Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
Corso di Laurea Magistrale in Matematica

The Complex Monge-Ampère Equation

Tesi di Laurea in Analisi non lineare

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I Sessione
Anno Accademico 2014-2015

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Introduzione

In questa trattazione studiamo la regolarità delle soluzioni viscose plurisubarmoniche dell'equazione di Monge-Ampère complessa

$$\det(\partial\overline{\partial}u) = f(z, u, Du)$$

dove $\partial\overline{\partial}u$ è la matrice Hessiana complessa di u, Du è il gradiente Euclideo di u e $f(z, s, p)$ è una funzione positiva assegnata regolare in $(z, s, p) \in \mathbb{C}^n \times \mathbb{R} \times R^{2n}$.

Con il termine equazione del tipo Monge-Ampère si indica in realtà una famiglia di equazioni alle derivate parziali del secondo ordine completamente non lineari il cui termine del secondo ordine è il determinante della matrice Hessiana di una funzione incognita a valori reali u. Questo tipo di equazioni ha una notevole importanza in geometria: ad esempio gioca un ruolo fondamentale sia nella dimostrazione della congettura di Calabi dimostrata da Shing-Tung Yau nel 1977-1979, sia in quella del teorema della mappa di Feffermann nella versione presentata da S.G. Krantz e Song-Ying Li [17]. Proprio per la sua rilevanza è stata oggetto di interesse da parte di un gran numero di autori. Un primo approccio per studiare tale classe di equazioni è quello di utilizzare le tecniche per lo studio delle equazioni ellittiche completamente non lineari (come ad esempio in [7], [5], per il caso complesso o in [12], [6] per il caso reale). E. Bedford e B.A. Taylor hanno invece introdotto in [2] soluzioni generalizzate (nel senso di misure Hessiane) per l'equazione di Monge-Ampère complessa quando il dominio Ω è un insieme limitato strettamente pseudoconvesso (si veda la Definizione 2.2). Inoltre in [2], [3], [4] hanno stabilito l'esistenza, l'unicità, e la regolarità Lipschitziana delle soluzioni generalizzate per il problema di Dirichlet. Molti altri autori hanno poi continuato il lavoro iniziato da E. Bedford e B.A. Taylor studiando, con gli strumenti della teoria del (pluri)potenziale, l'esistenza e la regolarità per le soluzioni delle equazioni del tipo Monge-Ampère.

Lo scopo della tesi è quello di dimostrare che, comunque si scelga la funzione f regolare e positiva, è possibile trovare una palla Euclidea $B_r \subset \mathbb{C}^n$ di raggio piccolo ed una soluzione viscosa plurisubarmonica $u \in \mathrm{Lip}(B_r)$

dell'equazione di Monge-Ampère complessa con secondo membro $f > 0$, tale che $u \notin C^{1,\alpha}(\overline{B}_r)$ per α maggiore di un indice critico legato alla dimensione. L'elaborato si articola in tre parti.

Il primo capitolo è dedicato alla teoria delle soluzioni viscose e riporta le definizioni ed i principali risultati della teoria: definizione di soluzione viscosa per mezzo di funzioni test C^2, principio del confronto, stabilità delle soluzioni rispetto alla convergenza uniforme, metodo di Perron.

Il secondo capitolo riporta un risultato di esistenza di soluzioni classiche per il problema di Dirichlet con assegnato dato al bordo ϕ regolare. Questo risultato è stato provato da Caffarelli, Kohn, Nirenberg, Spruck in [7, Teorema 1.3] utilizzando il metodo di continuità. In questo capitolo vengono inoltre ricordate alcune proprietà dell'equazione di Monge-Ampère complessa tra le quali l'ellitticità e l'invarianza dell'equazione rispetto a cambi di coordinate olomorfi.

L'ultimo capitolo è dedicato invece ad un nuovo risultato di non regolarità per l'equazione di Monge-Ampère complessa. Il teorema principale della tesi è il seguente

Teorema.
Sia $n \geq 2$ e sia f una funzione liscia limitata e strettamente positiva che soddisfi le condizioni di struttura

(H1) $f_u := \frac{\partial f}{\partial u} > 0.$

(H2) esiste una costante positiva C tale che

$$|\partial f|, |\overline{\partial} f|, |f_u|, |\frac{\partial f}{\partial p_j}| \leq C f^{1-\frac{1}{n}} \quad \textit{for } j = 1 \dots 2n$$

dove $\partial = (\partial_{z_1}, \dots, \partial_{z_n})$ e $\overline{\partial} = (\overline{\partial}_{z_1}, \dots, \overline{\partial}_{z_n})$ e per ogni $m \geq 0$ esiste una costante $C = C(m)$ tale che

$$\begin{aligned} |\partial f|, |\overline{\partial} f|, |f_u|, |\partial f_{p_j}|, |\overline{\partial} f_{p_j}| |f_{up_j}| &\leq C f^{1-1/2n} \\ |\partial(\partial f)|, |\overline{\partial}(\overline{\partial} f)|, |\partial f_u|, |\overline{\partial} f_u|, |f_u u| &\leq C f^{1-1/n} \\ |f_{p_j}|, |f_{p_i p_j}| &\leq C f \end{aligned} \tag{1}$$

per ogni $|u| + |p| \leq m$

Sia $f_\infty : B_1 \times \mathbb{R} \times \mathbb{C}^n \to \mathbb{R}$, $f_\infty := \lim_{\lambda \to \infty} \frac{f(z, \lambda s, \lambda p)}{\lambda^n}$,

(H3) Supponiamo che la funzione f_∞ esista in ogni punto e $f_\infty < \left(1 - \frac{1}{n}\right)^n$

Allora esistono $r \in (0,1)$ *ed una soluzione viscosa* u *dell'equazione*

$$-\det(\partial\overline{\partial}u) + f(z,u,Du) = 0 \quad \text{in } B_r \tag{2}$$

tale che $u \in Lip(\overline{B}_r)$ *con* $u \notin C^1(B_r)$ *se* $n = 2$ *e* $u \notin C^{1,\beta}(B_r)$ *per ogni* $\beta > 1 - \frac{2}{n}$

La dimostrazione utilizza il principio del confronto per soluzioni viscose, più una tecnica perturbativa introdotta in [23] per il Monge-Ampère Euclideo e recentemente utilizzata in [13], [14] e [19] per dimostrare l'esistenza di soluzioni non regolari per equazioni di tipo Monge-Ampère più generali e in ambiti diversi. Ricordiamo inoltre che un controesempio di questo tipo ma per la meno generale equazione $\det(\partial\overline{\partial}u) = 1$ è stato dimostrato in [5] e [15].

L'idea fondamentale che sta alla base della prova è quella di dimostrare l'esistenza di $r \in (0,1)$ e di una soluzione viscosa u_σ del problema di Dirichlet

$$\begin{cases} -\det(\partial\overline{\partial}u) + f(z,u,Du) = 0 & \text{in } B_r \\ u = \phi_\sigma & \text{su } \partial B_r \end{cases} \tag{3}$$

tale che

$$\|u\|_{L^\infty} + \|u\|_{\mathrm{Lip}(\overline{B_r})} \leq C \tag{4}$$

con C che dipende solo da r, $\sup_{B_r} |\phi_\sigma|$ e $\sup_{B_r} |D\phi_\sigma|$ (uniformemente in σ) e dove abbiamo scelto ϕ_σ in modo tale che sia una soprasoluzione classica di (3). Inoltre è possibile determinare in maniera esplicita una sottosoluzione classica ψ_σ la cui regolarità dipende da σ e da un parametro $\alpha = 1 - \frac{1}{n}$. Utilizzando il principio del confronto si ottiene quindi $\psi_\sigma \leq u_\sigma \leq \phi_\sigma$. La stima (4) assicura, a meno di considerare sottosuccessioni, l'esistenza di una funzione u tale che $u_\sigma \to u$ uniformemente, tale funzione risulta essere soluzione di (3) grazie alla proprietà di stabilità delle soluzioni viscose rispetto alla convergenza uniforme. Le considerazioni riguardanti la regolarità di tale soluzione si deducono passando al limite per $\sigma \to 0$ nell'ultima catena di disuguaglianze.

Introduction

In this dissertation we will study the regularity of viscosity pluri-subharmonic solutions of the complex Monge-Ampère equation

$$\det(\partial\overline{\partial}u) = f(z, u, Du) \tag{5}$$

where $\partial\overline{\partial}u$ is the complex Hessian matrix of u, Du is the Euclidean gradient of u and $f(z, s, p)$ is a positive and smooth given function.

The expression Monge-Ampère type equations denotes a family of fully nonlinear second order partial differential equations whose second order term is the determinant of the Hessian matrix of an unknown real valued function u. These equations are of considerable importance in some fields of geometry. For instance they play a crucial role in both the proof of the Calabi's conjecture (proved by Shing-Tung Yau between 1977 and 1979) and a proof of Feffermann's mapping theorem given by S.G. Krantz and Song-Ying Li in [17]. For these reasons they have been studied by a large number of authors using different techniques. L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck, in [7] and [6] obtained some existence results applying the methods of fully nonlinear elliptic equations. E. Bedford and B.A. Taylor in [2] introduced Monge-Ampère measure and the resultant notion of generalized solutions when Ω is a bounded strongly pseudoconvex set (see Definition 2.2). Moreover in [2], [3], [4] they established the existence, the uniqueness and the Lipschitz regularity for generalized solutions of the Dirichlet problem for the complex Monge-Ampère equation in a strictly pseudoconvex domain for continuous data. Many other authors have continued the work initiated by E. Bedford and B.A. Taylor and have investigated the existence and the regularity of solutions of complex Monge-Amperè type equations with the tools of pluripotential theory.

The purpose of this dissertation is to prove that for every given function f under some suitable structural assumptions there exist a small Euclidean ball $B_r \subset \mathbb{C}^n$, $n \geq 2$ and a pluri-subharmonic viscosity solution $u \in \mathrm{Lip}(B_r)$ of (5) such that $u \notin C^{1,\alpha}(\overline{B_r})$ for α larger than a critical index depending on the dimension n. The thesis is organized in three chapters.

The first chapter recalls some notions and main results of the theory of viscosity solutions including the definition of viscosity solutions, a comparison principle, the stability propriety with respect to the uniform convergence, the Perron's method.

In the second chapter we repeat the proof of the existence of smooth solutions for complex Monge-Ampère equation ([7, Theorem 1.3]) and recall some proprieties of the complex Monge-Ampère equation.

The third chapter is devoted to a new non-regularity result. The main theorem of this dissertation is the following:

Theorem.
Let $n \geq 2$, $\overline{B}_1 \subset \mathbb{C}^n$ and $f \in C^\infty(\overline{B}_1 \times \mathbb{R} \times \mathbb{R}^{2n})$ be a positive real valued function. Suppose moreover f satisfies the following structural assumptions:

(H1) $f_u := \frac{\partial f}{\partial u} > 0$.

(H2) There exists a positive constant C such that

$$|\partial f|, |\overline{\partial} f|, |f_u|, |\frac{\partial f}{\partial p_j}| \leq C f^{1-\frac{1}{n}} \quad \text{for } j = 1 \ldots 2n$$

where $\partial = (\partial_{z_1}, \ldots, \partial_{z_n})$ and $\overline{\partial} = (\overline{\partial}_{z_1}, \ldots, \overline{\partial}_{z_n})$ and for each $m \geq 0$ there exists a constant $C = C(m)$ such that

$$|\partial f|, |\overline{\partial} f|, |f_u|, |\partial f_{p_j}|, |\overline{\partial} f_{p_j}||f_{up_j}| \leq C f^{1-1/2n}$$
$$|\partial(\partial f)|, |\overline{\partial}(\overline{\partial} f)|, |\partial f_u|, |\overline{\partial} f_u|, |f_u u| \leq C f^{1-1/n}$$
$$|f_{p_j}|, |f_{p_i p_j}| \leq C f$$

whenever $|u| + |p| \leq m$

Let $f_\infty : B_1 \times \mathbb{R} \times \mathbb{C}^n \to \mathbb{R}$, $f_\infty := \lim_{\lambda \to \infty} \frac{f(z, \lambda s, \lambda p)}{\lambda^n}$, suppose

(H3) f_∞ exists and $f_\infty < \left(1 - \frac{1}{n}\right)^n$

Then there exist a small Euclidean ball B_r and a Lipschitz viscosity solution of (5) *such that $u \notin C^1(B_r)$ if $n = 2$ and $u \notin C^{1,\beta}(B_r)$ for every $\beta > 1 - \frac{2}{n}$*

The proof of this Pogorelov-type counterexample uses a comparison principle for viscosity solutions and a technique introduced in [23] for real Monge-Ampère equations and recently used in [13], [14] and [19] to prove the existence of non smooth solutions for more general Monge-Ampère type equations in different fields. We point out that a similar counterexample, for the equation $\det(\partial\overline{\partial} u) = 1$ has been proved in [5] and [15].

The key idea of the proof is to prove the existence of $r \in (0,1)$ and a viscosity solution u_σ of the Dirichlet problem

$$\begin{cases} -\det(\partial\overline{\partial}u) + f(z,u,Du) = 0 & \text{in } B_r \\ u = \phi_\sigma & \text{su } \partial B_r \end{cases} \tag{6}$$

such that

$$\|u\|_{L^\infty} + \|u\|_{\mathrm{Lip}(\overline{B_r})} \leq C \tag{7}$$

with C depending on r, $\sup_{B_r} |\phi_\sigma|$ and $\sup_{B_r} |D\phi_\sigma|$ (uniformly on σ) and where we chose ϕ_σ such that it is a classical supersolution of (6). Moreover it is possible to write explicitly a classical subsolution ψ_σ whose regularity depends on σ and on a parameter $\alpha = 1 - \frac{1}{n}$. By the comparison principle we deduce that $\psi_\sigma \leq u_\sigma \leq \phi_\sigma$. The estimate (7) ensures the existence of a function u such that u_σ uniformly converges to u. Such a function is a solution of (6). Considerations on the regularity of u are deduced taking the limit as $\sigma \to 0$ in the last chains of inequalities.

Chapter 1

Viscosity Solutions

In this chapter we want to introduce basic facts of theory of viscosity solutions of fully nonlinear scalar second order partial differential equations. We will consider certain equations of the form:

$$F(x, u, Du, D^2u) = 0$$

on open sets $\Omega \subset \mathbb{R}^N$, where F is a continuous map $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}(N) \to \mathbb{R}$, $\mathscr{S}(N)$ is the set of real $N \times N$ symmetric matrices and $u : \Omega \to \mathbb{R}$ is a real-valued function. This theory allows merely continuous functions to be weak solutions (in the sense of viscosity) of second order partial differential equations of the form $F(x, u, Du, D^2u)$, where Du and D^2u denote the gradient and the Hessian of u. In order to apply this theory, we require F to be degenerate elliptic and proper that is

Definition 1.1.
$F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}(N) \to \mathbb{R}$ is called *degenerate elliptic* if it is nonincreasing in its matrix argument

$$F(x, r, p, X) \leq F(x, r, p, Y) \quad \text{whenever } Y \leq X \tag{1.1}$$

Throughout this work $\mathscr{S}(N)$ will be equipped with its usual order that is:

$$Y \leq X \text{ if } \langle X\xi, \xi \rangle \leq \langle Y\xi, \xi \rangle \quad \text{for all } \xi \in \mathbb{R}^N$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.

Definition 1.2.
$F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}(N) \to \mathbb{R}$ is called *proper* if it is degenerate elliptic and nondecreasing in r, i.e. (1.1) holds and

$$F(x, s, p, X) \leq F(x, r, p, X) \quad \text{whenever } s \leq r$$

We always assume F proper and continuous.

In particular in this chapter we present the notion of viscosity solution, stability with respect to the uniform convergence, a comparison principle, and existence results via Perron's method. For further details we refer the reader to [8].

1.1 Some definitions of viscosity solution

In this section we give two different definitions of viscosity solution. This theory require to deal with semicontinuous functions, therefore we recall the following notions.

Definition 1.3.
If $u : \Omega \to \mathbb{R}$ is a function, we will call

$$u^*(x) := \limsup_{r \to 0} \{u(y) : y \in \Omega, |x-y| \leq r\}$$

its *upper semicontinuous envelope* and

$$u_*(x) := \liminf_{r \to 0} \{u(y) : y \in \Omega, |x-y| \leq r\}$$

its *lower semicontinuous envelope.*

Definition 1.4.
A function $u : \Omega \to \mathbb{R}$ is called *upper semicontinuous* if $u = u^*$ and *lower semicontinuous* if $u = u_*$

We are now ready to define the notions of viscosity subsolutions, supersolutions and solutions.

Definition 1.5.
Let F be proper and continuous, let $\Omega \subset \mathbb{R}^n$ be open. A *viscosity subsolution* of $F = 0$ (equivalently, a viscosity solution of $F \leq 0$) on Ω is an upper semicontinuous function $u : \Omega \to \mathbb{R}$ such that for every $\varphi \in C^2(\Omega)$ and for every local maximum point $x_0 \in \Omega$ of $u - \varphi$, we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0 \tag{1.2}$$

Likewise, a *viscosity supersolution* of $F = 0$ (equivalently, a viscosity solution of $F \geq 0$) on Ω is a lower semicontinuous function $u : \Omega \to \mathbb{R}$ such that for every $\varphi \in C^2(\Omega)$ and for every local minimum point $x_0 \in \Omega$ of $u - \varphi$, we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \tag{1.3}$$

Finally, a continuous function u is a *viscosity solution* of $F = 0$ in Ω if it is both a viscosity supersolution and a viscosity subsolution of $F = 0$ in Ω.

