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Bouncing Black Holes in Loop Quantum Gravity

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Abstract

In questo lavoro viene presentato un recente modello di buco nero che implementa le proprietà quantistiche di quelle regioni dello spaziotempo dove non possono essere ignorate, pena l'implicazione di paradossi concettuali e fenomenologici. In suddetto modello, la regione di spaziotempo dominata da comportamenti quantistici si estende oltre l'orizzonte del buco nero e suscita un'inversione, o più precisamente un effetto tunneling, della traiettoria di collasso della stella in una traiettoria di espansione simmetrica nel tempo. L'inversione, di natura quanto-gravitazionale, impiega un tempo molto lungo per chi assiste al fenomeno a grandi distanze, ma inferiore al tempo di evaporazione del buco nero tramite radiazione di Hawking, trascurata in prima istanza e considerata come un effetto dissipativo da studiarsi in un secondo tempo. Il resto dello spaziotempo, al di fuori della regione quantistica, soddisfa le equazioni di Einstein. Successivamente viene presentata la teoria della Gravità Quantistica a Loop (LQG) che permetterebbe di studiare la dinamica della regione quantistica senza far riferimento a una metrica classica, ma facendo leva sul contenuto relazionale del *tessuto* spaziotemporale. Il campo gravitazionale viene riformulato in termini di variabili hamiltoniane in uno spazio delle fasi vincolato e con simmetria di gauge, successivamente promosse a operatori su uno spazio di Hilbert finito-dimensionale, legato a una vantaggiosa discretizzazione dello spaziotempo operata a tal scopo. La teoria permette la definizione di un'ampiezza di transizione fra stati quantistici di geometria spaziotemporale e tale concetto è applicabile allo studio della regione quantistica nel modello di buco nero proposto. Infine vengono poste le basi preparatorie per un calcolo in LQG dell'ampiezza di transizione del fenomeno di rimbalzo quantistico all' interno del buco nero, e di conseguenza per un calcolo puramente quantistico del tempo di tale rimbalzo nel riferimento di osservatori statici a grande distanza da esso, utile per trattare a posteriori un modello che tenga conto della radiazione di Hawking e, auspicatamente, fornisca una possibile risoluzione dei problemi legati alla sua esistenza.

In this work I present a recent model of black hole that includes the quantum properties of those spacetime regions where they cannot be disregarded, if conceptual and phenomenological paradoxes are to be avoided. In such mode, the region of spacetime characterized by quantum behaviour extends slightly outside the black hole horizon and provokes the inversion, or more precisely the quantum tunnelling effect, of the trajectory of collapse of the star into a trajectory of expansion time-symmetric to the first one. The inversion, of quantum-gravitational nature, takes a very long time from the perspective of someone witnessing the phenomenon at large distance, but this time is shorter than the time of evaporation of the black hole due to Hawking's radiation, which is regarded as a dissipative phenomenon and is left for a further study. The rest of spacetime, outside the quantum region, satisfies Einstein's equations. Afterwards, I introduce the theory of Loop Quantum Gravity (LQG), that is supposed to allow the study of dynamics of the quantum region without referring to a classical metric, but instead appealing to the relational content in the *weave* of spacetime. The gravitational field is encoded into hamiltonian variables living in a constrained phase space featuring gauge symmetry, and such variables are then promoted to operators acting on a finite-dimensional Hilbert space related to a convenient discretization of classical spacetime performed to this aim. The theory allows for the definition of transition amplitude between quantum states of spacetime geometry, and this notion is usable in the study of the quantum region in our proposed model. Finally, the preparatory foundation is given for a LQG computation of the transition amplitude of the quantum bounce, and consequently for the computation of the time of such bounce with respect to static observers far from it in space, useful to analyse, *a posteriori*, a further model that may include Hawking's radiation and, hopefully, provide a possible solution to the problems it involves.

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Contents

Introduction

At present day, black holes have become conventional astrophysical objects. Yet, they still prove to resist any theoretical attempt to completely understand what happens inside them, which in addition is not supported by consolidated experimental practice of astrophysical domain, although encouraging parallelisms have been found between the behaviour of astrophysical black holes and specific condensed matter situations that would support experimental studies (e.g. acoustic black holes) to help overcome this difficulty.
Astrophysical observations indicate that general relativity (GR) describes well the region surrounding the event horizon of a black hole, together with a substantial region inside the horizon. But certainly GR fails to describe Nature at the smallest radii of the inner region, because nothing prevents quantum mechanics from affecting the zone of highest curvature, and because classical GR becomes ill-defined at the center of the black hole anyway.
Moreover, studying the evolution of quantum fields in a spacetime that is curved according to the laws of GR results in unpleasant conclusions that undermine the foundations of our knowledge of Nature in flat Minkowskian spacetime. It seems clear that, if we are not willing to question our predictive achievements in *quantum* mechanics in *flat* spacetime and, on the other hand, in *classically curved* spacetime, the merge of these two frameworks cannot be derived by their respective successfully understood theories, since the two points of view are intrinsically incomplete. We should call for a theory that might embrace the two but may not necessarily share their languages or a smart fusion of them. History of physics teaches us that the most significant advances in our modelling of physical phenomena have brought an evolution in its language too. This should be the case also when confronting the problem of compatibility between quantum mechanics and general relativity, a problem possibly due to the absence of a common foundation in terms of background, principles and formalism. However, the physics of black holes offers the invaluable chance of inspecting the roots of such desirable embracing theory, and test eventual candidates to this title.
In this work, I aim to explain a recently proposed model for the later stages of a gravitational collapse that, while avoiding the most relevant breakdowns of GR, remains a solution of the classical Einstein equations except where quantum effects cannot in principle be disregarded. Importantly, this model is naturally set for being studied within the framework of a promising quantum theory of gravity, called Loop Quantum Gravity, which I intend to introduce for the final purpose of giving the general outline of how, in practice, it should be put into action in the specific case of the above-mentioned new model.
The thesis is organized as follows:

1. A summary of the most relevant problems(in relation with black holes) of classical GR is given;

2. The model of the bouncing black hole is presented, featuring both quantum and classical behaviour and apparently overcoming the mentioned problems;

3. A brief analysis follows of the quantum principles that would serve as guidelines for the construction of a quantum theory of spacetime;

4. General relativity is reformulated in a fashion suitable for Dirac's quantization program: its hamiltonian analysis as a gauge theory is presented;
5. Quantization *á la* Dirac is performed and leads to an incomplete but strongly illustrative model of Loop Quantum Gravity (LQG);
6. The methods that should be borrowed from other theories to improve such model are rapidly presented and plugged into it;
7. A slightly different approach is then explained to enforce and make explicit the presence of local covariance that should be inherited from classical GR;
8. A theory of dynamic evolution is sketched within the framework of LQG;
9. Finally model of bouncing black holes is reformulated in the language so far presented. A first look at the expected approach is provided.

The following chapters mirror this ordering.

Chapter 1

Paradoxical aspects of semiclassical GR

General Relativity (GR) is a consistent theory of spacetime that certainly describes gravitational phenomena at large scales, i.e. length and energy scales much larger than those studied in particle physics, and appears to be in very good agreement with known experiments and experiences. However, it seems to leave space to unpalatable situations like matter compressed at huge densities in a microscopic volume (singularities). Moreover, a semiclassical study of quantum fields coupled to gravity produces unwanted phenomena that undermine the deepest principles on which quantum mechanics (the established theory of the microscopic world)is based, like unitary evolution of quantum processes.
In what follows we briefly review the "bad" consequences of GR that inspired and continue to drive the search for a quantum theory of gravity.

1.1 Gravitational collapse

According to Birkhoff's theorem, the most general solution to the vacuum Einstein's equation outside a static, chargless, spherically symmetric, self-gravitating astrophysical object is uniquely determined and given by the Schwarzschild line element (we use signature $(-,+,+,+)$):

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.1}$$

where:

- G is the gravitational constant and c the speed of light. Unless necessary, we will suppress them by working in natural units;
- $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ is the standard solid angle element;
- the coordinates (t, r, θ, ϕ) are referred to an observer asymptotically far from the star: $M/r \to 0$. At such distance, spacetime tends to the flat Minkowski space, coordinatized with the time coordinate t and spherical spatial coordinates. We say that Schwarzschild spacetime is *asymptotically flat.*
- the spherical coordinates (r, θ, ϕ) are centered in the center of the star, so as to recover the standard euclidean definition for the area of a 2-sphere of radius r as $A = 4\pi r^2$;

- M is the mass of the star measured by the observer at infinity, and we suppose it be completely localized in the center $r = 0$.

This line element is singular in two radii: $r = 0$ and $r = 2M$. The singularity in $r = 2M$ is only due to the chosen coordinates, and can instead be regularized using the well known extended coordinates of the Schwarzschild metric, such as the Eddington-Finkelstein coordinates and the (maximally extended) Kruskal coordinates. We will not talk about them, and we refer the reader to classical literature of general relativity. This radius is called the Schwarzschild radius of the star and the corresponding 2-sphere is a null trapped surface of spacetime, in the sense that both ingoing and outgoing light rays emanating inside it and oriented towards the future will never be able to reach greater radii, and will instead converge to the other singular point $r = 0$. For this reason, the region bounded by the 2-sphere $r = 2M$ is causally disconnected from the rest of spacetime and is called *black hole region*[1], while its boundary 2-sphere $r = 2M$ is called *event horizon*.
The other singularity $r = 0$, instead, cannot be removed by any choice of coordinates. It is an *essential singularity*, and every geodesic reaching this point with a finite affine parameter cannot be extended to a further affine parameter. A spacetime featuring such singularities is called *null geodesically incomplete*, with reference to the incapability of extending every geodesic to an arbitrary value of their affine parameter.
The singularity in $r = 0$ is by all means a breakdown in the theory of GR, or at least in its classical establishment. The theory at our current disposal is ill-defined in such situation. The efforts and intuitions by R. Penrose, S. Hawking, R. Geroch and others have led to a solid generalization of the occurrence of these breakdowns, in the form of the singularity theorems.
First of all, we can roughly distinguish the notion of singularity between two cases:

* *spacelike singularity*, a situation where matter and light are forced to be compressed to a point;
* *timelike singularity*, a situation where certain light rays come from a region with diverging curvature.

Next, we summarize here Penrose's theorem (1965) stating that in a spacetime $(\mathcal{M}, g_{\mu\nu})$ (where we denote each event with x) containing a trapped surface the three following conditions cannot hold together:

◇ For every time-like vector field $X^\mu(x)$, given the stress-energy tensor $T_{\mu\nu}(x)$:

$$\rho(x) = T_{\mu\nu}(x)X^\mu(x)X^\nu(x) \geq 0 \qquad \forall x \in \mathcal{M} \tag{1.2}$$

namely, the total mass-energy density measured by every observer in each event along her causal worldline is non negative;

◇ $\mathcal{M}$ admits a non compact Cauchy surface, namely $\mathcal{M}$ is globally hyperbolic[2];

◇ $\mathcal{M}$ is null geodesically complete.

The theorem states that, under suitable energy conditions, a globally hyperbolic spacetime containing a trapped region inevitably features a singularity, either spacelike (e.g. black holes) or timelike (e.g. white holes).
Actually, the assumption that the mass of our spherical object (the star) is entirely contained in the point $r = 0$ is in principle unphysical. Typical astrophysical objects have radius R much greater than their Schwarzschild radius $r = 2M$, and for a long time the actual absence of black hole regions was a common belief, because their existence would imply that some stars have collapsed until an unthinkably small volume. However, the theoretical basis for the existence of BH regions

[1] More precisely, in the representation of Penrose's conformal diagrams, the black hole region is defined as the entire spacetime manifold minus the causal past of the future null infinity: $\mathcal{B} = \mathcal{M}/\mathcal{I}^-(\mathscr{I}^+)$. We again refer the reader not familiar with this terminology to the classical literature.

[2] See appendix A for definition.

was given in the classical papers by Oppenheimer et al. in 1939, where the scenario is already presented in its essential features: a star is maintained in equilibrium by the positive pressure of the emitted radiation and the thermal energy of the burning nuclei, balanced by the negative pressure of the self-gravitational force; when the star exhausts its fuel, the only source of positive pressure is the Pauli exclusion principle; if the mass is greater than a critical limit, this pressure is not sufficient to counterbalance the gravitational force and the star collapses under its own weight, until it reaches the Schwarzschild radius: at this point nothing can prevent the star to contract indefinitely, and a BH forms. Again, classical GR does not exclude the possibility of the formation of ill-defined points like the final, infinitely dense point to which a star collapses.
The conformal diagram of the gravitational collapse of a star is depicted in the following picture.

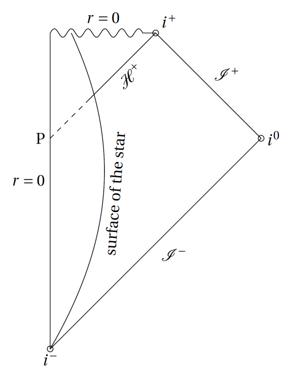


Figure 1.1: Conformal diagram of a collapsing star. The event horizon forms at point P, i.e. before the surface of the star reaches the Schwarzschild radius.

We can simplify this diagram by ignoring the kinematical details of the collapse, motivated by the fact that the resulting spacetime is only influenced by the mass of the object at study. The best way to do so is to look at the star as a null shell of total energy M collapsing at speed of light towards its center. The resulting line element describes the so-called Vaidya spacetime, named after its proposer. In advanced null coordinates it reads:

$$ds^2 = -\left(1 - \frac{2M}{r}\theta(v - v_0)\right) dv^2 + 2dvdr + r^2 d\Omega^2 \ , \tag{1.3}$$

where $\theta(v - v_0)$ is the Heaviside distribution, taking values 0 for $v < v_0$ and 1 for $v > v_0$. $v = v_0$ is nothing but the null ingoing trajectory of the star's spherical shell collapsing towards the origin of the coordinate system. Therefore, for $v < v_0$ spacetime is flat, as it should be inside a star, while for $v > v_0$ we have Schwarzschild spacetime. The conformal diagram of the simplified collapse is depicted in Figure 1.2.

1.2 Hawking radiation and trans-Planckian problem

In the environment of a curved spacetime produced by the presence of a non trivial stress-energy tensor, one can decide to bring along the teachings of quantum mechanics and study phenomena commonly located in smooth, flat Minkowskian spacetime. The resulting theory is called Quantum Field Theory on Curved Spacetime, and is by no means a fully quantum theory, since we lack a quantum field for the gravitational interaction: the presence of gravity is only encoded in adding a non trivial curvature to the still smooth, classically evolving spacetime hosting our quantum processes. We rather refer to it as a semiclassical theory.
A simple, and yet tremendously destabilizing, example of merging quantum mechanics with general

relativity was first sketched by Hawking in 1974. We are not going to explain it thoroughly, and will just give a simplified summary to grasp the problem it opens. As usual, we refer the reader to Hawking's original paper [18] and to the classical literature.

Pick a massless scalar field in a Vaidya spacetime[3], i.e. in a non static spacetime where a star is inexorably collapsing at speed of light to form a singularity in its center. In the two spacetime regions separated by the null trajectory of the shell, we can decompose the solutions of the covariant (i.e. coupled to gravity) Klein-Gordon equation into positive frequency and negative frequency modes, the sign of the frequency be related to the orientation of the killing vector in each of the two regions. What is different from QFT in flat spacetime is that such decomposition is not unique: positive frequency modes in the Minkowskian patch can be expressed as functions of both positive frequency and negative frequency modes in the Schwarzschild patch. In the second quantization language, this results in a striking ambiguity: the definition of a vacuum state is no more a global notion. In particular, if we consider a vacuum state at past null infinity in the Minkowski patch, and take into account the only normal modes with positive energy that, once the black hole is formed, propagate very closely to the horizon towards future null infinity, an observer waiting over there will argue (after a sufficiently long time) that, according to her "personal" decomposition in normal modes, the vacuum state is a collection of thermally distributed massless bosons with positive energy, whose negative energy partners have fallen into the black hole region. More precisely, the observer at future null infinity will expect a number of particles with frequency ω (we restore the physical units to make the thermal distribution evident):

$$\langle N_\omega \rangle = \frac{1}{\exp\left(\dfrac{\hbar\omega}{k_B T_H}\right) - 1} \tag{1.4}$$

where

$$T_H \equiv \frac{\hbar c^3}{8\pi k_B G M} = \frac{\hbar}{2\pi k_B c}\kappa \tag{1.5}$$

is the so-called *Hawking temperature* of the black hole, k_B is Boltzmann's constant and

$$\kappa \equiv \frac{c^4}{4GM} \tag{1.6}$$

is the *surface gravity* (it has the dimensions of an acceleration) on the event horizon as measured by the observer at future null infinity.

The result by Hawking is that black holes behave like black bodies, i.e. emit with a Planck spectrum at a temperature T_H inversely proportional to their mass. To confirm this statement, a further study by Wald [37] proved that the one-particle modes emitted at infinity are uncorrelated among each other, and the probability of observing N particles with frequency ω is:

$$P(N,\omega) = \frac{\exp\left(-\dfrac{N\hbar\omega}{k_B T_H}\right)}{\prod_\omega \left[1 - \exp\left(-\dfrac{\hbar\omega}{k_B T_H}\right)\right]} \ , \tag{1.7}$$

coherently with thermal emission.

Since the Hawking radiation may seem a purely mathematical speculation, one can ask what is the physical origin of the emitted particles reaching $\mathscr{I}^+$. We can interpret Hawking's radiation as a spontaneous creation of particle-antiparticle pairs occurring just outside the horizon immediately after the formation of the black hole region. Each pair includes a particle with positive energy and its antiparticle partner with negative energy, with respect to infinity: the negative energy member of the pair falls into the BH, where negative energy states exist, while the other reaches spatial infinity. To each particle received at future infinity corresponds a partner which falls into the BH.

[3]The case of a very general gravitational collapse can be found in [19].

One could ask if such radiation is empirically observable. First of all, the Hawking temperature of a black hole with mass M has magnitude:

$$T_H \sim 10^{-7} \frac{M_\odot}{M} \; K \tag{1.8}$$

where $M_\odot$ is the solar mass. Current estimates for realistic black holes suggest that they are too big to produce an observable effect. In addition, if we take into account the cosmic microwave background radiation, in order for a black hole to dissipate it must have a temperature greater than the present-day black-body radiation of the universe of $2.7K$. This implies that M must be less than 0.8% the mass of the Earth, approximately the mass of the Moon, otherwise the black hole would absorb more energy than it emits[4].
Now, if the observer at $\mathscr{I}^+$ receives a mode with a finite positive frequency ω_∞, she could imagine to trace that mode back to its source, extremely near to the horizon. The corresponding quantum particle has been subjected to a huge gravitational redshift while travelling towards spatial infinity, so that at the time it was "born" near the horizon the mode must have had an almost infinite frequency ω_H. In other words, if the particle reaches the observer with wavelength λ_∞, λ_H should have been much shorter than the Planck length, the minimal accessible length in physical processes (see §3). This is known as the trans-Planckian problem, and it raises the doubt whether the Hawking radiation can be legitimately based on a classical theory of gravity even in such highly energetic regimes.

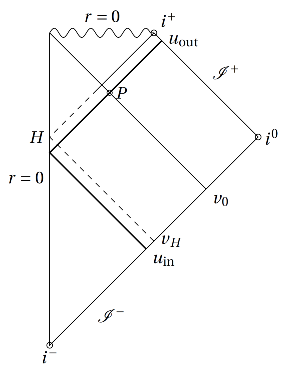


Figure 1.2: Vaidya spacetime: the shell collapses following a null trajectory at $v = v_0$: the spacetime is Minkowskian for $v < v_0$ and Schwarzschild for $v > v_0$. The thick line is the path followed by a positive frequency mode of a massless scalar field. The extremely high frequency with which it propagates from the horizon toward future null infinity allows us to draw it as a light ray i.e. a null mode (geometric optics approximation).

1.3 BH evaporation and information loss paradox

The spontaneous pair production immediately outside the event horizon implies the negative energy particles of each pair to fall into the black hole region. Here, the Killing energy of geodesics can be negative-valued and such objects are allowed to exist and move causally. Since these partners have negative energy, they effectively reduce the BH mass, as measured at infinity. Black holes lose their mass (as measured at infinity) as long as they radiate energy: ultimately, they are doomed to *evaporate* completely.
Consequently, the fixed background metric of Vaidya type in which we have described the Hawking

[4]Since such a little mass is not enough to form a black hole, it has been speculated that primordial black holes may have reduced their mass through Hawking radiation to the point that now they should emit an observable radiation, but no such radiation has been detected up to now.

effect is not faithful to reality. Spacetime changes as the black hole radiates energy, and we need a dynamical correction to the initial background geometry to keep track of the *backreaction* of the process: while the black hole emits, its mass will decrease and the resulting black hole will emit at greater temperature, thus will lose mass more rapidly than before, and so on until the mass becomes planckian. The easiest way to encode backreaction is based on the evidence that the Hawking temperature of macroscopic black holes is very low, with reference to equation (1.8). Therefore, one can make the plausible assumption that the evaporation process is *quasi-static*: it is a sequence of static photographies of Vaidya spacetimes at each time t, built on the mass at that time $M(t)$ thermally radiating with Hawking temperature $T_H(t) \sim 1/M(t)$.
Assuming quasi-static approximation, one can estimate the mass loss rate of a Schwarzschild black hole by the Stephan-Boltzmann's law

$$\frac{dM}{dt} = -\sigma A T_H^4 \ , \tag{1.9}$$

where σ is the Stephan-Boltzmann constant and $A = 4\pi(2MG/c^2)^2$ is the area of the event horizon. Inserting (1.6) we get:

$$\frac{dM}{dt} = -\alpha \frac{1}{M^2} \qquad ; \qquad \alpha \sim 10^{-5} \frac{M_{Planck}^3}{t_{Planck}} \tag{1.10}$$

where the Planck time $t_{Planck} \equiv \sqrt{\hbar G/c^5}$ is the time it takes to a photon to cover a Planck length, i.e. the shortest physically meaningful time lapse.
If we call t_{ev} the Schwarzschild time after which the black hole completely disappears, integration of this last equation yields:

$$t_{ev} = \frac{M_0^3}{3\alpha} \tag{1.11}$$

M_0 being the mass of the original black hole.
The above argument cannot be trusted when the mass reaches Planck scale. However, if we keep relying on the semiclassical approximation (on which Hawking radiation is based) until this scale is reached, for most of the time the black hole mass is much bigger than the Planck mass, and this estimation is likely to provide the correct order of magnitude[5].
The conformal diagram of spacetime where a black hole completely evaporates is shown in the following figure.

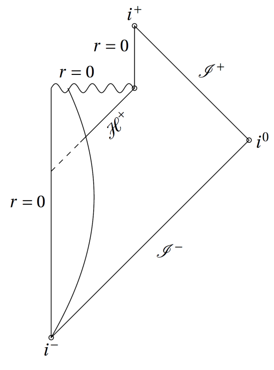


Figure 1.3: Penrose diagram of a collapsing star, forming a Schwarzschild black hole and completely evaporating by Hawking radiation.

[5]For a black hole of solar mass or higher, the evaporation time turns out to be much greater then the life of the Universe. The upper bound on the mass of a black hole that has undertaken a complete evaporation up to now is $M_0 \sim 5 \cdot 10^{14}\ g$. Celestial objects with this mass, however, are too "light" and their gravitational collapse cannot form black holes.

Hawking [20] recognized that such a scenario has a dramatic consequence, the so-called *information loss paradox*. To understand it, let us suppose to have a vacuum state $|0_{in}\rangle$ at $\mathscr{I}^-$, chosen as the Cauchy surface Σ_{in} where to define a scalar product. It is a pure state, with associated density matrix $\hat{\rho}_{in} = |0_{in}\rangle\langle 0_{in}|$ satisfying $\hat{\rho}_{in}^2 = \hat{\rho}_{in}$. During each of the quasi-static steps at time t before the BH has completely evaporated, we can decompose $|0_{in}\rangle$ on the basis of thermal pairs produced near the horizon once a Cauchy surface Σ_{out} is chosen in the Schwarzschild patch. Let us pick the surface $\Sigma_{out} = \mathscr{I}^+ \cup \mathscr{H}^+$, where $\mathscr{H}^+$ is the black hole event horizon. Here, we have:

$$|0_{in}\rangle = \frac{1}{\prod_\omega \sqrt{1 - \exp\left(-\frac{\hbar\omega}{k_B T_H(t)}\right)}} \sum_{N,\omega} \exp\left(-\frac{N\hbar\omega}{2k_B T_H(t)}\right) |N,\omega\rangle_{BH} \otimes |N,\omega\rangle_{out} \tag{1.12}$$

so that the density matrix $\hat{\rho}_{in}$ reads:

$$\begin{aligned}\hat{\rho}_{in} &= |0_{in}\rangle\langle 0_{in}| = \\ &= \frac{1}{\prod_\omega \left[1 - \exp\left(-\frac{\hbar\omega}{k_B T_H(t)}\right)\right]} \sum_{N,\omega}\sum_{N',\omega'} e^{-\frac{\hbar(N\omega+N'\omega')}{2k_B T_H(t)}} |N,\omega\rangle_{BH} \otimes |N,\omega\rangle_{out} \langle N',\omega'|_{BH} \otimes \langle N',\omega'|_{out}\end{aligned} \tag{1.13}$$

At each time t before the complete evaporation, the density matrix associated to the thermal state reaching $\mathscr{I}^+$ is obtained by tracing over the states that reach $\mathscr{H}^+$:

$$\begin{aligned}\hat{\rho}_{out} &= \frac{1}{\prod_\omega \left[1 - \exp\left(-\frac{\hbar\omega}{k_B T_H(t)}\right)\right]} \sum_{N,\omega} \exp\left(-\frac{N\hbar\omega}{k_B T_H(t)}\right) |N,\omega\rangle_{out} \langle N,\omega|_{out} \\ &= \sum_{N,\omega} P_t(N,\omega) |N,\omega\rangle_{out} \langle N,\omega|_{out} \quad .\end{aligned} \tag{1.14}$$

The thermal state at $\mathscr{I}^+$ is a mixed state: $\hat{\rho}_{out}^2 \neq \hat{\rho}_{out}$, and the observer at $\mathscr{I}^+$ can argue that the missing information (the correlations with the negative energy partners fallen in the BH) that would help reconstruct a pure state is just lying in the black hole region. But after the black hole has evaporated, this region exists no more and the Cauchy surface will be the remaining $\Sigma_{out} = \mathscr{I}^+$. The missing information is inevitably lost in the singularity. This implies that the pure density matrix $\hat{\rho}_{in}$ has evolved into a mixed density matrix $\hat{\rho}_{out}$ and nothing can be done to restore the lost correlations because they have disappeared from reality due to the evaporation. The evolution $\hat{\rho}_{in} \to \hat{\rho}_{out}$ is *not unitary*, in disagreement with one of the postulates of quantum mechanics. Unitarity assures the conservation of the probability current, or in other words that the information about the system is preserved by time evolution. When a BH evaporates information about the partners of the Hawking quanta is lost, vanished within the singularity: information is not preserved.
Lack of unitarity in Hawking radiation is cumbersome. As pointed out by Wald [38], two ways out are possible:

- the information is stored in a planck-size remnant, either stable or slowly evaporating after t_{ev};
- semiclassical approximation is violated well before the Planck scale and correlations find a way to escape the BH horizon during the evaporation, “riding” the Hawking quanta.

Objections to the first option are that a planck-size remnant is too small to contain a huge entropy (approximately one half of the initial Bekenstein entropy of the BH), while the second option seems to imply a strong violation of macroscopic causality.
Hawking and Wald originally gave up and admitted that unitarity is violated in quantum-gravitational processes. However, at present day the debate about if and how information can be recovered is still open and trust is put in a quantum theory of gravity, although we still do not have sufficiently understood theories to this aim.

Chapter 2

New proposal: bouncing black holes

In what follows we review a new model of black hole dynamics based on the idea that the formation of a singularity in a gravitational collapse may be prevented by a bounce of quantum origin that reverts the trajectory of the collapsing mass shell. The ideas are based on the recent paper by Rovelli and Haggard [16], which provides the departing point for the study of a particular quantum theory of gravity, as will be done in the following chapters of this thesis.

2.1 General features of quantum-based solutions

The problems we have briefly presented in the previous chapter are not expected, at least nowadays, to be solved in the framework of classical general relativity. The reason for hoping that the "cure" lies in a quantum theory of gravity is based on the evidence that GR is still an excellent theory for describing all the macroscopic phenomena of gravity, while it crumbles when we investigate extremely high curvature and energy regimes. The weakness of GR in these aspects is the same that affects classical mechanics and that, at its own time, led to the construction of quantum mechanics upon phenomenological suggestions. Nevertheless, classical mechanics is still a good mentor when dealing with problems of macroscopic entity.
Lacking a notion of quantum spacetime, we can sketch what kind of drawbacks rise from classical GR and how they should be addressed regardless of the specific formulation of an hypothetical quantum theory.
First of all, the breakdown of general relativity is generally due to the formation of singularities, which have been proven to be inevitable under the suitable energy conditions that we expect to hold in the physical world. The singularity theorems state that such paradoxical situations are the product of GR itself as we know it. Just like the avoidance, among the other issues of classical mechanics, of the ultraviolet catastrophe thanks to the foundation of quantum mechanics, we are authorized to assume that if a quantum theory for the gravitational field has to become relevant and non-negligible, it has to do so in the presence of singularities. In the frame of the spacetime portion where density and curvature become divergent quantities, we advance the hypothesis that quantum effects should dominate over the classical theory and save us from the notion of singularity by removing it and regularizing the processes that would lead to it.
The other paradox we have encountered manifests when studying quantum fields evolving near the event horizon of a black hole: evolution is not unitary since the information fallen inside the horizon is inevitably lost. Similarly to the problem of singularities, we expect the quantum theory to restore the unitarity of the evolution of quantum fields when these are studied in a background with non trivial curvature.
In [21], Hossenfelder and Smolin attempt a classification of the various models for the final stages

of the evolution of a black hole that have been put forward in order to resolve the information paradox. Their first rough classification is to divide the proposals depending on their theoretical assumptions. A proposed solution to the information loss paradox is labelled as *radical* if either or both of the following things are true:

- it attributes to the horizon or apparent horizon physical properties which are not also properties of arbitrary null surfaces, or which are not aparent in a semiclassical treatment;

- it calls on extreme forms of nonlocality in regions where spacetime with curvatures fra below the Planck scale, or requires transfer of quantum information over large spacelike intervals.

Generally speaking, according to the authors a radical quantum theory of gravity would attribute to weakly curved regions of spacetime properties very different from those found in the semiclassical description. Any approach to resolution of the black hole information puzzle that does not make either of these assumptions is in turn called *conservative*.
Sticking to a conservative approach, the authors point out that a plausible conservative quantum-based solution to the problems of classical gravitational collapse should answer the two questions:

- is there a real event horizon?

- is there a singularity?

The four possible ways of answering these two questions (yes or no) correspond to four related categories of quantum spacetime theories. Regardless of the philosophy and formalism of such theories, the authors assume that in general the *quantum spacetime* we are looking for must have a semiclassical extension that recovers, outside the genuinely quantum region, the classical Einstein equations for the expectation values of the quantum operators on which the theory is built.
A quantum spacetime, QST, may have a generalized definition of singularity and horizon, in particular:

- the QST is *quantum nonsingular* if for any two nonintersecting maximally extended spacelike surfaces there is a *reversible* map connecting the two Hilbert space representations of the local quantum fields defined on the two surfaces. The absence of singularities is therefore traced in the possibility to establish a meaningful dynamical evolution between any two of the above surfaces, instead of the absence of classically divergent densities or curvatures or even geodesical completeness, because we cannot say if a particular quantum theory of gravity will allow us to define either curvature or geodesics in the quantum regions. Notice that time reversibility is a necessary but not sufficient requirement for unitary evolution, but a heuristic necessity of defining a proper notion of quantum probability preserved on any basis of the Hilbert spaces naturally leads to unitarity;

- the QST has a *future event horizon* if there is a region in it that is not in the causal past of $\mathscr{I}^+$ of the classical approximation of QST. Again, the reader may have realized that we have generalized the definition of horizon by saying what is the condition for its presence, instead of postulating what it is in practice, since we cannot make precise statements about the quantum gravitational regions without knowing what theory we should apply.

With minimal assumptions about a quantum theory of gravity and these two generalized definitions for the classically problematic terms, the authors reach a simple and robust conclusion, i.e. that unitarity restoration requires a quantum theory of gravity that eliminates singularities, at least if we want to keep its classical limit to be the usual general relativity. When approaching the would-be singularity, we need to exit the classically relativistic frame and take into account the presence of a genuinely quantum region. We will apply these basic intuitions in the following, upon a condition: we are going to abandon the conservative approach.

2.2 Quantum bounce

The idea prompted by Rovelli and Haggard in [16] is that, in a spacetime where a star collapses to form a black hole, the quantum region around the singularity in $r = 0$, where the classical parameters of spacetime reach critical values, produces the effect of reverting the body's trajectory to an expanding one: quantum gravity is supposed to generate enough pressure to counterbalance the matter's weight, stop the collapse and make the star *bounce* back.
A number of indications make this scenario possible, and we summarize two of them below:

∗ *Loop Quantum Cosmology*
Using the machinery of Loop Quantum Gravity, Ashtekar et al. [2] showed that the wave packet representing a collapsing spatially-compact $k = 1$ FRW universe tunnels into a wave packet representing an expanding universe. The Big Bang singularity is replaced by a quantum bounce, and the effective Friedmann equation becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{crit}}\right) \tag{2.1}$$

where ρ_{crit} is the density of the Universe at one unit Planck time after its birth, of order $M_{Planck}/L^3_{Planck} = c^5/\hbar G^2$ (approximately $10^{96} kg/m^3$), also called *Planck density.* A matter dominated universe of total mass M will have, at the bounce, the volume:

$$V \sim \frac{M}{M_{Planck}} L^3_{Planck} \ . \tag{2.2}$$

While the common belief is that quantum gravitational effects arise when the scale of the phenomenon is of the order of the Planck length, now we discover that they manifest at Planck density, a condition that holds for dimensional scales still much larger than the Planck length. For example, in the case of a closed universe which re-collapses, the effect is to make gravity repulsive causing a bounce, at which the size of the universe is still much larger than planckian. Using current cosmological estimates, quantum gravity becomes relevant when the volume of the universe is some 75 orders of magnitudes larger than the Planck volume L^3_{Planck}. If we accept this result, we can reasonably suppose that the same happens when a star collapses: instead of forming a singularity, the repulsive character of quantum gravity stops the contraction and causes a bounce.

∗ *Null spherical collapsing shells*
Hájíček and Kiefer [17] have discussed a model describing exactly a thin spherically symetric shell of matter with zero rest mass coupled to gravity. The classical theory has two disconnected sets of solutions: those with the shell in-falling into a black hole, and those with the shell emerging from a white hole. The system is described by embedding variables and their conjugate momenta, which can be quantized exactly. The result is a non-perturbative quantum theory with a unitary dynamics. As a consequence of unitarity, a wave packet representing an in-falling quantum shell develops into a superposition of ingoing and outgoing shell if the region is reached where in the classical theory a singularity would form. The singularity is avoided by destructive interference in the quantum theory, so that in the limit $r \to 0$ the wave function vanishes: there is null probability for the shell to reach $r = 0$. If such a behaviour occurred for all collapsing matter, no information loss paradox would ever arise, since the horizon would become a superposition of a black hole and a white hole: it would be *grey*, and quantum oscillations would restore the otherwise lost correlations between Hawking quanta.

We learn that the quantum effects involved in the above examples may be the same that might take place during a gravitational collapse: here, they are consistently likely to produce a bounce of the collapsing star, and save the spacetime from the occurrence of a singularity.

2.3 Time reversibility

We have pointed out, following Hossenfelder' and Smolin's work, that a necessary condition for a quantum spacetime to have unitary evolution is the time reversal invariance of every process involving fields between two spacelike complete surfaces. How can we plug this assumption in the possible situation of bouncing stars? Let us consider, for instance, the trajectory of a ball that falls down to the ground due to Earth's gravity and then bounces up. This evolution is time symmetric if we disregard all dissipative effects like friction, or inelasticity of the bounce: without dissipation, the ball moves up after the bounce precisely in the same matter it fell down. In the same vein, we can ask what are the dissipative effects involved during a gravitational collapse: the first immediate answer is Hawking radiation. We are allowed to think this effect as being dissipative if we note that the mass-loss rate due to it is extremely low in macroscopic black holes, but, more radically, also if we do not trust the common assumption that Hawking radiation is capable of carrying away the entire energy of a collapsed star, and make it completely evaporate. This belief is shared by other proposals of quantum black holes, like Gidding's remnant scenario [15]. Furthermore, we must challenge the common intuition that no other mechanism intervenes before Hawking radiation has taken away a significant proportion of the available mass, as seen from infinity.
Moreover, Hawking radiation regards the horizon and its closest exterior: it has no major effect on what happens inside the black hole, apart from a very slow energy loss. Since we are interested in the fate of the star when it rapidly reaches the would-be singularity in $r = 0$, we feel allowed to study the process of a quantum bounce first, and then consider in a second stage the dissipative Hawking radiation as a correction to this model, in the same manner one can study the bounce of a ball on the floor first and then correct for friction and other dissipative phenomena.
The assumption of time reversal symmetry in our process results in a black hole-white hole transition: since the first part of the process describes the in-fall matter to form a black hole, the second part should describe the time reversed process, i.e. a white hole streaming out-going matter. This may be seem surprising at first, but it proves to be reasonable. If quantum gravity corrects the singularity yielding a region where the classical Einstein equations and the standard energy conditions do not hold, then the process of formation of a black hole continues into the future. Whatever emerges from such a region is then something that, if continued from the future backwards, would equally end in the quantum region: it must be a white hole. Therefore a white hole solution may not describe something completely unphysical: it is possible that it simply describes the portion of spacetime that emerges from quantum regions, in the same manner in which a black hole solution describes the portion of spacetime that evolves into a quantum region. Since the quantum region prevents the formation of singularity, the white hole should not be regarded any more as a singularity in the past of all geodesics, and so it gains its right to physically exist.

