Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
Corso di Laurea Magistrale in Fisica

Nonlinearity As a Resource for Nonclassicality

Relatore:
Prof. Cristian Degli Esposti Boschi

Presentata da:
Francesco Albarelli

Correlatore:
Prof. Matteo G. A. Paris

Sessione III
Anno Accademico 2013/2014

Sommario

Lo scopo di questo lavoro è cercare un'evidenza quantitativa a supporto dell'idea idea che la nonlinearità sia una risorsa per generare nonclassicità. Ci si concentrerà su sistemi unidimensionali bosonici, cercando soprattutto di connettere la nonlinearità di un oscillatore anarmonico, definito dalla forma del suo potenziale, alla nonclassicità del relativo ground state.

Tra le numerose misure di nonclassicità esistenti, verranno impiegate il volume della parte negativa della funzione di Wigner e l'entanglement potential, ovvero la misura dell'entanglement prodotto dallo stato dopo il passaggio attraverso un beam splitter bilanciato avente come altro stato in ingresso il vuoto. La nonlinearità di un potenziale verrà invece caratterizzata studiando alcune proprietà del suo ground state, in particolare se ne misurerà la non-Gaussianità e la distanza di Bures rispetto al ground state di un oscillatore armonico di riferimento. Come principale misura di non-Gaussianità verrà utilizzata l'entropia relativa fra lo stato e il corrispettivo stato di riferimento Gaussiano, avente la medesima matrice di covarianza.

Il primo caso che considereremo sarà quello di un potenziale armonico con due termini polinomiali aggiuntivi e il ground state ottenuto con la teoria perturbativa. Si analizzeranno poi alcuni potenziali il cui ground state è ottenibile analiticamente: l'oscillatore armonico modificato, il potenziale di Morse e il potenziale di Posch-Teller. Si andrà infine a studiare l'effetto della nonlinearità in un contesto dinamico, considerando l'evoluzione unitaria di uno stato in ingresso in un mezzo che presenta una nonlinearità di tipo Kerr.

Nell'insieme, i risultati ottenuti con tutti i potenziali analizzati forniscono una forte evidenza quantitativa a supporto dell'idea iniziale. Anche i risultati del caso dinamico, dove la nonlinearità costituisce una risorsa utile per generare nonclassicità solo se lo stato iniziale è classico, confermano la pittura complessiva.

Si sono inoltre studiate in dettaglio le differenze nel comportamento delle due misure di nonclassicità.

Abstract

The aim of the present work is to find a quantitative evidence to support the idea that nonlinearity is a resource to generate a nonclassicality. We will focus on unidimensional bosonic systems, mainly trying make a connection between the nonlinearity of an anharmonic oscillator, defined by the functional form of its potential, and the nonclassicality of the corresponding ground state.

Among the many nonclassicality measures in existence, we will use the volume of the negative part of the Wigner function and the entanglement potential, that is the amount of entanglement obtained after the state goes through a balanced beam splitter, with the vacuum as the other input state. The nonlinearity of a potential is characterized by properties of the ground state, in particular its non-Gaussianity and the Bures distance from the ground state of a reference harmonic oscillator. To measure non-Gaussianity we will use the relative entropy between the state and the corresponding reference Gaussian state, the one with the same covariance matrix.

The first model under consideration will be a harmonic potential with two added polynomial terms and the ground state obtained with perturbation theory. We will then analyse three potentials with an analytical ground state: the modified harmonic oscillator, the Morse potential and the Posch-Teller potential. We will also study the effect of nonlinearity in a dynamical context, focusing the attention on the unitary evolution of an input state entering in a medium with a Kerr nonlinearity.

The results obtained with all the potentials represent a strong quantitative evidence to support the thesis. The results for the dynamical case, where nonlinearity is a useful resource to generate nonclassicality only if the initial state is classical, confirm the general picture.

Moreover we studied in details the differences in the behaviour of the two nonclassicality measures.

Contents

Introduction

At the heart of the whole quantum technology research field lies the idea that quantum mechanical systems present features that have no counterpart in classical physics and that can be employed as resources to perform specific tasks better or faster than in the classical case. Coherence of quantum systems, together with its multipartite manifestation, entanglement, has been indeed recognized as one of the most important resources for quantum information processing.

The need of a proper definition of *nonclassicality* originated from quantum optics, in order to identify which states would produce effects not obtainable with classical light. Indeed, single-mode radiation fields and, more generally, single-mode bosonic systems are the natural playground to discuss, and test experimentally, the generation and the charaterization of nonclassicality.

The aim of this work is to support the idea that a nonlinearity is a general resource to generate nonclassicality. We will discuss some general features and work out some specific examples, with a particular emphasis on a quantitative visualization of the phenomenon. In order to do so, proper ways to quantify both nonlinearity and nonclassicality are reviewed and critically discussed. Finding a way to measure these quantities has proved to be a challenging task in and of itself and there exist different parameters which capture different aspects of the picture.

The idea to connect the nonlinear behaviour of a quantum system to the appearance of nonclassicality has recently been tested, in the context of nano-mechanical resonators, to the Duffing oscillator model [1]. Here we want to generalize that paper and check whether and to what extent this generalization can be done.

This Thesis is structured as follows.

- Chapter 1 is intended as a brief survey about the concept of nonclassicality in quantum optics, with particular emphasis on the methods developed to define and measure nonclassicality in a quantitative fashion. The first section is just a review about quasiprobability distributions, because they are fundamental to describe nonclassicality. Even though in the original part of this work only two measures of nonclassicality will actually be used (namely the negative volume of the Wigner function and the entanglement potential), more measures are presented, since it seems worth to recollect all these ideas together for comparison and for future reference.

- Chapter 2 is a review of a measure recently introduced [2] to quantify the nonlinearity of a quantum oscillator by analysing its ground state. This idea relies in turn on a measure of the non-Gaussian character of a continuous variable quantum state, so this concept is introduced and explored as well.

- Chapter 3 is devoted to the simple model of a harmonic oscillator with two perturbations, one proportional to x^4 and one to x^6. This model serves as a first example to check the correlation between nonclassicality and nonlinearity. The analysis shows that the intuition is correct and there is a quantitative connection between the two, even though they are not one to one when two parameters are present.

- Chapter 4 is an extension of the same idea to different potentials. In particular we choose potentials for which an exact solution is already known, in order to test the validity of the intuition not just through a perturbative analysis, but by using the true ground state. The main idea holds true, although we also highlight different behaviour between the two measures of nonclassicality in use.

- Chapter 5 contains the analysis in the case of a nonlinear term which commutes with the free Hamiltonian of the harmonic oscillator. The model under consideration is the single photon Kerr nonlinearity. In this chapter we do not study the ground state, but we examine the evolution of an input state under self-Kerr effect, looking at the behaviour of its non-Gaussianity and its nonclassicality. The analysis is carried out with two different input states.

1 Quantifying the Nonclassicality of a Single-mode Bosonic State

In quantum optics the expression "nonclassical light" is ubiquitous and it is used to address a wide range of phenomena that are considered truly quantistic in nature; this concept however applies equally well to other bosonic systems. In particular we will deal with single-mode bosonic systems, so that entanglement and other nonclassical correlations are kept out of the picture.

In the most general terms a quantum state is said to be nonclassical if the methods of classical statistics fail to describe its properties. To make this definition precise we need the concept of quasiprobability distributions in phase space, so we start by introducing them from scratch.

1.1 Characteristic Functions and Quasiprobability Distributions

In classical physics the state of a physical systems and its evolution can be visualized by a probability distribution in the phase space. In quantum mechanics however we have the Heisenberg's uncertainty relation that prevents a naive extension of this idea. It is nonetheless possible to visualize quantum mechanics in the phase space if we relax some of the axioms of probability theory, thus dealing with quasiprobability distributions rather than probability distributions.

In practical terms quasiprobability distributions are mainly used as a tool to calculate expectation values of functions of $\hat{a}$ and $\hat{a}^\dagger$, but they encode all the information on the quantum state, since the full density matrix can be reconstructed from them. In this sense they are an alternative representation of a quantum state.

Historically the first quasidistribution distribution introduced was the Wigner function [3] in the context of statistical mechanics; in quantum optics there are other well known examples: the Glauber-Sudarshan P function [4, 5] and the Husimi Q function [6]. These distributions can be derived in several different ways, but we will first introduce characteristic functions and then quasiprobability distributions as their Fourier transforms, roughly following Ref. [7]. This coherent visualization of quasiprobability distributions and ordering of the operators was first put forward in 1968 by Cahill and Glauber [8], the most general class of quasiprobability distributions was given by Agarwal and Wolf [9]. Moreover Ref. [10] is an excellent review on the subject which we partly refer to.

Since we aim to make this Thesis as self-contained as possible we start with a brief review

of coherent states and displacement operators for single mode bosonic systems. It can be useful to remark that all the concepts introduced in this chapter could be generalized to multi-mode systems, but it is not needed for our purposes.

1.1.1 Displacement Operator and Coherent States

We take in consideration a bosonic system with one degree of freedom, which may be a one dimensional harmonic oscillator or a single mode of the electromagnetic radiation. As customary we will describe this systems using its annihilation and creation operators

$$\begin{aligned} \hat{a} &= \left(\frac{1}{2\hbar}\right)\left(\lambda\hat{q} + \frac{\mathrm{i}}{\lambda}\hat{p}\right) \\ \hat{a}^\dagger &= \left(\frac{1}{2\hbar}\right)\left(\lambda\hat{q} - \frac{\mathrm{i}}{\lambda}\hat{p}\right) \end{aligned} \tag{1.1}$$

which satisfy the relation

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{1.2}$$

These operators act on the particle number states, which are the basis vectors $|n\rangle$ and have the following properties:

$$\begin{aligned} &\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \\ &\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \\ &\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle \\ &\hat{a}|0\rangle = 0. \end{aligned} \tag{1.3}$$

An additional and very useful property which can be easily demonstrated is the following

$$[\hat{a}, \hat{a}^{\dagger n}] = n\hat{a}^{\dagger(n-1)} \tag{1.4}$$

We note that the constant λ in (1.1) can become $\lambda = (m\omega)^{1/2}$ for a mechanical oscillator with mass m and angular frequency ω or $\lambda = (\hbar^{1/2}\omega/c)$ for a mode of the electromagnetic field with angular frequency ω. However in the following we will mostly work with appropriately rescaled units, so that $m = \hbar = \omega = 1$.

For every complex number ξ we can define the displacement operator:

$$\hat{D}(\xi) = \exp\left(\xi\hat{a}^\dagger - \xi^*\hat{a}\right). \tag{1.5}$$

It is a unitary operator, since evidently $\hat{D}(\xi)^\dagger = \hat{D}(\xi)^{-1} = \hat{D}(-\xi)$. It can also be written with a different ordering of the operators $\hat{a}$ and $\hat{a}^\dagger$, such as normal ordering, when all the creation operators come before the annihilation operators, or in antinormal ordering, which means the opposite; the way we have defined $\hat{D}$ in (1.5) is therefore its symmetrically ordered form. This task can be performed by disentangling the exponential operator, which means to express the exponential of a sum of operators as the product of the exponentials of operators; this topic is discussed more in depth in appendix A.1.

In this case however it is simply an application of the famous Backer-Campbell-Hausdorff formula, which reduces to $e^X e^Y = e^{X+Y} e^{1/2[X,Y]}$ for the property (1.2). The normal ordered form is the following

$$\hat{D}(\xi) = \exp\left(\xi \hat{a}^\dagger\right) \exp(-\xi^* \hat{a}) \exp\left(-\frac{|\xi|^2}{2}\right), \tag{1.6}$$

while the antinormal ordered form reads

$$\hat{D}(\xi) = \exp(-\xi^* \hat{a}) \exp\left(\xi \hat{a}^\dagger\right) \exp\left(\frac{|\xi|^2}{2}\right). \tag{1.7}$$

The Baker-Campbell-Hausdorff formula can also be used to verify that

$$\hat{D}(\alpha)\hat{D}(\beta) = e^{(\alpha\beta^* - \alpha^*\beta)/2} \hat{D}(\alpha + \beta). \tag{1.8}$$

The action as unitary similarity transformation of the creation and destruction operators yields this important property

$$\begin{aligned} \hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) &= \hat{a} + \alpha \\ \hat{D}(\alpha)\hat{a}\hat{D}^\dagger(\alpha) &= \hat{a} - \alpha, \end{aligned} \tag{1.9}$$

which can be easily proved using property (1.4).

The displacement operator can be used to define a important class of states, the coherent states:

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\hat{a}^{\dagger n}}{n!}|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{1.10}$$

they are eigenstates of the annihilation operator $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, as can be seen by direct inspection using again the property (1.4). This trivially implies that the real and imaginary parts of the complex variable α are proportional to $q = \langle\hat{q}\rangle$ and $p = \langle\hat{p}\rangle$ respectively.

These state have some important and well known properties which can all be proved by using their explicit expression in term of the number states. They are not orthogonal:

$$\langle\beta|\alpha\rangle = \exp\left[-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \beta^*\alpha\right] \tag{1.11}$$

but it is possible to express the identity operator as

$$I = \frac{1}{\pi} \int \mathrm{d}^2\alpha |\alpha\rangle\langle\alpha|, \tag{1.12}$$

where the integration is performed all over the complex plane and $\mathrm{d}^2\alpha = \mathrm{d}(\mathrm{Re}\,\alpha)\mathrm{d}(\mathrm{Im}\,\alpha)$. This means that the trace of any operator $\hat{A}$ can be computed as follows

$$\mathrm{Tr}(\hat{A}) = \frac{1}{\pi} \int \mathrm{d}^2\alpha \langle\alpha|\hat{A}|\alpha\rangle. \tag{1.13}$$

Another important property of the coherent states is that they are minimum uncertainty states, having the same uncertainty associated with $\hat{p}$ and $\hat{x}$. Moreover, thanks to property (1.8), the operator $\hat{D}(\beta)$ applied on a coherent state just gives another coherent states with displaced parameter, that is to say

$$\hat{D}(\beta)|\alpha\rangle = |\alpha + \beta\rangle. \tag{1.14}$$

1.1.2 p-ordered Characteristic Function

States of a quantum system are represented by normalized vectors $|\psi\rangle$ in a Hilbert space and every vector corresponds to a projection operator $\rho = |\psi\rangle\langle\psi|$, called the density operator of the state. The normalization of $|\psi\rangle$ is reflected in the property $\mathrm{Tr}[\rho] = 1$, while expectation values of other operators are computed with trace operation: $\langle\hat{A}\rangle = \mathrm{Tr}\left[\rho\hat{A}\right]$.

More generally any operator which is self-adjoint, positive semi-definite and of trace one represents a density operator even if it is not a projector, so that $\rho \neq \rho^2$. In this case we have a mixed state, which is a statistical mixture of pure states; in fact $\rho = \sum_n P_n|\psi_n\rangle\langle\psi_n|$, where P_n is a probability distribution. In the following chapters we will not use mixed states, nonetheless it has to be remembered that the concepts we are about to introduce apply to all kind of quantum states.

Given a generic quantum state ρ, the corresponding p-ordered characteristic function is defined as follows

$$\begin{aligned}\chi(\xi,p) &= \mathrm{Tr}\left[\rho\hat{D}(\xi)\right]\exp\left(p\frac{|\xi|^2}{2}\right)\\ &= \mathrm{Tr}\left[\rho\exp\left(\xi\hat{a}^\dagger - \xi^*\hat{a}\right)\right]\exp\left(p\frac{|\xi|^2}{2}\right).\end{aligned} \tag{1.15}$$

It is important to underline again that the characteristic function is an alternative way to represent a quantum state, in the sense that it encodes all the information contained in ρ. For the particular values $p = 1, 0, -1$ we get the normal, symmetrical and antinormal ordered characteristic functions respectively; if we use (1.6) and (1.7) we get

$$\begin{aligned}\chi(\xi,1) &= \mathrm{Tr}\left[\exp\left(\xi\hat{a}^\dagger\right)\exp(-\xi^*\hat{a})\right]\\ \chi(\xi,0) &= \mathrm{Tr}\left[\exp\left(\xi\hat{a}^\dagger - \xi^*\hat{a}\right)\right]\\ \chi(\xi,-1) &= \mathrm{Tr}\left[\exp(-\xi^*\hat{a})\exp\left(\xi\hat{a}^\dagger\right)\right].\end{aligned} \tag{1.16}$$

The characteristic function allows us to define the p-ordered expectation value of products of $\hat{a}^\dagger$ and $\hat{a}$, for a product of $\hat{a}^{\dagger m}$ and $\hat{a}^n$ we get

$$\langle\hat{a}^{\dagger m}\hat{a}^n\rangle_p = \left(\frac{\partial}{\partial\xi}\right)^m\left(\frac{\partial}{\partial\xi^*}\right)^n\chi(\xi,p)\Big|_{\xi=0} \tag{1.17}$$

The absolute value of p-ordered characteristic function is bounded, since $\hat{D}(\xi)$ is unitary we have that $|\chi(\xi,0)| \leq 1$, this and definition (1.15) imply that in general $|\chi(\xi,p)| \leq \exp\left(p|\xi|^2/2\right)$.

1.1.3 p-ordered Quasiprobability Distribution

To obtain the quasiprobability distribution from the characteristic function we need to perform a Fourier transform; a standard way to define the Fourier transform of a function of two real variables (in this case the real and imaginary parts: $\alpha = \alpha_1 + \mathrm{i}\alpha_2$) is the following

$$\tilde{f}(\alpha_1,\alpha_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\, f(x,y)\exp[\mathrm{i}(\alpha_1 x + \alpha_2 y)], \tag{1.18}$$

however if we perform the substitution $\xi = \pm(y - \mathrm{i}x)/2$ we can write it in a more convenient form

$$\tilde{f}(\alpha) = \frac{1}{\pi^2} \int \mathrm{d}^2\xi \, f(\xi) \exp(\alpha\xi^* - \alpha^*\xi), \tag{1.19}$$

where the integration is performed over the whole complex plane.

Applying this transform to the p-ordered characteristic function we get the p-ordered quasiprobability distribution, which reads

$$W(\alpha, p) = \frac{1}{\pi^2} \int \mathrm{d}^2\xi \, \chi(\xi, p) \exp(\alpha\xi^* - \alpha^*\xi). \tag{1.20}$$

This integral is not always well behaved, since for some specific states and for some values of the ordering parameter $\chi(\xi, p)$ can diverge for $|\xi| \to \infty$.

Properties

Some important properties of $W(\alpha, p)$ can be demonstrated from the definition (1.20). We can see that $W(\alpha, p)$ is always real, if we compute its complex conjugate, given by

$$\begin{aligned} [W(\alpha, p)]^* &= \frac{1}{\pi^2} \int \mathrm{d}^2\xi \, [\chi(\xi, p)]^* \exp(\alpha^*\xi - \alpha\xi^*) \\ &= \frac{1}{\pi^2} \int \mathrm{d}^2\xi \, \mathrm{Tr}\left[\rho \exp\left(\xi^*\hat{a} - \xi\hat{a}^\dagger\right)\right] \exp\left(p\frac{|\xi|^2}{2}\right) \exp(\alpha^*\xi - \alpha\xi^*). \end{aligned} \tag{1.21}$$

By substituting $\xi = -\eta$, we get

$$\begin{aligned} [W(\alpha, p)]^* &= \frac{1}{\pi^2} \int \mathrm{d}^2\eta \, \mathrm{Tr}\left[\rho \exp\left(\eta\hat{a}^\dagger - \eta^*\hat{a}\right)\right] \exp\left(p\frac{|\eta|^2}{2}\right) \exp(\alpha\eta - \alpha^*\eta) \\ &= W(\alpha, p), \end{aligned} \tag{1.22}$$

so the distribution is real for all α and p. The function is also normalized, since

$$\begin{aligned} \int \mathrm{d}^2\alpha \, W(\alpha, p) &= \frac{1}{\pi^2} \int \mathrm{d}^2\alpha \int \mathrm{d}^2\xi \, \chi(\xi, p) \exp(\alpha\xi^* - \alpha^*\xi) \\ &= \int \mathrm{d}^2\alpha \, \chi(\xi, p)\delta^{(2)}(\xi) = \chi(0, p) = 1, \end{aligned} \tag{1.23}$$

where the last passage follows from the normalisation of the state ρ, since from the definition (1.15) we have $\chi(0, p) = \mathrm{Tr}[\rho] = 1$.

The function $W(\alpha, p)$ can be used to obtain moments of p-ordered products of $\hat{a}$ and $\hat{a}^\dagger$, by integrating the appropriate powers of α and α^*:

$$\langle \hat{a}^{\dagger m} \hat{a}^n \rangle_p = \int \mathrm{d}^2\alpha W(\alpha, p) \alpha^{*m} \alpha^n. \tag{1.24}$$

To prove that this is true we plug the definition (1.20) in the r.h.s. of the above formula, so we get

$$\begin{aligned}
&\frac{1}{\pi^2}\int \mathrm{d}^2\alpha \int \mathrm{d}^2\xi\, \chi(\xi,p)\, \alpha^{*m}\alpha^n \exp(\alpha\xi^* - \alpha^*\xi) \\
&= \frac{1}{\pi^2}\int \mathrm{d}^2\alpha \int \mathrm{d}^2\xi\, \chi(\xi,p) \left(-\frac{\partial}{\partial\xi}\right)^m \left(\frac{\partial}{\partial\xi^*}\right)^n \exp(\alpha\xi^* - \alpha^*\xi) \\
&= \int \mathrm{d}^2\xi\, \chi(\xi,p) \left(-\frac{\partial}{\partial\xi}\right)^m \left(\frac{\partial}{\partial\xi^*}\right)^n \delta^{(2)}(\xi) \\
&= \left(\frac{\partial}{\partial\xi}\right)^m \left(-\frac{\partial}{\partial\xi^*}\right)^n \chi(\xi,p)\Big|_{\xi=0} = \langle \hat{a}^{\dagger m}\hat{a}^n\rangle_p,
\end{aligned} \tag{1.25}$$

where the derivatives of the Dirac δ are intended as distributional derivatives and the last equality is due to the definition (1.17).

All these properties make $W(\alpha,p)$ similar to a probability distribution, because it can be used to compute moments and it is real-valued and normalized. Anyhow it is not a true probability distribution because it can in general attain negative values and for this reason it is called a quasiprobability distribution.

Now we want to see that $W(\alpha,p)$ is just a Guassian convolution of $W(\alpha,p')$, with $p' < p$; from the definition (1.15) characteristic functions with different ordering parameters are related by

$$\chi(\alpha,p') = \chi(\xi,p)\exp\left[-(p-p')|\xi|^2/2\right]. \tag{1.26}$$

Thus the corresponding distributions are related by

$$W(\alpha,p') = \frac{1}{\pi^2}\int \mathrm{d}^2\xi\, \chi(\xi,p)\exp\left(-(p-p')|\xi|^2/2\right)\exp(\alpha\xi^* - \alpha^*\xi), \tag{1.27}$$

which is the Fourier transform of the product of $\chi(\xi,p)$ and a Gaussian function. The convolution theorem states that $\mathscr{F}[f\,g] = \mathscr{F}[f] * \mathscr{F}[g]$, where $\mathscr{F}$ denotes the Fourier transform and $*$ the convolution operation, so we get the following expression

$$W(\alpha,p') = \frac{2}{\pi(p-p')}\int \mathrm{d}^2\beta\, W(\beta,p)\exp\left[-\frac{2|\alpha-\beta|^2}{(p-p')}\right], \tag{1.28}$$

which is the convolution of $W(\beta,p)$ with a Gaussian distribution. The operational and intuitive meaning of this definition is that as p' decreases the distribution becomes smoother, since the convolution with a Gaussian function can be thought as a smoothness increasing operation which makes peaks of the function become broader.

1.1.4 P function and Q function

If we set $p = 1$ we get the Glauber-Sudarshan P function, which is the quasiprobability distribution corresponding to normal ordering. The P function is also used to express the density operator ρ as a diagonal sum over coherent states

$$\rho = \int \mathrm{d}^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|. \tag{1.29}$$

It has to be noted that this expression is not trivial at all, since generally one would need a double integration

$$\rho = \frac{1}{\pi^2}\int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, \rho(\alpha,\beta)|\alpha\rangle\langle\beta|, \tag{1.30}$$

it is possible to express states in diagonal form because coherent states are overcomplete, which means that is possible to write a resolution of the identity, but the states are not orthogonal. However it is important to note that $P(\alpha)$ exists as a proper function only for some states and it can fail to be interpreted even as a distribution. This is consistent with expression (1.28) since for $p' = 1$ no Gaussian smoothing is present, therefore in general P can be highly singular. For a mathematically precise derivation of this diagonal representation the reader should look at Ref. [11]. To explicitly see what we mean by "highly singular" we report from Ref. [12] a formal way to write the P-function of an arbitrary state:

$$P(\alpha) = \sum_{n,m} \rho_{nm}(-1)^{n+m} e^{|\alpha|^2} \frac{1}{\sqrt{n!\,m!}} \frac{\partial^{n+m}}{\partial\alpha^n \partial\alpha^{*m}}\delta(\alpha), \tag{1.31}$$

where $\rho_{nm} = \langle n|\rho|m\rangle$; it is evident that (1.31) is in general an highly singular expression and it is surprising that for some states, such as thermal states it can become a well behaved function.

We have now to show that the two formulations of the P function are equivalent, to do so we write $W(\alpha, 1)$ inserting ρ in the form (1.29), so we have

$$\begin{aligned} W(\alpha,1) &= \frac{1}{\pi^2}\int \mathrm{d}\xi\ \mathrm{Tr}\left[\int \mathrm{d}^2\beta\, P(\beta)|\beta\rangle\langle\beta| \exp(\xi\hat{a}^\dagger)\exp(-\xi^*\hat{a})\right]\exp(\alpha\xi^* - \alpha^*\xi) \\ &= \frac{1}{\pi^2}\int \mathrm{d}^2\xi \int \mathrm{d}^2\beta P(\beta)\exp\left[\xi(\beta^* - \alpha^*) - \xi^*(\beta - \alpha)\right] = P(\alpha), \end{aligned} \tag{1.32}$$

because the integration over ξ on the last line gives $\delta^{(2)}(\beta - \alpha)$.

In we choose the value $p = -1$ we get the so called Husimi Q-function, it will be the least interesting for the aim of the present work so it will not be studied in detail. This function has a simple representation in terms of the density matrix

$$Q(\alpha) = W(\alpha, -1) = \langle\alpha|\rho|\alpha\rangle; \tag{1.33}$$

it is always positive semi-definite function because ρ is a positive semi-definite operator.