Remark 1.6.
An equivalent definition of viscosity subsolution (respectively viscosity supersolution) is obtained by replacing local maximum point (respectively local minimum point) by strict local maximum point (respectively strict local minimum point)in the above statements.

Proof. It is clear that if u is a viscosity subsolution (respectively viscosity supersolution) according to the definition presented in the statement of Remark 1.6, then u satisfies (1.2) (respectively (1.3)). We want to prove the converse.

Suppose u is a viscosity subsolution according to Definition 1.5. Let $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum in x_0 that is $u(x) - \varphi(x) \leq u(x_0) - \varphi(x_0)$ for x near x_0. Now we replace $\varphi(x)$ with $\varphi(x) + |x - x_0|^4$ obtaining

$$\begin{aligned} u(x) - \varphi(x) - |x - x_0|^4 &\leq (u(x) - \varphi(x))|_{x=x_0} - |x - x_0|^4|_{x=x_0} \\ &< (u(x) - \varphi(x) - |x - x_0|^4)|_{x=x_0} \quad \text{for } x \neq x_0 \end{aligned}$$

We call $\psi(x) := \varphi(x) + |x - x_0|^4$. Obviously $\psi \in C^2$ and $u - \psi$ has a strict local maximum point in x_0, therefore, by hypothesis we get

$$F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \leq 0$$

Now we observe that $D\psi(x_0) = D\varphi(x_0)$ and $D^2\psi(x_0) = D^2\varphi(x_0)$ so we conclude

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0$$

that is u is a viscosity subsolution according to Definition 1.5. Similarly, replacing φ with $\varphi(x) + |x - x_0|^4$, it is possible to adapt the proof above in case u is a viscosity supersolution. Thus the two definitions are equivalent.

□

Note that we require a subsolution to be upper semicontinuous and a supersolution to be lower semicontinuous so that is straightforward to produce respectively maxima and minima. As the following Remark shows the notion of solution given in Definition 1.5 is consistent with the classical notion of solution of a partial differential equation.

Remark 1.7.
If F is proper, $u \in C^2(\Omega)$ and solves $F(x, u, Du, D^2u) = 0$ in classical sense, then u is a viscosity solution of $F = 0$ in Ω.

Proof. In order to prove that u is a viscosity solution we show that it is both a viscosity subsolution and a viscosity supersolution. For every $\varphi \in C^2(\Omega)$ and for every local maximum point $x_0 \in \Omega$ of $u - \varphi$, we have $Du(x_0) = D\varphi(x_0)$ and $D^2u(x_0) \leq D^2\varphi(x_0)$ so that, using the nonincreasing monotonicity of F in its matrix argument, we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \leq 0$$

so u is a viscosity subsolution of $F = 0$. Similarly, for every $\varphi \in C^2(\Omega)$ and for every local minimum point $x_0 \in \Omega$ of $u - \varphi$, we have $Du(x_0) = D\varphi(x_0)$ and $D^2u(x_0) \geq D^2\varphi(x_0)$ so that, using the nonincreasing monotonicity of F in its matrix argument, we get

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \geq 0$$

so u is a viscosity supersolution of $F = 0$. □

Now we introduce another definition of viscosity solution that is not based upon test function as it is Definition 1.5. This notion allows us to define "(Du, D^2u)" for nondifferentiable function u. To this purpose we present the concept of second order superjet and second order subjet.

Definition 1.8.
If $u : \Omega \to \mathbb{R}$, $x_0 \in \Omega$, we define the *second order superjet of* u *at* x_0 :

$$J_\Omega^{2,+}u(x_0) := \{(p, X) \in \mathbb{R}^n \times \mathscr{S}(N) : \text{ (1.4) holds as } x \to x_0\}$$

$$u(x) \leq u(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2}\langle X(x - x_0), x - x_0 \rangle + o(|x - x_0|^2) \qquad (1.4)$$

This defines a map $J_\Omega^{2,+}u : \Omega \to \mathbb{R}^N \times \mathscr{S}(N)$. Similarly, we define the *second order subjet of* u *at* x_0 :

$$J_\Omega^{2,-}u(x_0) := \{(p, X) \in \mathbb{R}^n \times \mathscr{S}(N) : \text{ (1.5) holds as } x \to x_0\}$$

$$u(x) \geq u(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2}\langle X(x - x_0), x - x_0 \rangle + o(|x - x_0|^2) \qquad (1.5)$$

Definition 1.9.
Let F be proper and $\Omega \subset \mathbb{R}^n$ be open. A *viscosity subsolution* of $F = 0$ on Ω is an upper semicontinuous function $u : \Omega \to \mathbb{R}$ such that

$$F(x, u(x), p, X) \leq 0 \quad \text{for all } x \in \Omega, \ (p, X) \in J_\Omega^{2,+}u(x) \qquad (1.6)$$

Likewise, a *viscosity supersolution* of $F = 0$ on Ω is an upper semicontinuous function $u : \Omega \to \mathbb{R}$ such that

$$F(x, u(x), p, X) \geq 0 \quad \text{for all } x \in \Omega,\ (p, X) \in J_{\Omega}^{2,-} u(x) \tag{1.7}$$

Finally, a continuous function u is a *viscosity solution* of $F = 0$ in Ω if it is both a viscosity supersolution and a viscosity subsolution of $F = 0$ in Ω.

We next provide definitions of closures of set-valued mappings needed in Section 1.3

Definition 1.10.

$$\begin{aligned}\overline{J}_{\Omega}^{2,+} u(x_0) := \{(p, X) \in \mathbb{R}^N \times \mathscr{S}^N : \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^N \times \mathscr{S}(N),\\ (p_n, X_n) \in J_{\Omega}^{2,+} u(x_n); (x_n, u(x_n), p_n, X_n) \to (x_0, u(x_0), p, X)\}\end{aligned}$$

$$\begin{aligned}\overline{J}_{\Omega}^{2,-} u(x_0) := \{(p, X) \in \mathbb{R}^N \times \mathscr{S}^N : \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^N \times \mathscr{S}(N),\\ (p_n, X_n) \in J_{\Omega}^{2,-} u(x_n); (x_n, u(x_n), p_n, X_n) \to (x_0, u(x_0), p, X)\}\end{aligned}$$

Remark 1.11.
If u is a viscosity subsolution (respectively viscosity supersolution) of $F = 0$ then for $x \in \Omega$, $(p, X) \in \overline{J}_{\Omega}^{2,+}(x_0)$ (respectively $(p, X) \in \overline{J}_{\Omega}^{2,-}(x_0)$) we have $F(x, u(x), p, X) \leq 0$ (respectively $F(x, u(x), p, X) \geq 0$). In fact, by definition of viscosity subsolution, for $x \in \Omega$, $(p, X) \in J_{\Omega}^{2,+}(x_0)$ we have $F(x, u(x), p, X) \leq 0$. Thus, by the continuity of F, it holds $F(x, u(x), p, X) \leq 0$ also in case $(p, X) \in \overline{J}_{\Omega}^{2,+}(x_0)$.

Now, we want to prove that notions given in Definitions 1.5 and in Definition 1.9 are equivalent, so we will be allowed to use either the first or the second depending on convenience in the purpose of demonstrating the following theorems.

Proof. Let u be a viscosity subsolution according to Definition 1.9. Then suppose $\varphi \in C^2(\Omega)$, such that $u - \varphi$ has a local maximum in x_0, so that

$$\begin{aligned}u(x) &\leq \varphi(x) + u(x_0) - \varphi(x_0)\\ &\leq u(x) + \langle D\varphi(x_0), x - x_0\rangle + \frac{1}{2}\langle D^2\varphi(x_0)(x - x_0), x - x_0\rangle + o(|x - x_0|^2)\end{aligned}$$

holds as $x \to x_0$. The inequality above implies $(D\varphi(x_0), D^2\varphi(x_0)) \in J_{\Omega}^{2,+} u(x_0)$, so, by (1.6) we can conclude that

$$F(x_0, u(x_0), D\varphi(x_0), D^2(x_0)) \leq 0$$

that is, u is a viscosity subsolution according to Definition 1.5. Now we show the converse. Suppose u is a viscosity subsolution according to Definition 1.5 and $(p, X) \in J^{2,+}u(x_0)$. For $\varepsilon > 0$ define

$$\phi(x) := u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + \frac{\varepsilon}{2}|x - x_0|^2$$

We want to check that $u - \phi$ has a maximum at x_0. Since (1.6) holds, for $x \to x_0$ and for every $\varepsilon > 0$ we get:

$$\begin{aligned} u(x) - \phi(x) &= u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + o(|x - x_0|^2) - \phi(x) \\ &\leq +o(|x - x_0|^2) - \frac{\varepsilon}{2}|x - x_0|^2 \\ &\leq 0 = u(x_0) - \phi(x_0) \end{aligned}$$

Therefore $F(x_0, u(x_0), D\phi(x_0), D^2(x_0)) = F(x_0, u(x_0), p, X - \varepsilon \mathrm{Id}_n) \leq 0$ for every $\varepsilon > 0$. Thanks to the continuity of F, passing to the limit for $\varepsilon \to 0$ we obtain $F(x_0, u(x_0), p, X) \leq 0$, so u is a viscosity subsolution according to Definition 1.6. Similarly it is possible to prove the equivalence between the two definitions of viscosity supersolution. □

1.2 Stability of viscosity solution with respect to the uniform convergence

In this section we prove one of the most useful propriety of viscosity solutions: the stability with respect to the uniform convergence. We proceed as in [18, Proposition I.3].

Theorem 1.12.
Let $\varepsilon > 0$, let (F_ε) be degenerate elliptic and continuous on $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}(N)$ and let u_ε be viscosity solutions of

$$F_\varepsilon(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) = 0 \quad \text{in } \Omega \tag{1.8}$$

If F_ε converges to some function F on compact subset of $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}(N)$ and u_ε uniformly converges on compact subset of Ω to some u, then u is a viscosity solution of

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega$$

Proof. We prove that u is a viscosity solution of $F = 0$ by checking that it satisfies the Definition 1.5. First of all we check (1.2) for some $\varphi \in C^2(\Omega)$

such that $u - \varphi$ has a strict local maximum at $x_0 \in \Omega$. Let $\delta > 0$ be small enough such that $\overline{B_\delta(x_0)} \subset \Omega$ and

$$(u - \varphi)(x_0) = \max_{B_\delta(x_0)} (u - \varphi)$$

Here and in the sequel $B_\delta(x_0)$ is the Eulidean ball with center x_0 and radius δ. Now consider x_ε the maximum point of $u_\varepsilon - \varphi$ in $\overline{B_{\delta/2}(x_0)}$, in particular it holds

$$(u_\varepsilon - \varphi)(x_0) \leq (u_\varepsilon - \varphi)(x_\varepsilon) \quad \text{in } \overline{B_{\delta/2}(x_0)} \tag{1.9}$$

The sequence $\{x_\varepsilon\}$ belongs to the compact set $\overline{B_{\delta/2}(x_0)}$, so there exists a subsequence (which we still denote by $\{x_\varepsilon\}$) such that $x_\varepsilon \to y$ as $\varepsilon \to 0^+$. Moreover the sequence u_ε uniformly converges to u hence, u is continuous. Now, passing to limit for $\varepsilon \to 0^+$ in (1.9) we get:

$$(u - \varphi)(x_0) \leq (u - \varphi)(y) \qquad \text{in } \overline{B_{\delta/2}(x_0)} \tag{1.10}$$

Since x_0 is a strict local maximum, it follows $y = x_0$, so we have: $x_\varepsilon \to x_0$, $u_\varepsilon(x_\varepsilon) \to u(x_0)$ as $\varepsilon \to 0^+$. Now we recall that u_ε is a subsolution:

$$F(x_\varepsilon, u(x_\varepsilon), D\varphi(x_\varepsilon), D^2\varphi(x_\varepsilon)) \leq 0$$

hence, by the continuity of F_ε and the C^2 regularity of φ, letting $\varepsilon \to 0^+$ we get

$$F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \leq 0$$

Similarly, we can check (1.3), for some $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a strict local minimum at $x_0 \in \Omega$. □

1.3 Comparison principle

The aim of this section is to prove a comparison principle: this one can be regarded as the nonlinear version of maximum principle. We proceed as in [8, Theorem 3.3]

Theorem 1.13.
Let $\Omega \subset \mathbb{R}^N$ be open and bounded, $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}(N))$ be proper. Suppose there exists $\gamma > 0$ such that

$$\gamma(r - s) \leq F(x, r, p, X) - F(x, s, p, X) \tag{1.11}$$

for $r \geq s, (x, p, X) \in \overline{\Omega} \times \mathbb{R} \times \mathscr{S}(N)$. Furthermore, suppose there exists a function $\omega : [0, \infty] \to [0, \infty]$ satisfying $\omega(0^+) = 0$ such that

$$F(y, r, \alpha(x - y), Y) - F(x, r, \alpha(x - y), X) \leq \omega(\alpha|x - y|^2 + |x - y|) \tag{1.12}$$

for all $x, y \in \Omega, r \in \mathbb{R}, X, Y \in \mathscr{S}(N)$ *whenever*

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

Let u, v *be a viscosity supersolution and a viscosity subsolution of* $F = 0$*, respectively. Then*

$$u \leq v \text{ on } \partial\Omega \Longrightarrow u \leq v \text{ in } \Omega$$

To prove the theorem we will use the following two Lemmas.

Lemma 1.14.
Let $\Omega \subset \mathbb{R}^N$*,* u *be upper semicontinuous in* $\overline{\Omega}$ *and* v *be lower semicontinuous in* $\overline{\Omega}$*. Define*

$$M_\alpha := sup_{\Omega\times\Omega}(u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2), \quad \alpha > 0 \tag{1.13}$$

Suppose $M_\alpha < \infty$ *for large* α *and* (x_α, y_α) *be such that*

$$\lim_{\alpha\to\infty} M_\alpha - (u(x) - v(y) - \frac{\alpha}{2}|x - y|^2)) = 0 \tag{1.14}$$

Then the following holds:

$$\lim_{\alpha\to\infty} \alpha|x_\alpha - y_\alpha|^2 = 0 \tag{1.15}$$

$$\lim_{\alpha\to\infty} M_\alpha = u(x_0) - v(x_0) = \sup_{\Omega}(u(x) - v(x)) \tag{1.16}$$

where $x_0 \in \Omega$ *and* $x_\alpha \to x_0$ *as* $\alpha \to \infty$*.*

Proof. Define $\delta_\alpha := M_\alpha - (u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2)$ so that (1.14) becomes $\lim_{\alpha\to\infty} \delta_\alpha = 0$. By assumptions, M_α decreases as α increases and $\lim_{\alpha\to\infty} M_\alpha$ exists finite. It holds

$$\begin{aligned} M_{\alpha/2} &= \sup_{\Omega\times\Omega}(u(x) - v(y) - \frac{\alpha}{2}|x - y|^2) \\ &\geq u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 = \\ &= u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 + \frac{\alpha}{4}|x_\alpha - y_\alpha|^2 \\ &= M_\alpha - \delta_\alpha + \frac{\alpha}{4}|x_\alpha - y_\alpha|^2 \end{aligned}$$

so $2(M_{\alpha/2} - M_\alpha + \delta_\alpha) \geq \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \geq 0$. Now, letting $\alpha \to \infty$ the left hand side of the inequality goes to 0, so (1.15) is proved. Suppose $x_\alpha, y_\alpha \to x_0$ as

$\alpha \to \infty$, $x_0 \in \Omega$.

$$\begin{aligned} u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 &= M_\alpha - \delta_\alpha = \\ &= \sup_{\Omega\times\Omega}(u(x) - v(y) - \frac{\alpha}{2}|x-y|^2) - \delta_\alpha \\ &\geq \sup_{\Omega}(u(x) - v(x)) - \delta_\alpha \end{aligned}$$

Now, recalling that $u - v$ is upper semicontinuous, we send $\alpha \to \infty$ to obtain

$$u(x_0) - v(x_0) \geq M_\alpha \geq \sup_{\Omega}(u(x) - v(x))$$

Since $u(x_0) - v(x_0) \leq \sup_\Omega(u(x) - v(x))$, inequalities above are actually equalities. □

Lemma 1.15.
Let Ω_i be locally compact subset of R_i^N for $i = 1, \ldots, k$, $\Omega = \Omega_1 \times \cdots \times \Omega_k$, let $u_i : \Omega_i \to \mathbb{R}$ be upper semicontinuous functions, and $\varphi \in C^2(\Omega_\varepsilon)$ (where Ω_ε is a neighbourhood of Ω). Set

$$w(x) = u_1(x_1) + \cdots + u_k(x_k) \quad \text{for } x = (x_1, \ldots, x_k) \in \Omega$$

and suppose $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_k) \in \Omega$ is a local maximum of $w - \varphi$ relative to Ω. Then for each $\varepsilon > 0$ there exists $X_i \in \mathscr{S}(N_i)$ such that

$$(D_{x_i}\varphi(\hat{x}), X_i) \in \overline{J}^{2,+}_{\Omega_i} u_i(\hat{x}_i) \quad \text{for } i = 1, \ldots, k$$

and the block diagonal matrix with entries X_i satisfies

$$-\Big(\frac{1}{\varepsilon} + \|A\|\Big)\mathrm{Id}_N \leq \begin{pmatrix} X_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & X_k \end{pmatrix} \leq A + \varepsilon A^2 \tag{1.17}$$

where $A = D^2\varphi(\hat{x}) \in \mathscr{S}(N)$, $N = N_1 + \cdots + N_k$,

$\|A\|$ denotes the norm of a symmetric matrix A

$$\|A\| := \sup\{|\lambda| : \lambda \text{ is an enginvalue of } A\} = \sup\{|\langle A\xi, \xi\rangle| : |\xi| \leq 1\} \tag{1.18}$$

this notation will be used throughout this work.

