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LIMITING THEOREMS IN STATISTICAL MECHANICS. THE MEAN-FIELD CASE.

Relatore:
Chiar.mo Prof.
PIERLUIGI CONTUCCI

Presentata da:
RACHELE LUZI

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"*Ohana* means *family*,
and *family*
means that nobody gets left behind or forgotten."

Ai miei genitori
e ai miei fratelli.

Introduzione

Lo studio della somma normalizzata di variabili aleatorie e il suo comportamento asintotico è argomento fondamentale per la scienza moderna. Tale questione compare infatti nella teoria della probabilità classica con il teorema del limite centrale ed è in relazione con i profondi risultati ottenuti nella fisica statistica per sistemi di particelle interagenti. In questa tesi viene esaminata una collezione di risultati a partire dal teorema del limite centrale ed alcune sue generalizzazioni a variabili aleatorie debolmente dipendenti. La tesi contiene inoltre un'analisi del teorema limite centrale e la sua violazione nella meccanica statistica per modelli ferromagnetici di campo medio di spin interagenti. La tesi è organizzata nei capitoli seguenti.

Nel primo capitolo studieremo alcune diverse versioni del teorema limite centrale e le loro dimostrazioni. Il teorema afferma che sotto determinate condizioni la media aritmetica di un numero abbastanza grande di variabili aleatorie indipendenti, ciascuna con attesa e varianza ben definite, è approssimativamente distribuito secondo una normale.

Il teorema limite centrale può essere formulato in vari modi: ogni versione suppone che le variabili siano indipendenti, mentre l'ipotesi che siano identicamente distribuite può essere sostituita da altre condizioni.

Una prima idea sul teorema limite centrale è data dal teorema di De Moivre-Laplace *che dà un'approssimazione normale alla distribuzione binomiale. Si afferma che la distribuzione binomiale del numero di successi di* n *prove indipendenti di Bernoulli ciascuna con probabilità di successo* p *è approssimativamente distribuita secondo una normale di media* np *e varianza* $np(1-p)$, *per* $n \to \infty$.

Si ha una generalizzazione del teorema di De Moivre-Laplace *lavorando con un campione di* n *variabili aleatorie indipendenti e identicamente distribuite* $X_1, \ldots, X_n$ *con aspettazione finita* $\mu = \mathbb{E}[X_i]$ *e varianza finita* $\sigma^2 = var[X_i]$; *indicando con* S_n *la loro somma, si ha che*

$$\frac{S_n}{\sqrt{n}} - \sqrt{n}\mu \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \quad per \quad n \to \infty.$$

Per la legge dei grandi numeri, la media del campione converge in probabilità quasi sicuramente al valore atteso μ per $n \to \infty$. Il teorema limite centrale classico descrive la forma distribuzionale delle fluttuazioni stocastiche attorno al numero deterministico μ durante tale convergenza.

Un'altra versione del teorema limite centrale fu studiata dal matematico russo Aleksandr Ljapunov. In questa versione le variabili X_i devono essere indipendenti ma non necessariamente identicamente distribuite. Il teorema richiede inoltre che le variabili X_i abbiano finiti i momenti di qualche ordine $2+\delta$, $\delta > 0$, e che la crescita di tali momenti sia limitata dalla condizione di Ljapunov, *data in dettaglio nel primo capitolo.*

Un'ulteriore versione del teorema limite centrale fu studiata da Lindeberg nel 1920: sotto le stesse ipotesi e notazioni date sopra, egli sostituì la condizione di Ljapunov con una più debole, detta condizione di Lindeberg, *la quale, in accordo con la precedente, richiede $\delta = 0$.*

Nel secondo capitolo proveremo diverse versioni del teorema limite centrale per variabili aleatorie interagenti.
Inizialmente definiamo il concetto di processo stazionario e diamo una caratterizzazione per la funzione di autocovarianza e per la densità spettrale.
In seguito diamo la definizione di strongly mixing, *proprietà che induce le variabili aleatorie ad essere asintoticamente indipendenti: tale proprietà deve essere necessariamente soddisfatta dal processo stazionario affinché la somma normalizzata delle variabili che lo definiscono converga in distribuzione ad una Gaussiana. Vedremo che in ogni versione del teorema le variabili aleatorie devono avere almeno la struttura di processo stazionario che verifica:*

$$\alpha(\tau) = \sup_{A \in \mathcal{F}^0_{-\infty}, B \in \mathcal{F}^{\infty}_{\tau}} |P(AB) - P(A)P(B)| \longrightarrow 0, \quad \text{per} \quad \tau \to \infty.$$

Infine, dopo aver fornito alcuni risultati preliminari utili alle varie dimostrazioni, vedremo alcune condizioni necessarie e sufficienti affinché una sequenza di variabili aleatorie debolmente dipendenti converga in distribuzione ad una Gaussiana.

Nel terzo capitolo studieremo delle variabili aleatorie, dette spins, la cui interazione è descritta da un'Hamiltoniana di campo medio $H_N(\overline{\sigma})$, dove $\overline{\sigma} = (\sigma_1, \ldots, \sigma_N)$ è una configurazione di N spins, e vedremo le condizioni che al limite termodinamico portano ad un comportamento gaussiano e quelle che portano ad una distribuzione esponenziale di ordine maggiore. Emerge che i punti in cui il teorema del limite centrale fallisce corrispondono ai valori critici in cui si ha la transizione di fase.

Una fase di un sistema termodinamico e lo stato di aggregazione della materia hanno proprietà fisiche uniformi; una transizione di fase è la trasformazione di un sistema termodinamico da una fase ad un'altra o da uno stato di materia ad un altro: in questo caso , in seguito ad una minima variazione di alcune condizioni esterne, tra cui le variabili termodinamiche, come temperatura, pressione e altre, si ha un brusco cambiamento di proprietà fisiche che avviene spesso in modo discontinuo. Ad esempio, un liquido può diventare gas in seguito al raggiungimento del punto di ebollizione, producendo un brusco cambiamento di volume, mentre può diventare solido raggiungendo il punto di congelamento. Un altro esempio è dato dai metalli magnetici, che passano dallo stato ferromagnetico allo stato paramagnetico quando raggiungono la temperatura di Curie.

Il modello più semplice in cui si può osservare una transizione di fase è il modello di Curie-Weiss. *Questo modello fu introdotto nel 1907 da Pierre Weiss per descrivere le osservazioni sperimentali di Pierre Curie del comportamento magnetico di alcuni metalli tra cui ferro e nickel a diverse temperature. Questi materiali, dopo essere stati esposti ad un campo magnetico esterno, sviluppano una magnetizzazione con lo stesso segno del campo. Curie notò che quando il campo si annullava, i due materiali mostravano due comportamenti diversi a seconda della temperatura in cui la magnetizzazione veniva indotta: se la temperatura era sotto il valore critico, i metalli continuavano a tenere un grado di magnetizzazione, detto magnetizzazione spontanea, mentre non erano capaci di farlo quando la temperatura raggiungeva o superava il punto critico, responsabile della transizione di fase. Non appena la temperatura raggiungeva tale punto critico, infatti, la magnetizzazione svaniva bruscamente.*

Nella prima sezione del terzo capitolo definiamo le principali osservabili del modello, come la probabilità di Boltzmann-Gibbs $P_{N,J,h}\{\overline{\sigma}\}$, *la* magnetizzazione $m_N\{\overline{\sigma}\}$ *e la* funzione pressione $p_N(\overline{\sigma})$. *Mostriamo l'esistenza del limite termodinamico della funzione pressione, associata all'Hamiltoniana, per un gran numero di spins, calcolando un limite superiore e uno inferiore di* $p_N(\overline{\sigma})$. *Infine calcoliamo la soluzione esatta di tale limite termodinamico usando particolari proprietà della funzione*

$$f(x) = -\frac{J}{2}x^2 + \ln\left(\int_{\mathbb{R}} \exp(s(Jx+h))d\rho(s)\right).$$

Nella seconda sezione calcoliamo il limite della somma normalizzata di un gran numero di spins. Costruiamo il risultato usando i punti di massimo globale $\mu_1, \dots, \mu_P$ *della funzione* f; *ciascuno di essi è caratterizzato da un intero positivo* k_p *e dal numero reale negativo* λ_p *rispettivamente chiamati* type *e* strength.

Inizialmente illustriamo il comportamento asintotico della magnetizzazione. Mostriamo che dati $\mu_1, \dots, \mu_P$, i punti di massimo globale della funzione $f(x)$ con maximal type k^* and strengths $\lambda_1, \dots, \lambda_P$, se $N \to \infty$, allora

$$m_N(\overline{\sigma}) \xrightarrow{\mathcal{D}} \frac{\sum_{p=1}^{P} \lambda_p^{-\frac{1}{2k^*}} \delta(x - \mu_p)}{\sum_{p=1}^{P} \lambda_p^{-\frac{1}{2k^*}}}.$$

In seguito illustriamo il comportamento asintotico della somma degli spin, Indicata con $S_N(\overline{\sigma})$ la somma dei primi N spins, mostriamo che se f ha un unico punto di massimo μ con type k e strength λ, allora

$$\frac{S_N(\overline{\sigma}) - N\mu}{N^{1-\frac{1}{2k}}} \xrightarrow{\mathcal{D}} \begin{cases} \mathcal{N}\left(0, -\left(\frac{1}{\lambda} + \frac{1}{J}\right)\right) & \text{se} \quad k = 1 \\ \exp\left(\frac{\lambda}{(2k)!} x^{2k}\right) & \text{se} \quad k > 1 \end{cases}$$

Questo teorema verrà poi esteso nel caso in cui la funzione f abbia più punti di massimo.
Da questi importanti risultati, possiamo vedere che la violazione del teorema limite centrale dipende dal tipo omogeneo del punto di massimo della funzione f e questi teoremi diventano strumenti importanti per ottenere informazioni riguardo la criticità di una fase.
Infine discutiamo in dettaglio il modello di Curie-Weiss. In particolare mostriamo che le fasi critiche, in probabilità, possono essere valutate analizzando la distribuzione della somma degli spins nel limite termodinamico.

Nella terza sezione mostriamo che il teorema limite centrale fallisce sempre quando il modello è definito dalla costante di imitazione $J = 1$ e dal campo magnetico $h = 0$. Per fare questo applichiamo al modello di Curie-Weiss alcuni dei risultati provati nella sezione precedente .

In conclusione, sotto particolari ipotesi, il comportamento asintotico della somma degli spins, normalizzata con la radice quadrata, tende ad essere simile al comportamento di una variabile Gaussiana.

Nel quarto capitolo presentiamo un problema aperto. Ci piacerebbe mostrare che nei risultati classici, la normalità di un processo limite può essere ottenuta, ad esempio nel modello ferromagnetico, sull'intero spazio delle *fasi fuori dal punto critico. Per fare questo abbiamo bisogno di identificare una nozione di processo probabilistico corrispondente al volume infinito nelle condizioni della meccanica statistica. In altre parole, per definire un processo stocastico abbiamo bisogno di una misura di probabilità che sia indipendente dalla lunghezza del vettore stocastico. Se lavoriamo con un modello in cui gli spins interagiscono gli uni con gli altri in accordo con l'Hamiltoniana definita sopra, la probabilità di una configurazione di spins è data dalla misura di Boltzmann-Gibbs, che dipende dal numero degli spins: ci piacerebbe estendere la misura di probabilità a volume infinito.*

I passaggi che siamo in grado di fare sono i seguenti.
Considerando una configurazione di N spins

$$\overline{\sigma} = (\sigma_1, \ldots, \sigma_N)$$

e ponendo

$$S_N(\overline{\sigma}) = \sigma_1 + \ldots + \sigma_N$$

la loro somma, proveremo due proprietà soddisfatte da tali configurazioni.
La prima proposizione fornisce un'idea sul comportamento della varianza della somma fuori dal punto critico: quando una configurazione è composta da un numero molto grande di spins, la varianza della loro somma cresce proporzionalmente a N. In particolare la proposizione afferma che, nel caso in cui $(J, h) \neq (1, 0)$:

$$var(S_N(\overline{\sigma})) = Nh(N),$$

dove $h(N)$ è una funzione slowly varying tale che $c_1 \leq h(N) \leq c_2$, con $c_1, c_2 \in \mathbb{R}$. *Proveremo questa proprietà scrivendo la varianza della somma come*

$$var(S_N(\overline{\sigma})) = N \left[var(\sigma_1) + (N-1) cov(\sigma_1, \sigma_2) \right].$$

Studiando il comportamento della covarianza tra due spins, vedremo che quando $(J, h) \neq (1, 0)$, per $N \to \infty$, si ha che $cov(\sigma_1, \sigma_2) = O\left(\frac{1}{N}\right)$, quindi riusciremo a scrivere la covarianza come richiesto.
La seconda proposizione dà un'alternativa alla condizione di Lindeberg per la configurazione di spins del modello e mostra un comportamento differente nel caso in cui stiamo lavorando al punto critico oppure no. Se il modello viene considerato fuori dal punto critico, la suscettività decresce a zero quando N diventa molto grande, mentre esplode se l'Hamiltoniana è definita nel punto critico: infatti nel primo caso le fluttuazioni diventano nulle. Più precisamente la proprietà afferma che, quando si ha un'unica soluzione soluzione

dell'equazione di campo medio:

$$\lim_{M\to\infty} \limsup_{N\to\infty} \int_{|z|>M} z^2 dF_N(z) = \begin{cases} 0 & se \quad (J,h) \neq (1,0) \\ +\infty & se \quad (J,h) = (1,0) \end{cases}$$

dove $F_N(z)$ è la funzione di distribuzione della variabile aleatoria $\dfrac{S_N(\overline{\sigma}) - N\mu}{\sqrt{N}}$ e μ è la soluzione dell'equazione di campo medio.
Proveremo questa proprietà osservando che fuori dal punto critico, la somma normalizzata con la radice quadrata del numero di spins converge in distribuzione ad una Gaussiana di media zero e varianza data dalla suscettività del modello, mentre al punto critico la sua distribuzione degenera a

$$\frac{S_N}{\sqrt{N}} \sim 1$$

e la suscettività porta l'integrale ad esplodere.
In seguito presteremo attenzione ad una specifica versione del teorema limite centrale per variabili aleatorie interagenti: vedremo che se è possibile identificare la configurazione di spins con un processo stocastico, le proprietà descritte sopra sono condizioni necessarie affinché la somma normalizzata con radice quadrata converga in distribuzione ad una Gaussiana.

Introduction

The study of the normalized sum of random variables and its asymptotic behaviour is a central topic of modern science. It has in fact appeared with the central limit theorem in classic probability theory and is related to the profound results obtained in statistical physics of interacting particles systems. This thesis is a review of some results starting from the classical central limit theorem and its extensions to weakly dependent random variables. It contains moreover an analysis of the central limit theorem and its breakdown in the statistical mechanics for the mean-field interacting ferromagnetic spin models. The thesis is organised in the following chapters.

In the first chapter we will see some different versions of the central limit theorem and their proofs. As we have told above, the central limit theorem states that, under certain conditions, the arithmetic mean of a sufficiently large number of iterates of independent random variables, each with a well-defined expected value and a well-defined variance, will be approximately normally distributed.

The central limit theorem has a number of variants: every version supposes that the variables are independent and identically distributed, even if this last one can be replaced with some other conditions.

A first idea of the central limit theorem is given by the *De Moivre-Laplace* theorem which gives a normal approximation to the binomial distribution. It states that the binomial distribution of the number of successes in n independent Bernoulli trials with probability p of success on each trial is approximately a normal distribution with mean np and variance $np(1-p)$, as $n \to \infty$.

We have a generalization of the *De Moivre-Laplace* theorem working with a sample of n independent and identically distributed random va-riables $X_1, \dots, X_n$ with finite expectation $\mu = \mathbb{E}[X_i]$ and finite variance $\sigma^2 = var[X_i]$; setting with S_n their sum, we have that:

$$\frac{S_n}{\sqrt{n}} - \sqrt{n}\mu \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \quad as \quad n \to \infty.$$

By the law of large numbers, the average of the sample converges in probability and almost surely to the expected value μ as $n \to \infty$. The classical central limit theorem describes the size and the distributional form of the stochastic fluctuations around the deterministic number μ during this convergence.

An other version of the central limit theorem was studied by the Russian mathematician Aleksandr Ljapunov. In this variant the random variables X_i have to be independent, but not necessarily identically distributed. The theorem also requires that random variables X_i have moments of some order $(2+\delta)$, and that the rate of growth of these moments is limited by the *Ljapunov's condition*, given in detail in the first chapter. An other version of the central limit theorem was given by Lindeberg in 1920: in the same setting and with the same notation as above, he replaced the Ljapunov's condition with a weaker one, called *Lindeberg's condition*, which, according to the previous one, takes $\delta = 0$.

In the second chapter we will prove some different versions of the central limit theorem for dependent random variables.
Firstly we define stationary processes and we give a characterization for the autocovariance function and for the spectral density.
Secondly we give the definition of strongly mixing, which leads the random variables to be asympthotically independent: this property must be necessary satisfied by the stationary process in order that it converges in distribution toward the Gaussian distribution. In each version of the theorem, we will see that, at least, the random variables must have the structure of a strongly mixing stationary process, i.e. it must hold:

$$\alpha(\tau) = \sup_{A \in \mathcal{F}^0_{-\infty}, B \in \mathcal{F}^\infty_\tau} |P(AB) - P(A)P(B)| \longrightarrow 0, \quad \text{as} \quad \tau \to \infty.$$

Finally, after having given some preliminary results useful for the proofs of the statements, we will see some necessary and sufficient conditions which ensure that a sequence of weakly-dependent random variables converges in distribution toward the Gaussian distribution.

In the third chapter we will study spin random variables whose interaction is described by a multi-species mean-field Hamiltonian $H_N(\overline{\sigma})$, where $\overline{\sigma} = (\sigma_1, \ldots, \sigma_N)$ is a configuration of N spins, and we will see the conditions that lead in the thermodynamic limit to a Gaussian behaviour and those who lead to a higher order exponential distribution. It emerges that the points where the central limit theorem breakes down correspond to the critical values in which we have the *phase transition*.

A phase of a thermodynamic system and the states of matter have uniform physical properties; a phase transition is the transformation of a thermodynamic system from one phase or state of matter to another one: in this case we have an abrupt change of physical properties, often discontinuously, as a result the minimal variation of some external condition, such as the thermodynamic variables, like the temperature, pressure, and others. For example, a liquid may become gas upon heating to the boiling point, resulting in an abrupt change in volume, instead it may become solid upon cooling down to the freezing point. An other example is given by magnetic metals, which have a phase transition between the ferromagnetic state and the paramagnetic state when they reach the Curie temperature. The measurement of the external conditions at which the transformation occurs is termed the phase transition.

The simplest model where we can see a phase transition is the *Curie Weiss model*. This model was introduces in 1907 by Pierre Weiss in the attempt to describe Pierre Curie's experimental observations of the magnetic behaviour of some metals such as iron and nickel at different temperature. These materials, after having been exposed to an external magnetic field, develope a magnetization with the same sign of the field. Curie noted that when the field switched off, the materials showed two different behaviours depending on the temperature at which the magnetization was induced: if the temperature was below a critical value, the materials retained a degree of magnetization, called spontaneous magnetization, whereas they were not capable of doing this when the temperature was greater or equal to the critical value, responsible of the phase transition. As the temperature approached the critical value from below, the spontaneous magnetization vanished abruptley.

In the first section we define the main observables of the model where spins interact one with each other according to the Hamiltonian $H_N(\overline{\sigma})$: we will talk about the *probability of Boltzmann-Gibbs* $P_{N,J,h}\{\overline{\sigma}\}$, the *magnetization* $m_N\{\overline{\sigma}\}$ and the *pressure function* $p_N(\overline{\sigma})$. Then we show the existence of the thermodynamic limit for a large number of spins of the pressure function associated to the Hamiltonian, computing an upper bound and a lower bound of $p_N(\overline{\sigma})$. Finally we compute the exact solution of the thermodynamic limit using particular properties of the function

$$f(x) = -\frac{J}{2}x^2 + \ln\left(\int_{\mathbb{R}} \exp(s(Jx+h))d\rho(s)\right).$$

In the second section we compute the limit for large number of spins of their normalized sum. We construct the results using the global maximum

points $\mu_1, \dots, \mu_P$ of the function f; each of them is characterized by the positive integer k_p and the negative real number λ_p respectively called *type* and *strength*.
Firstly we illustrate the asymptotic behaviour of the magnetization. We show that given $\mu_1, \dots, \mu_P$ the global maximum points of the function $f(x)$ with maximal type k^* and strengths $\lambda_1, \dots, \lambda_P$, as $N \to \infty$, then

$$m_N(\overline{\sigma}) \xrightarrow{\mathcal{D}} \frac{\sum_{p=1}^{P} \lambda_p^{-\frac{1}{2k^*}} \delta(x - \mu_p)}{\sum_{p=1}^{P} \lambda_p^{-\frac{1}{2k^*}}}.$$

Secondly we illustrate the asymptotic behaviour of the normalized sum of the spins. Indicated with $S_N(\overline{\sigma})$ the sum of the first N spins, we show that if f has a unique maximum point μ of type k and strength λ, then

$$\frac{S_N(\overline{\sigma}) - N\mu}{N^{1-\frac{1}{2k}}} \xrightarrow{\mathcal{D}} \begin{cases} \mathcal{N}\left(0, -\left(\frac{1}{\lambda} + \frac{1}{J}\right)\right) & if \quad k = 1 \\ \\ \exp\left(\frac{\lambda}{(2k)!} x^{2k}\right) & if \quad k > 1 \end{cases}$$

This theorem will be extendend in case that the function f has more maximum points.
By these important results, we can see that the breaking down of the central limit theorem depends on the homogeneous type of the maximum point of the function f and these theorems become potent tools to obtain information about the critically of a phase.
Finally we discuss in detail the Curie-Weiss model. In particular we show that the critically phases can be evaluated probabilistically analyzing the distribution of the sum of spins in the thermodynamic limit.

In the third section we show that the central limit theorem always breaks down in the case that the model is defined by the coupling constant $J = 1$ and the magnetic field $h = 0$. In order to do this we will use the results proved in the previous section and after having given some preliminary results we will apply them to the particular case of the Curie-Weiss model.

In conclusion, under particular hypothesis, the asymptotyc behaviour of their sum with square-root normalization tends to be similar to a Gaussian variable's behaviour.

In the fourth chapter we present an open problem problem. We would like to show that the classical results on the normality of a limiting process can be obtained, for instance in the mean-field ferromagnetic model, on the entire phase space outside the critical point. In order to do so we need to identify a notion of probability process correspondent to the infinite volume limit of the statistical mechanics setting. In other words, in order to define a stochastic process, we need a measure of probability which must be independent from the length of the stochastic vector. If we work with a model where spins interact one with each other according to the Hamiltonian defined above, the probability of a configuration of spins is given by the measure of Boltzmann-Gibbs, which depends on the number of the spins: we would like to extend the measure of probability to an infinite volume.

The steps that we were able to cover toward such result are the following. Considering a configuration of N spins

$$\overline{\sigma} = (\sigma_1, \dots, \sigma_N)$$

and setting

$$S_N(\overline{\sigma}) = \sigma_1 + \dots + \sigma_N$$

the sum of the spins, we will prove two properties fullfilled by such configuration.

The first proposition gives an idea of the behaviour of the variance of the sum outside of the critical point: when the configuration is composed by a very large number of spins, the variance of their sum grows proportionally to N. In particular the property says that, in the case that $(J, h) \neq (1, 0)$:

$$var(S_N(\overline{\sigma})) = Nh(N),$$

where $h(N)$ *is a slowly varying function such that* $c_1 \leq h(N) \leq c_2$*, with* $c_1, c_2 \in \mathbb{R}$.

We will prove this property writing the variance of the sum as

$$var(S_N(\overline{\sigma})) = N\left[var(\sigma_1) + (N-1)cov(\sigma_1, \sigma_2)\right].$$

Studying the behaviour of the covariance of two spins, we will see that when $(J, h) \neq (1, 0)$, as $N \to \infty$, yelds $cov(\sigma_1, \sigma_2) = O\left(\frac{1}{N}\right)$, hence will be able to write the variance as requested.

The second proposition gives an alternative of the Lindeberg's condition for the configuration of spins of the model and shows a different behaviour in the case that we are working at the critical point or not. If the model is considered outside of the critical point, the susceptibility decreases to zero when N, assumes large values, while it explodes if the Hamiltonian is defined at the critical point: infact in the first case the fluctuations become void. More precisely the property states that:

$$\lim_{M\to\infty} \limsup_{N\to\infty} \int_{|z|>M} z^2 dF_N(z) = \begin{cases} 0 & \text{if} \quad (J,h) \neq (1,0) \\ +\infty & \text{if} \quad (J,h) = (1,0) \end{cases}$$

where $F_N(z)$ *is the distribution function of the random variable* $\dfrac{S_N(\overline{\sigma}) - N\mu}{\sqrt{N}}$ *and* μ *is the solution of the mean-field equation.*
We will prove this property observing that outside of the critical point, the sum with square-root normalization converges in distribution toward a Gaussian with mean equal to zero and variance equal to the susceptibility of the model, while at the critical point, its distribution degenerates to

$$\frac{S_N}{\sqrt{N}} \sim 1$$

and the susceptibility leads the integral to explode.
After that we will pay attention to a specific version of the central limit theorem for interacting variables: we will see that if it is possible to identify the configuration of spins with a stocastic process, the properties described above are necessary conditions to have that the sum with square-root normalization converges toward the Gaussian distribution.

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Chapter 1

Central limit theorems

The central limit theorem, which is the foundation for all the classic probability, works with random variables which are independent and identically distributed and which have well defined expectation and variance: under these hypothesis it ensures that their sum with square-root normalization converges toward a Gaussian distribution. To understand how empirically the central limit theorem works, we suppose to have a sample obtained doing a large number of observations gene-rated in a way that does not depend on the values of the other observations and we suppose that the arithmetic mean of the observed values is computed. Performing this procedure many times, the central limit theorem says that the computed values of the mean will be distributed according to the normal distribution.

The central limit theorem has a number of variants: every version supposes that the variables are independent but we can replace the hypothesis of being identically distributed with other conditions; in this chapter we will prove some different versions and we will start with the De Moivre-Laplace theorem, which gives a normal approximation to the Binomial distribution.

1.1 De Moivre-Laplace theorem

In this section we will prove the *De Moivre-Laplace* theorem which is a special case of the central limit theorem and gives a normal approximation to the Binomial distribution.

Theorem 1.1.1. *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent random variables distributed according to the Bernoulli distribution of parameter $0 \leq p \leq 1$, i.e. $X_i \sim Be(p)$. Set $S_n = \sum_{i=1}^{n} X_i$.*

Then, $\forall \alpha > 0$,

$$P\left(-\alpha \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq \alpha\right) \xrightarrow{n\to\infty} P(-\alpha \leq Z \leq \alpha),$$

where Z *is a random variable distributed according to the normal distribution, i.e.* $Z \sim \mathcal{N}(0,1)$.
In other words

$$\frac{S_n - np}{\sqrt{np(1-p)}} \xrightarrow{\mathcal{D}} Z \sim \mathcal{N}(0,1).$$

Remark 1.1. Observe that the variables X_i have expectation $P(X_i) = p$ and variance $var(X_i) = p(1-p)$ $\forall i \in \mathbb{N}$.
Moreover the variable S_n is distributed according to the binomial distribution of parameters n, p, i.e. $S_n \sim Bin(n,p)$; it has expectation $P(S_n) = np$ and variance $var(S_n) = np(1-p)$.

Proof. Set $q = 1-p$ to simplify the notations. By *Remark 1.2*,

$$P(S_n = k) = \frac{n!}{k!(n-k)!}p^k q^{n-k} \quad \forall k = 1, \ldots, n.$$

Set

$$W_k = \frac{n!}{k!(n-k)!}p^k q^{n-k}$$

and

$$x_k = \frac{k-np}{\sqrt{npq}}.$$

Given $\alpha > 0$, we will prove that

$$\sum_{k,|x_k|\leq\alpha} W_k \xrightarrow{k\to\infty} \int_{-\alpha}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx. \tag{1.1}$$

Use the Stirling's formula

$$N! = e^{-N} N^N \sqrt{2\pi N}\left(1 + O\left(\frac{1}{N}\right)\right), \quad as \quad N \to \infty$$

to approximate W_k.

As $|x_k| \leq \alpha$, as $n \to \infty$,

$$
\begin{aligned}
W_k &= \\
&= \frac{e^{-n} n^n \sqrt{2\pi n} \left(1 + O\left(\frac{1}{n}\right)\right) \cdot p^k q^{n-k}}{e^{-k} k^k \sqrt{2\pi k} \left(1 + O\left(\frac{1}{k}\right)\right) e^{-n+k} (n-k)^{n-k} \sqrt{2\pi(n-k)} \left(1 + O\left(\frac{1}{n-k}\right)\right)} = \\
&= \frac{1}{\sqrt{2\pi n \cdot \frac{k}{n} \cdot \frac{n-k}{n}}} \left(\frac{k}{n}\right)^{-k} \left(\frac{n-k}{n}\right)^{-n+k} p^k q^{n-k} \cdot \\
&\cdot \left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{1}{k}\right) + O\left(\frac{1}{n-k}\right)\right) = \\
&= \frac{1}{\sqrt{2\pi n u_k (1-u_k)}} \cdot \exp\left(-n H(u_k)\right) \left(1 + O\left(\frac{1}{n}\right)\right), \qquad (1.2)
\end{aligned}
$$

where

$$u_k := \frac{k}{n}$$

and

$$H(u) := u \log\left(\frac{u}{p}\right) + (1-u) \log\left(\frac{1-u}{p}\right) \qquad \forall u \in [0,1].$$

Observe that

$$
\begin{aligned}
\frac{k - np}{\sqrt{npq}} \leq \alpha \Rightarrow k &\leq \alpha \sqrt{npq} + np \\
-k &\geq -\alpha \sqrt{npq} - np \\
n - k &\geq n(1-p) - \alpha \sqrt{npq} \xrightarrow{n \to \infty} \infty.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
& x_k = \frac{k - np}{\sqrt{npq}} = \frac{\frac{k}{n} - p}{\sqrt{\frac{pq}{n}}} \Longrightarrow \\
& u_k = x_k \sqrt{\frac{pq}{n}} + p, \qquad 1 - u_k = q - x_k \sqrt{\frac{pq}{n}}.
\end{aligned}
$$

Consider the first factor of (1.12).

$$\begin{aligned} u_k(1-u_k) &= \left(x_k\sqrt{\frac{pq}{n}}+p\right)\left(q-x_k\sqrt{\frac{pq}{n}}\right) = \\ &= pq + (q-p)x_k\sqrt{\frac{pq}{n}} - x_k^2\frac{pq}{n} = \\ &= pq\left(1+(q-p)x_k\sqrt{\frac{1}{npq}}-\frac{x_k^2}{n}\right). \end{aligned}$$

Thus

$$\begin{aligned} &(2\pi n u_k(1-u_k))^{-\frac{1}{2}} = \\ =&(2\pi npq)^{-\frac{1}{2}}\left(1+(q-p)x_k\sqrt{\frac{1}{npq}}-\frac{x_k^2}{n}\right)^{-\frac{1}{2}}. \end{aligned}$$

Setting

$$t := (q-p)x_k\sqrt{\frac{1}{npq}}-\frac{x_k^2}{n} \xrightarrow{n\to\infty} 0$$

and developing by Taylor's expansions we obtain

$$\begin{aligned} &(2\pi npq)^{-\frac{1}{2}}\left(1-\frac{1}{2}\left((q-p)x_k\sqrt{\frac{1}{npq}}-\frac{x_k^2}{n}\right)+O(t^2)\right) = \\ =&\frac{1}{\sqrt{2\pi npq}}\left(1-\frac{1}{2}\frac{(q-p)x_k}{\sqrt{npq}}+O\left(\frac{1}{n}\right)\right). \end{aligned} \tag{1.3}$$

Consider the second factor of (1.12) and use Taylor's expansions to expand

$H(u)$ at the point p.

$$\begin{aligned}
H(u) &:= u\log\left(\frac{u}{p}\right) + (1-u)\log\left(\frac{1-u}{p}\right) \\
&\Longrightarrow H(p) = 0, \\
H'(u) &= \log\left(\frac{u}{p}\right) + \frac{u}{p}\cdot\frac{p}{u} + \log\left(\frac{1-u}{p}\right) + (1-u)\frac{q}{1-u}\left(-\frac{1}{q}\right) = \\
&= log\left(\frac{u}{p}\right) + \log\left(\frac{1-u}{p}\right) \\
&\Longrightarrow H'(p) = 0, \\
H''(u) &= \frac{p}{u}\cdot\frac{1}{p} - \frac{q}{1-u}\left(-\frac{1}{q}\right) = \frac{1}{u} + \frac{1}{1-u} \\
&\Longrightarrow H''(p) = \frac{1}{p} + \frac{1}{q} = \frac{q+p}{pq} = \frac{1}{pq}, \\
H'''(u) &= -\frac{1}{u^2} + \frac{1}{(1-u)^2} \\
&\Longrightarrow H'''(p) = -\frac{1}{p^2} + \frac{1}{q^2} = \frac{p^2-q^2}{p^2q^2} = \frac{(p-q)(p+q)}{p^2q^2} = \frac{p-q}{p^2q^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
H(u) &= \frac{1}{2pq}(u-p)^2 + \frac{1}{3!}\frac{p-q}{p^2q^2}(u-p)^3 + O\left((u-p)^4\right) \quad as \quad u \to p \\
H(u_k) &= H\left(p + x_k\sqrt{\frac{pq}{n}}\right) = \\
&= \frac{1}{2pq}x^2\cdot\frac{pq}{n} + \frac{1}{6}\frac{p-q}{p^2q^2}x^3\left(\frac{pq}{n}\right)^{\frac{3}{2}} + O\left(\frac{1}{n^2}\right) = \\
&= \frac{x^2}{2n} + \frac{1}{6}\frac{(p-q)x_k^3}{n\sqrt{npq}} + O\left(\frac{1}{n^2}\right) \\
-nH(u_k) &= -\frac{x^2}{2} + \frac{1}{6}\frac{(q-p)x^3}{\sqrt{npq}} + O\left(\frac{1}{n}\right) \\
\exp\left(-nH(u_k)\right) &= \exp\left(-\frac{x^2}{2}\right)\exp\left(\frac{1}{6}\frac{(q-p)x^3}{\sqrt{npq}}\right) + O\left(\frac{1}{n}\right).
\end{aligned}$$

Observe that

$$\frac{1}{6}\frac{(q-p)x^3}{\sqrt{npq}} \xrightarrow{n\to\infty} 0.$$

Develop using the Taylor's expansion for the exponential,

$$e^t = 1 + t + O(t^2) \quad as \quad t \to 0,$$

and obtain

$$\exp\left(\frac{1}{6}\frac{(q-p)x^3}{\sqrt{npq}}\right)+O\left(\frac{1}{n}\right)=1+\frac{1}{6}\frac{(q-p)x^3}{\sqrt{npq}}+O\left(\frac{1}{n}\right)\quad as\quad n\to\infty.$$

Finally

$$\exp\left(-nH(u_k)\right)=\exp\left(-\frac{x^2}{2}\right)\cdot\left(1+\frac{1}{6}\frac{(q-p)x^3}{\sqrt{npq}}+O\left(\frac{1}{n}\right)\right). \tag{1.4}$$

Putting (1.13) and (1.14) in (1.12)

$$\begin{aligned}W_k&=\sqrt{\frac{1}{2\pi npq}}\left(1-\frac{3(q-p)x_k}{6\sqrt{npq}}+O\left(\frac{1}{n}\right)\right)e^{-\frac{x^2}{2}}\left(1+\frac{(q-p)x^3}{6\sqrt{npq}}+O\left(\frac{1}{n}\right)\right)=\\&=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi npq}}\left[1+\frac{(q-p)(x_k^2-3x_k)}{6\sqrt{npq}}+O\left(\frac{1}{n}\right)\right]\\&\longrightarrow W_k\xrightarrow{n\to\infty}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi npq}}\left(1+O\left(\frac{1}{n}\right)\right).\end{aligned} \tag{1.5}$$

Putting (1.15) in (1.11) we obtain

$$\begin{aligned}\sum_{k,|x_k|\le\alpha}W_k&=\\&=\sum_{k,|x_k|\le\alpha}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi npq}}\left(1+O\left(\frac{1}{n}\right)\right)=\\&=\sum_{k,|x_k|\le\alpha}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}(x_{k+1}-x_k)\left(1+O\left(\frac{1}{n}\right)\right),\end{aligned} \tag{1.6}$$

where the latter passage is due to the equality

$$x_{k+1}-x_k=\frac{k+1-np}{\sqrt{npq}}-\frac{k-np}{\sqrt{npq}}=\frac{1}{\sqrt{npq}}.$$

The sum in (1.16) is a Riemann sum, thus

$$x_{k+1}-x_k\to 0,\quad as\quad k\to\infty.$$

Finally

$$\sum_{k,|x_k|\le\alpha}W_k\xrightarrow{k\to\infty}\int_{-\alpha}^{\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx.$$

□

1.2 Lindeberg-Levy central limit theorem

The following version of the central limit theorem is probably the most known version: the proof uses the bijective correspondence between the characteristic function of a random variable and its distribution. It will be shown that the working charatheristic function of the sum of the random variables with square-root normalization converges toward the characteristic function of the Gaussian distribution.

Theorem 1.2.1. *Let $X_1, X_2, \dots,$ be independent and identically distributed random variables with expectation $P(X_i) = 0$ and variance $P(X_i^2) = 1$.*
Set $S_n = X_1 + \ldots + X_n$.
Then, the distribution of the sum S_n with square-root normalization converges toward the Gaussian distribution $\mathcal{N}(0,1)$.

The proof of the theorem needs the following preliminary result.

Lemma 1.2.2. *Let X be a real random variable such that $P(|X|^k) < \infty$, $\forall k \in \mathbb{N}$.*
The characteristic function of X can be written as

$$\varphi_X(u) = \sum_{j=0}^{k} \frac{(iu)^j}{j!} P(X^j) + \frac{(iu)^k}{k!}\delta(u),$$

where

$$\begin{cases} |\delta(u)| \leq 3P(|X|^k) \\ \lim\limits_{u\to\infty} \delta(u) = 0. \end{cases}$$

Proof. The characteristic function of a random variable X is defined as

$$\varphi_X(u) = P(e^{iuX});$$

expand the argument e^{iuX}:

$$\begin{aligned} e^{iuX} &= \sum_{j=0}^{k-1} \frac{(iu)^j}{j!} X^j + \left(Re\left(i^k X^k e^{i\xi X}\right) + iIm\left(i^k X^k e^{ii\eta X}\right)\right)\frac{u^k}{k!} = \\ &= \sum_{j=0}^{k} \frac{(iu)^j}{j!} X^j + \left(Re\left(i^k X^k e^{i\xi X}\right) + iIm\left(i^k X^k e^{ii\eta X}\right)\right)\frac{u^k}{k!} - \frac{(iu)^k}{k!}X^k. \end{aligned}$$

Observe that when $j \leq k$ we have $|X|^j \leq 1 + |X|^k$: then the hypothesis $P(|X|^k) < \infty$ implies that $P(|X|^j) < \infty$.

Hence, considering $0 \leq \xi \leq u$ and $0 \leq \eta \leq u$, by the dominated convergence theorem and by the latter observation:

$$\varphi_X(u) = P(e^{iuX}) = \sum_{j=0}^{k} \frac{(iu)^j}{j!} P(X^j) + \delta(u) \frac{(iu)^k}{k!},$$

with $\lim_{u \to \infty} \delta(u) = 0$. □

Now we proceede with the proof of the statement *1.1.1*.

Proof. Let $\varphi_X(u)$ be the characteristic function of the random variable X. Consider the characteristic function of the random variable X_1 and expand it according to the result in *1.1.2*:

$$\varphi_{X_1}(u) = 1 - \frac{u^2}{2} - \frac{u^2}{2} \delta(u),$$

where $|\delta(u)| \leq 3$ and $\lim_{u \to \infty} \delta(u) = 0$.
The random variables $X_1, X_2, \ldots$ are independent, thus by the properties of the characteristic function we have:

$$\varphi_{S_n}(u) = (\varphi_{X_1}(u))^n.$$

Making a linear transformation:

$$\varphi_{\frac{S_n}{\sqrt{n}}}(u) = \varphi_{X_1} \left(\frac{u}{\sqrt{n}} \right)^n = \left(1 - \frac{u^2}{2n} - \frac{u^2}{2n} \delta \left(\frac{u}{\sqrt{n}} \right) \right)^n.$$

By the continuity of the logarithm we have

$$\log \left(\varphi_{\frac{S_n}{\sqrt{n}}}(u) \right) = n \log \left(\varphi_{X_1} \left(\frac{u}{\sqrt{n}} \right) \right).$$

By Taylor's expansions, as $n \to \infty$, near the origin we have:

$$n \left(-\frac{u^2}{2n} - \frac{u^2}{2n} \delta \left(\frac{u}{\sqrt{n}} \right) + O \left(\frac{u^4}{n^2} \right) \right) \xrightarrow{n \to \infty} -\frac{u^2}{2}.$$

In conclusion:

$$\varphi_{\frac{S_n}{\sqrt{n}}}(u) \xrightarrow{n \to \infty} e^{-\frac{u^2}{2}}.$$

□

1.3 Central limit theorem (smooth functions)

The following version of the central limit theorem uses smooth functions in order to show the convergence of the sum of random variables with square-root normalization toward the Gaussian distribution.

Theorem 1.3.1. *Let $X_1, X_2, \ldots,$ be independent and identically distributed random variables with expectation $P(X_i) = 0$ and variance $P(X_i^2) = 1$. Set $S_n = X_1 + \ldots + X_n$.*
Then, the sum S_n with square-root normalization converges toward the Gaussian distribution $\mathcal{N}(0,1)$. In other words, taking a continuous function $f(x)$ with limited second and third derivatives, yelds:

$$P\left(f\left(\frac{S_n}{\sqrt{n}}\right)\right) \xrightarrow{n\to\infty} \int_{\mathbb{R}} f(x)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx.$$

The proof of the theorem needs the following preliminary result.

