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Tensorial Group Field Theories

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Abstract

In questa tesi il Gruppo di Rinormalizzazione non-perturbativo (FRG) viene applicato ad una particolare classe di modelli rilevanti in Gravità quantistica, conosciuti come Tensorial Group Field Theories (TGFT). Questa classe di modelli generalizza i più noti modelli matriciali che hanno dato prova di essere un formalismo di successo nella descrizione di una transizione dal discreto al continuo per gravità in due dimensioni. Inoltre, se arricchiti con ulteriori condizioni, le TGFT riescono ad implementare le stesse ampiezze di transizione che si riscontrano in Loop Quantum Gravity secondo il formalismo di Spin Foam Models. Le TGFT sono teorie di campo quantistiche definite sulla varietà di un gruppo $G^{\times d}$. In ogni dimensione esse possono essere espanse in grafici di Feynman duali a complessi simpliciali casuali e sono caratterizzate da interazioni che implementano una non-località combinatoriale. Le TGFT aspirano a generare uno spaziotempo in un contesto background independent e precisamente ad ottenere una descrizione continua della sua geometria attraverso meccanismi fisici come le transizioni di fase. Tra i metodi che meglio affrontano il problema di estrarre le transizioni di fase e un associato limite del continuo, uno dei più efficaci è il Gruppo di Rinormalizzazione non-perturbativo.

In questo elaborato ci concentriamo su TGFT definite sulla varietà di un gruppo non-compatto ($G = \mathbb{R}$) e studiamo il loro flusso di Rinormalizzazione attraverso l'equazione funzionale di Wetterich nella formulazione di Benedetti et al. [JHEP **03** (2015) 084]. La non-compattezza della varietà del gruppo su cui sono definiti i campi risulta cruciale per identificare i punti fissi del flusso di FRG come punti fissi Ultravioletti (UV) e Infrarossi (IR). Identifichiamo con successo punti fissi del flusso di tipo IR, e una superficie critica che suggerisce fortemente la presenza di transizioni di fase in regime Infrarosso. Ciò spinge ad uno studio per approfondire la comprensione di queste transizioni di fase e della fisica continua che vi è associata. Apportiamo inoltre una miglioria al precedente elaborato di Benedetti et al. nel senso che il nostro modello, che è definito su uno spazio non-compatto, fornisce direttamente un sistema autonomo di β-functions. Affrontiamo inoltre il problema delle divergenze Infrarosse, tramite un processo di regolarizzazione che definisce il limite termodinamico appropriato per le TGFT.

Infine, applichiamo i metodi precedentemente sviluppati ad un modello dotato di proiezione sull'insieme dei campi gauge invarianti. L'analisi, simile a quella applicata al modello precedente, conduce nuovamente all'identificazione di punti fissi (sia IR che UV) e di una superficie critica. La presenza di transizioni di fasi è, dunque, evidente ancora una volta ed è possibile confrontare il risultato col modello senza proiezione sulla dinamica gauge invariante.

Abstract

This master thesis deals with the study of the Functional Renormalization Group (FRG) approach for a particular class of models relevant for Quantum Gravity, called Tensorial Group Field Theories (TGFT's). This class of models generalizes the famous matrix models which prove to be a successful framework for addressing a discrete-to-continuum transition for Gravity in two dimensions. Furthermore, having implemented extra conditions, TGFT's are able to reproduce the same amplitudes of Spin Foams Models. TGFT's are quantum field theories defined on a group manifold $G^{\times d}$. They expand in Feynman graphs dual to random simplicial complexes in any dimensions and feature combinatorially non-local interactions. TGFT's aim at generating a space-time in a background independent context and precisely to recover a continuous description of its geometry through physical mechanisms such as a phase transition. In addressing the problem of extracting phase transitions and associated continuum limit, one of the most prominent methods is the Functional Renormalization Group.

In this work, we focus on TGFT's on a non-compact group manifold ($G = \mathbb{R}$) and study their RG flow through the Wetterich functional equation in the formulation of Benedetti et al. [JHEP **03** (2015) 084]. The non-compactness of the group manifold where the fields are defined is crucial to identify the fixed points of the FRG flow as Ultraviolet (UV) and Infrared (IR) fixed points. We successfully identify IR fixed points in the flow which strongly suggest the presence of phase transition(s) in the IR. This is encouraging for the next stage which is the understanding of these phases and their associated continuum physics. We improve the previous work of Benedetti in the sense that our model which is defined on a non-compact space yields directly an autonomous system of β-functions. We also tackle the issue of IR divergences by a regularization scheme defined through a proper thermodynamic limit for TGFT's. Finally, we apply the method developed to a different model endowed with a projection on the set of gauge invariant fields. The analysis, similar to the one made for the previous model, allows again to identify fixed points (IR and UV) and a critical surface. The presence of phase transition is, thus, pointed out and it is possible to compare the results with the one obtained from the model without gauge projection.

Contents

Chapter 1

Introduction

After Quantum Mechanics and Quantum Field Theory were developed and quite very well understood, many issues concerning the formulation of a quantum theory for General Relativity had been pointed out. In particular, the standard method used to quantize Standard Model interactions fail in providing a well defined and renormalizable model for gravity. Addressing this problematic, several alternative frameworks had been elaborated since then. We quote among these approaches, Asymptotic Safety Scenario, String Theory and the like considering higher dimensions (Super-Strings, Super-Gravity, etc...), Loop Quantum Gravity & Spin Foams, Analog Gravity, (Causal) Dynamical Triangulations, Causal Sets, etc. Let us emphasize at least two noteworthy aspects of these. Unlike String Theory, which provides a quantum gravity description in terms of a perturbative expansion around a flat background geometry, there exist other theories which rather address the same problem in a background independent context. This is the case of Loop Quantum Gravity (LQG) [1, 2], which finds its roots in the canonical analysis of General Relativity made in the ADM formalism [3]. A covariant definition of LQG is made by spin foams models [5, 6, 7], which, in terms of state sum models, create a link between LQG and simplicial gravity, recovering, in some limit, the Ponzano Regge calculus for discrete gravity [8, 9]. Among the attempts of providing a background independent theory of quantum gravity, one of the most promising, though young, framework appears in the Group Field Theories (GFT's) context [13, 14, 19, 20].

Developed in order to generalize frameworks coming from String Theory, such as matrix models [16] and tensor models [17, 18], GFT's also reproduce results coming from the spin foam formulation of LQG [9, 15] and offer a neat comparison between their description of the spacetime dynamics and the one provided by noncommutative geometry methods [12].

Group Field Theories are field theories endowed with a combinatorial nonlocal interaction where fields are defined over a group manifold. The Feynman rules, in the most simple situation, are given by an interaction that can be represented as a simplex, and a propagator providing a rule for gluing these simplices together. Then, Feynman graphs generated by these theories are dual to a cellular/simplicial decomposition of the spacetime manifold. Hence, because the combinatorics of their interaction terms reproduce (simplicial) building blocks, GFT's give a field theoretical setting and background independent formulation of simplicial gravity.

Mentioning discrete structures, one of the most fundamental problems arising in Quantum Gravity is to make sense of the transition from a discrete setup to a continuous one. In LQG and spin foams, as well as in tensor models and causal dynamical triangulations, the attempts to find an answer in this direction are based on a refinement of the lattice discretization leading to a semiclassical approximation where General Relativity is recovered. One the main tools that we possess to identify a regime where geometrical quantities emerge from a pre-geometric structure is the Renormalization Group (RG) [48]. Already applied in the spin foam context with a lattice gauge theory perspective [10, 21], the RG in its perturbative formulation [67] has been also applied to a special class of GFT called Tensorial GFT's [38][25]. These TGFTs have a recent history. Indeed, they emanates from the statistical study of colored tensor models [26, 27], a specific class of tensor models, which support a large N expansion [28, 29]. Consequently, many new results are today available in the study of critical behavior of colored tensor models [30, 31, 32, 33, 34, 35, 36, 37]. From the large N study, it appears possible to identify a class of interactions which are the dominant one in partition function of colored tensor models. These dominant terms, called melons [30] turned out to be the one interesting for performing a perturbative analysis of GFT's.

The advantage of GFT's is that, being a field theory, the RG can be applied in its full formalism. In [39], it was argued that, after the results obtained in tensor models and the perturbative RG applied to TGFT's, this subclass of GFT is enough general to provide interesting results. In particular, we are able to combine methods and results inherited from tensors with a non-perturbative formulation of the RG known as the Functional Renormalization Group. In short, representing a powerful tool to study the physics of a theory in a neighborhood of critical phenomena, the FRG formalism describes the flow of a theory as the energy scale changes, providing eventual regimes at which phase transitions occur. From this point of view, the emergence of a continuous geometry could be phrased in terms of a phase transition towards a condensate state of (T)GFT's [46] and we refer to this procedure as "geometrogenesis" or the "birth of geometry from basic structures" [47].

The FRG approach has been recently formulated in the context of matrix models with goal to probe their continuum limit [60, 61, 62, 63] and then, tested on TGFT's in the work by Benedetti et al. [66]. In the latter contribution, formulated on an underlying compact group manifold, i.e. $U(1)^d$, the β-functions obtained forms a non-autonomous system in the cut-off over tensor modes. This feature prevents from recognizing fixed points as IR and UV fixed points with respect to the cut-off in its small and large limits, respectively. It becomes a difficult matter, both at the technical and conceptual level, to understand the resulting phase diagrams within a full theory, in a unique consistent description. The present work mainly aims at improving this rather peculiar feature for GFT's.

In this thesis, the Functional Renormalization Group is addressed for particular models of Group Field Theories defined over a non-compact group manifold. The goal of this study is to obtain a picture describing the full flow of the two models, distinguished from the presence or not of a projection on the gauge invariant dynamics. Both models have similarities and differences that we emphasize along the text. In this end, for both cases, we identify a proper critical surface on the phase diagram which strongly suggest the existence of phase transitions. Both theories have several non-Gaussian IR fixed points and one Gaussian, UV fixed point. The neighborhood of each IR fixed point is similar to the one found in local scalar field theory in dimension 3, i.e. the vicinity of the Wilson-Fisher fixed point [58]. Thus, we have hints that phase transitions happen indeed. A tentative suggestion is that these phases refer again to the symmetric and broken phases, and the latter that we would like to associate with the condensed phase. Nevertheless, a rigorous proof of that statement about phase transition can be only achieved through a reparametrization of the system, just like in the ordinary scalar field theory. This change of dynamical variables depends on a resolution of the equation of motion of the GFT models which, in the case chosen to study, involve an integro-differential equation difficult to solve. This opens avenues of forthcoming investigations.

The plan of this thesis is as follows. Chapter 2 reviews the literature, putting at its center, the link between GFT's and other quantum gravity frameworks and following the historical path which led to their formulation. We start by explaining the crux ideas behind matrix models and tensor models, paying particular attention on applications to quantum gravity and quantum geometry, showing, in this way, the mathematical background of Group Field Theories. Then, we proceed further and explain the formulation of General Relativity on which Loop Quantum Gravity relies and the formal procedure leading to the spin foam formulation of LQG, pointing out the equivalence between the quantum amplitudes of spin foams and those obtained from particular models of GFT's. Finally, we discuss the peculiarities and the construction of a particular subclass of GFT's, known as Tensorial Group Field Theories (TGFT), on which this thesis particularly focuses.

Chapter 3 slightly reviews the Functional Renormalization Group. First, we illustrate the Wilsonian idea that gave birth to this formalism [48]. We point out the reasons leading to the necessity of a new formulation of the RG which constrats with the perturbative approach. Then, we motivate the equations describing the flow of a theory in this context, namely the Polchinski equation [51] and the Wetterich equation [52]. In the last section of chapter 3, we expand the formalism of the FRG to the case of TGFT's following the work by Benedetti et al. [66].

In chapter 4, start our investigations. We apply the constructions illustrated so far, to a specific rank 3 model of TGFT's defined over a non-compact group manifold, $\mathbb{R}$. This is the first attempt to address a renormalization procedure to the non-compact case of TGFT's. We introduce the model as well as the framework and single out problems arising from the infinite volume of the domain of fields. Then, we make an ansatz concerning the structure of the coupling constants which allows us to recover a well defined, autonomous, system of β-

functions in the non-compact limit. The key point of our regularization scheme is the introduction of a new parameter representing the dependence of couplings on the volume of the direct space. To implement this procedure, we compactify the volume in the x space obtaining a lattice regularization of the momentum space. Consequently, we use the system of β-functions found through this regularization to extract the flow equations of the model and its phase diagram and compare the result with usual models of local quantum field theories.

In chapter 5, we modify the model studied in chapter 4 introducing a projection on the gauge invariant dynamics, in order to recover the formulation of known quantum gravity models. We analyze the ambiguities arising from the construction of such a model and we discover that, willing to properly set up a non-trivial FRG formalism, we have no choice but to project both the kinetic and interaction terms of the action. Thus the resulting action becomes a new one never addressed before. Then, we apply the Wetterich equation to this second field theory using a similar regularization scheme adopted in chapter 4. This, again, allows us to obtain a proper non-compact limit and an autonomous system of β-functions. The flow of the gauge projected model turns out to have similarities but also differences compared with the one studied before.

Chapter 6 is our conclusion: we give a summary of our results and list important open problems for this approach that will be addressed in the future. Two appendices, namely appendix A and B, close the manuscript and provide details of our calculations, underlining their main features and leading to the system of β-functions for each model investigated.

Chapter 2

Group Field Theories

2.1 From matrix models to GFT's

A first attempt in describing quantum gravity through a dicretization of a manifold was implemented in the context of two dimensional gravity [16, 17]. In this framework, an integral over the intrinsic geometries of a $2d$ surface is regularized as a sum over randomly triangulated surfaces. This procedure allows to write the partition function of quantum gravity as the free energy of an associated hermitian matrix model [16]. In this subsection, in a streamline analysis, we give an overview of $2d$ quantum gravity and its link to matrix models.

In quantum gravity, the aim is to compute an integral over all 2-geometries and a sum over all 2-topologies of the type

$$\mathcal{Z} \simeq \sum_{\text{topologies}} \int \mathcal{D}g\mathcal{D}X \; e^{-S[g,X]} \,, \tag{2.1}$$

where X is a set of scalar fields defined over the $2d$ manifold with geometry determined by a metric g and S encodes their classical dynamics.

To get a better understanding of the correspondence with matrix models, let us consider a pure theory of two dimensional surfaces and the following partition function:

$$\mathcal{Z} = \sum_{h} \int \mathcal{D}g \; e^{-\beta A + \gamma\chi} \,, \tag{2.2}$$

where h is the number of handles of the surface, $A = \int \sqrt{g}$ its area, $\chi = \frac{1}{4\pi}\int \sqrt{g}R = 2 - 2h$ its Euler character, R is the corresponding the scalar curvature and β and γ are coupling constants. Already at this level, we can see that the structure of the action in (2.2) reflects the Einstein-Hilbert action for pure gravity with cosmological constant:

$$S_{EH}(g) = \int d^D x \; \sqrt{g}(KR + \Lambda) \,. \tag{2.3}$$

While in two dimensions the fact that $\int \sqrt{g}R$ is a topological term prevents it from influencing the equations of motion and makes classical General Relativity trivial, at the quantum level, strong fluctuations of the geometry may induce changes in the topology and the theory is no more trivial.

Let us consider now a triangulation of the surface. Without loss of generality, we can consider all triangles to be equilateral so that, if N_i is the number of incident triangles at the vertex i, then the curvature is positive (respectively negative) if $N_i > 6$ (resp. < 6) and null if $N_i = 6$. Call E, F and V the total number of edges, faces and vertices of the triangulation, respectively, then $2E = \sum_i N_i$ and $3F = 2E$. These relations are purely topological and allow to rewrite all quantities in terms of N_i in such a way that, in this discrete setting, the scalar curvature at the vertex i becomes $R_i = 2\pi(6 - N_i)/N_i$ and the square root of the determinant of metric tensor becomes $\sqrt{g}_i = N_i/3$ (the factor $\frac{1}{3}$ is due to the fact that every triangle, assumed to have unit area, has 3 vertices and thus is counted 3 times). The previous relations lead directly to the following correspondence:

$$\frac{1}{4\pi}\int \sqrt{g}R \;\longrightarrow\; \frac{4\pi}{4\pi}\sum_{i=1}^{V}(1-\frac{N_i}{6}) = V - \frac{1}{2}F = V - E + F = \chi\,, \tag{2.4}$$

where, in the last equality, we recognize the simplicial definition of the Euler character χ. This is of course Descartes' version of the Gauss-Bonnet theorem where curvature is seen as a discrete measure, i.e., is defined by counting the numbers of triangles converging toward a vertex.

As a result, the sum over all random triangulations becomes the discrete counterpart of the sum over all topologies and all geometries:

$$\sum_h \int \mathcal{D}g \;\longrightarrow \sum_{\text{random triangulations}} . \tag{2.5}$$

Up to this point, one formalism provides a discrete version of the other. We now define the sum over all possible triangulations using a matrix path integral as generating functional. Once expanded in perturbation theory, this integral will generate Feynman diagrams represented as ribbons graphs glued together, that we interpret as dual to the simplicial decomposition of the surface. We thus consider the following integral:

$$\mathcal{Z}(a) = \int dM\; e^{-\frac{1}{2}\mathrm{Tr}M^2 + \frac{a}{\sqrt{N}}\mathrm{Tr}M^3}\,, \tag{2.6}$$

where M is a $N \times N$ hermitian matrix, the measure is the invariant one $dM = \prod_i dM^i{}_i \prod_{i<j} d\mathrm{Re}M^i{}_j\, d\mathrm{Im}M^i{}_j$ and is normalized as $\int_M e^{-\frac{1}{2}\mathrm{Tr}M^2} = 1$ and where a is a coupling constant. Perturbatively, the a^n-term of the diagrammatic expansion of (2.6) generates graphs with n trivalent vertices which are dual to triangulations of a closed surface with area n, where each vertex, edge and face in a graph is associated, respectively, to a triangle, edge and vertex in the triangulation.

The diagrammatic representation of the expansion is built through the gluing of double stranded lines, rerouted by the vertices in order to reflect the convolution pattern of each graph (see Fig.2.1).

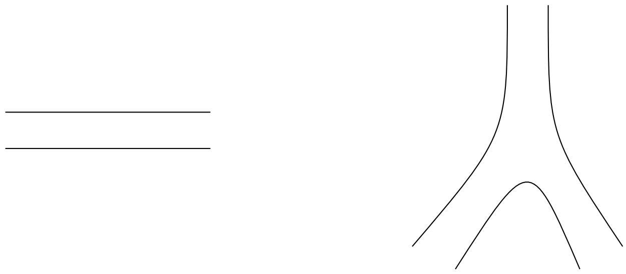


Figure 2.1: Diagrammatic representations of the propagator and vertex of the model (2.6).

We can extract important information from the size of the matrix: if we perform the following rescaling

$$M \longrightarrow \frac{M}{\sqrt{N}}\,, \tag{2.7}$$

the action becomes

$$S[M] = -\frac{N}{2}\mathrm{Tr}M^2 + aN\mathrm{Tr}M^3 = N\Big(-\frac{1}{2}\mathrm{Tr}M^2 + a\mathrm{Tr}M^3\Big)\,. \tag{2.8}$$

We notice that each vertex contributes to the amplitude of the graph with a factor N, each propagator or covariance (inverse of the kinetic term) with a factor N^{-1} and every face (loop of strands in the graph) generates a trace over a convolution of Kronecker δ's giving again a factor N. Thus, using a graph expansion, we write:

$$\mathcal{Z}(a) = \sum_{\Gamma} a^{V_\Gamma} N^{V_\Gamma - E_\Gamma + F_\Gamma} = \sum_{\Gamma} a^{V_\Gamma} N^{\chi_\Gamma}\,, \tag{2.9}$$

where Γ is a Feynman graph. Assuming analytic continuation, we can choose $N = e^{\gamma}$ and, identifying $\log a$ as a cosmological constant, we find the form of the action used in (2.2). From (2.9), the partition function finds the expansion in powers of N as:

$$\mathcal{Z}(a) = \sum_{h} N^{2-2h}\mathcal{Z}_h(a)\,. \tag{2.10}$$

In the large N limit, only the term proportional to $\mathcal{Z}_0(a)$ survives and we can expand it in powers of the coupling a to perform a continuum limit. One has:

$$\mathcal{Z}_0(a) = \sum_{V} V^{\sigma-3}\Big(\frac{a}{a_c}\Big)^{V} \simeq (a - a_c)^{2-\sigma}\,, \tag{2.11}$$

where σ is a critical exponent that describes the behavior of the theory when a approaches the critical value a_c. The previous expression diverges for the coupling constant going to a_c and computing the expectation value of the total area of the surface, it is possible to show that it diverges as well for $a \to a_c$, at least until each vertex, and thus, each triangle in the dual description, contributes with a finite non-zero value. Introducing a new parameter s representing the area of the triangles, the total area of the surface can be tuned to be finite in the critical regime if $s \to 0$, which means that we can perform a continuum limit.

In all above formulas, we deal with what, in the language of particle physics, are called vacuum diagrams, i.e., graphs without external data. The closed diagrams that are generated are dual to surfaces without boundaries. Nevertheless, the formalism holds in the case of open surfaces as well. The n-point Green's functions obtained from the quantum theory (2.6) admit an expansion in power series of the coupling a, where each term corresponds to a graph with n external links and number of vertices given by the exponent of a. In the dual triangulation, the set of external links becomes related with the boundary of the surface.

A last remark should be made on the framework of matrices. By choosing an interaction term proportional to $\mathrm{Tr}M^3$, we construct triangulations of surfaces. The choice of using triangles is standard because they are the smallest possible unit of cellular complexes and because the use of tools from simplicial geometry is then allowed. Nevertheless, nothing prevents from choosing other shapes, rather than triangles, for the building blocks for the discretized manifold and is possible to show that all the possible shapes belong to the same universality class.

The formalism of matrix models must be extended to higher dimensional manifolds [17, 18]. Already in the case of three dimensions, one encounters a lot of issues. A part the lack of a classification of topologies [17], the main problems which arose in the first formulation of tensor models were that:

- a procedure to identify the dominant triangulations was missing,
- there was not the possibility to identify the discretizations corresponding to pseudo-manifold and to check if they were suppressed or not,
- the non-trivial effects of a 3-dimensional theory of gravity were not reproduced.

The first two points were solved adding new degrees of freedom to these models, defining what we call "colored" tensor models [18, 25, 26, 27, 28, 29, 30]. The link with gravity is made by replacing labels of tensors with group theoretic data, i.e., through specific models of Group Field Theories [19, 20, 9].

Before getting in the details of the framework of tensor models, we must first introduce the concept of pseudo-manifold and simplicial-manifold following the conventions of [17, 26].

Definition 2.1.1. *A n-dimensional pseudo-manifold is a collection of n-simplices whose boundary $(n-1)$-simplices are pairwise identified.*

Definition 2.1.2. *A n-dimensional simplicial-manifold $\mathcal{M}$ is a pseudo-manifold where every point in $\mathcal{M}$ has a neighbourhood homeomorphic to the unit ball B^n in $\mathbb{R}^n$.*

For a three dimensional pseudo-manifold M, we define the Euler character in terms of the numbers l_d of d-simplices as:

$$\chi(M) = l_3(M) - l_2(M) + l_1(M) - l_0(M) \tag{2.12}$$

and we have $\chi(M) \leq 0$ with $\chi(M) = 0$ iff M is a simplicial-manifold.

Consider a three dimensional simplicial-manifold $\mathcal{T}$ and denote $E(\mathcal{T})$ its set of edges e, we can write the Einstein-Hilbert action as [8]

$$S_{EH}(\mathcal{T}) = \sum_{e \in E(\mathcal{T})} (\tau(e) - c) = 6l_3 - cl_1\,, \tag{2.13}$$

where $\tau(e)$ is the number of tetrahedra sharing e [1] and c an adjustable constant. We remember that, already in $3d$, there is no equilateral tessellation of the flat space as in $2d$. Adding a coupling constant to the model, we can absorb the contribution of c inside of it and write an action of the same form of the action for simplicial $3d$ gravity with cosmological constant (see [8]):

$$S(\mathcal{T}) = \lambda l_3(\mathcal{T}) + \kappa l_1(\mathcal{T}). \tag{2.14}$$

Hence, we define a model of simplicial quantum gravity through the following partition function:

$$Z_{\lambda,\kappa}(S_1, \ldots, S_n) = \sum_{\mathcal{T} \in \mathcal{M}(S_1,\ldots,S_n)} e^{-S(\mathcal{T})}\,, \tag{2.15}$$

where $\mathcal{M}(S_1, \ldots, S_n)$ is the set of simplicial-manifolds with boundary components $S_1, \ldots, S_n$. If we now look for a tool that allows us to use (2.15) for practical computations, we can generalize matrix models to (rank 3) tensor models using the following action:

$$S_T(t) = \sum_{\alpha,\beta,\gamma,\delta,\varepsilon,\rho} t_{\alpha\beta\gamma} t_{\gamma\delta\varepsilon} t_{\varepsilon\rho\alpha} t_{\beta\rho\delta}\,, \tag{2.16}$$

where $t_{\alpha\beta\gamma}$ is a tensor with indices running over a discrete set. Introducing the normalized Gaussian measure

$$d\mu_a(t) = C\,\mathcal{D}t\,\exp\left\{-\frac{a}{6}\sum_{\alpha\beta\gamma} |t_{\alpha\beta\gamma}|^2\right\}, \tag{2.17}$$

where $\mathcal{D}t = \prod_{\alpha\leq\beta\leq\gamma} d\mathrm{Re}\, t_{\alpha\beta\gamma} \prod_{\alpha<\beta<\gamma} d\mathrm{Im}\, t_{\alpha\beta\gamma}$, a is a coupling constant and C is a normalization factor, we construct a sum over histories of all possible random three dimensional pseudo-manifolds by writing the partition function

$$\mathcal{Z}(a, b) = \int d\mu_a(t) e^{-bS_T(t)}\,. \tag{2.18}$$

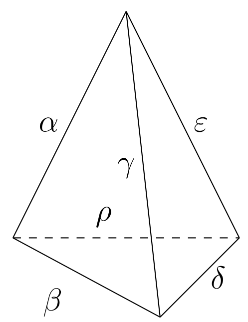


Figure 2.2: Labels of the edges of a tetrahedron

Again Feynman graphs coming from perturbative expansion of (2.18) are viewed as dual to a simplicial discretization of manifolds. This time, the tensor $t_{\alpha\beta\gamma}$ is equivalent to a triangle and the contractions of four tensors in the interaction term, see Fig.2.2, reflect the combinatorics of triangles combined to build up a tetrahedron, while the Gaussian measure describes the gluing of two tetrahedra by the identification of one of their boundary triangles.

As we will show in section 2.3, GFT's are the natural generalization of tensor models in which tensors become fields and the labels are defined over a group manifold. In this sense, they inherit from tensor models non-local interactions that describes a simplicial discretization of geometries.