The proof of the above Lemma is outlined in [8, Appendix]. We are now ready to prove the main result of this section.

Proof (Theorem 1.13). We want to prove the statement by contradiction, so we assume the existence of a point $z \in \Omega$ such that $u(z) > v(z)$. As in Lemma 1.14 we define

$$M_\alpha := \sup_{\overline{\Omega}\times\overline{\Omega}} (u(x) - v(y) - |x-y|^2\alpha/2)$$

it holds

$$M_\alpha \geq u(z) - v(z) = \delta > 0 \tag{1.19}$$

In view of the upper semicontinuity of $u - v$ and the compactness of $\overline{\Omega}$ we note that M_α is finite and the function M_α assumes its maximum at some point $(x_\alpha, y_\alpha) \in \overline{\Omega} \times \overline{\Omega}$. By (1.16), for $\sigma > 0$ small enough and α (depending on σ) large we have

$$u(x_\alpha) - v(y_\alpha) - |x_\alpha - y_\alpha|^2\alpha/2 = M_\alpha > \sup_{\overline{\Omega}}(u(x) - v(x)) - \sigma > 0$$

this inequality and the fact that $u \leq v$ on $\partial\Omega$ imply that $(x_\alpha, y_\alpha) \in \Omega \times \Omega$ for α large. Now we want to estimate M_α in order to contradict (1.19) for large α. In order to apply Lemma 1.15 we put $k = 2$, $\Omega_1 = \Omega_2 = \Omega$, $u_1 = u$, $u_2 = -v$, $\varphi(x,y) = |x-y|^2\alpha/2$ and recall that $\overline{J}^{2,-}_\Omega v = -\overline{J}^{2,-}_\Omega(-v)$. In this case $u_1 + u_2 - \varphi = u - v - |x-y|^2\alpha/2$ has a local maximum point in (x_α, y_α) and

$$D_x\varphi(x_\alpha, y_\alpha) = -D_y\varphi(x_\alpha, y_\alpha) = \alpha(x_\alpha - y_\alpha), \quad A = \alpha\begin{pmatrix} \mathrm{Id}_N & -\mathrm{Id}_N \\ -\mathrm{Id}_N & \mathrm{Id}_N \end{pmatrix}$$

$$A^2 = 2\alpha A, \quad \|A\| = 2\alpha$$

then we conclude that for every $\varepsilon > 0$ there exist $X, Y \in \mathscr{S}(N)$ such that

$$(\alpha(x_\alpha - y_\alpha), X) \in \overline{J}^{2,+}_\Omega u(x_\alpha), \quad (\alpha(x_\alpha - y_\alpha), Y) \in \overline{J}^{2,-}_\Omega v(y_\alpha) \tag{1.20}$$

and

$$-\Big(\frac{1}{\varepsilon} + 2\alpha\Big)\begin{pmatrix} \mathrm{Id}_N & \mathrm{Id}_N \\ \mathrm{Id}_N & \mathrm{Id}_N \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \alpha(1 + 2\varepsilon\alpha)\begin{pmatrix} \mathrm{Id}_N & -\mathrm{Id}_N \\ -\mathrm{Id}_N & \mathrm{Id}_N \end{pmatrix}$$

Choosing $\varepsilon = 1/\alpha$ we get

$$-3\alpha\begin{pmatrix} \mathrm{Id}_N & \mathrm{Id}_N \\ \mathrm{Id}_N & \mathrm{Id}_N \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha\begin{pmatrix} \mathrm{Id}_N & -\mathrm{Id}_N \\ -\mathrm{Id}_N & \mathrm{Id}_N \end{pmatrix} \tag{1.21}$$

summarizing, we conclude that at a maximum (x_α, y_α) of $u(x) - v(y) - |x - y^2\alpha/2|$ there exist $X, Y \in \mathscr{S}(N)$ such that (1.20) and (1.21) hold. Since u is a viscosity subsolution and v is a supersolution, from (1.20) we get

$$F(x_\alpha, u(x_\alpha), \alpha(x_\alpha - y_\alpha), X) \leq 0 \leq F(y_\alpha, v(y_\alpha), \alpha(x_\alpha - y_\alpha), Y) \tag{1.22}$$

Now we have all the necessary instruments to reach a contradiction and conclude the proof. We choose γ as in (1.11) to deduce that

$$\begin{aligned}
\gamma\delta &\overset{(1.19)}{\leq} \gamma M_\alpha \leq \gamma(u(x_\alpha) - v(y_\alpha)) \\
&\overset{(1.11)}{\leq} F(x_\alpha, u(x_\alpha), \alpha(x_\alpha - y_\alpha), X) - F(x_\alpha, v(y_\alpha), \alpha(x_\alpha - y_\alpha), X) \\
&= F(x_\alpha, u(x_\alpha), \alpha(x_\alpha - y_\alpha), X) - F(y_\alpha, v(y_\alpha), \alpha(x_\alpha - y_\alpha), Y) \\
&\qquad + F(y_\alpha, v(y_\alpha), \alpha(x_\alpha - y_\alpha), Y) - F(x_\alpha, v(x_\alpha), \alpha(x_\alpha - y_\alpha), X) \\
&\leq \omega(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|)
\end{aligned}$$

the last inequality follows from (1.22) and (1.12). Since $\omega(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|) \to 0$ as $\alpha \to \infty$, we conclude that $\gamma\delta \leq 0$, a contradiction. □

1.4 Perron's method

The classical Perron's method had been introduced in 1923 by Oskar Perron in order to find solutions for the Laplace equation; in 1987 H. Ishii applied it to solve nonlinear first-order equations ([16]). On one hand Perron's method is a powerful technique that allows to prove existence results for very general PDEs but, on the other hand does not give representative formula for the solution neither information on its regularity. The idea of this method consists in finding a solution as the supremum of a suitable family of viscosity subsolutions. We follow the proof given in [8, Theorem 4.1]

First of all we introduce the notion of viscosity solution of a Dirichlet problem.

Definition 1.16.
Let $\Omega \subset \mathbb{R}^n$ be open. We call u a *solution* (respectively viscosity subsolution, viscosity supersolution) *of the Dirichlet problem*

$$\begin{cases} F(x, u, Du, D^2) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.23}$$

a function $u \in C(\overline{\Omega})$ (respectively upper semicontinuous, lower semicontinuous) that is a viscosity solution (respectively subsolution, supersolution) of $F = 0$ in Ω and satisfies $u(x) = 0$ (respectively $u(x) \leq 0$, $u(x) \geq 0$) for $x \in \partial\Omega$.

Theorem 1.17 (Perron's Method)**.**
Let us assume that

1. *Comparison principle for* (1.23) *holds, i.e. given* $\underline{u}$ *viscosity subsolution and* $\overline{u}$ *viscosity supersolution of* (1.23), *then* $\underline{u} \leq \overline{u}$.

2. *There exist* $\underline{u}$ *and* $\overline{u}$ *which are, respectively, a viscosity subsolution and a viscosity supersolution of* (1.23) *that satisfy the boundary condition* $\underline{u}_*(x) = \overline{u}^*(x) = 0$ *for* $x \in \partial\Omega$.

We define

$$\mathscr{F} := \{w | \underline{u} \leq w \leq \overline{u} \text{ and } w \text{ is a viscosity subsolution of } (1.23)\}$$

$$W(x) = \sup_{w \in \mathscr{F}} \{w(x)\} \tag{1.24}$$

Then W *is a viscosity solution of* (1.23).

We recall that u^* and u_* denote the upper semicontinuous envelope of u and the lower semicontinuous envelope of u respectively. We will prove the Theorem with the help of two Lemmas.

Lemma 1.18.
Suppose Ω *be locally compact,* F *continuous and* $\mathscr{F}$ *be a family of subsolutions of* $F = 0$ *in* Ω. *Let* $w(x) = \sup\{u(x) : u \in \mathscr{F}\}$ *and assume that* $w^*(x) < \infty$ *for* $x \in \Omega$. *Then* w^* *is a subsolution in* Ω

Proof. Let $z \in \Omega$ and $(p, X) \in J^{2,+}_{\Omega} w^*(z)$, we show that $F(z, w^*(z), p, X) \leq 0$. Indeed, choose a sequence $(x_n, u_n) \in \Omega \times \mathscr{F}$ such that $(x_n, u_n(x_n)) \to (z, w^*(z))$ and

$$\limsup_{n \to \infty} u_n(x_n) \leq w^*(x) \tag{1.25}$$

We claim that

$$\begin{cases} \text{there exist } \hat{x}_n \in \Omega, \ (p_n, X_n) \in J^{2,+}_{\Omega} u_n(\hat{x}_n) \\ \text{such that } (\hat{x}_n, u_n(\hat{x}_n, p_n, X_n)) \to (z, w^*(z), p, X) \end{cases} \tag{1.26}$$

Assuming the claim for a moment and recalling that u_n is a viscosity subsolution, we pass to the limit in the relation $F(\hat{x}_n, u_n(\hat{x}_n), p_n, X_n)$ to conclude that $F(z, w^*(z), p, X) \leq 0$. Let us prove the claim. By the assumptions, for every $\delta > 0$ there is an $r > 0$ such that $N_r = \{x \in \Omega : |x - z| \leq r\}$ is compact and, for $x \in N_r$

$$w^*(x) \leq w^*(z) + \langle p, x - z \rangle + \frac{1}{2} \langle X(x - z), x - z \rangle + \delta |x - z|^2 \tag{1.27}$$

Define $\psi(x) = \langle p, x - z\rangle + \frac{1}{2}\langle Y(x-z), x-z\rangle$, where $Y = X + 4\delta \mathrm{Id}_N$, and let $\hat{x}_n \in N_r$ be a maximum point of the function $u_n(x) - \psi(x)$ over N_r so that

$$u_n(x) \leq u_n(\hat{x}_n) - \psi(\hat{x}_n) + \psi(x) \tag{1.28}$$

$$\leq u_n(\hat{x}_n) + \langle p, x - \hat{x}_n\rangle + \frac{1}{2}\langle Y(x-z), x-z\rangle - \frac{1}{2}\langle Y(\hat{x}_n - z), \hat{x}_n - z\rangle \tag{1.29}$$

suppose that, passing to a subsequence if necessary, $\hat{x}_n \to y$ as $n \to \infty$. Putting $x = x_n$, equation (1.29) becomes

$$u_n(x_n) \leq u_n(\hat{x}_n) + \langle p, x_n - \hat{x}_n\rangle + \frac{1}{2}\langle Y(x_n-z), x_n-z\rangle - \frac{1}{2}\langle Y(\hat{x}_n - z), \hat{x}_n - z\rangle \tag{1.30}$$

taking the limit inferior as $n \to \infty$, and using previous results we obtain

$$\begin{aligned} w^*(z) &\leq \liminf_{n\to\infty} u_n(\hat{x}_n) + \langle p, z - y\rangle - \frac{1}{2}\langle (X + 4\delta \mathrm{Id}_n)(y-z), y-z\rangle \\ &\overset{(1.25)}{\leq} w^*(y) - \langle p, y - z\rangle - \frac{1}{2}\langle X(y-z), y-z\rangle - 2\delta|y-z|^2 \\ &\overset{(1.27)}{\leq} w^*(z) + \delta|y-z|^2 - 2\delta|y-z|^2 \leq w^*(z) - \delta|y-z|^2 \end{aligned}$$

From the first and the last inequality we learn that $y = z$, so $\hat{x}_n \to z$ (without passing to a subsequence), then from the first inequality and (1.25) one sees that $w^*(z) \leq \liminf_{n\to\infty} u_n(\hat{x}_n) \leq \limsup_{n\to\infty} u_n(\hat{x}_n) \leq w^*(z)$, this implies

$$w^*(z) = \lim_{n\to\infty} u_n(\hat{x}_n) \tag{1.31}$$

Notice that

$$(p + Y(\hat{x}_n - z), Y) \in J^{2,+}_{\Omega} u_n(\hat{x}_n) \tag{1.32}$$

In fact, replacing $\frac{1}{2}\langle Y(x-z), x-z\rangle$ in (1.29) with the equivalent expression

$$\frac{1}{2}\langle Y(x - \hat{x}_n), x - \hat{x}_n\rangle + \frac{1}{2}\langle Y(\hat{x}_n - z), x - \hat{x}_n\rangle + \frac{1}{2}\langle Y(x-z), \hat{x}_n - z\rangle$$

we get

$$\begin{aligned} u_n(x) &\leq u_n(\hat{x}_n) + \langle p, x - \hat{x}_n\rangle + \frac{1}{2}\langle Y(x - \hat{x}_n), x - \hat{x}_n\rangle + \frac{1}{2}\langle Y(\hat{x}_n - z), x - \hat{x}_n\rangle \\ &\quad + \langle Y(x - \hat{x}_n), \hat{x}_n - z\rangle \\ &= u_n(\hat{x}_n) + \langle p + Y(\hat{x}_n - z), x - \hat{x}_n\rangle + \frac{1}{2}\langle Y(x - \hat{x}_n), x - \hat{x}_n\rangle \end{aligned}$$

To conclude we note that the set of $(q, Z) \in \mathbb{R}^N \times \mathscr{S}$ such that there exist $z_n \in \Omega$, $(p_n, X_n) \in J^{2,+}_{\Omega} u_n(z_n)$ such that $(z_n, u_n(z_n), p_n, X_n) \to (z, w^*(z), q, Y)$ is closed and, by (1.31) and (1.32), contains $(p, X + 4\delta \mathrm{Id}_N)$ for every $\delta > 0$. □

Lemma 1.19.

Let Ω be open and u be a subsolution of $F = 0$ in Ω. If u_* fails to be a supersolution at some point x_0, then for any small σ there is a subsolution U_σ of $F = 0$ in Ω satisfying:

$$\begin{cases} U_\sigma(x) \geq u(x) \text{ and } \sup_\Omega(U_\sigma - u) > 0 & \text{for } x \in \Omega \\ U_\sigma(x) = u(x) & \text{for } x \in \Omega,\ |x - x_0| \geq \sigma \end{cases} \tag{1.33}$$

Proof. Let assume u_* fails to be a supersolution at $x_0 \in \Omega$, that means there exists $(p, X) \in J^{2,-}_\Omega u_*(x_0)$ for which

$$F(x_0, u_*(x_0), p, X) < -\eta \quad \text{with } \eta > 0 \tag{1.34}$$

We are going to build the function U_σ of the statement using (p, X). Define

$$u_{\delta,\gamma}(x) := u_*(x_0) + \delta + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle - \gamma|x - x_0|^2 \tag{1.35}$$

and notice that, thanks to the continuity of F it is possible to choose $\delta, \gamma > 0$ small enough such that

$$\begin{aligned} F(x_0, u_{\delta,\gamma}(x_0), Du_{\delta,\gamma}u(x_0), D^2u_{\delta,\gamma}u(x_0)) = \\ = F(x_0, u_*(x_0) + \delta, p, X - 2\gamma) \leq 0 \end{aligned} \tag{1.36}$$

Moreover, since $u_{\delta,\gamma} \in C^2(\Omega)$, F is continuous and (1.34) holds, we can find $r > 0$ small enough such that $F(x, u_{\delta,\gamma}(x), Du_{\delta,\gamma}(x), D^2u_{\delta,\gamma}(x)) \leq 0$ in $B_r(x_0)$. In other words $u_{\delta,\gamma}$ is a classical subsolution of $F = 0$ in $B_r(x_0)$. Choosing $\delta = \frac{r^2}{8}\gamma$, for r sufficiently small and $\frac{r}{2} \leq |x - x_0| \leq r$ we have

$$\begin{aligned} u_{\gamma r^2/8,\gamma}(x) &= u_*(x_0) + \frac{r^2}{8}\gamma + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle - \gamma|x - x_0|^2 \\ &\leq u_*(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle - \frac{r^2}{8}\gamma \\ &< u_*(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + o(|x - x_0|^2) \\ &\leq u_*(x) \\ &\leq u(x) \end{aligned}$$

The last two inequalities follow from the fact that $(p, X) \in J^{2,-}_\Omega u_*(x_0)$ and the definition of u_*. By Lemma 1.18, for $\sigma > 0$ small enough, the function

$$U_\sigma(x) = \begin{cases} \max\{u(x), u_{\gamma\sigma^2/8,\gamma}(x)\} & \text{for } B_\sigma(x_0) \\ u(x) & \text{for } x \in \Omega \setminus B_\sigma(x_0) \end{cases}$$

is a viscosity subsolution of $F = 0$ in Ω. U_σ is the function we where looking for in fact, by definition of u_*, there is a sequence $(x_n, u(x_n))$ converging to $(x_0, u_*(x_0))$ and then

$$\lim_{n\to\infty}(U_\sigma(x_n) - x_n) = u_{\gamma\sigma^2/8,\gamma}(x_0) - u_*(x_0) = u_*(x_0) + \gamma\sigma^2/8 - u_*(x_0) > 0$$

so we conclude that $\sup_\Omega(U_\sigma - u) > 0$. □

Proof (Theorem 1.17, Perron's Method). First of all we prove that W is a viscosity subsolution of (1.23). By Lemma 1.18, W^* is a viscosity subsolution of $F = 0$ and it satisfies the boundary condition $W^* = 0$ by construction (in fact on $\partial\Omega$ we have $0 = \underline{u}_* \le W_* \le W \le W^* \le \overline{u}^* = 0$). Moreover applying the comparison principle to compare the viscosity supersolution $\overline{u}$ and W^* we obtain that $W^* \le \overline{u}$ in Ω. We deduce that $W^* \in \mathscr{F}$, and so $W = W^*$. Now we prove that W is also a viscosity supersolution showing that W_* is a viscosity supersolution and then observing that $W = W_*$. If W_* fails to be a viscosity supersolution at some point $x_0 \in \Omega$, consider the viscosity subsolution W_σ provided by Lemma 1.19. For σ small we have $W_\sigma = 0$ on $\partial\Omega$, then, by comparison we get $W_\sigma \le \overline{u}$. On the other hand, by definition of W_σ it holds $\underline{u} \le W \le W_\sigma$, and we recall that there are points of Ω for which the last inequality is strict. Since W is the maximal subsolution between $\underline{u}$ and $\overline{u}$ we arrive at the contradiction $W_\sigma \le W$. Hence W_* is a viscosity supersolution of (1.23). By comparison $W = W^* \le W_*$, clearly also the opposite inequality holds, so that we conclude $W = W_*$ □

Chapter 2

Existence of Smooth Solutions

In this chapter we aim to prove the existence of a pluri-subharmonic solution for the Dirichlet problem for the complex Monge-Ampère equation with smooth boundary data

$$\begin{cases} \det(\partial\overline{\partial}u) = f(z,u,Du) & \text{in } \Omega \\ u = \varphi \quad \text{on } \partial\Omega \end{cases} \tag{2.1}$$

This result is due to L. Caffarelli, J.J. Kohn, L. Nirenberg and J.Spruck. In our work we follow the proof they have given in [7, Theorem 1.3].

