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The Poincaré polynomial for the Hilbert scheme of points of $\mathbb{C}^2$

Tesi di Laurea Magistrale in Geometria

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Se fossi accademico, fossi maestro o dottore,
ti insignirei in toga di quindici lauree ad honorem,
ma a scuola ero scarso in latino e il pop non è fatto per me:
ti diplomerò in canti e in vino qui in via Paolo Fabbri 43
F.G.

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Introduzione

Dato uno schema quasi proiettivo X localmente Nötheriano, possiamo pensare allo schema di Hilbert, $\mathrm{Hilb}(X)$, come ad uno spazio di moduli che parametrizza i sottoschemi di X, tenendo in considerazione il loro polinomio di Hilbert. Se ad esempio fissiamo tale polinomio, considerandolo costante e uguale ad n, allora lo schema di Hilbert che otteniamo, $\mathrm{Hilb}^n(X) = X^{[n]}$ parametrizza i sottoschemi 0-dimensionali di X di lunghezza n. L'esempio più semplice di sottoschema zero dimensionale di questo tipo è rappresentato dai sottoinsiemi di n punti distinti di X, ma certamente essi non esauriscono tutte le possibilità: occorre infatti considerare i casi in cui gli n punti non sono più tutti distinti tra di loro, ed alcuni di essi coincidono e vengono contati con molteplicità. Si vede dunque come la struttura di schema assuma rilevanza e conferisca allo spazio una maggiore ricchezza.

Come sempre quando si trattano spazi di moduli, è di grande interesse studiare le possibili strutture e proprietà degli schemi di Hilbert: talune vengono ereditate dallo spazio X di partenza, ma ne potrebbero intervenire delle nuove, tipiche dello spazio di moduli in esame. Per prima cosa, la costruzione funtoriale degli schemi di Hilbert fornita da Grothendieck, ci assicura che $\mathrm{Hilb}(X)$ sia sempre uno schema e, di più, esso è proiettivo se X lo è. Inoltre se consideriamo il caso particolare di $X^{[n]}$, tale schema può essere dotato di una struttura simplettica, tutte le volte che X ne ha una; se poi ci limitiamo agli schemi X di dimensione 2, non singolari, allora è possibile dimostrare che $X^{[n]}$ è anch'esso non singolare ed il morfismo $X^{[n]} \to S^nX$, risulta essere una risoluzione delle singolarità del prodotto simmetrico.

Questo lavoro prende in esame il caso particolare in cui $X = \mathbb{C}^2$, considerandone lo schema di Hilbert di punti. Più nello specifico lo scopo della tesi è lo studio dei numeri di Betti di $(\mathbb{C}^2)^{[n]}$: ciò che si ottiene è una espressione del tipo serie di potenze, la quale è un caso particolare di una formula molto più generale, nota con il nome di formula di Göttsche. È interessante notare come tale formula descriva i numeri di Betti di tutti gli schemi di Hilbert di punti di $\mathbb{C}^2$ considerati simultaneamente, in termini dei numeri di Betti di X. La formula di Göttsche compare anche in un contesto totalmente diverso, mostrando una connessione tra ambiti distinti dell'algebra e della geometria: se infatti consideriamo un certo tipo di superalgebre infinito dimensionali, prodotto di algebre di Heisenberg e Clifford, e guardiamo alla loro formula dei caratteri, ritroviamo proprio la formula di Göttsche.

La tesi è organizzata come segue. Nel primo capitolo introduciamo gli schemi di Hilbert concentrandoci sul caso di nostro interesse. Diamo una semplice ed utile descrizione di $(\mathbb{C}^2)^{[n]}$ e ne esplorismo la strutture simplettica.
Il secondo capitolo è invece dedicato alla descrizione di $(\mathbb{C}^2)^{[n]}$ come quoziente iper Kähleriano: le varietà iper Kähleriane non sono facili da costruire, dunque da un lato tale risultato ha valore in sè, dall'altro sarà utile per poter utilizzare la teoria di Morse nel capitolo successivo.
Nel terzo capitolo infine determiniamo il polinimio di Poincaré dello schema di Hilbert di punti di $\mathbb{C}^2$, usando la teoria di Morse e l'azione naturale del toro su $\mathbb{C}^2$, insieme alla relativa mappa momento, che dimostreremo essere una funzione di Morse.

Introduction

Given a quasi projective locally Nötherian scheme X, the Hilbert scheme of X, $\mathrm{Hilb}(X)$ can be thought as the moduli space parametrizing the subschemes of X, coherently with the information provided by their Hilbert Polynomial. If we fix the Hilbert polynomial to be constant and equal to n ($\mathrm{Hilb}^n(X) = X^{[n]}$), then we restrict our attention to the 0-dimensional subschemes of X of length n. The simplest example of zero dimensional subschemes of this kind are the sets of n distinct points of X, but they of course do not run out all the possibilities: some of these points may collide, in other words some of them may coincide and be counted with multiplicity and then the structure of scheme becomes relevant.

As always when we deal with moduli spaces, it is of great interest to study all the possible structures and properties of the Hilbert schemes: some of them are inherited by the ones which exist on X, others can be typical of $\mathrm{Hilb}^P(X)$. First of all, Grothendieck's construction of the Hilbert scheme implies that $\mathrm{Hilb}(X)$ is a scheme and it is projective if X is. Moreover if we consider $X^{[n]}$, it can equipped with a symplectic structure if X has one; finally if $\dim X = 2$ and X is smooth, then $X^{[n]}$ has particularly mice properties: it is smooth, and the morphism $X^{[n]} \to S^n X$ turns out to be a resolution of singularities for the symmetric product.

This work is concerned with a very specific Hilbert scheme, which is the Hilbert scheme of points of $\mathbb{C}^2$; in particular our aim is to study the Betti numbers of $(\mathbb{C}^2)^{[n]}$: we will obtain a power series expression, which is a particular case of a more general formula, known as the Göttsche formula. The interesting fact is that it describes the Betti numbers of all the Hilbert schemes of points of $\mathbb{C}^2$ at once, in terms of the Betti numbers of X. This formula is also important because appears in a very different context, giving to us a striking connection between two fields that seem to be very far: indeed it coincide with the character formula for a representation of a type of infinite dimensional superalgebras, products of the Heisenberg and Clifford algebras.

The work is organized as follows. In the first chapter we introduce the Hilbert schemes, focusing on the case we are interested in. We give a useful description of $(\mathbb{C}^2)^{[n]}$ and explore the symplectic structure it is endowed with.
The second chapter is devoted to describe $(\mathbb{C}^2)^{[n]}$ as an hyper-Kähler quotient: the hyper-Kähler manifolds are not so easy to construct, hence this result has an interest

on its own, besides the hyper-Kähler structure will be needed after in order to apply Morse Theory .
In the third chapter we finally determine the Poincaré polynmial of the Hilbert scheme of points of $\mathbb{C}^2$ using Morse Theory and the natural torus action together with the related moment map, which will be proved to be a Morse function.

Chapter 1

Hilbert schemes, an overview

1.1 General results

In this section we give an introduction to Hilbert schemes, beginning with the general definition: all schemes are supposed locally Nötherian.

Definition 1.1. Let X be a projective scheme over an algebraically closed field K e $\mathcal{O}_X(1)$ an ample line bundle. For every scheme S the functor $\mathcal{H}ilb_X$ is defined as follows:

$$\mathcal{H}ilb_X(S) = \{Z \subset X \times S,\ Z \ closed\ subscheme,\ Z\ flat\ over\ S\}$$

$\mathcal{H}ilb_X(S)$ is a controvariant functor from the category of Schemes over K to the one of Sets which associates to each scheme S the set of families Z of closed subschemes $Z_x \subset X$ parametrized by S, so that, if we look at the diagram

$$\begin{array}{ccc} Z & \stackrel{i}{\hookrightarrow} & X \times S \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_S} \\ S & \stackrel{=}{\longrightarrow} & S \end{array}$$

the projection π is flat and $Z_x = \pi^{-1}(x)$, with $x \in S$
Let $P_{x,Z}(m)$ be the Hilbert polynomial

$$P_{x,Z}(m) = \chi(\mathcal{O}_{Z_x} \otimes \mathcal{O}_X(m))$$

Since Z is flat and projective over S, $P_{x,Z}(m)$ actually does not depend on $x \in S$ if S is connected, so for each family of subschemes Z the Hilbert polynomial is the same and it is well defined the subfunctor $\mathcal{H}ilb_X^P$ which associates S with the set of families of closed subschemes of X parametrized by S which have P as their Hilbert polynomial.

We are interested in $\mathcal{H}ilb_X^P$ since the following result holds:

Theorem 1.2 (Grothendieck)**.** *The functor* $\mathcal{H}ilb_X^P$ *is representable by a projective scheme* Hilb_X^P

This means that there are isomorphisms

$$\mathcal{H}ilb_X^P(S) \cong \mathrm{Hom}(S, \mathrm{Hilb}_X^P)$$

Equivalently, the fact that the functor is representable implies that there is a family of closed subschemes $\mathcal{Z}$ such that

$$\mathcal{Z} \subset X \times \mathrm{Hilb}_X^P$$

$\mathcal{Z}$ is flat over Hilb_X^P and it satisfies a universal property:
for every S and every closed subscheme $Z \subset X \times S$ (which has P as its Hilbert Polynomial) flat over S there is a unique morphism

$$\phi_Z : S \longrightarrow \mathrm{Hilb}_X^P$$

such that

$$Z = (1_X \times \phi_Z)^{-1}(\mathcal{Z})$$

A proof of the theorem above can be found in [8].

1.2 Hilbert schemes of points

Now we turn our attention to the special case in which P is a constant. Let us start with a motivating example.

Let $x_1, \dots, x_n$ be n distinct points in X, we consider the closed subset $Z = \{x_1, \dots, x_n\} \in X$ equipped with its reduced induced closed subscheme structure $(Z, \mathcal{O}_Z)$ where the structure sheaf $\mathcal{O}_Z$ is given by the quotient $\mathcal{O}_X/\mathcal{I}_Z$, $\mathcal{I}_Z$ being the sheaf of ideals defined by

$$\mathcal{I}_Z(U) \begin{cases} \mathcal{O}_X(U) & if\ x_i \notin U \ \forall\ i \\ \mathfrak{m}_{x_i} & if\ x_i \in U \end{cases}$$

where $\mathfrak{m}_{x_i}$ is the maximal ideal corresponding to x_i and U belongs to a basis of open sets $\{U_\alpha\}_{\alpha \in A}$ s.t. if $x_i \in U_\alpha$ and $x_j \in U_\alpha$ then $i = j$. It is always possible to construct such a basis, since $Z_1, \dots, Z_n$ are closed isolated points, hence, even if the space is not $T1$ in general, in this particular case the points can be separated, i.e there exists a

neighborhood of Z_i that does not contain Z_j for $j \neq i$.
Hence $\mathcal{O}_Z$ on the same basis is

$$\mathcal{O}_Z(U) \begin{cases} (0) & if\ x_i \notin U\ \forall\ i \\ K & if\ x_i \in U \end{cases}$$

or, equivalently

$$\mathcal{O}_Z = \bigoplus_{i=1}^{n} the\ skyscraper\ sheaf\ at\ x_i$$

We get

$$\mathcal{O}_Z \otimes \mathcal{O}_X(m) = \mathcal{O}_Z \quad \forall\ m$$

And, as a consequence we have that the Hilbert polynomial associated to Z is equal to

$$P_Z(m) = \chi(\mathcal{O}_Z \otimes \mathcal{O}_X(m)) = n$$

for all $m \in \mathbb{N}$. Hence $Z \in Hilb_X^P$, with $P = n$. This means that finite subsets of n distinct points of X are parametrized by a subset of Hilb_X^P if P is the constant polynomial: we are interested in studying these specific Hilbert schemes more in detail.

Definition 1.3. Let P be the constant polynomial $P(m) = n\ \forall\ m \in \mathbb{Z}$, with $n \in \mathbb{N}$, $n \geq 1$. We denote with $X^{[n]} := Hilb_X^P$ the corresponding Hilbert scheme, and define it the *Hilbert scheme of points*.

It is the moduli space that parametrizes 0-dimensional subschemes of length n in X. Recalling the example it is not difficult to understand the choice of its name; moreover it is quite natural to think about an analogy between $X^{[n]}$ and S^nX, where S^nX is the n-th symmetric product of X, that is

$$S^nX = \underbrace{X \times \cdots \times X}_{n \text{ times}} /\mathfrak{S}_n$$

and $\mathfrak{S}_n$ is the symmetric group of degree n.
The scheme S^nX parametrizes effective cycles of dimension zero and degree n; its elements can be thought as formal sums $\sum n_i[x_i]$, where $n_i \in \mathbb{N}\ \forall\ i$ and $\sum n_i = n$, $x_i \in X$. So when it comes to n distinct points they are parametrized both by $X^{[n]}$ and S^nX, but the Hilbert scheme of points of X is in general way more complex and rich than the symmetric product. However, there is a precise connection between the two objects: let us first recall that a zero-dimensional subscheme Z is the finite disjoint union $\coprod Z_x$ of subschemes supported on a single point. Then we have the following theorem:

Theorem 1.4. *There exist a morphism*

$$\pi : X^{[n]}_{\text{red}} \longrightarrow S^n X$$

given by

$$\pi(Z) = \sum_{x \in X} \text{length}(Z_x)[x]$$

This morphism is called the *Hilbert-Chow morphism*. We will study this map more explicitly for a particular case in the following section.

Let us consider now two interesting examples, which hopefully will shed some light on the relationship between $X^{[n]}$ and $S^n X$:

Example 1.5. Let X be a projective scheme. We suppose X to be non singular, and give a closer look to the points of $X^{[2]}$.
As we have seen before, we are interested in 0-dimensional subschemes and there are two different possible cases:

- If $Z = \{x_1, x_2\}$ and $x_1 \neq x_2$, it is clear that Z has length 2 and we've already shown that $Z \in X^{[2]}$. As for the Hilbert-Chow morphism π, it takes Z to the formal sum $[x_1] + [x_2]$.

- A more interesting case arises when $x_1 = x_2$, in other words x_1 collide with x_2 into a single point x, and we consider $Z = \{x\}$. Given $v \in T_x X$, $v \neq 0$ it is possible to define the ideal $\mathcal{I} \subset \mathcal{O}_X$

$$\mathcal{I} = \{f \in \mathcal{O}_X \ s.\ t.\ f(x) = 0,\ df_x(v) = 0\}$$

This ideal has colength 2, hence $\mathcal{O}_X/\mathcal{I}$ defines a 0-dimensional subscheme, of length 2. We observe that if $v' = \lambda v$, with $\lambda \neq 0$, then v' defines the same ideal, and as a consequence the same subscheme, so for every x, $X^{[2]}$ contains a set of points, one for each $v \in \mathbb{P}^1(K)$. Thinking about Z as given by two points x_1 and x_2 colliding, then the subschemes constructed keep track of the direction along which they collide. The image of Z through the morphism π is $2[x]$
Namely $X^{[2]}$ consists of couples of distinct points $\{x_1, x_2\}$, with no regard for their order and points identified by the couple $(\{x, x\}, v)$. More precisely

$$X^{[2]} = Blow_\Delta(X \times X)/\mathfrak{S}_2$$

where $Blow_\Delta(X \times X)$ is the blow-up of $(X \times X)$ along the diagonal Δ. This will be understood better in the next section.

Example 1.6. Let now $X = \mathbb{A}$, the affine line. Then

$$\begin{aligned}\mathbb{A}^{[n]} &= \{I \subset K[z] \mid I \text{ is an ideal}, \ \dim_K K[z]/I = n\} = \\ &= \{f \in K[z] | f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0, \ a_i \in K\} = S^n\mathbb{A}\end{aligned}$$

The equality $X^{[n]} = S^nX$ holds more generally if dim $X = 1$ for X nonsingular, roughly because in this case the tangent space T_xX has only one dimension, so there is no space to choose different directions v.

