse direction

Now we are going to study the evolution of $\rho_0 = diag\{k_1, k_2\}$ when k_1 and k_2 are not fixed but we will consider the basis fixed. As we explained before, we will evaluate the symmetric logarithmic derivative along this transversal direction $d_l^T \rho$ assuming:

$$d^T \rho = \frac{1}{2}\{d_l^T \rho, \rho\} \tag{4.36}$$

We use the condition $k_1 + k_2 = 1$ explicitly so that $k_1 = k$ and $k_2 = 1 - k$ where $k = k(\theta)$. Moreover we can rewrite:

$$\rho_0 = \frac{1}{2}\mathbb{I} + \frac{2k+1}{2}t_3 \tag{4.37}$$

In the first step we calculate the ordinary differential:

$$d^T \rho_0 = \frac{\partial \rho_0}{\partial k} dk = t_3 dk \tag{4.38}$$

Then, recalling the expansion on the Lie algebra generators we have only non-vanishing coefficient $D_3 = dk$. Using the definition $d^T \rho = \frac{1}{2}\{d_l^T \rho, \rho\}$ we have a matrix equation:

$$\begin{bmatrix} dk & 0 \\ 0 & -dk \end{bmatrix} = \frac{1}{2}\{\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \begin{bmatrix} k & 0 \\ 0 & 1-k \end{bmatrix}\} = \begin{bmatrix} kA & B \\ C & D(1-k) \end{bmatrix} \tag{4.39}$$

. We obtain:

$$A = \frac{dk}{k}, \quad B = 0, \quad C = 0, \quad D = -\frac{dk}{1-k}, \tag{4.40}$$

that is:

$$d_l^T \rho_0 = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & -\frac{1}{1-k} \end{bmatrix} dk \tag{4.41}$$

Using the usual scalar product $\langle . | . \rangle_{u^*}$ we cane rewrite $d_l^T \rho_0$ as:

$$d_l^T \rho_0 = L_{\mathbb{I}}\mathbb{I} + L_3 t_3 \tag{4.42}$$

with

$$L_{\mathbb{I}} = \frac{1}{2}(\frac{1}{k} - \frac{1}{1-k})dk \quad \text{and} \quad L_3 = \frac{1}{2}(\frac{1}{k} + \frac{1}{1-k})dk \tag{4.43}$$

Now we are ready to find the Fisher tensor on this transversal direction $\mathfrak{F}^T$:

$$\mathfrak{F}^T = Tr[\rho(L_{\mathbb{I}}\mathbb{I} + L_3 t_3) \otimes (L_{\mathbb{I}}\mathbb{I} + L_3 t_3)]$$

After a bit tedious, even if simple, calculation, one finds:

$$\mathfrak{F}^T = \frac{dk \otimes dk}{k(1-k)} \tag{4.44}$$

We observe that this term is symmetric, contributing to G_{FS}.

4.2 Three-level system

For a three level system or Q-trit the most general mixed state can always be expressed as:

$$\rho_0 = k_1 |\psi_1\rangle \langle\psi_1| + k_2 |\psi_2\rangle \langle\psi_2| + k_3 |\psi_3\rangle \langle\psi_3| \qquad \text{or} \qquad \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \tag{4.45}$$

where $|\psi_i\rangle$'s form an ortonormal basis, $k_1 + k_2 + k_3 = 1$ and $k_i \geq 0\ \forall i$. We choose as generators of $U(3)$ transformations the Gell-Mann matrices and the identity matrices:

$$t_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, t_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, t_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$t_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, t_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, t_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \tag{4.46}$$

$$t_{\mathbb{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, t_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, t_8 = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{bmatrix}.$$

It is quite easy, even if very tedious, to prove that the following relations hold:

$$\{t_j, t_k\} = \frac{4}{3}\delta_{ij}\mathbb{I} + 2\sum_{l=1}^{8} f_{jkl}, \tag{4.47}$$

$$[t_j, t_k] = 2i\sum_{l=1}^{8} c_{jkl}, \tag{4.48}$$

where the totally antisymmetric structure constants are:

$$c_{123} = 1; \qquad c_{458} = c_{678} = \frac{\sqrt{3}}{2};$$

$$c_{147} = c_{246} = c_{257} = c_{345} = -c_{156} = -c_{367} = \frac{1}{2}$$

and the totally symmetric symbols are:

$$f_{118} = f_{228} = f_{338} = -f_{888} = \frac{1}{\sqrt{3}},$$

$$f_{448} = f_{558} = f_{668} = f_{778} = -\frac{1}{2\sqrt{3}},$$

$$f_{416} = f_{157} = -f_{247} = f_{256} = \frac{1}{2},$$

$$f_{344} = f_{355} = -f_{366} = -f_{377} = \frac{1}{2}.$$

So we can decompose ρ_0 on this basis and we obtain:

$$\rho_0 = \frac{1}{3}t_{\mathbb{I}} + \frac{k_1 - k_2}{2}t_3 + \frac{k_1 + k_2 - 2k_3}{2\sqrt{3}}t_8 \tag{4.49}$$

and then, using the previous notation:

$$\rho_{\mathbb{I}} = \frac{1}{3}, \qquad \rho_3 = \frac{k_1 - k_2}{2}, \qquad \rho_8 = \frac{k_1 + k_2 - 2k_3}{2\sqrt{3}}, \tag{4.50}$$

all the other vanish. Note that, as before, ρ_0 has component only along the "directions" with diagonal generators.

4.2.1 Symmetric logarithmic derivative

Using the previous relations the homogeneous system (3.43) reads:

$$\begin{aligned} D_{\mathbb{I}} &= \rho_{\mathbb{I}} L_{\mathbb{I}} + \tfrac{2}{3}(\rho_3 L_3 + \rho_8 L_8) = 0 \\ D_3 &= \rho_3 L_{\mathbb{I}} + \rho_{\mathbb{I}} L_3 \tfrac{1}{\sqrt{3}}(\rho_3 L_8 + \rho_8 L_3) = 0 \\ D_8 &= \rho_3 L_{\mathbb{I}} + \rho_{\mathbb{I}} L_8 + \tfrac{1}{\sqrt{3}}(\rho_3 L_3 - \rho_8 L_8) = 0. \end{aligned} \tag{4.51}$$

Remark 14. We can evaluate the determinant of the matrix associated with the homogeneous system:

$$det \begin{bmatrix} \rho_{\mathbb{I}} & \frac{2}{3}\rho_3 & \frac{2}{3}\rho_8 \\ \rho_3 & \rho_{\mathbb{I}} + \frac{\rho_8}{\sqrt{3}} & \frac{\rho_3}{\sqrt{3}} \\ \rho_8 & \frac{\rho_3}{\sqrt{3}} & \rho_{\mathbb{I}} - \frac{\rho_8}{\sqrt{3}} \end{bmatrix} = k_1 k_2 k_3.$$

It is evident that if ρ_0 is full rank the determinant is non vanishing and then the solution of this system is the trivial one.

On the other hand the system (3.40) becomes:

$$D_l = \rho_{\mathbb{I}} L_l + \rho_l L_{\mathbb{I}} = \rho_{\mathbb{I}} L_l \qquad \text{with} \qquad l = 1, 2 \tag{4.52}$$

$$\begin{cases} D_1 = (\frac{k_1+k_2}{2})L_1 \\ D_2 = (\frac{k_1+k_2}{2})L_2 \\ D_4 = (\frac{k_1+k_3}{2})L_4 \\ D_5 = (\frac{k_1+k_3}{2})L_5 \\ D_6 = (\frac{k_2+k_3}{2})L_6 \\ D_7 = (\frac{k_2+k_3}{2})L_7 \end{cases} \Longleftrightarrow \begin{cases} L_1 = (\frac{2}{k_1+k_2})D_1 \\ L_2 = (\frac{2}{k_1+k_2})D_2 \\ L_4 = (\frac{2}{k_1+k_3})D_4 \\ L_5 = (\frac{2}{k_1+k_3})D_5 \\ L_6 = (\frac{2}{k_2+k_3})D_6 \\ L_7 = (\frac{2}{k_2+k_3})D_7 \end{cases} \tag{4.53}$$

In this way we have found the expression of the symmetric logarithmic derivative as a function of the coefficients D_i of the ordinary derivative, that is:

$$d_l\rho_0 = (\frac{2}{k_1+k_2})(D_1t_1+D_2t_2)+(\frac{2}{k_1+k_3})(D_4t_4+D_5t_5)+(\frac{2}{k_2+k_3})(D_7t_7+D_8t_8) \tag{4.54}$$

Before calculating the coefficients D_i we evaluate the Fisher Tensor as function of these.

4.2.2 Fisher tensor

Recall that the coefficients D_i and L_i are one-forms, then

$$\mathfrak{F}_{(3,3)} = Tr[\rho_0(d_l\rho_0 \otimes d_l\rho_0)] \tag{4.55}$$

after some easy calculations we have:

$$\begin{aligned} \mathfrak{F}_{(3,3)} = {} & (k_1+k_2)(L_1\otimes L_1 + L_2\otimes L_2) + i(k_1-k_2)(L_1\otimes L_2 - L_2\otimes L_1) \\ & (k_1+k_3)(L_4\otimes L_4 + L_5\otimes L_5) + i(k_1-k_3)(L_4\otimes L_5 - L_5\otimes L_4) \\ & (k_2+k_3)(L_6\otimes L_6 + L_7\otimes L_7) + i(k_2-k_3)(L_6\otimes L_7 - L_7\otimes L_6) \end{aligned} \tag{4.56}$$

and then:

$$\begin{aligned} \mathfrak{F}_{(3,3)} = {} & (k_1+k_2)(L_1\odot L_1 + L_2\odot L_2) - 2(k_1-k_2)(L_1\wedge L_2) \\ & (k_1+k_3)(L_4\odot L_4 + L_5\odot L_5) - 2(k_1-k_3)(L_4\wedge L_5) \\ & (k_2+k_3)(L_6\odot L_6 + L_7\odot L_7) - 2(k_2-k_3)(L_6\wedge L_7) \end{aligned} \tag{4.57}$$

Comparing with (4.13) we observe that this case is composed by three copies of the $\mathfrak{F}_{(2,2)}$.

4.2.3 Evaluation of the ordinary differential

To compute the standard differential we have to consider an orbit generated by the co-adjoint action of the unitary group acting on ρ_0. At the moment we are not considering the transversal directions. As explained before we have consider only unitary transformations that do not stabilize ρ_0 that are of the form

$$U = e^{(+ix_1t_1+ix_2t_2+ix_4t_4+ix_5t_5+ix_6t_6+ix_7t_7)} \tag{4.58}$$

In this way we have introduced a set of coordinates to identify

$$\rho = \rho(x_1, x_2, x_4, x_5, x_6, x_7)$$

and we can evaluate

$$d\rho_0 = (U(x_i)\rho_0 U^\dagger(x_i)|_{\mathbf{x}} \tag{4.59}$$

where the vector $\mathbf{x}$ has components x_i with $i = \{1, 2, 4, 5, 6, 7\}$. Evaluating explicitly we obtain the following expression:

$$\begin{aligned} d\rho_0 = [it_1\rho_0 + \rho_0(-it_1)]dx_1 + [it_2\rho_0 + \rho_0(-it_2)]dx_2 + \\ + [it_4\rho_0 + \rho_0(-it_4)]dx_4 + [it_5\rho_0 + \rho_0(-it_5)]dx_5 \\ [it_6\rho_0 + \rho_0(-it_6)]dx_6 + [it_7\rho_0 + \rho_0(-it_7)]dx_7 \end{aligned} \tag{4.60}$$

and then:

$$\begin{aligned} d\rho_0 = & \begin{bmatrix} 0 & r_1(dx_2 + idx_1) & 0 \\ r_1(dx_2 - idx_1) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \\ & + \begin{bmatrix} 0 & 0 & r_2(dx_5 + idx_4) \\ 0 & 0 & 0 \\ r_2(dx_5 - idx_4) & 0 & 0 \end{bmatrix} + \\ & + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r_3(dx_7 + idx_6) \\ 0 & r_3(dx_7 - idx_6) & 0 \end{bmatrix} \end{aligned}$$

where we have introduced:

$$r_1 = k_2 - k_1, \qquad r_2 = k_3 - k_1 \qquad r_3 = k_3 - k_2$$

Finally we have:

$$\begin{aligned} d\rho_0 = \begin{bmatrix} 0 & r_1(dx_2 + idx_1) & r_2(dx_5 + idx_4) \\ r_1(dx_2 - idx_1) & 0 & r_3(dx_7 + idx_6) \\ r_2(dx_5 - idx_4) & r_3(dx_7 - idx_6) & 0 \end{bmatrix} \\ \equiv \begin{bmatrix} 0 & r_1 dz_1^* & r_2 dz_2^* \\ r_1 dz_1 & 0 & r_3 dz_3^* \\ r_2 dz_2 & r_3 dz_3 & 0 \end{bmatrix} \end{aligned} \tag{4.61}$$

where in the last matrix we have defined 3 complex coordinates:

$$\begin{aligned} z_1 =& x_2 - ix_1 \\ z_2 =& x_5 - ix_4 \\ z_3 =& x_7 - ix_6 \end{aligned} \tag{4.62}$$

From these last matrix we can evaluate the coefficients D_i using the usual scalar product:

$$D_i = \langle t_i, d\rho_0 \rangle = \frac{1}{2} Tr[t_i d\rho_0] \tag{4.63}$$

Once evaluated the D_i's we obtain the L_i's using (4.53):

$$\begin{cases} D_1 = \dfrac{r_1}{2}(dz_1 + dz_1^*) = r_1 dx_2 \\ D_2 = \dfrac{r_1}{2i}(dz_1 - dz_1^*) = -r_1 dx_1 \\ D_4 = \dfrac{r_2}{2}(dz_2 + dz_2^*) = r_2 dx_5 \\ D_5 = \dfrac{r_2}{2i}(dz_2 - dz_2^*) = -r_2 dx_4 \\ D_6 = \dfrac{r_3}{2}(dz_3 + dz_3^*) = r_3 dx_7 \\ D_7 = \dfrac{r_3}{2i}(dz_3 - dz_3^*) = -r_3 dx_6 \end{cases} \Longleftrightarrow \begin{cases} L_1 = (\dfrac{2r_1 dx_2}{k_1 + k_2}) \\ L_2 = (\dfrac{-2r_1 dx_1}{k_1 + k_2}) \\ L_4 = (\dfrac{2r_2 dx_5}{k_1 + k_3}) \\ L_5 = (\dfrac{-2r_2 dx_4}{k_1 + k_3}) \\ L_6 = (\dfrac{2r_3 dx_7}{k_2 + k_3}) \\ L_7 = (\dfrac{-2r_3 dx_6}{k_2 + k_3}) D_7 \end{cases} \tag{4.64}$$

Now we are ready to write the Fisher Tensor putting the coefficients in (4.57):

$$\begin{aligned}\mathfrak{F}_{(3,3)} = 4\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\{(k_1+k_2)(dx_1\odot dx_1+dx_2\odot dx_2)+2(k_1-k_2)(dx_2\wedge dx_1)\}\\ +4\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\{(k_1+k_3)(dx_4\odot dx_4+dx_5\odot dx_5)+2(k_1-k_3)(dx_5\wedge dx_4)\}\\ +4\frac{(k_2-k_3)^2}{(k_2+k_3)^2}\{(k_2+k_3)(dx_6\odot dx_6+dx_7\odot dx_7)+2(k_2-k_3)(dx_7\wedge dx_6)\}\end{aligned} \tag{4.65}$$

or

$$\begin{aligned}\mathfrak{F}_{(3,3)} = 4\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\{(k_1+k_2)(dx_1\otimes dx_1+dx_2\otimes dx_2)+\\ +i(k_1-k_2)(dx_1\otimes dx_2-dx_2\otimes dx_1)\}+\\ +4\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\{(k_1+k_3)(dx_4\otimes dx_4+dx_5\otimes dx_5)+\\ +i(k_1-k_3)(dx_4\otimes dx_5-dx_5\otimes dx_4)\}+\\ +4\frac{(k_2-k_3)^2}{(k_2+k_3)^2}\{(k_2+k_3)(dx_6\otimes dx_6+dx_7\otimes dx_7)+\\ +i(k_2-k_3)(dx_6\otimes dx_7-dx_7\otimes dx_6)\}\end{aligned}$$

This expression can be rewritten in complex coordinates:

$$\begin{aligned}\mathfrak{F}_{(3,3)} =&4\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\{(k_1+k_2)dz_1\odot dz_1^*-i(k_1-k_2)dz_1\wedge dz_1^*\}\\ &4\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\{(k_1+k_3)dz_2\odot dz_2^*-i(k_1-k_3)dz_2\wedge dz_2^*\}\\ &4\frac{(k_2-k_3)^2}{(k_2+k_3)^2}\{(k_2+k_3)dz_3\odot dz_3^*-i(k_2-k_3)dz_3\wedge dz_3^*\}\end{aligned} \tag{4.66}$$

As before we note that the Fisher tensor has two parts a symmetric and an anti-symmetric one.

