ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

ORTHOGONAL GAMMA-BASED EXPANSIONS FOR VOLATILITY OPTION PRICES UNDER JUMP-DIFFUSION DYNAMICS

Tesi di Laurea in Finanza Quantitativa

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II **Sessione**
Anno Accademico 2013/2014

“DI COSTUI ALMENO IO SONO PIÚ SAPIENTE;
PUÓ BEN DARSI CHE NÉ LUI NÉ IO SAPPIAMO NIENTE DI BELLO E BUONO,
MA EGLI CREDE DI SAPERE NON SAPENDO,
IO INVECE NON SO, E NON CREDO DI SAPERE;
PARE DUNQUE CHE, ANCHE PER QUESTA PICCOLA COSA,
IO SIA PIÚ SAPIENTE DI COSTUI,
PERCHÉ NON RITENGO DI SAPERE QUELLO CHE NON SO. ”

Platone, “Apologia di Socrate”, Cap. VI.

Preface

In this work we derive closed-form pricing formulas for vanilla options on the CBOE VIX Index by suitably approximating the volatility process risk-neutral density function. We exploit and adapt the idea, which stands behind popular techniques already employed in the context of equity options such as Edgeworth or Gram-Charlier expansions, of approximating the underlying process by an alternate (and more tractable) distribution in terms of a series expansion. Jarrow and Rudd (1982) pioneered the density expansion approach to option pricing, deriving an option pricing formula from an Edgeworth series expansion of the log-normal probability density function to model the distribution of stock prices. Corrado and Su (1996) adopted the Jarrow-Rudd framework and derived a similar option pricing formula where the chief difference is that they employed a Gram-Charlier series expansion of the normal probability density function to model the distribution of stock log prices. A probability density function f can be represented as a Gram-Charlier series expansion in the following form:

$$f(x) = \sum_{k=0}^{+\infty} c_k H_k(x) z(x)$$

where $z(x)$ is the normal density function, $H_k(x)$ are Hermite polynomials of order k and the coefficients c_k are simple functions of the moments of the approximated distribution. More recently, Drimus, Necula and Farkas (2013) developed a new option pricing formula by embracing the Corrado and Su framework and employing a modified Gram-Charlier type A series expansion, replacing the "probabilists" Hermite polynomials by the "physicists" Hermite polynomials. These methodologies represent a valid alternative to the numerical integration techniques to obtain an option price in case the distribution function is not analytically tractable, but it may however be straightforward to estimate its moments. The aim of this thesis is to modestly generalize these techniques to be adapted to the context of volatility options. Indeed the expansions above-mentioned, which are successful in the context of equities, are not appropriate for approximating volatility densities as their support lies in the whole real line. Thus we propose an expansion based on a class of polynomials which are weighted by a Gamma distribution, instead of log-normal or Gaussian distributions, thus ensuring positive mass only in the positive real line: the polynomials in question are the Laguerre polynomials. We call this series expansion *Gamma-Laguerre expansion* and we write

$$f(x) = \sum_{k=0}^{+\infty} c_k L_k(x) \phi(x)$$

where $\phi(x)$ denotes the Gamma density function, $L_k(x)$ are Laguerre polynomials of order k and the coefficients c_k are now expressed in function of the characteristic function of the approximated volatility process risk-neutral distribution. The latter coefficients property moreover makes our "approximation recipe" an alternative procedure to the classic inverse Fourier transform methodology. The accuracy of this approximation is tested for the Heston model and closed-form pricing formulas for vanilla options on the VIX Index are developed for the Heston model as well as for the jump-diffusion SVJJ model, proposed by Duffie et al. (2000). Due to the empirical evidence that prices essentially move by jumps, manifesting a discontinuous behaviour, it is of interest to look at jump-diffusion models, such as the SVJJ model where both the stock and the variance are Lévy processes. Indeed, while diffusion models cannot generate sudden, discontinuous moves in prices, jump-diffusion models overlay continuous asset price changes with jumps.

At the beginning of any chapter there is a very short introduction about the topics analyzed therein. Here we want to give the outline of the thesis.

In *Chapter 1* we review some of the main results on the risk-neutral derivative valuation framework for continuous-time diffusion models. We show that, under this framework, the concept of Equivalent Martingale Measure Q is an essential ingredient for valuation. Indeed the value of a financial derivative corresponds, in mathematical terms, to the computation of the expected value, under the risk-neutral measure Q, of the payoff, discounted at the risk-free interest rate.
Chapter 2 is devoted to the study of the class of Affine-Jump-Diffusion processes. We turn towards applications of affine processes to the modeling of stochastic volatility, by presenting two standard examples given by the Heston model and the SVJJ model. Finally, we derive explicit expressions for the characteristic function under both the above-mentioned models.
In *Chapter 3* we provide the definition of the CBOE VIX Index, from both the economical and mathematical point of view. Once we have translated the VIX Index in probabilistic terms, we provide shorthand forms for the VIX squared under the Heston model as well as the SVJJ model.
In *Chapter 4* we describe in detail our approximation methodology, the *Gamma-Laguerre expansion*, and we provide some illustrative examples, based on the Inverse Gaussian distribution and the (simulated) Heston model distribution, to highlight the convergence of this expansion.
In *Chapter 5* we give a brief exposition of the contracts on the VIX Index and we derive interesting closed-form formulas for pricing them under the Heston model as well as the SVJJ model.
Chapter 6 contains the numerical tests of the pricing formulas provided in Chapter 5, based on the Heston model.
Finally, the Appendix gathers some classical results in stochastic calculus and Lévy process theory we consider relevant background material to the drafting of this thesis.

Prefazione

In questo lavoro ricaviamo formule di prezzo per opzioni vanilla sull'indice CBOE VIX in forma chiusa, approssimando opportunamente la funzione di densità neutrale al rischio del processo di volatilià. Utilizziamo e adattiamo l'idea che risiede dietro popolari tecniche, già impiegate nel contesto di opzioni sulle equity, come le espansioni di Edgeworth o di Gram-Charlier, di approssimare il processo sottostante con una distribuzione alternativa (e più trattabile) in termini di sviluppo in serie. Jarrow and Rudd (1982) hanno aperto la strada all'approccio basato su espansioni di densità per prezzare opzioni, derivando una formula di prezzo da una espansione in serie di Edgeworth della funzione di probabilità log-normale per modellare la distribuzione dei prezzi stock. Corrado and Su (1996) hanno adottato il quadro presentato da Jarrow e Rudd e derivato una simile formula di prezzo dove la principale differenza risiede nell'aver utilizzato uno sviluppo in serie di Gram-Charlier della densità di probabilità normale per modellare la distribuzione dei rendimenti logaritmici. Una funzione di densità di probabilità f può essere rappresentata come uno sviluppo in serie di Gram-Charlier nella seguente forma

$$f(x) = \sum_{k=0}^{+\infty} c_k H_k(x) z(x)$$

dove $z(x)$ è la funzione di densità normale, $H_k(x)$ sono i polinomi di Hermite di ordine$k$ e i coefficienti c_k sono semplici funzioni dei momenti della distribuzione approssimata. Più recentemente, Drimus, Necula and Farkas (2013) hanno sviluppato una nuova formula di prezzo abbracciando il contesto di Corrado e Su e utilizzando uno sviluppo in serie di Gram-Charlier di tipo A modificato, sostituendo i polinomi di Hermite "probabilistici" con i polinomi di Hermite "fisici". Queste metodologie rappresentano una valida alternativa alle tecniche di integrazione numerica usate per ottenere prezzi qualora la distribuzione non sia trattabile analiticamente, ma comunque risulti semplice valutare i suoi momenti. Lo scopo di questa tesi è di generalizzare, modestamente, queste tecniche cosicché possano essere adattate al contesto di opzioni sulla volatilità. Infatti le espansioni di cui sopra, che sono soddisfacenti nel contesto di equity, non sono appropriate per approssimare densità di volatilità in quanto supportate sull'intera linea reale. Proponiamo pertanto un'espansione basata su una classe di polinomi pesati da una distribuzione Gamma, anziché distribuzioni Gaussiane o log-normali, assicurando in questo modo massa positiva solo sulla linea reale positiva: i polinomi in questione sono i polinomi di Laguerre. Chiamiamo tale sviluppo in serie *Espansione Gamma-Laguerre* e scriviamo

$$f(x) = \sum_{k=0}^{+\infty} c_k L_k(x) \phi(x)$$

dove $\phi(x)$ denota la funzione di densità Gamma, $L_k(x)$ sono polinomi di Laguerre di ordine k and i coefficienti c_k sono ora espressi in funzione della funzione caratteristica della distribuzione neutrale al rischio del processo di volatilità che stiamo approssimando. Quest'ultima proprietà riguardante i coefficienti dell'espansione inoltre rende la nostra "ricetta" di approssimazione una procedura alternativa alla classica metodologia basata sulla inversione della trasformata di Fourier. L'accuratezza della suddetta approssimazione è testata sul modello a volatilità stocastica di Heston e formule di prezzo in forma chiusa sono sviluppate sia per il modello di Heston che per il modello diffusivo con salti, chiamato SVJJ, proposto da Duffie et al. (2000). Data l'evidenza empirica che i prezzi si muovono sostanzialmente con salti, manifestando un comportamento discontinuo, abbiamo trovato interessante anche trattare modelli di diffusione con salti, come il modello SVJJ nel quale sia il sottostante che la sua volatilità sono processi di Lévy. Infatti, mentre i modelli puramente diffusivi non possono generare repentini, discontinui movimenti nei prezzi, i modelli diffusivi con salti sovrappongono continui cambiamenti di prezzi con salti.

All'inizio di ogni capitolo si trova una breve introduzione circa gli argomenti ivi analizzati. Qui vogliamo fornire lo schema generale della tesi.

Nel *Capitolo 1* esaminiamo alcuni fra i risultati principali della teoria di valutazione neutrale al rischio di strumenti derivati in modelli a tempo continuo. Mostriamo come, in questo contesto, il concetto di Misura Martingala Equivalente Q sia un ingrediente essenziale per la valutazione. Infatti, il valore di un derivato finanziario corrisponde, in termini matematici, al calcolo del valore atteso, rispetto alla misura neutrale al rischio Q, del payoff, scontato al tasso di interesse privo di rischio.
Il *Capitolo 2* è dedicato allo studio della classe di processi di salto diffusivi affini. Ci spostiamo verso le applicazioni dei processi affini nella modellizzazione di volatilità stocastiche, presentando due esempi classici dati dal modello di Heston e dal modello SVJJ. Infine, deriviamo epressioni esplicite per la funzione caratteristica in entrambi i suddetti modelli a volatilità stocastica.
All'interno del *Capitolo 3* forniamo la defizione di Indice CBOE VIX, sia dal punto di vista economico che dal punto di vista matematico. Dopo aver tradotto l'indice VIX in termini probabilistici, forniamo forme abbreviate per il quadrato del VIX sia nel modello di Heston che nel modello SVJJ.
Nel *Capitolo 4* descriviamo dettagliatamente la nostra metodologia di approsimazione, l' *espansione Gamma-Laguerre*, e forniamo qualche esempio illustrativo, basato sulla distribuzione Inverse-Gamma e sulla distribuzione del modello di Heston (simulata), per sottolineare la convergenza della suddetta espansione.
All'interno del *Capitolo 5* forniamo una breve descrizione circa le opzioni sull'indice VIX e deriviamo formule in forma chiusa per valutarle, considerando sia il modello d Heston che il modello SVJJ.
Il Capitolo 6 contiene i test numerici delle formule di prezzo fornire nel precedente Capitolo 5, basate sul modello di Heston.
Infine, l'Appendice raccoglie alcuni classici risultati di calcolo stocastico e analisi di processi di Lévy che consideriamo materiale di supporto alla stesura di questa tesi.

CONTENTS

1

RISK-NEUTRAL PRICING AND MARTINGALE MEASURES

Two important concepts in the mathematical theory of option pricing are the *absence of arbitrage*, which imposes constraints on the way instruments are priced in a market and the notion of *risk-neutral price*, which represents the price of any derivative in an arbitrage-free market as its discounted expected payoff at the risk-free interest rate under an appropriate probability measure called the "risk-neutral" measure. Both of these notions are expressed in mathematical terms exploiting the concept of *Equivalent Martingale Measure (EMM)* which plays, in this chapter, a central role: in a market model defined by a probability measure P on market scenarios there is a one-to-one correspondence between risk-neutral pricing that avoids the introduction of arbitrage opportunities and risk-neutral probability measure Q, equivalent to P verifying a martingale property. Since this chapter is intended as an introduction for the theory of derivative pricing for continuous-time diffusion models, the proofs of the results we state are omitted: for a complete treatment of the theory we refer to [15].

1.1 Model assumptions

First of all, we set the assumptions on the model that are going to hold in the rest of the chapter. Thus, we consider a market whose possible evolutions between 0 and T are described by a probability space $\mathcal{P} := (\Omega, \mathcal{F}, P)$ and consisting of N risky assets, one non-risky asset and d sources of risk that are represented by a $d-$dimensional correlated Brownian motion $W = \left(W^1, \cdots, W^d\right)$ on the probability space $\mathcal{P}$ endowed with the Brownian filtration $\mathcal{F}_t^W = \left(\mathcal{F}_t^W\right)_{t\in[0,T]}$

[1].

