Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea in Matematica

REGULAR AND EXACT CATEGORIES

Tesi di Laurea in Teoria delle Categorie

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Al mi' Babbo e alla mi' Mamma.

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Introduction

The aim of this work is to present, in the context of algebraic category theory, some results concerning regular and exact categories. The notions of regularity and exactness recapture many of the exactness properties of abelian categories, but do not require additivity, which is a very strong condition on the Hom sets's structure. In particular we will prove that an abelian category is additive and exact (i.e. a regular category in which equivalence relations are effective). Finally, to provide the reader with a wide range of examples, we will prove that an elementary topos is an exact category.

The language of exact categories supplies a handy structure to work with. The usefulness of properties as the existence of an epi-mono factorization or the stability under pullback of epimorphisms becomes apparent as soon as we start working with this sort of categories. Moreover, such a language allows to talk about homology, cohomology and even homotopy (see [EC]).

Lo scopo di questo lavoro è quello di presentare, nell'ambito della teoria delle categorie algebrica, alcuni risultati riguardanti categorie regolari ed esatte. Le nozioni di esattezza e regolarità catturano molte delle proprietà di esattezza delle categorie abeliane, ma non richiedono l'additività che risulta essere una condizione molto forte sulla struttura degli Hom set. In particolare dimostremo che una categoria abeliana altro non è che una categoria esatta (i.e. regolare le cui relazioni di equivalenza sono effettive) e additiva. Infine, con lo scopo di fornire al lettore un ampio spettro di esempi, dimostreremo che un topos elementare è una categoria esatta.

Il linguaggio delle categorie esatte fornisce una struttura comoda su cui lavorare: l'utilità di proprietà come l'esistenza di una fattorizzazione epi-mono o la stabilità per pullback degli epimorfismi diviene chiara non appena si inizia a lavorare con categorie di questo tipo. Inoltre tale linguaggio permette di parlare di omologia e comologia ed addirittura di omotopia (si veda [EC]).

Chapter 1

Preliminaries

This chapter is intended to give some basic definitions and results in category theory. For all the definitions, results and notations which are taken for granted please refer to [H1].

1.1 Pullbacks

We will not give a wide view on limits and their properties. We will give a few results on pullbacks which may be useful for the reading of further chapters.

Diagram 1.1.

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \\ {\scriptstyle c}\downarrow & (I) & \downarrow{\scriptstyle d} & (II) & \downarrow{\scriptstyle e} \\ D & \xrightarrow[f]{} & E & \xrightarrow[g]{} & F \end{array}$$

Proposition 1.1.1 (see [H1] 2.5.9)**.** *In a category $\mathcal{C}$ consider Diagram 1.1, which is commutative.*

1. *If the square (I) and (II) are pullbacks, the outer rectangle is a pullback.*

2. *If $\mathcal{C}$ has pullbacks, if the square (II) is a pullback and the outer rectangle is a pullback, then (I) is a pullback.*

Proposition 1.1.2 (see [H1] 2.5.6)**.** *Consider a morphism $f : A \longrightarrow B$ in a category $\mathcal{C}$. The following conditions are equivalent:*

1. *f is a monomorphism;*

2. *the kernel pair of f exists and is given by $(A, 1_A, 1_A)$;*

3. *the kernel pair (P, α, β) of f exists and $\alpha = \beta$.*

Proposition 1.1.3 (see [H1] 2.5.7 and 2.5.8). *In a category $\mathcal{C}$, if e is a coequalizer and it has a kernel pair, then it is the coequalizer of its kernel pair. Furthermore, if a kernel pair (a, b) has a coequalizer, it is the kernel pair of its coequalizer.*

Proposition 1.1.4 (see [JC] 11.13). *If T is a terminal object, then the following are equivalent:*

$$\begin{array}{ccc} P & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & & \downarrow \\ B & \longrightarrow & T \end{array}$$

1. *the diagram above is a pullback square;*

2. *(P, p_A, p_B) is the product of A and B.*

1.2 Epi-mono factorizations

Definition 1.2.1. In a category, an epimorphism is called regular if it is the coequalizer of a pair of morphisms.

Diagram 1.2.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & \overset{w}{\swarrow} & \downarrow{\scriptstyle v} \\ X & \xrightarrow[z]{} & Y \end{array}$$

Definition 1.2.2. In a category $\mathcal{C}$, an epimorphism $f : A \longrightarrow B$ is called strong when, for every commutative square $v \circ f = z \circ u$ as in diagram 1.2, with $z : X \longrightarrow Y$ a monomorphism, there exists a unique arrow $w : B \longrightarrow X$ such that $w \circ f = u$ and $z \circ w = v$.

Proposition 1.2.3. *In a category $\mathcal{C}$,*

1. *the composite of two strong epimorphisms is a strong epimorphism;*

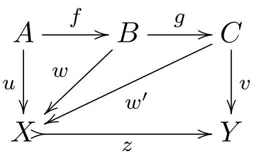
2. *if a composite $g \circ f$ is a strong epimorphism, g is a strong epimorphism;*

3. *a morphism which is both a strong epimorphism and a monomorphism is an isomorphism;*

4. *every regular epimorphism is strong.*

Proof. (1) Suppose f and g to be strong epimorphisms. In diagram 1.3 choose $z \circ u = v \circ g \circ f$ with z monomorphism. Since f is a strong epimorphism there exists $w : B \longrightarrow X$ such that $v \circ g = z \circ w$. Since g is a strong epimorphism there also exists $w' : C \longrightarrow X$ such that $v = z \circ w'$ and $w' \circ f = w$. From the last equation we get $w' \circ (g \circ f) = w \circ f = u$, therefore w' is the required factorization.

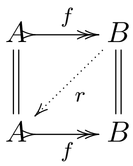
Diagram 1.3.



(2) Now suppose that $g \circ f$ is a strong epimorphism in Diagram 1.3 choose $z \circ w = v \circ g$, with z monomorphism. Putting $u = w \circ f$ we get a factorization w' such that $w' \circ g \circ f = u$, $z \circ w' = v$ since $g \circ f$ is a strong epimorphism. From $z \circ w' \circ g = v \circ g = z \circ w$ we deduce $w' \circ g = w$ since z is a monomorphism. Therefore w' is the required factorization.

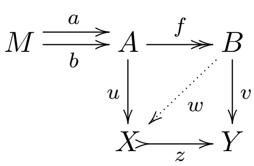
(3) If f is both a strong epimorphism and a monomorphism, we find, considering diagram 1.4, r such that $f \circ r = 1_B$ and $r \circ f = 1_A$.

Diagram 1.4.



(4) Finally, if $f = \mathbf{Coker}(a, b)$ and in diagram 1.5 we choose $z \circ u = v \circ f$ with z monomorphism, since $f \circ a = f \circ b$ we find $z \circ u \circ b = v \circ f \circ b = v \circ f \circ a = z \circ u \circ a$ thus $u \circ b = u \circ a$ since z is a monomorphism. Therefore we get some factorization w through $f = \mathbf{Coker}(a, b)$. From $w \circ f = u$ we deduce $z \circ w \circ f = u \circ z = v \circ f$ and thus $z \circ w = v$ since f is an epimorphism.

Diagram 1.5.



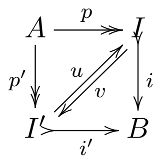
□

Definition 1.2.4. A category $\mathcal{C}$ has strong epi-mono factorization when every morphism f of $\mathcal{C}$ factors as $f = i \circ p$, with i a monomorphism and p a strong epimorphism.

Proposition 1.2.5. *Let $\mathcal{C}$ be a category with strong epi-mono factorizations. The strong epi-mono factorization of an arrow is unique up to an isomorphism.*

Proof.

Diagram 1.6.