4.2.4 Comparison with previous results

Recall that we have already found a metric tensor (2.43) and a symplectitc form on $u^*(3)$ (2.42) previously. Changing the eigenvalue from λ_i to k_i they read:

$$\eta_\xi^O = \frac{da_2^1 \otimes db_2^1 - db_2^1 \otimes da_2^1}{k_1 - k_2} + \frac{da_3^1 \otimes db_3^1 - db_3^1 \otimes da_3^1}{k_1 - k_3} + \frac{da_3^2 \otimes db_3^2 - db_3^2 \otimes da_3^2}{k_2 - k_3} \tag{4.67}$$

Finally

$$\gamma_\xi^O = \frac{db_2^1 \otimes db_2^1 + da_2^1 \otimes da_2^1}{|k_1 - k_2|} + \frac{db_3^1 \otimes db_3^1 + da_3^1 \otimes da_3^1}{|k_1 - k_3|} + \frac{db_3^2 \otimes db_3^2 + da_3^2 \otimes da_3^2}{|k_2 - k_3|} \tag{4.68}$$

Comparing (2.33) with (4.46) it is easy to see that the hermitian matrices, used as basis, are different in the two cases. In particular they are related according these relations:

$$A_2^1 = t_1, \quad B_2^1 = -t_2, \quad A_3^1 = t_4 \quad B_3^1 = -t_5 \quad A_3^2 = t_6 \quad B_3^2 = -t_7 \tag{4.69}$$

Then we have to make the following change of coordinates:

$$a_2^1 = x_1, \quad b_2^1 = -x_2, \quad a_3^1 = x_4 \quad b_3^1 = -x_5 \quad a_3^2 = x_6 \quad b_3^2 = -x_7 \tag{4.70}$$

So we can write the symplectic form as:

$$\begin{aligned}\eta_\xi^O = &\frac{1}{k_1 - k_2}\{dx_2 \otimes dx_1 - dx_1 \otimes dx_2\}+ \\ &+ \frac{1}{k_1 - k_3}\{dx_5 \otimes dx_4 - dx_4 \otimes dx_5\}+ \\ &+ \frac{1}{k_2 - k_3}\{dx_7 \otimes dx_6 - dx_6 \otimes dx_7\}\end{aligned} \tag{4.71}$$

As before we compare this expression with $\Omega_{FS} = -Im(\mathfrak{F})$

$$\begin{aligned}\Omega_{FS} = &4\frac{(k_1 - k_2)^3}{(k_1 + k_2)^2}\{(dx_2 \otimes dx_1 - dx_1 \otimes dx_2)\} \\ &+ 4\frac{(k_1 - k_3)^3}{(k_1 + k_3)^2}\{(dx_5 \otimes dx_4 - dx_5 \otimes dx_4)\} \\ &+ 4\frac{(k_2 - k_3)^3}{(k_2 + k_3)^2}\{(dx_6 \otimes dx_7 - dx_7 \otimes dx_6)\}\end{aligned} \tag{4.72}$$

We can also compare the metric tensor with $G_{FS} = Re(\mathfrak{F})$:

$$\begin{aligned}\gamma_\xi^O = &\frac{1}{|k_1 - k_2|}\{dx_2 \otimes dx_2 + dx_1 \otimes dx_1\}+ \\ &+ \frac{1}{|k_1 - k_3|}\{dx_5 \otimes dx_5 + dx_4 \otimes dx_4\}+ \\ &+ \frac{1}{|k_2 - k_3|}\{dx_7 \otimes dx_7 + dx_6 \otimes dx_6\},\end{aligned} \tag{4.73}$$

and

$$G_{FS} = 4\frac{(k_1-k_2)^2}{(k_1+k_2)}\{(dx_1 \otimes dx_1 + dx_2 \otimes dx_2)\} + 4\frac{(k_1-k_3)^2}{(k_1+k_3)}\{(dx_5 \otimes dx_4 - dx_5 \otimes dx_4)\} + 4\frac{(k_2-k_3)^2}{(k_2+k_3)}\{(dx_6 \otimes dx_6 + dx_7 \otimes dx_7)\} \quad (4.74)$$

It is evident that apart from some constant coefficients, the tensors derived from the Fisher Tensor have the same structure of the tensors derived before with the "geometrical" approach. As before one can build a complex structure J starting from a metric and a symplectic tensor $J = g^{-1} \circ \omega$ requiring $J^2 = -\mathbb{I}$. If we evaluate $\tilde{J}(v) = G_{FS}^{-1} \circ \Omega_{FS}(v,.)$ we have:

$$G_{FS}^{-1} = \frac{(k_1+k_2)}{4(k_1-k_2)^2}\{(\partial_{x_1} \otimes \partial_{x_1} + \partial_{x_2} \otimes \partial_{x_2})\} + \frac{(k_1+k_3)}{4(k_1-k_3)^2}\{\partial_{x_4} \otimes \partial_{x_4} + \partial_{x_5} \otimes \partial_{x_5})\} + \frac{(k_2+k_3)}{(k_2-k_3)^2}\{\partial_{x_6} \otimes \partial_{x_6} + \partial_{x_7} \otimes \partial_{x_7})\} \quad (4.75)$$

It is easy to check that:

$$\tilde{J} = \frac{(k_1-k_2)}{(k_1+k_2)}\{(dx_2 \otimes \partial_{x_1} - dx_1 \otimes \partial_{x_2})\} + \frac{(k_1-k_3)}{(k_1+k_3)}\{dx_5 \otimes \partial_{x_4} - dx_4 \otimes \partial_{x_5})\} + \frac{(k_2-k_3)}{(k_2+k_3)}\{dx_7 \otimes \partial_{x_6} - dx_6 \otimes \partial_{x_7})\} \quad (4.76)$$

To implement the normalisation condition it is better use the matrix notation:

$$\tilde{J} = \begin{bmatrix} 0 & \frac{k_1-k_2}{k_1+k_2} & 0 & 0 & 0 & 0 \\ -\frac{k_1-k_2}{k_1+k_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k_1-k_3}{k_1+k_3} & 0 & 0 \\ 0 & 0 & -\frac{k_1-k_3}{k_1+k_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{k_2-k_3}{k_2+k_3} \\ 0 & 0 & 0 & 0 & -\frac{k_2-k_3}{k_2+k_3} & 0 \end{bmatrix} \quad (4.77)$$

If we want that $J = N\tilde{J}$ is such that $J^2 = -\mathbb{I}$ we have to set N as:

$$N = \begin{bmatrix} |\frac{k_1+k_2}{k_1-k_2}| & 0 & 0 & 0 & 0 & 0 \\ 0 & |\frac{k_1+k_2}{k_1-k_2}| & 0 & 0 & 0 & 0 \\ 0 & 0 & |\frac{k_1+k_3}{k_1-k_3}| & 0 & 0 & 0 \\ 0 & 0 & 0 & |\frac{k_1+k_3}{k_1-k_3}| & 0 & 0 \\ 0 & 0 & 0 & 0 & |\frac{k_2+k_3}{k_2-k_3}| & 0 \\ 0 & 0 & 0 & 0 & 0 & |\frac{k_2+k_3}{k_2-k_3}| \end{bmatrix} \tag{4.78}$$

Coming back to the standard notation:

$$\begin{aligned} J_{FS} = sgn(k_1 - k_2)\{dx_2 \otimes \partial_{x_1} - dx_1 \otimes \partial_{x_2}\} \\ + sgn(k_1 - k_3)\{dx_5 \otimes \partial_{x_4} - dx_4 \otimes \partial_{x_5}\} \\ + sgn(k_2 - k_3)\{dx_7 \otimes \partial_{x_6} - dx_6 \otimes \partial_{x_7}\} \end{aligned} \tag{4.79}$$

We have already found in (2.45) a complex structure with another approach that reads:

$$\begin{aligned} J = sgn(\lambda_1 - \lambda_2)(da_2^1 \otimes \partial_{b_2^1} - db_2^1 \otimes \partial_{a_2^1}) \\ sgn(\lambda_1 - \lambda_2)(da_3^1 \otimes \partial_{b_3^1} - db_3^1 \otimes \partial_{a_3^1}) \\ sgn(\lambda_2 - \lambda_3)(da_3^2 \otimes \partial_{b_3^2} - db_3^2 \otimes \partial_{a_3^2}). \end{aligned} \tag{4.80}$$

Changing the coordinates it is easy to see that the two complex structures are the same with perfect agreement:

$$J = J_{FS} \tag{4.81}$$

4.2.5 Transverse direction

Now we are going to study the evolution of $\rho_0 = diag\{k_1, k_2, k_3\}$ when k_1,k_2 and k_3 are not fixed but we will consider the basis fixed. As we explained before, we will evaluate the symmetric logarithmic derivative along this transversal direction $d_l^T \rho$ assuming:

$$d^T \rho = \frac{1}{2}\{d_l^T \rho, \rho\} \tag{4.82}$$

We use the condition $k_1 + k_2 + k_3 = 1$ explicitly so that $k_3 = 1 - k_1 - k_2$ where $k_i = k_i(\theta)$ with $i \in \{1, 2\}$. Moreover we can rewrite:

$$\rho_0 = \frac{1}{2}\mathbb{I} + \frac{k_1 - k_2}{2} t_3 + \frac{3k_1 + 3k_2 - 2}{2\sqrt{3}} t_8 \tag{4.83}$$

In the first step we calculate the ordinary differential:

$$\begin{aligned} d^T\rho_0 =& \frac{\partial \rho_0}{\partial k_1} dk_1 + \frac{\partial \rho_0}{\partial k_2} dk_2 = \\ =& (\frac{1}{2} t_3 + \frac{\sqrt{3}}{2} t_8) dk_1 + (-\frac{1}{2} t_3 + \frac{\sqrt{3}}{2} t_8) dk_2 \end{aligned} \tag{4.84}$$

Then, recalling the expansion on the Lie algebra generators we have only two non-vanishing coefficients D_3 and D_8. We can also express the standard differential in a matrix form as:

$$d^T\rho_0 \begin{bmatrix} dk_1 & 0 & 0 \\ 0 & dk_2 & 0 \\ 0 & 0 & dk_1 - dk_2 \end{bmatrix} \tag{4.85}$$

Using the definition $d^T\rho = \frac{1}{2}\{d_l^T\rho, \rho\}$ we have a matrix equation:

$$\begin{bmatrix} dk_1 & 0 & 0 \\ 0 & dk_2 & 0 \\ 0 & 0 & dk_1 - dk_2 \end{bmatrix} = \frac{1}{2} \left\{ \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}, \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \right\} \tag{4.86}$$

i.e.

$$\begin{bmatrix} dk_1 & 0 & 0 \\ 0 & dk_2 & 0 \\ 0 & 0 & dk_1 - dk_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2k_1 A & B(k_1 + k_2) & C(k_3 + k_1) \\ D(k_1 + k_2) & 2Ek_2 & F(k_2 + k_3) \\ G(k_3 + k_1) & H(k_3 + k_2) & 2k_3 T \end{bmatrix} \tag{4.87}$$

Thus we have:

$$B = C = D = F = G = H = 0, \tag{4.88}$$

and

$$A = \frac{dk_1}{k_1}, \qquad E = \frac{dk_2}{k_2}, \qquad I = \frac{dk_1 + dk_2}{k_1 + k_2 - 1}. \tag{4.89}$$

We have found the matrix elements of the symmetric logarithmic derivative:

$$d_l\rho_0 = \begin{bmatrix} \frac{dk_1}{k_1} & 0 & 0 \\ 0 & \frac{dk_2}{k_2} & 0 \\ 0 & 0 & \frac{dk_1 + dk_2}{k_1 + k_2 - 1} \end{bmatrix}. \tag{4.90}$$

As usual we can extract from this matrix the coefficients L_i's, in particular

$$\begin{aligned} L_{\mathbb{I}} =& \frac{1}{3} \left(\frac{k_2 + 2k_1 - 1}{k_1(k_1 + k_2 - 1)} dk_1 + \frac{k_1 + 2k_2 - 1}{k_2(k_1 + k_2 - 1)} dk_2 \right) \\ L_3 =& \frac{1}{2} \left(\frac{dk_1}{k_1} - \frac{dk_2}{k_2} \right) \\ L_8 =& \frac{1}{2\sqrt{3}} \left(\frac{k_2 - k_1 - 1}{k_1(k_1 + k_2 - 1)} dk_1 + \frac{k_1 - k_2 - 1}{k_2(k_1 + k_2 - 1)} dk_2 \right) \end{aligned} \tag{4.91}$$

Now we are ready to find the Fisher tensor on this transversal direction $\mathfrak{F}^T$:

$$\mathfrak{F}^T = Tr[\rho(d_l^T\rho) \otimes (d_l^T\rho)]$$

that is:

$$\mathfrak{F}^T = Tr[\rho(t_{\mathbb{I}}L_{\mathbb{I}} + L_3t_3 + L_8t_8) \otimes (t_{\mathbb{I}}L_{\mathbb{I}} + L_3t_3 + L_8t_8)] \tag{4.92}$$

$$\begin{aligned}\mathfrak{F}^T &= Tr[\rho_0]L_{\mathbb{I}} \otimes L_{\mathbb{I}} + Tr[\rho_0t_3]L_{\mathbb{I}} \otimes L_3 + Tr[\rho_0t_8]L_{\mathbb{I}} \otimes L_8 + \\ &+ Tr[\rho_0t_3]L_3 \otimes L_{\mathbb{I}} + Tr[\rho_0t_3t_3]L_3 \otimes L_3 + Tr[\rho_0t_3t_8]L_3 \otimes L_8 + \\ &+ Tr[\rho_0t_8]L_8 \otimes L_{\mathbb{I}} + Tr[\rho_0t_8t_3]L_8 \otimes L_3 + Tr[\rho_0t_8t_8]L_8 \otimes L_8\end{aligned} \tag{4.93}$$

To evaluate explicitly this last expression the calculations are very long but easy so we will report only the example for $Tr[\rho_0]L_{\mathbb{I}}\otimes L_{\mathbb{I}}$. Obviously $Tr[\rho_0] = 1$, moreover:

$$\begin{aligned}L_{\mathbb{I}} \otimes L_{\mathbb{I}} =& \frac{1}{9}\left(\frac{k_2 + 2k_1 - 1}{k_1(k_1 + k_2 - 1)}\right)^2 dk_1 \otimes dk_1 + \\ &+ \frac{1}{9}\left(\frac{k_1 + 2k_2 - 1}{k_2(k_1 + k_2 - 1)}\right)^2 dk_2 \otimes dk_2 + \\ &+ \frac{1}{9}\left(\frac{(k_2 + 2k_1 - 1)(k_1 + 2k_2 - 1)}{k_1k_2(k_1 + k_2 - 1)}\right)(dk_1 \otimes dk_1 + dk_2 \otimes dk_2)\end{aligned} \tag{4.94}$$

After evaluated all the traces we obtain:

$$\begin{aligned}\mathfrak{F}^T =& (dk_1 \otimes dk_1)\left(\frac{1}{k_1} - \frac{1}{k_1 + k_2 - 1}\right) + \\ &+ (dk_2 \otimes dk_2)\left(\frac{1}{k_2} - \frac{1}{k_1 + k_2 - 1}\right) + \\ &(dk_1 \otimes dk_2 + dk_1 \otimes dk_1)\left(\frac{1}{1 - k_1 - k_2}\right)\end{aligned} \tag{4.95}$$

Remark 15. This part of the Fisher Tensor is symmetric under the exchange between the indices 1 and 2, so it will be contribute to the metric

Moreover one can repeat the same calculations without the condition $k_1 + k_2 + k_3 = 1$. After the same steps one obtains:

$$\mathfrak{F}^T = (dk_1 \otimes dk_1)\left(\frac{1}{k_1}\right) + (dk_2 \otimes dk_2)\left(\frac{1}{k_2}\right) + (dk_3 \otimes dk_3)\left(\frac{1}{k_3}\right) \tag{4.96}$$

From this last expression it is easy to obtain the previous one requiring $k_3 = 1 - k_2 - k_1$. So we can conclude that our Fisher Tensor it is well defined also when some eigenvalues k_i's vanish during the evolution, in fact in this case obviously $dk_i = 0$.