Underlying assets may then be described by a stochastic process:

$$\begin{aligned} S : [0,T] \times \Omega &\longrightarrow \mathbb{R}^N \\ (t,\omega) &\longrightarrow \left(S_t^1(\omega), \cdots, S_t^N(\omega)\right) \end{aligned}$$

where $S_t^i(\omega)$ represents the price of the risky asset i at time t in the market scenario ω whose dynamics is given by

$$dS_t^i = \mu_t^i S_t^i dt + \sigma_t^i S_t^i dW_t^i, \quad i = 1, \cdots, N, \; t \in [0,T]$$

with $\mu^i \in \mathbb{L}^1_{\text{loc}}$ and $\sigma^i \in \mathbb{L}^2_{\text{loc}}$. Concerning the non-risky asset B, we suppose it is a cash account with fixed (risk-free) interest rate r fulfilling the following formula of continuous compounding

$$B_t = e^{rt}, \quad B_0 = 1, \qquad t \in [0,T]$$

or, equivalently, in the "differential form"

$$dB_t = rB_t dt.$$

Before going any further, it is good to briefly recall some notions about derivative instruments. Discounting is done using the numeraire B_t: indeed, for any portfolio with value V_t, the *discounted value* is defined by

$$\tilde{V}_t = \frac{V_t}{B_t}.$$

An option with maturity T may be represented by specifying its terminal payoff $H(\omega)$ in each scenario: since H is revealed at T, the payoff is a $\mathcal{F}_T$−measurable map

$$H : \Omega \longrightarrow \mathbb{R}.$$

1.2 Change of measure

Definition 1.1. Let $\lambda \in \mathbb{L}^2_{\text{loc}}$ be a d−dimensional process. We call *exponential martingale associated to* λ the process

$$Z_t^\lambda = \exp\left(-\int_0^t \lambda_s \cdot dW_s - \frac{1}{2}\int_0^t |\lambda_s|^2 \, ds\right), \quad t \in [0,T].$$

[1]The natural filtration for W is defined by

$$\tilde{\mathcal{F}}_t^W = \sigma\left(W_s | 0 \le s \le t\right) := \sigma\left(\left\{W_s^{-1}(B) | 0 \le s \le t, \, B \in \mathcal{B}\right\}\right), \quad t \in [0,T].$$

We call *Brownian filtration*, and we denote it by $\mathcal{F}_t^W = \left(\mathcal{F}_t^W\right)_{t\in[0,T]}$, the filtration defined as the natural filtration completed by the collection of P-negligible events, i.e.

$$\mathcal{F}_t^W = \sigma\left(\tilde{\mathcal{F}}_t^W \cup \mathcal{N}\right)$$

where $\mathcal{N} = \left\{F \in \mathcal{F} | P(F) = 0\right\}$. The choice of considering the filtration containing negligible events stems from the need of avoiding the unpleasant situation in which $W_1 = W_2$ a.s., W_1 is $\mathcal{F}_t$-measurable but W_2 fails to be so.

Remark 1.2. The exponential martingale associated to λ can be written in the "differential form" as follows

$$dX_t^\lambda := d\ln(Z_t)^\lambda = -\lambda_t dW_t - \frac{1}{2}|\lambda_t|^2 dt$$

whence, by employing the Itô formula A.2 to the process $f(X_t^\lambda) = e^{X_t^\lambda} = Z_t^\lambda$, we get

$$\begin{aligned} dZ_t^\lambda = df &= e^{X_t^\lambda} dX_t^\lambda + \frac{1}{2}|\lambda_t|^2 e^{X_t^\lambda} dt \\ &= e^{X_t^\lambda}(-\lambda_t \cdot dW_t - \frac{1}{2}|\lambda_t|^2 dt) + \frac{1}{2}|\lambda_t|^2 e^{X_t^\lambda} dt \\ &= -Z_t^\lambda \lambda_t \cdot dW_t. \end{aligned}$$

Therefore Z^λ is a local martingale.

The following central theorem shows that it is possible to substitute "arbitrarily" the drift of an Itô process by modifying properly the considered probability measure and Brownian motion, while keeping unchanged the diffusion coefficient.

THEOREM - 1.2.1 (Girsanov's theorem)**.**
Let Z^λ be the exponential martingale associated to the process $\lambda \in \mathbb{L}^2_{loc}$. We assume that Z^λ is a $P-$martingale and we consider the measure Q defined by

$$\frac{dQ}{dP} = Z_T^\lambda.$$

Then the process

$$W_t^\lambda = W_t + \int_0^t \lambda_s \, ds, \quad t \in [0, T],$$

is a Brownian motion on $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t))$.

THEOREM - 1.2.2 (Change of drift)**.**
Let Q be a probability measure equivalent to P. The Radon-Nikodym derivative of Q with respect to P is an exponential martingale

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t^W} = Z_t^\lambda, \quad dZ_t^\lambda = -Z_t^\lambda \lambda_t \cdot dW_t$$

with $\lambda \in \mathbb{L}^2_{loc}$ and the process W^λ, defined by

$$dW_t = dW_t^\lambda - \lambda_t dt,$$

is a Brownian motion on $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t^W))$.

We now extend the previous result to the case of the correlated Brownian motion.

THEOREM - 1.2.3 (Change of drift with correlation)**.**
If Q is a probability measure equivalent to P then there exists a process $\lambda \in \mathbb{L}^2_{loc}$ such that

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t^W} = Z_t^\lambda, \quad dZ_t^\lambda = -Z_t^\lambda \lambda_t \cdot dW_t.$$

Moreover, the process W^λ, defined by

$$dW_t = dW_t^\lambda - \rho\lambda_t dt,$$

is a Brownian motion on $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t^W))$ with correlation matrix ρ.

Remark 1.3. Under the assumptions of Theorem 1.2.3, let X be an N-dimensional Itô process of the form

$$dX_t = b_t dt + \sigma_t dW_t.$$

Then the $Q-$dynamics of X is given by

$$dX_t = (b_t - \sigma_t \rho \lambda_t) dt + \sigma_t dW_t^\lambda.$$

Again, we emphasize the fundamental feature of the change of measure: it only affects the drift coefficient of the process X, whilst the diffusion coefficient (or volatility) does not vary.

1.3 Martingale measures

Definition 1.4. An *Equivalent Martingale Measure (EMM)* Q with numeraire B is a probability measure on (Ω, P) such that

(i) Q is equivalent to P, i.e.

$$P \sim Q \Longleftrightarrow \forall A \in \mathcal{F},\ P(A) = 0 \Leftrightarrow Q(A) = 0$$

namely that P and Q define the same set of (im)possible events.

(ii) The process of discounted prices

$$\tilde{S}_t = e^{-rt} S_t, \quad t \in [0, T]$$

is a $Q-$martingale. Therefore, in particular, the risk-neutral pricing formula

$$S_t = e^{-r(T-t)} E^Q \left[S_T | \mathcal{F}_t^W \right]$$

holds.

Now we consider an EMM Q and we use Theorem 1.2.3, in the form of Remark 1.3, to find the $Q-$dynamics of the price process. We recall that there exists a process $\lambda = (\lambda^1, \cdots, \lambda^d) \in \mathbb{L}^2_{\text{loc}}$ such that

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t^W} = Z_t$$

where

$$dZ_t = -Z_t(\rho^{-1}\lambda_t) \cdot dW_t, \quad Z_0 = 1. \tag{1.1}$$

Moreover the process $W^\lambda = (W^{\lambda,1}, \cdots, W^{\lambda,d})$ defined by

$$dW_t = dW_t^\lambda - \lambda_t dt$$

is a $Q-$Brownian motion with correlation matrix ρ. Therefore, for $i = 1, \cdots, N$, we have

$$\begin{aligned} d\tilde{S}_t^i &= (\mu_t^i - r_t)\tilde{S}_t^i dt + \sigma_t^i \tilde{S}_t^i dW_t^i \\ &= (\mu_t^i - r_t)\tilde{S}_t^i dt + \sigma_t^i \tilde{S}_t^i (dW_t^{\lambda,i} - \lambda_t^i dt) \\ &= (\mu_t^i - r_t - \sigma_t^i \lambda_t^i)\tilde{S}_t^i dt + \sigma_t^i \tilde{S}_t^i dW_t^{\lambda,i}. \end{aligned}$$

Now we recall that an Itô process is a local martingale if and only if it has null drift (cf. Remark A.6). Therefore, since Q is an EMM, the following drift condition necessarily holds:

$$\lambda_t^i = \frac{\mu_t^i - r_t}{\sigma_t^i}, \quad i = 1, \cdots, N. \tag{1.2}$$

Finally we give the following

Definition 1.5. A *market price of risk* is a $d-$dimensional process $\lambda \in \mathbb{L}^2_{\text{loc}}$ such that:

(i) the first N components of λ are given by (1.2);

(ii) the solution Z to the SDE (1.1) is a strict $P-$martingale.

1.4 Admissible strategies and arbitrage opportunities

Definition 1.6. A *strategy* (or *portfolio*) is a stochastic process in $\mathbb{R}^{N+1}$

$$(\alpha, \beta) = \left(\alpha_t^1, \cdots, \alpha_t^N, \beta_t\right), \quad t \in [0, T]$$

such that $\alpha, \beta \in \mathbb{L}^1_{\text{loc}}$. In financial terms, α_t^i (resp. β_t) represents the amount of the asset S^i (resp. bond) held in the portfolio at time t. The *value* of the portfolio (α, β) is the real-valued process

$$V_t^{(\alpha,\beta)} = \alpha_t \cdot S_t + \beta_t B_t = \sum_{i=1}^N \alpha_t^i S_t^i + \beta_t B_t, \quad t \in [0, T].$$

Definition 1.7. A strategy (α, β) is *self-financing* if

$$dV_t = \alpha_t \cdot dS_t + \beta_t dB_t. \tag{1.3}$$

From a purely intuitive point of view, (1.3) expresses the fact that the instantaneous variation of the value of the portfolio is caused uniquely by the changes of the prices of the assets, and not by injecting or withdrawing funds from outside. Therefore, in a self-financing strategy we establish the wealth we want to invest at the initial time and afterwards we do not inject or withdraw funds.

Proposition 1.4.1. *Let Q be an EMM and* (α, β) *a self-financing strategy such that*

$$\alpha^i \sigma^i \in \mathbb{L}^2_{loc}(\Omega, P), \quad i = 1, \cdots, N \tag{1.4}$$

then, $\tilde{V}_t^{(\alpha,\beta)}$ *is a* $Q-$*martingale. Therefore, in particular, the following risk-neutral pricing formula*

$$V_t^{(\alpha,\beta)} = e^{-r(T-t)} E^Q \left[V_T^{(\alpha,\beta)} | \mathcal{F}_t^W \right], \quad t \in [0, T]$$

holds.

Definition 1.8. A self-financing strategy (α, β) such that $\tilde{V}^{(\alpha,\beta)}$ is a $Q-$martingale for every EMM Q, is called an *admissible strategy*. We denote by $\mathcal{A}$ the collection of all admissible strategies.

Proposition 1.4.1 guarantees that the family $\mathcal{A}$ is not empty: indeed, any self-financing strategy (α, β) verifying condition (1.4) is admissible. Moreover we have the following version of the no-arbitrage principle.

Proposition 1.4.2 (No-arbitrage principle)**.**
If an EMM exists and (α, β), (α', β') *are admissible self-financing strategies such that*

$$V_T^{(\alpha,\beta)} = V_T^{(\alpha',\beta')} \quad P - a.s.$$

then $V^{(\alpha,\beta)}$ *and* $V^{(\alpha',\beta')}$ *are indistinguishable.*

Proof. If Q exists and (α, β), (α', β') are admissible, then $\tilde{V}^{(\alpha,\beta)}$ and $\tilde{V}^{(\alpha',\beta')}$ are $Q-$martingales with the same final value $Q-$a.s., because $Q \sim P$. Hence

$$\tilde{V}_t^{(\alpha,\beta)} = E^Q\left[\tilde{V}_T^{(\alpha,\beta)}\middle|\mathcal{F}_t\right] = E^Q\left[\tilde{V}_T^{(\alpha',\beta')}\middle|\mathcal{F}_t\right] = \tilde{V}_t^{(\alpha',\beta')}$$

for every $t \in [0, T]$.

□

1.5 Arbitrage pricing

We now analize the problem of pricing of a European derivative.

Definition 1.9. A derivative X is called *replicable* if there is an admissible strategy $(\alpha, \beta) \in \mathcal{A}$ such that

$$X = V_T^{(\alpha,\beta)} \quad P - a.s. \tag{1.5}$$

where the random variable X represents the payoff of the derivative. An admissible strategy (α, β) such that (1.5) holds, is called a *replicating strategy for X.*

Definition 1.10. The *risk-neutral price* of a European derivative X with respect to the EMM Q, is defined as

$$H_t^Q = e^{-r(T-t)} E^Q\left[X|\mathcal{F}_t^W\right], \quad t \in [0, T].$$

Next we introduce the collections of super and sub-replicating strategies:

$$\mathcal{A}_X^+ = \left\{(\alpha, \beta) \in \mathcal{A} | V_T^{(\alpha,\beta)} \geq X,\ P - a.s.\right\}$$
$$\mathcal{A}_X^- = \left\{(\alpha, \beta) \in \mathcal{A} | V_T^{(\alpha,\beta)} \leq X,\ P - a.s.\right\}$$

For a given $(\alpha, \beta) \in \mathcal{A}_X^+$ (resp. $(\alpha, \beta) \in \mathcal{A}_X^-$), the value $V_0^{(\alpha,\beta)}$ represents the initial wealth sufficient to build a strategy that super-replicates (resp. sub-replicates) the payoff X at maturity. The following result confirms the natural consistency relation among the initial values of the sub and super-replicating strategies and the risk-neutral price: this relation must necessarily hold true in any arbitrage-free market, otherwise arbitrage opportunities could be easily created.

Lemma 1.5.1. *Let X be a European derivative. For every EMM Q and* $t \in [0, T]$ *we have*

$$\sup_{(\alpha,\beta)\in\mathcal{A}_X^-} V_t^{(\alpha,\beta)} \leq e^{-r(T-t)} E^Q\left[X|\mathcal{F}_t^W\right] \leq \inf_{(\alpha,\beta)\in\mathcal{A}_X^+} V_t^{(\alpha,\beta)}.$$

Lemma 1.5.1 ensures that any risk-neutral price does not give rise to arbitrage opportunities since it is greater than the price of every sub-replicating strategy and smaller than the price of every super-replicating strategy. By definition, H^Q depends on the selected EMM Q; however, this is not the case if X is replicable. Indeed the following result shows that the risk-neutral price of a replicable derivative is uniquely defined and independent of Q.

THEOREM - 1.5.2. *Let X be a replicable European derivative. For every replicating strategy $(\alpha, \beta) \in \mathcal{A}$ and for every EMM Q, we have*

$$H_t := V_t^{(\alpha,\beta)} = e^{-r(T-t)} E^Q \left[X | \mathcal{F}_t^W \right].$$

The process H is called risk-neutral (or arbitrage) price of X.