Given $i \circ p = i' \circ p'$ with p, p' strong epimorphisms and i, i' monomorphisms, consider diagram 1.6. There exists u such that $u \circ p = p'$, $i' \circ u = i$ because p is a strong epimorphism and i' is a monomorphism. Since p' is a strong epimorphism and i is a monomorphism there exists v such that $i \circ v = i$ and $v \circ p' = p$. Therefore $i \circ v \circ u \circ p = i' \circ p' = i \circ p$ thus $v \circ u = 1_I$. In a similar way $u \circ v = 1_{I'}$ □

1.3 Additive and abelian categories

Definition 1.3.1. By a zero object in a category $\mathcal{C}$, we mean an object $\mathbf{0}$ which is both an initial and a terminal object. Moreover, in such a category a morphism is called zero morphism when it factors through the zero object.

Proposition 1.3.2. *Consider a category $\mathcal{C}$ with a zero object $\mathbf{0}$, there is exactly one zero morphism from each object A to each object B.*

Proof. This is just the composite of the unique morphisms $A \longrightarrow \mathbf{0}$, where $\mathbf{0}$ is considered as a terminal object, and $\mathbf{0} \longrightarrow B$, where $\mathbf{0}$ is considered as an initial object. □

Proposition 1.3.3. *In a category with zero object $\mathbf{0}$, the composite of a zero morphism with an arbitrary morphism is again a zero morphism.*

Proof. Of course, the composite factors through $\mathbf{0}$. □

Definition 1.3.4. In a category $\mathcal{C}$ with zero object $\mathbf{0}$, the kernel of an arrow $f : A \longrightarrow B$ is, when it exists, the equalizer of f and the zero morphism between A and B. The cokernel of f is defined dually.

Proposition 1.3.5. *Let f be a monomorphism in a category with a zero object. If $f \circ g = 0$ for some morphism g, then $g = 0$.*

Proof. $f \circ g = 0 = f \circ 0$, thus $g = 0$. □

Proposition 1.3.6. *In a category with zero object the kernel of a monomorphism $f : A \rightarrowtail B$ is just the zero morphism $\mathbf{0} \longrightarrow A$.*

Proof. The composite $\mathbf{0} \longrightarrow A \xrightarrow{f} B$ is the morphism between $\mathbf{0}$ and B, if another composite $f \circ g$ is zero,

$$X \underset{0}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B$$

then $f \circ g = f \circ 0$ and therefore $g = 0$, which means that g factors uniquely through $\mathbf{0}$. □

Proposition 1.3.7. *In a category with zero object, the kernel of a zero morphism* $0 : A \longrightarrow B$ *is just the identity on* A.

Proof. $0 \circ 1_A = 0$ and then, given $g : X \longrightarrow A$ with $0 \circ g = 0$; there exists a unique factorization of g through 1_A: it is g itself. □

Definition 1.3.8. By a preadditive category we mean a category $\mathcal{C}$ together with an abelian group structure on each set $\mathcal{C}(A, B)$ of morphisms, in such a way that the composition mappings

$$c_{ABC} : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$$

are group homomorphisms in each variables.

Proposition 1.3.9. *In a preadditive category* $\mathcal{C}$*, the following conditions are equivalent:*

1. *$\mathcal{C}$ has an initial object;*
2. *$\mathcal{C}$ has a terminal object;*
3. *$\mathcal{C}$ has a zero object.*
   In that case the zero morphisms are exactly the identities for the abelian group structures.

Proof. 3. implies 1. and 2. and, since preadditivity is autodual (see [H2] 1.1 and 1.2.2), it suffices to prove 1. implies 3. Let $\mathbf{0}$ be a initial object. The set $\mathcal{C}(\mathbf{0}, \mathbf{0})$ has a single element, which proves that $1_{\mathbf{0}}$ is the zero element of the group $\mathcal{C}(\mathbf{0}, \mathbf{0})$. Given an object C. $\mathcal{C}(C, \mathbf{0})$ has at least one element: the zero element of that group. But if $f : C \longrightarrow \mathbf{0}$ is any morphism, $f = 1_{\mathbf{0}} \circ f$ must be the zero element of $\mathcal{C}(C, \mathbf{0})$. Thus $\mathbf{0}$ is a terminal object as well.

Since $c_{C\mathbf{0}D} : \mathcal{C}(C, \mathbf{0}) \times \mathcal{C}(\mathbf{0}, D) \longrightarrow \mathcal{C}(C, D)$ is a group homomorphism, we have $\mathbf{0}_{(C,\mathbf{0})} \circ \mathbf{0}_{(\mathbf{0},D)} = (\mathbf{0}_{(C,\mathbf{0})} + \mathbf{0}_{(C,\mathbf{0})}) \circ \mathbf{0}_{(\mathbf{0},D)} = \mathbf{0}_{(C,\mathbf{0})} \circ \mathbf{0}_{(\mathbf{0},D)} + \mathbf{0}_{(C,\mathbf{0})} \circ \mathbf{0}_{(\mathbf{0},D)}$ which implies $\mathbf{0}_{(C,\mathbf{0})} \circ \mathbf{0}_{(\mathbf{0},D)} = \mathbf{0}_{(C,D)}$. □

Proposition 1.3.10. *Given two objects in a preadditive category* $\mathcal{C}$*, the following conditions are equivalent:*

1. *the product* (P, p_A, p_B) *of A,B exists;*
2. *the coproduct* (P, s_A, s_B) *of A,B exists;*
3. *there exists an object P and morphisms*

$$p_A : P \longrightarrow A\ ,\ \ p_B : P \longrightarrow B\ ,\ \ s_A : A \longrightarrow P\ ,\ \ s_B : B \longrightarrow P$$

with the properties

$$p_A \circ s_A = 1_A,\ p_B \circ s_B = 1_B,\ p_A \circ s_B = 0,\ p_B \circ s_A = 0,$$
$$s_A \circ p_A + s_B \circ p_B = 1_P.$$

Morever, under these conditions

$$s_A = \textbf{\textit{Ker}} p_B,\ s_B = \textbf{\textit{Ker}} p_A,\ p_A = \textbf{\textit{Coker}} s_B,\ p_B = \textbf{\textit{Coker}} s_A.$$

Proof. By duality it suffices to prove the equivalence between 1. and 3.

Given 1. define $s_A : A \longrightarrow P$ as the unique morphism with the properties $p_A \circ s_A = 1_A$, $p_B \circ s_A = 0$. In the same way $s_B : B \longrightarrow P$ is such that $p_B \circ s_B = 1_B$, $p_A \circ s_B = 0$. It is now easy to compute that

$$p_A \circ (s_A \circ p_A + s_B \circ p_B) = p_A + 0 = p_A,$$
$$p_B \circ (s_A \circ p_A + s_B \circ p_B) = 0 + p_B = p_B,$$

from which $s_A \circ p_A + s_B \circ p_B = 1_P$.

Given condition 3. consider $C \in \mathcal{C}$ and two morphisms $f : C \longrightarrow A$, $g : C \longrightarrow B$. Define $h : C \longrightarrow P$ as $h = s_A \circ f + s_B \circ g$. We have

$$p_A \circ h = p_A \circ s_A \circ f + p_A \circ s_B \circ g = f + 0 = f,$$
$$p_B \circ f = p_B \circ s_a \circ f + p_B \circ s_B \circ g = 0 + g = g.$$

Given $h' : C \longrightarrow P$ with the properties $p_A \circ h' = f$, $p_B \circ h' = g$, we deduce

$$h' = 1_P \circ h' = (s_A \circ p_A + s_B \circ p_B) \circ h' =$$
$$s_A \circ p_A \circ h' + s_B \circ p_B \circ h' = s_A \circ f + s_B \circ g = h.$$

Now assuming conditions 1. and 3. let us prove that $s_A = \mathbf{Ker} p_B$. We have already $p_B \circ s_A = 0$. Choose $x : X \longrightarrow P$ such that $p_B \circ x = 0$. The composite $p_A \circ x$ is the required factorization since the relations

$$p_A \circ s_A \circ p_A \circ x = p_A \circ x,$$
$$p_B \circ s_A \circ p_A \circ x = 0 \circ p_A \circ x = 0 = p_B \circ x$$

imply $s_A \circ p_A \circ x = x$. The factorization is unique because $p_A \circ s_A = 1_A$ and thus s_A is a monomorphism.