2.2 GFT's and Loop Quantum Gravity

Loop Quantum Gravity [1] (LQG) is a theory of quantum gravity that takes origin from the tetrad formulation of General Relativity [1, 2]. We define a tetrad as a quadrupole of 1-forms $e^I_\mu(x)$, $I = 0, 1, 2, 3$, such that:

$$g_{\mu\nu}(x) = e^I_\mu(x) e^J_\nu(x) \eta_{IJ}\,, \tag{2.19}$$

where η_{IJ} is the flat Minkowski metric. By definition, the tetrads provide an isomorphism between a general reference frame and an inertial one. The capital Latin indices thus carry representations of the Lorentz group and transform under the action of Lorentz matrices. Contracting vectors and tensors with tetrads, we obtain objects that again transform under the Lorentz group, i.e., we perform a mapping from a tangent bundle of the spacetime to a Lorentz principal bundle with connection ω^{IJ}_μ. We can use this connection to define a covariant derivative of the fibres as:

$$D_\mu v^I(x) = \partial_\mu v^I(x) + \omega^I_{\mu\,J}(x) v^J(x)\,, \tag{2.20}$$

and one for objects with both kinds of indices:

$$\mathcal{D}_\mu e^I_\nu(x) = \partial_\mu e^I_\nu(x) + \omega^I_{\mu\,J}(x) e^J_\nu(x) - \Gamma^\rho_{\mu\nu}(x) e^I_\rho(x)\,, \tag{2.21}$$

[1]In d-dimensions, the curvature is concentrated around $(d-2)$-simplices called links.

where $\Gamma^{\rho}_{\mu\nu}(x)$ is the Levi-Civita connection. Considering that $\Gamma^{\rho}_{\mu\nu}(x)$ is metric compatible, we need to require that $\mathcal{D}_\mu e^I_\nu \equiv 0$. In this case, we call ω a spin connection and it becomes a 1-form with values in the Lie algebra of the Spin group. We define the curvature associated to this connection as:

$$F^{IJ} = \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ} \,, \tag{2.22}$$

with components

$$F^{IJ}_{\mu\nu} = \partial_\mu \omega^{IJ}_\nu - \partial_\nu \omega^{IJ}_\mu + \omega^{I}_{\mu K}\omega^{KJ}_\nu - \omega^{J}_{\mu K}\omega^{KI}_\nu \,. \tag{2.23}$$

Now, we are able to rewrite the Einstein-Hilbert action in its tetrad formulation [2] (or Palatini formulation) as:

$$S_{EH}(e^I_\mu, \omega^{IJ}_\mu) = \frac{1}{2}\varepsilon_{IJKL}\int_{\mathcal{M}} e^I \wedge e^J \wedge F^{KL}(\omega)\,. \tag{2.24}$$

where $\mathcal{M}$ is the spacetime manifold. This action gives the same equation of motion of its conventional form but, apart from the usual invariance under diffeomorphism, presents a gauge symmetry under local Lorentz transformations. A remarkable feature is that the connection is raised to be an independent field, nevertheless, if the terad is non-degenerate, the equation of motion coming from varying the action with respect to ω just imposes the structure of spin connection and does not add any new element to the physics of spacetime. In a four dimensional spacetime, we can rewrite the action (2.24) in terms of a constrained BF theory (Plebanski formulation of GR) [5]:

$$S_1(\omega, B) = \int_{\mathcal{M}} \left[B^{IJ} \wedge F_{IJ}(\omega) - \frac{1}{2}\mu_{IJKL} B^{KL} \wedge B^{IJ} \right]. \tag{2.25}$$

The B field is a 2-form valued in the Lie algebra of the Lorentz group, while the Lagrange multipliers appearing in the second term of (2.25) satisfy $\mu_{[IJ][KL]} = \mu_{[KL][IJ]}$. Their equation of motion have four sector of solutions and, choosing one of the two sectors $B^{IJ} = \pm\frac{1}{2}\varepsilon^{IJ}{}_{KL} e^K \wedge e^L$, we recover the formulation (2.24).

Equation (2.25) can be generalized to include another term in the action encoding the dynamics of spacetime as follows:

$$S_2(\omega, B) = \int_{\mathcal{M}} \left[B^{IJ} \wedge F_{IJ}(\omega) + \frac{1}{\gamma}(*B)^{IJ} \wedge F_{IJ}(\omega) - \frac{1}{2}\mu_{IJKL} B^{KL} \wedge B^{IJ} \right]. \tag{2.26}$$

The action (2.26) is called the Plebanski-Holst action [4] and, at the classical level, it is equivalent to (2.24) because the additional terms vanish on shell. Nevertheless, the parameter γ, known as Immirzi parameter, turns out to be fundamental in the quantum theory.

The advantage of the formulation (2.26) comes from the fact that we know how to provide, a discretization and a covariant quantization procedure for a BF theory. Thus, from (2.26), we can construct a model of quantum gravity. Given a simplicial complex Δ, we should be able to reconstruct an associated discrete tetrad related to the bivectors B_f corresponding to the triangles of the complex. In order to do this, it is necessary to impose some constraints on the bivectors requiring the closure of tetrahedra and identifying the bivectors as orthogonal to them. The resulting constraints, known as simplicial constraints, that have to be imposed on the B fields read:

$$\forall \text{ tetrahedra } t \in \Delta, \quad \exists n_t \in S^3 \mid (B_f - \gamma * B_f)^{IJ} n_{tJ} = 0\,,\ \forall B_f,\ f \subset t\,. \quad (2.27)$$

On the other hand, General Relativity can be given a gauge formulation within the ADM formalism [3], which provides a formulation suitable for a canonical analysis of gravity. In this framework, one induces a splitting into a spatial $3d$ manifold Σ and a time coordinate which plays the role of gauge parameter. Now, one rewrites the Einstein-Hilbert action in terms of the following spatial quantities (a gauge fixing of the tetrad is implied):

$$A^I_a = \frac{1}{2}\omega^{JK}_a \varepsilon^I{}_{JK} + \gamma\omega^{0I}_a\,, \quad (2.28)$$

$$E^a_I = \det(e) e^a_I\,, \quad (2.29)$$

where A^I_a is called Ashtekar-Barbero-Immirzi connection and E^a_I is the densitized triad. Aiming at quantizing the theory, a useful change of coordinates is given by the holonomy-flux set, defined as

$$h_C(A) = \mathbf{P}\exp\left\{\int_C A\right\} \quad \in\ SU(2) \quad (2.30)$$

$$E(S) = \int_S (*E_J)n^J \quad \in\ su(2)\,, \quad (2.31)$$

where $\mathbf{P}$ is the path ordering operator, C is a generic path and n^J is the orthonormal vector to the surface S. Rising the parallel transports of the Ashtekar-Barbero-Immirzi connection and the fluxes of densitized triad to operator quantities, is the basic quantization procedure of LQG.

The holonomy-flux algebra still describes a continuous theory because one has to consider all possible paths in the manifold. A basis for the space of path is realizable by considering all possible graphs (without any restriction concerning the valence of vertices) embedded into the manifold. This is the main point of spin foam models [7], which is a proposal of a covariant form of LQG as state sum models. From the point of view of canonical LQG, the aim of spin foam models is to construct a physical inner product for quantum gravity states. Given a 4-manifold M with boundaries Σ_1 and Σ_2 and given a diffeomorphism class of 3-metrics $[\hat{g}_1]$ and $[\hat{g}_2]$ on these boundaries, we want to compute [14, 6, 7]:

$$\langle[\hat{g}_1]|\mathcal{P}|[\hat{g}_2]\rangle = \int_{\mathcal{M}} \mathcal{D}[g]e^{iS_2(\omega,B)}\,. \quad (2.32)$$

In this sum, $\mathcal{M}$ is the space of all 4-metrics modulo 4-diffeomorphism that have $[\hat{g}_1]$ and $[\hat{g}_2]$ as boundaries, while $\mathcal{P}$ is the projector on the kernel of the Hamiltonian constraint arising from a canonical analysis of General Relativity [3]. Equation (2.32) is purely formal at the non-perturbative level, but thanks to the mathematical structures of LQG, we can represent it as a sum over histories. In this framework, states are identified with spin networks Γ_j, where Γ represents a graph embedded in Σ and j the coloring of edges of the graph by representations of a Lorentz group G and the coloring of vertices of Γ by intertwiners of G. Thanks to the simplicity constraints, it is possible to rephrase the spin foams analysis in terms of representations of $SU(2)$. The type of states that we define are eigenstates of geometrical operators. Representations, labelling edges in Γ, give quanta of area $\propto \sqrt{j(j+1)}$ to the surfaces in the dual cellular decomposition intersecting the graph. Thus, we can represent a spacetime as a history between spin network states. The evolution process transforms edges of Γ in faces of the resulting spin foam $\mathcal{F}$ (thus colored by representations j of G), vertices becomes edges in $\mathcal{F}$ (thus colored by intertwiners i of G) and changes of topology occur at vertices of the foam. Basically, a spin foam is the set of faces of the $*$-dual of a cellular decomposition of spacetime. By suitable restrictions on $\mathcal{F}$, we can impose that this discretization is indeed a pseudo-manifold as defined in the previous section.

To characterize a spin foam model, we need to specify the local amplitudes $A_f(j_f)$, $A_e(j_{f_e}, i_e)$, $A_v(j_{f_v}, i_{e_v})$ assigned to faces, edges and vertices, respectively. Once this is done, we can define the transition amplitude associated to the spin foam $\mathcal{F}$ with boundaries Γ_1 and Γ_2 as [14, 7]

$$A(\mathcal{F}) = \langle \Gamma_1 | \Gamma_2 \rangle_{\mathcal{F}} = \sum_{j_f, i_e} \prod_f A_f(j_f) \prod_e A_e(j_{f_e}, i_e) \prod_v A_v(j_{f_v}, i_{e_v}) \,. \tag{2.33}$$

We recover again the contact with simplicial gravity in terms of the graph dual to the discretization.

The previous structure can be given in field theoretic sense by defining GFT models [13, 9] (see also the seminal works [22, 23]). Choosing particular GFT models we can match the quantum gravity framework of LQG and the respective spin foams formulation. In $3d$, the GFT action providing this link is known as the Boulatov model [19] while, in $4d$, it is known as the Ooguri model [20] and both models, originally were formulated to give a field theoretic quantization formalism for BF theory.

To be more explicit, let us discuss the $4d$ case. We define the fields to be real functions over four copies of the group $G = SU(2)$. Using a Peter-Weyl transform, the fields decompose as:

$$\phi(g_1, g_2, g_3, g_4) = \sum_{j_i, m_i, n_i} \phi^{m_1 n_1 \ldots m_4 n_4}_{j_1 \cdots j_4} D^{j_1}_{m_1 n_1}(g_1) \ldots D^{j_4}_{m_4 n_4}(g_4) \,, \tag{2.34}$$

with $g_i \in SU(2)$ and $D^j_{mn}(g)$ are the Wigner matrices for the j representation of the element g. We then require the fields to be invariant under the right action of the group:

$$\phi(g_1 h, g_2 h, g_3 h, g_4 h) = \phi(g_1, g_2, g_3, g_4) \,, \, \forall \, h \in SU(2) \,. \tag{2.35}$$

A way to implement (2.35) is to write

$$\phi(g_1, g_2, g_3, g_4) = \int_G dh\, \phi(g_1 h, g_2 h, g_3 h, g_4 h)\,, \tag{2.36}$$

where dh is the normalized invariant Haar measure on $SU(2)$.

We consider the following action [20]

$$S = \frac{1}{2}\int \prod_{i=1}^{4} dg_i\, \phi^2(g_1, g_2, g_3, g_4) + \frac{\lambda}{5!}\int \prod_{i=1}^{10} \phi(g_1, g_2, g_3, g_4) \times \phi(g_4, g_5, g_6, g_7)\phi(g_7, g_3, g_8, g_9)\phi(g_9, g_6, g_2, g_{10})\phi(g_{10}, g_8, g_5, g_1)\,. \tag{2.37}$$

At the quantum level, Feynman rules associated with the model (2.37) can be interpreted as follows: fields are associated with tetrahedra, each argument of the field is viewed as a boundary triangle of the tetrahedron. In this interpretation,

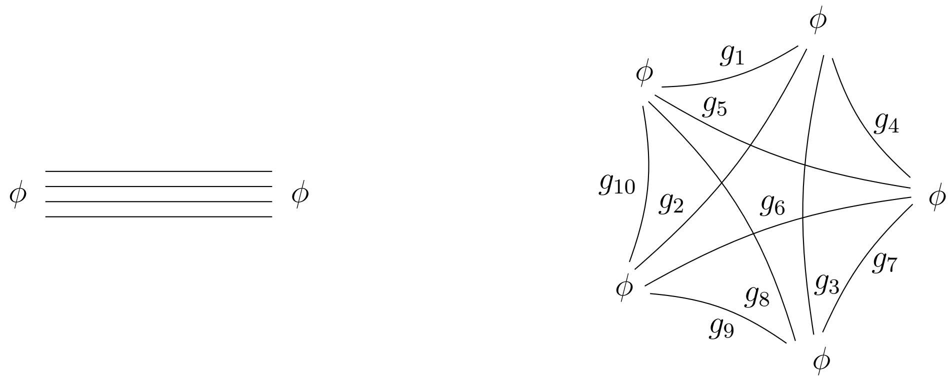


Figure 2.3: Feynman rules for the Ooguri model.

the interaction term follows the combinatoric structure of a 4-simplex, while the kinetic part describes the gluing of two 4-simplices by identifying one of their boundary tetrahedron (see Fig.2.3). This is a model of GFT whose Feynman graphs are dual to a simplicial decomposition of a manifold and where geometric variables are quantized according to the representations of $SU(2)$. Furthermore, the amplitudes of the Ooguri model have a precise form which can be shown equivalent to amplitudes of a simplicial path integral for BF theory.

2.3 Tensorial Group Field Theories

We introduce, at this point, the formalism of a special class of GFT's of particular interest for our work, known as Tensorial Group Field Theories (TGFT) [38]-[45][31, 34, 36, 37].

Consider a field ϕ defined over d-copies of a group manifold G, $\phi : G^{\times d} \longrightarrow \mathbb{C}$. For the moment, we focus just on the case of compact Lie group manifold.

Without assuming any symmetry under permutations of field labels and using Peter-Weyl theorem, the fields decompose as follows:

$$\phi(g_1, \ldots, g_d) = \sum_{\mathbf{P}} \phi_{\mathbf{P}} \prod_{i=1}^{d} D^{p_i}(g_i)\,, \tag{2.38}$$

with $\mathbf{P} = (p_1, \ldots, p_d)$, $g_i \in G$ and where the functions $D^{p_i}(g_i)$ form a complete orthonormal basis of functions on the group characterized by the labels p_i. In a TGFT model, we require fields to have tensorial properties under basis changes. We define a rank d covariant complex tensor $\phi_{\mathbf{P}}$ to transform through the action of the tensor product of unitary representations of the group $\bigotimes_{i=1}^{d} U^{(i)}$, each of them acting independently over the indices of labels of the fields, i.e.:

$$\phi_{p'_1,\ldots,p'_d} = \sum_{\mathbf{P}} U^{(1)}_{p'_1,p_1} \ldots U^{(d)}_{p'_d,p_d}\, \phi_{p_1,\ldots,p_d}\,. \tag{2.39}$$

The complex conjugate field will then be the contravariant tensor transforming as:

$$\overline{\phi}_{p'_1,\ldots,p'_d} = \sum_{\mathbf{P}} (U^{\dagger})^{(d)}_{p'_d,p_d} \ldots (U^{\dagger})^{(1)}_{p'_1,p_1}\, \overline{\phi}_{p_1,\ldots,p_d}\,. \tag{2.40}$$

Among the invariants built out of ϕ and $\overline{\phi}$, the “trace invariants” turn out to be an important class of invariants. They allow us to have a strong control on the combinatorial structure of convolutions, thus, trace invariants are relevant for the construction of TGFT’s renormalizable actions. The name trace is reminiscent of traces over matrices which indeed are classical unitary invariants. Hence the tensor trace invariants generalize traces over matrices. They are obtained contracting pairwise the indices with the same position of covariant and contravariant tensors and saturating all of them. In this way, they always show the same number of ϕ and $\overline{\phi}$. A simple example is the following:

$$\mathrm{Tr}(\phi\overline{\phi}) = \sum_{\mathbf{P},\mathbf{Q}} \phi_{\mathbf{P}} \overline{\phi}_{\mathbf{Q}} \prod_{i=1}^{d} \delta_{p_i,q_i}\,. \tag{2.41}$$

Considering that $\phi_{\mathbf{P}}$ (resp. $\overline{\phi}_{\mathbf{P}}$) transforms as a complex vector (resp. 1-form) under the action of the unitary representations of G on one single index, the fundamental theorem on classical invariants for U on each index entails that all invariant polynomials in field entries can be written as a linear combination of trace invariants [11]. This formulation of tensor models can be adapted to the real field case, where the unitary group is replaced by the orthogonal one.

As an interesting feature which, in turn, becomes an important computational tool in several contexts, tensor invariants can be given a graphical representation as bipartite colored graphs. A tensor ϕ is represented by a (white) node with exiting d-half lines with labels; its complex conjugate is a node with a different color (black). Feynman graphs obtained from a TGFT in rank d are obtained

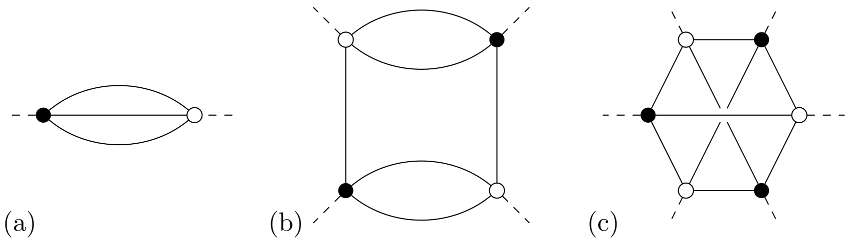


Figure 2.4: Three examples of Feynman graphs for a rank 3 TGFT's. The trace invariants used to build the interactions are: figure (a) $\mathrm{Tr}(\phi\overline{\phi})$, figure (b) an example of $\mathrm{Tr}(\phi\overline{\phi}\phi\overline{\phi})$, figure (c) an example of $\mathrm{Tr}(\phi\overline{\phi}\phi\overline{\phi}\phi\overline{\phi})$.

by attaching to trace invariants representations one propagator (dashed line) for each field obtaining a $d+1$ edge colored graph (some examples are depicted in Fig.2.4).

Trace invariants can be generalized as convolutions where the contractions are made by operators different from the delta distribution. In this case, the resulting object is not guaranteed to be an invariant. Hence, a generic convolution by non trivial kernels of tensors breaks the unitary invariance. We write a generic action for a TGFT model symbolically as:

$$S[\phi,\overline{\phi}] = \mathrm{Tr}(\overline{\phi}\cdot\mathcal{K}\cdot\phi) + S^{\mathrm{int}}[\phi,\overline{\phi}] \tag{2.42}$$
$$\mathrm{Tr}(\overline{\phi}\cdot\mathcal{K}\cdot\phi) = \sum_{\mathbf{P},\mathbf{Q}} \overline{\phi}_{\mathbf{P}}\,\mathcal{K}(\mathbf{P};\mathbf{Q})\,\phi_{\mathbf{Q}}\,, \qquad S^{\mathrm{int}}[\phi,\overline{\phi}] = \sum_{\{n_b\}} \lambda_{n_b}\mathrm{Tr}(\mathcal{V}_{n_b}\cdot\phi^n\cdot\overline{\phi}^n)\,.$$

Here $\mathcal{K}$ and $\mathcal{V}_n$ are kernels implementing the convolutions in the kinetic and interaction terms, respectively, where n indicates the numbers of covariant and contravariant fields appearing in the vertices, b labels the combinatorics of convolutions and λ_{n_b} is a coupling constant for the interaction n_b. The formalism presented so far can be easily generalized to the a non-compact group manifold G using the Plancherel formula to decompose fields and replacing the trace definition or sums over discrete indices by integrals over continuous variables.

Given an action $S[\phi,\overline{\phi}]$, the partition function is defined as usual:

$$\mathcal{Z}[J,\overline{J}] = e^{W[J]} = \int d\phi d\overline{\phi}\; e^{-S[\phi,\overline{\phi}]+\mathrm{Tr}(J\cdot\overline{\phi})+\mathrm{Tr}(\overline{J}\cdot\phi)}\,, \tag{2.43}$$

where J is a rank d complex source term and $\mathrm{Tr}(J\cdot\overline{\phi})$ is defined in (2.41).

Chapter 3

The Functional Renormalization Group

In Physics, one of the most relevant questions is whether a theory that applies in some context (in QFT, at some energy scale) is fundamental or not, that is, if it holds also in other contexts or not. The framework which allows us to address this problem is at the heart of Quantum Field Theory, giving sense of its inconsistencies, like infinities, and yielding its power of prediction. This is the renormalization group (RG), which pinpoints the relevant processes at some scale and, thus, the number of measures that we need to perform in order to have predictions. Along the evolution of Physics, many ways to implement this procedure have been developed. In this section, we present a non-perturbative formulation of the RG known as the Functional Renormalization Group (FRG). Introduced by Wilson in 1971 [48], the FRG has been developed by Wilson and Kogut [49] and finds also strong motivations after the works by Kadanoff [50], Polchinski [51] and Wetterich [52].

3.1 The Wilsonian idea of RG

The Renormalization Group is a group of transformations that describes the scaling properties of a given theory when energy scales are tuned (strictly speaking, it is a well-known fact that this is not a group, but a semi-group). In ordinary Quantum Field Theory, one of the most common approaches is the perturbative one. Perturbation theory rests on a regularization scheme for diverging Green's functions which consists in cutting-off the most "dangerous" sector of modes of the propagator and in the introduction, in the Lagrangian of the theory, of a set of counterterms which should cure pathological n-point functions.

The perturbative RG has led to a rich and deep understanding of QFT. Nevertheless, that study is clearly not without limitations. In 1971, Wilson pointed out two basic problems arising in the perturbative formulation of RG [48, 53, 54]:

- in general, we cannot compute exactly the contribution of quantum fluctuations and an approximation scheme is needed;

- in perturbation theory, fluctuations are summed in an inappropriate way since, in Feynman diagrams, they all are treated on the same footing, independently from their wavelengths.

Thus, Wilson's idea is, basically, to re-organize the way that fluctuations are summed over, in order to take into account which fluctuations are most relevant at a given scale. To understand this process, let us first consider a lattice theory. The short distance physics depends, of course, on the details of the lattice (spacing and shape of elementary cells) and the correlation functions, computed on a given site, are mainly affected by nearest neighbors. In other words, the most relevant fluctuations have wave length of the order of the lattice spacing. On the contrary, if we focus on the long-distance physics, we do not have any information on the lattice details, and their contributions are averaged out. Starting from this pedagogical example, we formalize a recipe to construct an effective theory for a subset of degrees of freedom, by integrating out all the others.

Varying the set of degrees of freedom, we obtain a description of the scaling of the theory that generates what we call the "renormalization flow". This is a dynamical system in the space of coupling constants. By definition, the critical surface in the coupling space is the set of points where the correlation length is diverging. For a second order phase transition, only one parameter has to be tuned to make the correlation length blowing at infinity, hence, the critical surface has co-dimension 1. It is important to point out that, while the physical trajectories lead the theory across the critical surface, the RG trajectories are totally non-physical, in fact they describe transformations that is not possible to perform over a system in a laboratory. The critical surface turns out to be stable under RG transformations, which means that RG trajectories lie on it. This is a consequence of the fact that any phase transition is a signal of a non-analytic discontinuity in the flow of a model, and the two subsets of the space of theories, corresponding to the two phases, cannot be continuously mapped one onto the other.

Studying the renormalization flow, it is possible to identify fixed points on the critical surface (of course their existence depends on the model under examination). In this case, we have a set of trajectories corresponding to theories with different short-distance physics, but leading to the same long-distance behavior.

We remark that all the formal quantities appearing in the FRG formalism, define the path integral for a quantum theory only a posteriori. After a solution for the flow, that is a fixed point, is found, we can make sense of the functional used in the procedure and specify the physics behind the partition function.

Although relying on the same concepts developed by Wilson, there are two different main implementations of the FRG, namely, the Wilson-Polchinski approach and the effective average action method. In this work, we mainly deal with the second, but, in order to get a better understanding of the formal aspects of FRG, we will present both of them.

3.1.1 Splitting of modes

The Wilson-Polchinski approach implements the FRG in a statistical field theory context and needs a clear ordering of momentum norms, thus it is usually applied on Euclidean spaces. The microscopic physics is supposed to correspond to a momentum scale Λ identified, up to a constant, with the inverse of the relevant length (in the discrete set the lattice spacing). We consider the following partition function (in loose notations, where integrals are over a given Euclidean space) [51]:

$$\mathcal{Z}[J] = \int d\mu_{C_\Lambda}(\phi)\ \exp\left\{-\int \mathcal{V}(\phi) + \int J\phi\right\}, \tag{3.1}$$

where J is a source field and $d\mu_{C_\Lambda}(\phi)$ is a functional Gaussian measure over fields with a covariance C, with cut-off at the scale Λ, namely,

$$d\mu_{C_\Lambda}(\phi) = \mathcal{D}\phi(x)\ \exp\left\{-\frac{1}{2}\int_{X,Y} \phi(x) C_\Lambda^{-1}(x-y)\phi(y)\right\}. \tag{3.2}$$

In momentum space, the cut-offed covariance is chosen of the form:

$$C_\Lambda(p) = (1-\theta_\varepsilon(p,\Lambda))C(p)\,, \tag{3.3}$$

$C(p)$ being the covariance of the full theory and $\theta_\varepsilon(p,\Lambda)$ is a cut-off function obtained by smoothing the step function centered around Λ in a neighbourhood of radius ε. Naturally, the sharp case in which $\varepsilon = 0$ is allowed, although less physical. Now, we want to implement a splitting of modes with respect to a ultraviolet cut-off k for the slow modes, in order to define an effective theory for a long-distance scale. We define:

$$\phi_p = \phi_{p,<} + \phi_{p,>}\,, \tag{3.4}$$

where $\phi_{p,<}$ are the slow modes with $|p| < k$, while $\phi_{p,>}$ are the rapid ones with $|p| > k$. We perform an association of the type:

$$\phi_p \longrightarrow C_\Lambda(p)\,, \tag{3.5}$$

$$\phi_{p,<} \longrightarrow C_k(p)\,, \tag{3.6}$$

$$\phi_{p,>} \longrightarrow C_\Lambda(p) - C_k(p)\,. \tag{3.7}$$

It is important to notice that (3.4) holds for every p and, in general, ϕ does not coincide neither with $\phi_<$ on $[0,k]$, nor with $\phi_>$ on $[k,\Lambda]$.

The partition function in terms of the split quantities has the form:

$$\mathcal{Z}[J] = \int d\mu_{C_k}(\phi_<) d\mu_{C_\Lambda - C_k}(\phi_>)\ \exp\left\{-\int \mathcal{V}(\phi_< + \phi_>) + \int J\phi_< + \int J\phi_>\right\} \tag{3.8}$$

and, at least formally, we perform the integration over the rapid modes:

$$e^{-\int \mathcal{V}_k(\phi_<)} = \int d\mu_{C_\Lambda - C_k}(\phi_>)\ e^{-\int \mathcal{V}(\phi_< + \phi_>) + \int J\phi_>}\,. \tag{3.9}$$

This is the formal framework that implements the Wilson-Polchinski approach and allows to formally define an effective theory for the slow modes. Thanks to this procedure, it is possible to derive a differential equation that describes the behavior of $\mathcal{V}_k$ with respect to k and to extract the renormalization flow of the theory.

3.1.2 Effective average action

The effective average action method presents conceptual and formal differences compared with the Wilson-Polchinski approach. While in the previous framework, we obtain a Hamiltonian for slow modes that are not integrated out, in the present approach, we obtain the Gibbs free energy $\Gamma_k[\varphi]$ of the rapid modes that are averaged [52], where we have $\varphi = \langle\phi\rangle$. The idea is to build a one-parameter family of models labeled by a scale k, that now plays the role of an infrared cut-off for the rapid modes. This forces us to impose conditions on the scale dependent Gibbs free energy, i.e.