2.1 Notations and definitions

Let us fix some notations: we denote by $B_r(z_0)$ the Euclidean ball with center at a point z_0 and radius r, i.e. $B_r(z_0) = \{z \in \mathbb{C}^n : \ |z - z_0| < r\}$. If the point z_0 is not specified we assume $z_0 = 0$. $\text{Re}(z)$ and $\text{Im}(z)$ denote the real and the imaginary part of a complex number z respectively. We identify $\mathbb{C}^n \approx \mathbb{R}^{2n}$, with $z = (z_1, \dots, z_n)$ and $z_j = x_j + iy_j \approx (x_j, y_j)$, for $j = 1, \dots, n$ and we set

$$u_j = \frac{\partial u}{\partial z_j} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} - i\frac{\partial u}{\partial y_j}\right) = \frac{1}{2}(u_{x_j} - iu_{y_j}), \quad u_{\bar{j}} = \overline{u}_j,$$

$$u_{j\bar{l}} = \frac{\partial^2 u}{\partial z_j \partial z_{\bar{l}}} = \frac{1}{4}\left(\left(\frac{\partial^2 u}{\partial x_j \partial x_l} + \frac{\partial^2 u}{\partial y_j \partial y_l}\right) + i\left(\frac{\partial^2 u}{\partial x_j \partial y_l} - \frac{\partial^2 u}{\partial y_j \partial x_l}\right)\right) =$$
$$= \frac{1}{4}\Big((u_{x_j x_l} + u_{y_j y_l}) + i(u_{x_j y_l} - u_{y_j x_l})\Big)$$

Moreover we define

$$D := (\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_n}, \partial_{y_n}), \quad \partial u := (\partial_1, \dots, \partial_n), \quad \overline{\partial} u := (\partial_{\overline{1}}, \dots, \partial_{\overline{n}})$$

Furthermore we give the following definitions

Definition 2.1.
Let $\Omega \subset \mathbb{C}^n$. A real valued smooth function $u : \Omega \to \mathbb{R}$, $u \in C^2(\Omega)$ is called *pluri-subharmonic* (respectively *strictly pluri-subharmonic*) if the Hessian $\partial\overline{\partial}u$ is non negative (respectively positive definite).

Definition 2.2.
A domain $\Omega \subset \mathbb{C}^n$, with a smooth boundary $\partial\Omega$ is called *strongly pseudo-convex* if there exists a C^∞ strictly pluri-subharmonic function r defined in a neighbourhood of $\partial\Omega$ such that $Dr \neq 0$, $r < 0$ in Ω, $r = 0$ on $\partial\Omega$ and $r > 0$ outside $\overline{\Omega}$.

Throughout this Chapter we denote by F the operator

$$F(u) := \det(\partial\overline{\partial}u) - f(z, u, Du) \tag{2.2}$$

notice that this operator is elliptic in the sense of [12] on the set of strictly pluri-subharmonic functions i.e.

Definition 2.3.
Let $O \subset \mathbb{R}^n$. The operator G in a subset $\mathscr{U}$ of $O \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ is *elliptic in the sense of [12]* if the matrix $(G_{ij}(\gamma))$, given by

$$G_{ij}(\gamma) = \frac{\partial G}{\partial r_{ij}}(\gamma) \quad i, j = 1, \dots, n$$

is positive definite for all $\gamma = (x, s, p, X) \in \mathscr{U}$, where $X = \{r_{ij}\}_{i,j=1,\dots,n}$.

Remark 2.4.
If the operator G is elliptic in the sense of [12] then $-G$ is degenerate elliptic according to Definition 1.1. Indeed, if $Y \leq X$

$$\begin{aligned} G(x, s, p, Y) - G(x, s, p, X) &= \int_0^1 \frac{d}{dt} G(x, s, p, X + t(Y - X))\, dt \\ &= \int_0^1 \frac{\partial G}{\partial r_{ij}}(x, s, p, X + t(Y - X))\, dt (Y - X)_{ij} \leq 0 \end{aligned}$$

In order to see that F is elliptic in the sense of [12] on the set of strictly pluri-subharmonic functions, we rewrite the operator (2.2) into an equivalent form to point out the dependence on the real Hessian D^2u instead of the dependence on the complex Hessian. We identify $\mathbb{C}^n \approx \mathbb{R}^{2n}$ and

$$z = (z_1, \dots, z_n) \approx (x_1, \dots, x_n, y_1, \dots, y_n)$$

so that

$$(\partial\overline{\partial}u)_{i,j} = \frac{1}{4}\Big((D^2u)_{i,j} + i(D^2u)_{i,n+j} - i(D^2u)_{n+i,j} + (D^2u)_{n+i,n+j}\Big)$$

where we use the notation $(A)_{i,j}$ to denote the entry in the i-th row and j-th column of a matrix A. Now we define the matrix

$$J := \frac{1}{2}(\mathrm{Id}_n, \sqrt{-1}\mathrm{Id}_n)$$

and observe that

$$\partial\overline{\partial}u = J(D^2u)\overline{J}^T$$

Hence (2.2) is equivalent to

$$F_J(z, u, Du, D^2u) := \det\big(J(D^2u)\overline{J}^T\big) - f(z, u, Du) \tag{2.3}$$

Let us prove that F_J is elliptic. Suppose $u \in C^2$ is strictly pluri-subharmonic, we denote by $u^{k,\bar{l}}$ the coefficients of $(\partial\overline{\partial}u)^{-1}$. Indices i, j go from 1 to $2n$ while k, l go from 1 to n. We have

$$\frac{\partial F_J}{\partial u_{ij}} = \big(\det(\partial\overline{\partial}u)\big)\sum_{k,l=1}^{n} u^{k\bar{l}}\frac{\partial u^{k\bar{l}}}{\partial u_{ij}} = \big(\det(\partial\overline{\partial}u)\big)\sum_{k,l=1}^{n} u^{k\bar{l}}J_{ki}\overline{J}^T_{jl}$$

moreover since $\partial\overline{\partial}u$ is positive definite, for every $\xi \in \mathbb{R}^{2n}$, $|\xi| \neq 0$ it holds $|J\xi| \neq 0$ so that

$$\sum_{i,j=1}^{2n}\frac{\partial F_J}{\partial u_{ij}}\xi_i\xi_j = \big(\det(\partial\overline{\partial}u)\big)\langle(\partial\overline{\partial}u)^{-1}J\xi, J\xi\rangle > 0$$

thus F_J is elliptic in the sense of [12]

One of the reasons for academic interest on the complex Monge-Ampère equation is that its expression changes in a simple way under a holomorphic change of variable.

Remark 2.5.
Let Ω_1 and Ω_2 be domains in C^n and

$$\phi : \Omega_1 \to \Omega_2, \quad z \mapsto w := \phi(z) = (\phi_1(z), \dots, \phi_n(z))$$

a holomorphic mapping. Suppose $u \in C^2(\Omega_2)$ and $v(z) := u \circ \phi(z)$. Then

$$\det(\partial\overline{\partial}_{z\bar{z}}v) = |\det(J_z\phi)|^2 \det(\partial\overline{\partial}_{w\overline{w}}u) \tag{2.4}$$

Proof. Notice that $\frac{\partial \overline{w}_k}{\partial \overline{z}_j} = \overline{\frac{\partial w_k}{\partial z_j}}$, for every $k, j = 1 \dots n$, furthermore since ϕ is a holomorphic function we have $\frac{\partial w_k}{\partial \overline{z}_j} = 0$ for every $k, j = 1 \dots n$. We compute

$$\frac{\partial v}{\partial \overline{z}_j}(z) = \sum_{k=1}^{n} \frac{\partial u}{\partial w_k}(w) \frac{\partial w_k}{\partial \overline{z}_j}(z) + \sum_{k=1}^{n} \frac{\partial u}{\partial \overline{w}_k}(w) \frac{\partial \overline{w}_k}{\partial \overline{z}_j}(z) = \sum_{k=1}^{n} \frac{\partial u}{\partial \overline{w}_k}(w) \frac{\partial \overline{w}_k}{\partial \overline{z}_j}(z)$$

and

$$\begin{aligned}\frac{\partial^2 v}{\partial z_i \partial \overline{z}_j}(z) &= \sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial w_l \partial \overline{w}_k}(w) \frac{\partial w_k}{\partial z_j}(z) \frac{\partial w_l}{\partial z_i}(z) + \sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial \overline{w}_l \partial \overline{w}_k}(w) \frac{\partial w_k}{\partial z_j}(z) \frac{\partial \overline{w}_l}{\partial z_i}(z) \\ &= \sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial w_l \partial \overline{w}_k}(w) \frac{\partial w_k}{\partial z_j}(z) \frac{\partial w_l}{\partial z_i}(z)\end{aligned}$$

therefore, denoting the Jacobian matrix of ϕ with $J_z\phi$, we have

$$\partial\overline{\partial}_{z\overline{z}} v = J_z\phi (J_z\phi)^T \partial\overline{\partial}_{w\overline{w}} u$$

from which we obtain (2.4) □

It will be useful the following comparison principle

Lemma 2.6.
Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and $v, w \in C^\infty(\overline{\Omega})$ be pluri-subharmonic, with v strictly pluri-subharmonic. Suppose

$$\begin{aligned}\det(\partial\overline{\partial} v) &\geq \det(\partial\overline{\partial} w) \quad \text{in } \Omega \\ v &\leq w \quad \text{on } \partial\Omega\end{aligned}$$

then $v \leq w$ in $\overline{\Omega}$

Proof. We define the linear constant coefficients operator

$$L := \sum_{i,j=1}^{n} \left(\int_0^1 B^{i\overline{j}}(t)\, dt \right) \frac{\partial^2}{\partial z_i \partial \overline{z}_j}$$

where $B^{i\overline{j}}(t)$ are cofactors of the matrix $\big(tv_{i\overline{j}} + (1-t)_{i\overline{j}} w\big)$. Since v is strictly pluri-subharmonic L is elliptic. Moreover we have

$$\begin{aligned}\det(\partial\overline{\partial} v) - \det(\partial\overline{\partial} w) &= \int_0^1 \frac{d}{dt} \det\big((tv_{i\overline{j}} + (1-t) w_{i\overline{j}})\big)\, dt \\ &= \sum_{i,j=1}^{n} \int_0^1 B^{i\overline{j}}(t)(v-w)_{i\overline{j}}\, dt \\ &= \sum_{i,j=1}^{n} \left(\int_0^1 B^{i\overline{j}}(t)\, dt \right) (v-w)_{i\overline{j}} \\ &= L(v-w) \geq 0\end{aligned}$$

Hence, by the maximum principle we infer that $v-w$ take its maximum over $\overline{\Omega}$ on $\partial\Omega$, that is

$$\max_{\overline{\Omega}}(v-w) = \max_{\partial\Omega}(v-w) \geq 0$$

□

Corollary 2.7.
Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and $u, w \in C^\infty(\overline{\Omega})$ be pluri-subharmonic functions with $u \leq w$ on $\partial\Omega$ and $\det(\partial\overline{\partial}u) \geq \det(\partial\overline{\partial}w)$. Then $u \leq w$ in Ω

Proof. Choose $v = u + \varepsilon(|z|^2 - \max_{\partial\Omega}|z|^2)$, $\varepsilon > 0$. Clearly v is strictly pluri-subharmonic, $v \leq w$ on $\partial\Omega$ and $\det(\partial\overline{\partial}v) \geq \det(\partial\overline{\partial}u) \geq \det(\partial\overline{\partial}w)$. By Lemma 2.6 we obtain $v \leq w$ for every $\varepsilon > 0$. Taking the limit as $\varepsilon \to 0$ we conclude. □

In treating the Monge-Ampère operator $F(z,u,p,R) = \det(R) - f(z,u,p)$ we use the following notation:

$$F_u := \frac{\partial F}{\partial u}, \quad F_i = \frac{\partial F}{\partial z_i}, \quad f_u = \frac{\partial f}{\partial u}, \quad f_{p_j} = \frac{\partial f}{\partial p_j}$$

The main result we aim to prove in this chapter is the following:

Theorem 2.8.
Let Ω be a strongly pseudo-convex domain in $\mathbb{C}^n$ with a C^∞ boundary $\partial\Omega$. Suppose $\phi \in C^\infty(\partial\Omega)$, $f \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{2n})$ and $f(z,u,p) > 0$ for all $z \in \overline{\Omega}$, $u \in \mathbb{R}$, $p \in \mathbb{R}^{2n}$. Suppose moreover that $f_u > 0$ and that there exists a positive constant C_1 such that

$$|\partial f|, |\overline{\partial} f|, |f_u|, |f_{p_j}| \leq C_1 f^{1-1/n} \quad \text{for } j = 1, \ldots, 2n \tag{2.5}$$

and for each $m \geq 0$ there exists a constant $C = C(m)$ such that

$$\begin{aligned} |\partial f|, |\overline{\partial} f|, |f_u|, |\partial f_{p_j}|, |\overline{\partial} f_{p_j}||f_{up_j}| &\leq C f^{1-1/2n} \\ |\partial(\partial f)|, |\overline{\partial}(\overline{\partial} f)|, |\partial f_u|, |\overline{\partial} f_u|, |f_u u| &\leq C f^{1-1/n} \\ |f_{p_j}|, |f_{p_i p_j}| &\leq C f \end{aligned} \tag{2.6}$$

whenever $|u| + |p| \leq m$. Then there exists a strictly pluri-subharmonic solution u of (2.1) such that $u \in C^\infty(\overline{\Omega})$ and which is the unique pluri-subharmonic solution in the space $C^{1,1}(\overline{\Omega})$

The theorem above is proved via the continuity method [12, Theorem 17.8]. To carry out this procedure we need a priori estimates $|u|_{C^{2+\alpha}(\overline{\Omega})}$ for C^∞ solution of (2.1) where

$$|u|_{C^{2,\alpha}(\overline{\Omega})} := |u|_{C^2(\overline{\Omega})} + \sum_{i,j=1}^{n} \sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}} \frac{|u_{i\bar{j}}(x) - u_{i\bar{j}}(y)|}{|x-y|^\alpha}$$

These estimates will follow from a priori estimates for both $|u|_{C^2(\overline{\Omega})}$ and the logarithmic modulus of continuity for the second derivatives of u.

2.2 Estimates for $|u|_{L^\infty(\overline{\Omega})}$ and $|Du|_{L^\infty(\overline{\Omega})}$

The purpose of this section is to establish a priori estimates for first derivatives of solutions of (2.1). To this end we first prove the existence of a subsolution of (2.1), then this result will easily imply bounds for $|Du|$ on $\partial\Omega$. Finally, thanks to the maximum principle, we show that $|Du|_{L^\infty}$ in the interior is controlled by $|Du|_{L^\infty}$ on the boundary.