1.1.5 Wigner Function

The Wigner function is the symmetrically ordered quasiprobability distribution $W(\alpha, 0) = W(\alpha)$ and it is useful to get expectation values of operators in symmetric order. From defi-

nition (1.20) we have

$$\begin{aligned} W(\alpha) &= \frac{1}{\pi^2}\int \mathrm{d}^2\xi\ \mathrm{Tr}[\rho D(\xi)]\exp(\alpha\xi^* - \alpha^*\xi) \\ &= \frac{1}{\pi^2}\int \mathrm{d}^2\xi\ \mathrm{Tr}\left\{\rho\exp\left[\xi(\hat{a}^\dagger - \alpha^*) - \xi^*(\hat{a}-\alpha)\right]\right\} \\ &= \frac{1}{\pi^2}\int \mathrm{d}^2\xi\ \mathrm{Tr}\left[\rho\hat{D}(\alpha)\hat{D}(\xi)\hat{D}^\dagger(\alpha)\right] \\ &= \mathrm{Tr}\left[\rho\hat{D}(\alpha)\hat{T}\hat{D}^\dagger(\alpha)\right] \end{aligned} \tag{1.34}$$

where we used the property to pass from the second to the third line and we defined $\hat{T} = \pi^{-2}\int \mathrm{d}^2\xi\,\hat{D}(\xi)$. To further simplify the form of the Wigner function it is convenient to express $\hat{T}$ as a function of $\hat{a}^\dagger\hat{a}$. Putting the displacement operator in normal ordering we get to the following result

$$\hat{T} = \frac{1}{\pi^2}\int \mathrm{d}^2\xi\ \exp(\xi\hat{a}^\dagger)\exp(-\xi\hat{a})\exp(-|\xi|^2/2), \tag{1.35}$$

if now we expand every exponential using the definition $\exp(\hat{A}) = \sum_{n=0}^{\infty}\hat{A}^n/n!$ and we perform the integration over the complex plane using polar coordinates we have the following expression

$$\begin{aligned} \hat{T} &= \frac{1}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{n!\,m!}\hat{a}^{\dagger n}\hat{a}^m\int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}\mathrm{d}|\xi|\,|\xi|^{n+m+1}\exp[\mathrm{i}\phi(n-m)]\exp(-|\xi|^2/2) \\ &= \frac{2}{\pi}\sum_{n=0}^{\infty}\hat{a}^{\dagger n}\hat{a}^n 2^n \end{aligned} \tag{1.36}$$

where we used the integral representation of the Kronecker delta $\delta_{m,n} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\, e^{\mathrm{i}(n-m)\phi}$ and the Gaussian integral $\int_0^{\infty}\mathrm{d}x\,x^{2n+1} = n!\,2^n$.

The general formula for the normally ordered exponential of $\hat{a}^\dagger\hat{a}$ is the following

$$\exp[\theta\hat{a}^\dagger\hat{a}] = \sum_{k=0}^{\infty}\frac{(e^\theta - 1)^k}{k!}\hat{a}^{\dagger k}\hat{a}^k, \tag{1.37}$$

(see appendix A.1 for more details), so we get to write $\hat{T}$ as follows

$$\hat{T} = \frac{2}{\pi}\exp\left(\mathrm{i}\pi\hat{a}^\dagger\hat{a}\right) = \frac{2}{\pi}(-1)^{\hat{a}^\dagger\hat{a}}. \tag{1.38}$$

The operator $(-1)^{\hat{a}^\dagger\hat{a}}$ can be easily decomposed on the number basis, since $|n\rangle$ are its eigenvectors, and it reads

$$(-1)^{\hat{a}^\dagger\hat{a}} = \sum_{n=0}^{\infty}(-1)^n|n\rangle\langle n|. \tag{1.39}$$

Now we can finally rewrite (1.34) to get the to this expression

$$W(\alpha) = \frac{2}{\pi} \operatorname{Tr}\left[\rho \hat{D}(\alpha)(-1)^{\hat{a}^\dagger \hat{a}} \hat{D}^\dagger\right], \tag{1.40}$$

which can be in turn still manipulated to get

$$W(\alpha) = \frac{2}{\pi} \operatorname{Tr}\left[\rho \hat{D}(2\alpha)(-1)^{\hat{a}^\dagger \hat{a}}\right]. \tag{1.41}$$

This last derivation follows from the effect of $\exp(\mathrm{i}\pi\hat{a}^\dagger\hat{a})$ as a reflection operator which is given by the following relations

$$\begin{aligned} \exp(\mathrm{i}\pi\hat{a}^\dagger\hat{a})\hat{a}\exp(-\mathrm{i}\pi\hat{a}^\dagger\hat{a}) &= -\hat{a} \\ \exp(\mathrm{i}\pi\hat{a}^\dagger\hat{a})\hat{a}^\dagger\exp(-\mathrm{i}\pi\hat{a}^\dagger\hat{a}) &= -\hat{a}^\dagger, \end{aligned} \tag{1.42}$$

they can be proved by explicit calculation, using properties (1.37) and (1.4). Equations (1.42) in turn imply that

$$\exp(\mathrm{i}\pi\hat{a}^\dagger\hat{a})\hat{D}(\alpha)\exp(-\mathrm{i}\pi\hat{a}^\dagger\hat{a}) = D(-\alpha). \tag{1.43}$$

When we put (1.43) into the expression (1.40) we finally get the compact form (1.41).

One more property of the Wigner function is the fact that it is bounded. From (1.41) when $\alpha = 0$ we get

$$W(0) = \frac{2}{\pi}\sum_n \rho_{nm}(-1)^n, \tag{1.44}$$

this clearly shows that $|W(0)| \leq 2/\pi$, and from definition (1.34) we also see that $|W(\alpha)| \leq |W(0)|$ so we have the general bound

$$|W(\alpha)| \leq |W(0)| \leq \frac{2}{\pi}. \tag{1.45}$$

The Wigner function always exists, even when the P function for the same state is singular, but in general it fails to be positive.

Wigner Function From Wave Functions

At this point we have to remark that (1.34) is not the definition originally introduced by Wigner in his paper. Writing W as in Eq. (1.41) is very useful when we deal with modes of the electromagnetic field (or a harmonic oscillator). On the other hand if we deal with other bosonic systems with finite mass, which in practical terms means anharmonic oscillators, it is more convenient to use the original expression written in term of the wave function in coordinate space. This will be of great importance to us, since in chapter 4 we will work with ground states of anharmonic quantum oscillators, for which the wave functions are found by solving the Schrödinger equation.

To get to this form we start by rewriting (1.34) using position and momentum operators:

$$W(u,v) = \frac{1}{2\pi^2}\int_{-\infty}^{+\infty} \mathrm{d}\xi \int_{-\infty}^{+\infty} \mathrm{d}\eta \operatorname{Tr}\left[\rho \exp(\mathrm{i}\hat{x}\xi + \mathrm{i}\hat{p}\eta)\exp(-\mathrm{i}u\xi - \mathrm{i}v\eta)\right], \tag{1.46}$$

then we express the density operator in coordinate-space representation as

$$\rho = \iint \mathrm{d}x\mathrm{d}x' \, |x\rangle\langle x \, |\langle x|\rho|x'\rangle; \tag{1.47}$$

using again the Baker-Campbell-Hausdorff identity for the exponential we have that $\exp(\mathrm{i}\hat{x}\xi + \mathrm{i}\hat{p}\eta) = \exp(\mathrm{i}\hat{x}\xi)\exp(\mathrm{i}\hat{p}\eta)\exp(\frac{1}{2}\mathrm{i}\eta\xi)$, since $[\hat{x},\hat{p}] = \mathrm{i}$, so we get to

$$W(u,v) = \frac{1}{(2\pi)^2}\int \mathrm{d}\xi \int \mathrm{d}\eta \int \mathrm{d}x \int \mathrm{d}x' \langle x|\rho|x'\rangle\langle x'|e^{\mathrm{i}\hat{x}\xi}e^{\mathrm{i}\hat{p}\eta}|x\rangle e^{\frac{1}{2}\mathrm{i}\xi\eta}e^{-\mathrm{i}u\xi-\mathrm{i}v\eta}. \tag{1.48}$$

If we remember that the operator $\exp(\mathrm{i}\hat{p})$ is the generator of the translations we can write that

$$\langle x'|e^{\mathrm{i}\hat{x}\xi}e^{\mathrm{i}\hat{p}\eta}|x\rangle = e^{\mathrm{i}x'\xi}\delta(x'-x+\eta), \tag{1.49}$$

moreover the ξ integral in (1.48) gives $2\pi\delta(x' + \frac{1}{2}\eta - u)$ so we can finally write

$$W(u,v) = \frac{1}{2\pi}\int \mathrm{d}\eta \, \langle u + \frac{1}{2}\eta|\rho|u - \frac{1}{2}\eta\rangle e^{-\mathrm{i}v\eta}, \tag{1.50}$$

which is the form introduced by Wigner [3] and for pure states it reduces to

$$W(u,v) = \frac{1}{2\pi}\int \mathrm{d}\eta \, \psi\left(u+\frac{1}{2}\eta\right)^{*}\psi\left(u-\frac{1}{2}\eta\right)e^{-\mathrm{i}v\eta}. \tag{1.51}$$

1.2 Definition of Nonclassical States

According to the generally accepted definition formulated by Titulaer and Glauber [13, 14] a quantum state is considered nonclassical when its P function fails to be interpreted as a probability distribution on the phase space. Using this definition, any correlation function of a classical state can be modelled using classical electrodynamics. It has been recently emphasized [15] that the P function is the only quasiprobability distribution which can give a description completely analogous to the classical case, therefore supporting again the idea that to fully identify a classical state it is necessary to use the P function.

This definition immediately creates a hierarchy of states: there are states which possess a well-behaved and positive P function, the most important example being the thermal state

$$\rho_T = \exp(-\beta\hbar\omega\hat{a}^{\dagger}\hat{a})/\operatorname{Tr}\left[\exp(-\beta\hbar\omega\hat{a}^{\dagger}\hat{a})\right], \tag{1.52}$$

whose P-function is a Gaussian:

$$P(\alpha) = \frac{1}{\pi\langle\hat{n}\rangle}\exp\left[-\frac{|\alpha|^2}{\langle\hat{n}\rangle}\right], \tag{1.53}$$

where the mean particle number is $\langle\hat{n}\rangle = 1/(\exp[\beta\hbar\omega] - 1)$.

Then we find that coherent states lie on the border between classical and nonclassical since their P function is just a δ, which can still be interpreted as a probability distribution even

if it is a singular function. We have to highlight that they are the only pure states which are classical according to this definition. Finally we have truly nonclassical states; for example a Fock state $\rho = |m\rangle\langle m|$, for which (1.31) gives a very singular P-function:

$$P(\alpha) = \frac{1}{m!} e^{|\alpha|^2} \left(\frac{\partial^2}{\partial\alpha\partial\alpha^*} \right) \delta(\alpha). \tag{1.54}$$

Therefore the signature of nonclassicality according to this definition is having a P function with negative values or more singular than a δ.

Even if nonclassicality defined in this way has a clear intuitive interpretation, it is unfortunately problematic to use practically because the P function can be a very difficult object to manipulate. For this reason a plethora of criteria to detect nonclassicality have been introduced, some more fundamental while others more applicative and experimental friendly. Our focus will not be on these *criteria* for nonclassicality, but on a *quantitative characterization* of it. With the expression quantitative characterization we mean the ability to express nonclassicality as a function of a paramater on which the state depends, in order to quantify *how* nonclassical the state is, not just if it is classical or not. From this fundamental definition it is not straight-forward how to define such a quantitative measure unambiguously; an overview of the different approaches introduced in the scientific literature will be presented in section 1.3.

1.2.1 Reformulation in Terms of Characteristic Functions

Over the last few years Vogel and collaborators have derived a series of conditions based on characteristic functions, which are equivalent to the P function definition, but easier to check because they do not involve singularities. For a complete treatment of the topic we refer to Ref. [16] and here we just want to sketch the main idea.

This reformulation is based on the Bochner theorem, which provides conditions for a continuous function to be a true probability density. It states that a continuous function $\chi(\xi)$, which obeys to $\chi(0) = 1$ and $\chi(\xi) = \chi^*(-\xi)$ is the characteristic function of a probability density iff $\chi(\xi)$ is non negative, which means that for two arbitrary vectors of complex numbers α_i and ξ_i the following relation has to be satisfied

$$\sum_{i,j=1}^{n} \chi(\xi_i - \xi_j)\alpha_j^*\alpha_i \geq 0, \tag{1.55}$$

for every n. According to definition (1.20) the P function is the Fourier transform of the characteristic function $\chi(\xi, 1)$, which in turn can be espressed as the (anti) Fourier transform of $P(\alpha) = W(\alpha, 1)$. This means that if the inequality (1.55) is violated for some α_i and ξ_j then P fails to be a probability distribution and therefore the state is nonclassical.

This condition is still very hard to check, since it involves an infinite number of inequalities; for $n = 1$ it can never be violated since $\chi(0) = 1$ and $|\xi|^2 \geq 0$, but in many cases it is enough to check the condition for $n = 2$. By determining the extreme values of the function in the

inequality (1.55) for $n = 2$, with respect to the phase difference and the ratio $|\alpha_1|/|\alpha_2|$, the following condition is derived

$$|\chi(\xi)| > 1, \tag{1.56}$$

if a state violates this condition is said to be nonclassical of first order. This is of course just a sufficient condition for nonclassicality and more conditions can be expressed based on the inequalities for growing n, thus defining nonclassicality of higher orders.

Even if the characteristic function itself cannot be measured it can be related to the characteristic function of the quadrature distributions or to the measurable moments of an operator. These quantities turn out to be very useful in experimental certification of nonclassicality and a lot of work has been done about them, notwithstanding we are not interested in this operational aspect. These criteria of nonclassicality based on characteristic functions can also be used to formulate ways to give a quantitative characterization [17]. This approach has not has not been used in literature and it will not be used in this Thesis, so it was left behind from the review in the following section.

1.3 Quantitive Characterization of Nonclassicality

As we said the problem of characterizing a nonclassical state in a quantitative manner consists in finding a way to measure how nonclassical a certain state is by find an appropriate nonclassicality measure which shows the correct dependence on the relevant parameters. For example a highly squeezed coherent state is generally considered more nonclassical than a slightly squeezed coherent state, they have the same analytical form but a different value of the squeezing parameter; a good measure should reflect this. Various ways to achieve this task have been developed, but every approach has its weaknesses, so we want to collect them all in this section, even if just two of them are actually used in the sequel, namely the negative volume of the Wigner function and the entanglement potential.

1.3.1 Distance Based Measures

As an attempt to attack the problem of quantifying nonclassicality, Hillery introduced the distance between two quantum states [18], an idea which has then been applied in many other scenarios. The distance introduced by Hillery was the trace norm of an operator:

$$\|\hat{O}\|_{\mathrm{Tr}} = \mathrm{Tr}\left[\sqrt{\hat{O}^\dagger \hat{O}}\right], \tag{1.57}$$

so that the distance between two states reads

$$d_{\mathrm{Tr}}(\rho_1, \rho_2) = \|\rho_1 - \rho_2\|_{\mathrm{Tr}}. \tag{1.58}$$

The definition of nonclassicality of a state that follows from here is then

$$D(\rho) = \inf_{\rho_{\mathrm{cl}}} \|\rho - \rho_{\mathrm{cl}}\|_{\mathrm{Tr}} \tag{1.59}$$

which represents the minimal distance of the state ρ from the set of nonclassical states ρ_{cl}.

This definition is appealing because in principle it allows us to quantify nonclassicality in a continuous way, nonetheless in general the task of minimizing this distance for arbitrary states it is very challenging at best. Hillery derived some upper and lower bounds for this quantity, but not much more can be done.

The idea of measuring nonclassicality using a distance is however conceptually very appealing, therefore other definitions of nonclassicality based on distances were given: the Hilbert-Schmidt norm has been used [19]

$$d_{\mathrm{HS}}(\rho_1, \rho_2) = \sqrt{\frac{1}{2}\operatorname{Tr}\left[(\rho_1 - \rho_2)^2\right]}, \tag{1.60}$$

as well as the Bures measure [20]

$$d_{\mathrm{B}}(\rho_1, \rho_2) = \sqrt{2 - 2\operatorname{Tr}\left\{\left(\sqrt{\rho_1}\rho_2\sqrt{\rho_1}\right)^2\right\}}. \tag{1.61}$$

Unfortunately none of these measures turned out to be particularly useful for practical purposes, and their application is restricted to some class of states [21]. It has also been argued that using topological distances to quantify nonclassicality can lead to ambiguous results, since the property of the chosen distance can have a dominant influence on the amount of nonclassicality obtained [22].

1.3.2 Nonclassical Depth

The way to quantify nonclassicality which has attracted more attention over the years is perhaps the nonclassical depth introduced by Lee [23, 24]. When we introduced quasiprobability distribution we noted that distributions for different values of p (for different orderings) could be related by Gaussian convolution through equation (1.28), if we write this relation for $p = 1$ we get

$$W(\alpha, p') = \frac{2}{\pi(1-p')}\int \mathrm{d}^2\beta P(\beta)\exp\left(-\frac{2|\alpha-\beta|^2}{1-p'}\right). \tag{1.62}$$

Nonclassical depth measures how much a singular P function has to be convoluted in order to become a well behaved probability distribution.

If we define the parameter $\tau = \frac{1-p}{2}$ then the previous equation becomes the R function introduced by Lee

$$R(\alpha, \tau) = \frac{1}{\tau}\frac{1}{\pi}\int \mathrm{d}^2\beta P(\beta)\exp\left(-\frac{1}{\tau}|\alpha-\beta|^2\right), \tag{1.63}$$

which corresponds to the P function for $\tau = 0$, to the Wigner function for $\tau = 1/2$ and to the Q function for $\tau = 1$.

The nonclassical depth is thus defined as

$$\tau_{\mathrm{m}} = \inf_{\tau\in\mathscr{C}}(\tau), \tag{1.64}$$

where $\mathscr{C}$ represents the set of all the τ which make the $\boldsymbol{R}$ function a proper probability distribution, thus completing the "smoothing" of the $\boldsymbol{P}$ function.

There is also a more practical way to interpret the nonclassical depth, following from the fact that the $\boldsymbol{P}$ function of the superposition of two states can be expressed by the convolution of the $\boldsymbol{P}$ functions of the two states [4]. Thanks to the fact that the $\boldsymbol{P}$ function of the thermal state is a Gaussian

$$P_{\mathrm{t}}(\alpha) = \frac{1}{\langle \hat{n} \rangle} \exp\left(-\frac{|\alpha|^2}{\langle \hat{n} \rangle} \right), \tag{1.65}$$

we can get to the result that the nonclassical depth of a quantum state in superposition with a thermal state, denoted by $\tau_{\mathrm{m}}^{\mathrm{t}}$, is reduced by the number of thermal photons. In formulas becomes $\tau_{\mathrm{m}}^{\mathrm{t}} = \tau_{\mathrm{m}} - \langle \hat{n} \rangle$, where τ_{m} is the the nonclassical depth of the state without thermal noise. This in turn implies that the nonclassical depth itself is the minimum number of thermal photons needed to destroy the nonclassical properties of the state.

Despite being a very appealing quantity because of its interpretation, nonclassical depth is quite hard to compute in general and has a defect which makes it sometimes unsuited for practical purposes: it very often saturates to its maximum value 1. For example all number state have $\tau_{\mathrm{m}} = 1$, despite of the number of photons. Actually if we restrict to pure states there is a quite general rule: it has been demonstrated [25] that the only pure states which have $\tau_{\mathrm{m}} < 1$ are states with a Gaussian $\boldsymbol{Q}$ function, which are only the squeezed coherent states, defined as follows

$$|\xi, \beta\rangle = \hat{D}(\beta)\hat{S}(\xi)|0\rangle = \exp\left(\beta \hat{a}^\dagger - \beta^* \hat{a}\right) \exp\left[\frac{1}{2}\left(\xi \hat{a}^{\dagger 2} - \xi^* \hat{a}^2 \right) \right] |0\rangle; \tag{1.66}$$

where $\hat{S}$ is the so called squeezing operator. This fact poses a great limitation for the aim of the present work, since we will deal with pure states only and τ_{m} would just tell us that the states are nonclassical, but not specify how much.

1.3.3 Negative Volume of the Wigner Function

Here we introduce a measure which will be very useful in the following chapters, which was proposed by Kenfack and Życzkowski [26]. As we stated previously the Wigner function is always a well behaved function, even if the $\boldsymbol{P}$ function is singular, but it can attain negative values. Checking for negativities of the Wigner function has long been a practical way to witness nonclassicality in quantum optics, since it can be measured experimentally.

The intuitive idea is that measuring the volume of the negative part of this function can be used as a way to quantify nonclassicality. The following quantity δ represents the double of the volume of the negative part of the Wigner function

$$\delta = \int \mathrm{d}^2\alpha \, [|W(\alpha)| - W(\alpha)] = \left(\int \mathrm{d}^2\alpha \, |W(\alpha)| \right) - 1, \tag{1.67}$$

where the equality follows from the normalization of the Wigner function and the integration is of course performed over the whole complex plane. A normalized version of this measure

can be defined in a straight-forward way:

$$\nu = \frac{\delta}{1+\delta}, \tag{1.68}$$

though in the following we will use δ rather than the normalized version, since there is no significant difference in their meaning. We can see that the calculation of this measure of nonclassicality boils down to the integration of the absolute value of the Wigner function, which is an achivable task (at least numerically) for a broader class of states than the previous measures. The main disadvantage of this way of characterizing nonclassicality is that negativities in the Wigner function are just a sufficient condition of nonclassicality, since we can have a positive Wigner functions even for singular P functions.

We also want to report that negativities in the Wigner function have been linked to other notions of nonclassicality, namely classical efficency in simulating a quantum system [27, 28].

A Remark About Nonclassicality of Pure States

If we concentrate just on pure states we have a theorem by Hudson [29] which affirms that the only states with a positive Wigner function are the squeezed coherent states (1.66). This is a quite relevant remark, because for the same class of states we have that the nonclassical depth is saturated to $\tau_{\mathrm{m}} = 1$. So the complete picture is that for pure states these two measures are complementary, in the sense that when nonclassical depth becomes useless in distinguishing the quantity of nonclassicality, at the same time the measure δ become useful. The fact that $\delta \neq 0$ remains just a sufficient condition will still be a thing to keep in mind, because squeezing effects which do not change the volume of the negative part of W can occur and not be captured by this measure.

1.3.4 Entanglement Potential

In quantum optics it has long been known that coherent states are the only states which produce uncorrelated outputs when going through a linear optics device [30], in particular nonclassicality has been identified as a prerequisite for having entangled states after a beam splitter [31]. From these considerations the idea of quantifying nonclassicality of a single mode state as the two mode entanglement at the output of a linear optic device was born, introduced by Asbóth et al.[32]. This measure is suitably called entanglement potential (*EP*) and it will be used extensively in the following analysis.

At first it seems that *EP* could be quite arbitrary, since in principle for any nonclassical state to be measured one would have to choose the auxiliary states and the optimal linear optics transformation to create as much entanglement as possible. It was shown in [32] that this is not the case and the optimal entangler is given just by a beam splitter with vacuum as an auxiliary state. This can be explained briefly by simple arguments.

Any passive linear optics transformation can be modelled by a generic circuit of beam splitters, see fig. 1.1. We have one nonclassical input mode σ and a number of auxiliary states ρ_n (ancillas). At the output we have an observer A receiving one output mode, an observer

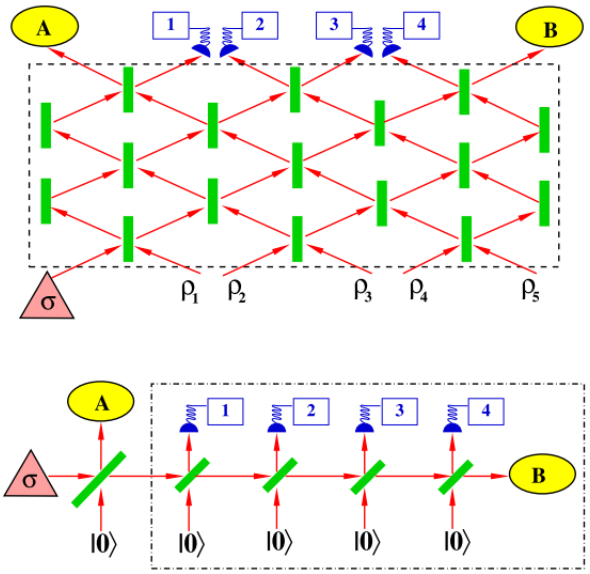


Figure 1.1: (top) The general optical circuit for creating entanglement between A and B, ρ_n are auxiliary classical states. (bottom) The circuit can be simplified by choosing vacuum ancillas, the dashed box is local to B. Figure taken from [32].

B who gets a second mode and all the others outputs are measured by ideals photodetectors. The auxiliary input states must be coherent states, otherwise entanglement would not be caused just by σ. A coherent state is just a displaced vacuum and displacing the auxiliary input states corresponds to the displacement of all the output states, in a way determined by the particular form of the circuit. Mixing the input modes states results in local mixing of the output states with additional classical communication. All these operations cannot increase entanglement, therefore we can choose $|0\rangle$ as ancillas. With this choice the circuit can be recast in a simpler form, see fig. 1.1, which is a single beam splitter splitting the input in two modes going to A and B, with the signal going to B split another arbitrary number of times. All measurements can be carried out using the modes of B, but local operations cannot increase entanglement, so there is no gain for B in splitting the beam and measuring just a part. This proves that the optimal entangler is a single beam splitter. The transmissivity of the beam splitter has yet to be choosen; even if there is no proof, it is argued in [32] that the 50:50 beam splitter is optimal independently of the input state.

As the last step we have to choose an appropriate measure for bipartite entanglement at the output; this is clearly a downside because different entanglement measures could lead to a different classification of nonclassical states. As long as we deal with a pure state ρ in a finite dimensional Hilbert space the Von Neumann entropy of the reduced density matrix is a good measure of the entanglement of a generic state ρ and it is defined as

$$\begin{aligned} E[\rho] &= S[\rho_B] = -\operatorname{Tr}_B[\rho_B \log \rho_B] \\ &= S[\rho_A] = -\operatorname{Tr}_A[\rho_A \log \rho_A], \end{aligned} \tag{1.69}$$

where $\rho_A = \operatorname{Tr}_B \rho$ and $\rho_B = \operatorname{Tr}_A \rho$ are the reduced density operators. In general they represent completely different states but they always possess the same amount of Von Neumann

entropy, which, for a generic state density operator σ, is defined as $S[\sigma] = -\operatorname{Tr}\left[\sigma \log \sigma\right]$. In the following we will deal with pure states so we choose to quantify entanglement with the measure $E(\rho)$. This choice corresponds to the *entropic entanglement potential* defined in [32], where entanglement is measured using the relative entropy of entanglement, which reduces to the Von Neumann entropy for pure states.

The evolution operator of the beam splitter has the following form

$$\hat{U}(\xi) = \exp\{\xi a^\dagger b - \xi^* a b^\dagger\} \tag{1.70}$$

which can be disentangled using the Schwinger two-mode boson representation of SU(2) (see for example Ref. [33]) in order to achieve the normal ordering in the mode $\hat{b}$:

$$\hat{U}(\xi) = \exp\{-e^{-\mathrm{i}\theta} \tan\phi\, \hat{a}\hat{b}^\dagger\} (\cos\phi)^{\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}} \exp\{e^{\mathrm{i}\theta} \tan\phi\, \hat{a}^\dagger \hat{b}\} \tag{1.71}$$

where $\xi = \phi e^{\mathrm{i}\theta}$. In this case $\hat{b}$ is the mode associated with the vacuum input, so that $\hat{b}|0\rangle = 0$ and the operator $\exp\{e^{\mathrm{i}\theta} \tan\phi\, \hat{a}^\dagger \hat{b}\}$ in (1.71) becomes the identity. To have a 50:50 beam splitter means choosing $\phi = \frac{\pi}{4}$; moreover we choose $\theta = 0$ in order not to add a phase to the output states. If we call $\hat{U}\left(\frac{\pi}{4}\right) = \hat{B}$ then we can finally define the entanglement potential of a generic state as

$$EP[\rho] = E\left[\hat{B}(\rho \otimes |0\rangle\langle 0|)\hat{B}^\dagger\right]. \tag{1.72}$$

1.3.5 Algebraic Measure

In a recent paper [34] a completely different approach to the problem of quantifying nonclassicality was proposed. This method is not related to quasiprobability distributions but it is based on the superposition principle. This is a fundamental feature of quantum mechanics and it is purely algebraic in nature, thus independent of geometrical features. For a pure single mode state this measure is defined as the minimum number r of coherent states necessary to write it as follow

$$|\Psi\rangle = \sum_{i=1}^{r} \kappa_i |\alpha_i\rangle. \tag{1.73}$$

Since coherent states are the only pure classical states, this definition tries to capture nonclassicality by counting how many classical states in superposition are needed to create the nonclassical state in exam.