2.4 Temporal extension of the quantum region: radical solution

GR is an efficient and consistent machinery wherever curvature is small. But this does not mean that the quantum gravitational content inside each of the "regular" events is completely suppressed. Quantum mechanics provides corrections to classical mechanics that simply become highly negligible in macroscopic regimes. That is, if there are quantum perturbations to the physical solutions of a macroscopic system, they are so small that the true dynamics is, locally, well approximated by the unperturbed classical solution. This approximation does not necessarily apply globally.
For instance, let us consider a particle with mass m subject to a very weak force ϵF where $\epsilon \ll 1$. It will move as $x = x_0 + v_0 t + \frac{1}{2}\epsilon\frac{F}{m}t^2$. For any small time interval, this is well approximated by a motion at constant speed, namely a solution of the unperturbed equation; but for any ϵ there is a $t \sim 1/\sqrt{\epsilon}$ long enough for the discrepance between the unperturbed solution and the true solution to be arbitrarily large.

Quantum effects can similarly pile up in the long term, and tunnelling is a prime example: with a very good approximation, quantum effects on the stability of a single atom of Uranium 238 in our lab are completely negligible. Still, after 4.5 billion years, the atom is likely to have decayed.
Let us see how we can apply this approach to our case: outside a macroscopic black hole, the curvature is small and quantum effects are negligible, today. But over a long enough time, they may drive the classical solution away from the exact global solution of the classical GR equations. After a sufficiently long time, the hole may tunnel from black to white. Does this mean that the bouncing process has to be very slow? Remember that there is no notion of absolute time in GR: (proper) time flows at a speed that depends on the vicinity to gravitational sources, and if compared with clocks that are faraway from the latters, it slows down near high density masses. The extreme gravitational time dilation is the conceptual turning point of our scenario: although the bounce may take a very short proper time according to an observer riding the collapsing shell, every observer far enough from the quantum region and safely living in a classically dominated world at large radii will argue that the bounce took place in a very long time. Such time can be comfortably long to allow quantum gravitational effects to pile up and influence small curvature regions. We will get a nice confirmation of this argument later on, when dealing with effective calculations.
This point of view is what classifies the proposed model, according to Hossenfelder and Smolin's distinction, as a *radical* solution: we expect that quantum effects may affect spacetime near the two apparent horizons related to the black hole patch and white hole patch in a way that does not occur in the surroundings of other arbitrary null surfaces. The long wait we have explained produces a quantum behaviour in a region of small curvature, something we would never see with a semiclassical description.

2.5 Spatial extension of the quantum region

Now, we want to address the task of understanding where the quantum region is supposed to take place. More precisely, we ask at what radius the collapsing shell should "dive" into it during its trajectory towards the would-be singularity, and to what extent the quantum region should affect, after letting the quantum effects pile up over a long time, regions of small curvature.

- ⊙ *Dive in the quantum*
  This estimation is inspired by the Loop quantum cosmology argument previously described. Quantum effects become predominant not only when the spatial extension of the star is comparable with the Planck length, as one would naively assume, but much before this situation, when the density reaches a critical value that breaks down the classicality of the around space. But how can we quantify the criticality of density. Heuristically, and motivated by Ashtekar et al. (see a full discussion in [3]), we can define the density to be critical when it produces a critical *curvature*. There are different ways of defining a curvature invariant quantity, and we choose the one we are most familiar with: the *Kretschmann scalar*, the trace of the Riemann tensor, which reads

$$K = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = \frac{48G^2M^2}{c^4r^6} \tag{2.3}$$

  M being as usual the mass of the star. It has the dimensions of an inverse length to the fourth power, so it is natural to define the critical value K_{crit} to be of order L^4_{Planck}. That is:

$$K_{crit} \sim \frac{G^2M^2}{c^4r^6_{crit}} \sim \frac{1}{L^4_{Planck}} \Longrightarrow r_{crit} \sim \left(\frac{M}{M_{Planck}}\right)^{\frac{1}{3}} L_{Planck} \ . \tag{2.4}$$

  This is the first radius where the shell, during its collapse, abandons the classical region and enters a spacetime portion with genuinely quantum dynamics. Clearly, it is inside the horizon, because when the shell enters the horizon the physics is still classical.

⊙ *Leaking of the quantum*
We have argued before that the quantum spacetime QST should be built in such a way that, outside the region where quantum effects are not negligible, the rest of the spacetime should be described classically by the expectation values of the quantum operators. The expectation values would act as the classical variables of the theory and obey the Einstein equations everywhere outside the quantum region. Let us assume that the quantum region lies entirely inside the radius $r = 2M$. This means that the whole region $r \geq 2M$ is classical, and classical GR predicts that it would be bound by the event horizon $r = 2M$. But if so, then people living in the classical spacetime will never experience the bounce of the collapsed matter, because it would violate the causality ruling this spacetime. If an event horizon forms, matter cannot bounce out and manifest in the classical extension of QST. Therefore, we need the quantum region to *leak* outside the horizon, in order to bridge the consequences of the inner quantum dynamics with the external classical dynamics and show its effects to the classical region. Quantum effects need to extend their range to an even little classical region where curvature is small and would not justify the rising of relevant quantum corrections. But we have seen that, actually, waiting for a long time rewards us with the possibility of observing a non-negligible quantum effect in phenomena where such effect is, locally in time, almost absent but can sum with the other forecast tiny corrections up to a macroscopic correction on the long term.
However, good sense suggests us that such piling should not be relevant everywhere in classical spacetime, otherwise the semiclassical approximation would abruptly break down: we foresee that they should become manifest when there is still a good reason for them to emerge after waiting a lot, i.e. where curvature is still the largest available. Therefore, the quantum region should leak up to a radius that slightly exceeds $r = 2M$, but no more than that. We thus call such radius $r = 2M + \delta$, $\delta > 0$ and sufficiently little.

2.6 Construction of the bouncing metric

We are ready to study if the bouncing model we have sketched is compatible with a realistic effective metric satisfying the Einstein equations everywhere outside a quantum region, as we would expect from a "good" QST. The following is largely taken from [16].
We summarize the assumptions we have made:

(i) Spherical symmetry, as usual;

(ii) To make the model the simplest possible, we choose a spherical shell of null matter, and disregard the thickness of this shell. Nevertheless, we expect these results to generalise to massive matter. In the solution the shell moves in from past null infinity, enters its own Schwarzschild radius, keeps ingoing, enters the quantum region, bounces, and then exits its Schwarzschild radius and moves outward to infinity;

(iii) We assume the process is invariant under time reversal;

(iv) We assume that the metric of the process is a solution of the classical Einstein equations for a portion of spacetime that includes the entire region outside $r = 2m + \delta$. In other words, the quantum process is local in space: it is confined in a finite region of space;

(v) We assume that the metric of the process is a solution of the classical Einstein equations for a portion of spacetime that includes all of space *before* a (proper) time ϵ preceding the bounce of the shell, and all of space *after* a (proper) time ϵ following the bounce of the shell. In other words, the quantum process is local in time: it lasts only for a finite time interval;

(*vi*) We assume the causal structure of spacetime is that of Minkowski spacetime, where no real event horizons appear because the quantum region is able to leak out and causally connects the inner and outer portions.

Because of spherical symmetry, we can use coorinates (u, v, θ, ϕ) with u and v null coordinates in the r-t plane and the metric is entirely determined by the two functions of u and v:

$$ds^2 = -F(u,v)dudv + r^2(u,v)(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.5}$$

In the following we will use different coordinate patches, but generally all of this form. Because of the assumption (*iv*), the conformal diagram of spacetime is trivial, just the Minkowski one, as in figure 2.1.

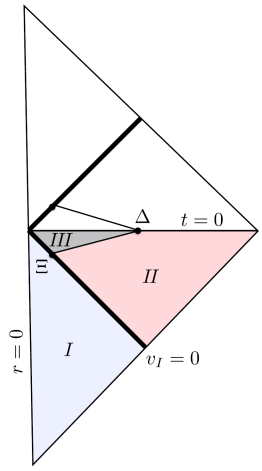


Figure 2.1: The spacetime of a bouncing star.

From assumption (*iii*) there must be a $t = 0$ hyperplane which is the surface of reflection of the time reversal symmetry. It is convenient to present it in the conformal diagram by an horizontal line as in figure 2.1. Now consider the incoming and outgoing null shells. By symmetry, the bounce must be at $t = 0$. For simplicity we assume (this is not crucial) that it is also at $r = 0$. These are represented by the two thick lines at 45 degrees in figure 2.1. In the figure, there are two significant points, Δ and Ξ, that lie on the boundary of the quantum region. The point Δ has $t = 0$ and is the maximal extension in space of the region where the Einstein equations are violated, therefore it has radius $r_\Delta = 2M + \delta$. The point Ξ is the first event in the trajectory of the shell where this happens, and we already pointed out that its radius should be $r_\Xi \equiv \epsilon \sim (M/M_{Planck})^{\frac{1}{3}} L_{Planck}$. We discuss later the geometry of the line joining Ξ and Δ.
Since the metric is invariant under time reversal, it is sufficient for us to construct the region below $t = 0$ and make sure it glues well with the future. The upper region will simply be the time reflection of the lower. The in-falling shell splits spacetime into a region interior to the shell, indicated as I in the figure, and an exterior part. The latter, in turn, is split into two regions, which we all II and III, by the line joining Ξ and Δ. Let us examin the metric of these three regions separately:

(I) The first region, inside the shell, must be flat by Birkhoff's theorem. We denote null Minkowski oordinates in this region (u_I, v_I, θ, ϕ).

(II) The second region, again by Birkhoff's theorem, must be a portion of the metric induced by a mass M, namely it must be a portion of the (maximal extension of the) Schwarzschild metric. We denote null Kruskal coordinates in this region (u, v, θ, ϕ) and the related radial coordinate r.

(III) Finally, the third region is where quantum gravity becomes non-negligible. WE know nothing about the metric of this region, except for the fact that it must join the rest of spacetime. We denote null coordinates for this quantum region (u_q, v_q, θ, ϕ) and the related radial coordinate r_q.

We can now start building the metric. Region I is easy: we have the Minkowski metric in null coordinates determined by

$$F(u_I, v_I) = 1 \qquad ; \qquad r_I(u_I, v_I) = \frac{v_I - u_I}{2} \ . \tag{2.6}$$

It is bounded by the past light cone of the origin, that is, by $v_I = 0$. In the coordinates of this patch, the ingoing shell is therefore given by $v_I = 0$.
Let us now consider region II. This must be a portion of the Kruskal spacetime. Which portion? Let us put a null ingoing shell in Kruskal spacetime, as in figure 2.2. The point Δ is a generic point in the region outside the horizon, which we take on the $t = 0$ surface, so that the gluing with the future is immediate.

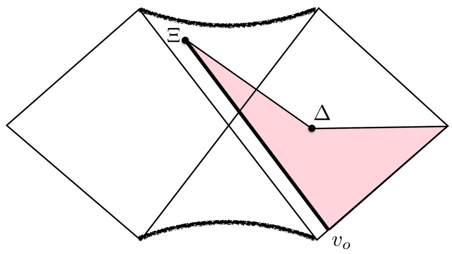


Figure 2.2: Classical black hole spacetime and the region II.

Meanwhile, Ξ lies well inside the horizon. Therefore the region that corresponds to region II is the shaded region of Kruskal spacetime depicted in figure 2.2.
In null Kruskal-Szekeres coordinates the metric of the Kruskal spacetime is given by

$$F(u,v) = \frac{32M^3}{r} \exp\left(-\frac{r}{2M}\right) \qquad ; \qquad r : \quad \left(1 - \frac{r}{2M}\right) \exp\left(\frac{r}{2M}\right) = uv \ . \tag{2.7}$$

The region of interest is bounded by a constant $v = v_0$ null line. The constant v_0 cannot vanish, because $v = 0$ is an horizon, which is not the case for the in-falling shell. Therefore v_0 is a constant that will enter in our metric.
The matching between the regions I and II is not difficult: the v coordinates match simply by identifying $v_I = 0$ with $v = v_0$. The matching of the u coordinate is determined by the obvious requirement that the radius must be equal across the matching, that is by:

$$r_I(u_I, v_I) = r(u, v) \ . \tag{2.8}$$

This gives

$$\left(1 - \frac{v_I - u_I}{4M}\right) \exp\left(\frac{v_I - u_I}{4M}\right) = uv \ , \tag{2.9}$$

which on the shell becomes

$$\left(1 + \frac{u_I}{4M}\right) \exp\left(-\frac{u_I}{4M}\right) = uv_0 \ . \tag{2.10}$$

Thus, the matching condition is:

$$u(u_I) = \frac{1}{v_0} \left(1 + \frac{u_I}{4M}\right) \exp\left(-\frac{u_I}{4M}\right) \ . \tag{2.11}$$

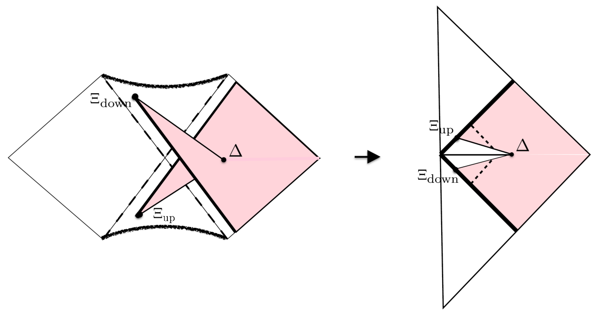


Figure 2.3: The portion of a classical black hole spacetime which is reproduced in the quantum case. The contours $r = 2M$ are indicated in both panels by dashed lines. They are apparent horizons, since they do not extend up to the future timelike infinity. In this sense, our QST does not feature a future event horizon as defined by Hossenfelder and Smolin. From future null infinity we can trace back any region of the spacetime.

So far we have glued the two intrinsic metrics along the boundaries[1]. The matching condition between region II and its symmetric, time reversed part along the $t = 0$ surface is immediate. Notice, however, that the ensemble of these two regions is not truly a portion of Kruskal space, but rather a portion of a double cover of it, as in figure 2.3: there are two distinct regions locally isomorphic to the Kruskal solution. The bouncing metric is obtained by "opening up" the two overlapping flaps in the figure and inserting a quantum region in between.

It remains to fix the coordinates for the points Δ and Ξ. The point Ξ has (u_I, v_I) coordinates $(-2\epsilon, 0)$, while the point Δ has Schwarzschild radius $r_\Delta = 2M + \delta$ and lies on the time reversal symmetry line $u + v = 0$. We cannot give it a unique couple of coordinates, since it lies in both patches of the double Kruskal cover. The constants ϵ and δ have dimensions of a length and determine the metric.

Lacking a better understanding of the quantum region, we take the line connecting Ξ and Δ to be a spacelike trajectory between the two. More generally, since the radius of Ξ marks the entrance in the quantum regime, we assume to be allowed to arbitrarily deform this boundary line by sliding from Ξ along the $r = \epsilon$ spacelike trajectory and then reaching Δ. See figure 2.4.

Finally, we have no reason to fix the metric in the region III too, since III hosts genuinely quantum dynamics and, in our picture, is not required to be endowed with a semiclassical limit in which to define a line element obeying Einstein equations. What is important, however, is that region III is outside the trapped region, which in turn is bounded by the incoming shell trajectory, the null $r = 2M$ horizon in the region II, and the boundary between region II and region III. The two trapped regions are blue-shaded in picture 2.5.

This concludes the construction of the metric, which is now completely defined. It satisfies all the assumptions in the beginning. It describes, in a first approximation and disregarding dissipative effects, the full process of a gravitational collapse, quantum bounce and explosion of a star of mass M. It depends on four constants: M, v_0, δ and ϵ. We are left to the duty of fixing v_0 and δ.

2.7 Slow motion and quantum leakage

Let us consider two observers, one at the center of the system, namely at $r = 0$, and one that remains at radius $r = R > 2M$. In the distant past, both observers are in the same Minkowski space. Notice that the entire process has a preferred Lorentz frame: the one where the center of the

[1] In order to truly define the metric over the whole region one should also need to specify how tangent vectors are identified along these boundaries, ensuring that this way the extrinsic geometries also glue. However, if the induced 3-metrics on the boundaries agree, it turns out that it is not necessary to impose further conditions [6], [11].

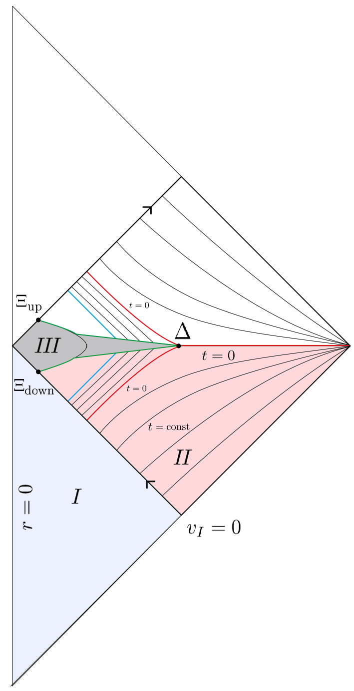


Figure 2.4: Some $t = const$ lines, t being the Schwarzschild time. The two patches of Kruskal spacetime are such that the $t = 0$ line (in red) splits into two symmetric lines. In both patches the time reaches infinity when approaching the horizons (blue lines), as we expect from the Schwarzschild observer's point of view. The quantum region III (we have also covered its time reversed for better clarity. It is surrounded by the green contour.) is deformed by partially sliding along $r = r_\Xi \equiv \epsilon$.

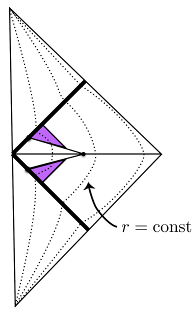


Figure 2.5: Some $r = const$ lines. A trapped region is a region where these lines become spacelike. There are two trapped regions in this metric, indicated by shading.

mass shell is not moving. Therefore the two observers can synchronise their clocks in this frame. In the distant future, the two observers find themselves again in a common Minkowski space with a preferred frame and therefore can synchronise their clocks again. However, there is no reason for the proper time τ_0 measured by one observer to be equal to the proper time τ_R measured by the other one, because of the conventional, general relativistic time dilation.

Let us compute the time difference accumulated between the two clocks during the full process. The two observers are both in a common Minkowski region until the shell reaches R while falling in (event A) and they are again both in this region after the shell reaches R while going out (event B). In the null coordinates (u_I, v_I) of region I, A has coordinates $(-2R, 0)$, while B has coordinates $(0, 2R)$. In (t_I, r_I) coordinates, where $t_I = (v_I + u_I)/2$, $r = (v_I - u_I)/2$, A is $(-R, R)$ and B is (R, R). The two events simultaneous to A and B for the inertial observer at $r = 0$ are $A'(-R, 0)$ and $B'(R, 0)$ and his proper time between the two is clearly:

$$\tau_0 = 2R \ . \tag{2.12}$$

Meanwhile, the observer at $r = R$ sits at constant radius in a Schwarzschild spacetime. The proper time between the two moments she crosses the shell is twice the proper time from the first crossing to the $t = 0$ surface. Since the observer is stationary, her proper time is given by:

$$\tau_R = -2\sqrt{1 - \frac{2M}{R}}t \tag{2.13}$$

where t is the Schwarzschild time from event A to $t = 0$, thus the Schwarzschild time of A. Therefore the proper time τ_R can simply be found by transforming the coordinates (u, v) to the Schwarzschild coordinates. The standard change of variables to the Schwarzschild coordinates in the exterior region $r > 2M$ is:

$$\frac{u+v}{2} = \exp\left(\frac{r}{4M}\right)\sqrt{\frac{r}{2M} - 1}\sinh\left(\frac{t}{4M}\right) \tag{2.14}$$

$$\frac{v-u}{2} = \exp\left(\frac{r}{4M}\right)\sqrt{\frac{r}{2M} - 1}\cosh\left(\frac{t}{4M}\right) \tag{2.15}$$

The event A lies along the shell's in-fall trajectory $v = v_0$ in region II, so:

$$t = 4M\log\left(\frac{v_0}{\sqrt{\frac{R}{2M} - 1}}\exp\left(-\frac{R}{4M}\right)\right) \quad . \tag{2.16}$$

Therefore the total time measured by the observer at radius R is:

$$\tau_R = \sqrt{1 - \frac{2M}{R}}\left(2R - 8M\log v_0 + 4M\log\frac{R - 2M}{2M}\right) \quad . \tag{2.17}$$

If the external observer is at large distance $R \gg 2M$, we obtain, to the first relevant order, the difference in the duration of the bounce measured outside and measured inside to be:

$$\tau = \tau_R - \tau_0 = -8M\log v_0 \ , \tag{2.18}$$

which can be arbitrarily large as v_0 is arbitrarily small. The process seen by an observer outside takes a time arbitrarily longer than the process measured by an observer inside the collapsing shell. The distant observer sees a dimming, frozen star that re-emerges bouncing out after a very long time.

In [16] the authors perform an heuristic argument to determine v_0 as that function of the only mass of the shell which allows enough time for quantum effects to pile up until Δ, and in turn they find an estimation of τ. Their result is that

$$\tau \sim \frac{M^2}{L_{Plank}} \ , \tag{2.19}$$

which is very long for a macroscopic black hole but still shorter than the Hawking evaporation time, which is of order M^3. Therefore the possibility of the bounce studied here affects radically the discussion about the black hole information puzzle, and the study of the perturbed case including Hawking radiation, in a second step, seems to open good chances to address the paradox.

Fixing the function $v_0(M)$ also allows to give an estimation for δ, the leaking range of the quantum region. The authors find:

$$\delta \in \left[\frac{M}{6}, \frac{2M}{3}\right] \ , \tag{2.20}$$

i.e. the quantum region extends only by a minimal part over the would-be horizon, as we expected. In the paper, a good guess is $\delta = M/3$, and we refer the reader to the source for the details on the heuristic argument about the classicality threshold that led to these estimations. In conclusion, the constants that determine the full dynamics of the process may be consistently dependent on the mass M in the shell, the only source of dynamics, and the Planck constants, which naturally arise when quantum regimes are entered. A tentative time reversal symmetric metric describing the quantum bounce of a star is entirely defined: it is an exact solution of the Einstein equations everywhere, including inside the Schwarzschild radius, except for a *finite* region, surrounding the points where the classical Einstein equations are likely to fail.

2.8 Relation with a full quantum gravity theory

We have constructed the metric of a black hole tunnelling into a white hole by using the classical equations outside the quantum region, an order of magnitude estimate for the onset of quantum gravitational phenomena, and some indirect indications on the effects of quantum gravity. This, of course, is not a first principle derivation, since we would need a full theory of quantum gravity to achieve it. However, the metric we have presented describes the process in such a way that the quantum content of the problem lies in computing a quantum transition amplitude in a finite portion of spacetime. Indeed, the quantum region that we have determined is bounded by a well defined classical geometry. Given the classical geometry, can we compute the corresponding quantum transition amplitude? Since there is no classical solution that matches the in and out geometries of this region, the calculation is conceptually a rather standard tunnelling calculation in quantum mechanics, like the one in [17].
Actually, there is a candidate theory for quantum gravity that is perfectly suitable for such a kind of computation: it is Loop Quantum Gravity. Although the theory is far from being complete, it already carries a notion of quantum amplitude between geometries: it could be possibly applied to our situation, where we essentially want to know what is the transition probability, at least to first order, between the metric on the lower boundary surface of III to the upper one. If this calculation can be done, we should then be able to replace the order of magnitudes estimates used here with a genuine quantum gravity calculation. In particular, we should be able to compute from first principles the duration τ of the bounce seen from the exterior.
In the next part, we aim to give a simple (and not complete) presentation of Loop Quantum Gravity, to understand why the above problem can be studied inside this theory.

Chapter 3

Spacetime as a quantum object

The problem we have presented in the previous chapter unveils the necessity of describing the quantum behaviour of the gravitational field. At current state, what we know is encapsulated into three major theories:

- Quantum Mechanics, which is the general theoretical framework for describing dynamics of matter in a completely probabilistic manner;

- the Standard Model of particle physics, which describes the non-gravitational interactions of all matter we have so far observed directly,in a fixed rigid spacetime;

- General Relativity, which describes gravity by relating the geometry of spacetime to its energy content through the Einstein equations.

Among the open issues in these theories, one of the most outstanding is the lack of a predictive theory capable of describing phenomena where both gravity and quantum theory play a role, such as the center of black holes (object of this discussion), the entropy of black holes, the short scale structure of nature, very early cosmology, or simply the scattering amplitude of two neutral particles at small impact parameter and high energy. In addition to this, our present day technologies do not support tentative theories with more than little direct empirical information. The goal of quantum gravity is to unify these three frameworks, both conceptually and mathematically, since they are based on different principles and make use of different mathematical languages. We need new mathematical tools in order to make the proper generalizations of these three worlds.
The main philosophy behind the Loop Quantum Gravity approach is that a suitable quantum theory of gravity should be *background independent*. The general covariance principle of GR suggests us that we cannot rely on a quantum gravitational field acting on a fixed background metric, as we do in QFT, since the metric itself becomes a fully dynamical object. Thus, a quantum theory of spacetime needs be defined without any underlying structure, and eventual quanta of spacetime are to be interpreted as the fibres that *weave* spacetime instead of living *in* a pre-existing environment. We are going to understand this point throughout the presentation of the theory.

First of all, to understand the quantum nature of spacetime, we give a simplified version of an enlighting argument brought by physicist Matvei Bronstein in 1936 ([30]). This example unveils the core of the entire theory of quantum gravity.
We want to measure some field value at a location x. For this we have to mark this location. Let us suppose we want to determine it with precision L. We can do this by having a particle at x. Being a quantum particle, there will be uncertainties Δx and Δp associated with its position

and momentum. To have localization determined with precision L, we want $\Delta x < L$, and since Heisenberg uncertainty gives $\Delta x > \hbar/\Delta p$, it follows that $\Delta p > \hbar/L$. We know that $\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2$, therefore $\langle p^2 \rangle > (\hbar/L)^2$. This a well known consequence of Heisenberg uncertainty: sharp location requires large momentum, which is the reason why at CERN high momentum particles are used to investigate small scales. In turn, large momentum implies large energy E. In the relativistic limit, where the rest mass is negligible, $E \sim cp$. Sharp localization requires large energy.

Now, in GR any form of energy acts as a gravitational mass $M \sim E/c^2$ and distorts spacetime around itself. The distortion increases when the energy is concentrated, to the point that a black hole forms when the mass M is concentrated in a sphere of radius $R \sim GM/c^2$, where G is the Newton constant. In our case, the particle we use to measure the precision L must have energy $E > c\hbar/L$, and so its related mass M has to satisfy $M > \hbar/Lc$. Consequently, its Schwarzschild radius becomes $R > G\hbar/Lc^3$. Let us suppose to take L arbitrarily small. There will be a point where $R > L$, and the region of size L that we wanted to mark will be hidden inside the black hole horizon of the particle that, to this purpose, we wanted to localize with precision L. Localization is lost.

What we conclude is that we cannot decrease L arbitrarily, but there is a minimal size L_{min} where we can localize a quantum particle without having it hidden by its own horizon. This size is reached when the horizon radius is of the same size of $L: R = L$. Combining the relations above, we have:

$$L_{min} = \frac{G\hbar}{L_{min} c^3} \Rightarrow L_{min} = \sqrt{\frac{G\hbar}{c^3}}. \tag{3.1}$$

We find that it is not possible to localize anything with a precision better than the length:

$$L_{Planck} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} cm \ , \tag{3.2}$$

which is called the *Planck length*. To measure this precision, we need a particle with mass:

$$M_{Planck} = \frac{\hbar}{L_{planck} c} = \sqrt{\frac{\hbar c}{G}} \sim 1.22 \cdot 10^{19} \ GeV/c^2 = 21.76 \ \mu g \ . \tag{3.3}$$

A particle with energy greater than $M_{Plank} c^2$ would be trapped in its own black hole.

Well above the Planck scale, we can treat spacetime as a smooth space. Below, it makes no sense to talk about distance. What happens at this scale is that the quantum fluctuations of the gravitational field, namely the metric, become wide, and spacetime can no longer be viewed as a smooth manifold.

This simple derivation is obtained by extrapolating semiclassical physics. But the conclusion is correct, and characterizes the physics of quantum spacetime.

Because of the presence of a scale below which spacetime is no longer smooth, we cannot treat the quantum gravitational field as a quantum field in space. The smooth metric geometry of physical space, which is the ground needed to define a standard quantum field, is itself affected by quantum theory. This implies that we have to look for a genuine quantum theory of geometry, where quantum states are not states *on* spacetime, but states *of* spacetime, whose semiclassical approximation are the smooth space and time we perceive in common life.

General relativity teaches us that geometry is a manifestation of the gravitational field, which determines quantities such as area, volume, length, angles etc. On the other hand, quantum theory teaches us that fields have quantum properties. The problem of quantum gravity is therefore to understand what are the quantum properties of geometrical quantities. Such properties are, in general:

- the possible discretization of the quantity itself (*quantization*);
- the short-scale "*fuzziness*" implied by uncertainty relation;

- the *probabilistic* nature of its evolution, given by transition amplitude.

In the following we focuss on the first two aspects of quantum nature.

3.1 Discreteness

To understand discreteness let us make use of three situations.

▷ Consider a mass m attached to a spring with elastic constant k. We describe its motion in terms of position q, velocity v, and momentum $p = mv$. The energy $E = \frac{1}{2}mv^2 + \frac{1}{2}kq^2$is a positve real number and is conserved. The quantization postulate from which quantum theory rises is the existence of a Hilbert space $\mathcal{H}$ where (p, q) are non-commuting self-adjoint operators satisfying:

$$[q, p] = i\hbar. \tag{3.4}$$

The energy operator $E(p, q) = \frac{p^2}{2m} + \frac{k}{2}q^2$ has discrete spectrum with eigenvalues $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ where $\omega = \sqrt{\frac{k}{m}}$. Energy is quantized.

▷ Now, consider a particle moving on a circle, subject to a potential $V(\alpha)$. Let its position be an angular variable $\alpha \in S^1 \sim [0; 2\pi[$ and its hamiltonian $H = \frac{p^2}{2m} + V(\alpha)$, where $p = m\frac{d\alpha}{dt}$ is the momentum. The quantum behaviour of the particle is described by the Hilbert space $\mathcal{L}_2(S^1)$ of the square-integrable functions $\psi(\alpha)$ on the circle and the momentum operator is $p = -i\hbar\frac{d}{d\alpha}$ in the α representation. This operator has a discrete spectrum with eigenvalues $p_n = n\hbar$, independently from the potential.

▷ Let $\vec{L} = (L^1, L^2, L^3)$ be the angular momentum of a system that can rotate in three dimensions. The total angular momentum is $L = |\vec{L}| = \sqrt{L^i L^i}$. Classical mechanics teaches us that $\vec{L}$ is the generator of infinitesimal rotations. Postulating that the corresponding quantum operator is also the generator of rotations in the Hilbert space, we have the quantization law:

$$\left[L^i, L^j\right] = i\hbar\epsilon^{ij}_k L^k. \tag{3.5}$$

$SU(2)$ representation theory then immediately gives the eigenvalues of L, if the operators L^i satisfy the above relations. These are:

$$L_j = \hbar\sqrt{j\,(j+1)} \;\; j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ... \tag{3.6}$$

That is, total angular momentum is quantized.

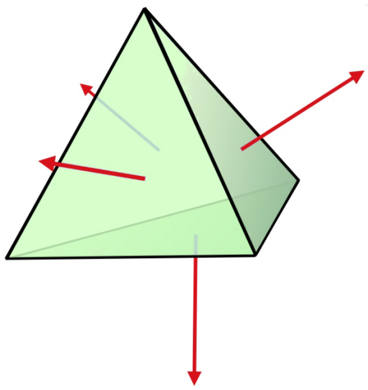
The three examples above show that discreteness is a direct consequence of a compact phase space. Quantization must take into account the global topology of phase space, and this feature has to be extended to a quantum theory of gravity.

3.2 Fuzziness

The commutation relations between observables are the basis for Heisenberg's uncertainty principle, asserting a fundamental limit in the precision of simultaneous measures of the observables involved. Thus, if geometry has to be coded into quantum observables, an intrinsic fuzziness of its properties is included too. Geometry can never be sharp in the quantum theory, in the same sense in which the three components of angular momentum can never be all sharp. But there is a substantial difference , coming from Bronstein's argument: while, for example, in the quantum theory of electromagnetic field the Heinsenberg uncertainty relations do not prevent a single component of the field at a spacetime point from being measured with arbitrary precision, the existence of the Planck scale induces an upper limit in the precision of geometrical measures. Geometry is maximally spread (in a quantum mechanical sense) at the Planck scale.

3.3 Toy model: the tetrahedron

Let us consider now a simple example of how discreteness and fuzziness can appear as quantum properties of space. We are interested in the elementary "quanta of space". Pick a simple geometrical object, an elementary portion of space. Say we pick a small tetrahedron τ, not necessarily regular. The geometry of a tetrahedron is characterized by the lengths of its sides, the areas of its faces, the volume, the angles and so on. These are all local functions of the gravitational field, because geometry is determined by it, and they are all related to one another. A set of independent quantities is provided for instance by the six lengths of the sides, but these are not appropriate for studying quantization, because they are constrained by inequalities.



The length of the three sides of a triangle, for instance, must satisfy the triangle inequalities. Non-trivial inequalities between dynamical variables are generally difficult to implement in quantum theory. Instead, we choose the four areas A_a, $a = 1, 2, 3, 4$ of its faces and the normals $\vec{n}_a$ to each face. From them, we can build up the four vectors $\vec{E}_a = A_a \vec{n}_a$ coming out of the faces of the tetrahedron. The total number of degrees of freedom is then given by the number of variables (eight) subtracted by the number of conditions they must satisfy.

Here, the geometry is defined up to global rotations (one constraint, the $SO(3)$ invariance) and the vectors $\vec{E}_a$ obey the so-called *closure condition* (our second condition), stating that the normals to the faces of every polyhedron sum up to zero:

$$\sum_{a=1}^{4} \vec{E}_a = 0. \tag{3.7}$$

We have six total degrees of freedom. If we consider the three edges emanating from a vertex of the tetrahedron, $\vec{e}_a$, $a = 1, 2, 3$, the vectors $\vec{E}^i_a$ normal to the triangles adjacent to the vertex can be written as:

$$\vec{E}_a = \frac{1}{2} \epsilon_a^{bc} \vec{e}_b \vec{e}_c \tag{3.8}$$

We can extend the index a to include the fourth normal to the remaining face of the tetrahedron. The quantities $\vec{E}_a$ determine all other geometrical quantities, for instance the dihedral angle between two triangles is measured by

$$\vec{E}_1 \cdot \vec{E}_2 = A_1 A_2 \cos\theta_{12}, \tag{3.9}$$

the four areas of the triangles that bound the tetrahedron are the lengths of the associated normal vectors,

$$|\vec{E}_a| = A_a, \tag{3.10}$$

and the volume V is determined by the oriented triple product of any three faces:

$$V^2 = \frac{2}{9} \epsilon^{abc} \left(\vec{E}_a \times \vec{E}_b \right) \cdot \vec{E}_c \tag{3.11}$$

We have all the ingredients for jumping to quantum gravity. The geometry of a real physical tetrahedron is determined by the gravitational filed, which is a quantum field. Therefore the normals $\vec{E}_a$ are to be described by quantum operators $\hat{\vec{E}}_a$, if we take the quantum nature of gravity into account. These will obey commutations relations obtained from the Poisson brackets in the hamiltonian analysis of their classical counterparts. Let us *postulate* the simplest possibility [5], namely to mimic the $SU(2)$ algebra of angular momentum operators. If we denote each vector $\vec{E}_a$ by its components in a 3-dimensional vector space, E^i_a, the commutation relations read:

$$\left[\hat{E}^i_a, \hat{E}^j_b\right] = i\delta_{ab} l_0^2 \epsilon^k_{ij} \hat{E}^k_a \tag{3.12}$$

where l_0^2 is a constant proportional to $\hbar$ and with the dimensions of an area. We leave this constant unspoken for now, and will let the quantum theory in the further chapters fix its scale.
One consequence of the commutation relations (3.12) is immediate [28] the quantum object $\hat{A}_a = |\hat{\vec{E}}_a|$, i.e. the area of the triangles bounding the tetrahedron, behaves as total angular momentum. Thus, it is quantized with eigenvalues:

$$A_a = l_0^2 \sqrt{j(j+1)}, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2} \dots \tag{3.13}$$

What we have built is a ***quantum tetrahedron***, a quantum state of geometry whose four areas are encoded in the eigenvalues j_1, j_2, j_3, j_4 of a quantum $SU(2)$ operator. Thus, we can build a Hilbert space of the quantum states of the tetrahedron at fixed values of the area of its faces as the tensor product of four representations of $SU(2)$, with respective spins $j_1, \dots, j_4$:

$$\mathcal{H}_\tau = \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4} \ . \tag{3.14}$$

If the states in $\mathcal{H}_\tau$ are to describe an effective tetrahedron, they must satisfy the quantum counterpart of the closure condition (3.7). This condition is now promoted to quantum theory by imposing the vanishing of the action of the operator:

$$\hat{C} = \hat{\vec{E}}_1 + \dots + \hat{\vec{E}}_4 \tag{3.15}$$

on the states ψ in $\mathcal{H}_\tau$.
Such constraint operator is nothing but the generator of a global (diagonal) action of $SU(2)$ rotations on the four representation spaces. Therefore, the states that solve $\hat{C}\psi = 0$ are the states that are invariant under its action, namely the states in the subspace

$$\mathcal{K} = Inv_{SU(2)}[\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}] \ . \tag{3.16}$$

This is consistent with the condition of rotational invariance we have imposed for the classical tetrahedron, and is a first hint that our quantization postulate is reasonable. One can also notice that the area operator, playing the role of total angular momentum in our construction, is invariant under $SU(2)$ rotations because it commutes with the constraint operator, an unavoidable condition if we want a physical observable.
If we now consider the quantum counterpart $\hat{V}$ of the volume in (3.11), this is well defined in $\mathcal{K}$ because it commutes with $\hat{C}$, namely it is rotationally invariant. Therefore we have a well-posed eigenvalue problem for the self-adjoint volume operator on the Hilbert space $\mathcal{K}$. As this space is finite-dimensional, it follows that its eigenvalues are discrete [28]. We have the result that the volume has discrete eigenvalues as well. In other words, there are *quanta of area* and *quanta of volume*: the space inside our quantum tetrahedron can grow only in discrete steps, just like the amplitude of a mode of the electromagnetic field.
At this stage, it is important not to confuse the discretization of geometry, namely the quantization of areas and volumes, with the discretization of space implied by focusing on a single tetrahedron. The first is the analogue of the fact that the energy of a mode of the electromagnetic field comes

in discrete quanta. It is a quantum phenomenon. The second is the analogue of the fact that it is convenient to decompose a field into discrete modes and study one mode at the time: it is a convenient isolation of degrees of freedom, completely independent from quantum theory. Geometry is not discrete because we focused on a single tetrahedron: geometry is discrete because the area and volume of *any* tetrahedron (in fact, every polyhedron, as we shall see) take only quantized values.