Lemma 2.9.
Suppose the same hypotheses of Theorem 2.8 hold, then there exists a plurisubharmonic subsolution $\underline{u} \in C^\infty(\overline{\Omega})$ which satisfies

$$\begin{cases} -\det(\partial\overline{\partial}u) + f(z,u,Du) \le 0 & in\ \Omega \\ u = \varphi & on\ \partial\Omega \end{cases} \tag{2.7}$$

Proof. First of all we show that if u is a solution of (2.1) and $u \le m$ then (2.5) implies the existence of a positive constant $C = C(m)$ such that

$$f(z,u,p) \le C(1+|p|^n)$$

In fact, when $v \le m$ it holds:

$$\begin{aligned}
f(z,v,q) &\le f(z,0,0) + \int_0^1 \frac{d}{dt} f(z,tv,tq)\,dt \\
&\le f(z,0,0) + \int_0^1 f_u(z,tv,tq)v\,dt + \int_0^1 \sum_{j=1}^{2n} f_{p_j}(z,tv,tq)q_j\,dt \\
&\le f(z,0,0) + m\int_0^1 f_u(z,tv,tq)\,dt + |q|\int_0^1 (\sum_{j=1}^{2n} |f_{p_j}(z,tv,tq)|^2)^{1/2}\,dt \\
&\le f(z,0,0) + mC\Big(\max_{t\in[0,1]} f(z,tv,tq)\Big)^{1-1/n} + C|q|\Big(\max_{t\in[0,1]} f(z,tv,tq)\Big)^{1-1/n} \\
&\le C\Big(1 + \Big(\max_{t\in[0,1]} f(z,tv,tq)\Big)^{1-1/n}(C+|q|)\Big)
\end{aligned}$$

defining $A := \max_{\substack{|q|\le p \\ v\le m}} f(z,v,q)$, from the above inequalities we get

$$A \le C\big(1 + A^{1-1/n}(C+|p|)\big)$$

from which we obtain $A \le C(1+|p|^n)$, so that

$$f(z,u,p) \le A \le C(m)(1+|p|^n) \quad \text{for } u \le m \tag{2.8}$$

Now we define

$$\underline{u} := \varphi + s(e^{kr}-1)$$

where r is a strictly pluri-subharmonic defining function for Ω and k, s are positive constants to be determined. It is clear that $\underline{u}$ satisfies the boundary condition (in fact $r = 0$ on $\partial\Omega$). Let's now prove that $\det \partial\overline{\partial}\underline{u} \le f(z,\underline{u},D\underline{u})$ in Ω. It holds :

$$\underline{u}_{i\bar{j}} = \varphi_{i\bar{j}} + ske^{kr}(r_{i\bar{j}} + kr_i r_{\bar{j}}) \quad \text{for } i,j = 1,\ldots,2n$$

then suppose to extend φ over $\overline{\Omega}$ as a pluri-subharmonic C^∞ function, so that we have

$$\partial\overline{\partial}\underline{u} = \partial\overline{\partial}\varphi + (r_{i\bar{j}} + kr_i r_{\bar{j}}) \ge (r_{i\bar{j}} + kr_i r_{\bar{j}}) \tag{2.9}$$

Choosing $\alpha > 0$ such that $\partial\overline{\partial}r \ge \alpha\mathrm{Id}_n$, we obtain

$$\det\big((r_{i\bar{j}} + kr_i r_{\bar{j}})\big) = \det(\partial\overline{\partial}r + \partial r \otimes \overline{\partial}r) \ge \det(\alpha\mathrm{Id}_n + \partial r \otimes \overline{\partial}r) =: \det(\Gamma)$$

where Γ is a $n \times n$ Hermitian matrix. It is easy to see that $\lambda_1 = \alpha$ is an eigenvalue of Γ with multiplicity $k-1$. Since

$$\mathrm{Tr}(\Gamma) = (n-1)\lambda_1 + \lambda_2 = n\alpha + k|\partial r|^2$$

we calculate $\lambda_2 = \alpha + k|\partial r|^2 = \alpha + k|Dr|^2$, thus

$$\det(\Gamma) = \lambda_1^{n-1}\lambda_2 = \alpha^{n-1}(1 + k|Dr|^2) \tag{2.10}$$

Now, (2.10) and (2.9) imply

$$\det(\partial\overline{\partial}u) \geq (ske^{kr})^n \det((r_{i\bar{j}} + kr_i r_{\bar{j}})) \geq (sk\alpha e^{kr})^n \left(1 + \frac{k}{\alpha}|Dr|^2\right) \tag{2.11}$$

and we have

$$\begin{aligned} |D\underline{u}| &\leq \max|D\varphi| + ske^{kr}|Dr| \\ |D\underline{u}|^n &\leq C_1 + C_2(ske^{kr})^n|Dr|^n \end{aligned}$$

Since $\underline{u}$ is pluri-subharmonic, it attains its maximum over $\overline{\Omega}$ on $\partial\Omega$, so $\underline{u} \leq m := \max\limits_{\partial\Omega}\varphi$, consequently

$$f(z, \underline{u}, D\underline{u}) \leq C(m)(1 + |D\underline{u}|^n)$$

Choosing k such that $C(m)C_2|Dr|^n \leq k\alpha^{n-1}|Dr|^2$, for $u \leq m$ we obtain

$$\begin{aligned} f(z, \underline{u}, D\underline{u}) &\leq C(m)(1 + |D\underline{u}|^n) \\ &\leq C(m)(1 + C_1 + C_2(ske^{kr})^n|Dr|^n) \\ &\leq C(m)(1 + C_1) + (ske^{kr})^n k\alpha^{n-1}|Dr|^2 \\ &\leq (sk\alpha e^{kr})^n\left(\frac{C(m)(1+C_1)}{(sk\alpha e^{kr})^n} + \frac{k}{\alpha}|Dr|^2\right) \end{aligned}$$

It suffices to choose $s \geq \frac{(C(m)(1+C_1))^{1/n}}{k\alpha e^{kr}}$ and use (2.11) to obtain

$$f(z, \underline{u}, D\underline{u}) \leq (sk\alpha e^{kr})^n\left(1 + \frac{k}{\alpha}|Dr|^2\right) \leq \det(\partial\overline{\partial}u)$$

and conclude the proof. □

Let h be an harmonic function which equals φ on $\partial\Omega$, then,

$$\frac{1}{4n}\Delta u \geq \left(\det(\partial\overline{\partial}u)\right)^{1/n} = f^{1/n} \geq 0 = \Delta h, \quad u = h \text{ on } \partial\Omega$$

so, the maximum principle [12, Corollary 3.2] implies $u \le h$. Analogously we obtain $\underline{u} \le u$. Hence

$$|u|_{L^\infty(\overline{\Omega})} \le \sup_{z\in\overline{\Omega}} \max\{|h(z)|, |\underline{u}(z)|\}$$

Write

$$Du(z) = D^\nu u(z) + D^\tau u(z)$$

where ν denotes the inward normal direction to $\partial\Omega$, $D^\nu u(z)$ and $D^\tau u(z)$ denote the normal and the tangent component of Du to $\partial\Omega$ respectively. Since it is clear that $D^\tau u = D^\tau \varphi$ we just need to find an estimate for the normal component $D^\nu u = \langle Du, \nu\rangle$. By the above considerations $\underline{u} \le u \le h$ in Ω and $\underline{u} = u = h = \varphi$ on $\partial\Omega$, so we conclude

$$D^\nu \underline{u}(z) \le D^\nu u(z) \le D^\nu h(z) \quad z \in \partial\Omega$$

This implies $|Du|$ is bounded on $\partial\Omega$. Now, we shall show that $|Du|_{L^\infty}$ in the interior is controlled, via maximum principle, by $|Du|_{L^\infty}$ on the boundary. It is convenient to rewrite (2.2) in the form

$$\widetilde{F}(z, u, Du, \partial\overline{\partial}u) := \log\det(\partial\overline{\partial}u) - \log f(z, u, Du) \tag{2.12}$$

and consider the equation $\widetilde{F} = 0$. We denote with $F^{i\bar{j}} = F_{u_{i\bar{j}}} = (u^{i\bar{j}})$ the inverse matrix of the matrix $\partial\overline{\partial}u$ and

$$\widetilde{L} := \sum_{i,j=1}^n u^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \tag{2.13}$$

Notice that $\widetilde{L}$ is elliptic if u is strictly pluri-subharmonic. If G is any first order constant coefficients operator of the form

$$S = \sum_{k=1}^n a_k \partial_k, \quad \text{with } \sum_{k=1}^n a_k^2 = 1 \tag{2.14}$$

we have

$$\begin{aligned}
S\log\det(\partial\overline{\partial}u) &= S\log f(z, u, Du) \\
\sum_{k=1}^n \sum_{i,j=1}^n u^{i\bar{j}} \partial_{i\bar{j}} a_k u_k &= \sum_{k=1}^n \frac{1}{f}\left(a_k f_k + f_u a_k u_k + \sum_{j=1}^n f_{p_i}\partial_i(a_k u_k) + \sum_{j=1}^n f_{p_{\bar{i}}}\partial_{\bar{i}}(a_k u_k)\right) \\
\widetilde{L}Su &= \left(\frac{f_u}{f} + \sum_{j=1}^n \frac{f_{p_i}}{f}\partial_i + \sum_{j=1}^n \frac{f_{p_{\bar{i}}}}{f}\partial_{\bar{i}}\right) Su + \frac{Sf}{f}
\end{aligned}$$

where

$$p = (p_1, \dots, p_n, p_{\overline{1}}, \dots, p_{\overline{n}}) \quad \text{corresponds to} \quad Du = (u_1, \dots, u_n, u_{\overline{1}}, \dots, u_{\overline{n}})$$

Hence, defining the following second order elliptic operator

$$L := \widetilde{L} - \sum_{j=1}^{n} \frac{f_{p_i}}{f} \partial_i - \sum_{j=1}^{n} \frac{f_{p_{\overline{i}}}}{f} \partial_{\overline{i}} - \frac{f_u}{f} \tag{2.15}$$

we have $L(Su) = \frac{Sf}{f}$. Consider the function

$$w := \pm Su + e^{\lambda |z|^2} \quad \text{for } \lambda \geq 1$$

and suppose the origin $0 \notin \overline{\Omega}$, we want to fix $\lambda > 0$ such that $Lw \geq 0$. With the aid of (2.5) we compute

$$\begin{aligned} J := Lw &= \pm \left(\frac{Sf}{f} \right)(z, u, Du) + \\ &+ e^{\lambda |z|^2} \left(\left(\sum_{i,j=1}^{n} u^{i\overline{j}} \overline{z}_i z_j \right) \lambda^2 + \left(\sum_{i}^{n} u^{i\overline{i}} - \frac{f_{p_i}}{f} \overline{z}_i - \frac{f_{\overline{p}_i}}{f} z_i \right) \lambda - \frac{f_u}{f} \right) \\ &\geq -Af^{-1/n} + e^{\lambda |z|^2} \left(\lambda^2 u^{i\overline{j}} \overline{z}_i z_j + \lambda \mathrm{Tr}((u^{i\overline{i}})) - B\lambda f^{-1/n} \right) \end{aligned}$$

where A and B are positive constants. Denoting the eigenvalues of $(u^{i\overline{j}})$ with $\lambda_1 \leq \cdots \leq \lambda_n$ we have $f^{-1} = \prod \lambda_k$ and $\mathrm{Tr}((u^{i\overline{j}})) = \sum \lambda_k$. In order to show that $J \geq 0$ we distinguish two cases:

Case i) $\lambda_1 \leq (2B)^{1-n} f^{-1/n}$. This implies, $\lambda_n \geq 2Bf^{-1/n}$ and consequently we have $\mathrm{Tr}((u^{i\overline{j}})) - Bf^{-1/n} \geq \lambda_n - Bf^{-1/n} \geq Bf^{-1/n}$. Therefore, choosing λ large enough

$$J \geq e^{\lambda |z|^2} B f^{n-1} - A f^{-1/n} \geq 0$$

Case ii) If $\lambda_1 \geq (2B)^{1-n} f^{-1/n}$, for λ sufficiently large

$$J \geq -Af^{n-1} + e^{\lambda |z|^2} B f^{-1/n} (\lambda^2 (2B)^{1-n} |z|^2 - B\lambda) \geq 0$$

In both cases, for λ large we have $Lw \geq 0$ and $-\frac{f_u}{f} \leq 0$, thus we can apply the maximum principle [12, Corollary 3.2] obtaining $\max_{\Omega} w \leq \max_{\partial\Omega} w^+$ thereby

$$\max_{\Omega} |Su| \leq \max_{\partial\Omega} |Su| + C$$

Since we have already shown that $|Du|_{L^\infty(\partial\Omega)} < C$ we conclude that $|Du|_{L^\infty(\overline{\Omega})}$ is bounded.

2.3 Estimates for second derivatives on the boundary

In this section we shall establish a priori estimates for second derivatives of u at any boundary point. We consider the easier case where $f = f(z, u(z))$ does not depend on Du (for a detailed proof of the more general case $f = f(z, u(z), Du(z))$ we refer the reader to [7, pages 218-220]). It is convenient to use the following notation:

$$(t_1, t_2, \dots, t_{2n-3}, t_{2n-2}, t_{2n-1}, t_{2n}) = (x_1, y_2, \dots, x_{n-1}, y_{n-1}, y_n, x_n)$$
$$t = y_n, \qquad t' = (t_1, \dots, t_{2n-1})$$

Let $P \in \partial\Omega$, and r be a defining function for Ω. Choose coordinates $z_1, \dots, z_n$ with origin at P such that $r_{z_\alpha}(0) = 0$ for $\alpha < n$ and $r_{y_n}(0) = 0$,$r_{x_n}(0) = -1$. Moreover we can suppose r to be strictly pluri-subharmonic in a neighbourhood of 0. For r near 0 we can find a function σ such that

$$u = \varphi + \sigma r$$

then

$$u_{x_n}(0) = \varphi(x_n)(0) - \sigma(0)$$

Since $|Du|_{L^\infty(\overline{\Omega})}$ is bounded, the equality above implies $|\sigma(0)| \le C$ so that, for $i, j = 1, \dots, 2n-1$,

$$u_{t_i t_j}(0) = \varphi_{t_i t_j}(0) + \sigma(0) r_{t_i t_j}$$

hence

$$|u_{t_i t_j}(0)| \le C \tag{2.16}$$

Next, we establish the estimate

$$|u_{t_i x_n}(0)| \le C \quad \text{for } i = 1, \dots, 2n-1 \tag{2.17}$$

To this end we introduce new coordinates.

$$z'_n = z_n - \sum_{i,j=1}^{n} a_{ij} z_i z_j$$
$$z'_k = z_k \quad \text{for } k \le n-1$$

where a_{ij} are constants which appear when writing the Taylor expansion of r in 0 up to second order

$$r = \mathrm{Re}\left(-z_n + \sum_{i,j=1}^{n} a_{ij} z_i z_j\right) + \sum b_{i\bar{j}} z_i z_{\bar{j}} + O(|z|^3)$$

using new coordinates we can rewrite the previous equation as follows

$$r = -\mathrm{Re}(z'_n) + \sum_{i,j=1}^{n} c_{i\bar{j}} z'_i z'_j + O(|z|^3) \tag{2.18}$$

Remark (2.5) shows that at points where the Jacobian of the transformation does not vanish, a function is pluri-subharmonic with respect to the z_j if and only if it is pluri-subharmonic with respect z'_j. Moreover it shows that the expression $\det(\partial\overline{\partial}_{z'_i \bar{z}'_j} u)$ is obtained multiplying $\det(\partial\overline{\partial}_{z_i \bar{z}_j})$ by the absolute squared value of the Jacobian. Thus, near 0, we can assume r to be of the form (2.18) and drop the coordinates z'_j. Since r is strictly pluri-subharmonic, the matrix $(c_{i\bar{j}})$ is positive definite. Now, let

$$\Omega_\varepsilon = \{z \text{ in a neighbourhood of the origin}; r(z) \leq 0,\ x_n \leq \varepsilon\} \tag{2.19}$$

define T_i near 0 by:

$$T_i = \frac{\partial}{\partial_{t_i}} - \frac{r_{t_i}}{r_{x_n}} \frac{\partial}{\partial_{x_n}} \quad \text{for } i = 1, \dots, 2n-1 \tag{2.20}$$

and

$$w = \pm T_i(u - \varphi) + (u_t - \varphi_t)^2 - Ax_n + B|z|^2 \tag{2.21}$$

Notice that $T_i(u - \varphi) = 0$ on $\partial\Omega$. Indeed here we can write $x_n = \rho(t')$ and $0 \equiv r(t', \rho(t')) = \rho(t') - x_n$ thus, differentiating with respect to t_i we find $0 = r_{t_i} - \rho_{t_i} r_{x_n}$ from which we get

$$\rho_{t_i} = \frac{r_{t_i}}{r_{x_n}}$$

therefore

$$T_i(u - \varphi)_{(t', \rho(t'))} = u_{t_i} - \varphi_{t_i} + \rho_{t_i} \partial_{x_n}(u - \varphi) - \frac{r_{t_i}}{r_{x_n}} \partial_{x_n}(u - \varphi) = 0$$

We shall show that, for ε sufficiently small and A, B large enough

$$\widetilde{L}(w) > 0 \quad \text{in } \Omega_\varepsilon \tag{2.22a}$$

$$w \leq 0 \quad \text{on } \partial\Omega_\varepsilon \tag{2.22b}$$

Thus, in view of the maximum principle [12, Corollary 3.2] $w \leq 0$ in Ω_ε. First we show that (2.22a) holds for B sufficiently large. Set $a = -\frac{r_{t_i}}{r_{x_n}}$,