Lemma 1.3.2. *Let X, Y, T be independent random variables such that X and Y have respectively expectation equal to zero $P(X) = P(Y) = 0$ and equal finite variance $P(X^2) = P(Y^2) < \infty$.*
Let $q(x)$ be a function defined as $q(x) = \min\{x^2, |x|^3\}$.
Let $f(x)$ be a continuous function with limited and continuous second and third derivatives.
Then

$$|P(f(T+X)) - P(f(T+Y))| \leq C_f P(q(X) + q(Y)),$$

where $C_f = \max\{\sup(|f''(x)|), \sup(|f'''(x)|)\} < \infty$.

Proof. Develop the difference $f(T+X) - f(T+Y)$ using Taylor's expansions and stop at the second order terms:

$$\begin{aligned} &f(T+X) - f(T+Y) = \\ =&f(0) + f'(0)X + \frac{f''(\xi)}{2}X^2 - f(0) - f'(0)Y - \frac{f''(\eta)}{2}Y^2 = \\ =&f'(0)(X-Y) + \frac{f''(0)}{2}(X^2 - Y^2) + \frac{f''(\xi) - f''(0)}{2}X^2 + \frac{f''(0) - f''(\eta)}{2}Y^2. \end{aligned} \tag{1.7}$$

In an analogous way, develop the difference $f(T+X) - f(T+Y)$ by Taylor's expansions, but now stop at the third order terms:

$$\begin{aligned}
&f(T+X)-f(T+Y)=\\
=&f(0)+f'(0)X+\frac{f''(0)}{2}X^2+\frac{f'''(\xi')}{6}X^3-f(0)-f'(0)Y-\frac{f''(0)}{2}Y^2-\frac{f'''(\eta')}{6}Y^3=\\
=&f'(0)(X-Y)+\frac{f''(0)}{2}(X^2-Y^2)+\frac{f'''(\xi')X^3-f'''(\eta')Y^3}{6}. \qquad (1.8)
\end{aligned}$$

Consider the difference

$$f(T+X)-f(T+Y)-\left(f'(0)(X-Y)+\frac{f''(0)}{2}(X^2-Y^2)\right).$$

By (1.1) we obtain the term

$$\frac{f''(\xi)-f''(0)}{2}X^2+\frac{f''(0)-f''(\eta)}{2}Y^2, \qquad (1.9)$$

while by (1.2) we obtain the term

$$\frac{f'''(\xi')X^3-f'''(\eta')Y^3}{6}. \qquad (1.10)$$

By hypothesis, the equation (1.3) has expectation equal to zero, hence

$$\begin{aligned}
&\left|P\left(f(T+X)-f(T+Y)-\left(f'(0)(X-Y)+\frac{f''(0)}{2}(X^2-Y^2)\right)\right)\right|=\\
=&|P\left(f(T+X)-f(T+Y)\right)|.
\end{aligned}$$

Finally

$$\begin{aligned}
|P\left(f(T+X)-f(T+Y)\right)| &\leq \min\{C_f X^2+C_f Y^2, C_f|X|^3+C_f|Y|^3\}=\\
&=C_f P(q(X)+q(Y)).
\end{aligned}$$

□

Now we proceede with the proof of the statement *1.2.1*.

Proof. Consider a sequence of auxiliar independent random Gaussian variables $Y_1, Y_2, \dots$, such that $P(Y_i)=0$ and $P(Y_i^2)=1\ \forall i=1,\dots,n$.
Consider a continuous function $f(x)$ with limited and continuous second and third derivatives.
Let $\mu_n(x)$ be the distribution of $\frac{S_n}{\sqrt{n}}$.
We want to prove that

$$\lim_{n\to\infty}\mu_n(f)=\nu(f),$$

where ν is the gaussian distribution.
We start observing that the sum $\dfrac{Y_1 + \ldots + Y_n}{\sqrt{n}}$ is distributed according to ν, since the sum of Gaussian random variables is a Gaussian random variable. Use the *lemma 1.2.2* to estimate the difference

$$|\mu_n(f) - \nu(f)| = \left| P\left(f\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}}\right) - f\left(\frac{Y_1 + \ldots + Y_n}{\sqrt{n}}\right)\right)\right|.$$

Define a random variable

$$T_k = \sum_{j=1}^{k-1} \frac{X_j}{\sqrt{n}} + \sum_{j=k+1}^{n} \frac{Y_j}{\sqrt{n}}$$

and observe that

$$f\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}}\right) - f\left(\frac{Y_1 + \ldots + Y_n}{\sqrt{n}}\right) = \sum_{k=1}^{n} \left(f\left(T_k + \frac{X_k}{\sqrt{n}}\right) - f\left(T_k + \frac{Y_k}{\sqrt{n}}\right)\right).$$

Moreover $T_k, \dfrac{X_k}{\sqrt{n}}, \dfrac{Y_k}{\sqrt{n}}$ are three independent variables: X_k and Y_k are independent by hypothesis and they are independent by T_k since they don't compare in the definition of T_k. Since $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are respectively identically distributed and independent and using the *lemma 1.1.2*, we find:

$$\begin{aligned}
&|\mu_n(f) - \nu(f)| = \\
=&\left| P\left(f\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}}\right) - f\left(\frac{Y_1 + \ldots + Y_n}{\sqrt{n}}\right)\right)\right| = \\
=&\left| P\left(\sum_{k=1}^{n} \left(f\left(T_k + \frac{X_k}{\sqrt{n}}\right) - f\left(T_k + \frac{Y_k}{\sqrt{n}}\right)\right)\right)\right| \leq \\
\leq& C_f \left[P\left(q\left(\frac{X_1}{\sqrt{n}}\right) + q\left(\frac{Y_1}{\sqrt{n}}\right)\right)\right] \cdot n = \\
=& C_f \left[P\left(\frac{X_1^2}{n} \wedge \frac{|X_1|^3}{n^{\frac{3}{2}}}\right) + P\left(\frac{Y_1^2}{n} \wedge \frac{|Y_1|^3}{n^{\frac{3}{2}}}\right)\right] \cdot n = \\
=& C_f \left[P\left(X_1^2 \wedge \frac{|X_1|^3}{\sqrt{n}}\right) + P\left(Y_1^2 \wedge \frac{|Y_1|^3}{\sqrt{n}}\right)\right] \xrightarrow{n\to\infty} 0
\end{aligned}$$

by the dominated convergence theorem. □

1.4 Central limit theorem with Lindeberg's condition

The following version of the central limit theorem supposes that the random variables are independent but not identically distributed: removing this hypothesis the central limit doesn't work, as we will see in the further examples, and it's necessary to suppose that the *Lindeberg's condition*, given below, is satisfied.

Example 1.1. Let $X_0, X_1, \dots,$ be independent random variables such that:

$$P(X_n = -2^n) = P(X_n = 2^n) = \frac{1}{2}.$$

Set $S_n = X_0 + \ldots + X_{n-1}$.
Obviously $P(X_n) = 0$ and $P(X_n^2) = 2^{2n}$ $\forall n \in \mathbb{N} \cup \{0\}$. Thus

$$P(S_n) = 0$$

and

$$var(S_n) = P(S_n^2) = \sum_{k=0}^{n-1} 2^{2k} = \frac{2^{2n}-1}{3}.$$

Obtain the standardized variable dividing S_n by the variance:

$$\frac{S_n}{var(S_n)} = \frac{3S_n}{2^{2n}-1}.$$

The distribution of this variable doesn't converge toward the Gaussian distribution since the variance of the variable X_n weigh on more for larger value of n than for smaller ones.

Example 1.2. Let $X_1, \dots, X_n$, be independent random variables such that:

$$P(X_k = 0) = P(X_k = 2^k) = \frac{1}{2}, \quad \forall k = 1, \dots, n.$$

Thus

$$\sum_{k=1}^{n} X_k = 2^n \left(\sum_{k=1}^{n} \frac{Y_k}{2^k} \right),$$

where

$$P(Y_k = 0) = P(Y_k = 1) = \frac{1}{2}, \quad \forall k = 1, \dots, n.$$

The variable $\frac{Y_k}{2^k}$ can be developed in binary form as a number chosen casually in the interval $[0, 1]$; hence Y_k has uniform distribution in $[0, 1]$, which doesn't converge toward the Gaussian distribution.

Theorem 1.4.1. *Let $X_1, X_2, \dots,$ be independent random variables with expectation $P(X_i) = 0$ and variance $P(X_i^2) = \sigma_i^2 < \infty$.*
Set $S_n = X_1 + \ldots + X_n$ and $s_n^2 = \sum_{k=1}^{n} = P(S_n^2) = var(S_n)$.
Let $\mu_i(x)$ be the distribution of the variable X_i.
If the Lindeberg's condition *is verified, i.e. if $\forall \epsilon > 0$*

$$\frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|x|>\epsilon s_n} x^2 d\mu_k(x) \xrightarrow{n\to\infty} 0,$$

then the distribution of $\frac{S_n}{s_n}$ converges toward the Gaussian distribution $\mathcal{N}(0,1)$.

The proof of the theorem needs the following preliminary results.

Remark 1.2. The *Lindeberg's condition* implies that

$$\max_{1\le k\le n} \frac{\sigma_k^2}{s_n^2} \xrightarrow{n\to\infty} 0.$$

Proof. By contradiction suppose it doesn't happen. Thus it would exist $\delta > 0$, a sequence k_j and a sequence n_j such that $\frac{\sigma_{k_j}^2}{s_{n_j}^2} > \delta$. Hence:

$$\int_{|x|>\frac{\sqrt{\delta}}{2} s_{n_j}^2} x^2 d\mu_{k_j}(x) > \delta - \frac{\delta}{4} = \frac{3\delta}{4} > \frac{\delta}{2},$$

but this contradicts the *Lindeberg's condition* since $\forall \epsilon > 0$ we would have had

$$\sum_{k=1}^{n} \int_{|x|>\epsilon s_n^2} x^2 d\mu_k(x) \xrightarrow{n\to\infty} 0.$$

□

Lemma 1.4.2. *It holds:*

$$\left| e^{it} - \sum_{j=0}^{k} \frac{(it)^j}{j!} \right| \le \frac{|t|^{k+1}}{(k+1)!}. \tag{1.11}$$

Proof. Proceede by induction on $k+1$.
If $k = 0$:

$$|t| \ge \left| \int_0^t e^{iu} du \right| = \left| \frac{e^{it}-1}{i} \right| = \left| e^{it} - 1 \right|.$$

Suppose

$$\left|e^{it} - \sum_{j=0}^{k-1} \frac{(it)^j}{j!}\right| \leq \frac{|t|^k}{k!}.$$

Consider $t > 0$:

$$\begin{aligned}
\frac{t^{k+1}}{(k+1)!} &= \int_0^t \frac{|u|^k}{k!} du \geq \left|\int_0^t \left(e^{iu} - \sum_{j=0}^{k-1} \frac{(iu)^j}{j!}\right) du\right| = \\
&= \left|\left[\frac{e^{iu}}{i} - \sum_{j=0}^{k-1} \frac{(iu)^{j+1}}{(j+1)!}\right]_0^t\right| = \left|\frac{e^{it}}{i} - \sum_{j=0}^{k} \frac{(it)^j}{j!}\right| = \\
&= \left|e^{it} - \sum_{j=0}^{k} \frac{(it)^j}{j!}\right|
\end{aligned}$$

□

Lemma 1.4.3. *Let $z_1, \dots, z_n \in \mathbb{C}$ and $z'_1, \dots, z'_n \in \mathbb{C}$ be such that $|z_i| \leq 1$ and $|z'_i| \leq 1$ $\forall i = 1, \dots, n$.*
Then

$$\left|\prod_{j=1}^n z_j \prod_{j=1}^n z'_j\right| \leq \sum_{j=1}^n |z_j - z'_j|. \tag{1.12}$$

Proof.

$$\begin{aligned}
\left|\prod_{j=1}^n z_j \prod_{j=1}^n z'_j\right| &= \left|\sum_{j=1}^n \left(\prod_{l=1}^{j-1} z_l \prod_{m=j+1}^n z'_m(z_j - z'_j)\right)\right| \leq \\
&\leq \sum_{j=1}^n |z_j - z'_j|.
\end{aligned}$$

□

Now we proceede with the proof of the statement *1.3.1*.

Proof. We want to estimate the difference

$$\left|\varphi_{\frac{S_n}{s_n}}(u) - e^{-\frac{u^2}{2}}\right|.$$

Since the variables $X_1, \dots, X_n$ are independent, we can write:

$$\begin{cases} \varphi_{\frac{S_n}{s_n}}(u) = \varphi_{S_n}\left(\frac{u}{s_n}\right) = \prod_{j=1}^{n} \varphi_{X_j}\left(\frac{u}{s_n}\right) \\ e^{-\frac{u^2}{2}} = \prod_{j=1}^{n} \exp\left(-\frac{u^2\sigma_j^2}{2s_n^2}\right) \end{cases}$$

Thus:

$$\begin{aligned} &\left|\varphi_{\frac{S_n}{s_n}}(u) - e^{-\frac{u^2}{2}}\right| = \\ &= \left|\prod_{j=1}^{n} \varphi_{X_j}\left(\frac{u}{s_n}\right) - \prod_{j=1}^{n} \exp\left(-\frac{u^2\sigma_j^2}{2s_n^2}\right)\right| = \\ &= \sum_{j=1}^{n} \left|\varphi_{X_j}\left(\frac{u}{s_n}\right) - \exp\left(-\frac{u^2\sigma_j^2}{2s_n^2}\right)\right| = \\ &= \sum_{j=1}^{n} \left|\varphi_{X_j}\left(\frac{u}{s_n}\right) - 1 + \frac{u^2\sigma_j^2}{2s_n^2} - \left(\exp\left(-\frac{u^2\sigma_j^2}{2s_n^2}\right) - 1 + \frac{u^2\sigma_j^2}{2s_n^2}\right)\right| \leq \\ &\leq \sum_{j=1}^{n} \left|\varphi_{X_j}\left(\frac{u}{s_n}\right) - 1 + \frac{u^2\sigma_j^2}{2s_n^2}\right| + \qquad (1.13) \\ &\quad + \sum_{j=1}^{n} \left|\exp\left(-\frac{u^2\sigma_j^2}{2s_n^2}\right) - 1 + \frac{u^2\sigma_j^2}{2s_n^2}\right|. \qquad (1.14) \end{aligned}$$

Consider the term (1.7).

$$\begin{aligned} &\left|\varphi_{X_j}\left(\frac{u}{s_n}\right) - 1 + \frac{u^2\sigma_j^2}{2s_n^2}\right| \leq \\ &\leq \int \left|\exp\left(\frac{iux}{s_n}\right) - 1 - \frac{iux}{s_n} + \frac{u^2x^2}{2s_n^2}\right| d\mu_j(x) \end{aligned}$$

The term $\frac{iux}{s_n}$ can be added since it has expectation equal to zero.
Split the integral into

$$\begin{aligned} &\int_{|x|\leq\epsilon s_n} \left|\exp\left(\frac{iux}{s_n}\right) - 1 - \frac{iux}{s_n} + \frac{u^2x^2}{2s_n^2}\right| d\mu_j(x) + \qquad (1.15) \\ &+ \int_{|x|>\epsilon s_n} \left|\exp\left(\frac{iux}{s_n}\right) - 1 - \frac{iux}{s_n} + \frac{u^2x^2}{2s_n^2}\right| d\mu_j(x) \qquad (1.16) \end{aligned}$$

Use the inequalities

$$\left| e^{ix} - 1 - ix + \frac{x^2}{2} \right| \le \frac{x^3}{6}$$

and

$$\left| e^{ix} - 1 - ix \right| \le \frac{x^2}{2}$$

given by *lemma 1.1.2* in order to estimate respectively (1.9) and (1.10):

$$\begin{aligned} &\int \left| \exp\left(\frac{iux}{s_n} \right) - 1 - \frac{iux}{s_n} + \frac{u^2 x^2}{2 s_n^2} \right| d\mu_j(x) \le \\ \le &\epsilon \frac{|u|^3 \sigma_j^2}{s_n^2} + \frac{u^2}{s_n^2} \int_{|x| > \epsilon s_n} x^2 d\mu_j(x). \end{aligned}$$

Finally, $\forall \epsilon > 0$ and for a fixed u:

$$\begin{aligned} &\sum_{j=1}^{n} \left| \varphi_{X_j}\left(\frac{u}{s_n} \right) - 1 + \frac{u^2 \sigma_j^2}{2 s_n^2} \right| \le \\ \le &\epsilon |u|^3 \sum_{j=1}^{n} \frac{\sigma_j^2}{s_n^2} + \frac{u^2}{s_n^2} \sum_{j=1}^{n} \int_{|x| > \epsilon s_n} x^2 d\mu_j(x) \xrightarrow{n \to \infty} 0 \end{aligned}$$

because of the *Lindeberg's condition*.

Consider the term (1.8).
By *lemma 1.1.2*, for $k = 1$,

$$\left| e^{-x} - 1 + x \right| \le \frac{x^2}{2},$$

hence:

$$\begin{aligned} &\sum_{j=1}^{n} \left| \exp\left(-\frac{u^2 \sigma_j^2}{2 s_n^2} \right) - 1 + \frac{u^2 \sigma_j^2}{2 s_n^2} \right| \le \sum_{j=1}^{n} \frac{u^4 \sigma_j^4}{4 s_n^4} \le \\ \le &\left(\max_{1 \le k \le n} \frac{\sigma_j^2}{s_n^2} \right) \frac{u^4}{4} \cdot \sum_{j=1}^{n} \frac{\sigma_j^2}{s_n^2} = \left(\max_{1 \le k \le n} \frac{\sigma_j^2}{s_n^2} \right) \frac{u^4}{4} \xrightarrow{n \to \infty} 0. \end{aligned}$$

In conclusion

$$\left| \varphi_{\frac{S_n}{s_n}}(u) - e^{-\frac{u^2}{2}} \right| \xrightarrow{n \to \infty} 0,$$

hence the distribution of $\dfrac{S_n}{s_n}$ converges toward the Gaussian distribution $\mathcal{N}(0, 1)$. □

1.5 Central limit theorem with Ljapunov's condition

As in the previous section, the following version of the central limit theorem supposes that the random variables are independent but not identically distributed: now this hypothesis is replaced by the *Ljapunov's condition*, given below. In order to prove the theorem, we will simply show that the *Ljapunov's condition* implies the *Lindeberg's condition*.

Theorem 1.5.1. *Let $X_1, X_2, \dots,$ be independent random variables with expectation $P(X_i) = 0$ and variance $P(X_i^2) = \sigma_i^2 < \infty$.*
Suppose that $P(X_i^{2+\delta}) < \infty$ for some $\delta \in \mathbb{R}^+$.
Set $S_n = X_1 + \ldots + X_n$ and $s_n^2 = \sum_{k=1}^{n} P(S_n^2) = var(S_n)$.
Let $\mu_i(x)$ be the distribution of the variable X_i.
If the Ljapunov's condition *is verified, i.e. if $\forall \epsilon > 0$*

$$\frac{1}{s_n^{2+\delta}} \sum_{k=1}^{n} \int x^{2+\delta} d\mu_k(x) \xrightarrow{n\to\infty} 0,$$

then the distribution of $\frac{S_n}{s_n}$ converges toward the gaussian distribution $\mathcal{N}(0,1)$.

Proof. According to the *theorem 1.3.1*, it's sufficient to show that the *Ljapunov's condition* implies the *Lindeberg's condition*.

$$\begin{aligned}\frac{1}{s_n^{2+\delta}} \sum_{k=1}^{n} \int x^{2+\delta} d\mu_k(x) &\geq \frac{1}{s_n^{2+\delta}} \sum_{k=1}^{n} \int_{|x|>\epsilon s_n} x^{2+\delta} d\mu_k(x) \geq \\ &\geq \frac{(\epsilon s_n)^\delta}{s_n^{2+\delta}} \sum_{k=1}^{n} \int_{|x|>\epsilon s_n} x^2 d\mu_k(x) = \\ &= \epsilon^\delta \frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|x|>\epsilon s_n} x^2 d\mu_k(x).\end{aligned}$$

Thus:

$$\frac{1}{s_n^{2+\delta}} \sum_{k=1}^{n} \int x^{2+\delta} d\mu_k(x) \xrightarrow{n\to\infty} 0 \Longrightarrow \frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|x|>\epsilon s_n} x^2 d\mu_k(x) \xrightarrow{n\to\infty} 0.$$

□

Chapter 2

Central limit theorem for interacting random variables

In this chapter we will prove some different versions of the central limit theorem for weakly-dependent random variables.

In general probability theory, a central limit theorem is a weakly-convergence theorem: a useful generalization of a sequence of independent, identically distributed random variables is a mixing random process in discrete time. *Mixing* means that random variables temporally far apart from one another are nearly independent. Several kinds of mixing are used in probability theory: we will expecially use *strong mixing* (also called *α-mixing*) defined by

$$\alpha(n) \longrightarrow 0, \quad as \quad n \to \infty$$

where $\alpha(n)$ is so-called *strong mixing coefficient*.

2.1 Stationary processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space defined by a sample place Ω, a σ-algebra $\mathcal{F}$ and a probability measure P.

A *random process* $(X_t)_{t\in T}$, $\quad T \subset \mathbb{R}$, is a collection of random variables and represents the evolution of some system of random values over the time.

A random process is called *stationary* (in the strict sense) if the distribution of the random vector

$$(X_{t_1+h}, X_{t_2+h}, \dots, X_{t_s+h})$$

does not depend on h, so long as the values $t_i + h$ belongs to T. In the wide sense, the random process is called stationary if

$$\mathbb{E}(X_t^2) < \infty, \quad \forall t \in T$$

and if $\mathbb{E}(X_s)$ and $\mathbb{E}(X_s\overline{X}_{s+t})$ do not depend on s.

With each random process X_t, where t is a finite real number, we can associate a σ-algebra

$$\mathcal{F}_a^b, \quad -\infty \leq a \leq b \leq \infty,$$

which is the σ-algebra generated by the events of the form

$$A = \{X_{t_1}, X_{t_2}, \ldots, X_{t_s}\}, \quad a \leq t_1 \leq \ldots \leq t_s \leq b.$$

The past of the process $(X_t)_{t\in T}$ is described by the σ-algebras $\mathcal{F}_{-\infty}^{t-s}$ and its future by the σ-algebras $\mathcal{F}_{t+s}^{\infty}$, with $s > 0$. It may be that these σ-algebras are independent, in the sense that,

$$\forall A \in \mathcal{F}_{-\infty}^{t-s}, \quad \forall B \in \mathcal{F}_{t+s}^{\infty}$$

it holds

$$P(AB) = P(A)P(B).$$

Now consider the stationary process X_t. Observe that, by definition, the expectation

$$\mathbb{E}(X_t\overline{X}_s)$$

depends only on the interval $t-s$. Indicated by R_{t-s}, the expectation

$$R_{t-s} = \mathbb{E}(X_t\overline{X}_s)$$

is called *autocovariance function* of the process X_t and has three properties:

1. $R_t = \overline{R_{-t}}$,

2. R_t is continuous,

3. R_t is positive definite.

Obviously the first two properties are satisfied. In order to show the third property consider the following estimation:

$$|R_t - R_s| = |\mathbb{E}(X_t\overline{X}_0) - \mathbb{E}(X_s\overline{X}_0)| \leq$$
$$\leq \left(\mathbb{E}|X_0|^2\mathbb{E}|X_t - \overline{X}_s|^2\right)^{\frac{1}{2}} \to 0, \quad as \quad s \to t.$$

If $z_1, \ldots, z_n$ are arbitrary complex numbers and if $t_1, \ldots, t_n$ are points of the parameters set T, it follows that

$$\sum_{i,j=1}^{n} R_{t_j - t_i} z_j \overline{z_i} = \sum_{i,j=1}^{n} z_j \overline{z_i} \mathbb{E}(X_{t_j} \overline{X}_{t_i}) = \\ = \mathbb{E} \left| \sum_{i=1}^{n} z_i X_{t_i} \right|^2 \geq 0,$$

hence the third property is verified.
These three properties imply that

$$\frac{R_t}{R_0}$$

is the characteristic function of some probability distribution. In the continuous time case, by *Bochner-Kinchin theorem*[1],

$$R_t = \int_{-\infty}^{\infty} e^{it\lambda} d\varsigma(\lambda),$$

while in the discrete case, by *Herglotz's theorem*[2],

$$R_t = \int_{-\pi}^{\pi} e^{it\lambda} d\varsigma(\lambda),$$

where in either cases the function $\varsigma(\lambda)$, which is called *spectral function* of the process X_t, is non-decreasing and bounded. It is absolutely continuous and its derivative $\zeta(\lambda) = \varsigma'(\lambda)$ is called *spectral density* of the stationary process. The relation between the autocovariance and spectral function is the same as that between characteristc and distribution function; in particular they determine one with another uniquely.

2.2 The strongly mixing condition

The stationary process $(X_t)_{t \in T}$ is said to be *strongly mixing* (or *completely regular*) if

$$\alpha(\tau) = \sup_{A \in \mathcal{F}_{-\infty}^{0}, B \in \mathcal{F}_{\tau}^{\infty}} |P(AB) - P(A)P(B)| \longrightarrow 0, \quad as \quad \tau \to \infty \tag{2.1}$$

[1]See *Appendix A* for the proof of the Bochner-Kinchin theorem.
[2]See *Appendix B* for the proof of the Herglotz's theorem.

through positive values. The non increasing function $\alpha(\tau)$ is called *mixing coefficient*. It is clear that a sequence of independent random variables is strongly mixing.

The stationary process $(X_t)_{t\in T}$ is said to be *uniformly mixing* if

$$\phi(\tau) = \sup_{A\in\mathcal{F}^0_{-\infty}, B\in\mathcal{F}^\infty_\tau} \frac{|P(AB) - P(A)P(B)|}{P(A)} \longrightarrow 0, \quad as \quad \tau \to \infty. \tag{2.2}$$

It is clear that $\phi(\tau)$, which is called *uniformly mixing coefficient*, is non-increasing and that a uniformly mixing process is strongly mixing (the converse is false).

The following results about the strong mixing condition will be used to show some different versions of the central limit theorem for dependent random variables.

Theorem 2.2.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space.*
Let the stationary process X_t satisfy the strong mixing condition.
Suppose that ξ is measurable with respect to $\mathcal{F}^t_{-\infty}$ and that η is measurable with respect to $\mathcal{F}^\infty_{t+\tau}$, for $\tau > 0$.
If $|\xi| \leq C_1$ and $|\eta| \leq C_2$, then

$$|\mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta)| \leq 4C_1C_2\alpha(\tau), \tag{2.3}$$

where $\alpha(\tau)$ is defined by (2.1).

Proof. We may assume that $t = 0$. Using the properties of conditional expectations, we have:

$$\begin{aligned} |\mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta)| &= |\mathbb{E}\{\xi[\mathbb{E}(\eta|\mathcal{F}^0_{-\infty}) - \mathbb{E}(\eta)]\}| \leq \\ &\leq C_1\mathbb{E}|\mathbb{E}(\eta|\mathcal{F}^0_{-\infty}) - \mathbb{E}(\eta)| = \\ &= C_1\mathbb{E}\{\xi_1[\mathbb{E}(\eta|\mathcal{F}^0_{-\infty}) - \mathbb{E}(\eta)]\}, \end{aligned}$$

where

$$\xi_1 = sgn\{\mathbb{E}(\eta|\mathcal{F}^0_{-\infty}) - \mathbb{E}(\eta)\}.$$

Clearly ξ_1 is measurable with respect to $\mathcal{F}^0_{-\infty}$ and therefore

$$|\mathbb{E}[\xi\eta] - \mathbb{E}[\xi]\mathbb{E}[\eta]| \leq C_1|\mathbb{E}[\xi_1\eta] - \mathbb{E}[\xi_1]\mathbb{E}[\eta]|.$$

Similarly, we may compare η with

$$\eta_1 = sgn\{\mathbb{E}(\xi_1|\mathcal{F}^\infty_t) - \mathbb{E}(\xi_1)\},$$

to give

$$|\mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta)| \leq C_1 C_2 |\mathbb{E}(\xi_1\eta_1) - \mathbb{E}(\xi_1)\mathbb{E}(\eta_1)|.$$

Introducing the events

$$A = \{\xi_1 = 1\} \in \mathcal{F}^0_{-\infty}$$

and

$$B = \{\eta_1 = 1\} \in \mathcal{F}^\infty_\tau,$$

the strong mixing condition in (2.1) gives

$$\begin{aligned} &|\mathbb{E}(\xi_1\eta_1) - \mathbb{E}(\xi_1)\mathbb{E}(\eta_1)| \leq \\ &\quad \leq |P(AB) + P(\overline{A}B) - P(\overline{A}B) - P(A\overline{B}) - P(A)P(B) + \\ &\qquad -P(\overline{A})P(\overline{B}) + P(\overline{A})P(B) + P(A)P(\overline{B})| \leq 4\alpha(\tau), \end{aligned}$$

hence (2.3) follows. □

Theorem 2.2.2. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space.*
Suppose that ξ *is measurable with respect to* $\mathcal{F}^t_{-\infty}$ *and that* η *is measurable with respect to* $\mathcal{F}^\infty_{t+\tau}, (\tau > 0)$.
Suppose that for some $\delta > 0$,

$$\mathbb{E}|\xi|^{2+\delta} < c_1 < \infty, \tag{2.4}$$

$$\mathbb{E}|\eta|^{2+\delta} < c_2 < \infty. \tag{2.5}$$

Then

$$|\mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta)| \leq \left[4 + 3\left(c_1^\beta c_2^{1-\beta} + c_1^{1-\beta} c_2^\beta\right)\right] \alpha(\tau)^{1-2\beta}, \tag{2.6}$$

where

$$\beta = \frac{1}{2+\delta}.$$

Proof. We can take $t = 0$ without loss of generality.
Introduce the variables ξ_N, $\overline{\xi}_N$ defined by

$$\xi_N = \begin{cases} \xi, & \text{if} \quad |\xi| \leq N \\ 0, & \text{if} \quad |\xi| > N, \end{cases}$$

and

$$\overline{\xi}_N = \xi - \xi_N$$

and the variables η_N, $\overline{\eta}_N$ similarly defined by

$$\eta_N = \begin{cases} \eta, & if \quad |\eta| \le N \\ 0, & if \quad |\eta| > N, \end{cases}$$

and

$$\overline{\eta}_N = \eta - \eta_N.$$

Then

$$\begin{aligned} |\mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta)| \le & |\mathbb{E}(\xi_N\eta_N) - \mathbb{E}(\xi_N)\mathbb{E}(\eta_N)| + \\ & + |\mathbb{E}(\xi_N\overline{\eta}_N)| + |\mathbb{E}(\xi_N)\mathbb{E}(\overline{\eta}_N)| + |\mathbb{E}(\overline{\xi}_N)\mathbb{E}(\eta_N)| + \\ & + |\mathbb{E}(\overline{\xi}_N\overline{\eta}_N)| + |\mathbb{E}(\overline{\xi}_N)\mathbb{E}(\overline{\eta}_N)|, \end{aligned} \tag{2.7}$$

and by *theorem* (2.2.1),

$$|\mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta)| \le 4N^2\alpha(\tau). \tag{2.8}$$

Because of the inequalities (2.4) and (2.5),

$$\begin{aligned} \mathbb{E}|\overline{\xi}_N| &\le \frac{\mathbb{E}|\overline{\xi}|^{1+\delta}}{N^\delta} \le \frac{c_1^{1-\beta}}{N^\delta}, \\ \mathbb{E}|\overline{\eta}_N| &\le \frac{c_2^{1-\beta}}{N^\delta}, \end{aligned}$$

so that

$$\begin{aligned} |\mathbb{E}(\overline{\xi}_N\eta_N)| &\le \left[\mathbb{E}|\overline{\xi}|^{(2+\delta)/(1+\delta)}\right]^{1-\beta} \left[\mathbb{E}|\eta_N|^{2+\delta}\right]^\beta \le \frac{c_1^{1-\beta}c_2^\beta}{N^\delta}, \\ |\mathbb{E}(\xi_N\overline{\eta}_N)| &\le \frac{c_1^\beta c_2^{1-\beta}}{N^\delta}, \\ |\mathbb{E}(\overline{\xi}_N\overline{\eta}_N)| &\le \frac{c_1^{1-\beta}c_2^{1-\beta}}{N^\delta}, \\ |\mathbb{E}(\xi_N\eta_N)| &\le \frac{c_1^\beta c_2^\beta}{N^\delta}. \end{aligned}$$

Combining the latter four inequalities, we have

$$|\mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta)| \le 4N^2\alpha(\tau) + 3N^{-\delta}\left(c_1^\beta c_2^{1-\beta} + c_1^{1-\beta}c_2^\beta\right).$$

Setting $N = \alpha(\tau)^{-\beta}$, (2.6) follows. □

Theorem 2.2.3. *Let $(\Omega, \mathcal{F}, P)$ be a probability space.*
Let the stationary process X_t satisfy the uniform mixing condition.
Suppose that ξ, η are respectively measurable with respect to the σ-algebras $\mathcal{F}_{-\infty}^{t}$ and $\mathcal{F}_{t+\tau}^{\infty}$. If

$$\mathbb{E}|\xi|^p < \infty$$

and

$$\mathbb{E}|\eta|^q < \infty,$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$|\mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta)| \le 2\phi(\tau) \left(\mathbb{E}|\xi|^p\right)^{1/p} \left(\mathbb{E}|\eta|^q\right)^{1/q}. \tag{2.9}$$

Proof. Suppose that ξ and η are represented by finite sums

$$\xi = \sum_j \lambda_j \chi(A_j) \tag{2.10}$$

and

$$\eta = \sum_i \mu_i \chi(B_i), \tag{2.11}$$

where the A_j are disjoint events in $\mathcal{F}_{-\infty}^{t}$ and B_i are disjoint events in $\mathcal{F}_{t+\tau}^{\infty}$. Then, using the Holder's inequality:

$$\begin{aligned}
&|\mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta)| = \\
=&\left|\sum_{i,j} \lambda_j \mu_i P(A_j B_i) - \sum_{i,j} \lambda_j \mu_i P(A_j) P(B_i)\right| = \\
=&\left|\sum_j \lambda_j P(A_j)^{1/p} \sum_i \left[P(B_i|A_j) - P(B_i)\right] \mu_i P(A_j)^{1/q}\right| \le \\
\le&\left\{\sum_j |\lambda_j|^p P(A_j)\right\}^{1/p} \left\{\sum_j P(A_j) \left|\sum_i \mu_i \left[P(B_i|A_j) - P(B_i)\right]\right|^q\right\}^{1/q} \le \\
\le&\left[\mathbb{E}|\xi|^p\right]^{1/p} \sum_j P(A_j) \cdot \\
&\cdot \left\{\sum_i |\mu_i|^q \left|P(B_i|A_j) - P(B_i)\right| \left[\sum_i \left|P(B_i|A_j) - P(B_i)\right|\right]^{p/q}\right\}^{1/q} \le \\
\le& 2^{1/p} \left[\mathbb{E}|\xi|^p\right]^{1/p} \left[\mathbb{E}|\eta|^q\right]^{1/q} \max_j \left\{\sum_i \left|P(B_i|A_j) - P(B_i)\right|\right\}^{1/q}.
\end{aligned} \tag{2.12}$$

Denoting the summations over positive terms by $\overset{+}{\sum}$ and over negative terms by $\overset{-}{\sum}$, we have:

$$
\begin{aligned}
&\sum_i |P(B_i|A_j) - P(B_i)| = \\
&= \overset{+}{\sum_i} \{P(B_i|A_j) - P(B_i)\} - \overset{-}{\sum_i} \{P(B_i|A_j) - P(B_i)\} = \\
&= \left\{ P\left(\overset{+}{\bigcup_i} (B_i|A_j) \right) - P\left(\overset{+}{\bigcup_i} B_i \right) \right\} + \left\{ P\left(\overset{-}{\bigcup_i} (B_i|A_j) \right) - P\left(\overset{-}{\bigcup_i} B_i \right) \right\} \leq \\
&\leq 2\phi(\tau).
\end{aligned}
\tag{2.13}
$$

Substituing (2.13) into (2.12) proves the theorem for variables of the form (2.10) and (2.11).
For the general case it suffices to remark that

$$\mathbb{E}|\xi - \xi_N|^p \xrightarrow{N\to\infty} 0$$

and

$$\mathbb{E}|\eta - \eta_N|^q \xrightarrow{N\to\infty} 0,$$

where ξ_N and η_N are random variables which are similarly respectively defined by

$$\xi_N = \begin{cases} \dfrac{k}{N} & as \quad \dfrac{k}{N} < \xi \leq \dfrac{k+1}{N}, -N^2 \leq k < N^2, \\ 0 & as \quad |\xi| > N, \end{cases}$$

and

$$\eta_N = \begin{cases} \dfrac{k}{N} & as \quad \dfrac{k}{N} < \eta \leq \dfrac{k+1}{N}, -N^2 \leq k < N^2, \\ 0 & as \quad |\eta| > N. \end{cases}$$

□

2.3 Preliminary results

In this section we will give some preliminary results useful to prove the different versions of the central limit for dependent variables given in the next section.

Theorem 2.3.1. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space.*
Let $A_n \in \mathcal{F}_{-\infty}^{0}$ *and* $B_n \in \mathcal{F}_t^{\infty}$*, such that*

$$\lim_{n\to\infty} B_n = \infty. \tag{2.14}$$

Let $F_n(x)$ *be the distribution function of*

$$B_n^{-1}\sum_{i=1}^{n} X_i - A_n, \tag{2.15}$$

where X_i *is a strongly mixing stationary sequence with mixing coefficient* $\alpha(n)$.
If $F_n(x)$ *converges weakly to a non-degenerate distribution function* $F(x)$*, then* $F(x)$ *is necessary* stable, *i.e. for any* $a_1, a_2 > 0$ *and for any* b_1, b_2 *there exists constants* $a > 0$ *and* b *such that*

$$F(a_1 x + b_1) * F(a_2 x + b_2) = F(ax + b).$$

If the latter distribution has exponent α*, then*

$$B_n = n^{\frac{1}{\alpha}} h(n),$$

where $h(n)$ *is* slowly varying *as* $n \to \infty$*, i.e. for all* $a \in \mathbb{R}^+$,

$$\lim_{x\to\infty} \frac{h(ax)}{h(x)} = 1.$$

Remark 2.1. Before proving the *theorem* (2.3.1), we make a general remark about the method of proof of this and other limit theorems for dependent variables. We represent the sum

$$S_n = X_1 + \ldots + X_n$$

in the form

$$S_n = \sum_{j=0}^{k} \xi_j + \sum_{j=0}^{k} \eta_j,$$

where

$$\xi_j = \sum_{s=jp+jq+1}^{(j+1)p+jq} X_s$$

and

$$\eta_j = \sum_{s=(j+1)p+jq+1}^{(j+1)p+(j+1)q} X_s.$$

Any two random variables ξ_i and ξ_j, for $i \neq j$, are separated by at least one variable η_j containing q terms. If q is sufficiently large, the mixing condition will ensure that the ξ_j are almost independent and the study of the sum $\sum \xi_j$ may be related to the well understood case of sums of indepedent random variables. If, however, q is small compared with p, the sum $\sum \eta_j$ will be small compared with S_n. Thus this method, called *Bernstein's method*, permits to reduce the dependent case to the independent case.

The proof of the *theorem* (2.3.1) needs the *lemma* (C.0.9)[3].

Proof. Firstly we prove that

$$\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = 1. \tag{2.16}$$

Suppose by contradiction that the hypothesys

$$\lim_{n\to\infty} B_n = \infty$$

does not hold, hence there is a subsequence (B_{n_k}) with limit $B \neq \infty$. Then

$$\left|\psi\left(\frac{t}{B_{n_k}}\right)\right|^{n_k} = |\nu(t)|(1+o(1)),$$

where ψ is the characteristic function of the variable (2.15) and ν is a Borel measurable function fron $\mathbb{R}^n$ to $\mathbb{R}$, so that, for all t,

$$|\psi(t)| = |\nu(tB_{n_k})|^{\frac{1}{n_k}}(1+o(1)).$$

This is possible if and only if $\psi(t) = 1$ for all t, which will imply that $F(t)$ is degenerate.
Thus we can state that necessarily

$$\lim_{n\to\infty} \left|\psi\left(\frac{t}{B_{n+1}}\right)\right| = 1.$$

Thus

$$\lim_{n\to\infty} \left|\psi\left(\frac{t}{B_{n+1}}\right)\right|^{n+1} = |\nu(t)|(1+o(1))$$

and

$$\lim_{n\to\infty} \left|\psi\left(\frac{t}{B_n}\right)\right|^{n} = |\nu(t)|(1+o(1)).$$

[3] See Appendix C.

Substituing $\frac{B_n}{B_{n+1}}t$ in the former and $\frac{B_n}{B_{n+1}}t$ in the latter, we deduce that

$$\lim_{n\to\infty}\left|\frac{\nu\left(\frac{B_{n+1}}{B_n}t\right)}{\nu(t)}\right| = \lim_{n\to\infty}\left|\frac{\nu\left(\frac{B_n}{B_{n+1}}t\right)}{\nu(t)}\right| = 1. \tag{2.17}$$

If

$$\lim_{n\to\infty}\frac{B_{n+1}}{B_n} \neq 1,$$

we can take a subsequence $\frac{B_{n+1}}{B_n}$ or $\frac{B_n}{B_{n+1}}$ converging to some $B < 1$. Going to the limit in (C.20), we arrive at the equation that

$$\nu(t) = \nu(Bt),$$

from which,

$$\nu(t) = \nu(B_n t) \xrightarrow{n\to\infty} |\nu(0)| = 1,$$

which is again impossible since the function $F(t)$ is non degenerate. Finally we can state that

$$\lim_{n\to\infty}\frac{B_{n+1}}{B_n} = 1.$$

Therefore, for any positive numbers a_1, a_2, there exists a sequence $m(n) \to \infty$ such that

$$\lim_{n\to\infty}\frac{B_m}{B_n} = \frac{a_1}{a_2}.$$

We can also choose a sequence $r(n)$ increasing so slowly that, in probability,

$$B_n^{-1}\sum_{i=1}^{r} X_i \xrightarrow{n\to\infty} 0.$$

Consider the sum

$$a_1^{-1}\left(B_n^{-1}\sum_{i=1}^{n} X_i - A_n - b_1\right) + \left(\frac{B_m}{a_1 B_n}\right)\left(B_m^{-1}\sum_{i=n+r+1}^{n+r+m} X_i - A_m - b_2\right) =$$

$$= \left((a_1 B_n)^{-1}\sum_{i=1}^{n+r+m} X_i - C_n\right) - (a_1 B_n)^{-1}\sum_{i=n+1}^{n+r} X_i. \tag{2.18}$$

By virtue of the strong mixing condition (2.1), the distribution function of the l.h.s. of (2.18) differs from

$$F_n(a_1 x + b_1) * F_m\left(a_1\frac{B_n}{B_m}x + b_2\right)$$

by at most $o(1)$ as $r \to \infty$. Because of the choice of r, the r.h.s. has the limiting distribution $F(ax+b)$, where $a > 0$ and b are constants. Consequently,

$$F(a_1x+b_1) * F(a_2x+b_2) = F(ax+b),$$

and $F(x)$ is stable.