Another subtle point needs to be caught in order to avoid confusion. One could ask: had we chosen a smaller tetrahedron to start with (a smaller chunk of space), would we had obtained smaller geometrical quanta? The answer is no, and the reason is at the core of the physics of general relativity: there is no notion of size independent from the one provided by the gravitational field itself. Indeed, the coordinates used in general relativity carry no meaning about the size of the problem in which we use them. If we repeat the above considerations for a smaller tetrahedron in coordinate space, we are not dealing with a physically smaller tetrahedron, but only with a different choice of coordinates. This is evident in the fact that the coordinates play no role in the derivation of the above formulas. Whatever tetrahedron we wish to draw, however small, its *physical* size will be determined by the gravitational field on it, and this is quantized, so that its physical size will be quantized with the *same* eigenvalues. The minimal tetrahedron is the first excitation state in quantum pattern of geometry, and its dimensions are arbitrary just like coordinates are. The important thing is that it is a fundamental grane of space that cannot be split in smaller parts, just like there is no way to split the minimal angular momentum in quantum mechanics.

Now that we have spotted the discreteness of quantum geometry in the simple example of the tetrahedron, fuzziness remains. To this end, it is possible to prove the four areas $\hat{A}_a$ of the four faces and the volume $\hat{V}$ of a tetrahedron form a maximally commuting set of operators in the sense of Dirac. Therefore they can be diagonalized together and quantum states of the geometry of the tetrahedron are uniquely characteized by their eigenvalues $|j_a, v\rangle$. Is the shape of such a *quantum* state truly a tetrahedron? The answer is no, for the following reason. The geometry of a classical tetrahedron is given by six degrees of freedom, as we have seen previously. For instance, by the six lengths of its edges. On the other hand, the quantum numbers that determine the quantum states of the tetrahedron are not six, they are only five: four areas and one volume.

The situation is exactly analogous to angular momentum, where a classical rotating system is determined by three numbers, the three components of the angular momentum, but only two quantities (say L^2, L_z) form a complete set. Because of this fact, as is well known, a quantum rotator has never a completely definite angular momentum $\vec{L}$, and we cannot really think of an electron as a ssmall rotating stone: if L_x is sharp, necessarily L_y is quantum spread, and this fuzziness is proportional to their commutator. For the very same reason, it is not possible to have all dihedral angles, all areas and all lengths sharply determined. In any real quantum state there will be a residual quantum fuzziness of the geometry.

In our postulate (3.12), the commutator of the vector operators was scaled by the factor l_0^2. Thus l_0^2 determines the amount of fuzziness of, say, the $\hat{E}^y$ component once $\hat{E}^x$ is maximally sharp. But we have already anticipated in Bronstein's argument that the Planck scale is an upper limit to the sharpness of the measure of a quantity. We are automatically led to conclude that l_0^2 has to be proportional to the Planck length, and we will prove this in the following.

3.4 Graphs and loops

In the previous section we have described the quantum geometry of a single grain of physical space. A region of possibly curved physical space can be described by a set of interconnected grains of space. The relations among these chunks of space can be encoded in the structure of a graph, where each node is a grain of space and the links relate adjacent grains. We will see in this work that this picture emerges naturally from the quantization of the gravitational field. The quantum states of the theory will have a natural graph structure of this kind.

The word *loops* in "Loop Quantum Gravity" refers to the loops formed by closed sequences of links in such graphs. The individual lines in the graph can be viewed as discrete lines of force of the gravitational field. Indeed LQG grew on the intuition that the quantum discreteness makes these lines discrete in the quantum theory, an idea first anticipated by Dirac [30]. Since the gravitational field is spacetime, its discrete quantum lines of force are not in space, but rather form the texture of space itself. This is the physical intuition of LQG.
The word *loops* will gain a precise mathematical meaning when, in order to define a quantum state of spacetime, we will make use of gauge theories to define a connection field ω which governs the parallel transport between two distinct spacetime points. The gauge symmetry comes from the necessity of describing the metric field with more general objects than the metric tensor: the tetrads. These are nothing but internal degrees of freedom in every spacetime point, carrying the local gauge symmetry of Lorentz transformations. Such gauge symmetry requires the existence of ω, living in the Lie algebra of the symmetry group. From ω, we can build a group element h, called holonomy, parallel transporting the tetrad space along the paths of the graph. ω can be derived from h by derivation, but here the theory breaks the relation: a derivation requires a notion of infinitesimal displacement, something that cannot be assumed in the presence of a Planck scale discreteness. Therefore, the existence of the Planck length induces us to only work with the group elements h, which keep well-defined. h is called *loop variable*, since we compute it on the closed paths of the graph, like Wilson loops.

In the following chapters, we are going to give a (limited) introduction to the beautiful journey towards quantum gravity. All that we have introduced in this chapter is meant to provide a first taste of where we are going to land.

Chapter 4

Hamiltonian general relativity

A canonical path towards quantization of GR starts by rewriting the Einstein-Hilbert action in hamiltonian form. This requires the identification of the variables which are canonically conjugated, followed by a Legendre transform. The resulting hamiltonian turns out to be proportional to Lagrange multipliers, whose equations of motion define constraints, that generate gauge transformations. The whole symplectic structure is then quantum promoted *à la* Dirac.

4.1 Einstein's formulation

In Einstein's original formulation, the gravitational field is symmetric tensor field $g_{\mu\nu}$ which can be interpreted as the pseudo-Riemannian metric of spacetime. The action, known as Einstein-Hilbert's, reads (we drop the cosmological constant since it does not contribute conceptually in what follows):

$$S[g] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu} \left(\Gamma(g)\right) \tag{4.1}$$

where g is the determinant of the metric and the Ricci scalar $R_{\mu\nu}$ depends on the Levi-Civita connection, that enters the covariant derivative of vectors and tensors:

$$\nabla_\mu u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\rho\mu}(g) u^\rho \tag{4.2}$$

$$\Gamma^\nu_{\rho\mu}(g) = \frac{1}{2} g^{\nu\lambda} \left[\frac{\partial g_{\lambda\mu}}{\partial x^\rho} + \frac{\partial g_{\lambda\rho}}{\partial x^\mu} - \frac{\partial g_{\rho\mu}}{\partial x^\lambda} \right] \ . \tag{4.3}$$

In the absence of matter, the equations of motion of the metric are:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{4.4}$$

referred to as Einstein's equations. They are a system of ten independent equations, and they are covariant with respect to arbitrary differentiable changes in the coordinate system, as a manifestation of the Principle of General Covariance. We hence say that the theory is diffeomorphism invariant, the diffeomorphism being the smooth maps from one coordinate system to another. The measured value of G is approximately $G \sim 6 \cdot 10^{-8}$ cm^3 g^{-1}s^{-1}.

4.2 ADM formalism

The canonical variables in terms of which we write the hamiltonian formulation of GR were introduced by Arnowit, Deser and Misner, and are called ADM variables after them. In the assumption

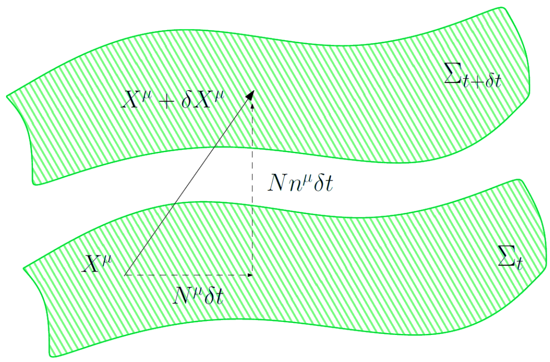
of a ***globally hyperbolic*** 4-dimensional spacetime $\mathcal{M}$ (roughly, if $\mathcal{M}$ has no causally disconnected regions), one can always foliate $\mathcal{M}$ in the topology $\mathcal{M} \cong \Re \times \Sigma$, where Σ is a fixed 3-dimensional manifold of arbitrary topology and spacelike signature. This foliation slices spacetime into a family of spacelike 3-dimensional hypersurfaces $\Sigma = X_t(\Sigma)$, embeddings of Σ in $\mathcal{M}$ labeled by their time coordinate $t \in \mathbb{R}$. This "time" should not be regarded as an absolute quantity, because a diffeomorphism $\phi \in \mathrm{Diff}(\mathcal{M})$ can map a foliation X into a new equivalent one $\phi = X' \circ X^{-1}$, meaning that the foliation is as arbitrary as the diffeomorphism and the physical quantities are independent of this choice.
Given a foliation X_t, we can define the time flow vector in the point x to be the tangent vector along the t coordinate ∂_t:

$$\tau = \partial_t = (1,0,0,0) = \frac{\partial X_t^\mu(x)}{\partial t}\partial_\mu = \tau^\mu \partial_\mu \quad \Rightarrow \quad \tau^\mu = \frac{\partial X_t^\mu(x)}{\partial t} \tag{4.5}$$

where $\{\partial_\mu\}$ is the basis that spans $T_x\mathcal{M}$ made of the tangent vectors related to the embedding ADM coordinates $X^\mu = (t, x^i)$ of x living in $\mathcal{M}$. τ^μ should not be confused with the normal unit vector of each spacelike sheet, n^μ. Although they are both timelike, $g_{\mu\nu}\tau^\mu\tau^\nu = g_{00}$ and $g_{\mu\nu}n^\mu n^\nu = -1$, they are not parallel in general. Thus we can decompose τ^μ into its normal and tangential components with respect to the surfaces:

$$\tau^\mu(x) = N(x)n^\mu(x) + N^\mu(x) \ . \tag{4.6}$$

It is convenient to parametrize $n^\mu(x) = (1/N, -N^a/N)$, so that $N^\mu = (0, N^a)$ and $\tau^\mu(x) = (1,0,0,0)$. N is called the *Lapse function*, and expresses the rate at which time elapses along the unit normal of a 3-surface. N^a is called the *Shift vector* and measures the spatial shift measured by a static observer who carries the coordinates of the t-surface and moves to the $t + dt$-surface.



In terms of Lapse and Shift we have:

$$\begin{aligned} g_{\mu\nu}\tau^\mu\tau^\nu = g_{00} &= g_{\mu\nu}(Nn^\mu + N^\mu)(Nn^\nu + N^\nu) = \\ &= N^2(g_{\mu\nu}n^\mu n^\nu) + g_{ab}N^aN^b + g_{\mu\nu}Nn^\mu N^\nu + g_{\mu\nu}N^\mu Nn^\nu = \\ &= -N^2 + g_{ab}N^aN^b + (g_{\mu\nu} - g_{\nu\mu})Nn^\mu N^\nu = \\ &= -N^2 + g_{ab}N^aN^b = -N^2 + N_aN^a \end{aligned}$$

and similarly:

$$g_{\mu\nu}\tau^\mu n^\nu = \frac{1}{N}(g_{00} - g_{0b}N^b) = -N + \frac{1}{N}g_{b0}N^b - \frac{1}{N}g_{ba}N^aN^b \tag{4.7}$$

$$\Rightarrow \quad g_{0b} = g_{ab}N^a = N_b \ . \tag{4.8}$$

Then we can rewrite the metric as:

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= g_{00}dt^2 + 2g_{a0}dtdx^a + g_{ab}dx^adx^b = \\ &- (N^2 - N_aN^a)dt^2 + 2N_adtdx^a + g_{ab}dx^adx^b \end{aligned}$$

where $a = 1, 2, 3$ are spatial indices and are contracted with the 3-dimensional metric g_{ab}. This 3-metric is not, in general, the spatial part of $q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ which is known as *intrinsic metric*[1]. However, since it lowers and rises indices in the same way as $g_{\mu\nu}$ along any sheet of the foliation, $q_{\mu\nu}$ can be used along with Lapse and Shift to replace the $g_{\mu\nu}$ variables.
When writing the action in terms of the intrinsic metric and the *extrinsic curvature* on the $t = const$ surfaces:

$$K_{\mu\nu} = g^{\alpha\beta} q_{\beta\mu} g^{\gamma\delta} q_{\delta\nu} \nabla_\alpha n_\beta \tag{4.9}$$

one finds out that the Lagrangian does not contain the time derivatives of N and N^a. This means that, among the triplet (q_{ab},N,N^a), N and N^a are Lagrange multipliers with null conjugate momenta:

$$\frac{\delta \mathcal{L}}{\delta \dot{N}} = \frac{\delta \mathcal{L}}{\delta \dot{N}^a} = 0 \ . \tag{4.10}$$

The true dynamical variable is therefore q_{ab} only, with conjugate momentum:

$$\pi^{ab} = \frac{\delta \mathcal{L}}{\delta \dot{q}_{ab}} = \sqrt{q}(K^{ab} - K q^{ab}) \ . \tag{4.11}$$

After performing the Legendre transform, the action becomes:

$$S_{EH}(q_{ab}, \pi^{ab}, N, N^a) = \frac{1}{16\pi G} \int dt \int d^3x \left[\pi^{ab} \dot{q}_{ab} - N^a H_a - NH\right] \tag{4.12}$$

where:

$$H_a = -2\sqrt{q}\, \nabla_b \left(\frac{\pi_a^b}{\sqrt{q}}\right) \ ; \tag{4.13}$$

$$H = \frac{1}{\sqrt{q}} G_{abcd} \pi^{ab} \pi^{cd} - \sqrt{q} R \ ; \tag{4.14}$$

$$G_{abcd} = q_{ac} q_{bd} + q_{ad} q_{bc} - q_{ab} q_{cd} \ . \tag{4.15}$$

The variation of the action with respect to the Lagrange multipliers gives:

$$H_a(q, \pi) = 0 \ ; \qquad H(q, \pi) = 0 \ , \tag{4.16}$$

called respectively *vector constraint* and *scalar* (or *Hamiltonian*) *constraint*, which bound physical configurations on the subspace where they are satisfied. They can be expressed in compact form as $H_\mu = (H, H_a)$.
The symplectic structure defined on the phase space (q_{ab},π^{ab}) is canonical and gives the canonical Poisson brackets for two generic functions $A(y)$ and $B(z)$:

$$\{A(y), B(z)\} = \int d^3x \frac{\delta A(y)}{\delta q_{ab}(x)} \frac{\delta B(z)}{\delta \pi^{ab}(x)} - \frac{\delta B(z)}{\delta q_{ab}(x)} \frac{\delta A(y)}{\delta \pi^{ab}(x)} \ . \tag{4.17}$$

In particular:

$$\{\pi^{ab}(t, \vec{x}), q_{cd}(t, \vec{y})\} = \delta^a_{(c} \delta^b_{d)} \delta^3(\vec{x} - \vec{y}) \tag{4.18}$$

which is the canonical result for the dynamic variables. The Poisson brackets of the constraints vanish on the constraint surface:

$$\{H_a(x), H_b(y)\}|_{H^\mu=0} = \{H_a(x), H(y)\}|_{H^\mu=0} = \{H(x), H(y)\}|_{H^\mu=0} = 0 \tag{4.19}$$

Such constraints are called first class constraints and their smeared versions:

$$H(\vec{N}) = \int_\Sigma d^3x H^a(x) N_a(x) \ ; \qquad H(N) = \int_\Sigma d^3x H(x) N(x) \tag{4.20}$$

[1] The intrinsic metric can be seen in a double way. It is either the pullback of the embedding function $i : \Sigma_t \to \mathcal{M}$ applied to the metric g, or the projector of the metric g on Σ_t. See [9] for details on foliations.

are respectively the infinitesimal generators of the space and time diffeomorphisms as we can see from the brackets:

$$\{H(\vec{N}), q_{ab}\} = \mathcal{L}_{\vec{N}} q_{ab} \ ; \qquad \{H(\vec{N}), \pi^{ab}\} = \mathcal{L}_{\vec{N}} \pi^{ab} \ . \tag{4.21}$$

$$\{H(N), q_{ab}\} = \mathcal{L}_{N\vec{n}} q_{ab} \ ; \qquad \{H(N), \pi^{ab}\} = \mathcal{L}_{N\vec{n}} \pi^{ab} \ . \tag{4.22}$$

The constraints are identified as the generators of the gauge symmetries that of change and deform the chosen foliation. Moreover these symmetries do not change the subspace where dynamics lives, thanks to them vanishing on it. To perform the gauge transformation, one should compute the Poisson brackets *before* repeating the physical condition of the null constraint. In jargon $\dot{H}_\mu$ is weakly null: $H_\mu \approx 0$, and this order is to be kept when quantizing, as we well see.
From the action we see that the hamitonian of general relatively is:

$$H = \frac{1}{16\pi G} \int d^3x N^a H_a + NH \ . \tag{4.23}$$

It is proportional to the Lagrange multipliers and thus vanishes on the physical space. Hence, there is no dynamics and no physical evolution in the time t. This puzzling absence of a physical Hamiltonian is in fact a consequence of the diffeomorphism invariance of the theory: t is a mere parameter devoid of an absolute physical meaning, thus there is no physical dynamics in t. This is the root of the so-called "problem of time" in general relativity. See [22],[27] for discussions on the problem.
To remark the importance of the constraints H_μ, it is possible to show (we will not do it here) that the quantity $G_{\mu\nu}n^\mu n^\nu$, where $G_{\mu\nu}$ is the Einstein tensor, is equivalent to a linear combination of the constraints. If $g_{\mu\nu}$ satisfies them on any spacelike Cauchy surface (i.e. on every sheet of the chosen foliation), it automatically satisfies the ten vacuum Einstein equations: $G_{\mu\nu} = 0$. This is a crucial point: the whole dynamical content of GR is in these four constraints.

4.3 Dirac's quantization program

The approach to quantization of fully constrained system proposed by Dirac is based on the fact that the dynamics is entirely encoded in the constraints. Basically, the physical quantum states are the ones that are annihilated by the operator counterpart of the constraints. The procedure is the following:

- ▷ find a representation of the phase space variables (q_{ab}, π^{ab}) as operators in an "auxiliary" kinematical Hilbert space $\mathcal{H}_{kin}$, satisfying the standard commutation relations:

$$\{\cdot,\cdot\} \rightarrow \frac{1}{\hbar}[\cdot,\cdot] \ . \tag{4.24}$$

- ▷ Promote the constraint to operators $\hat{H}^\mu$ acting on $\mathcal{H}_{kin}$.
- ▷ Define the physical Hilbert space $\mathcal{H}_{phys}$ as the space of solutions of the constraints: $\hat{H}^\mu \psi = 0 \ \forall \psi \ \in \ \mathcal{H}_{phys}$.

Nevertheless, we need to define explicitly a scalar product between states to equip $\mathcal{H}_{phys}$ with, and to give a physical interpretation of the quantum observables.
In gravity's case, this is not straightforward, and leads to some difficulties.
By analogy with better known cases, we choose a Schröedinger representation for the phase space variables $\hat{q}_{ab}(x) = q_{ab}(x)$; $\hat{\pi}^{ab}(x) = -i\hbar\delta/(\delta q_{ab}(x))$, acting on wave functionals $\psi[q_{ab}(x)]$ of the 3-metric. The resulting Hilbert space needs a scalar product, formally:

$$\langle\psi|\psi\rangle \equiv \int dq \overline{\psi[q]} \psi'[q] \tag{4.25}$$

but no Lebesgue measure on the space of metrics exists. Without this, we cannot check the hermiticity of $\hat{q}_{ab}$ and $\hat{\pi}^{ab}$, nor the spacelike character of $\hat{q}_{ab}$.
Even ignoring the mentioned problem, one can proceed to promote the constraints to operators, and find the space of solutions:

$$\mathcal{H}_{kin} \xrightarrow{\hat{H}_a = 0} \mathcal{H}_{diff} \xrightarrow{\hat{H} = 0} \mathcal{H}_{phys} \ . \tag{4.26}$$

While the promotion of $\hat{H}_a$ is straightforward, $\hat{H}$ is ill defined and the space that solves $\hat{H} = 0$ (known as Wheeler De Witt equation) after a tentative regularization of $\hat{H}$ is not provided with a proper measure and lacks any characterization.

4.4 Tetrads

Loop quantum gravity tries to improve the situation in a surprisingly simple way: instead of changing the gravitation theory or the quantizatiom paradigm, it only uses different variable, that allow us to reformulate general relativity in a more amenable way to Dirac's quantization procedure.
The starting point is building up an internal space of degrees of freedom at each point of the spacetime manifold $\mathcal{M}$. More precisely, we attach to each point a Lorentzian inner spacetime spanned by a set of 4 orthonormal vectors which will label with capital latin indices (e_I, $I = 0, 1, 2, 3$) one timelike and three spacelike:

$$g(e_I, e_J) = \eta_{IJ} \tag{4.27}$$

where g is the metric of the spacetime and η_{IJ} is the Minkowski metric with signature (-,+,+,+). The set of vectors we have introduced is called *tetrads* or *vierbeins* and can be seen as a set of four vector fields defined all over $\mathcal{M}$. Thus, in every point $x \in \mathcal{M}$ we can express each of the four tetrad vectors in terms of the natural basis $T_x\mathcal{M}$, the tangent vector space in x, i.e. $\{\partial_\mu\}$

$$e_I = e_I^\mu \partial_\mu \tag{4.28}$$

and viceversa, we can write $\{\partial_\mu\}$ on the tetrad basis:

$$\partial_\mu = e_\mu^I e_I \ . \tag{4.29}$$

The components e_μ^I form an $n \times n$ invertible matrix, and in terms of their inverse e_I^μ equation 4.27 becomes:

$$g_{\mu\nu}(x) e_I^\mu(x) e_J^\nu(x) = \eta_{IJ} \tag{4.30}$$

or equivalently:

$$g_{\mu\nu}(x) = e_\mu^I(x) e_\nu^J(x) \eta_{IJ} \ . \tag{4.31}$$

In jargon the tetrads are said to be the "square root" of the metric.
Similarly, we can set up an orthonormal basis of one-forms in the dual of the vector space of tetrads, which we denote e^I:

$$e^I(e_J) = e_J(e^I) = \delta_J^I \ . \tag{4.32}$$

As before, we can express the basis of 1-forms $\{dx^\mu\}$ in each $T_x^*\mathcal{M}$ on the basis of e^I and viceversa:

$$dx^\mu = e_I^\mu e^I \ ; \qquad e^I = e_\mu^I dx^\mu \ . \tag{4.33}$$

For simplicity, we will call e_μ^I the *tetrad* (1-form) and e_I^μ the *inverse tetrad* (vector), in accord with the fashion of blurring the distinction between objects and their components. They satisfy:

$$e_I^\mu e_\nu^I = \delta_\nu^\mu \ ; \qquad e_\mu^I e_J^\mu = \delta_J^I \ ; \qquad e_I^\mu = g^{\mu\nu} \eta_{IJ} e_\nu^J \ . \tag{4.34}$$

Any vector can be expressed in terms of its components in the tetrad basis:

$$V = V^\mu \partial_\mu \quad \to \quad V = V^I e_I \ ; \qquad \underline{V^I = e^I_\mu V^\mu} \ . \tag{4.35}$$

For a 1-form:

$$\theta = \theta_\mu dx^\mu \quad \to \quad \theta = \theta_I e^I \ ; \qquad \underline{\theta_I = e^\mu_I \theta_\mu} \ . \tag{4.36}$$

For a 1-tensor:

$$T = T^\mu_\nu \partial_\mu \otimes dx^\nu \quad \to \quad T = T^I_J e_I \otimes e^J \ ; \qquad \underline{T^I_J = e^I_\mu e^\nu_J T^\mu_\nu} \ . \tag{4.37}$$

It is now clear that we can see e^I_μ and e^μ_I as the components of local *identity maps* [10], i.e. that $(1,1)$-tensors act on objects by simply switching the basis on which they are written from the coordinate basis to the orthonormal basis, and viceversa:

$$e(x) = e^I_\mu(x) dx^\mu \otimes e_I \tag{4.38}$$

$$e^{-1}(x) = e^\mu_I(x) \partial_\mu \otimes e^I \tag{4.39}$$

$$e(x) \ : \ T_x\mathcal{M} \quad \to \quad T_x\mathbb{M} = \mathbb{M} \qquad \{\partial_\mu\} \quad \to \quad \{e_I\} \tag{4.40}$$

$$e(x) \ : \ T^*_x\mathbb{M} = \mathbb{M} \quad \to \quad T^*_x\mathcal{M} \qquad \{e_I\} \quad \to \quad \{dx^\mu\} \tag{4.41}$$

$$e^{-1}(x) \ : \ T_x\mathbb{M} = \mathbb{M} \quad \to \quad T_x\mathcal{M} \qquad \{e_I\} \quad \to \quad \{\partial_\mu\} \tag{4.42}$$

$$e^{-1}(x) \ : \ T^*_x\mathcal{M} \quad \to \quad T^*_x\mathbb{M} = \mathbb{M} \qquad \{dx^\mu\} \quad \to \quad \{e_I\} \ . \tag{4.43}$$

In this view equation 4.31 means that $g_{\mu\nu}(x)$ is the pullback of the Minkowski metric to the tangent space $T_x\mathcal{M}$. The tetrad map captures the equivalence principle of GR, asserting that locally we can always find a reference with respect to which spacetime is flat. The tetrad formulation is unavoidable when studying fermions in curved spacetime, since there is no finite dimensional spinorial representation of the general covariance group (the group of spacetime diffeomorphisms whose representation is carried by the greek indices in GR). Neverthless we can build spinorial finite-dimensional representations of the Lorentz group. The tetrads bridge the gap allowing us to couple fermions to gravity by implementing the metric in the components of the vierbeins. Fermions can then be defined in the local Minkowski spacetime, as usual.
The local inertial frame of tetrads is defined so that equation 4.30 holds. These leaves a freedom to arbitrary change the basis in each point with transformations that preserve the flat Minkowski metric. We know that such transformations are the Lorentz transformations:

$$e^I_\mu(x) \quad \to e^{I'}_\mu(x) = \Lambda^{I'}_I(x) e^I_\mu(x) \ . \tag{4.44}$$

The matrices $\Lambda^{I'}_I(x)$ represent position-dependent Lorentz transformations which, at each point, leave the flat metric unaltered:

$$g_{\mu\nu}(x) e^\mu_J(x) = \eta_{IJ} \quad \to \quad g_{\mu\nu}(x) e^\mu_{I'}(x) e^\nu_{J'}(x) = g_{\mu\nu}(x) \Lambda^I_{I'}(x) e^\mu_I(x) \Lambda^J_{J'}(x) e^\nu_J(x) = \\ = \Lambda^I_{I'}(x) \Lambda^J_{J'}(x) \eta_{IJ} = \eta_{I'J'} \ .$$

Conversely, the metric $g_{\mu\nu}$ is not affected by the local Lorentz transformations:

$$g_{\mu\nu}(x) = e^I_\mu(x) e^J_\nu(x) \eta_{IJ} = e^{I'}_\mu(x) e^{J'}_\nu(x) \eta_{I'J'} \tag{4.45}$$

so the action is invariant under such transformations. We conclude that the tetrad formalism is provided with a Lorentz $SO(3,1)$ gauge invariance, the Lorentz representation being carried by the "interval" index I.
We now have a spacetime manifold $\mathcal{M}$ over which we can build:

- the tangent bundle $T\mathcal{M} = \underset{x}{U}\{x \in \mathcal{M}, \ T_x\mathcal{M}\}$, whose fiber is the tangent space;

– the tetrad vector bundle $F = \underset{x}{U}\{x \in \mathcal{M},\ \{e_I\}\,\}$, whose fiber is a flat spacetime with a gauge freedom in the choiche of the basis.

The latter structure can be generalized with the notion of G- principal bundle, a bundle whose fibers carry a representation of a group G. The action of G on the fibers changes them locally but preserves the entire bundle, and G is denoted as the abstract fiber on the principle bundle. So in our case we have, along with the the tangent bundle, a Lorentz principal bundle $F = (\mathcal{M},\ SO(3,1))$. In Yang-Mills theories, we have a $SU(N)$-principal bundle, $SU(N)$ being the non Abelian Lie group that acts on the space of multiplets of N copies of the spinor field.
In a general G-principal bundle, the device that allows to differentiate objects moving along tangent directions in the base manifold (in our case, $\mathcal{M}$) is the G-connection, which keeps track of the gauge arbitrariness induced by G in each fibers and "connects" the fibers on nearby points to overcome this local amibiguity. The connection defines the gauge-covariant derivative along $\mathcal{M}$, a notion of parallel transport on the bundle, and then a curvature. For istance, the Levi-Civita connection keeps track of the freedom changing the local coordinates via diffeomorphism of the tangent (and cotangent) bundle. See appendix B for a general definition.
Although the connection form does not transform tensorially under the action of G, the gauge covariant derivative of objects defined in the vector fiber cannot sense, by its very definition, the difference in the choice of the basis, i.e. in the gauge fixing, and has to transform covariantly:

$$D'X' = g(DX)\ . \tag{4.46}$$

Let us now specialize what said to our case with tetrads on the spacetime $\mathcal{M}$. We already know that the connection form of the tangent bundle $T\mathcal{M}$ is the Levi-Civita connection form $\Gamma^{\alpha}_{\beta\gamma}$. Here, there is no difference in the tangent coordinate along which differentiating and the two "interval" indices of the vector fiber, since the latter is the tangent fiber itself.
Taking a vector field $X\ \in\ \Gamma(T\mathcal{M})$, where Γ denotes a section, we write:

$$\nabla X = \nabla\left(X^\mu\partial_\mu\right) = \left(\frac{\partial X^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\lambda}X^\lambda\right)\partial_\mu \otimes dx^\nu \tag{4.47}$$

as the covariant derivative related to the Levi-Civita connection.
Now we express the vector field V in terms of a tetrad base $\{e_I\}$:

$$V = V^\mu\partial_\mu = V^I e_I\ ; \qquad V^\mu = e^\mu_I V^I \tag{4.48}$$

and write its covariante derivative as:

$$DX = D\left(X^I e_I\right) = \left(\frac{\partial X^I}{\partial x^\nu} + X^J\omega^I_{\nu J}\right) e_I \otimes dx^\nu\ . \tag{4.49}$$

Let us convert this quantity to the $\{\partial_\mu\}$ basis:

$$\begin{aligned} DX &= \left(\frac{\partial}{\partial x^\nu}(e^I_\sigma X^\sigma) + (e^J_\lambda X^\lambda)\omega^I_{\nu J}\right)(e^\mu_I\partial_\mu)\otimes dx^\nu = \\ &= \left(\frac{\partial X^\mu}{\partial x^\nu} + e^\mu_I\partial_\nu e^I_\lambda X^\lambda + e^\mu_I e^J_\lambda\omega^I_{\nu J}X^\lambda\right)\partial_\mu\otimes dx^\nu \end{aligned} \tag{4.50}$$

Comparing equation 4.47 with 4.50 reveals:

$$\Gamma^\mu_{\nu\lambda} = e^\mu_I\partial_\nu e^I_\lambda + e^\mu_I e^J_\lambda\omega^I_{\nu J} \tag{4.51}$$

or equivalently:

$$\omega^I_{\nu J} = e^I_\mu e^\lambda_J\Gamma^\mu_{\nu\lambda} - e^\lambda_J\partial_\nu e^I_\lambda = e^I_\mu\nabla_\nu e^\mu_J\ . \tag{4.52}$$

A bit of manipulation allows us to write this relation as the vanishing of the covariant derivative of the tetrads:

$$D_\nu e^I_\mu = \partial_\nu e^I_\mu - \Gamma^\lambda_{\mu\nu} e^I_\lambda + \omega^I_{\nu J} e^J_\mu = 0 \tag{4.53}$$

which is sometimes known as the "tetrad postulate" or tetrad-compatibility and is analogous to the metric-compatibility of the Levi-Civita connection: $\nabla_\mu g_{\nu\rho} = 0$. If we define the torsion 2-form as the quantity:

$$T^I = de^I + \omega^I_J \wedge e^J = (\partial_\mu e^I_\nu + \omega^I_{\mu J} e^J_\nu) dx^\mu \wedge dx^\nu \tag{4.54}$$

namely the covariant exterior derivative of the tetrad (1-form), the tetrad postulate induces the vanishing of the torsion, something called the torsion-free condition.
The connection we have found dictates the covariant derivative of objects in the tetrad frame and is called *spin connection*, because it can be used to take derivatives of spinors, or *Lorentz connection*, since the gauge group is the Lorentz group. Under its action, this connection transforms as:

$$\omega^{I'}_{\mu J'} = \Lambda^{I'}_I \Lambda^J_{J'} \omega^I_{\mu J} - \Lambda^K_{J'} \partial_\mu \Lambda^{I'}_K \tag{4.55}$$

and so, as already said, the Lorentz connection takes values in the Lorentz algebra.
The metric compatibility condition $\nabla g = 0$ is translated in a further property of the connection form when passing to tetrads:

$$\begin{aligned} &0 = \nabla_\lambda(g_{\mu\nu}) = D_\lambda(e^I_\mu e^J_\nu \eta_{IJ}) = D_\lambda(\eta_{IJ}) = \partial_\lambda \eta_{IJ} - \omega^K_{\lambda I} \eta_{KJ} - e^L_{\lambda J} \eta_{IL} = -\omega_{\lambda JI} - \omega_{\lambda IJ} \\ &\Rightarrow \quad \omega_{\lambda IJ} = -\omega_{\lambda JI} \ . \end{aligned} \tag{4.56}$$

The tetrad 1-forms and connection 1-forms:

$$e^I = e^I_\mu dx^\mu \ ; \qquad \omega^I_J = \omega^I_{\mu J} dx^\mu \tag{4.57}$$

allow us to define the curvature, a tensor-valued 2-form:

$$F^I_J = d\omega^I_J + \omega^J_K \wedge \omega^K_J = F^I_{J\mu\nu} dx^\mu \wedge dx^\nu \tag{4.58}$$

where d is the exterior derivative and $\wedge$ is the wedge product (beware of not seeing it as a covariant derivative of the connection, since the latter is not a tensor). It can be seen as the field-strength of the connection and the I and J indices carry a representation of the Lorentz algebra, as the connection.
The curvature's components are:

$$F^I_{J\mu\nu} = \partial_\mu \omega^I_{\nu J} - \partial_\nu \omega^I_{\mu J} + \omega^I_{\mu K} \omega^K_{\nu J} - \omega^K_{\nu J} \omega^I_{\nu K} \tag{4.59}$$

Using the solution $\omega(e)$ of the tetrad postulate, we can express the latter equation as:

$$F^I_{J\mu\nu}(\omega(e)) = e^I_\rho e^\sigma_J R^\rho_{\sigma\mu\nu}(e) \tag{4.60}$$

where $R^\rho_{\sigma\mu\nu}(e)$ is the Riemann tenor coming from the metric g defined by the tetrad.
Thanks to the tetrad formalism, it is possible to re-write the Einstein-Hilbert action in the form (we set $8\pi G = 1$):

$$S[e] = \frac{1}{2} \epsilon_{IJKL} \int e^I \wedge e^J \wedge F^{KL}(\omega(e)) \ . \tag{4.61}$$

See appendix C for the explicit derivation.
On top of the invariance under diffeomorphisms, this reformulation of the theory possesses an additional gauge symmetry under local Lorentz transormations. The quantity:

$$\Sigma^{IJ} = e^I \wedge e^J \tag{4.62}$$

is called the Plebanski 2-form, with antisymmetric Minkowski indices and taking values in the Lorentz algebra.

It is important to notice that the tetradic formulation of the GR action is not fully equivalent to the Einstein-Hilbert action. The difference becomes clear when performing an internal time-reversal operation in the fiber basis:

$$T : \begin{cases} e^0 & \mapsto & -e^0 \\ e^i & \mapsto & e^i, \quad i = 1, 2, 3 \ . \end{cases} \tag{4.63}$$

Since $g = e^I e_I$, the Einstein-Hilbert action is not affected by this transformation, while the tetrad action flips sign:

$$S[Te] = -S[e] \ . \tag{4.64}$$

This has not much effect on the classical theory, but it is important in the quantum theory: in defining a path integral for the gravitational field, integration over tetrads includes integrating over configurations with backward oriented tetrads, which contribute to the Feynman integral with a term of the form:

$$e^{\frac{-i}{\hbar}S[e]} \tag{4.65}$$

together with

$$e^{\frac{i}{\hbar}S[e]} \ . \tag{4.66}$$

The two contributions are analogous to the forward propagation and the backward propagation paths integral for relativistic particles. The tetradic action appears to be the most natural starting point for an attempt to quantization.

4.5 First and second order formulation

The tetradic action we have found contains a term depending on the only spin connection ω that fulfils the the tetrad postulate, as remarked by writing ω as a function of e: $\omega(e)$. However, we can also consider a different action, where the tetrad and the spin connection are treated as indipendent fields:

$$S[e, \omega] = \frac{1}{2}\epsilon_{IJKL} \int e^I \wedge e^J \wedge F^{KL}(\omega) \ . \tag{4.67}$$

This action is polynomial, a welcome property for quantization, and is known as the *Palantini action.*

In the case of pure gravity, i.e. in absence of matter coupled to the tetrad such as fermions, this action gives the same equations of motion resulting from imposing the principle of least action with respect to the indipendent variable ω just recovers the tetradic postulate. In jargon, the spin connection is the on-shell connection for GR.

Such formulation of $S[e, \omega]$ is called *first order formulation* as distinguished from the formulation of $S[e]$, known as *second order formulation.* In both formulations, the variation of the action with respect to the tetrad gives the Einstein equations. If we decide to work in the first order formulation, there exists an additional term that can be added to the lagrangian which is compatible with all the symmetries of the theory and has mass dimension 4:

$$\delta_{I[K}\delta_{L]J} e^I \wedge e^J \wedge F^{KL}(\omega) \ . \tag{4.68}$$

This term does not affect the equations of motion for every value of γ, since it vanishes when the connection is on-shell, and can enter the action with a coupling constant $1/\gamma$ to form the so-called Host action (we suppress $1/(8\pi G)$)

$$\begin{aligned} S[e, \omega] &= \left(\frac{1}{2}\epsilon_{IJKL} + \frac{1}{\gamma}\delta_{I[K}\delta_{L]J}\right) \int e^I \wedge e^J \wedge F^{KL}(\omega) = \\ &= \int \left(\frac{1}{2}\epsilon_{IJKL} + \frac{1}{\gamma}\delta_{I[K}\delta_{L]J}\right) \Sigma^{IJ}(e) \wedge F^{KL}(\omega) \ . \end{aligned} \tag{4.69}$$

The Holst action is the most generic tetradic action and is the starting point of 4-dimensional quantum gravity. The coupling costant γ is called the Barbero-Immirzi parameter.