The following result shows that, if the number of risky assets is equal to the dimension of the underlying Brownian motion, i.e. $N = d$, then the market is complete and the martingale measure is unique. Roughly speaking, in a complete market every European derivative X is replicable and by Theorem 1.5.2 it can be priced in a unique way by arbitrage arguments: the price of X coincides with the value of any replicating strategy and with the risk-neutral price under the unique EMM.

THEOREM - 1.5.3. *When $N = d$, the market model (S, B) is complete, that is every European derivative is replicable. Moreover there exists only one EMM.*

Example 1.11 (Heston model)**.**
Heston [8] proposed the following stochastic volatility model:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^{(1)} \tag{1.6}$$

$$dv_t = k(\bar{v} - v_t)dt + \epsilon\sqrt{v_t} dW_t^{(2)} \tag{1.7}$$

where $\{S_t\}_{t\geq 0}$, $\{v_t\}_{t\geq 0}$ are the price and volatility processes, respectively, and $\left\{W_t^{(1)}\right\}_{t\geq 0}$, $\left\{W_t^{(2)}\right\}_{t\geq 0}$ are correlated Brownian motion processes (with correlation parameter ρ). $\{v_t\}_{t\geq 0}$ is a square root mean reverting process, previously suggested by Cox, Ingersoll and Ross (1985) as a model for the short rate dynamics in a fixed-income market, with long-run mean $\bar{v}$, and rate of reversion k. ϵ is referred to as the volatility of volatility. All the parameters, namely $\mu, k, \bar{v}, \epsilon, \rho$, are time and state homogenous. Finally, the interest rate r is supposed to be constant. By the Itô formula A.2, the solution of (1.6) is

$$S_t = S_0 \exp\left(\int_0^t \sqrt{v_s}\, dW_t^{(1)} + \int_0^t \left(\mu - \frac{v_s}{2}\right) ds \right).$$

A market price of risk is a two-dimensional process $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \mathbb{L}^2_{\text{loc}}$ such that

$$\lambda_t^{(1)} = \frac{\mu - r}{\sqrt{v_t}}$$

while there is no restriction on the second component $\lambda^{(2)}$ except for the fact that Z must be a martingale. If this is the case, we consider the corresponding EMM Q with respect to which the process W^λ, defined by

$$dW_t = dW_t^\lambda - \lambda_t dt = dW_t^\lambda - \begin{pmatrix} \frac{\mu - r}{\sqrt{v_t}} \\ \lambda_t^{(2)} \end{pmatrix} dt,$$

is a two-dimensional Brownian motion. Thus the $Q-$dynamics are given by

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^{(S)} \tag{1.8}$$

$$dv_t = \left(k(\bar{v} - v_t) - \epsilon\sqrt{v_t}\lambda_t^{(2)}\right) dt + \epsilon\sqrt{v_t} dW_t^{(v)}$$

where $dW_t^{(S)} := dW_t^{\lambda,(1)}$ and $dW_t^{(v)} = dW_t^{\lambda,(2)}$. We remark that by taking the process $\lambda^{(2)}$ of the form

$$\lambda_t^{(2)} = \frac{a v_t + b}{\sqrt{v_t}}$$

with some real constants a, b, the $Q-$dynamics of the volatility process reduces to

$$d v_t = \bar{k}(\theta - v_t) dt + \epsilon \sqrt{v_t} dW_t^{(v)} \tag{1.9}$$

where

$$\bar{k} := k + \epsilon a, \quad \theta := \frac{k\bar{v} - \epsilon b}{k + \epsilon a}$$

and therefore v is a square root process under Q as well.

We note that while the drift in (1.8) must be r under any EMM with the cash account as numeraire, we could use Girsanov's Theorem to change the drift in (1.9) in infinitely many different ways without changing the drift in (1.8). This means that the EMM is not unique, there are infinitely many EMM's depending on the value of $\lambda^{(2)}$, thus, in view of Theorem 1.5.3, the Heston stochastic volatility model is an incomplete model. This should not be too surprising as there are two sources of uncertainty in the Heston model, $W^{(S)}$ and $W^{(v)}$, but only one risky asset and so not every security is replicable. The implications are that the different EMM's will produce different option prices, depending on the value of $\lambda^{(2)}$: this, initially, poses a problem but we remark that from the economical point of view, the price of risk λ is determined by the market, namely, λ must be chosen on the basis of observations, by calibrating the parameters of the model to the available data. Therefore, once λ and the corresponding EMM Q have been selected, the risk neutral price of a derivative on S is defined as in Definition 1.10.

2

AFFINE JUMP-DIFFUSION PROCESSES

In this chapter we present Affine-Jump-Diffusion (AJD) processes and the Fourier transform calculation that will later be useful in option pricing. This class consists of all jump-diffusion processes, whose drift vector, covariance matrix and arrival rate of jumps all depend in an affine way on the state process. The attractiveness of affine processes for Finance stems from several reasons: firstly, a variety of models that have been proposed in the literature, and that are used by practitioners, fall into the class of affine models. For instance, in the area of interest rate models, prominent among affine models are the classical models of Vasicek [1977] and Cox, Ingersoll, and Ross [1985]; in the realm of asset price modelling, the Black-Scholes model, all exponential-Lévy models (cf. [3]), the model of Heston [1993], extensions of the Heston model, such as Bates [1996] and Bates [2000] are all based on affine processes. Secondly, affine processes exhibit a high degree of analytic tractability: the computation of the characteristic function can be reduced to a system of Riccati equations, which have in many cases explicit solutions. The explicit knowledge of the Fourier transform allows an analytical treatment of a range of valuation problems: Fourier inversion methods can be employed as well as alternative techniques, based on Fourier transform, such as the methodology provided by this work.

Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with an information filtration $(\mathcal{F}_t)$. Suppose that $X = (X_t)_{t\in[0,T]}$ is an $\mathcal{F}_t$-adapted continuous process solving the stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t + dZ_t \tag{2.1}$$

where

- W is a $d-$dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$
- $\mu = \mu(t, x) : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the *drift coefficient*, $\mu(t, X_t) \in \mathbb{L}^1_{\text{loc}}$
- $\sigma = \sigma(t, x) : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}$ is the *diffusion coefficient*, $\sigma(t, X_t) \in \mathbb{L}^2_{\text{loc}}$
- Z is a pure jump process whose jumps have a fixed probability distribution m and arrive with intensity λ.

Definition 2.1. We call *Affine-Jump-Diffusion (AJD) process* the stochastic process $X = (X_t)_{t\in[0,T]}$ satisfying (2.1) such that the parameter functions μ, σ and λ are determined by coefficients (K, H, l) defined as follows:

- $\mu(t, x) = K_0 + K_1 \cdot x$, for $K := (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$
- $\left(\sigma(x)\sigma(x)^T\right)_{ij} = (H_0)_{ij} + (H_1)_{ij} \cdot x$, for $H := (H_0, H_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}$
- $\lambda(x) = l_0 + l_1 \cdot x$, for $l := (l_0, l_1) \in \mathbb{R} \times \mathbb{R}^n$.

2.1 Two standard models

In this section we will look more closely at the most common affine one factor models, restricting our attention to the derivation of a closed-form expression for the Fourier transform.

2.1.1 Heston model

In the Heston stochastic volatility model, the risk-neutral dynamics for the joint process (S, v) is given by

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^{(S)} \\ dv_t = \bar{k}(\theta - v_t)dt + \epsilon\sqrt{v_t} dW_t^{(v)} \end{cases} \tag{2.2}$$

where $W := \left(W^{(S)}, W^{(v)}\right)$ is a two-dimensional correlated Brownian motion, with correlation parameter ρ, the constant parameters $\bar{k}, \theta$ are responsible for a mean-reverting ability of the process and ϵ is volatility of volatility v_t. To ensure that the process v is strictly positive, the parameters must obey the following condition

$$2\bar{k}\theta > \epsilon^2 \tag{2.3}$$

known as the *Feller condition*. Furthermore, we assume that both the stochastic processes $(\sqrt{v_t})_t$ and $(S_t\sqrt{v_t})_t$ belong to the class $\mathbb{L}^2$.

Starting from the dynamics (2.2) of the asset, by the Itô formula A.2, we can easily compute the equivalent risk-neutral dynamics for the joint process $(\ln(S), v)$

$$\begin{cases} d\ln(S_t) = \left(r - \frac{v_t}{2}\right) dt + \sqrt{v_t} dW_t^{(S)} \\ dv_t = \bar{k}(\theta - v_t)dt + \epsilon\sqrt{v_t} dW_t^{(v)}. \end{cases} \tag{2.4}$$

Furthermore, we shall prove that the discounted asset fulfills the martingale property: indeed, by applying the Itô lemma to $f(t, S_t) = e^{-rt}S_t$, we get

$$\begin{aligned} df &= -re^{-rt}S_t dt + e^{-rt}dS_t \\ &= -re^{-rt}S_t dt + e^{-rt}rS_t dt + e^{-rt}\sqrt{v_t}S_t dW_t^{(S)} \\ &= e^{-rt}\sqrt{v_t}S_t dW_t^{(S)} \end{aligned}$$

which corresponds to the following SDE:

$$e^{-rt}S_t = S_0 + \int_0^t e^{-r\tau}S_\tau\sqrt{v_\tau}dW_\tau.$$

Now, by assumption $(S_t\sqrt{v_t})_t \in \mathbb{L}^2$, then it follows that $(e^{-rt}S_t\sqrt{v_t})_t \in \mathbb{L}^2$ as well. Indeed we have

$$E\left[\int_0^T \left(e^{-rt}S_t\sqrt{v_t}\right)^2 dt\right] \le E\left[\int_0^T \left(S_t\sqrt{v_t}\right)^2 dt\right] < +\infty$$

since r and T are positive real constants. Therefore the discounted asset price is a martingale, by means of the Theorem A.1.1.

Among stochastic volatility models, the *Heston model* exhibits the affine property. The following result gives the formula for the Laplace transform in the Heston model:

Proposition 2.1.1 (Affine-type Laplace trasform)**.**
Let $\mathfrak{L}_{v_T}$ be the Laplace transform of v_T, conditional on the filtration $\mathcal{F}_t$ with time to expiration $\tau = T - t$, i.e.,

$$\mathfrak{L}_{v_T}(z;t,\tau,v_t) = E\left[e^{z\cdot v_T}\middle|\mathcal{F}_t\right]$$

then, for every $z \in \mathbb{C}$,

$$\mathfrak{L}_{v_T}(z;t,\tau,v_t) = e^{a_1(z,\tau)+a_2(z,\tau)v_t}$$

where

$$a_1(z,\tau) = \frac{-2\bar{k}\theta}{\epsilon^2}\ln\left(1+\frac{\epsilon^2 z}{2\bar{k}}\left(e^{-\bar{k}\tau}-1\right)\right)$$
$$a_2(z,\tau) = \frac{2\bar{k}z}{\epsilon^2 z + (2\bar{k}-\epsilon^2 z)e^{\bar{k}\tau}}.$$

Proof. The Feynman-Kac theorem A.1.4 implies that $\mathfrak{L}_{v_T}(z;t,\tau,v_t)$ is the solution of the (backward) Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\mathfrak{L}_{v_T} + \bar{k}(\theta - v)\frac{\partial}{\partial v}\mathfrak{L}_{v_T} + \frac{1}{2}\epsilon^2 v\frac{\partial^2}{\partial v^2}\mathfrak{L}_{v_T} = 0 \\ \mathfrak{L}_{v_T}(z;t+\tau,0,v) = e^{zv} \end{cases}$$

that is

$$\begin{cases} -\frac{\partial}{\partial \tau}\mathfrak{L}_{v_T} + \bar{k}(\theta - v)\frac{\partial}{\partial v}\mathfrak{L}_{v_T} + \frac{1}{2}\epsilon^2 v\frac{\partial^2}{\partial v^2}\mathfrak{L}_{v_T} = 0 \\ \mathfrak{L}_{v_T}(z;t+\tau,0,v) = e^{zv}. \end{cases} \tag{2.5}$$

Following the solution procedure used by [6], we can solve this Cauchy problem in closed-form by guessing that the affine-form solution is

$$\mathfrak{L}_{v_T}(z;t,\tau,v) = e^{a_1(z,\tau)+a_2(z,\tau)v}. \tag{2.6}$$

By substituting (2.6) into (2.5), we obtain:

$$-e^{a_1(z,\tau)+a_2(z,\tau)v}\left(\frac{\partial}{\partial\tau}a_1(z,\tau)+v\frac{\partial}{\partial\tau}a_2(z,\tau)\right)+\bar{k}(\theta-v)e^{a_1(z,\tau)+a_2(z,\tau)v}a_2(z,\tau)+\frac{\epsilon^2 v}{2}e^{a_1(z,\tau)+a_2(z,\tau)v}a_2(z,\tau)^2=0$$

that is

$$e^{a_1(z,\tau)+a_2(z,\tau)v}\left(-\frac{\partial}{\partial\tau}a_1(z,\tau)+\bar{k}\theta a_2(z,\tau)\right)+ve^{a_1(z,\tau)+a_2(z,\tau)v}\left(-\frac{\partial}{\partial\tau}a_2(z,\tau)+\frac{\epsilon^2}{2}a_2(z,\tau)^2-\bar{k}a_2\right)=0$$

whence we obtain two ordinary differential equations:

$$\begin{cases} \frac{\partial}{\partial\tau}a_2(z,\tau) = -\bar{k}a_2(z,\tau)+\frac{\epsilon^2}{2}a_2(z,\tau)^2 \\ \frac{\partial}{\partial\tau}a_1(z,\tau) = \bar{k}\theta a_2(z,\tau) \end{cases}$$

with initial conditions

$$\begin{cases} a_2(z,0) = z \\ a_1(z,0) = 0. \end{cases}$$

Finally, the solutions to these ODEs are given by

$$\begin{cases} a_2(z,\tau) = \frac{2\bar{k}z}{\epsilon^2 z + e^{\bar{k}\tau}(2\bar{k}-\epsilon^2 z)} \\ a_1(z,\tau) = -\frac{2\bar{k}\theta}{\epsilon^2}\ln\left(1+\frac{\epsilon^2 z}{2\bar{k}}\left(e^{-\bar{k}\tau}-1\right)\right) \end{cases}$$

hence the claim.