In order to prove the second part of the theorem it is sufficient to show that for all positive integers k,

$$\lim_{n\to\infty} \frac{B_{nk}}{B_n} = k^{\frac{1}{\alpha}}.$$

We denote by $\psi_n(\theta)$ the characteristic function of (2.15), so that by *lemma* (C.0.9)

$$\lim_{n\to\infty} |\psi_n(\theta)| = e^{-c|\theta|^\alpha}. \tag{2.19}$$

Let $r(n)$ be an unbounded increasing sequence, which will be chosen later, and write

$$\xi_j = \sum_{s=(j-1)n+(j-1)r+1}^{jn+(j-1)r} X_s, \quad j = 1, 2, \dots k.$$

The variables ξ_j are identically distributed, so that

$$\prod_j \mathbb{E}(e^{it\xi_j}) = \{\mathbb{E}(e^{it\xi_1})\}^k.$$

Let $r(n) \xrightarrow{n\to\infty} \infty$ so slowly that the limiting distribution of the sum

$$B_{nk}^{-1} \sum_{j=1}^{k} \xi_j - A_{nk}$$

coincides with that of the sum

$$B_{nk}^{-1} \sum_{j=1}^{nk} \xi_j - A_{nk}. \tag{2.20}$$

Since $r \to \infty$, the random variables ξ_j are weakly dependent; precisely, by the *theorem* (2.2.1),

$$\left|\mathbb{E}\exp\left(\frac{i\theta}{B_{nk}}\sum_{j=1}^{k}\xi_j\right)-\prod_{j=1}^{k}\mathbb{E}\exp\left(\frac{i\theta}{B_{nk}}\xi_j\right)\right|\leq$$
$$\leq\sum_{s=2}^{k}\left|\mathbb{E}\exp\left(\frac{i\theta}{B_{nk}}\sum_{j=1}^{s}\xi_j\right)-\mathbb{E}\exp\left(\frac{i\theta}{B_{nk}}\sum_{j=1}^{s-1}\xi_j\right)\mathbb{E}\exp\left(\frac{i\theta}{B_{nk}}\xi_s\right)\right|\leq$$
$$\leq 16(k-1)\alpha(n)\xrightarrow{n\to\infty}0.$$

Thus

$$\left||\psi_{nk}(t)|-\left|\psi_n\left(t\frac{B_n}{B_{nk}}\right)\right|^k\right|\xrightarrow{n\to\infty}0.$$

Hence it follows from (2.18) that

$$\lim_{n\to\infty}\left(\frac{B_n}{B_{nk}}\right)^{\alpha}k=1. \tag{2.21}$$

□

Theorem 2.3.2. *Let (X_n) be a stationary sequence with autocovariance function $R(n)$ and spectral function $\varsigma(\lambda)$. Suppose that $\mathbb{E}(X_i)=0$, $\forall i\in\mathbb{N}$.*
Set $S_n=X_1+\ldots+X_n$.

Then the variance of S_n is given, in terms of $R(n)$ and $\varsigma(\lambda)$ by the equations

$$var(S_n)=\sum_{|j|<n}(n-|j|)R(j)=$$
$$=\int_{-\pi}^{\pi}\frac{\sin^2\left(\frac{1}{2}n\lambda\right)}{\sin^2\left(\frac{1}{2}\lambda\right)}d\varsigma(\lambda).$$

If the spectral density exists and is continuous at $\lambda=0$,
then as $n\to\infty$,

$$var(S_n)=2\pi\varsigma(0)n+o(n). \tag{2.22}$$

Proof. In order to prove (2.22) consider

$$var(S_n)=\sum_{k,i=1}^{n}P(X_kX_i)=\sum_{k,i=1}^{n}R(k-i)=$$
$$=\sum_{|j|}\sum_{i-k=j}R(k-i)=\sum_{|j|<n}(n-|j|)R(j).$$

Since

$$R(j) = \int_{-\pi}^{\pi} e^{ij\lambda} d\varsigma(\lambda),$$

by the latter equation we have:

$$var(S_n) = \int_{-\pi}^{\pi} \sum_{|j|<n} (n - |j|) e^{ij\lambda} d\varsigma(\lambda)$$

Then consider

$$\begin{aligned}
&\sum_{|j|<n} (n - |j|) e^{ij\lambda} = \\
=& n + 2Re\left(n \sum_{j=1}^{n-1} (n - |j|) e^{ij\lambda} \right) - 2Re\left(\sum_{j=1}^{n-1} (n - |j|) e^{ij\lambda} \right) = \\
=& \frac{\sin^2\left(\frac{1}{2}n\lambda\right)}{\sin^2\left(\frac{1}{2}\lambda\right)}.
\end{aligned} \tag{2.23}$$

Finally, suppose that $\varsigma(\lambda)$ exists and is continuous at $\lambda = 0$.
Integrating (2.23) we have

$$\int_{-\pi}^{\pi} \frac{\sin^2\left(\frac{1}{2}n\lambda\right)}{\sin^2\left(\frac{1}{2}\lambda\right)} d\lambda = 2\pi n, \tag{2.24}$$

and hence

$$\begin{aligned}
&|var(S_n) - 2\pi\varsigma(0)n| = \\
=& \left| \int_{-\pi}^{\pi} \frac{\sin^2\left(\frac{1}{2}n\lambda\right)}{\sin^2\left(\frac{1}{2}\lambda\right)} [\varsigma(\lambda) - \varsigma(0)] d\lambda \right| \leq \\
\leq& \max_{|\lambda| \leq n^{-1/4}} |\varsigma(\lambda) - \varsigma(0)| \int_{-n^{-1/4}}^{n^{-1/4}} \frac{\sin^2\left(\frac{1}{2}n\lambda\right)}{\sin^2\left(\frac{1}{2}\lambda\right)} d\lambda + \\
&+ \frac{1}{\sin^2\left(\frac{1}{2}n^{-1/2}\right)} \int_{n^{-1/4} \leq |\lambda| \leq \pi} \sin^2\left(\frac{1}{2}n\lambda\right) |\varsigma(\lambda) - \varsigma(0)| d\lambda \leq \\
\leq& 2\pi n \max_{|\lambda| \leq n^{-1/4}} |\varsigma(\lambda) - \varsigma(0)| + O(n^{-1/2}) = o(n).
\end{aligned}$$

Thus the theorem is proved. □

Theorem 2.3.3. *Let (X_n) be a stationary sequence uniformly mixing. Set $S_n = X_1 + \ldots + X_n$. If*

$$\lim_{n\to\infty} var(S_n) = \infty,$$

then

$$var(S_n) = nh(n), \tag{2.25}$$

where $h(n)$ is a slowly varying function of the integral variable n. Moreover, $h(n)$ has an extension to the whole real line which is slowly varying.

Remark 2.2. The theorem therefore asserts that $var(S_n)$ is either bounded or almost linear.

Proof. We divide the proof of the theorem into several parts.
(I)
Set $\Psi(n) = var(S_n)$.
We first have to prove that, for any integer k,

$$\lim_{n\to\infty} \frac{\Psi(kn)}{\Psi(n)} = k. \tag{2.26}$$

We write

$$\xi_j = \sum_{s=1}^{n} X_{(j-1)n+(j-1)r+s}, \quad j = 1, 2, \ldots, k$$
$$\eta_j = \sum_{s=1}^{r} X_{jn+(j-1)r+s}, \quad j = 1, 2, \ldots, k-1$$
$$\eta_k = \sum_{s=1}^{(k-1)r} X_{nk+s}, \quad j = 1, 2, \ldots, k$$

where

$$r = \log\left(\Psi(n)\right).$$

Since by *theorem* (2.3.2),

$$\Psi(n) = \int_{-\pi}^{\pi} \frac{\sin^2\left(\frac{1}{2}n\lambda\right)}{\sin^2\left(\frac{1}{2}\lambda\right)} d\varsigma(\lambda) \leq n^2 \int_{-\pi}^{\pi} \varsigma(\lambda) d\lambda, \tag{2.27}$$

we have $r = O(\log n)$.
Clearly

$$S_{nk} = \sum_{j=1}^{k} \xi_j + \sum_{j=1}^{k} \eta_j,$$

and

$$\Psi(nk) = var(S_{nk}) =$$
$$= \sum_{j=1}^{k} \mathbb{E}\xi_j^2 + 2\sum_{i\neq j} \mathbb{E}\xi_i\xi_j + \sum_{i,j=1}^{k} \mathbb{E}\xi_j\eta_i + \sum_{i,j=1}^{k} \mathbb{E}\eta_i\eta_j. \tag{2.28}$$

Now we proceede bounding every term of the sums in (2.27).
Since X_n is stationary, the first term can be estimated with

$$\mathbb{E}\xi_j^2 = var(S_n) = \Psi(n). \tag{2.29}$$

Using *theorem* (2.2.3) with $p = q = 2$, we have, for $i \neq j$, the second term can be bounded by

$$|\mathbb{E}\xi_i\xi_j| \leq 2\phi(|i-j|)^{\frac{1}{2}}(\mathbb{E}\xi_i^2)^{\frac{1}{2}}(\mathbb{E}\xi_j^2)^{\frac{1}{2}} \leq 2\phi(r)^{\frac{1}{2}}\Psi(n), \tag{2.30}$$

where $\phi(\tau)$ is the uniform mixing coefficient.
Finally, by (2.29), the third term is estimated with

$$\begin{aligned} |\mathbb{E}\xi_j\eta_i| \leq &(\mathbb{E}\xi_j^2)^{\frac{1}{2}}(\mathbb{E}\eta_i^2)^{\frac{1}{2}} \leq \\ \leq &\Psi(n)^{\frac{1}{2}}\Psi(r)^{\frac{1}{2}} = \\ = &O\{\Psi(n)^{\frac{1}{2}}\log(\Psi(n))\}, \end{aligned} \tag{2.31}$$

and similarly the fourth term is estimated with

$$|\mathbb{E}\eta_i\eta_j| \leq \Psi(r) = O\{\log(\Psi(n))\}^2. \tag{2.32}$$

Since r increases with n, as $n \to \infty$, $\phi(r) = o(1)$.
The relations from (2.29) to (2.32) therefore show that

$$\Psi(nk) = k\Psi(n) + o(\Psi(n)),$$

so that $\Psi(n)$ is of the form (2.25) where $h(n)$ is slowly varying.

(II)
We now use the properties of $h(n)$ which admit its extension to a slowly varying function of a continuous variable.
We need to prove some following lemmas.

Lemma 2.3.4. *For fixed k,*

$$\lim_{n\to\infty} \frac{h(n+k)}{h(n)} = 1. \tag{2.33}$$

Proof. Since $\Psi(n) \to \infty$ as $n \to \infty$, the stationarity gives

$$\begin{aligned}\Psi(n+k) &= var(S_{n+k}) = \\ &= \mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^2 + \mathbb{E}\left(\sum_{j=n+1}^{n+k} X_j\right)^2 + 2\mathbb{E}\left(\sum_{i=1}^{n} X_i \sum_{j=n+1}^{n+k} X_j\right) = \\ &= \Psi(n) + \Psi(k) + O(\Psi(n)\Psi(k))^{\frac{1}{2}},\end{aligned}$$

so that

$$\frac{h(n+k)}{h(n)} = \frac{n}{n+k}\frac{\Psi(n+k)}{\Psi(n)} = \frac{n}{n+k}(1+o(1)) = 1+o(1).$$

□

Lemma 2.3.5. *For all $\epsilon > 0$,*

$$\lim_{n\to\infty} n^{\epsilon} h(n) = \infty \tag{2.34}$$

and

$$\lim_{n\to\infty} n^{-\epsilon} h(n) = 0. \tag{2.35}$$

Proof. Since

$$\lim_{n\to\infty} \frac{h(2n)}{h(n)} = 1$$

and using the result given by (2.33), we have

$$\log h(n) = \sum_j \log\left[\frac{h(2^{-j}n)}{h(2^{-j-1}n)}\right] = o(\log n).$$

Thus (2.34) and (2.35) are obviously verified. □

Lemma 2.3.6. *If n is sufficiently large, then*

$$\sup_{n\le r\le 2n} \frac{h(r)}{h(n)} \le 2. \tag{2.36}$$

Proof. Fix m so large that $\phi(m) \le \frac{1}{16}$.

We examine the case $r > \frac{3}{2}n$; the other case $r \le \frac{3}{2}n$ can be treated similarly. From the equation

$$\sum_{j=1}^{r+m} X_j = \sum_{j=1}^{n} X_j + \sum_{j=n+1}^{n+m} X_j + \sum_{j=n+m+1}^{r+m} X_j,$$

we find that

$$\Psi(r+m) = \Psi(n) - \Psi(r-n) + \theta,$$

where

$$\theta = 2\{[\Psi(m)\Psi(n)\Psi(r-n)]^{\frac{1}{2}} + [\Psi(m)\Psi(n)]^{\frac{1}{2}} + [\Psi(m)\Psi(r-n)]^{\frac{1}{2}}\} + \Psi(m).$$

Since

$$2[\Psi(n)\Psi(r-n)]^{\frac{1}{2}} \leq \Psi(n) + \Psi(r-n) = = nh(n) + (r-n)h(r-n),$$

we have, for large n,

$$h(r+m) = \theta_1 h(n) + \theta_2 h(r-n) + O(n^{-\frac{1}{4}}),$$

where $\theta_1 > \dfrac{15}{32}$ and $\theta_2 > 0$.
Consequently, for large n,

$$\theta_1 \frac{h(n)}{h(r+m)} < \frac{3}{2},$$

which implies that

$$\frac{h(n)}{h(r)} < 2.$$

□

Lemma 2.3.7. *For all sufficiently small c and all sufficiently large n, then*

$$\frac{h(cn)}{h(n)} \leq c^{\frac{1}{2}}. \tag{2.37}$$

(Observe that (2.37) *only holds if cn is an integer.)*

Proof. From what has been proved about $h(n)$,

$$\log\left(\frac{h(cn)}{h(n)}\right) = \sum_{k=0}^{[-\frac{\log c}{\log 2}]} \left\{\log\left[h(2^{-k-1}n)\right] - \log\left[h(2^{-k}n)\right]\right\} + \\ + \left\{\log\left[h(cn) -\right] - \log\left[h(2^{-[\log c/\log 2]}n)\right]\right\} < \\ < \frac{1}{2}\log\left(\frac{1}{c}\right).$$

□

We remark that (2.37) holds for all $c < c_0$, where c_0 does not depend on n.

(III)
Using *theorem* (2.3.2), we can now extend the functions $\Psi(n)$ and $h(n)$ to the interval $(0, \infty)$ by the equations

$$\Psi(x) = \int_{-\pi}^{\pi} \frac{\sin^2\left(\frac{1}{2}x\lambda\right)}{\sin^2\left(\frac{1}{2}\lambda\right)} d\varsigma(\lambda)$$

and

$$h(x) = \frac{\Psi(x)}{x}.$$

We have to prove that for all real $a > 0$,

$$\lim_{x\to\infty} \frac{\Psi(ax)}{\Psi(x)} = a. \tag{2.38}$$

As $x \to \infty$,

$$\Psi(x) = \Psi([x])(1 + o(1)),$$

so that, when $a = k$ is an integer, by *theorem* (2.3.2) obtain

$$\frac{\Psi(kx)}{\Psi(x)} = \frac{[kx]h([kx])}{[x]h([x])}(1 + o(1)) = k(1 + o(1)). \tag{2.39}$$

If $a = \dfrac{p}{q}$, where p, q are integers, then (2.39) gives

$$\lim_{x\to\infty} \frac{\Psi\left(\dfrac{p}{q}x\right)}{\Psi(x)} = \lim_{x\to\infty} \frac{\Psi\left(\dfrac{p}{q}x\right)}{\Psi\left(\dfrac{1}{q}x\right)} \frac{\Psi\left(\dfrac{1}{q}x\right)}{\Psi(x)} = \frac{p}{q}, \tag{2.40}$$

so that (2.38) is proved for rational values of a.
For any positive a, define

$$\Psi_1(a) = \liminf_{x\to\infty} \frac{\Psi(ax)}{\Psi(x)}$$
$$\Psi_2(a) = \limsup_{x\to\infty} \frac{\Psi(ax)}{\Psi(x)},$$

so that $\Psi_1(a) = \Psi_2(a) = a$ for rational a.
It suffices to prove that Ψ_1 and Ψ_2 are continuous functions. But

$$\frac{|\Psi((a+\epsilon)x) - \Psi(ax)|}{\Psi(x)} =$$
$$= \frac{1}{\Psi(x)} \left| \int_{-\pi}^{\pi} \frac{\sin^2\left(\frac{1}{2}\epsilon x\lambda\right)}{\sin^2\left(\frac{1}{2}\lambda\right)} \varsigma(\lambda) d\lambda + \frac{1}{2}\int_{-\pi}^{\pi} \frac{\sin(\epsilon x\lambda)\sin(ax\lambda)}{\sin^2\left(\frac{1}{2}\lambda\right)} \varsigma(\lambda) d\lambda \right| \leq$$
$$\leq \frac{\Psi(\epsilon x)}{\Psi(x)} + \left(\frac{\Psi(\epsilon x)}{\Psi(x)}\right)^{\frac{1}{2}},$$

so that it suffices to establish the continuity of Ψ_1 and Ψ_2 at zero.
Using (2.39) we have, if ϵ is sufficiently small, as $x \to \infty$,

$$\frac{\Psi(\epsilon x)}{\Psi(x)} = \frac{[\epsilon x]}{[x]} \frac{h\left(\frac{[\epsilon x]}{[x]}[x]\right)}{h([x])}(1 + o(1)) \leq \epsilon(1 + o(1)).$$

Consequently the functions Ψ_1 and Ψ_2 are continuous and the theorem is proved. □

Remark 2.3. Observe that in the proof of *theorem* (2.3.3) the full force of the uniform mixing condition wasn't used. We only used the inequality

$$\mathbb{E}\left|\sum_{i=1}^{n} X_i \sum_{j=n+p}^{n+m+p} X_j\right| \leq 2\phi(p)^{\frac{1}{2}} \left\{\mathbb{E}\left(\sum_{i=1}^{n} X_i\right)^2 \mathbb{E}\left(\sum_{j=1}^{m} X_j\right)^2\right\} \frac{1}{2}.$$

Thus the conclusions of the theorem remains true if one only assumes that

1. $var(S_n) = \Psi(n) \xrightarrow{n\to\infty} \infty$
2. For any $\epsilon > 0$, there exists numbers p, N such that for $n, m > N$,

$$\mathbb{E}\left|\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{j=n+p}^{n+m+p} X_j\right)\right| \leq \epsilon\Psi(n)\Psi(m).$$

Finally we remember the *Karamata's theorem*, which allows us to give the conclusion of the theorem in an other form.
Karamata's theorem states that a function $f(x)$ is slowly varying if and only if there exists $\alpha > 0$ such that for all $x \geq \alpha$, $f(x)$ can be written in the form

$$f(x) = \exp\left(c(x) + \int_{\alpha}^{x} \frac{\epsilon(t)}{t} dt\right),$$

where $c(x)$, which converges trough a finite number, and $\epsilon(x)$, which converges to zero as x goes to infinity, are measurable and bounded functions.

Thus:

Corollary 2.3.8. *Under the conditions of the* theorem (2.3.3)

$$var(S_n) = Cn(1+o(1))\exp\left(\int_1^n \frac{\epsilon(t)}{t}dt\right), \tag{2.41}$$

where $C > 0$ *and* $\epsilon(t) \xrightarrow{t\to\infty} 0$.

Lemma 2.3.9. *Let the sequence* $(X_i)_{i\in\mathbb{N}}$ *satisfy the strong mixing condition with mixing coefficient* $\alpha(n)$.
Let ξ_i *and* η_i *be two random variables defined respectively by the following equations:*

$$\xi_i = \sum_{j=ip+iq+1}^{(i+1)p+iq} X_j, \quad 0 \le i \le k-1$$

and

$$\eta_i = \begin{cases} \displaystyle\sum_{j=(i+1)p+iq+1}^{(i+1)p+(i+1)q} X_j, & 0 \le i \le k-1 \\ \displaystyle\sum_{j=kp+kq+1}^{n} X_j, & i = k \end{cases}$$

Suppose that for any pair of sequences $p = p(n)$, $q = q(n)$ *such that*

(a) $p \to \infty$, $q \to \infty$, $q = o(p)$, $p = o(n)$ *as* $n \to \infty$,

(b) $\lim_{n\to\infty} \frac{n^{1-\beta}q^{1+\beta}}{p^2} = 0, \quad \forall \beta > 0,$

(c) $\lim_{n\to\infty} \frac{n}{p}\alpha(q) = 0.$

and $\forall\epsilon > 0$, *the distribution function*

$$\overline{F}_n(z) = P(X_1 + \ldots + X_n < z)$$

satisfies

$$\lim_{n\to\infty} \frac{n}{p\sigma_n^2}\int_{|z|>\epsilon\sigma_n} z^2 d\overline{F}_p(z) = 0.$$

Suppose finally that $k = \left[\dfrac{n}{p+q}\right]$.
If the distribution function $F_n(x)$ *of the random variable* ξ_n *converges weakly as* $n \to \infty$ *to the distribution function* $F(x)$*, and if* η_n *converges to zero in probability, i.e. for all* $\epsilon > 0$*,*

$$\lim_{n\to\infty} P(|\eta_n| > \epsilon) \to 0,$$

then the distribution function of the random variable $\zeta_i = \xi_i + \eta_i$ *converges weakly to* $F(x)$*.*

Proof. Let $\psi(t)$ be the characteristic function of $F(x)$:

$$\lim_{n\to\infty} \mathbb{E}\left(e^{it\xi_n}\right) = \psi(t).$$

Thus

$$\begin{aligned}
&\limsup_{n\to\infty} \left|\mathbb{E}\left(e^{it(\xi_n+\eta_n)}\right) - \psi(t)\right| \le \\
\le &\lim_{n\to\infty} \left|\mathbb{E}\left(e^{it\xi_n}\right) - \psi(t)\right| + \limsup_{n\to\infty} \mathbb{E}\left|e^{it\eta_n} - 1\right| \le \\
\le &\limsup_{n\to\infty} \int_{|x|\le\epsilon} \left|e^{itx} - 1\right| dP(\eta_n < x) + 2 \lim_{n\to\infty} P(|\eta_n| > \epsilon) \le \\
\le &t\epsilon,
\end{aligned}$$

for any positive ϵ.
We represent the sum S_n in the form

$$S_n = \sum_{i=0}^{k-1} \xi_i + \sum_{i=0}^{k} \eta_i = S'_n + S''_n. \tag{2.42}$$

Consider the decomposition

$$Z_n = \frac{S_n}{\sigma_n} = \frac{S'_n}{\sigma_n} + \frac{S''_n}{\sigma_n} = Z'_n + Z''_n,$$

of the normalized sum Z_n.
To continue the proof along the lines suggested, we have to show that

$$Z''_n \xrightarrow{n\to\infty} 0$$

in probability, hence we prove that

$$\lim_{n\to\infty} \mathbb{E}|Z''_n|^2 = 0,$$

since

$$P(|Z_n''| > \epsilon) \leq \frac{\mathbb{E}|Z_n''|^2}{\epsilon^2}.$$

We have

$$\begin{aligned}
\mathbb{E}|Z_n''|^2 &= \frac{1}{\sigma_n^2}\sum_{i,j=1}^{k-1}\mathbb{E}(\eta_i\eta_j) + \frac{2}{\sigma_n^2}\sum_{i}^{k-1}\mathbb{E}(\eta_i\eta_k) + \frac{1}{\sigma_n^2}\mathbb{E}(\eta_k^2) \leq \\
&\leq \frac{1}{\sigma_n^2}k^2\mathbb{E}(\eta_0^2) + \frac{2}{\sigma_n^2}k\left[\mathbb{E}(\eta_0^2))\mathbb{E}(\eta_k^2))\right]^{\frac{1}{2}} + \frac{1}{\sigma_n^2}\mathbb{E}(\eta_k^2) \leq \\
&\leq \frac{k^2 qh(q)}{nh(n)} + \frac{2k\left[qh(q)q'h(q')\right]^{\frac{1}{2}}}{nh(n)} + \frac{q'h(q')}{nh(n)}, \qquad (2.43)
\end{aligned}$$

where $q' = n - (p+q)\left[\dfrac{n}{p+q}\right] \leq p+q$ is the number of terms in η_k.

From the properties of the function $h(n)$ seen in *lemma* (2.3.7) and the requirements imposed on k, p, q, we have that:

$$\frac{k^2 qh(q)}{nh(n)} \sim \frac{nqh(q)}{p^2 h(n)} = \left[\left(\frac{q}{n}\right)^{\beta}\frac{nqh(nq/n)}{h(n)}\right]\frac{n^{1+\beta}q^{1-\beta}}{p^2} \xrightarrow{n\to\infty} 0 \qquad (2.44)$$

by hypothesis (b). Similarly

$$\begin{aligned}
\frac{k\left[qh(q)q'h(q')\right]^{\frac{1}{2}}}{nh(n)} &= \left[\frac{kqh(q)}{nh(n)}\right]^{\frac{1}{2}}\left[\frac{kq'h(q')}{nh(n)}\right]^{\frac{1}{2}} \leq \\
&\leq \left[k\left(\frac{q}{n}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\left[k\left(\frac{q'}{n}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \xrightarrow{n\to\infty} 0 \qquad (2.45)
\end{aligned}$$

and

$$\frac{q'h(q')}{nh(n)} \leq \left(\frac{q'}{n}\right)^{\frac{1}{2}} \xrightarrow{n\to\infty} 0. \qquad (2.46)$$

Combining the conditions from (2.43) to (2.46), we see that

$$\lim_{n\to\infty}\mathbb{E}|Z_n''|^2 = 0,$$

as required. □

Lemma 2.3.10. *Let the uniformly mixing sequence X_j satisfy*

$$\mathbb{E}|X_j|^{2+\delta} < \infty$$

for some $\delta > 0$. *Suppose that*

$$\sigma_n^2 = \mathbb{E}(X_1 + X_2 + \ldots + X_n)^2 \xrightarrow{n\to\infty} \infty.$$

Then, if $\delta < 1$, *there exists a constant* a *such that*

$$\mathbb{E}\left|\sum_{j=1}^{n} X_j\right|^{2+\delta} \leq a\sigma_n^{2+\delta}.$$

Proof. We denote constants with $c_1, c_2, \ldots$.
Set

$$S_n = \sum_{j=1}^{n} X_j, \quad \hat{S}_n = \sum_{j=n+k+1}^{2n+k} X_j, \quad a_n = \mathbb{E}|S_n|^{2+\delta}.$$

We show that for any $\epsilon_1 > 0$, we can find a constant c_1 such that

$$\mathbb{E}|S_n + \hat{S}_n|^{2+\delta} \leq (2 + \epsilon_1)a_n + c_1\sigma^{2+\delta}. \tag{2.47}$$

Infact:

$$\begin{aligned}
&\mathbb{E}\left|S_n + \hat{S}_n\right|^{2+\delta} \leq \\
\leq &\mathbb{E}(S_n^2 + \hat{S}_n^2)\left(|S_n|^\delta + |\hat{S}_n|^\delta\right)^{2+\delta} \leq \\
\leq &\mathbb{E}|S_n|^{2+\delta} + \mathbb{E}|\hat{S}_n|^{2+\delta} + 2\mathbb{E}|S_n|^{1+\delta}|\hat{S}_n| + 2\mathbb{E}|S_n||\hat{S}_n|^{1+\delta}.
\end{aligned} \tag{2.48}$$

Because of the stationarity, S_n and $\hat{S}_n$ have the same distributions, and

$$\mathbb{E}|S_n|^{2+\delta} = \mathbb{E}|\hat{S}_n|^{2+\delta} = a_n.$$

By the *theorem* (2.2.3), with $p = \dfrac{2+\delta}{1+\delta}$,

$$\mathbb{E}|S_n|^{1+\delta}|\hat{S}_n| \leq 2\phi(k)^{\frac{1+\delta}{2+\delta}}a_n + \mathbb{E}|S_n|^{1+\delta}|\hat{S}_n|. \tag{2.49}$$

Using the *theorem* (2.2.3) again, but with $p = 2 + \delta$,

$$\mathbb{E}|S_n||\hat{S}_n|^{1+\delta} \leq 2\phi(k)^{\frac{1}{2+\delta}}a_n + \mathbb{E}|S_n||\hat{S}_n|^{1+\delta}. \tag{2.50}$$

By Ljapunov's inequality, according to which, given a random variable X, for any positive real numbers such that $0 < s < t$,

$$\left(\mathbb{E}(X^s)\right)^{\frac{1}{s}} \leq \left(\mathbb{E}(X^t)\right)^{\frac{1}{t}},$$

we obtain:

$$\mathbb{E}|S_n| \leq \sigma_n, \quad \mathbb{E}|S_n|^{1+\delta} \leq \sigma_n^{1+\delta}. \tag{2.51}$$

Inserting the inequalities (2.49), (2.50) and (2.51) in (2.48), we have

$$\mathbb{E}|S_n + \hat{S}_n|^{1+\delta} \leq \left[2 + 8\phi(k)^{\frac{1}{2+\delta}}\right] a_n + 4\sigma_n^{2+\delta}.$$

In order to prove (2.47) it suffices to take k so large that

$$8\phi(k)^{\frac{1}{2+\delta}} \leq \epsilon_1.$$

We now show that, for any $\epsilon_2 > 0$, there is a costant c_2 for which

$$a_{2n} \leq (2 + \epsilon_2)a_n + c_1\sigma_n^{2+\delta}. \tag{2.52}$$

Infact, using the Minkowski's inequality, according to which

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}}$$

for any real or complex numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, for any $1 < p < \infty$, and using the inequality (2.47), we have that for large n,

$$\begin{aligned}
& a_{2n} = \\
=& \mathbb{E}\left|S_n + \sum_{j=n+1}^{n+k} X_j + \hat{S}_n - \sum_{j=2n+1}^{2n+k} X_j\right|^{2+\delta} \leq \\
\leq& \left\{\left[\mathbb{E}|S_n + \hat{S}_n|^{2+\delta}\right]^{1/(2+\delta)} + \sum_{j=n+1}^{n+k} \left[\mathbb{E}|X_j|^{2+\delta}\right]^{1/(2+\delta)} + \sum_{j=2n+1}^{2n+k} \left[\mathbb{E}|X_j|^{2+\delta}\right]^{1/(2+\delta)}\right\}^{2+\delta} \leq \\
\leq& \left\{\left[(2+\epsilon_1)a_n + c_1\sigma^{2+\delta}\right]^{1/(2+\delta)} + 2ka_1^{1/(2+\delta)}\right\}^{2+\delta} \leq \\
\leq& (1+\epsilon')\left[(2+\epsilon_1)a_n + c_1\sigma^{2+\delta}\right] = \\
=& (1+\epsilon')\left[(2+\epsilon_1)a_n + (1+\epsilon')c_1\sigma^{2+\delta}\right],
\end{aligned}$$

where, since $\sigma_n \xrightarrow{n\to\infty} \infty$,

$$\epsilon' = \left[\frac{2ka_1}{(2+\epsilon_1)a_n + c_1\sigma^{2+\delta}}\right]^{1/(2+\delta)} \xrightarrow{n\to\infty} 0.$$

If we choose N so large that , for $n \geq N$, with $c_2' = 2c_1$ in place of c_2. But we can choose c_2 so that (2.52) holds also for $n > N$; hence (2.52) is proved.

Because of (2.52), for any integer r,

$$\sigma_{2^r} \le (2+\epsilon)^r a_1 + c_2 \sum_{j=1}^{r} (2+\epsilon)^{j-1} (\sigma_{2^{r-j}})^{2+\delta} \le$$
$$\le (2+\epsilon)^r a_1 + c_2 (\sigma_{2^{r-1}})^{2+\delta} \gamma_r,$$

where

$$\gamma_r = 1 + \frac{2+\epsilon}{2^{1+\frac{1}{2}\delta}} \left[\frac{h(2^{r-2})}{h(2^{r-1})}\right]^{1+\frac{1}{2}\delta} + \ldots + \left(\frac{2+\epsilon}{2^{1+\frac{1}{2}\delta}}\right)^{r-1} \left[\frac{1}{h(2^{r-1})}\right]^{1+\frac{1}{2}\delta}.$$

We show that, for sufficiently small ϵ, γ_r is bounded, i.e. $\gamma_r < c_3$.
The function $h(n)$ is slowly varying so that, for any $\epsilon_3 > 0$, there exists N such that, for $n \ge N$,

$$\frac{h(n)}{h(2n)} \le 1 + \epsilon_3.$$

For any integer l such that $2 \le l \le r-1$,

$$\frac{h(2^{r-1})}{h(2^{r-1})} = \left(\frac{h(2^{r-2})}{h(2^{r-1})} \cdots \frac{h(2^{r-s})}{h(2^{r-s+1})}\right) \left(\frac{h(2^{r-s+1})}{h(2^{r-s})} \cdots \frac{h(2^{r-1})}{h(2^{r-l+1})}\right).$$

Here we choose s so that $2^{r-s+1} \le N \le 2^{r-s}$, so that

$$\frac{h(2^{r-l})}{h(2^{r-1})} \le (1+\epsilon_3)^{s-1} c_4 \le c_4 (1+\epsilon_3)^{l-1}.$$

If ϵ_3 and ϵ are chosen so small that

$$\frac{(1+\epsilon_3)(2+\epsilon)}{2^{-1-\frac{1}{2}\delta}} > \rho < 1,$$

we obtain

$$\gamma_r \le \frac{c_5}{1-\rho} = c_3.$$

Thus, for any choice of ϵ,

$$a_{2^r} \le (2+\epsilon)^r a_1 + c_6 (\sigma_{2^{r-1}})^{2+\delta} =$$
$$= (\sigma_{2^r})^{2+\delta} \left(c_6 \frac{h(2^{r-1})}{2h(2^r)} + a_1 \left(\frac{2+t}{2^{1+\frac{1}{2}\delta}}\right)^r \frac{1}{h(2^r)}\right) \le c_7 (\sigma_{2^r})^{2+\delta}. \qquad (2.53)$$

Now let $2^r \le n \le 2^{r+1}$, and write n in binary form:

$$n = v_0 2^r + v_1 2^{r-1} + \ldots + v_r, \quad v_0 = 1, v_j = 0 \quad \text{or} \quad 1.$$

We write S_n in the form:

$$S_n = (X_1 + \ldots + X_{i_1}) + (X_{i_1+1} + \ldots + X_{i_2}) + \ldots + (X_{i_r+1} + \ldots + X_n),$$

where in the j-th parenthesis there are $v_j 2^{r-j}$ terms.
Using Minkowski's inequality and (2.52), remembering that X_j is stationary, we have:

$$\begin{aligned} a_n &\leq \left| \sum_{j=0}^{r} \left\{ \mathbb{E}|X_1 + \ldots + X_{v_j 2^{r-j}}|^{2+\delta} \right\}^{1+\frac{1}{2}\delta} \right|^{2+\delta} \leq \\ &\leq c_7 \left(\sum_{j=0}^{r} \sigma_{2^{r-j}} \right)^{2+\delta} = \\ &= c_7 \sigma^{2+\delta} \left(\sum_{j=0}^{r} \frac{\sigma_{2^{r-j}}}{\sigma_n} \right)^{2+\delta}. \end{aligned}$$

But

$$\sum_{j=0}^{r} \frac{\sigma_{2^{r-j}}}{\sigma_n} = \sum_{j=0}^{r} \frac{2^{\frac{1}{2}(r-j)}}{\sqrt{n}} \left(\frac{h(2^{r-j})}{h(2^r)} \frac{h(2^r)}{h(n)} \right)^{\frac{1}{2}}.$$

By *lemma* (2.3.6), we have that

$$\sup_r \sup_{2^r \leq n < 2^{r+1}} \frac{h(2^r)}{h(n)} < \infty,$$

thus we have only to prove that

$$\sum_{j=0}^{r} 2^{-\frac{1}{2}j} \left(\frac{h(2^{r-j})}{h(2^r)} \right)^{\frac{1}{2}} \tag{2.54}$$

is bounded. This is true because the j-th term is bounded by $c_8(\rho_1)^j$ for some $\rho_1 < 1$.
Thus the lemma is proved. □

Lemma 2.3.11. *Let the stationary sequence X_j be strongly mixing, with*

$$\sum_{n=1}^{\infty} \alpha(n) < \infty.$$

Let X_j be bounded, i.e. $P(|X_j| < c_0) = 1$, for some $c_0 \in \mathbb{R}$.
Then

$$\mathbb{E} \left(\sum_{j=1}^{n} X_j \right)^4 = o(n^3). \tag{2.55}$$

Proof. We have

$$\begin{aligned}\mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^4 =& n\mathbb{E}(X_0^4) + \sum_{i\neq j}\mathbb{E}(X_i^2X_j^2) + \sum_{i\neq j}\mathbb{E}(X_i^3X_j) + \\ &+ \sum_{i\neq j\neq k}\mathbb{E}(X_i^2X_jX_k) + \sum_{i\neq j\neq k\neq l}\mathbb{E}(X_iX_jX_kX_l). \qquad (2.56)\end{aligned}$$

The number of terms in the second and in the third sums is $O(n^2)$, thus it suffices to estimate the fourth and the fifth. By *theorem* (2.2.1),

$$\begin{aligned}\sum_{i\neq j\neq k}\mathbb{E}(X_i^2X_jX_k) &= O\left(\sum_{i<j<k}|\mathbb{E}(X_i^2X_jX_k)|\right) = \\ &= O\left(\sum_{i<j<k} c_0^4\alpha(k-j)\right) = \\ &= O(n^2),\end{aligned}$$

and

$$\begin{aligned}\sum_{i\neq j\neq k\neq l}\mathbb{E}(X_iX_jX_kX_l) &= O\left(\sum_{i<j<k<l}|\mathbb{E}(X_iX_jX_kX_l)|\right) = \\ &= O\left(\sum_{i<j<k<l} c_0^4\min(\alpha(j-i),\alpha(l-k))\right) = \\ &= O\left(n^2\sum_{j=1}^{n} j\alpha(j)\right).\end{aligned}$$

But

$$\sum_{j=1}^{n} j\alpha(j) \leq \sqrt{n}\sum_{j\leq\sqrt{n}}^{n}\alpha(j) + n\sum_{j>\sqrt{n}}^{n}\alpha(j) = o(n).$$

□

2.4 Central limit theorem for strongly mixing sequences

Let $(X_i)_{i\in\mathbb{N}}$ a stationary sequence with $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) < \infty\ \forall i \in \mathbb{N}$. Set

$$S_n^m = \sum_{i=m}^{n+m} X_i$$

and

$$\sigma_n^2 = \mathbb{E}((S_n^m)^2) = var(S_n^m).$$

We shall say that the sequence satisfies the central limit theorem if

$$\lim_{n\to\infty} P\left\{\frac{S_n^m}{\sigma_n} < z\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}u^2}du = n(z),$$

where $n(z)$ is the Gaussian probability density function.

Theorem 2.4.1. *Let the sequence $(X_i)_{i\in\mathbb{N}}$ satisfy the strong mixing condition with mixing coefficient $\alpha(n)$. In order that their sum with square-root normalization satisfies the central limit theorem it is necessary that:*

(i) $\sigma_n^2 = nh(n)$, where $h(x)$ is a slowly varying function of the continuous variable $x > 0$,

(ii) for any pair of sequences $p = p(n)$, $q = q(n)$ such that

(a) $p \to \infty$, $q \to \infty$, $q = o(p)$, $p = o(n)$ as $n \to \infty$,

(b) $\lim_{n\to\infty} \frac{n^{1-\beta}q^{1+\beta}}{p^2} = 0, \quad \forall\beta > 0$,

(c) $\lim_{n\to\infty} \frac{n}{p}\alpha(q) = 0$.

and $\forall\epsilon > 0$, the distribution function

$$\overline{F}_n(z) = P(X_1 + \ldots + X_n < z)$$

satisfies

$$\lim_{n\to\infty} \frac{n}{p\sigma_n^2}\int_{|z|>\epsilon\sigma_n} z^2 d\overline{F}_p(z) = 0. \tag{2.57}$$

Conversely, if (i) holds and if (2.57) is satisfied for some choice of the functions p, q satisfying the given conditions, the central limit theorem is satisfied.