The relation $s_B = \mathbf{Ker} p_A$ is true by analogy and the other two relations hold by duality. □

Definition 1.3.11. Given two objects A,B in a preadditive category, a quintuple (P, p_A, p_B, s_A, s_B) as in 1.3.10 3. is called a biproduct of A and B. The object P will generally be written $A \oplus B$.

Proposition 1.3.12. *Let* $f, g : A \rightrightarrows B$ *be two morphisms in a preadditive category. The following are equivalent:*

1. *the equalizer* $\textbf{\textit{Ker}}(f, g)$ *exists;*
2. *the kernel* $\textbf{\textit{Ker}}(f - g)$ *exists;*
3. *the kernel* $\textbf{\textit{Ker}}(g - f)$ *exists.*

When this is the case, those three objects are isomorphic.

Proof. Since in any case $\mathbf{Ker}(g, f) = \mathbf{Ker}(f, g)$, it suffices to prove (1.⇔2.). Given a morphism $x : X \longrightarrow A$, $f \circ x = g \circ x$ is equivalent to $(f - g) \circ x = 0$, from which the result follows. □

Definition 1.3.13. By an additive category we mean a preadditive category with a zero object and binary biproducts.

Definition 1.3.14. A category $\mathcal{C}$ is abelian when it satisfies the following properties:

1. $\mathcal{C}$ has a zero object;
2. every pair of object of $\mathcal{C}$ has a product and a coproduct;
3. every arrow of $\mathcal{C}$ has a kernel and a cokernel;
4. every monomorphism of $\mathcal{C}$ is a kernel; every epimorphism of $\mathcal{C}$ is a cokernel.

Theorem 1.3.15 (see [H2] 1.6.4)**.** *Every abelian category is additive.*

Chapter 2

Regular Categories

The aim of this chapter is to present some basic results concerning regular categories and also some "exactness properties" which are implied by regularity but do not require properties of additivity unlike abelian category theory. For an introduction to abelian categories the reader can refer to [H2].

Notation. From this point forward $\longrightarrow\!\!\!\!\!\rightarrow$ will denote only regular epimorphisms; no special notation will be used for ordinary epimorphisms.

2.1 Basic results and definitions

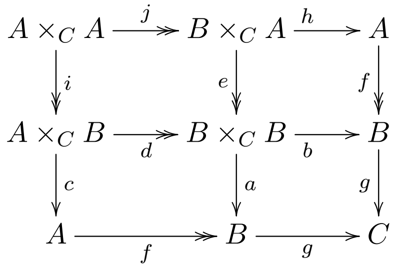
Definition 2.1.1. A category $\mathcal{C}$ is regular when it satisfies the following conditions:

1. every arrow has a kernel pair;
2. every kernel pair has a coequalizer;
3. regular epimorphisms are pullback stable, i.e. the pullback of a regular epimorphism along any morphism exists and is also a regular epimorphism.

Lemma 2.1.2. *In a regular category $\mathcal{C}$, consider $f : A \twoheadrightarrow B$ a regular epimorphism and an arbitrary morphism $g : B \longrightarrow C$. In these condition the factorization $f \times_C f : A \times_C A \longrightarrow B \times_C B$ exists and is an epimorphism.*

Proof. The pullback $B \times_C B$ of (g,g) is the kernel pair of g, thus it exists.

Diagram 2.1.



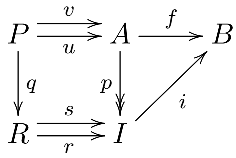
Since f is a regular epimorphism, the three other partial pullbacks involved in diagram 2.1 exist (from the third axiom of regular categories), yielding e, d, i, j regular epimorphisms. Then $f \times_C f = d \circ i = e \circ j$ exists and is an epimorphism as composite of two epimorphisms (see 1.2.3). □

Remark. In diagram 2.1 the notation of the partial pullback $B \times_C A$ instead of $A \times_B (B \times_C B)$ is coherent (see 1.1.1).

Theorem 2.1.3. *In a regular category, every morphism admits a regular epi-mono factorization which is unique up to isomorphism.*

Proof.

Diagram 2.2.



Consider a morphism f, its kernel pair (P, u, v) and the coequalizer $p = \mathbf{Coker}(u, v)$, as in diagram 2.2. Since $f \circ u = f \circ v$ there exists a factorization i through the coequalizer such that $i \circ p = f$. It remains to prove that i is a monomorphism. Let (R, s, r) be the kernel pair of i. Since $i \circ p \circ u = i \circ p \circ v$ there exists a unique morphism q such that $r \circ q = p \circ u$ and $s \circ q = p \circ v$. Applying the lemma to the regular epimorphism p and i, we get that

$$P = A \times_B A,\ R = I \times_B I,\ q = p \times_B p$$

thus q is an epimorphism. Then $r \circ q = p \circ u = p \circ v = s \circ q$ implies $s = r$ since q is an epimorphism. Therefore i is a monomorphism (see 1.1.2). p is a regular epimorphism as the coequalizer of a pair of morphisms, thus $f = i \circ p$ is the required factorization. The uniqueness follows from 1.2.5. □

Proposition 2.1.4. *In a regular category f is a regular epimorphism iff f is a strong epimorphism.*

Proof. ($\Rightarrow$) has been proved in 1.2.3. Conversely a strong epimorphism f factors as $f = i \circ p$ with p regular epimorphism and i monomorphism. But then i is both a strong epimorphism and a monomorphism, thus is an isomorphism (see 1.2.3). $\square$

Corollary 2.1.5. *In a regular category $\mathcal{C}$,*

1. *the composite of two regular epimorphisms is a regular epimorphism;*
2. *if a composite $g \circ f$ is a regular epimorphism, g is a regular epimorphism;*
3. *a morphism which is both a regular epimorphism and a monomorphism is an isomorphism.*

Proof. Via the proposition above, this follows directly from 1.2.3. $\square$

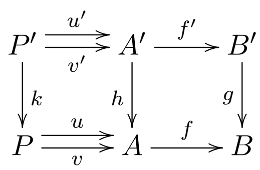
2.2 Exact sequences

Definition 2.2.1. A diagram $P \underset{v}{\overset{u}{\rightrightarrows}} A \xrightarrow{f} B$ is called exact sequence when (u, v) is the kernel pair of f and f is the coequalizer of (u, v).

Proposition 2.2.2. *In a regular category, pulling back along any arrow preserves exact sequences.*

Proof.

Diagram 2.3.



We consider the situation of diagram 2.3, where all the individual squares are pullbacks and $(f; u, v)$ is an exact sequence. Observe that since $f \circ u = f \circ v$, their pullbacks with g are the same and by associativity of pullbacks (see 1.1.1) this means the existence of a single morphism k such that (u', k) is the pullback of (u, h) and (v', k) is the pullback of (v, h). Now we have to prove that $(f'; u', v')$ is an exact sequence. $f' \circ x = f' \circ y$ implies $f \circ h \circ x = g \circ f' \circ x = g \circ f' \circ y = f \circ h \circ y$, from which there is a unique w such that $u \circ w = h \circ x$, $v \circ w = h \circ y$. This yields z_1, z_2 such that $u' \circ z_1 = x$, $k \circ z_1 = w$ and $v' \circ z_2 = y$, $k \circ z_2 = w$. The relations $k \circ z_1 = w = k \circ z_2$ and $f' \circ u' \circ z_1 = f' \circ x = f' \circ y = f' \circ v' \circ z_2$ imply $z_1 = z_2$, since the global diagram is a pullback. This yields a morphism $z = z_1 = z_2$ such that $u' \circ z = x$ and $v' \circ z = y$. The uniqueness of such a z is proved in the same way. Now f' is regular since f is. The pair (u', v') is the kernel pair of f' and f' is regular: this implies that f' is the coequalizer of (u', v') (see 1.1.3). $\square$

Proposition 2.2.3. *Let $\mathcal{C}$ be a regular category with binary products. Given two exact sequences*

$$P \underset{v}{\overset{u}{\rightrightarrows}} A \xrightarrow{f} B \qquad\qquad P' \underset{v'}{\overset{u'}{\rightrightarrows}} A' \xrightarrow{f'} B'$$

the product sequence

$$P \times P' \underset{v \times v'}{\overset{u \times u'}{\rightrightarrows}} A \times A' \xrightarrow{f \times f'} B \times B'$$

is again exact.