□

Corollary 2.1.2 (Affine-type characteristic function)**.**
Let ψ_{v_T} be the characteristic function of v_T, conditional on the filtration $\mathcal{F}_t$ with time to expiration $\tau = T-t$, i.e.,

$$\psi_{v_T}(\xi;t,\tau,v_t) = E\left[e^{i\xi\cdot v_T}\middle|\mathcal{F}_t\right]$$

then, for every $\xi \in \mathbb{R}$,

$$\psi_{v_T}(\xi;t,\tau,v_t) = e^{a_1(i\xi,\tau)+a_2(i\xi,\tau)v_t}$$

where

$$a_1(i\xi,\tau) = \frac{-2\bar{k}\theta}{\epsilon^2}\ln\left(1+\frac{\epsilon^2 i\xi}{2\bar{k}}\left(e^{-\bar{k}\tau}-1\right)\right)$$
$$a_2(i\xi,\tau) = \frac{2\bar{k}i\xi}{\epsilon^2 i\xi + (2\bar{k}-\epsilon^2 i\xi)e^{\bar{k}\tau}}.$$

Proof. The claim follows by combining the following equivalence

$$\psi_{v_T}(\xi;t,\tau,v_t) = \mathfrak{L}_{v_T}(z;t,\tau,v_t)\Big|_{z=i\xi}$$

with Proposition 2.1.1.

□

Remark 2.2. It follows from Corollary 2.1.2 that if the Feller condition is fulfilled, then $\psi_{v_T}(\xi;t,\tau,v_t)$ belongs to the class L^1. Moreover, if the condition

$$4\bar{k}\theta > \epsilon^2 \tag{2.7}$$

holds, then $\psi_{v_T}(\xi;t,\tau,v_t)$ belongs to the class L^2.

2.1.2 SVJJ model

The SVJJ model is the stochastic volatility model with simultaneous and correlated jumps in price and volatility, firstly introduced by Duffie et al. (2000) [6]. Roughly speaking, it corresponds to the Heston model with the addition of simultaneous and correlated jumps in both the price and volatility processes. The joint process (S,v) is driven by the following dynamics

$$\begin{cases} dX_t = \left(r - \frac{v_t}{2} - \lambda c\right)dt + \sqrt{v_t}dW_t^{(S)} + dZ_t^{(S)} \\ dv_t = \bar{k}(\theta - v_t)dt + \epsilon\sqrt{v_t}dW_t^{(v)} + dZ_t^{(v)} \\ X_t := \ln(S_t) \end{cases}$$

where $W := \left(W^{(S)}, W^{(v)}\right)$ is a bidimensional correlated Brownian motion, with correlation parameter ρ and $Z := \left(Z^{(S)}, Z^{(v)}\right)$ is a two-dimensional compound Poisson process with jump times process $N_t \sim \text{Poisson}(\lambda t)$ and correlated jump size processes $Y^{(S)}$, $Y^{(v)}$, independent from $\{N_t\}_{t\geq 0}$ and with correlation parameter ρ_Y

$$Z_t^{(S)} = \sum_{i=1}^{N_t} Y_i^{(S)}$$
$$Z_t^{(v)} = \sum_{i=1}^{N_t} Y_i^{(v)}.$$

The jump sizes in volatility are assumed to have an exponential distribution, i.e.

$$Y_i^{(v)} \sim \text{Exp}\left(\frac{1}{\mu_v}\right)$$

while jumps in asset log-prices are normally distributed conditionally on the realization of $Y_i^{(v)}$, formally

$$Y_i^{(S)} | Y_i^{(v)} \sim \mathcal{N}(\mu_S + \rho_Y Y_i^{(v)}, \sigma_S^2).$$

Finally,

$$c = \frac{e^{\mu_S + \frac{1}{2}\sigma_S^2}}{1 - \rho_Y \mu_v} - 1$$

is the compensator related to the jump component in the log-return process, that is the term that ensures that the discounted asset process is a martingale. To do so, with the same notations as above, let us compute the risk-neutral dynamics, under the general SVJJ model, of the asset S_t. By applying the Itô formula A.3.2 to the process

$$f(X_t) = e^{X_t} = e^{\ln(S_t)} = S_t$$

we get

$$\begin{aligned} df &= \left(r - \lambda c - \frac{v_t}{2}\right) e^{X_t}\, dt + \frac{v_t}{2} e^{X_t}\, dt + e^{X_t} \sqrt{v_t}\, dW_t^{(S)} + \left[e^{X_{t-} + \Delta X_t} - e^{X_{t-}}\right] \\ &= (r - \lambda c)\, e^{X_t}\, dt + e^{X_t} \sqrt{v_t}\, dW_t^{(S)} + e^{X_{t-}} \left[e^{\Delta X_t} - 1\right] \end{aligned}$$

whence

$$dS_t = rS_t dt + S_t \sqrt{v_t} dW_t^{(S)} + S_{t-}\left[e^{\Delta X_t} - 1\right] - S_t c\lambda dt$$

which corresponds to the following SDE

$$S_t = S_0 + \int_0^t (rS_s - c\lambda S_s)\, ds + \int_0^t S_s \sqrt{v_s} dW_s^{(S)} + \sum_{i\geq 1,\, T_i \leq t} S_{T_i-}\left(e^{\Delta X_i} - 1\right). \tag{2.8}$$

Furthermore, by using again Theorem A.3.2 to $f(t, S_t) = e^{-rt} S_t$, we obtain

$$\begin{aligned} df &= -re^{-rt} S_t\, dt + (rS_t - c\lambda S_t) e^{-rt}\, dt + e^{-rt} S_t \sqrt{v_t}\, dW_t + \left[e^{-rt}(S_{t-} + \Delta S_t) - e^{-rt} S_{t-}\right] \\ &= -c\lambda S_t e^{-rt}\, dt + e^{-rt} S_t \sqrt{v_t}\, dW_t + \left[e^{-rt} \Delta S_t\right] \\ &= -c\lambda S_t e^{-rt}\, dt + e^{-rt} S_t \sqrt{v_t}\, dW_t + \left[e^{-rt} S_{t-}\left(e^{\Delta X_t} - 1\right)\right] \end{aligned}$$

whence

$$d\left(e^{-rt} S_t\right) = e^{-rt} S_t \sqrt{v_t}\, dW_t + \left[e^{-rt} S_{t-}\left(e^{\Delta X_t} - 1\right)\right] - c\lambda S_t e^{-rt}\, dt.$$

which corresponds to the following SDE

$$e^{-rt} S_t = S_0 + \int_0^t e^{-rs} S_s \sqrt{v_s}\, dW_s + \sum_{i\geq 1,\, T_i \leq t} e^{-rT_i} S_{T_i-}\left(e^{\Delta X_i} - 1\right) - \int_0^t c\lambda S_s e^{-rs}\, ds.$$

Now, as we have already pointed out before, since the process $\left(e^{-rs}S_s\sqrt{v_s}\right)_{s\geq 0}$ belongs to $\mathbb{L}^2$, in view of Theorem A.1.1, the process

$$S_0 + \int_0^t e^{-rs}S_s\sqrt{v_s}\,dW_s$$

is a martingale. Therefore, in order to show that the discounted asset price is a martingale it remains to prove that the process

$$\sum_{i\geq 1,\, T_i\leq t} e^{-rT_i}S_{T_{i-}}\left(e^{\Delta X_i}-1\right) - \int_0^t c\lambda S_s e^{-rs}\,ds$$

is a martingale as well. By verifying that the compensator c is indeed the mean of the percentage price jump size $e^{Y_i^{(S)}}-1$, the claim easily follows from Theorem A.2.3.

Since the assumption

$$Y_i^{(S)}|Y_i^{(v)} \sim \mathcal{N}(\mu_S + \rho_Y Y_i^{(v)}, \sigma_S^2)$$

is equivalent to

$$Y_i^{(S)}|Y_i^{(v)} \sim \rho_Y Y_i^{(v)} + \mathcal{N}(\mu_S, \sigma_S^2)$$

we have

$$\begin{aligned}
E\left[e^{Y_i^{(S)}}-1\right] &= \int_{\mathbb{R}}\int_0^{+\infty}\left(e^{\rho_Y y + x}-1\right) f_{\text{Exp}}(y) f_{\mathcal{N}}(x)\,dy\,dx \\
&= \int_{\mathbb{R}}\int_0^{+\infty} e^{\rho_Y y}e^x f_{\text{Exp}}(y) f_{\mathcal{N}}(x)\,dy\,dx - \int_{\mathbb{R}}\int_0^{+\infty} f_{\text{Exp}}(y) f_{\mathcal{N}}(x)\,dy\,dx
\end{aligned}$$

(by Fubini's theorem)

$$\begin{aligned}
&= \int_0^{+\infty} e^{\rho_Y y} f_{\text{Exp}}(y)\,dy \int_{\mathbb{R}} e^x f_{\mathcal{N}}(x)\,dx - \int_0^{+\infty} f_{\text{Exp}}(y)\,dy\int_{\mathbb{R}} f_{\mathcal{N}}(x)\,dx \\
&= \int_0^{+\infty} e^{\rho_Y y} f_{\text{Exp}}(y)\,dy \int_{\mathbb{R}} e^x f_{\mathcal{N}}(x)\,dx - 1 \\
&= \frac{e^{\mu_s + \frac{\sigma_s^2}{2}}}{1-\rho_Y\mu_v} - 1 = c
\end{aligned}$$

and this proves the claim.

An explicit formula for the Laplace transform exists, the SVJJ model being an affine model, and it is stated in the following result.

Proposition 2.1.3 (Affine-type Laplace trasform)**.**
Let $\mathfrak{L}_{v_T}$ be the Laplace transform of v_T, conditional on the filtration $\mathcal{F}_t$ with time to expiration $\tau = T-t$, i.e.,

$$\mathfrak{L}_{v_T}(z;t,\tau,v_t) = E\left[e^{z\cdot v_T}\middle|\mathcal{F}_t\right]$$

then, for every $z\in\mathbb{C}$,

$$\mathfrak{L}_{v_T}(z;t,\tau,v_t) = e^{a_1(z,\tau)+a_2(z,\tau)v_t+a_3(z,\tau)}$$

where

$$\begin{aligned}
a_1(z,\tau) &= \frac{-2\bar{k}\theta}{\epsilon^2}\ln\left(1+\frac{\epsilon^2 z}{2\bar{k}}\left(e^{-\bar{k}\tau}-1\right)\right) \\
a_2(z,\tau) &= \frac{2\bar{k}z}{\epsilon^2 z + (2\bar{k}-\epsilon^2 z)e^{\bar{k}\tau}} \\
a_3(z,\tau) &= \frac{2\mu_v\lambda}{2\mu_v\bar{k}-\epsilon^2}\ln\left(1+\frac{\left(\epsilon^2-2\mu_v\bar{k}\right)z}{2\bar{k}\left(1-\mu_v z\right)}\left(e^{-\bar{k}\tau}-1\right)\right).
\end{aligned}$$

Proof. The Feynman-Kac theorem A.3.3 implies that $\mathfrak{L}_{v_T}(z;t,\tau,v_t)$ is the solution of the (backward) Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\mathfrak{L}_{v_T} + \bar{k}(\theta - v)\frac{\partial}{\partial v}\mathfrak{L}_{v_T} + \frac{1}{2}\epsilon^2 v \frac{\partial^2}{\partial v^2}\mathfrak{L}_{v_T} + \lambda \int_{\mathbb{R}} \left[\mathfrak{L}_{v_T}(z;t,\tau,v+y) - \mathfrak{L}_{v_T}(z;t,\tau,v)\right] m(dy) = 0 \\ \mathfrak{L}_{v_T}(z;t+\tau,0,v) = e^{zv} \end{cases}$$

that is

$$\begin{cases} -\frac{\partial}{\partial \tau}\mathfrak{L}_{v_T} + \bar{k}(\theta - v)\frac{\partial}{\partial v}\mathfrak{L}_{v_T} + \frac{1}{2}\epsilon^2 v \frac{\partial^2}{\partial v^2}\mathfrak{L}_{v_T} + \lambda E\left[\mathfrak{L}_{v_T}(z;t,\tau,v+Y^{(v)}) - \mathfrak{L}_{v_T}(z;t,\tau,v)|\mathcal{F}_t\right] = 0 \\ \mathfrak{L}_{v_T}(z;t+\tau,0,v) = e^{zv}. \end{cases} \tag{2.9}$$

Following the solution procedure used by [6], we can solve this Cauchy problem in closed-form by guessing that the affine-form solution is

$$\mathfrak{L}_{v_T}(z;t,\tau,v) = e^{a_1(z,\tau)+a_2(z,\tau)v+a_3(z,\tau)}. \tag{2.10}$$

By substituting (2.10) into (2.9), we obtain:

$$-e^{a_1(z,\tau)+a_2(z,\tau)v+a_3(z,\tau)}\left(\frac{\partial}{\partial \tau}a_1(z,\tau) + v\frac{\partial}{\partial \tau}a_2(z,\tau) + \frac{\partial}{\partial \tau}a_3(z,\tau)\right) + \bar{k}(\theta - v)e^{a_1(z,\tau)+a_2(z,\tau)v+a_3(z,\tau)}a_2(z,\tau)$$
$$+\frac{\epsilon^2 v}{2}e^{a_1(z,\tau)+a_2(z,\tau)v+a_3(z,\tau)}a_2(z,\tau)^2 + \lambda E\left[e^{a_1(z,\tau)+a_2(z,\tau)v+a_3(z,\tau)}\left(e^{a_2 Z^{(v)}} - 1\right)\Big|\mathcal{F}_t\right] = 0$$

that is

$$e^{a_1(z,\tau)+a_2(z,\tau)v+a_3(z,\tau)}\left(-\frac{\partial}{\partial \tau}a_1(z,\tau) - \frac{\partial}{\partial \tau}a_3(z,\tau) + \bar{k}\theta a_2(z,\tau) + \lambda E\left[e^{a_2 Z^{(v)}} - 1\Big|\mathcal{F}_t\right]\right)$$
$$+ve^{a_1(z,\tau)+a_2(z,\tau)v+a_3(z,\tau)}\left(-\frac{\partial}{\partial \tau}a_2(z,\tau) + \frac{\epsilon^2}{2}a_2(z,\tau)^2 - \bar{k}a_2\right) = 0$$

whence we obtain three ordinary differential equations:

$$\begin{cases} \frac{\partial}{\partial \tau}a_2(z,\tau) = -\bar{k}a_2(z,\tau) + \frac{\epsilon^2}{2}a_2(z,\tau)^2 \\ \frac{\partial}{\partial \tau}a_1(z,\tau) = \bar{k}\theta a_2(z,\tau) \\ \frac{\partial}{\partial \tau}a_3(z,\tau) = \lambda E\left[e^{a_2 Z^{(v)}} - 1\Big|\mathcal{F}_t\right] \end{cases}$$

with initial conditions

$$\begin{cases} a_2(z,0) = z \\ a_1(z,0) = 0 \\ a_3(z,0) = 0. \end{cases}$$

Finally, the solutions to these ODEs are given by

$$\begin{cases} a_2(z,\tau) = \frac{2\bar{k}z}{\epsilon^2 z + e^{\bar{k}\tau}(2\bar{k} - \epsilon^2 z)} \\ a_1(z,\tau) = -\frac{2\bar{k}\theta}{\epsilon^2}\ln\left(1 + \frac{\epsilon^2 z}{2\bar{k}}\left(e^{-\bar{k}\tau} - 1\right)\right) \\ a_3(z,\tau) = \frac{2\mu_v\lambda}{2\mu_v\bar{k} - \epsilon^2}\ln\left(1 + \frac{z(\epsilon^2 - 2\mu_v\bar{k})}{2\bar{k}(1-\mu_v z)}\left(e^{-\bar{k}\tau} - 1\right)\right) \end{cases}$$

and this is precisely the assertion of the proposition.