Proof. We first establish the necessity of (i). From *theorem* (2.3.1), it follows that $h(n)$ is slowly varying in its integral argument. Let the distribution function

$$F_n(z) = P\left(\frac{S_n}{\sigma_n} < z\right)$$

converge to $n(z)$ as $n \to \infty$, where $n(z)$ is the Gaussian probability density function.
Then, for fixed N,

$$\int_{|z|\le N} z^2 dF_n(z) \to \int_{|z|\le N} z^2 dn(z)$$

so that

$$\begin{aligned}
&\int_{|z|>N} z^2 dF_n(z) = \\
&=1 - \int_{|z|\leq N} z^2 dF_n(z) \xrightarrow{n\to\infty} 1 - \int_{|z|\leq N} z^2 dn(z) = \\
&= \int_{|z|>N} z^2 dn(z) \qquad (2.58)
\end{aligned}$$

and

$$\lim_{N\to\infty}\lim_{n\to\infty}\int_{|z|>N} z^2 dF_n(z) = 0.$$

Define the variables

$$\xi = \sum_{j=0}^{n-1} X_j$$

and

$$\eta = \sum_{j=n+1+p}^{2n+p} X_j,$$

and observe that

$$\mathbb{E}(\xi^2) = \mathbb{E}(\eta^2) = \sigma_n^2.$$

From the *theorem* (2.3.3) and from *remark* (2.3), we have only to show that for each $\epsilon > 0$, there exists $p = p(\epsilon)$ such that

$$|\mathbb{E}(\xi\eta)| \leq \epsilon\mathbb{E}(\xi^2). \qquad (2.59)$$

Using the arguments of the *theorem* (2.2.2), it is easy to show that for any $N_1 \in \mathbb{N}$,

$$|\mathbb{E}(\xi\eta)| \leq N_1^2\alpha(p) + 6\sigma_n\left(\int_{|z>N_1|} z^2 dP(\xi < z)\right)^{1/2}.$$

Choosing $N_1 = \dfrac{\sigma}{\sqrt{\alpha(p)}}$, we have:

$$|\mathbb{E}(\xi\eta)| \leq \sigma_n^2\sqrt{\alpha(p)} + 6\sigma_n^2\left(\int_{|z|>(\alpha(p))^{-1/4}} z^2 dF_n(z)\right)^{1/2}. \qquad (2.60)$$

The strong mixing condition shows that, by suitable choice of p, we can make $|\mathbb{E}(\xi\eta)|$ smaller than $\epsilon\sigma_n^2$ for sufficiently large n. Thus we have proved the necessity of the condition (i), which will hencefort be assumed.
Proceede with the remaining parts of the proof.

We represent the sum S_n in the form

$$S_n = \sum_{i=0}^{k-1} \xi_i + \sum_{i=0}^{k} \eta_i = S_n' + S_n'', \tag{2.61}$$

where the variables ξ_i and η_i are defined by the following respective equations:

$$\xi_i = \sum_{j=ip+iq+1}^{(i+1)p+iq} X_j, \quad 0 \le i \le k-1$$

and

$$\eta_i = \begin{cases} \displaystyle\sum_{j=(i+1)p+iq+1}^{(i+1)p+(i+1)q} X_j, & 0 \le i \le k-1 \\ \displaystyle\sum_{j=kp+kq+1}^{n} X_j, \quad i = k. \end{cases}$$

Suppose that p and q satisfy (ii) and that $k = \left[\dfrac{n}{p+q}\right]$.
Consider the decompositions

$$Z_n = \frac{S_n}{\sigma_n} = \frac{S_n'}{\sigma_n} + \frac{S_n''}{\sigma_n} = Z_n' + Z_n'',$$

of the normalized sum Z_n. Under the conditions imposed on p and q, we show that S_n'' is negligibile, and that the ξ_i are nearly independent.
Firstly, we verify that the conditions imposed on p and q can indeed be satisfied. In order to do this we set:

$$\begin{aligned} \lambda(n) &= \max\left\{\alpha[n^{1/4}]^{1/3}, \frac{1}{\log n}\right\}, \\ p &= \max\left\{\left[\frac{\alpha[n^{1/4}]}{\lambda(n)}\right], \left[\frac{n^{3/4}}{\lambda(n)}\right]\right\}, \\ q &= \left[n^{1/4}\right]. \end{aligned}$$

Then all the conditions (a),(b),(c) are satisfied:

(a) $p \to \infty$, $q \to \infty$, $p = o(n)$, $q = o(p)$ as $n \to \infty$,

(b) $\dfrac{n^{1-\beta} q^{1+\beta}}{p^2} = O\left(n^{-(1+3\beta)/4}\right) = o(1)$, if $\beta > 0$,

(c) $\dfrac{n\alpha(q)}{p} \leq \alpha[n^{1/4}]\dfrac{\lambda(n)}{\alpha[n^{1/4}]} \to 0, \quad as \quad n \to \infty.$

By *lemma* (2.3.9), we claim that the variable Z''_n is neglegible.
Thus it follows that the limiting distribution of Z_n is the same as that of Z'_n.
Denote with $\psi_n(t)$ the characteristic function of $\dfrac{\xi_0}{\sigma_n}$ and prove that

$$\left|\mathbb{E}\left(e^{itZ_n}\right) - \psi_n(t)^k\right| \xrightarrow{n\to\infty} 0. \tag{2.62}$$

The variable

$$\exp\left(\frac{it}{\sigma_n}(\xi_0 + \ldots + \xi_{k-2})\right)$$

is measurable with respect to the σ-algebra $\mathcal{F}_{-\infty}^{(k-1)p+(k-2)q}$ and the variable

$$\exp\left(\frac{it}{\sigma_n}(\xi_{k-1})\right)$$

is measurable with respect to the σ-algebra $\mathcal{F}_{(k-1)p+(k-1)q+1}^{\infty}$. By *theorem* (2.2.1),

$$\left|\mathbb{E}\exp\left[\frac{it}{\sigma_n}\sum_{j=0}^{k-1}\xi_j\right] - \mathbb{E}\exp\left[\frac{it}{\sigma_n}\sum_{j=0}^{k-2}\xi_j\right]\mathbb{E}\exp\left[\frac{it}{\sigma_n}\xi_{k-1}\right]\right| \leq 16\alpha(q),$$

and similarly, for $l \leq k-2$,

$$\left|\mathbb{E}\exp\left[\frac{it}{\sigma_n}\sum_{j=0}^{l}\xi_j\right] - \mathbb{E}\exp\left[\frac{it}{\sigma_n}\sum_{j=0}^{l-1}\xi_j\right]\mathbb{E}\exp\left[\frac{it}{\sigma_n}\xi_l\right]\right| \leq 16\alpha(q).$$

Hence

$$\left|\mathbb{E}\left(e^{itZ_n}\right) - \psi_n(t)^k\right| \leq 16\alpha(q),$$

which tends to zero by hypothesis (ii), and proves (2.62).
Now consider a collection of independent random variables

$$\xi'_{nj}, \quad n = 1, 2, \ldots, \quad j = 1, 2, \ldots, \quad k = k(n),$$

where ξ'_{nj} has the same distribution as $\dfrac{\xi_0}{\sigma_n}$.
Then (2.62) asserts that the limiting distribution of Z'_n is the same as that of

$$\xi'_{n1} + \xi'_{n2} + \ldots + \xi'_{nk},$$

which has characteristic function $\psi_n(t)^k$.
This limiting distribution is the Gaussian distribution if and only if

$$0 = \lim_{n\to\infty} \sum_{j=1}^{k} \int_{|z|>\epsilon} z^2 dP(\xi'_{nj} < z) =$$
$$= \lim_{n\to\infty} k \int_{|z|>\epsilon} z^2 dP\left(\frac{\xi_0}{\sigma_n} < z\right).$$

But

$$k \int_{|z|>\epsilon} z^2 dP\left(\frac{\xi_0}{\sigma_n} < z\right) = \frac{k}{\sigma_n^2} \int_{|z|>\epsilon} z^2 dP\left(\xi_0 < z\right) \sim$$
$$\sim \frac{n}{\sigma_n^2 p} \int_{|z|>\epsilon\sigma_n} z^2 d\overline{F}_p(z).$$

Thus the theorem is proved. □

Remark 2.4. We remark that the only part of the proof in which the necessary condition (b) was used, was in the proof that $\mathbb{E}|Z''_n|^2 = 0$.

Theorem 2.4.2. *Let X_i be a strongly mixing sequence of random variables such that $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) = \sigma^2 < \infty$. Suppose that*

$$var(S_n) = nh(n) \quad as \quad n \to \infty,$$

where $h(n)$ is a slowly varying function such that as $n \to \infty$, $c_1 \le h(n) \le c_2$, where c_1 and c_2 are constants. Then the sequence X_i satisfies the central limit theorem if and only if

$$\lim_{M\to\infty} \limsup_{n\to\infty} \int_{|x|>M} z^2 dF_n(z) = 0, \tag{2.63}$$

where $F_n(z)$ is the distribution function of the normalized sum

$$Z_n = \frac{1}{\sigma_n} \sum_{i=1}^{n} X_i.$$

Proof. Suppose that $F_n(x)$ converges weakly to the Gaussian probability density function $n(x)$. Then for fixed M,

$$\int_{|x|\le M} x^2 dF_n(x) \xrightarrow{n\to\infty} \int_{|x|\le M} x^2 dn(x).$$

By definition of Z_n, the variance of F_n is equal to one; this implies that

$$\int_{|x|>M} x^2 dF_n(x) \xrightarrow{n\to\infty} \int_{|x|>M} x^2 dn(x),$$

so that (2.63) is a necessary condition.

Conversely, if (2.63) is satisfied and $\sigma_n^2 = nh(n)$, we have

$$\begin{aligned}
&\frac{k}{\sigma_n^2}\int_{|z|>\epsilon\sigma_n} z^2 dP(S_p < z) = \\
=&\frac{k\sigma_p^2}{\sigma_n^2}\int_{|z|>\epsilon\sigma_n/\sigma_p} z^2 dF_p(z) = \\
=&\frac{h(p)}{h(n)}\int_{|z|>\epsilon k(1+o(1))} z^2 dF_p(z) \xrightarrow{n\to\infty} 0,
\end{aligned}$$

since $k \to \infty$ and $p \to \infty$. □

2.5 Sufficient conditions for the central limit theorem

Theorem 2.5.1. *Let the uniformly mixing sequence X_j satisfy*

$$\mathbb{E}|X_j|^{2+\delta} < \infty$$

for some $\delta > 0$. If

$$\sigma_n^2 = \mathbb{E}(X_1 + X_2 + \ldots + X_n)^2 \xrightarrow{n\to\infty} \infty,$$

then X_j satisfies the central limit theorem.

Proof. We show that all the conditions of *theorem* (2.2.1) are satisfied.
By *theorem* (2.3.3), it holds

$$\sigma_n^2 = nh(n),$$

so that the condition (i) is fulfilled.
By *lemma* (2.3.10), it exists a coonstant $a > 0$ such that it is verified the inequality

$$\mathbb{E}\left|\sum_{j=1}^{n} X_j\right|^{2+\delta} \leq a\sigma_n^{2+\delta}.$$

In order to complete the proof of the theorem, we have to prove that

$$\lim_{n\to\infty} \frac{n}{p\sigma_n^2} \int_{|z|>\epsilon\sigma_n} z^2 dF_p(z) = 0.$$

Using *lemma* (2.3.10) again,

$$\frac{n}{p\sigma_n^2} \int_{|z|>\epsilon\sigma_n} z^2 dF_p(z) \le \frac{n}{p\sigma_n^{2+\delta}\epsilon^\delta} \int_{-\infty}^{\infty} |z|^{2+\delta} dF_p(z) \le$$

$$\le \frac{an\sigma_p^{2+\delta}}{p\sigma_n^{2+\delta}\epsilon^\delta} = \frac{a}{\epsilon^\delta} \left(\frac{p}{n}\right)^\delta \left(\frac{h(p)}{h(n)}\right)^{1+\frac{1}{2}\delta} \xrightarrow{n\to\infty} 0, \quad (2.64)$$

The result of the limit (2.64) is due to the *theorem* (2.3.3) and to the necessary condition (a) of the *theorem* (2.2.1). □

Theorem 2.5.2. *Let the stationary sequence X_j satisfy the uniformly mixing condition.*
Let the mixing coefficient $\phi(n)$ satisfy

$$\sum_n \sqrt{\phi(n)} < \infty.$$

Then the sum

$$\sigma^2 = \mathbb{E}(X_0^2) + 2\sum_{j=1}^{\infty} \mathbb{E}(X_0 X_j) \quad (2.65)$$

converges. Moreover, if $\sigma \neq 0$, as $n \to \infty$,

$$P\left(\frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{n} X_j < z\right) \to \frac{1}{2\pi} \int_{-\infty}^{z} e^{-\frac{u^2}{2}} du. \quad (2.66)$$

Proof. By *theorem* (2.2.3),

$$|R(j)| = |\mathbb{E}(X_0 X_j)| \le 2\phi(n)^{\frac{1}{2}} \{\mathbb{E}(X_0^2)\mathbb{E}(X_j^2)\}^{\frac{1}{2}},$$

whence the convergence of (2.65) follows.
Moreover,

$$\sigma_n^2 = P\left(\sum_{j=1}^{n} X_j^2\right) =$$

$$= nR(0) + 2\sum_{j=1}^{n} (n-j)R(j) = \sigma^2 n(1 + o(1)),$$

so that, if $\sigma \neq 0$, $\sigma_n \to \infty$ as $n \to \infty$. We deduce the validity of (2.66) by the *theorem* (2.5.1).
Since the sequence X_j is uniformly mixing, so is the sequence $f_N(X_j)$, defined by

$$f_N(x) = \begin{cases} x, & as \quad |x| \leq N \\ 0, & as \quad |x| > N \end{cases}$$

and with mixing coefficient less or equal to $\phi(n)$. Clearly

$$\mathbb{E}|f_N(X_j)|^3 < \infty,$$

so that we can apply the *theorem* (2.5.1). Observe that as

$$\lim_{N\to\infty} \mathbb{E}\{f_N(X_j)\} = 0$$

and

$$\lim_{N\to\infty} \mathbb{E}\{f_N(X_0)f_N(X_j)\} = \mathbb{E}\{X_0 X_j\}.$$

Thus, since $\sigma \neq 0$, it follows that, for large N,

$$\begin{aligned}\sigma^2(N) =& \mathbb{E}\{f_N(X_0) - \mathbb{E}\{f_N(X_0)\}\}^2 + \\ &+ 2\sum_{j=1}^{\infty} \mathbb{E}\{f_N(X_0) - \mathbb{E}\{f_N(X_0)\}\}\left[f_N(X_j) - \mathbb{E}\{f_N(X_j)\}\right] > \\ &> \frac{1}{2}\sigma^2 > 0.\end{aligned}$$

For such N, as $n \to \infty$,

$$\begin{aligned}\sigma_n^2(N) &= \mathbb{E}\left\{\sum_{j=1}^{\infty} f(X_j) - \mathbb{E}\sum_{j=1}^{\infty} f(X_j)\right\}^2 = \\ &= n\sigma^2(N)(1 + o(1)) > \\ &> \frac{1}{2}n\sigma^2(N)(1 + o(1)) \to \infty\end{aligned}$$

Thus all the conditions of the *theorem* (2.5.1) are satisfied and consequently

$$\begin{aligned}&\lim_{n\to\infty} P\left\{\frac{1}{\sigma_n(N)}\sum_{j=1}^{n}[f_N(X_j) - \mathbb{E}\{f_N(X_j)\}] < z\right\} = \\ &= \lim_{n\to\infty} P\left\{\frac{1}{\sigma(N)\sqrt{n}}\sum_{j=1}^{n}[f_N(X_j) - \mathbb{E}\{f_N(X_j)\}] < z\right\} = \\ &= n(z),\end{aligned}$$

where $n(z)$ is the Gaussian probability density function.
Consider now the normalized sum

$$Z_n = \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{n} X_j = Z'_n + Z''_n,$$

where

$$Z'_n = \frac{\sum_{j=1}^{n} [f_N(X_j) - \mathbb{E}\{f_N(X_j)\}]}{\sigma(N)\sqrt{n}} \frac{\sigma(N)}{\sigma}$$

and

$$Z''_n = \frac{\sum_{j=1}^{n} [\overline{f}_N(X_j) - \mathbb{E}\{\overline{f}_N(X_j)\}]}{\sigma(N)\sqrt{n}},$$

with $\overline{f}_N(x) = x - f_N(x)$.
We first estimate $\mathbb{E}(Z''_n)^2$:

$$\begin{aligned}\mathbb{E}(Z''_n)^2 =& \frac{1}{\sigma^2 n} n\mathbb{E}[\overline{f}_N(X_0) - \mathbb{E}\{\overline{f}_N(X_0)\}]^2 + \\ &+ 2\frac{1}{\sigma^2 n} \sum_{j=1}^{n-1} (n-j)\mathbb{E}[\overline{f}_N(X_0) - \mathbb{E}\{\overline{f}_N(X_0)\}][\overline{f}_N(X_j) - \mathbb{E}\{\overline{f}_N(X_j)\}].\end{aligned}$$

By *theorem* (2.3.3), for $j \geq 0$,

$$\begin{aligned}&\left|\mathbb{E}[\overline{f}_N(X_0) - \mathbb{E}\{\overline{f}_N(X_0)\}][\overline{f}_N(X_j) - \mathbb{E}\{\overline{f}_N(X_j)\}]\right| \leq \\ \leq& 2\phi(j)^{\frac{1}{2}}\mathbb{E}[\overline{f}_N(X_0) - \mathbb{E}\{\overline{f}_N(X_0)\}]^2 \leq \\ \leq& r_N \phi(j)^{\frac{1}{2}},\end{aligned}$$

where $\phi(0) = 1$ and $r_N = 2\mathbb{E}[\overline{f}_N(X_0)]^2$.
Thus, since $r_N \xrightarrow{N\to\infty} 0$,

$$\mathbb{E}(Z''_n)^2 \leq \frac{r_N}{\sigma^2}\left\{1 + 2\sum_{j=1}^{\infty} \phi(j)^{\frac{1}{2}}\right\} \xrightarrow{N\to\infty} 0.$$

For a given $\epsilon > 0$, choose n so that

$$\mathbb{E}|Z''_n| \leq \epsilon$$

and

$$\left|1 - \frac{\sigma(n)}{\sigma}\right| \leq \epsilon.$$

Then the characteristic function $\psi_n(t)$ of Z_n satisfies

$$\begin{aligned}
&|\psi_n(t) - e^{-\frac{1}{2}t^2}| \leq \\
&\leq \left|\mathbb{E}(\exp(itZ_n')) - e^{-\frac{1}{2}t^2}\right| + |\mathbb{E}\,(\exp(itZ_n')(\exp(itZ_n'') - 1))| \leq \\
&\leq \left|\exp\left[-\frac{\sigma^2(N)}{\sigma^2}\cdot\frac{t^2}{2}\right] - e^{-\frac{1}{2}t^2}\right| + \mathbb{E}\,|\exp(itZ_n'') - 1| + \\
&\quad + \left|\mathbb{E}(\exp(itZ_n'') - \exp\left[-\frac{\sigma^2(N)}{\sigma^2}\cdot\frac{t^2}{2}\right]\right| \leq \\
&\leq \frac{1}{2}t^2\left|1 - \frac{\sigma^2(N)}{\sigma^2}\right| + t\mathbb{E}|Z_n''| + o(1) \leq \\
&\leq \epsilon t^2 + \epsilon t + o(1).
\end{aligned}$$

Thus the theorem is proved. □

We now turn to sequences which are strongly mixing without necessarily being uniformly mixing. Naturally, stronger conditions are necessary to ensure normal convergence.

Theorem 2.5.3. *Let the stationary sequence X_j satisfy the strong mixing condition with mixing coefficient $\alpha(n)$.*
Let $\mathbb{E}(X_j)^{2+\delta} < \infty$ for some $\delta > 0$. If

$$\sum_{n=1}^{\infty} \alpha(n)^{\frac{\delta}{2+\delta}} < \infty, \tag{2.67}$$

then

$$\sigma^2 = \mathbb{E}(X_0^2) + 2\sum_{j=1}^{\infty} \mathbb{E}(X_0 X_j) < \infty \tag{2.68}$$

and, if $\sigma \neq 0$, as $n \to \infty$,

$$P\left(\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n} X_j < z\right) \to n(z), \tag{2.69}$$

where $n(z)$ is the gaussian probability density function.

Before proving this theorem, it is convenient to deal with the case of bouned variables, to which the general theorem can be reduced.

Theorem 2.5.4. *Let the stationary sequence X_j be strongly mixing, with*

$$\sum_{n=1}^{\infty} \alpha(n) < \infty.$$

Let X_j be bounded, i.e. $P(|X_j| < c_0) = 1$, for some $c_0 \in \mathbb{R}$.
Then

$$\sigma^2 = \mathbb{E}(X_0^2) + 2\sum_{j=1}^{\infty} \mathbb{E}(X_0 X_j) < \infty \tag{2.70}$$

and, if $\sigma \neq 0$, as $n \to \infty$,

$$P\left(\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n} X_j < z\right) \to n(z). \tag{2.71}$$

Proof. The convergence of the series (2.70) follows from the inequality

$$|\mathbb{E}(X_0 X_j)| \leq 4c_0^2\alpha(j),$$

as we have seen in the theorem (2.2.2). From this, as in the proof of the *theorem* (2.5.2), it follows that

$$\sigma_n^2 = \mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^2 = \sigma^2 n(1 + o(1)),$$

and that consequently

$$\lim_{n\to\infty} P\left(\frac{X_1 + \ldots + X_n}{\sigma\sqrt{n}} < z\right) = \lim_{n\to\infty} P\left(\frac{X_1 + \ldots + X_n}{\sigma_n} < z\right),$$

so long as either limit exists. By *lemma* (2.3.11) we can estimate the moment $\mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^4$ using the inequality

$$\mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^4 = n^3\gamma(n) \leq n^3\tilde{\gamma}(n),$$

where $\gamma(n) \to 0$ and $\tilde{\gamma}(n) = \sup_{j\geq n} \gamma(j)$.
Define the variables ξ_i and η_i by the following respective equations:

$$\xi_i = \sum_{j=ip+iq+1}^{(i+1)p+iq} X_j, \quad 0 \leq i \leq k-1$$

and

$$\eta_i = \begin{cases} \displaystyle\sum_{j=(i+1)p+iq+1}^{(i+1)p+(i+1)q} X_j, & 0 \leq i \leq k-1 \\ \displaystyle\sum_{j=kp+kq+1}^{n} X_j, & i = k \end{cases}$$

where

$$p = p(n) = \min\{p, p \geq \sqrt{n}|\log\tilde{\gamma}(p)|\},$$
$$q = \left[\frac{n}{p}\right],$$
$$k = \left[\frac{n}{p+q}\right].$$

We show that p and q satisfy the necessary conditions (a) and (b) given by the *theorem* (2.2.1).

Consider the condition (a).
Clearly, as $n \to \infty$, $p \to \infty$. By Ljapunov's inequality:

$$\mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^4 \geq \left[\mathbb{E}\left(\sum_{j=1}^{n} X_j\right)^2\right]^2 = \sigma^4 n^2(1 + o(1)),$$

hence

$$\lim_{n\to\infty} n\tilde{\gamma(n)} > 0.$$

Thus for large n, yelds $p < \sqrt{n}(\log n)^2$, so that

$$p = o(n), \quad q \to \infty.$$

Since $p \to \infty$, then $q = o(n)$. Consider the condition (b).
Since $\alpha(n)$ is monotone and $\sum \alpha(n) < \infty$, then

$$\alpha(n) \leq \frac{2}{n} \sum_{j=\frac{1}{2}n}^{n} \alpha(j) = o\left(\frac{1}{n}\right),$$

so that

$$\frac{n\alpha(q)}{p} = o\left(\frac{n}{pq}\right) = o(1).$$

This condition is not in general satisfied. However, as observed in *remark* (2.3), this condition was only used to prove that

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{1}{\sigma_n}\sum_{i=0}^{k}\eta_i\right)^2 = 0, \tag{2.72}$$

and we can find some other way of proving this, the rest of the argument will go trough unchanged. Hence, in order to prove (2.72), it suffices to show that

$$\lim_{n\to\infty} \frac{n}{p\sigma_n^2} \int_{|z|>\epsilon\sigma_n} z^2 dP \left\{ \sum_{j=1}^{n} X_j < z \right\} = 0, \quad \forall \epsilon > 0. \tag{2.73}$$

We have

$$\mathbb{E}\left(\frac{1}{\sigma_n}\sum_{i=0}^{k}\eta_i\right)^2 = \frac{1+o(1)}{\sigma^2 n}\left(k\mathbb{E}(\eta_0^2) + 2\sum_{j=1}^{k-1}(k-j)\mathbb{E}(\eta_0\eta_1)\right) + \\ + \frac{1+o(1)}{\sigma^2 n}\left(2\sum_{i=0}^{k-1}(k-j)\mathbb{E}(\eta_i\eta_k) + \mathbb{E}(\eta_k^2)\right).$$

By *theorem* (2.2.1) yelds:

$$\mathbb{E}(\eta_i\eta_j) \leq c_0^2 q^2 \alpha(p|i-j|), \quad as \quad i,j \leq k-1, \tag{2.74}$$

$$\mathbb{E}(\eta_i\eta_k) \leq c_0^2 q(p+q)\alpha(p(k-i)), \quad as \quad j = k. \tag{2.75}$$

Moreover

$$\mathbb{E}(\eta_0^2) = \sigma^2 q(1+o(1))$$

and

$$\mathbb{E}(\eta_k^2) = O(p+q) = O(p),$$

so that

$$\frac{k}{n}\mathbb{E}(\eta_0^2) = O\left(\frac{kq}{n}\right) = o(1) \tag{2.76}$$

and

$$\frac{k}{n}\mathbb{E}(\eta_k^2) = O\left(\frac{p}{n}\right) = o(1). \tag{2.77}$$

Since $\alpha(n)$ is monotone,

$$\sum_{j=1}^{k-1}\alpha(pj) \leq \sum_{j=1}^{k-1}\frac{1}{p}\sum_{s=(j-1)p}^{jp-1}\alpha(s) \leq \frac{1}{p}\sum_{j=0}^{\infty}\alpha(j),$$

so that it follows from the inequalities (2.74) and (2.75) that

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{j=1}^{k-1}(k-j)\mathbb{E}(\eta_0\eta_j)\right| \leq \\
\leq& c_0^2\frac{kq^2}{n}\sum_{j=1}^{k-1}\alpha(pj) \leq \\
\leq& c_0^2\frac{kq^2}{np}\sum_{j=0}^{\infty}\alpha(j) = \\
=& O\left(\frac{n^2}{p^4}\right) = O\left(\frac{1}{\log^4\tilde{\gamma}(p)}\right) \xrightarrow{n\to\infty} 0.
\end{aligned} \tag{2.78}$$

Similarly

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i=0}^{k-1}\mathbb{E}(\eta_i\eta_k)\right| \\
=& O\left(\frac{pq}{n}\sum_{j=1}^{k-1}\alpha(pj)\right) = O\left(\frac{q}{n}\right)o(1).
\end{aligned} \tag{2.79}$$

Combining the results from (2.76) to (2.79), we obtain (2.72).
Finally, by *lemma* (2.3.11), as $n \to \infty$,

$$\begin{aligned}
&\frac{n}{p\sigma_n^2}\int_{|z|>\epsilon\sigma_n} z^2 dP\left(\sum_{j=1}^{p} X_j < z\right) \leq \\
\leq& \frac{n}{\epsilon\sigma_n^4}\int_{-\infty}^{\infty} z^4 dP\left(\sum_{j=1}^{p} X_j < z\right) \leq \\
\leq& \frac{np^3\tilde{\gamma}(p)}{\epsilon^2\sigma^4 n^2}(1+o(1)) = \\
=& O\left(\tilde{\gamma}(p)\log^3\frac{1}{\tilde{\gamma}(p)}\right) = o(1).
\end{aligned}$$

Thus the *theorem* (2.5.4) is proved. □

Now we extend the results to the *theorem* (2.5.3).

Proof. The convergence of the series (2.68) follows quickly from the *theorem* (2.2.2) and the convergence of $\sum_{j=1}^{\infty}\alpha(j)^{\frac{\delta}{2+\delta}}$. Introduce the functions f_N and

$\overline{f}_N$ as in the proof of the *theorem* (2.5.2) and consider the stationary sequence $f_N(X_j)$. Since $\sum_{j=1}^{\infty} \alpha(j)$ converges, this sequence satisfies all the conditions of the *theorem* (2.2.1) and thus satisfies the central limit theorem.
Now set

$$Z_n = \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{n} X_j = Z'_n + Z''_n,$$

where

$$Z'_n = \frac{\sum_{j=1}^{n}[\psi_N(X_j) - \mathbb{E}\{\psi_N(X_j)\}]}{\sigma(N)\sqrt{n}} \frac{\sigma(N)}{\sigma},$$
$$Z''_n = \frac{\sum_{j=1}^{n}[\overline{f}_N(X_j) - \mathbb{E}\{\overline{f}_N(X_j)\}]}{\sigma(N)\sqrt{n}}.$$

Observe that

$$\sigma^2(N) = \mathbb{E}\left[\psi_N(X_0) - \mathbb{E}(\psi_N(X_0))\right]^2 + \\ + 2\sum_{j=1}^{\infty} \mathbb{E}\left[\psi_N(X_0) - \mathbb{E}(\psi_N(X_0))\right]\left[\psi_N(X_j) - \mathbb{E}(\psi_N(X_j))\right].$$

Using (2.6), we have

$$\mathbb{E}|Z_N|^2 = \frac{1}{\sigma^2}\left\{\mathbb{E}\left[\overline{\psi}_N(X_0) - \mathbb{E}(\overline{\psi}_N(X_0))\right]\right\}\left[\overline{\psi}_N(X_j) - \mathbb{E}(\overline{\psi}_N(X_j))\right] \leq \\ \leq A\frac{8\mathbb{E}|\overline{\psi}_N(X_0)|^2}{\sigma^2} + C\sum_{j=A+1}^{\infty} \alpha(j)^{\frac{\delta}{2+\delta}},$$

where C is a constant and $A \leq n$ is a positive integer.
This implies that

$$\lim_{n\to\infty} \mathbb{E}|Z_N|^2 = 0$$

uniformly in n.
The proof can be completed in the same way as that of *theorem* (2.5.2). □

Chapter 3

The Mean-Field Model

In this chapter we are going to define the more general case of the mean-field model. After that we are going to compute the limit for large N of the pressure function and of the distribution of the normalized sum of spins, according to the results of Ellis, Newmann and Rosen. In the last section we will prove that, in general, the distribution of the normalized sum of spins of a model determined by the Hamiltonian defined by the coupling constant J equal to one and the magnetic field h equal to zero doesn't converge toward the Gaussian distribution.

3.1 The model

We consider a sistem composed by N particles that interact with each other (this interaction is indipendent from their distance) and with an external magnetic field. Such sistem is defined by the Hamiltonian

$$H_N(\overline{\sigma}) = -\frac{J}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i \tag{3.1}$$

where:

- σ_i is the spin of the particle i,
- $J > 0$ is a parameter called *coupling constant*,
- h is the *magnetic field*.

The distribution of a configuration of spins $\overline{\sigma} = (\sigma_1, \dots, \sigma_N)$ is given by the *measure of Boltmann-Gibbs*:

$$P_{N,J,h}\{\overline{\sigma}\} = \frac{e^{-H_N(\overline{\sigma})} \prod_{i=1}^{N} d\rho(\sigma_i)}{Z_N(J,h)} \tag{3.2}$$

where $Z_N(J,h)$ is the canonical *partition function* defined by:

$$Z_N(J,h) = \int_{\mathbb{R}^N} e^{-H_N(\overline{\sigma})} \prod_{i=1}^{N} d\rho(\sigma_i) \tag{3.3}$$

and ρ is the distribution of a single spin in absence of interaction with other spins.
We assume that ρ is a non degenerate Borel probability measure on $\mathbb{R}$ and satisfies

$$\int_{\mathbb{R}} e^{\frac{ax^2}{2}+bx} d\rho(x) < \infty, \quad \forall a, b \in \mathbb{R}, a > 0. \tag{3.4}$$

Remark 3.1. The measure

$$\overline{\rho}(x) = \frac{1}{2}\left(\delta(x-1) + \delta(x+1)\right),$$

where $\delta(x - x_0)$ with $x_0 \in \mathbb{R}$ denotes the unit point mass with support at x_0, verifies the condition (3.4).

Proof.

$$\int_{\mathbb{R}} e^{\frac{ax^2}{2}+bx} d\rho(x) = e^{\frac{a}{2}+b} + e^{\frac{a}{2}-b} = 2e^{\frac{a}{2}} cosh(b) < \infty$$

□

The model defined by the Hamiltonian (3.1) and the distribution (3.2) with $\rho = \bar{\rho}$ is called *model of Curie Weiss.*
Given a general observable $\psi(\overline{\sigma})$ of interest, we can compute its expected value respect to the distribution associated to the measure of Boltmann-Gibbs:

$$\langle \psi(\overline{\sigma}) \rangle_{BG} = \frac{\int_{\mathbb{R}^N} \psi(\overline{\sigma}) e^{-H_N(\overline{\sigma})} \prod_{i=1}^{N} d\rho(\sigma_i)}{\int_{\mathbb{R}^N} e^{-H_N(\overline{\sigma})} \prod_{i=1}^{N} d\rho(\sigma_i)}$$

which is called *Gibbs state of the observable* $\psi(\overline{\sigma})$.
The main observable of the mean-field model is the *magnetization* $m_N(\overline{\sigma})$ of a configuration σ:

$$m_N(\overline{\sigma}) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i.$$

Remark 3.2. The Hamiltonian (3.1) can be written as function of the magnetization $m_N(\overline{\sigma})$:

$$H_N(\overline{\sigma}) = -N\left[\frac{J}{2}m_N^2(\overline{\sigma}) + hm_N(\overline{\sigma})\right] \tag{3.5}$$

Proof. We start observing that

$$\sum_{i,j=1}^{N}\sigma_i\sigma_j = \sum_{i=1}^{N}\sigma_i\sum_{j=1}^{N}\sigma_j = \left(\sum_{i=1}^{N}\sigma_i\right)^2$$

Then the hamiltonian can be written as:

$$\begin{aligned}
H_N(\overline{\sigma}) &= -\frac{J}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i = \\
&= -\frac{J}{2N}\left(\sum_{i=1}^{N}\sigma_i\right)^2 - h\sum_{i=1}^{N}\sigma_i = \\
&= -\frac{J}{2N}N^2\frac{\left(\sum_{i=1}^{N}\sigma_i\right)^2}{N^2} - hN\frac{\sum_{i=1}^{N}\sigma_i}{N} = \\
&= -\frac{JN}{2}m_N(\overline{\sigma})^2 - hNm_N(\overline{\sigma}) = \\
&= -N\left[\frac{J}{2}m_N(\overline{\sigma})^2 + hm_N(\overline{\sigma})\right]
\end{aligned}$$

□

Instead of computing directly the Gibbs state, it will be useful to consider the *pressure function* associated to the model:

$$p_N(J,h) = \frac{1}{N}\ln(Z_N(J,h)).$$

Remark 3.3. This is possible because the Gibbs state of the magnetization can be obtained differentiating $p_N(J,h)$ with respect to h:

$$\frac{\partial p_N}{\partial h} = \frac{1}{N}\frac{\int_{\mathbb{R}^N}\left(\sum_{i=1}^{N}\sigma_i\right)e^{-H_N(\overline{\sigma})}\prod_{i=1}^{N}d\rho(\sigma_i)}{Z_N(J,h)} =$$
$$= \frac{1}{N}\frac{\int_{\mathbb{R}^N}\left(\sum_{i=1}^{N}\sigma_i\right)e^{-H_N(\overline{\sigma})}\prod_{i=1}^{N}d\rho(\sigma_i)}{Z_N(J,h)} =$$
$$= \frac{1}{N}\frac{\int_{\mathbb{R}^N} N m_N(\overline{\sigma})e^{-H_N(\overline{\sigma})}\prod_{i=1}^{N}d\rho(\sigma_i)}{\int_{\mathbb{R}^N} e^{-H_N(\overline{\sigma})}\prod_{i=1}^{N}d\rho(\sigma_i)} =$$
$$= \langle m_N(\overline{\sigma})\rangle_{BG}$$

3.2 Thermodynamic limit

3.2.1 Existence of the thermodynamic limit

After have defined the relevant obervable of the model, we have to check that the model is well defined, i.e. the Hamiltonian must be an intensive quantity of the number of spins: this property is verified if the pressure function $p_N(J,h)$, associated to the model, admits limit as $N \to \infty$. We are now going to show the existence of the thermodinamic limit of $p_N(J,h)$ associated to the Hamiltonian (3.1).

Upper bound
To semplify the notations, we denote with $m = m_N(\overline{\sigma})$ the trivial extimate of the magnetization, valid for all trial magnetizations M. Then the following inequality holds:

$$m^2 \geq 2mM - M^2.$$

We plug it in the definition of the partition function, so that we obtain:

$$Z_N(J,h) = \sum_{\sigma} e^{-H_N(\overline{\sigma})} = \sum_{\sigma} e^{\frac{JN}{2}m^2 + hNm} \geq \sum_{\sigma} e^{-\frac{JN}{2}M^2 + JNmM + Nhm}$$

The magnetization appears linearly and the sum factorizes in each spin; then, applying the definition of the pressure, we obtain:

$$\frac{1}{N}\ln(Z_N(J,h)) \geq \frac{1}{N}\ln\left(\sum_\sigma e^{-\frac{JN}{2}M^2+JNmM+Nhm}\right).$$

Calling

$$\frac{1}{N}\ln\left(\sum_\sigma e^{-\frac{JN}{2}M^2+NJmM+Nhm}\right) = \overline{p}(M,J,h)$$

we have the following inequality, which is true for every positive integers N and M:

$$p_N(J,h) = \frac{1}{N}\ln(Z_N(J,h)) \geq \overline{p}(M,J,h).$$

We observe that we can drive $\overline{p}$ to p_N computing its superior extremum:

$$p_N(J,h) \geq \sup_M \overline{p}(M,J,h).$$

Lower bound
Now we have to compute the opposite bound; we start observing that the magnetization can only take $N+1$ distinct values which belong to a set that we called S_N; we can pick $M \in S_N$.
Using the identity

$$\sum_M \delta_{m,M} = 1,$$

we can split the partition function into sums over configurations with constant magnetization in the following way:

$$Z_N(J,h) = \sum_\sigma \delta_{m,M} e^{-H_N(\overline{\sigma})}.$$

When $m = M$, we exactly have $(M-m)^2 = 0$, i.e. $m^2 = 2mM - M^2$. Plugging the latter equality into $Z_N(J,h)$ and using the fact that

$$\delta_{m,M} \leq 1$$

yelds

$$\begin{aligned}Z_N(J,h) &= \sum_M \sum_\sigma \delta_{m,M} e^{-\frac{JN}{2}M^2+JNmM+Nhm} \leq \\ &\leq \sum_M \sum_\sigma e^{-\frac{JN}{2}M^2+JNmM+Nhm} \leq \\ &\leq \sum_M e^{N\sup_M \overline{p}(M,J,h)} = \\ &= e^{N\sup_M \overline{p}(M,J,h)} \sum_M 1 = \\ &= e^{N\sup_M \overline{p}(M,J,h)} (N+1)\end{aligned}$$

The latter equality is due to the fact that $|S_N| = N+1$. Then we obtain:

$$\frac{1}{N}\ln(Z_N(J,h)) \leq \sup_M \overline{p}(M,J,h) + \frac{\ln(N+1)}{N}.$$

In conclusion:

$$\sup_M \overline{p}(M,J,h) \leq p_N(J,h) \leq \sup_M \overline{p}(M,J,h) + \frac{\ln(N+1)}{N},$$

Then, observing that

$$\frac{\ln(N+1)}{N} \xrightarrow{N\to\infty} 0,$$

we can say that the thermodynamic limit exists and it holds:

$$\lim_{N\to\infty} p_N(J,h) = \sup_M \overline{p}(M,J,h).$$

3.2.2 Exact solution of the thermodynamic limit

In this section we compute the exact solution of the thermodynamic limit. Using the expression of the Hamiltonian (3.5), we can write the partition function in the form

$$Z_N(J,h) = \int_{\mathbb{R}^2} e^{N\frac{J}{2}m^2+hm} d\nu_{m_N}(m)$$

where $d\nu_{m_N}(m)$ denotes the distribution of $m_N(\overline{\sigma})$ on $\left(\mathbb{R}^N, \prod_{i=1}^N \rho(\sigma_i)\right)$.

Remark 3.4. Since $J > 0$, the following identity holds:

$$\exp\left(\frac{NJ}{2}m^2\right) = \left(\frac{NJ}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left(NJ\left(xm - \frac{1}{2}x^2\right)\right) dx \tag{3.6}$$

Proof. We consider the r.h.s. of (3.6) and we apply a gaussian transform to the integral:

$$\begin{aligned}
&\int_{\mathbb{R}} \exp\left(NJ\left(xm - \frac{1}{2}x^2\right)\right) dx = \\
=&\int_{\mathbb{R}} \exp\left(-\frac{NJ}{2}\left(x-m\right)^2\right)\right) \exp\left(\frac{NJ}{2}m^2\right) dx = \\
=&\exp\left(\frac{NJ}{2}m^2\right) \int_{\mathbb{R}} \exp\left(-\frac{NJ}{2}\left(x-m\right)^2\right)\right) dx
\end{aligned}$$

Making the change of variable

$$y = \left(\frac{2}{NJ}\right)^{\frac{1}{2}} (x-m)$$

we can write:

$$\begin{aligned}
&\int_{\mathbb{R}} \exp\left(NJ\left(xm - \frac{1}{2}x^2\right)\right) dx = \\
=&\exp\left(\frac{NJ}{2}m^2\right) \int_{\mathbb{R}} \left(\frac{2}{NJ}\right)^{\frac{1}{2}} \exp(-y^2) dy = \\
=&\exp\left(\frac{NJ}{2}m^2\right) \left(\frac{2}{NJ}\right)^{\frac{1}{2}} (\pi)^{\frac{1}{2}}
\end{aligned}$$

Then

$$\exp\left(\frac{NJ}{2}m^2\right) = \left(\frac{NJ}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left(NJ\left(xm - \frac{1}{2}x^2\right)\right) dx$$

□

Using (3.6) we can write:

$$\begin{aligned}
Z_N(J,h) &= \left(\frac{NJ}{2\pi}\right)^{\frac{1}{2}} \int\int_{\mathbb{R}^2} \exp\left(NJ\left(xm - \frac{1}{2}x^2\right) + Nhm\right) d\nu_{m_N}(m) dx \\
&= \left(\frac{NJ}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{NJ}{2}x^2\right) \int_{\mathbb{R}} \exp\left(Nm(Jx+h) + Nhm\right) d\nu_{m_N}(m) dx
\end{aligned}$$

where, thank to the definition of m:

$$\int_{\mathbb{R}} \exp\left(Nm(Jx+h)+Nhm\right) d\nu_{m_N}(m) =$$
$$= \int_{\mathbb{R}^N} \exp\left(\sum_{i=1}^{N} \sigma_i(Jx+h)\right) \prod_{i=1}^{N} d\rho(\sigma_i) =$$
$$= \prod_{i=1}^{N} \int_{\mathbb{R}} \exp\left(\sum_{i=1}^{N} \sigma_i(Jx+h)\right) d\rho(\sigma_i).$$

Thus, considering the function

$$f(x) = -\frac{J}{2}x^2 + \ln\left(\int_{\mathbb{R}} \exp(s(Jx+h))d\rho(s)\right) \tag{3.7}$$

and integrating over the spins, we obtain:

$$Z_N(J,h) = \left(\frac{NJ}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp(Nf(x))dx.$$

Remark 3.5. The integral of the expression (3.7) is finite $\forall x \in \mathbb{R}$.