We conclude the paragraph with a final observation about S^nX
Let $\nu = (\nu_1, \ldots, \nu_k)$ be a partition of n, i.e. a finite sequence of non-increasing positive integers such that $\sum_{i=1}^{k} \nu_i = n$. For each partition ν of n, we define

$$S^n_\nu X = \{ \sum_{i=1}^{k} \nu_i[x_i] \in S^nX \mid x_i \neq x_j, \ for \ i \neq j \}$$

Then the S^n_ν form a stratification of S^nX into locally closed subschemes, and every point of S^nX lies in a unique S^n_ν. Each S^n_ν has dimension $k$dim X, where k is the length of the partition.
The Hilbert-Chow morphism implies that the above stratification of S^nX induces a stratification of $X^{[n]}$, defined by

$$X^{[n]}_\nu := \pi^{-1}(S^n_{-}\nu X)$$

For every partition $\nu = (\nu_1, \ldots, \nu_k)$ the geometric points of $X^{[n]}_\nu$ are the union of subschemes $(Z_1, \ldots, Z_k)$, where Z_i is a subscheme of length ν_i and support x_i, the points x_i being distinct.

1.2.1 Case $X = \mathbb{A}^2$

Previously we have described the structure of Hilbert schemes in their generality: from this point on we will focus on the simpler case in which dim $X = 2$. Let's start with a specific scheme, the affine plane $\mathbb{A}^2$. The following result gives an interesting and useful description of $(\mathbb{A}^2)^{[n]}$

Theorem 1.7. *Let*

$$\widehat{H} := \left\{ (B_1, B_2, i) \;\middle|\; \begin{array}{l} (i) \ [B_1, B_2] = 0 \\ (ii) \ \textit{There exists no proper subspace } S \subsetneq K^n \\ \textit{such that } B_\alpha(S) \subset S \ (\alpha = 1, 2) \textit{ and } \operatorname{im} i \subset S \end{array} \right\}$$

where $B_\alpha \in \operatorname{End}(K^n)$ and $i \in \operatorname{Hom}(K, K^n)$, i.e. i can be identified with a vector in K^n. Defining the action of $GL_n(K)$ on $\widehat{H}$ by

$$g \cdot (B_1, B_2, i) = (gB_1g^{-1}, gB_2g^{-1}, gi)$$

for $g \in GL_n(K)$ then the quotient space $H := \widehat{H}/GL_n(K)$ is a non singular variety and represents the functor $\mathcal{H}ilb_X^P$ for $\mathbb{A}^2$, $P = n$.

Remark 1.8. Clearly the set of elements (B_1, B_2, i) such that $[B_1, B_2] = 0$ is a Zariski closed subset of $\mathrm{End}(K^n) \times \mathrm{End}(K^n) \times K^n$. The second condition, which corresponds to the existence of a cyclic vector i, is a stability condition, and defines an open subset of it. The quotient is meant in the sense of geometric invariant theory.

Proof. Let us begin with the proof of the isomorphism between H and $(\mathbb{A}^2)^{[n]}$ as sets. If $p \in (\mathbb{A}^2)^{[n]}$ then we can associate it with the datum $(B_1, B_2, i) = \Phi(p)$ as follows.
From the definition, a point p in the Hilbert scheme of points of $\mathbb{A}^2$ corresponds to an ideal $I_p \subset K[z_1, z_2]$ s.t. $\dim_K K[z_1, z_2]/I_p = n$: we choose a basis to identify $K[z_1, z_2]/I_p \simeq K^n$, and we define B_α as the matrix of the multiplication by z_α mod I_p, $(\alpha = 1, 2)$

$$\begin{aligned} B_\alpha :\ K[z_1, z_2]/I_p &\longrightarrow K[z_1, z_2]/I_p \\ v &\longmapsto z_\alpha v \bmod I \end{aligned}$$

and i as the element of K^n corresponding to the identity $1 \in K[z_1, z_2]/I_p$. B_1 and B_2 are clearly endomorphisms of $K[z_1, z_2]/I_p$, while i is a homomorphism between K and $K[z_1, z_2]/I_p$; moreover multiplications by z_1 and z_2 commute with each other, so $[B_1, B_2] = 0$ and the stability condition holds, indeed:
Let S be a subset of $K[z_1, z_2]/I_p$, s.t. $i(K) \subset S$ and $B_\alpha(S) \subset S$ for $\alpha = 1, 2$, then $1 \bmod (I) \in S$ and $z_\alpha \in S$ for $\alpha = 1, 2$. But $1, z_1, z_2$ generate the whole $K[z_1, z_2]$, so S cannot be proper. Hence (B_1, B_2, i) lives in $\widehat{H}$ and the map

$$\begin{aligned} \Phi : (\mathbb{A}^2)^{[n]} &\longrightarrow \widehat{H} \\ p &\longmapsto (B_1, B_2, i) \end{aligned}$$

constructed above, is well defined. Choosing a different basis results precisely in the action of $GL_n(K)$ on the triple (B_1, B_2, i)
Now, let $h = (B_1, B_2, i) \in \widehat{H}$. Let φ_h be the morphism $\varphi_h : K[z_1, z_2] \longrightarrow K^n$ such that $\varphi_h(f) = f(B_1, B_2)i(1)$. Then the map

$$\begin{aligned} \Psi : \widehat{H} &\longrightarrow (\mathbb{A}^2)^{[n]} \\ h = (B_1, B_2, i) &\longmapsto \ker\ \varphi_h \end{aligned}$$

is well defined:
if $v \in \mathrm{im}\ \varphi_h \subset K^n$, then there is a $f \in K[z_1, z_2]$ s.t. $v = \varphi_h(f) = f(B_1, B_2)i(1)$, hence $B_\alpha v = B_\alpha f(B_1, B_2)i(1) = f'(B_1, B_2)i(1)$ with $f' \in K[z_1, z_2]$, and we have $B_\alpha v \in \mathrm{im}\ \varphi_h\ \forall\ v \in \mathrm{im}\ \varphi_h$ and $\alpha = 1, 2$. Besides $\mathrm{im}\ i = i(K) \subset \mathrm{im}\ \varphi_h$ (it is sufficient to take f as a constant polynomial), so $\mathrm{im}\ \varphi_h$ is B_α-invariant and contains $\mathrm{im}\ i$. By the

stability condition this implies that φ_h is surjective, and $\dim_K\ K[z_1, z_2]/\ker\ \varphi_h = n$. Finally two data in the same class of the quotient define similar endomorphisms, and as a consequence the kernels of the two morphisms are the same, as well as the correspondent points of $(\mathbb{A}^2)^{[n]}$, thus Ψ can be defined on the quotient space H and the two maps Φ and Ψ are mutually inverse on H.

The second step consists in proving the non-singularity of H.
The differential of the map $f : (B_1, B_2, i) \longmapsto [B_1, B_2]$ is

$$\mathrm{d}_{(B_1,B_2,i)} f(E, F, i') = [E, B_2] + [B_1, F]$$

We recall that, in general, there is a canonical isomorphism between $\mathrm{End}(K^n, K^n)$ and $\mathrm{End}(K^n, K^n)^*$ given by the non-degenerate canonical bilinear form

$$\mathrm{Hom}(K^n, K^n) \ni A, B \longmapsto \mathrm{tr}(AB) \in K$$

and that the annihilator of a linear subspace $W \in V$ is defined as

$$W^\perp := \{\varphi \in V^* \mid \varphi(w) = 0\ \forall\ w \in W\}$$

Finally, there is an isomorphism between $(W^*)^\perp$ and V/W. Then we can write the cokernel of $\mathrm{d}f$ as

$$\begin{aligned}\mathrm{coker}\ \mathrm{d}f &= \mathrm{Hom}(K^n, K^n)/\mathrm{im}\ \mathrm{d}f \cong (\mathrm{im}\ \mathrm{d}f^*)^\perp \\ &= \{C \in \mathrm{End}(K^n) \mid \mathrm{tr}(C([E, B_2] + [B_1, F])) = 0\ \forall\ E, F\}\end{aligned}$$

Besides we have $\mathrm{tr}(CEB_2 - CB_2E + CB_1F - CFB_1) = \mathrm{tr}(EB_2C) - \mathrm{tr}(ECB_2) + \mathrm{tr}(FCB_1) - \mathrm{tr}(FB_1C) = \mathrm{tr}(E[B_2, C]) + \mathrm{tr}(F[C, B_2])$; since this equality has to hold for all $E, F \in \mathrm{End}(K^n, K^n)$, we get

$$\mathrm{coker}\ \mathrm{d}f = \{C \in \mathrm{End}(K^n) \mid [C, B_1] = [C, B_2] = 0\}$$

Let R be the ring $\{B_1^k B_2^l i(1)\}_{k,l>0}$. If we consider it as an R-module on itself then the conditions $[C, B_1] = [C, B_2] = 0$ imply that

$$C : R \longrightarrow R$$

is a morphism of R-modules, since we get

$$\begin{aligned}C(v + w) &= C(B_1^{k_1} B_2^{l_1} i(1) + B_1^{k_2} B_2^{l_2} i(1)) = C((B_1^{k_1} B_2^{l_1} + B_1^{k_2} B_2^{l_2}) i(1)) \\ &= (B_1^{k_1} B_2^{l_1} + B_1^{k_2} B_2^{l_2}) C(i(1)) = B_1^{k_1} B_2^{l_1} C(i(1)) + B_1^{k_2} B_2^{l_2} C(i(1)) \\ &= C(v) + C(w)\end{aligned}$$

and similarly $C(\lambda v) = \lambda C(v)$ with $\lambda \in R$. Hence C is determined by the image of $i(1)$, and $C(i(1))$ can be any vector in K^n, since by the stability condition $B_1^k B_2^l i(1)$ span

K^n. Therefore there is a bijective correspondence between the endomorphisms which commute with B_1, B_2 and the vectors in K^n, so that the dimensions of the two spaces agree. Thus

$$\dim \text{ coker } \mathrm{d}f = \dim\{C \in \mathrm{End}(K^n) \mid [C, B_1] = [C, B_2] = 0\} = n$$

The constant rank of the differential implies that $\widehat{H}$ is non singular, what about the quotient space H?
Let us consider the action of $GL_n(K)$ on $\widehat{H}$: if $g \in GL_n(K)$ stabilizes $(B_1, B_2, i) \in \widehat{H}$ then $gB_1g^{-1} = B_1$, $gB_2g^{-1} = B_2$ and $gi = i$, so that

i) $(g - id)i = 0$, i.e. im $i \in$ ker $(g - id)$

ii) if $x \in$ker $(g - id)$ then $(g - id)(B_\alpha x) = gB_\alpha x - B_\alpha x = gB_\alpha g^{-1}x - B_\alpha g^{-1}x = B_\alpha x - B_\alpha g^{-1}x = B_\alpha(x - g^{-1}x) = 0$

Hence ker $(g - id)$ contains im i and is invariant under B_1, B_2: the stability condition implies that $g = id$, thus the stabilizer of $GL_n(K)$-action is trivial and by Luna's slice theorem [11] $H = \widehat{H}/GL_n(K)$ has a structure of non singular variety such that the map $\widehat{H} \to H$ is a principal étale fiber bundle for the group $GL_n(K)$; since $GL_n(K)$ is a special group, then the fact that the principal étale fiber bundle is trivial implies that the fiber bundle is locally trivial even in the Zariski topology. Moreover the map Ψ described before provides a flat family of 0-dimensional subschemes $\mathcal{H} \to H$.

Finally let us prove the universality of $\mathcal{H} \to H$, i.e. if $\pi : Z \to U$ is another flat family of 0-dimensional subschemes of $\mathbb{A}^2$ of length n, then there exists a unique morphism $\chi : U \to H$ such that the pullback $\chi^*\mathcal{H} \to U$ is $Z \to U$.
We observe that if $\pi : Z \to U$ is flat and of length n, then $\pi_*\mathcal{O}_Z$ is a $\mathcal{O}_U$-module and a locally free sheaf of rank n, hence locally it is possible to define B_1, B_2 as above from multiplication of coordinate functions z_1, z_2 and i from the constant polynomial 1: B_1, B_2 are commuting $\mathcal{O}_U$-linear endomorphisms of $\pi_*\mathcal{O}_Z$ and $i \in \mathrm{Hom}(\mathcal{O}_U, \pi_*\mathcal{O}_Z)$.
Fix an open covering $\{U_\alpha\}$ of U and trivializations of the restriction of $\pi_*\mathcal{O}_Z$ to U_α. Then (B_1, B_2, i) defines morphisms $U_\alpha \to \widehat{H}$. If we compose them with the projection $\widehat{H} \to H$, they glue together to define a morphism $\phi : U \to H$: it is precisely the morphism we were looking for because by construction $\phi^*\mathcal{H}$ is Z. The uniqueness is clear. □

Remark 1.9. We point out that according to the proof of the theorem, the ideal I corresponding to the point $h = (B_1, B_2, i)$ is given by the kernel of the map φ_h, thus

$$I = \{f \in K[z_1, z_2] \mid f(B_1, B_2)i(1) = 0\}$$

By the stability condition, however, it can be rewritten as

$$I = \{f \in K[z_1, z_2] \mid f(B_1, B_2) = 0\}$$

since ker $f(B_1, B_2)$ is B_α-invariant and contains im i, hence it cannot be proper.

Remark 1.10. Let us fix a point $[(B_1, B_2, i)]$ in $(\mathbb{C}^2)^{[n]}$. Consider the maps

$$\begin{array}{ccccc} & & \mathrm{End}(K^n, K^n) & & \\ & & \oplus & & \\ GL_n(K) & \xrightarrow{\psi} & \mathrm{End}(K^n, K^n) & \xrightarrow{\phi} & \mathrm{End}(K^n, K^n) \\ & & \oplus & & \\ & & \mathrm{Hom}(K, K^n) & & \end{array}$$

where ψ is the action of $GL_n(K)$ on the fixed point (B_1, B_2, i), ϕ is the map $(C_1, C_2, j) \mapsto [C_1, C_2]$ and their derivatives

$$\begin{array}{ccccc} & & \mathrm{Hom}(K^n, K^n) & & \\ & & \oplus & & \\ \mathrm{Hom}(K^n, K^n) & \xrightarrow{\mathrm{d}\psi} & \mathrm{Hom}(K^n, K^n) & \xrightarrow{\mathrm{d}\phi} & \mathrm{Hom}(K^n, K^n) \\ & & \oplus & & \\ & & K^n & & \end{array} \tag{1.1}$$

Then $\mathrm{d}\psi : G \mapsto ([G, B_1], [G, B_2], Gi)$ and $\mathrm{d}\phi : (C_1, C_2, j) \mapsto [B_1, C_2] + [C_1, B_2]$, moreover (1.1) is a complex, since

$$\begin{aligned} \mathrm{d}\phi(\mathrm{d}\psi(G)) &= \mathrm{d}\phi([G, B_1], [G, B_2], Gi) = [B_1, [G, B_2]] + [[G, B_1], B_2] = \\ &= -[B_2, [B_1, G]] - [G, [B_2, B_1]] + [[G, B_1], B_2] = \\ &= ([B_1, B_2] = 0) = [B_2, [G, B_1]] - [B_2, [G, B_1]] = 0 \end{aligned}$$

The description of H provided by theorem 1.7 implies that the tangent space of $(\mathbb{A}^2)^{[n]}$ at the point (B_1, B_2, i) is the middle cohomology group of this complex. We have already seen that the dimension of coker $\mathrm{d}\phi$ is n, besides if $G \in \ker \mathrm{d}\psi = \{G \mid [G, B_1] = [G, B_2] = 0,\ Gi = 0\}$, $G \neq 0$ then ker G is proper and

i) $Gi = 0 \ \Rightarrow \mathrm{im}\ i \in \ker G$

ii) given $x \in \ker G$ then $G(B_\alpha x) = B_\alpha(Gx) = 0 \ \Rightarrow \ B_\alpha(\ker G) \subseteq \ker G$

violating the stability condition. Hence $\mathrm{d}\psi$ is surjective and an easy calculation shows that the dimension of the tangent space is $2n$.