Chapter 5

Dynamical Evolution

In this chapter, starting from the quantum mechanics evolution postulate, we will review how systems evolve in time. In particular we will discuss the evolution of open quantum systems using the Kraus representation of a super-operator. This powerful instrument will allow us to implement explicitly the transversal direction studied in the previous chapter. Finally we will revisit the previous result obtained for 2×2*-density matrices using this formalism.*

5.1 General theory of evolution

5.1.1 Unitary evolution

We start by saying that we do not want to give a complete treatment on the evolution theory. The aim of this section is only to present the instruments that we will use later. Moreover we do not provide all the proofs but only the instructive ones. The interested reader can find all the proofs and more explanations in [14] and [16].
First of all we enunciate the **evolution postulate** for a quantum mechanical system [14, 16]:"*Time evolution of a closed quantum state is unitary; that is, the state* $|\psi(t_1)\rangle$ *of the system at time* t_1 *is related to the state* $|\psi(t_2)\rangle$ *at time* t_2 *by a unitary operator* U*.*

$$|\psi(t_2)\rangle = U(t_2, t_1)\,|\psi(t_1)\rangle$$

The evolution is described by the Schrödinger Equation:

$$i\hbar \frac{d\,|\psi(t)\rangle}{dt} = H\,|\psi(t)\rangle$$

where H *is an Hermitian operator known as the Hamiltonian of the closed quantum system.*"

Remark 16. The connection between U and H is very well known at least when H does not depend on time; first it is easy to verify that

$$|\psi(t)\rangle = exp(-i\frac{H(t-t_1)}{\hbar})\,|\psi(t_1)\rangle$$

is a solution of the Schrödinger Equation. Hence we have the identifications:

$$U(t,t_1) = exp(-i\frac{H(t-t_1)}{\hbar})$$

Remark 17. The postulate considers only *closed* quantum system; what does it mean? One could starts an epistemological discussion on this topic but for our aim is enough to consider a quantum system closed if it does not interact in any way with other systems. Actually all systems, except the whole Universe, interact with other systems. Nevertheless, there are a lot of systems that can be considered closed in a good approximation. Moreover, in principle, every open system can be considered as a part of a closed systems, the Universe, whose evolution is unitary.

We can also reformulate this postulate for a density matrix. In fact we can consider a pure state density matrix ρ, i.e. a matrix with only one eigenvalue $k = 1$. If we use the basis in which the matrix is diagonal we can rewrite $\rho = |\psi\rangle\langle\psi|$. According to the postulate $|\psi\rangle$ evolves as $|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle$ then, omitting the time dependence of the unitary operator, we have:

$$\rho(t) = U\,|\psi(0)\rangle\langle\psi(0)|\,U^\dagger = U\rho_0 U^\dagger \tag{5.1}$$

Because every mixed state can be written as a convex combination of pure states, the previous formula remains valid for every kind of density matrix.
To understand better what follows we now present rapidly the evolution of a pure bipartite system assuming no interaction between the two sub-systems. A pure bipartite state is a vector $|\psi(0)\rangle \in \mathcal{H} = \mathcal{H_A}\otimes\mathcal{H_B}$. If $\{|i(0)\rangle\}_i$ is a basis for $\mathcal{H_A}$ and $\{|\mu(0)\rangle\}_\mu$ is a basis for $\mathcal{H_B}$ then we can write:

$$|\psi(0)\rangle_{AB} = \sum_{i,\mu} a_{i\mu}\,|i(0)\rangle_A \otimes |\mu(0)\rangle_B \tag{5.2}$$

If there is no interaction between system A and system B we can use an Hamiltonian on $\mathcal{H_A}\otimes\mathcal{H_B}$ of the form:

$$H_{AB} = H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B \tag{5.3}$$

Since the Hamiltonian is time independent and has this particular form, the two sub-systems evolve independently and we have that the unitary operator for the combined system is:

$$U_{AB}(t) = U_A(t)\otimes U_B(t) \tag{5.4}$$

and then

$$|\psi(t)\rangle_{AB} = U_{AB}(t)\,|\psi(0)\rangle_{AB} = \sum_{i,\mu} a_{i\mu} U_A\,|i(0)\rangle_A \otimes U_B\,|\mu(0)\rangle_B$$

$$|\psi(t)\rangle_{AB} = \sum_{i,\mu} a_{i\mu}\,|i(t)\rangle_A \otimes |\mu(t)\rangle_B\,,$$

where $|i(t)\rangle_A$ and $|\mu(t)\rangle_B$ define a new orthonormal basis for $\mathcal{H_A}$ and $\mathcal{H_B}$, since U_A, U_B are unitary operators. We now focus on the evolution of the sub-system A. Noticing that we can obtain a sub-system A from the total one using

$$\rho_A(0) = Tr_B[\rho_{AB}(0)] = \sum_{i,j,\sigma} a_{i\sigma} a^*_{j\sigma}\,|i(0)\rangle_A\ {}_A\langle j(0)|\,,$$

and calling $\rho_{AB}(t) = |\psi(t)\rangle_{AB}\ {}_{AB}\langle\psi(t)|$; we have:

$$\begin{aligned}
\rho_A(t) = Tr_B[\rho_{AB}(t)] = &\sum_{i,j,\mu,\nu,\sigma} a_{i\mu} a^*_{j\nu}(|i(t)\rangle_A\ {}_A\langle j(t)| \otimes \langle\sigma(t)|\mu(t)\rangle_B\,\langle\nu(t)|\sigma(t)\rangle_B) \\
=&\sum_{i,j,\sigma} a_{i\sigma} a^*_{j\sigma}\,|i(t)\rangle_A\ {}_A\langle j(t)| \\
=&U_A(t)\rho_A(0)U_A^\dagger(t)
\end{aligned} \tag{5.5}$$

This result is in agreement with what we have said before, that is, if the system A does not interact with the system B then A evolves like a closed system.

5.1.2 Superoperators

In this section we would like to understand how a sub-system A of a bipartite system evolves when it interacts with another sub-system E. We have chosen the letter E to indicate the other sub-system because in this context the total system is considered the union between the system A and the environment E. In other words the system A is an open quantum system. Let us suppose that the initial composite state is described by the following tensor product:

$$\rho_{EA} = |e_0\rangle_E\ {}_E\langle e_0| \otimes \rho_A \tag{5.6}$$

In particular the system S is described by a density matrix and the environment is assumed to be in a pure state. Following the postulate, the system evolves for a finite time as:

$$\rho_{EA} = U_{EA}(|e_0\rangle_E\ {}_E\langle e_0| \otimes \rho_A)U_{EA}^\dagger \tag{5.7}$$

Performing the partial trace on the environment ($\mathcal{H}_E$) yields the density matrix of the system A after the evolution:

$$\begin{aligned}\rho'_A =& tr_E \left[U_{EA}(|e_0\rangle_E \ {}_E\langle e_0| \otimes \rho_A)U^\dagger_{EA}\right] \\ =& \sum_\mu \ {}_E\langle \mu|U_{EA}|e_0\rangle_E \, \rho_A \ {}_E\langle e_0|U_{EA}|\mu\rangle_E\end{aligned} \tag{5.8}$$

where $|\mu\rangle_E$ is an orthonormal basis for $\mathcal{H}_E$ and ${}_E\langle \mu|U_{EA}|e_0\rangle_E$ is an operator acting on $\mathcal{H}_A$. If $\{|\mu\rangle_E \otimes |i\rangle_A\}$ is an orthonormal basis for $\mathcal{H}_E \otimes \mathcal{H}_A$ then the operator ${}_E\langle \mu|U_{EA}|\nu\rangle_E$ on $\mathcal{H}_\mathcal{A}$ has as matrix elements:

$$ {}_A\langle i| \left({}_E\langle \mu|U_{EA}|\nu\rangle_E \right) |j\rangle_A \tag{5.9}$$

If we denote with:

$$E_\mu = \ {}_E\langle \mu|U_{EA}|e_0\rangle_E \tag{5.10}$$

we can rewrite

$$\rho'_A \equiv \mathfrak{S}(\rho_A) = \sum_\mu E_\mu \rho_A E^\dagger_\mu \tag{5.11}$$

Remark 18. Notice that from the unitarity of U_{EA} the E_μ's satisfy the property:

$$\begin{aligned}\sum_\mu E^\dagger_\mu E_\mu =& \sum_\mu \ {}_E\langle e_0|U_{EA}|\mu\rangle_E \ {}_E\langle \mu|U_{EA}|e_0\rangle_E \\ =& \ {}_E\langle e_0|U^\dagger_{EA}U_{EA}|e_0\rangle_E = \mathbb{I}_A\end{aligned} \tag{5.12}$$

Equation (5.11) defines a linear map $\mathfrak{S}$ that takes linear operators to linear operators. If this map $\mathfrak{S}$ has the property (5.12), is called *superoperator* and equation (5.11) defines the operator sum representation, also called *Kraus* representation of the superoperator. Moreover we can see a superoperator $\mathfrak{S}$ as a linear map between density operators; in fact from (5.11,5.12) we have that ρ'_A is a density matrix if ρ_A is, since:

1. ρ'_A is hermitian: $\rho'^\dagger_A = \sum_\mu E_\mu \rho^\dagger_A E^\dagger_\mu = \rho'_A$.
2. ρ'_A has unit trace $=\sum_\mu tr(\rho_A E^\dagger_\mu E_\mu) = tr(\rho_A) = 1$.
3. ρ'_A is positive: ${}_A\langle\psi|\ \rho'_A\ |\psi\rangle_A = \sum_\mu (\langle\psi|M_\mu\rangle)\rho_A(\langle M^\dagger_\mu|\psi\rangle) \geq 0$

We have showed that from a unitary transformation on the total system (the Universe) we obtain a Kraus representation on the sub-system (A). Moreover the inverse is true too: from a Kraus representation on the sub-system we could obtain the unitary transformation on the Universe.
We choose $\mathcal{H}_E$ to be an Hilbert space whose dimension is at least equal to the number of terms in the Kraus representation. If $\langle\psi|_A$ is a normalized vector in

$\mathcal{H}_A$ and $\{|\mu\rangle_E\}$ are orthonormal states of $\mathcal{H}_B$ and $|e_0\rangle_E$ is a normalized state, we can define the action of U_{EA} as:

$$U_{EA}(|e_0\rangle_E \otimes |\psi\rangle_A) = \sum_\mu |\mu\rangle_E \otimes E_\mu |\psi\rangle_A \tag{5.13}$$

It is easy to see [16] that U_{AE} preserves the inner product on $|e_0\rangle \otimes \mathcal{H}_A$ so we can extend U_{AB} to a unitary operator on $\mathcal{H}_E \otimes \mathcal{H}_A$. If we choose $\mathcal{H}_E$ with dimension equal to the number of terms in the Kraus representation and the states $\{|\mu\rangle_E\}$ such that the first vector is $|e_0\rangle_E$ and the other are chosen to complete the orthonormal basis, U_{EA} has a simple matrix representation. In fact, recalling that $E_\mu = \langle\mu| U_{EA} |e_0\rangle$, we have

$$U_{EA} = \begin{bmatrix} [E_0] & \dots & \dots & \dots \\ [E_1] & \dots & \dots & \dots \\ [E_2] & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ [E_\mu] & \dots & \dots & \dots \end{bmatrix} \tag{5.14}$$

In this notation is obvious that this definition does not determine completely the operator U_{EA}, if we want a unitary extension we could complete the matrix such that the columns, or equally the rows, form an orthonormal basis. Moreover it is immediate that:

$$\rho_A^{'} = tr_B[U_{EA}(|e_0\rangle_E \otimes |\psi\rangle_A)(\ _E\langle e_0| \otimes \ _A\langle\psi|)U_{EA}^\dagger] \tag{5.15}$$

Since any ρ_A can be expressed as a convex combination of pure states, we recover the Kraus representation on a general ρ_A. We have introduced the superoperators because they provide us with a powerful formalism very useful if we wish to describe the evolution of a pure state to a mixed one. From the definition, it is obvious that the unitary evolution of ρ_A can be recovered if the Kraus representation consist of only one operator E_μ. Only in this latter case if ρ_A is a pure state ρ_A' is still a pure state. In general even if ρ_A is a pure state we have that ρ_A' is a mixed one (that is; there are at least two component in the sum(5.11)). Moreover it is easy to see that if $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two superoperators also $\mathfrak{S}_2 \circ \mathfrak{S}_1$ is. In particular if $\mathfrak{S}_1$ describes the evolution from t_0 to t_1 and $\mathfrak{S}_2$ describes the evolution from t_1 to t_2, then $\mathfrak{S}_2 \circ \mathfrak{S}_1$ describes the evolution from t_0 to t_2. On the other hand one can show that a superoperator is invertible if and only if it is unitary. This means that while the unitary evolutions form a dynamical group the superoperator evolutions form only a dynamical semi-group. Physically speaking, if a system undergoes a genuine superoperator evolution it cannot go back to the initial state, from this point of view we could say that there is an arrow of time even at microscopic level.

Actually, to understand completely the powerful of the Kraus representation we have to recollect another important result, we will omit the proof that can be find in [16].

Theorem 5.1.1. *Any map $\mathfrak{S}$ such that:*

1. *$\mathfrak{S}$ is linear,*
2. *$\mathfrak{S}$ preserve hermiticity,*
3. *$\mathfrak{S}$ is trace preserving,*
4. *$\mathfrak{S}$ is completely positive,*

has Kraus representation.

Physically speaking, the theorem says that any "reasonable" time evolution has an operator-sum representation and then can be realized by a unitary transformation in a certain bipartite system. Let us explain the physic content of the hypothesis that define mathematically when an evolution is "reasonable". Hypothesis 2 and 3 are necessary if we want that a density matrix after the evolution still has an unitary trace and it is still Hermitian. Hypothesis 1 is, actually, an open question. From one hand this request permits us to maintain an easy ensemble interpretation. For example if the mixed state ρ is a convex combination of two pure state ρ_1 and ρ_2 then:

$$\mathfrak{S}(\rho(p)) \equiv \mathfrak{S}(p\rho_1 + (1-p)\rho_2) = p\mathfrak{S}(\rho_1) + (1-p)\mathfrak{S}(\rho_2)$$

So if at the initial time the state ρ has probability p (or $1-p$) to be in the state ρ_1 (or ρ_2), after the evolution it has a probability p (or $1-p$) to be in the state $\mathfrak{S}\rho_1$ (or $\mathfrak{S}\rho_2$). On the other hand there are version of "quantum mechanics" in which non linear evolution is still consistent with a probabilistic view but this could have strange, perhaps even absurd consequences, for example look at [15]. For our aim is enough to follow the tradition and to require $\mathfrak{S}$ linear. Finally, we have hypothesis 4. We say, by definition that $\mathfrak{S}_A$ is completely positive if, considering any extensions of $\mathcal{H}_A$ to the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$, $\mathfrak{S}_A \otimes \mathbb{I}_B$ is positive for all such extensions. This is a very physical condition; first of all this guarantees that the map takes positive operators to positive operators (i.e $\mathfrak{S}_A$ is positive), but we are asking for more. In fact if we are studying the evolution of the system A, there may be another system B that does not interact at all with A. Complete positivity, with the other assumptions, ensures that if the system A evolves and system B does not, any initial density matrix of the total system AB evolves to another density matrix.