Proof. We show it using the condition (3.4) on the measure ρ.
In the integral

$$\int_{\mathbb{R}} \exp\left(\sum_{i=1}^{N} \sigma_i(Jx+h)\right) d\rho(\sigma_i)$$

we have the product of two exponentials with rispective arguments Jxs and hs.

- If $x > 0$, we choose $a = Jx$ and $b = h$, so that (3.4) becomes:

  $$\int_{\mathbb{R}} e^{\frac{Jxs^2}{2}+hs} d\rho(s) < \infty.$$

  Now we can observe that

  $$Jxs < \frac{Jxs^2}{2} \Longrightarrow s < 0 \vee s > 2$$

  and we can use it to obtain the inequality

  $$\int_{\mathbb{R}} e^{Jxs+hs} d\rho(s) <$$
  $$< \int_{-\infty}^{0} e^{\frac{Jxs^2}{2}+hs} d\rho(s) + \int_{2}^{\infty} e^{\frac{Jxs^2}{2}+hs} d\rho(s) + \int_{0}^{2} e^{Jxs+hs} d\rho(s) <$$
  $$< 2\int_{\mathbb{R}} e^{\frac{Jxs^2}{2}+hs} d\rho(s) + \int_{0}^{2} e^{Jxs+hs} d\rho(s) < \infty.$$

- If $x < 0$, we choose $a = J$ and $b = h$, so that (3.4) becomes:

$$\int_{\mathbb{R}} e^{\frac{Js^2}{2}+hs} d\rho(s) < \infty.$$

Now, in an analogous way, we can observe that

$$Jxs < \frac{Js^2}{2} \Longrightarrow s < 2x \vee s > 0$$

and we can use it to obtain the inequality

$$\begin{aligned}
&\int_{\mathbb{R}} e^{Jxs+hs} d\rho(s) < \\
&< \int_{-\infty}^{2x} e^{\frac{Js^2}{2}+hs} d\rho(s) + \int_{0}^{\infty} e^{\frac{Js^2}{2}+hs} d\rho(s) + \int_{2x}^{0} e^{Jxs+hs} d\rho(s) < \\
&< 2\int_{\mathbb{R}} e^{\frac{Js^2}{2}+hs} d\rho(s) + \int_{2x}^{0} e^{Jxs+hs} d\rho(s) < \infty.
\end{aligned}$$

□

We can state the following:

Proposition 3.2.1. *Let* $f(x) = -\frac{J}{2}x^2 + \ln\left(\int_{\mathbb{R}} \exp(s(Jx+h))d\rho(s)\right)$ *the function defined in* (3.7)*. Then:*

1. *$f(x)$ is a real analytic function and* $\lim_{|x|\to\infty} f(x) = -\infty$*;*
2. *$f(x)$ admits a finite number of global maximum points;*
3. *for any positive $N \in \mathbb{N}$*

$$\int_{\mathbb{R}} \exp(Nf(x))dx < \infty; \tag{3.8}$$

4. *if μ is a global maximum point of $f(x)$,*

$$\lim_{N\to\infty} \frac{1}{N} \ln \int_{\mathbb{R}} \exp(Nf(x))dx = f(\mu). \tag{3.9}$$

Proof. 1. If we consider complex z and $L > 0$ we have:

$$\left| \int_{\mathbb{R}} \exp(s(Jz+h))d\rho(s) \right| \leq$$
$$\leq \int_{\mathbb{R}} |\exp(s(Jz+h))| d\rho(s) =$$
$$= \int_{|s|\leq L} \exp(|s(Jz+h)|)d\rho(s) + \int_{|s|>L} \exp(|sJz|)\exp(hs)d\rho(s)$$

This expression has order $o\left(\exp\left(\frac{J|z|^2}{2}\right)\right)$; infact, we can observe that:

a.
$$\int_{|s|\leq L} \exp(|s(Jz+h)|)d\rho(s) \leq \rho([-L,L])\exp(L|Jz+h|) \quad (3.10)$$

because $\rho([-L,L])$ is the interval of integration and L is the maximum value taken by the variable s;

b. rembering that $(|z|-|s|)^2 = |z|^2 - 2|sz| + |s|^2 \geq 0$, we have:

$$\int_{|s|>L} \exp(|sJz|)\exp(hs)d\rho(s) \leq$$
$$\leq \int_{|s|>L} \exp\left(\frac{J}{2}(s^2+|z|^2)\right)\exp(hs)d\rho(s) =$$
$$= \exp\left(\frac{J}{2}(|z|^2)\right)\int_{|s|>L} \exp\left(\frac{J}{2}s^2\right)\exp(hs)d\rho(s) \quad (3.11)$$

By the inequalities (3.10) and (3.11) and the condition (3.4) on the measure ρ, we can say that the function f is real analytic and

$$\lim_{|x|\to\infty} f(x) = -\infty.$$

2. To prove the second statement we take a sequence $x_l \in \mathbb{R}$ such that

$$\lim_{l\to\infty} f(x_l) = \sup_{x\in\mathbb{R}} f(x) = L \leq \infty.$$

Since $\lim_{|x|\to\infty} f(x) = -\infty$, the sequence x_l is bounded. Thus we can take a subsequence x_{k_l} such that $\lim_{l\to\infty} x_{k_l} = x_0$.
Hence, by continuity of $f(x)$ we have

$$f(x_0) = \lim_{l\to\infty} f(x_{k_l}) = \sup_{x\in\mathbb{R}} f(x) = L.$$

Morover, since $\lim_{|x|\to\infty} f(x) = -\infty$, the point x_0 and other possible global maximum points must belong to a compact set. The analyticity of $f(x)$ ensures that, inside that set, the global maximum points are finite in number.

3. To prove the statement (3.8) we proceede by induction on N.
For $N = 1$ we have:

$$\begin{aligned}\int_{\mathbb{R}^2} \exp(f(x))dx &= \int\int_{\mathbb{R}^2} \exp\left(-\frac{J}{2}x^2 + s(Jx+h)\right) d\rho(s)dx = \\ &= \int\int_{\mathbb{R}^2} \exp\left(-\frac{J}{2}(x-s)^2\right) \exp\left(\frac{J}{2}s^2 + hs\right) d\rho(s)dx = \\ &= \left(\frac{2\pi}{J}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left(\frac{J}{2}s^2 + hs\right) d\rho(s) \qquad (3.12)\end{aligned}$$

The result (3.8) is proved for $N = 1$ because the condition (3.4) on the measure ρ (with $a = J$ and $b = h$) ensures that the integral on the l.h.s. of (3.12) is finite.
Now we suppose true the inductive hypotesis

$$\int_{\mathbb{R}} \exp((N-1)f(x))dx < \infty. \qquad (3.13)$$

Defined $F = \max\{f(x)|x \in \mathbb{R}\}$, we have:

$$\int_{\mathbb{R}} \exp(Nf(x))dx = e^F \int_{\mathbb{R}} \exp((N-1)f(x))dx < \infty$$

thank to the result of the second statement of this proposition and (3.13).

4. To prove the statement (3.9) we write

$$I_N = \int_{\mathbb{R}} \exp(N(f(x) - f(\mu)))dx.$$

This allows us to write

$$\int_{\mathbb{R}} \exp(N(f(x))dx = e^{Nf(\mu)} \int_{\mathbb{R}} \exp(N(f(x) - f(\mu)))dx = e^{Nf(\mu)} I_N.$$

It holds $f(x) - f(\mu) \leq 0$ because μ is a maximum point for the function f, so the integral I_N is a decreasing function of N and in particular

$$I_N \leq I_1, \quad i.e. \quad \ln(I_N) \leq \ln(I_1)$$

thank to the increasing monotonicity of the logarithm.
We can observe that:

$$\ln\left(\int e^{Nf(x)}dx\right) = \ln\left(e^{Nf(\mu)}I_N\right) = \ln\left(\int_{\mathbb{R}} \exp(N(f(x))dx\right) \leq$$
$$\leq \ln\left(e^{Nf(\mu)}I_1\right) = Nf(\mu) + \ln(I_1).$$

Hence, as $N \to \infty$, we obtain that

$$\frac{1}{N}\ln\left(\int e^{Nf(x)}dx\right) \leq \frac{1}{N}\left(Nf(\mu) + \ln(I_1)\right) =$$
$$= f(\mu) + \frac{1}{N}\ln(I_1) \longrightarrow f(\mu). \tag{3.14}$$

The function $f(x)$ is continuous, so, given any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that as $|x-\mu| < \delta_\epsilon$ we have that $f(x) - f(\mu) > -\epsilon$.
Thus, integrating the positive function

$$e^{N(f(x)-f(\mu))}$$

over $[\mu-\delta, \mu+\delta]$, we have:

$$I_N \geq \int_{\mu-\delta}^{\mu+\delta} \exp(N(f(x)-f(\mu)) >$$
$$> \int_{\mu-\delta}^{\mu+\delta} e^{-N\epsilon}dx = 2\delta_\epsilon e^{-N\epsilon}.$$

We can observe that:

$$\ln\left(\int e^{Nf(x)}dx\right) = \ln\left(e^{Nf(\mu)}I_N\right) = \ln\left(\int_{\mathbb{R}} \exp(N(f(x))dx\right) \geq$$
$$\geq \ln\left(e^{Nf(\mu)}2\delta_\epsilon e^{-N\epsilon}\right) = Nf(\mu) + \ln(2\delta_\epsilon) - N\epsilon. \tag{3.15}$$

Hence, as $N \to \infty$, we obtain that

$$\frac{1}{N}\ln\left(\int e^{Nf(x)}dx\right) \geq \frac{1}{N}\left(Nf(\mu) + \ln(2\delta_\epsilon) - N\epsilon\right) =$$
$$= f(\mu) + \frac{1}{N}\ln(2\delta_\epsilon) - \epsilon \longrightarrow f(\mu) - \epsilon.$$

Since ϵ is arbitrary, the statement (3.9) follows from inequalities (3.14) and (3.15):

$$\lim_{N\to\infty} \frac{1}{N}\ln\int_{\mathbb{R}} \exp(Nf(x))dx = f(\mu),$$

where μ is a global maximum point for the function $f(x)$.

□

Remark 3.6. The *proposition* (3.2.1) implies that in the thermodinamic limit

$$\begin{aligned}
\lim_{N\to\infty} p_N(J,h) &= \lim_{N\to\infty} \frac{1}{N}\ln(Z_N(J,h)) = \\
&= \lim_{N\to\infty} \frac{1}{N}\ln\left[\left(\frac{NJ}{2\pi}\right)^{\frac{1}{2}}\int_{\mathbb{R}} \exp(Nf(x))dx\right] = \\
&= \lim_{N\to\infty} \frac{1}{N}\left[\frac{1}{2}\ln\left(\frac{NJ}{2\pi}\right) + \ln\left(\int_{\mathbb{R}} \exp(Nf(x))dx\right)\right] = \\
&= \lim_{N\to\infty} \left[\frac{1}{2N}\ln\left(\frac{NJ}{2\pi}\right) + \frac{1}{N}\ln\left(\int_{\mathbb{R}} \exp(Nf(x))dx\right)\right] = \\
&= \max_{x\in\mathbb{R}} f(x).
\end{aligned}$$

The derivative of $f(x)$ with respect to x vanishes as

$$x = \frac{\displaystyle\int_{\mathbb{R}} s\exp\left(s(Jx+h)\right)d\rho(s)}{\displaystyle\int_{\mathbb{R}} \exp\left(s(Jx+h)\right)d\rho(s)}, \tag{3.16}$$

infact:

$$\frac{\partial f(x)}{\partial x} = -Jx + \frac{J\displaystyle\int_{\mathbb{R}} s\exp\left(s(Jx+h)\right)d\rho(s)}{\displaystyle\int_{\mathbb{R}} \exp\left(s(Jx+h)\right)d\rho(s)} = 0 \iff$$

x is determined by (3.16), condition satisfied by every maximum point of f.

Remark 3.7. Let μ be a global maximum point of f. If we differentiate the thermodynamic limit of p_N with respect to h, we obtain μ, the *magnetization of the sistem in the thermodynamic limit.*

$$\begin{aligned}
\frac{\partial}{\partial h}\left(\lim_{N\to\infty} p_N(J,h)\right) &= -J\mu\frac{\partial\mu}{\partial h} + \left(J\mu\frac{\partial\mu}{\partial h}+1\right)\frac{\displaystyle\int_{\mathbb{R}} s\exp\left(s(Jx+h)\right)d\rho(s)}{\displaystyle\int_{\mathbb{R}} \exp\left(s(Jx+h)\right)d\rho(s)} = \\
&= -J\mu\frac{\partial\mu}{\partial h} + -J\mu\frac{\partial\mu}{\partial h} + \mu = \\
&= \mu
\end{aligned}$$

3.3 Asymptotic behaviour of the sum of spins

After having defined the more general case of the mean-field model in the third chapter and after having showed that the sum of N spins with square-root normalization doesn't satisfy the hypothesis of the central limit theorem for interacting random variables when the model is defined by $J = 1$ and $h = 0$ in the previous section, we are going to compute exactly the limit for large N of the distribution of the normalized sum of spins.

3.3.1 Ellis, Newmann and Rosen's results

The study of the normalized sum of random variables and its asymptotic behaviour is a central chapter in probability and statistical mechanics. The central limit theorem ensures that, if those variables are independent, the sum with square-root normalization converges toward a Gaussian distribution. Spins whose interaction is described by the hamiltonian (3.1) and which have distribution (3.2) are not indipendent random variables, so that the central limit theorem can't help us to understand the behaviour of their sum

$$S_N(\overline{\sigma}) = \sum_{i=1}^{N} \sigma_i.$$

Ellis, Newmann and Rosen performed the generalization of the central limit theorem to this type of random variables. They found that the behaviour in the thermodynamic limit of the probability distribution of $S_N(\overline{\sigma})$ depends on the number and on the type of the maximum points of the functional f given by (3.7).

We can start clarifying the meaning of type of a maximum point.
Let $\mu_1, \dots, \mu_P$ the global maximum points of the function $f(x)$ defined in (3.7). For each p there exists a positive integer k_p and a negative real number λ_p such that around μ_p we can write:

$$f(x) = f(\mu_p) + \lambda_p \frac{(x - \mu_p)^{2k_p}}{(2k_p)!} + o((x - \mu_p)^{2k_p}).$$

The numbers k_p and λ_p are called, respectively, the *type* and the *strength* of the maximum point μ_p; we define *maximal type* the number k^*, which is the largest of the k_p.

Remark 3.8. If a point μ_p has homogeneous type equal to 1, around μ we have:

$$f(x) = f(\mu_p) + \frac{1}{2} f''(\mu_p)(x - \mu_p)^2 + o((x - \mu_p)^2).$$

Hence, in this case $\lambda_p = f''(\mu)$.

Define the function

$$B(x, y) = f(x + y) - f(y).$$

For each $p = 1, \ldots, P$ there exists $\delta_p > 0$ sufficiently small such that for $|x| < \delta_p N^{\frac{1}{2k}}$ as $N \to \infty$ we have:

$$\begin{cases} NB\left(\frac{x}{N^{\frac{1}{2k}}}; \mu_p\right) = \frac{\lambda}{(2k)!}x^{2k} + o(1)P_{2k}(x) \\ \\ NB\left(\frac{x}{N^{\frac{1}{2k}}}; \mu_p\right) \leq \frac{1}{2}\frac{\lambda}{(2k)!}x^{2k} + P_{2k+1}(x) \end{cases} \tag{3.17}$$

Infact:

a.

$$\begin{aligned} NB\left(\frac{x}{N^{\frac{1}{2k}}}; \mu_p\right) &= N\left[f\left(\frac{x}{N^{\frac{1}{2k}}} + \mu_p\right) - f(\mu_p)\right] = \\ &= N\left[f(\mu_p) + \lambda_p \frac{\left(\frac{x}{N^{\frac{1}{2k}}} + \mu_p - \mu_p\right)^{2k_p}}{(2k_p)!} - f(\mu_p)\right] = \\ &= \lambda_p \frac{x^{2k_p}}{(2k_p)!} \longrightarrow \frac{\lambda}{(2k)!}x^{2k} + o(1)P_{2k}(x) \quad as \quad N \to \infty \end{aligned}$$

b.

$$\begin{aligned} NB\left(\frac{x}{N^{\frac{1}{2k}}}; \mu_p\right) \longrightarrow & \frac{\lambda}{(2k)!}x^{2k} + o(1)P_{2k}(x) = \\ &= \frac{1}{2}\frac{\lambda}{(2k)!}x^{2k} + \frac{1}{2}\frac{\lambda}{(2k)!}x^{2k} + o(1)P_{2k}(x) \leq \\ &\leq \frac{1}{2}\frac{\lambda}{(2k)!}x^{2k} + P_{2k+1}(x) \quad as \quad N \to \infty \end{aligned}$$

where $P_{2k}(x)$ and $P_{2k+1}(x)$ are polynomial respectively of degree $2k$ and $2k + 1$.

Normalizing $S_N(\overline{\sigma})$ by the total number of spins we obtain the magnetization:

$$\frac{S_N(\overline{\sigma})}{N} = \frac{\sum_{i=1}^{N} \sigma_i}{N} = m_N(\overline{\sigma}).$$

Its behaviour in the thermodynamic limit is specified by the following

Theorem 3.3.1. *Let $\mu_1, \dots, \mu_P$ the global maximum points of the function $f(x)$ defined in (3.7). Let k^* the maximal type of the points. Let $\lambda_1, \dots, \lambda_P$ the strengths of the maximum points.*
Then, as $N \to \infty$,

$$m_N(\overline{\sigma}) \xrightarrow{\mathcal{D}} \frac{\displaystyle\sum_{p=1}^{P} b_p \delta(x-\mu_p)}{\displaystyle\sum_{p=1}^{P} b_p},$$

where $b_p = \lambda_p^{-\frac{1}{2k^*}}$.

Remark 3.9. We are now going to do some observations about the distribution respect to the number of maximum points; this results are proved in [ENR80].

a. If $f(x)$ admits only one global maximum point μ of maximal type, the limiting distribution is a delta picked in μ; in other words, the variance of the magnetization vanishes for large N.

b. If $f(x)$ admits more global maximum points of maximal type, this result holds only locally around each maximum point.

Thus it is important to determinate a suitable normalization of $S_N(\overline{\sigma})$ such that in the thermodymanic limit it converges to a well define random variable.

If $f(x)$ has a unique maximum point, the problem is solved by the following

Theorem 3.3.2. *Suppose that the function $f(x)$ given by (3.7) has a unique maximum point μ of type k and strength λ. Then*

$$\overline{S}_k(\overline{\sigma}) = \frac{S_N(\overline{\sigma}) - N\mu}{N^{1-\frac{1}{2k}}} \xrightarrow{\mathcal{D}} \begin{cases} \mathcal{N}\left(0, -\left(\dfrac{1}{\lambda} + \dfrac{1}{J}\right)\right) & if \quad k=1 \\ \exp\left(\dfrac{\lambda}{(2k)!} x^{2k}\right) & if \quad k>1 \end{cases}$$

where $-\left(\dfrac{1}{\lambda} + \dfrac{1}{J}\right) > 0$ *for* $k = 1$.

If $f(x)$ has more than one maximum point, the problem is solved by the following

Theorem 3.3.3. *Suppose that the function $f(x)$ given by (3.7) has more maximum points; let μ a nonunique maximum point of type k and strength λ. Then there exists $A > 0$ such that for all $a \in (0, A)$, if $m_N(\overline{\sigma}) \in [\mu - a, \mu + a]$ then*

$$\overline{S}_k(\overline{\sigma}) = \frac{S_N(\overline{\sigma}) - N\mu}{N^{1-\frac{1}{2k}}} \xrightarrow{\mathcal{D}} \begin{cases} \mathcal{N}\left(0, -\left(\frac{1}{\lambda} + \frac{1}{J}\right)\right) & if \quad k = 1 \\ \exp\left(\frac{\lambda}{(2k)!}x^{2k}\right) & if \quad k > 1 \end{cases}$$

where $-\left(\frac{1}{\lambda} + \frac{1}{J}\right) > 0$ *for* $k = 1$.

The result of *theorem* (3.3.3) is valid also for local maximum of the function $f(x)$.

We have to give some results before proving the theorems.
First of all it is useful to define the function

$$\Phi_\rho(x) = \frac{1}{J} \ln\left(\int_{\mathbb{R}} \exp(s(Jx + h))d\rho(s)\right). \tag{3.18}$$

Remark 3.10. $\Phi''_{\rho_x}(x) > 0 \quad \forall x \in \mathbb{R}$.

Proof. We start observing that

$$\begin{aligned} \Phi_\rho(x) &= \frac{1}{J}\left[-\frac{J}{2}x^2 + \ln\left(\int_{\mathbb{R}} \exp(s(Jx + h))\right) d\rho(s) +\right] + \frac{1}{2}x^2 = \\ &= \frac{1}{J}f(x) + \frac{1}{2}x^2. \end{aligned} \tag{3.19}$$

The first statement of *proposition* (3.2.1) ensures that the function $\Phi_\rho(x)$ is real analytic because it is sum of real analytic functions. Now, we consider the second derivative.

Firstly we compute the first derivative:

$$\begin{aligned}
\Phi_\rho'(x) &= \frac{1}{J} f'(x) + x = \\
&= \frac{1}{J}\left[-Jx + J\frac{\int_{\mathbb{R}} s\exp\left(s(Jx+h)\right) d\rho(s)}{\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)}\right] + x = \\
&= J\frac{\int_{\mathbb{R}} s\exp\left(s(Jx+h)\right) d\rho(s)}{\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)}
\end{aligned}$$

Now we compute the second derivative:

$$\begin{aligned}
\Phi_\rho''(x) = &\frac{J\left(\int_{\mathbb{R}} s^2\exp\left(s(Jx+h)\right) d\rho(s)\right)\left(\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)\right)}{\left(\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)\right)^2} + \\
&- \frac{J\left(\int_{\mathbb{R}} s\exp\left(s(Jx+h)\right) d\rho(s)\right)\left(\int_{\mathbb{R}} s\exp\left(s(Jx+h)\right) d\rho(s)\right)}{\left(\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)\right)^2} = \\
&= J\left(\frac{\int_{\mathbb{R}} s^2\exp\left(s(Jx+h)\right) d\rho(s)}{\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)} - \left(\frac{\int_{\mathbb{R}} s\exp\left(s(Jx+h)\right) d\rho(s)}{\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)}\right)^2\right)
\end{aligned}$$

We define a variable Y whose distribution is

$$\rho_x(s) = \frac{\exp\left(s(Jx+h)\right) d\rho(s)}{\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)} \tag{3.20}$$

so that we can write

$$\begin{aligned}
\Phi_\rho''(x) &= J\left(\int_{\mathbb{R}} s^2 d\rho_x(s) - \left(\int_{\mathbb{R}} s d\rho_x(s)\right)^2\right) = \\
&= JVar_{\rho_x}(Y).
\end{aligned}$$

Since ρ is a nondegenerate measure, by definition of variance of a random variable, $\Phi_{\rho_x}''(x) > 0 \quad \forall x \in \mathbb{R}$. □

The proofs of the *theorems* (3.3.1) and (3.3.2) also need the following preliminary results:

Lemma 3.3.4. *Suppose that for each $N \in \mathbb{N}$, X_N and Y_N are independent random variables such that $X_N \xrightarrow{\mathcal{D}} \nu$, where $\forall a \in \mathbb{R}$*

$$\int e^{iax} d\nu(x) \neq 0.$$

Then

$$Y_N \xrightarrow{\mathcal{D}} \mu \iff X_N + Y_N \xrightarrow{\mathcal{D}} \nu * \mu,$$

where $\nu * \mu$ indicates the convolution of two distribution, that is:

$$\nu * \mu = \int_{-\infty}^{\infty} \nu(x-t)\mu(t)dt.$$

Proof. Weak convergence of measures is equivalent to pointwise convergence of characteristic functions.
The characteristic function of a random variable identifies its density, so we can prove the lemma using characteristic functions. We define with $\psi_X(t)$ and $\psi_Y(t)$ respectively the characteristic function of X and the characteristic function of Y. It holds

$$\psi_{X+Y}(t) = \psi_X(t)\psi_Y(t),$$

infact:

$$\psi_{X+Y}(t) = E[e^{i(X+Y)t}] = E[e^{iXt}]E[e^{iYt}] = \psi_X(t)\psi_Y(t).$$

$$\begin{aligned}
&Y_N \xrightarrow{\mathcal{D}} \mu \\
\iff &\psi_Y(t) = \left(\int e^{iby} d\nu(y)\right) \\
\iff &\psi_X(t)\psi_Y(t) = \left(\int e^{iax} d\mu(x)\right)\left(\int e^{iby} d\nu(y)\right) = \\
&= \psi_{X+Y}(t) = \int e^{icx} d(\nu * \mu)(x) \\
\iff &X_N + Y_N \xrightarrow{\mathcal{D}} \nu * \mu
\end{aligned}$$

□

Lemma 3.3.5. *Suppose that the random variable* $W \sim \mathcal{N}\left(0, \frac{1}{J}\right)$ *is independent of* $S_N(\overline{\sigma})$ $N \geq 1$. *Then given* $\gamma \in \mathbb{R}$ *and* $m \in \mathbb{R}$, *the distribution of*

$$\frac{W}{N^{\frac{1}{2}-\gamma}} + \frac{S_N(\overline{\sigma}) - Nm}{N^{1-\gamma}}$$

is given by

$$\frac{\exp\left(Nf\left(\frac{s}{N^{\gamma}} + m\right)\right) ds}{\int_{\mathbb{R}} \exp\left(Nf\left(\frac{s}{N^{\gamma}} + m\right)\right) ds} \tag{3.21}$$

where the function f *is given by* (3.7).

Proof. Given $\theta \in \mathbb{R}$,

$$P\left\{\frac{W}{N^{\frac{1}{2}-\gamma}} + \frac{S_N(\overline{\sigma}) - Nm}{N^{1-\gamma}} \leq \theta\right\} = P\left\{\sqrt{N}W + S_N(\overline{\sigma}) \in E\right\}$$

where $E = (-\infty, \theta N^{1-\gamma} + Nm]$. The distribution of $\sqrt{N}W + S_N(\overline{\sigma})$ is given by the convolution of the Gaussian $\mathcal{N}\left(0, \frac{N}{J}\right)$ with the distribution of $S_N(\overline{\sigma})$

$$\frac{1}{Z_N(J,h)} \exp\left(\frac{J}{2N}s^2 + hs\right) d\nu_S(s)$$

where $d\nu_S(s)$ denotes the distribution of $S_N(\overline{\sigma})$ on $\left(\mathbb{R}^N, \prod_{i=1}^{N} \rho(\sigma_i)\right)$.

Thus we have

$$P\left\{\sqrt{N}W + S_N(\overline{\sigma}) \in E\right\} =$$
$$= \frac{1}{Z_N(J,h)} \left(\frac{J}{2\pi N}\right)^{\frac{1}{2}} \int_E \exp\left(-\frac{J}{2N}t^2\right) \int_{\mathbb{R}} \exp\left(s\left(\frac{J}{N}t + h\right)\right) d\nu_S(s)dt$$

where

$$\int_{\mathbb{R}} \exp\left(s\left(\frac{J}{N}t + h\right)\right) d\nu_S(s) =$$
$$= \int_{\mathbb{R}^N} \exp\left(\sum_{i=1}^{N} \sigma_i \left(\frac{J}{N}t + h\right)\right) \prod_{i=1}^{N} d\rho(\sigma_i) =$$
$$= \prod_{i=1}^{N} \int_{\mathbb{R}} \exp\left(\sum_{i=1}^{N} \sigma_i \left(\frac{J}{N}t + h\right)\right) d\rho(\sigma_i).$$

If we make the following change of variable

$$x = \frac{t - Nm}{N^{1-\gamma}}$$

and we integrate over the spins, we obtain:

$$\begin{aligned}
&P\left\{\sqrt{N}W + S_N(\overline{\sigma}) \in E\right\} = \\
=&\frac{1}{Z_N(J,h)}\left(\frac{J}{2\pi N}\right)^{\frac{1}{2}} \int_E \exp\left(-\frac{J}{2N}t^2\right)\left(\prod_{i=1}^N \int_{\mathbb{R}} \exp\left(\sum_{i=1}^N \sigma_i \left(\frac{J}{N}t + h\right)\right) d\rho(\sigma_i)\right) dt = \\
=&\frac{1}{Z_N(J,h)}\left(\frac{J}{2\pi N}\right)^{\frac{1}{2}} N^{1-\gamma} \int_{-\infty}^{\theta} \exp\left(-\frac{J}{2N}\left(xN^{1-\gamma} + Nm\right)^2\right) \times \\
&\times \prod_{i=1}^N \left(\int_{\mathbb{R}} \exp\left(\sigma_i\left(\frac{J}{N}\left(xN^{1-\gamma} + Nm\right) + h\right)\right)\right) d\rho(\sigma_i) dx = \\
=&\frac{1}{Z_N(J,h)}\left(\frac{J}{2\pi N}\right)^{\frac{1}{2}} N^{1-\gamma} \int_{-\infty}^{\theta} \exp\left(-\frac{J}{2}\left(\frac{x}{N^\gamma} + m\right)^2\right) \times \\
&\times \prod_{i=1}^N \left(\int_{\mathbb{R}} \exp\left(\sigma_i\left(J\left(\frac{x}{N^\gamma} + m\right) + h\right)\right)\right) d\rho(\sigma_i) dx.
\end{aligned}$$

Observe that

$$\int_{\mathbb{R}} \exp\left(s\left(\frac{J}{N}t + h\right)\right) d\nu_S(s) = \prod_{i=1}^N \int_{\mathbb{R}} \exp\left(\sigma_i\left(\frac{J}{N}t + h\right)\right) d\rho(\sigma_i).$$

Hence we obtain

$$\begin{aligned}
&P\left\{\sqrt{N}W + S_N(\overline{\sigma}) \in E\right\} = \\
=&\frac{1}{Z_N(J,h)}\left(\frac{J}{2\pi N}\right)^{\frac{1}{2}} N^{1-\gamma} \int_{-\infty}^{\theta} \exp\left[N\left(-\frac{J}{2}\left(\frac{x}{N^\gamma} + m\right)^2\right)\right] \times \\
&\times \exp\left[N \ln\left(\int_{\mathbb{R}} e^{s\left(J\left(\frac{x}{N^\gamma}+m\right)+h\right)} d\nu_S(s)\right)\right] dx = \\
=&\frac{1}{Z_N(J,h)}\left(\frac{J}{2\pi N}\right)^{\frac{1}{2}} N^{1-\gamma} \times \\
&\times \int_{-\infty}^{\theta} \exp\left[N\left(-\frac{J}{2}\left(\frac{x}{N^\gamma} + m\right)^2\right) + \ln\left(\int_{\mathbb{R}} \exp\left(s\left(J\left(\frac{x}{N^\gamma} + m\right) + h\right)\right) d\nu_S(s)\right)\right] dx = \\
=&\frac{1}{Z_N(J,h)}\left(\frac{JN^{1-2\gamma}}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\theta} \exp\left[Nf\left(\frac{x}{N^\gamma} + m\right)\right] dx. \qquad (3.22)
\end{aligned}$$

Taking $\theta \longrightarrow \infty$, the (3.22) gives an equation for $Z_N(J,h)$ which when substitued back yelds the distribution (3.21). The integral in (3.21) gives an equation for $Z_N(J,h)$ finite by (3.8). □

Remark 3.11. We remark that for $\gamma < \frac{1}{2}$, the random variable W does not contribute to the limit of the distribution (3.21) as $N \longrightarrow \infty$.

Lemma 3.3.6. *Defined* $F = \max\{f(x)|x \in \mathbb{R}\}$*, let* V *be any closed (possibly unbounded) subset of* $\mathbb{R}$ *which contains no global maxima of f(x).*
Then there exists $\epsilon > 0$ *so that*

$$e^{-NF} \int_V e^{Nf(x)} dx = O(e^{-N\epsilon}). \tag{3.23}$$

Proof. For hypothesis V contains no global maxima of $f(x)$, thus:

$$\sup_{x \in V} f(x) \leq \sup_{x \in \mathbb{R}} f(x) - \epsilon = F - \epsilon.$$

Hence:

$$\begin{aligned} e^{-NF} \int_V e^{Nf(x)} dx &\leq e^{-NF} e^{(N-1)\sup_{x\in V} f(x)} \int_V e^{f(x)} dx \leq \\ &\leq e^{-NF} e^{(N-1)(F-\epsilon)} \int_V e^{f(x)} dx \leq \\ &\leq e^{-NF} e^{(N+1)(F-\epsilon)} \int_{\mathbb{R}} e^{f(x)} dx = \\ &= e^{-NF} e^{N(F-\epsilon)} \left(e^{F-\epsilon} \int_{\mathbb{R}} e^{f(x)} dx \right) = \end{aligned}$$

by definition of $f(x)$ in (3.7),

$$\begin{aligned} &= e^{-N\epsilon} \left(e^{F-\epsilon} \int_{\mathbb{R}} \exp\left(-\frac{J}{2} x^2 + \ln \left(\int_{\mathbb{R}} \exp(s(Jx+h)) d\rho(s) \right) \right) dx \right) = \\ &= e^{-N\epsilon} \left(e^{F-\epsilon} \int_{\mathbb{R}} \exp\left(-\frac{J}{2} x^2 \right) \left(\int_{\mathbb{R}} \exp(s(Jx+h)) d\rho(s) \right) dx = \right. \\ &= e^{-N\epsilon} \left(e^{F-\epsilon} \left(\frac{2\pi}{J} \right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left(\frac{J}{2} x^2 + hx \right) dx \right) \end{aligned} \tag{3.24}$$

The condition (3.3) on the measure ρ (with $a = J$ and $(b = h)$) assures that the latter passage of (3.24) is $O(e^{-N\epsilon})$ as $N \longrightarrow \infty$.
This proved the (3.23). □

At last we can proceed with the proof of the *theorem* (3.3.1).

Proof. By definition $m_N(\overline{\sigma}) = \dfrac{S_N(\overline{\sigma})}{N}$.
Thus, by *lemmas* (3.3.4) and (3.3.5), we know that

$$\frac{W}{N^{\frac{1}{2}-\gamma}} + \frac{S_N(\overline{\sigma}) - Nm}{N^{1-\gamma}} \sim \frac{\exp\left(Nf\left(\frac{x}{N^\gamma} + m\right)\right)dx}{\int_{\mathbb{R}} \exp\left(Nf\left(\frac{x}{N^\gamma} + m\right)\right)dx},$$

hence with $\gamma = 0$ and $m = 0$:

$$\frac{W}{N^{\frac{1}{2}}} + \frac{S_N(\overline{\sigma})}{N} \sim \frac{\exp\left(Nf(x)\right)ds}{\int_{\mathbb{R}} \exp\left(Nf(x)\right)dx}$$

where $W \sim \mathcal{N}\left(0, \dfrac{1}{J}\right)$. We have to prove that for any bounded continuous function $\phi(x)$

$$\frac{\int_{\mathbb{R}} e^{Nf(x)}\phi(x)dx}{\int_{\mathbb{R}} e^{Nf(x)}dx} \longrightarrow \frac{\sum_{p=1}^{P} \phi(\mu_p)b_p}{\sum_{p=1}^{P} b_p}. \tag{3.25}$$

Consider $\delta_1, \ldots, \delta_P$ such that the conditions expressed in (3.17) are satisfied, i.e. for $|x| < \delta_p N^{\frac{1}{2k}}$ as $N \to \infty$ we must have:

$$\begin{cases} NB\left(\dfrac{x}{N^{\frac{1}{2k}}}; \mu_p\right) = \dfrac{\lambda}{(2k)!}x^{2k} + o(1)P_{2k}(x) \\ \\ NB\left(\dfrac{x}{N^{\frac{1}{2k}}}; \mu_p\right) \leq \dfrac{1}{2}\dfrac{\lambda}{(2k)!}x^{2k} + P_{2k+1}(x) \end{cases}$$

We choose $\overline{\delta} = \min\{\delta_p | p = 1, \ldots, P\}$, decreasing it, if necessary, to assure that

$$0 < \overline{\delta} < \min\{|\mu_p - \mu_q| : 1 \leq p \neq q \leq P\}.$$

We denote by V the closet set

$$V = \mathbb{R} - \bigcup_{p=1}^{P} (\mu_p - \overline{\delta}, \mu_p + \overline{\delta}).$$

By (3.3.6) there exists $\epsilon > 0$ such that as $N \to \infty$

$$e^{-NF} \int_V e^{Nf(x)} \phi(x) dx = O(e^{-N\epsilon}). \tag{3.26}$$

For each $p = 1, \dots, P$, making the change of variable

$$x = u + \mu_p,$$

we have

$$\begin{aligned} &N^{\frac{1}{2k^*}} e^{-NF} \int_{\mu_p - \overline{\delta}}^{\mu_p + \overline{\delta}} e^{Nf(x)} \phi(x) dx = \\ =&N^{\frac{1}{2k^*}} e^{-Nf(\mu_p)} \int_{\mu_p - \overline{\delta}}^{\mu_p + \overline{\delta}} e^{Nf(x)} \phi(x) dx = \\ =&N^{\frac{1}{2k^*}} \int_{-\overline{\delta}}^{\overline{\delta}} e^{Nf(u+\mu_p)} e^{-Nf(\mu_p)} \phi(u + \mu_p) du = \\ =&N^{\frac{1}{2k^*}} \int_{-\overline{\delta}}^{\overline{\delta}} \exp\left(NB(u, \mu_p)\right) \phi(u + \mu_p) du. \end{aligned}$$

Making the change of variable

$$w = uN^{\frac{1}{2k^*}},$$

the latter equation becomes:

$$\begin{aligned} &N^{\frac{1}{2k^*}} \int_{|w| < \overline{\delta} N^{\frac{1}{2k^*}}} \exp\left(NB\left(\frac{w}{N^{\frac{1}{2k^*}}}, \mu_p\right)\right) \phi\left(\frac{w}{N^{\frac{1}{2k^*}}} + \mu_p\right) \frac{dw}{N^{\frac{1}{2k^*}}} = \\ =& \int_{|w| < \overline{\delta} N^{\frac{1}{2k^*}}} \exp\left(NB\left(\frac{w}{N^{\frac{1}{2k^*}}}, \mu_p\right)\right) \phi\left(\frac{w}{N^{\frac{1}{2k^*}}} + \mu_p\right) dw. \end{aligned}$$

Thus by (3.17) and the dominated convergence theorem we have:

$$\begin{aligned} &\lim_{N\to\infty} N^{\frac{1}{2k^*}} e^{-NF} \int_{\mu_p - \overline{\delta}}^{\mu_p + \overline{\delta}} e^{Nf(x)} \phi(x) dx = \\ =& \lim_{N\to\infty} \int_{|w| < \overline{\delta} N^{\frac{1}{2k^*}}} \exp\left(NB\left(\frac{w}{N^{\frac{1}{2k^*}}}, \mu_p\right)\right) \phi\left(\frac{w}{N^{\frac{1}{2k^*}}} + \mu_p\right) dw = \\ =& \int_{\mathbb{R}} \exp\left(NB(0, \mu_p)\right) \phi(0 + \mu_p) dw = \\ =&\phi(\mu_p) \int_{\mathbb{R}} \exp\left(\frac{\lambda_p}{(2k^*)!} w^{2k^*}\right) dw \end{aligned} \tag{3.27}$$

Since $\lambda_p < 0$, the integral (3.27) is finite.
Making the change of variable

$$x = w(-\lambda_p)^{\frac{1}{2k^*}}$$

we obtain that:

$$\lim_{N\to\infty} N^{\frac{1}{2k^*}} e^{-NF} \int_{\mu_p-\bar{\delta}}^{\mu_p+\bar{\delta}} e^{Nf(x)}\phi(x)dx =$$
$$= \frac{\phi(\mu_p)}{(-\lambda_p)^{\frac{1}{2k^*}}} \int_{\mathbb{R}} \exp\left(-\frac{x^{2k^*}}{(2k^*)!}\right) dx. \tag{3.28}$$

By (3.26) and (3.28)

$$\lim_{N\to\infty} N^{\frac{1}{2k^*}} e^{-NF} \int_{\mathbb{R}} e^{Nf(x)}\phi(x)dx =$$
$$= \sum_{p=1}^{P} \frac{\phi(\mu_p)}{(-\lambda_p)^{\frac{1}{2k^*}}} \int_{\mathbb{R}} \exp\left(-\frac{x^{2k^*}}{(2k^*)!}\right) dx. \tag{3.29}$$

In a similar way, for the denominator we have:

$$\lim_{N\to\infty} N^{\frac{1}{2k^*}} e^{-NF} \int_{\mathbb{R}} e^{Nf(x)}\phi(x)dx =$$
$$= \sum_{p=1}^{P} \frac{1}{(-\lambda_p)^{\frac{1}{2k^*}}} \int_{\mathbb{R}} \exp\left(-\frac{x^{2k^*}}{(2k^*)!}\right) dx. \tag{3.30}$$

By (3.29) and (3.30) we have the statement (3.25):

$$\frac{\int_{\mathbb{R}} e^{Nf(x)}\phi(x)dx}{\int_{\mathbb{R}} e^{Nf(x)}dx} \xrightarrow{N\to\infty} \frac{\sum_{p=1}^{P} \frac{\phi(\mu_p)}{(-\lambda_p)^{\frac{1}{2k^*}}} \int_{\mathbb{R}} \exp\left(-\frac{x^{2k^*}}{(2k^*)!}\right) dx}{\sum_{p=1}^{P} \frac{1}{(-\lambda_p)^{\frac{1}{2k^*}}} \int_{\mathbb{R}} \exp\left(-\frac{x^{2k^*}}{(2k^*)!}\right) dx} = \frac{\sum_{p=1}^{P} \phi(\mu_p) b_p}{\sum_{p=1}^{P} b_p}.$$

□

Finally we prove the *theorem* (3.3.2).