$y_n = t$. Notice that $\widetilde{L}(u_{x_k}) = \sum_{i,j=1}^n u^{i\bar{j}} u_{i\bar{j}x_k} = \partial_{x_k} \log\det(\partial\overline{\partial}u)$ where $\widetilde{L}$ is the operator defined in (2.13). Applying $\widetilde{L}$ to $T_i u$ we obtain

$$\begin{aligned}\widetilde{L}(T_i u) &= \widetilde{L}(u_{t_i}) + a\widetilde{L}(u_{x_n}) + \sum_{i,j=1}^n u^{i\bar{j}} a_i u_{x_n\bar{j}} + \sum_{i,j=1}^n u^{i\bar{j}} a_{\bar{j}} u_{x_n i} + \sum_{i,j=1}^n u^{i\bar{j}} a_{i\bar{j}} u_{x_n} \\ &= T_i(\log f(z,u,Du)) + \sum_{i,j=1}^n u^{i\bar{j}} a_i u_{x_n\bar{j}} + \sum_{i,j=1}^n u^{i\bar{j}} a_{\bar{j}} u_{x_n i} + \sum_{i,j=1}^n u^{i\bar{j}} a_{i\bar{j}} u_{x_n}\end{aligned}$$

We now estimate each term on the right hand side. By (2.5)

$$|T_i(\log f(z,u))| \le Cf^{-1/n}$$

We have

$$\sum_{j=1}^n u^{i\bar{j}} u_{n\bar{j}} = \delta_{in} \qquad \text{and} \qquad \partial_{x_n} = 2\partial_{z_n} + \sqrt{-1}\partial_t$$

consequently $u_{x_n\bar{j}} = 2u_{n\bar{j}} + \sqrt{-1}u_{t\bar{j}}$ and

$$\sum_{i,j=1}^n u^{i\bar{j}} a_i u_{x_n\bar{j}} = 2a_n + \sqrt{-1}\sum_{i,j=1}^n u^{i\bar{j}} a_i u_{t\bar{j}}$$

Similarly $\partial_{x_n} = 2\partial_{\overline{z}_n} - \sqrt{-1}\partial_t$, $u_{x_n i} = 2u_{\overline{n}i} - \sqrt{-1}u_{ti}$ and

$$\sum_{i,j=1}^n u^{i\bar{j}} a_{\bar{j}} u_{x_n i} = 2a_{\overline{n}} - \sqrt{-1}\sum_{i,j=1}^n u^{i\bar{j}} a_{\bar{j}} u_{ti}$$

Then applying Holder's inequality, we get

$$2a_n + \sqrt{-1}\sum_{i,j=1}^n u^{i\bar{j}} a_i u_{t\bar{j}} = O\left(1 + \left(\sum_{i=1}^n u^{i\bar{i}}\right)^{1/2}\left(\sum_{i,j=1}^n u^{i\bar{j}} u_{it} u_{t\bar{j}}\right)^{1/2}\right)$$

$$2a_{\overline{n}} - \sqrt{-1}\sum_{i,j=1}^n u^{i\bar{j}} a_{\bar{j}} u_{ti} = O\left(1 + \left(\sum_{i=1}^n u^{i\bar{i}}\right)^{1/2}\left(\sum_{i,j=1}^n u^{i\bar{j}} u_{t\bar{j}} u_{ti}\right)^{1/2}\right)$$

Moreover

$$\sum_{i,j=1}^n u^{i\bar{j}} a_{i\bar{j}} u_{x_n} = O\left(\sum_{i=1}^n u^{i\bar{i}}\right)$$

Thus, using (2.5), the inequality $\frac{1}{n}\sum_{i=1}^n u^{i\bar{i}} \ge f^{-1/n}$, and estimates above, we have

$$\pm\widetilde{L}(T_i u) \ge -cf^{-1/n} - \sum_{i,j=1}^n u^{i\bar{j}} u_{it} u_{\bar{j}t} \tag{2.23}$$

and

$$\pm\widetilde{L}T_i(u-\varphi) \geq -cf^{-1/n} - \sum_{i,j=1}^{n} u^{i\bar{j}}(u_{it}-\varphi_{it})(u_{\bar{j}t}-\varphi_{\bar{j}t}) \tag{2.24}$$

Further, by (2.5)

$$\begin{aligned}\widetilde{L}(u_t-\varphi_t)^2 &= 2(u_t-\varphi_t)(\widetilde{L}u_t-\widetilde{L}\varphi_t) + 2\sum_{i,j=1}^{n} u^{i\bar{j}}(u_{it}-\varphi_{it})(u_{\bar{j}t}-\varphi_{\bar{j}t})\\ &= 2(u_t-\varphi_t)(\partial_t \log\det(\partial\overline{\partial}u) - \widetilde{L}\varphi_t) + 2\sum_{i,j=1}^{n} u^{i\bar{j}}(u_{it}-\varphi_{it})(u_{\bar{j}t}-\varphi_{\bar{j}t})\\ &= 2(u_t-\varphi_t)(\partial_t \log f - \widetilde{L}\varphi_t) + 2\sum_{i,j=1}^{n} u^{i\bar{j}}(u_{it}-\varphi_{it})(u_{\bar{j}t}-\varphi_{\bar{j}t})\\ &\geq -c_1 f^{-1/n} - c_2\sum_{i=1}^{n} u^{i\bar{i}} + 2\sum_{i,j=1}^{n} u^{i\bar{j}}(u_{it}-\varphi_{it})(u_{\bar{j}t}-\varphi_{\bar{j}t})\end{aligned}$$

and

$$\widetilde{L}(-Ax_n + B|z|^2) = B\sum_{i=1}^{n} u^{i\bar{i}}$$

So that

$$\widetilde{L}w \geq (-c-c_1)f^{-1/n} + (B-c_2)\sum_{i=1}^{n} u^{i\bar{i}} + \sum_{i,j=1}^{n} u^{i\bar{j}}(u_{it}-\varphi_{it})(u_{\bar{j}t}-\varphi_{\bar{j}t})$$

Since the matrix $(u^{i\bar{j}})$ is positive definite, the last term of the right hand side is positive, thus

$$\widetilde{L}w \geq (-c-c_1)f^{-1/n} + (B-c_2)\sum_{i=1}^{n} u^{i\bar{i}} \geq 0$$

for B large proving (2.22a).

We now prove that (2.22b) holds for A large. On $\partial\Omega\cap\partial\Omega_\varepsilon$ we have $r=0$, so, by (2.18) we infer $x_n = \rho(t')$ and

$$\rho(t') = +\sum_{i,j=1}^{2n-1} b_{i\bar{j}} t_i t_j + O(|t'|^3)$$

where $(b_{i,j})$ is a $(2n-1)\times(2n-1)$ positive definite. Hence $x_n > a|z|^2$ with $a > 0$. Moreover

$$u(t', \rho(t')) = \varphi(t', \rho(t'))$$

so that

$$|u_{t_i} - \varphi_{t_i}| \le c|t'| \le c\rho^{1/2}$$

Since $\pm T_i(u-\varphi) = 0$ on $\partial\Omega$, on $\partial\Omega \cap \partial\Omega_\varepsilon$ it holds

$$w \le c\rho - Ax_n + B|z|^2 \le 0 \quad \text{for } A \text{ large}$$

On the other hand on $\Omega \cap \Omega_\varepsilon$, $T_i(u-\varphi)$, $(u_t - \varphi_t)^2$ are bounded and $x_n = \varepsilon$, so $w \le 0$ for A large. Thus we have proved (2.22b). Via maximum principle we find, $w \le 0$ in Ω_ε. Furthermore $w(0) = (u_t - \varphi_t)^2 \ge 0$, so that

$$w_{x_n}(0) = \lim_{h\to 0^+} \frac{w\big(h(0,\dots,0,x_n)\big) - w(0)}{h} \le 0$$

from which it follows the desired estimate $|u_{t_i x_n}| \le C$.

We point out that in case $f = f(z, u(z), Du(z))$ the proof is similar to the one presented above but it requires to consider the function

$$w = \pm T_i(u-\varphi) + (u_t - \varphi_t)^2 + A\left(\sum_{i,j=1}^{n}(c_{i\bar{j}} - \mu\delta_{ij} z_i \overline{z_j}) + 2Mx_n^2 - x_n\right)$$

where $c_{i\bar{j}}$ are defined in (2.18) and the operator L defined in (2.15)

To conclude the estimates of second derivatives on the boundary we have only to show that

$$|u_{x_n x_n}(0)| \le C \tag{2.25}$$

Since we can compute

$$u_{n\bar{n}} = \frac{1}{4}(u_{x_n x_n}\sqrt{-1}u_{x_n y_n} - \sqrt{-1}u_{y_n x_n} + u_{y_n y_n})$$

and we already have estimates for $|u_{t_i t_j}(0)|, |u_{t_i x_n}(0)|$ when $1 \le i, j \le 2n-1$ then (2.25) will immediately follow from

$$|u_{n\bar{n}}(0)| \le C \tag{2.26}$$

Solving equation $\det(\partial\overline{\partial}u) = f$ with respect to $u_{n\bar{n}}(0)$ we find

$$u_{n\bar{n}}(0) = \frac{f - \sum\limits_{\substack{\sigma(n)\neq n \\ \sigma\in S(n)}} u_{\sigma(1)\bar{1}}(0)\dots u_{\sigma(n)\bar{n}}(0)}{\sum\limits_{\sigma\in S(n-1)} u_{\sigma(1)\bar{1}}(0)\dots u_{\sigma(n-1)\overline{n-1}}(0)} \tag{2.27}$$

where $S(m)$ is the set of all the possible permutations of $\{1,\ldots,m\}$, hence, it suffices to prove

$$\left(u_{\alpha\overline{\beta}}(0)\right)_{\alpha,\beta\leq n-1} \geq c_1 \mathrm{Id}_{n-1}, \quad \text{with } c_1 > 0 \tag{2.28}$$

to obtain (2.26). After subtraction of the linear function $\sum_{j=1}^{2n-1}\varphi_{t_j}(0)t_j$ we may assume $\varphi_{t_j(0)} = 0$ for $j = 1,\ldots,2n-1$. To prove (2.28) it suffices to show that

$$\sum_{\alpha,\beta<n} \xi_\alpha \overline{\xi}_\beta u_{z_\alpha z_\beta}(0) \geq c_1|\xi|^2 \quad \text{for every } \xi \in \mathbb{C}^n,\ |\xi| \leq 1$$

There is no loss of generality in assuming $\xi = (1,0,\ldots,0)$, we wish then to prove

$$u_{1\overline{1}}(0) \geq c_2 > 0 \tag{2.29}$$

Let $\tilde{u} = u - \lambda x_n$ with λ chosen so that at 0

$$\Big(\frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2}\Big)\tilde{u}(t',\rho(t'))|_{t'=0} = 0$$

recalling that $\rho_{t_j}(0) = 0$ for every $j = 1,\ldots,2n-1$ the equation above is equivalent to

$$0 = u_{1\overline{1}}(0) + (u_{x_n} - \lambda)\rho_{1\overline{1}}(0) = u_{1\overline{1}}(0) + \tilde{u}(0)\rho_{1\overline{1}} \tag{2.30}$$

We expand $\tilde{u}|_{\partial\Omega}$ in a Taylor series, in $t_1,\ldots,t_{2n-1}$, and, using the fact that any real homogeneous cubic in (t_1,t_2) can be uniquely decomposed in $\mathrm{Re}\big(\alpha(t_1+it_2)^3 + \beta(t_1-it_2)(t_1+it_2)^2\big) = \mathrm{Re}(\alpha z_1^3 + \beta z_1|z_1|^2)$ we find

$$\tilde{u}|_{\partial\Omega} = \mathrm{Re}\left(\sum_{j=2}^{n-1} a_j z_1 \overline{z}_j\right) + \mathrm{Re}(a z_1 t) + \\ + \mathrm{Re}\big(p(z_1,\ldots,z_{n_1}) + \beta z_1|z_1|^2\big) + O(t_3^3 + \cdots + t_{2n-1}^2)$$

where p is a holomorphic cubic polynomial. Since

$$\rho(t') = \sum_{i,j=1}^{2n-1} b_{ij}t_i t_j + O(|t'|^3) \quad \text{on } \partial\Omega \cap \Omega_\varepsilon \tag{2.31}$$

by changing a_j, a and p appropriately we may obtain the inequality

$$\tilde{u}|_{\partial\Omega} \leq \mathrm{Re}p(z) + \mathrm{Re}\left(\sum_{j=2}^{n} a_j z_1 \overline{z}_j + C\sum_{j=2}^{n}|z_j|^2\right) \tag{2.32}$$

Define $\tilde{\tilde{u}} := \tilde{u} - \mathrm{Re}\big(p(z)\big)$. Since $p(z)$ is a holomorphic function, $\mathrm{Re}\big(p(z)\big)$ is harmonic so that

$$\det(\partial\overline{\partial}\tilde{\tilde{u}}) = \det(\partial\overline{\partial}\tilde{u}) = \det(\partial\overline{\partial}u) = f(z, u(z))$$

Consider the barrier function

$$h := -\delta_0 x_n + \delta_1 |z|^2 + \frac{1}{B}\sum_{j=2}^{n} |a_j z_1 + B z_j|^2$$

we claim that with a suitable choice of the positive constants δ_0, δ_1, B we have $h \geq \tilde{\tilde{u}}$ in Ω_ε. Let Ω_ε be defined by (2.19). Choose $0 < \delta_0 < \lambda$ small and ε such that

$$u - \lambda x_n < -\delta_0 \varepsilon \quad \text{on } \partial\Omega_\varepsilon \tag{2.33}$$

and

$$\delta_1 |z|^2 \geq \delta_0 x_n \quad \text{on } \partial\Omega_\varepsilon \cap \partial\Omega \tag{2.34}$$

Then choose B such that

$$-\mathrm{Re}(p) \leq \delta_1 |z|^2 + \frac{1}{B}\sum_{j=2}^{n} |a_j z_1 + B z_j|^2 \quad \text{on } \partial\Omega_\varepsilon \tag{2.35}$$

Now, on $\partial\Omega_\varepsilon \cap \partial\Omega$ we have

$$\begin{aligned} h &\overset{(2.35)}{\geq} -\delta_0\varepsilon - \mathrm{Re}(p) \\ &\overset{(2.33)}{\geq} u - \lambda x_n - \mathrm{Re}(p) \\ &= \tilde{\tilde{u}} \end{aligned}$$

Similarly on $\partial\Omega_\varepsilon \cap \Omega$ we have $h \geq \tilde{\tilde{u}}$. Hence the claim is proved. Now, recall that $f(z, u) > \delta > 0$ on $\overline{\Omega}$, moreover the function h is pluri-subharmonic and the lowest eigenvalue of $\partial\overline{\partial}h$ is δ_1 while the other eigenvalues are bounded independently of δ_1. Hence we can choose δ_1 so small that

$$\det(\partial\overline{\partial}h) \leq \delta \quad \text{in } \Omega_\varepsilon$$

By the comparison principle Lemma 2.6 we have

$$h \geq \tilde{\tilde{u}} \quad \text{in } \Omega_\varepsilon$$

and hence

$$\tilde{\tilde{u}}_{x_n}(0) \leq h_{x_n}(0) = -\delta_0$$

thus, from (2.30) we find the desired inequality (2.29).

2.4 Estimates for second derivatives in the interior

In this section we use the maximum principle to derive estimates for second derivatives of u in Ω. We consider the easier case where $f = f(z, u(z))$ does not depend on Du (for a detailed proof of the more general case $f = f(z, u(z), Du(z))$ we refer the reader to [7, pages 228-230]).

Remark 2.10.
$\widetilde{F}(z, u(z), Du(z), \partial\overline{\partial}u) = \log\det(\partial\overline{\partial}u) - \log f(z, u(z), Du(z))$ *is a concave function of the second derivatives* $u_{i\bar{j}}$ *for strictly pluri-subharmonic functions* u.