- given a ultraviolet cut-off Λ for the rapid modes, the free energy at the scale Λ has to coincide with the microscopical action:

$$\Gamma_{k=\Lambda}[\varphi] = S[\phi = \varphi]\,; \tag{3.10}$$

- when $k = 0$, we require the free energy to describe the full theory in order to recover the original model and, so

$$\Gamma_{k=0}[\varphi] = \Gamma[\varphi]\,. \tag{3.11}$$

Building of a one-parameter class of theories is implemented by adding to the action a regulator, mass-like, term that endows the slow modes with a large mass in order to approximately freeze their propagation:

$$\mathcal{Z}_k[J] = \int \mathcal{D}\phi(x)\ \exp\left\{-S[\phi] - \Delta S_k[\phi] + \int J\phi\right\}, \tag{3.12}$$

with

$$\Delta S_k[\phi] = \frac{1}{2}\int dq\ \phi_q\, R_k(q)\, \phi_{-q}\,. \tag{3.13}$$

The regulator function $R_k(q)$ needs, of course, to fit the condition that we impose on the free energy. Hence, it must satisfy the following requirements:

- $R_{k=0}(q) = 0$ identically ($\forall q$), so that the full theory is recovered when the cut-off is removed, and the following relation holds:

$$\mathcal{Z}_{k=0}[J] = \mathcal{Z}[J]\,; \tag{3.14}$$

- when $k = \Lambda$, all fluctuations are frozen and (3.10) holds. To achieve this condition, we demand $R_k(q)$ to diverge at the UV cut-off. In any case, a useful, though approximate, way to implement the freezing of the slow modes is to choose $R_k(q)$ of the order of Λ^2 for all momenta q at the scale Λ;

- for $k \in [0, \Lambda]$, the rapid modes are almost unaffected by the regulator, while the slow ones must get a mass big enough to decouple them from the long distance physics, i.e.:

$$R_k(q) \simeq 0\,, \quad \text{for} \quad |q| > k\,. \tag{3.15}$$

A last remark must be made. If we define the scale dependent generating functional of connected Green's functions, say $W_k[J] = \log \mathcal{Z}_k[J]$, one shows that it cannot be related with $\Gamma_k[\varphi]$ by the usual Legendre transform without violating (3.10). Hence, we define the following modified transform:

$$\Gamma_k[\varphi] = \sup_J \left[\int J\varphi - W_k[J] - \frac{1}{2}\int dq\; R_k(q)\,\varphi_q\varphi_{-q} \right]. \tag{3.16}$$

In this way, it becomes easy to verify that, if $R_k(q)$ satisfies the above conditions, then $\Gamma_{k=\Lambda}[\varphi] = S[\phi = \varphi]$.

3.2 Exact RG equations

In this section, we detail two practical ways to implement the above formulation resulting in equations of the flow of the couplings known as Exact Renormalization Group Equations. We follow the developments of [53].

3.2.1 Polchinski equation

The Polchinski equation is a partial differential equation that describes the scaling properties of the functional $W_k[J] = \log \mathcal{Z}_k[J]$ with respect to the cut-off scale k. Let us now focus on a theory with complex fields, which is of particular interest for our following analysis in the next chapter.

The scale dependent partition function can be written as:

$$\mathcal{Z}_k[J,\overline{J}] = \int \mathcal{D}\phi(x)\mathcal{D}\overline{\phi}(x)\; \exp\left\{ -S[\phi] - \int dq\; \overline{\phi}_q R_k(q)\phi_q + \int dq \left(\overline{J}_q\phi_q + J_q\overline{\phi}_q \right) \right\}. \tag{3.17}$$

Acting with the derivative with respect to k on (3.17), we find:

$$\begin{aligned} \partial_k e^{W_k} &= -\int d\phi d\overline{\phi}\, \left(\partial_k \int dq\; \overline{\phi}_q R_k(q)\phi_q \right) \exp\{-S - \triangle S_k + \int J\overline{\phi} + \int \overline{J}\phi\} \\ &= -\left(\int dq\; \partial_k R_k(q) \frac{\delta}{\delta J_q}\frac{\delta}{\delta \overline{J}_q} \right) e^{W_k[J,\overline{J}]} \\ &= \left(-\int dq\; \partial_k R_k(q) \left[\frac{\delta^2 W_k[J,\overline{J}]}{\delta J_q \delta \overline{J}_q} + \frac{\delta W_k[J,\overline{J}]}{\delta J_q}\frac{\delta W_k[J,\overline{J}]}{\delta \overline{J}_q} \right] \right) e^{W_k}\,. \end{aligned} \tag{3.18}$$

And, thus, we can write:

$$\partial_k W_k = -\int dq\, \partial_k R_k(q) \left[\frac{\delta^2 W_k}{\delta J_q \delta \overline{J}_q} + \frac{\delta W_k}{\delta J_q} \frac{\delta W_k}{\delta \overline{J}_q} \right] \tag{3.19}$$

which is the well-known Polchinski equation for this type of models.

3.2.2 Wetterich equation

In the previous section, we studied the scaling of the functional W_k, which is a functional of the sources. Our aim now is to encode the corresponding physical informations into an equation for the effective action, which is a functional of the mean values of the fields [52]. For that purpose, we rewrite the derivative with respect to k. Note that, in (3.19), the relation holds at fixed J and $\overline{J}$, while now, we need to express the same differential at fixed φ and $\overline{\varphi}$. To perform this, we use the following identity:

$$\partial_k|_{J,\overline{J}} \;=\; \partial_k|_{\varphi,\overline{\varphi}} + \int dq\, \partial_k \varphi_q \Big|_{J,\overline{J}} \frac{\delta}{\delta \varphi_q} + \int dq\, \partial_k \overline{\varphi}_q \Big|_{J,\overline{J}} \frac{\delta}{\delta \overline{\varphi}_q} \,. \tag{3.20}$$

The Legendre transform relating $W_k[J]$ and $\Gamma_k[\varphi]$ is given by

$$\Gamma_k[\varphi, \overline{\varphi}] = \sup_{J,\overline{J}} \left[\int dq \Big(\overline{J}_q \phi_q + J_q \overline{\phi}_q \Big) - W_k[J, \overline{J}] - \int dq\, \overline{\varphi}_q\, R_k(q)\, \varphi_q \right] . \tag{3.21}$$

Considering (3.21), one gets:

$$\begin{aligned} \partial_k \Gamma_k \Big|_{J,\overline{J}} + \partial_k W_k &= \int dq\, J_q\, \partial_k \overline{\varphi}_q + \int dq\, \overline{J}_q\, \partial_k \varphi_q \\ &- \int dq\, \partial_k \overline{\varphi}_q\, R_k(q)\, \varphi_q - \int dq\, \overline{\varphi}_q\, \partial_k R_k(q)\, \varphi_q - \int dq\, \overline{\varphi}_q\, R_k(q)\, \partial_k \varphi_q \,. \end{aligned} \tag{3.22}$$

On the other hand, using (3.20) and (3.19), we infer:

$$\begin{aligned} &\partial_k \Gamma_k|_{J,\overline{J}} + \partial_k W_k = \partial_k \Gamma_k|_{\varphi,\overline{\varphi}} \\ &+ \int dq\, \partial_k \varphi_q|_{J,\overline{J}} \frac{\delta \Gamma_k}{\delta \varphi_q} + \int dq\, \partial_k \overline{\varphi}_q|_{J,\overline{J}} \frac{\delta \Gamma_k}{\delta \overline{\varphi}_q} - \int dq\, \partial_k R_k(q) \left[\frac{\delta^2 W_k}{\delta J_q \delta \overline{J}_q} + \frac{\delta W_k}{\delta J_q} \frac{\delta W_k}{\delta \overline{J}_q} \right] \\ &= \partial_k \Gamma_k|_{\varphi,\overline{\varphi}} + \int dq\, \partial_k \varphi_q\, \overline{J}_q - \int dq\, \partial_k \varphi_q\, R_k(q)\, \overline{\varphi}_q \\ &+ \int dq\, \partial_k \overline{\varphi}_q\, J_q - \int dq\, \partial_k \overline{\varphi}_q\, R_k(q)\, \varphi_q - \int dq\, \partial_k R_k(q) \left[\frac{\delta^2 W_k}{\delta J_q \delta \overline{J}_q} + \overline{\varphi}_q \varphi_q \right] . \end{aligned} \tag{3.23}$$

Comparing the last two equations, the Wetterich equation follows [52]:

$$\partial_k \Gamma_k = \int dq \partial_k R_k(q) \left[\frac{\delta^2 W_k}{\delta J_q \delta \overline{J}_q} \right]$$

$$= \int dq\, \partial_k R_k(q) \left[\frac{\delta^2 \Gamma_k}{\delta \overline{\varphi}_q \delta \varphi_q} + R_k(q)\delta(0) \right]^{-1} . \tag{3.24}$$

To prove the last equality, consider the following small calculation:

$$\begin{aligned} \delta(p-q) = \frac{\delta \varphi_p}{\delta \varphi_q} &= \frac{\delta^2 W_k}{\delta \varphi_q \delta \overline{J}_p} \\ &= \int dl\, \frac{\delta^2 W_k}{\delta \overline{J}_p \delta J_l} \frac{\delta J_l}{\delta \varphi_q} \\ &= \int dl\, \frac{\delta^2 W_k}{\delta \overline{J}_p) \delta J_l} \left[\frac{\delta^2 \Gamma_k}{\delta \overline{\varphi}_l \delta \varphi_q} + R_k(l)\delta(l-q) \right] , \end{aligned} \tag{3.25}$$

which re-expresses as:

$$G_k^{(2)} \cdot [\Gamma_k^{(2)} + R_k] = \delta\,, \tag{3.26}$$

where $G_k^{(2)}$ is the scale dependent 2-point Green's function. Introducing the dimensionless time $t = \log k$, we write

$$\partial_t \Gamma_k \;\; = \int dq\, \partial_t R_k(q) \left[\Gamma_k^{(2)} + R_k \right]^{-1}_{q,q} . \tag{3.27}$$

From (3.26), one realizes that the Wetterich equation has a 1-loop structure. Considering that up to this point we did not perform any kind of approximation, we say that (3.27) is an exact functional equation.

Although we have expressed the problem of extracting the flow of the theory in terms of a partial differential equation in one parameter, we still have the issue that, in contrast with perturbative renormalization, all possible compatible couplings are allowed in Γ_k. If we want to perform practical computations, we need an approximation scheme for the form of the free energy. Usually, this is done by truncating Γ_k to a maximal power in the fields and in their derivative.

Already from (3.27), the Wetterich equation shows pathological IR divergences due to the presence of $\delta(0)$ arising from the two point Green's function computed at a single point $G_k^{(2)}(q,q)$[1]. A particular approximation procedure which allows to cure this problem is called the local potential approximation (LPA). Let us quickly review it.

First, we use the following ansatz for the free energy expressed in direct space:

$$\Gamma_k[\varphi, \overline{\varphi}] = \int dx \left(U(\varphi, \overline{\varphi}) + \frac{1}{2} (\nabla \varphi \nabla \overline{\varphi}) \right) , \tag{3.28}$$

and then define the local potential U_k as

$$U_k(\varphi_{\text{unif}}, \overline{\varphi}_{\text{unif}}) = \frac{1}{\Omega} \Gamma_k[\varphi_{\text{unif}}, \overline{\varphi}_{\text{unif}}] . \tag{3.29}$$

[1] While in the local case these divergent delta functions are homogeneous and proportional to the whole volume of the system, in non-local theories they arise, in general, in a non-homogeneous combination strictly dependent on the combinatorics of the interaction.

where Ω is the volume of the system. Now, using the fact that $\delta(0) = \Omega(2\pi)^{-d}$, we can write the Wetterich equation for the potential as:

$$\partial_t U_k(\rho) = \int dq \, \frac{\partial_t R_k(q)}{q^2 + R_k(q) + U'_k(\rho + 2\rho U''_k(\rho))} \,, \tag{3.30}$$

where $\rho = |\varphi|^2$ and U' and U'' are derivatives with respect to ρ.

Unfortunately, as we will explain in the next section, this procedure cannot be applied, at least, in the same straightforward way, to non-local theories as TGFT's. As we will show, this point and several other issues that we will list and emphasize, classify TGFT's as non-standard field theories of a new type.

3.3 FRG formulation for TGFT's

Now the generalization of the FRG formalism to tensor models and TGFT's is straightforward [66]. Given a partition function of the type (2.43), we choose a UV cut-off M and a IR cut-off N. Adding to the action a regulator term of the form:

$$\Delta S_N[\phi, \overline{\phi}] = \mathrm{Tr}(\overline{\phi} \cdot R_N \cdot \phi) = \sum_{\mathbf{P},\mathbf{P}'} \overline{\phi}_{\mathbf{P}} \, R_N(\mathbf{P};\mathbf{P}') \, \phi_{\mathbf{P}'} \,, \tag{3.31}$$

we can perform the splitting in high and low modes. In particular, given an action with a generic kernel depending on the derivative of the fields $\mathcal{K}(\nabla\phi)$ and a generalized Fourier transform $\mathcal{F}$, if we choose R_N to be of the specific form

$$R_N(\mathbf{P};\mathbf{P}') = N\delta_{\mathbf{P},\mathbf{P}'} R\left(\frac{\mathcal{F}(\mathcal{K}_{\mathbf{P}})}{N}\right) , \tag{3.32}$$

we need to impose on the profile function $R(z)$ the following conditions: positivity $R(z) \geq 0$, to indeed suppress and not emphasize modes; monotonicity $\frac{d}{dz}R(z) \leq 0$, so that high modes will not be suppressed more that low modes; $R(0) > 0$ and $\lim_{z\to+\infty} R(z) = 0$ to exclude constant profile functions. The last requirement, together with the form (3.32), guarantees that the regulator is removed for $Z \to 0$. Applying the same procedure shown in section 3.2, we define the scale dependent partition function as:

$$\mathcal{Z}_N[J,\overline{J}] = e^{W_N[J,\overline{J}]} = \int d\phi d\overline{\phi} \; e^{-S[\phi,\overline{\phi}] - \Delta S_N[\phi,\overline{\phi}] + \mathrm{Tr}(J\cdot\overline{\phi}) + \mathrm{Tr}(\overline{J}\cdot\phi)} \tag{3.33}$$

and the generating functional of 1PI correlation functions:

$$\Gamma_N[\varphi,\overline{\varphi}] = \sup_{J,\overline{J}} \left\{ \mathrm{Tr}(J\cdot\overline{\varphi}) + \mathrm{Tr}(\overline{J}\cdot\varphi) - W_N[J,\overline{J}] - \Delta S_N[\varphi,\overline{\varphi}] \right\} . \tag{3.34}$$

Under such circumstances, the Wetterich equation takes the form:

$$\partial_t \Gamma_N[\varphi,\overline{\varphi}] = \overline{\mathrm{Tr}}\Big(\partial_t R_N \cdot [\Gamma_N^{(2)} + R_N]^{-1}\Big) \,, \tag{3.35}$$

where $t = \log N$, so that $\partial_t = N\partial_N$, and the "super"-trace symbol $\overline{\mathrm{Tr}}$ means that we are summing over all momentum indices. Fully written, the trace reads:

$$\sum_{\mathbf{P},\mathbf{P}'} \partial_t R_N(\mathbf{P};\mathbf{P}')[\Gamma_N^{(2)} + R_N]^{-1}(\mathbf{P}';\mathbf{P})\,. \tag{3.36}$$

The presence of the $\partial_t R_N$ in the Wetterich equation for TGFT's, enforces the trace to be UV-finite if the profile function and its derivative go fast enough to 0, as $z \to +\infty$. In this way, we can basically forget about the UV cut-off M. In any case, as in any resolution of differential equation, we need an initial condition of the type

$$\Gamma_{N=M}[\varphi,\overline{\varphi}] = S[\varphi,\overline{\varphi}]\,, \tag{3.37}$$

for some scale M. The problem of solving the full quantum theory is now phrased in the one of pushing the initial condition to infinity, which usually requires the existence of a UV fixed point.

One of the most striking feature arising in the application of FRG to TGFT's, is that $\Gamma_N^{(2)}$ carries inside the Wetterich equation information about the non-locality of the theory. This will back react at the level of the β-functions, in the fact that, depending on the combinatorics of the interaction, the volume contributions appearing in (3.35) will be not homogeneous and, in general, a natural definition of effective local potential does not exist.

Chapter 4

A ϕ^4 model

In this section, we arrive at the heart of this work. We successfully apply the FRG method to a rank 3 TGFT defined over a non-compact group manifold, namely $G = \mathbb{R}$. Using a lattice regularization and then a thermodynamic limit, we address the problem of IR divergences in the model, introduce a neat notion of dimension of the coupling constants which yields a well-defined system of β-functions. From this point, we study the renormalization flow of the theory, list its fixed points and, finally, discuss how these fixed points suggest the existence of different phases appearing in the model. We stress that all the variables appearing in the following chapter are position and momenta only in the sense of the abstract field theory and do not have any spacetime interpretation, which is actually related to the fields.

4.1 The model

In TGFT's, there is a special class of interactions known in the literature as "melonic" interactions (or simply "melons") [30]. These terms can be built from rank-d colored theory where interactions are dual to d simplices. In $d = 3, 4$, these d-simplex interactions are also those considered by Ambjorn et al., Boulatov and Ooguri [17, 19, 20] with the extra feature of being defined with colored fields or tensors. Melons dually represent special triangulations of the d-sphere which can be represented as the trace invariants already introduced in section 2.3.

We consider a model defined by the following action:

$$S[\phi,\overline{\phi}] = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^{\times 3}}[dx_i]_{i=1}^3\;\overline{\phi}(x_1,x_2,x_3)\Big(-\sum_{s=1}^{3}\triangle_s+\mu\Big)\phi(x_1,x_2,x_3)$$
$$+\frac{\lambda}{2(2\pi)^6}\int_{\mathbb{R}^{\times 6}}[dx_i]_{i=1}^3[dx'_j]_{j=1}^3\Big[\phi(x_1,x_2,x_3)\overline{\phi}(x'_1,x_2,x_3)\phi(x'_1,x'_2,x'_3)\overline{\phi}(x_1,x'_2,x'_3)$$
$$+\text{sym}\Big\{1\to 2\to 3\Big\}\Big]\,, \tag{4.1}$$

where the symbol sym$\{\cdot\}$ represents the rest of the colored symmetric terms in the

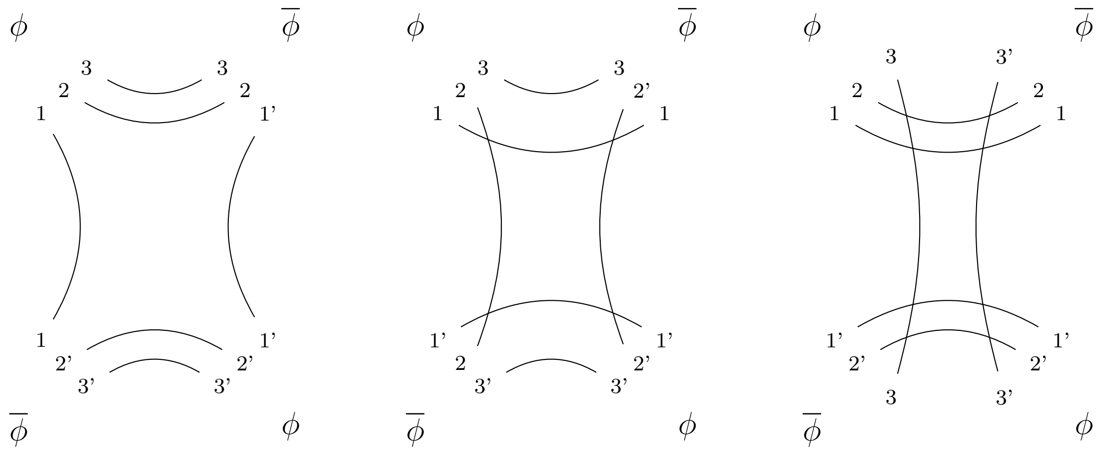


Figure 4.1: Colored symmetric interaction terms.

interaction (see Fig.4.1); μ and λ are coupling constants. As quickly realized, the interaction fully depends on all the six coordinates and this makes it non-local.

After Fourier transform, we write the action in momentum space as:

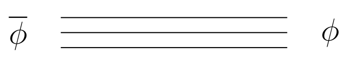
$$S[\phi,\overline{\phi}] = \int_{\mathbb{R}^{\times 3}} [dp_i]_{i=1}^3 \, \overline{\phi}_{123} \Big(\sum_s p_s^2 + \mu \Big) \phi_{123} \tag{4.2}$$
$$+ \frac{\lambda}{2} \int_{\mathbb{R}^{\times 6}} [dp_i]_{i=1}^3 [dp'_j]_{j=1}^3 \, \Big[\phi_{123}\overline{\phi}_{1'23}\phi_{1'2'3'}\overline{\phi}_{12'3'} + \mathrm{sym}\Big\{1 \to 2 \to 3\Big\} \Big] \,,$$

where we use the conventions

$$\phi_{123} = \phi_{p_1,p_2,p_3} = \phi(\mathbf{p}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^{\times 3}} [dx_i]_{i=1}^3 \; \phi(x_1,x_2,x_3)\, e^{-i\sum_i p_i x_i} \,, \tag{4.3}$$

$$\phi(x_1,x_2,x_3) = \int_{\mathbb{R}^{\times 3}} [dp_i]_{i=1}^3 \; \phi_{123}\, e^{i\sum_i p_i x_i} \,. \tag{4.4}$$

We represent the Feynman rule for the propagator as in Fig.4.2, while the combinatorics of the interaction is conserved by the Fourier transform.



$$= \Big(\sum_s p_s^2 + \mu \Big)^{-1}$$

Figure 4.2: Feynman rule for the propagator.

We can now proceed with the dimensional analysis to fix the dimensions of the coupling constants. In order to make sense of the exponentiation of the action in the partition function, we must set $[S] = 0$. Furthermore, we fix the dimensions to be in unit of the momentum, i.e., $[p] = [dp] = 1$. Now, for consistency reasons we have $[\mu] = 2$. This leads us to the following equations:

$$3 + 2[\phi] + 2 = 0 \;\Rightarrow\; [\phi] = -\frac{5}{2} \,, \tag{4.5}$$

$$[\lambda] + 6 + 4[\phi] = 0 \;\Rightarrow\; [\lambda] = 4\,. \tag{4.6}$$

4.2 Effective action and Wetterich equation

In order to proceed with the Functional Renormalization Group analysis, we introduce an IR cut-off k and a UV cut-off Λ. We need to perform a truncation on the form of the effective action that we define. To satisfy the condition (3.10), a natural choice is to write an effective action of the same form of the action itself, that is:

$$\begin{aligned}\Gamma_k[\varphi,\overline{\varphi}] = &\int_{\mathbb{R}^{\times 3}} [dp_i]_{i=1}^3\, \overline{\varphi}_{123}(Z_k \sum_s p_s^2 + \mu_k)\varphi_{123} \\ &+ \frac{\lambda_k}{2}\int_{\mathbb{R}^{\times 6}} [dp_i]_{i=1}^3 [dp'_j]_{j=1}^3 \left[\varphi_{123}\overline{\varphi}_{1'23}\varphi_{1'2'3'}\overline{\varphi}_{12'3'} + \mathrm{sym}\Big\{1\to 2\to 3\Big\}\right],\end{aligned} \tag{4.7}$$

where $\varphi = \langle\phi\rangle$. The upshot of the following analysis is to be tested by extending this truncation, including more invariants ($\mathrm{Tr}(\phi^4)$ of another type, higher order terms $\mathrm{Tr}(\phi^{2n})$, $n \geq 3$, in general, even disconnected invariants as multi-traces, $\mathrm{Tr}(\phi^{2n})\mathrm{Tr}(\phi^{2m})\dots$) and observing the convergence of the results. Enlarging the theory space is postponed for future investigations. However, as one will realize, even in the truncation given by (4.7), the calculations and the outcome of the present analysis remain highly non-trivial.

From the dimensional analysis of the previous section and from the fact that $[\Gamma_k] = 0$ and $[\varphi] = [\phi]$, one infers $[Z_k] = 0$, $[\mu_k] = [\mu] = 2$, $[\lambda_k] = [\lambda] = 4$.

We introduce a regulator kernel of the following form [56, 57]

$$R_k(\mathbf{p},\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}')Z_k(k^2 - \sum_s p_s^2)\theta(k^2 - \sum_s p_s^2)\,, \tag{4.8}$$

where θ stands for the Heaviside step function. This choice is standard because, due to its functional properties, a regulator of this form allows analytic solution for many spectral sums. It is easy to show that R_k satisfies the minimal requirements for a regulator kernel:

- As a consequence of the fact that $\theta(-|x|) = 0$,

$$R_{k=0}(\mathbf{p},\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}')Z_k(-\sum_s p_s^2)\theta(-\sum_s p_s^2) = 0\,; \tag{4.9}$$

- at the scale $k = \Lambda$, the regulator appears as:

$$R_{k=\Lambda}(\mathbf{p},\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}')Z_\Lambda(\Lambda^2 - \sum_s p_s^2)\theta(\Lambda^2 - \sum_s p_s^2)\,, \tag{4.10}$$

  which at the first order gives: $R_{k=\Lambda} \simeq Z_\Lambda \Lambda^2$;

- for $k \in [0, \Lambda]$, we have, for the same computation of the previous points:

$$R_k(\mathbf{p}, \mathbf{p}') = 0\,, \qquad \forall \mathbf{p}, \mathbf{p}',\ \text{such that}\ \ |\mathbf{p}|,\, |\mathbf{p}'| > k\,, \tag{4.11}$$
$$R_k(\mathbf{p}, \mathbf{p}') \simeq Z_k k^2\,, \qquad \forall\, \mathbf{p}, \mathbf{p}',\ \text{such that}\ \ |\mathbf{p}|, |\mathbf{p}'| < k\,. \tag{4.12}$$

The derivative of the regulator kernel with respect to the logarithmic scale $t = \log k$ evaluates as:

$$\partial_t R_k(\mathbf{p}, \mathbf{p}') = \theta(k^2 - \Sigma_s p_s^2)[\partial_t Z_k(k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]\delta(\mathbf{p} - \mathbf{p}')\,. \tag{4.13}$$

One notes that R_k and $\partial_t R_k$ are both symmetric kernels. This property matters during the convolutions induced by the Wetterich equation.