Finally we examine two simple examples:

Example 1.11. First, let us study $(\mathbb{A}^2)^{[1]}$. We fix

$$B_1 = \lambda, \quad B_2 = \mu \quad \lambda, \mu \in K$$

In this case the group $GL_1(K)$ acts as the multiplication on i:
the action is $(\lambda, \mu, i) \mapsto (g\lambda g^{-1}, g\mu g^{-1}, gi)$ but $g \in GL_1(K) \cong K^\times$ so that

$$(g\lambda g^{-1}, g\mu g^{-1}, gi) = (gg^{-1}\lambda, gg^{-1}\mu, gi) = (\lambda, \mu, gi)$$

Moreover we have seen that i has to be non-zero, so it must be a constant $i = k$, hence, applying the action of $GL_1(K)$ if necessary, we can always assume $i = 1$.
Remark 1.9 shows that the ideal I correspondent to the point $(\lambda, \mu, 1)$ is given by

$$I = \{f \in K[z_1, z_2] \mid f(\lambda, \mu) = 0\}$$

This is precisely the ideal of definition of the point $(\lambda, \mu) \in \mathbb{A}^2$. Thus it it defined the map $(\mathbb{A}^2)^{[1]} \to \mathbb{A}^2 : \ (\lambda, \mu, i) \mapsto (\lambda, \mu)$; it is clearly surjective, and it is injective, since if (λ, μ, i) and (λ', μ', i') are mapped into the same point (x, y) then obviously $x = \lambda = \lambda', \quad y = \mu = \mu'$ and $i' = gi$, therefore (λ, μ, i) and (λ', μ', i') coincide in the quotient space $H \cong (\mathbb{A}^2)^{[1]}$ and

$$(\mathbb{A}^2)^{[1]} \cong \mathbb{A}^2$$

Example 1.12. Now we consider $(\mathbb{A}^2)^{[2]}$: the points (B_1, B_2, i) can be of two different kinds.
(a) Both B_1 and B_2 have two distinct eigenvalues, therefore they are semisimple and since they commute , they can be simultaneously diagonalized. Then, remembering how $GL_2(K)$ acts, we can assume

$$B_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad B_2 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \quad i(1) = \begin{pmatrix} k \\ h \end{pmatrix} \quad \lambda_1, \lambda_2, \mu_1, \mu_2, k, h \in K$$

with $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$.
Let us focus on the characteristics of $i(1)$ for these particular matrices: if $h = 0$ then $S := \{(x, 0)^T, \ x \in K\}$ is proper, B_α-invariant and contains im i, since $i(\eta) = \eta i(1)$. Hence by the stability condition we get $h \neq 0$ and similarly $k \neq 0$. Moreover, given

$$g = \begin{pmatrix} 1/k & 0 \\ 0 & 1/h \end{pmatrix}$$

then

$$(gB_1g^{-1}, gB_2g^{-1}gi) = (B_1, B_2, gi), \quad \text{with } gi(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore we can always assume $i(1) = (1, 1)^T$. The corresponding ideal is

$$I = \left\{ f \in K[z_1, z_2] \mid f(B_1, B_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \right\}$$

In fact, if $f = \sum_{i,j=1}^{n} a_{ij} z_1^i z_2^j$ then

$$f(B_1, B_2) = \sum_{i,j=1}^{n} a_{ij} B_1^i B_2^j = \sum_{i,j=1}^{n} a_{ij} \begin{pmatrix} \lambda_1^i & 0 \\ 0 & \lambda_2^i \end{pmatrix} \begin{pmatrix} \mu_1^j & 0 \\ 0 & \mu_2^j \end{pmatrix} =$$
$$= \begin{pmatrix} \sum_{i,j=1}^{n} a_{ij} \lambda_1^i \mu_1^j & 0 \\ 0 & \sum_{i,j=1}^{n} a_{ij} \lambda_2^i \mu_2^j \end{pmatrix} = \begin{pmatrix} f(\lambda_1, \mu_1) & 0 \\ 0 & f(\lambda_2, \mu_2) \end{pmatrix}$$

thus the ideal I is given by

$$I = \{f \in K[z_1, z_2] \mid f(\lambda_1, \mu_1) = 0 = f(\lambda_2, \mu_2)\}$$

hence

$$I = I(\{(\lambda_1, \mu_1)\}) \cap I(\{(\lambda_2, \mu_2)\}) = I(\{(\lambda_1, \mu_1), (\lambda_2, \mu_2)\})$$

and it defines the subset of two distinct points in $\mathbb{A}^2$.

(b) Now we suppose both B_1 and B_2 have only one eigenvalue each. Since $[B_1, B_2] = 0$ we can make them into upper triangular matrices simultaneously, so we obtain

$$B_1 = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix} \quad B_2 = \begin{pmatrix} \mu & \beta \\ 0 & \mu \end{pmatrix} \quad i(1) = \begin{pmatrix} k \\ h \end{pmatrix} \quad \lambda, \alpha, \mu, \beta, k, h \in K,\ \alpha, \beta \neq 0$$

α and β must be non-zero, otherwise, posed $S :=\text{im } i$, then S is proper and it satisfies $B_1(S) = \lambda S \subseteq S$, $B_2(S) = \mu S \subseteq S$ violating the stability condition. Besides with an argument similar to the one made above we conclude that $h \neq 0$ and given

$$g = \begin{pmatrix} 1/h & -k/h^2 \\ 0 & 1/h \end{pmatrix}$$

it results

$$(gB_1g^{-1}, gB_2g^{-1}gi) = (B_1, B_2, gi), \quad \text{with } gi(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(is easy to check that $g \in GL_2(K)$ commutes with both B_1 and B_2) so we can always assume $i(1) = (0, 1)^T$. The ideal defined by this point is

$$I = \left\{ f \in K[z_1, z_2] \mid f(B_1, B_2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \right\}$$

More explicitly, if $f = \sum_{i,j=1}^{n} a_{ij} z_1^i z_2^j$ and

$$L = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \; M = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \; A = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \; B = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

it holds:

$$\begin{aligned}
f(B_1,B_2) &= \sum_{i,j=1}^{n} a_{ij}[L+A]^i[M+B]^j = \\
&= \sum_{i,j=1}^{n} a_{ij}\left[\sum_{k=0}^{i}\binom{i}{k} L^k A^{i-k}\right]\left[\sum_{h=0}^{j}\binom{j}{h} M^h A^{j-h}\right] = \\
&= \sum_{i,j=1}^{n} a_{ij}\left[L^i + iL^{i-1}A\right]\left[M^j + jM^{j-1}B\right] = \\
&= \sum_{i,j=1}^{n} a_{ij}\left[\begin{pmatrix}\lambda^i\mu^j & 0\\ 0 & \lambda^i\mu^j\end{pmatrix} + \begin{pmatrix}0 & \beta(j\lambda^i\mu^{j-1})\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \alpha(i\lambda^{i-1}\mu^j)\\ 0 & 0\end{pmatrix}\right] = \\
&= \begin{pmatrix} f(\lambda,\mu) & \alpha\frac{\partial f}{\partial z_1}(\lambda,\mu)+\beta\frac{\partial f}{\partial z_2}(\lambda,\mu)\\ 0 & f(\lambda,\mu)\end{pmatrix}
\end{aligned}$$

As a consequence the ideal I can be rewritten as follows:

$$\begin{aligned}
I &= \left\{ f\in K[z_1,z_2] \mid \begin{pmatrix} f(\lambda,\mu) & \alpha\frac{\partial f}{\partial z_1}(\lambda,\mu)+\beta\frac{\partial f}{\partial z_2}(\lambda,\mu)\\ 0 & f(\lambda,\mu)\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0\right\} \\
&= \left\{ f\in K[z_1,z_2] \mid f(\lambda,\mu)=0 \ \mathit{and}\ \alpha\frac{\partial f}{\partial z_1}(\lambda,\mu)+\beta\frac{\partial f}{\partial z_2}(\lambda,\mu)=0\right\}
\end{aligned}$$

Hence it corresponds to a subscheme concentrated at (λ,μ) of length 2, that can be thought as two points at (λ,μ) infinitesimally attached to each other in the direction of $\alpha\frac{\partial}{\partial z_1}+\beta\frac{\partial}{\partial z_2}$. These ideals are parametrized by the homogeneous coordinates $[\alpha,\beta]$ in the projectivezed tangent space $\mathbb{P}(T_{(\lambda,\mu)}\mathbb{A}^2)$, isomorphic to $\mathbb{P}^1$.
It is clear the identification

$$(\mathbb{A}^2)^{[2]} \cong \mathrm{Blow}_\Delta(\mathbb{A}^2\times\mathbb{A}^2)/\mathfrak{S}_2$$

Remark 1.13. the two examples above give us a hint to rewrite the Hilbert-Chow morphism according to the description of theorem 1.7:
let $[(B_1,B_2,i)]\in(\mathbb{A}^2)^{[n]}$; since $[B_1,B_2]=0$ we can always make them simultaneously into upper triangular matrices, with the eigenvalues $\{\lambda_1,\dots,\lambda_n\}$ and $\{\mu_1,\dots,\mu_n\}$ on the diagonals. Then, similarly to the case $n=2$, the subscheme identified by $[(B_1,B_2,i)]$ is concentrated on the points $\{(\lambda_1,\mu_1),\dots,(\lambda_k,\mu_k)\}$ where each distinct couple $(\lambda_{i_h},\mu_{i_h})$, appears l_{i_h} times, so that

$$\pi([(B_1,B_2,i)]) = \sum_{i_h} l_{i_h}[(\lambda_{i_h},1\mu_{i_h})] = \{(\lambda_1,\mu_1),\dots,(\lambda_k,\mu_k)\} = (p_{B_1},p_{B_2})$$

where p_{B_1},p_{B_2} are the characteristic polynomials of B_1,B_2

Remark 1.14. The non singularity of $X^{[n]}$ holds every time X is non singular and dim $X = 2$. More precisely:

Theorem 1.15. *Suppose X is non singular and dim $X = 2$ then the following holds.*

1. *$X^{[n]}$ is non singular and has dimension $2n$*
2. *$\pi : X^{[n]} \to S^n X$ is a resolution of singularities.*

The proof of (1) comes easily from theorem 1.7:
let $Z \in X^{[n]}$ and $\mathcal{I}_Z$ the corresponding ideal. Suppose $\pi(Z) = \sum_i \nu_i[x_i]$, where the points x_i are pair wise distinct. Let $Z = \coprod_i Z_i$ the corresponding decomposition. Then locally (in the classical topology) $X^{[n]}$ decomposes into a product $\prod_i X[\nu_i]$ Thus it is enough to show that $X^{[n]}$ is non singular at Z when Z is supported at a single point. Hence we may replace X by the affine plane $\mathbb{A}^2$.

1.3 Further facts and results

1.3.1 Framed moduli space of torsion free sheaves on $\mathbb{P}^2$

Let $K = \mathbb{C}$ and $\mathcal{M}(r,n)$ be the framed moduli space of torsion free sheaves on $\mathbb{P}^2$ with rank r and $c_2 = n$, i.e.

$$\mathcal{M}(r,n) := \left\{ (E,\Phi) \;\middle|\; \begin{array}{l} E \text{ is a torsion free sheaf of rank} \\ E = r,\ c_2(E) = n \text{ which is locally free in a} \\ \text{neighborhood of } l_\infty, \\ \Phi : E|_{l_\infty} \to \mathcal{O}_{l_\infty}^{\oplus r} \text{framing at infinity} \end{array} \right\} / \text{isomorphism}$$

where $l_\infty = \{[0 : z_1 : z_2] \in \mathbb{P}^2\} \subset \mathbb{P}^2$ is the line at infinity: Notice that the existence of framing Φ implies $c_1(E) = 0$.
There is an interesting relation between this space and $(\mathbb{C}^2)^{[n]}$; more precisely, theorem 1.7 can be seen in fact as a particular case of a more general theorem which gives a description of $\mathcal{M}(r,n)$. We would like to outline how this connection arise, although we shall leave out most of the proofs. The following theorem is due to Barth [2].

Theorem 1.16. *There exists a bijection*

$$\mathcal{M}(r,n) \cong \left\{ (B_1,B_2,i,j) \;\middle|\; \begin{array}{l} (i)\ [B_1,B_2] + ij = 0 \\ (ii)\ \textit{There exists no proper subspace } S \subsetneq \mathbb{C}^n \\ \textit{such that } B_\alpha(S) \subset S\ (\alpha = 1,2) \textit{ and } \operatorname{im} \subset S \end{array} \right\} / GL_n(\mathbb{C})$$

where $B_\alpha \in \mathrm{End}(\mathbb{C}^n)$*,* $i \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ *and* $j \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ *and the action of* $GL_n(\mathbb{C})$ *is given by*

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1})$$

for $g \in GL_n(\mathbb{C})$*.*

Here we give only a set theoretical bijection, but it is possible to prove that the bijection is actually an isomorphism between algebraic varieties. In the case $r = 1$ we have an isomorphism

$$\mathcal{M}(1, n) \cong (\mathbb{P}^2 \setminus l_\infty)^{[n]} = (\mathbb{C}^2)^{[n]}$$

hence the theorem 1.16 gives a description of the Hilbert scheme of n points of $\mathbb{C}^2$ as a special case. The next proposition shows that this is exactly the description of $(\mathbb{C}^2)^{[n]}$ provided by theorem 1.7.

Proposition 1.17. *Assume* $r = 1$*. Suppose a quadruple* (B_1, B_2, i, j) *satisfying conditions* $(i), (ii)$ *in theorem 1.16 is given. Then* $j = 0$

Proof. Let $S \subset \mathbb{C}^n$ be a subspace defined by

$$S = \sum B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_k} i(\mathbb{C})$$

where $\alpha_i = 1, 2$; we prove that the restriction $j|_S$of j to S vanishes by induction on k. If $k = 0$ then $\widehat{B} = B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_k} = 1$ and we have

$$ji = \mathrm{tr}(ji) = \mathrm{tr}(ij) = -\mathrm{tr}([B_1, B_2]) = 0$$

Now, suppose the claim is true for $k \leq m - 1$. If $\widehat{B}$ contains a sequence $\cdots B_2 B_1 \cdots$ we get

$$\begin{aligned} j\widehat{B} &= jB_{\alpha_1} \cdots B_2 B_1 \cdots B_{\alpha_m} \\ &= jB_{\alpha_1} \cdots ([B_2, B_1] + B_1 B_2) \cdots B_{\alpha_m} \\ &= (jB_{\alpha_1} \cdots i)j \cdots B_{\alpha_m} + jB_{\alpha_1} \cdots B_1 B_2 \cdots B_{\alpha_m} \\ &= jB_{\alpha_1} \cdots B_1 B_2 \cdots B_{\alpha_m} \end{aligned}$$

where the last equality comes from the fact that $(jB_{\alpha_1} \cdots i)$ has length less or equal to $m - 1$, thus it is zero by induction hypothesis. Hence every time we have a sequence $B_2 B_1$ we can switch B_2 and B_1, so that we obtain

$$jB_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_m} = jB_1^{m_1} B_2^{m_2}$$

with $m_1 + m_2 = m$ and $m_s = \#\{l|\alpha_l = s\}$, $s = 1, 2$ and it is sufficient to show the claim for $\widehat{B} = B_1^{m_1} B_2^{m_2}$. In this case we have

$$\begin{aligned}
&= j\widehat{B}i = \mathrm{tr}(\widehat{B}ij) = -\mathrm{tr}(B_1^{m_1}B_2^{m_2}[B_1, B_2]) \\
&= -\mathrm{tr}([B_1^{m_1}B_2^{m_2}, B_1], B_2) = -\mathrm{tr}(B_1^{m_1}[B_2^{m_2}, B_1]B_2) \\
&= -\sum_{l=0}^{m_2-1} \mathrm{tr}(B_1^{m_1}B_2^{l}[B_2, B_1]B_2^{m_2-l-1}B_2) \\
&= -\sum_{l=0}^{m_2-1} \mathrm{tr}(B_2^{m_2-l}B_1^{m_1}B_2^{l}[B_2, B_1]) \\
&= -\sum_{l=0}^{m_2-1} \mathrm{tr}(B_2^{m_2-l}B_1^{m_1}B_2^{l}ij) = -\sum_{l=0}^{m_2-1} jB_2^{m_2-l}B_1^{m_1}B_2^{l}i
\end{aligned}$$

Since $jB_2^{m_2-l}B_1^{m_1}B_2^{l} = jB_1^{m_1}B_2^{m_2}$ we have

$$j\widehat{B}i = -m_2\widehat{ji}$$

Hence $j\widehat{B}i = 0$.
Since S is B_α-invariant and $\mathrm{im}i \subset S$, we must have $S = \mathbb{C}^n$. Therefore $j = 0$. □

The difference between the two descriptions is the appearance of j, which turns out to be 0 when $r = 1$. We point out that the auxiliary datum j is not actually artificial and it will play an important role when we will construct a hyper-Kähler structure on $(\mathbb{C}^2)^{[n]}$ in chapter 2.