Proof. As in the proof of *theorem* (3.3.1), by *lemmas* (3.3.6) and (3.3.7) with $\gamma = \dfrac{1}{2k}$ and $m = \mu$, we have:

$$\frac{W}{N^{\frac{1}{2}-\frac{1}{2k}}} + \frac{S_N(\overline{\sigma}) - N\mu}{N^{1-\frac{1}{2k}}} \sim \frac{\exp\left(Nf\left(\frac{x}{N^{\frac{1}{2k}}} + \mu\right)\right) dx}{\int_{\mathbb{R}} \exp\left(Nf\left(\frac{x}{N^{\frac{1}{2k}}} + \mu\right)\right) dx}$$

where $W \sim \mathcal{N}\left(0, \frac{1}{J}\right)$. If $k > 1$,
we have to prove that for any bounded continuous function $\phi(x) : \mathbb{R} \longrightarrow \mathbb{R}$

$$\frac{\int_{\mathbb{R}} \exp\left(Nf\left(\frac{x}{N^{\frac{1}{2k}}} + \mu\right)\right)\phi(x)dx}{\int_{\mathbb{R}} \exp\left(Nf\left(\frac{x}{N^{\frac{1}{2k}}} + \mu\right)\right)dx} \xrightarrow{N\to\infty} \frac{\int_{\mathbb{R}} \exp\left(\frac{\lambda}{(2k)!}x^{2k}\right)\phi(x)dx}{\int_{\mathbb{R}} \exp\left(\frac{\lambda}{(2k)!}x^{2k}\right)dx}. \tag{3.31}$$

We pick $\delta > 0$ such that it satisfies the conditions (3.17).
By *lemma* (3.3.6) there exists $\epsilon > 0$ so that

$$\lim_{N\to\infty} e^{-NF} \int_{|x|\geq \delta N^{\frac{1}{2k}}} \exp\left(Nf\left(\frac{x}{N^{\frac{1}{2k}}} + \mu\right)\right)\phi(x)dx = O(N^{\frac{1}{2k}}e^{-N\epsilon}) \tag{3.32}$$

where $F = \max\{f(x)|x \in \mathbb{R}\}$.
On the other hand, as $|x| < \delta N^{\frac{1}{2k}}$

$$\begin{aligned}
&e^{-NF} \int_{|x|<\delta N^{\frac{1}{2k}}} \exp\left(Nf\left(\frac{x}{N^{\frac{1}{2k}}} + \mu\right)\right)\phi(x)dx = \\
=&e^{-N(F-f(\mu))} \int_{|x|<\delta N^{\frac{1}{2k}}} \exp\left(Nf\left(\frac{x}{N^{\frac{1}{2k}}} + \mu\right) - Nf(\mu)\right)\phi(x)dx = \\
=&e^{-N(F-f(\mu))} \int_{|x|<\delta N^{\frac{1}{2k}}} \exp\left(NB\left(\frac{x}{N^{\frac{1}{2k}}}, \mu\right)\right)\phi(x)dx \\
=&\int_{|x|<\delta N^{\frac{1}{2k}}} \exp\left(NB\left(\frac{x}{N^{\frac{1}{2k}}}, \mu\right)\right)\phi(x)dx
\end{aligned}$$

By (3.17) and the dominate convergence theorem we have that:

$$\begin{aligned}
&\lim_{N\to\infty} e^{-Nf(\mu)} \int_{|x|<\delta N^{\frac{1}{2k}}} \exp\left(Nf\left(\frac{x}{N^{\frac{1}{2k}}} + \mu\right)\right)\phi(x)dx = \\
=&\lim_{N\to\infty} \int_{|x|<\delta N^{\frac{1}{2k}}} \exp\left(NB\left(\frac{x}{N^{\frac{1}{2k}}}, \mu\right)\right)\phi(x)dx = \\
=&\int_{\mathbb{R}} \exp\left(\frac{\lambda}{(2k)!}x^{2k}\right)\phi(x)dx
\end{aligned} \tag{3.33}$$

where the integral of the r.h.s. is finite because $\lambda < 0$.
By (3.32) and (3.33), the statement (3.31) follows for $k > 1$.
If $k = 1$,
we obtain in an analogous way that for any bounded continuous function

$\phi : \mathbb{R} \to \mathbb{R}$:

$$\frac{\int_{\mathbb{R}} \exp\left(Nf\left(\frac{x}{\sqrt{N}}\mu\right)\right)\phi(x)dx}{\int_{\mathbb{R}} \exp\left(Nf\left(\frac{x}{\sqrt{N}}+\mu\right)\right)dx} \xrightarrow{N\to\infty} \frac{\int_{\mathbb{R}} \exp\left(\frac{\lambda}{2}x^2\right)\phi(x)dx}{\int_{\mathbb{R}} \exp\left(\frac{\lambda}{2}x^2\right)dx}.$$

Both the random variables W and $W + \overline{S}_1(\overline{\sigma})$ have Gaussian distribution:

- $W \sim \mathcal{N}\left(0, \frac{1}{J}\right)$ by hypothesis,
- $W+\overline{S}_1(\overline{\sigma}) \sim \mathcal{N}\left(0, -\frac{1}{\lambda}\right)$ by the convolution of the limiting distribution of the random variables W and $\overline{S}_1(\overline{\sigma})$.

Hence the random variable $\overline{S}_1(\overline{\sigma})$ as $N \to \infty$ has to converge to a Gaussian whose covariance is $-\left(\frac{1}{\lambda}+\frac{1}{J}\right)$.
Indicated with $\psi_W(t)$, $\psi_{\overline{S}_1}$ and $\psi_{W+\overline{S}_1}$ respectively the characteristic functions of $W, \overline{S}_1$ and their sum, the following inequality holds:

$$\psi_{W+\overline{S}_1} = \psi_W(t)\psi_{\overline{S}_1}(t).$$

We can write:

$$\psi_{\overline{S}_1} = \psi_{(W+\overline{S}_1)-W}(t) = \frac{\psi_{W+\overline{S}_1}(t)}{\psi_W(t)} = \frac{e^{-\frac{x^2}{\lambda}}}{e^{\frac{x^2}{J}}} = e^{-\left(\frac{1}{\lambda}+\frac{1}{J}\right)x^2}$$

Remark 3.12. To complete the proof, we must check that

$$-\left(\frac{1}{\lambda}+\frac{1}{J}\right) = -\frac{1}{\lambda} - \frac{1}{J} = \frac{\lambda + J}{-\lambda J} > 0. \tag{3.34}$$

When $k = 1$, $\lambda = f''(\mu)$ as we have seen before in *remark* (3.8).
The denominator is positive because $J > 0$ by hypothesis and $f''(\mu) < 0$ because μ is a maximum point.
Now we consider the numerator. We have just computed the first derivative of the function $f(x)$, we are now going to compute the second derivative of

the function $g(x) = f(x) + \frac{J}{2}x^2$.

$$g''(x) = \frac{\left(\int_{\mathbb{R}} s^2 \exp\left(s(Jx+h)\right) d\rho(s)\right)\left(\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)\right)}{\left(\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)\right)^2} +$$

$$- \frac{\left(\int_{\mathbb{R}} s \exp\left(s(Jx+h)\right) d\rho(s)\right)\left(\int_{\mathbb{R}} s \exp\left(s(Jx+h)\right) d\rho(s)\right)}{\left(\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)\right)^2} =$$

$$= \frac{\int_{\mathbb{R}} s^2 \exp\left(s(Jx+h)\right) d\rho(s)}{\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)} - \left(\frac{\int_{\mathbb{R}} s \exp\left(s(Jx+h)\right) d\rho(s)}{\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)}\right)^2$$

We define a variable Y whose distribution is

$$\rho_x(s) = \frac{\exp\left(s(Jx+h)\right) d\rho(s)}{\int_{\mathbb{R}} \exp\left(s(Jx+h)\right) d\rho(s)}$$

so that we can write

$$g''(x) = \int_{\mathbb{R}} s^2 d\rho_x(s) - \left(\int_{\mathbb{R}} s d\rho_x(s)\right)^2 = Var_{\rho_x}(Y) > 0 \quad \forall x \in \mathbb{R}$$

since ρ is a nondegenerate measure and by definition of variance of a random variable. So at the numerator we have

$$\lambda + J = f''(\mu) + J = g''(\mu) > 0.$$

□

To prove *theorem* (3.3.3) it's useful to consider the Legendre transformation of the function ϕ_ρ defined in (3.18):

$$\phi_\rho^*(y) = \sup_{x \in \mathbb{R}} \{xy - \phi_\rho(x)\}. \tag{3.35}$$

We claim that it's possible to define the function ϕ_ρ^* because $\phi_\rho''(x) > 0$ $\forall x \in \mathbb{R}$.

Lemma 3.3.7. *Let* ϕ_ρ^* *the function defined in* (3.35) *and* μ *a maximum point of the function* $f(x)$ *given by* (3.7). *Then:*

1. *There exists an open (possibily unbounded) interval I containing μ such that $\phi_\rho^*$is finite, real analytic and convex (with $(\phi_\rho^*)''(x) > 0$) on I and $\phi_\rho^* = +\infty$ on I^C.*

2. *Consider the random variable $U_N(\overline{\sigma}) = m_N(\overline{\sigma}) - \mu$. Denote by ν_U its distribution on $(\mathbb{R}^N, \prod_{i=1}^N \rho_\mu(\sigma_i))$ where ρ_μ is given by (3.20) with $x = \mu$. For any $u > 0$:*

$$P\{U_N(\overline{\sigma}) > u\} \leq \exp\left(-NJ(\phi_\rho^*(\mu+u) - \phi_\rho^*(\mu)) - (\phi_\rho^*)'(\mu)u\right). \quad (3.36)$$

3. *There exists a number $u_0 > 0$ such that $\forall u \in (0, u_0)$*

$$(\phi_\rho^*)'(\mu+u) - (\phi_\rho^*)'(\mu) = u + \xi(u) \quad \xi(u) > 0. \quad (3.37)$$

Proof. 1. Since $\phi_\rho''(x) > 0 \quad \forall x \in \mathbb{R}$, the function ϕ_ρ' is strictly increasing and hence admits inverse (ϕ_ρ'). By (3.35) the function ϕ_ρ^* is bounded if and only if there is a point $x_0 \in \mathbb{R}$ such that $y = \phi_\rho'(x_0)$: infact, computing the first derivative of ϕ_ρ^* respect to x, we find:

$$\frac{d}{dx}(xy - \phi_\rho(x)) = y - \phi_\rho'(x) = 0 \iff \exists \quad x_0 \in \mathbb{R}: \quad y = \phi_\rho'(x_0).$$

This condition is verified when y belongs to the image of ϕ_ρ'. In this case we have:

$$\begin{cases} \phi_\rho^*(y) & = yx_0 - \phi_\rho(x_0) \\ (\phi_\rho^*)'(y) & = \dfrac{d}{dx}\sup\limits_{x\in\mathbb{R}}\{xy - \phi_\rho(x)\} = \sup\limits_{x\in\mathbb{R}}\{y - \phi_\rho'(x)\} \Longrightarrow \\ & \Longrightarrow y = \phi_\rho'(x) \iff x = (\phi_\rho')^{-1}(y) \iff (\phi_\rho^*)'(y) = (\phi_\rho')^{-1}(y) \\ (\phi_\rho^*)''(y) & = \dfrac{d}{dx}(\phi_\rho')^{-1}(x) = \dfrac{1}{\phi_\rho''(x_0)} \end{cases} \quad (3.38)$$

Thus ϕ_ρ^* is real analytic and convex, in particular with $(\phi_\rho^*)''(y) > 0$. By (3.19) and (3.16) we have $\phi_\rho'(\mu) = \frac{1}{J}f'(\mu) + \mu = \mu$; hence μ is inside the image of ϕ_ρ'. On the other hand, for y in the complement of the closure of the image of ϕ_ρ', we have $\phi_\rho^*(y) = +\infty$. This shows that the first sentence of the lemma is proved taken I equal to the image of ϕ_ρ'

2. Let ν be any measure on $\mathcal{B}$. Choosing $Jy + h > 0$, by the monotonicity of the exponential we have the equality:

$$\begin{aligned} &P\left\{\sum_{i=1}^{N} x_i > N\omega\right\} = \\ =&P\left\{\exp\left(\sum_{i=1}^{N} x_i\beta(Jy+h)\right) > \exp\left(N\omega\beta(Jy+h)\right)\right\}. \end{aligned}$$

Using the exponential Chebychev's inequality and the definition of expectation we can write:

$$\begin{aligned} &P\left\{\sum_{i=1}^{N} x_i > N\omega\right\} \leq \\ \leq&\frac{E\left[\exp\left(x_i(Jy+h)\right)\right]^N}{\exp\left(N\omega(Jy+h)\right)} = \\ =&\exp\left(-N\omega(Jy+h)\right)\prod_{i=1}^{N}\int_{\mathbb{R}}\exp\left(x_i(Jy+h)\right)d\nu(x_i) \leq \\ \leq&\exp\left(-Nh\omega - NJ\left(\omega y - \frac{1}{J}\int_{\mathbb{R}}\exp\left(x_i(Jy+h)\right)d\nu(x_i)\right)\right) \leq \\ \leq&\exp\left(-Nh\omega - NJ\sup\left\{\omega y - \phi_\nu(y) | Jy+h > 0\right\}\right) \end{aligned}$$

where ϕ_ν is given by (3.18) with $\rho = \nu$ and $E[\cdot]$ denotes the expectation value with respect to the measure ρ. By convexity of the function ϕ_ν, whenever $\omega > \int_{\mathbb{R}} x d_{\nu(x)}$, the superior value of $\{\omega y - \phi_\nu(y) | y \in \mathbb{R}\}$ is reached for $Jy + h > 0$. This shows that:

$$P\left\{\sum_{i=1}^{N} x_i > N\omega\right\} \leq \exp\left(-Nh\omega - NJ\phi_\nu^*(\omega)\right)$$

whenever $\omega > \int_{\mathbb{R}} x d_{\nu(x)}$.

Since μ is a maximum point, of the function $f(x)$, by the condition (3.16) and the definition of the measure ρ_μ in (3.20) with $x = \mu$, thus:

$$\int_{\mathbb{R}} x d\rho_\mu(x) = \frac{\displaystyle\int_{\mathbb{R}} x\exp\left(x(J\mu+h)\right)d\rho(x)}{\displaystyle\int_{\mathbb{R}}\exp\left(x(J\mu+h)\right)d\rho_\mu(x)} = \mu < \mu + u.$$

Thus

$$
\begin{aligned}
P\{U_N(\overline{\sigma}) > u\} &= P\{S_N(\overline{\sigma}) > N(\mu+u)\} \leq \\
&\leq \exp\left(-Nh(\mu+u) - NJ\phi^*_{\rho_\mu}(\mu+u)\right),
\end{aligned}
$$

where by definition of ϕ_ρ and ϕ^*_ρ:

$$
\phi^*_{\rho_\mu}(\mu+u) = \sup_{y\in\mathbb{R}} \left\{ (\mu+u)y - \frac{1}{J}\ln\int_{\mathbb{R}} \exp\left(s(Jy+h)\right) d\rho_\mu(s) \right\}.
$$

Adding and subtracting the term $\mu + \dfrac{h}{J}$, the last expression becomes:

$$
\begin{aligned}
=&\sup_{y\in\mathbb{R}} \left\{ (\mu+u)\left(y+\mu+\frac{h}{J}\right) - \frac{1}{J}\ln\int_{\mathbb{R}} \exp\left(s\left(J\left(y+\mu+\frac{h}{J}\right)+h\right)\right) d\rho_\mu(s) \right\} + \\
&+ \sup_{y\in\mathbb{R}} \left\{ -(\mu+u)\left(\mu+\frac{h}{J}\right) + \frac{1}{J}\ln\int_{\mathbb{R}} \exp\left(s\left(J\mu+h\right)\right) d\rho_\mu(s) \right\} = \\
=&\sup_{y\in\mathbb{R}} \left\{ (\mu+u)\left(y+\mu+\frac{h}{J}\right) - \phi_\rho\left(y+\mu+\frac{h}{J}\right) \right\} + \\
&- \sup_{y\in\mathbb{R}} \left\{ (\mu+u)\left(\mu+\frac{h}{J}\right) - \phi_\rho\left(\mu+\frac{h}{J}\right) \right\} = \\
=&\phi^*_\rho(\mu+u) - \mu^2 - \mu u - \frac{h}{J}(\mu+u) + \phi_\rho(\mu).
\end{aligned}
$$

Since $(\phi'_\rho)^{-1}(\mu) = \mu$, by (3.38) we have:

$$
\begin{cases}
\phi^*_\rho(\mu) = \mu^2 - \phi_\rho(\mu) \\
(\phi^*_\rho)'(\mu) = \mu.
\end{cases}
$$

Thus

$$
\begin{aligned}
P\{U_N(\overline{\sigma}) > u\} \leq& \exp(-Nh(\mu+u) - NJ(\phi^*_\rho(\mu+u) - \phi^*_\rho(\mu) + \\
&- (\phi^*_\rho)'(\mu)u - \frac{h}{J}(\mu+u)) = \\
=& \exp(-NJ\left(\phi^*_\rho(\mu+u) - \phi^*_\rho(\mu) - (\phi^*_\rho)'(\mu)u\right) \quad (3.39)
\end{aligned}
$$

This proves the statement (3.36).

3. Since μ is a maximum point of $f(x)$, there exists $u_0 > 0$ such that $x > (\phi_\rho)'(x)$ as $x \in (\mu, \mu+u_0)$. Thus:

$$
(\phi^*_\rho)'(\mu+u) > \mu+u
$$

is true for any $u \in (0, u_0)$. Since $(\phi_\rho^*)'(\mu) = \mu$, the statement (3.37) is proved.

□

Lemma 3.3.8 (Transfer Principle). *Let ν_U be the distribution of the random variable $U_N(\overline{\sigma}) = m_N(\overline{\sigma}) - \mu$ on $\left(\mathbb{R}^N, \prod_{i=1}^N d\rho_\mu(\sigma_i)\right)$. There exists $\hat{B} > 0$ only depending on ρ such that for each $B \in (0, \hat{B})$ and for each $a \in (0, \frac{B}{2})$ and each $r \in \mathbb{R}$, there exists $\overline{\delta} = \overline{\delta}(a, B) > 0$ such that as $N \to \infty$:*

$$\int_{\mathbb{R}} \exp\left(irN^\gamma w - \frac{NJ}{2}w^2\right) \int_{|u|\le a} \exp(NJuw)d\nu_U(u)dw =$$
$$= \int_{|w|\le B} \exp\left(irN^\gamma w - \frac{NJ}{2}w^2\right) \int_{\mathbb{R}} \exp(NJuw)d\nu_U(u)dw + O(e^{-N\delta}).$$

Proof. We shall find $\hat{B} > 0$ such that for each $B \in (0, \hat{B})$ and each $a \in (0, \frac{B}{2})$, there exists $\overline{\delta} = \overline{\delta}(a, B) > 0$ such that as $N \longrightarrow \infty$

$$\int_{|w|>B} \exp\left(-\frac{NJ}{2}w^2\right) \int_{|u|\le a} \exp(NJuw)d\nu_U(u)dw = O(e^{-N\delta}) \tag{3.40}$$

and

$$\int_{|w|\le B} \exp\left(-\frac{NJ}{2}w^2\right) \int_{|u|>a} \exp(NJuw)d\nu_U(u)dw = O(e^{-N\delta}). \tag{3.41}$$

We start by equality (3.40).
For any $B > 0$ and $a \in (0, \frac{B}{2})$ we have:

$$\int_{|w|>B} \exp\left(-\frac{NJ}{2}w^2\right) \int_{|u|\le a} \exp(NJuw)d\nu_U(u)dw \le$$
$$\le 2\int_B^\infty \exp\left(-NJ\left(\frac{w^2}{2} - aw\right)\right) dw \le$$
$$\le 2\int_B^\infty \exp\left(-NJw\left(\frac{B}{2} - a\right)\right) dw \tag{3.42}$$

since we can use a as upper bound for u and B as lower bound for B. As $N \to \infty$, the latter integral (3.42) is $O(e^{-N\overline{\delta_1}})$, with $\overline{\delta_1} = B(\frac{B}{2} - a)$; thus the equality (3.40) is proved.

Now we proceede with the equality (3.41).
In the proof of the identity we exploit the following result:

$$E\left[Y\mathbb{I}_{\{a\leq Y\leq b\}}\right]\leq aP(Y\geq a)+\int_a^b P(Y\geq t)dt \tag{3.43}$$

where:

- Y is a random variable whose distribution is given by ρ_Y
- $E[\cdot]$ denotes the expectation value with respect to the distribution ρ_Y
- $\mathbb{I}_{\{a\leq Y\leq b\}}$ is the indicator function of the set $\{a\leq Y\leq b\}$.

The inequality (3.43) is obtained integrating by parts the l.h.s. of the following:

$$\begin{aligned}\int_a^b P(Y\geq t)dt &= bP(Y\geq b)-aP(Y\geq a)-\int_a^b tP(Y\geq t)'dt=\\ &= bP(Y\geq b)-aP(Y\geq a)+\int_a^b t\rho_Y(t)dt=\\ &= bP(Y\geq b)-aP(Y\geq a)+E\left[Y\mathbb{I}_{\{a\leq Y\leq b\}}\right];\end{aligned}$$

then

$$\begin{aligned}E\left[Y\mathbb{I}_{\{a\leq Y\leq b\}}\right] &= -bP(Y\geq b)+aP(Y\geq a)+\int_a^b P(Y\geq t)dt\leq\\ &\leq aP(Y\geq a)+\int_a^b P(Y\geq t)dt\end{aligned}$$

The l.h.s. of the equality (3.41) is upper bounded by

$$2B\sup_{|w|\leq B}\int_{|u|>a}\exp\left(-NJ\left(\frac{w^2}{2}-uw\right)\right)d\nu_U(u). \tag{3.44}$$

The integral in (3.44) breaks up into one over $(a,+\infty)$ and another over $(-\infty,a)$. For the first, using (3.43), we obtain:

$$\begin{aligned}&\sup_{|w|\leq B}\int_a^{+\infty}\exp\left(-NJ\left(\frac{w^2}{2}-uw\right)\right)d\nu_U(u)\leq\\ &\leq\sup_{|w|\leq B}\exp\left(-NJ\left(\frac{w^2}{2}-wa\right)\right)P\{U_N(\overline{\sigma})>a\}+\\ &+JNB\sup_{|w|\leq B}\int_a^{+\infty}\exp\left(-NJ\left(\frac{w^2}{2}-uw\right)\right)P\{U_N(\overline{\sigma})>u\}du \qquad (3.45)\end{aligned}$$

By (3.36) we can bound $P\{U_N(\overline{\sigma}) > u\}$, where $u \geq a$ with

$$P\{U_N(\overline{\sigma}) > u\} \leq \exp(-NJ\left(\phi_\rho^*(\mu+u) - \phi_\rho^*(\mu) - (\phi_\rho^*)'(\mu)u\right).$$

In particular for $u \geq a$, for the third statement of the *lemma* (3.3.7), it holds:

$$\phi_\rho^*(\mu+u) - \phi_\rho^*(\mu) - (\phi_\rho^*)'(\mu)u \geq \begin{cases} u^2 + \theta_1 & for \quad a \leq u \leq u_0 \\ u\theta_2 & for \quad u > u_0 \end{cases} \tag{3.46}$$

where $\theta_1 = \int_0^a \xi(t)dt > 0$ and $\theta_2 = \dfrac{\xi\left(\frac{u_0}{2}\right)}{2}$.
We consider an interval I such that *lemma* (3.3.7) is verified.
For all $\mu + u \in \overline{I}^C$, the (3.46) holds since $\phi_\rho^*(\mu+u) = \infty$.
For $\mu + u \in \overline{I}$, if $a \leq u \leq u_0$, by the fundamental theorem of calculus and by (3.37) we have:

$$\begin{aligned}\phi_\rho^*(\mu+u) - \phi_\rho^*(\mu) - (\phi_\rho^*)'(\mu)u &= \int_0^u \left[(\phi_\rho^*)'(\mu+t) - (\phi_\rho^*)'(\mu)\right] dt = \\ &= \int_0^u \left[t + \xi(t)\right] dt;\end{aligned}$$

integrating this expression we obtain:

$$\phi_\rho^*(\mu+u) - \phi_\rho^*(\mu) - (\phi_\rho^*)'(\mu)u = \frac{u^2}{2} + \int_0^u \xi(t)dt \geq \frac{u^2}{2} + \theta_1.$$

This proves the first line of (3.46).
If $u > u_0$, for $\dfrac{u_0}{2} \leq t \leq u$, since ϕ_ρ^* is monotonically increasing, we have:

$$\begin{aligned}(\phi_\rho^*)'(\mu+t) - (\phi_\rho^*)'(\mu) &\geq (\phi_\rho^*)'\left(\mu + \frac{u_0}{2}\right) - (\phi_\rho^*)'(\mu) = \\ &= \frac{u_0}{2} + \xi\left(\frac{u_0}{2}\right) \geq \\ &\geq \xi\left(\frac{u_0}{2}\right).\end{aligned}$$

Thus, if $u \geq u_0$

$$\begin{aligned}\int_0^u \left[(\phi_\rho^*)'(\mu+t) - (\phi_\rho^*)'(\mu)\right] dt &\geq \int_{u_0/2}^u \left[(\phi_\rho^*)'(\mu+t) - (\phi_\rho^*)'(\mu)\right] dt \geq \\ &\geq \left(u - \frac{u_0}{2}\right)\xi\left(\frac{u_0}{2}\right) \geq \\ &\geq u\theta_2.\end{aligned}$$

This proves the second line of (3.46).
Choose $\hat{B}$ such that $0 < \hat{B} < \theta_2$, for any $B \in (0, \hat{B})$. The integral in (3.45) breaks up into one over (a, u_0) and another over $(u_0, +\infty)$. Using (3.36) and (3.46) we have:

$$\begin{aligned} JNB \sup_{|w|\leq B} \int_a^{+\infty} \exp\left(-NJ\left(\frac{w^2}{2} - uw\right)\right) P\{U_N(\overline{\sigma}) > u\} du \leq \\ \leq JNB \sup_{|w|\leq B} \int_a^{u_0} \exp\left(-NJ\left(\frac{w^2}{2} - uw + \frac{u^2}{2} + \theta_1\right)\right) du + \\ + JNB \sup_{|w|\leq B} \int_{u_0}^{+\infty} \exp\left(-NJ\left(\frac{w^2}{2} - uw + \theta_2\right)\right) du. \end{aligned}$$

Since

$$\begin{aligned} \int_a^{u_0} \exp\left(-NJ\left(\frac{w^2}{2} - uw + \frac{u^2}{2} + \theta_1\right)\right) du = \\ = e^{-NJ\theta_1} \int_a^{u_0} \exp\left(-\frac{NJ}{2}\left(w^2 - 2uw + u^2\right)\right) du = \\ = e^{-NJ\theta_1} \int_a^{u_0} \exp\left(-\frac{NJ}{2}(w-u)^2\right) du \end{aligned}$$

and

$$\int_{u_0}^{+\infty} \exp\left(-N\left(\frac{J}{2}w^2 - Juw + Ju\theta_2\right)\right) du = \frac{\exp\left(-NJ\left(\frac{w^2}{2} + u_0(\theta_2 - w)\right)\right)}{NJ(\theta_2 - w)}.$$

We obtain:

$$\begin{aligned} JNB \sup_{|w|\leq B} \int_a^{+\infty} \exp\left(-NJ\left(\frac{w^2}{2} - uw\right)\right) P\{U_N(\overline{\sigma}) > u\} du = \\ = O\left(Ne^{-N\theta_1}\right) + O\left(e^{-Nu_0(\theta_2 - B)}\right). \end{aligned}$$

Thus the last line of (3.45) is $O\left(Ne^{-N\overline{\delta}_2}\right)$ where $\overline{\delta}_2 = \min\left\{\frac{\theta_1}{2}, u_0(\theta_2 - B)\right\}$.
Concerning the term of (3.45) involving $P\{U_N(\overline{\sigma}) > a\}$ we have

$$\begin{aligned} \sup_{|w|\leq B} \exp\left(-NJ\left(\frac{w^2}{2} - wa\right)\right) P\{U_N(\overline{\sigma}) > a\} \leq \\ \leq \sup_{|w|\leq B} \exp\left(-NJ\left(\frac{w^2}{2} - wa + \frac{a^2}{2} + \theta_1\right)\right) P\{U_N(\overline{\sigma}) > a\} = O\left(e^{-N\theta_1}\right). \end{aligned}$$

The integral over $(-\infty, a)$ is handled in the same way. Thus we have proved identities (3.40) and (3.41) with $\overline{\delta} = \min\{\overline{\delta_1}, \overline{\delta_2}\}$. □

Now we can prove the *theorem* (3.3.3).

Proof. If $k > 1$, to prove the statement, we must find $A > 0$ such that for each $r \in \mathbb{R}$ and any $a \in (0, A)$ when the magnetization $m_N(\overline{\sigma})$ is inside $[\mu - a, \mu + a]$, the Gibbs value of the characteristic function of the random variable $\overline{S}_k(\overline{\sigma})$:

$$\left\langle e^{ir\overline{S}_k(\overline{\sigma})} \Big| |m_N(\overline{\sigma}) - \mu| \leq a \right\rangle_{BG} = \frac{\displaystyle\int_{|m_N(\overline{\sigma})-\mu|\leq a} e^{ir\overline{S}_k(\overline{\sigma})} e^{-H_N(\overline{\sigma})} \prod_{i=1}^{N} d\rho(\sigma_i)}{\displaystyle\int_{|m_N(\overline{\sigma})-\mu|\leq a} e^{-H_N(\overline{\sigma})} \prod_{i=1}^{N} d\rho(\sigma_i)} \tag{3.47}$$

tends as $N \longrightarrow \infty$ to

$$\frac{\displaystyle\int_{\mathbb{R}} \exp(irs) \exp\left(\frac{\lambda}{(2k)!} s^{2k}\right) ds}{\displaystyle\int_{\mathbb{R}} \exp\left(\frac{\lambda}{(2k)!} s^{2k}\right) ds}. \tag{3.48}$$

Defining

$$\tilde{H_N}(\overline{\sigma}) = -\frac{J}{2} \left(\frac{S_N(\overline{\sigma}) - N\mu}{\sqrt{N}}\right)^2$$

we can write (3.47) as

$$\left\langle e^{ir\overline{S}_k(\overline{\sigma})} \Big| |m_N(\overline{\sigma}) - \mu| \leq a \right\rangle_{BG} = \frac{\displaystyle\int_{|m_N(\overline{\sigma})-\mu|\leq a} e^{ir\overline{S}_k(\overline{\sigma})} e^{-\tilde{H}_N(\overline{\sigma})} \prod_{i=1}^{N} d\rho_\mu(\sigma_i)}{\displaystyle\int_{|m_N(\overline{\sigma})-\mu|\leq a} e^{-\tilde{H}_N(\overline{\sigma})} \prod_{i=1}^{N} d\rho_\mu(\sigma_i)}$$

where ρ_μ is the function defined by (3.20) with $x = \mu$.
Consider the variable

$$U_N(\overline{\sigma}) = \frac{S_N(\overline{\sigma}) - N\mu}{N}$$

and let ν_U be its distribution on $\left(\mathbb{R}^N, \prod_{i=1}^N d\rho_\mu(\sigma_i)\right)$. Making the change of variable, since

$$\overline{S}_k(\overline{\sigma}) = \frac{S_N(\overline{\sigma}) - N\mu}{N^{1-\frac{1}{2k}}} = U_N(\overline{\sigma}) N^{\frac{1}{2k}} = U_N(\overline{\sigma}) N^{\gamma},$$

we can write:

$$\left\langle e^{ir\overline{S}_k(\overline{\sigma})} \middle| |m_N(\overline{\sigma}) - \mu| \leq a \right\rangle_{BG} = \frac{\displaystyle\int_{u\leq a} \exp(irN^{\gamma}u) \exp\left(\frac{NJ}{2}u^2\right) d\nu_U(u) \prod_{i=1}^{N} d\rho(\sigma_i)}{\displaystyle\int_{u\leq a} e^{-H_N(\overline{\sigma})} \prod_{i=1}^{N} d\rho(\sigma_i)}. \tag{3.49}$$

By identity (3.6), we have that

$$\exp\left(\frac{NJ}{2}u^2\right) = \left(\sqrt{\frac{NJ}{2\pi}}\right) \int_{\mathbb{R}} \exp\left(-\frac{NJ}{2}x^2 + NJxu\right) dx;$$

considering $m = u$ and after the simplification of the term $\sqrt{\frac{NJ}{2\pi}}$, the r.h.s. of (3.49) becomes

$$\frac{\displaystyle\int_{|u|\leq a} \exp(irN^{\gamma}u) \int_{\mathbb{R}} \exp\left(-\frac{NJ}{2}x^2 + NJxu\right) d\nu_U(u)dx}{\displaystyle\int_{\{|u|\leq a\}\times\mathbb{R}} \exp\left(-\frac{NJ}{2}x^2 + NJxu\right) d\nu_U(u)dx}. \tag{3.50}$$

Making the change of variable

$$w = x + \frac{ir}{JN^{1-\gamma}}, \tag{3.51}$$

the (3.50) becomes:

$$\frac{\exp\left(\frac{r^2}{2JN^{1-2\gamma}}\right) \displaystyle\int_{\mathbb{R}} \exp\left(irN^{\gamma}w - \frac{NJ}{2}w^2\right) \int_{|u|\leq a} \exp\left(NJwu\right) d\nu_U(u)dw}{\displaystyle\int_{\mathbb{R}} \exp\left(\frac{NJ}{2}w^2\right) \int_{|u|\leq a} \exp\left(NJwu\right) d\nu_U(u)dw}. \tag{3.52}$$

The change of variable (3.51) is justified by the analiticity of the integrand in (3.52) as function of w complex and the rapid decrease of this integrand to 0 as $|Re(w) \to \infty|$ and $|Im(w)| \leq |r|N^{\gamma}$. Since $k > 1$, we have that

$$\exp\left(\frac{r^2}{2JN^{1-2\gamma}}\right) \to 1 \; as \; N \to \infty,$$

hence we can neglect this term for the rest of the proof.
Using the *Transfer principle 3.1.8* we can find $\hat{B} > 0$ such that (3.53) can be written as

$$\frac{\exp\left(\frac{r^2}{2JN^{1-2\gamma}}\right)\int_{|w|\le\hat{B}}\exp\left(irN^{\gamma}w-\frac{NJ}{2}w^2\right)\int_{\mathbb{R}}\exp\left(NJwu\right)d\nu_U(u)dw}{\int_{|w|\le\hat{B}}\exp\left(\frac{NJ}{2}w^2\right)\int_{\mathbb{R}}\exp\left(NJwu\right)d\nu_U(u)dw}+ \\ +O(e^{-N\overline{\delta}}). \tag{3.53}$$

Making the change of variable $s = N^{\gamma}w$ and picking $\overline{B} = \min\{\delta, \hat{B}\}$, where δ is taken such that the conditions (3.17) are verified, we have for (3.53):

$$\frac{\exp\left(\frac{r^2}{2JN^{1-2\gamma}}\right)\int_{|s|\le\hat{B}N^{\gamma}}\exp\left(irs-JN^{1-2\gamma}\frac{s^2}{2}\right)\int_{\mathbb{R}}\exp\left(JN^{1-\gamma}us\right)d\nu_U(u)ds}{\int_{|s|\le\hat{B}N^{\gamma}}\exp\left(-JN^{1-2\gamma}\frac{s^2}{2}\right)\int_{\mathbb{R}}\exp\left(JN^{1-\gamma}us\right)d\nu_U(u)ds}+ \\ +O(e^{-N\overline{\delta}}). \tag{3.54}$$

We remember that

$$U_N(\overline{\sigma}) = \frac{S_N(\overline{\sigma}) - N\mu}{N};$$

hence we can write:

$$\begin{aligned}
&\int_{\mathbb{R}}\exp\left(JN^{1-\gamma}us\right)d\nu_U(u) = \\
&=\int_{\mathbb{R}^N}\exp\left(\frac{J}{N^{\gamma}}s(S_N(\overline{\sigma})-N\mu)\right)\prod_{i=1}^{N}d\rho(\sigma_i) = \\
&=\frac{\int_{\mathbb{R}^N}\exp\left(\frac{J}{N^{\gamma}}s(S_N(\overline{\sigma})-N\mu)\right)\exp\left(S_N(\overline{\sigma})(J\mu+h)\right)\prod_{i=1}^{N}d\rho(\sigma_i)}{\int_{\mathbb{R}^N}\exp\left(S_N(\overline{\sigma})(J\mu+h)\right)\prod_{i=1}^{N}d\rho(\sigma_i)} = \\
&=\frac{\int_{\mathbb{R}^N}\exp\left[S_N(\overline{\sigma})\left(J\left(\frac{s}{N^{\gamma}}+\mu\right)+h\right)\right]\prod_{i=1}^{N}d\rho(\sigma_i)}{\int_{\mathbb{R}^N}\exp\left(NJ\mu\frac{s}{N^{\gamma}}\right)\prod_{i=1}^{N}d\rho(\sigma_i)\int_{\mathbb{R}^N}\exp\left(S_N(\overline{\sigma})(J\mu+h)\right)\prod_{i=1}^{N}d\rho(\sigma_i)} = \\
&=\exp\left(NJ\left(\phi\left(\mu+\frac{s}{N^{\gamma}}\right)-\phi(\mu)-\mu\frac{s}{N^{\gamma}}\right)\right).
\end{aligned}$$

The latter equality is obtained as follows. We rememeber the definition of $\phi(x)$:

$$\phi(x) = \frac{1}{J} \ln \int_{\mathbb{R}} \exp\left(s(J\mu + h)\right) d\rho(s).$$

Thus we have:

- $$\int_{\mathbb{R}^N} \exp\left[S_N(\overline{\sigma})\left(J\left(\frac{s}{N^\gamma} + \mu\right) + h\right)\right] \prod_{i=1}^N d\rho(\sigma_i) =$$
$$= \exp\left(NJ\left(\frac{1}{J} \ln \int_{\mathbb{R}} \exp\left[s\left(J\left(\frac{s}{N^\gamma} + \mu\right) + h\right)\right] d\rho(s)\right)\right) =$$
$$= \exp\left(NJ\phi\left(\frac{s}{N^\gamma} + \mu\right)\right)$$

- $$\int_{\mathbb{R}^N} \exp\left(NJ\mu\frac{s}{N^\gamma}\right) \prod_{i=1}^N d\rho(\sigma_i) = \exp\left(\frac{NJ\mu}{N^\gamma}\right) \int_{\mathbb{R}^N} s \prod_{i=1}^N d\rho(\sigma_i) = \exp\left(\frac{NJ\mu s}{N^\gamma}\right)$$

- $$\int_{\mathbb{R}^N} \exp\left(S_N(\overline{\sigma})(J\mu + h)\right) \prod_{i=1}^N d\rho(\sigma_i) =$$
$$= \exp\left(NJ\left(\frac{1}{J} \ln \int_{\mathbb{R}} \exp\left[s\left(J\mu + h\right)\right] d\rho(s)\right)\right) =$$
$$= \exp\left(NJ\phi(\mu)\right)$$

Thus:

$$\exp\left(-JN^{1-2\gamma}\frac{s^2}{2}\right) \int_{\mathbb{R}} \exp(JN^{1-\gamma}us) d\nu_U(u) =$$
$$= \exp\left(NJ\left(-\frac{s^2}{2N^{2\gamma}} + \phi\left(\mu + \frac{s}{N^\gamma}\right) - \phi(\mu) - \mu\frac{s}{N^\gamma}\right)\right) =$$

by the definition of the function $\phi(x)$:

$$= \exp\left[N\left(f\left(\mu + \frac{s}{N^\gamma}\right) - f(\mu)\right)\right] =$$

by the definition of the function $NB(x, \mu)$:

$$= \exp\left(NB\left(\frac{s}{N^\gamma}, \mu\right)\right).$$

By condition expressed in (3.17) and the dominated convergence theorem the statement follows.
If $k = 1$ in an analogous way we obtain that

$$\overline{S}_1(\overline{\sigma}) \sim \mathcal{N}\left(0, -\left(\frac{1}{\lambda} + \frac{1}{J}\right)\right)$$

(as for the *theorem* (3.3.2) we can show that $-\left(\frac{1}{\lambda} + \frac{1}{J}\right) > 0$).
Thus we have that:

$$\overline{S}_k(\overline{\sigma}) = \frac{S_N(\overline{\sigma}) - N\mu}{N^{1-\frac{1}{2k}}} \xrightarrow{\mathcal{D}} \begin{cases} \mathcal{N}\left(0, -\left(\frac{1}{\lambda} + \frac{1}{J}\right)\right) & if \quad k = 1 \\ \exp\left(\frac{\lambda}{(2k)!}x^{2k}\right) & if \quad k > 1 \end{cases}$$

□

3.3.2 Example: the Curie-Weiss model

Now we describe the Curie-Weiss model, that is defined by Hamiltonian (3.1) and distribution (3.2) where ρ is given by

$$\rho(x) = \frac{1}{2}\left(\delta(x-1) + \delta(x+1)\right).$$

For further arguments related to this model see [Ell05].
The definition of ρ implies that the space of all configurations is

$$\Omega_N = \{-1, +1\}^N.$$

The function f given by (3.7) becomes

$$f(x) = -\frac{J}{2}x^2 + \ln\cosh(Jx + h) \tag{3.55}$$

whose estremality condition is given by the so called mean-field equation

$$\mu = \tanh(J\mu + h). \tag{3.56}$$

The solutions of this equations are the intersections between the hyperbolic tangent $y = \tanh(J\mu + h)$ and the line $y = \mu$.
As $h \neq 0$, for any positive value of J, the equation (3.56) can admit one

solution (in this case the point is the unique maximum of the function f) or more (in this second case only one of them is a maximum point of the function f). In both the cases the maximum μ_h has the same sign as the field h and it holds $\mu_h \neq 0$.
On the other hand, as $h = 0$, the number of solutions of equations (3.56) depends on the slope J of the the hyperbolic tangent.

a. if $J \leq 1$, there is a unique solution, the origin, which is the maximum point of the function f.

b. if $J > 1$, the equation (3.56) admits other two solutions $\pm\mu_0$.