5.2 Q-bit density matrix evolution

Now we are ready to use this formalism to clarify the meaning of transversal direction and revise what we have done until now for 2×2 density matrix. But first let us introduce an useful instrument to visualize concretely the construction we are going to build.

5.2.1 The Bloch ball

We will now give an illuminating parametrisation of the convex cone of the density matrix. As we have already done, we can express every 2×2 matrix using the Pauli Matrices and the identity, so we can do the same with 2×2 density matrices:

$$\rho = \frac{1}{2}(\mathbb{I} + R_1\sigma_1 + R_2\sigma_2 + R_3\sigma_3) \tag{5.16}$$

where the coefficient associated with the identity is fixed and equal to $\frac{1}{2}$; in fact ρ must have trace equal to one and all the Pauli matrices are traceless. Moreover, we have factorized $\frac{1}{2}$ so that the scalar product that defines the coefficients is $R_i = Tr[\sigma_i\rho]$. So we have a correspondence between a density matrices and $\mathbb{R}^3$ vectors $\boldsymbol{R} = (R_1, R_2, R_3)$. If we write, in matrix notation, the previous decomposition we have:

$$\rho(\boldsymbol{R}) = \frac{1}{2}\begin{bmatrix} 1 + R_3 & R_1 - iR_2 \\ R_1 + iR_2 & 1 - R_3 \end{bmatrix} \tag{5.17}$$

To be sure that the matrix is positive a necessary condition is

$$det\rho = \frac{1}{4}(1 - |\boldsymbol{R}|^2) \geq 0 \Longleftrightarrow |\boldsymbol{R}|^2 \leq 1 \tag{5.18}$$

Note that this condition is also sufficient: because $tr[\rho] = 1$ then ρ cannot have both the eigenvalues negative. Thus we have a $1-1$ correspondence between density matrices of a single q-bit (i.e two level quantum system) and the points on the unit $3-$ball: $0 \leq |\boldsymbol{R}| \leq 1$. This ball is usually called Bloch Sphere, but actually it is a ball. The boundary of the ball, defined by the equation $|\boldsymbol{R}|^2 = 1$, is a $2-$sphere; for these density matrices we have two conditions:

$$det\rho = 0 \qquad \text{and} \qquad tr[\rho] = 1 \tag{5.19}$$

this implies that the density matrices can have eigenvalues 0 and 1. In other words they are one dimensional projectors and also they are pure state. There is also a special point in this ball: the centre. The centre is defined to have $|\mathbf{R}| = 0$ in this case the eigenvalues must satisfy:

$$p_1 + p_2 = 1 \qquad \text{and} \qquad p_1 p_2 = \frac{1}{4} \Longrightarrow p_1 = p_2 = \frac{1}{2} \tag{5.20}$$

or in matrix notation:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{5.21}$$

This matrix represents the state called maximally entangled. Notice also that the maximally entangled state, being proportional to the identity matrix, is invariant under the change of basis defined by the adjoint action of an unitary matrix U, hence it is a fixed point of the co-adjoint action.

Remark 19. What we have said is an explicit application of the theorem (2.2.3). In fact each 2×2 matrix is unitarily equivalent to $\text{diag}(k, 1-k)$.

1. if $k = 0$ or $k = 1$, i.e. the matrix represents a pure state, we have two eigenvalues with multiplicity equal to 1, then the orbit of ρ is homeomorphic to

$$U(2)/[U(1) \times U(1)] \tag{5.22}$$

2. if $0 < k < 1$ and $k \neq \frac{1}{2}$ we are in the same situation as before

3. if $k = \frac{1}{2}$ the orbit of ρ is omeomorphic to

$$U(2)/U(2) \tag{5.23}$$

   i.e a point, the centre of the sphere.

Recalling that, from group theory, one has the diffeomorphism $U(2)/[U(1) \times U(1)] \sim S^2$, we have proved that any $U(2)$ orbit is omeomorphic to a two-sphere, where the centre is a sphere of null radius. Furthermore, we shall see that, varying the value of k from 0 to 1, the union of all the co-adjoint unitary orbits, the set of 2×2 density matrix, is homeomorphic to a closed ball in $\mathbb{R}^3$: the Bloch Ball.

We will represent the Bloch ball drawing one of its great circles as shown in Fig.(5.1). Along the grey circumference there are pure states, among them we have called $\mathbf{N}$ the one that in a certain basis $\{|0\rangle, |1\rangle\}$ can be represented by:

$$\rho_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.24}$$

Remark 20. The Bloch vector $\boldsymbol{R}_{\rho_0}$ associated with ρ_0 reads:

$$\boldsymbol{R}_{\rho_0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Longrightarrow |\boldsymbol{R}_{\boldsymbol{\rho_0}}|^2 = 1 \tag{5.25}$$

Moreover in Fig.(5.1) we have called **ME** the representative point of the maximally entangled state.

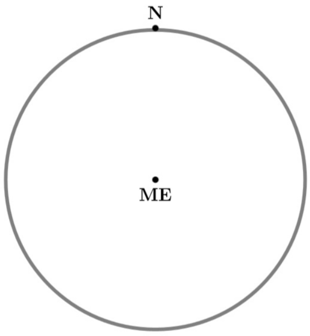

Figure 5.1: *Representation of the Bloch Ball*

5.2.2 Paths on the Bloch ball

We are now ready to introduce the evolution. In what follows we consider as starting point the matrix ρ_0, corresponding to N, and we investigate what happens when evolution occurs.

Unitary evolution First of all if ρ_0 is a closed system it could evolve with an unitary evolution:

$$\rho_0 \mapsto \rho_1 = U\rho_0 U^\dagger \tag{5.26}$$

where $U \in U(2)$. We can see every $U \in U(2)$ as usual:

$$U = exp[\alpha_0 \mathbb{I} + \sum_{i=1}^{3} \alpha_i \sigma_i] \tag{5.27}$$

In matrix notation it is easy to prove that:

$$U = e^{i\alpha_0} \begin{bmatrix} cos(|\boldsymbol{\alpha}|) + i\frac{\sin(|\boldsymbol{\alpha}|)}{|\boldsymbol{\alpha}|}\alpha_3 & i\frac{\sin(|\boldsymbol{\alpha}|)}{|\boldsymbol{\alpha}|}(\alpha_1 - i\alpha_2) \\ i\frac{\sin(|\boldsymbol{\alpha}|)}{|\boldsymbol{\alpha}|}(\alpha_1 + i\alpha_2) & \cos(|\boldsymbol{\alpha}|) - i\frac{\sin(|\boldsymbol{\alpha}|)}{|\boldsymbol{\alpha}|}\alpha_3 \end{bmatrix} \tag{5.28}$$

Where $|\boldsymbol{\alpha}| = \sqrt{\sum_{i=1}^{3} \alpha_i^2}$.
Obviously the phase factor, $e^{i\alpha_0}$, goes away since U acts according the co-ajoint action, i.e. the transformations "along" the $\mathbb{I}$ direction belong to the stabilizer of any ρ. So we can ignore $e^{i\alpha_0}$ and write the most general U as:

$$U = \begin{bmatrix} v & u \\ -\bar{u} & \bar{v} \end{bmatrix} \quad \text{where} \quad u, v \in \mathbb{C} \quad \text{such that} \quad |v|^2 + |u|^1 = 1 \tag{5.29}$$

In this way ρ_1 reads as:

$$\rho_1 = \begin{bmatrix} |v|^2 & -uv \\ -\bar{u}\bar{v} & |u|^2 \end{bmatrix} \tag{5.30}$$

Moreover we can evaluate the Bloch vector $\boldsymbol{R}_{\rho_1}$:

$$\boldsymbol{R}_{\rho_1} = \begin{pmatrix} -uv - \bar{u}\bar{v} \\ -iuv + i\bar{u}\bar{v} \\ |v|^2 - |u|^2 \end{pmatrix} \Rightarrow |\boldsymbol{R}_{\boldsymbol{\rho_1}}|^2 = (|v|^2 + |u|^2)^2 = 1 \tag{5.31}$$

So we have obtained another pure state, i.e $|\boldsymbol{R}_{\boldsymbol{\rho_1}}|^2 = 1$. This result is quite obvious: the co-adjoint action of the unitary group is simply a change of basis, and the rank of a matrix is invariant under this transformation. Of course the state ρ_1 is different from ρ_0 but it is still on the surface of the Bloch ball. We can represent this path, as in 5.2, like a rotation.

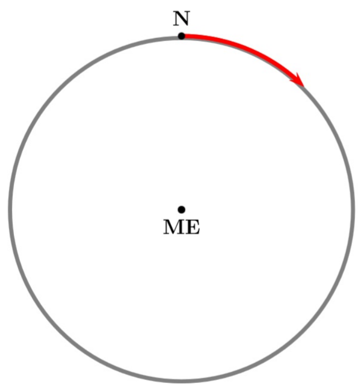

Figure 5.2: *Representation of a unitary transformation on the Bloch Sphere*

Open Evolution Now we are interested in studying an evolution that starting from a pure state ends in a mixed state; the simplest case the one can imagine is:

$$\rho_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \longmapsto \rho_1 = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix} \tag{5.32}$$

where $0 \leq p \leq 1$. We have just seen that the superoperator formalism provides us the instruments to face this problem. In particular we can explicitly build a Kraus representation of the superoperator $\mathfrak{S}$ such that $\mathfrak{S}(\rho_0) = \rho_1$. We can choose:

$$E_1 = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_2 = \sqrt{1-p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5.33}$$

that are defined up to unitary transformations that leave ρ_0 invariant, that is up to phases. It very easy to show that:

$$\mathfrak{S}(\rho_0) = \rho_1 = E_1\rho_0 E_1^\dagger + E_2\rho_0 E_2^\dagger \tag{5.34}$$

and that

$$E_1^\dagger E_1 + E_2^\dagger E_2 = \mathbb{I} \tag{5.35}$$

In this case the Bloch Vector reads:

$$\boldsymbol{R}_{\rho_1} = \begin{pmatrix} 0 \\ 0 \\ 2p-1 \end{pmatrix} \Longrightarrow |\boldsymbol{R}_{\boldsymbol{\rho_1}}|^2 = (2p-1)^2 \tag{5.36}$$

Being $p \in [0,1]$ we have $|\boldsymbol{R}_{\boldsymbol{\rho_1}}|^2 \leq 1$, then, as we have explained before, ρ_1 is a mixed state as expected, but we conclude also that the state ρ_1 is inside the Bloch Ball, (see fig.5.3).

Remark 21. In this way we have found an explicit way to implement the transversal direction, that we have studied before. In fact taking $p = p(t)$ as a function of a real parameter t, we have that the eigenvalues of the density matrix are variable. If we consider only the eigenvalues variable and ignore the unitary transformation (i.e. the basis in which the matrix is written is fixed), we are in the same conditions that we used to evaluate the $d^T\rho$ in the previous chapter.

Remark 22. If one sets $p = 0$ or $p = 1$ the superoperator evolution becomes a trivial unitary evolution:

1. if $p = 1$ we have $E_2 = 0$ and $E_1 = \mathbb{I}$. The state is unchanged after the evolution

2. if $p = 0$ we have $E_1 = 0$ and $E_2 = \sigma_1$. Since $\sigma_1\sigma_1^\dagger = \mathbb{I}$ the pure state $\rho_0 = |0\rangle\langle 0|$ becomes $\rho_1 = |1\rangle\langle 1|$ that it is still pure.

As we have said, superoperator evolution is a more general evolution than the unitary one; in particular if a superaoperator can be represented with a single Kraus Operator E then the superoperator evolution is a unitary one,

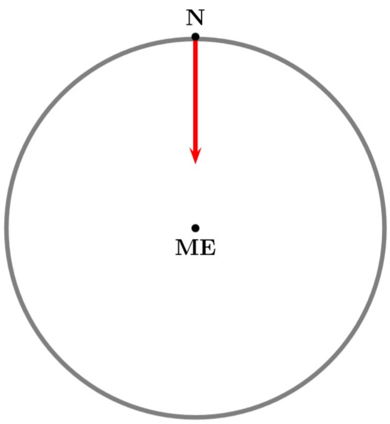

Figure 5.3: *Representation of a "Kraus" transformation on the Bloch Sphere*

and only in this case a pure state remains pure after the evolution. Moreover, we have also said that, when superoperator evolution occurs, one can reach the initial state again with another evolution if and only if the superoperator is unitary; otherwise it is impossible. We are going to show this result using the Bloch vector. Let F_1 and F_2, a Kraus Representation of a superoperator $\mathfrak{S}_1$, in the same basis of the E_i, such that:

$$F_1 = \sqrt{q}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \qquad F_2 = \sqrt{1-q}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \tag{5.37}$$

with $q \in [0, 1]$. Let us compose this $\mathfrak{S}_1$ with the previous $\mathfrak{S}$, as in fig.5.4

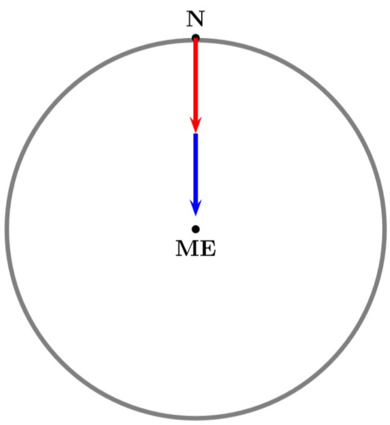

Figure 5.4: *Composition of two "Kraus" transformations on the Bloch Sphere*

$$\mathfrak{S}_1 \circ \mathfrak{S}(\rho_0) = \rho_2 = F_1\rho_1F_1^\dagger + F_2\rho_1F_2^\dagger = \\ F_1E_1\rho_0E_1^\dagger F_1^\dagger + F_2E_1\rho_0E_1^\dagger F_2^\dagger + F_1E_2\rho_0E_2^\dagger F_1^\dagger + F_2E_2\rho_0E_2^\dagger F_2^\dagger \quad (5.38)$$

after some easy calculations we obtain:

$$\rho_2 = \begin{bmatrix} qp + (1-p)(1-q) & 0 \\ 0 & q(1-p) + (1-q)p \end{bmatrix} \quad (5.39)$$

First of all one can check that trace is still equal to one, and that the Block vector is:

$$\boldsymbol{R}_{\rho_2} = \begin{pmatrix} 0 \\ 0 \\ (2p-1)(2q-1) \end{pmatrix} \Longrightarrow |\boldsymbol{R}_{\boldsymbol{\rho_2}}|^2 = (2p-1)^2(2q-1)^2 \quad (5.40)$$

Keeping in mind that $q \in [0,1]$ it is immediate see that $|\boldsymbol{R}_{\boldsymbol{\rho_2}}|^2 \leq |\boldsymbol{R}_{\boldsymbol{\rho_1}}|^2$. From this, we understand that any other superoperator can make the state evolve only toward the centre and thus the second transformation is not able to "bring" the mixed state back to the surface of the Block Sphere. We underline also that if the first superoperator $\mathfrak{S}$ takes ρ in the centre of the ball, i.e. $p = \frac{1}{2}$ then $\mathfrak{S}_2$ has no effects on the state. Note that, we have made an explicit proof that the composition of two superoperators is still a superoperator, in fact you can check that:

$$A_1 = F_1E_1 \qquad A_2 = F_1E_2 \qquad A_3 = F_2E_1 \qquad A_4 = F_2E_2 \quad (5.41)$$

form a Kraus representation of the superoperators :$\mathfrak{S}_1 \circ \mathfrak{S}$.

Remark 23. Notice that one could find transformations that bring the state ρ_1 back to ρ_0; for example:

$$G = \frac{1}{\sqrt{p}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (5.42)$$

such that:

$$G\rho_1G^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (5.43)$$

But it is easy to see that this is not a Kraus representation of a superoperator in fact $GG^\dagger \neq \mathbb{I}$. We conclude that this transformation has no physical meaning.