Let us now compute the 1PI 2-point function. Differentiating Γ_k once with respect to the field, we obtain:

$$\begin{aligned}
\frac{\delta\Gamma_k[\varphi, \overline{\varphi}]}{\delta\varphi(\mathbf{q})} &= \int_{\mathbb{R}^{\times 3}} [dp_i]_{i=1}^3 \overline{\varphi}(\mathbf{p})(Z_k \sum_s p_s^2 + \mu_k)\delta(\mathbf{p} - \mathbf{q}) \\
&+ \frac{\lambda_k}{2} \int_{\mathbb{R}^{\times 6}} [dp_i]_{i=1}^3 [dp'_j]_{j=1}^3 \Big[\overline{\varphi}_{p'_1 p_2 p_3} \varphi_{p'_1 p'_2 p'_3} \overline{\varphi}_{p_1 p'_2 p'_3} \delta(\mathbf{p} - \mathbf{q}) \\
&+ \varphi_{p_1 p_2 p_3} \overline{\varphi}_{p'_1 p_2 p_3} \overline{\varphi}_{p_1 p'_2 p'_3} \delta(\mathbf{p'} - \mathbf{q}) + \text{sym}\Big\{1 \to 2 \to 3\Big\}\Big] \\
&= \overline{\varphi}(\mathbf{q})(Z_k \sum_s q_s^2 + \mu_k) \\
&+ \lambda_k \int_{\mathbb{R}^{\times 3}} [dp_i]_{i=1}^3 \Big[\varphi_{p_1 p_2 p_3} \overline{\varphi}_{q_1 p_2 p_3} \overline{\varphi}_{p_1 q_2 q_3} + \text{sym}\Big\{1 \to 2 \to 3\Big\}\Big]
\end{aligned} \tag{4.14}$$

And differentiating with respect to the second field yields:

$$\begin{aligned}
\Gamma_k^{(2)}(\mathbf{q}, \mathbf{q}') &= \frac{\delta^2\Gamma_k}{\delta\overline{\varphi}(\mathbf{q}')\delta\varphi(\mathbf{q})} = (Z_k \sum_s q_s^2 + \mu_k)\delta(\mathbf{q} - \mathbf{q}') \\
&+ \lambda_k \Big[\int_{\mathbb{R}^{\times 3}} [dp_i]_{i=1}^3\ \varphi_{p_1 p_2 p_3} \overline{\varphi}_{p_1 q_2 q_3} \delta(q_1 - q'_1)\delta(p_2 - q'_2)\delta(p_3 - q'_3) \\
&+ \int_{\mathbb{R}^{\times 3}} [dp_i]_{i=1}^3\ \overline{\varphi}_{p_1 p_2 p_3} \varphi_{q_1 p_2 p_3} \delta(p_1 - q'_1)\delta(q_2 - q'_2)\delta(q_3 - q'_3) \\
&+ \text{sym}\Big\{1 \to 2 \to 3\Big\}\Big] \\
&= (Z_k \sum_s q_s^2 + \mu_k)\delta(\mathbf{q} - \mathbf{q}') + \lambda_k \Big[\int_{\mathbb{R}} dp_1\ \varphi_{p_1 q'_2 q'_3} \overline{\varphi}_{p_1 q_2 q_3} \delta(q_1 - q'_1) \\
&+ \int_{\mathbb{R}^{\times 2}} dp_2 dp_3\ \varphi_{q'_1 p_2 p_3} \overline{\varphi}_{q_1 p_2 p_3} \delta(q_2 - q'_2)\delta(q_3 - q'_3) + \text{sym}\Big\{1 \to 2 \to 3\Big\}\Big] \\
&= (Z_k \sum_s q_s^2 + \mu_k)\delta(\mathbf{q} - \mathbf{q}') + F_k(\mathbf{q}, \mathbf{q'})\,.
\end{aligned} \tag{4.15}$$

There is a simple graphical way to picture the terms of F_k. Each index summed can be represented by a segment and each fixed index (not summed) by a dot. Then Fig.4.3 displays two terms coming from the second variation of the interaction labeled by color 1 (the ones which appear explicitly in (4.15)). The rest of the terms appearing in sym$\{\cdot\}$ can be inferred by color permutation.

Figure 4.3: Terms of the second variation of Γ_k.

Defining the operator P_k with kernel

$$P_k(\mathbf{p}, \mathbf{p'}) = R_k(\mathbf{p}, \mathbf{p'}) + (Z_k \sum_s p_s^2 + \mu_k)\delta(\mathbf{p} - \mathbf{p'})\,, \tag{4.16}$$

the Wetterich equation can be recast as:

$$\partial_t \Gamma_k = \mathrm{Tr}[\partial_t R_k \cdot (P_k + F_k)^{-1}]\,. \tag{4.17}$$

The r.h.s. of (4.17) generates an infinite series of terms with arbitrary number of fields convoluted. In order to compare the two sides of (4.17), we must therefore perform a truncation in this series to match with the l.h.s. of that equation. This may be achieved expanding (4.17) in powers of $F_k \cdot (P_k)^{-1}$, that is, in powers of $\varphi\overline{\varphi}$, and considering only the terms up to the power 2:

$$\begin{aligned} \partial_t \Gamma_k &= \mathrm{Tr}[\partial_t R_k \cdot (P_k)^{-1} \cdot (1 + F_k \cdot (P_k)^{-1})^{-1}] \\ &= \mathrm{Tr}[\partial_t R_k \cdot (P_k)^{-1} \cdot (1 - F_k \cdot (P_k)^{-1} + F_k \cdot (P_k)^{-1} \cdot F_k \cdot (P_k)^{-1}) + o((\varphi\overline{\varphi})^3)]\,. \end{aligned} \tag{4.18}$$

The vacuum term proportional to the 0-th order in the above expansion will be discarded because it does not reflect any term in the l.h.s. of (4.17). As an example, fully written, the trace at linear order appears as:

$$\partial_t \Gamma_k^{\mathrm{kin}} = \int_{\mathbb{R}^{\times 12}} \partial_t R_k(\mathbf{p}, \mathbf{p}')(P_k)^{-1}(\mathbf{p}', \mathbf{q}) F_k(\mathbf{q}, \mathbf{q}')(P_k)^{-1}(\mathbf{q}', \mathbf{p})\,. \tag{4.19}$$

Already, from the structure of the operators, $\partial_t R_k$, P_k and F_k, we expect the presence of singular δ-functions which need to be regularized. The appearance of $\delta(0)$-terms reflects the fact that we have infinite volume effects which have to be treated. It must be pointed out immediately that the presence of such infinities is not a peculiar fact of TGFT, simply because this also arise in standard QFT. What is rather peculiar is the fact that, due to the combinatorics of the vertex operators, the way that these singular delta-functions occur cannot be addressed by the so-called projection on the constant fields. Roughly speaking, projecting on constant fields allows to factorize out the full volume of the space as a power of $\delta(0)$ dependent on the order of the field. Such a procedure cannot be applied in the present setting for the main reason that the TGFT interactions follows a precise convolution pattern which must be checked term-by-term in the expansion of (4.17). In other words, not all ϕ^4-terms generated from the r.h.s. are those included in the l.h.s. of that equation. In order to solve our present issue, one must resort in a proper lattice regularization and a thermodynamic limit in the conjugate space. This is the goal of the next section.

4.3 IR divergences and thermodynamic limit

In order to regularize volume divergences, we perform a compactification of the direct space and a lattice regularization in the conjugate space, following the conventions of [68]. Defining the model (4.1) over a compact set $D \subset \mathbb{R}^{\times 3}$ with volume $L^3 = (2\pi r)^3$, we must map the theory onto the p-space through a Fourier series transform. The domain of integration, actually summation, of the effective action becomes the lattice

$$D^* = \left(\frac{2\pi}{L}\mathbb{Z}\right)^{\times 3} = \left(\frac{1}{r}\mathbb{Z}\right)^{\times 3} := \left(l\mathbb{Z}\right)^{\times 3}, \tag{4.20}$$

so that we have, for any function $F(\mathbf{p})$,

$$\int_{D^*} [dp_i]_{i=1}^3 \, F(\mathbf{p}) = l^3 \sum_{p_1,p_2,p_3 \in D^*} F(\mathbf{p}) \,. \tag{4.21}$$

We define the delta distribution in D^* (identity in the space of operators) as:

$$\delta_{D^*}(\mathbf{p},\mathbf{q}) = l^{-3}\delta_{\mathbf{p},\mathbf{q}} \,. \tag{4.22}$$

with $\delta_{\mathbf{p},\mathbf{q}} = \prod_s \delta_{p_s,q_s}$, the Kronecker delta. Choosing an orthonormal base $(e_{\mathbf{p}})_{\mathbf{p}\in D^*}$ for the space of fields such that $e_{\mathbf{p}}(\mathbf{q}) = \delta_{D^*}(\mathbf{p},\mathbf{q})$, we have:

$$\phi(\mathbf{p}) = \langle e_{\mathbf{p}}, \phi\rangle_{D^*} \,. \tag{4.23}$$

For some observable A, it is direct to get

$$(A\phi)(\mathbf{p}) = \int_{D^*} [dq_i]_{i=1}^3 \, A(\mathbf{q},\mathbf{p})\phi(\mathbf{p}) = \int_{D^*} [dq_i]_{i=1}^3 \, \langle e_{\mathbf{q}}, Ae_{\mathbf{p}}\rangle_{D^*}\phi(\mathbf{p}) \,. \tag{4.24}$$

In the case A is invertible, then the inverse operator satisfies

$$\int_{D^*} [dr_i]_{i=1}^3 \, A(\mathbf{p},\mathbf{r})A^{-1}(\mathbf{r},\mathbf{q}) = \delta_{D^*}(\mathbf{p},\mathbf{q}) \,. \tag{4.25}$$

We also define the derivative with respect to the field as:

$$\frac{\delta}{\delta\phi(\mathbf{p})} = l^{-3}\frac{\partial}{\partial\phi(\mathbf{p})} \,, \tag{4.26}$$

so that the following relations hold:

$$\frac{\delta}{\delta\phi(\mathbf{p})}\,\phi(\mathbf{q}) = \delta_{D^*}(\mathbf{p},\mathbf{q}) \,, \qquad \frac{\delta}{\delta J(\mathbf{p})}\, e^{\langle J,\phi\rangle_{D^*}} = J(\mathbf{p})\, e^{\langle J,\phi\rangle_{D^*}} \,. \tag{4.27}$$

This set of conventions is fully consistent with the continuous version of field theory, where δ_{D^*} becomes the Dirac δ-distribution and the derivative (4.26) becomes the standard functional derivative.

Using this regularization prescription, the effective action of the model is of the form:

$$\begin{aligned}\Gamma_k[\varphi,\overline{\varphi};l] = l^3 \sum_{\mathbf{p}\in D^*} \overline{\varphi}_{123}\Big(Z_k \sum_s p_s^2 + \mu_k\Big)\varphi_{123} \\ + l^6\,\frac{\lambda_k}{2} \sum_{\mathbf{p},\mathbf{p}'\in D^*}\Big[\varphi_{123}\overline{\varphi}_{1'23}\varphi_{1'2'3'}\overline{\varphi}_{12'3'} + \mathrm{sym}\Big\{1\to 2\to 3\Big\}\Big],\end{aligned} \quad (4.28)$$

where, using the same notation φ for the field and its Fourier transformed, one has:

$$\begin{aligned}\varphi(x_1,x_2,x_3) &= (2\pi)^{-3} l^3 \sum_{\mathbf{p}\in D^*} e^{i\sum_i p_i x_i}\varphi(\mathbf{p})\,, \\ \varphi(\mathbf{p}) &= \int_D [dx_i]_{i=1}^3\, e^{-i\sum_i p_i x_i}\varphi(x_1,x_2,x_3)\,.\end{aligned} \quad (4.29)$$

Now use the relations (4.29) to transform δ_{D^*} and obtain

$$(2\pi)^{-3} l^3 \sum_{\mathbf{p}\in D^*} \delta_{D^*}(\mathbf{p},\mathbf{q}) e^{i\sum_i p_i x_i} = (2\pi)^{-3} e^{i\sum_i q_i x_i}\,. \quad (4.30)$$

Thus, an integral representation of the delta distribution over D^* makes sense if defined as

$$\delta_{D^*}(\mathbf{p},\mathbf{q}) = (2\pi)^{-3}\int_D [dx_i]_{i=1}^3\, e^{-i\sum_i (p_i-q_i)x_i}\,. \quad (4.31)$$

As a result, we finally have:

$$\delta_{D^*}(\mathbf{p},\mathbf{p}) = \frac{(2\pi r)^3}{(2\pi)^3} = r^3 = \frac{1}{l^3}\,. \quad (4.32)$$

In the end, the continuous description will be recovered in the thermodynamic limit $l\to 0$.

With this procedure, highlighting the dependence of the volume of the direct space of the model, it is natural to incorporate in the coupling constants a dependence on that volume. We will use and tune this dependence in such a way that the non-compact limit of the theory becomes well defined and all divergences are consistently removed.

4.4 β-functions and RG flows

We introduce a lattice regularization in the sense of section 4.3, and write the corresponding effective action as:

$$\Gamma_k[\varphi,\overline{\varphi}] = \int_{D^*} [dp_i]_{i=1}^3\, \overline{\varphi}_{123}(Z_k \sum_s p_s^2 + \mu_k)\varphi_{123} \quad (4.33)$$

$$+\frac{\lambda_k}{2}\int_{D^{*\times 2}}[dp_i]_{i=1}^3[dp'_j]_{j=1}^3\ \Big[\varphi_{123}\overline{\varphi}_{1'23}\varphi_{1'2'3'}\overline{\varphi}_{12'3'}+\text{sym}\Big\{1\to 2\to 3\Big\}\Big].$$

We can study the Wetterich equation of (4.33) and perform a thermodynamic limit at the end of the computation to extract the coefficients valid in the non-compact case. The pathological divergences arise as negative powers of the parameter l which, in the limit $l \to 0$, reproduce divergent Dirac delta functions. In this section, we illustrate how to cure these divergences and make sense of the non-compact limit by incorporating a dependence on the volume in the coupling constants.

The set of β-functions that we obtain from the discretized model is (important steps of the calculation are detailed in appendix A):

$$\begin{cases}\beta(Z_k)=\dfrac{\lambda_k}{(Z_kk^2+\mu_k)^2}\Big\{\partial_tZ_k\Big[\pi\dfrac{k^2}{l^2}+4\dfrac{k}{l}\Big]+2Z_k\Big[\pi\dfrac{k^2}{l^2}+2\dfrac{k}{l}\Big]\Big\}\\ \beta(\mu_k)=\dfrac{-3\lambda_k}{(Z_kk^2+\mu_k)^2}\Big\{\partial_tZ_k\Big[\dfrac{\pi}{2}\dfrac{k^4}{l^2}+\dfrac{4}{3}\dfrac{k^3}{l}\Big]+2Z_k\Big[\dfrac{k^4}{l^2}\pi+2\dfrac{k^3}{l}\Big]\Big\}\\ \beta(\lambda_k)=\dfrac{2\lambda_k^2}{(Z_kk^2+\mu_k)^3}\Big\{\partial_tZ_k\Big[\dfrac{\pi}{2}\dfrac{k^4}{l^2}+\dfrac{20}{3}\dfrac{k^3}{l}+2k^2\Big]+2Z_k\Big[\pi\dfrac{k^4}{l^2}+10\dfrac{k^3}{l}+2k^2\Big]\Big\}\end{cases}\tag{4.34}$$

It must be stressed that the coefficients appearing in (4.34) are computed with integrals like in the continuous setup. This is however not an issue, once the volume dependence totally factorized, the order of taking the limit and performing the integral does not matter.

At least, two features of the system (4.34) must be stressed: at this intermediate step (the limit $\lim_{l\to 0}$ still has to be taken), this is a non-autonomous system and it involves different exponents in the cut-off k (that we refer to "non-homogeneity" in k). Non-autonomous systems are known to occur in other contexts, for example quantum field theory at finite temperature [55], or on a curved [64] and non-commutative spacetime [65]. The non-homogeneity in k of the system is an effect of the particular combinatorics of the vertices of the theory which, after differentiated yields 1PI 2-point function with terms with different volume contributions. If the l parameter is kept finite, we see two different system arising in the UV and IR limits, coming from different leading terms. Such a feature has been found in previous work [66] and the two limits and intermediate regime investigated. Fixed points were found, but difficult to interpret as UV and IR fixed points in the ordinary sense. In our present situation, we can fix, and so improve, this issue by taking advantage of the presence of the two parameters (k, l).

To make sense of the above system, consider the following ansatz:

$$Z_k=\overline{Z}_kl^{\chi}k^{-\chi},\quad \mu_k=\overline{\mu}_k\overline{Z}_kl^{\chi}k^{2-\chi},\quad \lambda_k=\overline{\lambda}_k\overline{Z}_k^2l^{\xi}k^{\sigma}\,,\tag{4.35}$$

where $[\overline{Z}_k]=[\overline{\mu}_k]=[\overline{\lambda}_k]=0$, $[\varphi]=-\frac{5}{2}$ and $\xi+\sigma=4$. Considering that, at fixed scale, we are interested in the modes with wave length relevant for the physics of the system, we look for the scaling of dimensionless coupling constants, i.e., we look for dimensionless β-functions. From (4.35), one must find:

$$\begin{aligned}
\eta_k &= \frac{1}{\overline{Z}_k}\beta(\overline{Z}_k) = \frac{1}{Z_k}\beta(Z_k) + \chi\,, \\
\beta(\mu_k) &= \frac{1}{\overline{Z}_k l^\chi k^{2-\chi}}\beta(\mu_k) - \eta_k\overline{\mu}_k - (2-\chi)\overline{\mu}_k\,, \\
\beta(\overline{\lambda}_k) &= \frac{1}{l^\xi k^\sigma \overline{Z}_k^2}\beta(\lambda_k) - 2\eta_k\overline{\lambda}_k - \sigma\overline{\lambda}_k\,,
\end{aligned} \tag{4.36}$$

and reach the following expressions:

$$\begin{aligned}
\eta_k =& \frac{\overline{\lambda}_k l^\xi k^\sigma}{l^{2\chi}k^{4-2\chi}(1+\overline{\mu}_k)^2}\Big[(\eta_k-\chi)\Big(\pi\frac{k^2}{l^2}+4\frac{k}{l}\Big)+2\Big(\pi\frac{k^2}{l^2}+2\frac{k}{l}\Big)\Big]+\chi\,; \\
\beta(\overline{\mu}_k) =& -\frac{3\overline{\lambda}_k l^\xi k^\sigma}{l^{3\chi}k^{6-3\chi}(1+\overline{\mu}_k)^2}\Big[(\eta_k-\chi)\Big(\frac{\pi}{2}\frac{k^4}{l^2}+\frac{4}{3}\frac{k^3}{l}\Big)+2\Big(\pi\frac{k^4}{l^2}+2\frac{k^3}{l}\Big)\Big] \\
& - \eta_k\overline{\mu}_k - (2-\chi)\overline{\mu}_k\,; \\
\beta(\overline{\lambda}_k) =& \frac{2\overline{\lambda}_k^2 l^\xi k^\sigma}{l^{3\chi}k^{6-3\chi}(1+\overline{\mu}_k)^3}\Big[(\eta_k-\chi)\Big(\frac{\pi}{2}\frac{k^4}{l^2}+\frac{20}{3}\frac{k^3}{l}+2k^2\Big)+2\Big(\pi\frac{k^4}{l^2}+10\frac{k^3}{l}+2k^2\Big)\Big] \\
& - 2\eta_k\overline{\lambda}_k - \sigma\overline{\lambda}_k\,.
\end{aligned} \tag{4.37}$$

Aiming at obtaining a regularized non-compact limit, we must solve the system in the variables ξ and χ by requiring that the highest degree of divergence (highest negative power of l) is regularized and all the sub-leading infinities sent to zero. This is achieved by solving

$$\begin{cases} \xi - 2\chi - 2 = 0 \\ \xi - 3\chi - 2 = 0 \end{cases} \quad\Rightarrow\quad \begin{cases} \chi = 0 \\ \xi = 2 \end{cases} \quad\Rightarrow\quad \sigma = 2\,. \tag{4.38}$$

The resulting system of equations for the theory is:

$$\begin{cases}
\eta_k = \dfrac{\pi\overline{\lambda}_k}{(1+\overline{\mu}_k)^2}(\eta_k+2) \\
\beta(\overline{\mu}_k) = -\dfrac{3\pi\overline{\lambda}_k}{(1+\overline{\mu}_k)^2}(\dfrac{\eta_k}{2}+2) - \eta_k\overline{\mu}_k - 2\overline{\mu}_k \\
\beta(\overline{\lambda}_k) = \dfrac{\pi\overline{\lambda}_k^2}{(1+\overline{\mu}_k)^3}(\eta_k+4) - 2\eta_k\overline{\lambda}_k - 2\overline{\lambda}_k
\end{cases} \tag{4.39}$$

which defines an autonomous system of coupled differential equations describing the flow of dimensionless couplings constants.

Before proceeding with the standard analysis, which consists in finding fixed points of the flow and studying the linearized equations around them, we point out that, because of the non-linear nature of the β-functions, we have a singularity at $\overline{\mu} = -1$ and $\overline{\lambda} = (1+\overline{\mu})^2/\pi$. This is a common feature in dealing with a truncated Wetterich equation. In a neighborhood of those singularities, we do not trust the linear approximation, and being interested to the part connected with the Gaussian fixed point, we will not study the flow beyond the divergence of the β-functions.

By numerical integration, we find a Gaussian fixed point and three non-Gaussian fixed points in the plane $(\overline{\mu}, \overline{\lambda})$ at:

$$P_1 = (8.619, -47.049), \qquad P_2 = 10^{-1}(-6.518, 0.096), \tag{4.40}$$
$$P_3 = 10^{-1}(-8.010, 0.212)\,. \tag{4.41}$$

A quick inspection proves that P_3 lies in the sector disconnected from the origin, we will not perform any analysis around it.

We linearize the system of equations by evaluating the stability matrix around fixed points:

$$\begin{pmatrix} \beta(\overline{\mu}_k) \\ \beta(\overline{\lambda}_k) \end{pmatrix} = \begin{pmatrix} \partial_{\overline{\mu}_k}\beta(\overline{\mu}_k) & \partial_{\overline{\lambda}_k}\beta(\overline{\mu}_k) \\ \partial_{\overline{\mu}_k}\beta(\overline{\lambda}_k) & \partial_{\overline{\lambda}_k}\beta(\overline{\lambda}_k) \end{pmatrix}_{\text{F.P.}} \begin{pmatrix} \overline{\mu}_k \\ \overline{\lambda}_k \end{pmatrix} . \tag{4.42}$$

In a neighborhood of the Gaussian fixed point, we have an eigenvalue with algebraic multiplicity 2, corresponding to the canonical scaling dimensions of the couplings λ_k and μ_k: $\theta^G = -2$. The geometric multiplicity of θ^G is 1, hence, the matrix of the linearized system turns out to be not diagonalizable and has a single eigenvector $(1, 0)$.

In a neighbourhood of the non-Gaussian fixed points(NGFP) we have:

$$\theta^1_+ \sim 0.351 \text{ for } \mathbf{v}^1_+ \sim 10^{-1}(0.65, -9.98), \tag{4.43}$$
$$\theta^1_- \sim -2.548 \text{ for } \mathbf{v}^1_- \sim 10^{-1}(-6.88, 7.26), \tag{4.44}$$
$$\theta^2_+ \sim 10.066 \text{ for } \mathbf{v}^2_+ \sim 10^{-1}(9.996, -0.269), \tag{4.45}$$
$$\theta^2_- \sim -1.988 \text{ for } \mathbf{v}^2_- \sim 10^{-1}(9.987, 0.506). \tag{4.46}$$

Because of the difference in their magnitudes (distance at the origin), it becomes difficult to plot the two NGFP's with enough precision on their vicinity. We plot two sectors of the RG flow in the plane $(\overline{\mu}_k, \overline{\lambda}_k)$ (see Fig.4.4).

At a vicinity of a fixed point, we define relevant directions those eigendirections that are UV attractive with respect to the cut-off, while we call irrelevant the UV repulsive eigendirections. Marginal directions can be both attractive or repulsive depending on the initial condition of the trajectory. The origin is a "great" attractor and has one relevant direction connecting it to the other two fixed points. The absence of a second eigenvector for the stability matrix around the Gaussian fixed point requires an approximation beyond the linear order and is a signal of the presence of a marginal perturbation. Even if we do not present this analysis here, we know from the plots that this will be still UV attractive which means that it corresponds to a marginally relevant direction. The fact that the GFP is a sink for the flow, means that this model is asymptotically free with respect to the cut-off. Both non-Gaussian fixed points have one relevant and one

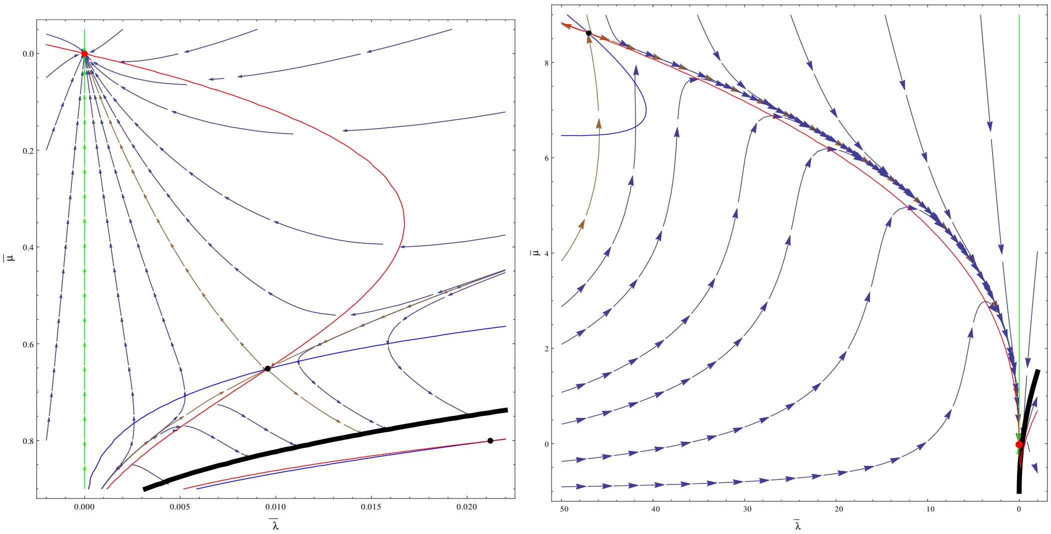


Figure 4.4: Flow of the theory. The red and blue lines represent respectively the zeros of $\beta(\overline{\mu}_k)$ and $\beta(\overline{\lambda}_k)$, the brown arrows are the eigenperturbations of the non-Gaussian fixed points (represented in black), and the green ones those of the Gaussian fixed point (in red). Arrows point in the UV direction. The thick black line is the singularity of the flow.

irrelevant directions. The eigendirections connecting the three fixed points turn out to be stable under RG transformations and they are characterized by an effect known as "large river effect". This means that all the RG trajectories in a neighborhood of these perturbations get closer and closer to them while pointing in the UV. This effect shows a splitting of the space of coupling in two regions not connected by any RG trajectory. Thus, the relevant directions for the Gaussian fixed point reflect the properties of a critical surface and suggest the presence of phase transitions in the model. In the $\overline{\lambda}_k > 0$ plane, the flow is similar to the one of standard local scalar field theory in a neighborhood of the Wilson-Fisher fixed point, but the presence of a second non-Gaussian fixed point in the $\overline{\lambda}_k < 0$ plane makes the theory radically different. Nevertheless, the properties of this second NGFP are basically the same as the former one. In this context, we do have strong hint to a phase transition with two phases: a symmetric and a broken one. The spontaneous symmetry breaking would happen while crossing the critical surface and it assumes to generate a condensed state which, in a simplicial gravity context, might be interpreted as a geometric phase [47]. To confirm this interpretation of the broken phase, one should change parametrization for the effective potential and study the theory around the new (degenerate) ground state solving the equation of motion in the saddle point approximation. Unfortunately, because of the particular combinatorial character of the interaction, a proper analysis in this way is complicated and, at the present days, totally missing. We can only see that, in the constant modes approximation, which forgets about the non-locality features, we recover the usual formulation of the Ginzburg-Landau phase transition for a ϕ^4 scalar complex theory.

Chapter 5

Gauging the model

In section 2.2, we discuss that GFT finds some its origins and motivations in the Boulatov and Ooguri models [19, 20], which are models for simplicial gravity. The way that one implements the mentioned link with simplicial gravity is defined via a gauge projection on the tensor fields. It is therefore relevant to investigate a model in GFT endowed with a gauge projection, under the lense of the FRG, for discussing a possible phase transition towards a continuum geometry for this class of GFT. Furthermore, it is also significant to make a comparison with the previous study of TGFT models, in the same rank 3 and same class of interactions, to understand, at the level of the FRG, what distinguishes these models. We emphasize that this study is completely original because this is the first time that one applies FRG methods to a GFT model with gauge projection and the first time that one performs a Renormalization analysis as well this type of models over a noncompact group.

5.1 The gauge projection

In GFT, a gauge invariant field defined over a group manifold G is a field satisfying

$$\phi(g_1, g_2, g_3) = \phi(g_1 h, g_2 h, g_3 h)\,, \tag{5.1}$$

with $g_i, h \in G$.