1.3.2 Symplectic structure

Assume $k = \mathbb{C}$ and that X has a holomorphic symplectic form ω, i.e. ω is an element in $H^0(X, \Omega_X^2)$ which is non degenerate at every point $x \in X$; we wonder if $X^{[n]}$ inherits this property. Actually this is true as the following theorem shows

Theorem 1.18. *Suppose X has a holomorphic symplectic form ω. Then $X^{[n]}$ has a holomorphic symplectic form.*

Before proving it we focus on the particular case $X^{[n]} = (\mathbb{C}^2)^{[n]}$; It is possible to prove that the description in theorem 1.7 can be thought as a holomorphic symplectic quotient (actually we will prove that it has a structore of hyper-Kähler quotient), therefore we have a holomorphic symplectic form ω on $(\mathbb{C}^2)^{[n]}$.
As an application of the existence of ω, we give an estimate of the dimension of fibers of the Hilbert-Chow map as follows: the parallel traslation of $\mathbb{C}^2$ provides the factorization $(\mathbb{C}^2)^{[n]} = \mathbb{C}^2 \times ((\mathbb{C}^2)^{[n]}/\mathbb{C}^2)$. In our description a point in $(\mathbb{C}^2)^{[n]}/\mathbb{C}^2$ corresponds to (B_1, B_2, i) with $\mathrm{tr}(B_1) = \mathrm{tr}(B_2) = 0$. We have

Theorem 1.19. *The subvariety $\pi^{-1}(n[0])$ is isotropic with respect to the holomorphic symplectic form on $(\mathbb{C}^2)^{[n]}/\mathbb{C}^2$, i.e. the symplectic form restricts to zero on $\pi^{-1}(n[0])$. In particular, dim $\pi^{-1}(n[0]) \leq n-1$. Moreover there exist at least one $(n-1)$-dimensional component.*

Proof. Let us consider the torus action on $\mathbb{C}^2$ given by

$$\phi_{t_1,t_2} : (z_1, z_2) \mapsto (t_1 z_1, t_2 z_2) \quad for\ (t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$$

This action lifts to $(C^2)^{[n]}$ and $\pi^{-1}(n[0])$ is preserved under the resulting action. We observe that, as t_1, t_2 goes to infinity, any point in $\pi^{-1}(n[0])$ converges to a fixed point of the torus action: if Z is a non singular point of $\pi^{-1}(n[0])$ and v, w are two vectors in $(\phi_{t_1,t_2})_*(v)$, $(\phi_{t_1,t_2})_*(w)$ converge as $t_1, t_2 \to \infty$. On the other hand if we consider the pullback of the symplectic form ω by ϕ_{t_1,t_2} we obtain

$$t_1 t_2 \omega(v, w) = \omega((\phi_{t_1 t_2})_*(v), (\phi_{t_1 t_2})_*(w))$$

and, when $t_1, t_2 \to \infty$ it converges only if $\omega(v, w) = 0$. Hence $\pi^{-1}(n[0])$ is isotropic.
A $n-1$-component is given by:

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{n-1} \\ & 0 & a_1 & \cdots & a_{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & a_1 \\ & & & & 0 \end{pmatrix} \quad i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

where $a_1, \ldots, a_{n-1}$ are parameters in $\mathbb{C}$. The correspondent ideal is

$$I = (z_1^n, z_2 - (a_1 z_1 + \cdots + a_{n-1} z_{n-1}))$$

and clearly ideals of this type corresponding to different choices of the parameters are not isomorphic, hence this represents a $n-1$-dimensional subset of $\pi^{-1}(n[0])$ □

We conclude the paragraph proving theorem 1.18

Proof. We follow the proof that can be found in [3]. Let $S^n_* X$ be the subset of $S^n X$ consisting of $\sum \nu_i [x_i]$, with x_i distinct and $\nu_1 \leq 2$, $\nu_2 = \cdots = \nu_k = 1$. We denote by $X^{[n]}_*$ the inverse image by the Hilbert-Chow morphism π and X^n_* the one by the quotient map q, and consider $\Delta = \{(x_1, \ldots, x_n),\ with\ x_i = x_j\ for\ such\ i \neq j\}$. then $\Delta \cap X^n_*$ is smooth of codimension 2 in X^n_*, where the codimension can be estimated using the previous theorem. Moreover, if we generalize the result of example 1.12, we have the following commutative diagram

$$\begin{array}{ccc} Blow_\Delta(X^n_*) & \xrightarrow{\quad\eta\quad} & X^n_* \\ \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow \\ X^{[n]}_* & \xrightarrow{\quad\pi\quad} & S^n_*X \end{array}$$

where $\eta : Blow_\Delta(X^n_*) \to X^n_*$ denotes the blow-up of X^n_* along Δ and ρ is the quotient map given by the action of $\mathfrak{S}_n$. The map ρ is a covering ramified along the exceptional divisor E of η.
The holomorphic symplectic form on X induces one on X^n, $\sum p_i^*\omega$, if $p_i : X^n \to X$ is the projection on the i-th factor, still denoted by ω. Its pullback $\eta^*\omega$ is invariant under the action of $\mathfrak{S}^n$, therefore it defines a holomorphic 2-form $\widehat{\omega}$ on $X^{[n]}_*$, such that $\rho^*\widehat{\omega} = \eta^*\omega$. Then

$$\mathrm{div}(\rho^*\widehat{\omega}^n) = \rho^*\mathrm{div}(\widehat{\omega}^n) + E$$

and

$$\mathrm{div}(\eta^*\omega^n) = \eta^*\mathrm{div}(\omega^n) + E = E$$

Hence $\mathrm{div}(\widehat{\omega}^n) = 0$ and $\widehat{\omega}$ is a holomorphic symplectic form on $X^{[n]}_*$.
The following lemma shows that $X^{[n]} \setminus X^{[n]}_*$ has codimension 2, hence $\widehat{\omega}$ extends to the whole $X^{[n]}$ by the Hartogs theorem and it is still non degenerate. □

Lemma 1.20. *$X^{[n]} \setminus X^{[n]}_*$ has codimension 2 in $X^{[n]}$.*

Proof. Recalling the stratification $X^{[n]} = \bigcup_\nu X^{[n]}_\nu$ where ν runs over the partitions of n, we can write $X^{[n]}_*$ as $X^{[n]}_* = X^{[n]}_{(1,\dots,1)} \cup X^{[n]}_{(2,1,\dots,1)}$. Let us take $X^{[n]}_\nu$ with $\nu \neq (1,\dots,1), (2,1,\dots,1)$. Then $X^{[n]}_\nu \to S^nX_\nu$ is a locally trivial fiber bundle, which has fiber $Z_{\nu_1} \times \cdots \times Z_{\nu_k}$ where the Z_{ν_i} are punctual Hilbert schemes, i.e. schemes of length ν_i supported on one point. Therefore

$$\begin{aligned} \dim X^{[n]}_\nu &= \dim S^nX_\nu + \dim(Z_{\nu_1} \times \cdots \times Z_{\nu_k}) \\ &= 2k + (\nu_1 - 1) + \cdots + (\nu_k - 1) \\ &= 2k + n - k = n + k \end{aligned}$$

Hence $X^{[n]}_\nu$ has dimension $n + k$ where k is the length of ν. In our hypotesis ν has length less or equal to $n - 2$ so that $X^{[n]}_\nu$ has codimension less or equal to 2, since $X^{[n]}$ has dimension $2n$.
Finally if we choose $\nu = (3, 1, \dots, 1)$, then the codimension of $X^{[n]}_\nu$ is exactly $2n - (2n - 2) = 2$, and this completes the proof. □

1.3.3 The Douady space

We always assumed X to be projective, but Hilbert schemes can be generalized to the case X is a complex analytic space: the objects arising are called Douady spaces. We limit ourself to consider the zero-dimensional case, and denote them still by $X^{[n]}$. We would like to observe that many of the results in this first chapter can be generalized to Douady spaces, for instance:

- $X^{[n]}$ is a complex space;
- the Hilbert-Chow morphism $\pi : X^{[n]} \to S^n X$ is still defined and it is a holomorphic function;
- theorem 1.15 is still true in the complex analytic case (the proof still works);
- finally $X^{[n]}$ has a Kähler metric if X is compact and has a Kähler metric.

In fact M.A. de Cataldo and L. Migliorini in [5] give an explicit description of the Douady space $X^{[n]}$ and the Hilbert-Chow morphism. Here it is a sketch of their argument.
The idea is based on the construction of the Douady space $\Delta^{[m]}$, with $m \leq n$, for the bi-disk

$$\Delta = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_\alpha| < 1\}$$

By theorem 1.7 and using the Hilbert-Chow morphism, we obtain

$$\begin{aligned} \Delta^{[n]} &= \{Z \in (\mathbb{C}^2)^{[n]} \mid \pi(Z) \in S^n\Delta\} = \\ &= \left\{ (B_1, B_2, i) \bmod GL_n(\mathbb{C}) \in (\mathbb{C}^2)^{[n]} \;\middle|\; \begin{array}{l} \textit{the absolute value of the eigenvalues} \\ \textit{of } B_1, B_2 \textit{ are smaller than 1} \end{array} \right\} \end{aligned}$$

We point out that a consequence of the model above is that the manifolds $\mathbb{C}^{2[n]}$ and $\Delta^{[n]}$ are homeomorphic to each other and in particular they have the same Betti numbers.

Now we consider a non singular complex surface X, the nth symmetric product and its stratification $S^n X = \coprod S^n_\nu X$. For $\sum_i \nu_i[x_i] \in S^n_\nu X$ we take a collection of coordinate neighborhoods Δ_i of x_i such that

i. they are pairwise disjoint

ii. each Δ_i is biholomorphic to the bi-disk $\Delta \subset \mathbb{C}^2$

If we consider the complex manifold $\prod_i (\Delta_i)^{[\nu_i]}$ then we have a set of charts which glue by the universal property of the Douady space for Δ and get a complex manifold $X^{[n]}$: it follows by the construction that $X^{[n]}$ carries a universal family $\mathcal{Z} \to X^{[n]}$ and thus it represents the functor $\mathcal{H}ilb_X^P$ for $P = n$. The local (Douady-Barlet) $\prod_i (\Delta_i)^{[\nu_i]} \to \prod_i S^{\nu_i}(\Delta_i)$ also glue, defining a global map $\pi : X^{[n]} \to S^n X$ (the analogous of the Hilbert-Chow morphism in the algebraic case).

Chapter 2

Hyper-Kähler metric on $(\mathbb{C}^2)^{[n]}$

The aim of the chapter is to construct an Hyper-Kähler metric on $(\mathbb{C}^2)^{[n]}$, identifying it with an Hyper-Kähler quotient. We will use the description provided by theorem 1.16.
Let us start with a brief introduction to geometric invariant theory quotients in the affine case.

2.1 Geometric invariant theory quotients

Let G be a reductive algebraic group and let X be an affine variety, i.e. there exists $n > 0$ such that $X \subset \mathbb{A}^n$ as a closed subset. Equivalently, X can be recovered form its coordinate ring. With a slight abuse we will write $X = \text{Spec } R$. We assume that G acts linearly on $\mathbb{A}^n$ and hence on X. We would like to consider the quotient space of X under the action of G, but the set theoretical quotient X/G usually behaves badly and it is not even Hausdorff in general. This is due to the fact that G is only rarely compact, hence the orbits of its action may not be closed and contain orbits of smaller dimension in their closures. In order to clarify what can happen, we begin with a simple example.

Example 2.1. Consider the action of $\mathbb{C}^\times$ on $\mathbb{C}^2$ given by matrices

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in SL_2(\mathbb{C}), \quad \lambda \in \mathbb{C}^\times$$

Hence the action takes (z_1, z_2) to $(\lambda z_1, \lambda^{-1} z_2)$ and the orbits are:

- the hyperbola $h_\alpha = \{(x, y) \mid xy = \alpha\}$ for $\alpha \neq 0$
- the punctured x-axis $\{(x, y) \mid y = 0, x \neq 0\}$
- the punctured y-axis $\{(x, y) \mid y \neq 0, x = 0\}$
- the origin.

We observe that the orbits h_α are closed and in bijection with $\mathbb{C}^\times$. The punctured axes instead are not closed and their closures intersect in the origin, which is an orbit of smaller dimension, therefore if we consider the set theoretic quotient these orbits correspond to points that cannot be separated, making the quotient space non Hausdorff: the idea of the geometric invariant theory is to identify the three orbits with each other in an equivalence class.

Let us construct the quotient in the affine case: given the affine variety $X = \text{Spec } R$, the action of G on X induces a G-action on R. Let R^G be the ring of invariants. A theorem of Nagata ensures that this is a finitely generated algebra. We define

$$X//G = \text{Spec}\left(R^G\right)$$

This is called the *affine geometric invariant theory quotient* of X by G. The principal result of geometric invariant theory states that

Theorem 2.2 ([15],[13])**.** *There exists a surjective morphism*

$$\phi : X \longrightarrow X//G$$

induced by the inclusion $R^G \subset R$. Moreover $\phi(x) = \phi(y)$ if and only if

$$\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset \tag{2.1}$$

The underlying space of $X//G$ is the set of closed G-orbits modulo the equivalence relation defined by $x \sim y$ if and only if 2.1 holds.

Retrieving the example we have made before we get:

$$\mathbb{C}^2 = \text{Spec } \mathbb{C}[x, y]$$

and

$$\mathbb{C}[x, y]^G = \mathbb{C}[xy]$$

since $\lambda x \lambda^{-1} y = xy$, hence

$$\mathbb{C}^2//\mathbb{C}^\times = \text{Spec } \mathbb{C}[xy] \cong \mathbb{C}$$

It is possible to consider even a slightly different construction; the idea is to mimic the construction used in the projective case for an affine variety. Namely we consider the action of G on X and, after choosing a character of G, we lift it to the trivial fiber bundle $X \times \mathbb{C}$, using it to obtain a graded algebra. More formally we choose a character $\chi : G \to \mathbb{C}^*$, then the lifting of the action is defined by

$$g \cdot (x, z) = (g \cdot x, \chi(g)^{-1} z) \quad for\ (x, z) \in X \times \mathbb{C}$$

If $X = \text{Spec } R$, then let R^{G,χ^n} be the space of functions satisfying

$$f(g \cdot x) = \chi(g)^n f(x)$$

It can be identified with the space of G invariant functions on $X \times \mathbb{C}$. Hence the direct sum $\bigoplus_{n\geq 0} R^{G,\chi^n}$ is a finitely generated graded algebra. Therefore we can define

$$X//_\chi G := \text{Proj}\left(\bigoplus_{n\geq 0} R^{G,\chi^n}\right)$$

We still call it the *geometric invariant theory quotient* of X.

In geometric language $V//_\chi G$ can be described as follows. We say that $x \in X$ is χ-semistable if there exists $f \in R^{G,\chi^n}$ with $n \geq 0$ such that $f(x) \neq 0$. This happens if and only if the closure of $G \cdot (x,z)$ does not intersect with $X \times \{0\}$ for $z \neq 0$. Let $X^{ss}(\chi)$ be the set of χ semistable points. We introduce an equivalence relation $\sim$ on $V^{ss}(\chi)$ by defining $x \sim y$ if and only if $\overline{G \cdot x} \cap \overline{G \cdot y}$ is non empty in $X^{ss}(\chi)$. It is always possible to take a representative x so that $G \cdot (x,z)$ is closed for $z \neq 0$ and $G \cdot x$ is closed in $X^{ss}(\chi)$ for such a representative x. Therefore the quotient space $X^{ss}(\chi)/\sim$ is bijective to the set of orbits $G \cdot x$ such that $G \cdot (x,z)$ is closed for $z \neq 0$. Then $X//_\chi G$ is $X^{ss}(\chi)/\sim$.