The main feature of this measure is the fact that it can be used to quantify bipartite entanglement and nonclassicality in a unified way [35]. Exploiting the beam splitter duality between nonclassicality of the input state and entanglement of the output it can be proved that the minimum number of coherent states necessary to represent the input state is equal to the Schmidt rank of the output state. This results can also be extended to multimode states and multipartite entanglement.

This idea is conceptually very neat but it is in completely different from the other measures introduced and it gives a completely different hierarchy of states. For example it is known that the nonclassical depth of squeezed states is bounded from above by $\tau_{\mathrm{m}} = 1/2$, while number states have $\tau_{\mathrm{m}} = 1$; using this algebraic measure we have a completely different

result. It was shown in fact that $r = \infty$ for squeezed states, while $r = n + 1$ for the Fock state $|n\rangle$. Another example is the famous cat state $|\alpha\rangle + |-\alpha\rangle$, which has negative parts in the Wigner function, therefore being more nonclassical than a squeezed state (with a Gaussian Wigner function); on the other hand according to this new definition the cat state has $r = 2$ so it is minimally nonclassical.

We will not use this measure because it seems to capture a different notion of nonclassicality, giving such a different categorization. On a more practical level there is also another problem, which lies in the fact that r is an natural number, so even if it quantitatively characterize some features correctly (number states with increasing n are more and more nonclassical) it is unsuited to express a proper dependence on a continuous real parameter.

2 Quantifying the Nonlinearity of a One-dimensional Potential

Oscillators are one of the most fundamental concepts in physics and they are used to represent and model a plethora of different physical situations, both in the classical and quantum regime. The harmonic oscillator is the best known example and the easiest one to study, but nonlinear oscillators have attracted a lot of interest, both from a mathematical point of view and from a more applicative one.

In particular in the context of discrete variable quantum information nonlinear oscillators could be useful because they produce unequally spaced energy levels so they allow us to engineer two-level systems. On the other hand in continuous variable systems the concept of Gaussian states is fundamental, they are simply quantum states with a Gaussian Wigner function. Much of the efforts done so far in continuous variable quantum information involve this kind of states, even though in recent years non-Gaussian states have increasingly been recognized as an important resource for various quantum technology processes. In this context nonlinear oscillators could play an important role since their ground states and their equilibrium states are not Gaussian, as opposed to the ones of a quantum harmonic oscillator.

These arguments are an indication that finding a method to quantify the nonlinear character of a quantum oscillator would be useful and interesting, since nonlinearity can represent a resource in various applications. One idea to do so could be to define a distance between potential functions and the reference harmonic potential, but this turns out to be not feasible in general, since potentials do not need to be integrable functions. In this chapter we will present some ideas useful to quantify the the *anharmonic character* of a potential, by studying its ground state.

2.1 Quantifying Non-Gaussianity of a State

We now proceed to define a measure of the non-Gaussian character of a quantum state, as introduced by Genoni and Paris [36, 37]. In doing so we will briefly review these ideas in a classical context and also review some properties of Gaussian states.

2.1.1 Classical Probability Distributions

In classical probability theory the Gaussian distribution is of paramount importance, thanks to the central limit theorem it is used to describe countless natural phenomena. Thus finding a way to quantify deviations from a perfect Gaussian behaviour is an important problem and

there are mainly two approaches. The former one is about evaluating the moments of the distribution, while the latter makes use of Shannon entropy.

The kth central moments of random variable x with a probability density $p(x)$ are defined as

$$E[(x-\mu)^k] = \int_{-\infty}^{\infty} \mathrm{d}x' \, (x'-\mu)^k p(x'), \tag{2.1}$$

where

$$\mu = \int_{-\infty}^{\infty} \mathrm{d}x' \, x' p(x'). \tag{2.2}$$

The variable x is Gaussian distributed if

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{x-\mu}{2\sigma^2} \right\}, \tag{2.3}$$

where $\sigma^2 = E[(x-\mu)^2]$ is the variance.

The first way to quantify non-Gaussianity is the kurtosis, defined as follows

$$K(x) = E[(x-\mu)^4] - 3\sigma^2. \tag{2.4}$$

This quantity is zero for Gaussian variables and different from zero for most non-Gaussian variables, however it is not considered a robust measure since it may strongly depend on the observed data.

A more robust way to quantify non-Gaussianity is using the so-called differential entropy, which is the continuous version of the Shannon entropy and is defined as

$$H(x) = -\int \mathrm{d}x' \, p(x') \ln p(x'). \tag{2.5}$$

It is widely known that Gaussian variables are the ones that maximize this entropy at fixed variance, so this allows us to define a measure of non-Gaussianity called negentropy

$$N(x) = H(g) - H(x), \tag{2.6}$$

where g is a Gaussian variable with the same variance of x; negentropy is always non negative and equal to zero only for Gaussian variables.

2.1.2 Quantum Gaussian States

Gaussian states are n modes bosonic states with a Gaussian Wigner function, a general treatment of their properties is beyond our goals, but a complete review of the subject can be found in Ref. [33]. We will deal only with the simple case of a single mode state, so we just have a single pair of creation and destruction operators $\hat{a}$ and $\hat{a}^\dagger$, and a single pair of canonical operators $\hat{q}$ and $\hat{p}$.

Introducing the vector $\boldsymbol{R} = (\hat{x}, \hat{p})^T$ we can define the covariance matrix of a single mode states as follows

$$\sigma_{jk} = \frac{1}{2}\langle \{\hat{R}_j, \hat{R}_k\} \rangle - \langle R_j \rangle \langle R_k \rangle, \tag{2.7}$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is the anticommutator. We also define the mean vector $\bar{\boldsymbol{X}}$, its components are $X_k = \langle R_k \rangle$.

A state is said to be Gaussian if its Wigner function is Gaussian and therefore can be written in the following manner

$$W(\boldsymbol{X}) = \frac{1}{2\pi\sqrt{\det[\sigma]}} \exp\Big[-\frac{1}{2}\big(\boldsymbol{X} - \bar{\boldsymbol{X}}\big)^T \sigma^{-1} \big(\boldsymbol{X} - \bar{\boldsymbol{X}}\big)\Big], \tag{2.8}$$

where $\boldsymbol{X} = (\mathrm{Re}\, z, \mathrm{Im}\, z)$. This definition means that, even if Gaussian states are continuous variable states, they are the simplest ones because they are fully determined by the knowledge of $\bar{\boldsymbol{X}}$ and $\boldsymbol{\sigma}$. Gaussian states are in general generated by Hamiltonians at most bilinear in the mode operators, in particular the most general one mode Gaussian state can be written as

$$\rho_G = \hat{D}(\alpha)\hat{S}(\xi)\nu(n)\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha), \tag{2.9}$$

with ν being the single mode thermal state: $\nu(n) = (1+n)^{-1}[n/(1+n)]^{\hat{a}^\dagger\hat{a}}$, where $n = \mathrm{Tr}\left[\hat{a}^\dagger\hat{a}\nu(n)\right]$. This statement is valid in general for states with more than one mode: every Gaussian state can always be written as an unitary transformation generated by an Hamiltonian at most bilinear in the creation and destruction operators applied to a thermal state.

2.1.3 Distance From a Reference Gaussian State

Similarly to what has been shown in section 1.3.1 for nonclassicality one can quantify non-Gaussianity by measuring the distance from a reference Gaussian state. The reference state τ is a Gaussian state having the same first and second moments of the state ρ in examination, that is to say

$$\begin{aligned} \boldsymbol{X}[\tau] &= \boldsymbol{X}[\rho] \\ \sigma[\tau] &= \sigma[\rho]. \end{aligned} \tag{2.10}$$

If we use the Hilbert-Schmidt distance d_{HS} in (1.60), we can define the degree of non-Gaussianity of a state ρ as

$$NG_{\mathrm{HS}}[\rho] = \frac{d_{\mathrm{HS}}(\rho, \tau)}{\mu[\rho]}, \tag{2.11}$$

where $\mu[\rho]$ is the purity of the state defined as $\mu[\rho] = \mathrm{Tr}\left[\rho^2\right]$. This measure can be thought as a quantum version of quantifying classical non-Gaussianity by using the moments of the distribution. It enjoys some properties (see [37] for more details): NG_{HS} is equal to zero if and only if ρ is Gaussian and it is invariant for unitary evolutions derived by Hamiltonians at most quadratical in the creation operators. Moreover this measure is proportional to the L^2 distance of the Wigner functions (or analogously the characteristic functions) of ρ and τ, this property can be written as

$$\begin{aligned} NG_{\mathrm{HS}}[\rho] &\propto \int \mathrm{d}^2\alpha \, \{W[\rho](\alpha) - W[\tau](\alpha)\}^2 \\ NG_{\mathrm{HS}}[\rho] &\propto \int \mathrm{d}^2\lambda \, \{\chi[\rho](\lambda) - \chi[\tau](\lambda)\}^2 \end{aligned} \tag{2.12}$$

There is also a conjecture affirming that for single mode states the upper bound for NG_{HS} is the value 1/2.

2.1.4 Entropic Measure

A different approach to measure non-Gaussianity is based on quantum relative entropy; for two quantum states it is defined as

$$S\left(\rho_1 \middle\| \rho_2\right) = \mathrm{Tr}\left[\rho_1(\ln \rho_1 - \ln \rho_2)\right]. \tag{2.13}$$

It can be proved that $0 \leq S\left(\rho_1 \middle\| \rho_2\right) < \infty$, when it is properly defined, which means that the support of ρ_1 is contained in the support of ρ_2. It has the important property that $S\left(\rho_1 \middle\| \rho_2\right) = 0$ if and only if $\rho_1 = \rho_2$. While the relative entropy is not a proper metric, because it is not symmetric in its arguments, it has been used widely to quantify the distinguishability of two states. As a matter of fact the probability of confusing ρ_1 with ρ_2 after N measurements for $N \to \infty$ becomes proportional to $\exp\left[-N S\left(\rho_1 \middle\| \rho_2\right)\right]$

This leads to the definition of the entropic measure of non-Gaussianity

$$NG_{\mathrm{E}}(\rho) = S(\rho \| \tau), \tag{2.14}$$

that is to say the quantum relative entropy between a state and the corresponding reference Gaussian state. This measure becomes

$$NG_{\mathrm{E}}(\rho) = \mathrm{Tr}[\rho \ln \rho] - \mathrm{Tr}[\rho \ln \tau] = S(\tau) - S(\rho), \tag{2.15}$$

where S denotes the standard von Neumann entropy and the equality follows because $\mathrm{Tr}[\tau \ln \tau] = \mathrm{Tr}[\rho \ln \tau]$ for the way τ is defined. This measure can be considered as the quantum version of the negentropy (2.6) and it satisfies all the properties of the Hilbert-Schmidt measure plus some additional ones that will not be needed in the present work. However it is worth to mention that whereas the Bures measure is not additive under the tensor product operation, the entropic measure satisfies this property.

The von Neumann entropy of a single mode Gaussian state assumes a simple form

$$S(\rho_{\mathrm{G}}) = h\left(\sqrt{\det \boldsymbol{\sigma}}\right), \tag{2.16}$$

where $h(x)$ is a function defined as follows

$$h(x) = (x + \frac{1}{2}) \ln(x + \frac{1}{2}) - (x - \frac{1}{2}) \ln(x - \frac{1}{2}). \tag{2.17}$$

Thanks to this form the entropic non-Gaussianity becomes

$$NG_{\mathrm{E}}(\rho) = h\left(\sqrt{\det \boldsymbol{\sigma}}\right) - S(\rho), \tag{2.18}$$

which is further simplified for pure states, since $S(\rho) = 0$.

2.2 Nonlinearity of a Potential From Its Ground State

In order to measure the nonlinearity of a potential we are going to study its ground state [2]. In principle also the state at thermal equilibrium could be a good state to examine, but this makes computations more complex and it is yet to see if it is possible to work out such a measure explicitly for interesting cases.

The quantum harmonic oscillator has been partly reviewed in section 1.1.1, in particular we have that its ground state wave function is a Gaussian:

$$\psi_{\mathrm{H}}(x) = \langle x \mid 0 \rangle = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\omega x^2}, \tag{2.19}$$

where the mass m and the Planck constant have been rescaled to unity. We see that a harmonic oscillator is completely specified by the value of its frequency ω.

On the other hand if we consider a generic potential $V(x)$ which leads to an oscillatory behaviour the wave function can be obtained by solving the Schrödinger equation

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right]\phi(x) = E\phi(x), \tag{2.20}$$

which in general is not a Gaussian function. We denote by $|0\rangle_V$ the ground state of the system with the potential $V(x)$.

Distance From the Ground State

The first idea to quantify nonlinearity relies again on the concept of geometrical distances between quantum states. In particular for this purpose the Bures metric (1.61) has been employed. If we denote the ground state of the reference harmonic oscillator by $|0\rangle_H$, the corresponding measure to quantify nonlinearity is defined as

$$\eta_{\mathrm{B}}[V] = d_{\mathrm{B}}\big(|0\rangle_V, |0\rangle_H\big), \tag{2.21}$$

since the ground states are pure states this just reduces to

$$\eta_{\mathrm{B}}[V] = \sqrt{1 - |_H\langle 0 \mid 0\rangle_V|}. \tag{2.22}$$

To have a proper definition we still have to choose a value for the frequency ω of the reference harmonic oscillator, because the value of η_{B} depends on it. The most natural choice is expanding the potential near its minimum and finding ω as a function of the nonlinear parameters of the potential, however determining this frequency is not always straight-forward and for some potentials it can even be misleading.

Nonlinearity Through Non-Gaussianity

We have stated that determining a reference harmonic potential might not always be possible so it would be useful to have a measure not dependent on the choice of a reference potential.

This is achieved by employing the entropic non-Gaussianity NG_{E}, so that the measure of nonlinearity is defined as

$$\eta_{\mathrm{NG}}[V] = NG_{\mathrm{E}}\left(|0\rangle_{VV}\langle 0|\right) = h\left(\sqrt{\det \boldsymbol{\sigma}}\right), \tag{2.23}$$

where the equality holds because the ground state is pure; $\boldsymbol{\sigma}$ represents the covariance matrix of the ground state.

This definition is more appealing than η_{B} because it does not require the determination of a reference potential for $V(x)$, but just a reference Gaussian state for the ground state of $V(x)$. This renders η_{NG} independent of the specific features of the potential, since we do not need to know the behaviour of $V(x)$ near its minimum to computer the reference frequency.

3 Harmonic Oscillator With Polynomial Perturbations

We want to study the behaviour of a physical system composed by a one dimensional harmonic oscillator with two perturbations proportional to x^4 and x^6, we work with unitary mass and units rescaled in such a way that $m = 1$ and $\hbar = 1$. The classical Hamiltonian of this system is

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 + \epsilon_4 x^4 + \epsilon_6 x^6 \tag{3.1}$$

We will also be interested in the two limiting cases where only one of the perturbations is present, that is to say $\epsilon_4 = 0$ or $\epsilon_6 = 0$; we will refer to quantities related to these two cases with the subscript 4 (6) when there is only the term $\epsilon_4 x^4$ ($\epsilon_6 x^6$). This system will be studied using perturbation theory since it is not exactly solvable.

The aim of studying this simple system is to gather some evidence for the idea that nonlinearity is a resource for nonclassicality before moving to more interesting and realistic potentials. For this purpose it is also very useful to work with first order perturbative ground states, since it is relatively easy to compute all the relevant quantities for vectors with a small number of components in the particle number basis (also called Fock basis).

Terms proportional to x and to x^2 could be treated in a perturbative way as well, but they are exactly solvable and actually give rise to a harmonic oscillator with a different frequency, so we do not consider this perturbations because they do not cause real anharmonic behaviour. On the other hand a perturbation proportional to x^3 can be indeed considered, but this poses some problems for the convergence of the perturbative series, so it was not considered in order to avoid technical difficulties that could influence the results and their interpretation.

We note that part of the analytical calculations was carried out with the help of the *Quantum*[1] add-on for Mathematica, developed by José Luis Gómez-Muñoz and Francisco Delgado.

3.1 Perturbative states

In order to compute the ground states for these Hamiltonians we use time-independent perturbation theory (Rayleigh-Schrödinger perturbation theory), as found in standard Quantum Mechanics textbooks [38]. We are going to need the matrix elements of the perturbation on the basis of the energy eigenstates of the unperturbed system, which in this case are the

[1] The package can be found on-line at the address http://homepage.cem.itesm.mx/lgomez/quantum/.

Fock states $|n\rangle$. This can be done in an explicit fashion using the wave functions of the harmonic oscillator and performing integrals but it is more convenient to use the creation and annihilation operators already introduced in section 1.1.1.

With our choices of units ($m = \hbar = 1$) these operator are defined this way

$$\begin{aligned} \hat{a} &= \sqrt{\frac{\omega}{2}}\left(\hat{x} + \frac{\mathrm{i}}{\omega}\hat{p}\right) \\ \hat{a}^\dagger &= \sqrt{\frac{\omega}{2}}\left(\hat{x} - \frac{\mathrm{i}}{\omega}\hat{p}\right). \end{aligned} \tag{3.2}$$

One can express the position and momentum operators in terms of $\hat{a}$ and $\hat{a}^\dagger$

$$\begin{aligned} \hat{x} &= \sqrt{\frac{1}{2\omega}}(\hat{a} + \hat{a}^\dagger) \\ \hat{p} &= \sqrt{\frac{\omega}{2}}(\hat{a}^\dagger - \hat{a}). \end{aligned} \tag{3.3}$$

We can now compute the matrix elements, expressing the perturbations in function of $\hat{a}$ and $\hat{a}^\dagger$ and using the properties (1.3). For x^4 the only elements that are not zero are

$$\begin{aligned} \langle n|\hat{x}^4|n\rangle &= \frac{6n^2 + 6n + 3}{4\omega^2} \\ \langle n|\hat{x}^4|n+4\rangle = \langle n+4|\hat{x}^4|n\rangle &= \frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{4\omega^2} \\ \langle n|\hat{x}^4|n+2\rangle = \langle n+2|\hat{x}^4|n\rangle &= \frac{(4n+6)\sqrt{n(n-1)}}{4\omega^2}, \end{aligned} \tag{3.4}$$

while for x^6 we have

$$\begin{aligned} \langle n|\hat{x}^6|n\rangle &= \frac{5\left(4n^3 + 6n^2 + 8n + 3\right)}{8\omega^3} \\ \langle n|\hat{x}^6|n+6\rangle = \langle n+6|\hat{x}^4|n\rangle &= \frac{\sqrt{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}}{8\omega^3} \\ \langle n|\hat{x}^6|n+4\rangle = \langle n+4|\hat{x}^4|n\rangle &= \frac{3(2n+5)\sqrt{(n+1)(n+2)(n+3)(n+4)}}{8\omega^3} \\ \langle n|\hat{x}^6|n+2\rangle = \langle n+2|\hat{x}^4|n\rangle &= \frac{15\left(n^2 + 3n + 3\right)\sqrt{(n+1)(n+2)}}{8\omega^3}. \end{aligned} \tag{3.5}$$

3.1.1 First Order

We want now to calculate the ground states at the first order in the powers of ϵ (which is supposed to be a small parameter). The standard formula for the nth perturbed level is

$$|n\rangle = |n^{(0)}\rangle + \epsilon \sum_{k \neq n} |k^{(0)}\rangle \frac{V_{k0}}{E_n^{(0)} - E_k^{(0)}} \tag{3.6}$$

where $H = H_0 + \epsilon V$, $H_0|n^{(0)}\rangle = E_n^{(0)}|n^{(0)}\rangle$ and $V_{nk} = \langle n^{(0)}|V|k^{(0)}\rangle$

Our notation will be slightly different, since we are just interested in the ground state we are going to call $|n\rangle$ the eigenvectors of the unperturbed Hamiltonian ($H_0|n\rangle = E_n|n\rangle = \omega(\frac{1}{2} + n)|n\rangle$) and $|\psi\rangle$ the ground state obtained with first order perturbation theory. Thus the above formula reduces to

$$|\psi\rangle = |0\rangle + \epsilon \sum_{k\neq 0} |k\rangle \frac{V_{k0}}{-\omega k} \tag{3.7}$$

Using this formula and the matrix elements found in the last section we readily find the normalized ground state ψ, which reads

$$|\psi\rangle = \frac{1}{C}\left[|0\rangle - \left(\frac{45\epsilon_6}{4\sqrt{2}\omega^3} + \frac{3\epsilon_4}{\sqrt{2}\omega^2}\right)|2\rangle - \left(\frac{15\sqrt{\frac{3}{2}}\epsilon_6}{2\omega^3} + \frac{\sqrt{\frac{3}{2}}\epsilon_4}{\omega^2}\right)|4\rangle - \sqrt{5}\epsilon_6|6\rangle\right], \tag{3.8}$$

where the normalization constant C is

$$C = \frac{\sqrt{\omega^2\left(96\omega^6 + 117\epsilon_4^2\right) + 945\omega\epsilon_4\epsilon_6 + 2055\epsilon_6^2}}{4\sqrt{6}\omega^4}. \tag{3.9}$$

For the sake of brevity and order from now on the coefficents of the first order perturbed ground state will be named γ, so that the state assumes the following form

$$|\psi\rangle = \gamma_0|0\rangle + \gamma_2|2\rangle + \gamma_4|4\rangle + \gamma_6|6\rangle. \tag{3.10}$$

The two limiting states are called ψ_4 and ψ_6 respectively and they are written explicitly as follows

$$|\psi_4\rangle = \frac{1}{C_4}\left(|0\rangle - \frac{3\epsilon_4}{2\sqrt{2}\omega^3}|2\rangle - \frac{\sqrt{3}\epsilon_4}{4\sqrt{2}\omega^3}|4\rangle\right) \tag{3.11}$$

$$|\psi_6\rangle = \frac{1}{C_6}\left(|0\rangle - \frac{45\epsilon_6}{8\sqrt{2}\omega^4}|2\rangle - \frac{15\sqrt{3}\epsilon_6}{8\sqrt{2}\omega^4}|4\rangle - \frac{\sqrt{5}\epsilon_6}{4\omega^4}|6\rangle\right), \tag{3.12}$$

where the normalization constants are given by

$$C_4 = \sqrt{1 + \frac{39{\epsilon_4}^2}{32\omega^6}} \qquad C_6 = \sqrt{1 + \frac{2055{\epsilon_6}^2}{96\omega^8}}. \tag{3.13}$$

3.1.2 Approximate Solution

A possible way to obtain a good approximation of the ground state of the harmonic oscillator with polynomial perturbations is to work in a finite dimensional Fock space. In this setting al the operators $\hat{x}$, $\hat{p}$, $\hat{a}$, $\hat{a}^\dagger$ and $\hat{H}$ are finite dimensional operators representable as matrices.

If we choose the maximum number N of dimensions of the Fock space, every pure state can be represented as $\sum_{n=0}^{N} c_n|n\rangle$. The matrix elements of the operator $\hat{x}$ are

$$x_{ij} = \frac{\sqrt{j}\delta_{i-j+1} + \sqrt{j+1}\delta_{-i+j+1}}{\sqrt{2\omega}}, \tag{3.14}$$

where i and j go from 0 to N; the operators $\hat{x}^4$ and $\hat{x}^6$ can be easily obtained by standard matrix multiplication. In this basis the matrix representation of the unperturbed Hamiltonian operator $\hat{H}^{(0)}$ is diagonal and its elements are

$$H^{(0)}_{jj} = \omega\left(j + \frac{1}{2}\right). \tag{3.15}$$

The ground state of the system can thus be calculated by diagonalizing the matrix $H^{(0)} + \epsilon_4 x^4 + \epsilon_6 x^6$, which can always be done, at least numerically. The resulting energy eigenstates and eigenvalues will be of course dependent on the dimension N of the Fock space, but they should converge for N big enough. This is indeed the case and we choose $N = 60$ to obtain a ground state ϕ to be confronted with the perturbative state ψ using the fidelity $|\langle\phi|\psi\rangle|$. In

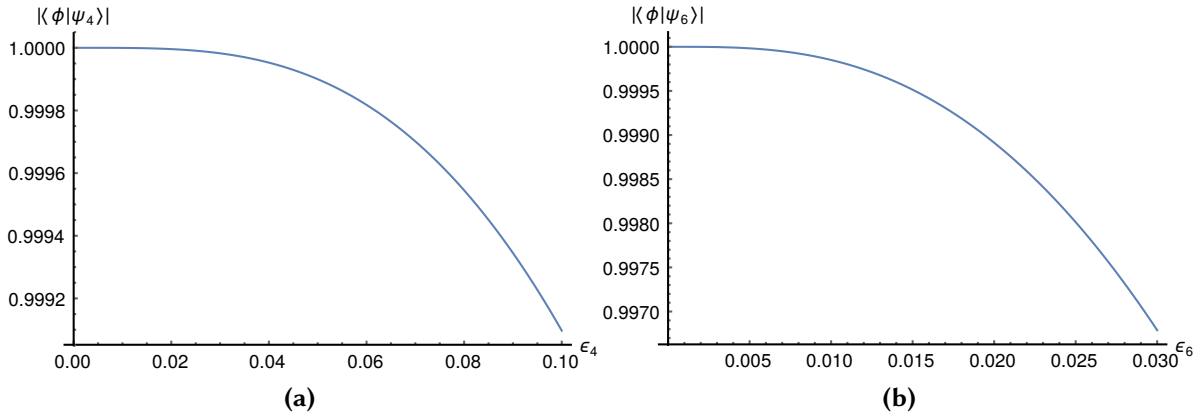


Figure 3.1: Plot of the overlap between the perturbative and the approximate ground states for $\omega = 1$. In figure (a) we have $\epsilon_6 = 0$, while in figure (b) we have $\epsilon_4 = 0$.

fig. 3.1 we can see the overlap decreasing as the perturbative parameters increases for the two limiting states ψ_4 and ψ_6, while in fig. 3.2 we have a contour plot of the overlap as a function of both parameters. This analysis gives us a range of values of the parameters ϵ_4 and ϵ_6 where the first order perturbative state represents a good approximation for the real solution. We chose to make ϵ_4 vary between 0 and 0.1, while ϵ_6 stays from 0 to 0.03, this choice leads to the minimum overlap being ≈ 0.988 for the maximum value of both parameters (the top right corner of the contour plot in fig. 3.2).