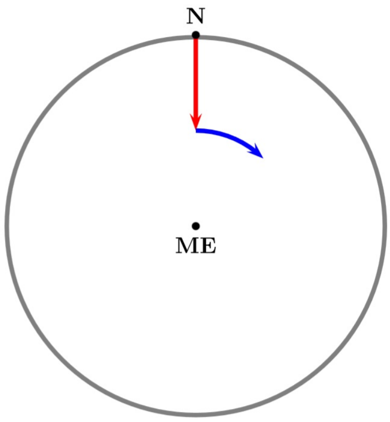

Figure 5.5: *A "Kraus" transformations on the Bloch Sphere, followed by an unitary one*

Mixing the evolutions We are interested in mixing the previous paths on the Bloch sphere, to generate the most general evolution, i.e. an evolution that allows us to reach every point in the Ball. We can apply , at first, the superoperator $\mathfrak{S}$, to choose in which sphere we want the transformed state to be, then with an unitary evolution we can move the state along this sphere (fig.5.5). Of course this construction makes sense inly if the unitary transformation leaves unchanged the radius (i.e. the modulus of the Bloch vector). We have just proved this result for a pure state, and we are going to generalize this result also for the mixed ones.

Indeed:

$$\begin{aligned}\rho_0 \mapsto \rho_1 = E_1\rho_0E_1^\dagger + E_2\rho_0E_2^\dagger \mapsto \rho_2 =& U\rho_1U^\dagger = \\ =& UE_1\rho_0E_1^\dagger U^\dagger + UE_2\rho_0E_2^\dagger U^\dagger\end{aligned} \tag{5.44}$$

In matrix form:

$$\rho_0 \mapsto \rho_1 = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix} \mapsto \rho_2 = \begin{bmatrix} |v|^2p + |u|^2(1-p) & -uv(2p-1) \\ -\bar{u}\bar{v}(2p-1) & |u|^2p + |v|^2(1-p) \end{bmatrix} \tag{5.45}$$

Note that ρ_2 is still hermitian and has trace equal to one. Moreover we can evaluate the Bloch vector:

$$\boldsymbol{R}_{\rho_0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \boldsymbol{R}_{\rho_1} = \begin{pmatrix} 0 \\ 0 \\ 2p-1 \end{pmatrix} \mapsto \boldsymbol{R}_{\rho_2} = (2p-1) \begin{pmatrix} -uv - \bar{u}\bar{v} \\ -iuv + i\bar{u}\bar{v} \\ |v|^2 - |u|^2 \end{pmatrix} \tag{5.46}$$

This yields:

$$|\boldsymbol{R}_{\boldsymbol{\rho_0}}|^2 = 1 \geq |\boldsymbol{R}_{\boldsymbol{\rho_1}}|^2 = |\boldsymbol{R}_{\boldsymbol{\rho_2}}|^2 = (2p-1)^2 \tag{5.47}$$

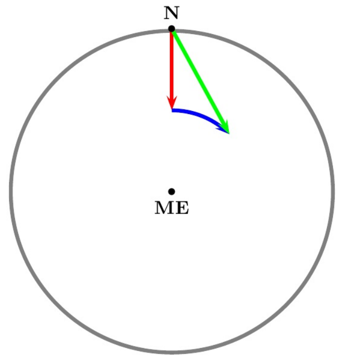

Figure 5.6: *An E-transformations on the Bloch Sphere, followed by an unitary one compared with an F-transformation (green arrow)*

We have followed this argument to be clear, but we want to underline that everything would have been the same if we started from the following Kraus operator (see fig.5.6):

$$F_1 = \sqrt{p}\begin{bmatrix} v & u \\ -\bar{u} & \bar{v} \end{bmatrix} \qquad F_2 = \sqrt{1-p}\begin{bmatrix} u & v \\ \bar{v} & -\bar{u} \end{bmatrix} \tag{5.48}$$

In particular it is easy to check that $\rho_2 = \sum_{i=1}^{2} F_i \rho_0 F_i^\dagger$ and

$$F_1^\dagger F_1 + F_2^\dagger F_2 = E_1^\dagger E_1 + E_2^\dagger E_2 = \mathbb{I}$$

One may wonder what happens if we exchange the order of the evolutions, that is, if at first we have an unitary evolution and then a superoperator evolution as in (fig. 5.7). To answer this question we have to keep in mind two facts:

1. the Kraus operator E_i's , used till now, are written in the same basis in which $\rho_0 = |0\rangle\langle 0|$;
2. acting with an unitary evolution means changing the basis.

Hence, if ρ_0 becomes $\rho_1 = U\rho_0 U^\dagger$ then, to be consistent, we have to change our E_i's in:

$$\tilde{E}_1 = UE_1U^\dagger \qquad \tilde{E}_2 = UE_2U^\dagger \tag{5.49}$$

First we have to check that these are still Kraus operator:

$$\begin{aligned} \tilde{E}_1^\dagger \tilde{E}_1 + \tilde{E}_2^\dagger \tilde{E}_2 =& UE_1^\dagger U^\dagger UE_1U^\dagger + UE_2^\dagger U^\dagger UE_2U^\dagger = \\ =& E_1^\dagger E_1 + E_2^\dagger E_2 = \mathbb{I} \end{aligned} \tag{5.50}$$

Then we can calculate what is the evolved state:

$$\tilde{E}_1 U\rho_0 U^\dagger \tilde{E}_1^\dagger + \tilde{E}_2 U\rho_0 U^\dagger \tilde{E}_2^\dagger = F_1\rho_0 F_1^\dagger + F_2\rho_0 F_2^\dagger \tag{5.51}$$

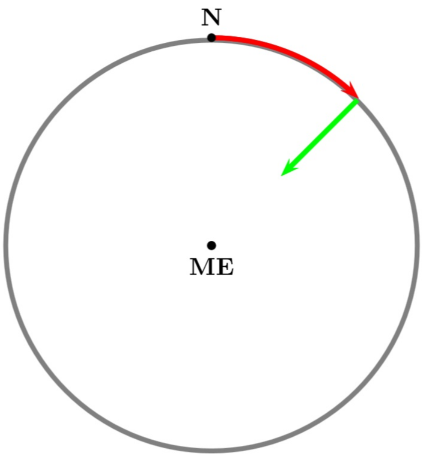

Figure 5.7: *An Unitary transformations on the Bloch Sphere, followed by superoperator evolution*

We have just proved that if we perform first an unitary evolution then a superoperator one, we obtain the same result than before, with the prescription of change the basis of the Kraus operator. This result makes us able to evaluate all the paths we would.

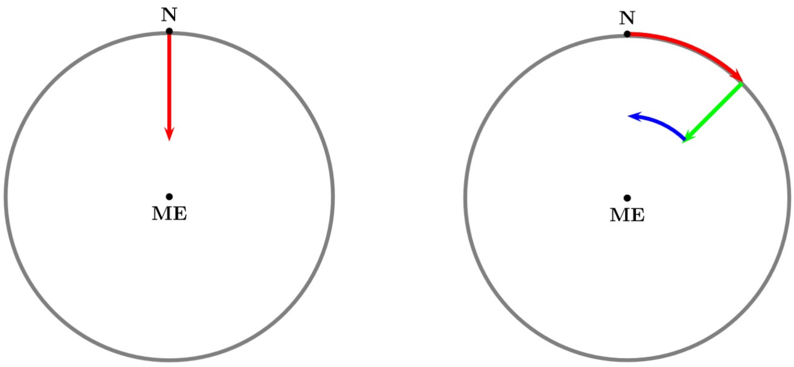

Figure 5.8: *equivalence between two paths*

For example we can prove the equivalence between the paths showed in (fig. 5.8). On one hand we have the genuine superoperator evolution:

$$\rho_1 = E_1 \rho_0 E_1^\dagger + E_2 \rho_0 E_2^\dagger. \tag{5.52}$$

On the other hand we have an unitary evolution followed by a Kraus Evolution, that is followed, in turn, by another unitary evolution that is inverse with respect to the first one:

$$\begin{aligned} U\rho_0U^\dagger \mapsto & \tilde{E}_1U\rho_0U^\dagger\tilde{E}_1^\dagger + \tilde{E}_2U\rho_0U^\dagger\tilde{E}_2^\dagger \mapsto \\ & \mapsto U^\dagger\tilde{E}_1U\rho_0U^\dagger\tilde{E}_1^\dagger U + U^\dagger\tilde{E}_2U\rho_0U^\dagger\tilde{E}_2^\dagger U = \\ & = U^\dagger(UE_1U^\dagger)U\rho_0U^\dagger(UE_1^\dagger U^\dagger)U + U^\dagger(UE_2U^\dagger)U\rho_0U^\dagger(UE_2^\dagger U^\dagger)U = \\ & = E_1\rho_0E_1^\dagger + E_2\rho_0E_2^\dagger \end{aligned} \tag{5.53}$$

5.2.3 The differential for the "Transverse Direction"

With this formalism we can also check explicitly that the differential, along the transversal direction evaluated in any point ρ, can be obtained, directly, by applying the co-adjoint action to the differential evaluated in ρ_0 where, as usual, ρ_0 is the diagonal density matrix representing the orbit of ρ. In particular we have in :

$$d^T\rho_0 = d(\sum_i E_i\rho_0E_i^\dagger)|_{p=0} \tag{5.54}$$

where we have used a generic sum representation of a certain superoperator $E_i(p)$, expressed in the same basis of ρ_0. If we would like to evaluate the same differential at the point $\rho = U\rho U^\dagger$, we have to change the Kraus operators as explained before:

$$E_i \mapsto \tilde{E}_i = UE_iU^\dagger. \tag{5.55}$$

Hence:

$$d^T\rho = d(\sum_i \tilde{E}_i\rho\tilde{E}_i^\dagger)|_{p=0} = d(\sum_i \tilde{E}_iU\rho_0U^\dagger\tilde{E}_i^\dagger)|_{p=0} = Ud^T\rho_0U^\dagger \tag{5.56}$$

5.2.4 Lindblad operator and tangent vectors

In this section we would like to find the generator of a Kraus evolution for a 2×2 density matrix ρ_0. In fact, we have seen that the unitary evolutions that do not stabilize ρ_0 are generated by σ_1 and σ_2. Introducing a "transversal" direction of evolution and ignoring the identity generator, we expect to use the whole SU(2) algebra. This intuition is motivated by what we have obtained evaluating the ordinary differential. In particular in chapter 3 we have seen that the differential for an unitary transformation was of the form:

$$d\rho_0 = D_1\sigma_1 + D_2\sigma_2, \tag{5.57}$$

while for the transversal evolution the differential reads:

$$d\rho_0 = D_3\sigma_3. \tag{5.58}$$

To answer this question we could use a Lindblad operator. Let us consider a quantum dynamical semigroup [1].

Definition 5.1. A quantum Dynamical Semigroup is a family of linear maps $\{\mathfrak{S}_t, t \geq 0\}$ such that:

1. $\mathfrak{S}_t$ has a Kraus representation or equivalently $\mathfrak{S}_t$ is a dynamical map
2. $\mathfrak{S}_t \circ \mathfrak{S}_s = \mathfrak{S}_{t+s}$
3. $\mathrm{Tr}[(\mathfrak{S}_t(\rho))A]$ is a continuous function of t for any density matrix ρ of the system (without the environment) and for any hermitian and bounded operator A defined on the Hilbert space of the system.

One can show that there exists a densely defined linear map L, called *a generator of a semigroup* such that:

$$\frac{d}{dt}\rho(t) = L\rho(t) \tag{5.59}$$

where $\rho(t) = \mathfrak{S}_t(\rho(0))$. This equation has the formal solution:

$$\rho(t) = e^{Lt}\rho(0). \tag{5.60}$$

The problem of finding L has been solved in general [16], yielding:

$$L\rho = -i[H,\rho] + \sum_a L_a \rho L_a^\dagger - \frac{1}{2}L_a^\dagger L_a \rho - \frac{1}{2}\rho L_a^\dagger L_a \rho \tag{5.61}$$

where H is the effective Hamiltonian of the system; i.e. the generator of unitary transformations, and the L_a's are connected with the operators C_a that form the Kraus representation of the superoperator.

Remark 24. We can naively motivate the form of the operator L. On one hand we already know that:

$$\rho(t) = \sum_a C_a(t)\rho_0 C_a^\dagger(t), \tag{5.62}$$

with the C_a's operators that form a Kraus representation, on the other hand we are looking for the Lindblad operator that can be obtained considering an infinitesimal variation of the parameter t, that is:

$$\rho(t) = e^{Lt}\rho_0 \Rightarrow \rho(t) = (1 - Lt)\rho_0 \Rightarrow \lim_{t \mapsto 0} \frac{\rho(t) - \rho_0}{t} = L\rho_0 \tag{5.63}$$

So we can write an infinitesimal evolution in the vicinity of $t = 0$ as:

$$\rho(dt) = \sum_a C_a(dt)\rho_0 C_a^\dagger(dt) = \rho_0 + O(dt) \tag{5.64}$$

Then all our Kraus operators $C_a(dt)$ are of order $\sqrt{dt}$, except one that is $C_0 = \mathbb{I} + O(dt)$. So we can write:

$$\begin{aligned} C_a =& \sqrt{dt} L_a, \\ C_0 =& \mathbb{I} + (-iH + K)dt \end{aligned} \tag{5.65}$$

Where H and K are Hermitian operators and H, K and L_a are all t-independent. Using the Kraus normalisation condition it is easy show that:

$$K = \sum_{a \neq 0} L_a^\dagger L_a \tag{5.66}$$

Putting all these last result in (5.62), and expressing $\rho(dt) = \rho_0 + dt \frac{d\rho}{dt}|_{t=0}$, we obtain (5.61).

First we have to check explicitly the semi-group condition for our Superoperator in Kraus representation, that is:
if $\mathfrak{S}_t$ and $\mathfrak{S}_s$ are superoperators we have to check that:

$$\mathfrak{S}_s \circ \mathfrak{S}_t = \mathfrak{S}_{t+s} \tag{5.67}$$

This is not a trivial statement because we are required to make explicit the dependence of the Kraus operators E_i from the variable parameter. In the two-level system case we can find a possible solution. In particular, we can ignore the unitary transformations, knowing that they cause only a change of basis but they do not make the orbit change, and rewrite only a "pure" Kraus transformation of the form (5.33) as:

$$E_1(t) = \sqrt{\frac{1}{2} + \frac{e^{-t}}{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_2(t) = \sqrt{\frac{1}{2} - \frac{e^{-t}}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5.68}$$

Remark 25. In simple words we have chosen the parameter p of the transformation (5.33) as:

$$p(t) = \frac{1}{2} + \frac{e^{-t}}{2} \tag{5.69}$$

With a simple algebra one can show that:

$$\begin{aligned} \mathfrak{S}_s \circ \mathfrak{S}_t(\rho_0) =& \mathfrak{S}_s(E_1(t)\rho_0 E_1(t)^\dagger + E_2(t)\rho_0 E_2(t)^\dagger) = \\ =& E_1(s)E_1(t)\rho_0 E_1(t)^\dagger E_1(s)^\dagger + E_1(s)E_2(t)\rho_0 E_2(t)^\dagger E_1(s)^\dagger + \\ &+ E_2(s)E_1(t)\rho_0 E_1(t)^\dagger E_2(s)^\dagger + E_2(s)E_2(t)\rho_0 E_2(t)^\dagger E_2(s)^\dagger = \\ =& E_1(t+s)\rho_0 E_1(t+s)^\dagger + E_2(t+s)\rho_0 E_2(t+s)^\dagger = \mathfrak{S}_{t+s}(\rho_0) \end{aligned}$$

for any ρ_0 of the form:

$$\rho_0 = \begin{bmatrix} p_0 & 0 \\ 0 & 1-p_0 \end{bmatrix}.$$

As t changes, this transformation evolves any starting point $\rho_0 = \rho_0(p(t=0))$ toward the centre of the Bloch sphere, along its radius. We notice that with this Kraus representation the maximally entangled state can be reached only asymptotically:

$$\lim_{t\mapsto\infty} \begin{bmatrix} \frac{1}{2}+\frac{e^{-t}}{2} & 0 \\ 0 & \frac{1}{2}-\frac{e^{-t}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{5.70}$$