Finally we observe that R^{G,χ^0} is the ring of the invariants on X, hence the inclusion $R^G = R^{G,\chi^0} \subset R^{G,\chi^n}$ induces a morphism

$$\pi : X//_\chi G \longrightarrow X//G \tag{2.2}$$

2.1.1 Geometric invariant theory and the moment map

Let V be a vector space over $\mathbb{C}$ with an hermitian metric, G be a connected closed Lie subgroup of $U(V)$ and $G^{\mathbb{C}}$ its complexification; the Lie algebra of G is denoted by $\mathfrak{g}$. We point out that since G is compact then its complexification is a reductive group, hence it is possible to apply in this particular case the second construction made in the previous paragraph.

Let $\chi : G \to U(1)$ be a character and let χ also denotes its complexification $\chi : G^{\mathbb{C}} \to \mathbb{C}^*$. Consider the trivial line bundle $V \times \mathbb{C}$ over V. We use χ to construct the GIT quotient $V//_\chi G^{\mathbb{C}}$ and we define the map $\mu : V \to \mathfrak{g}^*$ by

$$\langle \mu(x), \xi \rangle = \frac{1}{2}(\sqrt{-1}\xi x, x) \quad for\ x \in V,\ \xi \in \mathfrak{g}$$

It is a special case of the *moment map*, which is defined for an action on a symplectic manifold (we will use it again in the following chapter). Now we consider the function

$$p_{(x,z)}(g) = logN(g \cdot (x,z)) \quad for\ z \neq 0$$

where $N(x,z) = |z|e^{\frac{1}{2}||x||^2}$. This map has the following properties

Proposition 2.3. *For $z \neq 0$, the map $p_{(x,z)}$ has the following properties*

1. $p_{(x,z)}$ *descends to a function on* $G \setminus G^{\mathbb{C}}/G^{\mathbb{C}}_{(x,z)}$, *where* $G^{\mathbb{C}}_{(x,z)}$ *is the stabilizer of* (x, z).
2. $p_{(x,z)}$ *is a convex function on* $G \setminus G^{\mathbb{C}}$.
3. g *is a critical point if and only if* $\langle \mu(g \cdot x), \xi \rangle = \sqrt{-1} d\chi(\xi)$.
4. *All critical points are minima of* $p_{(x,z)}$.
5. *If* $p_{(x,z)}$ *attains its minimum, it does so on exactly one double coset* $G \setminus g/G^{\mathbb{C}}_{(x,z)}$
6. $p_{(x,z)}$ *attains minimum if and only if* $G^{\mathbb{C}} \cdot (x, z)$ *is closed in* $V \times \mathbb{C}$.

Proof. For $\xi \in \mathfrak{g}$ it holds:

$$i.\ \frac{d}{dt} p_{(x,z)}(\exp t\sqrt{-1}\xi g) = \langle \mu(\exp t\sqrt{-1}\xi gx), \xi \rangle - d\chi(\sqrt{-1}\xi)$$
$$ii.\ \frac{d^2}{dt^2} p_{(x,z)}(\exp t\sqrt{-1}\xi g) = 2||\xi \exp t\sqrt{-1}\xi gx||^2$$

Assertion 2. comes from $ii.$, since $2||\xi \exp t\sqrt{-1}\xi gx||^2 \geq 0$; assertion 3. instead comes from $i.$ and the fact that ξ is generic, because g is critical if and only if $(i.) = 0$ for $t = 0$; this is equal to require that $\langle \mu(gx), \xi \rangle - d\chi(\sqrt{-1}\xi) = 0$, hence the assertion.
The fourth statement is true since $p_{(x,z)}$ is convex and any two points in $G \setminus G^{\mathbb{C}}$ can be joined by a geodesic.
To prove assertion 5. suppose $p_{(x,z)}$ attains minimum at g and $\exp\sqrt{-1}\xi \cdot g$. Then the convexity implies

$$p_{(x,z)}(\exp t\sqrt{-1}\xi g) = const$$

Therefore we get $\xi gx = 0$ by setting $t = 0$ in $ii.$. Hence we have $\exp\sqrt{-1}\xi gx = gx$, i.e. $g^{-1} \exp \sqrt{-1}\xi g \in G^{\mathbb{C}}_x$. Now we prove assertion 6. Suppose $G^{\mathbb{C}} \cdot (x, z)$ is closed. Since $G^{\mathbb{C}} \cdot (x, z)$ and $V \times \{0\}$ are mutually disjoint, closed subsets, there exists an invariant polynomial $P = zP_1(x) + \cdots + z^n P_n(x)$ which satisfies

$$P \equiv \begin{cases} 1 \ on \ G^{\mathbb{C}} \cdot (x, z) \\ 0 \ on \ V \times \{0\} \end{cases}$$

Suppose $N(\widetilde{x}, \widetilde{z}) = |\widetilde{z}| e^{\frac{1}{2}||\widetilde{x}||^2} \leq C$. Then $|\widetilde{z}|$ is bounded. Moreover

$$\begin{aligned} 1 &= |\widetilde{z} P_1(\widetilde{x}) + \cdots + \widetilde{z}^n P_n(\widetilde{x})| \\ &\leq C|P_1(\widetilde{x})| e^{-\frac{1}{2}||\widetilde{x}||^2} + \cdots + C^n |P_1(\widetilde{x})| e^{-\frac{n}{2}||\widetilde{x}||^2} \leq C' e^{-\frac{1}{4}||\widetilde{x}||^2} \end{aligned}$$

Thus $||\widetilde{x}||$ is bounded. Therefore $p_{(x,z)}$ attains a minimum. Conversely suppose $p_{(x,z)}$ attains a minimum. We may assume it does so at $g = e$, replacing x if necessary. Let $\mathfrak{g}_x^\perp$ be the orthogonal complement of $\mathfrak{g}_x$ in $\mathfrak{g}$. By *ii.* we have

$$\frac{d^2}{dt^2} p_{(x,z)}(\exp t\sqrt{-1}\xi) > 0$$

for any $0 \neq \xi \in \mathfrak{g}_x^\perp$. Hence we can choose a positive constant ε so that

$$\frac{d^2}{dt^2} p_{(x,z)}(\exp t\sqrt{-1}\xi) \geq \varepsilon$$

for $\xi \in \mathfrak{g}_x^\perp$ with $||\xi|| = 1$ and $t \in [0, 1]$. Therefore we have

$$\frac{d^2}{dt^2} p_{(x,z)}(\exp t\sqrt{-1}\xi) \geq \varepsilon$$

for $\xi \in \mathfrak{g}_x^\perp$ with $||\xi|| = 1$ and $t = 1$. The same inequality holds for $t \geq 1$ since $p_{(x,z)}$ is convex. It implies

$$p_{(x,z)}(\exp t\sqrt{-1}\xi) \geq \varepsilon(t-1) + p_{(x,z)}(\exp \sqrt{-1}\xi) \quad for\ t \geq$$

Thus $p_{(x,z)}(\exp t\sqrt{-1}\xi)$ diverges as $t \to \infty$. This implies the orbit $G^{\mathbb{C}} \cdot x$ is closed. □

Corollary 2.4. *There exists a bijection between* $\mu^{-1}(\sqrt{-1}d\chi)/G$ *and the set* $\{x \in V \mid G^{\mathbb{C}} \cdot (x, z)$ *is closed for* $z \neq 0\}$

Proof. If $x \in \mu^{-1}(0)/G$ then $G^{\mathbb{C}} \cdot (x, z)$ is closed for $z \neq 0$ by assertions 3., 4., 6. of proposition 2.3 : $x \in \mu^{-1}(0)/G$ implies that $p_{(x,z)}$ attains its minimum and this happens if and only if $G^{\mathbb{C}} \cdot (x, z)$ is closed. Hence it is well posed the map

$$\mu^{-1}(\sqrt{-1}d\chi)/G \longrightarrow \{x \in V \mid G^{\mathbb{C}} \cdot (x, z) \text{ is closed for } z \neq 0\}$$

The surjectivity of the map follows from 3. and 6. while injectivity comes from 5. □

2.1.2 Description of $(\mathbb{C}^2)^{[n]}$ as a GIT quotient

Looking at theorem 1.16 we consider Hermitian vector spaces V and W whose dimensions are n and 1 respectively. Then $M = \mathrm{End}(V, V) \oplus \mathrm{End}(V, V) \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W)$ becomes a vector space with an Hermitian product. We consider the action of $U(V)$ given by

$$(B_1, B_2, i, j) \longmapsto (g^{-1}B_1 g, g^{-1}B_2 g, g^{-1}i, jg) \tag{2.3}$$

The correspondent moment map $\mu_1 : V \to \mathfrak{u}(V)^*$ is defined by

$$\mu_1(B_1, B_2, i, j) = \frac{\sqrt{-1}}{2}([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j)$$

We introduce a map $\mu_\mathbb{C}$ given by

$$\mu_\mathbb{C}(B_1, B_2, i) = [B_1, B_2] + ij$$

Then this is a holomorphic function from M to $\mathfrak{gl}(V)$ and $\mu_\mathbb{C}$ is $GL(V)$ invariant. Finally we define χ by

$$\chi(g) = (\det g)^l$$

where l is an arbitrary positive integer. Then the following theorem holds

Theorem 2.5.

$$(\mathbb{C}^2)^{[n]} = \mu_\mathbb{C}^{-1}(0)//_\chi GL_n(\mathbb{C}) \cong \mu_1^{-1}(\sqrt{-1}d_\chi) \cap \mu_\mathbb{C}^{-1}(0)/U(n)$$

Proof. The second equality comes from corollary 2.4. Besides $(B_1, B_2, i, j) \in \mu_\mathbb{C}^{-1}(0)$ belongs to $\widehat{H}$ (where $\widehat{H}$ is the one defined in theorem 1.7) if and only if $j = 0$ and the stability condition in theorem 1.16 holds. Hence the only thing we need to prove is the next lemma. □

Lemma 2.6. (B_1, B_2, i, j) *satisfy the stability condition in theorem 1.16 if and only if* $G^\mathbb{C} \cdot (x, z)$ *is closed for* $z \neq 0$

Proof. Suppose $G^\mathbb{C} \cdot (x, z)$ is closed for $z \neq 0$. And suppose there exists a subspace S which satisfies the following
i. S is B_α-invariant ($\alpha = 1, 2$)
ii. im $i \subset S$.
Taking a complementary subspace $S^\perp$ we decompose V as $S \oplus S^\perp$. Then we have

$$B_\alpha = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad i = \begin{pmatrix} * \\ 0 \end{pmatrix}$$

If we consider $g(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix}$, then we have

$$g(t) B_\alpha g(t)^{-1} = \begin{pmatrix} * & t* \\ 0 & * \end{pmatrix} \quad g(t)i = i$$

On the other hand we have $(\det g)^{-l} z = t^{l \dim S^\perp} z \to 0$ as $t \to 0$, but this contradicts the closedness of $G^\mathbb{C} \cdot (x, z)$.

Now we suppose the stability condition is satisfied. If $G^{\mathbb{C}} \cdot (x,z)$ is not closed, then by the Hilbert criterion Theorem (Birkes [4]), there exists a map $\lambda : \mathbb{C}^* \to GL(V)$ which satisfies the condition: $\lim_{t\to} \lambda(t)\cdot(x,z)$ exists and this limit is contained within $G^{\mathbb{C}} \setminus G^{\mathbb{C}} \cdot (x,z)$. Let us take weight decomposition of V with respect to λ

$$V = \bigoplus_m V(m)$$

with $V(m) = \{v \in V \mid \lambda(t)\cdot v = t^m v\}$. The existence of limit implies

$$B_\alpha(V(m)) \subseteq \bigoplus_{l\geq m} V(l) \quad \text{im } i \subset \bigoplus_{m\geq 0} V(m)$$

Hence by the stability condition we have

$$\bigoplus_{m\geq 0} V(m)$$

Therefore $\det\lambda(t) = t^N$ for some $N \geq 0$.
If $N = 0$ then $V = V(0)$, so that $\lambda \equiv 1$ and $\lambda(t)\cdot(x,z) = (x,z)$, but this is impossible because $\lim_{t\to 0} \notin G^{\mathbb{C}} \cdot (x,z)$.
Otherwise, if $N > 0$ then

$$\lambda(t)\cdot(x,z) = (\lambda(t)x\lambda(t)^{-1}, (\det\lambda(t))^{-l}z) = (\lambda(t)x\lambda(t)^{-1}, t^{-lN}z)$$

diverges as $t \to 0$. Hence the contradiction. □

Remark 2.7. The morphism 2.2 in this specific case can be rewritten as

$$\pi : \mu_{\mathbb{C}}^{-1}(0)//_\chi GL_n(\mathbb{C}) \longrightarrow \mu_{\mathbb{C}}^{-1}(0)//GL_n(\mathbb{C})$$

and it is possible to prove ([14]) that $\mu_{\mathbb{C}}^{-1}(0)//GL_n(\mathbb{C})$ is (isomorphic to?) $S^n\mathbb{C}^2$, hence we recover the Hilbert-Chow morphism.

2.2 Hyper-Kähler quotients

This section is devoted to show that the quotient in theorem 2.5 is in fact a hyper-Kähler quotient. Let us begin with a brief review on the Kähler and hyper-Kähler structures.

Definition 2.8. Let X be a $2n$-dimensional manifold. A *Kähler structure* of X is a pair given by a Riemannian metric g and by an almost complex structure I, which satisfies the following conditions:

1. g is hermitian for I, i.e. $g(Iv, Iw) = g(v, w)$ for $v, w \in TX$
2. I is integrable
3. If we define a 2-form ω by
$$\omega(v, w) = g(Iv, w) \quad for\ v, w \in TX$$
then $d\omega = 0$.

The 2-form ω is called the *Kähler form* associated with (g, I).

It is known (see e.g. [10]) that the above conditions are equivalent to requiring that I is parallel with respect to the Levi-Civita connection of g, i.e. $\nabla I = 0$. This is also equivalent to the condition
$$(\textit{the holonomy group of}\ \nabla) \subseteq U(n)$$
The hyper-Kähler structure is a quaternionic version of the Kähler structure, with the difference that there is no good definition of integrability for the almost hyper-Kähler structure. Hence the definition is given generalizing the equivalent definition we have just discuss.

Definition 2.9. Let X be $4n$-dimensional manifold. A *hyper-Kähler structure* of X consists of a Riemannian metric g and a triple of almost complex structures I, J, K which satisfy the following conditions:

1. $g(Iv, Iw) = g(Jv, Jw) = g(Kv, Kw) = g(v, w)$ for $v, w \in TX$
2. (I, J, K) satisfies $I^2 = J^2 = K^2 = IJK = -1$
3. (I, J, K) are parallel with respect to the Levi-Civita connection of g, i.e. $\nabla I = \nabla J = \nabla K = 0$

The above conditions are equivalent to the condition
$$(\textit{the holonomy group of}\ \nabla) \subseteq Sp(n)$$
.

Remark 2.10. Each one of $(g, I), (g, J), (g, K)$ defines a Kähler structure, and it is always possible to construct a holomorphic symplectic form: let us pick up I and combine the other Kähler forms as $\omega_{\mathbb{C}} = \omega_2 + \sqrt{-1}\omega_3$. Then
$$\begin{aligned}\omega_{\mathbb{C}}(Iv, w) &= g(JIv, w) + \sqrt{-1}g(KIv, w) = \\ &= \sqrt{-1}(g(Jv, w) + \sqrt{-1}g(Kv, w)) = \sqrt{-1}\omega_{\mathbb{C}}(v, w)\end{aligned}$$
This means that $\omega_{\mathbb{C}}$ is of type (2,0). Moreover it is clear that $d\omega_{\mathbb{C}} = 0$ and $\omega_{\mathbb{C}}$ is not degenerate. Then $\omega_{\mathbb{C}}$ is a holomorphic symplectic form.