3.2 Nonlinearity

Now that we have the perturbed ground state (3.8) we want to use it to compute the nonlinearity of the perturbation. As discussed in chapter 2, this quantity can be measured by two

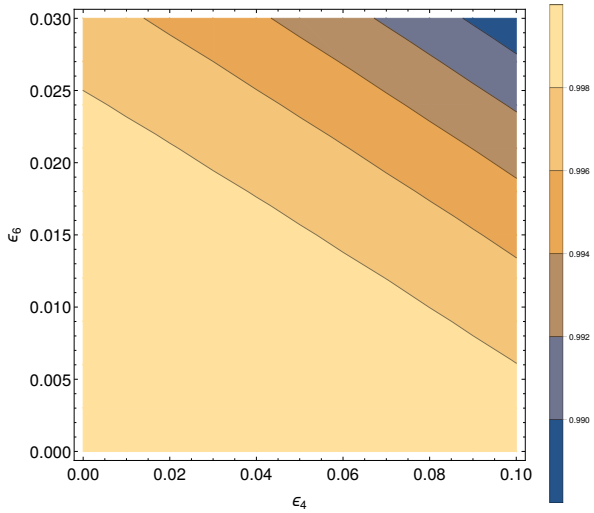


Figure 3.2: Contour plot of the overlap between the perturbative and the approximate ground states for $\omega = 1$. In figure (a) we have $\epsilon_6 = 0$, while in figure (b) we have $\epsilon_4 = 0$.

different parameters η_B and η_{NG}, the former is usually easier to compute but dependent on a reference harmonic oscillator, while the second can be more general. We are going to check both these indices; the system under consideration is a a perturbed harmonic oscillator so the reference harmonic frequency needed for η_B is defined with no ambiguity.

3.2.1 Measure Based on Bures Distance

The measure η_B is just the renormalized Bures distance between the ground state $|0\rangle_H = |\psi\rangle$ of the reference quantum harmonic oscillator and the ground state $|0\rangle_V = |0\rangle$ of the perturbed oscillator obtained. For this system the definition (2.22) yields

$$\begin{aligned}\eta_B[\epsilon_4 x^4 + \epsilon_6 x^6] &= \sqrt{1 - |\langle 0|\psi\rangle|} = \sqrt{1 - |\gamma_0|} = \sqrt{1 - \frac{1}{C}} \\ &= \sqrt{1 - \frac{4\sqrt{6}\omega^4}{\sqrt{\omega^2\left(96\omega^6 + 117\epsilon_4^2\right) + 945\omega\epsilon_4\epsilon_6 + 2055\epsilon_6^2}}}\end{aligned} \tag{3.16}$$

3.2.2 Measure Based On Non-Gaussianity

The covariance matrix (2.7) can be explicitly written out as follows

$$\boldsymbol{\sigma} = \begin{pmatrix} \langle \hat{x}^2\rangle - \langle \hat{x}\rangle^2 & \frac{1}{2}\langle\{\hat{x},\hat{p}\}\rangle - \langle \hat{x}\rangle\langle \hat{p}\rangle \\ \frac{1}{2}\langle\{\hat{p},\hat{x}\}\rangle - \langle \hat{p}\rangle\langle \hat{x}\rangle & \langle \hat{p}^2\rangle - \langle \hat{p}\rangle^2 \end{pmatrix} \tag{3.17}$$

and it can also be rewritten in terms of the expectation values of creation and destruction operators. First we note that

$$\{\hat{x},\hat{p}\} = [\hat{x},\hat{p}] + 2\hat{p}\hat{x} = \mathrm{i} + 2\hat{p}\hat{x} \tag{3.18}$$

and then we use the definitions (3.3) to get

$$\hat{p}\hat{x} = \frac{\mathrm{i}}{2}(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a} - 1), \tag{3.19}$$

this in turn yields

$$\langle\{\hat{x},\hat{p}\}\rangle = \mathrm{i}\big(\langle\hat{a}^\dagger\hat{a}^\dagger\rangle - \langle\hat{a}\hat{a}\rangle\big). \tag{3.20}$$

Since $[\hat{a},\hat{a}^\dagger] = 1$ we are going to keep just the terms $\hat{a}^\dagger\hat{a}$ expressing the others as $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a}+1$. In this fashion we get

$$\langle\hat{x}^2\rangle = -\frac{1}{2\omega}\big(\langle\hat{a}^\dagger\hat{a}^\dagger\rangle + 2\langle\hat{a}^\dagger\hat{a}\rangle + 1 + \langle\hat{a}\hat{a}\rangle\big) \tag{3.21}$$

and

$$\langle\hat{p}^2\rangle = -\frac{\omega}{2}\big(\langle\hat{a}^\dagger\hat{a}^\dagger\rangle - 2\langle\hat{a}^\dagger\hat{a}\rangle - 1 + \langle\hat{a}\hat{a}\rangle\big). \tag{3.22}$$

Substituting these values the covariance matrix becomes

$$\begin{pmatrix} \frac{1}{2\omega}(1+\langle\hat{a}^2\rangle + \langle(\hat{a}^\dagger)^2\rangle + 2\langle\hat{a}^\dagger\hat{a}\rangle - 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle - \langle\hat{a}\rangle^2 - \langle\hat{a}^\dagger\rangle^2) & \frac{\mathrm{i}}{2}(\langle(\hat{a}^\dagger)^2\rangle - \langle\hat{a}^2\rangle - \langle\hat{a}^\dagger\rangle^2 + \langle\hat{a}\rangle^2) \\ \frac{\mathrm{i}}{2}(\langle(\hat{a}^\dagger)^2\rangle - \langle\hat{a}^2\rangle - \langle\hat{a}^\dagger\rangle^2 + \langle\hat{a}\rangle^2) & \frac{\omega}{2}(1 - \langle\hat{a}^2\rangle - \langle(\hat{a}^\dagger)^2\rangle) + 2\langle\hat{a}^\dagger\hat{a}\rangle - 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle + \langle\hat{a}\rangle^2 + \langle\hat{a}^\dagger\rangle^2) \end{pmatrix} \tag{3.23}$$

We can calculate the expectation values explicitly in terms of the coefficients (3.8).

$$\begin{aligned} \langle\psi|\hat{a}|\psi\rangle &= 0 \\ \langle\psi|\hat{a}^\dagger|\psi\rangle &= \langle\psi|\hat{a}|\psi\rangle^* = 0 \\ \langle\psi|\hat{a}^\dagger a|\psi\rangle &= 2|\gamma_2|^2 + 4|\gamma_4|^2 + 6|\gamma_4|^2 \\ \langle\psi|\hat{a}^2|\psi\rangle &= \sqrt{2}\gamma_2{\gamma_0}^* + 2\sqrt{3}\gamma_4{\gamma_2}^* + \sqrt{30}\gamma_6{\gamma_4}^* \\ \langle\psi|(\hat{a}^\dagger)^2|\psi\rangle &= \langle\psi|\hat{a}^2|\psi\rangle^* = \sqrt{2}\gamma_0{\gamma_2}^* + 2\sqrt{3}\gamma_2{\gamma_4}^* + \sqrt{30}\gamma_4{\gamma_6}^* \end{aligned} \tag{3.24}$$

These values turn out to be real, since the coefficients γ_i of the ground state are real. For this reason we have that $\langle a\rangle = \langle a^\dagger\rangle$ and $\langle a^2\rangle = \langle(a^\dagger)^2\rangle$. This fact leads to a simplified correlation matrix which is diagonal

$$\begin{pmatrix} \frac{1}{2\omega}(1 + 2\langle\hat{a}^2\rangle + 2\langle\hat{a}^\dagger a\rangle - 4\langle\hat{a}\rangle^2) & 0 \\ 0 & \frac{\omega}{2}(1 + 2\langle\hat{a}^\dagger\hat{a}\rangle - 2\langle\hat{a}^2\rangle), \end{pmatrix} \tag{3.25}$$

so that the determinant is easily computed

$$\det\boldsymbol{\sigma} = \frac{1}{4} - \langle\hat{a}^2\rangle^2 - \langle\hat{a}\rangle^2 + \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{a}\rangle^2 + 2\langle\hat{a}\rangle^2\langle\hat{a}^2\rangle - 2\langle\hat{a}\rangle^2\langle\hat{a}^\dagger\hat{a}\rangle, \tag{3.26}$$

it can be made explicit as a function of the coefficients inserting the values (3.24)

$$\begin{aligned} \det\boldsymbol{\sigma} = \Big(& 4\gamma_2^4 - 2\gamma_0^2\gamma_2^2 + 4\gamma_4^2\gamma_2^2 + 24\gamma_6^2\gamma_2^2 - 4\sqrt{6}\gamma_0\gamma_4\gamma_2^2 \\ & + 2\gamma_2^2 - 12\sqrt{10}\gamma_4^2\gamma_6\gamma_2 - 4\sqrt{15}\gamma_0\gamma_4\gamma_6\gamma_2 \\ & + 16\gamma_4^4 + 36\gamma_6^4 + 4\gamma_4^2 + 18\gamma_4^2\gamma_6^2 + 6\gamma_6^2 + \frac{1}{4}\Big). \end{aligned} \tag{3.27}$$

We write explicitly the nonlinearity as a function of the perturbative parameters ϵ_4, ϵ_6 just for the two limiting cases $\epsilon_4 = 0$ and $\epsilon_6 = 0$, since the full expression is very involved.

$$\eta_{\mathrm{NG}}(\epsilon_4) = h\left(-\frac{20736\omega^6\epsilon_4^2}{\left(96\omega^6+117\epsilon_4^2\right)^2}+\frac{252\epsilon_4^2}{96\omega^6+117\epsilon_4^2}+\right.$$
$$\left.+\frac{51840\epsilon_4^4}{\left(96\omega^6+117\epsilon_4^2\right)^2}+\frac{31104\omega^3\epsilon_4^3}{\left(96\omega^6+117\epsilon_4^2\right)^2}+\frac{1}{4}\right) \tag{3.28}$$

$$\eta_{\mathrm{NG}}(\epsilon_6) = h\left(-\frac{291600\omega^8\epsilon_6^2}{\left(96\omega^8+2055\epsilon_6^2\right)^2}+\frac{10485\epsilon_6^2}{2\left(96\omega^8+2055\epsilon_6^2\right)}+\right.$$
$$\left.+\frac{13701150\epsilon_6^4}{\left(96\omega^8+2055\epsilon_6^2\right)^2}+\frac{4009500\omega^4\epsilon_6^3}{\left(96\omega^8+2055\epsilon_6^2\right)^2}+\frac{1}{4}\right) \tag{3.29}$$

The index η_{NG} is plotted parametrically with the nonlinearity index η_{B} and for both cases we see a positive correlation as expected, an indication that the physical meaning of the two measures is equivalent. In the following we will use both η_{NG} and η_{B} as it turns out they have sometimes slightly different behaviours.

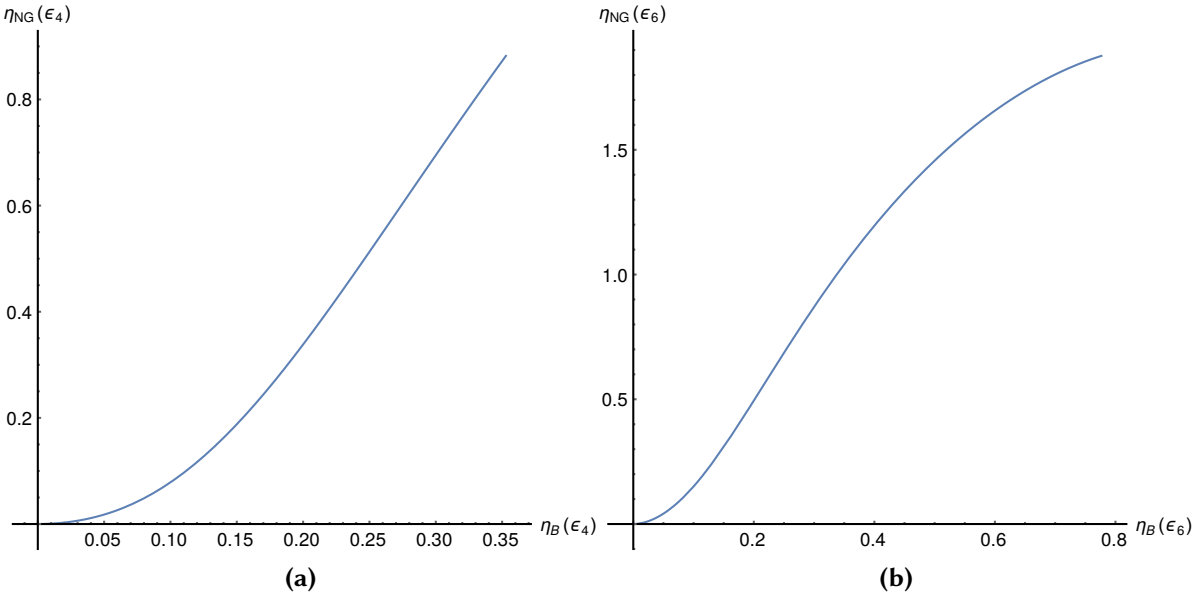


Figure 3.3: Parametric plot of the two nonlinearity measures the paramater ϵ_4 and ϵ_6 going from 0 to $\frac{1}{2}$; we work with $\omega = 1$.

3.3 Nonclassicality

We are interested in quantifying the nonclassicality of the ground state (3.8) of the perturbed system. As explained in chapter 1 there exists a plethora of ways to quantify how nonclassical a quantum state is.

The first measure we will use is the negative volume of the Wigner function δ, discussed in section 1.3.3 and defined by equation (1.67). Then we will characterize the nonclassicality using the *entropic entanglement potential* which was discussed in section 1.3.4.

3.3.1 Wigner Function

The Wigner function for the state $|\psi\rangle$ has to be calculated. Among the various ways one can define the Wigner function, the most suitable for a linear combination of Fock states is the definition (1.41). In this case the trace become just the expectation value on the state so we have

$$\begin{aligned} W(z) = \frac{2}{\pi}\langle\psi|\hat{D}(2z)(-1)^{\hat{a}^\dagger\hat{a}}|\psi\rangle = \\ \frac{2}{\pi}\big[{\gamma_0}^2\langle 0|\hat{D}(2z)|0\rangle + {\gamma_2}^2\langle 2|\hat{D}(2z)|2\rangle + {\gamma_4}^2\langle 4|\hat{D}(2z)|4\rangle + {\gamma_6}^2\langle 6|D(2z)|6\rangle \\ + \gamma_0\gamma_2\big(\langle 2|\hat{D}(2z)|0\rangle + \langle 0|\hat{D}(2z)|2\rangle\big) + \gamma_0\gamma_4\big(\langle 4|\hat{D}(2z)|0\rangle + \langle 0|\hat{D}(2z)|4\rangle\big) + \\ + \gamma_0\gamma_6\big(\langle 6|\hat{D}(2z)|0\rangle + \langle 0|\hat{D}(2z)|6\rangle\big) + \gamma_2\gamma_4\big(\langle 4|\hat{D}(2z)|2\rangle + \langle 2|D(2z)|4\rangle\big) + \\ + \gamma_2\gamma_6\big(\langle 6|\hat{D}(2z)|2\rangle + \langle 2|\hat{D}(2z)|6\rangle\big) + \gamma_4\gamma_6\big(\langle 6|\hat{D}(2z)|4\rangle + \langle 4|\hat{D}(2z)|6\rangle\big)\big] \end{aligned} \tag{3.30}$$

To have the explicit Wigner function we need the expectation value of the displacement operator between number states; it turns out to be (see appendix A.2 for an explicit derivation):

$$\langle n'|\hat{D}(z)|n\rangle = \begin{cases} \sqrt{\frac{n!}{n'!}}e^{-\frac{|z|^2}{2}}(-z)^{n'-n}L_{n'}^{(n'-n)}(|z|^2) & \text{if} \quad n' > n \\ \sqrt{\frac{n'!}{n!}}e^{-\frac{|z|^2}{2}}(z^*)^{n-n'}L_{n}^{(n-n')}(|z|^2) & \text{if} \quad n > n' \end{cases} \tag{3.31}$$

where $L_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}$ are the associated Laguerre polynomials. The Wigner function then becomes

$$\begin{aligned} W(z) = \frac{2}{\pi}e^{-2|z|^2}\big[{\gamma_0}^2L_0(4|z|^2) + {\gamma_2}^2L_2(4|z|^2) + {\gamma_4}^2L_4(4|z|^2) + {\gamma_6}^2L_6(4|z|^2) + \\ + 4\sqrt{2}\gamma_0\gamma_2\,\mathrm{Re}(z^2)L_2^2(4|z|^2) + \frac{16}{\sqrt{3}}\gamma_0\gamma_4\,\mathrm{Re}(z^4)L_4^4(4|z|^2) + \frac{32}{3\sqrt{5}}\gamma_0\gamma_6\,\mathrm{Re}(z^6)L_6^6(4|z|^2) \\ + \frac{4}{\sqrt{3}}\gamma_2\gamma_4\,\mathrm{Re}(z^2)L_4^2(4|z|^2) + \frac{16}{3\sqrt{10}}\gamma_2\gamma_6\,\mathrm{Re}(z^4)L_6^4(4|z|^2) + \frac{8}{\sqrt{30}}\gamma_4\gamma_6\,\mathrm{Re}(z^2)L_6^2(4|z|^2)\big]. \end{aligned} \tag{3.32}$$

In order to have the measure of nonclassicality δ we need to integrate the absolute value of (3.32). This integration cannot be carried out analytically so it was performed numerically using the software Mathematica.

3.3.2 Entanglement Potential

Determining the entanglement potential introduced in section 1.3.4 means quantifying the entanglement of the output state when $|\psi\rangle$ interacts with the vacuum through a beam splitter. The initial state is the following tensor product

$$|\Psi_{\text{IN}}\rangle\rangle = |0\rangle \otimes |\psi\rangle = \gamma_0|0,0\rangle + \gamma_2|0,2\rangle + \gamma_4|0,4\rangle + \gamma_6|0,6\rangle, \tag{3.33}$$

where $|n,m\rangle$ means the tensor product $|n\rangle \otimes |m\rangle$. We can get the output state applying the unitary evolution operation $\hat{B}$ introduced in section 1.3.4

$$|\Psi_{\text{OUT}}\rangle\rangle = \hat{B}|\Psi_{\text{IN}}\rangle\rangle. \tag{3.34}$$

To get the output state we could work out all the calculations explicitly using expression (1.71), however the effect of $\hat{U}(\xi)$ on a state of the form $|n\rangle \otimes |m\rangle$ is given by the following general formula [39]:

$$\hat{U}(\xi)|n\rangle \otimes |m\rangle = \sum_{k=0}^{n}\sum_{l=0}^{m} A_{kl}^{nm}|k+l\rangle \otimes |n+m-k-l\rangle, \tag{3.35}$$

where A_{kl}^{nm} is the transfer matrix given by

$$A_{kl}^{nm} = \sqrt{\frac{(k+l)!\,(n+m-k-l)!}{n!\,m!}}(-1)^l \binom{n}{k}\binom{m}{l} \sin\phi^{n-k+l}\cos\phi^{m+k-l}. \tag{3.36}$$

Remembering that $\hat{U}\left(\frac{\pi}{4}\right) = \hat{B}$, the formula we need for the entanglement potential is just the following

$$\hat{B}|0,n\rangle = \left(\frac{1}{\sqrt{2}}\right)^n \sum_{k=0}^{n}\binom{n}{k}^{\frac{1}{2}}|k,n-k\rangle, \tag{3.37}$$

so that for the linearity of the operator $\hat{B}$ we have

$$|\Psi_{\text{OUT}}\rangle\rangle = \gamma_0\hat{B}|0,0\rangle + \gamma_2\hat{B}|0,2\rangle + \gamma_4\hat{B}|0,4\rangle + \gamma_6\hat{B}|0,6\rangle. \tag{3.38}$$

Following definition (1.72) the entanglement of the state $|\Psi_{\text{OUT}}\rangle\rangle$ is then quantified with the Von-Neumann entropy of the partial trace[2], so we have

$$EP(\epsilon_4,\epsilon_6) = S\left(\text{Tr}_B\left[|\Psi_{\text{OUT}}\rangle\rangle\langle\langle\Psi_{\text{OUT}}|\right]\right) = S\left(\rho_{\text{OUT}}\right), \tag{3.39}$$

where we denoted ρ_{OUT} as the reduced density matrix of the system after the interaction with the beam splitter. In order to compute entropy in actual calculations it has to be rewritten using the eigenvalues p_i of the density matrix ρ_{OUT}. So, if we choose to measure the entropy in bits, it becomes

$$S(\rho_{\text{OUT}}) = -\sum_{i=0}^{n} p_i \log_2 p_i. \tag{3.40}$$

[2]The Mathematica code used to compute the partial trace was written by Mark Tame and can be found here http://library.wolfram.com/infocenter/MathSource/8763/

3.4 Nonclassicality Versus Nonlinearity

Now that we have investigated the two measures of nonlinearity η_{NG} and η_{B} and the two measures of nonclassicality δ and EP for the perturbed ground state ψ we are in the position to compare these quantities. We want to check if increasing the nonlinearity of the potential renders the ground state more nonclassical.

This will be done mainly by using parametric plots or scatter plots of nonclassicality against nonlinearity; in this plots nonlinearity plays the role of the independent variable represented on the horizontal axis. The fact that there are two different measures for both quantities makes possible to draw four possible graphs, however if the underlying idea is correct the interpretation should not be too different. Nonetheless every measure has not the exact same properties so some differences will indeed arise and they will be pointed out.

For nonclassicality both measures have to be used, because they capture nonclassicality in different ways, as explained in chapter 1. Regarding nonlinearity we could just use η_{NG} since this measure has more desirable properties, in fact we have already pointed out in section 3.2.2 that η_{NG} and η_{B} have a positive correlation for the system under consideration. We will however check also η_{B}, because a different behaviour arises when we deal with two parameters.

All the calculations in this part are obtained for the choice $\omega = 1$ because we want to highlight the dependence on the pertubative parameters. The range of the parameters is the one discussed in section 3.1.2, that is to say $0 < \epsilon_4 < 0.1$ and $0 < \epsilon_6 < 0.03$, in order to have perturbed ground states that are a good approximation of the real solution.

3.4.1 Parametric Plots

If we deal with the ground states ψ_4 or ψ_6 we have just one parameter on which nonlinearity and nonclassicality depend. In this case there is not much difference in comparing the various measures: it turns out they are all monotonous to each other for increasing values of the perturbation parameters.

The parametric plots are actually made of discrete points since both the nonclassicality measures need to be computed numerically and are not known as functions of the parameters ϵ_4 and ϵ_6. The measure δ requires a time consuming numerical integration for every different value of the parameter just 100 discrete points are considered, whereas to compute EP the eigenvalues of the density matrix need to be computed numerically; this is a faster process so 1000 points are considered.

The plot of nonclassicality δ against the entropic nonlinearity η_{NG} is shown in fig. 3.4, the curve can be fitted approximatively as $\delta = b\sqrt{\eta_{\mathrm{NG}}}$, with $b = 3.72955$ for x^4 and $b = 6.22686$ for x^6. If instead we use the entanglement potential, as shown in fig. 3.5, the correlation is approximatively linear, in the form $EP = c\eta_{\mathrm{NG}}$, with $c = 1.40685$ for x^4 and $c = 0.718866$ for x^6. We see that the form of the curve is not the exactly the same, nevertheless the important thing is that a positive correlation is present in both cases.

As we already pointed out the two measures of nonlinearity are monotonous so we do not expect different results when using η_{B}, this is indeed the case as shown in fig. 3.6. Even in

this case the curves are not the same, but we always see a positive correlation that makes possible to interpret nonlinearity as a resource to generate nonclassicality.

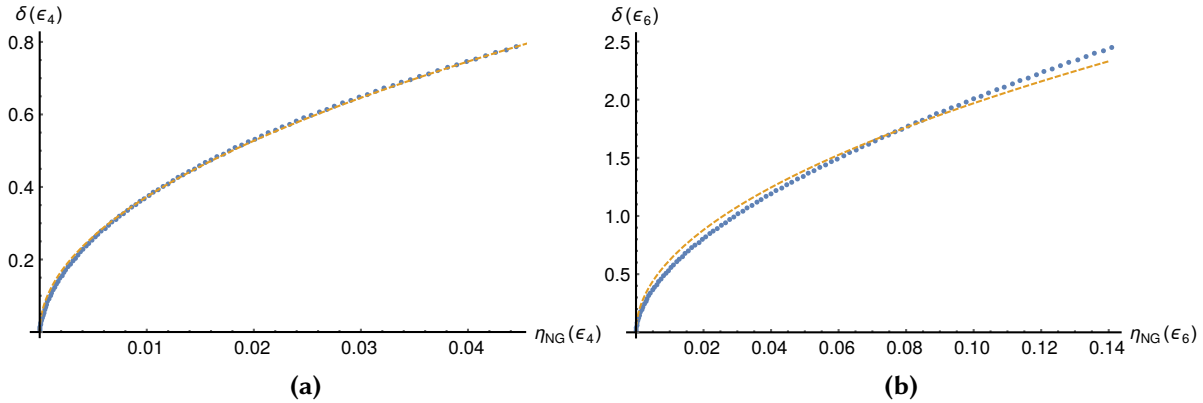


Figure 3.4: Parametric scatter plot of the negative volume of the Wigner function δ versus the entropic nonlinearity η_{NG} for the x^4 perturbation (a) and for the x^6 perturbation (b), the fitted function is shown as a orange dashed line.

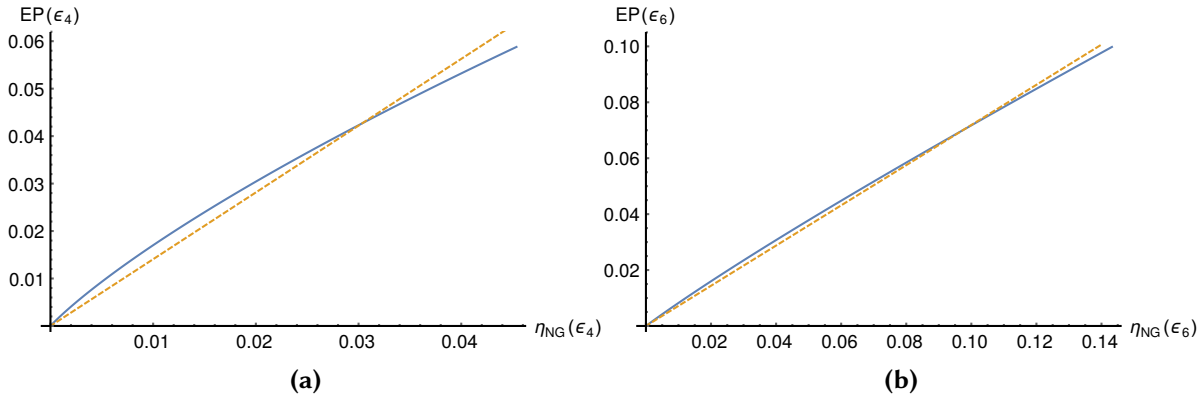


Figure 3.5: Parametric plot of the entanglement potential *EP* versus the entropic nonlinearity η_{NG} for the x^4 perturbation (a) and for the x^6 perturbation (b), , the fitted function is shown as a orange dashed line.