Proof. It is just an obvious exercise on pullbacks and products to check that $(u \times u', v \times v')$ is the kernel pair of $f \times f'$. By 1.1.3, it remains to prove that $f \times f'$ is a regular epimorphism.

Diagram 2.4.

$$\begin{array}{ccc} A \times A' & \xrightarrow{f \times 1} & B \times A' \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p_B} \\ A & \xrightarrow[f]{} & B \end{array} \qquad\qquad \begin{array}{ccc} B \times A' & \xrightarrow{1 \times f'} & B \times B' \\ {\scriptstyle p_{A'}}\downarrow & & \downarrow{\scriptstyle p_{B'}} \\ A' & \xrightarrow[f']{} & B' \end{array}$$

Observing that the squares of diagram 2.4 are pullbacks we can conclude that $f \times 1$ and $1 \times f'$ are regular epimorphisms thus $f \times f' = f \times 1 \circ 1 \times f'$ is a regular epimorphism (see 2.1.5). $\square$

Now we will define a class of functors between regular categories which preserve these "exactness properties"

Definition 2.2.4. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between regular categories $\mathcal{C}$, $\mathcal{D}$. F is exact when it preserves:

1. all finite limits which happen to exist in $\mathcal{C}$;
2. exact sequences.

Proposition 2.2.5. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor between regular categories $\mathcal{C}$, $\mathcal{D}$. The functor F preserves:*

1. *regular epimorphisms;*
2. *kernel pairs;*

3. *coequalizers of kernel pairs;*

4. *regular epi-mono factorizations.*

Proposition 2.2.6. *Let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be a functor between regular categories. The following conditions are equivalent:*

1. *F is exact;*

2. *F preserves finite limits and regular epimorphisms.*

Proof. (1) $\Rightarrow$ (2) by 2.2.5. Conversely, with the notation of 2.2.1, the regular epimorphism $F(f)$ has a kernel pair $(F(u), F(v))$, thus is its coequalizer (see 1.1.3). □

2.3 Examples

In this section we will give some examples of regular and also non-regular categories. First of all, it is necessary to prove a couple of results which give a possible definition of regular categories in terms of regular/strong epimorphisms.

Proposition 2.3.1. *A category* $\mathcal{C}$ *is regular iff it satisfies the following conditions:*

1. *every arrows has a kernel pair;*

2. *every arrow* f *can be factored as* $f = i \circ p$ *with* i *a monomorphism and* p *a regular epimorphism;*

3. *the pullback of a regular epimorphism along any morphism exists and is a regular epimorphism.*

Proof. 2.1.3 proves the necessary condition. Conversely, consider an arrow f, its kernel pair (u, v) and its regular epi-mono factorization $f = i \circ p$. Since i is a monomorphism, (u, v) is still the kernel pair of p and since p is a regular epimorphism, p is the coequalizer of (u, v) (see 1.1.3). □

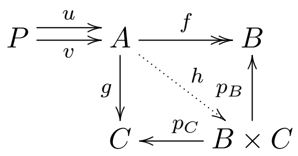
Proposition 2.3.2. *Let* $\mathcal{C}$ *be a category with all finite limits. The category* $\mathcal{C}$ *is regular iff it satisfies the following condition:*

1. *every arrow* f *can be factored as* $f = i \circ p$ *with* i *a monomorphism and* p *a strong epimorphism;*

2. *the pullback of a strong epimorphism along any morphism is again a strong epimorphism.*

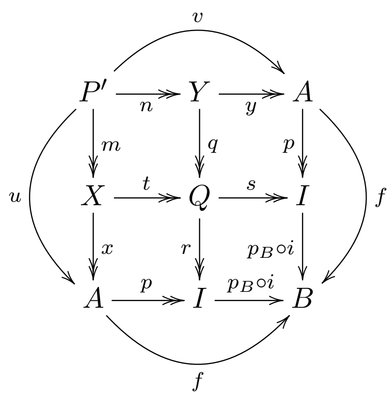
Proof. The necessity of conditions 1. and 2. follows from 2.1.5 and 2.3.1. Conversely, it suffices by 2.3.1 to prove the coincidence between strong and regular epimorphisms. To do this, let consider $f : A \longrightarrow B$, the kernel pair (u, v) of f and a morphism g such that $g \circ u = g \circ v$; we shall prove the existence of $w : B \longrightarrow C$ such that $g = w \circ f$ (such a w is necessarily unique since f is an epimorphism).

Diagram 2.5.



We consider the product of B, C and the unique factorization $h : A \longrightarrow B \times C$ as in diagram 2.5, such that $p_B \circ h = f$, $p_C \circ h = g$. The morphism h can be factored as $h = i \circ p$ with i a monomorphism and p a strong epimorphism. We shall prove that $p_B \circ i$ is an isomorphism and $w = p_C \circ i \circ (p_B \circ i)^{-1}$ is the required factorization.

Diagram 2.6.



Let us consider diagram 2.6, where all the individual squares are pullbacks. Since $p_B \circ i \circ p = p_B \circ h = f$, we can identify the global pullback with the kernel pair of f, yielding $P' = P$, $x \circ m = u, y \circ n = v$ Since p is a strong epimorphism, so are t, q, m, n; r, s are strong epimorphisms as well as parts of a kernel pair, but it is not really relevant here. By commutativity of diagram, we have immediately $p_B \circ i \circ r \circ t \circ m = p_B \circ i \circ s \circ t \circ m$.
On the other hand

$p_C \circ i \circ r \circ t \circ m =$
$p_C \circ i \circ p \circ x \circ m =$
$p_C \circ h \circ x \circ m =$
$g \circ u =$
$g \circ v =$

$p_C \circ h \circ y \circ n =$
$p_C \circ i \circ p \circ y \circ n =$
$p_C \circ i \circ s \circ q \circ n =$
$p_C \circ i \circ s \circ t \circ m$

By definition of product, this yelds $i \circ r \circ t \circ m = i \circ s \circ t \circ m$, thus $r = s$ since i is a monomorphism and t, m are epimorphisms. This proves that $p_B \circ i$ is a monomorphism (see 1.1.2). But $f = (p_B \circ i) \circ p$ and f is a strong epimorphism, $p_B \circ i$ is a strong epimorphism as well and finally $p_B \circ i$ is an epimorphism (see 1.2.3). So we put $w = p_C \circ i \circ (p_B \circ i)^{-1}$. It is straightforward to observe that

$w \circ f =$
$p_C \circ i \circ (p_B \circ i)^{-1} \circ f =$
$p_C \circ i \circ (p_B \circ i)^{-1} \circ p_B \circ i \circ p =$
$p_C \circ i \circ p \circ = g$

On the other hand we have noticed already that such a factorization w is unique since f is an epimorphism. This proves that $f = \textbf{Coker}(u, v)$. In particular every strong epimorphism is regular and thus regular epimorphisms coincide with strong epimorphisms. □

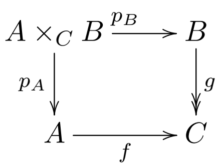
2.3.1 The category of sets and functions

Theorem 2.3.3 (see [H1], 2.8.6)**.** *The category of sets is complete and cocomplete.*

Therefore we can apply 2.3.2, moreover the pullback of a strong epimorphism along any arrow does exist.