Hyper-Kähler structures are not easy to construct or flexible: the following quotient, which was introduced by Hitchin et al.[9] as an analogue of Marsden-Weinstein quotients for symplectic manifolds is a powerful way to construct new hyper-Kähler manifolds.

Let (X, g, I, J, K) be a hyper-Kähler manifold, and $\omega_1, \omega_2, \omega_3$ the Kähler forms corresponding to I, J, K. Suppose that a compact Lie group G acts on X preserving g, I, J, K.

Definition 2.11. A map

$$\mu = (\mu_1, \mu_2, \mu_3) : X \to \mathbb{R}^3 \times \mathfrak{g}^*$$

is said to be a *hyper-Kähler moment map* if we have the following:

1. μ is G-equivariant, i.e. $\mu(g\cdot) = \mathrm{Ad}^*_{g^{-1}}\mu(x)$
2. $\langle d\mu(v), \xi \rangle = \omega_i(\xi^*, v)$ for any $v \in TX$, any $\xi \in \mathfrak{g}$ and $i = 1, 2, 3$, where ξ^* is a vector field generated by ξ

We take $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3 \otimes \mathfrak{g}^*$ which satisfies $\mathrm{Ad}^*_g(\zeta_i) = \zeta_i$ for any $g \in G$, $(i = 1, 2, 3)$. Then $\mu^{-1}(\zeta)$ is invariant under the G-action. So we can consider the quotient space $\mu^{-1}(\zeta)/G$.

Theorem 2.12. *Suppose G-action on $\mu^{-1}(\zeta)$ is free. Then the quotient space $\mu^{-1}(\zeta)/G$ is a smooth manifold and has a Riemannian metric and a hyper-Kähler structure induced from those on X.*

This quotient space is called a *hyper-Kähler quotient.*

Remark 2.13. Let i be the natural inclusion $\mu^{-1}(\zeta) \hookrightarrow X$ and π the natural projection $\mu^{-1}(\zeta) \to \mu^{-1}(\zeta)/G$. Let $\omega_1, \omega_2, \omega_3$ be the Kähler forms associated with the hyper-Kähler structure on X and $\omega'_1, \omega'_2, \omega'_3$ the forms associated with the hyper-Kähler structure on $\mu^{-1}(\zeta)/G$. Then we say that the hyper-Kähler structure on $\mu^{-1}(\zeta)/G$ is induced by that on X if $\pi^*\omega'_\alpha = i^*\omega_\alpha$, with $\alpha = 1, 2, 3$.

Remark 2.14. Take $x \in \mu^{-1}(\zeta)$ and consider the differential

$$d\mu_x : T_x X \longrightarrow \mathbb{R}^3 \otimes \mathfrak{g}^*$$

If the G-action is free on $\mu^{-1}(\zeta)$ the tangent space of the orbit through x, denoted by V_x is isomorphic to $\mathfrak{g}$ under the identification

$$\xi \longmapsto \xi^*_x \in V_x, \quad \xi \in \mathfrak{g}$$

Before proving the theorem, we give proof of the following lemmas.

Lemma 2.15. *V_x, IV_x, JV_x, KV_x are orthogonal to each other.*

Proof. Let $\xi, \eta \in \mathfrak{g}$. Since μ_i is equivariant, we have $\mu(exp(t\eta)x) = \zeta$ for any $t \in \mathbb{R}$. Differentiating with respect to t we have

$$d\mu_x(\eta^*_x) = 0$$

hence we have

$$g(I\xi^*_x, \eta^*_x) = \omega_1(\xi^*_x, \eta^*_x) = \langle d\mu_{i,x}(\eta^*_x), \xi \rangle = 0$$

Thus V_x is orthogonal to IV_x; the same argument proves that it is orthogonal to JV_x and KV_x. Moreover I, J, K are hermitian, hence

$$\begin{aligned} g(I\xi^*_x, J\eta^*_x) &= g(I^2\xi^*_x, IJ\eta^*_x) = -g(\xi^*_x, K\eta^*_x) = 0 \\ g(I\xi^*_x, K\eta^*_x) &= g(\xi^*_x, J\eta^*_x) = 0 \\ g(J\xi^*_x, K\eta^*_x) &= -g(\xi^*_x, I\eta^*_x) = 0 \end{aligned}$$

This completes the proof. □

Lemma 2.16. *Let (X, g) a Riemannian manifold with skew adjoint endomorphisms I, J, K of the tangent bundle TX satisfying conditions 1.,2. of definition 2.9. Then (g, I, J, K) is hyper-Kähler if and only if the associated Kähler forms $\omega_1, \omega_2, \omega_3$ are closed.*

Proof. If (g, I, J) is hyper-Kähler, then clearly $\omega_1, \omega_2, \omega_3$ are closed, since (g, I), (g, J), (g, K) are Kähler structures. Hence we need to show only the converse: we shall prove the integrability of I using the Newlander-Niremberg theorem. Let v, w be complex-valued vector fields, then

$$\omega_2(v, w) = g(Jv, w) = g(KIv, w) = \omega_3(Iv, w)$$

Hence v is of type (1,0) with respect to I, i.e. $Iv = \sqrt{-1}v$ if and only if

$$i_v\overline{\omega_{\mathbb{C}}} = 0 \tag{2.4}$$

where $\overline{\omega_{\mathbb{C}}} = \omega_2 - \sqrt{-1}\omega_3$.
Now we choose v, w of type (1,0) with respect to I and denote by L_v the Lie derivative with respect to the vector field v. Then we have

$$\begin{aligned} & i_{[v,w]}\overline{\omega_{\mathbb{C}}} \\ =& L_v i_w\overline{\omega_{\mathbb{C}}} - i_w L_v\overline{\omega_{\mathbb{C}}} && \text{by the naturality of the Lie derivative} \\ =& - i_w d(i_v\overline{\omega_{\mathbb{C}}}) && \text{by 2.4 for } w \text{ and the closedness of } \overline{\omega_{\mathbb{C}}} \\ =& 0 && \text{by 2.4 for } v \end{aligned}$$

Therefore $[v, w]$ is of type (1,0). The Newlander-Niremberg theorem implies that I is integrable and the same argument shows that the same holds for J, K. □

Proof of theorem 2.12. Let ξ be in $\mathfrak{g}$, and consider a tangent vector $I\xi_x^* \in T_xX$. Then we have

$$\begin{aligned} d\langle \mu_x(I\xi_x^*), \eta\rangle &= (\omega_1(\eta_x^*, I\xi_x^*), \omega_2(\eta_x^*, I\xi_x^*), \omega_3(\eta_x^*, I\xi_x^*)) \\ &= (g(I\eta_x^*, I\xi_x^*), g(J\eta_x^*, I\xi_x^*), g(K\eta_x^*, I\xi_x^*)) \\ &= (g(\eta_x^*, \xi_x^*), 0, 0) \end{aligned}$$

where the last equality comes from lemma 2.15. Similarly we get

$$\begin{aligned} d\langle \mu_x(J\xi_x^*), \eta\rangle &= (0, g(\eta_x^*, \xi_x^*), 0) \\ d\langle \mu_x(K\xi_x^*), \eta\rangle &= (0, 0, g(\eta_x^*, \xi_x^*)) \end{aligned}$$

Hence $d\mu_x$ is surjective, which implies that ζ is a regular value and $\mu^{-1}(\zeta)$ is a submanifold of X whose tangent space is $\ker d\mu_x$. On the other hand it holds

$$\begin{aligned} d\langle \mu_x(v), \eta\rangle &= (\omega_1(\eta_x^*, v), \omega_2(\eta_x^*, v), \omega_3(\eta_x^*, v)) \\ &= (g(I\eta_x^*, v), g(J\eta_x^*, v), g(K\eta_x^*, v)) \end{aligned}$$

therefore the kernel of $d\mu_x$ is the orthogonal complement of $IV_x \oplus JV_x \oplus KV_x$.
Since the G-action on $\mu^{-1}(\zeta)$ is free, the slice theorem implies that the quotient space $\mu^{-1}(\zeta)/G$ has a structure of a C^∞-manifold such that the tangent space $T_{G\cdot x}\mu^{-1}(\zeta)/G$ at the orbit $G\cdot x$ is isomorphic to the orthogonal complement of V_x in $T_x\mu^{-1}(\zeta)$. Thus the tangent space is the orthogonal complement of $V_x \oplus IV_x \oplus JV_x \oplus KV_x$ in T_xX, which is invariant under I, J, K, hence the induced almost complex structure. The restriction on the Riemannian metric g induces a Riemannian metric on the quotient $\mu^{-1}(\zeta)/G$. In order to show that these define a hyper-Kähler structure, it is enough to check that the associated Kähler forms $\omega_1, \omega_2, \omega_3$ are closed by lemma 2.16.
Let $i : \mu^{-1}(\zeta) \hookrightarrow X$ be the inclusion and $\pi : \mu^{-1}(\zeta) \to \mu^{-1}(\zeta)/G$ the projection.
By definition, it holds $i^*\omega_i = \pi^*\omega_i'$ for $i = 1, 2, 3$. Therefore

$$\pi^*(d\omega_i') = d(\pi^*\omega_i') = d(i^*\omega_i) = i^*(d\omega_i) = 0$$

But π is a submersion, hence $\pi^*(d\omega_i') = 0$ implies $d\omega_i' = 0$ □

Let us apply this construction to the description in theorem 2.5: let M be as in section 2.1.1, then the antilinear endomorphism

$$J(B_1, B_2, i, j) = (B_2^\dagger, -B_1^\dagger, j^\dagger, -i^\dagger)$$

makes M into a quaternion vector space ($J^2 = -Id$). Hence M is a (flat) hyper-Kähler manifold and the action 2.3 of $U(V)$ preserves the hyper-Kähler structure. If we consider the maps

$$\mu_1(B_1, B_2, i, j) = \frac{\sqrt{-1}}{2}([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j)$$

$$\mu_{\mathbb{C}}(B_1, B_2, i) = [B_1, B_2] + ij$$

and decompose the latter as

$$\mu_{\mathbb{C}} = \mu_2 + \sqrt{-1}\mu_3$$

considering $\mathfrak{gl}_n(\mathbb{C})$ as the complexification of $\mathfrak{u}(V)$, then the map

$$\mu = (\mu_1, \mu_2, \mu_3) : M \longrightarrow \mathbb{R}^3 \otimes \mathfrak{u}(V)$$

is a hyper-Kähler moment map and the action of $U(V)$ on $\mu^{-1}(\sqrt{-1}d\chi, 0, 0)$ is free. Thus by theorem 2.12 we have

Corollary 2.17. *$\mu_1^{-1}(\sqrt{-1}d\chi) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/U(V) = (\mathbb{C}^2)^{[n]}$ is a hyper-Kähler quotient. In particular, $(\mathbb{C}^2)^{[n]}$ has a hyper-Kähler structure.*

Remark 2.18. Let us take $\sqrt{-1}\zeta\mathrm{id}_V \in \mathbb{R}^3 \otimes \mathfrak{u}(V)$ for any $\zeta \in \mathbb{R}^3$. Then the $U(V)$-action is free on $\mu^{-1}(\sqrt{-1}\zeta\mathrm{id}_V)$ if $\zeta \neq 0$ and we get essentially the same hyper-Kähler manifolds. More precisely there exists a map from $\mu^{-1}(\sqrt{-1}\zeta\mathrm{id}_V)/U(V)$ to $\mu_1^{-1}(\sqrt{-1}|\zeta|\mathrm{id}_V)$ which is an isometry and transforms the hyper-Kähler structure (I, J, K) into

$$\begin{pmatrix} I' \\ J' \\ K' \end{pmatrix} = R \begin{pmatrix} I \\ J \\ K \end{pmatrix}$$

for some $R \in SO(3)$ satisfying $(|\zeta|, 0, 0)^T = R\zeta$.
We observe that $\mu^{-1}(\sqrt{-1}\zeta\mathrm{id}_V)/U(V)$ is not isomorphic to $(\mathbb{C}^2)^{[n]}$ as a complex manifold in general. Let us decompose $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ by choosing an identification $\mathbb{R}^3 \cong \mathbb{R} \oplus \mathbb{C}$. We decompose the hyper-Kähler moment map into $(\mu_{\mathbb{R}}, \mu_{\mathbb{C}})$. If $\zeta_{\mathbb{C}} \neq 0$ it is possible to prove that

i. $\mu_{\mathbb{C}}^{-1}(\zeta)$ is non singular

ii. every $GL(V)$-orbit in $\mu_{\mathbb{C}}^{-1}(\zeta)$ is closed

iii. the stabilizer of any point in $\mu_{\mathbb{C}}^{-1}(\zeta)/U(V) = \mu_{\mathbb{C}}^{-1}(\zeta)/GL(V)$ is trivial

In particular we have $\mu^{-1}(\zeta)/U(V) = \mu_{\mathbb{C}}^{-1}(\zeta)/GL(V)$. The right hand side is an affine algebro geometric quotient and in particular it is an affine algebraic variety. As a corollary we have

Theorem 2.19. *$(\mathbb{C}^2)^{[n]}$ is diffeomorphic to an affine algebraic manifold.*

Chapter 3

The Poincaré polynomial of $(\mathbb{C}^2)^{[n]}$

In this chapter we shall calculate the Poincaré polynomial of $(\mathbb{C}^2)^{[n]}$: given a manifold X, its general definition is the following

$$P_t(X) := \sum_{n\geq 0} t^n \mathrm{dim} H^n(X)$$

where $H^*()$ is the cohomology group with rational coefficients. We will use the Bialynicki-Birula decomposition associated with the torus action on $(\mathbb{C}^2)^{[n]}$ and compute the Poincaré polynomial using Morse theory. Thus, as a first step, we need to prove that the Morse function arising from the moment map connected to the torus action is perfect. In fact we will prove the statement for a generic compact symplectic manifold (considering the case in which the critical submanifolds are points) and then use it for our specific case.

3.1 Perfectness of the Morse function

Let (X, ω) be a compact symplectic manifold and T a compact torus. We suppose that there exist a T-action on X preserving the 2-form ω: it is defined the pairing

$$\langle , \rangle : \mathfrak{t}^* \times \mathfrak{t} \longrightarrow \mathbb{R}$$

where $\mathfrak{t}$ is the Lie algebra of T; besides any ξ in $\mathfrak{t}$ induces a vector field ξ^* on X describing the infinitesimal action of ξ. The corresponding moment map for the T-action on X is the map

$$\mu : X \longrightarrow \mathfrak{t}^*$$

satisfying

$$d \langle \mu, \xi \rangle = i_{\xi^*} \omega \quad for\ \xi \in \mathfrak{t}$$

here $i_{\xi^*}\omega$ is the contraction of the vector field ξ^* with ω and $\langle\mu,\xi\rangle$ is the function from X to $\mathbb{R}$ defined by $\langle\mu,\xi\rangle(x) = \langle\mu(x),\xi\rangle$. The moment map is uniquely defined up to an additive constant $c \in \mathfrak{t}$ of integration.

We take a non-zero element $\xi \in \mathfrak{t}$ and use $f = \langle\mu,\xi\rangle$ as a Morse function.
We observe that $x \in X$ is a critical point of f if and only if $d\langle\mu(x),\xi\rangle = 0$, that is, for avery $v \in T_xX$ we have $d\langle\mu(x),\xi\rangle(v) = 0$; but $d\langle\mu(x),\xi\rangle(v) = i_{\xi^*_x}\omega(v) = \omega(\xi^*_x, v)$ and $\omega(\xi^*_x, v) = 0$ for evry $v \in T_xX$ if and only if $\xi^*_x = 0$, since ω is non degenerate. Hence

$$g \cdot x = x \quad \text{for any } g \in \overline{\exp\mathbb{R}\xi}$$

If we choose a generic element ξ in $\mathfrak{t}$ then we get $\overline{\exp\mathbb{R}\xi} = T$. Thus in such a case the critical points of f coincide with the fixed points of the action of T.
Now we introduce a Riemannian metric g which is invariant under the action of T: this metric together with the symplectic form ω gives an almost complex structure I defined by

$$\omega(v,w) = g(Iv,w)$$

that allow us to regard the tangent space T_xX as a complex vector space.
Now let us consider the decomposition into connected components of X^T, $X^T = \coprod_\nu C_\nu$.
For each $x \in C_\nu$ the action of T induces the weight decomposition

$$T_xX = \bigoplus_{\lambda \in \mathrm{Hom}(T,U(1))} V(\lambda)$$

where $V(\lambda) = \{v \in T_xX \mid t \cdot v = \lambda(t)v \text{ for any } t \in T\}$. We define

$$N_x^+ = \sum_{\langle\sqrt{-1}d\lambda,\xi\rangle>0} V(\lambda) \qquad N_x^- = \sum_{\langle\sqrt{-1}d\lambda,\xi\rangle<0} V(\lambda)$$

Since ξ is generic, then $\langle\sqrt{-1}d\lambda,\xi\rangle = 0$ if and only if $d\lambda = 0$, hence

$$T_xX = N_x^+ \oplus V(0) \oplus N_x^-$$

The exponential map gives a T-equivariant isomorphism between a neighborhood of $0 \in T_xX$ and a neighborhood of $x \in X$.