Some technical ambiguities arise when one tries to realize the condition (5.1). A possible (formal) way to implement this feature would be to allow only the propagation of modes satisfying (5.1) by inserting in the kinetic kernel a projector on the space of these modes. Defining the projector P and a kinetic kernel $\mathcal{K}$, one encounters an ambiguity regarding the order in which they should be convoluted in the action. The three main possibilities are:

$$K_1 = \mathcal{K} \cdot P\,, \tag{5.2}$$

$$K_2 = P \cdot \mathcal{K} \cdot P\,, \tag{5.3}$$

$$K_3 = P \cdot \mathcal{K}\,. \tag{5.4}$$

If $\mathcal{K}$ is a differential operator and P has the form of a delta-function (this is precisely what will happen in the following), the differentiation of the δ, in the weak sense, enforces the convolution $\mathcal{K} \cdot P(\phi)$ to be zero. For this reason, we might discard both K_1 and K_2. As a consequence, this means that we do not consider the dynamics of the constraint, we just constraint the dynamics of the fields. Thus, we write an action of the form:

$$\begin{aligned} S[\phi,\overline{\phi}] &= \int_{G^{\times 3}} [dg_i]_{i=1}^3 [dg_i']_{i=1}^3 \, \overline{\phi}(g_1,g_2,g_3)(P\cdot\mathcal{K})(\{g_i\}_{i=1}^3;\{g_i'\}_{i=1}^3)\phi(g_1',g_2',g_3') \\ &\quad +\mathcal{V}[\phi,\overline{\phi}]\,, \end{aligned} \tag{5.5}$$

where $\mathcal{V}$ is the interaction term. The main issue of this formulation is that a projector is by definition not invertible, thus, a kinetic kernel built out of such an operator cannot, in general, define a covariance of a field theory measure. We partially avoid this problem by inverting the kinetic kernel in the operatorial sense, so that the same constraint will define the covariance.

Now we restrict our description to the case of an Abelian additive group ($\mathbb{R}$) and consider $\mathcal{V}$ with same combinatorics used in chapter 4:

$$\begin{aligned} S_1[\phi,\overline{\phi}] &= \int_{\mathbb{R}^{\times 7}} d\mathbf{x}d\mathbf{y}dh\, \overline{\phi}(\mathbf{x})\prod_{i=1}^3\delta(x_i+h-y_i)(-\Delta_y+\mu)\phi(\mathbf{y}) \\ &+ \frac{\lambda}{2}\int_{\mathbb{R}^{\times 6}} d\mathbf{x}d\mathbf{x}'\Big[\phi(x_1,x_2,x_3)\overline{\phi}(x_1',x_2,x_3)\phi(x_1',x_2',x_3')\overline{\phi}(x_1,x_2',x_3') \\ &+ \text{sym}\Big\{1\to 2\to 3\Big\}\Big]\,. \end{aligned} \tag{5.6}$$

We expect that the Wetterich equation will exhibit IR divergences of the same type encountered in the non-projected model. Then, we must introduce a regularization scheme. To this purpose, we consider a compact subset D of $\mathbb{R}$ homeomorphic to a 1 dimensional ring and write a regularized action as:

$$\begin{aligned} S_1[\phi,\overline{\phi}] &= \int_{D^{\times 7}} d\mathbf{x}d\mathbf{y}dh\, \overline{\phi}(\mathbf{x})\prod_{i=1}^3\delta(x_i+h-y_i)(-\Delta_y+\mu)\phi(\mathbf{y}) \\ &+ \frac{\lambda}{2}\int_{D^{\times 6}} d\mathbf{x}d\mathbf{x}'\phi(x_1,x_2,x_3)\overline{\phi}(x_1',x_2,x_3)\phi(x_1',x_2',x_3')\overline{\phi}(x_1,x_2',x_3') \\ &+ \text{sym}\Big\{1\to 2\to 3\Big\}\,, \end{aligned} \tag{5.7}$$

where we used the same notations introduced in section 4.3.

The computation will be performed in momentum space, hence, keeping the same previous notation for the lattice as $D^* = \mathcal{D}^{\times 3}$, the Fourier series of the model (5.7) reads (in this chapter, we denote the constraint on a 1D lattice as $\delta_{\mathcal{D}}(X) := \delta_{\mathcal{D}}(X,0)$ for simplicity):

$$S_1[\phi,\overline{\phi}] = l^3\sum_{\mathbf{p}\in D^*}\overline{\phi}(\mathbf{p})\Big[\Sigma_s p_s^2+\mu\Big]\phi(\mathbf{p})\delta_{\mathcal{D}}(\Sigma p) + \frac{\lambda}{2}\,l^6\sum_{\mathbf{p},\mathbf{p}'\in D^*}\Big[\phi_{123}\overline{\phi}_{1'23}\phi_{1'2'3'}\overline{\phi}_{12'3'}$$

$$+ \quad \mathrm{sym}\Big\{1 \to 2 \to 3\Big\}\Big] \, . \tag{5.8}$$

The FRG formalism introduced in section 3.3 is generic and applies to this model, roughly, in the same way we did in the previous chapter. In particular, the regulator kernel will incorporate the same gauge constraint appearing in the kinetic term. The overall Wetterich equation has the same structure and expands the same of way as (4.18).

Testing the type of ansatz which might have interesting properties, we choose an effective action of the form:

$$\begin{aligned} \Gamma^1_k[\varphi,\overline{\varphi}] \;&=\; l^3 \sum_{\mathbf{p}\in D^*} \overline{\varphi}(\mathbf{p}) \Big[Z_k \Sigma_s p_s^2 + \mu_k \Big] \varphi(\mathbf{p}) \delta_{\mathcal{D}}(\Sigma p) \\ &+\; \frac{\lambda_k}{2} l^6 \sum_{\mathbf{p},\mathbf{p}'\in D^*} \Big[\varphi_{123}\overline{\varphi}_{1'23}\varphi_{1'2'3'}\overline{\varphi}_{12'3'} + \mathrm{sym}\Big\{1\to 2\to 3\Big\}\Big] \, , \end{aligned} \tag{5.9}$$

Then, we introduce the kernels (using the same notation as (4.18)):

$$R_k(\mathbf{q},\mathbf{q'}) \;=\; \Theta(k^2 - \Sigma_s q_s^2) Z_k (k^2 - \Sigma_s q_s^2) \delta_{\mathcal{D}}(\Sigma q) \prod \delta_{\mathcal{D}}(\mathbf{q},\mathbf{q'}) \, , \tag{5.10}$$

$$F^1_k(\mathbf{q},\mathbf{q}') \;=\; \frac{\delta^2}{\delta\overline{\varphi}_{\mathbf{q}'}\delta\varphi_{\mathbf{q}}} \mathcal{V}^1[\varphi,\overline{\varphi}] \, , \tag{5.11}$$

where $\mathcal{V}^1_k$ refers to the interaction part of Γ^1_k. This is a natural choice following from the FRG application to (5.7). Performing the computation of the Wetterich equation, one realizes that this proposal drastically fails: the delta's enforcing the gauge constraints do not convolute properly with fields. This feature comes from the fact that, if one evaluates (4.18) using (5.10) and (5.11), the fields appearing in the r.h.s. come from the F^1_k operator, while the constraints always from the mass-like terms. This prevents the comparison of the two sides of the Wetterich equation for this model and trivializes to 0 all β's functions.

There is another way to choose the interaction term which will produce a more sensible result. We simply insert gauge projections also in all fields in the interaction. A interaction satisfying this requirement expresses as:

$$\begin{aligned} \mathcal{V}[\phi,\overline{\phi}] =& \frac{\lambda_k}{2}\frac{1}{(2\pi)^6} \int_{D^{\times 22}} \{d\mathbf{w}^i\}_{i=1}^4 d\mathbf{x} d\mathbf{x}' \{dh_j\}_{j=1}^4 \phi(\mathbf{w}^1)\overline{\phi}(\mathbf{w}^2)\phi(\mathbf{w}^3)\overline{\phi}(\mathbf{w}^4) \\ & \times \delta(x_1 + h_1 - w^1_1)\delta(x_2 + h_1 - w^1_2)\delta(x_3 + h_1 - w^1_3) \\ & \times \delta(x'_1 + h_2 - w^2_1)\delta(x_2 + h_2 - w^2_2)\delta(x_3 + h_2 - w^2_3) \\ & \times \delta(x'_1 + h_3 - w^3_1)\delta(x'_2 + h_3 - w^3_2)\delta(x'_3 + h_3 - w^3_3) \\ & \times \delta(x_1 + h_4 - w^4_1)\delta(x'_2 + h_4 - w^4_2)\delta(x'_3 + h_4 - w^4_3) \\ & + \mathrm{sym}\Big\{1 \to 2 \to 3\Big\} \\ =& \frac{\lambda_k}{2} l^6 \sum_{\mathbf{p}\mathbf{p}'} \phi_{123}\overline{\phi}_{1'23}\phi_{1'2'3'}\overline{\phi}_{12'3'}\delta_{\mathcal{D}}(\Sigma p)\delta_{\mathcal{D}}(\Sigma p') \end{aligned}$$

$$\times\,\delta_{\mathcal{D}}(p_1'+p_2+p_3)\delta_{\mathcal{D}}(p_1+p_2'+p_3')+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\,. \qquad (5.12)$$

Hence, re-starting the analysis from the beginning, we define a model with gauge constraints on both the kinetic and interaction kernels via the action:

$$\begin{aligned}
S[\phi,\overline{\phi}] &= l^3\sum_{\mathbf{p}}\overline{\phi}(\mathbf{p})\Big[\Sigma_s p_s^2+\mu\Big]\phi(\mathbf{p})\delta_{\mathcal{D}}(\Sigma p)\\
&+\frac{\lambda_k}{2}l^6\sum_{\mathbf{p}\mathbf{p}'}\phi_{123}\overline{\phi}_{1'23}\phi_{1'2'3'}\overline{\phi}_{12'3'}\delta_{\mathcal{D}}(\Sigma p)\delta_{\mathcal{D}}(\Sigma p')\delta_{\mathcal{D}}(p_1'+p_2+p_3)\delta_{\mathcal{D}}(p_1+p_2'+p_3')\\
&+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\,, \qquad (5.13)
\end{aligned}$$

with corresponding continuous model defined by

$$\begin{aligned}
S[\phi,\overline{\phi}] &= \int d\mathbf{p}\;\overline{\phi}(\mathbf{p})\Big[\Sigma_s p_s^2+\mu\Big]\phi(\mathbf{p})\delta(\Sigma p)\\
&+\frac{\lambda_k}{2}\int d\mathbf{p}d\mathbf{p}'\;\phi_{123}\overline{\phi}_{1'23}\phi_{1'2'3'}\overline{\phi}_{12'3'}\delta(\Sigma p)\delta(\Sigma p')\delta(p_1'+p_2+p_3)\delta(p_1+p_2'+p_3')\\
&+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\,. \qquad (5.14)
\end{aligned}$$

In fact, the result obtained in this section could have been guessed from a more general consideration. Even if the quantum amplitudes of the theory do not depend on whether the gauge projection appears in the kinetic term, in the interaction or in both, the non-perturbative analysis is of course radically different. From this point of view, a model which presents this constraint in only one of the two terms cannot be consistent. This directly reflects in the analysis, we just presented.

We can now proceed further using model (5.14).

5.2 Effective action and Wetterich equation

Having defined the model ingredients, we are in position to evaluate and analyse its FRG equation. We shall again restrict to a simple and non-trivial ansatz for the effective action for the model (5.13) which reads:

$$\begin{aligned}
\Gamma_k[\varphi,\overline{\varphi}] &= l^3\sum_{\mathbf{p}}\overline{\varphi}(\mathbf{p})\Big[\Sigma_s p_s^2+\mu\Big]\varphi(\mathbf{p})\delta_{\mathcal{D}}(\Sigma p)\\
&+\frac{\lambda_k}{2}l^6\sum_{\mathbf{p}\mathbf{p}'}\varphi_{123}\overline{\varphi}_{1'23}\varphi_{1'2'3'}\overline{\varphi}_{12'3'}\delta_{\mathcal{D}}(\Sigma p)\delta_{\mathcal{D}}(\Sigma p')\\
&\times\;\delta_{\mathcal{D}}(p_1'+p_2+p_3)\delta_{\mathcal{D}}(p_1+p_2'+p_3')+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\,. \qquad (5.15)
\end{aligned}$$

Considering that $[\delta_{\mathcal{D}}(p)]=-1$, the dimensional analysis for the coupling constants gives different results from the model of chapter 4. We have:

$$[Z_k]=0 \quad\Rightarrow\quad [\mu_k]=2$$

$$\begin{aligned} 2[\varphi] + 3 + 2 - 1 = 0 \quad &\Rightarrow \quad [\varphi] = -2 \\ [\lambda_k] + 6 + 4[\varphi] - 4 = 0 \quad &\Rightarrow \quad [\lambda_k] = 6\,, \end{aligned} \tag{5.16}$$

where, again, we set the canonical dimensions by requiring $[S] = [\Gamma_k] = 0$.

In the same conventions introduced in the previous chapter, one introduces:

$$R_k(\mathbf{q}, \mathbf{q'}) = \Theta(k^2 - \Sigma_s q_s^2) Z_k (k^2 - \Sigma_s q_s^2) \delta_{\mathcal{D}}(\Sigma q) \delta_{D^*}(\mathbf{q}, \mathbf{q}')\,, \tag{5.17}$$

$$\partial_t R_k(\mathbf{q}, \mathbf{q'}) = \Theta(k^2 - \Sigma_s q_s^2)[\partial_t Z_k (k^2 - \Sigma_s q_s^2) + 2k^2 Z_k] \delta_{\mathcal{D}}(\Sigma q) \delta_{D^*}(\mathbf{q}, \mathbf{q}')\,, \tag{5.18}$$

$$\begin{aligned} F_k(\mathbf{q}, \mathbf{q}') &= \lambda_k \Bigg[l^2 \sum_{m_2, m_3} \varphi_{q_1', m_2 m_3} \overline{\varphi}_{q_1, m_2 m_3} \delta_{\mathcal{D}}(\Sigma q) \delta_{\mathcal{D}}(q_1' + m_2 + m_3) \\ &\times \delta_{\mathcal{D}}(q_1' + q_2 + q_3) \delta_{\mathcal{D}}(q_1 + m_2 + m_3) \delta_{\mathcal{D}}(q_2 - q_2') \delta_{\mathcal{D}}(q_3 - q_3') \\ &+ l \sum_{m_1} \varphi_{m_1 q_2' q_3'} \overline{\varphi}_{m_1 q_2 q_3} \delta_{\mathcal{D}}(\Sigma q) \delta_{\mathcal{D}}(m_1 + q_2' + q_3') \\ &\times \delta_{\mathcal{D}}(m_1 + q_2 + q_3) \delta_{\mathcal{D}}(q_1 + q_2' + q_3') \delta_{\mathcal{D}}(q_1 - q_1') \\ &\quad \mathrm{sym}\Big\{ 1 \to 2 \to 3 \Big\} \Bigg]\,, \end{aligned} \tag{5.19}$$

$$P_k(\mathbf{q}, \mathbf{q}') = \Big(Z_k \sum_s q_s^2 + \mu_k \Big) \delta_{\mathcal{D}}(\Sigma q) \delta_{D^*}(\mathbf{q}, \mathbf{q}')\,, \tag{5.20}$$

which enter in the Wetterich equation:

$$\begin{aligned} \partial_t \Gamma_k &= \mathrm{Tr}[\partial_t R_k \cdot (P_k + F_k)^{-1}] \\ &= l^6 \sum_{\mathbf{p}, \mathbf{p}'} \partial_t R_k(\mathbf{p}, \mathbf{p}') \Big(P_k + F_k \Big)^{-1} (\mathbf{p}', \mathbf{p})\,. \end{aligned} \tag{5.21}$$

On the left hand side, as in chapter 4, we have a truncation at the level of the quartic interaction, thus the expansion on the r.h.s. of the Wetterich equation will be the same shown in (4.18), where now the operators involved are given by (5.18), (5.19) and (5.20).

We must pay attention to a subtlety occurring while extracting the β-functions of this model: the δ's implementing the convolutions which appear in the P_k operators can be inverted using (4.25) and summing over their indices we do not modify the dimensions of the whole expression. This is, however, not true for the δ's coming from the gauge constraints because they are not summed, so we need to keep them in the denominator. In any case, these constraints, turn out to be redundant with other delta functions coming from the F_k and $\partial_t R_k$ operators, in such a way that their contribution, because of the regularization, is equivalent to some power of l, which is naturally well defined.

5.3 β-functions and RG flows

Expanding the FRG equation (5.21), we find the following system of dimensionful β-functions (the main steps of the calculations have been given in appendix

B):

$$\begin{cases} \beta(Z_k) = \dfrac{\lambda_k}{(Z_k k^2+\mu_k)^2}\Big[\dfrac{3}{\sqrt{2}}\dfrac{k}{l^3}(1+\partial_t)Z_k + \dfrac{4}{l^2}\partial_t Z_k\Big] \\ \beta(\mu_k) = -\dfrac{3\lambda_k}{(Z_k k^2+\mu_k)^2}\Big[\sqrt{2}\dfrac{k^3}{l^3}\Big(2+\dfrac{2}{3}\partial_t\Big)Z_k + \dfrac{k^2}{l^2}(2+\partial_t)Z_k\Big] \\ \beta(\lambda_k) = \dfrac{2\lambda_k^2}{(Z_k k^2+\mu_k)^3}\Big[2\sqrt{2}\dfrac{k^3}{l^3}\Big(1+\dfrac{1}{3}\partial_t\Big)Z_k + 7\dfrac{k^2}{l^2}(2+\partial_t)Z_k\Big] \end{cases} \tag{5.22}$$

In order to obtain a well defined non-compact limit of the model, we use a modified ansatz (different from in section 4.4):

$$Z_k = \overline{Z}_k k^{-\chi} l^{\chi}\,, \quad \mu_k = \overline{\mu}_k \overline{Z}_k k^{2-\chi} l^{\chi}\,, \quad \lambda_k = \overline{\lambda}_k \overline{Z}_k^2 k^{6-\xi} l^{\xi}\,, \tag{5.23}$$

from which we obtain the dimensionless β-functions according to the following small calculation:

$$\begin{aligned} \eta_k &= \frac{1}{\overline{Z}_k}\beta(\overline{Z}_k) = \frac{k^{\chi} l^{-\chi}}{\overline{Z}_k}\beta(Z_k) + \chi\,, \\ \beta(\overline{\mu}_k) &= \frac{k^{\chi-2} l^{-\chi}}{\overline{Z}_k}\beta(\mu_k) - \eta_k\overline{\mu}_k + (\chi-2)\overline{\mu}_k\,, \\ \beta(\overline{\lambda}_k) &= \frac{k^{\xi-6} l^{-\xi}}{\overline{Z}_k^2}\beta(\lambda_k) - 2\eta_k\overline{\lambda}_k + (\xi-6)\overline{\lambda}_k\,. \end{aligned} \tag{5.24}$$

Inserting the above in (5.22), we deduce the dimensionless coupling constant equations:

$$\begin{cases} \eta_k = \dfrac{\overline{\lambda}_k k^{2+2\chi-\xi} l^{\xi-2\chi}}{(1+\overline{\mu}_k)^2}\Big[(\eta_k-\chi)\Big(\dfrac{3}{\sqrt{2}}\dfrac{k}{l^3}+\dfrac{4}{l^2}\Big) + \dfrac{3}{\sqrt{2}}\dfrac{k}{l^3}\Big] + \chi \\ \beta(\overline{\mu}_k) = -\dfrac{3\overline{\lambda}_k k^{2\chi-\xi} l^{\xi-2\chi}}{(1+\overline{\mu}_k)^2}\Big[(\eta_k-\chi)\Big(\dfrac{2\sqrt{2}}{3}\dfrac{k^3}{l^3}+\dfrac{k^2}{l^2}\Big) + 2\Big(\sqrt{2}\dfrac{k^3}{l^3}+\dfrac{k^2}{l^2}\Big)\Big] \\ \qquad - \eta_k\overline{\mu}_k + (\chi-2)\overline{\mu}_k \\ \beta(\overline{\lambda}_k) = \dfrac{2\overline{\lambda}_k^2 k^{2\chi-\xi} l^{\xi-2\chi}}{(1+\overline{\mu}_k)^3}\Big[(\eta_k-\chi)\Big(\dfrac{2\sqrt{2}}{3}\dfrac{k^3}{l^3}+7\dfrac{k^2}{l^2}\Big) + 2\Big(\sqrt{2}\dfrac{k^3}{l^3}+7\dfrac{k^2}{l^2}\Big)\Big] \\ \qquad - 2\eta_k\overline{\lambda}_k + (\xi-6)\overline{\lambda}_k \end{cases} \tag{5.25}$$

As in chapter 4, the system of β-functions is non-autonomous in the IR cut-off k, as long as l is kept finite. This is again due to the non-local feature of the interaction part, but we notice a different dependence on the parameters k and l with respect to (4.37). The difference is of course a consequence of the presence of the delta functions which, having non-trivial dimensions, change both the canonical and scaling dimensions of couplings and fields and remove degrees of freedom of the space of dynamical fields by imposing constraints.

Concerning this peculiarity, we must point out that, in the case we introduced one delta for each field appearing in both the kinetic and interaction kernels, we could have absorbed, from the point of view of the dimensions, the contribution of deltas inside a redefinition of fields. In that case we would expect the couplings to have the same (canonical) dimensions of those appearing in the previous model.

Regarding the parameters ξ and χ, one infers that there is a unique independent equation which allows the regularization of the infinite volume contributions in the non-compact limit, that is:

$$\xi = 3 + 2\chi \,. \tag{5.26}$$

Taking into account this relation, we obtain an autonomous system with an explicitly dependence on χ:

$$\begin{cases} \eta_k = \dfrac{3\overline{\lambda}_k}{\sqrt{2}(1+\overline{\mu}_k)^2}(\eta_k - \chi + 1) + \chi \\ \beta(\overline{\mu}_k) = -\dfrac{6\overline{\lambda}_k\sqrt{2}}{(1+\overline{\mu}_k)^2}\Big(\dfrac{\eta_k - \chi}{3} + 1\Big) - (\eta_k - \chi)\overline{\mu}_k - 2\overline{\mu}_k \\ \beta(\overline{\lambda}_k) = \dfrac{4\overline{\lambda}_k^2\sqrt{2}}{(1+\overline{\mu}_k)^3}\Big(\dfrac{\eta_k - \chi}{3} + 1\Big) - 2(\eta_k - \chi)\overline{\lambda}_k - 3\overline{\lambda}_k \end{cases} \tag{5.27}$$

However, this dependence in χ can be merely re-absorbed by a redefinition of η_k as $\eta'_k = \eta_k - \chi$. We therefore have finally a system of dimensionless β-functions given by

$$\begin{cases} \eta'_k = \dfrac{3\overline{\lambda}_k}{\sqrt{2}(1+\overline{\mu}_k)^2 - 3\overline{\lambda}_k} \\ \beta(\overline{\mu}_k) = -\dfrac{6\overline{\lambda}_k\sqrt{2}}{(1+\overline{\mu}_k)^2}\Big(\dfrac{\eta'_k}{3} + 1\Big) - \eta'_k\overline{\mu}_k - 2\overline{\mu}_k \\ \beta(\overline{\lambda}_k) = \dfrac{4\overline{\lambda}_k^2\sqrt{2}}{(1+\overline{\mu}_k)^3}\Big(\dfrac{\eta'_k}{3} + 1\Big) - 2\eta'_k\overline{\lambda}_k - 3\overline{\lambda}_k \end{cases} \tag{5.28}$$

Like in the model without gauge projection, the system presents a divergence in the flow due to the truncation scheme. Here the singularity occurs at $\overline{\mu} = -1$ and $\overline{\lambda} = \frac{\sqrt{2}}{3}(1+\overline{\mu})^2$. In the plane $(\overline{\mu}, \overline{\lambda})$, we find four fixed points, the Gaussian (GFP) and three non-Gaussian fixed points (NGFP) at:

$$P_1 = (10)^{-1}(-7.083, 0.154), \qquad P_2 = 10^{-1}(-7.935, 0.273), \tag{5.29}$$
$$P_3 = (-12.809, 169.635)\,. \tag{5.30}$$

Both P_2 and P_3 lie in the sector disconnected from the origin. We restrict the analysis and linearize the system only around P_1 and the Gaussian fixed point. The following eigenvalues and eigenvectors can be found by simple calculation from the stability matrix:

$$\theta_1^G = -3 \text{ for } \mathbf{v}_1^G = (6\sqrt{2}, 1), \tag{5.31}$$
$$\theta_2^G = -2 \text{ for } \mathbf{v}_2^G = (1, 0), \tag{5.32}$$
$$\theta_+^{NG} \sim 14.47 \text{ for } \mathbf{v}_+^{NG} \sim 10^{-1}(9.986, -0.529), \tag{5.33}$$

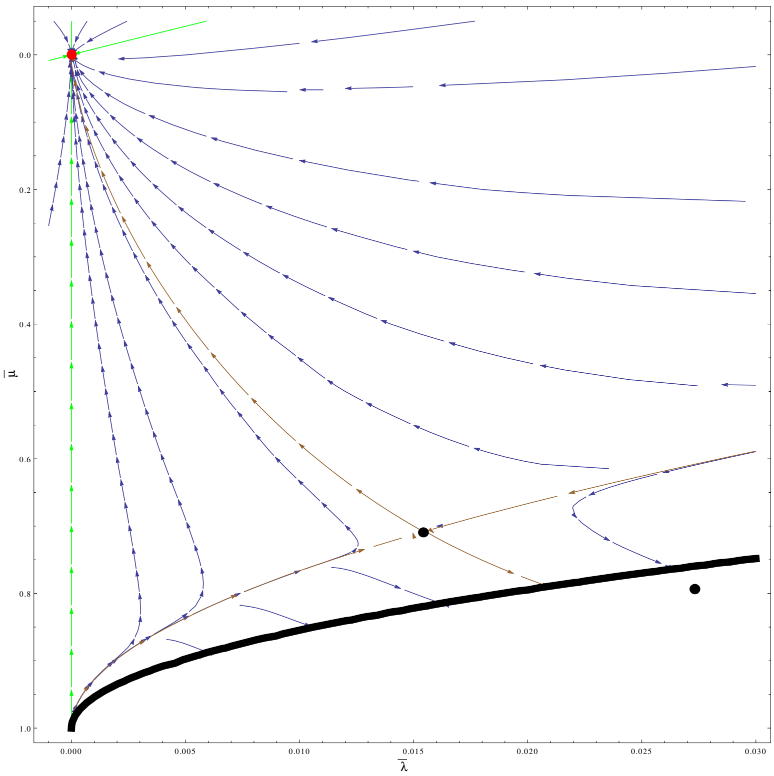


Figure 5.1: Flow for the rank 3 model with gauge projection. Brown arrows represent the eigendirections of the NGFP (in black), while green arrows are the eigendirections of GFP (in red). The thick black line manifests the singularity of the system.

$$\theta_-^{NG} \sim -2.29 \text{ for } \mathbf{v}_-^{NG} \sim 10^{-1}(9.948, 1.022). \tag{5.34}$$

Negative eigenvalues represents UV-attractive eigendirections, while positive eigenvalues correspond to UV-repulsive eigendirections. From the plot in Fig. 5.1, we see that the Gaussian fixed point, where we have two negative eigenvalues corresponding to the scaling dimensions of the couplings, is a UV-attractor and has two relevant directions. Thus, we infer that the model is asymptotically free in the UV. Meanwhile the NGFP has a relevant direction and an irrelevant direction. In this model, there are no marginal directions in the flow and, qualitatively, the structure of the plot is again reminiscent of the Wilson-Fisher fixed point in standard scalar field theory in three dimensions. This is again a strong hint to a phase transition between a symmetric and a broken phase. If this spontaneous symmetry breaking is proved, then we could interpret the broken phase with a condensed state identifiable with a geometric phase.