Lemma 2.3.4 (see [H1], 4.3.10)**.** *In* `Set` *the strong epimorphisms are exactly the surjective functions.*

Diagram 2.7.



We will now prove the regularity of `Set`. Since every function in `Set` factors as a surjection followed by an injection, the second condition of 2.3.2 holds. Furthermore, the pullback of a surjection is a surjection. Indeed if the diagram 2.7 is a pullback of sets with g a surjection,

$$A \times_C B = \{(a, b) | a \in A; b \in B, f(a) = g(b)\}$$

Given $a \in A$, there exists $b \in B$ such that $f(a) = g(b)$, just because g is a surjection. Therefore $(a, b) \in A \times_C B$ and $p_A(a, b) = a$. This proves that p_A is surjective. So `Set` is regular.

2.3.2 The category of topological spaces and continuous maps

The category `Top` of topological spaces and continuous maps is not regular.

Proposition 2.3.5 (see [H1], 4.3.10b)**.** *In `Top` strong epimorphisms are just the quotient maps $f : A \longrightarrow B$, i.e. the surjections f where B is provided with the corresponding quotient topology.*

But quotient maps are not pullback stable, it is shown by the following counterexample:
Let us put

$$A = \{a, b, c, d\} \text{ with } \{a, b\} \text{ open,}$$
$$B = \{l, m, n\} \text{ with } \{m, n\} \text{ open;}$$
$$C = \{x, y, z\} \text{ with the indiscrete topology.}$$

We define $f : A \longrightarrow C$, $g : B \longrightarrow C$ by

$$f(a) = x,\ f(b) = y = f(c),\ f(d) = z \text{ and } g(l) = x,\ g(m) = z = g(n).$$

Now f is surjective and no subset of C has $\{a, b\}$ as inverse image; thus f is a quotient map. The product $A \times B$ has a single non-trivial open subset, namely $\{a, b\} \times \{l, m\}$. The pullback of f, g is thus given by

$$P = \{(a, l); (d, m); (d, n)\} \text{ with } \{(a, l)\} \text{ open}$$

The projection $p_B : P \longrightarrow B$ is not a quotient map since $p_C^{-1}(l) = \{(a, l)\}$ while $\{l\}$ is not. So `Top` is not a regular category.

2.3.3 Abelian categories

Theorem 2.3.6 (see [H2] 1.5.3)**.** *Abelian categories are finitely complete and cocomplete*

Proposition 2.3.7 (see [H2] 1.5.8)**.** *In an abelian category*

1. *every monomorphism is the kernel of its cokernel;*

2. *every epimorphism is the cokernel of its kernel.*

Corollary 2.3.8. *In an abelian category every epimorphism is regular.*

Proof. Via 2.3.7, directly by 1.3.4. □

Proposition 2.3.9 (see [H2] 1.7.6)**.** *In an abelian category the pullback of an epimorphism is still an epimorphism and the corresponding pullback square is also a pushout.*

The propositions above prove that abelian categories are regular.

2.4 Equivalence relations

Definition 2.4.1. By a relation on an object A of a category $\mathcal{C}$, we mean an object $R \in \mathcal{C}$ together with a monomorphic pair of arrows

$$r_1, r_2 : R \rightrightarrows A$$

(i.e. given arrows $x, y : X \rightrightarrows R$, $x = y$ iff $r_i \circ x = r_i \circ y \ \forall i \in \{1, 2\}$). For every object $X \in \mathcal{C}$ we write

$$R_X = \{(r_1 \circ x, r_2 \circ x) | x \in \mathcal{C}(X, R)\}$$

for the corresponding relation (in the usual sense) generated by R on the set $\mathcal{C}(X, A)$.

Definition 2.4.2. By an equivalence relation on an object A in a category $\mathcal{C}$, we mean a relation (R, r_1, r_2) on A such that, for every object $X \in \mathcal{C}$, the corresponding relation R_X on the set $\mathcal{C}(X, A)$ is an equivalence relation. More generally, the relation R is reflexive (transitive, symmetric, antisymmetric, etc.) when each relation R_X is.

Diagram 2.8.

$$\begin{array}{ccc} R \times_A R & \xrightarrow{\rho_2} & R \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow[r_2]{} & A \end{array}$$

Proposition 2.4.3. *Let $\mathcal{C}$ be a category admitting pullbacks of strong epimorphisms. A relation (R, r_1, r_2) on an object $A \in \mathcal{C}$ is an equivalence relation precisely when there exist:*

1. *a morphism $\delta : A \longrightarrow R$ such that $r_1 \circ \delta = 1_A = r_2 \circ \delta$;*
2. *a morphism $\sigma : R \longrightarrow R$ such that $r_1 \circ \sigma = r_2$, $r_2 \circ \sigma = r_1$;*
3. *a morphism $\tau : R \times_A R \longrightarrow R$ such that $r_1 \circ \tau = r_1 \circ \rho_1$, $r_2 \circ \tau = r_2 \circ \rho_2$ where the pullback is that of diagram 2.8.*

Such morphisms δ, σ, τ are necessarily unique.

Proof. The reflexivity of the relation (R, r_1, r_2) implies that given the pair $(1_A, 1_A) : A \rightrightarrows A$, there exists a morphism $\delta : A \longrightarrow R$ such that $r_1 \circ \delta = 1_A = r_2 \circ \delta$. Conversely, if given a relation (R, r_1, r_2) such a morphism δ exists, then for every arrow $x : X \longrightarrow A$ one has

$$x = r_1 \circ \delta \circ x, \ x = r_2 \circ \delta \circ x,$$

which proves that $(x, x) \in R_X$ and thus R is reflexive.

The pair $(r_1, r_2) : R \rightrightarrows A$ is obviously in R_R, since $r_1 = r_1 \circ 1_R$ and $r_2 = r_2 \circ 1_R$. By symmetry of (R, r_1, r_2), the pair $(r_2, r_1) : R \rightrightarrows A$ is in R_R, yielding a morphism $\sigma : R \longrightarrow R$ such that $r_1 \circ \sigma = r_2$, $r_2 \circ \sigma = r_1$. Conversely, if given the relation (R, r_1, r_2) such a morphism σ exists, then for every pair of arrows $(x, y) : X \rightrightarrows A$ in R_X one has a morphism $z : X \longrightarrow R$ such that $r_1 \circ z = x$, $r_2 \circ z = y$; as consequence,

$$r_1 \circ \sigma \circ z = r_2 \circ z = y,\ r_2 \circ \sigma \circ z = r_1 \circ z = x$$

and the pair (y, x) is in R_X as well, proving the symmetry of R.

Let us consider the pullback of diagram 2.8, which exists since the relations $r_1 \circ \delta = 1_A = r_2 \circ \delta$ imply that r_1, r_2 are strong epimorphisms (1.2.3). $R \times_A R$ represents the pairs $((a_1, a_2), (a_2, a_3))$ where $a_1 \approx a_2$ and $a_2 \approx a_3$. Considering the diagrams

$$R \times_A R \xrightarrow{\rho_1} A \underset{r_2}{\overset{r_1}{\rightrightarrows}} A\ ,\quad R \times_A R \xrightarrow{\rho_2} A \underset{r_2}{\overset{r_1}{\rightrightarrows}} A$$

we conclude that $(r_1 \circ \rho_1, r_2 \circ \rho_1)$ and $(r_1 \circ \rho_2, r_2 \circ \rho_2)$ are in $R_{(R \times_A R)}$. Since $r_1 \circ \rho_1 = r_2 \circ \rho_2$, this implies that $(r_1 \circ \rho_1, r_2 \circ \rho_2)$ are in $R_{(R \times_A R)}$, yielding an arrow $\tau : R \times_A R \longrightarrow R$ such that $r_1 \circ \tau = r_1 \circ \rho_1$, $r_2 \circ \tau = r_2 \circ \rho_2$. Conversely, suppose we are given a relation (R, r_1, r_2) with the property that such a morphism exists. Given three arrows $x, y, z : X \longrightarrow A$ with $(z, y), (y, z) \in R_X$, we get two morphisms $u, v : X \rightrightarrows R$ such that $r_1 \circ u = x$, $r_2 \circ u = y$, $r_1 \circ v = y$ and $r_2 \circ v = z$. From the relation $r_1 \circ v = r_2 \circ u$ we get a morphism $w : X \longrightarrow R \times_A R$ such that $\rho_1 \circ w = u$, $\rho_2 \circ w = v$. Finally one has

$$x = r_1 \circ u = r_1 \circ \rho_1 \circ w = r_1 \circ \tau \circ w,$$
$$z = r_2 \circ u = r_2 \circ \rho_2 \circ w = r_2 \circ \tau \circ w$$

so that $(x, z) \in R_X$ and R is transitive.