This implies that C_ν is a submanifold of X whose tangent space is $T_xC_\nu = V(0)$. Moreover f is approximated around x by the map

$$v = \sum_\lambda v_\lambda \longmapsto \frac{1}{2}\langle\sqrt{-1}d\lambda,\xi\rangle||v_\lambda||^2$$

where v is in T_xX and v_λ is the component of v in the weight space $V(\lambda)$. Therefore we have

$$\mathrm{Hess}f(v,v) = \frac{1}{2}\langle\sqrt{-1}d\lambda,\xi\rangle||v_\lambda||^2$$

Hence f is non-degenerate in the sense of Bott, since the set of critical points is a disjoint union of submanifolds of X and the Hessian of f is positive defined on N_x^+ and negative defined on N_x^-, therefore it is non degenerate in the normal direction at any critical point. We define $d_\nu = \dim_{\mathbb{R}} N_x^- = 2\dim_{\mathbb{C}} N_x^-$ which is the index of f at the critical manifold C_ν
We denote by W_ν^+ the stable manifold of C_ν and by W_ν^- the unstable manifold of C_ν, defined by

$$W_\nu^+ := \{x \in X \mid \lim_{t\to-\infty} \phi_t(x) \in C_\nu\}$$

$$W_\nu^- := \{x \in X \mid \lim_{t\to+\infty} \phi_t(x) \in C_\nu\}$$

where ϕ_t is a gradient flow of f with respect to the T-invariant metric g on X. We observe that stable manifold W_ν^+ is diffeomorphic to the positive normal bundle $\bigcup_{x\in C_n u} N_x^+ \to C_\nu$, while the unstable manifold W_ν^- is diffeomorphic to the negative normal bundle $\bigcup_{x\in C_\nu} N_x^- \to C_\nu$; besides W_ν^- is an orientable real vector bundle of rank d_ν on C_ν since N_x^- consist of non-zero weight spaces for the T-action.

It can be proved [1] that there exists a partial ordering, $<$, on the index set of the critical manifolds with the property

i. $\overline{W_\nu^+} \subset \bigcup_{\nu\leq\mu} W_\mu^+$

ii. $\mu \leq \nu$ implies $f(C_\mu) \leq f(C_\nu)$

We suppose that for $c \in \mathbb{R}$ there exists only one critical manifold C_ν with $f(C_\nu) = c$ (the argument or the general case is essentially the same). We define $X_{c,-} = \bigcup_{\mu<\nu} W_\mu^+$ and $X_{c,+} = \bigcup_{\mu\leq=\nu} W_\mu^+ = X_{c,-} \cup W_\mu^+$. Then the cohomology exact sequence for the pair $(X_{c,+}, X_{c,-})$ gives

$$\ldots \to H^q(X_{c,+}, X_{c,-}) \xrightarrow{\jmath_q} H^q(X_{c,+}) \to H^q(X_{c,-}) \to H^{q+1}(X_{c,+}, X_{c,-}) \xrightarrow{\jmath_{q+1}} \ldots$$

We split that sequence into the following short exact sequences

$$\begin{cases} 0 \to \mathrm{im} j_q \xrightarrow{f} H^q(X_{c,+}) \xrightarrow{g} H^q(X_{c,-}) \xrightarrow{h} \ker j_{q+1} \to 0 \\ 0 \to \ker j_q \xrightarrow{f'} H^q(X_{c,+}, X_{c,-}) \xrightarrow{g'} \mathrm{im} j_q \to 0 \end{cases}$$

Note that $X_{c,+}$ is an open submanifold of X and W_ν^+ is a closed submanifold of $X_{c,+}$. If N_ν is the normal bundle of W_ν^+ in $X_{c,+}$, then the restriction of N_ν to C_ν is the unstable manifold W_ν^- and the inclusion $(W_\nu^-, C_\nu) \hookrightarrow (N_\nu, W_\nu^+)$ becomes a homotopy equivalence. Since W_ν^- is orientable so is N_ν and thanks to the Thom isomorphism we have the identification

$$H^q(X_{c,+}, X_{c,-}) \cong H^q(N_\nu, N_\nu \setminus W_\nu^+) \cong H^q(W_\nu^-, W_\nu^- \setminus C_\nu) \cong H^{q-d_\nu}(C_\nu)$$

We set $a_q = \dim(\ker j_q)$ and $c_q = \dim(\mathrm{im} j_q)$. Then from the equalities above we get

$$\begin{cases} c_q = \dim(\mathrm{im} j_q) = \dim(\ker f) + \dim(\mathrm{im} f) = \dim(\mathrm{im} f) \\ b_q(X_{c,+}) = \dim(\ker g) + \dim(\mathrm{im} g) = \dim(\mathrm{im} f) + \dim(\mathrm{im} g) = c_q + \dim(\mathrm{im} g) \\ b_q(X_{c,-}) = \dim(\ker h) + \dim(\mathrm{im} h) = \dim(\mathrm{im} g) + \dim(\ker j_{q+1}) = \dim(\mathrm{im} g) + a_{q+1} \end{cases}$$

and

$$\begin{cases} a_q = \dim(\ker j_q) = \dim(\mathrm{im} f') \\ b_q(X_{c,+}, X_{c,-}) = \dim(\ker g') + \dim(\mathrm{im} g') = \dim(\mathrm{im} f') + \dim(\mathrm{im} j_q) = a_q + c_q \end{cases}$$

Therefore we have

$$\begin{cases} b_q(X_{c,+}) = b_q(X_{c,-}) + c_q - a_{q+1} \\ b_{q-d_\nu}(C_\nu) = a_q + c_q \end{cases}$$

where b_q's are the qth Betti numbers. Finally it holds

$$\begin{aligned} P_t(X_{c,+}) &= \sum_q t^q b_q(X_{c,+}) = \sum_q t^q (b_q(X_{c,-}) + c_q - a_{q+1}) = \\ &= \sum_q t^q b_q(X_{c,-}) + \sum_q t^q (a_q + c_q) - \sum_q t^q (a_q + a_{q+1}) = \\ &= P_t(X_{c,-}) + \sum_q t^q b_{q-d_\nu}(C_\nu) - (1+t) \sum_q t^q a_{q+1} = \\ &= P_t(X_{c,-}) + t^{d_\nu} P_t(C_\nu) - (1+t) R_\nu(t) \end{aligned}$$

and, summing over the critical values $c \in \mathbb{R}$, we obtain the Morse inequality

$$P_t(X) = \sum_\nu t^{d_\nu} P_t(C_\nu) - (1+t) R(t)$$

where $R(t) = \sum_\nu R_\nu(t)$. If C_ν is a point (or more generally $H^{odd}(C_\nu) = 0$) the cohomology long exact sequence above splits into short exact sequences and as a consequence $R(t) = 0$, i.e. the Morse function is perfect.
We point out that the proof holds even in the case of a non compact symplectic manifold if the appropriate conditions on f are satisfied. For example the condition that $f^{-1}((-\infty, c])$ is compact for all $c \in \mathbb{R}$ is sufficient and this is the case for $(\mathbb{C}^2)^{[n]}$.

3.2 Case $X = (\mathbb{C}^2)^{[n]}$

In the previews chapters we have discussed three different descriptions of $(\mathbb{C}^2)^{[n]}$: the first one is the definition, the second is the one provided by theorem 1.7 and the last

is the one analyzed in theorem 2.5. In the next section we will use the third one, for we need a kähler structure in order to apply Morse theory and we'll switch, when it is possible, to the second one, easier to manage.
Let us consider the action of the compact 2-dimensional torus T^2 on $\mathbb{C}^2$:

$$(z_1, z_2) \longmapsto (t_1 z_1, t_2 z_2), \qquad (z_1, z_2) \in \mathbb{C}^2,\ (t_1, t_2) \in T^2$$

It induces an action of T^2 on the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ given by

$$[(B_1, B_2, i)] \longmapsto [(t_1 B_1, t_2 B_2, i)], \qquad [(B_1, B_2, i)] \in (\mathbb{C}^2)^{[n]}\ (t_1, t_2) \in T^2$$

The corresponding moment map $\mu : (\mathbb{C}^2)^{[n]} \to (t^2)^*$ is defined as

$$\mu([B_1, B_2, i]) = \left(\frac{\sqrt{-1}}{2}||B_1||^2, \frac{\sqrt{-1}}{2}||B_2||^2\right)$$

Here the norm $||B_\alpha||$ is well defined since here we are using the description of theorem 2.5. Indeed in this description, we consider the action of $U(n)$, hence given $[B_1, B_2, i] = [B_1', B_2', i']$, it holds $B_\alpha' = G^{-1} B_\alpha G,\ G \in U(n)$ and we have

$$\begin{aligned}\mu([B_1', B_2', i']) &= \left(\frac{\sqrt{-1}}{2}||G^{-1}B_1G||^2, \frac{\sqrt{-1}}{2}||G^{-1}B_2G||^2\right) = \\ &= \left(\frac{\sqrt{-1}}{2}||B_1||^2, \frac{\sqrt{-1}}{2}||B_2||^2\right) = \mu([B_1, B_2, i])\end{aligned}$$

Now we choose an element in $\mathfrak{t}^2$, $\xi = -2\sqrt{-1}(1, \varepsilon)$, so that the Morse function, defined as in section 3.1 becomes

$$f([(B_1, B_2, i)]) = \langle \mu([(B_1, B_2, i)]), \xi \rangle = ||B_1||^2 + \varepsilon ||B_2||^2$$

As we've already seen, if ε is generic ($0 < \varepsilon \ll 1$) the critical points of f coincide with the fixed points of the action of T^2. Moreover $f^{-1}((-\infty, c])$ is compact, because $f \leq c$ together with the equation $\frac{\sqrt{-1}}{2}([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger) = \sqrt{-1} d\chi$ implies a bound on $||B_1||^2 + ||B_2||^2 + ||i||^2$. Hence it is possible to apply Morse theory in our case.

The first step consist of identifying the fixed point set: we search for points that satisfy

$$[t_1 B_1, t_2 B_2, i] = [B_1, B_2, i]$$

for all $(t_1, t_2) \in T^2$, but if we regard $(\mathbb{C}^2)^{[n]}$ as

$$(\mathbb{C}^2)^{[n]} = \mu_{\mathbb{C}}^{-1}(0)//_\chi GL_n(\mathbb{C}) \cong \mu_1^{-1}(\sqrt{-1}d_\chi) \cap \mu_{\mathbb{C}}^{-1}(0)/U(n)$$

then the condition above is equal to require that exists a map $\lambda : T^2 \to U(V)$ such that

$$\begin{aligned} t_1 B_1 &= \lambda(t)^{-1} B_1 \lambda(t) \\ t_2 B_2 &= \lambda(t)^{-1} B_2 \lambda(t) \\ i &= \lambda(t)^{-1} i \end{aligned} \tag{3.1}$$

So if $[(B_1, B_2, i)]$ is a fixed point we have the weight decomposition of V with respect to $\lambda(t)$

$$V = \bigoplus_{k,l} V(k,l)$$

with

$$V(k,l) = \{v \in V \mid \lambda(t) \cdot v = t_1^k t_2^l v\}$$

Remark 3.1. We observe that condition 3.1 implies that $B_1 = t_1^{-1} \lambda(t)^{-1} B_1 \lambda(t)$ if and only if $\lambda(t) B_1 = t_1^{-1} B_1 \lambda(t) = B_1 t_1^{-1} \lambda(t)$, thus if $v \in V(k,l)$ then

$$\lambda(t) B_1 v = B_1 t_1^{-1} \lambda(t) = B_1 t_1^{-1} t_1^k t_2^l = t_1^{k-1} t_2^l B_1 v$$

and $B_1 v$ lies in $V(k-1, l)$. Similarly we get

$$\lambda(t) B_2 v = t_1^k t_2^{l-1} B_2 v$$

and $B_2 v \in V(k, l-1)$. Besides $\lambda(t) i(1) = i(1)$, therefore the only components of B_1, B_2, i which might survive are

$$\begin{aligned} B_1 &: V(k,l) \longrightarrow V(k-1,l) \\ B_2 &: V(k,l) \longrightarrow V(k,l-1) \\ i &: W \to V(0,0) \end{aligned} \tag{3.2}$$

where W has dimension 1.

As a consequence we obtain the diagram

$$\begin{array}{ccccccc} & & \uparrow & & \uparrow & & \\ \longrightarrow & & V(k,l-1) & \xrightarrow{B_1} & V(k-1,l-1) & & \longrightarrow \\ & & {\scriptstyle B_2}\uparrow & & {\scriptstyle B_2}\uparrow & & \\ \longrightarrow & & V(k,l) & \xrightarrow{B_1} & V(k,l-1) & & \longrightarrow \\ & & \uparrow & & \uparrow & & \end{array} \tag{3.3}$$

It is a commutative diagram, thanks to the condition $[B_1, B_2] = 0$, moreover we have

Proposition 3.2. *The commutative diagram in 3.3 has the following properties:*

1. *If* $k > 0$ *or* $l > 0$ *then* $V(k,l) = 0$
2. *For every* $k, l \in \mathbb{Z}$*, it holds* $\dim V(k,l) \leq 1$
3. *If* $k \leq 0$ *and* $l \leq 0$ *then* $\dim V(k,l) \geq \dim V(k, l-1)$ *and* $\dim V(k,l) \geq \dim V(k-1, l)$.
4. *Maps between non-zero vector spaces in 3.3 are non-zero.*

Proof. Let us begin with 1.
The stability condition of theorem 1.7 implies that V is spanned by vectors of the form $B_1^i B_2^j$, with $i, j > 0$; besides remark 3.1 shows that $i(1) \in V(0,0)$ and $B_1^i B_2^j i(1) \in V(-i,-j)$ for all $k, l > 0$. Therefore

$$V = \bigoplus_{k,l<0} V(k,l)$$

and $V(k,l)$ is forced to be trivial if $k, l > 0$.