Comparing the model analyzed in the present chapter with the one studied in chapter 4, we can list some similarities and differences.

From the computational point of view of the FRG, there are no fundamental differences, the presence of the delta gauge constraints has an influence on the end result like in the dependence on the parameters k and l of the equations. The type of ansatz that we use to make the RG equations autonomous is similar, even though fixing the parameters of the ansatz differs from one model to the other. Hence, as a first difference, the canonical dimension of the ϕ^4-coupling changes from one model to the other. We claim that these models are not in the same universality class. From a qualitative point of view, we find in both models the same number of non-Gaussian fixed points, but their distribution on the plain $(\overline{\mu}, \overline{\lambda})$ appears to be different. On one hand, the TGFT model without gauge projection has two interesting NGFP's in the region of the plane $(\overline{\mu}, \overline{\lambda})$ connected to the origin, whereas the gauge projected model has a unique NGFP. Finally, the linearized theory around the Gaussian fixed point turns out to be slightly different. While in the previous chapter, we have found a non-diagonalizable stability matrix with only one strictly relevant directions, for the gauge projected case, we have two relevant directions and the eigenperturbations form indeed a basis for the linearized system. Both GFP's are sink in these models and so, both models can be called asymptotically free.

Chapter 6

Conclusion and outlook

In this thesis, we have undertaken the Functional Renormalization Group analysis of two models in Tensorial Group Field Theories. Endowed with a melonic combinatorial interaction and distinguished from the presence (or absence) of a projection on the gauge invariant dynamics, both models are pertinent for probing a continuum limit for geometry, as this is the main objective of these quantum gravity theories. The particular aim of our study was to identify the presence of phase transitions in GFT setup and to enlighten the peculiarities coming from the non-compactness of the underlying group manifold to be able to consistently identify fixed points in the flow in the IR and UV regimes. The existence of these fixed points make phase transition very likely and could make consistent a geometrogenesis scenario. We have successfully identified IR fixed points, hence have improved the formulation of [66] however the full characterization of the phases is still to be obtained. Let us be a little bit more precise on these points.

The FRG approach that we apply turns out to have several implications that we have properly adjusted to our context. First of all, from the fact that the group manifold over which we define the fields is non-compact, new properties of IR divergences during the expansion of the Wetterich equation occur. These are different compared with infinite volume factors in local field theories. We attribute this feature to the particular combinatorics of TGFT's interaction terms. The lattice regularization of these pathologies and the thermodynamic limit become fundamental ingredients to obtain an autonomous system of β-functions in both models. The system of β-functions is analysed and, we recognize the fixed points as IR and UV fixed points with respect to the cut-off. Thus one of our goal is reached.

In both models studied, the flows of the theories show similarities of the neighborhood of the non-Gaussian fixed points in comparison with the Wilson-Fisher fixed point. This is a hint toward an interpretation of the phases in term of spontaneous symmetry breaking which might lead to a condensate state. If this claim is proved, then our results would suggest, in a simplicial gravity context, the emergence of a continuous spacetime from discrete pre-geometric structures.

A clear understanding of the different phases in both models require a solution for the equations of motion and a change of parametrization for the effective potential. This point must be sorted. Because of the combinatorial non-local structure of the interactions, a resolution of the equation of motion turns out to highly non-trivial, to the best of our knowledge. In order to achieve the description of the phase transition for the present class of models, further investigations on the solution of that equation is required, this might be addressed in forthcoming investigations. Finally, one hopes that the FRG methods might prove to be a consistent and powerful way to address a background independent quantum gravity scenario.

Appendix A

Evaluation of the β-functions for rank 3 tensorial GFT

In this appendix, we provide the detailed calculation of the β equations and emphasize its peculiarities. Note that, this computation of the β-functions is performed in the regularized framework and only, at the end, we take the thermodynamic limit. The system of equations that we obtain is an autonomous system in a continuous non-compact space.

Notations. Given the regularization prescription introduced in section 4.3, we set the notation $\delta_{D^*}(\mathbf{p},\mathbf{q}) = \delta(\mathbf{p}-\mathbf{q})$ not to be confused with the continuous Dirac delta that we do not use in this appendix. We also define $\mathcal{D}$ to be the one dimensional lattice, that is, the domain of a single component of objects in D^*. We have $D^* = \mathcal{D}^{\times 3}$ so that:

$$l\sum_{p_i} = \int_{\mathcal{D}} dp_i\,. \tag{A.1}$$

Let us recall the second variation of the effective action

$$\begin{aligned}
\Gamma_k^{(2)} &= (Z_k\sum_s p_s^2 + \mu_k)\delta(\mathbf{p}-\mathbf{p'})\\
&+ \lambda_k\Big[\int_{\mathcal{D}^{\times 2}} dq_2 dq_3\ \varphi_{p'_1 q_2 q_3}\overline{\varphi}_{p_1 q_2 q_3}\delta(p_2-p'_2)\delta(p_3-p'_3)\\
&+ \int_{\mathcal{D}} dq_1\ \varphi_{q_1 p'_2 p'_3}\overline{\varphi}_{q_1 p_2 p_3}\delta(p_1-p'_1) + \mathrm{sym}\Big\{1\to 2\to 3\Big\}\Big]\\
&= (Z_k\sum_s p_s^2 + \mu_k)\delta(\mathbf{p}-\mathbf{p'}) + F_k(\mathbf{p},\mathbf{p'})\,,
\end{aligned}$$

and choose a regulator of the following form:

$$R_k(\mathbf{p},\mathbf{p'}) = \delta(\mathbf{p}-\mathbf{p'})Z_k(k^2-\sum_s p_s^2)\Theta(k^2-\sum_s p_s^2)\,,$$

where $\Theta(f(\mathbf{p}))$ is the discrete step function. This implies:

$$\partial_t R_k = \delta(\mathbf{p}-\mathbf{p'})\Theta(k^2 - \sum_s p_s^2)[\partial_t Z_k(k^2 - \sum_s p_s^2) + Z_k 2k^2]\,.$$

Defining

$$P_k(\mathbf{p},\mathbf{p'}) = R_k(\mathbf{p}-\mathbf{p'}) + (Z_k \sum_s p_s^2 + \mu_k)\delta(\mathbf{p}-\mathbf{p'})\,,$$

we expand and truncate the Wetterich equation as follows:

$$\begin{aligned}\partial_t \Gamma_k &= \mathrm{Tr}[\partial_t R_k \cdot (P_k + F_k)^{-1}] = \mathrm{Tr}[(\partial_t R_k \cdot (P_k)^{-1}) \cdot (1 + F_k \cdot (P_k)^{-1})^{-1}] \\ &= \mathrm{Tr}[(\partial_t R_k \cdot (P_k)^{-1}) \cdot (1 - F_k \cdot (P_k)^{-1} + (F_k \cdot (P_k)^{-1})^2) + \dots]\,. \end{aligned} \quad \text{(A.2)}$$

The zeroth order of the previous expansion is the vacuum term and does not provide us any useful information. On the other hand, the first and the second order, will provide us with the flow of the kinetic (φ^2-) and interaction (φ^4-) terms, respectively, from which we can compute the β-functions for the couplings μ_k, Z_k and λ_k.

A.1 φ^2-terms

To compute the flow of couplings in the quadratic term of Γ_k, namely, the β-functions for μ_k and Z_k, we focus on the first order of (A.2). A change of notation helps during the calculation:

$$\mathbf{q} = (q_1, q_2, q_3) \quad \Rightarrow \quad q_1 := q_1\,; \qquad \mathbf{q}_1^{(2)} := (q_2, q_3)\,; \quad q_1^{(2)} := \sqrt{q_2^2 + q_3^2}\,,$$

for a generic 3-dimensional momentum $\mathbf{q}$. When there is no possible confusion, we will simply forget the subscript 1 of $\mathbf{q}_1^{(2)}$ and $q_1^{(2)}$, and use $\mathbf{q}^{(2)}$ and $q^{(2)}$, respectively.

To have more compact notations, let us introduce the first convolution appearing in the expansion:

$$\begin{aligned}\tilde{\partial}_t R_k(\mathbf{p},\mathbf{p}'') &= \int_{D^*} d\mathbf{p}'\; \partial_t R_k(\mathbf{p},\mathbf{p}')(P_k)^{-1}(\mathbf{p}',\mathbf{p}'') \\ &= \int_{D^*} d\mathbf{p}'\; \delta(\mathbf{p}-\mathbf{p}')\delta(\mathbf{p}'-\mathbf{p}'')\Theta(k^2 - \sum_s p_s^2)\frac{\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k}{Z_k(k^2 - \sum_s p_s'^2)\Theta(k^2 - \sum_s p_s'^2) + Z_k \sum_s p_s'^2 + \mu_k} \\ &= \delta(\mathbf{p}-\mathbf{p}'')\Theta(k^2 - \sum_s p_s^2)\frac{\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k}{(Z_k k^2 + \mu_k)}\,,\end{aligned}$$

where we used the fact that, after integration over $\mathbf{p}'$, the two Θ's appearing in the expression are redundant and, the one in the denominator can be set to 1.

Thus, calling $(I)_W$ the first order of the Wetterich equation, we write

$$-(I)_W = \overline{\mathrm{Tr}}[\tilde{\partial}_t R_k \cdot F_k \cdot (P_k)^{-1}]$$

$$= \int_{D^{*\times 2}} d\mathbf{p}d\mathbf{p}' \, \tilde{\partial}_t R_k(\mathbf{p},\mathbf{p}') \int_{D^*} d\mathbf{q} \, F_k(\mathbf{p}',\mathbf{q})(P_k)^{-1}(\mathbf{q},\mathbf{p})$$
$$= \int_{D^*} d\mathbf{p} \, \Theta(k^2 - \sum_s p_s^2) \frac{\partial_t Z_k (k^2 - \sum_s p_s^2) + 2k^2 Z_k}{(Z_k k^2 + \mu_k)^2} F_k(\mathbf{p},\mathbf{p}) \,,$$

where, once again, we used the redundancy of the Θ-functions to factorize their contribution. We can now split the integral in two pieces, namely:

$$A = \frac{\partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} \, \Theta(k^2 - \sum_s p_s^2) \Big(\sum_s p_s^2 \Big) F_k(\mathbf{p},\mathbf{p}) \,,$$

$$B = \frac{k^2 (2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} \, \Theta(k^2 - \sum_s p_s^2) \, F_k(\mathbf{p},\mathbf{p}) \,, \tag{A.3}$$

and so, we have $(I)_W = A - B$. Let us treat the first term and recall that $\delta_{\mathcal{D}}(0) = \delta(0) = \frac{1}{l}$:

$$A = \frac{\lambda_k \, \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} \, \Theta(k^2 - \sum_s p_s^2) \Big(\sum_s p_s^2 \Big)$$
$$\times \left[\frac{1}{l^2} \int_{\mathcal{D}^{\times 2}} d\mathbf{q}^{(2)} \, |\varphi_{p_1 q_2 q_3}|^2 + \frac{1}{l} \int_{\mathcal{D}} dq_1 \, |\varphi_{q_1 p_2 p_3}|^2 + \mathrm{sym}\Big\{ 1 \to 2 \to 3 \Big\} \right]$$
$$= \frac{\lambda_k \, \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \times$$
$$\Bigg\{ \frac{1}{l^2} \int_{D^*} dp_1 d\mathbf{q}^{(2)} \, |\varphi_{p_1 q_2 q_3}|^2 \int_{\mathcal{D}^{\times 2}} d\mathbf{p}^{(2)} \, \Theta[(k^2 - p_1^2) - (p^{(2)})^2] \Big[(p^{(2)})^2 + p_1^2 \Big]$$
$$+ \frac{1}{l} \int_{D^*} dq_1 d\mathbf{p}^{(2)} \, |\varphi_{q_1 p_2 p_3}|^2 \int_{\mathcal{D}} dp_1 \, \Theta[(k^2 - (p^{(2)})^2) - p_1^2] \Big[(p^{(2)})^2 + p_1^2 \Big] \Bigg\}$$
$$+ \mathrm{sym}\Big\{ 1 \to 2 \to 3 \Big\} .$$

Now we perform the continuum limit $l \to \infty$ and this corresponds to:

$$\int_{\mathcal{D}} \longrightarrow \int_{\mathbb{R}} , \qquad \Theta \longrightarrow \theta \, . \tag{A.4}$$

The negative powers of l appearing in the expressions keep track of the former IR divergences of the continuous model. Extracting an l dependence from the couplings, we will address them at the end. In order to simplify the notation, we drop the limit symbol $\lim_{l \to \infty}$ and get

$$A = \frac{\lambda_k \, \partial_t Z_k}{(Z_k k^2 + \mu_k)^2}$$
$$\times \Bigg\{ \frac{1}{l} \int_{\mathbb{R}^3} dq_1 d\mathbf{p}^{(2)} \, \theta(k^2 - (p^{(2)})^2) |\varphi_{q_1 p_2 p_3}|^2 \int_{-\sqrt{k^2 - (p^{(2)})^2}}^{\sqrt{k^2 - (p^{(2)})^2}} dp_1 \, [(p^{(2)})^2 + p_1^2]$$

$$
+\frac{1}{l^2}\int_{\mathbb{R}^3} dp_1 d\mathbf{q}^{(2)}\,\theta(k^2-p_1^2)|\varphi_{p_1q_2q_3}|^2\int_0^{2\pi}d\vartheta\int_0^{\sqrt{k^2-p_1^2}}dp^{(2)}\,p^{(2)}[(p^{(2)})^2+p_1^2]\Bigg\}
$$

$$
+\,\mathrm{sym}\Big\{1\to2\to3\Big\}
$$

$$
=\frac{\lambda_k\,\partial_t Z_k}{(Z_kk^2+\mu_k)^2}\Bigg\{\frac{1}{l}\int_{\mathbb{R}^3}dq_1d\mathbf{p}^{(2)}\,|\varphi_{q_1p_2p_3}|^2\,\theta(k^2-(p^{(2)})^2)
$$

$$
\times\left[2(p^{(2)})^2\sqrt{k^2-(p^{(2)})^2}+\frac{2}{3}(k^2-(p^{(2)})^2)^{3/2}\right]
$$

$$
+2\pi\frac{1}{l^2}\int_{\mathbb{R}^3}dp_1d\mathbf{q}^{(2)}\,|\varphi_{p_1q_2q_3}|^2\,\theta(k^2-p_1^2)\left[\frac{(k^2-p_1^2)^2}{4}+\frac{p_1^2}{2}(k^2-p_1^2)\right]\Bigg\}
$$

$$
+\,\mathrm{sym}\Big\{1\to2\to3\Big\}\,.
$$

Expanding the term B, we find:

$$
B=\lambda_k\frac{k^2(2+\partial_t)Z_k}{(Z_kk^2+\mu_k)^2}
$$

$$
\times\int_{D^*}d\mathbf{p}\;\Theta(k^2-\sum_s p_s^2)\left[\frac{1}{l^2}\int_{\mathcal{D}^{\times2}}d\mathbf{q}^{(2)}\,|\varphi_{p_1q_2q_3}|^2+\frac{1}{l}\int_{\mathcal{D}}dq_1\,|\varphi_{q_1p_2p_3}|^2\right]
$$

$$
+\mathrm{sym}\Big\{1\to2\to3\Big\}\,, \tag{A.5}
$$

which, in the limit, gives

$$
B=\lambda_k\frac{k^2(2+\partial_t)Z_k}{(Z_kk^2+\mu_k)^2}\Bigg\{\frac{2\pi}{l^2}\int_{\mathbb{R}^3}dp_1d\mathbf{q}^{(2)}\,\theta(k^2-p_1^2)|\varphi_{p_1q_2q_3}|^2\int_0^{\sqrt{k^2-p_1^2}}dp^{(2)}\,p^{(2)}
$$

$$
+\frac{1}{l}\int_{\mathbb{R}^3}dq_1d\mathbf{p}^{(2)}\,\theta(k^2-(p^{(2)})^2)|\varphi_{q_1p_2p_3}|^2\int_{-\sqrt{k^2-(p^{(2)})^2}}^{\sqrt{k^2-(p^{(2)})^2}}dp_1\Bigg\}
$$

$$
+\,\mathrm{sym}\Big\{1\to2\to3\Big\}
$$

$$
=\lambda_k\frac{k^2(2+\partial_t)Z_k}{(Z_kk^2+\mu_k)^2}\Bigg\{\frac{\pi}{l^2}\int_{\mathbb{R}^3}dp_1d\mathbf{q}^{(2)}\,|\varphi_{p_1q_2q_3}|^2\theta(k^2-p_1^2)(k^2-p_1^2)
$$

$$
+\frac{2}{l}\int_{\mathbb{R}^3}dq_1d\mathbf{p}^{(2)}\,|\varphi_{q_1p_2p_3}|^2\theta(k^2-(p^{(2)})^2)\sqrt{k^2-(p^{(2)})^2}\Bigg\}
$$

$$
+\,\mathrm{sym}\Big\{1\to2\to3\Big\}\,.
$$

β-functions. To find the β-functions of the coupling constants, we rely on the fact that the l.h.s. of (A.2) is of the form:

$$
\partial_t\Gamma_k=\int d\mathbf{p}\,|\varphi(\mathbf{p})|^2\left(\beta(Z_k)\sum_s p_s^2+\beta(\mu_k)\right).
$$

We must expand the r.h.s. up to $o(p^3)$. The terms with momenta of order p_i^2 convoluted with the fields $\varphi_{\dots,p_i,\dots}$ will contribute to the flow of the wave function renormalization, while the zeroth order will be proportional to the scaling of the mass. All remaining terms, falling out of the truncation, must be discarded. Hence, we have

$$
\begin{aligned}
A \simeq\, & 2\lambda_k \frac{\partial_t Z_k}{(Z_k k^2+\mu_k)^2}\Bigg\{\frac{\pi}{l^2}\int d\mathbf{q}^{(2)}dp_1\; |\varphi_{p_1 q_2 q_3}|^2\;\theta(k^2-p_1^2)\,\frac{k^4}{4} \\
& +\frac{1}{l}\int dq_1 d\mathbf{p}^{(2)}\; |\varphi_{q_1 p_2 p_3}|^2\;\theta(k^2-(p^{(2)})^2)[k(p^{(2)})^2+\frac{1}{3}(k^3-\frac{3}{2}k(p^{(2)})^2)]\Bigg\} \\
& +\mathrm{sym}\Big\{1\to 2\to 3\Big\} \\
\simeq\, & \lambda_k \frac{\partial_t Z_k}{(Z_k k^2+\mu_k)^2}\Bigg\{\frac{2\pi}{l^2}\int d\mathbf{q}^{(2)}dp_1\; |\varphi_{p_1 q_2 q_3}|^2\frac{1}{4}[\theta(k^2)k^2-\delta(k^2)k^2p_1^2]k^2 \\
& +\frac{2}{l}\int dq_1 d\mathbf{p}^{(2)}\; |\varphi_{q_1 p_2 p_3}|^2\;\theta(k^2-(p^{(2)})^2)[\frac{1}{3}k^2+\frac{1}{2}(p^{(2)})^2]k\Bigg\} \\
& +\mathrm{sym}\Big\{1\to 2\to 3\Big\} \\
\simeq\, & \lambda_k \frac{\partial_t Z_k}{(Z_k k^2+\mu_k)^2}\Bigg\{\frac{\pi}{2l^2}\int d\mathbf{q}^{(2)}dp_1\; |\varphi_{p_1 q_2 q_3}|^2 k^4 \\
& +\frac{2}{l}\int dq_1 d\mathbf{p}^{(2)}\; |\varphi_{q_1 p_2 p_3}|^2\Big[\frac{1}{3}k^2+\frac{1}{2}(p^{(2)})^2\Big]k\Bigg\}+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\,,
\end{aligned}
$$

where the following relations have been used:

$$
\begin{aligned}
&\theta(0)=1 \quad\Rightarrow\quad \theta(k^2)=1\,,\ \forall k\,,\\
&\delta(k^2)k^2=\delta(\frac{k^2}{k^2})=\delta(1)=0\,.
\end{aligned}
$$

For the B terms, one finds:

$$
\begin{aligned}
B \simeq\, & \lambda_k \frac{k^2(2+\partial_t)Z_k}{(Z_k k^2+\mu_k)^2}\Bigg\{\frac{\pi}{l^2}\int d\mathbf{q}^{(2)}dp_1\; |\varphi_{p_1 q_2 q_3}|^2(k^2-p_1^2) \\
& +\frac{2}{l}\int dq_1 d\mathbf{p}^{(2)}\; |\varphi_{q_1 p_2 p_3}|^2[k-\frac{1}{2k}(p^{(2)})^2]\Bigg\}+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\,. \qquad (A.6)
\end{aligned}
$$

Now, we concentrate on the colored symmetric terms. Note that the procedure and result of the above integrals will not change for each colored term in sym$\{\cdot\}$, up to a simple relabeling. Concerning the scaling of μ_k, involving "constant" terms $C(k)$, we collect a factor 3 in front of what we have already computed, but for the Z_k term, which involves p_i-labeled terms, this is not trivial. For each term in sym, there is a $C_1(k)$ times only a simple p_i^2 momentum square, and another term $C_2(k)$ multiplied by the sum of the other two momenta square, i.e. $(p_i^{(2)})^2$.

We have an expression of the form:

$$\begin{aligned}\beta(Z_k) &= C_1(k)\delta^2(0)(p_1^2+p_2^2+p_3^2) + C_2(k)\delta(0)[(p_2^2+p_3^2)+(p_1^2+p_3^2)+(p_1^2+p_2^2)] \\ &= C_1(k)\delta^2(0)(p_1^2+p_2^2+p_3^2) + 2C_2(k)\delta(0)(p_1^2+p_2^2+p_3^2)\,.\end{aligned}$$

Taking this combinatorial subtlety into account, putting all the pieces together and recalling that $(I)_W = A - B$, we write the dimensionful β-functions of the mass and wave function as:

$$\begin{aligned}\beta(Z_k) &= \frac{\lambda_k}{(Z_k k^2+\mu_k)^2}\left[\partial_t Z_k\left(\pi\frac{k^2}{l^2}+4\frac{k}{l}\right)+2Z_k\left(\pi\frac{k^2}{l^2}+2\frac{k}{l}\right)\right],\\ \beta(\mu_k) &= 3\frac{\lambda_k}{(Z_k k^2+\mu_k)^2}\left[\partial_t Z_k\left(\frac{\pi}{2}\frac{k^4}{l^2}+\frac{2}{3}\frac{k^3}{l}-\pi\frac{k^4}{l^2}-2\frac{k^3}{l}\right)-2Z_k\left(\frac{k^4}{l^2}\pi+2\frac{k^3}{l}\right)\right]\\ &= -3\frac{\lambda_k}{(Z_k k^2+\mu_k)^2}\left[\partial_t Z_k\left(\frac{\pi}{2}\frac{k^4}{l^2}+\frac{4}{3}\frac{k^3}{l}\right)+2Z_k\left(\frac{k^4}{l^2}\pi+2\frac{k^3}{l}\right)\right].\end{aligned}$$

Already at this level, one realizes that each β-function does not have homogeneous scaling in k and dimensions in l. This feature clearly comes from the pattern of the convolution of the interaction which is specific to TGFTs.

A.2 φ^4-terms

The second order $(II)_W$ of (A.2) will provide the β-function for λ_k, which completes the set of β-functions of the model. Defining:

$$\begin{aligned}&R_k' \quad \text{s.t.} \quad R_k(\mathbf{p},\mathbf{p'}) = R_k'(\mathbf{p})\Theta(k^2-\sum_s p_s^2)\delta(\mathbf{p}-\mathbf{p}')\,,\\ &P_k' \quad \text{s.t.} \quad P_k(\mathbf{p},\mathbf{p'}) = P_k'(\mathbf{p})\delta(\mathbf{p}-\mathbf{p}')\,,\end{aligned} \tag{A.7}$$

the terms of interest take the form:

$$\begin{aligned}&(II)_W = \mathrm{Tr}[\partial_t R_k\cdot(P_k)^{-1}\cdot F_k\cdot(P_k)^{-1}\cdot F_k\cdot(P_k)^{-1}]\\ &= \int_{D^{*\times 6}} d\mathbf{p}d\mathbf{p}'d\mathbf{p}''d\mathbf{q}d\mathbf{q}'d\mathbf{q}''\;\partial_t R_k'(\mathbf{p})\Theta(k^2-\sum_s p_s^2)\delta(\mathbf{p}-\mathbf{p}')\\ &(P_k')^{-1}(\mathbf{p}')\delta(\mathbf{p}'-\mathbf{p}'')F_k(\mathbf{p}'',\mathbf{q})(P_k')^{-1}(\mathbf{q})\delta(\mathbf{q}-\mathbf{q}')F_k(\mathbf{q}',\mathbf{q}'')(P_k')^{-1}(\mathbf{q}'')\delta(\mathbf{q}''-\mathbf{p})\\ &= \int_{D^{*\times 5}} d\mathbf{p}d\mathbf{p}'d\mathbf{p}''d\mathbf{q}d\mathbf{q}'\;\partial_t R_k'(\mathbf{p})\Theta(k^2-\sum_s p_s^2)\delta(\mathbf{p}-\mathbf{p}')(P_k')^{-1}(\mathbf{p}')\delta(\mathbf{p}'-\mathbf{p}'')\\ &F_k(\mathbf{p}'',\mathbf{q})(P_k')^{-1}(\mathbf{q})\delta(\mathbf{q}-\mathbf{q}')F_k(\mathbf{q}',\mathbf{p})(P_k')^{-1}(\mathbf{p})\\ &= \int_{D^{*\times 4}} d\mathbf{p}d\mathbf{p}''d\mathbf{q}d\mathbf{q}'\;\partial_t R_k'(\mathbf{p})\Theta(k^2-\sum_s p_s^2)(P_k')^{-1}(\mathbf{p})\delta(\mathbf{p}-\mathbf{p}'')\\ &F_k(\mathbf{p}'',\mathbf{q})(P_k')^{-1}(\mathbf{q})\delta(\mathbf{q}-\mathbf{q}')F_k(\mathbf{q}',\mathbf{p})(P_k')^{-1}(\mathbf{p})\end{aligned}$$

$$
= \int_{D^*} d\mathbf{p}\, \partial_t R'_k(\mathbf{p}) \Theta(k^2 - \sum_s p_s^2)(P'_k)^{-1}(\mathbf{p})
$$
$$
\times \int_{D^*} d\mathbf{q}\, F_k(\mathbf{p},\mathbf{q})(P'_k)^{-1}(\mathbf{q}) F_k(\mathbf{q},\mathbf{p})(P'_k)^{-1}(\mathbf{p})\,. \tag{A.8}
$$

We focus on the intermediate convolution $F_k \cdot P_k^{-1} \cdot F_k$ which expands as:

$$
(F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p},\mathbf{p}) = \lambda_k^2 \int_{D^*} d\mathbf{q} F(\mathbf{p},\mathbf{q})(P'_k)^{-1}(\mathbf{q}) F_k(\mathbf{q},\mathbf{p})
$$
$$
= \lambda_k^2 \int_{D^*} dq_1 d\mathbf{q}^{(2)} \Big[\int_{\mathcal{D}} dm_1\, \varphi_{m_1 p_2 p_3} \overline{\varphi}_{m_1 q_2 q_3} \delta(p_1 - q_1)
$$
$$
+ \int_{\mathcal{D}^{\times 2}} d\mathbf{m}^{(2)}\, \varphi_{p_1 m_2 m_3} \overline{\varphi}_{q_1 m_2 m_3} \delta(\mathbf{p}^{(2)} - \mathbf{q}^{(2)}) + \mathrm{sym}\Big\{1 \to 2 \to 3\Big\} \Big]
$$
$$
(P'_k)^{-1}(\mathbf{q}) \Big[\int_{\mathcal{D}} dm'_1\, \varphi_{m'_1 q_2 q_3} \overline{\varphi}_{m'_1 p_2 p_3} \delta(p_1 - q_1)
$$
$$
+ \int_{\mathcal{D}^{\times 2}} d\mathbf{m}'^{(2)}\, \varphi_{q_1 m'_2 m'_3} \overline{\varphi}_{p_1 m'_2 m'_3} \delta(\mathbf{p}^{(2)} - \mathbf{q}^{(2)}) + \mathrm{sym}\Big\{1 \to 2 \to 3\Big\} \Big]\,.
$$

At this level, the product of colored symmetric terms generate a list of terms (among which cross terms) that we must all carefully analyse. First, we deal the case when the product involves two terms of the same color, then we will treat the crossed-colored case. Below, we further specialize the study to the product of terms of color 1 and, then on the cross term 1-2 in the above expansion. We refer to the first type of term as $(F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p},\mathbf{p})|_{1,1}$ and to the overal contribution after tracing over remaining indices as $(II)_W|_{1,1}$ (respectively, the symbol $|_{1,2}$ will stand for the cross term product of the colors 1 and 2). This evaluation is, of course, without loss of generality because one can quickly infer the result coming from the other product with different colors. All these contributions, at the end, must be summed.