To determinate the type and the strength of the maximum points of f as parameters J and h, we compute the even derivatives of f in the points until we obtain a value different from 0.

$$\begin{aligned}
f'(x) &= -\,Jx + J\tanh(Jx+h)\\
f''(x) &= -\,J + J^2(1-\tanh^2(Jx+h)) = -J(1-J(1-\tanh^2(Jx+h)))\\
f'''(x) &= -\,2J^3\tanh(Jx+h)(1-\tanh^2(Jx+h))\\
f^{(iv)}(x) &= -\,2J^4(1-\tanh^2(Jx+h))(1-3\tanh^2(Jx+h))
\end{aligned}$$

obtain:

a. if $h \neq 0$ and $J > 0$,
the maximum point μ_h is of type $k = 1$ and strength $\lambda = -J(1-J(1-\mu_h^2))$:

$$\begin{aligned}
f'(\mu_h) &= -\,J(\mu + \tanh(J\mu + h)) = 0\\
f''(\mu_h) &= -\,J + J^2(1-\tanh^2(J\mu+h)) = -J(1-J(1-\mu_h^2));
\end{aligned}$$

b. if $h = 0$ and $J < 1$,
the maximum point 0 is of type $k = 1$ and strength $\lambda = -J(1-J)$:

$$\begin{aligned}
f'(0) &= -\,J(0 + \tanh(J\cdot 0)) = 0\\
f''(0) &= -\,J + J^2(1-\tanh^2(0)) = -J(1-J);
\end{aligned}$$

c. if $h = 0$ and $J > 1$,
maximum points $\pm\mu_0$ is of type $k = 1$ and strength $\lambda = -J(1-J(1-\mu_0^2))$:

$$\begin{aligned}
f'(\mu_0) &= -\,J(\mu_0 + \tanh(J\mu_0)) = 0\\
f''(\mu_0) &= -\,J + J^2(1-\tanh^2(\mu_0)) = -J(1-J(1-\mu_0^2));
\end{aligned}$$

d. if $h = 0$ and $J = 1$,
the maximum point 0 is of type $k = 2$ and strength $\lambda = -2$:

$$\begin{aligned} f'(0) &= -\left(0 + \tanh(0)\right) = 0 \\ f''(0) &= -1 + 1 - \tanh^2(0) = 0 \\ f'''(0) &= -2\tanh(0)(1 - \tanh^2(0)) = 0 \\ f^{(iv)}(0) &= -2(1 - \tanh^2(0))(1 - 3\tanh^2(0)) = -2. \end{aligned}$$

By *theorem* (3.3.1) we get the distribution in the thermodynamic limit of the magnetization:

$$m_N(\overline{\sigma}) \xrightarrow{\mathcal{D}} \begin{cases} \delta(x - \mu_h) & h \neq 0, J > 0 \\ \delta(x) & h = 0, J \leq 1 \\ \frac{1}{2}\delta(x - \mu_0) + \frac{1}{2}\delta(x + \mu_0) & h = 0, J > 1 \end{cases}$$

We define the susceptibility of the model as

$$\chi = \frac{\partial \mu}{\partial h} = \left(1 - \tanh^2(J\mu + h)\right)\left(J\frac{\partial \mu}{\partial h} + 1\right) = \left(1 - \mu^2\right)\left(J\chi + 1\right),$$

by the mean-field equation (3.55) we obtain

$$\chi = \frac{1 - \mu^2}{1 - J(1 - \mu^2)}.$$

By the *theorem* (3.3.2) it's easy to check that in the thermodynamic limit

$$\begin{aligned} \frac{S_N(\overline{\sigma}) - N\mu}{\sqrt{N}} &\xrightarrow{\mathcal{D}} \mathcal{N}(0,\chi) \quad \textit{as} \quad J > 0 \quad \textit{and} \quad h \neq 0 \\ \frac{S_N(\overline{\sigma})}{\sqrt{N}} &\xrightarrow{\mathcal{D}} \mathcal{N}(0,\chi) \quad \textit{as} \quad 0 < J < 1 \quad \textit{and} \quad h = 0 \\ \frac{S_N(\overline{\sigma})}{N^{\frac{3}{4}}} &\xrightarrow{\mathcal{D}} \frac{\exp\left(-\frac{x^4}{12}\right)dx}{\int_{\mathbb{R}} \exp\left(-\frac{x^4}{12}\right)dx} \quad \textit{as} \quad J = 1 \quad \textit{and} \quad h = 0. \end{aligned}$$

If $J > 1$ and $h = 0$, the function f admits two global maximum points $\pm\mu_0$. Considering the point μ_0, by the *theorem* (3.3.3) there exists $A > 0$ such that $\forall a \in (0, A)$, if $m_N(\overline{\sigma}) \in [\mu_0 - a, \mu_0 + a]$

$$\frac{S_N(\overline{\sigma}) - N\mu_0}{\sqrt{N}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\chi).$$

An analogous result holds for the point $-\mu_0$.

To complete the description of the Curie-Weiss model we analyze its phase transition. A phase transition point is any point of non-analyticity of the thermodynamic limit of the pressure occurring for real h and/or real positive J.
If $h \neq 0$ it is easy to show that there is not any phase transition, infact:

$$\lim_{N\to\infty} p_N(J,h) = -\frac{J}{2}\mu_h^2 + \ln\cosh(J\mu_h + h),$$

which is a differentiable function.
Differentiating this limit with respect to J we obtain:

$$\begin{aligned}\frac{\partial}{\partial J}\left(\lim_{N\to\infty} p_N(J,h)\right) &= -\frac{1}{2}\mu_h^2 - J\mu_h\frac{\partial\mu_h}{\partial J} + \tanh(J\mu_h + h)\left(\mu_h + J\frac{\partial\mu_h}{\partial J}\right) = \\ &= -\frac{1}{2}\mu_h^2 - J\mu_h\frac{\partial\mu_h}{\partial J} + \mu_h^2 + J\mu_h\frac{\partial\mu_h}{\partial J} = \\ &= \frac{1}{2}\mu_h^2.\end{aligned}$$

Differentiating this limit with respect to J a second time we obtain:

$$\begin{aligned}\frac{\partial^2}{\partial J^2}\left(-\frac{J}{2}\mu_h^2 + \ln\cosh(J\mu_h + h)\right) &= \frac{\partial}{\partial J}\frac{1}{2}\mu_h^2 = \\ &= \mu_h\frac{\partial\mu_h}{\partial J} = \frac{1}{2}\frac{\partial\mu_h^2}{\partial J}.\end{aligned}$$

On the other hand, defferentiating this limit with respect to h a first time we obtain:

$$\frac{\partial}{\partial h}\left(-\frac{J}{2}\mu_h^2 + \ln\cosh(J\mu_h + h)\right) = \mu_h.$$

On the other hand, defferentiating this limit with respect to h a second time we obtain:

$$\frac{\partial^2}{\partial h^2}\left(-\frac{J}{2}\mu_h^2 + \ln\cosh(J\mu_h + h)\right) = \frac{\partial}{\partial h}\mu_h = \chi.$$

The hyperbolic tangent is an analytic function, thus, if $h \neq 0$, for any values of J we don't have any phase transition.
The situation is totally different as $h = 0$. In absence of the field h we have:

$$\lim_{N\to\infty} p_N(J,0) = \begin{cases} 0 & \text{when} \quad J \leq 1 \\ -\frac{J}{2}\mu_0^2 + \ln\cosh(J\mu_0) & \text{when} \quad J > 1 \end{cases}$$

As $J \to 1^+$ the spontaneous magnetization $\mu_0 \to 0$, thus the limit of the pressure is continuous for every values of J.
Thus, in an analogous way, in 0 the magnetic field

$$\frac{\partial}{\partial J}\left(\lim_{N\to\infty} p_N(J,0)\right) = \begin{cases} 0 & when \quad J \le 1 \\ -\frac{J}{2}\mu_0^2 & when \quad J > 1 \end{cases}$$

Also this function is continuous in J. If we differentiate another time the limit of the pressure we get:

$$\frac{\partial^2}{\partial J^2}\left(\lim_{N\to\infty} p_N(J,0)\right) = \mu\frac{\partial \mu}{\partial J}.$$

Since

$$\mu\frac{\partial \mu}{\partial J} = \frac{1}{2}\frac{\partial \mu^2}{\partial J} \tag{3.57}$$

in zero field we have:

$$\frac{\partial^2}{\partial J^2}\left(\lim_{N\to\infty} p_N(J,0)\right) = \begin{cases} 0 & when \quad J \le 1 \\ \\ \frac{1}{2}\frac{\partial \mu_0^2}{\partial J} & when \quad J > 1 \end{cases} \tag{3.58}$$

Just below $J = 1$ the value of μ_0 is small, thus we can expand the hyperbolic tangent of the mean-field equation (3.55):

$$\mu_0 = J\mu_0 - \frac{(J\mu_0)^3}{3} + O(\mu_0^5) \quad as \quad J \to 1^+. \tag{3.59}$$

Since $\mu_0 \neq 0$ as $J > 1$, we can divide by $J\mu_0$ the equation (3.59). We obtain

$$\frac{1}{J} = 1 - \frac{(J\mu_0)^2}{3} + O(\mu_0^4) \quad as \quad J \to 1^+. \tag{3.60}$$

Thus

$$\mu_0 \sim \left(\frac{3}{J^2}\left(1-\frac{1}{J}\right)\right)^{\frac{1}{2}} \sim \left(3\left(1-\frac{1}{J}\right)\right)^{\frac{1}{2}} \quad as \quad J \to 1^+$$

and the second line of (3.57) can be approximate in the following way:

$$\frac{1}{2}\frac{d\mu_0^2}{dJ} \sim \frac{1}{2}\frac{d}{dJ}\left(3\left(1-\frac{1}{J}\right)\right) = \frac{3}{2J^2} \quad as \quad J \to 1^+. \tag{3.61}$$

By (3.61) it follows that the second derivative of the thermodynamic limit of (3.58) is discontinuous. The model exhibits a phase transition of the second order for $h = 0$ and $J = 1$. We claim that for this choice of the parameters the normalize sum of spins does not converge to a Gaussian distribution in the thermodynamic limit. Thus the *theorems* (3.3.2) and (3.3.3) are potent tools to obtain information about the critically of a phase.

3.4 Hamiltonian defined by J=1 and h=0

In this section we will prove that the central limit theorem always breaks down when it is applied on a model determined by the coupling constant J equal to one and the magnetic field h equal to zero. We will use the results given in the previous section. After have given some general preliminary results, we will apply them to the Curie-Weiss model.

3.4.1 Preliminary results

Consider the Hamiltonian defined by the coupling constant J equal to one and the magnetic field h equal to zero:

$$H_N(\overline{\sigma}) = -\frac{1}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j. \tag{3.62}$$

Observe that, under these hypothesis, the function $f(x)$ given by (3.7) can be written as

$$f(x) = -\frac{1}{2}x^2 + \ln\left(\int_{\mathbb{R}}\exp(sx)d\rho(s)\right). \tag{3.63}$$

By *theorem* (3.3.2), we can see that if the function $f(x)$ has a unique maximum point μ of type k and strength λ, we assist to the breaking down of the central limit theorem when $k>1$: then, according to this theorem and to the *theorem* (3.3.3), we will prove that when $J=1$ and $h=0$, the function $f(x)$ admits one or more maximum points of type $k>1$.

To compute the maximum of the function $f(x)$ consider its first derivative:

$$f'(x) = -x + \frac{\int_{\mathbb{R}} s\exp(sx)d\rho(s)}{\int_{\mathbb{R}}\exp(sx)d\rho(s)}$$

and observe that a maximum point μ solves the equation

$$\mu = \frac{\int_{\mathbb{R}} s\exp(s\mu)d\rho(s)}{\int_{\mathbb{R}}\exp(s\mu)d\rho(s)}. \tag{3.64}$$

In order to prove that under these hypothesis the central limit theorem breaks down, we have to show that $f''(\mu)=0$, hence we compute the second derivative of $f(x)$.

$$f''(x) = -1 + \frac{\left(\int_{\mathbb{R}} s^2 \exp(sx)\, d\rho(s)\right)\left(\int_{\mathbb{R}} \exp(sx)\, d\rho(s))\right)}{\left(\int_{\mathbb{R}} \exp(sx)\, d\rho(s)\right)^2} + \\ - \frac{\left(\int_{\mathbb{R}} s \exp(sx)\, d\rho(s)\right)\left(\int_{\mathbb{R}} s\exp(sx)\, d\rho(s)\right)}{\left(\int_{\mathbb{R}} \exp(sx)\, d\rho(s)\right)^2} = \\ = -1 + \frac{\int_{\mathbb{R}} s^2 \exp(sx)\, d\rho(s)}{\int_{\mathbb{R}} \exp(sx)\, d\rho(s)} - \left(\frac{\int_{\mathbb{R}} s \exp(sx)\, d\rho(s)}{\int_{\mathbb{R}} \exp(sx)\, d\rho(s)}\right)^2.$$

Defining a variable Y whose distribution is

$$\rho_x(s) = \frac{\exp(sx)\, d\rho(s)}{\int_{\mathbb{R}} \exp(sx)\, d\rho(s)} \tag{3.65}$$

we can write

$$f''(x) = -1 + \int_{\mathbb{R}} s^2 d\rho_x(s) - \left(\int_{\mathbb{R}} s d\rho_x(s)\right)^2 = -1 + Var_{\rho_x}(Y).$$

Finally we have to show that $Var_{\rho_\mu}(Y) = 1$.

In order to show our purpouse we recall the definition of the *moment-generating function* of a random variable which, in probability, is an alternative specification of its probability distribution. In probability theory and statistics, the moment-generating function of a random variable X is defined as

$$M_X(t) = \mathbb{E}[e^{Xt}], \quad t \in \mathbb{R}$$

whenever the expectation of the random variable exists.
Observe the correlation between the characteristic function and the moment-generating function of a random variable X: the characteristic function $\varphi_X(t)$ is related to the moment-generating function via

$$\varphi_X(t) = M_{iX}(t) = M_X(it).$$

We can consider the characteristic function as the moment-generating function of iX or as the moment generating function of X evaluated on the imaginary axis. A key problem with moment-generating functions is that moments and the moment-generating function may not exist, as the integrals need not converge absolutely. By contrast, the characteristic function always exists (because it is the integral of a bounded function on a space of finite measure), and thus may be used instead.
The reason for defining the moment-generating function is that it can be used to find all the *moments* of the distribution of the random variable. Consider the series expansion of e^{tX}:

$$\begin{aligned} e^{tX} &= \sum_{n=0}^{\infty} \frac{t^n X^n}{n!} = \\ &= 1 + tX + \frac{t^2 X^2}{2} + \frac{t^3 X^3}{3!} + \ldots + \frac{t^n X^n}{n!} + o\left((tX)^n\right). \end{aligned}$$

Hence

$$\begin{aligned} M_X(t) &= \mathbb{E}[e^{Xt}] = \\ &= \sum_{n=0}^{\infty} \frac{t^n \mathbb{E}[X^n]}{n!} = \\ &= 1 + t\mathbb{E}[X] + \frac{t^2 \mathbb{E}[X^2]}{2} + \frac{t^3 \mathbb{E}[X^3]}{3!} + \ldots + \frac{t^n \mathbb{E}[X^n]}{n!} + o\left(t^n \mathbb{E}[X^n]\right) = \\ &= 1 + tm_1 + \frac{t^2 m_2}{2} + \frac{t^3 m_3}{3!} + \ldots + \frac{t^n m_n}{n!} + o\left(t^n m_n\right), \end{aligned}$$

where m_n is the nth moment. The moment-generating function is so called because if it exists on an open interval around $t = 0$, then it is the exponential generating function of the moments of the probability distribution:

$$m_n(t) = \frac{\partial^n}{\partial t^n} M_X(t)\Big|_{t=0}.$$

Moreover, by the moment-generating function, we can define the *cumulants-generating function* as

$$K_X(t) = \log\left(M_X(t)\right) = \log\left(\mathbb{E}[e^{Xt}]\right).$$

In an analogous way as for the moments, we can obtain the *cumulants* k_n from a power series expansion of the cumulant generating function

$$K_X(t) = \sum_{n=0} k_n \frac{t^n X^n}{n!}. \qquad (3.66)$$

the nth cumulant can be obtained by differentiating the above expansion n times and evaluating the result at zero

$$k_n(t) = \frac{\partial^n}{\partial t^n} K_X(t)\Big|_{t=0}.$$

Observe that, using the cumulants of the random variable X and according to (3.66), the integral in (3.63) can be written as

$$\int_{\mathbb{R}} \exp(sx) d\rho(s) = \exp\left(\sum_{n=0} k_n \frac{s^n}{n!}\right). \tag{3.67}$$

Consider the following results.

Remark 3.13. Let $f(x)$ be the function defined in (3.63).
By (3.67) the function $f(x)$ has an expansion near the origin given by

$$\begin{aligned} f(x) &= -\frac{1}{2}x^2 + \ln\left(\exp\left(\sum_{n=0} k_n \frac{t^n s^n}{n!}\right)\right) = \\ &= k_1 s + (k_2 - 1)\frac{s^2}{2} + \sum_{n=3} k_n \frac{s^n}{n!}. \end{aligned}$$

Thus a (local) maximum point of type k and strength λ occurs at the origin if and only if

$$\begin{cases} k_2 = 1 \\ k_1 = k_3 = \ldots = k_{2k-2} = k_{2k-1} = 0 \\ k_{2k} = \lambda \end{cases}$$

Remark 3.14. Without loss of generality, it's sufficient to consider that the maximum μ defined in (3.64) is equal to zero.

Proof. If $\mu \neq 0$, define the measure

$$d\bar{\rho}(s) = \frac{\exp(\mu s) d\rho(s+\mu)}{\int_{\mathbb{R}} \exp(\mu s) d\rho(s+\mu)}.$$

Set

$$\begin{aligned} f_{\bar{\rho}}(x) &= -\frac{1}{2}x^2 + \ln\left(\int_{\mathbb{R}} \exp(sx) d\bar{\rho}(s)\right) = \\ &= -\frac{1}{2}x^2 + \ln\left(\frac{\int_{\mathbb{R}} \exp(sx)\exp(\mu s) d\rho(s+\mu)}{\int_{\mathbb{R}} \exp(\mu s) d\rho(s+\mu)}\right) \end{aligned}$$

and

$$f_\rho(x) = -\frac{1}{2}x^2 + \ln\left(\int_{\mathbb{R}} \exp(sx)d\rho(s)\right).$$

We have to prove that

$$f_\rho(x+\mu) - f_\rho(\mu) = f_{\bar{\rho}}(x),$$

i.e. that

$$-\frac{1}{2}(x+\mu)^2 + \ln\left(\int_{\mathbb{R}} \exp(s(x+\mu))d\rho(s)\right) + \frac{1}{2}\mu^2 - \ln\left(\int_{\mathbb{R}} \exp(s\mu)d\rho(s)\right) =$$

$$= -\frac{1}{2}x^2 + \ln\left(\frac{\displaystyle\int_{\mathbb{R}} \exp(sx)\exp(\mu s)d\rho(s+\mu)}{\displaystyle\int_{\mathbb{R}} \exp(\mu s)d\rho(s+\mu)}\right).$$

Making some simplifications and by logarithm's properties we obtain

$$x\mu + \ln\left(\frac{\displaystyle\int_{\mathbb{R}} \exp(s(x+\mu))d\rho(s)}{\displaystyle\int_{\mathbb{R}} \exp(s\mu)d\rho(s)}\right) = \ln\left(\frac{\displaystyle\int_{\mathbb{R}} \exp(sx)\exp(\mu s)d\rho(s+\mu)}{\displaystyle\int_{\mathbb{R}} \exp(\mu s)d\rho(s+\mu)}\right). \tag{3.68}$$

Making the change of variable

$$s + \mu = y,$$

the r.h.s. of (3.68) becomes

$$\ln\left(\frac{\displaystyle\int_{\mathbb{R}} \exp((y-\mu)(x+\mu))d\rho(y)}{\displaystyle\int_{\mathbb{R}} \exp(\mu(y-\mu))d\rho(y)}\right) =$$

$$= \ln\left(\frac{\displaystyle\int_{\mathbb{R}} \exp(xy - x\mu - \mu^2 + y\mu)d\rho(y)}{\displaystyle\int_{\mathbb{R}} \exp(y\mu - \mu^2)d\rho(y)}\right) =$$

$$= \ln\left(\frac{e^{-x\mu-\mu^2}\displaystyle\int_{\mathbb{R}} \exp(xy + y\mu)d\rho(y)}{e^{-\mu^2}\displaystyle\int_{\mathbb{R}} \exp(y\mu)d\rho(y)}\right) =$$

$$= -x\mu + \ln\left(\frac{\displaystyle\int_{\mathbb{R}} \exp(y(x+\mu))d\rho(y)}{\displaystyle\int_{\mathbb{R}} \exp(y\mu)d\rho(y)}\right).$$

Thus the existence and the nature of a maximum of $f_{\bar{\rho}}(x)$ at μ is equivalent to the corresponding facts about $f_\rho(x)$ at the origin. $\square$

In conclusion, by *remark* (3.13) and by *remark* (4.1) we can say that the va-riance, which corresponds to the moment of the second order, of the random variable Y with distribution (3.65) is equal to one. Thus

$$f''(\mu) = -1 + Var_{\rho_\mu}(Y) = -1 + 1 = 0,$$

i.e. μ is a (local) maximum point of the function $f(x)$, defined for $J = 1$ and $h = 0$, of type $k \geq 2$. According to the *theorem* (3.3.2) and to the *theorem* (3.3.3) we claim that, when the Hamiltonian is defined for $J = 1$ and $h = 0$, the central limit theorem breaks down.

3.4.2 Example: the Curie Weiss Model

In this section we will apply the results which have been discussed above on a Curie Weiss Model: when the coupling constant J is equal to one and the magnetic field h is equal to zero, we assist to a phase transition which occurs at the values where the central limit theorem breaks down.

When $J = 1$ and $h = 0$, the Curie Weiss model is defined by the Hamiltonian

$$H_N(\overline{\sigma}) = -\frac{1}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j$$

and the spins are identically distributed according to the distribution

$$\rho(x) = \frac{1}{2}(\delta(x+1) + \delta(x-1)).$$

The space of all configurations is $\Omega_N = \{-1, +1\}^N$, thus the probability of a configuration of N spins is given by

$$P_{N,J=1,h=0}\{\overline{\sigma}\} = \frac{\exp\left(-N\frac{x^2}{2} + N\ln(\cosh(x))\right) dx}{\displaystyle\int_{\mathbb{R}^N} \exp\left(-N\frac{x^2}{2} + N\ln(\cosh(x))\right) dx}.$$

The function $f(x)$ given in (3.63) becomes

$$f(x) = -\frac{1}{2}x^2 + \ln\left(\cosh(x)\right). \tag{3.69}$$

We can observe that $\mu = 0$ is the maximum point of the function in (3.69) and it's the only solution of the mean-field equation

$$\mu = \tanh(\mu).$$

If the central limit theorem was valid when $J = 1$ and $h = 0$, we would find that the sum of spins with square-root normalization had converged toward the Gaussian distribution, but it doesn't happen.
Proceede according to the results seen above and compute the first and the second derivative of $f(x)$:

$$\begin{aligned} f'(x) &= -x + \tanh(x) \\ f''(x) &= -1 + 1 + \tanh^2(x) = \tanh^2(x) \end{aligned}$$

Thus

$$f''(\mu) = f''(0) = 0$$

and the central limit theorem breaks down.

Infact, according to the results in the previous section, yelds:

$$\frac{S_N(\overline{\sigma})}{N^{\frac{3}{4}}} \xrightarrow{\mathcal{D}} \frac{\exp\left(-\dfrac{x^4}{12}\right) dx}{\displaystyle\int_{\mathbb{R}} \exp\left(-\dfrac{x^4}{12}\right) dx}.$$

Chapter 4

Conclusions ans perspectives

In this thesis we have investigated some limiting theorems for interacting particle systems. While we have shown that, under suitable hipothesis, the limiting distribution exists and is normal (central limit theorem), we have also shown how specific cases of statistical mechanics models do violate the central limit theorem and converge, suitably normalised, to a different probability density at the so called *critical point*. It would be interesting to investigate if it is possible to recover the classical central limit theorem everywhere but at the critical point using its extensions to the interacting variables. In order to do so, for instance for the mean-field models, one should identify a notion of infinite volume equilibrium state (like in [LST07] and [Sta06]) to test hypothesis like the condition of strongly mixing. The following considerations are a first step on such direction.
In this chapter, we will firstly prove some properties fullfilled by a configuration of spins and then we will make some considerations about the unsolved problem mentioned above.

In order to work with the Curie-Weiss model, we will briefly recall some results and definitions given in the previous chapter.

The Curie Weiss model is defined by the Hamiltonian

$$H_N(\overline{\sigma}) = -\frac{J}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i$$

where the spins are identically distributed according to the distribution

$$\rho(x) = \frac{1}{2}(\delta(x+1) + \delta(x-1)).$$

The space of all configurations is $\Omega_N = \{-1,+1\}^N$, thus the probability of a configuration

$$\overline{\sigma} = (\sigma_1, \ldots, \sigma_N)$$

is given by the measure of Boltzmann-Gibbs

$$P_{N,J,h}\{\overline{\sigma}\} = \frac{e^{-H_N(\sigma)} \prod_{i=1}^{N} d\rho(\sigma_i)}{Z_N(J,h)},$$

which, in the case of the model of Curie-Weiss, has the following continuous formulation:

$$P_{N,J,h}\{\overline{\sigma}\} = \frac{\exp\left(-N\frac{Jx^2}{2} + N\ln(\cosh(Jx+h))\right)dx}{\int_{\mathbb{R}^N} \exp\left(-N\frac{Jx^2}{2} + N\ln(\cosh(Jx+h))\right)dx}.$$

We saw in the third chapter that the main observable of the model is the magnetization $m_N(\overline{\sigma})$ of a configuration $\overline{\sigma} = (\sigma_1, \ldots, \sigma_N)$, which is defined by

$$m_N(\overline{\sigma}) = \frac{1}{N}\sum_{i=1}^{N} \sigma_i.$$

By the definition of the magnetization, we have that for each $i = 1, \ldots, N$:

$$\langle \sigma_i \rangle_{BG} = \langle m_N(\overline{\sigma}) \rangle_{BG}.$$

Indicate with $m(J,h)$ the stable solution of the mean-field equation,

$$m(J,h) = \tanh(Jm(J,h) + h), \tag{4.1}$$

which admits:

a. one solution μ_h, if $h \neq 0$ and $J > 0$,

b. one solution $\mu_0 = 0$, if $h = 0$ and $J < 1$,

c. two solutions $\pm\mu_0$, if $h = 0$ and $J > 1$,

d. one solution $\mu_0 = 0$, if $h = 0$ and $J = 1$.

When $J > 0$ and $h \neq 0$ or $J \leq 1$ and $h = 0$, the Curie-Weiss model satisfies the following property (see [Ell05])

$$\lim_{N\to\infty} \langle m_N(\overline{\sigma})\rangle_{BG} = m(J,h), \tag{4.2}$$

On the other hand, when $h = 0$ and $J > 1$, (4.1) has two different stable solutions $\pm m(J,0)$ and the identity (4.2) is not verified. Anyway, there exists $\epsilon > 0$ such that, whenever

$$m_N(\overline{\sigma}) \in (\pm m(J,0) - \epsilon, \pm m(J,0) + \epsilon),$$

the following limit holds (see [ENR80])

$$\lim_{N\to\infty} \langle m_N(\overline{\sigma})\rangle_{BG} = \pm m(J,0).$$

In order to simplify the notations we will denote $m(J,h)$ with μ.

The variance of a spin is finite and can be computed, for each $i = 1, \ldots N$, as

$$var(\sigma_i) = \langle \sigma_i^2\rangle_{BG} - \langle \sigma_i\rangle_{BG}^2 = 1 - \langle m_N(\overline{\sigma})\rangle_{BG}^2,$$

thus at the thermodynamic limit we have:

$$\lim_{N\to\infty} (1 - \langle m_N(\overline{\sigma})\rangle_{BG}^2) = 1 - \mu^2.$$

We will proceede proving some technical properties of the configuration of N spins.

Proposition 4.0.1. *Let $\overline{\sigma} = (\sigma_1, \ldots, \sigma_N)$ be a configuration of N spins. Define $S_N(\overline{\sigma}) = \sigma_1 + \ldots + \sigma_N$.*
Then when $J > 0$ and $h \neq 0$ or $J < 1$ and $h = 0$, we can write

$$var(S_N(\overline{\sigma})) = Nh(N),$$

where $h(N)$ is a slowly varying function such that $c_1 \leq h(N) \leq c_2$, with $c_1, c_2 \in \mathbb{R}$.

Proof. Using the definition of the variance for a sum of random variables we have that

$$\begin{aligned} var(S_N(\overline{\sigma})) &= \sum_{i=1}^{N} var(\sigma_i) + \sum_{i\neq j} cov(\sigma_i,\sigma_j) = \\ &= N\left(var(\sigma_i) + 2\frac{\sum_{i<j} cov(\sigma_i,\sigma_j)}{N}\right) = \\ &= N\left(var(\sigma_i) + \frac{\frac{2N(N-1)}{2} cov(\sigma_i,\sigma_j)}{N}\right) = \\ &= N\left(var(\sigma_i) + (N-1)cov(\sigma_i,\sigma_j)\right). \end{aligned}$$

Set

$$h(N) = \left(var(\sigma_i) + (N-1)cov(\sigma_i,\sigma_j)\right).$$

We want to prove that $h(N)$ is slowly varying. An example of slowly varying function, by definition, is given by $f:(0,+\infty)\to(0,+\infty)$ such that

$$\lim_{n\to\infty} f(x) = c \in \mathbb{R},$$

infact, for any choice of the real number $a \in \mathbb{R}$, we will have:

$$\lim_{x\to\infty} \frac{f(ax)}{f(x)} = \frac{c}{c} = 1.$$

We will prove that $(N-1)cov(\sigma_i,\sigma_j)$ is constant as $N\to\infty$.
Make the following considerations.
Using the definition of covariance, we find that:

$$\begin{aligned} cov(\sigma_i,\sigma_j) &= \mathbb{E}\left[(\sigma_i - \mathbb{E}(\sigma_i))(\sigma_j - \mathbb{E}(\sigma_j))\right] = \\ &= \mathbb{E}\left[\sigma_i\sigma_j - \mathbb{E}(\sigma_i)\sigma_j - \sigma_i\mathbb{E}(\sigma_j) + \mathbb{E}(\sigma_i)\mathbb{E}(\sigma_j)\right] = \\ &= \langle\sigma_i\sigma_j\rangle_{BG} - \langle m_N\rangle_{BG}^2 - \langle m_N\rangle_{BG}^2 + \langle m_N\rangle_{BG}^2 = \\ &= \langle\sigma_i\sigma_j\rangle_{BG} - \langle m_N\rangle_{BG}^2. \end{aligned} \tag{4.3}$$

Remembering the definition of magnetizazion and splitting up the sum over the spins in the following way, we find an expression for $\langle\sigma_i\sigma_j\rangle_{BG}$:

$$\begin{aligned}
\langle m_N^2\rangle_{BG} &= \langle\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\right)^2\rangle_{BG} = \\
&= \frac{1}{N^2}\langle\sum_{i,j=1}^{N}\sigma_i\sigma_j\rangle_{BG} = \\
&= \frac{1}{N^2}\langle\sum_{i\neq j=1}^{N}\sigma_i\sigma_j\rangle_{BG} + \frac{1}{N^2}\langle\sum_{i=1}^{N}\sigma_i^2\rangle_{BG} = \\
&= \frac{N(N-1)}{N^2}\langle\sigma_i\sigma_j\rangle_{BG} + \frac{N}{N^2} = \\
&= \frac{N-1}{N}\langle\sigma_i\sigma_j\rangle_{BG} + \frac{1}{N} \\
\Longrightarrow \langle\sigma_i\sigma_j\rangle_{BG} &= \left(\langle m_N^2\rangle_{BG} - \frac{1}{N}\right)\frac{N}{N-1} = \\
&= \frac{N}{N-1}\langle m_N^2\rangle_{BG} - \frac{1}{N-1},
\end{aligned} \tag{4.4}$$

hence, as $N\to\infty$,

$$\begin{aligned}
cov(\sigma_i,\sigma_j) &= \frac{N}{N-1}\langle m_N^2\rangle_{BG} - \frac{1}{N-1} - \langle m_N\rangle_{BG}^2 \sim \\
&\sim var(m_N) = \frac{1}{N}\frac{\partial^2 p_N(J,h)}{\partial h^2}.
\end{aligned}$$

Thus we have that the second derivative of the pressure function corresponds to the variance of the magnetization multiplied by N:

$$\frac{\partial^2 p_N(J,h)}{\partial h^2} = N\langle m_N^2(\overline{\sigma})\rangle_{BG} - \langle m_N(\overline{\sigma})\rangle_{BG}^2.$$

Remark 4.1. Working in finite volume, we find that $\dfrac{\partial^2 p_N(J,h)}{\partial h^2}$ is a finite quantity.

Proof. We will proceede by induction on N.
For $N=1$, we have only one spin $\overline{\sigma}=\sigma_1$ and the hamiltonian becomes

$$H_1(\overline{\sigma}) = -J - h\sigma_1.$$

Thus

$$\frac{\partial^2 p_1(J,h)}{\partial h^2} = var(m_1) = 1 - 1 = 0.$$

Suppose true for N and show for $N+1$.
Considering a configuration of $N+1$ spins $\overline{\sigma} = (\sigma_1, \ldots, \sigma_{N+1})$, we find that

$$\begin{aligned}\frac{\partial^2 p_{N+1}(J,h)}{\partial h^2} &= (N+1)var(m_{N+1}) = \\ &= Nvar(m_{N+1}) + var(m_{N+1}).\end{aligned}$$

Using the following identity

$$\begin{aligned}m_{N+1} &= \frac{\sum_{i=1}^{N+1}\sigma_i}{N+1} = \\ &= \frac{\sum_{i=1}^{N}\sigma_i + \sigma_{N+1}}{N+1} = \\ &= \frac{N}{N+1}\frac{\sum_{i=1}^{N}\sigma_i}{N} + \frac{\sigma_{N+1}}{N+1} = \\ &= \frac{N}{N+1}m_N + \frac{\sigma_{N+1}}{N+1},\end{aligned}$$

we can say that

$$\begin{aligned}var(m_{N+1}) =& \langle\left(\frac{N}{N+1}m_N + \frac{\sigma_{N+1}}{N+1}\right)^2\rangle_{BG} - \langle\frac{N}{N+1}m_N + \frac{\sigma_{N+1}}{N+1}\rangle_{BG}^2 = \\ =& \frac{1}{(N+1)^2}\left(N^2\langle m_N^2\rangle_{BG} + 1 + 2N\langle m_N\sigma_{N+1}\rangle_{BG} + \right. \\ & \left. -N^2\langle m_N\rangle_{BG}^2 - \langle\sigma_{N+1}\rangle_{BG}^2 - 2N\langle m_N\rangle_{BG}\langle\sigma_{N+1}\rangle_{BG}\right) = \\ =& \frac{1}{(N+1)^2}\left(N^2 var(m_N) + var(\sigma_{N+1}) + \right. \\ & \left. +2N(\langle m_N\sigma_{N+1}\rangle_{BG} - \langle m_N\rangle_{BG}\langle\sigma_{N+1}\rangle_{BG})\right).\end{aligned}$$

Observing that

$$\langle m_N\sigma_{N+1}\rangle_{BG} \leq \langle m_N\rangle_{BG}$$

and

$$\langle m_N\rangle_{BG}\langle\sigma_{N+1}\rangle_{BG} \geq -\langle m_N\rangle_{BG},$$

we can say that:

$$var(m_{N+1}) \leq \frac{1}{(N+1)^2}\left(N^2 var(m_N) + var(\sigma_{N+1}) + 4N(\langle m_N\rangle_{BG})\right).$$

Hence:

$$\begin{aligned}\frac{\partial^2 p_{N+1}(J,h)}{\partial h^2} &= (N+1)var(m_{N+1}) \leq \\ &\leq \frac{1}{N+1}\left(N^2 var(m_N) + var(\sigma_{N+1}) + 4N(\langle m_N \rangle_{BG})\right) \leq \\ &\leq \frac{N^2+4N+1}{N+1}.\end{aligned}$$

□

At the thermodynamic limit, consider the susceptibility of the model which can be computed differentiating (4.2) with respect to h:

$$\begin{aligned}\chi(J,h) &= \lim_{N\to\infty} \frac{\partial}{\partial h}\langle m_N(\overline{\sigma})\rangle_{BG} = \\ &= \frac{\partial}{\partial h}\lim_{N\to\infty}\langle m_N(\overline{\sigma})\rangle_{BG} = \\ &= \frac{\partial m(J,h)}{\partial h} = \\ &= \frac{1-m^2(J,h)}{1-J(1-m^2(J,h))} = \\ &= \frac{1-\mu^2}{1-J(1-\mu^2)}.\end{aligned} \tag{4.5}$$

Using (3.60), we want to understand the behaviour of the susceptibility for different values of the coupling constant J and of the magnetic field h. When the magnetic field is not equal to zero, as we saw in *section* 3.3.2, there is not any phase transition. According to the notations used above, we have that

$$\chi = \frac{\partial^2 p_N(J,h)}{\partial h^2} = \frac{1-\mu_h^2}{1-J(1-m_h^2)} < \infty.$$

When $h \neq 0$, the pressure function is an analytic function, hence it doesn't have points of discontinuity, thus the second derivative of the pressure function with respect to h is a finite quantity.

When the magnetic field is equal to zero, we find that

$$\chi = \frac{1 - m^2(J,0)}{1 - J(1 - m^2(J,0))} \sim$$
$$\sim \frac{1 - 3\left(1 - \frac{1}{J}\right)}{1 - J\left(1 - 3\left(1 - \frac{1}{J}\right)\right)} =$$
$$= \frac{-2 + \frac{3}{J}}{2J - 2}.$$

When $J \to 1^+$, we have a phase transition and the susceptibility explodes:

$$\chi = \frac{3 - 2J}{J(2J-2)} \xrightarrow{J \to 1^+} \infty. \tag{4.6}$$

When $(J,h) = (1,0)$, the pressure function is not an analytic function: in-fact its second derivative with respect to h presents a point of discontinuity as $J \to 1^+$. On the other side, when $h = 0$ and $J > 1$, we can't observe any phase transtion and the second derivative of the pressure function with respect to h is a finite quantity.

In conclusion, using *remark* (4.1) and the equations (4.3), (4.4) and (4.6) we can say that, when $(J,h) \neq (1,0)$, $h(N)$ is a slowly varying function.
Moreover, according to the hypothesis of the theorem, we can choose as lower bound $c_1 = var(\sigma_i)$ and as upper bound $c_2 = var(\sigma_i) + C$, where C is bigger than the variance of the magnetization. □

Proposition 4.0.2. *Let $\overline{\sigma} = (\sigma_1, \dots, \sigma_N)$ be a configuration of N spins. Define $S_N(\overline{\sigma}) = \sigma_1 + \dots + \sigma_N$.*
Then, when $J > 0$ and $h \neq 0$ or $J \leq 1$ and $h = 0$

$$\lim_{M \to \infty} \limsup_{N \to \infty} \int_{|z|>M} z^2 dF_N(z) = \begin{cases} 0 & if \quad (J,h) \neq (1,0) \\ +\infty & if \quad (J,h) = (1,0) \end{cases}$$

where $F_N(z)$ is the distribution function of the random variable $\dfrac{S_N(\overline{\sigma}) - N\mu}{\sqrt{N}}$ and μ is the solution of the mean-field equation.

Proof. As it has been shown, the distribution function of $S_N(\overline{\sigma})$ is given by

$$\frac{1}{Z_N(J,h)} \exp\left(\frac{J}{2N}s^2 + hs\right) d\nu_S(s)$$

where $d\nu_S(s)$ denotes the distribution of $S_N(\overline{\sigma})$ on $\left(\mathbb{R}^N, \prod_{i=1}^{N} \rho(\sigma_i)\right)$.
Ellis, Newmann and Rosen proved that as $N \to \infty$,

$$\frac{S_N(\overline{\sigma}) - N\mu}{N^{1-\frac{1}{2k}}} \xrightarrow{\mathcal{D}} \begin{cases} \mathcal{N}\left(0, -\left(\dfrac{1}{\lambda} + \dfrac{1}{J}\right)\right) & if \quad k = 1 \\ \exp\left(\dfrac{\lambda}{(2k)!} x^{2k}\right) & if \quad k > 1 \end{cases}$$

where $f(x)$ is the function defined in (3.7) and k and λ are respectively the type and the strength of the solution μ of the mean-field equation.
In order to verify the hypothesis of the theorem, we will work in the case that $k = 1$, since we have to normalize with $\sqrt{N}$.

At the thermodynamic limit, we find that:

$$\begin{aligned} \chi = \lim_{N\to\infty} \frac{\partial^2 p_N(J,h)}{\partial h^2} &= \\ &= \frac{1 - m^2(J,h)}{1 - J(1 - m^2(J,h))}, \end{aligned}$$

where $m(J,h)$ is the solution of the mean-field equation.

When $(J,h) \neq (1,0)$, as we proved in the *section* 3.3.2, the point μ has type $k = 1$ and the sum with square-root normalization converges toward the gaussian distribution with mean equal to 0 and variance equal to the susceptibility χ, which is finite.
Hence it holds:

$$0 \leq \lim_{M\to\infty} \int_{|z|>M} z^2 d\mathcal{N}(0,\chi)(z) = 0.$$

When $(J,h) = (1,0)$, the maximum point $\mu = 0$ has type $k = 2$ and $f''(\mu) = 0$, as we computed in the *section* 3.3.2. Hence the random variable $\dfrac{S_N(\overline{\sigma})}{N^{\frac{1}{2}}}$, has the following distribution:

$$F_N(x) = \exp\left(\frac{f''(\mu)}{2!} x^2\right) = \exp(0) = 1.$$

Consider now the equation (4.6) studied above.