□

Corollary 2.1.4 (Affine-type characteristic function)**.**
Let ψ_{v_T} be the characteristic function of v_T, conditional on the filtration $\mathcal{F}_t$ with time to expiration $\tau = T - t$, i.e.,

$$\psi_{v_T}(\xi; t, \tau, v_t) = E\left[e^{i\xi \cdot v_T} \middle| \mathcal{F}_t\right]$$

then, for every $\xi \in \mathbb{R}$,

$$\psi_{v_T}(\xi; t, \tau, v_t) = e^{a_1(i\xi,\tau) + a_2(i\xi,\tau)v_t + a_3(i\xi,\tau)}$$

where

$$\begin{aligned}
a_1(i\xi, \tau) &= \frac{-2\bar{k}\theta}{\epsilon^2} \ln\left(1 + \frac{\epsilon^2 i\xi}{2\bar{k}}\left(e^{-\bar{k}\tau} - 1\right)\right) \\
a_2(i\xi, \tau) &= \frac{2\bar{k} i\xi}{\epsilon^2 i\xi + (2\bar{k} - \epsilon^2 i\xi)e^{\bar{k}\tau}} \\
a_3(i\xi, \tau) &= \frac{2\mu_v \lambda}{2\mu_v \bar{k} - \epsilon^2} \ln\left(1 + \frac{\left(\epsilon^2 - 2\mu_v \bar{k}\right) i\xi}{2\bar{k}\left(1 - \mu_v i\xi\right)}\left(e^{-\bar{k}\tau} - 1\right)\right).
\end{aligned}$$

Proof. The claim follows by combining the following equivalence

$$\psi_{v_T}(\xi; t, \tau, v_t) = \mathfrak{L}_{v_T}(z; t, \tau, v_t)\Big|_{z = i\xi}$$

with Proposition 2.1.3.

□

3

THE CBOE VOLATILITY INDEX - VIX

In 1993, the Chicago Board Options Exchange (CBOE) introduced the CBOE Volatility Index, VIX, which was originally designed to measure the market's expectation of the 30-day volatility implied by at-the-money *S*&*P* 100 Index (OEX) [1] option prices. VIX soon became a benchmark barometer of U.S. stock market volatility.

Ten years later, in 2003, trading of *S*&*P* 500 (SPX) options was more active, hence the VIX index calculation was changed and based on the *S*&*P* 500 Index, the core index for U.S. equities. The VIX index formula was altered to reflect a new way to estimate expected volatility by averaging the weighted prices of SPX puts and calls over a wide range of strike prices.

On March 24, 2004, CBOE introduced the first exchange-traded VIX futures contract on its new, all-electronic CBOE Futures Exchange. Two years later in February 2006, CBOE launched VIX options, the most successful new product in Exchange history: in less than five years, the combined trading activity in VIX options and futures has grown to more than 100,000 contracts per day.

3.1 The VIX calculation step-by-step

Stock indexes, such as the *S*&*P* 500, are calculated using the prices of their component stocks. Each index employs rules for selecting component options and a formula to calculate index values. VIX is a volatility index comprised of *options* rather than stocks, with the price of each option reflecting the market's expectation of future volatility. Like conventional indexes, VIX employs rules that govern the selection of component options and a formula to compute index values.

[1] The Standard & Poor's 100 Index is a capitalization-weighted index of 100 stocks from a broad range of industries. The component stocks are weighted according to the total market value of their outstanding shares. The impact of a component's price change is proportional to the issue's total market value, which is the share price times the number of shares outstanding. These are summed for all 100 stocks and divided by a predetermined base value. The base value for the *S*&*P* 100 Index is adjusted to reflect changes in capitalization resulting from mergers, acquisitions, stock rights, substitutions, etc. Index options on the *S*&*P* 100 are traded with the ticker symbol "OEX".

The generalized formula used in the VIX calculation is:

$$\sigma^2 = \left\{ \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i) - \frac{1}{T} \left[\frac{F}{K_0} - 1 \right]^2 \right\} \tag{3.1}$$

where

σ	$\frac{\text{VIX}}{100}$, i.e. $\text{VIX} = \sigma \times 100$
T	Time to expiration
F	Forward index level derived from index option prices
K_0	First strike below the forward index level, F
K_i	Strike price of the i^{th} out-of-the-money option: – a call if $K_i > K_0$ – a put if $K_i < K_0$ – both put and call if $K_i = K_0$.
ΔK_i	Interval between strike prices: $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$ (*Note.* ΔK for the lowest strike is simply the difference between the lowest strike and the next higher strike. Likewise, ΔK for the highest strike is the difference between the highest strike and the next lower strike.)
r	Risk-free interest rate to expiration
$Q(K_i)$	The midpoint of the bid-ask spread for each option with strike K_i.

Figure 3.1 below depicts the VIX Index between September 2010 and September 2014. By reading the chart backwards we observe that by 2014 to early 2013 it tended to stay between 10 and 20; then it increased gradually until it spiked at over 40 in September and August 2011. By mid-2011 it had declined to more normal levels, but in April 2010 it reached a spike of 40. By March 2010 it finally declined to lower levels.

Figure 3.1: The VIX index, September 2010 to September 2014.

Hereafter we provide all the necessary information about the way the VIX Index is calculated.

The components of VIX are near- and next-term put and call options, usually in the first and second SPX contract months. “Near-term” options must have at least one week to expiration; a requirement intended to minimize pricing anomalies that might occur close to expiration. When the near-term options have less than a week to expiration, VIX “rolls” to the second and third SPX contract months. For the purpose of calculating time to expiration, SPX options are deemed to expire at the open of trading on SPX settlement day - the third Friday of the month. The VIX calculation measures time to expiration T in calendar days and divides each day into minutes, indeed it is given by the following expression:

$$T = \frac{\{M_{\text{Current day}} + M_{\text{Settlement day}} + M_{\text{Other days}}\}}{\text{Minutes in a year}}$$

where

$M_{\text{Current day}}$	Minutes remaining until midnight of the current day
$M_{\text{Settlement day}}$	Minutes from midnight until 8:30 a.m. on the SPX settlement day
$M_{\text{Other days}}$	Total minutes in the days between current day and settlement day.

For example, if we assume that the near-term and the next-term options have 9 days and 37 days to expiration, respectively, using 8:30 a.m. as the time of the calculation T, the time for the near-term and next-term options, denoted by T_1 and T_2, respectively, is calculated as follows

$$T_1 = \frac{930 + 510 + 11520}{525600} = 0.0246575$$
$$T_2 = \frac{930 + 510 + 51840}{525600} = 0.1013699.$$

The risk-free interest rate r is the bond-equivalent yield of the U.S. T-bill maturing closest to the expiration dates of relevant SPX options. As such, the VIX calculation may use different risk-free interest rates for near- and next-term options. In this example, however, we assume that $r = 0.38\%$ for both sets of options.

Hereafter we present a representative sample of the VIX computation, the interim calculations will be a repetition of it.

STEP 1 - Select the options to be used in the VIX calculation.

The selected options are out-of-the-money SPX calls and out-of-the-money SPX puts centered around an at-the-money strike price, K_0. Besides, only SPX options quoted with non-zero bid prices are used in the VIX calculation.

For each contract month:

- Determine the forward SPX level F by identifying the strike price at which the absolute difference between the call and put prices is smallest: the call and put prices reflect the average of each option’s bid/ask quotation.

Near-Term Options				Next-Term Options			
Strike Price	Call	Put	Absolute Difference	Strike Price	Call	Put	Absolute Difference
.	.	.	.	.	.	.	.
900	48.95	27.25	21.70	900	73.6	52.8	20.80
905	46.15	29.75	16.40	905	70.35	54.7	15.65
910	42.55	31.70	10.85	910	67.35	56.75	10.60
915	40.05	33.55	6.50	915	64.75	58.9	5.85
920	***37.15***	***36.65***	***0.50***	***920***	***61.55***	***60.55***	***1.00***
925	33.30	37.70	4.40	925	58.95	63.05	4.10
930	32.45	40.15	7.70	930	55.75	65.4	9.65
935	28.75	42.70	13.95	935	53.05	67.35	14.30
940	27.50	45.30	17.80	940	50.15	69.8	19.65
.	.	.	.	.	.	.	.

In this example, the difference between the call and put prices is smallest at the 920 strike for both the near- and next-term options, thus using the put-call parity formula

$$F = \text{Strike price} + e^{r\tau}(\text{Call price} - \text{Put price})$$

the forward index prices, F_1 and F_2 for the near- and next-term options, respectively, are

$$F_1 = 920 + e^{0.0038\times 0.0246575}(37.15 - 36.65) = 920.50005$$
$$F_2 = 920 + e^{0.0038\times 0.1013699}(61.55 - 60.55) = 921.00039.$$

- Determine K_0, the strike immediately below the forward index level F for the near- and next-term options. In this example $K_{0,1} = K_{0,2} = 920$.
- Select out-of-the-money put options with strike smaller than K_0. Start with the put strike immediately lower than K_0 and move to successively lower strike prices, excluding any put options that have a bid price equal to zero. Finally, once two puts with consecutive strike prices are found to have zero bid prices, no puts with lower strikes are considered.

Put Strike	Bid	Ask	Include?
200	0.00	0.05	*Not considered following two zero bids*
250	0.00	0.05	
300	0.00	0.05	
350	0.00	0.05	No
375	0.00	0.10	No
400	0.05	0.20	Yes
425	0.05	0.20	Yes
450	0.05	0.20	Yes
.	.	.	.

Then, select out-of-the-money call options with strike greater than K_0. Start with the call strike immediately higher than K_0 and move to successively higher strike prices, excluding any call options that have a bid price equal to zero. Equally to the puts, once two calls with consecutive strike prices are found to have zero bid prices, no calls with higher strikes are considered.

Call Strike	Bid	Ask	Include?
.	.	.	.
1215	0.05	0.50	Yes
1220	0.05	1.00	Yes
1225	0.00	1.00	No
1230	0.00	1.00	No
1235	0.00	0.75	*Not considered following two zero bids*
1240	0.00	0.50	
1245	0.00	0.15	
1250	***0.05***	***0.10***	
1255	0.00	1.00	
.	.	.	

Finally, select both the put and call with strike price K_0. The K_0 put and call prices are averaged to produce a single value. In our example, the price used for the 920 strike in the near-term and in the next-term are, respectively,

$$(37.15 + 36.65)/2 = 36.90$$

$$(61.55 + 60.55)/2 = 61.05.$$

Near-term Strike	Option Type	Mid-quote Price		Next-term Strike	Option Type	Mid-quote Price
400	Put	0.125		200	Put	0.325
425	Put	0.125		300	Put	0.30
450	Put	0.125		350	Put	0.50
.	.	.		.	.	.
910	Put	31.70		910	Put	56.75
915	Put	33.55		915	Put	58.90
920	***Put/Call Average***	***36.90***		***920***	***Put/Call Average***	***61.05***
925	Call	33.30		925	Call	58.95
930	Call	32.45		930	Call	55.75
.	.	.		.	.	.
1210	Call	0.275		1150	Call	0.825
1215	Call	0.275		1155	Call	0.725
1220	Call	0.525		1160	Call	0.60

STEP 2 - Calculate volatility for both near-term and next-term options.

Applying the VIX formula (3.1) to the near-term and next-term options with time to expiration of T_1 and T_2, respectively, yields:

$$\sigma_1{}^2 = \frac{2}{T_1}\sum_i \frac{\Delta K_i}{K_i^2} e^{rT_1} Q(K_i) - \frac{1}{T_1}\left[\frac{F}{K_0} - 1\right]^2$$

$$\sigma_2{}^2 = \frac{2}{T_2}\sum_i \frac{\Delta K_i}{K_i^2} e^{rT_2} Q(K_i) - \frac{1}{T_2}\left[\frac{F}{K_0} - 1\right]^2$$

VIX is an amalgam of the information reflected in the prices of all of the selected options. The contribution of a single option to the VIX value is proportional to ΔK and the price of that option, and inversely proportional to the square of the option's strike price.

In our example the contribution of the near-term 400 Put is given by:

$$\frac{\Delta K_{400\text{ Put}}}{K^2_{400\text{ Put}}} e^{rT_1} Q(400\text{Put}) = \frac{25}{400^2} e^{0.0038\times 0.0246575} \times 0.125 = 0.0000195$$

and a similar calculation is performed for each option. The resulting values for the near-term options are then summed and multiplied by $\frac{2}{T_1}$. Likewise, the resulting values for the next-term options are summed and multiplied by $\frac{2}{T_2}$. The table below summarizes the results for each strip of options.