We have

$$
(F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p},\mathbf{p})|_{1,1} =
$$
$$
\lambda_k^2 \int_{D^*} dq_1 d\mathbf{q}^{(2)} \int_{\mathcal{D}} dm_1 \varphi_{m_1 p_2 p_3} \overline{\varphi}_{m_1 q_2 q_3} \delta(p_1 - q_1)(P'_k)^{-1}(\mathbf{q})
$$
$$
\times \int_{\mathcal{D}^{\times 2}} d\mathbf{m}'^{(2)} \varphi_{q_1 m'_2 m'_3} \overline{\varphi}_{p_1 m'_2 m'_3} \delta(\mathbf{p}^{(2)} - \mathbf{q}^{(2)})
$$
$$
+ \lambda_k^2 \int_{D^*} dq_1 d\mathbf{q}^{(2)} \int_{\mathcal{D}^{\times 2}} d\mathbf{m}^{(2)} \varphi_{p_1 m_2 m_3} \overline{\varphi}_{q_1 m_2 m_3} \delta(\mathbf{p}^{(2)} - \mathbf{q}^{(2)})(P'_k)^{-1}(\mathbf{q})
$$
$$
\times \int_{\mathcal{D}} dm'_1 \varphi_{m'_1 q_2 q_3} \overline{\varphi}_{m'_1 p_2 p_3} \delta(p_1 - q_1)
$$
$$
+ \lambda_k^2 \int_{D^*} dq_1 d\mathbf{q}^{(2)} \int_{\mathcal{D}} dm_1 \varphi_{m_1 p_2 p_3} \overline{\varphi}_{m_1 q_2 q_3} \delta(p_1 - q_1)(P'_k)^{-1}(\mathbf{q})
$$
$$
\times \int_{\mathcal{D}} dm'_1 \varphi_{m'_1 q_2 q_3} \overline{\varphi}_{m'_1 p_2 p_3} \delta(p_1 - q_1)
$$

$$
+ \lambda_k^2 \int_{D^*} dq_1 d\mathbf{q}^{(2)} \int_{\mathcal{D}^{\times 2}} d\mathbf{m}^{(2)} \varphi_{p_1 m_2 m_3} \overline{\varphi}_{q_1 m_2 m_3} \delta(\mathbf{p}^{(2)} - \mathbf{q}^{(2)}) (P'_k)^{-1}(\mathbf{q})
\times \int_{\mathcal{D}^{\times 2}} d\mathbf{m}'^{(2)} \varphi_{q_1 m'_2 m'_3} \overline{\varphi}_{p_1 m'_2 m'_3} \delta(\mathbf{p}^{(2)} - \mathbf{q}^{(2)}) \,.
$$

The first two terms, once that the δ's in $\mathbf{q}$ are integrated out, become proportional to the product of two square modulus of the fields, thus they represent disconnected interactions. They can be discarded for the same reasons invoked above. As a remainder, we get:

$$
(F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p},\mathbf{p})|_{1,1} \simeq \tag{A.9}
$$
$$
\frac{\lambda_k^2}{l} \int_{\mathcal{D}^{\times 2}} d\mathbf{q}^{(2)} \int_{\mathcal{D}^{\times 2}} dm_1 dm'_1 \varphi_{m_1 p_2 p_3} \overline{\varphi}_{m_1 q_2 q_3} \varphi_{m'_1 q_2 q_3} \overline{\varphi}_{m'_1 p_2 p_3} (P'_k)^{-1}(p_1, \mathbf{q}^{(2)})
+ \frac{\lambda_k^2}{l^2} \int_{\mathcal{D}} dq_1 \int_{\mathcal{D}^{\times 4}} d\mathbf{m}^{(2)} d\mathbf{m}'^{(2)} \varphi_{p_1 m_2 m_3} \overline{\varphi}_{q_1 m_2 m_3} \varphi_{q_1 m'_2 m'_3} \overline{\varphi}_{p_1 m'_2 m'_3} (P'_k)^{-1}(q_1, \mathbf{p}^{(2)}) \,.
$$

Then plugging back (A.9) in $(II)_W$, and concentrating on the contribution of this term, one finds:

$$
(II)_W|_{1,1} = \lambda_k^2 \int_{D^*} d\mathbf{p}\, \Theta(k^2 - \sum_s p_s^2) \tfrac{[\partial_t Z_k (k^2 - \sum_s p_s^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2}
\Bigg\{ \frac{1}{l} \int_{\mathcal{D}^{\times 2}} d\mathbf{q}^{(2)} \int_{\mathcal{D}^{\times 2}} dm_1 dm'_1 \varphi_{m_1 p_2 p_3} \overline{\varphi}_{m_1 q_2 q_3} \varphi_{m'_1 q_2 q_3} \overline{\varphi}_{m'_1 p_2 p_3}
\left[Z_k (k^2 - p_1^2 - (q^{(2)})^2) \Theta(k^2 - p_1^2 - (q^{(2)})^2) + Z_k (p_1^2 + (q^{(2)})^2) + \mu_k \right]^{-1}
+ \frac{1}{l^2} \int_{\mathcal{D}} dq_1 \int_{\mathcal{D}^{\times 4}} d\mathbf{m}^{(2)} d\mathbf{m}'^{(2)} \; \varphi_{p_1 m_2 m_3} \overline{\varphi}_{q_1 m_2 m_3} \varphi_{q_1 m'_2 m'_3} \overline{\varphi}_{p_1 m'_2 m'_3}
\left[Z_k (k^2 - q_1^2 - (p^{(2)})^2) \Theta(k^2 - q_1^2 - (p^{(2)})^2) + Z_k (q_1^2 + (p^{(2)})^2) + \mu_k \right]^{-1} \Bigg\} \,.
$$

With the same principle used for evaluation of the β-functions of Z_k and μ_k, any explicit dependence on the six momenta involved in the four fields in the spectral sums of (A.8) must be discarded. In other words, any term of the form $p_i^\alpha \varphi_{\ldots p_i \ldots} \bar{\varphi}_{\ldots p_i \ldots} \cdot (\varphi \bar{\varphi} \varphi \bar{\varphi})$ falls out of the truncation. After taking the limit (again we drop the symbol $\lim_{l \to 0}$), we expand the expression at zeroth order and get:

$$
(II)_W|_{1,1} \simeq \frac{\lambda_k^2}{l} \int_{\mathbb{R}^6} dm_1 dm'_1 d\mathbf{p}^{(2)} d\mathbf{q}^{(2)} \; \varphi_{m_1 p_2 p_3} \overline{\varphi}_{m_1 q_2 q_3} \varphi_{m'_1 q_2 q_3} \overline{\varphi}_{m'_1 p_2 p_3}
\times \int_{\mathbb{R}} dp_1 \; \tfrac{[\partial_t Z_k (k^2 - p_1^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2} \tfrac{\theta(k^2 - p_1^2)}{Z_k (k^2 - p_1^2) \theta(k^2 - p_1^2) + Z_k p_1^2 + \mu_k}
+ \frac{\lambda_k^2}{l^2} \int_{\mathbb{R}^6} dp_1 dq_1 d\mathbf{m}^{(2)} d\mathbf{m}'^{(2)} \; \varphi_{p_1 m_2 m_3} \overline{\varphi}_{q_1 m_2 m_3} \varphi_{q_1 m'_2 m'_3} \overline{\varphi}_{p_1 m'_2 m'_3}
\times \int_{\mathbb{R}^2} d\mathbf{p}^{(2)} \; \tfrac{[\partial_t Z_k (k^2 - (p^{(2)})^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2} \tfrac{\theta(k^2 - (p^{(2)})^2)}{Z_k (k^2 - (p^{(2)})^2) \theta(k^2 - (p^{(2)})^2) + Z_k (p^{(2)})^2 + \mu_k} \,.
$$

The θ's turn out to be redundant in both the terms and we can simplify their contributions. Call $\mathcal{V}_i$ the vertex of color i of the effective interaction. Rather than using the explicit form of that vertex, we will simply use $\mathcal{V}_i$ in the following, when no confusion might arise.

We split the previous terms in two pieces:

$$
\begin{aligned}
&(II)'_W|_{1,1} = \\
&\tfrac{1}{l}\tfrac{\lambda_k^2 k^2(2+\partial_t)Z_k}{(Z_k k^2+\mu_k)^3} \int d\mathbf{q}^{(2)} d\mathbf{p}^{(2)} dm_1 dm'_1 \; \varphi_{m_1 p_2 p_3}\overline{\varphi}_{m_1 q_2 q_3}\varphi_{m'_1 q_2 q_3}\overline{\varphi}_{m'_1 p_2 p_3} \\
&\times \int dp_1 \; \theta(k^2 - p_1^2) \\
&- \tfrac{1}{l}\tfrac{\lambda_k^2 \partial_t Z_k}{(Z_k k^2+\mu_k)^3} \int d\mathbf{q}^{(2)} d\mathbf{p}^{(2)} dm_1 dm'_1 \; \varphi_{m_1 p_2 p_3}\overline{\varphi}_{m_1 q_2 q_3}\varphi_{m'_1 q_2 q_3}\overline{\varphi}_{m'_1 p_2 p_3} \\
&\times \int dp_1 \; p_1^2 \theta(k^2 - p_1^2) \\
&= 2\frac{\lambda_k^2 k^3}{l}\left[\frac{(2+\partial_t)Z_k}{(Z_k k^2+\mu_k)^3} - \frac{1}{3}\frac{\partial_t Z_k}{(Z_k k^2+\mu_k)^3}\right]\mathcal{V}_1 \\
&= \frac{2\lambda_k^2 k^3}{l\,(Z_k k^2+\mu_k)^3}\left[2Z_k + \frac{2}{3}\partial_t Z_k\right]\mathcal{V}_1\,.
\end{aligned}
$$

The second piece now can be computed as:

$$
\begin{aligned}
&(II)''_W|_{1,1} = \\
&\tfrac{1}{l^2}\tfrac{\lambda_k^2 k^2(2+\partial_t)Z_k}{(Z_k k^2+\mu_k)^3} \int dp_1 dq_1 d\mathbf{m}^{(2)} d\mathbf{m}'^{(2)} \; \varphi_{p_1 m_2 m_3}\overline{\varphi}_{q_1 m_2 m_3}\varphi_{q_1 m'_2 m'_3}\overline{\varphi}_{p_1 m'_2 m'_3} \\
&\times \int d\mathbf{p}^{(2)} \theta(k^2 - p^{(2)}) \\
&- \tfrac{1}{l^2}\tfrac{\lambda_k^2 \partial_t Z_k}{(Z_k k^2+\mu_k)^3} \int dp_1 dq_1 d\mathbf{m}^{(2)} d\mathbf{m}'^{(2)} \; \varphi_{p_1 m_2 m_3}\overline{\varphi}_{q_1 m_2 m_3}\varphi_{q_1 m'_2 m'_3}\overline{\varphi}_{p_1 m'_2 m'_3} \\
&\times \int d\mathbf{p}^{(2)} \; (p^{(2)})^2 \theta(k^2 - p^{(2)}) \\
&= \frac{\pi\lambda_k^2 k^4}{l^2(Z_k k^2+\mu_k)^3}\left[(2+\partial_t)Z_k - \frac{1}{2}\partial_t Z_k\right]\mathcal{V}_1 \\
&= \frac{\pi\lambda_k^2 k^4}{l^2(Z_k k^2+\mu_k)^3}\left[2Z_k + \frac{1}{2}\partial_t Z_k\right]\mathcal{V}_1\,.
\end{aligned}
$$

A simple check of the dimensions of these terms and the dimension of the interaction term of the effective action can be given as

$$[(II)'_W] = [(II)''_W] = 2[\lambda] + 2 + 4[\varphi]\,,$$

which fixes $[\lambda] = 4$ as expected.

Let us now concentrate on the cross term given by the product of the contribution of color 1 and 2, this is:

$$(II)_W|_{1,2} = \lambda_k^2 \int_{D^{*\times 2}} d\mathbf{p} d\mathbf{j} \; \tfrac{\Theta(k^2 - \sum_s p_s^2)}{(Z_k k^2+\mu_k)^2} \tfrac{\partial_t Z_k (k^2 - \sum_s p_s^2) + 2k^2 Z_k}{\Theta(k^2 - \sum_s j_s^2) Z_k (k^2 - \sum_s j_s^2) + Z_k \sum_s j_s^2 + \mu_k}$$

$$\Big[\int_{\mathcal{D}^{\times 2}} dm_1 dn_2\ \varphi_{m_1j_2j_3}\overline{\varphi}_{m_1p_2p_3}\varphi_{p_1n_2p_3}\overline{\varphi}_{j_1n_2j_3}\delta(p_1-j_1)\delta(p_2-j_2)$$
$$+\int_{\mathcal{D}^4} dm_2dm_3dn_1dn_3\ \varphi_{j_1m_2m_3}\overline{\varphi}_{p_1m_2m_3}\varphi_{n_1p_2n_3}\overline{\varphi}_{n_1j_2n_3}\delta(p_2-j_2)\delta^2(p_3-j_3)\delta(p_1-j_1)$$
$$+\int_{D^*} dm_1dn_1dn_3\ \varphi_{m_1j_2j_3}\overline{\varphi}_{m_1p_2p_3}\varphi_{n_1p_2n_3}\overline{\varphi}_{n_1j_2n_3}\delta(p_1-j_1)\delta(p_1-j_1)\delta(p_3-j_3)$$
$$+\int_{D^*} dm_2dm_3dn_3\ \varphi_{j_1m_2m_3}\overline{\varphi}_{p_1m_2m_3}\varphi_{p_1n_2p_3}\overline{\varphi}_{j_1n_2j_3}\delta(p_2-j_2)\delta(p_2-j_2)\delta(p_3-j_3)\Big].$$

If we integrate the deltas over the j variables, the second term is again a disconnected 4-point function that we neglect. Meanwhile, for the other terms, we find:

$$(II)_W|_{1,2} = \frac{\lambda_k^2}{(Z_kk^2+\mu_k)^2}\int_{D^*} dp_1dp_2dp_3dm_1dn_2dj_3\ \varphi_{m_1p_2j_3}\overline{\varphi}_{m_1p_2p_3}\varphi_{p_1n_2p_3}\overline{\varphi}_{p_1n_2j_3}$$
$$\frac{\Theta(k^2-\sum_s p_s^2)[\partial_t Z_k(k^2-\sum_s p_s^2)+2k^2Z_k]}{\Theta(k^2-p_1^2-p_2^2-j_3^2)Z_k(k^2-p_1^2-p_2^2-j_3^2)+Z_k(p_1^2+p_2^2+j_3^2)+\mu_k}$$
$$+\frac{\lambda_k^2}{(Z_kk^2+\mu_k)^2}\frac{1}{l}\int_{D^*\times\mathcal{D}} dp_1dp_2dp_3dj_2dm_1dn_1dn_3\ \varphi_{m_1j_2p_3}\overline{\varphi}_{m_1p_2p_3}\varphi_{n_1p_2n_3}\overline{\varphi}_{n_1j_2n_3}$$
$$\frac{\Theta(k^2-\sum_s p_s^2)[\partial_t Z_k(k^2-\sum_s p_s^2)+2k^2Z_k]}{\Theta(k^2-p_1^2-j_2^2-p_3^2)Z_k(k^2-p_1^2-j_2^2-p_3^2)+Z_k(p_1^2+j_2^2+p_3^2)+\mu_k}$$
$$+\frac{\lambda_k^2}{(Z_kk^2+\mu_k)^2}\frac{1}{l}\int_{D^*\times\mathcal{D}} dp_1dp_2dp_3dj_1dm_2dm_3dn_2\ \varphi_{j_1m_2m_3}\overline{\varphi}_{p_1m_2m_3}\varphi_{p_1n_2p_3}\overline{\varphi}_{j_1n_2p_3}$$
$$\frac{\Theta(k^2-\sum_s p_s^2)[\partial_t Z_k(k^2-\sum_s p_s^2)+2k^2Z_k]}{\Theta(k^2-j_1^2-p_2^2-p_3^2)Z_k(k^2-j_1^2-p_2^2-p_3^2)+Z_k(j_1^2+p_2^2+p_3^2)+\mu_k}.$$

In the continuum limit, the previous integrals can be evaluated at 0-momentum truncation and the Θ in the denominator, put to 1. One realizes that the first term is proportional $\mathcal{V}_3$, the second term to $\mathcal{V}_2$ and the third term to $\mathcal{V}_1$. By casting away the $p_i^2\varphi_{p_i}^4$-terms, t can be inferred,

$$(II)_W|_{1,2} \simeq \frac{\lambda_k^2k^2(2+\partial_t)Z_k}{(Z_kk^2+\mu_k)^3}\mathcal{V}_3$$
$$+\frac{\lambda_k^2}{(Z_kk^2+\mu_k)^3}\frac{1}{l}\mathcal{V}_2\int dp_1\theta(k^2-p_1^2)[\partial_t Z_k(k^2-p_1^2)+2k^2Z_k]$$
$$+\frac{\lambda_k^2}{(Z_kk^2+\mu_k)^3}\frac{1}{l}\mathcal{V}_1\int dp_2\ \theta(k^2-p_2^2)[\partial_t Z_k(k^2-p_2^2)+2k^2Z_k]$$

Now we can perform the integrals over the external momenta and find:

$$(II)_W|_{1,2} = \frac{\lambda_k^2k^2(2+\partial_t)Z_k}{(Z_kk^2+\mu_k)^3}\mathcal{V}_3$$
$$+\frac{\lambda_k^2k^3}{l(Z_kk^2+\mu_k)^3}\Big[-\frac{2}{3}\partial_t Z_k+2(2+\partial_t)Z_k\Big]\Big(\mathcal{V}_2+\mathcal{V}_1\Big)$$

Dimensionful β-functions. Adding all contributions, in each sector, taking care about the symmetric terms, the calculation can be easily performed. We write the full set of dimensionful β-functions for the model as:

$$\begin{cases} \beta(Z_k) = \dfrac{\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[\pi\dfrac{k^2}{l^2}+4\dfrac{k}{l}\right]+2Z_k\left[\pi\dfrac{k^2}{l^2}+2\dfrac{k}{l}\right]\right\} \\ \beta(\mu_k) = \dfrac{-3\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[\dfrac{\pi}{2}\dfrac{k^4}{l^2}+\dfrac{4}{3}\dfrac{k^3}{l}\right]+2Z_k\left[\dfrac{k^4}{l^2}\pi+2\dfrac{k^3}{l}\right]\right\} \\ \beta(\lambda_k) = \dfrac{2\lambda_k^2}{(Z_k k^2+\mu_k)^3}\left\{\partial_t Z_k\left[\dfrac{\pi}{2}\dfrac{k^4}{l^2}+\dfrac{20}{3}\dfrac{k^3}{l}+2k^2\right]+2Z_k\left[\pi\dfrac{k^4}{l^2}+10\dfrac{k^3}{l}+2k^2\right]\right\} \end{cases} \tag{A.10}$$

which is reported in section 4.4, (4.34).

Appendix B

Evaluation of the β-functions for TGFT with gauge projection

The computation of the dimensionful β-functions for the gauge projected model follows roughly the same steps of the calculations of the model without constraints. However, due to the presence of the extra delta's of the gauge projection, the analysis requires, at some point, a different technique. In this appendix, we provide details of the procedure for obtaining the system of the dimensionful RG equations, namely (5.22) of section 5.3, and underline the differences with the previous calculus.

We start by expanding equation (5.21) of section 5.2 and focus, first on the φ^2-terms and then calculate higher order terms.

B.1 φ^2-term

Referring to the conventions introduced at the begining of section 5.2, say (5.17)–(5.20), for the scaling of the kinetic term, we have:

$$(I^g)_W = -\mathrm{Tr}[\partial_t R_k \cdot (P_k)^{-1} \cdot F_k \cdot (P_k)^{-1}]$$

$$= -\lambda_k \int_{D^{*\times 4}} d\mathbf{p}d\mathbf{p}'d\mathbf{q}d\mathbf{q}'\ \Theta(k^2 - \Sigma_s p_s^2)[\partial_t Z_k(k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]\delta(\Sigma p)\delta(\mathbf{p}-\mathbf{p}')$$

$$\times \frac{\delta(\mathbf{p}'-\mathbf{q})}{[Z_k\Sigma_s p'^2_s + \mu_k + \Theta(k^2 - \Sigma_s p'^2_s)Z_k(k^2 - \Sigma_s p'^2_s)]\delta(\Sigma p')}$$

$$\times \Bigg[\int_{\mathcal{D}^{\times 2}} dm_2 dm_3\ \varphi_{q'_1 m_2 m_3}\overline{\varphi}_{q_1 m_2 m_3}\delta(\Sigma q)\delta(q'_1 + m_2 + m_3)\delta(q'_1 + q_2 + q_3)$$

$$\times \delta(q_1 + m_2 + m_3)\delta(q_2 - q'_2)\delta(q_3 - q'_3)$$

$$+ \int_{\mathcal{D}} dm_1\ \varphi_{m_1 q'_2 q'_3}\overline{\varphi}_{m_1 q_2 q_3}\delta(\Sigma q)\delta(m_1 + q'_2 + q'_3)\delta(m_1 + q_2 + q_3)$$

$$\times \delta(q_1 + q'_2 + q'_3)\delta(q_1 - q'_1) + \mathrm{sym}\Big\{1 \to 2 \to 3\Big\}\Bigg]$$

$$\frac{\delta(\mathbf{q}'-\mathbf{p})}{[Z_k\Sigma_s q'^2_s + \mu_k + \Theta(k^2 - \Sigma_s q'^2_s)Z_k(k^2 - \Sigma_s q'^2_s)]\delta(\Sigma q')}$$

$$= -\lambda_k \int_{D^*} d\mathbf{p}\ \Theta(k^2 - \Sigma_s p_s^2) \frac{[\partial_t Z_k(k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2} \frac{\delta(\Sigma p)}{\delta^2(\Sigma p)}$$
$$\times \left[\frac{1}{l^2} \int_{\mathcal{D}^{\times 2}} dm_2 dm_3\ |\varphi_{p_1 m_2 m_3}|^2 \delta^2(\Sigma p) \delta^2(p_1 + m_2 + m_3) \right.$$
$$\left. + \frac{1}{l} \int_{\mathcal{D}} dm_1\ |\varphi_{m_1 p_2 p_3}|^2 \delta^2(\Sigma p) \delta^2(m_1 + p_2 + p_3) + \mathrm{sym}\Big\{1 \to 2 \to 3\Big\} \right], \qquad \text{(B.1)}$$

where, in the last passage, after integration, we set to 1 the redundant Θ. In the same perspective, the square delta's can be reduced as $\delta^2(p) = \delta(p)\delta(0) = \frac{1}{l}\delta(p)$. Computing the integrals over some variables which are not involved in the field convolutions, we can further simplify the expression as

$$(I^g)_W = -\lambda_k \frac{1}{l^2} \int_{D^*} dm_1 dp_2 dp_3\ \frac{|\varphi_{m_1 p_2 p_3}|^2 \delta(m_1 + p_2 + p_3)}{(Z_k k^2 + \mu_k)^2}$$
$$\Theta[k^2 - 2(p_2^2 + p_3^2 + p_2 p_3)] \{\partial_t Z_k [k^2 - 2(p_2^2 + p_3^2 + p_2 p_3)] + 2k^2 Z_k\}$$
$$- \lambda_k \frac{1}{l^3} \int_{D^*} dp_1 dm_2 dm_3\ \frac{|\varphi_{p_1 m_2 m_3}|^2 \delta(p_1 + m_2 + m_3)}{(Z_k k^2 + \mu_k)^2}$$
$$\int_{\mathcal{D}} dp_2\ \Theta[k^2 - 2(p_2^2 + p_1^2 + p_2 p_1)] \{\partial_t Z_k [k^2 - 2(p_2^2 + p_1^2 + p_2 p_1)] + 2k^2 Z_k\}$$
$$+ \mathrm{sym}\Big\{1 \to 2 \to 3\Big\}. \qquad \text{(B.2)}$$

At this stage, the continuum limit is well defined: we assign integrals over the lattice to integrals over $\mathbb{R}^d$, and the discrete step function to the Heaviside θ-function.