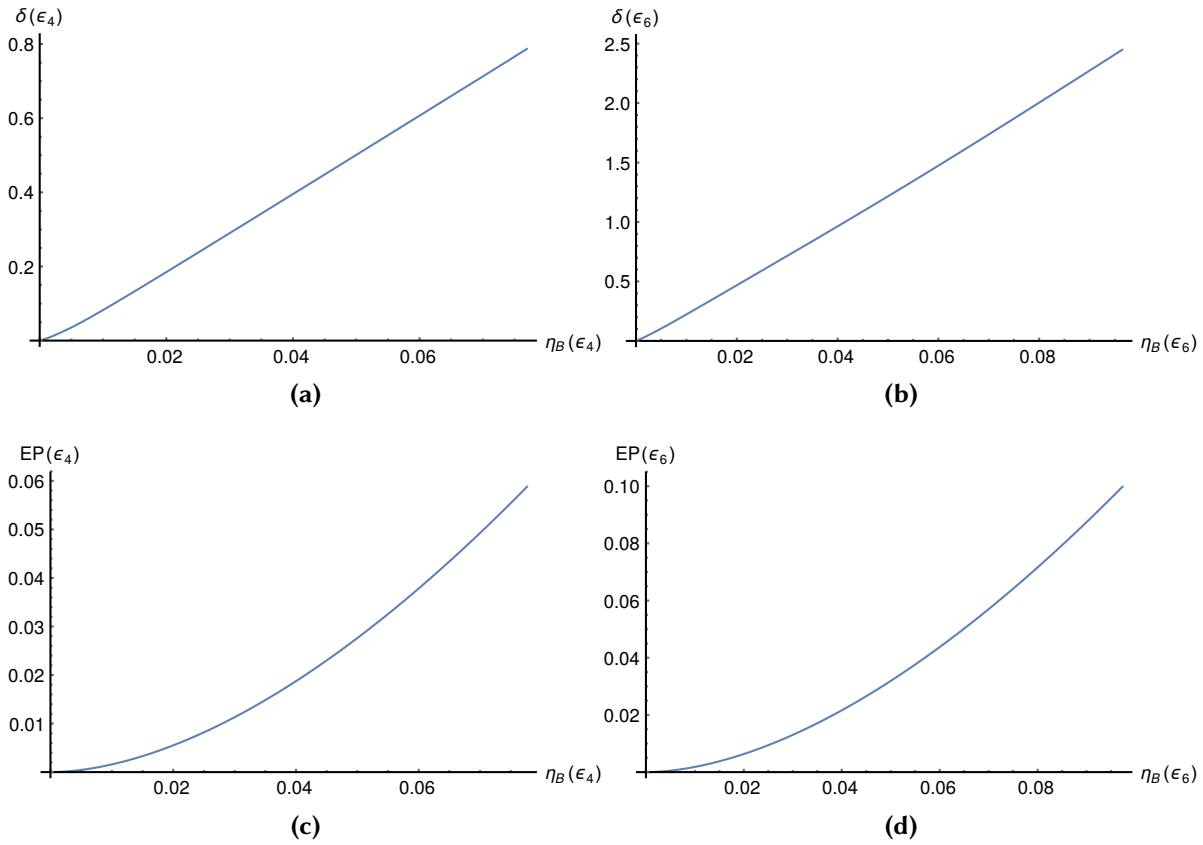


Figure 3.6: Parametric plots using the Bures nonlinearity η_{B}, with δ in (a) and (b) and with EP in (c) and (d).

3.4.2 Random Scatter Plots

The full ground state ψ depends on two perturbative parameters, it is then interesting to see how nonclassicality and nonlinearity behave when treated as functions of both parameters. This goal is achieved by generating random couple of parameters (ϵ_4, ϵ_6) in the appropriate range and then computing nonclassicality and nonlinearity for that values.

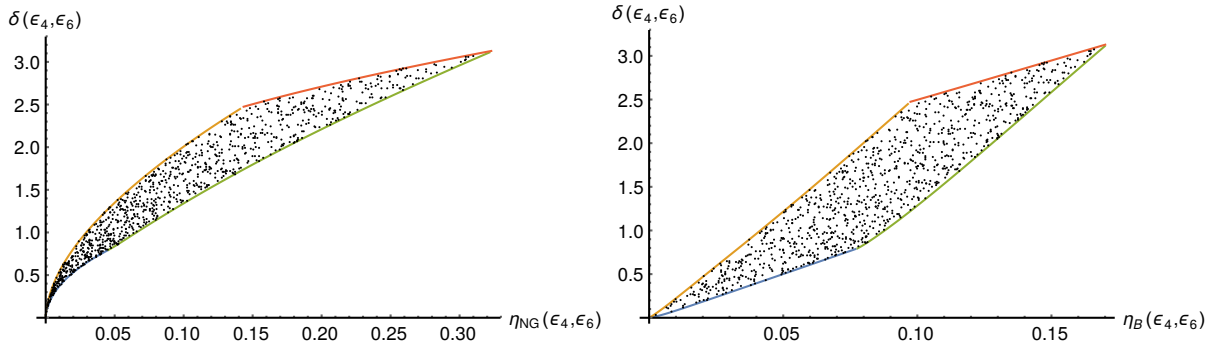
This highlights that nonclassicality and nonlinearity are dependent on the details of the system under consideration. If these two quantities were intrinsically the same regardless of the potential we would expect a one-to-one relationship, but this is not the case. As a matter of fact the scatter plot of nonclassicality versus nonlinearity does not form a line but a whole region. This means that the different combinations of the parameters which give a fixed value of nonlinearity instead produce different nonclassicality values.

The region spanned by the random points in the nonlinearity against nonlinearity plot is limited by four curves. The first two are the ones already obtained in the previous section, the curve for $\epsilon_6 = 0$ is represented in blue, while the curve for $\epsilon_4 = 0$ is represented in yellow. The curve where ϵ_4 varies but ϵ_6 is fixed to its maximum value 0.03 is the orange one while the curve for varying ϵ_6 and $\epsilon_4 = 0.1$ is the green one.

If we consider δ to quantify nonclassicality the situation is the following: if we choose the perturbative parameters to get the same amount of nonlinearity then the x^6 potential yields a more nonclassical ground state for the corresponding value of ϵ_6. As a matter of fact the two graphs in fig. 3.7 have the same structure, the yellow and the orange curves are always above the blue and the green ones, with all the random points confined inside the region delimited by them.

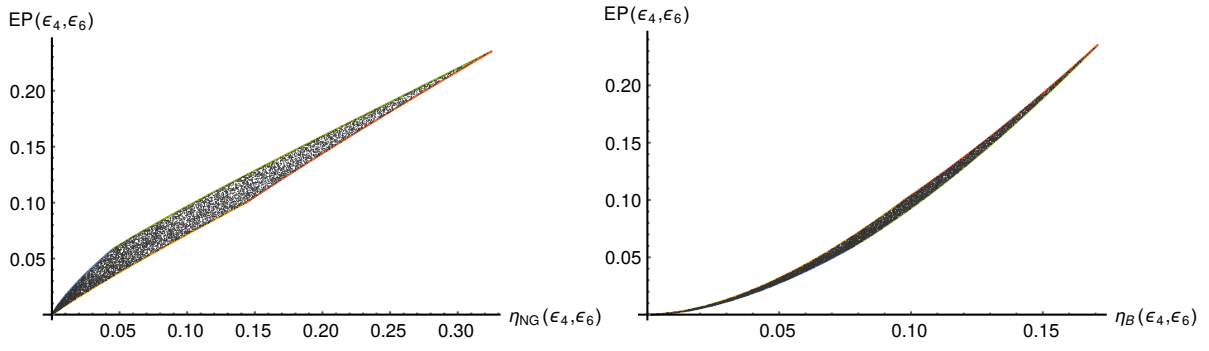
The situation is different if we look at the entanglement potential in fig. 3.8. The random points are always delimited by the four curves, but the two graphs have a different structure, as illustrated in fig. 3.9. We see that in the $\boldsymbol{EP}$ against η_{NG} plot the blue and green curves are above, while in the $\boldsymbol{EP}$ versus η_{B} plot the situation is the opposite (and the same as the graphs with δ). If we consider just the states ψ_4 and ψ_4 this means that, after choosing the parameters ϵ_4 and ϵ_6 in such a way that the entropic linearity is fixed, the state ψ_4 generates more entanglement than ψ_6 for the chosen values of the parameters. A possible interpretation of this results is that the "right" nonlinearity measure to be associated to $\boldsymbol{EP}$ is η_{B}, while the one for δ is η_{NG} since they both depend on the form of the Wigner function.

From the analysis of the harmonic oscillator with perturbations we can conclude that in this case the nonlinearity of the potential is a resource to generate nonclassicality in its ground state. Though we have to specify that these quantities have one-to-one correspondence only if we consider the potential and the ground state as function of just one parameter.



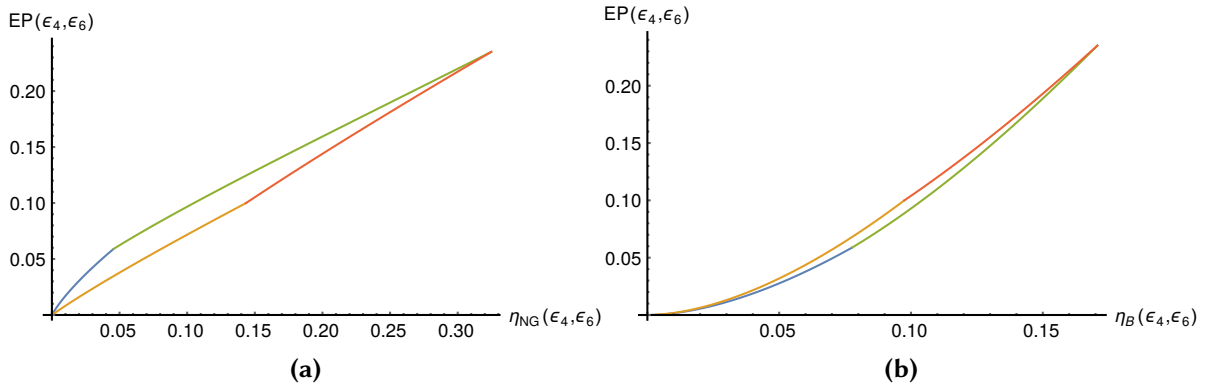
(a) Entropic nonlinearity **(b)** Bures nonlinearity

Figure 3.7: Random scatter plot of δ versus nonlinearity when both parameters are present, 1000 random points were generated.



(a) Entropic nonlinearity **(b)** Bures nonlinearity

Figure 3.8: Random scatter plot of *EP* versus nonlinearity when both parameters are present, 10000 random points were generated.



(a) **(b)**

Figure 3.9: The limiting curves of fig. 3.8a in (a) and of fig. 3.8b in (b).

4 Exactly Solvable Non Linear Oscillators

The perturbed harmonic oscillator confirmed the intuition that nonlinearity can be a resource for nonclassicality. We now propose to stress this idea using more realistic potentials for which the ground state can be found analytically.

4.1 Modified Harmonic Oscillator

The Modified Harmonic Oscillator (MHO) potential (choosing units appropriately to get $\hbar = 1 = m$ as usual) is defined as

$$V_{\mathrm{MHO}}(x) = \frac{\alpha^2 x^2}{2} - 2\beta x \tanh(\beta x), \tag{4.1}$$

and it depends on two parameters: α, which correspond to the frequency of the unmodified harmonic oscillator, and β.

The normalized wave function of the ground state of this potential can be found analytically [40] and it is the following one

$$\phi_{\mathrm{MHO}}(x) = \frac{\sqrt{2} e^{-\frac{1}{2}(\alpha x^2)} \cosh(\beta x)}{\sqrt[4]{\pi}\sqrt{\sqrt{\frac{1}{\alpha}}\left(e^{\frac{\beta^2}{\alpha}} + 1\right)}}. \tag{4.2}$$

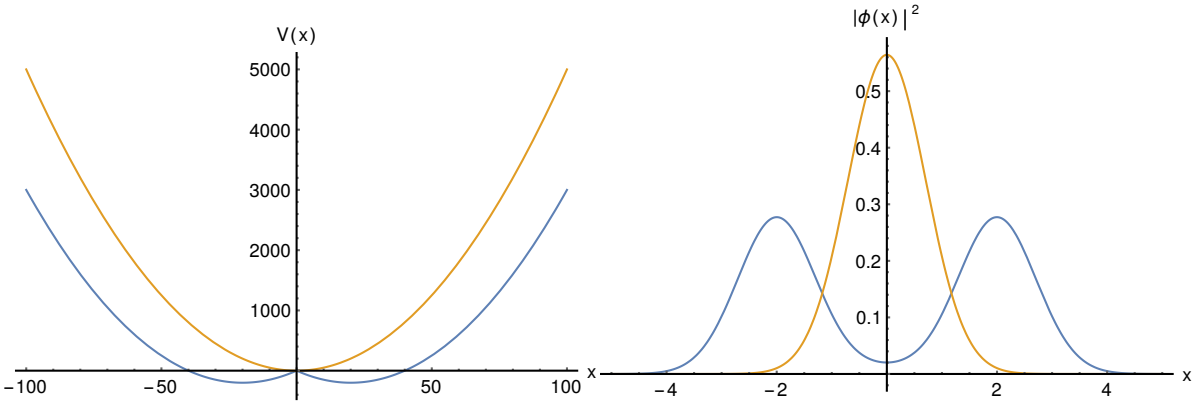
To get an idea the MHO potential and its ground state are represented in fig. 4.1 for a particular choice of the parameters, the reference harmonic oscillator and its wave function are also represented for comparison.

4.1.1 Nonlinearity

Entropic Nonlinearity

The expectation values of the operators can be carried out analytically by integrating the wave function (4.2) and they are:

$$\langle \hat{x} \rangle = 0 \qquad \langle \hat{p} \rangle = 0 \qquad \langle \hat{x}^2 \rangle = \frac{2\beta^2\left(1 - \frac{1}{e^{\frac{\beta^2}{\alpha}}+1}\right) + \alpha}{2\alpha^2} \qquad \langle \hat{p}^2 \rangle = \frac{\alpha}{2} - \frac{\beta^2}{e^{\frac{\beta^2}{\alpha}} + 1} \qquad \langle \hat{p}\hat{x} \rangle = 0, \tag{4.3}$$



(a) The potential **(b)** The modulus squared of the wave function

Figure 4.1: The MHO potential and its ground state wave function for $\alpha = 1$ and $\beta = 2$ are represented in blue, the harmonic potential with unitary frequency and mass and its ground state wave function in orange.

so we get the full correlation matrix (diagonal, because the coefficients of the wave function are real)

$$\boldsymbol{\sigma}_{\text{MHO}} = \begin{pmatrix} \frac{2\left(1-\frac{1}{1+e^{\frac{\beta^2}{\alpha}}}\right)\beta^2+\alpha}{2\alpha^2} & 0 \\ 0 & \frac{\alpha}{2}-\frac{\beta^2}{1+e^{\frac{\beta^2}{\alpha}}} \end{pmatrix} \tag{4.4}$$

The nonlinearity based on the entropic measure of non-Gaussianity is as usual $\eta_{\text{NG}} = h\left(\sqrt{\det \boldsymbol{\sigma}_{\text{MHO}}}\right)$, where the determinant of the correlation matrix is the following

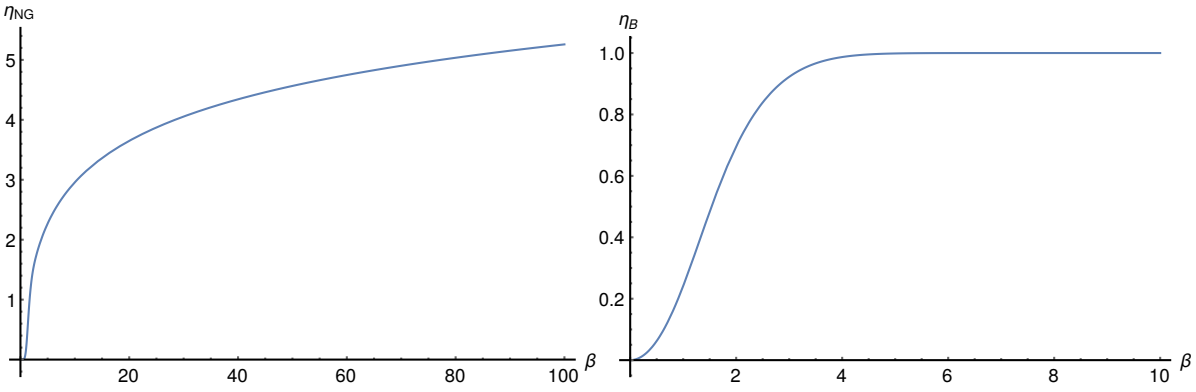
$$\det \boldsymbol{\sigma}_{\text{MHO}} = -\frac{\beta^4}{\alpha^2\left(e^{\frac{\beta^2}{\alpha}}+1\right)} + \frac{\beta^4}{\alpha^2\left(e^{\frac{\beta^2}{\alpha}}+1\right)^2} + \frac{\beta^2}{2\alpha} - \frac{\beta^2}{\alpha\left(e^{\frac{\beta^2}{\alpha}}+1\right)} + \frac{1}{4}. \tag{4.5}$$

We can see from this expression that the determinant (and thus the nonlinearity) depends only on the parameter $\frac{\beta^2}{\alpha}$ and not upon the two parameters independently, so we can choose $\alpha = 1$ and just make β change to get all the values of $\frac{\beta^2}{\alpha}$.

Bures nonlinearity

To get the Bures nonlinearity we have to compute the scalar product between the wave function (4.2) and the wave function of the reference harmonic oscillator. We choose the frequency ω of the reference harmonic oscillator equal to α. Performing the integration we eventually get

$$\eta_{\text{B}} = \sqrt{1 - |\langle \phi_{\text{MHO}}|0\rangle|} = \sqrt{1 - \frac{\sqrt{2}e^{\frac{\beta^2}{4\alpha}}}{\sqrt{e^{\frac{\beta^2}{\alpha}}+1}}}. \tag{4.6}$$



(a) Entropic nonlinearity **(b)** Bures nonlinearity

Figure 4.2: The two measure of nonlinearity as a function of the parameter $\frac{\beta^2}{\alpha}$. The computation is actually performed by fixing $\alpha = 1$ and varying β from 0 to 100 in (a) and from 0 to 10 in (b).

Again we can note that the only parameter that comes into play is $\frac{\beta^2}{\alpha}$. In fig. 4.2 we can see that as β increases the measure η_{B} saturates to one, which means that the ground state of the MHO becomes perpendicular to the ground state of the harmonic oscillator. On the other hand η_{NG} continues to grow even when η_{B} saturates to one, since $\lim_{\beta\to\infty} \eta_{\mathrm{NG}}(\beta) = \infty$, for this reason the parametric plot in fig. 4.3 turns flat after the parameter passes the value $\frac{\beta^2}{\alpha} \approx 25$.

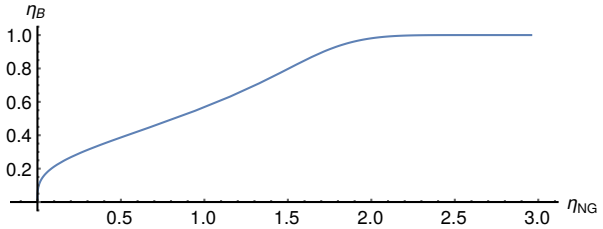


Figure 4.3: Parametric plot of the entropic nonlinearity versus the Bures nonlinearity for $0 < \frac{\beta^2}{\alpha} < 100$.

4.1.2 Wigner Function Nonclassicality

The Wigner function for the ground state (4.2) can be found analytically by explicitly integrating the wave function using the original definition given by Wigner (1.51). This integration was carried out in Ref. [40], the Wigner function is the following

$$W_{\mathrm{MHO}}(x,y) = \frac{e^{-\alpha x^2 - \frac{y^2}{\alpha}}\left(e^{-\frac{\beta^2}{\alpha}}\cosh(2\beta x) + \cos\left(\frac{2\beta y}{\alpha}\right)\right)}{\pi\left(e^{-\frac{\beta^2}{\alpha}} + 1\right)}. \tag{4.7}$$

In order to get the measure δ (1.67) we have to integrate the absolute value of (4.7):

$$\iint \mathrm{d}x\mathrm{d}y \left| \frac{e^{-\alpha x^2 - \frac{y^2}{\alpha}} \left(e^{-\frac{\beta^2}{\alpha}} \cosh(2\beta x) + \cos\left(\frac{2\beta y}{\alpha}\right) \right)}{\pi \left(e^{-\frac{\beta^2}{\alpha}} + 1 \right)} \right|$$

changing the variables as $q = \beta x$ and $p = \frac{\beta}{\alpha} y$ the previous integral becomes

$$\frac{\frac{\alpha}{\beta^2} e^{-\frac{\alpha}{\beta^2}}}{\pi \left(1 + e^{-\frac{\alpha}{\beta^2}} \right)} \iint \mathrm{d}q\mathrm{d}p \left| e^{q^2+p^2} \left[e^{-\frac{\beta^2}{\alpha}} \cosh(2q) + \cos(2p) \right] \right|. \tag{4.8}$$

It is again evident that this expression depends only on the combination $\frac{\alpha}{\beta^2}$, which comes to be the only real paremeter governing this measure of nonclassicality.

4.1.3 Entanglement Potential

For the perturbed harmonic oscillator obtaining the entanglement potential was straightforward, since the perturbed ground states were restricted to a finite-dimensional subspace of the Hilbert space in the Fock basis. In the case of the MHO, and for other potentials for which we have just a ground state wave function $\phi(x)$, visualising the effect of the beam splitter and quantifying the entanglement requires more thought.

One possible approach to the problem of sending the ground state (4.2) into a beam splitter is to restrict the analysis to a finite dimensional subspace in the Fock basis. As a matter of fact, since the wave functions $u_n(x)$ of the harmonic oscillator form a basis for the functional vector space L^2 we can write

$$\phi(x) = \sum_{i=0}^{\infty} \phi_n u_n(x) \tag{4.9}$$

where

$$\phi_n = \int_{-\infty}^{\infty} \mathrm{d}x \phi(x) u_n(x). \tag{4.10}$$

In bra-ket notation this corresponds to

$$|\phi\rangle = \sum_{n=0}^{\infty} \phi_n |n\rangle. \tag{4.11}$$

The approximation which has to be used is truncating this series to a finite number N of terms that will determine the dimension of the subspace of the Fock space. As a control for the approximation we have to make sure that the norm (squared) of the vector (4.11), that is to say the sum $\sum_{n=0}^{N} \phi_n^2$, is sufficently close to unity. This means that we have enough terms and we are just neglecting parts of the Fock space whose vectors have a very small norm.

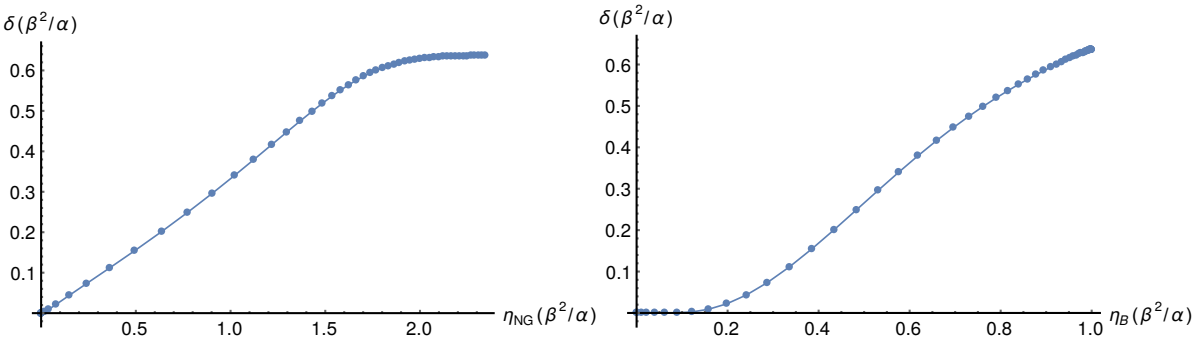
In case of the ground state ϕ_{MHO} of the modified harmonic oscillator we choose to set $N = 35$, which leads to a norm which is good up to the fourth decimal digit (it is greater

than 0.9999), as long as β stays in the range $[0, 6]$ (for fixed α). The choice of this range is imposed by the fact that the coefficients (4.10) are computed through numerical integrations for every value of β. As the value of β increases the number of terms needed to write the ground state in the Fock basis grows and the integrals needed to compute the coefficients are increasingly hard to estimate numerically with precision.

Once we have a state which is a finite linear combination of Fock states we can proceed exactly as in section 3.3.2 by using equation (3.37) to get the state after the beam splitter and then computing the entropy of the reduced density matrix.

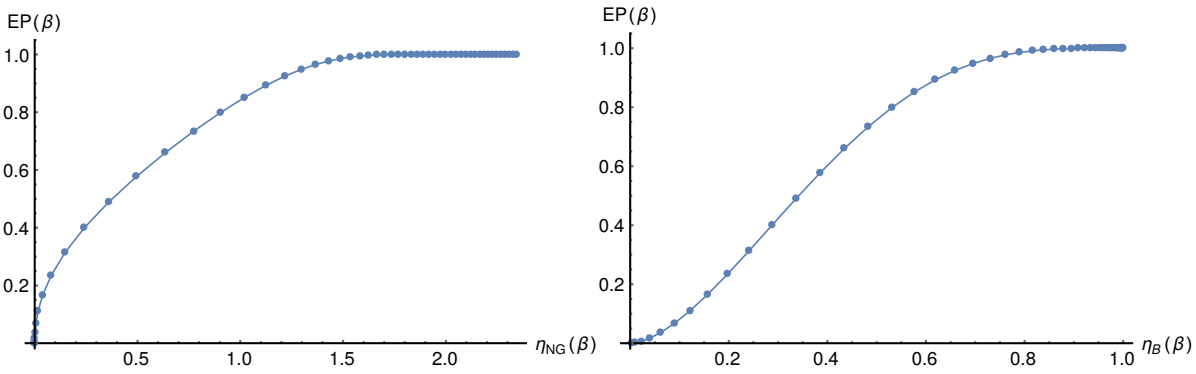
4.1.4 Nonclassicality Versus Nonlinearity

To check how nonclassicality and nonlinearity are related we proceed to create parametric plots. Both the nonlinearity measures depend on the combination of parameters $\frac{\beta^2}{\alpha}$ as well as δ and just the entanglement truly depends on both parameters. In fig. 4.4 and fig. 4.5 we see



(a) Entropic nonlinearity **(b)** Bures nonlinearity

Figure 4.4: Parametric plot of the nonclassicality δ versus nonlinearity for the parameter $\frac{\beta^2}{\alpha}$ going from 0 to 100.



(a) Entropic nonlinearity **(b)** Bures nonlinearity

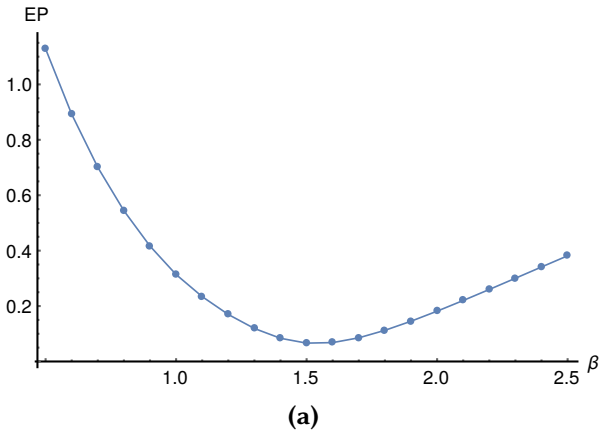
Figure 4.5: Parametric plot of the entanglement potential versus nonlinearity for $\alpha = 1$ and $0 \leq \beta \leq 6$.

that all the plots enjoy the same behaviour for the first part of the graphs, which shows the expected correlation between nonclassicality and nonlinearity. As the value of β increases we see that the two nonclassicality measures saturate, with EP saturating before δ. When they are plotted against η_{NG} the curve becomes flat, while when plotted against η_{B} the points accumulate since both quantities do not grow any more even if we increase β.