It remains to observe that the morphisms δ, σ, τ are unique because (r_1, r_2) is monomorphic. □

Definition 2.4.4. An equivalence relation (R, r_1, r_2) on an object A of a category $\mathcal{C}$ is effective when the coequalizer q of (r_1, r_2) exists and (r_1, r_2) is the kernel pair of q.

Chapter 3

Exact Categories

After the categorical formulation of equivalence relations given in the previous chapter, we can now define exact categories. Furthermore, in this chapter we will show some examples of exact categories and emphasize how they are a sort of "non-additive version of abelian categories".

Definition 3.0.5. An exact category is a regular category in which equivalence relations are effective.

3.1 A non-additive version of abelian categories

Lemma 3.1.1. *In a non-empty and preadditive regular category $\mathcal{C}$, the biproduct $A \oplus A$ exists for every object A.*

Proof. Consider an arbitrary object $A \in \mathcal{C}$, the zero map $0 : A \longrightarrow A$ and its kernel pair $u, v : P \rightrightarrows A$. Given arbitrary morphisms $x, y : X \rightrightarrows A$, one has $0 \circ x = 0 \circ y$, from which there is a unique morphism $z : X \longrightarrow P$ such that $u \circ z = x$, $v \circ z = y$ This proves that (P, u, v) is the product $A \times A$. One derives the conclusion by 1.3.10. □

Lemma 3.1.2. *A non-empty and preadditive regular category $\mathcal{C}$ has a zero object.*

Proof. Choose an arbitrary object A. By 3.1.1, the zero map $0 : A \longrightarrow A$ admits $p_1, p_2 : A \oplus A \rightrightarrows A$ as kernel pair. Let $q : A \longrightarrow Q$ be the coequalizer of p_1, p_2. Given a morphism $x : Q \longrightarrow X$ we have

$$x \circ q = x \circ q \circ p_1 \circ s_1 = x \circ q \circ p_2 \circ s_1 = x \circ q \circ 0 = 0$$

from which $x = 0$ since q is an epimorphism. Thus 0 is the unique morphism from Q to X and Q is an initial object. One derives the conclusion by 1.3.9. □

Lemma 3.1.3. *A non-empty and preadditive regular category $\mathcal{C}$ has biproducts.*

Proof. Given objects A, B the morphisms $A \longrightarrow \mathbf{0}$, $B \longrightarrow \mathbf{0}$ to the zero object are retractions, with zero as a section. Therefore they are regular epimorphisms and the pullback of those two arrows exists, yielding the product $A \times B$ (see 1.1.4). One derives the conclusion by 1.3.10. $\square$

Lemma 3.1.4. *A non-empty and preadditive regular category $\mathcal{C}$ has kernels.*

Proof. Take a morphism $f : A \longrightarrow B$ and its kernel pair $u, v : P \rightrightarrows A$. The morphism $u - v : P \longrightarrow A$ can be factored as $u - v = i \circ p$ with i a monomorphism and p a regular epimorphism. We shall prove that i is the kernel of f.

First $f \circ i \circ p = f \circ (u - v) = f \circ u - f \circ v = 0$, which proves $f \circ i = 0$ since p is an epimorphism. Next if $x : X \longrightarrow A$ is such that $f \circ x = 0 = f \circ 0$, we get a factorization $y : X \longrightarrow P$ such that $u \circ y = x$, $v \circ y = 0$. This yields

$$i \circ p \circ y = (u - v) \circ y = u \circ y - v \circ y = x - 0 = x$$

and $p \circ y$ is a factorization of x through i. This factorization is unique since i is a monomorphism. $\square$

Lemma 3.1.5. *A non-empty preadditive regular category $\mathcal{C}$ is finitely complete (i.e. equivalently $\mathcal{C}$ has equalizers and binary products; see [H1] 2.8.1).*

Proof. By 1.3.12 and 3.1.4 $\mathcal{C}$ has equalizers. We derive the conclusion by 3.1.3 and 1.3.10. $\square$

Lemma 3.1.6. *In a preadditive category $\mathcal{C}$, every reflexive relation is necessarily an equivalence relation.*

Proof. Consider a reflexive relation $s_1, s_2 : S \rightrightarrows A$. Given an object $X \in \mathcal{C}$, the relation $S_X = \{(s_1 \circ x, s_2 \circ x) | x \in \mathcal{C}(X, S)\}$ on the abelian group $\mathcal{C}(X, A)$ contains the diagonal, just by assumption. Since S_X is obviously a subgroup of $\mathcal{C}(X, A) \times \mathcal{C}(X, A)$, this reduce the problem to proving the lemma in the category of abelian groups.

Consider `Ab` the category of abelian groups and their homomorphisms, we already know that all the pairs (a, a) belong to S, since S is reflexive. Next if $(a, b) \in S$ we get

$$(b, a) = (a, a) - (a, b) + (b, b) \in S$$

proving the symmetry of S. Finally if $(a, b), (b, c) \in S$ we get

$$(a, c) = (a, b) - (b, b) + (b, c) \in S$$

which proves the transitivity of S. □

Notation. If A_1, A_2, B_1 ,B_2 are four object in an additive category $\mathcal{C}$ a morphism

$$f : A_1 \oplus A_2 \longrightarrow B_1 \oplus B_2$$

is completely characterized by the following four morphisms

$$\begin{aligned} f_{11} &= p_1 \circ f \circ s_1 : A_1 \longrightarrow B_1 \\ f_{12} &= p_1 \circ f \circ s_2 : A_2 \longrightarrow B_1 \\ f_{21} &= p_2 \circ f \circ s_1 : A_1 \longrightarrow B_2 \\ f_{22} &= p_2 \circ f \circ s_2 : A_2 \longrightarrow B_2 \end{aligned}$$

so that it makes sense to use the notation

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

Moreover, the composite of two morphisms is represented by the product of their matrices (see [H2] 1.2).

Lemma 3.1.7. *In an additive exact category $\mathcal{C}$, every monomorphism has a cokernel and is the kernel of its cokernel.*

Proof. Let $f : A \rightarrowtail B$ be a monomorphism. Applying 3.1.3 and using the matrix notation, let us consider the morphism

$$r = \begin{bmatrix} f & 1_B \\ 0 & 1_B \end{bmatrix} : A \oplus B \longrightarrow B \oplus B$$

This is a monomorphism, since given morphisms

$$a, a' : X \rightrightarrows A\ , \ \ b, b' : X \rightrightarrows B$$

$$\begin{bmatrix} f & 1_B \\ 0 & 1_B \end{bmatrix} \circ \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (f \circ a) + b \\ b \end{bmatrix}$$

so from the relation

$$\begin{bmatrix} f & 1_B \\ 0 & 1_B \end{bmatrix} \circ \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} f & 1_B \\ 0 & 1_B \end{bmatrix} \circ \begin{bmatrix} a' \\ b \end{bmatrix}$$

We deduce $b = b'$ and $(f \circ a) + b = (f \circ a') + b'$, thus $f \circ a = f \circ a'$ and finally $a = a'$ since f is a monomorphism. Observing moreover that

$$\begin{bmatrix} f & 1_B \\ 0 & 1_B \end{bmatrix} \circ \begin{bmatrix} 0 \\ 1_B \end{bmatrix} = \begin{bmatrix} 1_B \\ 1_B \end{bmatrix} = \Delta_B$$

we conclude that the monomorphism r, seen as a relation on B, contains the diagonal Δ_B. Therefore r is an equivalence relation by 3.1.6 and thus an effective relation by the exactness of $\mathcal{C}$.