4.6 Hamiltonian analysis of tetradic GR action

We are now ready to extract again the hamiltonian formulation of the theory with the aim of quantizing it following Dirac's program. What is different from before is the use of the tetrad formalism, which is expected to overcome the past difficulties.
We proceed as before by assuming a 3+1 splitting of the spacetime ($\mathcal{M} = \Sigma_t \times \mathbb{R}$) with coordinates (t, x^a) and decomposing the metric in terms of the ADM variables, i.e. the lapse function, the shift vector and the spatial metric:

$$ds^2 = -(N^2 - N_a N^a)dt^2 + 2N_a dt dx^a + g_{ab}dx^a dx^b \qquad a, b = 1, 2, 3 \ . \tag{4.70}$$

The time-flow vector is again written as:

$$\tau^\mu(x) = N(x)n^\mu(x) + N^\mu(x) \tag{4.71}$$

where $n^\mu(x)$ is the unit length vector normal to all vectors tangent to Σ. If we call $\{\sigma^i\}$ with $i = 1, 2, 3$ the coordinates of a point $\sigma \in \Sigma_t$ and $x^\mu(\sigma)$ the embedding of Σ_t into 4d spacetime, the tangent vectors that span the tangent space of each spacelike 3d manifold Σ_t, $T_\sigma \Sigma_t$ are:

$$\partial_i = \frac{\partial x^\mu}{\partial \sigma^i}\partial_\mu \ , \qquad i = 1, 2, 3 \tag{4.72}$$

where $\partial_\mu = (\partial_0, \partial_a; \ a = 1, 2, 3)$ are the vectors that span $T_{x(\sigma)}\mathcal{M}$ and so:

$$n^\mu \sim \epsilon_{\mu\nu\lambda\kappa}\frac{\partial x^\nu}{\partial \sigma^1}\frac{\partial x^\lambda}{\partial \sigma^2}\frac{\partial x^\kappa}{\partial \sigma^3} \tag{4.73}$$

while: $\tau^\mu = \dfrac{\partial x^\mu}{\partial t} = (1, 0, 0, 0)$ as usual.
Given a point $\sigma \in \Sigma_t$ and its coordinates $x^\mu(\sigma)$ in $\mathcal{M}$, we now plug tetrads in order to map the tangent space $T_x\mathcal{M}$ into Minkowski space $T_x\mathbb{M} = \mathbb{M}$.
The basis of the Minkowski internal space $e_I = (e_0, e_i \ ; \ i = 1, 2, 3)$ is made of:

$$e_0 = e_0^\mu \partial_\mu \tag{4.74}$$
$$e_i = e_i^\mu \partial_\mu \qquad i = 1, 2, 3 \tag{4.75}$$

where $g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ}$.
As already said, the basis comes provided with a gauge $SL(2, \mathbb{C})$ symmetry given by the arbitrariness in the choice of the local Lorentz frame in each point of the manifold.
In the tetradic space we get:

$$\tau^I = e_\mu^I \tau^\mu = N e_\mu^I n^\mu + e_\mu^I N^\mu = N n^I + N^I \tag{4.76}$$

where:

$$n^I = e_\mu^I n^\mu = \epsilon^I_{\ JKL} e_\nu^J e_\lambda^K e_\kappa^L \frac{\partial x^\nu}{\partial \sigma^1}\frac{\partial x^\lambda}{\partial \sigma^2}\frac{\partial x^\kappa}{\partial \sigma^3} \ . \tag{4.77}$$

As before, we want to identify canonically conjugated variables and perform the Legendre transform, but we have two new features: the first one is the symmetry given by the tetrads, which we expect to result in an additional constraint; the second one comes from working in first order formulation, i.e. treating the tetrads and the connection as indipendent fields, so that the conjugated variables we are looking for will be functions of both of them (and of their time derivatives). As a consequence, the constraint algebra turns out to be of second class, but this difficulty can be overcome with two smart moves.
The first step is to take advantage of the tetradic gauge symmetry by choosing the local Lorentz frames in such a way that the discussion simplifies. For example, it is customary to orient the Lorentz frames in such way that in each of them, all spacelike surface Σ_t are fixed-time surfaces.

This is a partial gauge fixing called *time gauge*, where we set n^I to be $(1, 0, 0, 0)$, i.e. $n \equiv e_0$. This way, the spatial part e^i_a of the tetrad called *triad*, maps the 3-dimensional tangent space $T_x\Sigma$ into the euclidean 3-d spatial subspace of the Minkowski space:

$$g_{ab} = e^i_a e^j_b \delta_{ij} \ . \tag{4.78}$$

The original $SL(2, \mathbb{C})$ gauge symmetry breaks down to $SO(3)$ subgroup of spatial rotations that preserves the choice of aligning every local tetradic time vector e_0 along n.
The second step is to perform a change of variables, introduced for the first time by Ashtekar: we define the *Ashtekar electric field*:

$$\begin{aligned} E^a_i = (det e) e^a_i &= \frac{1}{2}\epsilon_{ljk}\epsilon^{dbc} e^l_d e^j_b e^k_c e^a_i = \\ &= \frac{1}{2}\epsilon_{jkl}\epsilon^{dbc} e^j_b e^k_c \delta^l_i \delta^a_d = \\ &= \frac{1}{2}\epsilon_{ijk}\epsilon^{abc} e^j_b e^k_c \end{aligned} \tag{4.79}$$

namely the densitized inverse triad (inverse triad multiplied by its determinant), and the *Ashtekar-Barbero connection*:

$$A^i_a = \gamma\omega^{0i}_a[e] + \frac{1}{2}\epsilon^i_{jk}\omega^{jk}_a[e] \tag{4.80}$$

where γ is the Barbero-Immirzi parameter.
This new connection satisfies the Poisson brackets:

$$\{A^i_a(x),\ A^j_b(y)\} = 0 \tag{4.81}$$

while:

$$\{A^i_a(x), E^b_j(y)\} = \gamma\delta^b_a\delta^i_j\delta^3(x, y) \tag{4.82}$$

and so these variables are canonically conjugate.
In terms of them, the Holst action reads (we suppress $1/16\pi G$):

$$S[A, E, N, N^a] = \frac{1}{\gamma}\int dt \int_\Sigma d^3x \left[\dot{A}^i_a E^a_i - A^i_0 D_a E^a_i - NH - N^a H_a\right] \tag{4.83}$$

where:

$$H_a = \frac{1}{\gamma}F^j_{ab}E^b_j - \frac{1+\gamma^2}{\gamma}K^i_a G_i \tag{4.84}$$

$$H = \left[F^j_{ab} - (\gamma^2+1)\epsilon^j_{mn}K^m_a K^n_b\right]\frac{1}{\det E}\epsilon^{kl}_j E^a_k E^b_l + \frac{1+\gamma^2}{\gamma}G^i\partial_a\left(\frac{E^a_i}{\det E}\right) \tag{4.85}$$

are as usual the constraints that generate diffeomorphisms of the foliation while:

$$G_i \equiv D_a E^a_i = \partial_a E^a_i + \epsilon_{ijk}A^j_a E^{ak} \tag{4.86}$$

is understood as the covariant derivative of the densitized triad, and is the new expected constraint which generates the $SO(3) \sim SU(2)$ gauge transformations of the triads, the so-called Gauss constraint. Under the action of its smeared version:

$$G(\Lambda) = \int d^3x G_i(x)\Lambda^i(x) \tag{4.87}$$

through the Poisson brackets, E^a_i transforms as an $SU(2)$ vector, while A^i_a as an $SU(2)$ connection:

$$\left\{\int d^3x\Lambda^j(x)G_j(x)\ ,\ E^a_i(y)\right\} = \gamma\epsilon_{ijk}\Lambda^j(y)E^{ak}(y) \tag{4.88}$$

$$\left\{\int d^3x\Lambda^j(x)G_j(x)\ ,\ A^i_a(y)\right\} = \gamma\partial_a\Lambda^i(y) + \epsilon^i_{jk}\Lambda^j(y)A^k_a(y)\ . \tag{4.89}$$

$$\tag{4.90}$$

4.7 Linear simplicity constraint

At this stage it is important to gather a simple result which has major importance for the quantum theory. Let us go back to the Holst action written in terms of the Plebanski 2-form and the curvature tensor:

$$S\left[e,\omega\right]=\frac{1}{16\pi G}\int\left(\frac{1}{2}\epsilon_{IJKL}+\frac{1}{\gamma}\delta_{I[K}\delta_{L]J}\right)\Sigma^{IJ}\left(e\right)\wedge F^{KL}\left(\omega\right) \tag{4.91}$$

and let us call:

$$B_{KL}\left(e\right)\equiv\frac{1}{8\pi G}\left(\frac{1}{2}\epsilon_{IJKL}+\frac{1}{\gamma}\delta_{I[K}\delta_{L]J}\right)\Sigma^{IJ}\left(e\right) \tag{4.92}$$

Given a foliation and working in *time gauge*, the pullback of the Lorentz connection ω to each of the $t = const$ hypersurface is $\omega = \omega_a^{IJ} dx^a$; $a = 1, 2, 3$ and the pullback of B to each surface is the momentum conjugate to ω, since the quadratic part of the action becomes proportional to the term $B \wedge d\omega$. Therefore B is a 2-form on the spacelike hypersurfaces taking values in the Lorentz algebra. But there is something more: in the local Lorentz frame we have chosen via the *time gauge*, B_{IJ} can be decomposed into its electric $K_I = B_{IJ}n^J$ and magnetic $L_I = \frac{1}{2}\epsilon_{IJ}^{KL}B_{KL}n^J$ parts, similarly to the decomposition of the electromagnetic tensor once a Lorentz frame is picked. And since B is antisymmetric in its Lorentz indices, just like the electromagnetic tensor, both K and L do not have components along n

$$K_I n^I = B_{IJ}n^J n^I = 0 \qquad L_I n^I = \frac{1}{2}\epsilon_{IJ}^{KL}B_{KL}n^J n^I = 0 \tag{4.93}$$

and lie as 3d vectors in the 3d fixed-time hypersurface Σ normal to n, with 3d components:

$$\vec{K} = K_i = B_{i0} \qquad \vec{L} = L_i = \frac{1}{2}\epsilon_i^{jk}B_{jk} \qquad i,j,k = 1,2,3. \tag{4.94}$$

Keeping in mind that B takes values in the Lorentz algebra, K_i are nothing but the boost generators of the Lorentz group, while L_i are the 3d rotation generators. If we express them in terms of the Plebanski form we get:

$$\begin{aligned}\vec{K} = K_I = B_{IJ}n^J &= \left(\frac{1}{2}\epsilon_{IJKL}+\frac{1}{\gamma}\delta_{K[I}\delta_{J]L}\right)\Sigma^{KL}n^J = \\ &= \frac{1}{2}\epsilon_{IJKL}\Sigma^{KL}n^J+\frac{1}{\gamma}\Sigma_{IJ}n^J = \\ &= \frac{1}{2}\epsilon_{IJKL}\left(e^K\wedge e^L\right)n^J+\frac{1}{\gamma}\left(e_I\wedge e_J\right)n^J;\end{aligned} \tag{4.95}$$

$$\begin{aligned}\vec{L} = L_I = \frac{1}{2}\epsilon_{IJ}^{KL}B_{KL}n^J &= \frac{1}{2}\epsilon_{IJ}^{KL}\left(\frac{1}{2}\epsilon_{KLMN}\Sigma^{MN}+\frac{1}{\gamma}\Sigma_{KL}\right)n^J = \\ &= \frac{1}{4}\delta_{IM}\delta_{JN}\left(e^M\wedge e^N\right)n^J+\frac{1}{2\gamma}\epsilon_{IJ}^{KL}\left(e_K\wedge e_L\right)n^J = \\ &= \frac{1}{4}\left(e_I\wedge e_J\right)n^J+\frac{1}{2\gamma}\epsilon_{IJ}^{KL}\left(e_K\wedge e_L\right)n^J.\end{aligned} \tag{4.96}$$

On the surfaces at constant time Σ the *time gauge* is such that $e_I n^I|_\Sigma = 0$ and so

$$\vec{K} = \frac{1}{2}\epsilon_{IJKL}\left(e^K\wedge e^L\right)n^J \quad ; \quad \vec{L} = \frac{1}{2\gamma}\epsilon_{IJ}^{KL}\left(e_K\wedge e_L\right)n^J \tag{4.97}$$

which means that $\vec{K}$ and $\vec{L}$ are proportional, the constant being the Immirzi parameter:

$$\boxed{\vec{K} = \gamma\vec{L}} \tag{4.98}$$

Such relation is called *linear simplicity constraint*. Actually, its very simple form is due to the rather useful *time gauge*, but it could be given in a covariant fashion as well, leading us to conclude that it puts an actual physical restriction on the 2-form B, regardless of the mathematical choice (the tetradic frame) we make to describe it (in our specific case, the time gauge). So far, we have understood that the *time gauge* paves the way to two different and equivalent approaches towards a possible quantization of the theory:

- either we perform the smart change of variables introduced by Ashtekar and break the gauge symmetry group $SL(2,\mathbb{C})$ down to $SU(2)$, then we identify the canonical variables with the new $SU(2)$ ones and quantize the theory following Dirac's procedure - this approach leads to *canonical loop quantum gravity*
- or we keep the Lorentz covariance explicit and, when quantizing *á la* Dirac, we implement the linear simplicity constraint on a boundary spacelike 3d hypersurface to properly recover the physical states - this approach leads to *covariant loop quantum gravity* .

However, the two approaches are not strictly separated: we will find out that we can nicely bridge them when studying the dynamics of LQG, in order to exploit the advantages of both formalisms.

4.8 Smearing of the algebra and geometrical interpretation

Since the Ashtekar $SU(2)$ variables are going to represent the canonical variables in the GR phace space that we wish to quantize, it is important to grasp their geometrical meaning. Not surprisingly, the following short analysis will shed a first light on the profound link existing between the canonical ($SU(2)$) and covariant ($SL(2,\mathbb{C})$) formalisms, a link that will appear every time we plug the linear simplicity constraint.
The field E^a_i turns out to be closely related to an area element. A simple calculation, indeed, shows that the area of a 2-surface S in a $t = const$ hypersurface is:

$$A(S) = \int_S d^2\sigma \sqrt{E^a_i n_a E^{ib} n_b} \ , \tag{4.99}$$

where $n^a = \epsilon_{abc} \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^c}{\partial \sigma^2}$ is the vector normal to the surface.
To prove this, we start from the definition of an area in terms of the metric:

$$A(S) = \int_S d\sigma^1 d\sigma^2 \sqrt{\det\left(g_{ab} \frac{\partial x^a}{\partial \sigma^\alpha} \frac{\partial x^b}{\partial \sigma^\beta}\right)} \ ; \ \alpha, \beta = 1, 2 \ . \tag{4.100}$$

Solving the determinant and using the definition of densitized triad we have:

$$A(S) = \int_S d\sigma^1 d\sigma^2 \sqrt{\det(g) g^{ab} n_a n_b} = \int_S d\sigma^1 d\sigma^2 \sqrt{det^2(e) e^a_i e^{ib} n_a n_b} = \int_S d^2\sigma \sqrt{E^a_i n_a E^{ib} n_b} \ . \tag{4.101}$$

This means that, in the approximation in which the metric is constant on S, i.e. that S is small enough to be flat compared to the local curvature, the area of S is given by the modulus of a vector $E_i(S)$ which is normal to the surface and has components:

$$E_i\left(S\right) = \int_S E^a_i n_a d^2\sigma \ ; \ |E_i(S)| = A(S) \, . \tag{4.102}$$

$E_i(S)$ is the smearing on S of the vector-valued 2-form $E_i = E^a_i n_a$ or, in a more direct way, the *flux* of the Ashtekar electric field E^a_i across S[2]. In terms of the triad, the flux variable $E_i\left(S\right)$

[2] With this definition the flux does not transform covariantly under gauge transformation. Nevertheless, this problem can be solved choosing $E_i(S)$ in an appropriate gauge (where the connection is constant in a certain small region) or (see [35]) with a different definition of the smearing.

reads:

$$E_i\,(S)=\frac{1}{2}\epsilon_i^{jk}\int_S e_j\wedge e_k\ . \tag{4.103}$$

Equation (4.103) reminds us of another expression: in the previous section, the components of the generator of rotational symmetries were exactly the same(we restore the physical constants):

$$L_I=\frac{1}{16\pi\gamma G}\epsilon_{IJ}^{KL}(e_K\wedge e_L)n^J\Rightarrow L_i=\frac{1}{16\pi\gamma G}\epsilon_i^{kl}e_k\wedge e_l\quad i,k,l=1,2,3. \tag{4.104}$$

We learn that L_i is proportional to the *area element* of S, just like E_i, but now the proportionality constant being $1/8\pi\gamma G$. The same relation holds between their smeared versions on S. If we call such smearing $L_i(S)$ coherently with the notation used for E, the norm of this vector is:

$$|L_i(S)|=\frac{1}{8\pi\gamma G}|E_i(S)|=\frac{1}{8\pi\gamma G}A(S)\ . \tag{4.105}$$

This simple relation unveils the fact that L_i from the covariant formalism and E_i from the Ashtekar variables play the same role as 3d rotation generators and area elements. Our quantization postulate (3.12) is now rigorously confirmed in both formalisms and applies to a generic surface S.
Moving onto the Ashetkar-Barbero connection, it is convenient to see it as a vector-valued 1-form A_a^i, thus as an element of the $su(2)$ algebra written in the basis of the $SU(2)$ generators J_i in a representation j (if we consider the fundamental representation $j=1/2$ the generators are $\tau_i=-\frac{i}{2}\sigma_i$, σ_i being the Pauli matrices):

$$A_a\equiv A_a^iJ_i\ . \tag{4.106}$$

Recall that a connection defines a notion of parallel transport of the fiber over the base manifold. Consider a path γ and a parametrization of it $x^a(s):[0,1]\to\Sigma$. Then we can integrate A_a along γ as a line integral:

$$\int_\gamma A\equiv\int_0^1 dsA_a^i\,(x\,(s))\,\frac{dx^a\,(s)}{ds}J_i\ . \tag{4.107}$$

Next, we define the *holonomy* [3] of A along γ to be the path ordered exponential

$$\begin{aligned} h_\gamma\,(A)&=\mathcal{P}\exp\left(\int_\gamma A\right)\equiv\\ &\equiv\sum_{n=0}^{\infty}\underbrace{\int\cdots\int}_{1>s_n>\ldots>s_1>0}A_a\,(x\,(s_1))\,\frac{dx^a\,(s_1)}{ds_1}\ldots A_b\,(x\,(s_n))\,\frac{dx^b\,(s_n)}{ds_n}ds_1\ldots ds_n\ . \end{aligned} \tag{4.108}$$

Analitically, the holonomy is defined as the value for $s=1$ of the solution of the differential equation

$$\frac{d}{ds}h\,(s)=\frac{dx^a\,(s)}{ds}A_a\,(x\,(s)) \tag{4.109}$$

with initial condition $h\,(0)=\mathbb{I}_{SU(2)}$. Geometrically, $h_\gamma\in SU(2)$ is the group transformation of a vector in the fiber obtained by following the path $\gamma=x^a(s)$ and rotating as dictated by the connection A. In other words, the holonomy is the rotation that each vector V undergoes in the fiber space when it is moved along γ and kept covariantly constant (i.e. parallel transported):

$$V(s=1)=h_\gamma\,(A)\,V(s=0)\ . \tag{4.110}$$

Let us list some useful properties of the holonomy:

- the holonomy of the composition of two paths is the product of the holonomies of each path:

$$h_{\beta\alpha}=h_\beta h_\alpha\quad ; \tag{4.111}$$

[3] This definition is slightly different from the one commonly given in mathematics.

- under local gauge transformations $g(x) \in SU(2)$, the holonomy transforms as:

$$h_\gamma^g = g_{s(\gamma)} h_\gamma g_{t(\gamma)}^{-1} \tag{4.112}$$

  where $s(\gamma)$ and $t(\gamma)$ are respectively the initial (*source*) and final (*target*) points of the path γ. The holonomy is insensitive to gauge transformations localized in intermediate points along γ, and only cares about the gauge transformations at the extremities.

The *flux* and *holonomy* variables are regularized versions of the original Ashtekar variables, and the resulting algebra is known as *holonomy-flux algebra*, providing the most regular version of the Poisson algebra (no delta functions appear). Its Poisson brackets are:

$$\{h_\gamma, E_i(S)\} = \sum_{p \in \gamma \cap S} h_{\gamma_p}^- J_i h_{\gamma_p}^+ \tag{4.113}$$

where $h_{\gamma_p}^{\pm}$ are the holonomies along the two halves of the path γ cut by the point p where it intersects the surface S. In case of no intersection between γ and S, the brackets vanish.

Chapter 5

Canonical loop quantum gravity

In this chapter we briefly review the program of canonical loop quantum gravity, based on the formulation of GR as a $SU(2)$ gauge theory in terms of the densitized triad and the Ashtekar connection and shaped on Dirac's quantization procedure for constrained systems. We will come up with the construction of a kinematical Hilbert space from which the physical space can be extracted by imposing the three sets of constraints:

- $G_i = 0$: Gauss constraint;
- $H_a = 0$: spatial diffeomorphim constraint;
- $H = 0$: hamiltonian constraint.

The latters, as we already know, completely define the dynamical content of the theory and their quantum operator counterparts will act as projectors onto the physical space.

5.1 Quantization program

We follow Dirac's procedure for quantizing GR as a constrained hamiltonian system. The main formal steps are [25]:

1. We define the kinematical Hilbert space of GR, we choose the polarization where the connection is regarded as the configuration variable. The kinematical Hilbert space $\mathcal{H}_{kin}$ consists of suitable functionals of the connection $\psi[A]$ which are square-integrable with respect to a suitable gaussian measure δA, defining a scalar product which makes the space a Hilbert one. On this space the representation of the Poisson algebra is defined to be a Schrödinger one:
$$\hat{A}^i_a \psi[A] = A^i_a \psi[A] \tag{5.1}$$
$$\hat{E}^a_i \psi[A] = -i\hbar\gamma \frac{\delta}{\delta A^i_a} \psi[A] \tag{5.2}$$
with the connection acting by multiplication and the flux variable as a derivative; they satisfy the canonical commutation relations:
$$\left[\hat{A}^i_a, \hat{E}^b_j\right] = i\hbar\gamma\delta^b_a\delta^i_j\delta^{(3)}(x,y) \quad . \tag{5.3}$$

2. Next, we want to the define the constraints as operators acting on this space. It turns out that the Gauss and the spatial diffeomorphism constraints have a natural action on the states in $\mathcal{H}_{kin}$. There is instead some ambiguity in the definition of the hamiltonian constraint, mainly due to its highly non linear structure.

3. Finally, we define the physical Hilbert space $\mathcal{H}_{phys}$ as the space annihilating all the constraint operators:

$$\mathcal{H}_{kin} \xrightarrow{\hat{G}\psi=0} \mathcal{H}^G_{kin} \xrightarrow{\hat{H}_a\psi=0} \mathcal{H}_{diff} \xrightarrow{\hat{H}\psi=0} \mathcal{H}_{phys} \tag{5.4}$$

$$\psi[A] \in \mathcal{H}_{phys} \iff \hat{G}\psi = \hat{H}_\mu \psi = 0 \ . \tag{5.5}$$

5.2 Spin networks

The problem with the quantization program adapted to hamiltonian GR is that we do not have a background metric at disposal to define the integration measure δA, since now the metric is a fully dynamical quantity. We need to define a measure on the space of functions of the connection $\psi[A]$ without relying on any fixed background metric. To this aim, a key notion is the one of *cylindrical functions* [12]. Roughly speaking, they are functionals of a field that depend only on some subset of the components of the field itself. In the case at hand, we consider functionals of the connection A that depend solely on its holonomies $h_e[A]$ along some finite set of paths $\{e\}$.
Now, we define an *embedded graph* $\Gamma \subset \Sigma$ a collection of oriented paths e in the hypersurface Σ (we will call these paths *links* of the graph) meeting at most at their endpoints i.e. never crossing each other. Being L the total number of links, a cylindrical function is a couple (Γ, f) of a graph and a smooth function $f : SU(2)^L \to \mathbb{C}$ defined as

$$\psi_{(\Gamma,f)}[A] \equiv \langle A|\Gamma, f\rangle = f\left(h_{e_1}[A], \dots, h_{e_L}[A]\right) \in Cyl_\Gamma \tag{5.6}$$

where e_l ; $l = 1, \dots, L$ are the links of the corresponding graph Γ.
This space of functionals can be turned into a Hilbert space if we equip it with a scalar product.

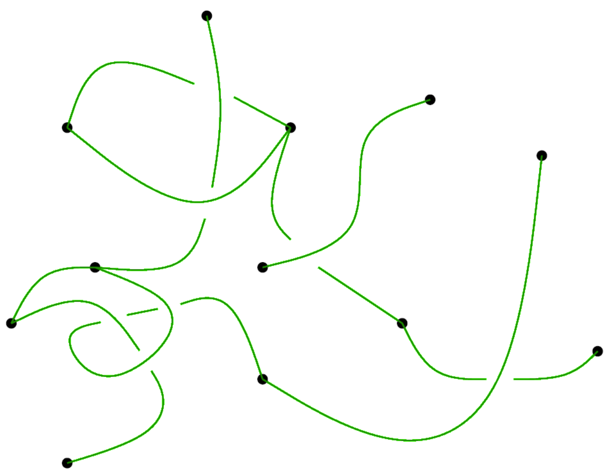


Figure 5.1: Collection of paths: $\Gamma = \{e_1, ..., e_L\}$

The switch from the connection to the holonomy is crucial in this respect, because the holonomy is an element of $SU(2)$, and on a compact group there is a unique gauge-invariant and normalized measure, called the *Haar measure* $d\mu_{Haar} = dh$. Using L copies of the Haar measure, we can define on Cyl_Γ the following scalar product:

$$\langle \psi_{(\Gamma,f)} | \psi_{(\Gamma,f')} \rangle \equiv \int \prod_{l=1}^{L} dh_{e_l} \overline{f\left(h_{e_1}[A], \dots, h_{e_L}[A]\right)} f'\left(h_{e_1}[A], \dots, h_{e_L}[A]\right) \ . \tag{5.7}$$

This turns Cyl_Γ into a Hilbert space $\mathcal{H}_\Gamma$ associated with a given graph Γ. Hence, the full Hilbert space can be defined as the direct sum of Hilbert spaces on all possible graphs living in Σ:

$$\mathcal{H}_{kin} = \bigoplus_{\Gamma \subset \Sigma} \mathcal{H}_\Gamma = \mathcal{L}_2 \left[A, d\mu_{Haar}\right] \ . \tag{5.8}$$

The scalar product on $\mathcal{H}_{kin}$ is easily induced from (5.7) in the following manner: if ψ and ψ' share the same graph, then (5.7) immediately applies. If they have different graphs, say Γ_1 and Γ_2, we consider a further graph $\Gamma_3 \equiv \Gamma_1 \cup \Gamma_2$, we extend f_1 and f_2 trivially on Γ_3 and define the scalar product as (5.7) on Γ_3:

$$\langle \psi_{(\Gamma_1, f_1)} | \psi_{(\Gamma_2, f_2)} \rangle \equiv \langle \psi_{(\Gamma_1 \cup \Gamma_2, f_1)} | \psi_{(\Gamma_1 \cup \Gamma_2, f_2)} \rangle \ . \tag{5.9}$$

Once we have a Hilbert space $\mathcal{H}_\Gamma$ associated with a graph Γ, we can look for a representation of the holonomy-flux algebra on it. To this purpose, it is convenient to introduce an orthogonal basis in the space. This is possible thanks to the *Peter-Weyl theorem*. It states that a basis on the Hilbert space $\mathcal{L}_2 \left[G, d\mu_{Haar}\right]$ of functions on a compact group G is given by the matrix elements of the unitary irreducible representations of the group. In our case, G is $SU(2)$ and every function can be expanded as

$$f(g) = \sum_j \sum_{m=-j}^{j} \tilde{f}^j_{mn} D^j_{mn}(g) \quad j = 0, \frac{1}{2}, 1, \dots \tag{5.10}$$

where the *Wigner matrices* $D^j_{mn}(g)$ give the spin-j irreducible matrix representations of the $SU(2)$ element g. They are orthogonal with respect to the scalar product defined by the Haar measure, specifically:

$$\int dg \overline{D^{j'}_{m'n'}(g)} D^j_{mn}(g) = \frac{1}{2j+1} \delta^{jj'} \delta_{mm'} \delta_{nn'} \ . \tag{5.11}$$

Such expansion immediately applies to $\mathcal{H}_\Gamma$. To begin with, this space can be seen as the tensor product of L Hilbert spaces associated with each of the links that form the graph, if we regard the links as the trivial graphs of themselves. Let us call these spaces $\mathcal{H}_{e_l} = \mathcal{L}_2 \left[SU(2)\right]$, $l = 1, \dots, L$. A functional belonging to $\mathcal{H}_{e_l}$ is then expanded as

$$\psi_{(e_l, f)}[A] \equiv \langle A | e_l, f \rangle = \sum_{j_l} \sum_{m_l = -j_l}^{j_l} \tilde{f}^{j_l}_{m_l n_l} D^{j_l}_{m_l n_l} \left(h_{e_l}[A]\right) \ , \tag{5.12}$$

therefore a state in $\mathcal{H}_\Gamma = \mathcal{L}_2[SU(2)^L]$ is of the form

$$\begin{aligned} \psi_{(\Gamma, f)}[A] \equiv \langle A | \Gamma, f \rangle = \langle A | \left(\bigotimes_{l=1}^{L} |e_l, f\rangle \right) = \\ = \sum_{\{j_l, m_l, n_l\}_l} \tilde{f}^{j_1 \dots j_L}_{m_1 n_1 \dots m_L n_L} D^{j_1}_{m_1 n_1} \left(h_{e_1}[A]\right) \dots D^{j_L}_{m_L n_L} \left(h_{e_L}[A]\right) \end{aligned} \tag{5.13}$$

where the notation $\{j_l, m_l, n_l\}_l$ means that the sum is made varying separately j_l, m_l, n_l for each link l.

Let us step back to $\mathcal{H}_{e_l} = \mathcal{L}_2 \left[SU(2)\right]$, the space defined on a single link. In the holonomy representation, its basis elements are the Wigner matrices D^j_{mn}. The latters rotate vectors in, say, a spin j representation into another of the same representation. In other words, they are maps from the Hilbert space $\mathcal{H}_j$ of the spin j representation to itself. Such maps can be viewed as elements of $\mathcal{H}_j \otimes \mathcal{H}_j$, where the two Hilbert spaces $\mathcal{H}_j$ belong to the two ends of the link, and therefore $\mathcal{H}_{e_l} = \mathcal{L}_2 \left[SU(2)\right]$ can be decomposed into the direct sum of the finite-dimensional subspaces at fixed representation j_l:

$$\mathcal{H}_{e_l} = \mathcal{L}_2 \left[SU(2)\right] = \bigoplus_{j_l} \left(\mathcal{H}_{j_l} \otimes \mathcal{H}_{j_l}\right) \ . \tag{5.14}$$

Since $\mathcal{H}_\Gamma$ is made of L such copies we have:

$$\begin{aligned}\mathcal{H}_\Gamma = \mathcal{L}_2\left[SU(2)^L\right] &= \bigotimes_{l=1}^{L}\bigoplus_{j_l}\left(\mathcal{H}_{j_l}\otimes\mathcal{H}_{j_l}\right) = \\ &= \bigoplus_{\{j_l\}_l}\bigotimes_{l=1}^{L}\left(\mathcal{H}_{j_l}\otimes\mathcal{H}_{j_l}\right)\end{aligned} \tag{5.15}$$

where $\{j_l\}_l$ means that the sum is made varying separately the representation in the extremities of each link l.
On this space, we can define the action of the holonomy-flux algebra (4.113) by promoting the holonomy to an operator acting by multiplication, the flux to the operator acting through the derivative and their Poisson brackets to commutator relations (multiplied by $i\hbar$):

$$\hat{h}_\gamma[A]h_e[A] = h_\gamma[A]h_e[A] \ , \tag{5.16}$$

$$\hat{E}_i(S)h_e[A] = -i\hbar\gamma\int_S d^2\sigma n^b\frac{\delta h_e[A]}{\delta A_i^b(x(\sigma))} = \pm i\hbar\gamma h_{e_1}[A]J_i h_{e_2}[A] \ , \tag{5.17}$$

$$\left[\hat{h}_\gamma, \hat{E}_i(S)\right] = i\hbar\gamma\sum_{p\in\gamma\cap S}\hat{h}^-_{\gamma_p}J_i\hat{h}^+_{\gamma_p} \ . \tag{5.18}$$

If the intersection between the surface S and the link e is not empty, the flux splits the holonomy in the two halves cut by the surface and inserts a $SU(2)$ generator in between. The sign is defined by the relative orientation between S and e. If the surface does not cut the link, the action of the flux vanishes, as well as the commutator.
The action of this algebra trivially extends to a generic basis element $D^j_{mn}(h)$ (just replace $h_e[A]$ with the Wigner matrix) and, starting from the definition in $\mathcal{H}_\Gamma$, is extended by linearity over the whole $\mathcal{H}_{kin}$.
With the definition of the algebra operators, we have fulfilled the definition of the kinematical Hilbert space of LQG. The physical one, on the other hand, is defined by imposing the constraints. The solutions of the Gauss constraint $\hat{G}\psi = 0$ are states living in what we have denoted $\mathcal{H}^G_{kin}$ and are defined as the $SU(2)$ invariant elements of the kinematical Hilbert space. They are called *spin networks*, because they are states with spins assigned to the links, and eventually nodes, of a graph.
In order to characterize these solutions, let us recall how the gauge symmetry acts on the holonomy. Given $g\in SU(2)$ and a holonomy along a link e_l, $h_{e_l}[A]$:

$$g\triangleright h_{e_l} = g_{s(e_l)}h_{e_l}g^{-1}_{t(e_l)} \tag{5.19}$$

where $s(e_l)$ and $t(e_l)$ are the source and target of the link e_l, that is the initial and final point. If we consider a graph Γ with L links connecting N points, which we call ***nodes***, the continuous $SU(2)$ gauge symmetry is restricted to a discrete $SU(2)$ symmetry at the nodes. Thus, the gauge group of the theory is $SU(2)^N$.The gauge invariant states are those satisfying

$$\begin{aligned}g\triangleright\psi_{(\Gamma,f)}[A] &= f\left(g_{s(e_1)}h_{e_1}[A]g^{-1}_{t(e_1)},\ldots,g_{s(e_L)}h_{e_L}[A]g^{-1}_{t(e_L)}\right) = \\ &= f\left(h_{e_1}[A],\ldots,h_{e_L}[A]\right) = \psi_{(\Gamma,f)}[A]\end{aligned} \tag{5.20}$$

and they form a proper subspace of $\mathcal{H}_\Gamma$ which we call $\mathcal{K}_\Gamma$ (the $\mathcal{K}$ means *kinematical space*):

$$\mathcal{K}_\Gamma = \mathcal{L}_2\left[SU(2)^L/SU(2)^N\right] \ , \tag{5.21}$$

so that $\mathcal{H}^G_{kin} = \bigoplus_{\Gamma\subset\Sigma}\mathcal{K}_\Gamma$.
In the Wigner matrix basis, a transformation g_n at the node n acts on the group elements (the Wigner matrices) of the links that meet at the node. Being the Wigner matrices representation

matrices, the gauge transformation acts on their indices. Therefore, for the state to be invariant, the coefficient $\tilde{f}^{j_1 \dots j_L}_{m_1 n_1 \dots m_L n_L}$ in equation (5.13) must be an invariant tensor when acted upon by a group transformation on the indices related to that node.
Since the gauge group is made of transformations localized at the nodes, it is convenient to rearrange $\mathcal{H}_\Gamma$ by grouping together the Hilbert spaces at the same node, which transform together under a gauge transformation. For each node $n = 1, \dots, N$ we label with $v(n) = 1, \dots, V(n)$ each of the $V(n)$ links coming out of n (n is called V-valent node). We obtain:

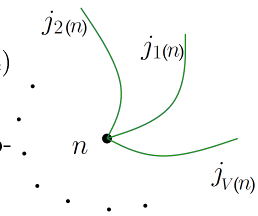
$$\mathcal{H}_\Gamma = \bigoplus_{\{j_{v(n)}\}_{v(n)}} \bigotimes_{n=1}^{N} \left(\mathcal{H}_{j_{1(n)}} \otimes \dots \otimes \mathcal{H}_{j_{V(n)}}\right) \tag{5.22}$$

where $\mathcal{H}_{j_{v(n)}}$ is the spin j representation Hilbert space at the extremity of the link $v(n)$ that meets the other links in a node n, and as usual the sum is made varying separately each $j_{v(n)}$.
Next, we want the gauge invariant states. They will live in the space

$$\mathcal{K}_\Gamma = \bigoplus_{\{j_{v(n)}\}_{v(n)}} \bigotimes_{n=1}^{N} Inv_{SU(2)} \left(\mathcal{H}_{j_{1(n)}} \otimes \dots \otimes \mathcal{H}_{j_{V(n)}}\right) \ . \tag{5.23}$$

Given a state ψ in the space on each node $\left(\mathcal{H}_{j_{1(n)}} \otimes \dots \otimes \mathcal{H}_{j_{V(n)}}\right)$, the projection on the gauge invariant subspace is achieved by the *group averaging* procedure. Given an arbitrary function $f \in Cyl_\Gamma$, the function

$$f_0\left(h_{e_1}, \dots, h_{e_L}\right) \equiv \int \prod_n dg_n f(g_{s(e_1)} h_{e_1} g^{-1}_{t(e_1)}, \dots, g_{s(e_L)} h_{e_L} g^{-1}_{t(e_L)}) \tag{5.24}$$



clearly satisfies (5.20). Without going into details, one realizes that this procedure is equivalent to applying a projector to each node:

$$\mathcal{P} = \int dg \prod_{v(n)=1}^{V(n)} D^{(j_{v(n)})}(g) \tag{5.25}$$

i.e. plugging the $SU(2)$ transformation in the representations of each of the links going into the node and averaging over the possible elements of the gauge group.
Let us specialize to the cases $V = 3$, which is the minimal number of links one needs for the invariant subspace to exist, and $V = 4$, which will be useful in the following chapters.