The other assertions are proven by induction, using 1. for the base cases:
2. is true if $k = 0$ or $l = 0$
3. holds if $k = 0$ or $l = 0$, more precisely
$\dim V(0,l) \geq \dim V(0, l-1)$
$\dim V(k,0) \geq \dim V(k-1, 0)$
4. is verified for maps between spaces in 0-th row or 0-column
$\to V(0, l+1) \to V(0,l) \to V(0,l) \to$
$\to V(k+1, 0) \to V(k,0) \to V(k-1,0) \to$
On the other hand commutativity of 3.3 and the stability condition exclude the following diagrams

$$\begin{array}{ccc} V(k,l-1) & \xrightarrow{B_1} & V(k-1,l-1) \neq 0 \\ \uparrow{\scriptstyle B_2} & & \uparrow{\scriptstyle B_2} \\ V(k,l) & \xrightarrow{B_1} & 0 \end{array}$$

where $V(k,l)$ and $V(k,l-1)$ are supposed 1-dimensional,

$$\begin{array}{ccc} 0 & \xrightarrow{B_1} & V(k-1,l-1) \neq 0 \\ \uparrow{\scriptstyle B_2} & & \uparrow{\scriptstyle B_2} \\ V(k,l) & \xrightarrow{B_1} & V(k-1,l) \end{array}$$

here $V(k,l)$ and $V(k-1,l)$ are supposed 1-dimensional, hence 4. follows by induction. Moreover if $V(k,l)$, $V(k-1,l)$, $V(k,l-1)$ are 1-dimensional, then $V(k-1,l-1)$ has dimension 1 or 0 by commutativity and stability:

$$\begin{array}{ccc} V(k,l-1) & \xrightarrow{B_1} & V(k-1,l-1) \\ {\scriptstyle B_2}\uparrow & & \uparrow{\scriptstyle B_2} \\ V(k,l) & \xrightarrow{B_1} & V(k-1,l) \end{array}$$

Thus we can conclude by induction even the proof of the other assertions. □

Now we use the descriptions in theorem 1.7 and normalize all the maps in 3.3 by the action of $\coprod_{k,l} GL(V(k,l))$, so that the critical point is uniquely determined by the commutative diagram. Besides, given a critical point, it is possible to associate it to a Young diagram as follows.
Let $[(B_1, B_2, i)]$ be a critical point; this point gives a weight decomposition, as shown in remark 3.1, which provides a diagram as the one in 3.3. Thanks to proposition 3.2 this diagram has some special properties, which together with the fact that the direct sum of the spaces $V(k,l)$ is V, whose dimension is n, guarantee that if we put a box when $\dim V(k,l) = 1$ we always obtain a Young diagram of weight n. Conversely, given a Young diagram we can construct a unique commutative diagram satisfying proposition 3.2.
Hence there is the bijective correspondence

$$\{critical\ points\} \leftrightarrow \{commutative\ diagrams\} \leftrightarrow \{Young\ diagrams\}$$

For a Young diagram of weight n we denote by

$$\nu_i = \text{the number of boxes in the } i\text{th column}$$
$$\nu'_j = \text{the number of boxes in the } j\text{th row}$$

We observe that in terms of the critical point they are equal to

$$\nu_i = \sum_k \dim V(k, 1-i) \qquad \nu'_j = \sum_l \dim V(1-j, l)$$

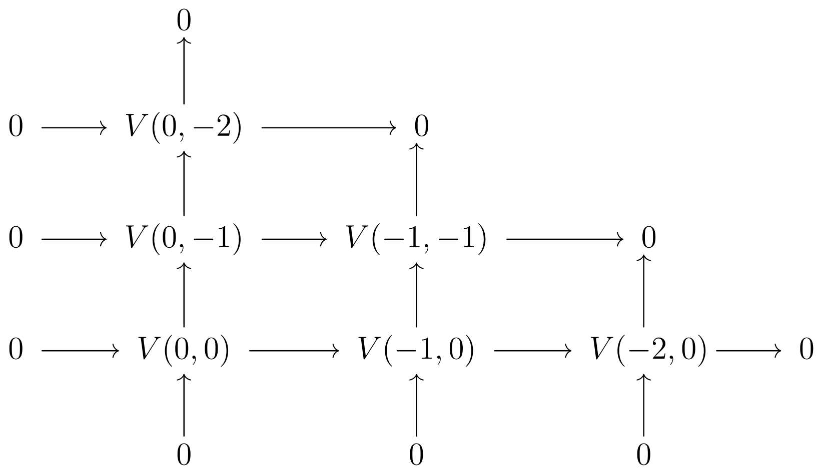
As a consequence a Young diagram corresponds to the partition $\nu = (\nu_1, \nu_2, \dots)$ and to its conjugate $\nu' = (\nu'_1, \nu'_2, \dots)$.

Remark 3.3. Using theorem 1.7, it is easy to describe the ideal I of $C[z_1, z_2]$ that defines the critical point $[(B_1 B_2, i)]$. We recall that given $\varphi_h : K[z_1, z_2] \longrightarrow K^n$, such that $\varphi_h(f) = f(B_1, B_2)i(1)$, then the ideal I is $\ker \varphi_h$.

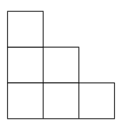
The commutative diagram shows that $B_2^{\nu_1} i(1) = 0$, $B_1 B_2^{\nu_2} i(1) = 0$, $\ldots$, $B_1^{\nu'_1} i(1) = 0$, $B_2 B_1^{\nu'_2} i(1) = 0$. Therefore the ideal I is given by

$$I = (z_2^{\nu_1}, z_1 z_2^{\nu_2}, \ldots, z_2 z_1^{\nu'_2}, z_1^{\nu'_1})$$

Example 3.4. If we suppose that the fixed point $[(B_1, B_2, i)]$ corresponds to the diagram



Then the Young diagram associated to it is simply



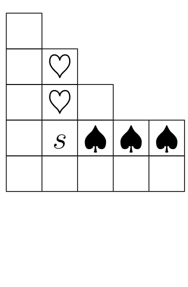
with

$$\nu = (3, 2, 1) \qquad \nu' = (3, 2, 1)$$

And the point is determined by the ideal

$$I = (z_1^3, z_1^2 z_2, z_1 z_2^2, z_2^3)$$

Finally we will calculate the character of $T_Z(\mathbb{C}^2)^{[n]}$ as a T^2 module for Z lying in $((\mathbb{C}^2)^{[n]})^{T^2} = Crit(f))$.
We start defining $l(s) = \nu_i - j$ and $a(s) = \nu'_j - i$ if s is the box which sits at the ith row and jth column. Graphically we have



$a(s) =$ number of $\heartsuit$
$l(s) =$ number of $\spadesuit$

Let $R[T^1, T^2]$ be the representation ring of T^2, where T_i denotes the one dimensional representation defined as

$$T_i : (t_1, t_2) \longmapsto t_i \quad i = 1, 2$$

Then the character of $T_Z(\mathbb{C}^2)^{[n]}$ is given by the following proposition.

Proposition 3.5. *Let D be the Young diagram corresponding to the fixed point $Z \in \mathrm{Fix}(T^2)$. Then the character of $T_Z(\mathbb{C}^2)^{[n]}$ is given by*

$$[T_Z(\mathbb{C}^2)^{[n]}] = \sum_{s\in D}(T_1^{l(s)+1}T_2^{-a(s)} + T_1^{-l(s)}T_2^{a(s)+1})$$

Proof. Let $[(B_1, B_2, i, 0)]$ be the data corresponding to the point Z in $(\mathbb{C}^2)^{[n]}$. We consider the complex

$$\begin{array}{ccccc} & & \mathrm{Hom}(V, Q\otimes V) & & \\ & & \oplus & & \\ \mathrm{Hom}(V,V) & \xrightarrow{a} & \mathrm{Hom}(W,V) & \xrightarrow{b} & \mathrm{Hom}(V,V)\otimes \bigwedge^2 Q \\ & & \oplus & & \\ & & \mathrm{Hom}(V, \bigwedge^2 Q\otimes W) & & \end{array} \tag{3.4}$$

where Q is a 2-dimensional T^2-module and a and b are defined by

$$a(\xi) = \begin{pmatrix} \xi B_1 - B_1\xi \\ \xi B_1 - B_1\xi \\ \xi i \\ 0 \end{pmatrix} \qquad b\begin{pmatrix} C_1 \\ C_2 \\ I \\ J \end{pmatrix} = [B_1, C_2] + [C_1, B_2] + iJ$$

Since a is the differential of the $GL_n(\mathbb{C})$-action and b is the differential of $\mu_{\mathbb{C}}$, the tangent space of $(\mathbb{C}^2)^{[n]}$ in Z can be identified with $\ker b/\mathrm{im}\ a$. We put the one dimensional representation $\bigwedge^2 Q$ to make the complex T^2-equivariant. Since a is injective and b is surjective (see ch.1) than we obtain a short sequence of the form $0 \to V_1 \to V_2 \to V_3 \to 0$ where V_i are T^2-modules and we can write $\ker b/\mathrm{im} a = -V_1 + V_2 - V_3$. In our case we have

$$\begin{aligned} T_Z(\mathbb{C}^2)^{[n]} &= \mathrm{Hom}(V,V)\otimes(Q - \bigwedge^2 Q - 1) + \mathrm{Hom}(W,V) + \mathrm{Hom}(V,W)\otimes\bigwedge^2 Q \\ &= V^*\otimes V\otimes(Q - \bigwedge^2 Q - 1) + V + V^*\otimes\bigwedge^2 Q \end{aligned}$$

We know from the previous analysis of V and its weight decomposition that

$$V = \sum_{j=1}^{\nu_1}\sum_{i=1}^{\nu'_j} T_1^{-j+1}T_2^{-i+1} = \sum_{i=1}^{\nu'_1}\sum_{j=1}^{\nu_i} T_1^{-j+1}T_2^{-i+1}$$

$$Q - \bigwedge^2 Q - 1 = T_1 + T_2 - T_1T_2 - 1 = (T_1 - 1)(T_2 - 1)$$

therefore

$$\begin{aligned} V \otimes (Q - \bigwedge^2 Q - 1) &= \sum_{j=1}^{\nu_1} T_1^{-j+1}(T_1 - 1) \sum_{i=1}^{\nu'_j} T_2^{-i+1}(1 - T_2) \\ &= \sum_{j=1}^{\nu_1} T_1^{-j+1}(T_1 - 1)(T_2^{-\nu_j+1} - T_2) \end{aligned}$$

Hence we get

$$\begin{aligned} & V^* \otimes V \otimes (Q - \bigwedge^2 Q - 1) = \\ &= \sum_{i=1}^{\nu'_1} \sum_{j'=1}^{\nu_i} T_1^{j'-1} T_2^{i-1} \sum_{j=1}^{\nu_1} T_1^{-j+1}(T_1 - 1)(T_2^{-\nu'_j+1} - T_2) \\ &= \sum_{i=1}^{\nu'_1} \sum_{j=1}^{\nu_1} T_1^{-j} \left[\sum_{j'=1}^{\nu_i} (T_1^{j'+1} - T_1^{j'}) \right] (T_2^{i-\nu'_j} - T_2^i) \\ &= \sum_{i=1}^{\nu'_1} \sum_{j=1}^{\nu_1} (T_1^{-j+\nu_i+1} - T_1^{-j+1})(T_2^{i-\nu'_j} - T_2^i) \\ &= \sum_{i=1}^{\nu'_1} \sum_{j=1}^{\nu_1} [(T_1^{-j+\nu_i+1} T_2^{i-\nu'_j} - T_1^{-j+1} T_2^i) \\ &\qquad - T_1^{-j+1}(T_2^{i-\nu'_j} - T_2^i) - (T_1^{-j+\nu_i+1} - T_1^{-j+1}) T_2^i] \end{aligned}$$

Moreover we have

$$\begin{aligned} \sum_{i=1}^{\nu'_1} \sum_{j=1}^{\nu_1} T_1^{-j+1}(T_2^{i-\nu'_j} - T_2^i) &= \sum_{j=1}^{\nu_1} T_1^{-j+1} \sum_{i=1}^{\nu'_j} (T_2^{-i+1} - T_2^{\nu'_1-i+1}) \\ &= V - \sum_{j=1}^{\nu_1} \sum_{i=1}^{\nu'_j} T_1^{-j+1} T_2^{\nu'_1-i+1} \end{aligned}$$

and

$$\sum_{i=1}^{\nu'_1} \sum_{j=1}^{\nu_1} (T_1^{-j+\nu_i+1} - T_1^{-j+1}) T_2^i = V^* \otimes \bigwedge^2 Q - \sum_{j=1}^{\nu_1} \sum_{i=1}^{\nu'_j} T_1^{-\nu_1+j} T_2^i$$

Hence if we put $R = T_Z(\mathbb{C}^2)^{[n]}$ we obtain

$$\begin{aligned} R &= V^* \otimes V \otimes (Q - \bigwedge^2 Q - 1) + V + V^* \otimes \bigwedge^2 Q \\ &= \sum_{i=1}^{\nu_1'} \sum_{j=1}^{\nu_1} (T_1^{-j+\nu_i+1} T_2^{i-\nu_j'} - T_1^{-j+1} T_2^{i}) + \sum_{j=1}^{\nu_1} \sum_{i=1}^{\nu_j'} (T_1^{-j+1} T_2^{\nu_1'-i+1} + T_1^{-\nu_1+j} T_2^{i}) \end{aligned}$$

We write $R = R(T_1, T_2) = \sum_{(k,l)\in\mathbb{Z}^2} c_{k,l} T_1^k T_2^l$ and define $R_{>0} = \sum_{k>0} c_{k,l} T_1^k T_2^l$ and $R_{\leq 0} = \sum_{k\leq 0} c_{k,l} T_1^k T_2^l$. Therefore we have

$$\begin{aligned} R_{>0} &= \sum_{i=1}^{\nu_1'} \sum_{j=1}^{\nu_i} T_1^{-j+\nu_i+1} T_2^{i-\nu_j'} \\ &= \sum_{s\in D} T_1^{l(s)+1} T_2^{-a(s)} \end{aligned}$$

But $R = R^* \otimes \bigwedge^2 Q$, thus $R(T_1, T_2) = R(T_1^{-1}, T_2^{-1} T_1 T_2$ so that R can be written even as

$$R = \sum_{i=1}^{\nu_1'} \sum_{j=1}^{\nu_1} (T_1^{j-\nu_i} T_2^{-i+\nu_j'+1} - T_1^{j} T_2^{-i+1}) + \sum_{j=1}^{\nu_1} \sum_{i=1}^{\nu_j'} (T_1^{j} T_2^{-\nu_1'+i} + T_1^{\nu_1-j+1} T_2^{-i+1})$$

Hence we get

$$\begin{aligned} R_{\leq 0} &= \sum_{i=1}^{\nu_1'} \sum_{j=1}^{\nu_i} T_1^{j-\nu_i} T_2^{-i+\nu_j'+1} \\ &= \sum_{s\in D} T_1^{-l(s)} T_2^{a(s)+1} \end{aligned}$$

and

$$R = R_{>0} + R_{\leq 0} = \sum_{s\in D} (T_1^{l(s)+1} T_2^{-a(s)} + T_1^{-l(s)} T_2^{a(s)+1})$$

□

Now we recall that the Morse function we have chosen is perfect, hence it holds

$$P_t(X) = \sum_{\nu} t^{d_\nu}$$

and it is possible to determine the Poincaré polynomial computing the indexes of the critical points. Therefore, thanks to the proposition above we get

Corollary 3.6. *The Poincaré polynomial of* $(\mathbb{C}^2)^{[n]}$ *is given by*

$$P_t((\mathbb{C}^2)^{[n]}) = \sum_{\nu} t^{2(n-l(\nu'))}$$

where ν' *runs over all partitions of* n *and* $l(\nu')$ *is the length of* ν'*.*

Proof. The only thing we have to do is calculate the index of the critical point Z. Remembering how we have chosen the Morse function, the index is equal to the sum of dimensions of the weight spaces such that the weight of t_1 is negative or the weight of t_1 is zero and the one of t_2 is negative. Using the proposition above this index is equal to twice the number of boxes satisfying $l(s) > 0$. Hence the index of f in Z is $2(n - l(\nu'))$. □

If we introduce the generating function of $P_t((\mathbb{C}^2)^{[n]})$ the corollary implies that

$$\sum_{n=0}^{\infty} q^n P_t((\mathbb{C}^2)^{[n]})) \prod_{m=1}^{\infty} \frac{1}{(1 - t^{2m-2}q^m)}$$

This is a special case of a more general formula, proved for the first time by Göttsche, that holds for any quasi-projective non singular surface X.

Finally in chapter 1 we have proved that $\dim_{\mathbb{C}} \pi^{-1}(n[0]) \leq n-1$ and there exists at least one $(n-1)$-dimensional component. By the corollary the highest power of t is $2(n-1)$ corresponding to the partition (n) of length 1 and this is the unique partition which has length 1, therefore $H_{2(n-1)}((\mathbb{C}^2)^{[n]})$ has rank 1, and $H_k((\mathbb{C}^2)^{[n]}) = 0$ for all $k \geq 2n-1$. Besides $H_{2(n-1)}((\mathbb{C}^2)^{[n]})$ is generated by the closure of the unstable manifold l_ν corresponding to the partition $\nu = (n)$, hence we have the following theorem

Theorem 3.7. *There is exactly one* $(n-1)$*-dimensional irreducible component in* $\pi^{-1}(n \cdot 0)$

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