It holds:

$$\begin{aligned}
&\lim_{M\to\infty}\limsup_{N\to\infty}\int_{|z|>M} z^2 dF_N(z) = \\
&= \lim_{M\to\infty}\int_{|z|>M}\limsup_{N\to\infty} z^2 dF_N(z) \geq \\
&\geq \lim_{M\to\infty}\int_{|z|>M}\lim_{N\to\infty} z^2 dF_N(z) = \\
&= \lim_{M\to\infty}\int_{|z|>M} z^2 dz = +\infty.
\end{aligned}$$

□

Consider in particular the sufficient and necessary conditions given by *theorem* (2.4.2), which states that:

Let X_i be a strongly mixing sequence of random variables such that $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) = var(X_i) < \infty$. Suppose that

$$var(S_n) = nh(n) \quad as \quad n \to \infty, \tag{4.7}$$

where $h(n)$ is a slowly varying function such that as $n \to \infty$, $c_1 \leq h(n) \leq c_2$, where $c_1 \leq c_2$ are constants. Then the sequence X_i satisfies the central limit theorem if and only if

$$\lim_{M\to\infty}\limsup_{n\to\infty}\int_{|z|>M} z^2 dF_n(z) = 0, \tag{4.8}$$

where $F_n(z)$ is the distribution function of the normalized sum

$$Z_n = \frac{1}{\sqrt{var(S_n)}}\sum_{i=1}^{n} X_i.$$

Remark 4.2. More in general, if we suppose that the variables are identically distributed with expectation $\mathbb{E}(X) \neq 0$, we have to consider the asymptotic behaviour of the variable

$$\frac{S_n - n\mathbb{E}(X)}{\sqrt{var(S_n)}}.$$

Remark 4.3. Observe that the conditions (4.7) and (4.8) are proved in the propositions (4.0.1) and (4.0.2). In particular we find that they are fullfilled if and only if $(J, h) \neq (1, 0)$: this is a first signal that at the critical point the sum of the spins with square-root normalization can't converge toward the Gaussian distribution.

Following the introduction of this chapter, in order to identify a configuration of N spins with a random process, we can imagine that the configuration presents the evolution in time of the values assumed by a single spin during descrete instants. Anyway it is necessary that the probability associated to the random process does not depend by its length: in the case of the Curie-Weiss model, we have a different probability for every configuration with different length, defined by the measure of Boltzmann-Gibbs. Thus, working in a finite volume, we don't have problems for defining the probability of the configuration of spins but we need to extend it at the thermodynamic limit.

If we fix a natural N, a configuration of spins of the Curie-Weiss model, and more in general the spins which interact one with each other according to the Hamiltonian (3.1), define a stationary process.
Consider a positive integer τ and consider the configuration

$$\overline{\sigma}_\tau = (\sigma_{1+\tau}, \ldots, \sigma_{N+\tau}).$$

Let M be a positive integer such that $M \geq N + \tau$. The space of all configurations Ω_M contains the space Ω_N, hence it contains all the configurations of length less or equal to M: hence both the configurations $\overline{\sigma}$ and $\overline{\sigma}_\tau$ can be found in Ω_M. The probability of Boltzmann-Gibbs only depends on the number of spins of the configuration, then, if we shift the spins of a parameter τ, the probability of the configuration doesn't change: thus it may be identified as a stationary process.

It remains to prove that the stationary process satisfies the property of *strongly mixing*: in order to do this we need to prove that there exists a probability measure P, which is the extension of the measure of Boltzmann-Gibbs at the thermodynamic limit.

After having proved this property, it is possible to apply the hypothesis of the *theorem* (2.4.2) to a configuration of the Curie-Weiss model, in order to see that the sum of the spins with square-root normalization converges toward the Gaussian distribution if and only if $(J, h) \neq (1, 0)$.

Appendix A

Bochner-Kinchin's theorem

Let $m_n(x)$ be the normalized Lebesgue measure on $\mathbb{R}^n$ such that

$$m_n(x) = \frac{1}{(2\pi)^n} dx.$$

If μ is a finite positive Borel measure on $\mathbb{R}^n$, the Fourier transform of μ is the function $\hat{\mu} : \mathbb{R}^n \longrightarrow \mathbb{C}$ defined by:

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-i<x,\xi>} d\mu(x), \quad \xi \in \mathbb{R}^n.$$

Using the dominated convergence theorem, it's easy to prove that $\hat{\mu}$ is a continuous function. If $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is the function $\hat{f} : \mathbb{R}^n \longrightarrow \mathbb{C}$ defined by:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i<x,\xi>} f(x) dm_n(x), \quad \xi \in \mathbb{R}^n.$$

Likewise, using the dominated convergence theorem, it's easy to prove that $\hat{f}$ is a continuous function. One proves that if $f \in L^1(\mathbb{R}^n)$ and if $\hat{f} \in L^1(\mathbb{R}^n)$, then, for almost all $x \in (\mathbb{R}^n)$

$$f(x) = \int_{\mathbb{R}^n} e^{i<x,\xi>} \hat{f}(\xi) dm_n(\xi).$$

Moreover, observe that as

$$\hat{\mu}(0) = \int_{\mathbb{R}^n} d\mu(x) = \mu(\mathbb{R}^n),$$

μ is a probability measure if an only if $\hat{\mu}(0) = 1$.

Theorem A.0.3 (Bochner-Kinchin's theorem)**.** *Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a positive-definite and continuous function that satisfies the condition $\phi(0) = 1$. Then there is some Borel probability measure μ on $\mathbb{R}^n$ such that $\phi = \hat{\mu}$.*

Proof. Let $\{\psi_U\}$ be an approximate identity, that is, for each neighborhoo U of 0, ψ_U is a function such that:

(i) suppψ_U is compact and contained in U,

(ii) $\psi \geq 0$ and $\psi_U(x) = \psi_U(-x)$,

(iii) $\displaystyle\int_{\mathbb{R}^n} \psi_U(x) dm_n(x) = 1$.

For every $f \in L^1(\mathbb{R}^n)$, an approximate identity satisfies

$$||f * \psi_U - f||_{L^1} \longrightarrow 0, \quad as \quad U \to \{0\}.$$

We have that $\psi_U^* := \overline{\psi_U(-x)} = \psi_{-U}(x)$, so

$$supp(\psi_U^* * \psi_U) \subset \overline{supp\psi_{-U} + supp\psi_U} = supp\psi_{-U} + supp\psi_U = -U + U.$$

Moreover

$$\int_{\mathbb{R}^n} (f * g) dm_n = \int_{\mathbb{R}^n} f dm_n \int_{\mathbb{R}^n} g dm_n.$$

Therefore $\{\psi_U^* * \psi_U\}$ is an approximate identity.
For all $f, g \in L^1(\mathbb{R}^n)$, define

$$< f, g >_\phi = \int_{\mathbb{R}^n} (g^* * f) \phi dm_n :$$

this is a positive Hermitian form, i.e. $< f, f >_\phi \geq 0$ for all $f \in L^1(\mathbb{R}^n$. Using the Cauchy-Schwartz inequality,

$$| < f, g >_\phi |^2 \leq < f, f >_\phi < g, g >_\phi .$$

We have laid out the tools that we will use for the proof.
Let $f \in L^1(\mathbb{R}^n)$, $\psi_U * f \to f$ as $U \to \{0\}$; as ϕ is bounded this gives

$$\int_{\mathbb{R}^n} (\psi_U * f) \phi dm_n \to \int_{\mathbb{R}^n} f \phi dm_n \quad as \quad U \to \{0\}.$$

Because $\{\psi_U^* * \psi_U\}$ is an approximate identity,

$$\int_{\mathbb{R}^n} (\psi_U^* * \psi_U) \phi dm_n \to \phi(0) \quad as \quad U \to \{0\},$$

that is we have

$$< f, \psi_U >_\phi \to \int_{\mathbb{R}^n} f\phi dm_n$$

and

$$< \psi_U, \psi_U >_\phi \to \phi(0)$$

as $U \to \{0\}$ and as $\phi(0) = 1$; thus the Cauchy-Schwartz inequality produces

$$\left| \int_{\mathbb{R}^n} f\phi dm_n \right|^2 \leq \int_{\mathbb{R}^n} (f^* * f)\phi dm_n. \tag{A.1}$$

With $h = f^* * f$, the inequality (A.1) reads

$$\left| \int_{\mathbb{R}^n} f\phi dm_n \right|^2 \leq \int_{\mathbb{R}^n} h\phi dm_n.$$

Define $h^{(1)} = h$, $h^{(2)} = h * h$, $h^{(3)} = h * h * h$, etc., apply (A.1) to the function h and obtain, because $h^* = h$,

$$\left| \int_{\mathbb{R}^n} h\phi dm_n \right|^2 \leq \int_{\mathbb{R}^n} h^{(2)}\phi dm_n.$$

Applying (A.1) to $h^{(2)}$, which satisfies $\left(h^{(2)}\right)^* = h^{(2)}$,

$$\left| \int_{\mathbb{R}^n} h^{(2)}\phi dm_n \right|^2 \leq \int_{\mathbb{R}^n}^{(4)} \phi dm_n.$$

Thus, for any $m \geq 0$ we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f\phi dm_n \right|^2 \leq \left| \int_{\mathbb{R}^n} h^{(2^m)}\phi dm_n \right|^{2^{-(m+1)}} &\leq \\ \leq \left\| h^{(2^m)} \right\|_{L^1}^{2^{-(m+1)}} &\leq \\ \leq \left(\left\| h^{(2^m)} \right\|_{L^1}^{2^{-m}} \right)^{\frac{1}{2}}&, \end{aligned}$$

since $\|\Phi\|_\infty = \phi(0) = 1$. With convolution as multiplication, $L^1(\mathbb{R}^n)$ is a commutative Banach algebra. The Gelfand transform is an algebra homomorphism $L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ that satisfies:

$$\|\hat{g}\|_\infty = \lim_{k\to\infty} \left\| g^{(k)} \right\|_{L^1}^{\frac{1}{k}}, \quad g \in L^1(\mathbb{R}^n);$$

for $f \in L^1(\mathbb{R}^n)$, the Gelfand transform is the Fourier transform.
Write the Fourier transform as $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$.
Stating that the Gelfand transform is an homomorphism means that

$$\mathcal{F}(g_1 * g_2) = \mathcal{F}(g_1)\mathcal{F}(g_2),$$

because multiplication in the Banach algebra $C_0(\mathbb{R}^n)$ is pointwise multiplication. Then, since a subsequence of a convergent sequence converges to the same limit,

$$\lim_{m\to\infty}\left(\|h^{(2^m)}\|_{L^1}^{2^{-m}}\right)^{\frac{1}{2}} = \left(\left\|\hat{h}\right\|_\infty\right)^{\frac{1}{2}}.$$

But

$$\hat{h} = \mathcal{F}(f^* * f) = \mathcal{F}(f^*)\mathcal{F}(f) = \overline{\mathcal{F}(f)}\mathcal{F}(f) = |\mathcal{F}(f)|^2.$$

so

$$\left(\left\|\hat{h}\right\|_\infty\right)^{\frac{1}{2}} = \left(\left\||\hat{f}|^2\right\|_\infty\right)^{\frac{1}{2}} = \left\|\hat{f}\right\|_\infty.$$

Putting things together, we have that for any $f \in L^1(\mathbb{R}^n)$,

$$\left|\int_{\mathbb{R}^n} f\phi dm_n\right| \le \left\|\hat{f}\right\|_\infty.$$

Therefore

$$\hat{f} \longmapsto \int_{\mathbb{R}^n} f\phi dm_n$$

is a bounded linear functional $\mathcal{F}(L^1(\mathbb{R}^n))) \to \mathbb{C}$.
Moreover it has norm ≤ 1.
To prove it remember that $\phi(0) = 1$; appling the inequality to $\mathcal{F}(\delta)$, we can see that the two sides are equal, hence, appling the inequality to a sequence of functions that converge weakly to δ, we obtain that the norm of the functional satisfies the requested bound.
Remember that $\mathcal{F}(L^1(\mathbb{R}^n))$ is dense in the Banach space $C_0(\mathbb{R}^n)$, so there is a bounded linear functional $\Phi : C_0(\mathbb{R}^n) \to \mathbb{C}$ whose restriction to $\mathcal{F}(L^1(\mathbb{R}^n))$ is equal to

$$\hat{f} \longmapsto \int_{\mathbb{R}^n} f\phi dm_n \quad and \quad ||\Phi|| = 1.$$

Using the Riesz-Markov theorem, there is a regular complex Borel measure μ on $\mathbb{R}^n$ such that

$$\Phi(g) = \int_{\mathbb{R}^n} g d\mu, \quad g \in C_0(\mathbb{R}^n),$$

and $||\mu|| = ||\Phi||$; $|\mu|$ is the total variation norm of μ: $||\mu|| = |\mu|(\mathbb{R}^n)$. Then, for $f \in L^1(\mathbb{R}^n)$, we have:

$$\begin{aligned}
\int_{\mathbb{R}^n} f\phi dm_n = \Phi(\hat{f}) = \\
&= \int_{\mathbb{R}^n} f d\mu = \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-i<x,\xi>} f(x) dm_n(x) \right) d\mu(\xi) = \\
&= \int_{\mathbb{R}^n} f(x) \left(\int_{\mathbb{R}^n} e^{-i<x,\xi>} d\mu(\xi) \right) dm_n(x) = \\
&= \int_{\mathbb{R}^n} f(x)\hat{\mu}(x) dm_n(x).
\end{aligned}$$

This is true for all $f \in L^1(\mathbb{R}^n)$ implies that $\phi = \hat{mu}$. As

$$\hat{\mu}(\mathbb{R}^n) = \hat{\mu}(0) = \phi(0) = 1$$

and

$$||\mu|| = ||\Phi|| = 1,$$

we have that $\mu(\mathbb{R}^n) = ||\mu||$ and implies that μ is the positive, hence, as $\mu(\mathbb{R}^n) = 1$, is a probability measure. $\square$

Appendix B

Herglotz's theorem

Herglotz's theorem characterizes the complex-valued autocovariance functions on the integers as the function which can be written in the form

$$R_t = \int_{-\pi}^{\pi} e^{it\lambda} dF(\lambda), \tag{B.1}$$

for some bounded distribution function F with mass concentrated in the interval $[-\pi, \pi]$.

Theorem B.0.4 (Herglotz's theorem)**.** *A complex-valued function R_t defined on the integers is non-negative definite if an only if*

$$R_t = \int_{-\pi}^{\pi} e^{it\lambda} dF(\lambda), \quad \forall t = 0, \pm 1, \ldots, \tag{B.2}$$

where $F(\cdot)$ is a right-continuous, non-decreasing, bounded function on $[-\pi, \pi]$ and $F(-\pi) = 0$. The function F is called spectral istribution function of R and if

$$F(\lambda) = \int_{-\infty}^{\lambda} f(v) dv, \quad -\pi \leq \lambda \leq \pi,$$

the f is called spectral density function of R.

Proof. If R_t has the representation (B.2), then it is clear that R_t is Hermitian, i.e. $R_{-t} = \overline{R_t}$. Moreover, if $a_r \in \mathbb{C}$, $r = 1, \ldots, n$, then

$$\sum_{r,s=1}^{n} a_r R_{r-s} \bar{a}_s = \int_{-\pi}^{\pi} \sum_{r,s=1}^{n} a_r \bar{a}_s \exp\left(iv(r-s)\right) dF(v) =$$
$$= \int_{-\pi}^{\pi} \left| \sum_{r=1}^{n} a_r \exp\left(ivr\right) \right|^2 dF(v) \geq 0$$

so that R_t is also non-negative definite an therefore is an autocovariance function by the properties defined in *Chapter 4*.

Conversely suppose that R_t is a non-negative definite function on the integers. Then, defining

$$\begin{aligned} f_N(v) &= \frac{1}{2\pi N}\sum_{r,s=1}^{N}\exp\left(-ivr\right)R_{r-s}\exp(ivs) = \\ &= \frac{1}{2\pi N}\sum_{m<N}(N-|m|)\exp\left(-ivm\right)R_m, \end{aligned}$$

we see from the negative definiteness of R_t that

$$f_N(v) \geq 0 \quad \forall v \in [-\pi, \pi].$$

Let $F_N(\cdot)$ be the distribution function corresponding to the density $f_N(\cdot)\mathbb{I}_{[-\pi,\pi]}(\cdot)$. Thus

$$\begin{cases} F_N(\lambda) = 0 & as \quad \lambda \leq -\pi \\ F_N(\lambda) = F_N(\pi) & as \quad \lambda \geq \pi \\ F_N(\lambda) = \displaystyle\int_{-\pi}^{\pi} f_N(v)dv & as \quad -\pi \leq \lambda\pi. \end{cases}$$

Then for any integer t,

$$\int_{-\pi}^{\pi}\exp(ivt)dF_N(v) = \frac{1}{2\pi}\sum_{|m<N|}\left(1-\frac{|m|}{N}\right)R_m\int_{-\pi}^{\pi}\exp(i(t-m)v)dF_N(v),$$

i.e.

$$\int_{-\pi}^{\pi}\exp(ivt)dF_N(v) = \begin{cases}\left(1-\dfrac{|t|}{N}\right)R_t, & as \quad |t|<N, \\ 0 & otherwise. \end{cases} \tag{B.3}$$

Since

$$F_N(\pi) = \int_{-\pi}^{\pi} dF_N(v) = R_0 < \infty, \quad \forall N,$$

we can fin a distribution function F and a subsequence $\{F_{N_k}\}$ of the sequence $\{F_N\}$ such that for any bounded continuous function g, with $g(\pi) = g(-\pi)$,

$$\int_{-\pi}^{\pi} g(v)dF_{N_k}(v) \xrightarrow{k\to\infty} \int_{-\pi}^{\pi} g(v)dF_N(v).$$

Replacing N by N_k in (B.3) and letting $k \to \infty$, we obtain

$$R_t = \int_{-\pi}^{\pi} \exp(ivt) dF(v),$$

which is the required spectral representation of R_t. □

Appendix C

Infinitely divisible distributions and stable distributions

A distribution function $F(x)$ is said to be *infinitely divisible* if, for each $n \in \mathbb{N}$, there exists a distribution F_n such that

$$F(x) = F_n(x)^{*n}.$$

Thus a random variable X with an infinitely divisible distribution can be expressed, for every n, in the form

$$X = X_{1,n} + X_{2,n} + \ldots + X_{n,n}$$

where $X_{j,n}, \quad j \in \{1, 2, \ldots, n\}$ are identically distributed.

Theorem C.0.5. [1] *In order that the function $\psi_X(t)$ be the characteristic function of any infinitely divisible distribution it is necessary and sufficient that*

$$\log(\psi_X(t)) = i\gamma t - \frac{1}{2}\sigma^2 t^2 + \int_{-\infty}^{0} \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) dM(u) + \\ + \int_{0}^{\infty} \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) dN(u), \qquad \text{(C.1)}$$

where $\sigma \geq 0$, $\gamma \in \mathbb{R}$, M and N are non-decreasing functions with

$$M(-\infty) = N(\infty) = 0$$

and

$$\int_{-\epsilon}^{0} u^2 dM(u) \int_{0}^{\epsilon} u^2 dN(u) < \infty$$

[1] The proof of this theorem can be find in the text *Limit distributions for sums of independent random variables*, written by Gnedenko and Kolmogorov in 1954.

for all $\epsilon > 0$. *The representation is unique.*
The equation (C.1) *is called* Levy's formula.

Theorem C.0.6. [2] *In order that the distribution* $F(x)$ *should be, for an appropriate choice of constants* A_n, *the weak limit of the distribution of*

$$Z_n = X_{n,1} + X_{n,2} + \ldots + X_{n,k_n} - A_n, \quad as \quad n \to \infty, \tag{C.2}$$

where the $X_{n,k}$ *are uniformly asymptotically negligible, it is necessary and sufficient that* $F(x)$ *is infinitely divisible.*

Conditions for the convergence to a particular $F(x)$ can be expressed in the following way.

Theorem C.0.7. [3] *In order that, for an appropriate choice of the* A_n, *the distribution of* (C.2) *should converge to* $F(x)$, *it is necessary and sufficient that*

$$\sum_{k=1}^{k_n} F_{n,k}(x) \to M(x), \quad x < 0$$

$$\sum_{k=1}^{k_n} (1 - F_{n,k}(x)) \to N(x), \quad x > 0$$

at every point of continuity of $M(x)$ *and* $N(x)$ *and*

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \sum_{k=1}^{k_n} \left[\int_{|x|<\epsilon} x^2 dF_{n,k}(x) - (x dF_{n,k}(x))^2 \right] =$$

$$= \lim_{\epsilon \to 0} \liminf_{n \to \infty} \sum_{k=1}^{k_n} \left[\int_{|x|<\epsilon} x^2 dF_{n,k}(x) - (x dF_{n,k}(x))^2 \right] = \sigma^2,$$

where $M(x), N(x)$ *and* σ^2 *are as in the Levy formula for* $F(x)$ *and* $F_{n,k}$ *is the distribution of* $X_{n,k}$.

A distribution function $F(x)$ is said to be *stable* if, for any $a_1, a_2 > 0$ and any b_1, b_2, there exists constants $a > 0$ and b such that

$$F(a_1 x + b_1) * F(a_2 x + b_2) = F(ax + b). \tag{C.3}$$

In terms of the characteristic function $\psi(t)$ of $F(x)$, (C.3) becomes:

$$\psi\left(\frac{t}{a_1}\right) \psi\left(\frac{t}{a_2}\right) = \psi\left(\frac{t}{a}\right) e^{-ibt}. \tag{C.4}$$

[2] See [1].
[3] See [1].

Theorem C.0.8. *In order that a distribution $F(x)$ is stable, it is necessary and sufficient that $F(x)$ is infinitely divisible, with Levy representation either*

$$\log(\psi_X(t)) = i\gamma t + \int_{-\infty}^{0} \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) dM(u) + \\ + \int_{0}^{\infty} \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) dN(u), \tag{C.5}$$

with

$$M(u) = c_1(-u)^{-\alpha}, \quad N(u) = -c_2^{\alpha}, \\ 0 < \alpha < 2, \quad c_1 \geq 0, c_2 \geq 0, c_1 + c_2 > 0$$

or

$$\log(\psi_X(t)) = i\gamma t - \frac{1}{2}\sigma^2 t^2. \tag{C.6}$$

Proof. The infinite divisibility of $F(x)$ follows from the results above. Consequently $\log(\psi(t))$ has the Levy representation (C.1). The equation (C.4) gives

$$\log\left(\psi\left(\frac{t}{a_1}\right)\right) + \log\left(\psi\left(\frac{t}{a_2}\right)\right) = \log\left(\psi\left(\frac{t}{a}\right)e^{-ibt}\right). \tag{C.7}$$

Comparing this with (C.1) we have

$$\begin{aligned} & i\gamma a^{-1}t - \frac{1}{2}\sigma^2 a^{-2}t^2 + \int_{-\infty}^{0} \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) dM(au) + \\ & + \int_{0}^{\infty} \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) dN(au) = \\ =& i\gamma a_1^{-1}t - \frac{1}{2}\sigma^2 a_1^{-2}t^2 + \int_{-\infty}^{0} \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) dM(a_1 u) + \\ & + \int_{0}^{\infty} \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) dN(a_1 u) + \\ & + i\gamma a_2^{-1}t - \frac{1}{2}\sigma^2 a_2^{-2}t^2 + \int_{-\infty}^{0} \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) dM(a_2 u) + \\ & + \int_{0}^{\infty} \left(e^{iut} - 1 - \frac{iut}{1+u^2}\right) dN(a_2 u) + \\ & + ibt. \end{aligned}$$

The uniqueness of the Levy representation therefore implies that

$$\sigma^2(a^{-2} - a_1^{-2} - a_2^{-2}) = 0, \tag{C.8}$$

$$M(au) = M(a_1 u) + M(a_2 u), \quad \text{if} \quad u < 0, \tag{C.9}$$

$$N(au) = N(a_1 u) + N(a_2 u), \quad \text{if} \quad u > 0. \tag{C.10}$$

Suppose that M is not identically zero and write

$$m(x) = M(e^{-x}), x \in \mathbb{R}.$$

From the second equation in (C.8) it follows that, for any $\lambda_1, \lambda_2, \ldots, \lambda_n$, there exists $\lambda = \lambda(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that, for all x,

$$m(x + \lambda) = m(x + \lambda_1) + \ldots + m(x + \lambda_n). \tag{C.11}$$

Setting $\lambda_1 = \ldots = \lambda_n = 0$, there exists $\lambda = \lambda(n)$ such that

$$m(x + \lambda)nm(x). \tag{C.12}$$

If $\frac{p}{q}$ is any positive rational in its lowest terms, define

$$\lambda\left(\frac{p}{q}\right) = \lambda(p) - \lambda(q);$$

then (C.12) implies that

$$\frac{p}{q}m(x) = pm\left(x - \lambda(q)\right) = m\left(x + \lambda(p) - \lambda(q)\right) = m\left(x + \lambda(\frac{p}{q})\right).$$

Thus, for any rational $r > 0$,

$$m\left(x + \lambda(r)\right) = rm(x). \tag{C.13}$$

Since M is non increasing, m is non increasing, and so therefore is the function λ define on the positive rationals. Consequently, λ has right and left limits at all $s > 0$. From (C.13) these are equal and $\lambda(s)$ is define as a non increasing continuous function on $s > 0$, satisfying

$$m\left(x + \lambda(s)\right) = sm(x). \tag{C.14}$$

Moreover, it follows from this equation that

$$\lim_{s \to 0} \lambda(s) = \infty,$$
$$\lim_{s \to \infty} \lambda(s) = -\infty.$$

Since m is not identically zero, we may assume that $m(0) \neq 0$ and write $m_1(x) = \dfrac{m(x)}{m(0)}$. Let x_1, x_2 arbitrary, and choose s_1, s_2 so that

$$\lambda(s_1) = x_1, \quad \lambda(s_2) = x_2.$$

Then

$$s_1 m(0) = m(x_1), \quad s_2 m(0) = m(x_2), \quad s2m(x_1) = m(x_1 + x_2),$$

so that

$$m(x_1 + x_2) = m(x_1)m(x_2). \tag{C.15}$$

Since m_1 is non negative, non increasing and not identically zero, (C.15) shows that $m_1 > 0$ and then $m_2 = \log(m_1)$ is monotonic and satisfies

$$m(x_1 + x_2) = m(x_1) + m(x_2). \tag{C.16}$$

The only monotonic functions satisfying this equation are of the form

$$m_2(x) = \alpha x.$$

Since $M(-\infty) = 0$, this implies that

$$\begin{aligned} &m_1(x) = e^{-\alpha x}, \\ &M(u) = c_1(-u)^{-\alpha}, \quad \alpha > 0, c_1 > 0. \end{aligned}$$

As the integral

$$\int_{-1}^{0} u^2 dM(u) = c_1 \alpha \int_0^1 u^{1-\alpha} du$$

must converge, we have $\alpha < 2$.
Thus finally

$$M(u) = c_1(-u)^{-\alpha}, \quad 0 < \alpha < 2, c_1 \geq 0. \tag{C.17}$$

In a similar way

$$N(u) = -c_1(u)^{-\beta}, \quad 0 < \beta < 2, c_2 \geq 0. \tag{C.18}$$

Taking $a_1 = a_2 = 1$ in the second and in the third equation in (C.8) we have

$$a^{-\alpha} = a^{-\beta} = 2, \tag{C.19}$$

whence $\alpha = \beta$.
Moreover the first equation in (C.8) becomes

$$\sigma^2(a^{-2} - 2) = 0.$$

This is incompatible with (C.19) unless $\sigma^2 = 0$, so that either $\sigma^2 = 0$ or $M(u) = N(u) = 0$ for all u. □

Lemma C.0.9. *Let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$ with distribution $F(x)$.*
$F(x)$ is stable if and only if the characteristic function of X can be expressible in the form

$$\log(\psi(t)) = i\gamma t - c|t|^{\alpha}\left(1 - i\beta\frac{t}{|t|}\omega(t,\alpha)\right), \tag{C.20}$$

where α, β, γ and c are constants such that $c \geq 0$, $0 < \alpha \leq 2$, $|\beta| \leq 1$ and

$$\omega(t,\alpha) = \begin{cases} \tan\left(\frac{1}{2}\pi\alpha\right), & \alpha \neq 1, \\ \frac{2}{\pi}\log|t|, & \alpha = 1. \end{cases}$$

Proof. Examine (C.20) in three different cases as we did in the previous theorem.

1. $0 < \alpha < 1$
   In this case the integrals

   $$\int_{-\infty}^{0} \frac{u}{1+u^2}\frac{du}{|u|^{1+\alpha}}$$

   and

   $$\int_{0}^{\infty} \frac{u}{1+u^2}\frac{du}{u^{1+\alpha}}$$

   are finite and $\log(\psi(t))$ can be written, for some γ', as

   $$\log(\psi(t)) = i\gamma' t + \alpha c_1 \int_{-\infty}^{0}(e^{itu}-1)\frac{du}{|u|^{1+\alpha}} + \alpha c_2 \int_{0}^{\infty}(e^{itu}-1)\frac{du}{u^{1+\alpha}}.$$

   Therefore, in $t > 0$,

   $$\log(\psi(t)) = i\gamma' t + \alpha t^{\alpha}\left[c_1 \int_{0}^{\infty}(e^{iu}-1)\frac{du}{u^{1+\alpha}} + c_2 \int_{0}^{\infty}(e^{iu}-1)\frac{du}{u^{1+\alpha}}\right].$$

   The function

   $$\frac{e^{iu}-1}{u^{1+\alpha}}$$

   is analytic in the complex plane cut along the positive half of the real axis. Integrating it round a contour consisting of the segment (r, R), $0 < r < R$, the circular arc with centre 0 from R to iR, the line segment

(iR, ir) and the circular arc from ir to r, we obtain, as $R \to \infty$ and as $r \to 0$,

$$\int_0^{\infty} (e^{iu} - 1) \frac{du}{u^{1+\alpha}} = \int_0^{i\infty} (e^{iu} - 1) \frac{du}{u^{1+\alpha}} =$$
$$= \exp\left(-\frac{1}{2} i\pi\alpha\right) L(\alpha),$$

where

$$L(\alpha) = \int_0^{\infty} (e^{-u} - 1) \frac{du}{u^{1+\alpha}} = -\frac{\Gamma(1-\alpha)}{\alpha} < 0.$$

Similarly

$$\int_0^{\infty} (e^{-iu} - 1) \frac{du}{u^{1+\alpha}} = \int_0^{i\infty} (e^{-iu} - 1) \frac{du}{u^{1+\alpha}} =$$
$$= \exp\left(\frac{1}{2} i\pi\alpha\right) L(\alpha).$$

Therefore, for $t > 0$,

$$\log(\psi(t)) = i\gamma' t + \alpha L(\alpha) t^{\alpha} \left[(c_1 + c_2) \cos\left(\frac{1}{2}\pi\alpha\right) + (c_1 - c_2) \sin\left(\frac{1}{2}\pi\alpha\right) \right] =$$
$$= i\gamma' t - c t^{\alpha} \left[1 - i\beta \tan\left(\frac{1}{2}\pi\alpha\right) \right],$$

where

$$c = -\alpha L(\alpha)(c_1 + c_2) \cos\left(\frac{1}{2}\pi\alpha\right) \geq 0,$$
$$\beta = \frac{c_1 - c_2}{c_1 + c_2}, \quad |\beta| \leq 1.$$

For $t < 0$,

$$\log(\psi(t)) = \log \overline{f(-t)} = i\gamma' t - c t^{\alpha} \left[1 - i\beta \tan\left(\frac{1}{2}\pi\alpha\right) \right].$$

Hence (C.20) holds for all t.

2. $1 < \alpha < 2$

For this case we can throw (C.20) into the form, for $t > 0$,

$$\begin{aligned}
\log(\psi(t)) = i\gamma' t + c_1\alpha \int_{-\infty}^{0} (e^{itu} - 1 - itu)\frac{du}{|u|^{1+\alpha}} + \\
+ c_2\alpha \int_{0}^{\infty} (e^{itu} - 1 - itu)\frac{du}{u^{1+\alpha}} = \\
= i\gamma' t + \alpha t^{\alpha}\left[c_1 \int_{0}^{\infty} (e^{-iu} - 1 - itu)\frac{du}{u^{1+\alpha}}\right] + \\
+ \alpha t^{\alpha}\left[c_2 \int_{0}^{\infty} (e^{-iu} - 1 - itu)\frac{du}{u^{1+\alpha}}\right].
\end{aligned}$$

Integrating the function

$$\frac{e^{-iu} - 1 - itu}{u^{1+\alpha}}$$

round the same contour as above we obtain

$$\int_{0}^{\infty} (e^{-iu} - 1 - itu)\frac{du}{u^{1+\alpha}} = \exp\left(-\frac{1}{2}i\pi\alpha\right) M(\alpha)$$

and

$$\int_{0}^{\infty} (e^{iu} - 1 - itu)\frac{du}{u^{1+\alpha}} = \exp\left(\frac{1}{2}i\pi\alpha\right) M(\alpha).$$

where

$$M(\alpha) = \int_{0}^{\infty} (e^{-u} - 1 + u)\frac{du}{u^{1+\alpha}} = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)} > 0.$$

Proceeding as before, we deduce that (C.20) holds with

$$\begin{aligned}
c &= -\alpha M(\alpha)(c_1 + c_2)\cos\left(\frac{1}{2}\pi\alpha\right) \geq 0, \\
\beta &= \frac{c_1 - c_2}{c_1 + c_2}, \quad |\beta| \leq 1.
\end{aligned}$$

3. $\alpha = 1$
   Using the fact that

$$\int_{0}^{\infty} \frac{1 - \cos u}{u^2} du = \frac{1}{2}\pi,$$

we have

$$\int_0^\infty (e^{itu} - 1 - \frac{itu}{1+u^2})\frac{du}{u^2} =$$

$$= \int_0^\infty \frac{\cos tu - 1}{u^2} du + i\int_0^\infty \left(\sin tu - \frac{ut}{1+u^2}\right)\frac{du}{u^2} =$$

$$= -\frac{1}{2}\pi + i\lim_{\epsilon\to 0}\left[\int_\epsilon^\infty \frac{\sin tu}{u^2}du - t\int_\epsilon^\infty \frac{du}{u(1+u^2)}\right] =$$

$$= -\frac{1}{2}\pi - i\lim_{\epsilon\to 0}\left[-t\int_\epsilon^\infty \frac{\sin u}{u^2}du + t\int_\epsilon^\infty \left(\frac{\sin u}{u^2} - \frac{1}{1+u^2}\right)du\right] =$$

$$= -\frac{1}{2}\pi - it\lim_{\epsilon\to 0}\int_\epsilon^{\epsilon t}\frac{du}{u} + it\int_0^\infty \left(\frac{\sin u}{u^2} - \frac{1}{1+u^2}\right)du =$$

$$= -\frac{1}{2}\pi - it\log t + it\Gamma.$$

Thus (C.20) is satisfied with

$$c = \frac{1}{2}(c_1 + c_2)$$

$$\beta = \frac{c_1 - c_2}{c_1 + c_2}, \quad |\beta| \le 1.$$

□

Appendix D

The asympthotic behaviour of the Hamiltonian

Let $N \in \mathbb{N}$. Let $\overline{\sigma}$ be a configuration over N spins which take values in the set $\{-1, 1\}$.
Consider the hamiltonian defined for the Curie Weiss model by parameters $J > 0$ and h:

$$H_N(\overline{\sigma}) = -\frac{J}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i.$$

Without loss of generality, we will work under the hypothesis that $h = 0$.
Suppose to divide the spins in two sets P_1 and P_2 respectively with cardinality N_1 and N_2.
Set the relative sizes of the sets as

$$\alpha = \frac{N_1}{N} \quad \text{and} \quad 1 - \alpha = \frac{N_2}{N}.$$

We start working with the following hamiltonian:

$$\tilde{H_N}(\overline{\sigma}) = \tilde{H_N}^{(1)} + \tilde{H_N}^{(2)} + \tilde{H_N}^{(12)}, \tag{D.1}$$

where:

$$
\begin{aligned}
\tilde{H_N}^{(1)} &= -\frac{\alpha J}{2(\alpha N-1)} \sum_{i\neq j\in P_1} \sigma_i\sigma_j,\\
\tilde{H_N}^{(2)} &= -\frac{(1-\alpha) J}{2((1-\alpha) N-1)} \sum_{i\neq j\in P_2} \sigma_i\sigma_j,\\
\tilde{H_N}^{(12)} &= -\frac{J}{N} \sum_{i\in P_1, j\in P_2} \sigma_i\sigma_j.
\end{aligned}
$$

Lemma D.0.10. *Under the notations above:*

$$
\langle \tilde{H_N}\rangle_{BG} = \langle \tilde{H_N}^{(1)}\rangle_{BG} + \langle \tilde{H_N}^{(2)}\rangle_{BG} + \langle \tilde{H_N}^{(12)}\rangle_{BG}
$$

Proof.

$$
\begin{aligned}
\langle \tilde{H_N}^{(1)}\rangle_{BG} =& -\frac{\alpha J}{2(\alpha N-1)}\langle \sum_{i\neq j\in P_1} \sigma_i\sigma_j\rangle_{BG} =\\
=& -\frac{\alpha J(\alpha N-1)(\alpha N)}{2(\alpha N-1)}\langle\sigma_i\sigma_j\rangle_{BG} =\\
=& -\frac{\alpha^2 JN}{2}\langle\sigma_i\sigma_j\rangle_{BG}\\
\langle \tilde{H_N}^{(2)}\rangle_{BG} =& -\frac{(1-\alpha) J}{2((1-\alpha) N-1)}\langle \sum_{i\neq j\in P_2} \sigma_i\sigma_j\rangle_{BG} =\\
=& -\frac{(1-\alpha)J((1-\alpha)N-1)((1-\alpha)N)}{2((1-\alpha)N-1)}\langle\sigma_i\sigma_j\rangle_{BG} =\\
=& -\frac{(1-\alpha)^2 JN}{2}\langle\sigma_i\sigma_j\rangle_{BG}\\
\langle \tilde{H_N}^{(12)}\rangle_{BG} =& -\frac{J}{N}\langle \sum_{i\neq j\in P_1} \sigma_i\sigma_j\rangle_{BG} =\\
=& -\frac{J(\alpha N)((1-\alpha)N)}{N}\langle\sigma_i\sigma_j\rangle_{BG} =\\
=& -J\alpha(1-\alpha)N\langle\sigma_i\sigma_j\rangle_{BG}
\end{aligned}
$$

In conclusion:

$$
\begin{aligned}
&\langle \tilde{H_N}^{(1)} \rangle_{BG} + \langle \tilde{H_N}^{(2)} \rangle_{BG} + \langle \tilde{H_N}^{(12)} \rangle_{BG} = \\
=& -\frac{\alpha^2 JN}{2} \langle \sigma_i \sigma_j \rangle_{BG} - \frac{(1-\alpha)^2 JN}{2} \langle \sigma_i \sigma_j \rangle_{BG} - J\alpha(1-\alpha) N \langle \sigma_i \sigma_j \rangle_{BG} = \\
=& -\frac{JN}{2} \langle \sigma_i \sigma_j \rangle_{BG} \left[\alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha) \right] = \\
=& -\frac{JN}{2} \langle \sigma_i \sigma_j \rangle_{BG} = \\
=& \langle \tilde{H_N} \rangle_{BG}.
\end{aligned}
$$

□

We proceede working with the following hamiltonian:

$$
\hat{H_N}(\overline{\sigma}) = \hat{H_N}^{(1)} + \hat{H_N}^{(2)} + \hat{H_N}^{(12)}, \tag{D.2}
$$

where:

$$
\begin{aligned}
\hat{H_N}^{(1)} &= -\frac{J}{2N} \sum_{i,j \in P_1} \sigma_i \sigma_j, \\
\hat{H_N}^{(2)} &= -\frac{J}{2N} \sum_{i,j \in P_2} \sigma_i \sigma_j, \\
\hat{H_N}^{(12)} &= -\frac{J}{N} sum_{i \in P_1, j \in P_2} \sigma_i \sigma_j.
\end{aligned}
$$

Lemma D.0.11. *Under the notations above, it holds:*

$$
\lim_{N \to \infty} \frac{\hat{H}_N}{N} = \lim N \to \infty \frac{\tilde{H}_N}{N}, \tag{D.3}
$$

or equivalently

$$
\hat{H}_N = \tilde{H}_N + o(1).
$$

Proof.

$$
\begin{aligned}
\hat{H_N}^{(1)} &= -\frac{J\alpha(\alpha N - 1)}{2\alpha N(\alpha N - 1)} \sum_{i \neq j \in P_1} \sigma_i \sigma_j - \frac{J\alpha N}{2N} = \\
&= -\frac{J\alpha^2 N}{2\alpha N(\alpha N - 1)} \sum_{i \neq j \in P_1} \sigma_i \sigma_j - \frac{J\alpha}{2\alpha N(\alpha N - 1)} \sum_{i \neq j \in P_1} \sigma_i \sigma_j - \frac{J\alpha}{2} = \\
&= -\frac{J\alpha}{2(\alpha N - 1)} \sum_{i \neq j \in P_1} \sigma_i \sigma_j - \frac{J}{2N(\alpha N - 1)} \sum_{i \neq j \in P_1} \sigma_i \sigma_j - \frac{J\alpha}{2} = \\
&= \tilde{H_N}^{(1)} + o(1)
\end{aligned}
$$

and

$$\begin{aligned}
\hat{H_N}^{(2)} &= -\frac{J(1-\alpha)((1-\alpha)N-1)}{2(1-\alpha)N((1-\alpha)N-1)}\sum_{i\neq j\in P_2}\sigma_i\sigma_j - \frac{J(1-\alpha)N}{2N} = \\
&= -\frac{J(1-\alpha)^2N}{2(1-\alpha)N((1-\alpha)N-1)}\sum_{i\neq j\in P_2}\sigma_i\sigma_j - \\
&\quad -\frac{J(1-\alpha)}{2(1-\alpha)N((1-\alpha)N-1)}\sum_{i\neq j\in P_2}\sigma_i\sigma_j - \frac{J(1-\alpha)}{2} = \\
&= -\frac{J(1-\alpha)}{2((1-\alpha)N-1)}\sum_{i\neq j\in P_2}\sigma_i\sigma_j - \frac{J}{2N((1-\alpha)N-1)}\sum_{i\neq j\in P_2}\sigma_i\sigma_j - \frac{J(1-\alpha)}{2} = \\
&= \tilde{H_N}^{(2)} + o(1).
\end{aligned}$$

Thus

$$\hat{H_N} = \tilde{H_N} + o(1).$$

□

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