$\boxed{V = 3}$ For a 3-valent node we have the space $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$. Recall from the quantum theory of composition of angular momenta that we write the tensor product of two angular representations as a sum over representations in the form

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_j \ . \tag{5.26}$$

Therefore:

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} = \left(\bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_j \right) \otimes \mathcal{H}_{j_3} = \bigoplus_{k=|j-j_3|}^{j+j_3} \mathcal{H}_k \ . \tag{5.27}$$

The invariant subspace is nothing but the singlet, i.e. the spin zero representation $k = 0$. This means that:

- of the three spaces in the original tensor product, only one of them can be a spin zero representation;
- the sum of the three spins must be integer;

– the three spins must satisfy the *triangular inequality*

$$|j_1 - j_2| < j_3 < j_1 + j_2. \tag{5.28}$$

The resulting invariant space is 1-dimensional, so that we do not need a quantum number to resolve its degeneracy and any state of the basis of $\mathcal{K}_\Gamma$ will be completely determined by assigning a representation $j_{v(n)}$ to each link. In the basis of the original triple tensor product, a gauge invariant state on the node is, up to normalization:

$$|i\rangle = \sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle \tag{5.29}$$

where the coefficient

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \iota^{m_1 m_2 m_3} \tag{5.30}$$

is called *3j-symbol* and is the generalized symmetric form of the Clebsh-Gordan coefficients [32]. As we have requested, it is invariant under the diagonal action of $SU(2)$:

$$D^{(j_1)m_1}_{n_1} D^{(j_2)m_2}_{n_2} D^{(j_3)m_3}_{n_3} \iota^{n_1 n_2 n_3} = \iota^{m_1 m_2 m_3} \quad . \tag{5.31}$$

A generic gauge invariant state in K_Γ, where Γ is a graph made of N 3-valent nodes and L links is a linear combination

$$\psi[A] = \sum_{j_1 \cdots j_L} C_{j_1 \ldots j_L} \psi_{j_1 \ldots j_L}[A] \tag{5.32}$$

of the orthogonal states labelled by a spin associated with each link:

$$\begin{aligned} \psi_{j_1 \ldots j_L}[A] =& \iota_1^{m_1 m_2 m_3} \ldots \iota_{N/2}^{m_{L-2} m_{L-1} m_L} \iota_{N/2+1}^{n_1 n_2 n_3} \ldots \iota_N^{n_{L-2} n_{L-1} n_L} \\ & D^{j_1}_{m_1 n_1}(h_{e_1}[A]) \ldots D^{j_L}_{m_L n_L}(h_{e_L}[A]) \end{aligned} \tag{5.33}$$

where all indices m are contracted between the 3j-symbols and the Wigner matrices. How these indices are contracted is dictated by the structure of the graph Γ.

$$\psi_{j_1 \ldots j_6} = \sum_{\{m_i n_i\}_i}$$

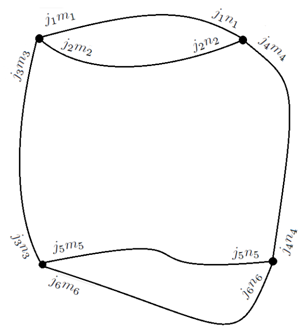


Figure 5.2: Graph with six links and four 3-valent nodes, with one of the orthogonal states in the related kinematical space. Each couple $|j_l m_l\rangle \otimes |j_l n_l\rangle$ represents the Wigner matrix element $D^{j_l}_{m_l n_l}[h_{e_l}(A)]$, while the triples around each node represent the $SU(2)$ invariant tensor products of the three states.

$\boxed{V = 4}$ For a 4-valent node, the space $Inv_{SU(2)}(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4})$ is not 1-dimensional anymore. Now a gauge invariant state on the node takes the form

$$|i\rangle = \sum_{m_1,m_2,m_3,m_4} \iota^{m_1 m_2 m_3 m_4} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle \otimes |j_4 m_4\rangle \tag{5.34}$$

where $\iota^{m_1 m_2 m_3 m_4}$ is an invariant tensor with four indices, named *intertwiner*. It is invariant under the diagonal action of $SU(2)$:

$$D^{(j_1)m_1}_{n_1} D^{(j_2)m_2}_{n_2} D^{(j_3)m_3}_{n_3} D^{(j_4)m_4}_{n_4} \iota^{n_1 n_2 n_3 n_4} = \iota^{m_1 m_2 m_3 m_4} \quad . \tag{5.35}$$

If we call k the quantum number resolving the degeneracy of this space, the linear independent states that span it have, as coefficients, the invariant tensors

$$\iota_k^{m_1 m_2 m_3 m_4} = \sqrt{2k+1} \begin{pmatrix} j_1 & j_2 & k \\ m_1 & m_2 & m \end{pmatrix} g_{mm'} \begin{pmatrix} k & j_3 & j_4 \\ m' & m_3 & m_4 \end{pmatrix} \tag{5.36}$$

where $g_{mm'} = (-1)^{j-m}\delta_{m,-m'}$ is the antisymmetric symbol (the unique invariant tensor with two indices).
The quantum number k acts like a virtual link connecting two 3-valent nodes.

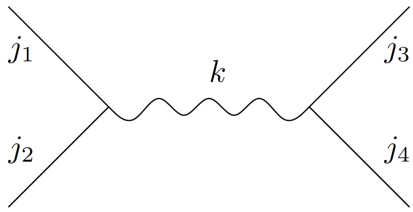


Figure 5.3: Decomposition over a virtual link k of a 4-valent node.

Its possible values are those satisfying the triangular inequalities with (j_1, j_2) and (j_3, j_4):

$$\max\left[|j_1 - j_2|, |j_3 - j_4|\right] \leq k \leq \min\left[(j_1 + j_2), (j_3 + j_4)\right] \tag{5.37}$$

and therefore the dimension of $Inv_{SU(2)}\left(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}\right)$ is

$$\min\left[(j_1 + j_2), (j_3 + j_4)\right] - \max\left[|j_1 - j_2|, |j_3 - j_4|\right] + 1 \quad . \tag{5.38}$$

A generic gauge invariant state in K_Γ where Γ is a graph made of N 4-valent nodes and L links is a linear combination

$$\psi[A] = \sum_{\{j_l, k_n\}_{l,n}} C^{k_1 \dots k_N}_{j_1 \dots j_L} \psi^{k_1 \dots k_N}_{j_1 \dots j_L}[A] \tag{5.39}$$

of the orthogonal states now labelled by a spin j_l assigned to each link l and a number k_n associated with each node:

$$\begin{aligned} \psi^{k_1 \dots k_N}_{j_1 \dots j_L}[A] = & \iota_{k_1}^{m_1 m_2 m_3 m_4} \dots \iota_{k_{N/2}}^{m_{L-3} m_{L-2} m_{L-1} m_L} \iota_{k_{N/2+1}}^{n_1 n_2 n_3 n_4} \dots \iota_{k_N}^{n_{L-3} n_{L-2} n_{L-1} n_L} \\ & D^{j_1}_{m_1 n_1}\left(h_{e_1}[A]\right) \dots D^{j_L}_{m_L n_L}\left(h_{e_L}[A]\right) \ , \end{aligned} \tag{5.40}$$

where again all indices m are contracted according to the structure of the graph.

5.3 Quanta of area

The definition of spin networks as the basis for the gauge invariant Hilbert space built on a graph embedded in a spatial hypersurface Σ is one of the most interesting results of LQG since it leads to the quantization of space. As we show now, to each spin network it is possible to attach a notion of discrete geometry characterized by the Planck length as the minimal length.

$$\psi^{k,k'}_{j_1\ldots j_4} = \sum_{\{m_i n_i\}_i}$$

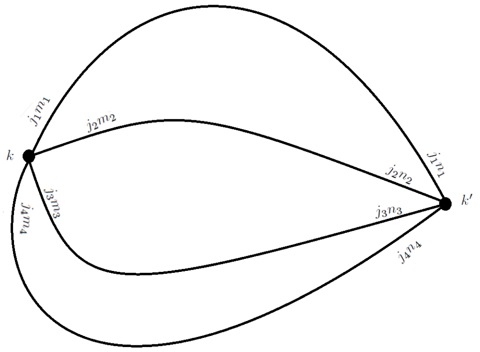


Figure 5.4: Graph with six links and two 4-valent nodes, with one of the orthogonal states in the related kinematical space. Each couple $|j_l m_l\rangle \otimes |j_l n_l\rangle$ represents the Wigner matrix element $D^{j_l}_{m_l n_l}[h_{e_l}(A)]$, while the quartets around each node represent the $SU(2)$ invariant intertwiners, labelled by the virtual link k_n.

Let us recall that the classical flux variable $E_i(S)$ defined in equation (4.103) is a 3d vector normal to the surface S whose length is equal to the area of S:

$$A(S) = \sqrt{E_i(S)E^i(S)}. \tag{5.41}$$

The strategy to correctly find the quantum version of this quantity is to chop S into N 2-dimensional cells $S_k; k = 1, \ldots, N$ (this operation is called *triangulation*) and write $A(S)$ as the limit of a Riemann sum:

$$A(S) = \lim_{N\to\infty} A_N(S) \equiv \lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{E_i(S_k)E^i(S_k)}\ . \tag{5.42}$$

In this limit the cells S_k become infinitesimal, allowing us to consider the densitized triad constant all over each of them:

$$E_i(S_k) = \int_{S_k} d^2\sigma n_a E^a_i \approx n_a E^a_i \int_{S_k} d^2\sigma = E^a_i n_a S_k \tag{5.43}$$

and to recover the original formula (4.99):

$$\lim_{N\to\infty} A_N(S) = \lim_{N\to\infty} \sum_{k=1}^{N} S_k \sqrt{E^a_i n_a E^b_i n_b} = \int_S d^2\sigma \sqrt{E^a_i n_a E^b_i n_b} = A(S)\ . \tag{5.44}$$

The area operator is then simply given by replacing $E_i(S_k)$ with its quantum counterpart $\hat{E}_i(S_k)$:

$$\hat{A}(S) = \lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{\hat{E}_i(S_k)\hat{E}^i(S_k)}. \tag{5.45}$$

This operator now acts on a generic spin network state ψ_Γ, where the graph Γ is generic and can intersect S many times. Let us therefore consider the action of the scalar product of two fluxes on the holonomy along a link e:

$$\hat{E}_i(S_k)\hat{E}^i(S_k)h_e[A] = -\hbar^2\gamma^2 h_{e_1}[A] J_i J^i h_{e_2}[A]. \tag{5.46}$$

On the right hand side, we see the appearance of the scalar product of the algebra generators, something known as the Casimir operator of the algebra $SU(2)$ $C^2 \equiv J_i J^i = -j(j+1)\mathbb{I}_{2j+1}$ in the representation j. The Casimir clearly commutes with all the group elements, so we can write

$$\hat{E}_i(S_k)\hat{E}^i(S_k)h_e[A] = -\hbar^2\gamma^2 C^2 h_{e_1}[A]h_{e_2}[A] = -\hbar^2\gamma^2 C^2 h_e[A] \tag{5.47}$$

and, if we act on a basis element $D^j_{mn}(h_e[A])$:

$$\hat{E}_i(S_k)\hat{E}^i(S_k)D^j_{mn}(h_e[A]) = \hbar^2\gamma^2 j(j+1)D^j_{mn}(h_e[A]) \ . \tag{5.48}$$

We also know that $\hat{E}_i(S_k)\hat{E}^i(S_k)$ gives zero if S_k is not intersected by any link of the graph. Therefore, once the triangulation is sufficiently fine to have each surface S_k punctured by a link, taking further refinements has no consequences. Thus the limit simply amounts to sum the contributions of the finite number of punctures p of S caused by the links of Γ. That is:

$$\hat{A}(S)\psi_\Gamma = \lim_{N\to\infty}\sum_{k=1}^{N}\sqrt{\hat{E}_i(S_k)\hat{E}^i(S_k)}\psi_\Gamma = \sum_{p\in S\cap\Gamma}\hbar\gamma\sqrt{j_p(j_p+1)}\psi_\Gamma \tag{5.49}$$

where j_p are the representations attached to each link of the graph that punctures S[1]. Restoring the constants for E ($8\pi G$):

$$\hat{A}(S)\psi_\Gamma = \sum_{p\in S\cap\Gamma} 8\pi\gamma\frac{\hbar G}{c^3}\sqrt{j_p(j_p+1)}\psi_\Gamma = \sum_{p\in S\cap\Gamma} 8\pi\gamma l_P^2\sqrt{j_p(j_p+1)}\psi_\Gamma \ . \tag{5.50}$$

We have learned that the spectrum of the area operator is diagonal in the spin network basis, but more importantly it is *discrete* and the minimal scale, as anticipated, is given by the squared Planck length $l_P^2 = \hbar G/c^3 \sim 10^{-33}cm$.

5.4 Scale of quantum gravity

Going back to the estimation of l_0^2 in §3.3, from comparison between (3.13) and (5.50) we can finally fix, as promised, the value of l_0^2 to be:

$$l_0^2 = \gamma 8\pi\frac{G\hbar}{c^3} = 8\pi\gamma l_P^2 \ . \tag{5.51}$$

Actually, the reader may already have smelt this relation from equation (4.105).
All our early considerations are now consistent with the results of the canonical approach (and the covariant approach too, as we will see later): quantum geometry is discrete and its scale is completely given by the Planck length, the only dimensionful constant in quantum gravity. The other quantities appearing in l_0^2 are 8π, which comes from the the historical choice of using $8\pi G$ as coupling constant in Einstein's equations, and the dimensionless Immirzi parameter γ (presumably of the order of unity). The precise scale of the theory is thus determined by the action of general relativity and, in particular, by the coupling constant in the Holst action.

5.5 Quanta of volume

The volume operator is more difficult to approach and a general expression does not exist. Nevertheless it is always possible to compute its spectrum and extract some general properties. In literature there are two proposals, both acting non trivially only on the nodes of a graph on which we act with the volume operator. These proposals agree up to a constant for the 3-valent and

[1] In this expression, we assume that each puncture is caused by a link crossing the surface. However, it could also happen that S is punctured by a node of the graph. A closed expression for the area operator is also known in this general case, see [27] for details.

4-valent graph cases. We will briefly describe the one introduced by Rovelli and Smolin [28].
Let us rewrite the 3d volume of GR, usually given in metric variables, through our new variables:

$$V(\Sigma) = \int_\Sigma \sqrt{h} d^3x = \int_\Sigma \sqrt{|\det E|} d^3x \ . \tag{5.52}$$

The strategy is the same as before: we chop the region Σ in 3d cells and refine the decomposition until each cell contains at most one node, thus the integral can be replaced by the Riemann sum over the cells.
The proposal by Rovelli and Smolin for the volume operator of a region R is given by:

$$\hat{V}_{RS}(R) = \lim_{\epsilon \to 0} \sum_n \sqrt{\frac{1}{48} \sum_{\alpha,\beta,\gamma} |\epsilon_{ijk} \hat{E}^i(S_n^\alpha) \hat{E}^j(S_n^\beta) \hat{E}^k(S_n^\gamma)|} \tag{5.53}$$

where ϵ is the size of the cells and the three E are the fluxes across the surfaces that 1)envelope each node, 2)are punctured by a link each and 3)bound the cell C_n centered in the node: $\partial C_n = \cup_\alpha S_n^\alpha$. Indeed, due to the presence of ϵ_{ijk} the operator vanishes if the three fluxes are not different, which means that it acts non trivially only at the nodes because on the links we have at most two fluxes associated with the source and the target. Actually, the action of $\hat{V}$ vanishes on gauge invariant states defined on a 3-valent node, and non trivial contributions come from nodes of valency 4 or higher.
Moreover the flux variables are proportional to the $SU(2)$ generators, which at the quantum level do not commute. Therefore from the above expression it is clear that the spectrum of the volume operator is also discrete and its minimal excitation is proportional to l_P^3.

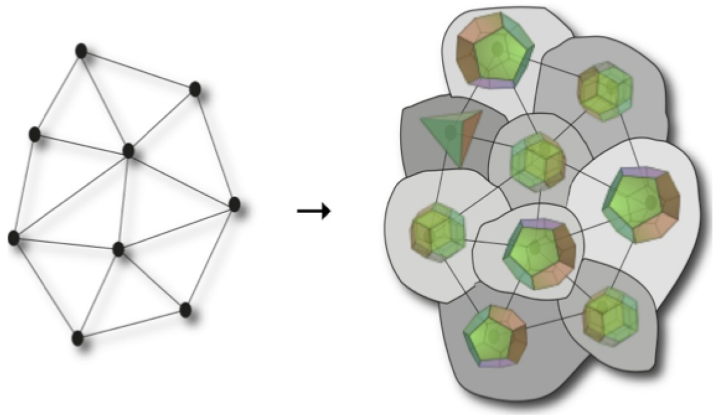


Figure 5.5: The discrete geometry determined by a graph

5.6 Diffeomorphism constraints

The next step of Dirac's procedure is to characterize the solution of the spatial and time diffeomorphism constraint: $\hat{H}_\mu \psi = 0$. Let us begin with $\hat{H}_a \psi = 0$, the spatial constraint. As before, it is important to remember the action of the diffeomorphisms on the holonomy:

$$\hat{\phi} \triangleright \hat{h}_e = \hat{h}_{\phi \circ e}. \tag{5.54}$$

Since our space is the direct sum of the Hilbert spaces on each graph, the action of the spatial diffeomorphism projects $\mathcal{H}_\Gamma$ into $\mathcal{H}_{\phi \circ \Gamma}$, which are orthogonal subspaces of $\mathcal{H}_{kin}$. This means that

we cannot define the action of an infinitesimal diffeomorphism, but this is not really a restriction: one can proceed to the group averaging as we did with the Gauss constraint and build $\mathcal{H}_{diff}$ from the states invariant under *finite* diffeomorphisms.
We only need to worry about the trivial diffeomorphism transformations which leave each link untouched and just shuffle the points inside. They have to be taken out because they would ruin the group averaging procedure.
Another feature to be taken into account is that, being the diffeomorphism group non compact, the request of invariance under its action over the elements of $\mathcal{H}_{kin}^{G}$ will not result in some subspace of this space. The solution of this constraint must be considered in the larger space of the linear functionals over the gauge invariant space $\mathcal{H}_{kin}^{G*}$.
The result of the group averaging procedure are spin-network states defined on *equivalence classes* of graphs under diffeomorphisms, called *knots*.
Since a diffeomorphism transformation changes the way in which a graph is embedded in Σ, it is possible to interpret these spin networks as defined on graphs which are not embedded into a manifold: they are *abstract graphs*, only defined by their combinatorial structures. This way the graph is not a collection of real paths, but rather a collection of relational information, which can be seen as representing all the graphs equivalent to a given one but with different embeddings in Σ.
Therefore the Hilbert space invariant under diffeomorphisms can be related to the kinematical space of abstract graphs:

$$\mathcal{H}_{diff} = \bigoplus_{[\Gamma]} \mathcal{K}_{[\Gamma]} \tag{5.55}$$

where the sum is only over the equivalence classes defined before and we already took into account the Gauss constraint.
The definition of abstract graph will return in the definition of a truncated theory, even if it will be introduced from another point of view.
The last step of Dirac's procedure is to solve the hamiltonian constraint. The classical constraint is highly non linear and this makes it difficult to turn it into an operator. Anyway a trick due to Thiemann allows us to rewrite it in a way which is more useful for quantization. Without going into details (we refer the reader to [33], [34]), we sketch some results: a Hamiltonian operator $\hat{H}$ is well defined and its action is understood. Moreover, it is possible to find an infinite number of solutions, at least formally. Nevertheless, the program of quantization is far from being complete: neither the complete characterization of $\mathcal{H}_{phys}$ nor the full spectrum of $\hat{H}$ are known. And it should be underlined that, in spite of the success of this approach, some ambiguities remain.
At this stage, the research has split into many lines, among which the main line follows the idea of a *Master Constraint*, which is an attempt of implementing simultaneously the spatial and scalar diffeomorphism constraints.
The covariant approach we presented as an alternative to the canonical approach, instead, seeks a functional integral description of transition amplitudes between spin-network states. We will briefly present this approach in the following chapters.

Chapter 6

Truncated theories

So far, the reader may start to have a feeling about what is the point in using graphs. The quantum holonomy-flux operators act on gauge invariant states defined on a graph that is embedded in space, and given a surface S in that space, its associated area operator $\hat{A}(S)$ acts on such states by "asking" them how many links of the graph (on which they are built) cross a suitable triangulation of S, and from the $SU(2)$ representation on those links it computes the area of S. The same happens with the volume operator $\hat{V}$ related to a 3d volume, which instead "asks" the states how many nodes of their graph are contained in the cells into which we have discretized the volume. If we fix a chosen surface and a chosen volume, their related operators of area and volume can act on all the possible states defined on all possible embedded graphs, regardless of how the graph and the spacetime chunks (surfaces and cells) of the discretization are arranged in the spatial hypersurface. This is quite of a freedom, and we can take great advantage from it. What if, for example, the chunks themselves gave us the rules for building the most natural embedded graph we would need to study their geometry? If this was possible, the only freedom we would retain would be that of choosing different arrangements of chunks, i.e. different discretizations of spacetime objects.

Being discretization the core ingredient for defining a quantum theory for the gravitational field, we need to study how to discretize general relativity. A discretization is an approximation: a truncation in the number of degrees of freedom where we disregard those likely to be irrelevant for a given problem. In quantum field theories, truncations play a constructive role. Indeed, recall the construction of Fock space: one starts from the one particle Hilbert space $\mathcal{H}_1$ and then builds the n-particle Hilbert space as the symmetric/antisymmetric tensor product of n copies of $\mathcal{H}_1$. The space of N or less particles is then given by $\mathcal{H}_N = \oplus_{n=1}^{N} \mathcal{H}_n$ and finally the Fock space is defined by the $N \to \infty$ limit. Nevertheless, most calculations are done at N finite, so that in most cases this limit is just performed formally and does not play any role. For example, in QED all the scattering amplitudes are computed among a finite number of particles, and the Feynman diagrams are considered up to a finite order. The same happens in lattice QCD, where real calculations are done on a fixed discrete lattice. And again, the same can be done in quantum gravity: we can build up a discretized theory , capturing the relevant physics of a given problem, and use this theory for all real calculations.

In this chapter, we review two classic discretizations: lattice Yang-Mills theory and Regge calculus. Then we introduce the discretization of general relativity which we use in the following, borrowing features from the first two.

6.1 Lattice Yang-Mills theory

Yang-Mills theory is a gauge theory based on the $SU(N)$ group or, more generally, on any compact Lie group. The main example is QCD, with $SU(3)$ as gauge group, which describes the strong interaction. The corresponding truncated theory lives on a lattice and, in the case of QCD, is known as Lattice QCD.
We consider for simplicity a $SU(2)$ Yang-Mills theory in 4 dimensions. This case is useful since it shares the same gauge group as the canonical approach of LQG, allowing a straightforward parallelism.
The field variable in the continuous theory is an $SU(2)$ connection $A^i_\mu(x)$, where i is the index in the Lie algebra of $SU(2)$, $su(2)$. Inside this algebra, we can decompose A on the basis of $SU(2)$ generators J_i and write it as the 1-form

$$A(x) = A^i_\mu(x) J_i dx^\mu \ . \tag{6.1}$$

The central idea of Wilson, on which loop quantum gravity relies, is that we must view the algebra as the tangent space to the group, and A as the log of a group variable.
In order to discretize the theory, let us fix a cubic lattice with N vertices connected by L oriented edges embedded in flat spacetime. This of course breaks the rotation and Lorentz invariance of the theory, which will only be recovered in a suitable limit. Let a be the length of lattices edges, which is determined by the spacetime flat metric, here fixed. We associate a ***group element*** $U_e \in SU(2)$ to each oriented edge e of the lattice (and the inverse U_e^{-1} to the same edge with opposite orientation). The set of group elements U_e on all edges of the lattice provides a natural discretization of the continuous field A.
Wilson's idea is that the quantum theory is better defined starting from these group variables than from the algebra variables. Physical quantities must then be studied in the simultaneous limits $N \to \infty$ and $a \to 0$.
The formal relation between U_e from the discretization and the continuous field A is:

$$U_e = P e^{\int_e A}, \tag{6.2}$$

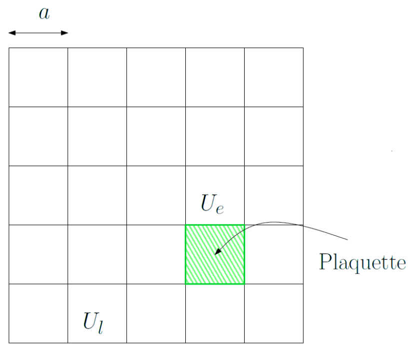
namely U_e is the path ordered exponential of the connection along the edge e, i.e. the holonomy of A. Expanding in the length a of the edge, (6.2) gives at first order:

$$U_e = \mathbb{I} + a A_\mu(s_e) \dot{e}^\mu \ , \tag{6.3}$$

being s_e the source of the link e and $\dot{e}^\mu$ the unit vector tangent to e. The discretization reduces the internal local continuous $SU(2)$ rotational symmetry to a discrete $SU(2)$ symmetry at the vertices. The gauge group of the lattice theory is therefore $SU(2)^N$. Under such gauge transformation, the group variables (holonomies), transform as

$$U_e \to g_{s_e} U_e g_{t_e}^{-1} \ , \ g_v \in SU(2) \ . \tag{6.4}$$

We can restore the continuous symmetry sending $N \to \infty$ and $a \to 0$. This is a general feature of discretizations, and we will see that the structural difference between gravity and Yang-Mills theory lies in the different way these limits enters when recovering the continuous.
The ordered product of four group elements around a plaquette f, namely an elementary square in the lattice:



$$U_f = U_{e_1} U_{e_2} U_{e_3} U_{e_4} \tag{6.5}$$

is a discrete version of the curvature. Its trace is gauge invariant and Wilson has shown that the discrete action:

$$S = \beta \sum_f TrU_f + c.c. \tag{6.6}$$

approximates the continuous action in the limit in which a is small. The coupling constant β is a number that depends on a and goes to zero when a goes to zero.

6.2 Analogies between lattice Yang-Mills theory and gravity

The hamiltonian formulation of this theory shows an astonishing amount of similarities with the hamiltonian analysis of gravity in chapter 4. Of course, this is not casual, for historical and physical reasons.
The hamiltonian formulation of lattice Yang-Mills theory lives on the boundary of the lattice, which we assume to be spacelike. The hamiltonian coordinates are the group elements U_l defined on the L edges of the boundary, which we call *links*. The N vertices of the boundary are instead called *nodes*. In the time gauge, the group elements correspond to the holonomies of the spatial connection on the links of the boundary. They form the configuration space $SU(2)^L$. In the corresponding phase space $T^*SU(2)^L$, their conjugate momenta L^i_l take values in the $SU(2)$ algebra and are as well associated each with a link. More precisely, each L^i_l is the smearing on the surface S_l (associated, in the dual boundary lattice, with the link l and punctured by the latter) of the time-space component (electric component) of the field strength 2-form. Their Poisson brackets are:

$$\{U_l, L^i_{l'}\} = \delta_{ll'} U_l J^i \tag{6.7}$$

and define a symplectic structure with residual gauge transformations at the nodes of the boundary. When quantizing, the Poisson brackets are promoted to commutator relations and the canonical variables act as operators on the Hilbert space of the discrete theory, which is made of states $\psi(\{U_l\})$, functions on the configuration space belonging to $\mathcal{L}_2[SU(2)^L]$. The Hilbert space carries a natural scalar product, which is invariant under the gauge transformations on the boundary: the one defined by the $SU(2)$ Haar measure

$$\langle\phi|\psi\rangle = \int_{SU(2)} dU_l \overline{\phi(\{U_l\})}\psi(\{U_l\}) \ . \tag{6.8}$$

The boundary gauge transformations act at the nodes of the boundary and transform the states as follows:

$$\psi(\{U_l\}) \to \psi(\{\lambda_{s_l} U_l \lambda_{t_l}\}), \quad \lambda \in SU(2)^N \ . \tag{6.9}$$

The states invariant under this transformation form the Hilbert space of the gauge invariant states which has the structure $\mathcal{L}_2\left[SU(2)^L/SU(2)^N\right]$. The operator corresponding to the group elements is diagonal in this basis and act by multiplication:

$$\hat{U}_{l'}\psi(U_l) = U_{l'}\psi(U_l) \ , \tag{6.10}$$

while the conjugate momentum operator acts as a derivative operator on the states [1]

$$\hat{L}^i_{l'}\psi(\{U_l\}) = -i\hbar\frac{d}{dt}\psi(\{U_l\}_{l\neq l'}, U_{l'}e^{tJ^i})|_{t=0} \ . \tag{6.11}$$

The reader may have noticed the whole similarity with the quantum kinematical space of LQG.

[1] Actually, there exist two possible derivatives to be defined on a Lie group. They are known as *left-invariant* vector field, L^i_l, and the *right-invariant* vector field R^i_l, the first acting at the source of the link, the second at the target. We choose the set of left-invariant vector fields, which forms a Lie algebra.

6.3 Differences between lattice Yang-Mills theory and gravity

We have already pointed out that the main difference between discretized lattice YM theory and discretized gravity lies in the limits one should perform to recover the continuous theories. In what follows we will show this difference in some detail: the point of comparison is the discretization of the action in both theories.
In lattice YM theory, the discretized action (6.6) becomes continuous if one sends the lattice spacing a to zero as well as increasing the number of lattice "steps", i.e. the number of plaquettes N in the sum, up to infinity. In GR the action:

$$S[g] = \int d^4x \sqrt{-\det g} R[g] \tag{6.12}$$

is invariant under any reparametrization of x. The canonical hamiltonian vanishes and the information about the dynamics is coded in the constraints, to which the hamiltonian is completely proportional. This means that the dynamics does not describe the evolution of the gravitational field $g_{\mu\nu}(x)$, and other matter fields, as functions of x, but rather the evolution of the fields with respect to one another. The variable x is just a gauge, a tool we use to parametrize the system.
It is possible to generalize this situation to a wide category of physical systems, which we call *reparametrization invariant* or *generally covariant*. Consider the generic action:

$$S[q] = \int dt \mathcal{L}\left(q(t), \dot{q}(t)\right) \tag{6.13}$$

where $\mathcal{L}$ is the lagrangian depending on the configuration variable q and its time derivative $\dot{q}$. The most immediate interpretation is that the evolution of our system is coded in the equation of motion of q, from the Principle of Least Action, with respect to the time coordinate t. But we are not forbidden to build a different formalism where q and t are treated as variables on the same footing: t can lose its role as the privileged parameter (independent variable) and join the "democracy" of the lagrangian variables (dependent variables) if we express *parametrically* the function $q(t)$ in terms of a couple of functions:

$$q(t) \to \begin{cases} q(\tau) \\ t(\tau) \end{cases} \tag{6.14}$$

where each variable depends on a new parameter τ. Given a function $q(t)$, we can always decompose it into two parametric equations where both q and t depend on τ. The opposite is not always true: for example, if $q = \sin\tau$ and $t = \cos\tau$, we cannot recover a proper function $q(t)$. Therefore the parametric representation is more general than the $q(t)$ representation, and carries a large redundancy. For instance, the function $q(t) = t^{2/3}$ can be written in terms of $q(\tau) = \tau^2$ and $t(\tau) = \tau^3$, but also in terms of $q(\tau) = f(\tau)^2$ and $t(\tau) = f(\tau)^3$ for any invertible function $f(\tau)$. This is a gauge invariance, the gauge being $\tau \to f(\tau)$, f *invertible*, and shares the nature of the diffeomorphism invariance in GR.
Now the action reads:

$$S[q,t] = \int d\tau \frac{dt(\tau)}{d\tau} \mathcal{L}\left(q(\tau), \frac{dq(\tau)/d\tau}{dt(\tau)/d\tau}\right) \ . \tag{6.15}$$

The motions $(q(\tau), t(\tau))$ that minimize this action determine the motion $q(t)$ that minimizes the original action.
If we choose the lagrangian:

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 - V(q) \tag{6.16}$$

its parametric form in the parameter τ is:

$$\mathcal{L}^{par}(q, t, \dot{q}, \dot{t}) = \frac{1}{2} m \frac{\dot{q}^2}{\dot{t}} - \dot{t} V(q) \tag{6.17}$$

where now the dot means the τ derivative. The resulting equations of motion for $q(\tau)$ and $t(\tau)$ are not independent and this is the sign of a redundancy in the description of the system, given by the freedom in replacing the parameter τ with any invertible differentiable function $f(\tau)$. τ is pure gauge and the physics lies in the relation between q and t.
Turning to the hamiltonian formulation, one discovers that the relations between the momenta and the velocities cannot be inverted to express the velocities in terms of the momenta. In compact form:

$$\det\left[\frac{\partial^2\mathcal{L}}{\partial\dot{q}_i\partial\dot{q}_j}\right] = 0 \ , \tag{6.18}$$

where the matrix is in the indices i, j and each one explores the lagrangian variables, i.e. q and t. The resulting theory is the well-known constrained-systems theory, where (first class) constraints emerge to bound the dynamics on a subspace of the original phase space and to generate the gauge transformations of τ, while the hamiltonian becomes fully proportional to the constraints and thus vanishes "on shell". This is coherent with the fact that the hamiltonian would generate evolution with respect to an evolution parameter τ which is devoid of physical meaning. The whole dynamics is expressed in this formalism by the relation between the dependent variables q and t, rather than the individual evolution of these in the gauge parameter τ. The physical content of the evolution is transferred to the constraints, sometimes called *hamiltonian constraints* for this reason.
How can we discretize the action of (6.16)? We already know a natural discretization thanks to Feynman's transition amplitude. This is built from dividing the time interval between the initial and final states in N steps of duration ϵ (*time slices*), then performing a sum over N paths $q_n(t)$, each weighted by the exponential of the N pieces of action times $i/\hbar$, and taking N to infinity. The discretized action of lagrangian (6.16) reads:

$$S_N(q_n) = \sum_{n=1}^{N} \frac{m\left(q_{n+1}-q_n\right)^2}{2\epsilon} - \epsilon V\left(q_n\right) \tag{6.19}$$

and after replacing $q_n \to q_n/\sqrt{\epsilon}$:

$$S_N[q_n] = \sum_{n=1}^{N} \frac{m\left(q_{n+1}-q_n\right)^2}{2} - \epsilon V\left(\sqrt{\epsilon}q_n\right) \tag{6.20}$$

The continuous action is restored by sending N to infinity and ϵ to zero, just like in lattice YM. But if we remember of the parametric formulation, the discretized action of the parametric lagrangian (6.17) is instead:

$$S_N^{par}[q_n] = \sum_{n=1}^{N} \frac{m}{2}\frac{\left(q_{n+1}-q_n\right)^2}{t_{n+1}-t_n} - (t_{n+1}-t_n)V\left(q_n\right) \ . \tag{6.21}$$

The scaling ϵ has disappeared. The continuous theory is recovered by the only limit $N \to \infty$. Just like rotational invariance in YM, the reparametrization invariance is only recovered in the continuous limit, but now such limit is performed by *only* increasing the number of steps, with no additional scaling required.
This is a common feature of reparametrization invariant systems, and of general relativity as well. A suitable discretization of GR must show this behaviour. Regge calculus does.

6.4 Regge Calculus

The elegant discretization of GR introduced by Tullio Regge, called *Regge calculus*, can be defined as follows. A d-simplex, defined as the convex hull of $d+1$ vertices, is the generalization of the triangle in 2d or the tetrahedron in 3d to higher dimensions. Its vertices are connected by segments, whose lengths L_s fully specify the shape of the simplex, i.e. fully specify the metric geometry.

A Regge space $(\mathcal{M}, L_s)$ in d dimensions is a d-dimensional space obtained by gluing flat d-simplices along matching boundary $(d-1)$-simplices. For instance in 3d we chop the space in *tetrahedra*, bounded by *triangles*, in turn bounded by *segments*, which meet at *points*. And this can be done in any dimension. These structures are called *triangulations* Δ and a scheme of them for the lowest-dimensional cases is given in Table 6.1.

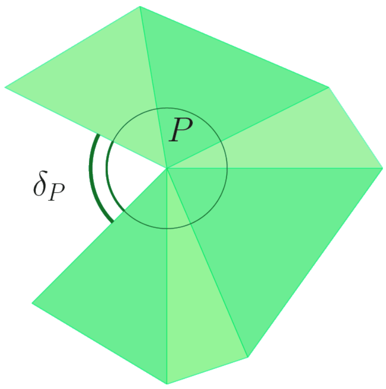
Gluing together flat d-simplices can generate curvature on the $(d-2)$-simplices, called *hinges*.

Triangulations					
2d			triangle	segment	**point**
3d		tetrahedron	triangle	**segment**	point
4d	4-simplex	tetrahedron	**triangle**	segment	point

Table 6.1: The simplex carrying curvature, or *hinge*, is underlined.

This is easily understood in the 2d case: consider a point P and the triangles t around it and let the angles at P of these triangles be θ_t. Now if the sum of the angles is 2π, the manifold $\mathcal{M}$ which we have paved with the triangles is flat in P, otherwise there is curvature. From this example we can deduce that the Regge curvature in 2d can be defined as the angle:

$$\delta_P(L_s) = 2\pi - \sum_t \theta_t(L_s), \tag{6.22}$$



where $\theta_t(L_s)$ means that we can deduce the angles of the triangles in P by simple geometrical computations once the lengths of the segments are known. δ_P is called *deficit angle* at P. The same logic can be used in higher dimensions. In d dimensions, the Regge curvature is still given by a single deficit angle, but now P is replaced by a $(d-2)$-simplex (a segment in 3d and a triangle in 4d) and the angles become the dihedral angles of the flat d-simplices, which can be computed from the L_s using elementary formulas of geometry.