Proof. Indeed we have

$$\frac{\partial^2 \widetilde{F}}{\partial u_{i\bar{j}} \partial u_{k\bar{l}}} = \frac{\partial}{\partial u_{k\bar{l}}} u^{i\bar{j}} = -u^{i\bar{k}} u^{l\bar{j}}$$

and the corresponding quadratic form is negative definite. Let us prove the equality $\frac{\partial}{\partial u_{k\bar{l}}} u^{i\bar{j}} = -u^{i\bar{k}} u^{l\bar{j}}$. Consider the identity

$$\sum_{s=1}^{n} u^{s\bar{i}} u_{s\bar{t}} = \delta_{i\bar{t}}$$

Differentiating with respect to $u_{k\bar{l}}$ we find

$$\sum_{s=1}^{n} \frac{\partial u^{i\bar{s}}}{\partial u_{k\bar{l}}} u_{s\bar{t}} + \sum_{s=1}^{n} u^{i\bar{s}} \frac{\partial u_{s\bar{t}}}{\partial u_{k\bar{l}}} = 0$$

i.e.

$$\sum_{s=1}^{n} \frac{\partial u^{i\bar{s}}}{\partial u_{k\bar{l}}} u_{s\bar{t}} = -u^{i\bar{k}} \delta_{l,t} \quad \text{for every } i, k, t, l = 1 \ldots, n$$

Multiplying by $u^{t\bar{j}}$ and summing over t we find

$$\sum_{s,t=1}^{n} \frac{\partial u^{i\bar{s}}}{\partial u_{k\bar{l}}} u_{s\bar{t}} u^{t\bar{j}} = -u^{i\bar{k}} \sum_{t=1}^{n} \delta_{l,t} u^{t\bar{j}}$$

$$\sum_{s=1}^{n} \frac{\partial u^{i\bar{s}}}{\partial u_{k\bar{l}}} \delta_{s,\bar{j}} = -u^{i\bar{k}} u^{l\bar{j}}$$

from which we deduce the desired equality. □

Consider the real constant coefficients operator S defined as follows

$$S := \sum_{i=1}^{2n} a_{t_i}\partial_{t_i}, \quad \text{with} \sum_{i=1}^{2n} a_{t_i}^2 = 1$$

Similarly to Section 2.2 we have

$$S(\widetilde{F}) = 0$$
$$S(\log\det(\partial\overline{\partial}u) - \log f(z, u(z))) = 0$$
$$\sum_{k=1}^{n}\sum_{i,j=1}^{n} \widetilde{F}_{u_{i\bar{j}}}\partial_{i\bar{j}}a_{t_k}u_{t_k} - \sum_{k=1}^{n}\frac{a_{t_k}f_{t_k} + f_u a_{t_k}u_{t_k}}{f} = 0$$

applying S again we get

$$\sum_{i,j,k,l=1}^{n} \widetilde{F}_{u_{i\bar{j}}}\partial_{i\bar{j}}a_{t_k}a_{t_l}u_{t_kt_l} + \sum_{i,j,k,l}^{n} F_{u_{i\bar{j}}u_{t_kt_l}}\partial_{i\bar{j}}a_{t_k}u_{t_k} - \frac{1}{f^2}\sum_{k=1}^{n}\Big(fS^2f - (Sf)^2 - f_uSfSu\Big) +$$
$$-\frac{1}{f^2}\sum_{k=1}^{n}\Big(fSf_uSu + ff_{uu}(Su)^2 - f_uSfSu - (f_uSu)^2 - ff_uS^2u\Big) = 0$$

thus, by concavity of $\widetilde{F}$

$$(\widetilde{L} - \frac{f_u}{f})(S^2u) \geq -\frac{1}{f^2}\sum_{k=1}^{n}\Big(fS^2f - (Sf)^2 - f_uSfSu + fSf_uSu +$$
$$+ ff_{uu}(Su)^2 - f_uSfSu - (f_uSu)^2\Big)$$

Moreover, defining $L := \widetilde{L} - \frac{f_u}{f}$ and using (2.5), (2.6) and bounds for Su obtained in Section 2.2 we get

$$L(S^2u) \geq -Cf^{-1/n} \tag{2.36}$$

Consider now the function

$$w = S^2u + e^{\lambda|z|^2}$$

using, (2.36), (2.5) and (2.6) it is possible to fix $\lambda > 0$ so that $Lw \geq 0$ in Ω. Hence, by the maximum principle and estimates of second derivatives on $\partial\Omega$, we find an upper bound for every S^2u

$$\max_{\overline{\Omega}} S^2u \leq \max_{\partial\Omega} S^2u + C$$

In particular if we consider $S_k = \partial_{t_k}$ we find $u_{t_k t_k} = S_k^2 u \leq C$ (if indices go from 1 to $2n$ we denote them with k, l otherwise if they go from 1 to n we denote them by i, j). The lower bound $u_{x_i x_i} + u_{y_i y_i} \geq 0$ implies $u_{t_k t_k} \geq -C$ so that

$$|u_{t_k t_k}| \leq C \quad \text{for every } k = 1, \dots, 2n$$

Then defining $E_i = \frac{1}{\sqrt{2}}(\partial_{x_i} \pm \partial_{y_i})$ we obtain

$$u_{x_i y_i} \leq \frac{1}{2}(u_{x_i x_i} + u_{y_i y_i} \pm 2u_{x_i y_i}) = E_i^2 u \leq C$$

from which we get

$$|u_{x_i y_i}| \leq C \quad \text{for every } i = 1, \dots, n$$

Finally consider $H = \frac{1}{\sqrt{2}}(\partial_{t_k} \pm \partial_{t_l})$ to find

$$\frac{1}{2}(u_{t_k t_k} + u_{t_l t_l} \pm 2u_{t_k t_l}) = H^2 u \leq C$$

hence, $\pm u_{t_k t_l} \leq 3C$ from which we conclude

$$|u_{t_k t_l}| \leq C \quad \text{for every } k, l = 1, \dots, 2n$$

as desired.

2.5 Proof of Theorem 2.8

In this section we finally prove Theorem 2.8 with the help of the following two theorems

Theorem 2.11.
Suppose $u \in C^\infty(\overline{\Omega})$ is a pluri-subharmonic solution of (2.1) *and let $\underline{u}$ be as in Lemma 2.9. Suppose moreover $\underline{u} \leq u$, then there exists a positive constant C such that*

$$|u|_2 \leq C \tag{2.37}$$

Theorem 2.11 follows directly from estimates derived in previous sections.

Theorem 2.12.
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let u be a smooth solution in $\overline{\Omega}$ of

$$\begin{cases} G(z, u, Du, D^2u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \Omega \end{cases} \tag{2.38}$$

with G elliptic in the sense of [12]. Suppose moreover to have a priori estimates $|u|_{C^2(\overline{\Omega})} \leq C_1$ and G is a concave function of the second derivatives $u_{i\bar{j}}$. Then u also satisfies the a priori estimate in $\overline{\Omega}$

$$|u|_{C^{2,\alpha}(\overline{\Omega})} \leq K \quad \textit{for some positive } \alpha < 1 \tag{2.39}$$

and K depends only on Ω, C_1, G and $|\varphi|_{C^4_{(\overline{\Omega})}}$

Proof. In the paper [7] the proof of this result makes use of estimates (2.37) and the following inequality

$$|u_{ij}(x) - u_{ij}(y)| \leq \frac{K}{1 - |\log|x-y||} \quad \text{for } x \in \partial\Omega,\ y \in \overline{\Omega} \tag{2.40}$$

We consider (2.40) valid and refer the reader to [7, pages 231-235] for a detailed proof.

We wish to prove that

$$|u_{ij}(x) - u_{ij}(y)| \leq K|x-y|^\alpha \quad \text{for } x, y \in \overline{\Omega} \text{ for some } 0 < \alpha < 1 \tag{2.41}$$

L.C. Evans in his papers [10], [11] and N. Trudinger in [21], [22] established strictly interior $C^{2,\alpha}$ estimates of the form

$$|u|_{C^{2,\alpha}(\Omega')} \quad \text{for all } \Omega' \subset \overline{\Omega}' \subset\subset \Omega$$

Thanks to their results and the bounds $|u_{ij}| \leq C$ we may suppose both x and y to be close to some boundary point, e.g. the origin. Furthermore, by a local transformation of variables we may also suppose the boundary to be the hyperplane $x_n = 0$ near the origin (observe that inequality (2.40) still holds in the new variables with a new constant K). Now, any derivative u_α, for $\alpha < N$ satisfies the linearized equation

$$|Lu_\alpha| = |G_\alpha + G_u u_\alpha| \leq C$$

By (2.40), for z near 0, we have

$$|u_{ij}(z) - u_{ij}(0)| \leq \frac{K}{1 + |\log|z||} \tag{2.42}$$

this implies that the coefficients of L have small oscillations in a ball $B_R(0) \cap \overline{\Omega}$ provided R is sufficiently small. We remark now that to derive (2.39) it suffices to have a priori estimates $|u|_{C^2(\overline{\Omega})} \leq C_1$ and a weaker form of (2.40), that is: there exists a constant $\varepsilon > 0$, depending only on C_1 and on equation (2.38) such that if for some $\delta > 0$ it holds

$$\sum_{i,j=1}^{n} |u_{ij}(x) - u_{ij}(y)| \leq \varepsilon \quad \text{for } x, y \in \overline{\Omega},\ |x-y| < \delta$$

then the estimate (2.39) holds with K depending on δ, Ω, C, G and $|\varphi|_{C^4_{(\overline{\Omega})}}$. This estimate follows from the proof of the results in [20] and the L_p theory of [1]. This is, in fact, a local result. From these observations and (2.42) we deduce that the first derivatives of u_α, (i.e. $u_{\alpha i}$, $\alpha < n$, $1 \leq n$) satisfy a fixed Holder condition in $B_{R/2}(0) \cap \overline{\Omega}$ provided R is sufficiently small but fixed. Applying the mean value theorem $G(x, u(x), \ldots, D^2u(x)) - G(y, u(y), \ldots, D^2u(y))$ we estimate

$$|u_{nn}(x) - u_{nn}(y)| \leq C_2|x-y| + C_3 \sum_{\alpha=1}^{n-1} |u_{\alpha i}(x) - u_{\alpha i}(y)|$$

Hence (2.41) holds whenever $x, y \in B_{R/2}$ and the proof is completed. □

Notice that this result does not apply directly to (2.1) but to the equivalent Dirichlet Problem (recall notation used in (2.12))

$$\begin{cases} \widetilde{F} = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{2.43}$$

Indeed $\log\det(\partial\overline{\partial}u)$ is a concave function of the second derivatives for strictly pluri-subharmonic functions u

To prove Theorem 2.8 we use the continuity method. This technique consists in deforming the given PDE by a continuous map into a simpler one for which one already knows the existence of a solution. We present only the main steps of this method applied to the particular case of the complex Monge-Ampère equation. For a more general and straightforward proof we refer the reader to [12, Section 17.2]. Our notation will be coherent with the one used in [12], so that Theorem 2.8 will be an easy consequence of [12, Theorem 17.8].

Proof (Theorem 2.8). Let $\mathscr{B}_1 = C^{2,\alpha}(\overline{\Omega})$, $\mathscr{B}_2 = C^{0,\alpha}(\overline{\Omega})$, and suppose $u^0 \in \mathscr{B}_1$ is a classical pluri-subharmonic subsolution of $F = 0$ i.e

$$f_0 := \det(\partial\overline{\partial}u^0) \geq f(z, u^0, Du^0)$$

(for example we can choose u^0 as in Lemma 2.9). We would like to show that the Dirichlet problem

$$\begin{cases} \det(\partial\overline{\partial}u^t) = tf(z, u^t, Du^t) + (1-t)f_0 & \text{in } \Omega \\ u^t = \varphi \text{on } \partial\Omega \end{cases} \tag{2.44}$$

has a strictly pluri-subharmonic solution for every $t \in [0,1]$. In particular, for $t = 1$ we obtain a solution of (2.1).
Define

$$\mathscr{U} := \{u \in C^{2,\alpha}(\overline{\Omega}) : \ u \text{ is strictly pluri-subharmonic}\}$$
$$G := \mathscr{U} \times [0,1] \to \mathscr{B}_2, \qquad G[u,t] := \det(\partial\overline{\partial}u) - tf(z,u,Du) - (1-t)f_0(x)$$
$$E := \{u \in \mathscr{U} : \ u \text{ is a solution of (2.44) for some } t \in [0,1]\}$$
$$S := \{t \in [0,1] : \ G[u,t] = 0 \text{ for some } u \in \mathscr{U}\}$$

Notice that $\mathscr{U}$ is open. Since $G[u^0, 0] = \det(\partial\overline{\partial}u^0) - f_0 = 0$, we have $0 \in S$, $u^0 \in E$ so that S and E are non empty, moreover $F_u = -f_u < 0$ consequently, by the implicit function theorem [12, Theorem 17.6], we infer that S is open. Now we want to show that S is closed so that necessarily $S = [0,1]$ i.e. the Dirichlet problem (2.44) has a strictly pluri-subharmonic solution for every $t \in [0,1]$. The closure of S follows from the boundedness of E in $C^{2,\alpha}(\overline{\Omega})$ (proved in Theorem 2.12) and the fact that $\overline{E} \subseteq \mathscr{U}$. Let us prove the last assertion. Suppose $(u_j)_{j\in\mathbb{N}} \in C^{2,\alpha}(\overline{\Omega})$, $u_j \to u$ in $C^{2,\alpha}$. Since $\partial\overline{\partial}u = \lim_{j\to\infty} \partial\overline{\partial}u_j \geq 0$, u is pluri-subharmonic therefore $G[u,t] = \lim_{j\to\infty} G[u_j, t_j] = 0$ thus u is strictly pluri-subharmonic and $u \in E$. □

Chapter 3

A Pogorelov-Type Counterexample

In this chapter we show a new Pogorelov-type counterexample for the complex Monge-Ampère equation

$$MA(u) := -\det(\partial\overline{\partial}u) + f(z, u, Du) = 0 \tag{3.1}$$

where f is an arbitrary smooth positive function under some suitable structural assumptions. We show that for every given f there exist a small Euclidean ball $B_r \subset \mathbb{C}^n$, $n \geq 2$ and a pluri-subharmonic viscosity solution $u \in \mathrm{Lip}(B_r)$ of (3.1) such that $u \notin C^{1,\alpha}(\overline{B_r})$ for $\alpha > 1 - \frac{1}{n}$. We point out that recently a similar counterexample, for the equation $\det(\partial\overline{\partial}u) = 1$ has been proved in [5] and [15]. In order to show the existence of a non classical solution, we will use the comparison principle and the property of stability of viscosity solutions with respect to the uniform convergence which we obtained in Chapter 1. Indeed our Monge-Ampère operator MA is the operator F that we defined in Chapter 2 with a change of sign. So MA is degenerate elliptic on the set of strictly pluri-subharmonic functions by Remark 2.4. If f is monotone increasing with respect to u, MA is proper.

3.1 Existence of non classical solutions

Here we want to prove the following Theorem concerning the existence of non classical solutions for the complex Monge-Ampère equation.

Theorem 3.1.
Let $n \geq 2$, $\overline{B}_1 \subset \mathbb{C}^n$ and $f \in C^\infty(\overline{B}_1 \times \mathbb{R} \times \mathbb{R}^{2n})$ be a positive real valued function. Suppose moreover f satisfies the following structural assumptions:

(H1) $\dfrac{\partial f}{\partial u} > 0.$

(H2) There exists a positive constant C *such that*

$$|\partial f|, |\overline{\partial} f|, |f_u|, |\frac{\partial f}{\partial p_j}| \leq C f^{1-\frac{1}{n}} \quad \textit{for } j = 1 \ldots 2n$$

and for each $m \geq 0$ *there exists a constant* $C = C(m)$ *such that*

$$\begin{aligned} |\partial f|, |\overline{\partial} f|, |f_u|, |\partial f_{p_j}|, |\overline{\partial} f_{p_j}| |f_{up_j}| &\leq C f^{1-1/2n} \\ |\partial(\partial f)|, |\overline{\partial}(\overline{\partial} f)|, |\partial f_u|, |\overline{\partial} f_u|, |f_u u| &\leq C f^{1-1/n} \\ |f_{p_j}|, |f_{p_i p_j}| &\leq C f \end{aligned} \tag{3.2}$$

whenever $|u| + |p| \leq m$

Let $f_\infty : B_1 \times \mathbb{R} \times \mathbb{C}^n \to \mathbb{R}$, $f_\infty := \lim_{\lambda \to \infty} \frac{f(z, \lambda s, \lambda p)}{\lambda^n}$, *suppose*

(H3) f_∞ *exists and* $f_\infty < \left(1 - \frac{1}{n}\right)^n$

Then there exist a small Euclidean ball B_r *and a Lipschitz viscosity solution of*

$$-\det(\partial \overline{\partial} u) + f(z, u, Du) = 0$$

such that $u \notin C^1(B_r)$ *if* $n = 2$ *and* $u \notin C^{1,\beta}(B_r)$ *for every* $\beta > 1 - \frac{2}{n}$

We denote by $z = (z_1, z')$; $z' = (z_2, \ldots, z_n)$. For $0 \leq \sigma < 1$, in $\overline{B_1}$ we define

$$\begin{gathered} \omega_\sigma(z) := (1 + |z_1|^2)(\sigma + |z'|^2)^\alpha, \quad \alpha = 1 - \frac{1}{n} \\ \psi_\sigma(z) := K \omega_\sigma(z) \\ \phi_\sigma(z) := 2K(\sigma + |z'|^2)^\alpha \end{gathered}$$

where K is a positive constant to be determined.