Dependence on squeezing

So far we have outlined a similar behaviour for all the quantities computed, that is to say both the nonlinearity measures and both the ways to quantify nonclassicality grow as the parameter $\frac{\beta^2}{\alpha}$ assumes greater values. This is what we expected and it fits nicely in our idea of nonlinearity as a way to generate nonclassicality. Nonetheless there are some more remarks to make, mainly due to the different ways to define nonclassicality in quantitative way.

We already stated that EP is the only quantity which depends on α and β separately. If we choose $\beta^2 = \alpha$ and we make β change we can see in fig. 4.6 that EP is not constant. But this choice means fixing $\frac{\beta^2}{\alpha} = 1$ and therefore the other quantities do not change. This feature of

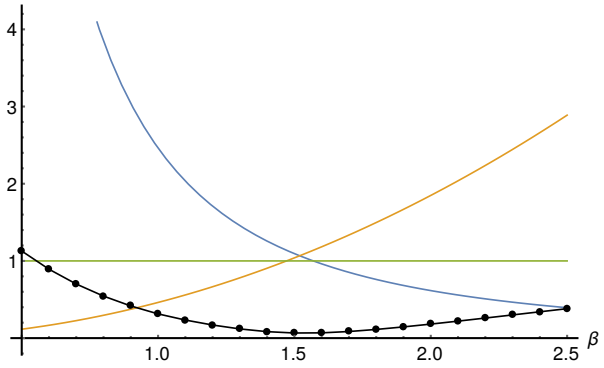


(a)

Figure 4.6: Entanglement potential in the case $\alpha = \beta^2$ for $0.5 < \beta < 2$.

the entanglement potential is due to the fact that it detects also nonclassicality derived from sub-vacuum squeezing, a feature which is not captured by the volume of negative part of the Wigner function.

By inspecting the expressions (4.3) and (4.4), we note that since $\langle \hat{p} \rangle = 0 = \langle \hat{x} \rangle$ the determinant of the correlation matrix is equal to the product of the square of the two variances $(\Delta \hat{p})^2 = \langle \hat{p}^2 \rangle$ and $(\Delta \hat{x})^2 = \langle \hat{x}^2 \rangle$. This means that when we fix $\alpha = \beta^2$ we also fix this product to have the value $\Delta \hat{p} \Delta \hat{x} \approx 0.284$. On the other hand $\Delta \hat{x}$ and $\Delta \hat{p}$ depend on α and β separately, not on the combination $\frac{\beta^2}{\alpha}$. All of this can be seen in fig. 4.7, where the ratio between the variances and the variances of the vacuum state (which are equal to $\frac{1}{2}$) is shown, when this ratio is less than one it means that we have sub-vacuum squeezing. The interpretation of this situation is that the increase in the entanglement potential for small values of β is due to the squeezing in p.



(a)

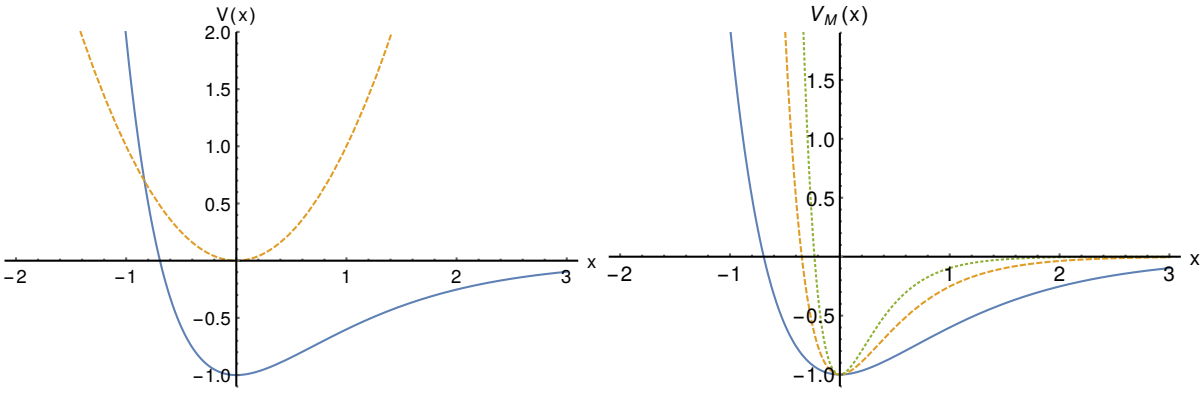
Figure 4.7: This graph shows $\frac{(\Delta x)^2}{0.5}$ in blue, $\frac{(\Delta p)^2}{0.5}$ in orange, the green line is the constant 1; when one of the curves assumes values below 1 it means we have sub-vacuum squeezing. The points are the ones of fig. 4.6 and represent the entanglement potential as a function of β. Everything is carried out choosing $\alpha = \beta^2$ with $0.5 < \beta < 2.5$.

4.2 Morse potential

The Morse potential was introduced by Morse [41]; it is a good approximation to the potential energy of diatomic molecules and it provides a better description of the vibrational structure than the (quantum) harmonic oscillator. The potential is not symmetrical and its expression is the following:

$$V_{\mathrm{M}} = D\left(e^{-2\alpha x} - 2e^{-\alpha x}\right), \tag{4.12}$$

the coordinate x represents the distance from the minimum of the potential, the parameter $D > 0$ controls the depth of the well, while α controls its width. Expanding the two exponentials for $\alpha \to 0$ at fixed D we get the harmonic limit, which is an oscillator with a frequency equal to $\omega_{\mathrm{M}} = \sqrt{2D}\alpha$. The potential is plotted in fig. 4.8 with the parameters set to unity.



(a) $D = 1$,$\alpha = 1$ and $\omega_{\mathrm{M}} = \sqrt{2}$ **(b)** $D = 1$ and $\alpha = 1, 2, 3$

Figure 4.8: In (a) we have a representation of the Morse Potential for $D = 1 = \alpha$ (solid blue), together with its reference harmonic potential for comparison (dashed orange). In (b) the potential for $D = 1$ and $\alpha = 1$ (solid blue), $\alpha = 2$ (dashed orange), $\alpha = 3$ (dotted green).

The unidimensional Schrödinger equation associated with this potential can be solved analytically, the energy eigenvectors are labelled by two parameters N, ν, connected to the su(2) algebra. The quantum number N is connected to the parameters of the potential, we have that

$$N = -\frac{1}{2} + \frac{\sqrt{2D}}{\alpha}, \tag{4.13}$$

while ν counts the anharmonic excitations and it can be $\nu = 0, 1, ..., \lfloor N \rfloor$ (where $\lfloor N \rfloor$ means the largest integer not greater than N) and the number of bound states is $N + 1$. Since we want at least one bound state we need $N > 0$ so we have the constraint $\alpha < 2\sqrt{2D}$; the limiting case where we have just one bound state (the ground state) is achieved for $D \to 0$ or $\alpha \to 2\sqrt{2D}$.

In the following we will need only the wave function of the ground state, which is given by

$$\phi_{\mathrm{M}}(x) = (2N+1)^N \sqrt{\frac{N\alpha}{N!}} e^{-\alpha x N - (N+\frac{1}{2})e^{-\alpha x}}, \tag{4.14}$$

and describes a bound state with energy $E = -\frac{1}{2}\alpha N^2$. Since N can be real valued actually $N!$ is interpreted as $\Gamma(N+1)$, where $\Gamma(z)$ is the gamma function.

4.2.1 Nonlinearity

The behaviour of the nonlinearity of the Morse potential (already analysed in Ref. [2]) can be partially explained by looking at the form of the potential in fig. 4.8b. For any fixed value of D we would expect a more anharmonic behaviour for increasing α and a harmonic behaviour for vanishing α.

Entropic nonlineraity

The correlation matrix calculated with the state (4.14) is the following

$$\boldsymbol{\sigma}_{\mathrm{M}} = \begin{pmatrix} \frac{\psi^{(1)}(2N)+\left(\log(2N+1)-\psi^{(0)}(2N)\right)^2}{\alpha^2} - \frac{\left(\log(2N+1)-\psi^{(0)}(2N)\right)^2}{\alpha^2} & 0 \\ 0 & \frac{\alpha^2 N}{2} \end{pmatrix}. \tag{4.15}$$

Where $\psi^{(n)}(z)$ is the polygamma function, defined as the nth derivative of the digamma function, which in turn is just the logarithmic derivative of the gamma function: $\psi^{(n)}(z) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} \log \Gamma(z)$.

The determinant of this correlation matrix is $\det \boldsymbol{\sigma}_{\mathrm{M}} = \frac{1}{2} N \psi^{(1)}(2N)$, so the entropic nonlinearity $\eta_{\mathrm{NG}} = h\left(\sqrt{\det \boldsymbol{\sigma}_{\mathrm{M}}}\right)$ will depend just on the parameter N, or equivalently on the combination $\frac{\sqrt{D}}{\alpha} = N + \frac{1}{2}$.

Bures nonlinearity

The scalar product between the ground state (4.14) and the ground state $|0\rangle$ of the harmonic oscillator with the reference frequency $\omega = \sqrt{2D\alpha}$ is needed to computer the Bures nonlinearity (2.22). This scalar product is carried out in the position representation and thus becomes the following integral

$$\int \mathrm{d}x\, (2N+1)^N \sqrt{\frac{\alpha N}{N!}} e^{-\alpha x N - (N+\frac{1}{2})e^{-\alpha x}} \left(\frac{\sqrt{2D}\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\sqrt{D}x^2\alpha}{\sqrt{2}}}; \tag{4.16}$$

if we change the integration variable to $y = \alpha x$ we get

$$\sqrt{2}(2N+1)^N \sqrt{\frac{N}{N!}} \left(\frac{\sqrt{2D}}{\pi\alpha}\right)^{\frac{1}{4}} \int \mathrm{d}y\, e^{-yN-(N+\frac{1}{2})e^{-y}-\frac{\sqrt{D}y^2}{\sqrt{2}\alpha}}, \tag{4.17}$$

which again depends solely on N, since $\frac{\sqrt{D}}{\alpha} = \frac{1}{2} + N$. The actual integration in (4.17) has to be carried out numerically for each different value of the parameter N.

We see from fig. 4.9a that both η_{NG} and η_{B} capture the intuitive ideas we got from looking at the shape of the potential: they grow for increasing α and decrease for increasing D, this

is due to the fact that they both are monotonically decreasing function of N, as shown in fig. 4.9b. The difference between the two measures lies in the fact that while η_B is bounded and saturates to 1, η_{NG} diverges for $N \to 0$. In fig. 4.10 a parametric plot between the two measures is shown, we get the maximum anharmonicity for $N \to 0$ and the minimum for $N \to \infty$.

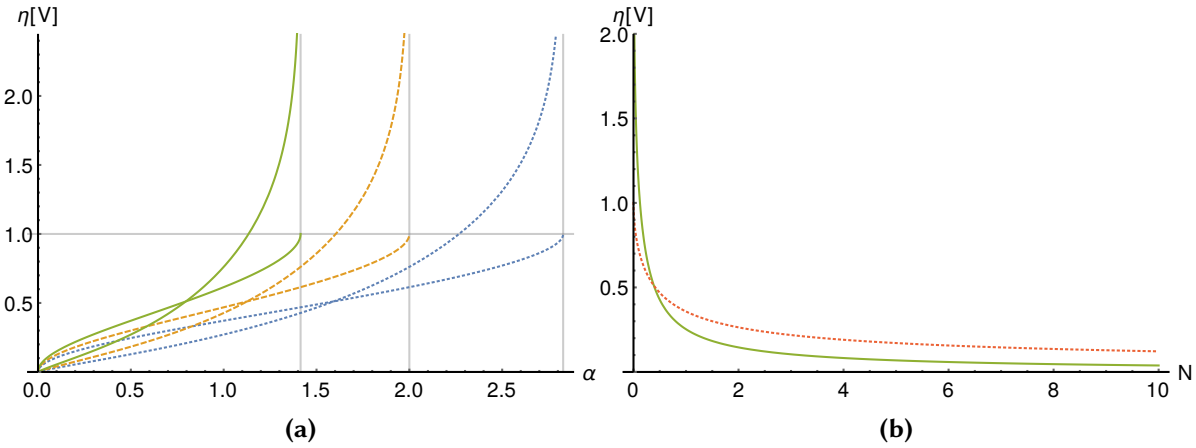


Figure 4.9: Figure (a): the two nonlinearity measures as a function of α for $D = 0.25$ (solid blue), $D = 0.5$ (dashed red) and $D = 1$ (dotted green). η_B is the curve that ends at 1, while η_{NG} is the unbounded one; the vertical lines represent the limit $N = 0$ (or $\alpha = 2\sqrt{2D}$). Figure (b): η_{NG} (solid green) and η_B (dotted orange) as a function of N.

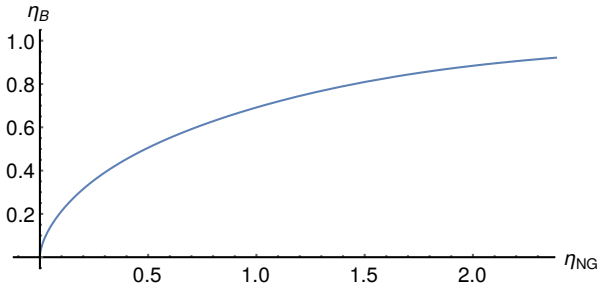


Figure 4.10: Parametric plot of the two nonlinearity measures for $0 < N < \infty$.

4.2.2 Nonclassicality

The Wigner function for the ground state of the Morse potential, as reported in Ref. [42], reads as follow

$$W_M(x, y) = \frac{2}{\pi\Gamma(2N)}(2N+1)^{2N} e^{-2N\alpha x} K_{-2iy/\alpha}((2N+1)e^{-\alpha x}), \tag{4.18}$$

where $K_\alpha(z)$ is the Bessel K function.

In order to calculate δ, as defined in (1.67), we need to computer the following integration

$$\iint \mathrm{d}x\mathrm{d}y \left|W_{\mathrm{M}}(x,y)\right| = \iint \mathrm{d}q\mathrm{d}p \left| \frac{2}{\pi\Gamma(2N)} (2N+1)^{2N} e^{-2Nq} K_{-2\mathrm{i}p}((2N+1)e^{-q}) \right|, \tag{4.19}$$

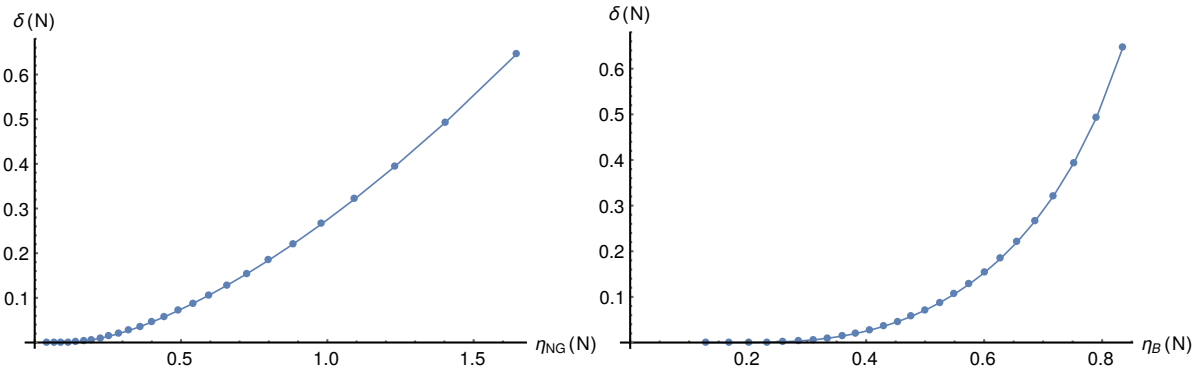
where we changed variables to $q = \alpha x$ and $p = \frac{y}{\alpha}$. Once again we see that the only relevant parameter is N and there is no dependence on α alone.

Due to the behaviour of the K functions the numerical integration of (4.19) is particularly difficult and it was carried out with the aid of the CUBA libraries [43], in particular we used the integration function *Cuhre*.

For the entanglement potential we proceeded exactly as in section 4.1.3 by expanding the ground state on the Fock basis. In this case however far more terms are needed to keep the normalization of the state close to unity, we had to keep 65 terms in the series for the parameters $D = 1$ and $0.1 < \alpha < 2.4$, depicted in fig. 4.12.

4.2.3 Nonclassicality Versus Nonlinearity

As we did for the MHO we can examine the correlation between nonclassicality and nonlinearity by first looking at the parametric plots of δ versus the two measures of nonlinearity, both shown in fig. 4.11. In this case all the quantities depends on the parameter N, but if we

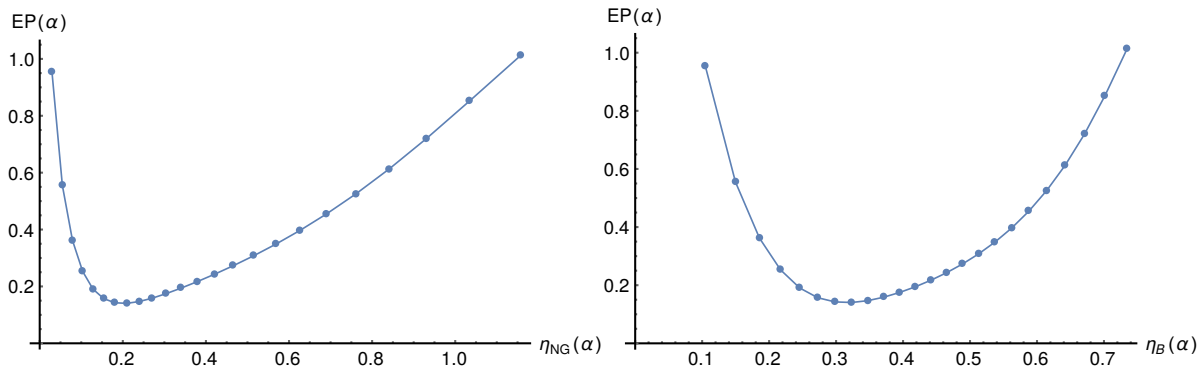


(a) Entropic nonlinearity **(b)** Bures nonlinearity

Figure 4.11: Parametric plot of the nonclassicality δ versus nonlinearity for $D = 1$ and $0.15 < \alpha < 2.7$, which in terms of N means $0.0238 < N < 8.928$.

use the entanglement potential we have a different behaviour. In fig. 4.12 we represent the entanglement potential versus the two nonlinearity measures as a function of α for a fixed value of D. This time we have a different behaviour and EP is not a monotonous function of the nonlinearity.

By studying the MHO we highlighted that the entanglement potential depends also on the squeezing of the state. In fig. 4.13 we represent the ratio between the variances Δx, Δp and the variances of the vacuum, so that when the curves assume values below 1 the state has sub-vacuum squeezing. Even in this case we have that the entanglement potential for small values of α is due to the squeezing in p. This figure is slightly different because in this case



(a) Entropic nonlinearity **(b)** Bures nonlinearity

Figure 4.12: Parametric plot of the entanglement potential EP versus nonlinearity for $D = 1$ and $0.1 < \alpha < 2.4$.

the product between the two uncertainties is not fixed, so that value $(\Delta x)^2(\Delta p)^2$ as a function of α is reported together with the minimum value allowed by the the uncertainty principle, which is reached for $\alpha \to 0$.

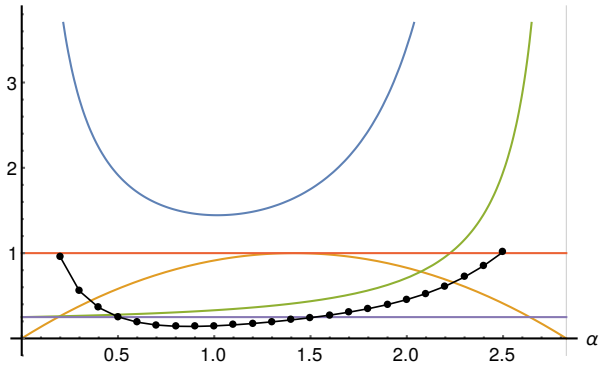


Figure 4.13: This graph shows $\frac{(\Delta x)^2}{0.5}$ in blue, $\frac{(\Delta p)^2}{0.5}$ in yellow, the orange line is the constant 1. When the blue or the yellow curve assumes values below 1 it means we have sub-vacuum squeezing. The green curve represent the product of the two variances $(\Delta x)^2(\Delta p)^2$, while the purple line has the value $\frac{1}{4}$ which is the minimal value of $(\Delta x)^2(\Delta p)^2$ allowed by the uncertainty principle. The range of the parameter is $0.5 < \alpha < 3$, while $D = 1$ is fixed. The points represent the entanglement potential as a function of α.

4.3 Posch-Teller

We devote just a small section to the Posch-Teller potential, because it happens to confirm some of the ideas we gathered so far, but a complete treatment is not possible since the Wigner function of the ground state is just known for a special case.

This potential is defined in general as

$$V_{\mathrm{PT}}(x) = -\frac{D}{\cosh^2(\alpha x)}, \tag{4.20}$$

where $D > 0$ represents the depth of the potential and α is connected to its range. The harmonic limit is obtained at fixed D for $\alpha \to 0$ and the frequency of the reference harmonic oscillator is then $\omega = \sqrt{2D}\alpha$. The nonlinearity of this potential was studied in Ref. [2], however an explicit form of the Wigner function was found only for the choice of the parameters $D = \alpha^2$, so the potential we will be working with is the following

$$V_{\mathrm{PT}}(x) = -\frac{\alpha^2}{\cosh^2(\alpha x)}. \tag{4.21}$$

With this choice the reference harmonic oscillator has a frequency $\omega = \sqrt{2}\alpha^2$; the potential (4.21) and its reference harmonic oscillator are shown in fig. 4.14 for the value $\alpha = 1$. The

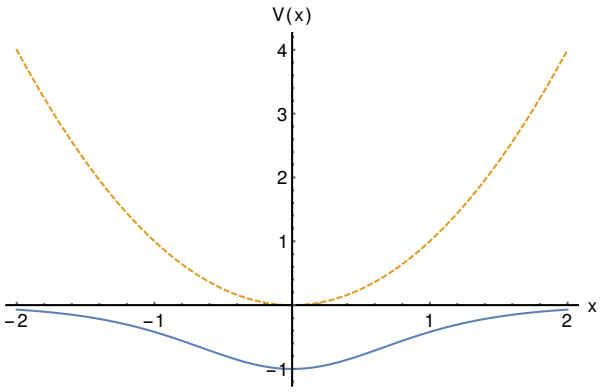


Figure 4.14: The Posh-Teller potential with $D = \alpha^2$ and $\alpha = 1$ (solid blue) and its reference harmonic potential (dashed orange) with $\omega = \sqrt{2}\alpha^2 = \sqrt{2}$.

ground state of potential (4.21) reads

$$\phi_{\mathrm{PT}}(x) = \frac{\sqrt{\alpha}\,\mathrm{sech}(\alpha x)}{\sqrt{2}}. \tag{4.22}$$

The correlation matrix of this state is diagonal

$$\boldsymbol{\sigma}_{\mathrm{PT}} = \begin{pmatrix} \frac{\pi^2}{12\alpha^2} & 0 \\ 0 & \frac{\alpha^2}{3} \end{pmatrix}, \tag{4.23}$$

this shows immediately that while $\Delta\hat{x} = \langle\hat{x}^2\rangle$ and $\Delta\hat{p} = \langle\hat{p}^2\rangle$ depend on α, the product of the variances is the fixed constant $\frac{\pi}{36}$, so the determinant of $\boldsymbol{\sigma}_{\mathrm{PT}}$ and η_{NG} are independent on the parameter as well.

The nonlinearity η_{B} is obtained by the following integration

$$\begin{aligned}\langle\phi|0\rangle &= \int_{-\infty}^{+\infty} \mathrm{d}x\, \alpha\, \mathrm{sech}(\alpha x) \exp\left(-\frac{x^2\alpha^2}{\sqrt{2}}\right) \\ &= \int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{sech}(t) \exp\left(-\frac{t^2}{\sqrt{2}}\right),\end{aligned} \tag{4.24}$$

where the last step follows by changing the integration variable to $t = \alpha x$. The result does not depend on α so neither η_{B} will depend on it.

The Wigner function of the state (4.22) is taken from Ref. [40] and it reads

$$W_{\mathrm{PT}}(x,p) = \frac{\sin(2px)}{\sinh\left(\frac{\pi p}{\alpha}\right)\sinh(2\alpha x)} \tag{4.25}$$

and again we can see that if we try to integrate the absolute value of this function it does not depend on α, in fact

$$\iint \mathrm{d}x\mathrm{d}p \left|\frac{\sin(2px)}{\sinh\left(\frac{\pi p}{\alpha}\right)\sinh(2\alpha x)}\right| = \iint \mathrm{d}t\mathrm{d}q \left|\frac{\sin(2qt)}{\sinh(\pi q)\sinh(2t)}\right|, \tag{4.26}$$

where the last equality follows by changing variables to $t = \alpha x$ and $q = \frac{p}{\alpha}$.

The entanglement potential is computed exactly as it was done for the Morse potential and for the MHO ground states. So we have that δ and the two nonlinearity measures are constant while EP depends on α. Everything is the same as for the case of the MHO and a graph analogous to fig. 4.7 can be drawn to show that the entanglement potential for small values of α depends on the amount of squeezing in p, this is shown in fig. 4.15. To compute the entanglement we kept 40 terms in the expansion on the Fock basis, in order to have the state with a normalization equal to unity up to the fourth decimal digit.

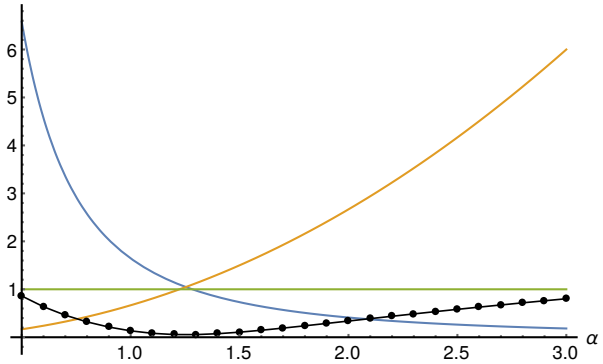


Figure 4.15: This graph shows $\frac{(\Delta x)^2}{0.5}$ in blue, $\frac{(\Delta p)^2}{0.5}$ in orange, the green line is the constant 1; when one of the curves assumes values below 1 it means we have sub-vacuum squeezing. The range of the parameter is $0.5 < \alpha < 3$. The points represent the entanglement potential as a function of α.