The geometrical interpretation of the deficit angle is simple: if we parallel transport a vector in a loop around a $(d-2)$-simplex, the vector gets back rotated by the deficit angle[2]. Using this, Regge defines the Regge action of a Regge manifold $(\mathcal{M}, L_s)$ to be:

$$S_{\mathcal{M}}(L_s) = \sum_h A_h(L_s)\delta_h(L_s) \ , \tag{6.23}$$

where the sum is over the hinges and A_h is the $(d-2)$-volume of the hinge h[3]. The remarkable result obtained by Regge is then that the Regge action converges to the Einstein-Hilbert action $S[g]$ when the Regge manifold $(\mathcal{M}, L_s)$ converges to the Riemannian manifold $(\mathcal{M}, g)$, and this condition is achieved by simply refining the triangulation up to the continuum. Thus the Regge theory is a good discretization of general relativity.

The Regge equations of motion are obtained by varying the action with respect to the length. This

[2]Actually, we have omitted to say that the rotation is around the axis parallel to the hinge. This is an algebraic restriction of the Regge curvature to the only rotations that preserve the hinges. However, this limitation is not inherited by the average Regge curvature over a region, so we are not worried.

[3]Notice the resemblance with the tetradic action, where the curvature Riemann tensor F is traced with the Lorentz connection ω. In a sense, also the lattice YM action features a trace over a curvature, if we interpret the product of the group elements around the plaquette f, U_f, as the counterpart of the curvature, the identity element being the analogue of flatness. All these theories can be generalized in a category of topological theories called *BF theories*.

gives two terms: the first from the variation of the A's and the second from the variation of the angles. The second term vanishes, in analogy with the vanishing of the variation of the Riemann tensor in the Einstein-Hilbert action. Thus, Regge equations are:

$$\sum_h \frac{\partial A_h}{\partial L_s} \delta_h(L_s) = 0 \ , \tag{6.24}$$

where the sum is over the hinges adjacent to the segment s, and there is one equation per segment. We can identify the left hand side of this equation as a measure of the discrete Ricci tensor.
In three spacetime dimensions the hinges are the same things as the segments and therefore the Regge equations reduce to $\delta_h(L_s) = 0$, that is flatness, as in the continuous 3d case.
The action (6.23) can be rewritten as a sum over the d-simplices v of the triangulation, if we recall the definition (6.22):

$$S_{\mathcal{M}}(L_s) = 2\pi \sum_h A_h(L_s) - \sum_v S_v(L_s) \ , \tag{6.25}$$

where the action of a d-simplex is

$$S_v(L_s) = \sum_h A_h(L_s)\theta_h(L_s) \ . \tag{6.26}$$

6.5 Discretization on a two-complex

The Regge discretization is not very good for the quantum theory, for two reasons. First, it is based on the metric variables (lengths, angles, etc). The existence of fermions indicates that we need tetrads at the fundamental level. Second, the segments of a Regge triangulation are subjected to triangular inequalities, which make the configuration space rather complicated.
We now introduce a slightly different discretization which is closer to that of Yang-Mills theory, while at the same time retaining the specific feature of a discrete generally covariant theory. This discretization is described in terms of two-complexes, which bridge between a Yang-Mills lattice and a Regge triangulation, and is the one we use in LQG.

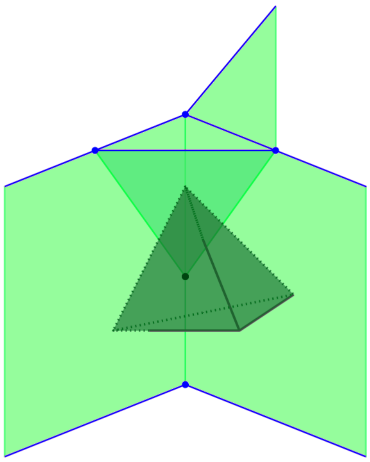


Figure 6.1: 3d duality

The key notion we need is the *dual* of a triangulation. Let us start with the 3d case, shown in Figure 6.1, where the tetrahedron belongs to the original triangulation, while the light-green faces meeting along the edges, in turn meeting at points, form the dual. More precisely, if Δ is a 3d triangulation, its dual Δ^* is obtained by:

- placing a *vertex* within each tetrahedron, dual to the related tetrahedron;
- joining the vertices of two adjacent tetrahedra by an *edge*, dual to the triangle that separates the two tetrahedra;
- associating a *face* to each segment of the triangulation, dual to the segment and bounded by the edges that surround the segment.

If the triangulation has an orientation, the dual of the triangulation inherits univocally an orientation. The set of vertices, edges and faces is called a *two-complex* $\mathcal{C} = \Delta^*$.
The discretization of a spacetime region $\mathcal{R}$ induces a discretization on the boundary $\Sigma = \partial\mathcal{R}$ of this region. In the 3d case, Σ is discretized by the boundary triangles, separated by the boundary segments of Δ. The dual of the boundary is given by the end points of the edges dual to those triangles, called *nodes*, or boundary vertices, and by the boundaries of the faces dual to those segments, which we call *links*, or boundary edges. Nodes and links together form the *graph* of the boundary, $\Gamma = \partial\mathcal{C}$. Note that the boundary graph is at the same time the boundary of the two-complex and the dual of the boundary of the triangulation:

$$\Gamma = \partial(\Delta^*) = (\partial\Delta)^* \ . \tag{6.27}$$

Does this graph have the same meaning as the one introduced in §5.2 ? This graph, being in the dual space of the manifold, is an abstract object which is only defined by its combinatorial structure. It should not be thought as embedded in the manifold, with the links being real paths in it, because it is meant to represent the manifold itself. Instead, starting from the embedded one, we could see it as the equivalence class of embedded graphs $[\Gamma]$ (that, with an abuse of notation we still call Γ) which differ from each other for a diffeomorphism transformation, in the same manner in which we study the only physical properties of a manifold that can describe it as the same object even if we change the language (embedding coordinates) we use to speak about it. Just think of tensorial (covariant) and invariant observables, the core ingredients of any relativistic problem. As anticipated in §5.6, the abstract graph naturally encodes the invariance under spacetime diffeomorphisms in our theory.

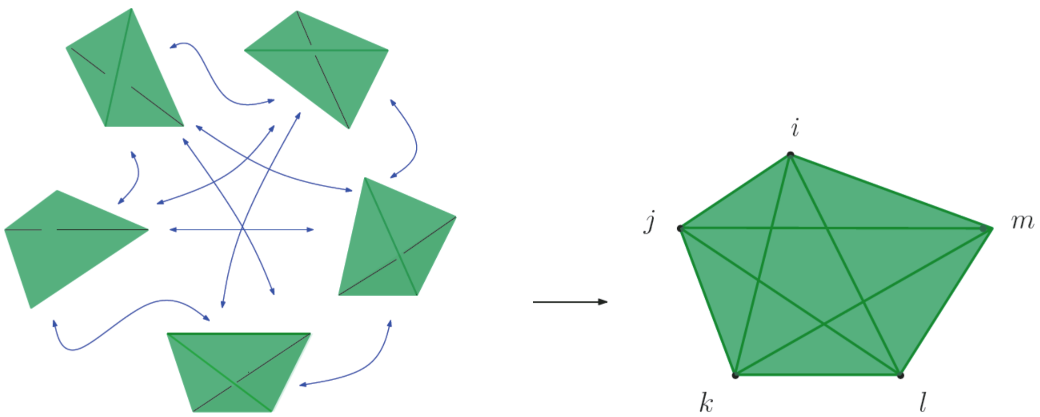


Figure 6.2: 4-Simplex

However, the interesting theory is the 4d case, where we consider the dual of a triangulation of a region of real spacetime. All the dual construction can be easily extended: now we triangulate (Δ) a 4d region, chopping it into 4-simplices (which can be thought as convex regions in $\mathbb{R}^4$ delimited by five points), bounded by five tetrahedra each (Figure 6.2). As before, we consider the two-complex $\mathcal{C} = \Delta^*$ dual of the triangulation and focus on its *vertices*, *edges* and *faces*.
Now, each vertex is dual to a chunk of the manifold, which now is a 4-simplex. In turn, every edge

is dual to a tetrahedron and each face is dual to a triangle.
Similarly, on the boundary we still have a graph Γ, whose nodes are now associated with chunks of boundary space (tetrahedra), and whose links connect adjacent tetrahedra and are therefore dual to the triangles of the boundary triangulation.[4] The choice of triangulating the boundary with tetrahedra (3-simplices) gives rise to graphs with only 4-valent nodes. Had we chosen different polyhedra with V faces, we would have obtained V-valent nodes.

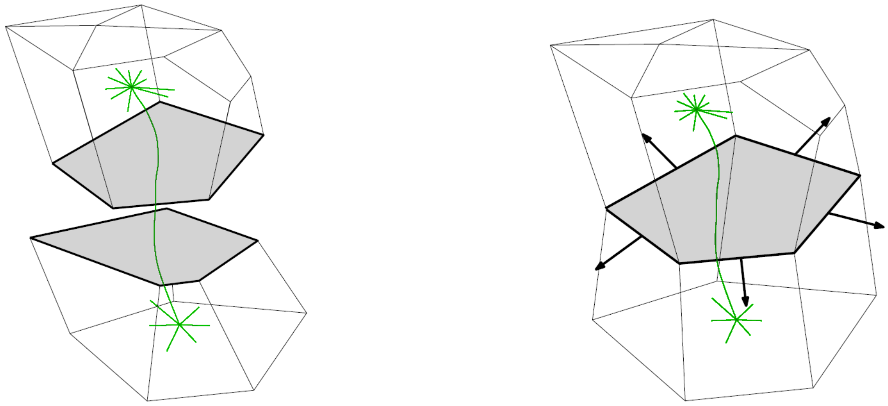


Figure 6.3: Two high-valency nodes and the link connecting them. Left panel : In general, the two adjacent polyhedra, defined by holonomies and fluxes on the nodes, don't glue well. Even if the area of a polygonal face is the same because there is a unique spin associated with the link, the shapes will be different in general. Right panel : in order for the shapes to match, one needs to impose appropriate conditions on the polyhedra such as the matching of the normals to the edge in the plane of the face. These conditions, studied in [8], affect the global shape of the polyhedron.

The bulk and boundary terminology in the 4d case is listed in the tables below.

Bulk terminology (4d)

Bulk triangulation Δ	**Two-complex** $\Delta*$
4-simplex (v)	vertex (v)
tetrahedron (τ)	edge (e)
trangle (t)	face (f)
segment (s)	
point (p)	

Boundary terminology (4d)

Boundary triangulation $\partial\Delta$	**Boundary graph** Γ
teatrahedron (τ)	node (boundary vertex) (n)
trangle (t)	link (boundary edge) (l)

6.6 Truncated LQG

Before moving to the covariant approach of LQG, we are already prepared to build the truncated theory from the canonical approach. All the ingredients have been defined previously, we just need

[4]Recalling (6.27) the nodes can be seen as the boundaries of the links that, in the bulk triangulation, are dual to the tetrahedra, while the links are also the boundaries of the faces that in the bulk are dual to triangles.

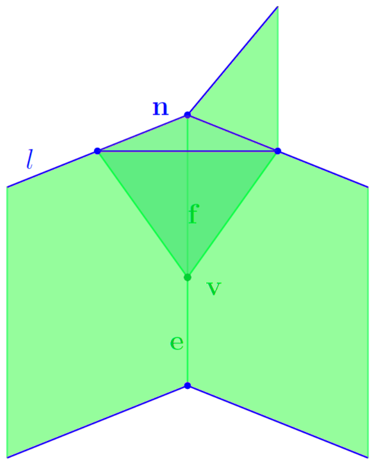


Figure 6.4: Two-complex structure and nomenclature

to connect the apparatus with the final idea of an abstract boundary graph. We summarize the reasons behind the words *abstract* and *boundary*.

abstract The discretization of spacetime d-dimensional regions ($d = 1, \dots, 4$) on two-complexes starts with the idea of truncating GR into discrete chunks, and lands to the definition of a diffeomorphism invariant abstract graph whose combinatorial structure encodes the way we have chosen to chop the theory, i.e. the level of approximation performed. We have created a method for associating a *unique* abstract structure to the *arbitrary* triangulation of the given (sub)manifold. Going to the quantum theory and recalling what we have learnt in the canonical path towards quantum gravity, instead of working with geometrical observables (area, volume and so on) built from operators that are defined on embedded graphs which are not necessarily correlated to the truncated spacetime objects we want to measure, now the truncated spacetime objects can tell us what dual graphs we have to build, thus what quantum states we have to define, then what operators act on them from which to build the geometrical observables that we use to describe the given truncation and study quantum gravity problems.

boundary The reader may have noticed that lattice YM theory works with operators on the boundary of the lattice, and it is this environment that shows great resemblance with canonical loop quantum gravity: same gauge group, some holonomy-flux algebra, same nature of discrete invariances and of the resulting Hilbert space. But why on the boundary? And how is this aspect recognized in our theory? The answer is, again, predictable: the hamiltonian formulation of general relativity, even before quantization, is based on foliations, on tetrads, and the time gauge to link the two and extend the 3+1 splitting to the inner vector space (the tetradic Minkowskian fiber) where we have decided to define the variables of the theory. Everything we have talked about in the two previous chapters is built on spacelike 3d hypersurfaces at most, boundary spaces of the real 4d world, and the literature shows how this choice rewards us: the theory becomes an $SU(2)$ gauge theory, where we can make use of the achievements in quantum mechanics in a surprisingly coherent way. Another reason for the boundary formalism will be explained when we focus on dynamics, recalling what is a transition amplitude. We leave it to that part.

As we have just seen, the boundary truncation of a 4d spacetime region is a 3d collection of tetrahedric chunks glued together. The dual graph Γ $(= [\Gamma])$ is therefore an abstract collection of links (say L in total) meeting in 4-valent nodes (say N in total). Recall from §5.2 that the Hilbert space of gauge-invariant states on this graph is (5.23):

$$\mathcal{K}_\Gamma = \mathcal{L}_2\left[SU(2)^L/SU(2)^N\right] = \bigoplus_{\{j_{v(n)}\}_{v(n)}} \bigotimes_{n=1}^{N} Inv_{SU(2)}\left(\mathcal{H}_{j_{1(n)}} \otimes \ldots \otimes \mathcal{H}_{j_{V(n)}}\right) ,$$

and a state in it is the combination (5.39) of the orthogonal states (5.40), which we can write in the compact form:

$$\psi^{k_1 \ldots k_N}_{j_1 \ldots j_L}[\{U_l\}] = \langle \{U_l\} | \{j_l, k_n\} \rangle = \bigotimes_{n=1}^{N} \iota_{k_n} \overset{\Gamma}{\cdot} \bigotimes_{l=1}^{L} D^{j_l}(U_l) , \tag{6.28}$$

where the product $\overset{\Gamma}{\cdot}$ means that contractions among indices are dictated by the combinatorial structure of the graph.
In analogy with lattice YM theory, we can use this apparatus to define, on the boundary graph Γ, a discretized version of the algebra of GR, now univocally associated with the graph. For each link of the graph, we assign:

- an $SU(2)$ group element U_l, the analogous of Yang-Mills' configuration variable;
- an $su(2)$ algebra element L^i_l, the momentum conjugate to U_l.

They form the same closed algebra as in $SU(2)$ lattice YM theory, with Poisson brackets:

$$\{U_l, U_{l'}\} = 0 \quad ; \quad \{L^i_l, L^j_{l'}\} = \delta_{ll'} \epsilon^{ij}_k L^k_l \quad ; \quad \{U_l, L^i_{l'}\} = \delta_{ll'} U_l J^i . \tag{6.29}$$

The matching with the canonical LQG states is straightforward, if we look at Γ as the equivalence class of its embeddings in the boundary manifold: U_l stands for the holonomy $h_{e_l}[A]$ of the Ashtekar connection A on the link e_l (now a real path) belonging to one of the possible embeddings of the abstract graph in spacetime (one of the embedded graphs in the equivalence class). The quantum version $\hat{U}_l$ is the holonomy multiplicative operator along the embedded link e_l. The conjugate momentum L^i_l ($su(2)$ algebra element), corresponds to the flux variable $E^i(S_l)$ where S_l is now the triangle in the boundary triangulation that is punctured by its dual link e_l, embedding in the boundary 3d space of the abstract link l. The triangle is shared by the two adjacent tetrahedra whose dual nodes are the source and target of the link l. The corresponding quantum operator $\hat{L}^i_l$ will act as the derivative operator[5]. In conclusion:

$$\hat{U}_l = \hat{h}_{\gamma = e_l} \quad ; \quad \hat{L}^i_l = \frac{1}{8\pi G\hbar\gamma} \hat{E}^i(tr_l) . \tag{6.30}$$

In these variables, the Gauss constraint for a node n is now the operator:

$$\hat{G}^i_n = \sum_{l(n)=1}^{4} \hat{L}^i_{l(n)} , \tag{6.31}$$

which reminds us of (3.15) and generates $SU(2)$ rotations around n. The physical constants behind $\hat{E}^i$ are meant to recover the definition of the area operator as:

$$\hat{A}_l \equiv \hat{A}(tr_l) = \sqrt{\hat{E}_i(tr_l)\hat{E}^i(tr_l)} = 8\pi G\hbar\gamma \sqrt{\hat{L}^l_i \hat{L}^i_l} \tag{6.32}$$

with eigenvalues:

$$A(tr_l) = 8\pi G\hbar\gamma \sqrt{j_l(j_l+1)} . \tag{6.33}$$

[5]More precisely, in the discretized theory, we define a ***left-invariant*** vector field $L^i_{l,s}$ as the flux operator across tr_l parallel transported to the source node n_s of the link l, and a ***right-invariant*** vector field $L^i_{l,s}$ as the flux operator across tr_l parallel transported to the target node n_t of the link l. The two operators are related by the holonomy defined on l, $\hat{U}_l$. We define L as the left-invariant vector fields.

The eigenvalues of the area operator, along with the eigenvalues of the volume operator:

$$\hat{V}_n^2 = \frac{2}{9}\epsilon_{ijk}\hat{L}^i_{a(n)}\hat{L}^j_{b(n)}\hat{L}^k_{c(n)} \tag{6.34}$$

(where $a(n), b(n), c(n)$ are three of the links that go into the node dual to the tetrahedron of which we are computing the volume) can be used to label an orthonormal basis in $\mathcal{K}_\Gamma$: $\{|\{j_l, v_n\}\rangle\}_{\{l,n\}}$ each with a spin j_l assigned to every link l and a quantum number v_n assigned to every node.
On the other hand, the volume operator is not diagonal in the basis $\{|\{j_l, k_n\}\rangle\}_{\{l,n\}}$ we have defined in (5.40) and (6.28). Such basis is called *recoupling basis* and diagonalizes the two rotation invariant operators $\hat{L}_l^2 = \hat{L}_i^l\hat{L}_l^i$ and $(\hat{L}^i_{a(n)}+\hat{L}^i_{b(n)})^2 = (\hat{L}_i^{a(n)}+\hat{L}_i^{b(n)})(\hat{L}^i_{a(n)}+\hat{L}^i_{b(n)})$ where $a(n), b(n)$ are two of the four links that go into a generic node n. Indeed, their action on a basis element $|\{j_l, k_n\}\rangle$ is:

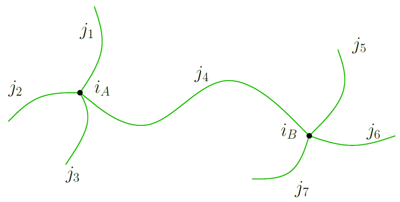
$$\hat{L}^2_{l'}\,|\{j_l, k_n\}\rangle = j_{l'}(j_{l'}+1)\,|\{j_l, k_n\}\rangle \tag{6.35}$$

$$(\hat{L}^i_{a(n')} + \hat{L}^i_{b(n')})^2\,|\{j_l, k_n\}\rangle = k_{n'}(k_{n'}+1)\,|\{j_l, k_n\}\rangle \quad . \tag{6.36}$$

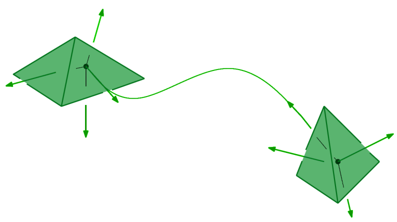
We already know that j_l is a measure of the area of the triangle dual to the link l. Now we also discover that k_n is instead a measure of the dihedral angle $\theta_{a(n)b(n)}$ between the triangles dual to the links $a(n)$, $b(n)$ that meet in the node n:

$$(\hat{L}^i_{a(n)} + \hat{L}^i_{b(n)})^2 = (\hat{L}_i^{a(n)} + \hat{L}_i^{b(n)})(\hat{L}^i_{a(n)} + \hat{L}^i_{b(n)}) = \hat{L}^2_{a(n)} + \hat{L}^2_{b(n)} + 2\hat{L}_i^{a(n)}\cdot\hat{L}^i_{b(n)} \tag{6.37}$$

and $\hat{L}_i^{a(n)}\cdot\hat{L}^i_{b(n)} \propto \theta_{a(n)b(n)}$. This is coherent with (3.9) when we studied the geometrical quantities of the tetrahedron as functions of the normals to its faces.



(a)



(b)

Figure 6.5: Graph (a) - Triangulation (b) duality

Chapter 7

Covariant loop quantum gravity

The canonical approach to LQG apparently breaks the local Lorentz invariance of the tetradic formalism on which it is based. This is due to the time gauge and to the definition of the holonomy-flux operators thanks to works by Ashtekar, Lewandoski, Barbero et al. Such definition reduces the total $SL(2,\mathbb{C})$ symmetry down to $SU(2)$.
Actually, Ashtekar's first proposal [1] for his connection was slightly different from the one modified by Barbero more recently. It is possible to rewrite the Holst action using the connection (4.80) with the imaginary unit instead of the Immirzi parameter γ (Ashtekar's first connection was a complex connection). It turns out [39] that the Holst action gains two types of constraints: those being $SL(2,\mathbb{C})$ gauge invariant (the collection of Gauss and diffeomorphism constraints), which are first class, and those breaking this symmetry, that happen to be instead of second class. The latters, which do not appear in the formulation with the real valued Ashtekar-Barbero connection, restrict the gauge transformations allowed to those preserving the time gauge and, *ipso facto*, break the $SL(2,\mathbb{C})$ invariance. One of these so-called *reality conditions*, in turn, will match the linear simplicity constraint.
The reader may have noticed that the group algebra of the problem leads to the very same dynamics either if we directly land to $SU(2)$ or if we keep the bigger $SL(2,\mathbb{C})$ symmetry and let the constraints work. We have already pointed this out when describing the completely covariant path, which appears at this stage as a third possibility : we choose to keep the Lorentz invariance explicit and do not perform Ashtekar's change of variables. The following happens: when pulling back the connection and the 2-form B defined in (4.92) to spacelike hypersurfaces Σ, thanks to the time gauge (which is just the choice of the particular Lorentz frame where the tetrad has one vector aligned with the normal vector n of Σ) we still recover the flux operator (and its geometrical interpretation as area vector) by quantum promoting the magnetic part of B in Σ, and this will live in the $SU(2)$ algebra. Since we build the geometrical observables from the flux operator, in any case their core algebra is $SU(2)$. But while the holonomy of the Ashtekar-Barbero connection is a $SU(2)$ group element, the holonomy of the Lorentz connection stays a $SL(2,\mathbb{C})$ group element. And since LQG states are functionals of the holonomy, the Hilbert spaces and, thus, the gauge invariances are again different. Nevertheless, we still have to take one last thing into account: the time gauge allows to express the *physical* simplicity constraint in a nicely simple way, where the magnetic and electric parts of B are proportional one to the other through the Immirzi parameter. This feature is the last piece in the puzzle: it bridges this formalism with the $SU(2)$ one so tightly that we will come up with a one-to-one correspondence between states in the two Hilbert states.
In this chapter we briefly review the covariant version of truncated LQG, that leaves the local Lorentz invariance explicit. If the reader feels that the truncated theory in §6.6 has become useless, now they are relieved: we will find out that we do not have to throw away all the tools we have created until now, but we can land to them in a beautiful manner.

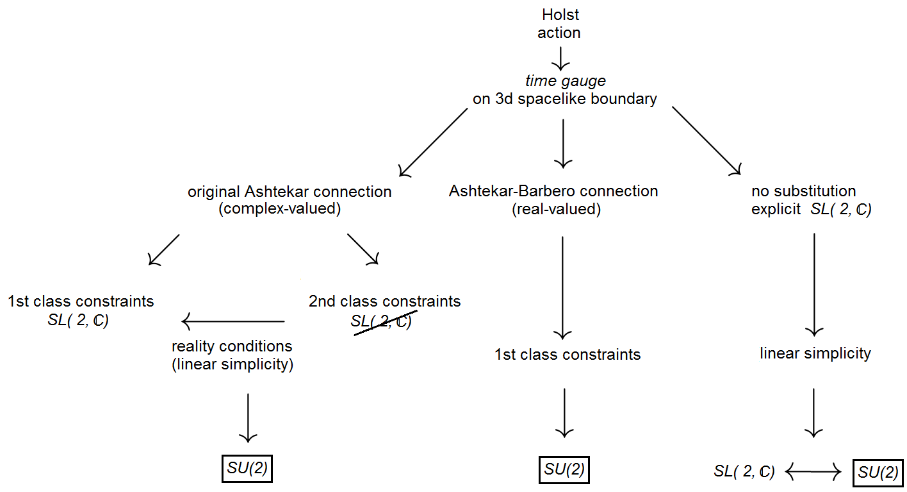


Figure 7.1: The possible canonical (first two columns) and covariant (last column) approaches

7.1 $SL(2,\mathbb{C})$ variables

We want to build a Lorentz locally invariant theory following the example of lattice YM theory. We know what to do: picked a spacetime 4d region (bulk) bounded by a 3d spacelike hypersurface (boundary), we triangulate the bulk as explained in §6.5. The triangulation will induce a triangulation of the boundary. Next, we draw the dual two-complex of both bulk and boundary. We will have a graph of 5-valent nodes in the bulk and a graph of 4-valent nodes in the boundary. Now, imitating lattice YM, we assign in the bulk:

- a group element g_e for each edge e of the two-complex dual to a tetrahedron, corresponding to the holonomy of the Lorentz connection $\omega \in sl(2,\mathbb{C})$ along an embedding of e in the bulk:

$$g_e = \mathcal{P}e^{\int_e \omega} \in SL(2,\mathbb{C}) \ ; \tag{7.1}$$

- an algebra element B_f for each face f of the two-complex, corresponding to the smearing of the 2-form B^{IJ} on the triangle of the bulk triangulation which is dual to the face f:

$$B_f = \int_{tr_f} B^{IJ} \in sl(2,\mathbb{C}) \ . \tag{7.2}$$

On the boundary, we assign instead:

- a group element g_l for each link l (of the boundary graph) dual to a boundary triangle, corresponding to the holonomy of the Lorentz connection $\omega \in sl(2,\mathbb{C})$ along an embedding of l in the boundary:

$$g_l = \mathcal{P}e^{\int_l \omega} \in SL(2,\mathbb{C}) \ ; \tag{7.3}$$

- an algebra element B_l for each link l of the boundary graph, corresponding to the smearing of the 2-form B^{IJ} (taking values in the Lorentz algebra) on the triangle of the boundary triangulation that is punctured by l:

$$B_l = \int_{tr_l} B^{IJ} \in sl(2,\mathbb{C}) \ . \tag{7.4}$$

We already know that, in the time gauge, its smeared magnetic part $L^i(tr_l)$ is a vector normal to tr_l with modulus proportional to the area of tr_l, (generator of $SO(3) \sim SU(2)$ rotations). Recall (4.105).

We are ready for quantizing the discretized covariant theory.

7.2 The simplicity map

Suppose we have a triangulation of a spacetime region with boundary graph made of L links and N nodes. The configuration space is the group $SL(2,\mathbb{C})^L$. Moving to the quantum theory, the quantum states of the graph will be functionals of the group elements g_l, $\psi(\{g_l\})$ belonging to the Hilbert space $\mathcal{L}_2[SL(2,\mathbb{C})^L]$, while $\hat{g}_l$ and $\hat{B}_l$ are now operators acting respectively by multiplication and derivation on such functionals. Moreover, the components of $\hat{B}_l$ must satisfy the linear simplicity condition, at least in the classical limit, and some limitation of the states is needed for this condition to hold.
Functionals of $SL(2,\mathbb{C})$ elements can be expanded on a basis of irreducible *unitary* representations of $SL(2,\mathbb{C})$ elements. Such representations should not be confused with the Lorentz representations we are familiar with in physics, such as:

- the spinor representation, with rotation generators: $L^i = -i\sigma^i/2$ (hermitian, belonging to the fundamental $SU(2)$ representation) and boost generators : $K^i = -\sigma^i/2$ (antihermitian);
- the vector representation, the fundamental of $SO(3,1)$, formed by 4-vectors;
- the adjoint representation, given by tensors with two antisymmetric Minkowski indices, like the electromagnetic field $F^{\mu\nu}$. Picked a frame, the generators of the algebra, in the adjoint representation, split into boost and rotations.

These representations are all non-unitary. Thus there is no positive definite scalar product defined on them, since it depends on the choice of a Lorentz frame in spacetime. Once this choice is made, the resulting scalar product becomes invariant for the residual gauge symmetry in the frame.
A detailed presentation of the unitary representations of $SL(2,\mathbb{C})$ is given in [31]. The facts that we need are the following: the two Casimirs of $SL(2,\mathbb{C})$, functions of the boost and rotation generators:

$$C_1 = |\vec{K}|^2 - |\vec{L}|^2 \ , \tag{7.5}$$

$$C_2 = \vec{K} \cdot \vec{L} \tag{7.6}$$

label the unitary representations with a positive real number p and a non negative half-integer k. The Hilbert space $\mathcal{V}^{(p,k)}$ of the (p,k) representation decomposes into irreducible representations of the subgroup $SU(2) \subset SL(2,\mathbb{C})$ as follows:

$$\mathcal{V}^{(p,k)} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j^{(p)} \tag{7.7}$$

where $\mathcal{H}_j^{(p)}$ is the $2j+1$ dimensional space that carries the spin j irreducible representation of $SU(2)$. $\mathcal{V}^{(p,k)}$ is thus infinite-dimensional. In this space, a basis is therefore given by the states:

$$|p,k;j,m\rangle \quad ; \quad j = k, k+1, \dots \quad , \quad m = -j, \dots, j \ . \tag{7.8}$$

The quantum numbers are related to the Casimirs by the relations:

$$|\vec{K}|^2 - |\vec{L}|^2 = p^2 - k^2 + 1 \ , \tag{7.9}$$

$$\vec{K} \cdot \vec{L} = pk \ , \tag{7.10}$$

while j and m are the quantum numbers of $|\vec{L}|^2$ and L_z respectively.
Now, we want to satisfy (4.98) in the classical limit, so we plug it in (7.9) and (7.10):

$$|\vec{K}|^2 - |\vec{L}|^2 = (\gamma^2 - 1)|\vec{L}|^2 \Rightarrow p^2 - k^2 + 1 = (\gamma^2 - 1)j(j+1) \ , \tag{7.11}$$

$$\vec{K} \cdot \vec{L} = \gamma|\vec{L}|^2 \Rightarrow pk = \gamma j(j+1) \ . \tag{7.12}$$

Solving the resulting equations in the limit of large j (i.e. the classical limit) one gets [1]

$$p = \gamma k \tag{7.13}$$

$$k = j \tag{7.14}$$

The first equation selects the only p's that are half integer multiples of γ. The second equation picks out the lowest spin subspace in the sum (7.7).Thus, the only Hilbert spaces allowed are those of the representations $(\gamma j, j)$:

$$\mathcal{V}^{(\gamma j, j)} = \mathcal{H}_j \ . \tag{7.15}$$

They are now finite-dimensional ($(2j+1)$-dimensional) and a basis for them is given by the $2j+1$ states:

$$|\gamma j, j; j, m\rangle \quad ; \quad m = -j, \ldots, j \ . \tag{7.16}$$

These states are in one-to-one correspondence with the states in the representations of $SU(2)$. We can introduce a *simplicity map* $\mathcal{Y}_\gamma$ as:

$$\mathcal{Y}_\gamma : \qquad \mathcal{H}_j \longrightarrow \mathcal{V}^{(p=\gamma j, k=j)} \tag{7.17}$$

$$|j, m\rangle \longmapsto |\gamma j, j; j, m\rangle \tag{7.18}$$

This map extends immediately to a map from $SU(2)$ group elements to $SL(2,\mathbb{C})$ elements. Recall that, given $h \in SU(2)$, a functional $f(h)$ decomposes like (5.10). But the Wigner matrix $D^j_{mn}(h)$ can be interpreted as a map from $\mathcal{H}_j$ to itself that sends $|j,m\rangle$ into $|j,n\rangle$. Therefore, $\mathcal{Y}_\gamma$ acts on it as:

$$\mathcal{Y}_\gamma : \qquad \mathcal{H}_j \otimes \mathcal{H}_j \longrightarrow \mathcal{V}^{(\gamma j, j)} \otimes \mathcal{V}^{(\gamma j, j)} \tag{7.19}$$

$$D^j_{mn}(h) = |j,m\rangle \otimes |j,n\rangle \longmapsto \mathcal{D}^{(\gamma j, j)}_{jm\ jn}(g) \equiv |\gamma j, j; j, m\rangle \otimes |\gamma j, j; j, n\rangle \tag{7.20}$$

where $\mathcal{D}^{(\gamma j, j)}_{jm\ jn}(g)$ is the corresponding matrix element of the $(\gamma j, j)$ irreducible representation of $g \in SL(2,\mathbb{C})$. Finally, the map between functionals of group elements is:

$$\mathcal{Y}_\gamma : \qquad \mathcal{L}_2[SU(2)] \longrightarrow \mathcal{F}[SL(2,\mathbb{C})] \tag{7.21}$$

$$\psi(h) = \sum_{jmn} c_{jmn} D^j_{mn}(h) \longmapsto \psi(g) = \sum_{jmn} c_{jmn} \mathcal{D}^{(\gamma j, j)}_{jm\ jn}(g) \ . \tag{7.22}$$

As anticipated, we have a map from $SU(2)$ spin networks to $SL(2,\mathbb{C})$ spin networks. Notice that the image of $\mathcal{L}_2[SU(2)]$ under $\mathcal{Y}_\gamma$ is a space of functionals on $SL(2,\mathbb{C})$ which are not square-integrable, but form a more general linear space. The scalar product carried by this space is the one determined by the $SU(2)$ Haar measure: it is $SU(2)$ invariant and not $SL(2,\mathbb{C})$ invariant (it is only covariant under $SL(2,\mathbb{C})$). This happens because the correspondence to $SU(2)$ spin networks is made thanks to the linear simplicity condition written in the form dictated by the Lorentz frame where the time gauge is preserved on the boundary (we remind the reader that the linear simplicity condition could be given in a covariant form). Picking a Lorentz frame means breaking the $SL(2,\mathbb{C})$ invariance.
The physical states of quantum gravity are those in $\mathcal{L}_2[SU(2)]$ or equivalently their image under $\mathcal{Y}_\gamma$. The latters satisfy the simplicity constraint in weak sense, i.e. :

$$\langle \mathcal{Y}_\gamma \psi | \vec{K} - \gamma \vec{L} | \mathcal{Y}_\gamma \phi \rangle = 0 \qquad \forall \psi, \phi \in \mathcal{L}_2[SU(2)] \tag{7.23}$$

[1] Actually, this solution satisfies the linear simplicity condition only in the classical limit, but there is also a solution that satisfies *exactly* our request : $p = \gamma k(k+1)$ and $k = j$. We choose the first.

We have found a bridge of consistency between the canonical truncated LQG and the covariant counterpart: we can still work with $SU(2)$ spin networks and their geometrical operators, and use $\mathcal{Y}_\gamma$ whenever we are requested to make the $SL(2,\mathbb{C})$ local symmetry explicit. Basically, we can define the boundary graph states as done in §6.6. We will see soon that the map $\mathcal{Y}_\gamma$ is the core ingredient of quantum gravity dynamics. It depends on the Einstein-Hilbert action and codes the way $SU(2)$ states transform under $SL(2,\mathbb{C})$ transformations in the theory. Such transformations, in turn, code the dynamical evolution of the quantum states of space.

Chapter 8

Transition amplitude

In this chapter we briefly review the general definition of a transition amplitude between quantum gravity state. The transition amplitude is the core ingredient for studying dynamical processes in quantum theories: it computes the amplitude for an initial state to evolve into a final state. The words *initial* and *final*, however, are meaningful if we consider time as the real parameter with respect to which the states of a quantum theory evolve. Being GR a generally covariant theory, we have seen in §6.3 that we cannot base the dynamics of the gravitational field on these words, since time becomes just a coordinate, treated at the same level as the others and evolving in a pure gauge parameter. In what follows, we are going to deal with this new feature.

8.1 Definition in quantum mechanics

A general quantum theory is in principle defined by a Hilbert space $\mathcal{H}$, the operators $\hat{q}$ and $\hat{p}$ corresponding to the classical canonical variables, the time variable t and the generator of time evolution, namely the hamiltonian $\hat{H}$.
Let us now consider a maximally commuting set of operators $\{\hat{x}\}$, in the sense of Dirac, and the basis in $\mathcal{H}$ that diagonalizes these operators: $\hat{x}\,|x\rangle = x\,|x\rangle$. Then the transition amplitude is defined by

$$W(x,t,x',t') = \langle x'|e^{-\frac{i}{\hbar}\hat{H}(t-t')}|x\rangle \quad , \tag{8.1}$$

i.e. the transition amplitude is given by the matrix elements of the evolution operator $U(t'-t) = e^{-\frac{i}{\hbar}\hat{H}(t-t')}$. It is a function of the initial and final states, and computes the amplitude of the pair. As well known, this definition can be rewritten through a path integral *à la* Feynman:

$$W(x,t,x',t') = \int \mathcal{D}[x(t)]e^{\frac{i}{\hbar}S[x]} \tag{8.2}$$

providing the intuition of the quantum transition between two states as a "sum over paths". When $\hbar$ can be considered small, we can use the saddle point approximation to show that

$$W(x,t,x',t') \sim e^{\frac{i}{\hbar}S[x(x,t,x',t')]} \quad , \tag{8.3}$$

where $S[x(x,t,x',t')]$ is known as the *Hamilton function*, which is the value of the classical action computed on the classical trajectory that links (x,t) with (x',t'). In other words, the Hamilton function is the least value of the action between the initial and final states, and we know that the action has a minimum in the physical trajectory that is solution of the equations of motion. Just like the transition amplitude, it is a function of the only initial and final states, and thus can be considered the best classical counterpart of W. How can we adapt these objects to a theory where time is a dependent variable?