Near-term Strike	Option Type	Mid-quote Price	Contribution by Strike	Next-term Strike	Option Type	Mid-quote Price	Contribution by Strike
400	Put	0.125	0.0000195	200	Put	0.325	0.0008128
425	Put	0.125	0.0000173	300	Put	0.300	0.0002501
450	Put	0.125	0.0000139	350	Put	0.500	0.0001531
.	.	.	.	.	.	.	.
910	Put	31.70	0.0001914	910	Put	56.75	0.0003428
915	Put	33.55	0.0002004	915	Put	58.90	0.0003519
920	*Put/Call Average*	*36.90*	*0.0002180*	*920*	*Put/Call Average*	*61.05*	*0.0003608*
925	Call	33.30	0.0001946	925	Call	58.95	0.0003446
930	Call	32.45	0.0001876	930	Call	55.75	0.0003224
.	.	.	.	.	.	.	.
1210	Call	0.275	0.0000009	1150	Call	0.825	0.0000031
1215	Call	0.275	0.0000009	1155	Call	0.725	0.0000027
1220	Call	0.525	0.0000018	1160	Call	0.600	0.0000022
$\frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i)$			**0.4727799**				**0.3668297**

Next, we calculate $\frac{1}{T}\left[\frac{F}{K_0}-1\right]^2$ for the near-term T_1 and the next-term T_2.

$$\frac{1}{T_1}\left[\frac{F_1}{K_0}-1\right]^2 = \frac{1}{0.0246575}\left[\frac{920.50005}{920}-1\right]^2 = 0.0000120$$

$$\frac{1}{T_2}\left[\frac{F_2}{K_0}-1\right]^2 = \frac{1}{0.1013699}\left[\frac{921.00039}{920}-1\right]^2 = 0.0000117.$$

Finally, we compute σ_1^2 and σ_2^2:

$$\sigma_1{}^2 = \frac{2}{T_1}\sum_i \frac{\Delta K_i}{K_i^2} e^{rT_1} Q(K_i) - \frac{1}{T_1}\left[\frac{F}{K_0}-1\right]^2 = 0.4727799 - 0.0000120 = 0.4727679$$

$$\sigma_2{}^2 = \frac{2}{T_2}\sum_i \frac{\Delta K_i}{K_i^2} e^{rT_2} Q(K_i) - \frac{1}{T_2}\left[\frac{F}{K_0}-1\right]^2 = 0.3668297 - 0.0000117 = 0.3668180.$$

STEP 3 - Calculate the 30-day weighted average of σ_1^2 and σ_2^2, take the square root of that value and multiply by 100 to get the VIX.

$$\text{VIX} = 100 \times \sqrt{\left\{T_1\sigma_1^2\left[\frac{N_{T_2}-N_{30}}{N_{T_2}-N_{T_1}}\right] + T_2\sigma_2^2\left[\frac{N_{30}-N_{T_1}}{N_{T_2}-N_{T_1}}\right] \times \frac{N_{365}}{N_{30}}\right\}}.$$

When the near-term options have less than 30 days to expiration and the next-term options have more than 30 days to expiration, the resulting VIX value reflects an interpolation of σ_1^2 and σ_2^2; i.e., each individual weight is less than or equal to 1 and the sum of the weights equals 1. At the time of the VIX "roll", instead, both the near-term and next-term options have more than 30 days to expiration: the same formula is used to calculate the 30-day weighted average, but the result is an extrapolation of σ_1^2 and σ_2^2; i.e., the sum of the weights is still 1, but the near-term weight is greater than 1 and the next-term weight is negative.

Returning to our example we finally get

N_{T_1}	Number of minutes to settlement of the near-term options $(12,960)$
N_{T_2}	Number of minutes to settlement of the next-term options $(53,280)$
N_{30}	Number of minutes in 30 days $(30 \times 1,440 = 43,200)$
N_{365}	Number of minutes in a 365-day year $(365 \times 1,440 = 525,600)$

and

$$\mathrm{VIX} = 100 \times \sqrt{\left\{0.0246575 \times 0.4727679 \times \left[\frac{53,280 - 43,200}{53,280 - 12,960}\right] + 0.1013699 \times 0.3668180 \right.} \\ \times \sqrt{\left.\left[\frac{43,200 - 12,960}{53,280 - 12,960}\right] \times \frac{525,600}{43,200}\right\}}$$

whence

$$\mathrm{VIX} = 100 \times 0.612179986 = 61.22.$$

3.2 VIX Squared and Forward Price of Integrated Variance

In this section we derive the probabilistic representation of the square of the VIX, indeed we will prove that it can be interpreted as the conditional risk-neutral expectation of a log contract. Before proceeding, we state beforehand the definition of the VIX squared we are going to use hereafter. Since the purpose of this work is to price European options on the VIX under continuous-time jump-diffusion models, we should be able to provide the corresponding continuous-time version for the definition of VIX squared (3.1). As a matter of fact, it is straightforward to extend the previous discrete definition (3.1) to the continuous case, simply by assuming to take the limit as $\Delta K \longrightarrow 0$. Indeed we have

$$\mathrm{VIX}^2 = 100^2 \times \left\{ \frac{2}{\tau} \left[\int_0^F \frac{1}{y^2} \tilde{P}(y)\, dy + \int_F^\infty \frac{1}{y^2} \tilde{C}(y)\, dy \right] \right\} \tag{3.2}$$

where $\tilde{P}(y)$ and $\tilde{C}(y)$ represent forward put and call prices with strike y, respectively. We notice that the term $\left[\frac{F}{K_0} - 1\right]^2$ has disappeared from the new expression for the square of the VIX, since K_0, being the first strike immediately below the forward index level F, tends to equalize the value F as ΔK tends to zero.

THEOREM - 3.2.1. *The risk-neutral probability density function of the stock price S at time T is given by*

$$f(S_T, T; S_t, t) = \frac{\partial^2 \tilde{C}(S_t, x, t, T)}{\partial x^2}\bigg|_{x=S_T} \tag{3.3}$$

or, equivalently,

$$f(S_T, T; S_t, t) = \frac{\partial^2 \tilde{P}(S_t, x; t, T)}{\partial x^2}\bigg|_{x=S_T} \tag{3.4}$$

where $\tilde{C}$ and $\tilde{P}$ represent forward call and put prices, respectively:

$$\begin{aligned}\tilde{C}(S_t, x; t, T) &= e^{r(T-t)} C(S_t, x; t, T)\\ \tilde{P}(S_t, x; t, T) &= e^{r(T-t)} P(S_t, x; t, T).\end{aligned}$$

Proof. For the sake of simplicity, in the following, we denote the risk-neutral probability density function of S_T, conditional on $\mathcal{F}_t$, as follows

$$f(\cdot) := f(\cdot, T; S_t, t).$$

For every measurable function ϕ, we have

$$\begin{aligned}\int_0^\infty \phi(x) \frac{\partial^2 \tilde{C}(S_t, x; t, T)}{\partial x^2}\, dx &= \int_0^\infty \phi(x) e^{r(T-t)} \frac{\partial^2 C(S_t, x; t, T)}{\partial x^2}\, dx\\ &= \int_0^\infty \phi(x) e^{r(T-t)} \frac{\partial^2}{\partial x^2} e^{-r(T-t)} E^Q\left[(S_T - x)^+ | S_t\right]\, dx\\ &= \int_0^\infty \phi(x) \frac{\partial^2}{\partial x^2} \int_\Omega (S_T - x)^+\, dQ\, dx\\ &= \int_0^\infty \phi(x) \frac{\partial^2}{\partial x^2} \int_0^\infty (y - x)^+ f(y)\, dy\, dx\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \phi(x) \frac{\partial^2}{\partial x^2} \int_x^\infty (y-x) f(y)\, dy\, dx \\
&= \int_0^\infty \phi(x) \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \int_x^\infty y f(y)\, dy - \frac{\partial}{\partial x} \int_x^\infty x f(y)\, dy \right) dx \\
&= \int_0^\infty \phi(x) \frac{\partial}{\partial x} \left(-\frac{\partial}{\partial x} \int_\infty^x y f(y)\, dy + \frac{\partial}{\partial x} x \int_\infty^x f(y)\, dy \right) dx
\end{aligned}
$$

(by the fundamental theorem of calculus)

$$
\begin{aligned}
&= \int_0^\infty \phi(x) \frac{\partial}{\partial x} \left(-x f(x) + \int_\infty^x f(y)\, dy + x f(x) \right) dx \\
&= \int_0^\infty \phi(x) \frac{\partial}{\partial x} \left(\int_\infty^x f(y)\, dy \right) dx \\
&= \int_0^\infty \phi(x) f(x)\, dx.
\end{aligned}
$$

Hence

$$
\frac{\partial^2 \tilde{C}(S_t, x; t, T)}{\partial x^2}\Big|_{x=S_T}
$$

is the probability density function of S_T. Analogously, in the case of put options, we have, for every measurable function ϕ

$$
\begin{aligned}
\int_0^\infty \phi(x) \frac{\partial^2 \tilde{P}(S_t, x; t, T)}{\partial x^2}\, dx &= \int_0^\infty \phi(x) e^{r(T-t)} \frac{\partial^2 P(S_t, x; t, T)}{\partial x^2}\, dx \\
&= \int_0^\infty \phi(x) e^{r(T-t)} \frac{\partial^2}{\partial x^2} e^{-r(T-t)} E^Q \left[(x - S_T)^+ | S_t \right] dx \\
&= \int_0^\infty \phi(x) \frac{\partial^2}{\partial x^2} \int_\Omega (x - S_T)^+\, dQ\, dx \\
&= \int_0^\infty \phi(x) \frac{\partial^2}{\partial x^2} \int_\Omega (x - y)^+\, P^{S_T}(dy)\, dx \\
&= \int_0^\infty \phi(x) \frac{\partial^2}{\partial x^2} \int_0^\infty (x - y)^+ f(y)\, dy\, dx \\
&= \int_0^\infty \phi(x) \frac{\partial^2}{\partial x^2} \int_0^x (x - y) f(y)\, dy\, dx \\
&= \int_0^\infty \phi(x) \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \int_0^x x f(y)\, dy - \frac{\partial}{\partial x} \int_0^x y f(y) \right) dx
\end{aligned}
$$

(by the fundamental theorem of calculus)

$$
\begin{aligned}
&= \int_0^\infty \phi(x) \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \left(x \int_0^x f(y)\, dy \right) - x f(x) \right) dx \\
&= \int_0^\infty \phi(x) \frac{\partial}{\partial x} \left(\int_0^x f(y)\, dy + x f(x) - x f(x) \right) dx \\
&= \int_0^\infty \phi(x) f(x)\, dx.
\end{aligned}
$$

Thus

$$
\frac{\partial^2 \tilde{P}(S_t, x; t, T)}{\partial x^2}\Big|_{x=S_T}
$$

is the probability density function of S_T.

□

Proposition 3.2.2. *Let g be a measurable function, then there exists a unique Borel-measurable function h such that*

$$E^Q\left[g(S_T)|S_t\right] = h(S_t).$$

Proof. Firstly, we remark that, since g is measurable, also $g(S_T)$ is $\sigma(S_T)-$measurable. Moreover, by the defining properties of the conditional expectation, we have that

(i) $Y_t := E^Q\left[g(S_T)|S_t\right]$ is $\sigma(S_T)-$measurable;

(ii) $\int_A Y_t\, dQ = \int_A g(S_T)\, dQ \quad \forall A \in \sigma(S_T)$.

Thus, from (i), exploiting a known result (see Corollary A.10 in [15]), there exists a Borel-measurable function h such that $Y_t = h(S_t)$ and from (ii) it follows that h is unique Q-a.s.

□

By means of the following proposition, we shall provide the analytical expression of the above function h.

Proposition 3.2.3. *Let g be a measurable function. The value of a claim with generalized terminal payoff $g(S,T)$ is calculated as*

$$E^Q\left[g(S_T)|S_t\right] = g(F) + \int_0^F \tilde{P}(S_t, y; t, T)g''(y)\, dy + \int_F^\infty \tilde{C}(S_t, y; t, T)g''(y)\, dx \tag{3.5}$$

where $F = S_t e^{(r-q)(T-t)}$ denotes the forward price of the stock with risk-free interest rate r and dividend yield q.