With this explicit expression we can calculate the tangent vector at the curve $\rho(t) = \mathfrak{S}_t(\rho_0)$ at the point $\rho(0) = \rho_0$. Considering an infinitesimal variation δt of the parameter t, we have:

$$\rho(\delta t) = \begin{bmatrix} (\frac{1}{2}+\frac{e^{-\delta t}}{2})p_0 & 0 \\ 0 & (\frac{1}{2}+\frac{e^{-\delta t}}{2})(1-p_0) \end{bmatrix} + \begin{bmatrix} (\frac{1}{2}-\frac{e^{-\delta t}}{2})(1-p_0) & 0 \\ 0 & (\frac{1}{2}-\frac{e^{-\delta t}}{2})p_0 \end{bmatrix}$$

That at the first order in Taylor expansion reads:

$$\rho(\delta t) = \frac{1}{2}\begin{bmatrix} (2p_0 - 2p_0\delta t + \delta t) & 0 \\ 0 & (2 - 2p_0 + 2p_0\delta t - \delta t). \end{bmatrix}$$

Hence:

$$\rho(\delta t) - \rho_0 = \frac{1}{2}\begin{bmatrix} \delta t(1-2p_0) & 0 \\ 0 & \delta t(2p_0 - 1) \end{bmatrix}$$

Finally:

$$\frac{1}{2}(1-2p_0)\sigma_3 = \frac{1}{2}(1-2p_0)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{d\rho(t)}{dt}|_{t=0} \tag{5.71}$$

From this result is easy to calculate the associated one-form:

$$d\rho = \frac{1}{2}(1-2p_0)\sigma_3 dt. \tag{5.72}$$

We can compare this result with what we have found in (4.38), that is:

$$d^T\rho_0 = \frac{\partial \rho_0}{\partial k}dp = t_3 dp = \sigma_3 dp \tag{5.73}$$

but:

$$dp = \frac{1}{2}\frac{de^{-t}}{dt}|_{t=0}dt = -\frac{1}{2}dt \tag{5.74}$$

and then:

$$d^T\rho_0 = -\frac{\sigma_3}{2}dt. \tag{5.75}$$

This latter result is the same as (5.72) if we set $p_0 = 1$. This means that we can reinterpret the one-form evaluated in (4.38) as the one-form at the point $\rho_0 = diag(1, 0)$. The next step is the evaluation of the Lindblad operator related to this kind of transversal direction. In particular, to use the formula (5.61), putting $H = 0$ because we are not interested in unitary evolution, we need to find the operators L_a's:

$$\begin{aligned}\rho(dt) =& p(dt)\mathbb{I}\rho_0\mathbb{I} + (1 - p(dt))\sigma_1\rho_0\sigma_1 \\ =& (1 - \frac{dt}{2})\mathbb{I}\rho_0 + \frac{dt}{2}\sigma_1\rho_0\sigma_1.\end{aligned} \tag{5.76}$$

Looking at this last expression it is easy to see that the first term is proportional to F_0 in (5.65), and that we are left with only one L_a: $L_2 = \sqrt{\frac{1}{2}}\sigma_1$ and then

$$L\rho_0 = +L_2\rho_0 L_2^\dagger - \frac{1}{2}L_2^\dagger L_2\rho_0 - \frac{1}{2}\rho_0 L_2^\dagger L_2; \tag{5.77}$$

and finally

$$L\rho_0 = \frac{(1 - 2p_0)}{2}\sigma_3 \tag{5.78}$$

that is in perfect agreement with the tangent vector to the transversal orbit at ρ_0 previously found, as we expected. Nevertheless it is important to underline that the integral curve of the one parameter semi-group are not smooth curve on the Hermitian matrices algebra.

5.3 Open questions

An interesting question comes from Quantum Estimation Theory. In this context It was pointed out in [4] that the Quantum Fisher Information constitutes an upper bound of the Classical Fisher Information; that is:

$$i(\theta) \leq I(\theta) \tag{5.79}$$

After, some authors wondered which features a measure must to have in order to reach the equality in (5.79),i.e. to optimize the measurement. It turns out that the optimal path followed by $\rho(\theta)$ coincides with a geodetic of the Fisher metric [3, 12]. This holds when considering unitary orbits, i.e. to so called Projection-Valued Measure.

It would be interesting to see if this result could be extended also to a generic evolution defined by Kraus operators, i.e. to POVM measures.

The answer to this question was pointed out for a general mixed state of a two-level quantum system in. It turns out that the inequality is saturated for measurements that are proportional to one-projectors onto states which lies in the geodetics of the metric that comes from the Fisher Tensor. Actually, this result refers only to the restriction of the metric on the unitary orbit. So it could be interesting to investigate if we can extend this result for the geodetics of the whole metric (i.e. considering the metric defined on the unitary orbits and on transversal directions).

Chapter 6

Some Results for Q-trits

In this chapter we will point out how we would continue this research and what are the difficulties that arise when one tries to extend this framework in more that two dimensions.

6.1 The q-trit

In this section we recollect some results on the space of 3×3 density matrices, highlighting the connections with our work. First we can consider one more time the theorem (2.2.3) to have an intuitive way to classify the orbits in this case. Let us consider a 3×3 matrix ρ:

1. if ρ has only one eigenvalue with multiplicity 3, that is $\rho = \frac{1}{3}\mathbb{I}_{3\times 3}$, its co-adjoint orbit is homeomorphic to
$$U(3)/U(3),$$
that is a point. It has a null dimension.

2. if ρ has two eigenvalues, one with multiplicity equals to 1 and the other with multiplicity equals to 2, then its co-adjoint orbit is homeomorphic to
$$U(3)/[U(2) \times U(1)]$$
which is a 4-dimensional manifold.

3. if ρ has three distinct eigenvalues then its orbit co-adjoint is homemorphic to
$$U(3)/[U(1) \times U(1) \times U(1)]$$
which is a 6-dimensional manifold.

The difference between the 2×2 case and this case is clear: while all the orbits of the two-level systems are homeomorphic to a 2-sphere, the orbits of a three-level are not. We can motivate this statement in an intuitive way. First of all we can define the generalized block vector; in fact any 3×3 density matrix can be decomposed on a basis of Hermitian matrices t_k plus the identity as we have seen in chapter 3:

$$\rho = \frac{1}{3}\mathbb{I} + \frac{1}{2}\sum_{k=1}^{8} s_k t_k \tag{6.1}$$

Then we can associate to each density matrix the vector $\mathbf{S} \in \mathbb{R}^8$ such that $\mathbf{S}= (s_1, ..., s_8)$. It is easy to see [17] that the distance of each unitary orbit from the centre in $\mathbb{R}^8$ remains completely determined by:

$$|\mathbf{S}|^2 = \sum_{k=1}^{8} s_k^2 \tag{6.2}$$

Looking at the dimension of the manifold homeomorphic to the orbit we can easily prove the previous statement; the unitary orbit associated to a pure state is homeomorphic to 4-dimensional manifold. Because in $\mathbb{R}^8$ this 4-dimensional manifold is equidistant from the origin (i.e. $|\mathbf{S}|^2$ is invariant under the unitary action) this orbit is only a sub-manifold of the 7-sphere with radius $|\mathbf{S}|$. Moreover, knowing that there is only one orbit of pure state it is evident that there are some points in the 7-sphere that do not correspond to physical states. In [17] you can find a rigorous proof that shows why for $n > 2$ the correspondence between physical states and spheres is no more surjective. From these naive intuitions one can understand that, starting from 3-level systems, the space of density states is much more complicated, and we have to leave the intuitive Bloch sphere picture to continue with our study.

6.1.1 Superoperators

Even if it is not possible to visualize the whole state space of a three-level system in a simple way, we can find some Kraus representations of superoperators. As in the previous case, again we are not interested in unitary evolution, knowing that it is only a change of basis. This means that we will define the Kraus operators in the basis in which the matrix is diagonal. It is the same trick that we used in (5.33) for a 2-level system. that case was particularly simple because we have seen that to reach any state in the Bloch Ball one should act first with a non-unitary evolution (5.33), to choose the “radius” of the orbit, and than make an unitary evolution to reach the desired state on the Spherical Orbit. Now there are new possibilities of Kraus evolutions. First of all we introduce the most general Kraus evolution that preserves the diagonal form of the density matrix. Let us start from a maximal point,i.e. a pure state. The Kraus representation of the superoperator $\mathfrak{S}$ is such that:

$$\rho_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \mathfrak{S}(\rho_0) = \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1-p-q \end{bmatrix} \tag{6.3}$$

where $0 \leq q, p \leq 1$. A possible Kraus Representation of this superoperator is:

$$E_1 = \sqrt{p} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \sqrt{q} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \sqrt{1-p-q} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{6.4}$$

again defined up to unitary matrices that leaveρ_0 invariant. It is very easy to verify the normalisation condition: $\sum_{a=1}^{3} E_a^\dagger E_a = \mathbb{I}$. Initially, we would like to focus on Kraus representation with only one parameter, that can be seen as a subset of the most general case (6.4), to obtain a simple way to visualize the space of diagonal 3×3 density matrices. In fact we have seen in (4.50) that every diagonal density matrix $\rho = diag(k_1, k_2, k_3)$ can be decomposed on the basis made of the Gell-Mann matrices as:

$$\rho_{\mathbb{I}} = \frac{1}{3}, \qquad \rho_3 = \frac{k_1 - k_2}{2}, \qquad \rho_8 = \frac{k_1 + k_2 - 2k_3}{2\sqrt{3}} \tag{6.5}$$

where $\rho_{\mathbb{I}}$,ρ_3 and ρ_8 are the coefficients of the matrix $\mathbb{I}$, t_3 and t_8 respectively. So we can associate to each diagonal density matrix ρ a two dimensional vector that is:

$$\mathbf{S} = (k_1 - k_2, \frac{k_1 + k_2 - 2k_3}{\sqrt{3}}) \tag{6.6}$$

Compared with the generalized Bloch vector that we have introduced previously, it is clear that $\mathbf{S}$ contains only the components s_3 and s_8. This means that all the diagonal density matrices can be represented in a 2-dimensional subspace of $\mathbb{R}^8$. As a first example we can represent the maximally entangled matrix as:

$$\rho_{ME} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \Rightarrow \mathbf{S} = (0, 0). \tag{6.7}$$

Hence $\mathbf{S}_{ME}$ will be the origin of our 2-dimensional space. We will now act with a one-parameter Kraus operators starting from the pure state $\rho_{P1} =$

$diag(1,0,0)$ and we will represent the transformed states in $\mathbb{R}^2$ using the formula (6.6). First of all we can write the three pure states:

$$\rho_{P1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \rho_{P2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0, \\ 0 & 0 & 0 \end{bmatrix}, \quad \rho_{P3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.8}$$

and the respective $\mathbf{S}$ vectors:

$$\mathbf{S}_{P1} = (1, \frac{1}{\sqrt{3}}), \quad \mathbf{S}_{P2} = (-1, \frac{1}{\sqrt{3}}), \quad \mathbf{S}_{P1} = (0, -\frac{2}{\sqrt{3}}). \tag{6.9}$$

So we draw Fig.6.1. Pure states are in the same orbit of the U(3) group.

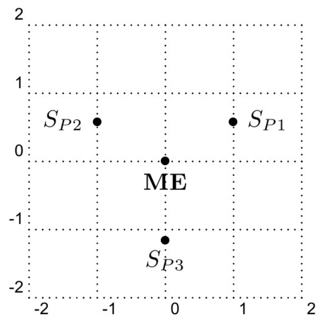

Figure 6.1: *Pure states and maximally entangled state in* $\mathbb{R}^2$

In fact starting from ρ_{P1} one can reach the others with the following unitary transformations:

$$\rho_{P1} \mapsto \rho_{P2} = U_{12}\rho_{P1}U_{12}^{\dagger} \quad \text{where} \quad U_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6.10}$$

$$\rho_{P1} \mapsto \rho_{P3} = U_{13}\rho_{P1}U_{13}^{\dagger} \quad \text{where} \quad U_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{6.11}$$

$$\rho_{P2} \mapsto \rho_{P3} = U_{23}\rho_{P2}U_{23}^{\dagger} \quad \text{where} \quad U_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{6.12}$$

that permute the eigenvalues of ρ_{Pj} Moreover it is easy to check that the representative vectors of the pure states have the same euclidean length, that is:

$$|\mathbf{S}_{P1}|^2 = |\mathbf{S}_{P2}|^2 = |\mathbf{S}_{P3}|^2 = \frac{4}{3}. \tag{6.13}$$

We are now ready to study the first superoperator evolution that depends on only one parameter;

$$\rho_{P1} \mapsto \mathfrak{S}_1(\rho_{P1}) = \begin{bmatrix} p & 0 & 0 \\ 0 & 1-p & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{6.14}$$

that can be represented with the Kraus operators:

$$E_1 = \sqrt{p}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \sqrt{1-p}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.15}$$

with the condition $\frac{1}{2} \leq p \leq 1$.

The choice on the range of p is not casual but respects the physics of our system as we are going to explain. One can easily prove that if $p = \frac{1}{2}$ any other Kraus transformation of the same form of (6.15) cannot modify the state:

$$\rho_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \mathbf{S}_1 = (0, \frac{1}{\sqrt{3}}) \tag{6.16}$$

Remark 26. This is the same reasoning the we have done for 2×2 matrices, and reflects the physical property that a superoperator evolution that is not unitary is neither invertible.

On the other hand we note that the matrix (6.14) can be obtained by making the convex combination between ρ_{P1} and ρ_{P2}:

$$\rho_{P1}(p) + \rho_{P2}(1-p). \tag{6.17}$$

In this notation it is clear that if $p \in [0,1]$ we are parametrizing the segment between ρ_{P1} and ρ_{P2}. Nevertheless this convex combination has no physical meaning (i.e. it does not represent a physical transformation). In fact, on one hand if p could pass from $p(0) = 0$ to $p(T) = 1$, the state $\rho(0) = \rho_{P1}$ will evolve toward the state $\rho(T) = \rho_{P2}$; on the other hand we have seen that for $p = \frac{1}{2}$ there is no Kraus transformation able to decrease further p. This motivates the choice of the range of p, and implies that with our Kraus Transformation we parametrize the segment between ρ_{P1} and ρ_1.

Remark 27. Actually we can parametrize the segment between ρ_{P2} and ρ_2 acting with a Kraus transformation on ρ_{P2} of the same form of (6.15) but with p that flows from $\frac{1}{2}$ and 0 .

As a last property we underline that the segment between ρ_{P1} and ρ_1 is oriented; this means that under Physical evolutions the states can only get closer to ρ_1, or at most remain equidistant from ρ_1, but they never get closer to ρ_{P1}. To show this we act with two superoperators $\mathfrak{S}_1$ starting from ρ_{P1} we have:

$$\rho_{P1} \mapsto \begin{bmatrix} p & 0 & 0 \\ 0 & 1-p & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} pq+(1-q)(1-p) & 0 & 0 \\ 0 & q(1-p)+p(1-q) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.18}$$

Hence p becomes $pq+(1-q)(1-p)$, if we check when $pq+(1-q)(1-p) \leq p$, we obtain $p \geq \frac{1}{2}$ that is always true in this segment. This prove that the orientation of the segment is from ρ_{P1} to ρ_1. We sum up all these results in the Fig 6.2 . We can repeat all this reasoning for the superoperator $\mathfrak{S}_2$ such

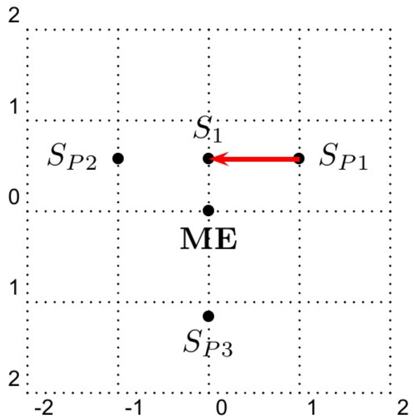

Figure 6.2: *The oriented segment from ρ_{P1} to ρ_1*

that:

$$\mathfrak{S}_2(\rho_{P1}) = \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-p \end{bmatrix}. \tag{6.19}$$

with Kraus Representation:

$$E_1 = \sqrt{p} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \sqrt{1-p} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{6.20}$$

with the condition $\frac{1}{2} \leq p \leq 1$.