Writing q for the coequalizer of the effective equivalence relation r on B, thus we have an exact sequence

$$A \oplus B \underset{(0,1_B)}{\overset{(f,1_B)}{\rightrightarrows}} B \xrightarrow{q} Q$$

In particular $q \circ f = q \circ (f, 1_B) \circ s_A = f \circ (0, 1_B) \circ s_A = q \circ 0 = 0$. Now given a morphism $x : B \longrightarrow X$ such that $x \circ f = 0$

$$x \circ (f, 1_B) = (x \circ f, x) = (0, x) = x \circ (0, 1_B)$$

from which we get a unique factorization $z : Q \longrightarrow X$ such that $z \circ q = x$. This proves that $q = \mathbf{Coker} f$.

It remains to prove that f is the kernel of q. Given $y : Y \longrightarrow B$ such that $q \circ y = 0 = q \circ 0$, we find a unique $z : Y \longrightarrow A \oplus B$ such that $(f, 1_B) \circ z = y$, $(0, 1_B) \circ z = 0$. The arrow z has the form $\begin{bmatrix} u \\ v \end{bmatrix}$ for some morphism $u : Y \longrightarrow A$, $v : Y \longrightarrow B$. The arrow u is the required factorization since

$$0 = (0, 1_b) \circ \begin{bmatrix} u \\ v \end{bmatrix} = (0 \circ u) + (1_B \circ v) = v$$

$$y = (f, 1_B) \circ \begin{bmatrix} u \\ v \end{bmatrix} = (f \circ u) + (1_B \circ v) = f \circ u + 0 = f \circ u$$

Such a factorization is unique since f is a monomorphism. □

Lemma 3.1.8. *An additive exact category is finitely cocomplete.*

Proof. By 3.1.3 we get the existence of binary coproducts, thus it suffices to prove the existence of coequalizers, which is equivalent to the existence of cokernels (1.3.12). Given a morphism $f : A \longrightarrow B$, we factor it as $f = i \circ p$ with i a monomorphism and p a regular epimorphism. Since p is an epimorphism, the cokernel of i, which exists by 3.1.7, is also the cokernel of f. □

Lemma 3.1.9. *In an additive exact category, every epimorphism is a cokernel*

Proof. Let $f : A \longrightarrow B$ be an epimorphism. Since the category is regular, we can factor f as $f = i \circ p$ with i a monomorphism and p a regular epimorphism. But the monomorphism i is a kernel by 3.1.7; so it is a strong monomorphism by 1.2.3 but i is also an epimorphism, since f is. Then i is an isomorphism and f is a regular epimorphism. Thus $f = \mathbf{Coker}(u, v)$ for some pair $u, v : P \rightrightarrows A$ and therefore $f = \mathbf{Coker}(u - v)$, see 1.3.12. □

Theorem 3.1.10. *The following conditions are equivalent:*

1. $\mathcal{C}$ *is an abelian category;*
2. $\mathcal{C}$ *is an additive exact category;*
3. $\mathcal{C}$ *is a non-empty preadditive regular category.*

Proof. We have proved in the previous chapter that an abelian category is regular. By [H2] 2.5.6d we get that abelian categories are even exact. Then the result follows by lemmas 3.1.1, 3.1.9. □

3.2 Examples

We have just proved that every abelian category is exact. In this section we will provide a list of examples of exact categories. The main references for this section are [EC] and [CF].

3.2.1 The category of sets and functions

In 2.3.1 we have seen that `Set` is a regular category. The proposition below proves that `Set` is even exact.

Proposition 3.2.1 (See [CF] 2.28 (1))**.** *In* `Set` *equivalence relations are effective.*

3.2.2 The category of compact Hausdorff spaces

`CH` (compact Hausdorff spaces) is complete and cocomplete, since it is reflexive in `Top`. The regularity and the exactness follow by the fact that any epimorphism is a topological quotient (see, [SS] pg. 100).

3.2.3 The category of torsion-free abelian groups

The category $\mathtt{Ab}_{tf}$ is an example of a regular category which is not exact. An abelian group G is torsion-free when for every $g \in G$ and every non-zero natural number n

$$ng = g + g + ... + g = 0 \text{ iff } g = 0$$

let us first remark that the category of torsion-free abelian groups is closed under finite limits and subobjects in `Ab` (see [CF] 2.28 (4)). The regularity of `Ab` (it is abelian seen as the category $\mathtt{Mod}_{\mathbb{Z}}$, see [H2] 1.4.6a) then implies that $\mathtt{Ab}_{tf}$ is regular. Since $\mathtt{Ab}_{tf}$ is additive, the exactness would imply its abelianness. In particular we would have a short exact sequence, in the sense of [H2] 1.8,

$$\mathbf{0} \longrightarrow 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{q} Q \longrightarrow \mathbf{0}$$

where i is the canonical inclusion of the even integers in $\mathbb{Z}$ and q is the cokernel of i in $\mathtt{Ab}_{tf}$. But from $q(2) = 0$ we get $q(1) + q(1) = 0$ which implies $q(1) = 0$ since Q is a torsion-free group. Thus $q = 0$, with therefore the identity on $\mathbb{Z}$ as kernel (see 1.3.7). This contradicts the fact that i should be the kernel of q (see [H2] 1.8.5).

Chapter 4

Elementary Topoi as Exact Categories

The aim of this chapter is to provide the reader with a wide range of examples of exact categories, proving that every elementary topos is an exact category.

Diagram 4.1.

$$\begin{array}{ccc} Y & \longrightarrow & \mathbf{1} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle t} \\ X & \underset{\Phi_\sigma}{\longrightarrow} & \Omega \end{array}$$

Definition 4.0.2. A category $\mathcal{E}$ is called elementary topos if:

1. $\mathcal{E}$ is finitely complete (i.e $\mathcal{E}$ has pullbacks and a terminal object $\mathbf{1}$);

2. $\mathcal{E}$ is cartesian closed (i.e. for each object X we have an exponential functor $(-)^X : \mathcal{E} \longrightarrow \mathcal{E}$ which is right adjoint to the functor $(-)\times X$);

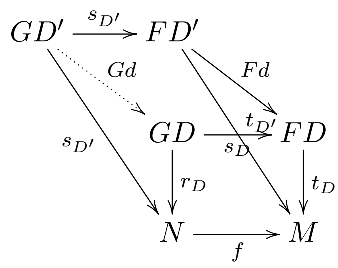
3. $\mathcal{E}$ has a subobject classifier (i.e. an object Ω and a morphism $\mathbf{1} \xrightarrow{t} \Omega$ (called "true") such that, for each monomorphism $Y \overset{\sigma}{\rightarrowtail} X$ in $\mathcal{E}$, there is a unique $\Phi_\sigma : X \longrightarrow \Omega$ (the classifying map of σ) making a pullback diagram as in 4.1).

Proposition 4.0.3 (see [TT] 1.22)**.** *A topos is balanced (i.e. a morphism which is both a monomorphism and an epimorphism is an isomorphism).*

The original definition of elementary topos, given by Lawvere and Tierney, included the existence of finite colimits, but it was a redundant condition as proved by the following proposition.

Proposition 4.0.4 (see [TT] 1.36)**.** *A topos has finite colimits.*

Diagram 4.2.