Proof. For the sake of brevity, in the following we will simply denote the forward call and put prices by

$$\tilde{C}(y) := \tilde{C}(S_t, y; t, T) \quad \text{and} \quad \tilde{P}(y) := \tilde{P}(S_t, y; t, T)$$

respectively. The value of a claim with generalized terminal payoff $g(S,T)$ is calculated as the conditional expectation of the quantity $g(S,T)$ given all the information available on the asset at time t, under the risk-neutral probability measure, that is

$$E^Q\left[g(S_T)|S_t\right] = \int_0^\infty g(y) f(y, T; S_t, t) dy$$

whence, in view of (3.3) and (3.4), we get

$$E^Q\left[g(S_T)|S_t\right] = \int_0^F g(y)\frac{\partial^2 \tilde{P}(y)}{\partial y^2} dy + \int_F^\infty g(y)\frac{\partial^2 \tilde{C}(y)}{\partial y^2} dy.$$

Now, integrating by parts twice the expression above, we obtain

$$
\begin{aligned}
E^Q\left[g(S_T)|S_t\right] =& g(y)\frac{\partial\tilde{P}(y)}{\partial y}\bigg|_0^F - \int_0^F g'(y)\frac{\partial\tilde{P}(y)}{\partial y}\,dy + g(y)\frac{\partial\tilde{C}(y)}{\partial y}\bigg|_F^\infty - \int_F^\infty g'(y)\frac{\partial\tilde{C}(y)}{\partial y}\,dy\\
=& g(y)\frac{\partial\tilde{P}(y)}{\partial y}\bigg|_0^F - \left(g'(y)\tilde{P}(y)\bigg|_0^F - \int_0^F g''(y)\tilde{P}(y)\,dy\right) + g(y)\frac{\partial\tilde{C}(y)}{\partial y}\bigg|_F^\infty\\
&- \left(g'(y)\tilde{C}(y)\bigg|_F^\infty - \int_F^\infty g''(y)\tilde{C}(y)\,dy\right)\\
=& g(F)\frac{\partial\tilde{P}(F)}{\partial y} - g(0)\frac{\partial\tilde{P}(0)}{\partial y} - g'(F)\tilde{P}(F) + g'(0)\tilde{P}(0) + \int_0^F g''(y)\tilde{P}(y)\,dy\\
&+ \lim_{y\to\infty} g(y)\frac{\partial\tilde{C}(y)}{\partial y} - g(F)\frac{\partial\tilde{C}(F)}{\partial y} - \lim_{y\to\infty} g'(y)\tilde{C}(y) + g'(F)\tilde{C}(F) + \int_F^\infty g''(y)\tilde{C}(y)\,dy\\
=& g(F)\left(\frac{\partial\tilde{P}(F)}{\partial y} - \frac{\partial\tilde{C}(F)}{\partial y}\right) + g'(F)\left(\tilde{C}(F) - \tilde{P}(F)\right) - g(0)\frac{\partial\tilde{P}(0)}{\partial y} + g'(0)\tilde{P}(0)\\
&+ \int_0^F g''(y)\tilde{P}(y)\,dy + \lim_{y\to\infty} g(y)\frac{\partial\tilde{C}(y)}{\partial y} - \lim_{y\to\infty} g'(y)\tilde{C}(y) + \int_F^\infty g''(y)\tilde{C}(y)\,dy.
\end{aligned}
$$

From the put-call parity formula

$$
C(S_t, K; t, T) - P(S_t, K; t, T) = S_t e^{-q(T-t)} - K e^{-r(T-t)}
$$

we get

$$
\left(C(S_t, K; t, T) - P(S_t, K; t, T)\right) e^{r(T-t)} = S_t e^{(r-q)(T-t)} - K
$$

which, converted to our notations, corresponds to

$$
\begin{aligned}
\tilde{C}(S_t, K; t, T) - \tilde{P}(S_t, K; t, T) &= S_t e^{(r-q)(T-t)} - K\\
&= F - K.
\end{aligned}
$$

Then it follows directly that

$$
\frac{\partial(\tilde{C}(y) - \tilde{P}(y))}{\partial y} = \frac{\partial(F-y)}{\partial y} = -1.
$$

Hence we have

$$
\begin{aligned}
E^Q\left[g(S_T)|S_t\right] =& g(F) - g(0)\frac{\partial\tilde{P}(0)}{\partial y} + g'(0)\tilde{P}(0) + \int_0^F g''(y)\tilde{P}(y)\,dy + \lim_{y\to\infty} g(y)\frac{\partial\tilde{C}(y)}{\partial y}\\
&- \lim_{y\to\infty} g'(y)\tilde{C}(y) + \int_F^\infty g''(y)\tilde{C}(y)\,dy.
\end{aligned}
$$

Now, since

$$
\tilde{C}(y) = E^Q\left[(S_T - y)^+|S_t\right]
$$

we observe that

$$
\lim_{y\to\infty} E^Q\left[(S_T - y)^+|S_t\right] = 0
$$

whence

$$
\lim_{y\to\infty} g'(y)\tilde{C}(y) = 0.
$$

Moreover

$$\begin{aligned}
\frac{\partial \tilde{C}(y)}{\partial y} &= \frac{\partial}{\partial y} E^Q\left[(S_T - y)^+ | S_t\right] \\
&= \frac{\partial}{\partial y} \int_0^\infty (S_T - y)^+ \, dQ \\
&= \frac{\partial}{\partial y} \int_0^\infty (x - y)^+ f(x)\, dx \\
&= \frac{\partial}{\partial y} \int_y^\infty (x - y) f(x)\, dx \\
&= -\frac{\partial}{\partial y} \int_\infty^y x f(x)\, dx + \frac{\partial}{\partial y} y \int_\infty^y f(x)\, dx \\
&= -y f(y) + \int_\infty^y f(x)\, dx + y f(y) \\
&= \int_\infty^y f(x)\, dx
\end{aligned}$$

whence

$$\lim_{y \to \infty} g(y) \frac{\partial \tilde{C}(y)}{\partial y} = 0.$$

Analogously, considering put options, since

$$\tilde{P}(y) = E^Q\left[(y - S_T)^+ | S_t\right]$$

we observe that

$$g'(0)\tilde{P}(0) = 0$$

and

$$\begin{aligned}
\frac{\partial \tilde{P}(y)}{\partial y} &= \frac{\partial}{\partial y} E^Q\left[(y - S_T)^+ | S_t\right] \\
&= \frac{\partial}{\partial y} \int_0^\infty (y - S_T)^+ \, dQ \\
&= \frac{\partial}{\partial y} \int_0^\infty (y - x)^+ f(x)\, dx \\
&= \frac{\partial}{\partial y} \int_0^y (y - x) f(x)\, dx \\
&= \frac{\partial}{\partial y} y \int_0^y f(x)\, dx - \frac{\partial}{\partial y} \int_0^y x f(x)\, dx \\
&= \int_0^y f(x)\, dx + y f(y) - y f(y) \\
&= \int_0^y f(x)\, dx
\end{aligned}$$

whence

$$g(0) \frac{\partial \tilde{P}(0)}{\partial y} = 0.$$

Therefore, it follows that

$$E^Q\left[(S_T - y)^+ | S_t\right] = g(F) + \int_0^F g''(y)\tilde{P}(y)\, dy + \int_F^\infty g''(y)\tilde{C}(y)\, dy$$

hence the claim.

□

Note. We note that equation (3.5) is completely model-independent.

The following result exhibits an expression for the VIX squared in terms of the risk-neutral expectation of a log contract.

Proposition 3.2.4. *The VIX squared can be expressed as*

$$VIX_t^2 = -\frac{2}{\tau} E^Q \left[\ln\left(\frac{S_T}{F}\right) \middle| S_t \right] \cdot 100^2. \tag{3.6}$$

Proof. We consider a log contract, that is, using the notations of Proposition 3.2.3,

$$g(S_T) = \ln\left(\frac{S_T}{F}\right).$$

Then

$$g'(S_T) = \frac{d}{dS_T} \ln\left(\frac{S_T}{F}\right) = \frac{1}{S_T}$$

and

$$g''(S_T) = \frac{d}{dS_T}\left(\frac{1}{S_T}\right) = -\frac{1}{S_T^2}.$$

Therefore, in view of Proposition 3.2.3 , we get

$$\begin{aligned} E^Q \left[\ln\left(\frac{S_T}{F}\right) \middle| S_t \right] &= \ln\left(\frac{F}{F}\right) - \int_0^F \frac{1}{y^2} \tilde{P}(y)\, dy - \int_F^\infty \frac{1}{y^2} \tilde{C}(y)\, dy \\ &= -\int_0^F \frac{1}{y^2} \tilde{P}(y)\, dy - \int_F^\infty \frac{1}{y^2} \tilde{C}(y)\, dy. \end{aligned} \tag{3.7}$$

Finally, by the definition of VIX squared and (3.7), we have

$$\begin{aligned} VIX_t^2 &= \frac{2}{\tau}\left[\int_0^F \frac{1}{y^2} \tilde{P}(y)\, dy + \int_F^\infty \frac{1}{y^2} \tilde{C}(y)\, dy \right] \cdot 100^2 \\ &= -\frac{2}{\tau} E^Q \left[\ln\left(\frac{S_T}{F}\right) \middle| S_t \right] \cdot 100^2 \\ &= -\frac{2}{\tau} E^Q \left[\ln(S_T) - \ln(F) \middle| S_t \right] \cdot 100^2 \\ &= -\frac{2}{\tau} E^Q \left[\ln(S_T) - \ln(S_t e^{(r-q)\tau}) \middle| S_t \right] \cdot 100^2 \\ &= -\frac{2}{\tau} E^Q \left[\ln(S_T) - \ln(S_t) - (r-q)\tau \middle| S_t \right] \cdot 100^2 \\ &= 2(r-q) - \frac{2}{\tau} E^Q \left[\ln(S_T) - \ln(S_t) \middle| S_t \right] \cdot 100^2 \end{aligned}$$

whence the claim.

□

3.3 VIX under Heston model

By examining (3.6), it is clear that different dynamics for the asset price S will result in various expressions for VIX squared. In this section we provide a shorthand form for the VIX squared under the Heston model. Later we shall exploit this more manageable form for pricing vanilla options on the VIX, assuming that the volatility of the asset is driven by square-root diffusion (Heston model).

THEOREM - 3.3.1. *The VIX squared, under the Heston model, is expressed as*

$$VIX_t^2 = (a_\tau v_t + b_\tau) \times 100^2 \tag{3.8}$$

where

$$a_\tau = \frac{1 - e^{-\bar{k}\tau}}{\bar{k}\tau}$$
$$b_\tau = \theta(1 - a_\tau)$$

and $\tau := T - t = \frac{30}{365}$.

Proof. By Proposition 3.6, we have

$$\begin{aligned} VIX_t^2 &= -\frac{2}{T-t} E\left[\ln\left(\frac{S_T}{F}\right)\middle|\mathcal{F}_t\right] \times 100^2 \\ &= -\frac{2}{T-t} E\left[\ln(S_T) - \ln\left(S_t e^{(r-q)(T-t)}\right)\middle|\mathcal{F}_t\right] \times 100^2. \end{aligned}$$

Now, applying Itô's lemma A.2 to the stochastic process

$$f(t, S_t) = \ln(F)$$

under Heston model, we obtain

$$\begin{aligned} d\ln(F) &= \frac{1}{S_t e^{(r-q)(T-t)}} S_t e^{(r-q)(T-t)} (q-r) dt \\ &+ \frac{1}{S_t e^{(r-q)(T-t)}} e^{(r-q)(T-t)} dS_t - \frac{1}{2} v_t S_t^2 \frac{1}{S_t^2} dt \\ &= -(r-q) dt + \frac{1}{S_t} dS_t - \frac{1}{2} v_t dt \\ &= -\left(r - q + \frac{1}{2} v_t\right) dt + \frac{1}{S_t}\left((r-q) S_t dt + S_t \sqrt{v_t} dW_t^{(S)}\right) \\ &= -\left(r - q + \frac{1}{2} v_t\right) dt + (r-q) dt + \sqrt{v_t} dW_t^{(S)} \\ &= -\frac{1}{2} v_t dt + \sqrt{v_t} dW_t^{(S)} \end{aligned}$$

that is

$$\ln(S_T) - \ln(S_t e^{(r-q)(T-t)}) = -\frac{1}{2}\int_t^T v_s ds + \int_t^T \sqrt{v_s} dW_s^{(S)}.$$

Hence

$$\begin{aligned} E\left[\ln(S_T) - \ln(S_t e^{(r-q)(T-t)}) | \mathcal{F}_t\right] &= E\left[-\frac{1}{2}\int_t^T v_s ds + \int_t^T \sqrt{v_s} dW_s^{(S)} \middle| \mathcal{F}_t\right] \\ &= E\left[-\frac{1}{2}\int_t^T v_s ds \middle| \mathcal{F}_t\right] + E\left[\int_t^T \sqrt{v_s} dW_s^{(S)}\right] \end{aligned}$$

(by Theorem A.1.1, as $(\sqrt{v_s})_s \in \mathbb{L}^2$)

$$= E\left[-\frac{1}{2}\int_t^T v_s ds \middle| \mathcal{F}_t\right].$$

Therefore

$$VIX_t^2 = \frac{1}{T-t} E\left[\int_t^T v_s ds \middle| \mathcal{F}_t\right] \times 100^2.$$

By applying Itô's lemma to to the stochastic process

$$f(t, v_t) = e^{\bar{k}t} v_t$$

under the Heston model, we have

$$\begin{aligned}
d\left(e^{\bar{k}t} v_t\right) &= \bar{k} e^{\bar{k}t} v_t dt + e^{\bar{k}t} dv_t \\
&= \bar{k} e^{\bar{k}t} v_t dt + e^{\bar{k}t}\left(\bar{k}(\theta - v_t)\, dt + \epsilon\sqrt{v_t} dW_t^{(v)}\right) \\
&= \bar{k} e^{\bar{k}t} v_t dt + \bar{k}\theta e^{\bar{k}t} dt - \bar{k} e^{\bar{k}t} v_t dt + \epsilon e^{\bar{k}t}\sqrt{v_t} dW_t^{(v)} \\
&= \bar{k}\theta e^{\bar{k}t} dt + \epsilon e^{\bar{k}t}\sqrt{v_t} dW_t^{(v)}
\end{aligned}$$

that is

$$\begin{aligned}
e^{\bar{k}T} v_T - e^{\bar{k}t} v_t &= \theta\int_t^T \bar{k} e^{\bar{k}s} ds + \epsilon\int_t^T e^{\bar{k}s}\sqrt{v_s} dW_s^{(v)} \\
&= \theta\left(e^{\bar{k}T} - e^{\bar{k}t}\right) + \epsilon\int_t^T e^{\bar{k}s}\sqrt{v_s} dW_s^{(v)}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
v_T &= e^{-\bar{k}(T-t)} v_t + \theta e^{-\bar{k}T}\left(e^{\bar{k}T} - e^{\bar{k}t}\right) + \epsilon e^{-\bar{k}T}\int_t^T e^{\bar{k}s}\sqrt{v_s} dW_s^{(v)} \\
&= e^{-\bar{k}(T-t)} v_t + \theta\left(1 - e^{-\bar{k}(T-t)}\right) + \epsilon e^{-\bar{k}T}\int_t^T e^{\bar{k}s}\sqrt{v_s} dW_s^{(v)}
\end{aligned}$$

that leads to the following expression of v_T

$$v_T = \alpha_\tau v_t + \beta_\tau + \epsilon e^{-\bar{k}T}\int_t^T e^{\bar{k}s}\sqrt{v_s} dW_s^{(v)}$$

where

$$\alpha_\tau = e^{-\bar{k}\tau}$$

and

$$\beta_\tau = \theta\left(1 - e^{-\bar{k}\tau}\right).$$

The mean of the instantaneous variance is thus

$$E[v_T|\mathcal{F}_t] = E\left[\alpha_\tau v_t + \beta_\tau + \epsilon e^{-\bar{k}T}\int_t^T e^{\bar{k}s}\sqrt{v_s} dW_s^{(v)} \middle| \mathcal{F}_t\right]$$