8.2 Quantum theory of covariant systems

In classical covariant systems, the lagrangian variables q and the time t are treated on equal footing, and we refer to them as *partial observables*. Physics is about the relative evolution of partial observables, and the term "partial" stresses the fact that they include the t variable, and allows the distinction from the functions on the configuration space or phase space, which we refer to as *observables*. In a general sense, in covariant systems the observables are those quantities that, together with being measured with some apparatus, can be predicted from the knowledge of the *relations* among partial observables. The number of partial observables is always greater than the number of observables, which in turn are the physical degrees of freedom of a theory.
The space of partial observables is called the *extended configuration space* $\mathcal{C}_{ext} = \mathcal{C} \times \mathbb{R}$. We denote $x \in \mathcal{C}_{ext}$ a generic point in this space. In the example discussed, $x = (q,t)$. Given the momenta p_x conjugate to the set x of partial observables, we have already pointed out that the hamiltonian vanishes and the conjugate pair (x, p_x) satisfies a constraint $C(x, p_x) = 0$ that encodes the dynamical information.
Moving to the quantum theory of covariant systems, we understand that we need:

- a Hilbert space $\mathcal{K}$ called the *kinematical* Hilbert space, where self-adjoint operators $\hat{x}$ and $\hat{p}_x$ corresponding to the classical variables (x, p_x) are defined;
- a *constraint operator* $\hat{C}$, whose classical limit is the constraint $C(x, p_x)$.

By the way, in canonical LQG the constraint operator is nothing but the hamiltonian constraint $\hat{H}$. The states that are annihilated by this operators belong to the physical state $\mathcal{H}_{phys}$.

In order to gather the space of physical states, we need the operator $\hat{C}$ to have zero in its spectrum, be it discrete or continuous (the latter case requires more refined mathematics). Then, the subspace of $\mathcal{K}$ formed by the states ψ that satisfy the equation

$$\hat{C}\psi = 0 \tag{8.4}$$

is the Hilbert space of physical states $\mathcal{H}_{phys}$. This last equation, known as *Wheeler-deWitt equation*, generalizes the Schrödinger equation to parametrized covariant systems[1]. Equation (8.4) defines the transition amplitude in the following way: we can build a projective map P that sends $\mathcal{K}$ to $\mathcal{H}_{phys}$ by orthogonal projection. the transition amplitude is defined by the matrix elements of P in the basis that diagonalizes the operators $\hat{x}$:

$$W(x, x') = \langle x'|\hat{P}|x\rangle \,. \tag{8.5}$$

Formally, we can rewrite the definition as

$$W(x, x') = \langle x'|\delta(\hat{C})|x\rangle \sim \int_{-\infty}^{+\infty} d\tau \, \langle x'|e^{i\tau\hat{C}}|x\rangle \quad , \tag{8.6}$$

and since C generates evolution in the parameter τ, we can translate this into the Feynman language in the form:

$$W(x, x') = \int_x^{x'} \mathcal{D}[x(\tau)] e^{\frac{i}{\hbar} S[x]} \quad , \tag{8.7}$$

where the integration is over all paths that start at x and end at x', for any parametrization of these. Since the action does not depend on the parametrization, the integration includes a large gauge redundancy that must be factored out.
Equation (8.5) can be interpreted as follows: the unphysical state $|x\rangle = |q,t\rangle$ in $\mathcal{K}$ is a delta-function centered on the point $x = (q,t)$. It represents the system being in the configuration q

[1] For instance, a system with the lagrangian (6.17) shows the constraint $C = p_t + \frac{p_q^2}{2m} + V(q) = p_t + H(p_q, q) = 0$. The reader may easily recognize a Schrödinger equation.

at time t. The operator $\hat{P}$ projects this kinematical state down to a solution of the Schrödinger equation, namely to a wave function which solves the Schrödinger equation and is centered in the point x. The contraction with the state $\langle x'| = \langle q', t'|$ gives the value of this wave function at point x', namely the amplitude for the system that was in x to be at x'. This is the *physical* overlap of the *kinematical* state representing the event "system in x" on the *kinematical* state representing the event "system in x' ".
Such covariant formalism treats q and t on equal footing, and is unavoidable when studying systems whose dynamics is fully relational, instead of being marked by an independent evolution parameter. Relativity teaches us that there is no absolute time to observe: it is always measured with respect to some arbitrary variable. A general quantum theory of relativity must be written in this covariant fashion.

8.3 Boundary formalism

The formalism for a quantum field theory of gravity has been championed and developed by Robert Oeckl [23], [24]. It supplies a counterpart for the "initial" and "final" notions through the notion of *boundary*, and for this it is called *boundary formalism*.
First of all, let us try to define what is a *process* in quantum mechanics [29] a process is what happens to a system between an initial and a final interaction with the measuring apparatus. The dynamics of a quantum system between the two interactions is a probabilistic esteem of the finite portion of its trajectory enclosed in the *boundaries* of the process, i.e. the previous and following measures. Thus, the transition amplitude W determines the probability for a set of boundary values to be grouped together, and we can further generalize this approach.
In field theory, when we refer to a *finite* portion of the trajectory of a system we mean that such portion is finite in time as well as in space. We therefore assume the process to take place in a compact region of spacetime $\mathcal{M}$.
The argument proposed by Oeckl is that the transition amplitude is a function of the field values on initial and final spacelike surfaces, but also on the "sides" of $\mathcal{M}$: eventual timelike surfaces that bound the box. In other words, W is a function of the field on the *entire* boundary of the spacetime region $\mathcal{M}$. Formally, W can be expressed as the Feynman path integral of the field in $\mathcal{M}$, with fixed values on the boundary $\Sigma = \partial\mathcal{M}$. The quantum state of the field on the entire boundary is an element of the boundary Hilbert space $\mathcal{H}_\Sigma$, so that the transition amplitude W is a linear functional $\langle W|$ on this space, whose components along the boundary states are the amplitudes for a process to occurr with those states as boundary.
As an example, in the non relativistic case the boundary Hilbert space can be identified with the tensor product of the initial and final Hilbert spaces $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_t$, and the transition amplitude is:

$$\langle W_t|\psi \otimes \phi\rangle \equiv \langle \phi|e^{-\frac{i}{\hbar}\hat{H}t}|\psi\rangle \quad , \quad |\psi \otimes \phi\rangle \in \mathcal{H} \ . \tag{8.8}$$

Now, given a field theory on a fixed spacetime, W depends on the shape and geometry of Σ, for instance on the time elapsed between its initial and final sides, as we can see from this last equation. But it is not so for gravity: here, the transition amplitude cannot be a function over the geometrical features of a boundary "environment" hosting the process, because such geometrical features are already determined by the state of the gravitational field on the boundary Σ. If we call the state $|\Psi\rangle$, a transition amplitude $\langle W|\Psi\rangle$ depends on $|\Psi\rangle$ and nothing else, since $|\Psi\rangle$ includes the entire relevant geometrical information about the boundary.
We now (almost) fully understand the reason behind using boundary graph variables in LQG: in quantum gravity, the dynamics of a process is captured by a transition amplitude W that is a function uniquely of the quantum state of the surface Σ bounding the process. Intuitively, W is a *sum over geometries* of a finite bulk region $\mathcal{M}$ bounded by Σ, i.e. over all the geometries of $\mathcal{M}$

that are compatible with the fixed boundary state $|\Psi\rangle$ of the geometry of Σ.

$$\langle W|\Psi\rangle = \int_{\Sigma(|\Psi\rangle)} dg_{\mathcal{M}}\ e^{\frac{i}{\hbar}S[g]} \ . \tag{8.9}$$

Finally, we understand that the spacetime region *is* a process: it is actually a Feynman sum of everything that can happen inside its boundary, i.e. the state.

Quantum mechanics		Quantum gravity
Process	$\longleftrightarrow$	Spacetime finite region
State	$\longleftrightarrow$	Boundary (spacelike) region

Table 8.1: Boundary formalism

8.4 3d LQG amplitude

Before discussing the transition amplitude in four dimensions, it is a useful exercise to start with the 3d case. We already have all the machinery to define the quantum state of a two-complex of a triangulation and the quantum state of the boundary graph dual to the triangulation induced on the boundary.
Let us fix a triangulation Δ of a 3d region and its related two-complex. Knowing that the continuous variables in the 3d case are the triad e^i and the connection ω, we assign in the bulk:

- a $SU(2)$ group element U_e to each edge e of the two-complex (dual to a triangle), defined as the holonomy of ω along the edge;
- a $su(2)$ algebra element L^i_f for each face f of the two-complex, defined as the smearing of the triad 1-form e^i along the segment s_f dual to f. It is therefore a vector in the adjoint representation of the $SU(2)$ algebra, $L_f = L^i_f J_i$.

The states on the boundary graph, which we assume to have L links and N (3-valent) nodes, will be functions in $\mathcal{L}[SU(2)^L/SU(2)^N]$ of the form (5.32).
Given a face f bounded by the edges $e_1, \dots, e_n$, we can multiply the group elements U_e around it and obtain a single group element U_f associated with the face itself:

$$U_f = U_{e_1} \dots U_{e_n} \ . \tag{8.10}$$

This is the holonomy of the connection going around the segment s_f dual to the face f. If U_f is different from the identity, $U_f \neq \mathbb{I}$, then there is curvature, and we can associate this curvature with the segment, as in Regge calculus (remember that the curvature is a property localized in the $(d-2)$-dimensional hinges: in the 3d case, it is localized in the segments). The discretized action reads:

$$S = \frac{1}{8\pi G} \sum_f Tr[L_f U_f] \tag{8.11}$$

where $L_f \in su(2)$ and $U_f \in SU(2)$. Notice the similarity with the discrete Regge action. On the boundary, we must close the perimeter of the facees in order to write the quantity U_f for the faces that end on the boundary, and write the related contributions to the action. Therefore, there must also be group quantities U_l associated to the links of the boundary graph. In other words, the links of the boundary graph are treated just like the edges in the bulk, and are assigned group variables as well.
Now, from what said in the previous section, the transition amplitude is a function of the state of the boundary graph. If such state is $|\{U_l\}\rangle$, we have:

$$W_\Delta(\{U_l\}) = \langle W|\{U_l\}\rangle \ . \tag{8.12}$$

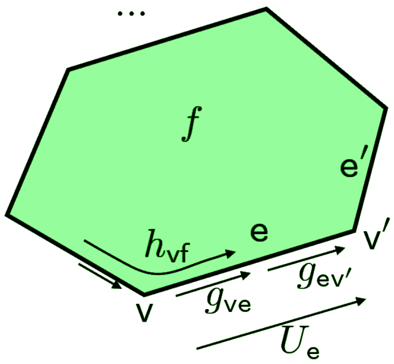
To compute it, we use the Feynman path integral, interpreting the amplitude as the sum over all the geometries in the 3d bulk that can occurr once we have fixed the graph state on the 2d boundary. Such geometries are weighted by the exponential of the action, that is:

$$W_\Delta(\{U_l\}) = N \int dU_e \int dL_f e^{\frac{i}{8\pi\hbar G}\sum_f Tr[U_f L_f]} \ , \tag{8.13}$$

where N is a renormalization factor, where we will absorb various constant contributions, and the integration measure is meant as the product of all the integration measures of each edge. The integral over the momenta is easy to perform, since it is an integral of an exponential which gives a delta function. Therefore we obtain:

$$W_\Delta(\{U_l\}) = N \int dU_e \prod_f \delta(U_f) \tag{8.14}$$

where the delta function is over $SU(2)$.



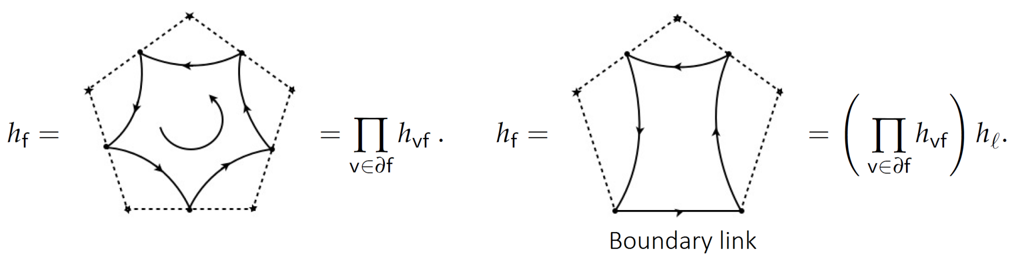
Now we introduce two variables per each edge e, namely $U_e = g_{ve}g_{ev'}$, where v is the source vertex and v' the target vertex of e (remember that the edges always carry an orientation induced by the triangulation). Each variable $g_{ev} = g_{ve}^{-1}$ is associated with the couple vertex-edge.
Then we can rewrite each face element in this fashion and get:

$$W_\Delta(\{U_l\}) = N \int dg_{ve} \prod_f \delta(g_{ve}g_{ev'}g_{v'e'}g_{e'v''}\dots) \ . \tag{8.15}$$

Now imagine to regroup the g_{ev} variables around each vertex $v(f)$ belonging to f, by defining $h_{v(f)} = g_{ev}g_{ve'}$, where e is the edge going into v and e' the one that emerges from v. Than we can plug an integral over $h_{v(f)}$ and a delta function fixing its definition:

$$W_\Delta(\{U_l\}) = N \int dh_{v(f)}dg_{ve} \prod_f \delta(h_{v(f)}h_{v'(f)}\dots) \prod_{v(f)} \delta(g_{e'v}g_{ve}h_{v(f)}) \ . \tag{8.16}$$

The integration in $dh_{v(f)}$ fixes $g_{e'v}g_{ve}h_{v(f)} = \mathbb{I} \Rightarrow h_{v(f)} = g_{ev}g_{ve'}$, as we have requested. If we call $h_f = \prod_{v(f)} h_{v(f)}$ the product of the vertex elements around the face f, including those faces bounded by links, as in figure:



and define the *vertex amplitude* as the amplitude related to the faces sharing the vertex v:

$$A_v(h_{v(f)}) \equiv \int dg_{ve} \prod_f \delta(g_{e'v}g_{ve}h_{v(f)}) \ , \tag{8.17}$$

we get, renaming U_l as h_l for uniformity:

$$W(\{h_l\}) = \int dh_{v(f)} \prod_f \delta(h_f) \prod_v A_v(h_{v(f)}) \ . \tag{8.18}$$

The $SU(2)$ integrals in the vertex amplitude are four in total: one group element per each of the four edges coming out of the vertex. This is a bit redundant, because a moment of reflection shows that if we perform three of these integrals, the result is invariant under the last remaining integration variable- Therefore, we can drop one of these integrations without affecting the result. We replace the integration measure dg_{ve} with:

$$\int_{SU(2)^4} dg'_{ve} \equiv \int_{SU(2)^3} dg_{ve_1} \dots dg_{ve_3} \ . \tag{8.19}$$

The last step is to expand the δ into $SU(2)$ representations. This decomposition is easily understood if one thinks of the correspondent decomposition in $U(1)$:

$$\delta(\phi) = \frac{1}{2\pi} \sum_n e^{in\phi} \tag{8.20}$$

where the inequivalent irreducible representations are labelled by an integer and are 1-dimensional element, i.e. the exponentials. In the $SU(2)$ reresentations, the delta reads:

$$\delta(U) = \sum_j (2j+1) Tr D^j(U) \ , \tag{8.21}$$

so that we finally have

$$W(\{h_l\}) = \int_{SU(2)} dh_{v(f)} \prod_f \delta(h_f) \prod_v A_v(h_{v(f)}) \tag{8.22}$$

$$A_v(h_{v(f)}) = \sum_{j_f} \int_{SU(2)} dg'_{ve} \prod_f (2j_f+1) Tr_{j_f}[g_{e'v} g_{ve} h_{v(f)}] \ . \tag{8.23}$$

Now, pick a vertex v in the two-complex of the 3d bulk triangulation, and imagine drawing a small sphere around the vertex: the intersection between this sphere and the two-complex is the boundary graph of the vertex, Γ_v, with four nodes and six links. Therefore the vertex amplitude A_v related to v is a function of the states in:

$$\mathcal{H}_{\Gamma_v} = \mathcal{L}_2[SU(2)^6/SU(2)^4]_{\Gamma_v} \ , \tag{8.24}$$

or, in alternative, the projection on the $SU(2)$ invariant part of the state on the boundary graph of the vertex, that is

$$A_v = (P_{SU(2)} \psi_{\Gamma_v})(\mathbb{I}) \ , \tag{8.25}$$

where ψ_{Γ_v} is the boundary state of the vertex v, $P_{SU(2)}$ is the projection on its locally $SU(2)$ invariant component and $(\mathbb{I})$ indicates the evaluation of the spin network boundary state on the identity $h_l = \mathbb{I}$.

8.5 4d LQG amplitude

Now that we have the structure for building transition amplitudes in 3-dimensional space, the extension to the 4d case is almost direct. Indeed, the kinematical Hilbert space of state in the boundary graph is just the same as in the 3d case, with the difference that now the boundary graph is made of 4-valent nodes. Equation (8.22) applies unchanged to the 4d world:

$$W(\{h_l\}) = \int_{SU(2)} dh_{v(f)} \prod_f \delta(h_f) \prod_v A_v(h_{v(f)}) \ . \tag{8.26}$$

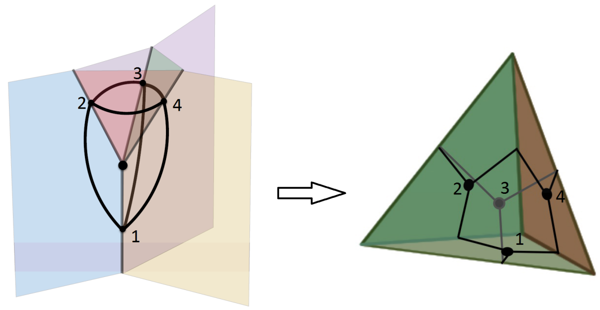


Figure 8.1: The vertex boundary graph Γ_v in 3d. The vertex at the center of the left figure is dual to the tetrahedron in the right figure.

What is different is the vertex amplitude, because it is related to the vertex (it is a function of one $SU(2)$ variable per each face around the vertex) and it is naturally built to be invariant under the discrete gauge symmetry localized at the vertex. The vertex amplitude in equation (8.23) is only invariant under $SU(2)$, but does not know anything about $SL(2,\mathbb{C})$. In order to obtain an amplitude which is $SL(2,\mathbb{C})$ invariant, we must replace the $SU(2)$ integrals in (8.23) with $SL(2,\mathbb{C})$ integrals, and somehow map the $SU(2)$ group elements into $SL(2,\mathbb{C})$ ones.
We do have the tool for this: it is the simplicity map. $\mathcal{Y}_\gamma$ can be used to map a function of $SU(2)$ variables into a function of $SL(2,\mathbb{C})$ variables. These can then be projected on $SL(2,\mathbb{C})$ locally invariant states and evaluated: in the group representation, A_v has the explicit form

$$A_v(h_{v(f)}) = A_v(h_{v(f)}) = \sum_{j_f} \int_{SL(2,\mathbb{C})} dg'_{ve} \prod_f (2j_f+1) Tr_{j_f}[\mathcal{Y}_\gamma^\dagger g_{e'v} g_{ve} \mathcal{Y}_\gamma h_{v(f)}] \tag{8.27}$$

where the meaning of the trace notation is:

$$Tr_{j_f}[\mathcal{Y}_\gamma^\dagger g \mathcal{Y}_\gamma h] = Tr[\mathcal{Y}_\gamma^\dagger \mathcal{D}^{(\gamma j,j)}_{jm\ jn}(g) \mathcal{Y}_\gamma D^j_{mn}(h)] = \sum_{mn} \mathcal{D}^{(\gamma j,j)}_{jm\ jn}(g) D^j_{mn}(h) \ . \tag{8.28}$$

The vertex amplitude A_v related to the vertex v is a function of one $SU(2)$ variable per face around the vertex. Now, as in the 3d case, imagine drawing small sphere surrounding the vertex: the intersection between this sphere and the two-complex is a boundary graph Γ_v of the vertex v, with ten links and five nodes.

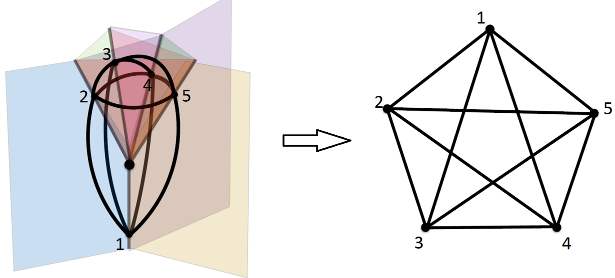


Figure 8.2: The vertex boundary graph Γ_v in 4d.

The vertex amplitude is then a function of the states in:

$$\mathcal{H}_{\Gamma_v} = \mathcal{L}_2[SU(2)^{10}/SU(2)^4]_{\Gamma_v} \ , \tag{8.29}$$

or, in the same fashion as the 3d case, the projection on the $SL(2,\mathbb{C})$ invariant part of the state on the boundary graph of the vertex, that is:

$$A_v = \left(P_{SL(2,\mathbb{C})}\mathcal{Y}_\gamma \psi_{\Gamma_v}\right)(\mathbb{I}) \ , \tag{8.30}$$

where $P_{SL(2,\mathbb{C})}$ is a projector on the $SL(2,\mathbb{C})$ locally invariant subspace and $(\mathbb{I})$ indicates the evaluation of the spin network boundary state on the identity $h_l = \mathbb{I}$.
In (8.27), the prime in the $SL(2,\mathbb{C})$ integrations indicates that we do not integrate in the five group elements related to the five faces around each vertex, but only in four of them. The result of the integration turns out to be independent on the fifth. Moreover, this procedure is required because the integration on the last group element would give a divergence, being $SL(2,\mathbb{C})$ non compact.
The way we have written the transition amplitude reflects:

- the *superposition principle* of quantum mechanics, for which amplitudes are obtained by summing the amplitudes of individual stories (in quantum mechanics, this sum is expressed by Feynman's sum over paths);
- the *locality principle*, for which the amplitude of a process is the product of the individual local amplitudes (A_v) associated with separated regions in spacetime (in standard quantum field theory, this product is expressed as the exponential of an integral over spacetime points).
- the *local Lorentz invariance* in each vertex, as can be seen by the presence of the projector $P_{SL(2,\mathbb{C})}$ on the locally invariant component of the boundary state of the vertex.

8.6 Continuum limit

The equations above define a transition amplitude determined by a given two-complex $\mathcal{C}$ dual to a given triangulation. This means that the theory has a finite number of degrees of freedom, since we start with a truncation of classical general relativity, which is a theory with an infinite number of degrees of freedom. The full theory is better approximated by choosing increasingly refined two-complexes $\mathcal{C}$ and thus increasingly refined graphs $\Gamma = \partial\mathcal{C}$, towards the *continuum limit*.
The refinement is performed in the following way: Let Γ' be a subgraph of Γ, namely a graph formed by a subset of the nodes and links of Γ. Then it is immediate to see that there is a subspace $\mathcal{H}_{\Gamma'} \subset \mathcal{H}_\Gamma$ which is precisely isomorphic to the loop gravity Hilbert space of the graph Γ': indeed this is formed by all states $\psi(\{U_l\}) \in \mathcal{H}_\Gamma$ which are independent of the group elements U_l associated with the links l that are in Γ but not in Γ'. Equivalently, $\mathcal{H}_{\Gamma'}$ is the linear span of the spin-network states in $\mathcal{H}_\Gamma$ characterized by $j_l = 0$ for any link l that is in Γ but not in Γ'.
Therefore if we define the theory on Γ, we have at our disposal a subset of states that captures the theory defined on the smaller graph Γ'. In this sense, the step from Γ' to Γ is a refinement of the theory.
One can define a Hilbert space $\mathcal{H}$ that contains all $\mathcal{H}_\Gamma$'s by using projective limit techniques. This is in close similarity with the space of N or fewer particles, $\mathcal{H}_N$, subspace of the Fock space. The space $\mathcal{H}_\Gamma$ where Γ has N nodes, is the LQG counterpart of $\mathcal{H}_N$. In the formal limit $N \to \infty$, the space $\mathcal{H}$ is the counterpart of the Fock space, and they are both separable.
This refinement procedure also applies to transition amplitudes. Again, we can formally define the transition amplitude of a refined boundary graph, and take the limit where the refinement is taken to an infinite degree of detail. As far as physics is concerned , the theory is useful to the extent of a subclass of graphs and two-complexes, sufficient to capture the relevant physics.
This procedure is a common one in quantum field theory. Concrete physical calculations are mostly performed on *finite* lattices in lattice QCD, and at a finite order of perturbation theory in QED.

In the first case, a finite lattice means that we choose a lattice with a finite number N of vertices. In QED, a finite order means that all Feynman graphs considered have at most a finite number of particles N. The fact that arbitrarily high momenta can enter a single Feynman graph means that the graph involves an infinite number of modes, but contains in fact a finite number of degrees of freedom, i.e. a finite number of relevant excitations of these modes, namely a finite number of particles at play. In both cases (lattice QCD and QED) one formally considers the limit $N \to \infty$, but concretely one always computes at finite N, with some arguments that the rest be smaller and negligible.

In LQG, the two pictures of QED and lattice QCD merge: the two-complex is at the same time a Feynman graph where the nodes are quanta of space interacting among each others, and a lattice of quanta, i.e. a collection of chunks of spacetime. The coincidence is made by seeing the gravitational field as a dynamical and quantized field, carrying a quantum geometry glued in some specific way dictated by its "history", namely its graph/lattice. In other words, spacetime is a dynamical patchwork of regions adjacent to one another at their boundaries and whose reciprocal interaction is a relation of contiguity, i.e. of being next to one another. Under this light, the bond between locality and interactions is explicit: interactions are local, in the sense that they require spacetime contiguity, but it is also true, in the opposite sense, that the only way to ascertain that two objects are contiguous in spacetime is to have them interacting. In the case of spacetime, the objects that interact are chunks of spacetime itself and *interacting* is to read as *gluing together*.

The analogy with QED is strongly reinforced by the fact that a transition amplitude in LQG can actually be concretely obtained as a term in a Feynman expansion of an auxiliary field theory, called *group field theory*. We refer the reader to the growing literature on the topic.

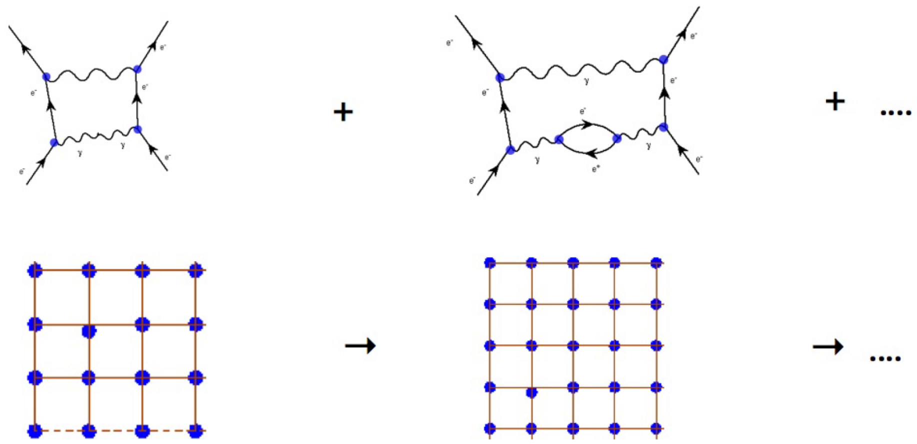


Figure 8.3: QED Feynman graphs and lattice QCD lattices. The two pictures converge in LQG.

Chapter 9

Preliminary study of the boundary state

This chapter concerns how the problem of finding a quantum-nature transition amplitude for the BH-WH bounce, which we left as a proposal in §2.8, can be at least in principle addressed in the framework of LQG that we have shortly presented in the previous part.
In the language of boundary formalism specialized to a quantum theory of gravity, we want to find the amplitude for the process (i.e. the quantum spacetime region, which we called III) between the states of geometry that bound it, namely the upper and lower boundary 3d spacelike surfaces of III. What follows is a partial list of what should be done to achieve this purpose.

9.1 Equation for the boundary surface

First of all, we have to explicitly choose the two time-symmetric surfaces that are to bound the quantum region. In §2.5 we specified how such surfaces should be like: focussing on the lower one (in the black hole patch), it must join the event in the trajectory of the null collapsing shell where quantum effects become non negligible for the the first (proper) time of the shell, Ξ, and the event when they are non negligible at the largest radius, which lies outside the horizon, Δ. Apart from this condition, in addition to it being spacelike, we do not have sketched any other restriction. So the first question we have posed ourselves is: is it really necessary to compute the quantum transition amplitude for the entire (closed) boundary surface? In fact no, because if the region extends to radii slightly larger than the apparent horizons, it is in that area that the semiclassical limit holds most, because it carries the lowest curvature of the entire region III. If a quantum bounce is to happen, we can study what is its probability where it is least likely to do so. Such probability will be a lower limit to the amplitude of the hole region. If the bounce proves to be reasonably expectable at the largest radii covered by III, then the hypothesis of a transition involving the entire region will be nothing but reinforced.
Following this argument, we reduce the problem to that of finding a suitable equation for the surface $\Xi - \Delta$, because it is this surface that punctures the apparent horizon and reaches the lowest Riemannian curvatures. This surface carries, in turn, the only limitation of being spacelike, since we cannot arrange suitable matching conditions between region III and the other semiclassical regions, I and II along $\Xi - \Delta$, lacking the notion of a metric in region III that would be the semiclassical limit of its quantum process. Therefore, we are not prevented from conveniently deforming $\Xi - \Delta$, for example connecting Δ with an event sharing the same radius as Ξ, but a different Schwarzschild time, which we call Γ, and then joining Γ with Ξ by sliding along $r = \epsilon$.
Now, the problem is to find a suitable equation for a spacelike surface connecting Γ and Δ. Three

arguments suggest us that we should ask this surface to cover a Planck-sized proper 4-distance between the two events:

- it seems reasonable to demand extreme vicinity (in terms of spatial-temporal distance) of the event Δ to the region where quantum effects are predominant(radii $r < \epsilon$), in such a way that the quantum region III may largely belong to the Cauchy development of the portion $r < \epsilon$ of any Cauchy surface of III;
- since the surface $\Gamma - \Delta$ bounds the region where a meaningful classical limit still holds and features the largest curvature, the least we can make in order to increase the amplitude for a quantum bounce process is to make the extension of this region comparable to the typical size of quantum gravity, i.e. to the Planck length, in the same fashion where quantum mechanics diverges from classical results when the action becomes comparable to the (reduced) Planck constant;
- we want to start with an easy model, where we can grasp the most relevant physical properties within few variables. This is possible, keeping in mind the formalism of LQG, if we can describe the surface with a very simple triangulation which, still, can be considered a good approximation of the continuum limit. For example, since the Riemannian curvature increases enormously within a relatively short interval of radii, it seems wise to choose a surface whose triangulation keeps track of this sudden change inside few chunks of spacetime, instead than in the details of a complicated collection, which would be hardly controllable.

The 3d hypersurface $\Sigma \equiv \Gamma - \Delta$, to which we restrict our attention, is characterized by the following: it is topologically a hypercylinder (geometrically, if we suppress the θ dimension, a 3d truncated cone), that is of the form $\Sigma \sim S^2 \times C$ where C is some 1-dimensional sub-manifold (curve) with topology of a finite interval of $\mathbb{R}$ and S^2 is the 2-sphere. The large end of the cylinder is the 2-sphere S_Δ that lies at the event Δ. The small end is the 2-sphere S_Γ taken to lie at the event Γ somewhere on the surface $r = \epsilon$. The analytic form of the side C of the cylinder is not specified for our purposes: it will only be important that it be a curve of proper length of the order of the Planck length l_P. We calculate a candidate curve C here for the lower Σ, lying in the BH region. First of all, we need a metric to define a line element in order to perform a factual computation of C. The surface Σ is a part of the border between regions II and III. Therefore, we can use the Schwarzschild metric of region II to describe it mathematically. A simple choice for the functional form of C can be made when using ingoing Eddington-Finkelstein coordinates, in which the line element reads:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \ . \tag{9.1}$$

The idea is that null ingoing curves are given, in Eddington-Finkelstein coordinates, by $v = const$, where $v = \tilde{t} + r$, r is the physical radius and

$$\tilde{t} = t + 2M \log \left| \frac{r}{2M} - 1 \right| \ . \tag{9.2}$$

Thus, a general null ingoing trajectory is given, in coordinates $(\tilde{t}, r)$[1], by $\tilde{t} = -r + const$, and the related worldline in a graph $\tilde{t} - r$ is a straight line at -45 degrees with respect to the r-axis. Since we want C to be spacelike but with a very small proper length, we imagine to tilt this line to an angle slightly larger than -45 degrees, namely $\tilde{t} = -r + Ar + const$, where A is a positive dimensionless constant of small value. Restoring $v = \tilde{t} + r$ we get

$$\Sigma \quad : \quad v(r) = Ar + const \ . \tag{9.3}$$

For simplicity, we can choose $const = 0$, being it just a shift in the time axis.
The reader should not feel disturbed by the fact that the surface Σ is not slightly tilted from

[1]Usually, the substitution performed for the E-F coordinates send the physical radius in the so-called *tortoise radius*, but we do this for the time since we want to keep control over the physical radii.

45 degrees of inclination in the conformal diagram of the bounce, since the gluing of the two Schwarzschild patches inside the Minkowski one implies a strong deformation of non-lightlike trajectories.
Consequently, $dv = Adr$ and the infinitesimal proper length along this curve is given by (all constants included):

$$ds^2 = \left[-\left(1 - \frac{2GM}{c^2 r}\right) A^2 + 2A \right] dr^2 \ . \tag{9.4}$$

The sphere S_Γ has radius $r_\Gamma = \epsilon$, while we assume S_Δ to have radius equal to the most favorable guess in [16] : $r_\Delta = 2M + \delta$; $\delta = M/3$. Therefore we have:

$$r_\Gamma \sim \left(\frac{M}{m_P}\right)^{\frac{1}{3}} l_P \ ; \tag{9.5}$$

$$r_\Delta \sim \frac{7MG}{3c^2} \ . \tag{9.6}$$

The proper length of C is then

$$L_C(A, M) = \int_{r_\Gamma}^{r_\Delta} dr \sqrt{ -\left(1 - \frac{2GM}{c^2 r}\right) A^2 + 2A } \ . \tag{9.7}$$

Although A could be guessed, we want to be as precise as possible since the form of C will be a starting point for an LQG calculation. To do things properly, we should set $L_C \sim l_P$ and, after the integration, solve for A as a function of M. This is more complicated than it seems because the integral yields a not easily reversible function and there are also a lot of constants around, of enormously small and large values which make a brute numerical estimation unreliable. There are a couple of things that we can do to simplify the calculation and convince ourselves that we are doing things correctly. First, we can define a new dimensionless coordinate

$$R = \frac{c^2 r}{GM} \tag{9.8}$$

and express the line element as

$$ds^2 = \left(\frac{GM}{c^2}\right)^2 \left[-\left(1 - \frac{2}{R}\right) A^2 + 2A \right] dR^2 \ . \tag{9.9}$$

Knowing that $l_P = \sqrt{\hbar G/c^3}$ and $m_P = \sqrt{\hbar c/G}$ we can simplify things further. The proper length becomes

$$L_C(A, M) = l_P \left(\frac{M}{m_P}\right) \int_{R_\Gamma}^{R_\Delta} dR \sqrt{ -\left(1 - \frac{2}{R}\right) A^2 + 2A } \ , \tag{9.10}$$

the extremes of integration being now

$$R_\Gamma \sim \left(\frac{m_P}{M}\right)^{\frac{2}{3}} \ ; \tag{9.11}$$

$$R_\Delta \sim \frac{7}{3} \ . \tag{9.12}$$

Now we have a good candidate for a variable with respect to which we can take a reliable series expansion: m_P/M is small for any black hole resulting from a macroscopic object of mass M. Setting $L_C(A, M) \sim l_P$ and setting $w^3 = m_P/M$, we get

$$w^3 = \int_{w^2}^{R_\Delta} dR \sqrt{ -\left(1 - \frac{2}{R}\right) A^2 + 2A } \ . \tag{9.13}$$

The idea then is to try and revert this relation to find $A(w) = A(M)$. The direct integral yields a complicated result which is not easy to handle or expand since we have quantities that vary in different manners. The problem is near R_Γ where the integrand becomes large. We try to identify some suitable small parameter that allows us to expand the right hand side and simplify things. The suitable parameter turns out to be $K \equiv Aw^2$ (both A and w are small). The above equation becomes

$$w^4 = \int_{w^2}^{R_\Delta} dR \sqrt{-\left(\frac{1}{w^2} - \frac{2}{w^2 R}\right) K^2 + 2K} \ . \tag{9.14}$$

Now the factor $-\left(\frac{1}{w^2} - \frac{2}{w^2 R}\right)$ is large close to R_Γ and we may legitimately expand the integrand in powers of K (there is no problem near the horizon for this integral as can be seen from its original form). Using Mathematica we get that the integrand is $\sqrt{2M}$ to the lowest order in M. Happily this does not depend on R. Reinstating $K \to Aw^2$ the equation becomes:

$$w^4 = \sqrt{2Aw^2}(R_\Delta - w^2) \ . \tag{9.15}$$

Solving for A and expanding in powers of w we get

$$A = \frac{w^6}{2R_\Delta^2} \tag{9.16}$$

plus higher $O(w^8)$ terms. Thus we have found that in order for the proper length of the curve defined by $v = Ar + const$ to be of the order of l_P

$$A \sim \left(\frac{m_P}{M}\right)^2 \ . \tag{9.17}$$

In order to check what has been done, we integrate with *Mathematica* equation (9.10) and then replace A with the value we found in (9.16) (we keep R_Δ explicit). The integral is now a function of w and the proper length looks like ($w^3 \equiv (m_P/M)$)

$$L_C = l_P w^{-3} F(w) \tag{9.18}$$

where $F(w)$ is some complicated function of w. Expanding $F(w)$ in powers of w we get:

$$F(w) = w^3 \tag{9.19}$$

plus higher terms $O(w^5)$. Thus the proper length is $L_C = l_P$ with the first correction being of the order of $O(w^2)l_P$. Our curve with this value of A has indeed a proper length $\sim l_P$.
We found that $A \sim (m_P/M)^2$. Now let us insert this back into the curve equation:

$$\Sigma \quad : \quad v(r) = Ar = (m_P/M)^2 r = \left(\frac{l_P c^2}{GM}\right)^2 r \ . \tag{9.20}$$

In turn, the equation for the upper $\Gamma - \Delta$ surface Σ', time reversed of Σ, reads, in the white hole patch with outgoing null Eddington-Finkelstein coordinates:

$$\Sigma' \quad : \quad u(r) = -Ar = -(m_P/M)^2 r = -\left(\frac{l_P c^2}{GM}\right)^2 r \ . \tag{9.21}$$

To check that the two hypercylinders are actually spacelike 3d surfaces, we see that this is true if

$$2 - Af(r) > 0 \ . \tag{9.22}$$

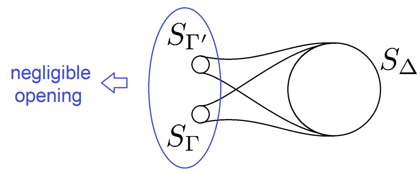
This yields:

$$r > \frac{2MA}{A-2} \tag{9.23}$$

which is always true for macroscopic black/white holes since the r.h.s. is negative and the radius is always positive.