5 Evolution in the Presence of Self-Kerr Interaction

So far we have examined systems with a static potential and in particular we have been interested in their ground state. We can say that the idea we have explored in the previous chapters was related to mechanical systems and the variable x of the potential $V(x)$ could represent an actual spatial position (albeit rescaled to be dimensionless). The very fact that the ground state is different from the vacuum of the harmonic oscillator is a consequence of $V(\hat{x})$ being an operator which does not commute with the number operator $\hat{a}^\dagger\hat{a}$.

Now we want to extend our main idea to a different domain, namely to the framework of quantum optics. In quantum optics we usually have a given initial state of light interacting with a particular medium for a finite amount of time and then one or more output states after the interaction. This is conceptually very different from dealing with potentials and their ground states obtained by solving the Schrödinger equation.

The idea we want to test is that an interaction (rather than a potential) which is more non-linear gives rise to a more nonclassical output state. To perform this analysis we will examine the simple model of a bosonic field going through a third-order nonlinear $\chi^{(3)}$ medium, which is one of the simplest and most interesting examples of non-linear interaction in quantum optics and gives rise to the so-called self-Kerr effect. The Hamiltonian which describes the self-Kerr interaction is the following

$$H_{\text{Kerr}} = \Gamma(\hat{a}^\dagger\hat{a})^2. \tag{5.1}$$

The corresponding evolution operator in interaction picture is $U = \exp\{-\mathrm{i}\gamma(a^\dagger a)^2\}$, where $\gamma = \Gamma t$ is a dimensionless coupling constant that encodes both the interaction time and strength. The evolution corresponding to this Hamiltonian gives rise to non-Gaussian states because H_{Kerr} is more than quadratic in the destruction and creator operators. Moreover the evolution is periodical with period 2π as for $\gamma = 2\pi$ we get the initial state. It is worth noting that this kind of nonlinearity was studied in Ref. [44] both classically and on the quantum level as an exactly solvable anharmonic oscillator. We emphasize that this Hamiltonian commutes with the Hamiltonian of the harmonic oscillator, this is the reason why the model is solvable, but it also implies that the ground state of the system remains $|0\rangle$, so it would be useless to treat this interaction as we did for the other anharmonic oscillators in the previous chapters.

The figures of merit that we will study are the nonclassicality and non-Gaussianity of the output state, forgetting completely the ground state. In particular non-Gaussianity will be used to quantify the nonlinearity of the interaction at different values of γ, but in order to do so the input state has to be Gaussian. For this reason a coherent state is used as the initial

state; this choice makes sense even from a physical point of view since it can be thought as a very simplified model of a monochromatic laser going trough an optical fibre. Two ways to quantify non-Gaussianity were already introduced in section 2.1.

One might also ask if non-Gaussianity is in general related to nonclassicality, regardless of the input state. For this reason we analyse also an input state which is already highly nonclassical and non-Gaussian.

5.1 Coherent State Input

5.1.1 Generation of Cat and Kitten States

Cat states are usually intended as a superposition of two macroscopically distinguishable states, while kitten states are superpositions of more than two of those states. In quantum optics these macroscopically distinguishable states are coherent states $|\alpha\rangle$ with a large mean photon number $|\alpha|^2$. Kerr interaction has long been proposed as a method to generate optical cat states [45], an experimental goal which has been achieved only recently by engineering an artificial Kerr medium [46].

The evolution of a coherent state in a Kerr medium has been studied extensively in the quantum optics literature [47–49]. If we write the initial coherent state in the Fock basis (1.10) the evolved state is obtained easily by applying the evolution operator:

$$|\alpha_\gamma\rangle = \exp\{-\mathrm{i}\gamma(\hat{a}^\dagger\hat{a})^2\}\left(e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\right) = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-\mathrm{i}\gamma n^2}|n\rangle. \tag{5.2}$$

This state is sometimes called Kerr state and it is a particular example of the so-called generalized coherent states [13], which are the most general states which exhibit full coherence and differ from ordinary coherent states because of additional phase factors in their Fock decomposition. It was recognized early that under certain conditions these state can be expressed as linear combinations of coherent states [50, 51].

The first and easiest simmetry to note is that for $\gamma = \pi$ the coefficients in (5.2) become just $(-1)^{n^2}$ thus we have $|\alpha_\pi\rangle = |-\alpha\rangle$. In general at a time $\gamma = 2\pi m/N$, with m and N coprimes, the state will be given by a superposition of N coherent states if N is odd, or $N/2$ states if N is even (details are left in Ref. [47]). The states for different values of m will have however different coefficients and the coefficients of the state obtained for $\gamma = 2\pi(N-k)/N$ are the complex conjugates of the ones obtained for $\gamma = 2\pi k/N$.

5.1.2 Non-Gaussianity

For this analysis only the entropic non-Gaussianity (2.18) is computed, since to compute the Hilbert-Schmdit distance (2.18) we would need to integrate the Wigner function as shown in (2.12). It can be done numerically, but it is computationally hard and we believe that not much insight is gained by studying this measure as well.

Entropic non-Gaussianity

To compute the entropic non-Gaussianity NG_{E} we need the correlation matrix. If we use the form (3.23) written in terms of the creation and destruction operators, then we just need to compute the expectation values for the state (5.2).

We have that $\langle \hat{a}^\dagger \hat{a} \rangle = |\alpha|^2$ because the interaction Hamiltonian (5.1) commutes with the number operator, so the expectation value is the same of a coherent state. On the other hand $\langle \hat{a}^2 \rangle$ and $\langle \hat{a} \rangle$ have to be carried out explicitly, while $\langle (\hat{a}^\dagger)^2 \rangle$ and $\langle \hat{a}^\dagger \rangle$ are of course just their complex conjugates. We get the following results

$$\begin{aligned} \langle \alpha_\gamma | \hat{a}^2 | \alpha_\gamma \rangle &= e^{-|\alpha|^2} \sum_{n,m=0}^{\infty} \frac{\alpha^{*m} \alpha^n}{\sqrt{n!\, m!}} e^{-\mathrm{i}\gamma(n^2-m^2)} \sqrt{n}\sqrt{n-1} \langle m | n-2 \rangle \\ &= e^{-|\alpha|^2} \sum_{m=0}^{\infty} \frac{\alpha^{*m} \alpha^{m+2}}{m!} e^{-\mathrm{i}\gamma(4m+4)} \\ &= \alpha^2 e^{-|\alpha|^2(1-e^{-4\mathrm{i}\gamma})-4\mathrm{i}\gamma} \end{aligned} \tag{5.3}$$

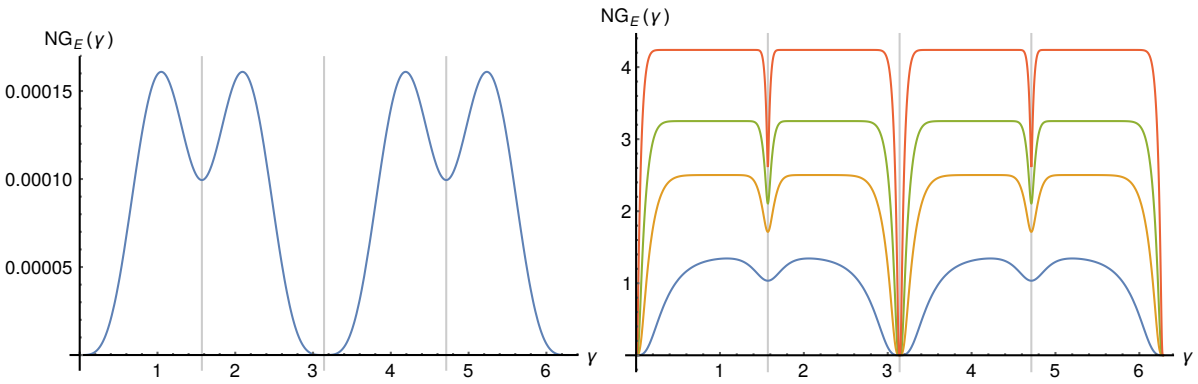
$$\begin{aligned} \langle \alpha_\gamma | \hat{a} | \alpha_\gamma \rangle &= e^{-|\alpha|^2} \sum_{n,m=0}^{\infty} \frac{\alpha^{*m} \alpha^n}{\sqrt{n!\, m!}} e^{-\mathrm{i}\gamma(n^2-m^2)} \sqrt{n} \langle m | n-1 \rangle \\ &= e^{-|\alpha|^2} \sum_{m=0}^{\infty} \frac{\alpha^{*m} \alpha^{m+1}}{m!} e^{-\mathrm{i}\gamma(2m+1)} \\ &= \alpha e^{-|\alpha|^2(1-e^{-2\mathrm{i}\gamma})-\mathrm{i}\gamma}. \end{aligned} \tag{5.4}$$

With this results the non-Gaussianity can be computed as usual $NG_{\mathrm{E}} = h\left(\sqrt{\det \sigma}\right)$ and it is depicted in fig. 5.1 for some different values of α. As the mean photon number gets bigger the non-Gaussianity tends to have larger and larger plateaus, even though the absolute minima for $\gamma = 0, \pi, 2\pi$ and the relative minima for $\gamma = \frac{\pi}{2}, \frac{3\pi}{2}$ are always present. The absolute minima correspond to times at which only one coherent state is present, while the relative minima are at the times when cat states appear. The symmetry present in fig. 5.1 can be explained by noting that the evolution from 0 to π and from π to 2π involves Wigner functions with the same form, even if they are centred in different points of the phase space. For this reason we expect the relevant properties to have the same behaviour in the two halves of the period.

We also notice that as $|\alpha|$ (and thus the mean photon number) increases the maximum value of non-Gaussianity becomes bigger and it is reached for smaller values of γ.

5.1.3 Nonclassicality

In this case the only measure of nonclassicality we actually use is the entanglement potential, because of technical difficulties in the integration of the Wigner function.



(a) $\alpha = 0.1$ **(b)** $\alpha = 1, 2, 3, 5$

Figure 5.1: The entropic non-Gaussianity over a whole period, for different values of α. In figure (b) we have $\alpha = 1$ in blue, $\alpha = 2$ in yellow, $\alpha = 3$ in green and $\alpha = 5$ in orange. The case $\alpha = 0.1$ is plotted separately because it has a much smaller scale. The vertical lines correspond the values $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

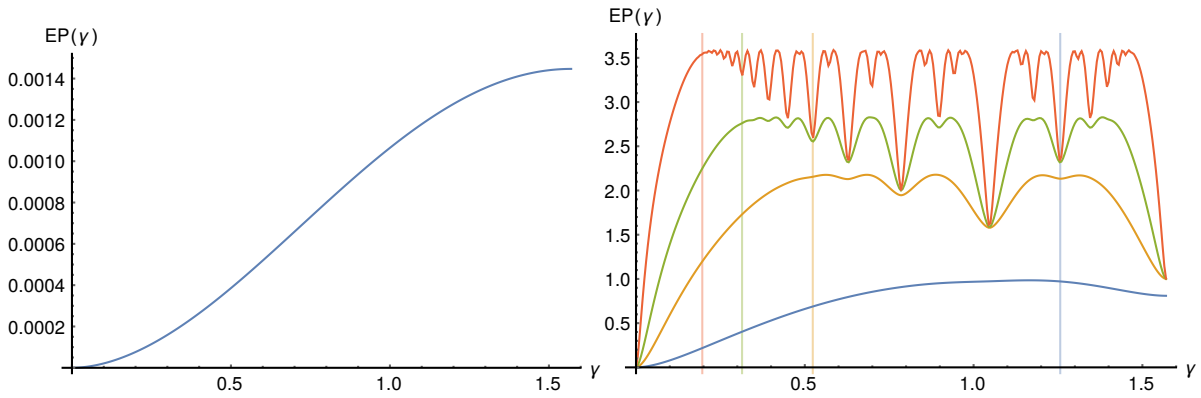
Entanglement Potential

The entanglement potential was obtained exactly like we did for the ground states of the potentials, that is by expanding the state on the Fock basis and applying the evolution operator of the beam-splitter. In this case we already have the expansion on the Fock basis (5.2), but finding the eigenvalues of the density matrix to compute the entropy still has to be done numerically.

The entanglement potential is depicted in fig. 5.2 and we choose to represent just a quarter of the period. The part of the graph between $\frac{\pi}{2}$ and π is symmetrical to the first quarter while from π to 2π it is exactly the same as the first half: this is the same periodicity of the non-Gaussianity in fig. 5.1. This is the expected behaviour, according to the considerations of the previous section.

Like for the non-Gaussianity we can observe a trend for growing mean photon number, but this time it is partly different. Similarly to the non-Gaussianity, we see that the maximum entanglement potential obtained during the evolution is bigger and it is reached sooner if we increase α. But we can see that the graph also becomes more and more oscillating, it always retains the absolute minima for $\gamma = 0, \pi$, while the situation for the relative minima is more complex and their number grows with $|\alpha|$. The relative minima actually happen when $\gamma = 2\pi m/N$, that is when the state becomes a kitten state. This means that for growing photon number the number of distinguishable superposed coherent states increases, coherently with the phase space analysis in Ref. [47].

The entanglement potential was obtained by keeping enough terms in the series (5.2) to have the norm of the state greater than 0.99999; for $\alpha = 0.1, 1$ we kept the first ten terms, for $\alpha = 3$ the first twenty-five terms and for $\alpha = 5$ the first fifty terms.

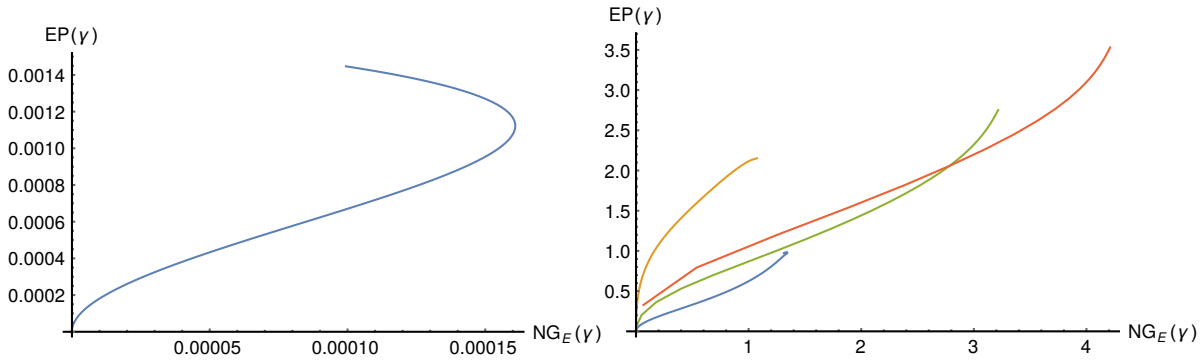


(a) $\alpha = 0.1$ **(b)** $\alpha = 1, 2, 3, 5$

Figure 5.2: The entropic non-Gaussianity for $0 \leq \gamma \leq \frac{\pi}{2}$ and different values of α. In figure (b) we have $\alpha = 1$ in blue, $\alpha = 2$ in yellow, $\alpha = 3$ in green and $\alpha = 5$ in orange. The vertical lines have the following values of γ: $\frac{2\pi}{5}$ (blue), $\frac{\pi}{6}$ (yellow), $\frac{\pi}{10}$ (green) and $\frac{\pi}{16}$ (orange). The case $\alpha = 0.1$ is plotted separately in figure (a) because it has a much smaller scale.

Nonclassicality Versus Non-Gaussianity

The behaviour of these two quantities is somewhat similar, but not completely the same; in fig. 5.3 we have the nonclassicality versus non-Gaussianity parametric plot. The value of the parameters is bounded by choosing the values of the vertical lines in (5.2b) as maximums. Non-Gaussianity and nonclassicality are positively correlated for the initial part of the evolution, but the same behaviour does not hold for the whole period as they cease to be monotonous with respect to each other. This behaviour suggests the idea that adding some kind of nonlinearity (and thus non-Gaussianity of the corresponding output state) let a classical state become nonclassical, but after it reaches its maximum nonclassicality the relation is not one to one any more.



(a) $\alpha = 0.1$ **(b)** $\alpha = 1, 2, 3, 5$

Figure 5.3: Parametric plots of the entanglement entropic versus the non-Gaussianity for different values of α. In figure (b) we have $\alpha = 1$ in blue, $\alpha = 2$ in yellow, $\alpha = 3$ in green and $\alpha = 5$ in orange. The parameter γ goes from 0 to $\frac{2\pi}{5}, \frac{\pi}{6}, \frac{\pi}{10}$ and $\frac{\pi}{16}$ respectively. The case $\alpha = 0.1$ is plotted separately in figure (a) because it has a much smaller scale with $0 < \gamma < \frac{\pi}{2}$.

5.2 Finite Superposition of Fock States

To conclude our analysis we want to check an unrealistic (nonetheless conceptually interesting) scenario, obtained by choosing a simple and already non-Gaussian and nonclassical state as the input state of the unitary evolution given by the self-Kerr interaction. The simplest state to use would be a single Fock state, but of course it would be unaffected, since it would just gain a phase factor $e^{-i\gamma n^2}$, we thus have to choose a finite superposition of Fock states. The state we have chosen is $|\Psi\rangle = \frac{|0\rangle+|1\rangle+2\rangle}{\sqrt{3}}$, in this way the effects of the term n^2 are relevant, because of the presence of $|2\rangle$. The evolved state is the following

$$|\Psi_\gamma\rangle = \frac{|0\rangle + e^{-i\gamma}|1\rangle + e^{-4i\gamma}|2\rangle}{\sqrt{3}}. \tag{5.5}$$

This state is clearly not classical and not Gaussian, so we want to check if the relationship between non-Gaussianity and nonclassicality holds true in this case as well.

5.2.1 Non-Gaussianity

Entropic Non-Gaussianity

Using again the covariance matrix in the form (3.23), with these expectation values

$$\langle\Psi_\gamma|\hat{a}^\dagger\hat{a}|\Psi_\gamma\rangle = 1 \qquad \langle\Psi_\gamma|\hat{a}^2|\Psi_\gamma\rangle = \frac{\sqrt{2}e^{-4i\gamma}}{3} \qquad \langle\Psi_\gamma|\hat{a}|\Psi_\gamma\rangle = \frac{e^{-i\gamma} + \sqrt{2}e^{-3i\gamma}}{3} \tag{5.6}$$

we obtain the result represented in fig. 5.4 for the non-Gaussianity. As one would expect this state is highly non Gaussian even before the self-Kerr interaction (for $\gamma = 0$), but it becomes even more non-Gaussian thanks to the self-Kerr effect.

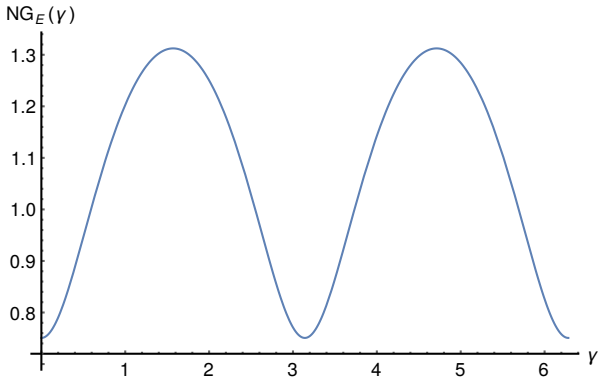
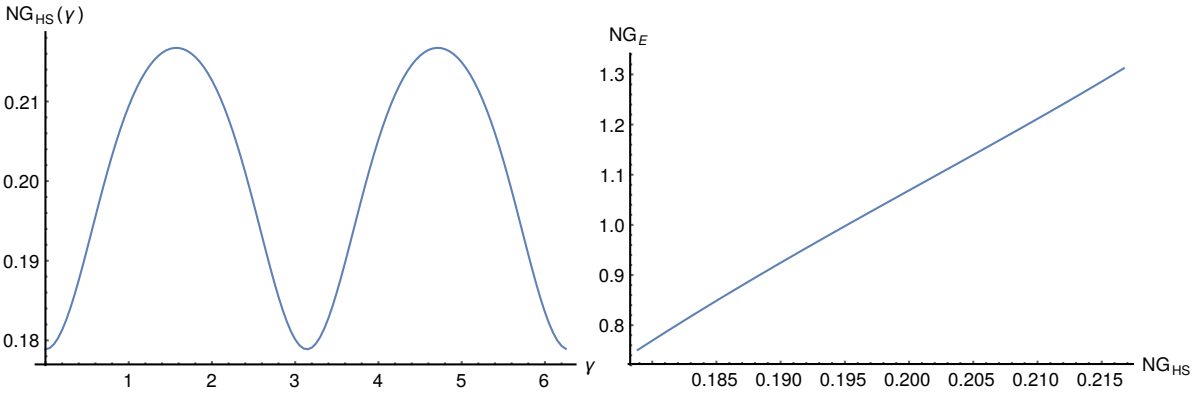


Figure 5.4: The entropic non-Gaussianity over a whole period.

Hilbert-Schmidt Non-Gaussianity

The Hilbert-Schmidt distance from a reference Gaussian state can be obtained by computing the L^2 distance between the Wigner function of the state under examination and the

reference Gaussian state, as reported in (2.11). Since this state is quite simple NG_{HS} could be computed without too much effort. The two measures of non-Gaussianity have a very similar behaviour, as can be seen in fig. 5.5. The scale of the values assumed by the two measures is different but they can be used interchangeably in this case, since they are perfectly correlated.



(a) Hilbert-Schmidt non-Gaussianity **(b)** Parametric plot of non-Gaussianities

Figure 5.5: Hilbert-Schmidt non-Gaussianity as a function of γ for a whole period and parametric plot of NG_{E} and NG_{HS}.

5.2.2 Nonclassicality

For this simple input state we can calculate the nonclassicality both via entanglement potential and via negative volume of the Wigner function, without any kind of approximations.

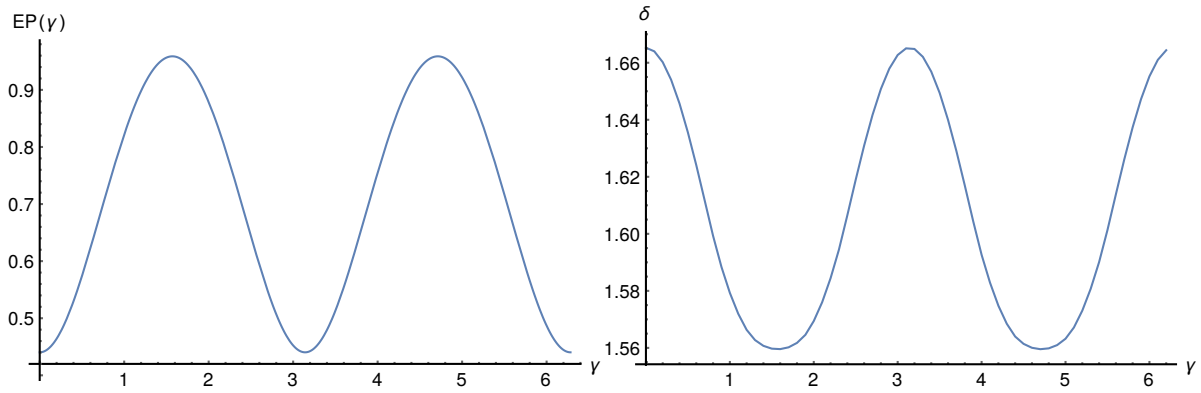
The entanglement potential is computed exactly like we did in chapter 3 for the perturbative states, with the little difference that now the coefficients are complex numbers. The result is in fig. 5.6a and it shows perfect agreement with the behaviour of the non-Gaussianity: state begins as already nonclassical and becomes even more so due to the self-Kerr interaction.

The Wigner function is computed exactly in the same manner used for the perturbative states in chapter 3 and it is a linear combination of Legendre polynomials. The nonclassicality index δ (twice the volume of the negative part of the Wigner function), is obtained by a numerical integration of the absolute value of the Wigner function over the whole complex plane.

This time the result is quite peculiar: every quantity has the same periodicity, but δ has an opposite phase. We can see in fig. 5.6b that δ has its greatest value before interacting, and the interaction makes the negative volume of the Wigner become smaller.

Nonclassicality and Non-Gaussianity

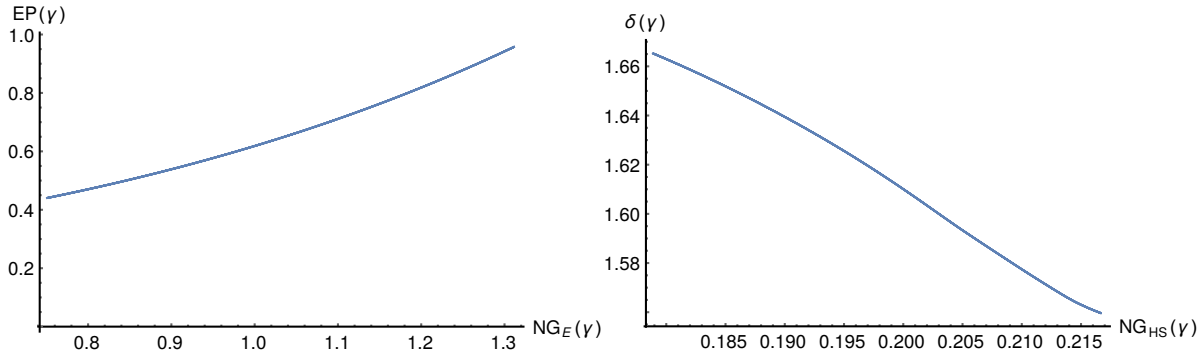
Differently than in the previous cases we observe that the relationship between non-Gaussianity and nonclassicality is completely dependent on the definition of nonclassicality that we choose. This was already evident by looking at fig. 5.7b, but nonetheless we can see in a



(a) Entanglement potential **(b)** Wigner volume nonclassicality

Figure 5.6: Plot of both the nonclassicality measures for a whole period.

parametric plot in fig. 5.7 that depending on the nonclassicality measure we get a positive correlation or a negative correlation.



(a) Entanglement potential **(b)** Wigner volume nonclassicality

Figure 5.7: Parametric plot of nonclassicality versus nonlinearity.

Conclusions and Outlook

In this work we have addressed the possible connection between the nonclassical features of a quantum oscillator and its nonlinearity, either due to its static potential or encountered during its dynamical evolution. In particular, we have analysed in detail how and whether the nonlinearity of a confining potential is linked to the nonclassicality of its oscillatory ground state. Our results provide strong evidence to support the idea that nonlinearity is indeed a resource to generate nonclassicality and that this relationship is present either using entropic or metric quantifiers for both the quantities.

At first we have considered a harmonic oscillator with polynomial perturbations and found that the nonclassicality of the (perturbative) ground state is quantitatively related to the nonlinearity of the potential. The link is present using the Bures based measure of the non linearity of the potential as well as the entropic one and using both both the negative volume of the Wigner function and the entanglement potential to quantify nonclassicality of the ground state.

This model also highlighted that when more than one parameter is present the relationship between the quantities is not in general one to one. We also found that using different measures of nonclassicality we can have different hierarchies. In fact, at a fixed nonlinearity, the ground state of the potential with the perturbative term x^4 is more nonclassical in the sense of the Wigner function than the one with the term x^6. On the other hand we have the opposite situation if we look at the entanglement potential. This behaviour however does not change the validity of the main statement, because if we change just one parameter at a time nonlinearity and nonclassicality increase or decrease together.

We then considered potentials that are analytically solvable and again we confirmed our main idea also finding an additional scaling property: even if the potentials and the ground states do depend on two different parameters both the measures of nonlinearity and the nonclassicality turned out to depend on a single parameter.