The key idea of the proof of Theorem 3.1 is to find a non classical solution of $MA(u) = 0$ as the uniform limit of solutions u_σ of the Dirichlet problem $MA(u) = 0$ in B_r, $u = \phi_\sigma$ on ∂B_r, therefore it will be useful to prove the following:

Proposition 3.2.
Let f *be as in Theorem 3.1, then there exists* $0 < r < 1$ *such that the following Dirichlet problem*

$$MA(u) = 0 \quad \textit{in } B_r, \quad u = \phi_\sigma \quad \textit{on } \partial B_r \tag{3.3}$$

has a viscosity solution $u \in Lip(\overline{B_r})$ *such that*

$$\|u\|_{L^\infty(\overline{B}_r)} + \|u\|_{Lip(\overline{B_r})} \leq C \tag{3.4}$$

where C *depends only on* r, $\sup_{B_r} |D\phi_\sigma|$ *e* $\sup_{B_r} |\phi_\sigma|$

We prove it with the help of two Lemmas

Lemma 3.3.
Let f *be as in Theorem 3.1, and* $\varphi \in C^2(B_1) \cap Lip(\overline{B_1})$ *be a pluri-subharmonic function. Then there exists a positive constant c depending only on* $\sup_{B_1} |D\varphi|$ *and* $\sup_{\partial B_1} |\varphi|$*, such that for every* $0 < r < R := \min\{1/\lambda, 1\}$ *in* B_r *it holds:*

$$f(z, \varphi + \lambda d, D\varphi + \lambda Dd(z)) < \lambda^n \tag{3.5}$$

where $d(z) := (\|z\|^2 - r^2)/2$ *with* $z \in B_1$

Proof. Define $\rho := \sup_{B_1} |D\varphi| + 1$ and choose $\lambda > 0$ such that

$$\sup_{(z,p) \in B_1 \times B_\rho} f(z, \sup_{\partial B_1} |\varphi|, p) < \lambda^n$$

let us prove that λ satisfies (3.5). Since $f_u > 0$, d is negative in B_r and φ is pluri-subharmonic, for $0 < r < R$ in B_r we have:

$$\begin{aligned} f(z, \varphi + \lambda d, D\varphi + \lambda Dd(z)) &\leq f(z, \sup_{\partial B_1} |\varphi| + \lambda d, D\varphi + \lambda Dd(z)) \\ &\leq f(z, \sup_{\partial B_1} |\varphi|, D\varphi + c\overline{z}) \\ &\leq \sup_{(z,p) \in B_1 \times B_\rho} f(z, \sup_{\partial B_1} \varphi, p) \\ &\leq \lambda^n \end{aligned}$$

□

Lemma 3.4.
Let f *be as in Theorem 3.1, and* $\varphi \in C^2(B_1) \cap Lip(\overline{B_1})$ *be a pluri-subharmonic function. Define*

$$u_\lambda(z) := \varphi(z) + \lambda d(z), \quad z \in B_1$$

where $d(z)$ *and* λ *are defined as in Lemma 3.3. Then*

$$MA(u)_\lambda < 0 \quad \text{in } B_r \tag{3.6}$$

for every $0 < r < R := \min\{1/\lambda, 1\}$

Proof. Since φ is pluri-subharmonic we have $\partial\overline{\partial}u_\lambda = \partial\overline{\partial}\varphi + \lambda \mathrm{Id}_n \geq \lambda \mathrm{Id}_n$. Thus in B_r it holds:

$$\begin{aligned} MA(u)_\lambda &= -\det(\partial\overline{\partial}u_\lambda) + f(z, u_\lambda, Du_\lambda) \\ &\leq -\lambda^n + f(z, u_\lambda, Du_\lambda) \\ &= -\lambda^n + f(z, \varphi + \lambda d, D\varphi + \lambda \overline{z}) \end{aligned} \tag{3.7}$$

Since λ and R are defined as in Lemma 3.3, in B_r we have:

$$f(z, \varphi + \lambda d, D\varphi + \lambda Dd(z)) < \lambda^n$$

This inequality and (3.7) imply (3.6). □

Proof (Theorem 3.2). Let $u_\lambda(z)$ be the function defined in Lemma 3.4 with $\varphi = \phi_\sigma$. So, if $r < R := \min\{1/\lambda, 1\}$, then $u_\lambda \in C^2(\overline{B}_r)$ is a classical subsolution of $MA = 0$ in B_r. Moreover $u_\lambda = \phi_\sigma$ on ∂B_r. On the other hand, $\phi_\sigma(z)$ is independent of z_1 so $\partial\overline{\partial}\phi_\sigma$ has a null eigenvalue. Therefore

$$MA(\phi_\sigma) = f(z, \phi_\sigma, D\phi_\sigma) \geq 0 \quad \text{in } B_r \tag{3.8}$$

thus ϕ_σ is a classical supersolution of $MA = 0$ in B_r.
Since $\phi_\sigma \in C^\infty(\partial B_r)$, B_r is a pseudoconvex domain and f is smooth, positive and satisfies structural assumptions *(H1)* and *(H2)*, Theorem 2.8 ensures the existence of a classical strictly pluri-subharmonic solution $u_\sigma \in C^\infty(\overline{B_r})$ of (3.3). Now by the comparison principle,

$$u_\lambda \leq u_\sigma \leq \varphi \quad \text{in } \overline{B_r}$$

hence

$$\sup_{B_r} |u_\sigma| \leq \sup_{B_r} |\varphi| + \lambda r$$

Let us now estimate Du_σ on ∂B_r, we write:

$$Du_\sigma = D^\tau u_\sigma + D^\nu u_\sigma$$

where ν denotes the inward normal direction to ∂B_r, $|\nu| < 1$ and $D^\nu u_\sigma$, $D^\tau u_\sigma$ denote the normal and the tangent component of Du to ∂B_r respectively. Since $u = \phi_\sigma$ on ∂B_r, we have

$$D^\tau u_\sigma = D^\tau \phi_\sigma$$

Now consider the normal component $D^\nu u_\sigma = \langle Du_\sigma, \nu \rangle$. We have $u_\lambda = u_\sigma = \phi_\sigma$ on ∂B_r and $u_\lambda \leq u_\sigma \leq \phi_\sigma$ in B_r, so that for every $z_0 \in \partial B_r$, and $0 \leq t < 2r$ the following inequality holds

$$\frac{u_\lambda(z_0 + t\nu) - u_\lambda(z_0)}{t} \leq \frac{u_\sigma(z_0 + t\nu) - u(z_0)}{t} \leq \frac{\phi(z_0 + t\nu) - \phi(z_0)}{t}$$

Taking the limit for $t \to 0^+$ we get

$$D^\nu u_\lambda \leq D^\nu u_\sigma \leq D^\nu \phi_\sigma$$

as desired.

Notice that $\sup_{B_r} |Du_\lambda|$ is bounded by a constant depending only on r, $\sup_{B_r} |D\phi_\sigma|$. Indeed

$$\sup_{B_r} |Du_\lambda| = \sup_{B_r} \left| D\phi_\sigma + \lambda \overline{z} \right| \leq \sup_{B_r} |D\phi_\sigma| + 1$$

We conclude that $u_\sigma \in \mathrm{Lip}(\overline{B_r})$ where $\|u_\sigma\|_{\mathrm{Lip}(\overline{B_r})}$ depends only on r, $\sup_{B_r} |D\phi_\sigma|$ and $\sup_{B_r} |\phi_\sigma|$ $\square$

Now we show that it is possible to choose the constant C in Proposition 3.2 independent of σ. Indeed C depends on σ only through

$$C_{\phi_\sigma} := \sup_{B_1} |\phi_\sigma| + \sup_{B_1} |D\phi_\sigma|$$

and direct computations show

$$|D\phi_\sigma| = 4K\alpha(\sigma + |z'|^2)^{-\frac{1}{n}} |z'|$$

Since $0 \leq \sigma < 1$ it holds

$$|z'| = |z'|^{\frac{2}{n}+1-\frac{2}{n}} \leq (|z'|^2 + \sigma)^{\frac{1}{n}} |z'|^{1-\frac{2}{n}}$$

thus

$$\sup_{B_1} |D\phi_\sigma| \leq 4K$$

Moreover

$$\sup_{B_1} |\phi_\sigma| \leq 4K$$

so that $C_{\phi_\sigma} < 8K$

Let us now choose the constant K

Lemma 3.5.
There exists K such that

$$MA(\psi_\sigma) < 0 \quad \text{in } B_1, \text{ for all } \sigma \in]0, 1[$$

Proof. First of all we show that

$$\det(\partial\overline{\partial}_{z_1,z'}\omega_\sigma) = f_\sigma \tag{3.9}$$

with

$$f_\sigma = \alpha^n(1+|z_1|^2)^{n-2}\frac{(\alpha^{-1}\sigma + |z'|^2) + \alpha^{-1}\sigma|z'|^2}{\sigma+|z'|^2}$$

In fact, computations yield

$$\partial\overline{\partial}_{z_1,z'}\omega_\sigma = \begin{pmatrix} \partial\overline{\partial}_{z_1\overline{z}_1}\omega_\sigma & \partial\overline{\partial}^2_{z_1\overline{z}'}\omega_\sigma \\ \partial\overline{\partial}^2_{z'\overline{z}_1}\omega_\sigma & \partial\overline{\partial}^2_{z'\overline{z}'}\omega_\sigma \end{pmatrix} =$$
$$= (\sigma+|z'|^2)^{n(\alpha-1)}\det\begin{pmatrix} (\sigma+|z'|^2) & \alpha\overline{z}_1 z' \\ \alpha z_1(\overline{z}')^T & \alpha(1+|z_1|^2)\Big(Id_{n-1} + (\alpha-1)\frac{\overline{z'}\otimes z'}{(\sigma+|z'|^2)}\Big) \end{pmatrix}$$

Since $(\alpha-1)n = -1$ we get

$$\det(\partial\overline{\partial}_{z_1,z'}\omega_\sigma) =$$
$$= (\sigma+|z'|^2)^{-1}\det\begin{pmatrix} (\sigma+|z'|^2) & \alpha\overline{z}_1 z' \\ \alpha(\overline{z}')^T z_1 & \alpha(1+|z_1|^2)\Big(Id_{n-1} + (\alpha-1)\frac{\overline{z'}\otimes z'}{(\sigma+|z'|^2)}\Big) \end{pmatrix}$$
$$= \det\begin{pmatrix} 1 & \alpha\overline{z}_1\frac{z'}{(\sigma+|z'|^2)^{1/2}} \\ \alpha z_1\frac{(\overline{z}')^T}{(\sigma+|z'|^2)^{1/2}} & \alpha(1+|z_1|^2)\Big(Id_{n-1} + (\alpha-1)\frac{\overline{z'}\otimes z'}{\sigma+|z'|^2}\Big) \end{pmatrix}$$
$$= \det\begin{pmatrix} 1 & 0 \\ \alpha z_1\frac{\overline{z}_m}{(\sigma+|z'|^2)^{1/2}} & \alpha(1+|z_1|^2)\Big(Id_{n-1} + (\alpha-1)\frac{\overline{z'}\otimes z'}{\sigma+|z'|^2}\Big) - \alpha^2|z_1|^2\frac{\overline{z'}\otimes z'}{\sigma+|z'|^2} \end{pmatrix}$$
$$= \det\Big(\alpha(1+|z_1|^2)\Big(Id_{n-1} + (\alpha-1)\frac{\overline{z'}\otimes z'}{\sigma+|z'|^2}\Big) - \alpha^2|z_1|^2\frac{\overline{z'}\otimes z'}{\sigma+|z'|^2}\Big)$$
$$= \det\Big(\alpha(1+|z_1|^2)Id_{n-1} + \frac{\overline{z'}\otimes z'}{\sigma+|z'|^2}\Big(\alpha(\alpha-1)(1+|z_1|^2) - \alpha^2|z_1|^2\Big)\Big)$$
$$:= \det\Gamma$$

Where Γ is a $(n-1)\times(n-1)$ Hermitian matrix. It is easy to see that $\lambda_1 = \alpha(1+|z_1|^2)$ is an eigenvalue of Γ with multiplicity $n-2$. Now, $\mathrm{Tr}(\Gamma) = (n-2)\lambda_1 + \lambda_2$ with

$$\mathrm{Tr}(\Gamma) = (n-1)\alpha(1+|z_1|^2) + \frac{|z'|}{\sigma+|z'|^2}\Big(\alpha(\alpha-1)(1+|z_1|^2) - \alpha^2|z_1|^2\Big)$$

from which we infer

$$\lambda_2 = \alpha(1+|z_1|^2) + \frac{|z'|}{\sigma+|z'|^2}\Big(\alpha(\alpha-1)(1+|z_1|^2) - \alpha^2|z_1|^2\Big)$$
$$= \alpha^2\frac{\Big(\frac{\sigma}{\alpha} + |z'|^2\Big) + \frac{\sigma}{\alpha}|z'|^2}{\sigma+|z'|^2}$$

Thus

$$\det\Gamma = \lambda_1^{n-2}\lambda_2 = \alpha^n(1+|z_1|^2)^{n-2}\frac{\left(\frac{\sigma}{\alpha}+|z'|^2\right)+\frac{\sigma}{\alpha}|z'|^2}{\sigma+|z'|^2} = f_\sigma$$

which completes the proof of (3.9).

For every $\sigma \in\,]0;1[$ we have

$$f_\sigma \geq \alpha^n \quad \text{in } B_1$$

now, since $\psi_\sigma = K\omega_\sigma$, $\det(\partial\overline{\partial}\psi_\sigma) = K^n \det(\partial\overline{\partial}\omega_\sigma)$ we get

$$\begin{aligned} MA(\psi_\sigma) &= -K^n f_\sigma + f(z, K\omega_\sigma, KD\omega_\sigma) \\ &\leq -K^n\alpha^n + f(z, K\omega_\sigma, KD\omega_\sigma) \end{aligned}$$

Thus, by assumption $(H3)$ we can choose $K>0$ so large to obtain $MA(\psi_\sigma) \leq 0$ as desired. □

Finally we prove Theorem 3.1

Proof (Theorem 3.1). Applying Proposition 3.2 there exists $0<r<1$ such that the Dirichlet problem

$$\begin{cases} MA(u) = 0 & \text{in } B_r \\ u = \phi_\sigma & \text{on } \partial B_r \text{ with } \sigma \in]0,1[\end{cases}$$

has a viscosity solution u_σ such that

$$\|u_\sigma\|_{L^\infty(\overline{B}_r)} + \|u_\sigma\|_{\mathrm{Lip}(\overline{B_r})} \leq C_{\phi_\sigma}(r,\sigma,K)$$

We have already shown that we can choose $C(r,\sigma,K)$ independent of σ and so

$$\|u_\sigma\|_{L^\infty(\overline{B}_r)} + \|u_\sigma\|_{\mathrm{Lip}(\overline{B_r})} \leq C_{\phi_\sigma}(r,K)$$

Now we use the comparison principle (Theorem 1.13) to compare u_σ with ϕ_σ and ψ_σ. Indeed by (3.8) and Lemma 3.5, ϕ_σ and ψ_σ are, respectively, classical supersolution and subsolution to $MA=0$ in B_r. On the other hand $\psi_\sigma \leq \phi_\sigma$ in B_1, in particular, $\psi_\sigma \leq \phi_\sigma$ on ∂B_r. Thus, by the comparison principle,

$$\psi_\sigma \leq u_\sigma \leq \phi_\sigma \text{ in } B_r,\ \forall\sigma \in]0,1[\tag{3.10}$$

The uniform estimate (3.4) implies that $(u_\sigma)_{\sigma\in]0,1[}$ is a sequence of equicontinuous and uniformly bounded functions, thus Ascoli-Arzelà theorem ensures the existence of a subsequence $\sigma_j \searrow 0$ such that $(u_{\sigma_j})_{j\in\mathbb{N}}$ uniformly converges to a function $u \in \text{Lip}(\overline{B_r})$. Now, since MA is proper, by Theorem 1.12, we have that u is a viscosity solution to $MA = 0$ in B_r. Moreover on ∂B_r, $u_\sigma = \phi_\sigma \to \phi_0$ as $\sigma \to 0^+$, hence u is a viscosity solution of the Dirichlet problem

$$\begin{cases} MA = 0 & \text{in } B_r \\ u = \phi_0 & \text{on } \partial B_r \end{cases} \tag{3.11}$$

Therefore, from the comparison principle,

$$\psi_0 \leq u \leq \phi_0 \text{ in } B_r$$

So,

$$K|z_2|^{2\alpha} \leq u(0, z_2, 0, \ldots, 0) \leq 2K|z_2|^{2\alpha} \tag{3.12}$$

and in particular

$$K|x_2|^{2\alpha} \leq u(0, (x_2, 0), 0, \ldots, 0) \leq 2K|x_2|^{2\alpha} \tag{3.13}$$

Inequalities (3.13) imply

$$u \notin C^1, \quad \text{if } 2\alpha = 1 \text{ (i.e. } n = 2)$$

and

$$u \notin C^{1,\beta}, \quad \forall \beta > 2\alpha - 1 = 1 - \frac{2}{n} \text{ if } 2\alpha > 1 \text{(i.e. } n > 2)$$

Indeed if $2\alpha > 1$, then $\partial_{x_2} u(0, 0, \ldots, 0) = 0 = u(0, 0, \ldots, 0)$ so that if u was $C^{1,\beta}$, with $\beta > 2\alpha - 1$, we would have $u(0, (x_2, 0), \ldots, 0) \leq C|x_2|^{1+\beta}$ for a suitable $C > 0$ and for every $|x_2|$ sufficiently small. Hence, by the first inequality in (3.13), we would have $\beta \leq 2\alpha - 1$, a contradiction.
If $k = 2$, in the same way, we see that $\partial_{x_2} u$ is not continuous and this ends the proof. □

Conclusions

We point out that the theory of viscosity solutions used in this dissertation is only one of the possible approaches in treating the Monge-Ampère equations. This theory led us to prove the existence of a non classical viscosity solution for the complex Monge-Ampère equation with smooth right hand side. Such a solution has been found solving the Dirichlet problem with prescribed boundary condition $u = \varphi_0$ where $\varphi_0 \in C^{1,1-\frac{2}{n}}$. It is still an unsolved issue whether a greater regularity of the boundary condition (i.e $\phi \in C^{1,\beta}$ with $\beta > 1 - \frac{2}{n}$) would imply the C^∞ regularity of solutions.

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Ringraziamenti

Desidero ringraziare la Prof.ssa Montanari per la disponibilità e la gentilezza con la quale mi ha sempre ricevuta. Vorrei anche ringraziarla per la passione che mette nel suo lavoro e che mi ha trasmesso in questi anni dandomi la voglia di continuare gli studi.

Ringrazio il Dott. Vittorio Martino per il tempo che mi ha dedicato le approfondite spiegazioni, i preziosi consigli e per tutto l'aiuto fornito durante la stesura della tesi.

Vorrei poi ringraziare Lorenzo e la mia famiglia che mi hanno sostenuta in questi anni di studio e che sono sempre stati presenti nei momenti difficili.

Infine vorrei ringraziare Riccardo per i suoi consigli tecnici e i miei amici e compagni di corso Giulia, Stefano e Lorenzo che hanno condiviso con me le gioie e le fatiche di questi cinque anni. Grazie di cuore a tutti.