In this case the oriented segment must end at:

$$\rho_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \Longrightarrow \mathbf{S}_2 = (\frac{1}{2}, -\frac{1}{2\sqrt{3}}) \tag{6.21}$$

Finally it is quite intuitive that starting from ρ_{P2} or ρ_{P3} one can find another special point at:

$$\rho_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \Longrightarrow \mathbf{S}_3 = (-\frac{1}{2}, -\frac{1}{2\sqrt{3}}) \tag{6.22}$$

So we can draw physical paths on $\mathbb{R}^2$ as in fig.6.3

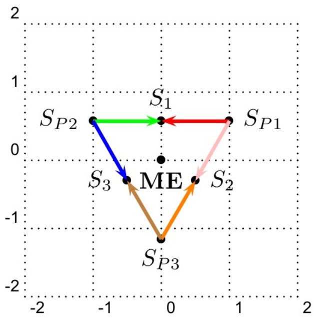

Figure 6.3: *The oriented segments between pure states.*

Remark 28. The euclidean length of the vectors $\mathbf{S}_1$,$\mathbf{S}_2$ and $\mathbf{S}_3$ is the same, that is:

$$|\mathbf{S}_1|^2 = |\mathbf{S}_2|^2 = |\mathbf{S}_3|^2 = \frac{1}{3} \tag{6.23}$$

Remark 29. We would like to stress that also the points ρ_1, ρ_2 and ρ_3 are connected each other by the unitary transformations defined in (6.10),(6.11) and (6.12), that is, they belong to the same unitary orbit. This reasoning is general in fact each matrix in the space of diagonal density matrices can be written as:

$$\rho = p\rho_{P1} + q\rho_{P2} + (1-p-q)\rho_{P3}. \tag{6.24}$$

Acting with one of (6.10),(6.11) and (6.12) we can exchange for example p with q or $(1-p-q)$ and so on. So, it is sufficient to study a piece of the triangle, for example the "sub-triangle" $\rho_1\rho_{P1}\rho_{ME}$ and act with unitary transformations to study the other pieces of the triangle $\rho_{P1}\rho_{P2}\rho_{P3}$.

We can repeat all this construction starting from $\mathbf{S}_1$,$\mathbf{S}_2$ and $\mathbf{S}_3$ or equivalently from ρ_1,ρ_2 and ρ_3. We will make only an example: we can act with $\mathfrak{S}_2$ defined as (6.20) and we find:

$$\mathfrak{S}_2(\rho_1) = \frac{1}{2}\begin{bmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-p \end{bmatrix}. \tag{6.25}$$

As before this is the parametrization of the segment between ρ_1 and

$$\rho_A = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \Longrightarrow \mathbf{S}_A = (\frac{1}{4}, +\frac{1}{4\sqrt{3}}) \tag{6.26}$$

Finding the appropriate Kraus representation to parametrize the oriented segments between ρ_1,ρ_2 and ρ_3 is quite simple, so we will immediately report the result: there are other two special point, i.e. where the segments end, that are (fig.6.4 and fig.6.5):

$$\rho_B = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \Longrightarrow \mathbf{S}_B = (0, -\frac{1}{2\sqrt{3}}) \tag{6.27}$$

and

$$\rho_C = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \Longrightarrow \mathbf{S}_C = (-\frac{1}{4}, +\frac{1}{4\sqrt{3}}) \tag{6.28}$$

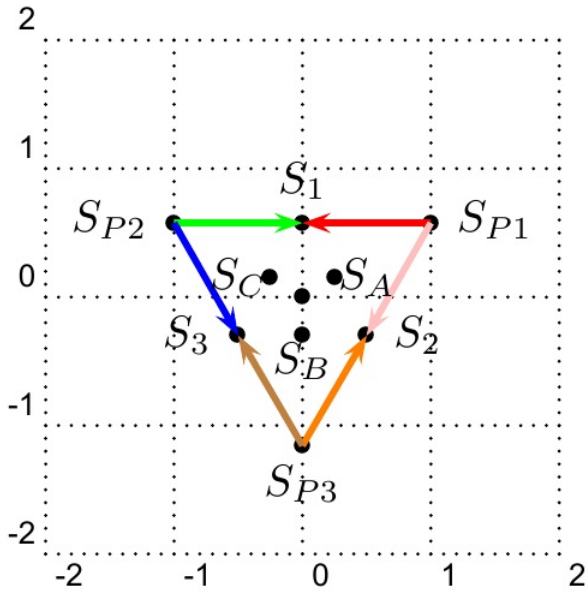

Figure 6.4: *Other "special" points.*

Following this pattern we can find other points and other oriented segments,closer and closer (with respect to the euclidean metric) to the origin. We can sum up the construction: starting from the pure states we can build a triangle. The midpoints of each side of the triangle (i.e. ρ_1,ρ_2 and ρ_3) are

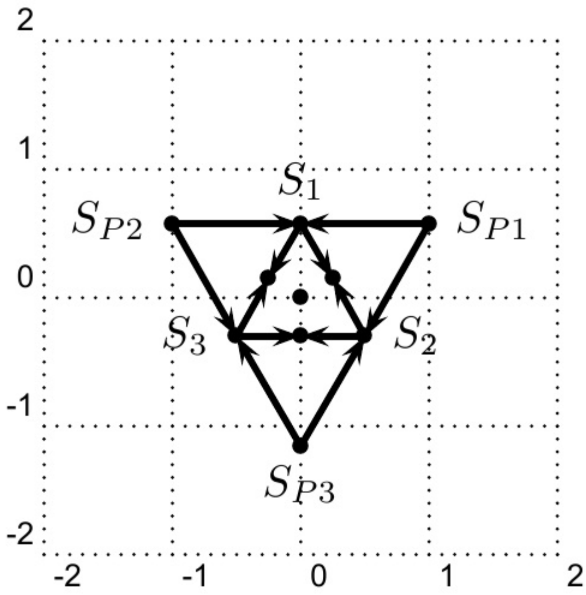

Figure 6.5: *The oriented segment*

special points from a physical point of view; in fact, as we have shown, acting with a Kraus transformation on a pure state, the transformed state can only flow toward the midpoint, it cannot flow in the opposite direction, and it cannot cross the midpoint. Then one has to repeat the same actions considering the three special points and act on these with other Kraus transformations different from the ones that have been used to reach the three midpoints.

Until now we have only analysed particular evolutions (i.e. only the evolutions that depends on one parameter) so our next step in this research will be to study the most general Kraus evolution which depends on two parameters. We will try to visualize the allowed evolutions using the pattern that we have found in the 3×3 space of diagonal density states. In fact if we understood how diagonal 3×3 density matrices evolve, we could recover the dynamics of every density matrix (i.e. even non-diagonal) by unitary acting. Moreover, we would like to reinterpret the Fisher metric on these transversal directions; this question is not trivial in fact in this case there are two independent directions. Furthermore we would like to answer to the Quantum Estimation question presented at the end of the previous chapter relatively to the case of 3×3 density matrices. As a final step we would like to generalize this study for an n-level systems.

Conclusions

To sum up what we have done in this work:

1. We have recalled some results, from [5, 10], that show how to relate the standard Hilbert space to the Projective Hilbert Space endowed with a Kähler structure. Moreover, following [10], we have showed how to built a Kähler structure on the orbit of the co-adjoint action of the unitary group starting from the Hermitian structure which the whole space of Hermitian operators is endowed of. We have also reported the explicit formulas of the Complex structure J, the metric tensor and symplectic form on this Co-adjoint orbits.

2. We have explicitly evaluated the Complex structure, the metric tensor and symplectic form, of the previous point , on the orbits passing through a 2×2 and 3×3 density matrix.

3. We have recollected some results about Fisher Tensor ([8, 6, 7]), in particular we have presented an algebraic method to evaluate it. Following the work [8], we have evaluated the real and the imaginary part of the Fisher tensor, because they are proportional to a metric tensor and a symlectic form. Hence, we have calculated the metric tensor and the symplectic form, obtained from the Fisher Tensor, in the case of 2×2 and 3×3 density matrices, and we have shown that these are the same tensors founded in the previous point up to a constant normalisation factor.

4. We have noticed that with the Fisher Tensor we can find a metric tensor also on non-unitary paths; i.e. curves that are not generated by the action of the unitary group. Because this kind of paths links different unitary orbits we have called these curves transversal directions. Then we have evaluated the metric tensor on these transversal directions for 2×2 and 3×3 density matrix.

5. We reflected upon what is the Physical meaning of the " transversal directions". This have led us to the dynamics of open system and to the

Kraus operators formalism. We have studied in detail the different paths on the 2×2 density states space using the Bloch ball to visualize them easily.

6. In the latter case we have found an explicit formula to parametrize the Kraus operator that make clear the semi-group structure of the "transversal transformations". This reflects the Physical property for which if the system undergoes a non-unitary evolution there is an arrow of time.

7. We have calculated the tangent vector to a Transversal path and we have found that the Lindblad generator of this kind of evolution is proportional to the Pauli Matrix σ_3, while the generators of unitary transformations,that do not stabilize the density matrices, are the Pauli Matrices σ_1 and σ_2.

8. We have presented a new scheme to study the 3×3 density matrices, that will be the starting point of our further research.

These results are only a small step in this new research. We hope that, studying deeply the 3×3 case, we can find a way to generalize these result for a generic n-level system.

Appendix A

Complex Manifolds

In this appendix and in the following ones we recollect some definition and results about some mathematical structures that are used in the work. There is no completeness demand but we only want build a "ready to use" and intuitive collection of theoretical tools.

A.1 Almost complex manifolds

First of all we recall the definition of complex manifold.

Definition A.1 (Complex Manifold)**.** A complex manifold is a differentiable manifold with a holomorphic atlas. They are necessarily of even dimension,i.e. 2n, they can be endowed with a collection of charts (U_j, z_j) that are in one to one correspondence with $\mathbb{C}^n$ such that for every non-empty intersection $U_j \cap U_i$,the map composition $z_j \circ z_k^{-1}$ are holomorphic.

In other words a complex manifold is like a real one but with another request: the functions which relate the coordinates in overlapping patches are holomorphic. An almost complex manifold, as the word says, is "not quite" complex in fact:

Definition A.2 (Almost Complex Manifold)**.** A real manifold M with dimension m that can be endowed with a globally defined tensor J of rank (1,1) with the property:

$$J^2 = -\mathbb{I} \tag{A.1}$$

is called an almost complex manifold. Moreover J is called an *almost complex structure.*

To be clear we specify that both J and $\mathbb{I}$ are (1,1)-tensor fields, then they map the tangent bundle TM into itself. Making everything more explicit, we can consider J locally. Chosen a $p \in M$ there is a endomorphism $J_p : T_pM \mapsto T_pM$ such that $(J_p)^2 = -\mathbb{I}_p$ and that depend smoothly on p. Here $\mathbb{I}_p$ indicates the identity operator acting on the tangent space T_pM at the point p.Introducing a basis for the tangent space $\frac{\partial}{\partial x^\mu}$ and a basis for the cotangent space dx^μ, where x^μ with $\mu = 1, 2..., m$ are the coordinates of p; we can write:

$$J_p = J^\nu_\mu(p)\frac{\partial}{\partial x^\nu} \otimes dx^\mu \tag{A.2}$$

with $J^\nu_\mu(p)$ real. Given a vector field $X = X^\nu \frac{\partial}{\partial x^\nu}$ Then

$$J(X) = (X^\mu J^\nu_\mu)\frac{\partial}{\partial x^\nu}$$

and

$$J^2(X) = (X^\rho J^\mu_\rho J^\nu_\mu)\frac{\partial}{\partial x^\nu}$$

In this way we can rewrite the condition for an almost complex structure in local coordinates:

$$J(p)^\rho_\mu J(p)^\nu_\rho = -\delta^\nu_\mu \qquad \forall p \in M \tag{A.3}$$

Having an almost complex structure globally defined, means that J_p is well defined in every patch and we could join them together with no singularities. From the property of the almost complex structure it is easy to prove that

Theorem A.1.1. *Almost complex manifolds have even dimension*

Now we are ready to introduce and important operation that we use extensively in this work

A.1.1 Complexification of the tangent space

It is possible to complexify the tangent space, introducing linear combinations of vector fields with complex coefficients. Given X and Y, vector fields in TM, we can write:

$$Z = \frac{1}{2}(X + iY) \qquad \bar{Z} = \frac{1}{2}(X - iY) \tag{A.4}$$

These kind of vector fields generate at point p the complexified tangent space that we denote as $T_pM^{\mathbb{C}}$ or $T_pM \otimes \mathbb{C}$

Theorem A.1.2. *The eigenvalues of J_p can only be $\pm i$ on $T_pM^{\mathbb{C}}$*
Proof

On $TM^{\mathbb{C}}$ one can define the projectors operators

$$P^{\pm} = \frac{1}{2}(\mathbb{I} \mp iJ) \tag{A.5}$$

such that

$$(P^{\mp})^2 = p^{\pm} \quad P^{+} + P^{-} = \mathbb{I} \quad P^{+}P^{-} = 0$$

Considering an arbitrary element $T \in T_pM^{\mathbb{C}}$ and defining:

$$Z \equiv P^{+}(W) = \frac{1}{2}(W - iJ(W)) \quad \bar{Z} \equiv P^{-}(W) = \frac{1}{2}(W + iJ(W))$$

it is straightforward to show that $J(Z) = iZ$ and $J(\bar{Z}) = -i\bar{Z}$ □

This theorem allows us to consider $T_pM^{\mathbb{C}}$ as a direct sum:

$$T_pM^{\mathbb{C}} = T_pM^{+} \oplus T_pM^{-}$$

with $T_pM^{\pm} = \{Z \in T_pM^{\mathbb{C}} | J_p(Z) = \pm iZ\}$.
We call the elements in $T_pM^{\pm}$ holomorphic and anti-holomorphic vectors respectively.

A.2 Complex and almost complex manifolds

Definition A.3. Let (M, J) be an almost complex manifold. The almost complex structure J is said *integrable* if the Lie Brackets of any two Holomorphic vector fields is again a holomorphic vector field.

Definition A.4 (Nijenhuis tensor). for any two vector fields X, Y we define Nijenhuis tensor N:

$$N(X;Y) \equiv [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]. \tag{A.6}$$

Theorem A.2.1. *An almost complex structure J on a M is integrable if and only if $N(X,Y) = 0$*

All the previous definitions and theorem are necessary to understand the following and fundamental result

Theorem A.2.2. *Let (M, J) be an almost complex manifold. J is integrable if and only if the manifold M is complex*

For this reason an almost complex structure is said complex structure if it is integrable.

A.3 Kähler manifolds

Definition A.5 (Kähler Manifold)**.** Let K be a real and even-dimensional manifold with:

- a complex structure J such that $J^2 = -\mathbb{I}$
- a closed, non-degenerate two-form satisfying:

$$\omega(x, Jy) + \omega(Jx, y) = 0 \tag{A.7}$$

  with $x, y \in TK$. In other words ω is a symplectic structure.

- a positive (0,2)-tensor $g(.,.)$ such that:

$$g(.,.) =: \omega(., J(.)) \qquad g(x, y) =: \omega(x, Jy) \tag{A.8}$$

  Note that equation (1.16) implies that g is symmetric and non-degenerate iff ω is non-degenerate. In this latter case g is a metric.

In this case K is said Kähler Manifold.

Appendix B

Distributions and Foliations

We give some basic and intuitive definitions and results about smooth manifold theory. The first section introduces the notation and terminology; moreover we try to build an intuitive idea of Distribution via an analogy with the concept of integral curves. Then we present the Frobenius theorem and we conclude defining the concept of Foliation and its connection to the Frobenius theorem.

B.1 Preliminaries

Let M be and $m-$dimensional manifold,(everything in this appendix is implicitly assumed smooth), T_pM the tangent space to $p \in M$, and TM the tangent bundle of M. Recalling that a vector field on M is a section of TM, it can be thought of as a choice of a tangent vector at every point in the manifold M. Given a vector field one can consider the integral curve associated to it.