9.2 Triangulation of the hypercylinder

We have understood that the process we want to compute the amplitude of is bounded by two time-symmetric hypercylinders sharing, as the base, the 2-sphere S_Δ and bounded respectively by the smaller 2-spheres S_Γ in the BH region and $S_{\Gamma'}$ in the WH region. Both cylinders are very short, i.e. of Planckian proper height.
We notice that with this choice of simplification, the boundary of the process III is no longer closed. However, a quick look at the shape of the boundary convinces us that we should not worry about the "hole" we have created, because the proper length of a spacelike trajectory at constant radius $r = \epsilon$ between the events Γ and Γ' is short enough to make the hole almost negligible:



The next step prescribed by LQG is the choice of a suitable triangulation of the bulk spacetime portion of III whose related two-complex has, as boundary, a graph that is dual to the boundary triangulation of the two hypercylinders Σ, Σ'. If we use 4-simplexes, they will be triangulated with 3d tetrahedra, so the boundary graph must feature only 4-valent nodes. However, it is not straightforward to derive the boundary graph, since there is no one-to-one correspondence between a triangulation with a given number of chunks and the topology of the object: being the abstract graph a collection of relational information without any care about the embedding of the manifold under study, the same triangulation can refer to manifolds with different topologies. This ambiguity can be solved in two dimensions if we define a notion of ordered matching between the chunks of the triangulation, but is not sufficient in higher dimensions.
As an example, imagine that we want to triangulate a 2d cylinder and a 2d disk with two flat triangles. Regardless of how they are embedded in spacetime (apart from a flat embedding of the disk, which is not triangulable with two flat triangles), they share the same abstract graph. To grasp this, cut the cylinder along its side and "open" it into a plane parallelogram, whose sides along the cut are to be identified, then triangulate it with two triangles. Finally, pick the disk and triangulate it with two triangles. The dual abstract graph you should get is the following for both manifolds:

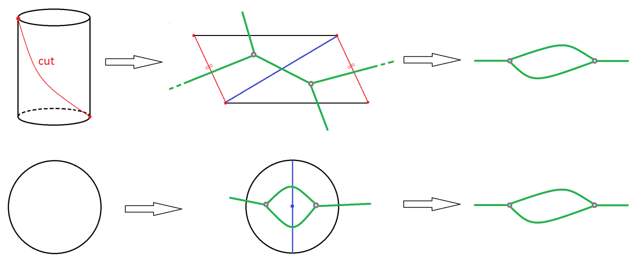


Figure 9.1: Manifold→triangulation→graph sequence for the 2d cylinder and disk. Two objects with different topology may share the same abstract graph. In the 2d case, it is possible to distinguish the two object by assigning an orientation to the triangles: when drawing the dual graph, if we take the orientation into account, the graph of the cylinder shows two links crossing each other, while in the disk's graph the links do not cross. This is no more sufficient in three dimensions, where a 2-sphere and a hypercylinder share the same graph dual to triangulation with two flat tetrahedra.

Apart from this ambiguity, which one could possibly feel authorized to ignore, there is the already mentioned arbitrariness in choosing different triangulations of the same object, depending on how many links and nodes we want in the dual graph, and how many details we want to keep trace of. This happens when we try to triangulate each of the two hypercylinders in three dimensions. Three of the possible resulting abstract graphs are given in figure 9.2. We kept a topological symmetry between the two 2-spheres bounding the hypercylinder, which translates into a symmetric way of triangulating both of them. In addition to the many possibilities connected with a triangulation in

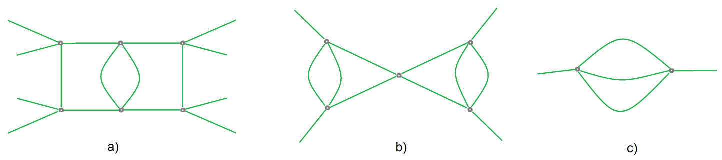
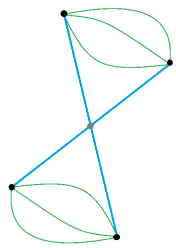


Figure 9.2: Different proposals for the triangulation of *each* of the boundary hypercylinders. a) This is the most refined of the three triangulations, with each 2-sphere triangulated by two tetrahedra (on the sides) and two tetrahedra in between to join them. b) This is a simplified version of the previous graph, with only one tetrahedron in between, fully matched with the two 2-spheres. c) This is a brute simplification of the hypercylinder, when the 2-spheres are triangulated with just one tetrahedron and are directly linked together. It takes advantage of the fact that the hypercylinder has Planckian length, so that there is no need to catch further inner details.

d-simplices, it is also possible to discretize Σ (Σ') with polytopes instead of tetrahedra, so that the dual graph will have nodes of valency higher than four. Indeed, the closure condition that assures rotational invariance in each node can be generalized to every convex polyhedron: such polyhedron is closed if and only if the normals to the faces of every polyhedron sum up to zero. This strategy is particularly useful in situations where we want to condense a large number of degrees of freedom inside very few chunks of spacetime. We leave this to further studies.
However, if we go with the simplest choice, namely option c) in figure, we can set a transition amplitude between two copies of that boundary graph with expansion up to one vertex in the bulk. Since the bulk does not feature a semiclassical approximation in terms of geometry, we are not compelled to refer this vertex to a dual chunk of spacetime. Therefore, we are not bounded by the condition of using a 5-valent vertex. Indeed, we could assume to triangulate the spacetime region II, which instead features a semiclassical geometry, with 4-simplexes and derive the graph of Σ and Σ' as induced by the triangulation of that region, instead of the hypothetical one in region III, which does not necessarily become a classical geometry in the limit of an infinitely refined triangulation, i.e. in the continuum limit. As a result, the vertex in III can feature as many outgoing edges as needed to completely close the graph.
The whole structure is called *spin foam*, and since the spin foam with a 2-valent vertex has already been proved to diverge in a first trial computation, the first foam we would use to attack our problem would be as depicted in figure 9.3.

Figure 9.3: The spin foam whose amplitude one should compute to solve the problem of the BH-WH quantum bounce. The 4-valent vertex in the bulk is not meant to be dual to an effective triangulation of spacetime.



9.3 Coherent states in LQG

From what we have said until now, once we have defined the graph of our process we should assign a group element and an algebra element to each link of the graph, in such a way that the spin network state obeys local invariance from the action of the symmetry group localized at its nodes. Then, we would be able to compute the amplitude for the process with the tools introduced in chapter 8. But we have not told the full story. The reader may have noticed that in our short presentation of LQG there is no trace of how the notions of curvature and of the Riemannian geometry of spacetime play a role when dealing with spin networks and spin foams. The key problem is that our quantum theory of gravity does not enclose the full classical information of the smooth spacetime which we have abandoned in its favor, since the uncertainty principle that comes when implementing a quantum behaviour inside a theory of geometry prevents us from simultaneously fixing all the degrees of freedom carried by a classical geometrical object. We have already noticed this when quantizing a simple geometric structure like a flat tetrahedron. It is not possible to fix, for instance, the four areas of its faces and the dihedral angles formed by two distinct couples of its faces, which together would uniquely determine the shape of the tetrahedron. In the recoupling basis, the maximally commuting set of observables is diagonalized over the the four areas and only *one* of the two chosen dihedral angles. The other dihedral angle is completely fuzzy. Therefore a quantum state of a geometric structure is highly non classical, in the sense that some variables of the intrinsic geometry it describes are maximally quantum-spread. The question arising is: how can we define a quantum state of geometry which is minimally spread around *all* the classical values? The answer to this question is represented by intrinsic and extrinsic coherent states in LQG.
Before briefly explaining intrinsic coherent states, it is a useful example to build coherent states for a single rotating particle. Suppose we want to write a state for which the dispersion of its angular momentum is minimized. If j is the quantum number associated to its total angular momentum, a basis of states in $\mathcal{H}_j$ is $\{|j,m\rangle \;\; , m=-j,\dots,j\}$. The commutation relations defining the theory are given in terms of the angular momentum operators: $\left[L^i,L^j\right]=i\hbar\epsilon^{ij}_k L^k$. Given a state $|\psi\rangle$ we know from Heisenberg's uncertainty principle that, for example:

$$\Delta_{|\psi\rangle}L_x\Delta_{|\psi\rangle}L_y \geq \frac{1}{2}\left|\langle\psi|\left[L_x,L_y\right]|\psi\rangle\right| = \frac{1}{2}\left|\langle\psi|L_z|\psi\rangle\right| \;\; , \tag{9.24}$$

where as usual

$$\Delta_{|\psi\rangle}L_i = \sqrt{\langle\psi|L_i^2|\psi\rangle - \left(\langle\psi|L_i|\psi\rangle\right)^2} \; . \tag{9.25}$$

Thus we cannot have a really classical state in the theory, where all the variables are well defined (i.e. with zero uncertainty), but the best we can do is saturating this relation, namely finding states such that $\Delta L_x \Delta L_y = \frac{1}{2}\left|\langle L_z\rangle\right|$. A state saturating its uncertainty is found to be $|j,j\rangle$. This can be shown as follows: $|j,j\rangle$ is an eigenstate of L_z, therefore L_z carries no uncertainty on $|j,j\rangle$. On the other hand:

$$\langle L_x\rangle = \langle L_y\rangle = 0\,;\; \langle L_x^2\rangle = \langle L_y^2\rangle = \frac{1}{2}\langle L^2 - L_z^2\rangle = \frac{j}{2} \Rightarrow \Delta_{|j,j\rangle}L_x = \Delta_{|j,j\rangle}L_y = \sqrt{\frac{j}{2}} \tag{9.26}$$

and the uncertainty relation between L_x and L_y is saturated. In the limit of large j this state becomes sharp, as we can see below:

$$\frac{\Delta^2_{|j,j\rangle}L_x}{\langle j,j|L^2|j,j\rangle} = \frac{j/2}{j(j+1)} = \frac{1}{2(j+1)} \xrightarrow{j\to\infty} 0 \; . \tag{9.27}$$

Now from this state we want to construct a family of states in $\mathcal{H}_j$ with the same property. This can be easily obtained by rotating $|j,j\rangle$ into an arbitrary direction $\vec{n}$, defining $R_{\vec{n}}$ as the element of $SU(2)$ that rotates the z axis into $\vec{n}$:

$$|j,\vec{n}\rangle = D(R_{\vec{n}})\,|j,j\rangle = \sum_m |j,m\rangle\,\langle j,m|\,D^j(R_{\vec{n}})\,|j,j\rangle = \sum_m D^j_{mj}(R_{\vec{n}})\,|j,m\rangle \tag{9.28}$$

where $D(R_{\vec{n}})$ is the corresponding Wigner matrix in the representation j. The family of states $|j,\vec{n}\rangle$ is labelled by the continuous parameter $\vec{n}$, expressing the axis of rotation whose related angular momentum has eigenvalue j, and all the states are minimally spread around the values of the angular momenta of the axes orthogonal to $\vec{n}$. They are coherent states and form an overcomplete basis of $\mathcal{H}_j$, providing a resolution of the identity:

$$\mathbb{I}_j = \frac{2j+1}{4\pi}\int_{S^2} d^2\vec{n}\,|j,\vec{n}\rangle\langle j,\vec{n}| \quad , \tag{9.29}$$

where the integral is over all normalized 3d vectors, hence over a 2-sphere, with the standard $\mathbb{R}^3$ measure restricted to the unit sphere.
This procedure can be plugged in a very natural way into the spin network states defined over graphs, and leads to the definition of *Livine-Speziale intrinsic coherent states*[30]. As an introductory example, let us consider the state on a 4-valent node n of a graph Γ dual to a 3d triangulation. Calling $j_1 \dots j_4$ the four spin representations attached to the outcoming links, a state in the node belongs to the Hilbert space $\mathcal{K}_n = Inv_{SU(2)}(\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}\otimes\mathcal{H}_{j_4})$. If we pick a coherent state in each of the four Hilbert spaces, we can write a coherent state of the node starting from:

$$|j_1,\vec{n}_1\rangle\otimes|j_2,\vec{n}_2\rangle\otimes|j_3,\vec{n}_3\rangle\otimes|j_4,\vec{n}_4\rangle \tag{9.30}$$

by imposing that the normals $j_1\vec{n}_1 \dots j_4\vec{n}_4$ sum up to zero. This means projecting the above state on its $SU(2)$ invariant part, belonging to $\mathcal{K}_n$, via the projector:

$$P:\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}\otimes\mathcal{H}_{j_4}\to\mathcal{K}_n\equiv Inv_{SU(2)}(\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}\otimes\mathcal{H}_{j_4}) \ . \tag{9.31}$$

This projector can be either explicitly written as

$$P\equiv\sum_k |j_1,\dots,j_4,k_n\rangle\langle j_1,\dots,j_4,k_n| \tag{9.32}$$

in the recoupling basis, or is implemented by group averaging techniques. With the latter procedure, we get:

$$|\,|j_a,\vec{n}_a\rangle = \int_{SU(2)} dh D^{j_1}(h)\,|j_1,\vec{n}_1\rangle\otimes\ldots\otimes D^{j_4}(h)\,|j_4,\vec{n}_4\rangle \tag{9.33}$$

where dh is the Haar measure of the group, with the property $dh = d(gh) = d(hg)\forall g\in SU(2)$. Making use of this property allows to check the invariance of $|\,|j_a,\vec{n}_a\rangle$ under $SU(2)$ transformations at the nodes. This is an intrinsic coherent state on the node, also called *coherent intertwiner*. It can be written on any intertwiner basis $\{|\iota_k\rangle\}_k$: for large j it becomes a Gaussian packet concentrated around a single value k_0 which determines the value of the corresponding dihedral angle, and carries a phase such that when changing basis to a different intertwined basis, we still obtain a state concentrated around that value.
Now, if we want to define a state for the whole spin network of the 3d triangulation, the states on each node keep track of their intrinsic geometry but do not know anything about the extrinsic geometry of the 3d manifold, namely the way the 3d manifold is embedded in four dimensions. To catch a minimally spread extrinsic geometry, we look for semiclassical states peaked both on the intrinsic and extrinsic classical geometry: they will be "classical" in the same sense in which a wave packet which is gaussian both in the position variable and in the momentum one, is the best quantum approximation of a classical particle. Such wave packet, peaked around a point in the phase space, can be written in the Schrödinger representation (up to renormalization) as

$$\langle k|q,p\rangle\equiv\psi_{q,p}(k)\sim e^{-\frac{(k-p/\hbar)^2}{2\sigma^2}+iqk} \tag{9.34}$$

$$\langle x|q,p\rangle\equiv\psi_{q,p}(x)\sim e^{-\frac{(x-q)^2}{2\sigma^2}+\frac{i}{\hbar}px} \tag{9.35}$$

with spread $\Delta x\sim\sigma$ in position and spread $\Delta k\sim\hbar/\sigma$ in momentum. If we introduce the complex variable

$$z = q - \frac{i}{\hbar}\sigma^2 p \tag{9.36}$$

the state $|q, p\rangle$ reads

$$\psi_{q,p}(x) \sim e^{-\frac{(x-z)^2}{2\sigma^2}} \quad , \tag{9.37}$$

That is, a wave packet peaked on a phase space point (q, p) can be written as a Gaussian function peaked around a *complex* position. This method is called *complexifier method* and is generalizable to all systems whose symplectic structure is a cotangent bundle (which is always the case for unconstrained systems whereas is not guaranteed for constrained ones). The resulting states are coherent in both conjugate variables. We refer the reader to [36] for a detailed discussion. Generally speaking, the complexifier method can be applied to LQG leading to different definitions of an *extrinsic coherent state*. Work is still to be done to grasp a globally accepted overview on the topic, but all definitions are meant to have it peaked over the parameters that allow to determine the extrinsic and intrinsic geometry of the discretized manifold it represents. We suggest useful works about the definition of extrinsic coherent states:[4], [7], [13], [14]. It is also possible to define a transition amplitude for extrinsic coherent boundary states [30].

9.4 Preliminary computation of the curvature for the bouncing problem

In preparation for a practical computation of a spin foam amplitude for an extrinsic coherent state defined on the boundary graph of the surfaces Σ and Σ', we give here an introductory study of the geometrical properties of the surfaces and in particular their intrinsic metric and extrinsic curvature. These classical quantities must serve as guidance when building the corresponding quantum coherent states.
To do so, it is convenient to use the ADM (3+1) formalism in order to adapt our results to a further tetradic formalism in a natural way. Tetrads are the key formulation from which we build the phase space of GR, both in the canonical and in the covariant approaches. Therefore what follows is of primary importance for defining a proper graph for the boundary of the quantum region and a suitable coherent state on it. We refer the reader to [26] for a useful guide of the mathematical tools needed.
In order to study the BH boundary surface Σ, we foliate the region II as $II = \mathbb{R} \times \Sigma_T$ where each Σ_T is an infinite hypercylinder. The equation of each spacelike sheet is, in E-F ingoing null coordinates:

$$v(r) = Ar + T; \quad A = \left(\frac{m_P}{M}\right)^2 \tag{9.38}$$

where T is the parameter which labels the hypercylinders and the Σ_0 contains our boundary surface Σ passing through the event Δ. T should not be confused with the Schwarzschild time t.
The line element expressed in coordinates (T, r, θ, ϕ) becomes:

$$ds^2 = -f(r)dT^2 + 2[1 - f(r)A]drdT + [2A - f(r)A^2]dr^2 + r^2 d\Omega^2 \tag{9.39}$$

where $f(r) = 1 - 2M/r$ (we use natural units). The metric in these coordinates is:

$$g_{\mu\nu} = \begin{bmatrix} -f & 1 - fA & 0 & 0 \\ 1 - fA & 2A - fA^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{9.40}$$

and its inverse:

$$g^{\mu\nu} = \begin{bmatrix} fA^2 - 2A & 1 - fA & 0 & 0 \\ 1 - fA & f & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 sin^2\theta} \end{bmatrix} \tag{9.41}$$

In the ADM formalism we have:

$$ds^2 = -(N^2 - N_a N^a)dt^2 + 2N_a dt dx^a + g_{ab}dx^a dx^b \tag{9.42}$$

where the 3-vector N^a is the Shift vector and N is the Lapse function.
It follows that

$$\begin{aligned} N_a &= \begin{bmatrix} 1 - fA \\ 0 \\ 0 \end{bmatrix} \\ N_a N^a &= \frac{1}{2A - fA^2}[1 - fA]^2 \\ N^2 &= f + N_a N^a = \frac{1}{2A - fA^2} \end{aligned} \tag{9.43}$$

For the sake of simplicity, we set:

$$\begin{aligned} N &= \frac{1}{\sqrt{2A - fA^2}} \\ L &= 1 - fA \end{aligned} \tag{9.44}$$

so that we can rewrite:

$$g_{\mu\nu} = \begin{bmatrix} N^2\left(L^2 - 1\right) & L & 0 & 0 \\ L & \frac{1}{N^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{9.45}$$

$$g^{\mu\nu} = \begin{bmatrix} -\frac{1}{N^2} & L & 0 & 0 \\ L & N^2\left(1 - L^2\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 sin^2\theta} \end{bmatrix} \tag{9.46}$$

Knowing that the unit time vector can be decomposed into its components along the normal vector of the hypersurface and the tangent one which has the same spatial coordinates as the Shift vector:

$$\begin{aligned} \tau^\mu(x) &= N(x)n^\mu(x) + N^\mu(x) = \\ &= N(x) \cdot \begin{bmatrix} n^T \\ n^r \\ n^\theta \\ n^\varphi \end{bmatrix} + \begin{bmatrix} 0 \\ N^r \\ N^\theta \\ N^\varphi \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{9.47}$$

we have:

$$n^\mu = \begin{bmatrix} \frac{1}{N} \\ -LN \\ 0 \\ 0 \end{bmatrix} \quad n_\mu = \begin{bmatrix} -N \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad N^\mu = \begin{bmatrix} 0 \\ LN^2 \\ 0 \\ 0 \end{bmatrix} \quad N_\mu = \begin{bmatrix} L^2N^2 \\ L \\ 0 \\ 0 \end{bmatrix} \tag{9.48}$$

In conclusion:

$$\begin{aligned} &\text{Lapse function:} \;\; N = \sqrt{\frac{1}{2A - fA^2}} \\ &\text{Shift vector:} \;\; N^a = \begin{bmatrix} LN^2 \\ 0 \\ 0 \end{bmatrix} \qquad L = 1 - fA \\ &N_a = \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{9.49}$$

The intrinsic metric is

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu = \begin{bmatrix} N^2 L^2 & L & 0 & 0 \\ L & \frac{1}{N^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{9.50}$$

Its inverse $q^{\mu\nu}$ can be computed either by $q^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu$ or by $q^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} q_{\rho\sigma}$. We get:

$$q^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & N^2 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 sin^2\theta} \end{bmatrix} \tag{9.51}$$

Since n_r vanishes the intrinsic metric has the same spatial part as the original metric: $q_{ab} = g_{ab}$.
The tensor $q^\mu_\nu = g^{\mu\rho} q_{\rho\nu}$ acts as a projector onto Σ_T.
Now, the extrinsic curvature on the spatial hypersurface is defined as:

$$K_{ab} = q^\alpha_a q^\beta_b \nabla_\alpha n_\beta \tag{9.52}$$

where ∇ is the covariant derivative compatible with the original metric g. It can be shown that we can write it as a function of q and Shift:

$$K_{ab} = \frac{1}{2N} \left(\dot{q}_{ab} - D_a N_b - D_b N_a \right) = \frac{1}{2N} \left(\mathcal{L}_T q_{ab} - \partial_b N_a - \partial_a N_b + 2\Gamma^c_{ab} N_c \right) \tag{9.53}$$

In the last equation, D and Γ are respectively the covariant derivative and the connection related to the induced 3-metric q_{ab} : $D_a N_b \equiv \partial_a N_b - \Gamma^c_{ab} N_c$; $\Gamma^c_{ab} = \frac{1}{2} q^{cd} (\partial_a q_{db} + \partial_b q_{ad} - \partial_d q_{ab})$. The term $\dot{q}_{ab}$ is the Lie derivative of the 3-metric with respect to the coordinate T.
The only non vanishing Christoffel symbols of the 3-metric are:

$$\begin{aligned} \Gamma^r_{rr} &= -\frac{1}{N} \frac{dN}{dr} \\ \Gamma^r_{\theta\theta} &= -N^2 r \\ \Gamma^r_{\varphi\varphi} &= -N^2 r \sin\theta \\ \Gamma^\theta_{r\theta} &= \Gamma^\theta_{\theta r} = \frac{1}{r} \\ \Gamma^\theta_{\varphi\varphi} &= -sin\theta cos\theta \\ \Gamma^\varphi_{\theta\varphi} &= \Gamma^\varphi_{\varphi\theta} = \frac{cos\theta}{sin\theta} \\ \Gamma^\varphi_{r\varphi} &= \Gamma^\varphi_{\varphi r} = \frac{1}{r} \end{aligned} \tag{9.54}$$

We now want to compute the components of the extrinsic curvature tensor. To do this, we first notice that:

- ∂_T is a Killing field, so $\dot{q}_{ab}$ vanishes;
- the Christoffels are contracted with the Shift vector, so we only need those with upper r index;
- The only nonvanishing component of Shift is N_r, which only depends on r, and the Γ^r_{ab} is nonvanishing only if $a = b$.

Thus we only need to compute K_{rr}, $K_{\varphi\varphi}$, $K_{\theta\theta}$:

$$\begin{aligned} K_{rr} &= -\frac{1}{N}\left(\frac{dL}{dr} + \frac{L}{N}\frac{dN}{dr}\right) \\ K_{\varphi\varphi} &= -LNr\sin^2\theta \\ K_{\theta\theta} &= -LNr \end{aligned} \tag{9.55}$$

For the WH boundary surface Σ', we foliate the time reversed of region II, II' as $II' = \mathbb{R} \times \Sigma_T$ where each Σ_T is an infinite hypercylinder. The equation of each spacelike sheet is:

$$u(r) = -Ar + T; \quad A = \left(\frac{m_P}{M}\right)^2 \tag{9.56}$$

where T is the parameter which labels the hypercylinders and the Σ_0 contains our boundary surface Σ' passing through the event Δ. T should not be confused with the Schwarzschild time t.
The line element expressed in coordinates (T, r, θ, ϕ) becomes:

$$ds^2 = -f(r)dT^2 + 2[f(r)A - 1]drdT + [2A - f(r)A^2]dr^2 + r^2 d\Omega^2 \tag{9.57}$$

where $f(r) = 1 - 2M/r$. The metric in these coordinates is:

$$g_{\mu\nu} = \begin{bmatrix} -f & fA-1 & 0 & 0 \\ fA-1 & 2A - fA^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{9.58}$$

$$g^{\mu\nu} = \begin{bmatrix} fA^2 - 2A & fA-1 & 0 & 0 \\ fA-1 & f & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 sin^2\theta} \end{bmatrix} \tag{9.59}$$

In the ADM formalism we have:

$$ds^2 = -(N^2 - N_a N^a)dt^2 + 2N_a dt dx^a + g_{ab}dx^a dx^b \tag{9.60}$$

where the 3-vector N^a is the Shift vector and N is the Lapse function.
It follows that

$$\begin{aligned} N_a &= \begin{bmatrix} fA-1 \\ 0 \\ 0 \end{bmatrix} \\ N_a N^a &= \frac{1}{2A - fA^2}[fA-1]^2 \\ N^2 &= f + N_a N^a = \frac{1}{2A - fA^2} \end{aligned} \tag{9.61}$$

We find that the Laspe function is the same as the black hole's Lapse function, while the Shift vector has the opposite sign of the black hole's one. Thus the following computation remains the same as in the BH case, with the only difference of adding a minus sign to every L. The extrinsic curvature of the white hole boundary surface turns out to be just the opposite of the black hole boundary surface, as expected.

Conclusions

In this work we have reviewed a very recent proposal of non singular black hole, whose spacetime is no more affected by the classical "illnesses" of GR and is cured by a surprisingly intuitive and heuristically coherent implementation of quantum gravitational dynamics, causing a bounce in the otherwise unstoppable collapse of a star and potentially saving, at least in principle, what remains of the classical spacetime from the paradox of information loss. The beauty that in the author's opinion lies in this model is its great level of predictability without actually controlling all the details of such implementation, something that would require a solid and complete theoretical framework of quantum gravity. Moreover, source of interest can be found in its rather simple assumptions, motivated by lucid arguments and inspired by only what, at present day, seems to be the consolidated knowledge of the physical behaviour of Nature. The rather inviting simplicity of the model proposed in [16] pays the price of it being classified as *radical*, according to a recent qualitative study of the different proposals of non singular black holes [21], in contrast with more *conservative* solutions that confine quantum gravity to play a role uniquely inside the BH region and upon non peculiar behaviour of its horizon.

The model of bouncing black holes appears as naturally ready to be studied in the formalism of Loop Quantum Gravity, a background independent theory of quantization of the gravitational field largely based on the relational content lying in our perception of the physical world through observables. We have then reviewed the basics of the formulation of this promising theory of quantum gravity, underlining the key aspects of this approach and gradually building the machinery that one can use to study the bouncing model in a genuinely quantum formalism.

Finally, we have prompted the general outline of our model in the perspective of LQG and studied the geometrical features that the quantum state of gravity involved in the process should encode if we want it to have the right classical limit (dictated by the solution of the Einstein equations) wherever this is expected in spacetime, namely everywhere outside the purely quantum region.

In conclusion, we have set the problem of computing the amplitude for the quantum bounce of a BH into a WH to happen disregarding dissipative effects like, first of all, Hawking's radiation. This problem is, first, simplified as much as possible and then formulated as the amplitude of the quantum spacetime (seen as the process itself) given the quantum state of geometry that bounds it in space and in time. Such state has to be extrinsic coherent in the sense that it needs to be "peaked" around the classical parameters of extrinsic geometry that describe the classical limit of the boundary region.

In trying to give a flavour of what we aim to find, we can say that the extrinsic coherent state of the boundary will depend on its classical geometrical limit, which in turn is determined by the mass of the star experiencing the bounce, M, and the time t of the bounce as seen by an external observer. By imposing the amplitude of the process to be exactly 1, we want to trace back a relation $t(M)$ that can help us confront, in a second step, the issue of information loss and non unitary evolution of quantum fields in curved spacetime due to the dissipative phenomenon of Hawking radiation:

$$\langle W | extr.\ coherent\ state(M, t_{bounce}) \rangle = 1 \Rightarrow t_{bounce}(M) \ . \tag{9.62}$$

We leave this as work in progress.

Appendix A

Globally hyperbolic spacetime

Global hyperbolicity is a condition of the causal structure of a spacetime without boundaries. First we say that:

- $\mathcal{M}$ is *causal* if it does not admit closed timelike curves;
- Given a point P in $\mathcal{M}$, we call $\mathscr{I}^{+}(P)$ ($\mathscr{I}^{+}(P)$) as the future (past) development of P and define it as the set of all points in $\mathcal{M}$ that can be reached by causal continuous trajectories directed from P to the future (past);
- Given a subset S in $\mathcal{M}$, the *Cauchy development* of S is defined as the set of all points P in $\mathcal{M}$ through which any causal non-extendible curve intersects S. This definition can be divided in past and future developments in an obvious way;
- A subset S in $\mathcal{M}$ is said *achronal* if no timelike curve intersects S more than once;
- A *Cauchy surface* in $\mathcal{M}$ is a closed achronal subset of $\mathcal{M}$ whose Cauchy development is $\mathcal{M}$ itself.

Now, we can define a spacetime $\mathcal{M}$ to be *globally hyperbolic* if it satisfies one of the following equivalent properties:

⋆ $\mathcal{M}$ is causal, and for every couple of points P and Q in $\mathcal{M}$ $\mathscr{I}^{-}(P) \cup \mathscr{I}^{+}(Q)$ is compact;

⋆ $\mathcal{M}$ is causal, and for every couple of points P and Q in $\mathcal{M}$ the set of all causal curves that are continuous from P to Q and directed to the future is compact;

⋆ $\mathcal{M}$ has a Cauchy surface.

Appendix B

Connection of a G-principal bundle

Given a fiber bundle with $\mathcal{M}$ as base manifold and whose fibers carry a k-dimensional representation of a group G, namely a G-principal bundle, a G-connection is a type of differential operator

$$D \;:\; \Gamma(F) \to \Gamma(F \otimes \Omega^1(\mathcal{M})) \tag{B.1}$$

where F is a k-dimensional vector bundle over the base manifold $\mathcal{M}$, and $\Gamma(F \otimes \Omega(\mathcal{M}))$ is a local section of the product of F with the space of 1-forms over $\mathcal{M}$, $\Omega^1(\mathcal{M})$. The G-connection allows to define a covariant derivative insensitive of the local gauge symmetries in the fibers, and consequently a parallel transport of objects defined in the fiber of one point in $\mathcal{M}$ to the fiber of another point of $\mathcal{M}$. If v is a vector field in F and f is a smooth function (0-form $\in \Omega^0(\mathcal{M})$):

$$D(fv) \equiv v \otimes df + fDv \;, \tag{B.2}$$

where d is the exterior derivative of f.
If we consider a local basis vector e_I in the vector fiber, we have:

$$D(e_I) = \sum_J e_J \otimes \omega_I^J = \sum_J \sum_\gamma \omega_{I\gamma}^J e_J \otimes dx^\gamma \;, \tag{B.3}$$

where the $k \times k$ matrix of 1-forms $\omega_{\gamma I}^J$ is called *connection form.* The index γ runs over the local coordinates of the base manifold $\mathcal{M}$, while I and J are the internal indices of the vector fiber. Taking a vector field $V = V^I e_I$ (we assume sum over repeated indices):

$$\begin{aligned} D(V) = D(V^I e_I) &= e_I \otimes dV^I + V^I e_J \otimes \omega_I^J \\ &= \frac{\partial V^I}{\partial x^\gamma} e_I \otimes dx^\gamma + V^I \omega_{\gamma I}^J e_J \otimes dx^\gamma \\ &= \left[\frac{\partial V^I}{\partial x^\gamma} + V^J \omega_{\gamma J}^I\right] e_I \otimes dx^\gamma \;. \end{aligned} \tag{B.4}$$

Retaining components on both sides we have:

$$DV = dV + \omega V = (d + \omega)V \tag{B.5}$$

which highlights the role of ω in "correcting" the standard derivative.
If we choose the tangent direction ∂_α along which we want to differentiate:

$$D_\alpha(V) = \left[\frac{\partial V^I}{\partial x^\gamma} + V^J \omega_{\gamma J}^I\right] e_I \otimes dx^\gamma(\partial_\alpha) = \left[\frac{\partial V^I}{\partial x^\alpha} + V^J \omega_{\alpha J}^I\right] e_I \;. \tag{B.6}$$

For general tensors in the vector fiber their gauge-covariant derivative will show a ω for each internal index. When we change the bsis of the fiber through the action of an element of the gauge group $g \in G$:

$$\{e\} \to \{e'\} = \{ge\} \ , \tag{B.7}$$

we get an inhomogeneous transformation law for the connection:

$$\omega_{e'} = \omega_{ge} = g^{-1}dg + g^{-1}\omega_e \ , \tag{B.8}$$

where the labels e, e' indicate that the connection form depends on the local choice of the basis in the fiber.
This transformation law tells us that the connection form $\omega^I_{\gamma J}$ is a 1-form (lower index γ), with values in the generators of G (whose representation is in the indices I, J) or, in case of a Lie group G, in its Lie algebra. Although the connection form does not transform tensorially under the action of G, the gauge-covariant derivative of objects defined in the vector fiber cannot sense, by its ver definition, the difference in the choice of the basis, i.e. in the local gauge fixing, and has to transform covariantly:

$$D'\bullet = g(D\bullet) \ . \tag{B.9}$$

Appendix C

Tetradic action of General Relativity

Here is a list of useful relations:

$$dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \epsilon^{\mu\nu\rho\sigma} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{C.1}$$

$$e^I_\mu e^K_\nu e^M_\rho e^N_\sigma = \epsilon^{IKMN} e^0_\mu e^1_\nu e^2_\rho e^3_\sigma \tag{C.2}$$

$$e^0_\mu e^1_\nu e^2_\rho e^3_\sigma \epsilon^{\mu\nu\rho\sigma} = \det e \tag{C.3}$$

$$\epsilon^{\mu\nu\rho\sigma} e^I_\mu e^K_\nu e^M_\rho e^N_\sigma = (\det e)\epsilon^{IKMN} \tag{C.4}$$

$$dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma e^0_\mu e^1_\nu e^2_\rho e^3_\sigma = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \tag{C.5}$$

Finally, we remind Cailey's formula for the determinant of a 4×4 matrix A^I_J:

$$\det A = \frac{1}{4}\epsilon_{IJKL}\epsilon^{MNOP} A^I_M A^J_N A^K_O A^L_P \ . \tag{C.6}$$

Then, posing $c^4/16\pi G = 1$, we have:

$$\begin{aligned}
S &= \int d^4x\sqrt{-g}R \\
&= \int d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu} = \int d^4x\sqrt{-g}g^{\mu\nu}g^{\rho\sigma}R_{\rho\mu\sigma\nu} = \int d^4x\sqrt{-g}g^{\mu\nu}g^{\rho\sigma}g_{\rho\alpha}R^\alpha_{\mu\sigma\nu} = \\
&= \int d^4x(\det e)e^\mu_I e^\nu_J \eta^{IJ} e^\rho_L e^\sigma_K \eta^{LK} e^M_\rho e^N_\alpha \eta_{MN} e^\alpha_P e^Q_\mu F^P_{Q\sigma\nu} = \\
&= \int d^4x(\det e)e^\nu_J \eta^{IJ} e^\sigma_K \eta^{LK} e^N_\alpha \eta_{LN} e^\alpha_P F^P_{I\sigma\nu} \\
&= \int d^4x(\det e)e^\nu_J e^\sigma_K F^{KJ}_{\sigma\nu} = \\
&= \int d^4x \left[\frac{1}{4}\epsilon_{ABCD}\epsilon^{\alpha\beta\gamma\delta}e^A_\alpha e^B_\beta e^C_\gamma e^D_\delta\right] e^\nu_J e^\sigma_K F^{KJ}_{\sigma\nu} = \\
&= \int dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \frac{1}{4}\epsilon^{\nu\sigma\gamma\delta}\epsilon_{JKCD}e^C_\gamma e^D_\delta F^{KJ}_{\sigma\nu} = \\
&= \int dx^\nu \wedge dx^\sigma \wedge dx^\gamma \wedge dx^\delta \frac{1}{4}\epsilon_{JKCD}e^C_\gamma e^D_\delta F^{KJ}_{\sigma\nu} = \\
&= \int \frac{1}{2}\epsilon_{IJKL}e^I \wedge e^J \wedge F^{KL} \ .
\end{aligned}$$

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