The domain of integration of p_2 in the second term of (B.2) can be sorted as the θ distribution is non-zero when $-2p_2^2 - 2p_2 p_1 + (k^2 - 2p_1^2) \geq 0$. The boundary of this inequality in the variable p_2 in given by the roots

$$p_2^{\pm} = \frac{1}{2}\Big(-p_1 \pm \sqrt{2k^2 - 3p_1^2}\Big). \qquad \text{(B.3)}$$

The previous function is therefore positive when $p_2 \in [p_2^-, p_2^+]$. There is still a residual constraint over p_1 which has to be imposed in order to keep real the square root appearing in (B.3), that is, $3p_1^2 \leq 2k^2$. Thus, (B.2) becomes

$$(I^g)_W = -\frac{\lambda_k}{l^2 (Z_k k^2 + \mu_k)^2} \int dm_1 dp_2 dp_3\ |\varphi_{m_1 p_2 p_3}|^2 \delta(m_1 + p_2 + p_3)$$
$$\times \theta[k^2 - 2(p_2^2 + p_3^2 + p_2 p_3)] \{\partial_t Z_k [k^2 - 2(p_2^2 + p_3^2 + p_2 p_3)] + 2k^2 Z_k\}$$
$$- \frac{\lambda_k}{l^3 (Z_k k^2 + \mu_k)^2} \int dp_1 dm_2 dm_3\ |\varphi_{p_1 m_2 m_3}|^2 \delta(p_1 + m_2 + m_3)$$
$$\times \theta(2k^2 - 3p_1^2) \int_{\frac{1}{2}(-p_1 - \sqrt{2k^2 - 3p_1^2})}^{\frac{1}{2}(-p_1 + \sqrt{2k^2 - 3p_1^2})} dp_2\ \{\partial_t Z_k [k^2 - 2(p_2^2 + p_1^2 + p_2 p_1)] + 2k^2 Z_k\}$$

$$
\begin{aligned}
&+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\\
&= -\frac{\lambda_k}{l^2(Z_k k^2+\mu_k)^2}\int dm_1 dp_2 dp_3\,|\varphi_{m_1 p_2 p_3}|^2\delta(m_1+p_2+p_3)\\
&\times\theta[k^2-2(p_2^2+p_3^2+p_2p_3)]\{\partial_t Z_k[k^2-2(p_2^2+p_3^2+p_2p_3)]+2k^2 Z_k\}\\
&-\frac{\lambda_k}{l^3(Z_k k^2+\mu_k)^2}\int dp_1 dm_2 dm_3\,|\varphi_{p_1 m_2 m_3}|^2\delta(p_1+m_2+m_3)\theta(2k^2-3p_1^2)\\
&\times\Big\{k^2\sqrt{2k^2-3p_1^2}(2+\partial_t)Z_k-\frac{3}{2}\sqrt{2k^2-3p_1^2}\partial_t Z_k p_1^2-\frac{1}{6}(2k^2-3p_1^2)^{3/2}\partial_t Z_k\Big\}\\
&+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\,. \qquad \text{(B.4)}
\end{aligned}
$$

Expanding the last result up to the third order in momenta, one obtains

$$
\begin{aligned}
(I^g)_W \;\simeq\; & -\frac{\lambda_k}{(Z_k k^2+\mu_k)^2}\Bigg\{\\
& \frac{1}{l^2}\int dm_1 dp_2 dp_3\,|\varphi_{m_1 p_2 p_3}|^2\delta(m_1+p_2+p_3)\\
\times\; & [k^2(2+\partial_t)Z_k-2\partial_t Z_k(p_2^2+p_3^2)]\\
+\; & \frac{1}{l^3}\int dp_1 dm_2 dm_3\,|\varphi_{p_1 m_2 m_3}|^2\delta(p_1+m_2+m_3)\\
\times\; & \Big[k^3\Big(\sqrt{2}-\frac{\sqrt{8}}{6}\Big)\partial_t Z_k+2\sqrt{2}k^3 Z_k-\frac{3}{\sqrt{2}}k(1+\partial_t)Z_k p_1^2\Big]\Bigg\}\\
+\; & \mathrm{sym}\Big\{1\to 2\to 3\Big\}\,. \qquad \text{(B.5)}
\end{aligned}
$$

Taking in account the color symmetry, we write the β-functions for the couplings μ_k and Z_k as:

$$
\begin{aligned}
\beta(Z_k) &= \frac{\lambda_k}{(Z_k k^2+\mu_k)^2}\Big[\frac{3}{\sqrt{2}}\frac{k}{l^3}(1+\partial_t)Z_k+\frac{4}{l^2}\partial_t Z_k\Big]\,;\\
\beta(\mu_k) &= -\frac{3\lambda_k}{(Z_k k^2+\mu_k)^2}\Big[\sqrt{2}\frac{k^3}{l^3}\Big(2+\frac{2}{3}\partial_t\Big)Z_k+\frac{k^2}{l^2}(2+\partial_t)Z_k\Big]\,. \qquad \text{(B.6)}
\end{aligned}
$$

B.2 φ^4-terms

The next order of the truncation made on the Wetterich equation, i.e. $(II^g)_W = \mathrm{Tr}[\partial_t R_k\cdot(P_k)^{-1}\cdot F_k\cdot(P_k)^{-1}\cdot F_k\cdot(P_k)^{-1}]$, provides the β-function for the coupling λ_k:

$$
\begin{aligned}
&(II^g)_W =\\
&\lambda_k^2\int_{D^{*\times 6}} d\mathbf{p}d\mathbf{p}'d\mathbf{q}d\mathbf{q}'d\mathbf{r}d\mathbf{r}'\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]\delta(\Sigma p)\delta(\mathbf{p}-\mathbf{p}')
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{\delta(\mathbf{p}'-\mathbf{q})}{[Z_k\Sigma_s p_s'^2+\mu_k+\Theta(k^2-\Sigma_s p_s'^2)Z_k(k^2-\Sigma_s p_s'^2)]\delta(\Sigma p')}\\
&\times \Bigg[\int_{\mathcal{D}^{\times 2}} dm_2 dm_3\ \varphi_{q_1' m_2 m_3}\overline{\varphi}_{q_1 m_2 m_3}\delta(\Sigma q)\delta(q_1'+m_2+m_3)\delta(q_1'+q_2+q_3)\\
&\times \delta(q_1+m_2+m_3)\delta(q_2-q_2')\delta(q_3-q_3')\\
&+\int_{\mathcal{D}} dm_1\ \varphi_{m_1 q_2' q_3'}\overline{\varphi}_{m_1 q_2 q_3}\delta(\Sigma q)\delta(m_1+q_2'+q_3')\delta(m_1+q_2+q_3)\\
&\times \delta(q_1+q_2'+q_3')\delta(q_1-q_1')+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\Bigg]\\
&\times \frac{\delta(\mathbf{q}'-\mathbf{r})}{[Z_k\Sigma_s q_s'^2+\mu_k+\Theta(k^2-\Sigma_s q_s'^2)Z_k(k^2-\Sigma_s q_s'^2)]\delta(\Sigma q')}\\
&\times \Bigg[\int_{\mathcal{D}^{\times 2}} dn_2 dn_3\ \varphi_{r_1' n_2 n_3}\overline{\varphi}_{r_1 n_2 n_3}\delta(\Sigma r)\delta(r_1'+n_2+n_3)\delta(r_1'+r_2+r_3)\\
&\times \delta(r_1+n_2+n_3)\delta(r_2-r_2')\delta(r_3-r_3')\\
&+\int_{\mathcal{D}} dn_1\ \varphi_{n_1 r_2' r_3'}\overline{\varphi}_{n_1 r_2 r_3}\delta(\Sigma r)\delta(n_1+r_2'+r_3')\delta(n_1+r_2+r_3)\\
&\times \delta(r_1+r_2'+r_3')\delta(r_1-r_1')+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\Bigg]\\
&\times \frac{\delta(\mathbf{r}'-\mathbf{p})}{[Z_k\Sigma_s r_s'^2+\mu_k+\Theta(k^2-\Sigma_s r_s'^2)Z_k(k^2-\Sigma_s r_s'^2)]\delta(\Sigma r')}\\
\\
&=\lambda_k^2\int_{D^{*\times 2}} d\mathbf{p}d\mathbf{r}\ \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]\delta(\Sigma p)}{(Z_k k^2+\mu_k)^2[Z_k\Sigma_s r_s^2+\mu_k+\Theta(k^2-\Sigma_s r_s^2)Z_k(k^2-\Sigma_s r_s^2)]\delta(\Sigma r)\delta^2(\Sigma p)}\\
&\times \Bigg[\int_{\mathcal{D}^{\times 2}} dm_2 dm_3\ \varphi_{r_1 m_2 m_3}\overline{\varphi}_{p_1 m_2 m_3}\delta(\Sigma p)\delta(r_1+m_2+m_3)\delta(r_1+p_2+p_3)\\
&\times \delta(p_1+m_2+m_3)\delta(p_2-r_2)\delta(p_3-r_3)\\
&+\int_{\mathcal{D}} dm_1\ \varphi_{m_1 r_2 r_3}\overline{\varphi}_{m_1 p_2 p_3}\delta(\Sigma p)\delta(m_1+r_2+r_3)\delta(m_1+p_2+p_3)\\
&\times \delta(p_1+r_2+r_3)\delta(p_1-r_1)+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\Bigg]\\
&\times \Bigg[\int_{\mathcal{D}^{\times 2}} dn_2 dn_3\ \varphi_{p_1 n_2 n_3}\overline{\varphi}_{r_1 n_2 n_3}\delta(\Sigma r)\delta(p_1+n_2+n_3)\delta(p_1+r_2+r_3)\\
&\times \delta(r_1+n_2+n_3)\delta(r_2-p_2)\delta(r_3-p_3)\\
&+\int_{\mathcal{D}} dn_1\ \varphi_{n_1 p_2 p_3}\overline{\varphi}_{n_1 r_2 r_3}\delta(\Sigma r)\delta(n_1+p_2+p_3)\delta(n_1+r_2+r_3)\\
&\times \delta(r_1+p_2+p_3)\delta(r_1-p_1)+\mathrm{sym}\Big\{1\to 2\to 3\Big\}\Bigg], \qquad \text{(B.7)}
\end{aligned}
$$

where the redundant Θ-functions are set to 1. The combinatorics of the present model is the same studied in the previous appendix, we therefore proceed in the same way by collecting different types of colored contributions. We first discuss the contribution obtained by the product of color 1-1:

$$
\begin{aligned}
(II^g)_W|_{1,1} = \lambda_k^2 \int_{D^{*\times 2}} d\mathbf{p} d\mathbf{r}\ & \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]\cancel{\delta^2(\Sigma p)\delta(\Sigma r)}}{(Z_k k^2+\mu_k)^2[Z_k \Sigma_s r_s^2+\mu_k+\Theta(k^2-\Sigma_s r_s^2) Z_k(k^2-\Sigma_s r_s^2)]\cancel{\delta(\Sigma r)\delta^2(\Sigma p)}} \\
\times \Bigg[\int_{\mathcal{D}^{\times 2}} & dm_2 dm_3\ \varphi_{r_1 m_2 m_3} \overline{\varphi}_{p_1 m_2 m_3} \delta(r_1+m_2+m_3)\delta(r_1+p_2+p_3) \\
\times\, & \delta(p_1+m_2+m_3)\delta(p_2-r_2)\delta(p_3-r_3) \\
+ \int_{\mathcal{D}} & dm_1\ \varphi_{m_1 r_2 r_3} \overline{\varphi}_{m_1 p_2 p_3} \delta(m_1+r_2+r_3)\delta(m_1+p_2+p_3) \\
\times\, & \delta(p_1+r_2+r_3)\delta(p_1-r_1) \Bigg] \\
\times \Bigg[\int_{\mathcal{D}^{\times 2}} & dn_2 dn_3\ \varphi_{p_1 n_2 n_3} \overline{\varphi}_{r_1 n_2 n_3} \delta(p_1+n_2+n_3)\delta(p_1+r_2+r_3) \\
\times\, & \delta(r_1+n_2+n_3)\delta(r_2-p_2)\delta(r_3-p_3) \\
+ \int_{\mathcal{D}} & dn_1\ \varphi_{n_1 p_2 p_3} \overline{\varphi}_{n_1 r_2 r_3} \delta(n_1+p_2+p_3)\delta(n_1+r_2+r_3) \\
\times\, & \delta(r_1+p_2+p_3)\delta(r_1-p_1) \Bigg] \\
\\
= \lambda_k^2 \int_{D^{*\times 2}} d\mathbf{p} d\mathbf{r}\ & \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{(Z_k k^2+\mu_k)^2[Z_k \Sigma_s r_s^2+\mu_k+\Theta(k^2-\Sigma_s r_s^2) Z_k(k^2-\Sigma_s r_s^2)]} \\
\times \Bigg[\int_{\mathcal{D}^{\times 2}} & dm_1 dn_1\ \varphi_{m_1 r_2 r_3} \overline{\varphi}_{m_1 p_2 p_3} \varphi_{n_1 p_2 p_3} \overline{\varphi}_{n_1 r_2 r_3} \delta(m_1+r_2+r_3) \\
\times\, & \delta(m_1+p_2+p_3)\delta(p_1+r_2+r_3)\delta(n_1+p_2+p_3)\delta(n_1+r_2+r_3) \\
\times\, & \delta(r_1+p_2+p_3)\delta(r_1-p_1)\delta(p_1-r_1) \\
+ \int_{\mathcal{D}^{\times 4}} & dm_2 dm_3 dn_2 dn_3\ \varphi_{r_1 m_2 m_3} \overline{\varphi}_{p_1 m_2 m_3} \varphi_{p_1 n_2 n_3} \overline{\varphi}_{r_1 n_2 n_3} \\
\times\, & \delta(r_1+p_2+p_3)\delta(p_1+m_2+m_3)\delta(p_1+n_2+n_3)\delta(p_1+r_2+r_3) \\
\times\, & \delta(r_1+n_2+n_3)\delta(r_1+m_2+m_3)\delta(r_2-p_2)\delta(r_3-p_3)\delta(p_2-r_2)\delta(p_3-r_3) \\
+\, & \text{disconnected} \Bigg],
\end{aligned}
\tag{B.8}
$$

where the remaining terms "disconnected" describe disconnected interactions which we discard. Integrating over r_i, in the delta functions which are not convoluted with the fields, and replacing the redundant δ by $1/l$, one gets:

$$
\begin{aligned}
(II^g)_W|_{1,1} \simeq \lambda_k^2 \int_{D^{*\times 2}} & dm_1 dp_2 dp_3 dn_1 dr_2 dr_3\ \frac{\varphi_{m_1 r_2 r_3} \overline{\varphi}_{m_1 p_2 p_3} \varphi_{n_1 p_2 p_3} \overline{\varphi}_{n_1 r_2 r_3}}{(Z_k k^2+\mu_k)^2} \\
\times\, & \delta(m_1+r_2+r_3)\delta(m_1+p_2+p_3)\delta(n_1+p_2+p_3)\delta(n_1+r_2+r_3) \\
\times\, & \frac{1}{l} \int_{\mathcal{D}} dp_1\ \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{Z_k(p_1^2+r_2^2+r_3^2)+\mu_k+\Theta[k^2-(p_1^2+r_2^2+r_3^2)] Z_k[k^2-(p_1^2+r_2^2+r_3^2)]}
\end{aligned}
$$

$$
\begin{aligned}
&\times\,\delta(\Sigma p)\delta(p_1+r_2+r_3)\\
&+\lambda_k^2\int_{D^{*\times 2}} dp_1dm_2dm_3dr_1dn_2dn_3\;\frac{\varphi_{r_1m_2m_3}\overline{\varphi}_{p_1m_2m_3}\varphi_{p_1n_2n_3}\overline{\varphi}_{r_1n_2n_3}}{(Z_kk^2+\mu_k)^2}\\
&\times\,\delta(p_1+m_2+m_3)\delta(p_1+n_2+n_3)\delta(r_1+n_2+n_3)\delta(r_1+m_2+m_3)\\
&\times\frac{1}{l^2}\int_{\mathcal{D}^{\times 2}} dp_2dp_3\;\frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2Z_k]}{Z_k(r_1^2+p_2^2+p_3^2)+\mu_k+\Theta[k^2-(r_1^2+p_2^2+p_3^2)]Z_k[k^2-(r_1^2+p_2^2+p_3^2)]}\\
&\times\,\delta(\Sigma p)\delta(r_1+p_2+p_3)\,. \qquad\qquad (B.9)
\end{aligned}
$$

Integrating over p_1, the first term, and over p_3, the second term, we obtain (using explicit powers of l in order to keep track of the former $\delta(0)$):

$$
\begin{aligned}
&(II^g)_W|_{1,1}\simeq\frac{\lambda_k^2}{(Z_kk^2+\mu_k)^2}\Bigg\{\\
&\frac{1}{l}\int_{D^{*\times 2}} dm_1dp_2dp_3dn_1dr_2dr_3\;\varphi_{m_1r_2r_3}\overline{\varphi}_{m_1p_2p_3}\varphi_{n_1p_2p_3}\overline{\varphi}_{n_1r_2r_3}\\
&\times\,\delta(m_1+r_2+r_3)\delta(m_1+p_2+p_3)\delta(n_1+p_2+p_3)\delta(n_1+r_2+r_3)\\
&\times\,\frac{\Theta[k^2-2(p_2^2+p_3^2+p_2p_3)]\{\partial_t Z_k[k^2-2(p_2^2+p_3^2+p_2p_3)]+2k^2Z_k\}}{Z_k((-p_2-p_3)^2+r_2^2+r_3^2)+\mu_k+\Theta[k^2-((-p_2-p_3)^2+r_2^2+r_3^2)]Z_k[k^2-((-p_2-p_3)^2+r_2^2+r_3^2)]}\\
&\times\,\delta(r_2+r_3-(p_2+p_3))\\
&+\frac{1}{l^2}\int_{D^{*\times 2}} dp_1dm_2dm_3dr_1dn_2dn_3\;\varphi_{r_1m_2m_3}\overline{\varphi}_{p_1m_2m_3}\varphi_{p_1n_2n_3}\overline{\varphi}_{r_1n_2n_3} \qquad\qquad (B.10)\\
&\times\,\delta(p_1+m_2+m_3)\delta(p_1+n_2+n_3)\delta(r_1+n_2+n_3)\delta(r_1+m_2+m_3)\\
&\times\int_{\mathcal{D}} dp_3\;\frac{\Theta[k^2-2(p_2^2+p_1^2+p_2p_1)]\{\partial_t Z_k[k^2-2(p_2^2+p_1^2+p_2p_1)]+2k^2Z_k\}}{Z_k(r_1+p_2^2+(-p_2-p_1)^2)+\mu_k+\Theta[k^2-(r_1+p_2^2+(-p_2-p_1)^2)]Z_k[k^2-(r_1+p_2^2+(-p_2-p_1)^2)]}\\
&\times\,\delta(r_1-p_1)\Bigg\}\,.
\end{aligned}
$$

To perform the proper truncation scheme, we evaluate (B.10) at 0-momentum and take the limit. As usual, we drop the symbol $\lim_{l\to 0}$ and keep the dependence on l explicit, hence, we basically compute the coefficients through "normal" integrals. We use $\mathcal{V}_i$ to denote, once again, the type of colored vertex involved in the convolution of fields. One arrives to the following expression:

$$
(II^g)_W|_{1,1}\simeq\frac{\lambda_k^2}{(Z_kk^2+\mu_k)^3}\Big\{\frac{k^2}{l^2}(2+\partial_t)Z_k+\frac{k^3}{l^3}[\sqrt{2}(2+\partial_t)Z_k-\frac{\sqrt{2}}{3}\partial_t Z_k]\Big\}\mathcal{V}_1
$$

$$\simeq \frac{\lambda_k^2}{(Z_k k^2+\mu_k)^3}\Big\{\Big[\frac{k^2}{l^2}+\frac{2\sqrt{2}}{3}\frac{k^3}{l^3}\Big]\partial_t Z_k+\Big[\frac{2k^2}{l^2}+2\sqrt{2}\frac{k^3}{l^3}\Big]Z_k\Big\}\mathcal{V}_1\,. \tag{B.11}$$

Inspecting the 2-color cross terms, we focus on the product of terms 1-2 and have:

$$\begin{aligned}
(II^g)_W|_{1,2} = {}& \lambda_k^2 \int_{D^{*\times 2}} d\mathbf{p}d\mathbf{r}\ \tfrac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]\cancel{\delta^2(\Sigma p)\delta(\Sigma r)}}{(Z_k k^2+\mu_k)^2[Z_k\Sigma_s r_s^2+\mu_k+\Theta(k^2-\Sigma_s r_s^2)Z_k(k^2-\Sigma_s r_s^2)]\cancel{\delta(\Sigma r)\delta^2(\Sigma p)}} \\
& \Big[\int_{\mathcal{D}^{\times 2}} dm_2 dm_3\ \varphi_{r_1 m_2 m_3}\overline{\varphi}_{p_1 m_2 m_3}\delta(r_1+m_2+m_3)\delta(r_1+p_2+p_3) \\
& \times\delta(p_1+m_2+m_3)\delta(p_2-r_2)\delta(p_3-r_3) \\
& +\int_{\mathcal{D}} dm_1\ \varphi_{m_1 r_2 r_3}\overline{\varphi}_{m_1 p_2 p_3}\delta(m_1+r_2+r_3)\delta(m_1+p_2+p_3) \\
& \times\delta(p_1+r_2+r_3)\delta(p_1-r_1)\Big] \\
& \times\Big[\int_{\mathcal{D}^{\times 2}} dn_1 dn_3\ \varphi_{n_1 p_2 n_3}\overline{\varphi}_{n_1 r_2 n_3}\delta(n_1+p_2+n_3)\delta(r_1+p_2+r_3) \\
& \times\delta(n_1+r_2+n_3)\delta(r_1-p_1)\delta(r_3-p_3) \\
& +\int_{\mathcal{D}} dn_2\ \varphi_{p_1 n_2 p_3}\overline{\varphi}_{r_1 n_2 r_3}\delta(p_1+n_2+p_3)\delta(r_1+n_2+r_3) \\
& \times\delta(p_1+r_2+p_3)\delta(r_2-p_2)\Big]\,.
\end{aligned} \tag{B.12}$$

Discarding the disconnected interactions, the expression simplifies as

$$\begin{aligned}
(II^g)_W|_{1,2} \simeq {}& \lambda_k^2 \int_{D^{*\times 2}} d\mathbf{p}d\mathbf{r}\ \tfrac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{(Z_k k^2+\mu_k)^2[Z_k\Sigma_s r_s^2+\mu_k+\Theta(k^2-\Sigma_s r_s^2)Z_k(k^2-\Sigma_s r_s^2)]} \\
& \times\Big[\int_{\mathcal{D}^{\times 2}} dm_1 dn_2\ \varphi_{m_1 r_2 r_3}\overline{\varphi}_{m_1 p_2 p_3}\varphi_{p_1 n_2 p_3}\overline{\varphi}_{r_1 n_2 r_3} \\
& \times\delta(m_1+r_2+r_3)\delta(m_1+p_2+p_3) \\
& \times\delta(p_1+r_2+r_3)\delta(p_1+n_2+p_3)\delta(r_1+n_2+r_3) \\
& \times\delta(p_1+r_2+p_3)\delta(r_2-p_2)\delta(p_1-r_1) \\
& +\int_{\mathcal{D}^{\times 3}} dm_1 dn_1 dn_3\ \varphi_{m_1 r_2 r_3}\overline{\varphi}_{m_1 p_2 p_3}\varphi_{n_1 p_2 n_3}\overline{\varphi}_{n_1 r_2 n_3} \\
& \times\delta(m_1+r_2+r_3)\delta(m_1+p_2+p_3) \\
& \times\delta(p_1+r_2+r_3)\delta(n_1+p_2+n_3)\delta(r_1+p_2+r_3) \\
& \times\delta(n_1+r_2+n_3)\delta(r_1-p_1)\delta(r_3-p_3)\delta(p_1-r_1) \\
& +\int_{\mathcal{D}^{\times 3}} dm_2 dm_3 dn_2\ \varphi_{r_1 m_2 m_3}\overline{\varphi}_{p_1 m_2 m_3}\varphi_{p_1 n_2 p_3}\overline{\varphi}_{r_1 n_2 r_3} \\
& \times\delta(r_1+m_2+m_3)\delta(r_1+p_2+p_3) \\
& \times\delta(p_1+m_2+m_3)\delta(p_1+n_2+p_3)\delta(r_1+n_2+r_3)
\end{aligned}$$

$$
\begin{aligned}
&\times \delta(p_1+r_2+p_3)\delta(p_2-r_2)\delta(r_2-p_2)\delta(p_3-r_3)\Big] \\
&\simeq \frac{\lambda_k^2}{(Z_k k^2+\mu_k)^2}\Bigg\{ \int_{D^{*\times 2}} d\mathbf{p}\, dm_1 dn_2 dr_3\; \varphi_{m_1p_2r_3}\overline{\varphi}_{m_1p_2p_3}\varphi_{p_1n_2p_3}\overline{\varphi}_{p_1n_2r_3} \\
&\times \delta(m_1+p_2+r_3)\delta(m_1+p_2+p_3)\delta(p_1+n_2+p_3)\delta(p_1+n_2+r_3) \\
&\times \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{Z_k(p_1^2+p_2^2+r_3^2)+\mu_k+\Theta[k^2-(p_1^2+p_2^2+r_3^2)]Z_k[k^2-(p_1^2+p_2^2+r_3^2)]} \\
&\times \delta(p_1+p_2+r_3)\delta(\Sigma p) \\
&+\frac{1}{l}\int_{D^{*\times 2}} dm_1dp_2dp_3dn_1dr_2dn_3\; \varphi_{m_1r_2p_3}\overline{\varphi}_{m_1p_2p_3}\varphi_{n_1p_2n_3}\overline{\varphi}_{n_1r_2n_3} \\
&\times \delta(m_1+r_2+p_3)\delta(m_1+p_2+p_3)\delta(p_1+r_2+p_3)\delta(n_1+r_2+n_3) \\
&\times \int_{\mathcal{D}} dp_1\; \delta(p_1+r_2+p_3)\delta(\Sigma p) \\
&\times \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{Z_k(p_1^2+r_2^2+p_3^2)+\mu_k+\Theta[k^2-(p_1^2+r_2^2+p_3^2)]Z_k[k^2-(p_1^2+r_2^2+p_3^2)]} \\
&+\frac{1}{l}\int_{D^{*\times 2}} dr_1dm_2dm_3dp_1dn_2dp_3\; \varphi_{r_1m_2m_3}\overline{\varphi}_{p_1m_2m_3}\varphi_{p_1n_2p_3}\overline{\varphi}_{r_1n_2p_3} \\
&\times \delta(r_1+m_2+m_3)\delta(p_1+m_2+m_3)\delta(p_1+n_2+p_3)\delta(r_1+n_2+p_3) \\
&\times \int_{\mathcal{D}} dp_2\; \delta(r_1+p_2+p_3)\delta(\Sigma p) \\
&\times \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{Z_k(r_1^2+p_2^2+p_3^2)+\mu_k+\Theta[k^2-(r_1^2+p_2^2+p_3^2)]Z_k[k^2-(r_1^2+p_2^2+p_3^2)]}\Bigg\}.
\end{aligned}
\tag{B.13}
$$

Performing the integral over p_1 and p_2 in the last two terms, removing the corresponding δ's and evaluating at the 0-momentum we find:

$$
(II^g)_W|_{1,2} \simeq \frac{\lambda_k^2}{(Z_k k^2+\mu_k)^3}\frac{k^2}{l^2}(2+\partial_t)Z_k\Big[\mathcal{V}_3+\mathcal{V}_2+\mathcal{V}_1\Big] \tag{B.14}
$$

Collecting all contributions, $(II^g)_W|_{i,i}$ (B.11), $i=1,2,3$, and $(II^g)_W|_{i,j}$ (B.14), $i<j$, $i,j=1,2,3$, the β-function for λ_k expresses as

$$
\begin{aligned}
\frac{1}{2}\beta(\lambda_k) &= \frac{\lambda_k^2}{(Z_k k^2+\mu_k)^3}\Big[2\sqrt{2}\frac{k^3}{l^3}\Big(1+\frac{1}{3}\partial_t\Big)Z_k+7\frac{k^2}{l^2}(2+\partial_t)Z_k\Big] \\
&= \frac{\lambda_k^2}{(Z_k k^2+\mu_k)^3}\Big\{\Big[\frac{2\sqrt{2}}{3}\frac{k^3}{l^3}+7\frac{k^2}{l^2}\Big]\partial_t Z_k+\Big[2\sqrt{2}\frac{k^3}{l^3}+14\frac{k^2}{l^2}\Big]Z_k\Big\}\,.
\end{aligned}
\tag{B.15}
$$

Dimensionful β-functions. We gather (B.6) and (B.15) for the complete system of β-functions for the gauge invariant rank 3 TGFT model which expresses as:

$$\begin{cases} \beta(Z_k) = \dfrac{\lambda_k}{(Z_k k^2 + \mu_k)^2}\Big[\dfrac{3}{\sqrt{2}}\dfrac{k}{l^3}(1+\partial_t)Z_k + \dfrac{4}{l^2}\partial_t Z_k\Big] \\ \beta(\mu_k) = -\dfrac{3\lambda_k}{(Z_k k^2 + \mu_k)^2}\Big[\sqrt{2}\dfrac{k^3}{l^3}\Big(2+\dfrac{2}{3}\partial_t\Big)Z_k + \dfrac{k^2}{l^2}(2+\partial_t)Z_k\Big] \\ \beta(\lambda_k) = \dfrac{2\lambda_k^2}{(Z_k k^2 + \mu_k)^3}\Big[2\sqrt{2}\dfrac{k^3}{l^3}\Big(1+\dfrac{1}{3}\partial_t\Big)Z_k + 7\dfrac{k^2}{l^2}(2+\partial_t)Z_k\Big] \end{cases} \tag{B.16}$$

which is reported in (5.22) in section 5.3.

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