The entanglement potential does not enjoy this property and therefore, in this case, it is not a monotonous function of nonlinearity. In turn, this behaviour corresponds to the fact that the state becomes highly squeezed for values of the parameter that correspond instead to a small nonlinearity. We note that the Wigner function remains Gaussian for squeezed states, but purely squeezed states are generated by Hamiltonians which are quadratical in the creation and destruction operators, therefore they do not correspond to proper nonlinearities. If we deal only with the squeezed vacuum this fact is trivial, but in the more realistic case of the ground state of an anharmonic potential the squeezing is mixed with all the other nonlinear effects. By showing the scaling property we thus highlight that measuring nonclassicality via the Wigner function negative volume actually addresses what we might call "nonlinear nonclassicality", that is to say nonlinearity generated by nonlinear effects rather than linear ones like squeezing. The validity of this statement is not yet general, but rooted in these

particular examples, the issue will be subject to further investigations. For the moment no general underlying rule has been found to explain the existence of the scaling property for all the potentials; it remains an open question and paves the way for further developments.

As the last step we found that the connection is present also in a dynamical context. This was done by slightly changing our framework and studying the evolution of an input state under self-Kerr interaction rather than the ground state of the Hamiltonian. As an additional result we found that the connection between nonclassicality and nonlinearity basically disappears if the input state is already nonclassical and non-Gaussian. This fact actually fits nicely with the idea that nonlinearity is a resource to generate nonclassicality, and underlines that it is a different concept than having a general connection between non-Gaussianity and nonclassicality of a quantum state.

A Appendices

A.1 Disentangling the Exponential Operator

To disentangle the exponential operator means to express the exponential of a sum of operators as the product of the exponentials of operators, this appendix follows the methods of Ref. [52]. This problem, given the operators $\hat{A}$ and $\hat{B}$, means to find operators $\hat{C}_1, \hat{C}_2, \ldots$ which satisfy this relationship

$$\exp\left[\hat{A} + \hat{B}\right] = \exp\left(\hat{C}_1\right) \exp\left(\hat{C}_2\right) \ldots \tag{A.1}$$

In general we need an infinite number of $\hat{C}_n$ operators which are combinations of repeated commutators of $\hat{A}$ and $\hat{B}$, and finding an analytical expression might not be possible in general. The situation is much simpler if we restrict to the case where $\hat{A}$ and $\hat{B}$ are elements of a finite-dimensional Lie algebra.

A Lie algebra is a linear vector space of operators which is also closed under the operation of commutation (the commutator of two operators is still a vector of the space). If we have a vector space spanned by the complete set of operators $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$, which are called the generators of the algebra, then

$$\left[\hat{X}_i, \hat{X}_j\right] = \sum_{k=1}^{n} c_{ijk} \hat{X}_k, \tag{A.2}$$

where the c_{ijk} are in general complex numbers and are called the structure constants of the algebra.

In this case we can reduce the problem (A.1) to the following situation

$$\exp\left[\theta \sum_{i=1}^{n} \alpha_i \hat{X}_i\right] = \exp\left[f_1(\theta)\hat{X}_1\right] \ldots \exp\left[f_n(\theta)\hat{X}_n\right], \tag{A.3}$$

where we have the condition $f_i(0) = 0$ and the solution is found by obtaining expressions for the f_i. We remind that a differential operator enjoys the property $\exp\left(\hat{A}\right)^{-1} = \exp\left(-\hat{A}\right)$ and this extends also to a product of exponentials in the following way

$$\left[\exp\left(\hat{A}_1\right) \ldots \exp\left(\hat{A}_m\right)\right]^{-1} = \exp\left(-\hat{A}_m\right) \ldots \exp\left(-\hat{A}_1\right). \tag{A.4}$$

If we differentiate equation (A.3) with respect to θ and then we then multiply the resulting

expression for the inverse of (A.3) (where the inverse is defined by (A.4)) we get

$$\begin{aligned}\sum_{i=1}^{n}\alpha_i\hat{X}_i =& \dot{f}_1(\theta)\hat{X}_1 + \dot{f}_2(\theta)\exp\big[f_1(\theta)\big]\hat{X}_2\exp\big[-f_1(\theta)\hat{X}_1\big] + \dots \\ &+ \dot{f}_n(\theta)\Big\{\exp\big[f_1(\theta)\hat{X}_1\big]\dots\exp\big[f_{n-1}(\theta)\hat{X}_{n-1}\big]\hat{X}_n \\ &\times\exp\big[-f_{n-1}(\theta)\hat{X}_{n-1}\big]\dots\exp\big[-f_1(\theta)\hat{X}_1\big]\Big\}.\end{aligned} \tag{A.5}$$

In (A.5) transformations of the following kind are present

$$\hat{X}_i(\theta) = \exp\big(-\theta\hat{Z}\big)\hat{X}_i\exp\big(\theta\hat{Z}\big), \tag{A.6}$$

they are called similarity transformations. If $\hat{Z} = \sum_{i=1}^{n}\beta_i\hat{X}_i$ (it is an element of the algebra), we can differentiate (A.6) with respect to θ and then use the definition of $\hat{Z}$ to get the following result

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{X}_i(\theta) = \sum_{k=1}^{n}\left[\sum_{j=1}^{n}c_{ijk}\beta_j\right]\hat{X}_k(\theta). \tag{A.7}$$

The solution of (A.7) gives $\hat{X}_i(\theta)$ in terms of a linear combination of $\hat{X}_1(0),\dots,\hat{X}_n(0) \equiv \hat{X}_1,\dots,\hat{X}_n$.

By solving the similarity transformations with (A.7) the right hand side of (A.5) reduces to a linear combination of the $\hat{X}_i$s. A comparison of the coefficients in the equation then leads to differential equations for the $\{f_i(\theta)\}$; their solution gives the solution to the problem of disentangling the exponential operator. This general procedure can be carried out for different algebras, but we will focus on the only one needed in this Thesis, i.e. the harmonic oscillator algebra.

A.1.1 Harmonic Oscillator Algebra

The set of operators $\big(\hat{a}^\dagger, \hat{a}, \hat{N} = \hat{a}^\dagger\hat{a}, I\big)$ which we have already introduced in section 1.1.1 obeys the following commutation relations

$$\big[\hat{a},\hat{a}^\dagger\big] = 1 \qquad \big[\hat{N},\hat{a}\big] = -\hat{a} \qquad \big[\hat{N},\hat{a}^\dagger\big] = \hat{a}^\dagger, \tag{A.8}$$

the second two follow from the first one, which is the standard bosonic commutation relation (1.2). For this property these operators are the generators of the harmonic oscillator Lie algebra; the name is given by the fact that the hamiltonian of a harmonic oscillator with a linear driving force can be expressed as a linear combination of the elements of this algebra.

A general element of the algebra is expressible as

$$\hat{Z}_{\mathrm{ho}} = \alpha_1\hat{a} + \alpha_2\hat{a}^\dagger + \alpha_3\hat{a}^\dagger\hat{a}, \tag{A.9}$$

where the α_i are complex coefficients. We have the following general similarity transformation

$$\hat{a}(\theta) = \exp\big(-\theta\hat{Z}_{\mathrm{ho}}\big)\hat{a}\exp\big(-\theta\hat{Z}_{\mathrm{ho}}\big), \tag{A.10}$$

which, following the general arguments, leads to

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{a}(\theta) = \alpha_3\hat{a}(\theta) + \alpha_2, \tag{A.11}$$

which is solved by

$$\hat{a}(\theta) = \exp\big(\alpha_3\theta\big)\hat{a} + \frac{\alpha_2}{\alpha_3}\big[\exp\big(\alpha_3\theta\big) - 1\big]. \tag{A.12}$$

With this result we can address the problem of disentangling the exponential of a sum of generators of the harmonic oscillator algebra, so we focus on the following equation

$$\exp\big[\theta\big\{\alpha_1\hat{a} + \alpha_2\hat{a}^\dagger\hat{a} + \alpha_3\hat{a}^\dagger\big\}\big] = \exp\big[f_1(\theta)\hat{a}^\dagger\big]\exp\big[f_2(\theta)\hat{a}^\dagger\hat{a}\big]\exp\big[f_3(\theta)\hat{a}\big]\exp\big[f_4\big]. \tag{A.13}$$

A long but straight-forward calculation, using the procedure outlined for the general case along with the result (A.12), yields the following differential equations

$$\dot{f}_1 - f_1\dot{f}_2 = \alpha 3, \quad \dot{f}_2 = \alpha_2, \quad \dot{f}_3 = \alpha_1\exp\big(f_2\big), \quad \dot{f}_4 - f_1\dot{f}_3\exp\big(-f_2\big), \tag{A.14}$$

their solutions reads

$$\begin{aligned} f_1 &= \frac{\alpha_3}{\alpha_2}\big[\exp(\alpha_2\theta) - 1\big], \qquad & f_2 &= \alpha_2\theta, \\ f_3 &= \frac{\alpha_1}{\alpha_2}\big[\exp(\alpha_2\theta) - 1\big], \qquad & f_4 &= \frac{\alpha_1\alpha_3}{\alpha_2^2}\big[\exp(\alpha_2\theta) - \alpha_2\theta - 1\big]. \end{aligned} \tag{A.15}$$

If we choose the particular solution $\alpha_2 = 0$, $\theta = 1$ and we substitute the solutions (A.15) into equation (A.1) we have

$$\begin{aligned} \exp\big[\alpha_1\hat{a} + \alpha_3\hat{a}^\dagger\big] &= \exp\big[\alpha_1\hat{a}\big]\exp\big[\alpha_3\hat{a}^\dagger\big]\exp\Big[-\frac{1}{2}\alpha_1\alpha_3\Big] \\ &= \exp\big[\alpha_3\hat{a}^\dagger\big]\exp\big[\alpha_1\hat{a}\big]\exp\Big[\frac{1}{2}\alpha_1\alpha_3\Big], \end{aligned} \tag{A.16}$$

these expressions readily become the normally and antinormally ordered forms of the displacement operator (1.6) and (1.7).

We often deal with a function $F\big(\hat{a}, \hat{a}^\dagger\big)$ which has to be expanded in powers of $\hat{a}$ and $\hat{a}^\dagger$ and this expansion can be done in normal or antinormal order. In particular we want to expand the exponential of the number operator $\exp\big[\theta\hat{a}^\dagger\hat{a}\big]$ in its normally ordered form to get the expression (1.37). We start from

$$\exp\big[\theta\hat{a}^\dagger\hat{a}\big] = \sum_{m=0}^{\infty}\frac{x^m(\theta)}{m!}\hat{a}^{\dagger m}\hat{a}^m, \tag{A.17}$$

where the function $x(\theta)$ is unknown and has to be determined. We differentiate (A.17) with respect to θ and get

$$\hat{a}^\dagger\hat{a}\exp\big[\theta\hat{a}^\dagger\hat{a}\big] = \frac{\mathrm{d}}{\mathrm{d}\theta}\sum_{m=0}^{\infty}\frac{x^m(\theta))}{m!}\hat{a}^{\dagger m+1}\hat{a}^{m+1}. \tag{A.18}$$

Now a few manipulations have to be performed: we have to rewrite the expansion (A.17) in the last expression, then we have to repeatedly use the relation $\hat{a}\hat{a}^{\dagger m} = \hat{a}^{\dagger m}\hat{a} + m\hat{a}^{\dagger m-1}$ (a consequence of property (1.4)). The comparison of the coefficients of the resulting equation gives rise to the following differential equation for $x(\theta)$

$$\frac{\mathrm{d}x(\theta)}{\mathrm{d}\theta} = x(\theta) + 1, \tag{A.19}$$

if we choose the initial condition $x(0) = 0$ the solution reads

$$x(\theta) = \exp(\theta) - 1, \tag{A.20}$$

which finally gives the wanted result

$$\exp\left[\theta\hat{a}^{\dagger}\hat{a}\right] = \sum_{m=0}^{\infty} \frac{(\exp(\theta) - 1)^m}{m!} \hat{a}^{\dagger m}\hat{a}^m. \tag{A.21}$$

We remark that the procedure to get the antinormally ordered form is analogous.

A.2 Expectation Value of The Displacement Operator

In this appendix we want to show an explicit derivation of the result (3.31) which is the expectation value of the displacement operator between two Fock states. We start by making use of the normally ordered form of the displacement operator (1.6), so we get

$$\langle n'|\hat{D}(z)|n\rangle = e^{-\frac{1}{2}|z|^2}\langle n'|e^{z\hat{a}^\dagger}e^{-z^*\hat{a}}|n\rangle. \tag{A.22}$$

We can now focus just on the product of right part of the scalar product $e^{-z^*\hat{a}}|n\rangle$, this becomes

$$e^{-z^*\hat{a}}|n\rangle = \sum_{k=0}^{\infty}\frac{(-z^*)^k}{k!}\hat{a}^k|n\rangle = \sum_{k=0}^{n}\frac{(-z^*)^k}{k!}\sqrt{n(n-1)\dots(n-k+1)}|n-k\rangle, \tag{A.23}$$

where we used the properties of the bosonic creation and destruction operators eq. (1.1). By noting that $n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$ we can do the same manipulation for $\langle n'|e^{z\hat{a}^\dagger}$ and put all together to rewrite (A.22) as

$$\langle n'|\hat{D}(z)|n\rangle = e^{-\frac{1}{2}|z|^2}\sum_{j=0}^{n'}\sum_{k=0}^{n}\frac{z^j(-z^*)^k}{j!\,k!}\sqrt{\frac{n'!\,n!}{(n'-j)!\,(n-k)!}}\langle n'-j|n-k\rangle. \tag{A.24}$$

We have that $\langle n'-j|n-k\rangle = \delta_{n'-j,n-k}$ and if we choose the case where $n' > n$ then we can carry one summation and get

$$\begin{aligned}\langle n'|\hat{D}(z)|n\rangle &= e^{-\frac{1}{2}|z|^2}\sqrt{n'!\,n!}z^{n'-n}\sum_{k=0}^{n}\frac{(-1)^k|z|^{2k}}{k!\,(k+n'-n)!\,(n-k)!}\\ &= e^{-\frac{1}{2}|z|^2}\sqrt{(n+d)!\,n!}z^d\sum_{k=0}^{n}\frac{(-1)^k|z|^{2k}}{k!\,(k+d)!\,(n-k)!},\end{aligned} \tag{A.25}$$

where $d = n' - n$. By noting that the following relationship holds true

$$\binom{n+d}{n-k} = \frac{(n+d)!}{(n-k)!\,(n+d-n+k)!} = \frac{(n+d)!}{(n-k)!\,(k+d)!} \tag{A.26}$$

and reminding the explicit expression for the generalized Laguerre polynomials

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}, \tag{A.27}$$

we can finally write

$$\langle n'|\hat{D}(z)|n\rangle = e^{-\frac{1}{2}|z|^2}z^d\sqrt{\frac{n!}{(n+d)!}}L_k^d(|z|^2). \tag{A.28}$$

If instead of $n' > n$ the opposite is true, then everything can be done in the same way but we get

$$\langle n'|\hat{D}(z)|n\rangle = e^{-\frac{1}{2}|z|^2}(-z^*)^d\sqrt{\frac{n!}{(n+d)!}}L_n^d(|z|^2). \tag{A.29}$$

Bibliography

[1] B. Teklu, A. Ferraro, M. Paternostro, and M. G. A. Paris, "Nonlinearity and nonclassicality in a nanomechanical resonator," Jan. 2015 (cit. on p. 1).

[2] M. G. A. Paris, M. G. Genoni, N. Shammah, and B. Teklu, "Quantifying the nonlinearity of a quantum oscillator," *Physical Review A*, vol. 90, no. 1, p. 012 104, Jul. 2014 (cit. on pp. 1, 25, 49, 53).

[3] E. P. Wigner, "On the quantum correction for thermodynamic equilibrium," *Physical Review*, 1932 (cit. on pp. 3, 12).

[4] R. J. Glauber, "Coherent and incoherent states of the radiation field," *Physical Review*, vol. 49, no. 638, 1963 (cit. on pp. 3, 16).

[5] E. C. G. Sudarshan, "Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams," *Physical Review Letters*, vol. 10, no. 7, pp. 1963–1965, 1963 (cit. on p. 3).

[6] K. Husimi, "Some formal properties of the density matrix," *Proc. Phys. Math. Soc. Jpn*, vol. 22, 1940 (cit. on p. 3).

[7] S. M. Barnett and P. M. Radmore, *Methods in theoretical quantum optics*. Oxford, New York: Oxford University Press, 1997 (cit. on p. 3).

[8] K. E. Cahill and R. J. Glauber, "Density operators and quasiprobability distributions," *Physical Review*, 1969 (cit. on p. 3).

[9] G. S. Agarwal and E. Wolf, "Calculus for functions of noncommuting operators and general phase-space methods in quantum mechanics. I. Mapping theorems and ordering of functions of noncommuting operators," *Physical Review D*, vol. II, 1970 (cit. on p. 3).

[10] M. Hillery, R. O'Connell, M. Scully, and E. Wigner, *Distribution functions in physics: Fundamentals*, 1984 (cit. on p. 3).

[11] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics*. New York: Benjamin, 1968 (cit. on p. 9).

[12] G. S. Agarwal, *Quantum Optics*. Cambridge University Press, 2013 (cit. on p. 9).

[13] U. M. Titulaer and R. J. Glauber, "Correlation functions for coherent fields," *Physical Review*, vol. 2529, no. 3, 1965 (cit. on pp. 12, 57).

[14] L. Mandel, "Non-classical states of the electromagnetic field," *Physica scripta*, vol. 34, 1986 (cit. on p. 12).

[15] T. Kiesel, "Classical and quantum-mechanical phase space distributions," p. 7, Mar. 2013. arXiv: 1303.4718 (cit. on p. 12).

[16] W. Vogel and D.-G. Welsch, *Quantum Optics, 3rd, Revised and Extended Edition*. Verlang Berlin GmbH: Wiley-Vch, 2006, p. 520 (cit. on p. 13).

[17] C. Gehrke, "Quantitative Characterization of Nonclassicality," PhD thesis, University of Rostock, 2008 (cit. on p. 14).

[18] M. Hillery, "Nonclassical distance in quantum optics," *Physical Review A*, vol. 35, no. 2, 1987 (cit. on p. 14).

[19] V. V. Dodonov, O. V. Man'ko, V. I. Man'ko, and A. Wünsche, "Hilbert-Schmidt distance and non-classicality of states in quantum optics," *...of Modern Optics*, no. March 2013, pp. 37–41, 2000 (cit. on p. 15).

[20] P. Marian, T. A. Marian, and H. Scutaru, "Quantifying Nonclassicality of One-Mode Gaussian States of the Radiation Field," *Physical Review Letters*, vol. 88, no. 15, p. 153 601, Mar. 2002 (cit. on p. 15).

[21] J. Malbouisson and B. Baseia, "On the measure of nonclassicality of field states," *Physica Scripta*, vol. 93, 2003 (cit. on p. 15).

[22] J. Sperling and W. Vogel, "Ordering and quantification of quantumness," Apr. 2010. arXiv: 1004.1944 (cit. on p. 15).

[23] C. T. Lee, "Measure of the nonclassicality of nonclassical states," *Physical Review A*, no. 2, pp. 2775–2778, 1991 (cit. on p. 15).

[24] C. T. Lee, "Moments of P functions and nonclassical depths of quantum states," *Physical Review A*, vol. 45, no. 9, pp. 6586–6595, 1992 (cit. on p. 15).

[25] N. Lütkenhaus and S. M. Barnett, "Nonclassical effects in phase space," *Physical Review A*, vol. 51, no. 4, pp. 3340–3342, 1995 (cit. on p. 16).

[26] A. Kenfack and K. Życzkowski, "Negativity of the Wigner function as an indicator of non-classicality," *Journal of Optics B: Quantum and Semiclassical Optics*, vol. 6, no. 10, pp. 396–404, Oct. 2004. arXiv: 0406015 [quant-ph] (cit. on p. 16).

[27] A. Mari and J. Eisert, "Positive Wigner Functions Render Classical Simulation of Quantum Computation Efficient," *Physical Review Letters*, vol. 109, no. 23, p. 230 503, Dec. 2012 (cit. on p. 17).

[28] V. Veitch, N. Wiebe, C. Ferrie, and J. Emerson, "Efficient simulation scheme for a class of quantum optics experiments with non-negative Wigner representation," *New Journal of Physics*, vol. 15, no. 1, p. 013 037, Jan. 2013 (cit. on p. 17).

[29] R. L. Hudson, "When is the Wigner quasi-probability density non-negative?" *Reports on Mathematical Physics*, vol. 6, no. 2, pp. –3, 1974 (cit. on p. 17).

[30] Y. Aharonov, D. Falkoff, E. Lerner, and H. Pendleton, "A quantum characterization of classical radiation," *Annals of Physics*, no. 1, pp. 498–512, 1966 (cit. on p. 17).

[31] M. S. Kim, W. Son, V. Bužek, and P. L. Knight, "Entanglement by a beam splitter: Nonclassicality as a prerequisite for entanglement," *Physical Review A*, vol. 65, no. 3, p. 032 323, Feb. 2002 (cit. on p. 17).

[32] J. Asbóth, J. Calsamiglia, and H. Ritsch, "Computable Measure of Nonclassicality for Light," *Physical Review Letters*, vol. 94, no. 17, p. 173 602, May 2005 (cit. on pp. 17–19).

[33] A. Ferraro, S. Olivares, and M. G. A. Paris, "Gaussian states in continuous variable quantum information," *arXiv preprint quant-ph/0503237*, p. 100, Mar. 2005. arXiv: 0503237 [quant-ph] (cit. on pp. 19, 22).

[34] C. Gehrke, J. Sperling, and W. Vogel, "Quantification of nonclassicality," *Physical Review A*, vol. 86, no. 5, p. 052 118, Nov. 2012 (cit. on p. 19).

[35] W. Vogel and J. Sperling, "Unified quantification of nonclassicality and entanglement," *Physical Review A*, vol. 89, no. 5, p. 052 302, May 2014 (cit. on p. 19).

[36] M. G. Genoni, M. G. A. Paris, and K. Banaszek, "Quantifying the non-Gaussian character of a quantum state by quantum relative entropy," *Physical Review A*, vol. 78, no. 6, p. 060 303, Dec. 2008 (cit. on p. 21).

[37] M. G. Genoni and M. G. A. Paris, "Quantifying non-Gaussianity for quantum information," *Physical Review A*, vol. 82, no. 5, p. 052 341, Nov. 2010 (cit. on pp. 21, 23).

[38] J. J. Sakurai and J. Napolitano, *Modern quantum mechanics*, 2nd. Boston: Addison-Wesley, 2011 (cit. on p. 27).

[39] V. Usenko and M. G. A. Paris, "Quantum communication with photon-number entangled states and realistic photodetection," *Physics Letters A*, 2010 (cit. on p. 35).

[40] G. W. Bund and M. C. Tijero, "Mapping Wigner distribution functions into semiclassical distribution functions," *Physical Review A*, vol. 61, pp. 1–8, 2000 (cit. on pp. 41, 43, 54).

[41] P. Morse, "Diatomic molecules according to the wave mechanics. II. vibrational levels," *Physical Review*, vol. 457, no. 1927, 1929 (cit. on p. 48).

[42] A. Frank, A. Rivera, and K. Wolf, "Wigner function of Morse potential eigenstates," *Physical Review A*, vol. 61, no. 5, p. 054 102, Apr. 2000 (cit. on p. 50).

[43] T. Hahn, "Cuba—a library for multidimensional numerical integration," *Computer Physics Communications*, vol. 168, no. 2, pp. 78–95, Jun. 2005. arXiv: 0404043v2 [arXiv:hep-ph] (cit. on p. 51).

[44] G. Milburn, "Quantum and classical Liouville dynamics of the anharmonic oscillator," *Physical Review A*, vol. 33, no. 1, 1986 (cit. on p. 56).

[45] B. Yurke and D. Stoler, *The dynamic generation of Schrödinger cats and their detection*, 1988 (cit. on p. 57).

[46] G. Kirchmair, B. Vlastakis, Z. Leghtas, S. E. Nigg, H. Paik, E. Ginossar, M. Mirrahimi, L. Frunzio, S. M. Girvin, and R. J. Schoelkopf, "Observation of quantum state collapse and revival due to the single-photon Kerr effect.," *Nature*, vol. 495, no. 7440, pp. 205–9, Mar. 2013 (cit. on p. 57).

[47] a. Miranowicz, R. Tanas, and S. Kielich, "Generation of discrete superpositions of coherent states in the anharmonic oscillator model," *Quantum Optics*, vol. 2, no. 3, pp. 253–256, Nov. 1990. arXiv: 1111.0866 (cit. on pp. 57, 59).

[48] R. Tanas, "Nonclassical states of light propagating in Kerr media," in *Theory of Nonclassical States of Light*, 2003 (cit. on p. 57).

[49] M. Stobińska, G. J. Milburn, and K. Wódkiewicz, "Wigner function evolution of quantum states in the presence of self-Kerr interaction," *Physical Review A*, vol. 78, no. 1, p. 013 810, Jul. 2008 (cit. on p. 57).

[50] Z. Bialynicka—Birula, "Properties of the generalized coherent state," *Physical Review*, pp. 3–5, 1968 (cit. on p. 57).

[51] D. Stoler, "Generalized coherent states," *Physical Review D*, vol. 4, no. 8, pp. 2309–2312, 1971 (cit. on p. 57).

[52] R. Puri, *Mathematical methods of quantum optics*. Berlin ;;New York: Springer, 2001 (cit. on p. 67).

Acknowledgements

First of all I want to thank Professor Matteo Paris for many reasons: introducing me to the ideas discussed in this Thesis, guiding me throughout all the work, organizing the Quantum Expo workshops in Milan and in general for his support during the last eight months. I want to thank Professor Cristian Degli Esposti Boschi for his attentive kindness and support and for his availability; I also thank Professor Elisa Ercolessi for pointing me in the right direction when searching for a Thesis and for always being supportive and helping in every possible way. I feel grateful to Professor Mauro Paternostro for the useful conversation and advice and to all the people in the Applied Quantum Mechanics group in Milan for always making me feel welcome, in particular to all the PhD students.

A particular mention goes to my friend Alessandro who got me involved in physics in the first place and has always helped me with his physical intuition and wisdom ever since.

I want to express my gratitude towards all the *regaz* of the Physics Department in Bologna for all the good, average, or bad, but always unforgettable, times together (in no particular order): Davide, Francesco, Simone, Riccardo, Francesca, Marina, Camilla, Grazia, Edoardo, Alessandra, Riccardo, Lisa C., Giulia, Leonardo, Nicola, Onofrio, Alesandro, Silvia, Federica, Lisa B., Bernardo, Tommaso, Marco, Mauro, Luigi, Luca, Elena and all the ones I am too forgetful to mention.

A mention of honour goes to Francesca, Martello and Dimitri for standing my presence at home as well as in the department. I also feel the need to thank my bandmates Giacomo, Andrea, Matteo, Paride, Michele, Fabio as well as my friends in Bologna: Martino, Silvia, Marco; the emotive support of all these people was invaluable.

The only reason I was able to get this far was because of the support of my parents, who have always, always, been there for me; I will never thank them enough.

Last but not least I want to thank Tiziana for everything, and possibly more.