Definition B.1. An *integral curve of a vector field V is a curve* $\gamma : [a, b] \mapsto M$, *where a and b are real numbers, such that:*

$$\gamma'(t) = V_{\gamma(t)} \qquad \forall t \in [a, b] \tag{B.1}$$

That is, the tangent vector at any point of the curve, with respect to time is precisely the value of the vector field at that point. It is important to remember that the integral curves are connected and 1-dimensional submanifold of M. In what follows we try to generalize this idea to connected $k-$dimensional submanifold. Firstly we take the concept of vector field and increase its dimension.

Definition B.2. A $k-$dimensional (tangent) distribution on M is a choice of a $k-$dimensional linear subspace $D_p \subset T_pM$ at each point $p \in M$. We will denote this with D, where

$$D = \bigcup_{p \in M} D_p \subset TM \tag{B.2}$$

.

If D is a $k-$dimensional distribution it is possible to find k vector fields, $V^1, ..., V^k$ such that the collection of vectors,$V_p^1, ..., V_p^k$ forms a basis for D_p at each $p \in U$, where U is neighborhood in M. Now, let D be a $k-$dimensional distribution and consider a $k-$dimensional submanifold $S \subset M$.Then for $s \in S$, D_s is a $k-$dimensional linear subspace of T_sM. A natural question arises: does T_sS correspond to D_s, and going further, does there exist a submanifold such that $T_sS = D_s$. Such S would be analogous to the integral curves but with a higher dimension.

Definition B.3 (Integral Manifold and Integrable Distribution)**.** An immersed submanifold S is an integral manifold of the distribution D if $T_sS = D_s$ for all $s \in S$, and D is said integrable if each point of M is contained in an integral manifold of D.

To answer the previous questions we need another definition:

Definition B.4 (Involutive distribution)**.** A distribution is called involutive if given to vector fields V and W with $V_p, W_p \in D_p$ for all p in some neighborhood U, it follows that $[V_p, W_p] \in D_p$

Now we are ready to introduce the following:

B.1.1. *If D is an integrable distribution, then D is necessarily involutive.*

B.2 The Frobenius Theorem

Let $U \subset M$ be an open neighbothood with parametrization $\phi : U \mapsto \mathbb{R}^m$, and let D a $k-$dimensional distribution:

Definition B.5 (Flat Parametrization and Completely Integrable Distribution)**.** A parametrization ϕ is flat for D if $\phi(U) \subset \mathbb{R}^m$ is a product of connected open sets in $\mathbb{R}^k \times \mathbb{R}^{m-k}$ and for each $p \in U$, D_p is spanned by the first k basis vector fields. A distribution D is completely integrable if there exists a flat parametrization for D in a neighborhood of every point of M.

Than we can report the fundamental result, known as The Frobenius Theorem

Theorem B.2.1. *If a distribution D is involutive, then D is completely integrable.*

This result can be formulated in different terms, for example using the concept of foliation, this motivates the next section.

B.3 Foliations

Naively, a foliation is an equivalence relation on an $m-$manifold M in which the equivalence classes are connected immersed submanifolds of the same dimension k.

Definition B.6. A $k-$dimensional foliation on an $m-$manifold M is a collection of disjoint, connected,immersed $k-$dimensional submanifolds of M (called the leaves of the foliation) such that:

- the union of the leaves forms all M
- there is a parametrization ϕ in every neighbourhood U of $p \in M$, such that $\phi(U)$ is a product of connected open sets in $\mathbb{R}^k \times \mathbb{R}^{m-k}$. Moreover, the intersection of each leaf with U has the last $m - k$ local coordinates constant.

We are ready to see the Foliation version of the Frobenius theorem, starting from the following lemma.

B.3.1. *If F is a $k-$dimensional foliation of M, then the collection of tangent spaces to the leaves of F form an involutive distribution.*

We can also re-write the Frobenius theorem as:

Theorem B.3.2. *If D is an involutive distribution on M, then the collection of all maximal connected integral manifold of D forms a foliation of M*

Appendix C

Lie Group Actions

C.1 Smooth action

Let M a smooth manifold and Diff(M) the group of diffeomorphisms from M to M

Definition C.1. An action of a Lie group G on M is a homomorphism of groups $\tau : G \mapsto Diff(M)$. We will use equivalently the notation $\tau(g)$ and τ_g.

Explicitly, for any $g \in G$, $\tau(g) \equiv \tau_g : M \mapsto M$ is a diffeomorphism such that $\tau(g_1 g_2) = \tau(g_1) \circ \tau(g_2)$. We also say that the action is smooth if the map:

$$G \times M \mapsto M, \qquad (g, m) \mapsto \tau_g(m) \tag{C.1}$$

is smooth. We will denote for brevity $\tau(g)(m) \equiv \tau_g(m)$ by $g \cdot m$. We have just defined the *left action* of a Lie group. We could also define a *right action* to be an *anti*-homomorphism of groups i.e.

$$\hat{\tau} : G \mapsto Diff(M) \qquad \text{such that} \qquad \hat{\tau}(g_1 g_2) = \hat{\tau}(g_1)\hat{\tau}(g_2)$$

Any left action τ can be converted to a right action $\hat{\tau}$ by requiring $\hat{\tau}_g(m) = \tau(g^{-1})(m) \equiv g^{-1} \cdot m$

C.2 Orbits and stabilizers

Let $\tau : G \mapsto Diff(M)$ be a smooth action.

Definition C.2 (Orbit and Stabilizer)**.** We have:

1. the Orbit of G through $m \in M$ is

$$G \cdot m = \{g \cdot m | \forall g \in G\}$$

2. The Stabilizer (also called isotropic subgroup) of $m \in M$ is the subgroup

$$G_m = \{g \in G | g \cdot m = m\}$$

It is useful, for our aims, to recall that any orbit $G \cdot m$ is an immersed submanifold of M. We denote the set of orbits by M/G. In what follows we would like to put conditions on actions to obtain interesting results.

Definition C.3. Let Lie group G acts smoothly on M.

1. The G-action is *transitive* if there is only one orbit, i.e. $M = G \cdot m$ for any $m \in M$
2. The G-action is *effective* or *faithful* if $\bigcap_{m \in M} G_m = e$.
3. The G-action is *proper* if the action map

$$\alpha : G \times M \mapsto M \times M, \qquad (g, m) \mapsto (g \cdots m, m)$$

   is proper, that is the inverse image of any compact set is compact.
4. The G-action is *free* if $G_m = \{e\}$ $\forall m \in M$ where e is the identity element,

We are ready to enunciate some important results.

Theorem C.2.1. *If the action is proper and free then M/G is a smooth manifold.*

Theorem C.2.2. *If the action is transitive then $\forall m \in M$ the map:*

$$F : G/G_m \mapsto M, \qquad gG_m \mapsto g \cdot m$$

is a diffeomorphism

In particular if the $G-$action on M is transitive, then $\forall m \in M$,

$$M \simeq G/G_m$$

Such a manifold is called a homogeneous space. For example we can consider the action of $O(n)$ on S^{n-1}. It is note that it is transitive. So S^{n-1} is an homogeneous space. Moreover, if we choose a point $m \in M$ it is easy to check that the isotropy group G_m is $O(n-1)$.Then

$$S^{n-1} \simeq O(n)/O(n-1)$$

The same result hold in the unitary case.

Appendix D

Hamiltonian Actions

In this appendix we will use some definition and notion of Appendix C.

D.1 Fundamental vector fields

First of all we recall that the Lie Algebra $\mathfrak{g}$ of a Group G can be defined as $\mathfrak{g} := T_e G$, that is the tangent space to G at the identity $e \in G$.

Definition D.1 (Fundamental Vector Field)**.** For each $\xi \in \mathfrak{g}$, the fundamental vector field on M, a smooth manifold, induced by ξ is the vector field defined by:

$$(\xi_M)_x := \frac{d}{dt}\tau_{exp(-t\xi)}(x)|_{t=0}$$

$\forall x \in M$

Intuitively, if we start with the curve $t \mapsto exp(t\xi) \in G$, whose tangent vector at $t = 0$ is ξ, and transport it in M via the action τ, we obtain for each $x \in M$ the curve $t \mapsto \tau_{exp(t\xi)}$ with tangent vector at $t = 0$ equal to the Fundamental vector field. In other word we can think the fundamental vector field as the vector field tangent to the curve passing from x at $t = 0$ generated from the action $\tau_{exp(t\xi)}$

D.2 Symplectic manifolds

Definition D.2. A symplectic structure on M is a smooth differential 2-form ω that is closed and non-degenerate. A symplectic manifold is a pair (M, ω) consisting of a smooth manifold M and a choice of symplectic structure ω on it.

Let $f : M \mapsto N$ be a smooth map between manifolds. f defines a pull-back map f^* from differential forms on N to differential forms on M. In particular is α is a differential $k-$form on N, then $f^*\alpha$ is the $k - form$ on M defined by:

$$(f^*\alpha)_x(X_1, ..., X_k) = \alpha_{f(x)}((T_x f)X_1, ..., (T_x f)X_k) \qquad \forall x \in M$$

where $X_1, ..., X_k \in T_x M$ and Tf is the induced map on tangent bundles, that is $Tf = TM \mapsto TN$.

Definition D.3 (symplectic map and symplectomorphism)**.** Let (M, ω) and (N, σ) be symplectic manifolds, a smooth map $f : M \mapsto N$ is said symplectic if $f^*\sigma = \omega$. Moreover f is said a symplectomorphism if it is a symplectic diffeomorphism.

D.3 Hamiltonian vector fields

Let us consider the bundle isomorphism induced by the symplectic form ω on M such that $\tilde{\omega} : TM \mapsto T^*M$. This $\tilde{\omega}$ induces an isomorphism between smooth sections of these two bundles,which are vector field and differential one-forms respectively,defined by:

$$X \mapsto \tilde{\omega} \circ X \equiv i_X \omega \tag{D.1}$$

For each vector field $X \in TM$. Fixed a point $x \in M$, the one-form $\tilde{\omega}(X_x) \in T_x^*M$ associated to the vector $X_x \in T_x M$ is:

$$\tilde{\omega}(X_x) = i_{X_x}\omega_x = \omega_x(X_x, .)$$

where the " . " indicates that we are waiting for a tangent vector to be plugged in.

Definition D.4 (Hamiltonian Vector Field)**.** Let $f : M \mapsto \mathbb{R}$ be a smooth function. The Hamiltonian vector field of f is the smooth vector field X_f corresponding to the differential one-form df via the bundle map:

$$\tilde{\omega}^{-1} : T^*M \mapsto TM. \tag{D.2}$$

Explicitly:

$$\tilde{\omega}^{-1}(., df) := X_f \Rightarrow df = \tilde{\omega} \circ \tilde{\omega}^{-1}(., df) = \tilde{\omega} \circ X_f = \omega_x(X_f, .) = i_{X_f}\omega$$

Moreover an arbitrary vector field X on M is called Hamiltonian if it is the Hamiltonian vector field of some smooth function $f : M \mapsto \mathbb{R} : X = X_f$

D.4 Hamiltonian group actions

In this section we use, for brevity, the notation $\langle .|. \rangle$ to denote the pairing between covectors and vectors;that is $\langle .|. \rangle : \mathfrak{g}^* \times \mathfrak{g} \mapsto \mathbb{R}$. So if $\lambda \in \mathfrak{g}$ and $\xi \in \mathfrak{g}$ then:

$$\lambda(\xi) = \xi(\lambda) = \langle \lambda, \xi \rangle$$

Recall that the conjugation action of G on itself

$$\phi : G \mapsto Aut(G) \qquad g \mapsto \phi_g$$

with $g, h \in G$, induces a linear action of G on $\mathfrak{g}$. In fact it is easy to check that the differential $d(\phi_g) \in Aut(\mathfrak{g})$. In symbol

$$d(\phi_g)|_e := Ad_g : \mathfrak{g} \mapsto \mathfrak{g}$$

The map $Ad : G \mapsto Aut(\mathfrak{g})$ is called Adjoint representation, where $Aut(\mathfrak{g})$ is the collection of isomorphism from $\mathfrak{g}$ to itself. This induces, in turn, a linear action of G on the dual space $\mathfrak{g}^*$, denoted by $Coad : G \mapsto Aut(\mathfrak{g}^*)$ defined by:

$$Coad_g(\lambda) := \lambda \circ Ad_{g^{-1}}$$

for all $g \in G$ and $\lambda \in \mathfrak{g}^*$. To be clearer:

$$\langle Coad_g(\lambda), \xi \rangle = \langle \lambda, Ad_{g^{-1}}(\xi) \rangle$$

where $\xi \in \mathfrak{g}$

Definition D.5 (Momentum Map)**.** Let $\phi : M \mapsto \mathfrak{g}^*$ be a smooth map. $\forall \xi \in \mathfrak{g}$ we denote $\phi^\xi = \langle \phi(.), \xi \rangle$ the smooth map $M \mapsto \mathbb{R}$ defined by $x \mapsto \langle \phi(x), \xi \rangle$.

The map ϕ is called momentum map for the action τ of G on (M, ω) is it satisfies the following properties:

1. The map ϕ is $G-$equivariant, that is for all $g \in G$ the diagram:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & \mathfrak{g}^* \\ {\scriptstyle Ad_g}\downarrow & & \downarrow{\scriptstyle Coad_g} \\ M & \xrightarrow{\phi} & \mathfrak{g}^* \end{array} \tag{D.3}$$

   commutes

2. $\forall \xi \in \mathfrak{g}$, the fundamental vector field ξ_M is the Hamiltonian vector field corresponding to the function $\phi^\xi : M \mapsto \mathbb{R}$ that is:

$$d\phi^\xi = d \langle \phi, \xi \rangle = \omega(\xi_M, .).$$

Finally we say that a group action is Hamiltonian if it is symplectic and there exist a moment map $\phi : M \mapsto \mathfrak{g}^*$ for it.

Bibliography

[1] Aliki, R., and Lendi, K. *Quantum Dynamical semigroup and applications.* Springer, 2007.

[2] Bengtsson, I., and Zyczkowski, K. *Geometry of quantum states.* Cambridge Press, 2006.

[3] Brandorff-Nielsen, O. E., and D., G. R. Fisher information in quantum statistics. *J. Phys. A 33* (2000), 4481–4490.

[4] Braunstein, S. L., and Caves, C. M. Statistical distance and the geometry of quantum states. *Phys. Rev. Lett 37* (1994), 3439–3443.

[5] Ercolessi, E., Marmo, G., and Morandi, G. *La rivista del nuovo cimento 33*, 401 (2010).

[6] Ercolessi, E., and Schiavina, M. *J. Phys. A:Math.Theor. 45*, 365303 (2012).

[7] Ercolessi, E., and Schiavina, M. *Phys. Lett. A 377*, 1996 (2013).

[8] Ercolessi, E., Schiavina, M., and Contreras, I. On the geometry of mixed states and the fisher information tensor. *ArXiv:1501.00054v1 [Math-ph]* (31 Dec 2014).

[9] Facchi, P., Kulkarni, R., Man'ko, V. I., Marmo, G., Sudarshan, E., and Ventriglia, F. On the geometry of mixed states and the fisher information tensor. *Phy. Let A 374*, 4801-4803 (2010).

[10] Grabowski, J., Kus, M., and Marmo, G. *J. Phys. A 381*, 10217 (2005).

[11] Kirillov, A. A. *Lectures on the orbit method*, vol. 64. American mathematical society, 2004.

[12] Luati, A. Maximum fisher information in mixed quantum system. *The Ann. of Stat 32*, 4 (2004), 1770–1779.

[13] Morandi, G., Ercolessi, E., and Napoli, F. *Statistical Mechanics: An Intermediate Course*. World Scientific, 2001.

[14] Nielsen, M. A., and Chuang, I. L. *Quantum Computation and Quantum Information*. Cambridge University Press, 2010.

[15] Polchinski, J. Weinberg's nonlinear quantum mechanics and the epr paradox. *Phy. Rev. Let.* (1991).

[16] Preskill, J. Lecture notes for physics229: Quantum information and computation. California Institute of Technology, September 1998.

[17] Schirmer, S. G., Zhang, T., and Leahy, J. V. Orbits of quantum states and geometry of bloch vectors for n-level system. *J.Phys.A: Math. Gen 37* (1389 2004).