Definition 4.0.5. Consider a category $\mathcal{C}$ with pullbacks and an arbitrary functor $F : \mathcal{D} \longrightarrow \mathcal{C}$. Given a cocone $(t_D : FD \longrightarrow M)_{D \in \mathcal{D}}$ on F and a morphism $f : N \longrightarrow M$ in $\mathcal{C}$, we can compute the various pullbacks (GD, r_D, s_D) of t_D along f. Moreover, given a morphism $d : D' \longrightarrow D$ in $\mathcal{D}$, the equalities

$$t_D \circ Fd \circ s_{D'} = t_{D'} \circ s_{D'} = f \circ r_{D'}$$

imply the existence of a unique factorization Gd making the diagram above commutative. In particular we have defined a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ and a cocone $(r_D : GD \longrightarrow N)_{D \in \mathcal{D}}$ on this functor.

A colimit $(M, (t_{D \in \mathcal{D}}))$ is said to be universal when for every morphism $f : N \longrightarrow M$ in $\mathcal{C}$, the cocone constructed above is a colimit of the corresponding functor G.

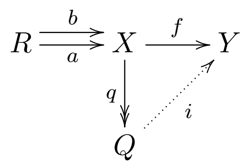
In other words, colimits are universal in $\mathcal{C}$ if any colimit remains a colimit in $\mathcal{C}$ after pulling back along an arrow (see, [HT] 6.1.1 (ii)).

Proposition 4.0.6 (see [H3], 5.9.1)**.** *In a topos, finite colimits are universal.*

Theorem 4.0.7. *Any morphism in a topos can be factored as an epimorphism followed by a monomorphism.*

Proof. Given $f : X \longrightarrow Y$, form the diagram

Diagram 4.3.



where (a, b) is the kernel pair of f, q is the coequalizer of (a, b) and i is the unique map between the coequalizer and Y. We need to show that i is a monomorphism. Consider the pullback in the following diagram

Diagram 4.4.

$$\begin{array}{ccc} S & \xrightarrow{e} & P' \\ c \downdownarrows d & & a \downdownarrows b \\ X & \xrightarrow[q]{} & Q \end{array}$$

where (c, d) is the kernel pair of i. Since q is a coequalizer and finite colimit are universal in a topos, e is a coequalizer and then it is an epimorphism. The equalities $f \circ g = i \circ q \circ g = i \circ c \circ e = i \circ d \circ e = i \circ q \circ h = f \circ h$ implies the existence of a unique factorization k of (g, h) through (a, b). Thus we get $c \circ e = q \circ g = q \circ a \circ k = q \circ b \circ k = q \circ h = d \circ e$ and then $c = d$. The result follows directly by 1.1.2. □

Proposition 4.0.8. *In a topos, every epimorphism is a coequalizer.*

Proof. Suppose $f = i \circ q$ is an epimorphism, then so is i, and hence it is an isomorphism by 4.0.3. □

Corollary 4.0.9. *A topos is a regular category.*

Proof. Via 2.3.2, it follows directly by 4.0.8, 4.0.7 and 4.0.6. □

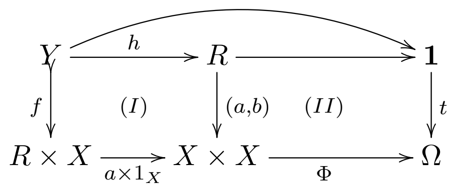
Proposition 4.0.10. *In a topos, equivalence relations are effective.*

Proof. Let $R \overset{a}{\underset{b}{\rightrightarrows}} X$ be an equivalence relation. Since a, b is a monomorphic pair, $R \xrightarrow{(a,b)} X \times X$ is a monomorphism: let $X \times X \xrightarrow{\Phi} \Omega$ be its classifying map and $X \xrightarrow{\overline{\Phi}} \Omega^X$ the exponential transpose of Φ. We will show that $R \overset{a}{\underset{b}{\rightrightarrows}} X$ is a kernel pair of $\overline{\Phi}$.

Let $U \overset{f}{\underset{g}{\rightrightarrows}} X$ be a pair of arrows such that $\overline{\Phi} \circ f = \overline{\Phi} \circ g$. Then applying the exponential adjuction, we have $\Phi(f \times 1_X) = \Phi(g \times 1_X) : U \times X \longrightarrow \Omega$; and composing with $(1_U, g) : U \longrightarrow U \times X$, we obtain $\Phi(f, g) = \Phi(g, g)$. But $U \xrightarrow{(g,g)} X \times X$ factors through R since R is reflexive, and so $\Phi(g, g)$ classifies the maximal subobject $U \xrightarrow{1} U$. Hence $\Phi(f, g)$ also classifies this subobject; so (f, g) factors through R.

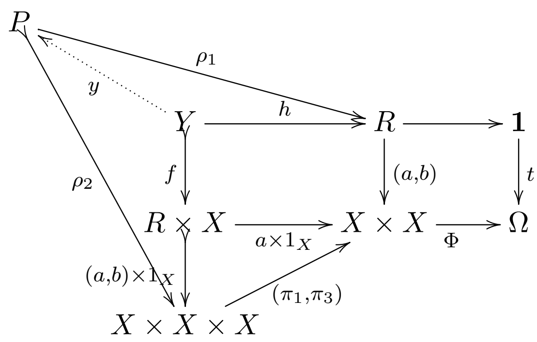
Conversely, we must show that $\overline{\Phi} \circ a = \overline{\Phi} \circ b$, or equivalently that the subobjects of $R \times X$ classified by $\Phi(a \times 1_X)$ and $\Phi(b \times 1_X)$ (resp. Y and Y') are isomorphic.

Diagram 4.5.



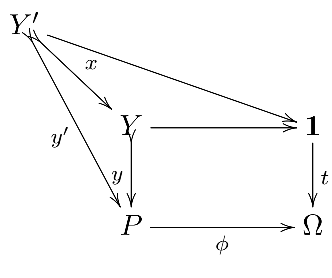
As in diagram 4.5, since (a, b) is classified by Φ, we get that $(R, (a, b), c_R)$ is a pullback, where c_R is the (unique) arrow to the terminal object 1. Then there exists a unique map $h : Y \longrightarrow R$ such that $c_R \circ h = c_Y$ and $(a, b) \circ h = f \circ (a \times 1_X)$. $\Phi(a \times 1_X)$ classifies f, thus $(Y, c_R \circ h, f)$ is a pullback; since (II) is also a pullback, by 1.1.1, we get that (I) is a pullback.

Diagram 4.6.



If we compose the subobject f with the monomorphism $(a, b) \times 1_X : R \times X \longrightarrow X \times X \times X$ we obtain the situation in diagram 4.6, where P is the pullback of $((\pi_1, \pi_3), (a, b))$. Since $(a, b) \circ h = (a \times 1_X) \circ f = (\pi_1, \pi_3) \circ ((a, b) \times 1_X) \circ f$, there exists a unique arrow $y : Y \longrightarrow P$. Furthermore, y is a monomorphism since $((a, b) \times 1_X) \circ f = \rho_2 \circ y$ is. In the same way we get $y' : Y' \rightarrowtail P$.

Diagram 4.7.



Finally, y is classified by a morphism ϕ, then as in diagram 4.7, since such a square is a pullback, there exist a map $x : Y' \longrightarrow Y$ such that $c_Y \circ x = c_{Y'}$ and $y \circ x = y'$. In the same way there exists a unique x' such that $c_{Y'} \circ x' = c_Y$ and $y' \circ x' = y$. $y \circ x \circ x' = y' \circ x' = y$, thus $x \circ x' = 1_Y$, since y is a monomorphism. Then x is both a monomorphism, since $y' = y \circ x$

is a monomorphism, and an epimorphism, as a retraction of x'. A topos is balanced, thus x is the required isomorphism, see 4.0.3.

Since a topos is a regular category, every kernel pair has a coequalizer and then is the kernel pair of that coequalizer. Thus every equivalence relation in a topos is effective. □

With the proposition above we have finally proved the exactness of an elementary topos.

Corollary 4.0.11. *A topos is an exact category.*

Proof. It follows directly by 4.0.9 and 4.0.10. □

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