(by Theorem A.1.1)

$$= \alpha_\tau v_t + \beta_\tau.$$

Now, we can compute the expectation of the integrated variance: since the variance is strictly positive, by Tonelli's theorem, we have

$$
\begin{aligned}
E\left[\int_t^T v_s ds \middle| \mathcal{F}_t\right] &= \int_t^T E\left[v_s|\mathcal{F}_t\right] ds \\
&= \int_t^T E\left[\alpha_{s-t} v_t + \beta_{s-t}|\mathcal{F}_t\right] ds \\
&= \int_t^T \left(\alpha_{s-t} v_t + \beta_{s-t}\right) ds \\
&= \int_t^T \alpha_{s-t} ds\, v_t + \int_t^T \beta_{s-t} ds \\
&= \int_t^T e^{-\bar{k}(s-t)} ds\, v_t + \int_t^T \theta\left(1 - e^{-\bar{k}(s-t)}\right) ds \\
&= -v_t \frac{1}{\bar{k}}\left[e^{-\bar{k}(s-t)}\right]_t^T + \theta(T-t) + \frac{\theta}{\bar{k}}\left[e^{-\bar{k}(s-t)}\right]_t^T \\
&= -v_t \frac{1}{\bar{k}} e^{-\bar{k}(T-t)} + \frac{1}{\bar{k}} v_t + \theta(T-t) + \frac{\theta}{\bar{k}} e^{-\bar{k}(T-t)} - \frac{\theta}{\bar{k}} \\
&= v_t\left(\frac{1 - e^{-\bar{k}(T-t)}}{\bar{k}}\right) + \theta\left((T-t) - \frac{1 - e^{-\bar{k}(T-t)}}{\bar{k}}\right)
\end{aligned}
$$

that leads to the following expression

$$
E\left[\int_t^T v_s ds \middle| \mathcal{F}_t\right] = \bar{a}_\tau v_t + \bar{b}_\tau
$$

where

$$
\bar{a}_\tau = \frac{1 - e^{-\bar{k}\tau}}{\bar{k}}
$$

and

$$
\bar{b}_\tau = \theta\left(\tau - \frac{1 - e^{-\bar{k}\tau}}{\bar{k}}\right) = \theta\left(\tau - \bar{a}_\tau\right).
$$

Finally, the VIX squared is expressed by

$$
\mathrm{VIX}_t^2 = \left(\frac{\bar{a}_\tau}{\tau} v_t + \frac{\bar{b}_\tau}{\tau}\right) \times 100^2
$$

whence the claim.

□

3.4 VIX under SVJJ model

We now state and prove the analogue expression for the VIX squared under the SVJJ model.

THEOREM - 3.4.1. *The VIX squared, under the SVJJ model, is expressed as*

$$VIX_t^2 = (a_\tau v_t + b_\tau + c_\tau) \times 100^2 \tag{3.9}$$

where

$$a_\tau = \frac{1-e^{-\bar{k}\tau}}{\bar{k}\tau}$$
$$b_\tau = \left(\theta + \frac{\lambda\mu_v}{\bar{k}}\right)(1-a_\tau)$$
$$c_\tau = 2\lambda(c-\mu_S-\rho_Y\mu_v)$$

and $\tau := T-t = \frac{30}{365}$.

Proof. By Proposition 3.6, we have

$$\begin{aligned} VIX_t^2 &= -\frac{2}{T-t}E\left[\ln\left(\frac{S_T}{F}\right)\Big|\mathcal{F}_t\right] \times 100^2 \\ &= -\frac{2}{T-t}E\left[\ln(S_T) - \ln\left(S_t e^{(r-q)(T-t)}\right)\Big|\mathcal{F}_t\right] \times 100^2. \end{aligned}$$

Now, applying Itô's lemma A.3.2 to the stochastic process

$$f(t, S_t) = \ln(F)$$

under SVJJ model, we obtain

$$\begin{aligned} d\ln(F) &= \frac{1}{S_t e^{(r-q)(T-t)}} S_t e^{(r-q)(T-t)}(q-r)dt + \left((r-q)S_t - S_t c\lambda\right)\frac{1}{S_t e^{(r-q)(T-t)}} e^{(r-q)(T-t)} dt \\ &\quad + \frac{S_t^2 v_t}{2}\left(-\frac{1}{S_t^2}\right)dt + \frac{1}{S_t}S_t\sqrt{v_t}dW_t + \ln\left(e^{(r-q)(T-t)}(S_{t-}+\Delta S_t)\right) - \ln\left(e^{(r-q)(T-t)}S_{t-}\right) \\ &= -c\lambda dt - \frac{v_t}{2}dt + \sqrt{v_t}dW_t + \ln\left(1+\frac{\Delta S_t}{S_{t-}}\right) \\ &= \left(-c\lambda - \frac{v_t}{2}\right)dt + \sqrt{v_t}dW_t + \ln\left(1 + \frac{S_{t-}\left(e^{\Delta X_t}-1\right)}{S_{t-}}\right) \\ &= \left(-c\lambda - \frac{v_t}{2}\right)dt + \sqrt{v_t}dW_t + \Delta X_t \end{aligned}$$

that is

$$\ln(S_T) - \ln(S_t e^{(r-q)(T-t)}) = -\frac{1}{2}\int_t^T v_s\, ds + \int_t^T \sqrt{v_s}\, dW_s^{(S)} + \sum_{i\geq 1,\, t\leq T_i\leq T}\Delta X_i - c\lambda(T-t).$$

Hence

$$\begin{aligned} E\left[\ln(S_T) - \ln(S_t e^{(r-q)(T-t)})|\mathcal{F}_t\right] &= E\left[-\frac{1}{2}\int_t^T v_s\, ds + \int_t^T \sqrt{v_s}\, dW_s^{(S)} + \sum_{i\geq 1,\, t\leq T_i\leq T}\Delta X_i - c\lambda(T-t)\Big|\mathcal{F}_t\right] \\ &= E\left[-\frac{1}{2}\int_t^T v_s\, ds\Big|\mathcal{F}_t\right] + E\left[\int_t^T \sqrt{v_s}\, dW_s^{(S)}\Big|\mathcal{F}_t\right] \\ &\quad + E\left[\sum_{i\geq 1,\, t\leq T_i\leq T}\Delta X_i\Big|\mathcal{F}_t\right] - c\lambda(T-t) \end{aligned}$$

(by Theorem A.1.1 and Example A.18)

$$= E\left[-\frac{1}{2}\int_t^T v_s ds\middle|\mathcal{F}_t\right] + \lambda(T-t)E\left[\Delta X_T|\mathcal{F}_t\right] - c\lambda(T-t)$$

(since $\Delta X_T = Y^{(S)}$ and $Y^{(S)}|Y^{(v)} \sim \mathcal{N}(\mu_S + \rho_Y Y^{(v)}, \sigma_S^2)$)

$$\begin{aligned}
&= E\left[-\frac{1}{2}\int_t^T v_s ds\middle|\mathcal{F}_t\right] + \lambda(T-t)\left(\mu_S + \rho_Y\mu_v\right) - c\lambda(T-t) \\
&= E\left[-\frac{1}{2}\int_t^T v_s ds\middle|\mathcal{F}_t\right] + \lambda(T-t)\left(\mu_S + \rho_Y\mu_v - c\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
VIX_t^2 &= \left(\frac{1}{T-t}E\left[\int_t^T v_s ds\middle|\mathcal{F}_t\right] + 2\lambda(c - \mu_S - \rho_Y\mu_v)\right) \times 100^2 \\
&= \left(\frac{1}{T-t}E\left[\int_t^T v_s ds\middle|\mathcal{F}_t\right] + c_\tau\right) \times 100^2.
\end{aligned}$$

By applying Itô's lemma A.3.2 to to the stochastic process

$$f(t, v_t) = e^{\bar{k}t}v_t$$

under the SVJJ model, we have

$$\begin{aligned}
d\left(e^{\bar{k}t}v_t\right) &= \bar{k}e^{\bar{k}t}v_t dt + \bar{k}(\theta - v_t)e^{\bar{k}t}dt + \epsilon\sqrt{v_t}e^{\bar{k}t}dW_t + e^{\bar{k}t}\Delta v_t \\
&= (\bar{k}\theta + \lambda\mu_v)e^{\bar{k}t}dt + \epsilon\sqrt{v_t}e^{\bar{k}t}dW_t + e^{\bar{k}t}\Delta v_t - e^{\bar{k}t}\lambda\mu_v
\end{aligned}$$

that is

$$\begin{aligned}
e^{\bar{k}T}v_T - e^{\bar{k}t}v_t &= \theta\int_t^T \bar{k}e^{\bar{k}s}ds + \lambda\mu_v\int_t^T e^{\bar{k}s}\,ds + \epsilon\int_t^T e^{\bar{k}s}\sqrt{v_s}dW_s^{(v)} + \sum_{i\geq 1,\, t\leq T_i\leq T} e^{\bar{k}T_i}\Delta v_i - \lambda\mu_v\int_t^T e^{\bar{k}s}\,ds \\
&= \theta\left(e^{\bar{k}T} - e^{\bar{k}t}\right) + \frac{\lambda\mu_v}{\bar{k}}\left(e^{\bar{k}T} - e^{\bar{k}t}\right) + \epsilon\int_t^T e^{\bar{k}s}\sqrt{v_s}dW_s^{(v)} + \sum_{i\geq 1,\, t\leq T_i\leq T} e^{\bar{k}T_i}\Delta v_i - \frac{\lambda\mu_v}{\bar{k}}\left(e^{\bar{k}T} - e^{\bar{k}t}\right) \\
&= \left(e^{\bar{k}T} - e^{\bar{k}t}\right)\left(\theta + \frac{\lambda\mu_v}{\bar{k}}\right) + \epsilon\int_t^T e^{\bar{k}s}\sqrt{v_s}dW_s^{(v)} + \sum_{i\geq 1,\, t\leq T_i\leq T} e^{\bar{k}T_i}\Delta v_i - \frac{\lambda\mu_v}{\bar{k}}\left(e^{\bar{k}T} - e^{\bar{k}t}\right).
\end{aligned}$$

Hence we get

$$\begin{aligned}
v_T &= e^{-\bar{k}(T-t)}v_t + \left(\theta + \frac{\lambda\mu_v}{\bar{k}}\right)e^{-\bar{k}T}\left(e^{\bar{k}T} - e^{\bar{k}t}\right) + \epsilon e^{-\bar{k}T}\int_t^T e^{\bar{k}s}\sqrt{v_s}dW_s^{(v)} + e^{-\bar{k}T}\sum_{i\geq 1,\, t\leq T_i\leq T} e^{\bar{k}T_i}\Delta v_i \\
&\quad - e^{-\bar{k}T}\frac{\lambda\mu_v}{\bar{k}}\left(e^{\bar{k}T} - e^{\bar{k}t}\right) \\
&= e^{-\bar{k}(T-t)}v_t + \left(\theta + \frac{\lambda\mu_v}{\bar{k}}\right)\left(1 - e^{-\bar{k}(T-t)}\right) + \epsilon e^{-\bar{k}T}\int_t^T e^{\bar{k}s}\sqrt{v_s}dW_s^{(v)} + e^{-\bar{k}T}\sum_{i\geq 1,\, t\leq T_i\leq T} e^{\bar{k}T_i}\Delta v_i \\
&\quad - \frac{\lambda\mu_v}{\bar{k}}\left(1 - e^{\bar{k}(T-t)}\right)
\end{aligned}$$

that leads to the following expression of v_T

$$v_T = \alpha_\tau v_t + \beta_\tau + \epsilon e^{-\bar{k}T}\int_t^T e^{\bar{k}s}\sqrt{v_s}dW_s^{(v)} + e^{-\bar{k}T}\sum_{i\geq 1,\, t\leq T_i\leq T} e^{\bar{k}T_i}\Delta v_i - \frac{\lambda\mu_v}{\bar{k}}\left(1 - e^{\bar{k}(T-t)}\right)$$

where

$$\alpha_\tau = e^{-\bar{k}\tau}$$

and

$$\beta_\tau = \frac{\bar{k}\theta + \lambda\mu_v}{\bar{k}}\left(1 - e^{-\bar{k}\tau}\right).$$

The mean of the instantaneous variance is thus

$$\begin{aligned}
E\left[v_T|\mathcal{F}_t\right] &= E\left[\alpha_\tau v_t + \beta_\tau + \epsilon e^{-\bar{k}T}\int_t^T e^{\bar{k}s}\sqrt{v_s}dW_s^{(v)} + e^{-\bar{k}T}\sum_{i\geq 1,\ t\leq T_i\leq T} e^{\bar{k}T_i}\Delta v_i - \frac{\lambda\mu_v}{\bar{k}}\left(1-e^{\bar{k}(T-t)}\right)\Big|\mathcal{F}_t\right] \\
&= E\left[\alpha_\tau v_t + \beta_\tau|\mathcal{F}_t\right] + \epsilon e^{-\bar{k}T}E\left[\int_t^T e^{\bar{k}s}\sqrt{v_s}dW_s^{(v)}\Big|\mathcal{F}_t\right] + e^{-\bar{k}T}E\left[\sum_{i\geq 1,\ t\leq T_i\leq T} e^{\bar{k}T_i}\Delta v_i\Big|\mathcal{F}_t\right] \\
&\quad - \frac{\lambda\mu_v}{\bar{k}}\left(1-e^{\bar{k}(T-t)}\right)
\end{aligned}$$

(by Theorem A.1.1 [2], since $E\left[e^{\bar{k}T_i}Y^{(v)}|\mathcal{F}_t\right] = E\left[e^{\bar{k}T_i}|\mathcal{F}_t\right]E\left[Y^{(v)}|\mathcal{F}_t\right] = \frac{e^{\bar{k}\tau}}{\bar{k}}\mu_v$ and then by Example A.18)

$$= \alpha_\tau v_t + \beta_\tau.$$

We now proceed with the computation of the expectation of the integrated variance. By Tonelli's theorem, being the variance strictly positive, we get

$$\begin{aligned}
E\left[\int_t^T v_s ds\Big|\mathcal{F}_t\right] &= \int_t^T E\left[v_s|\mathcal{F}_t\right]ds \\
&= \int_t^T E\left[\alpha_{s-t}v_t + \beta_{s-t}|\mathcal{F}_t\right]ds \\
&= \int_t^T \left(\alpha_{s-t}v_t + \beta_{s-t}\right)ds \\
&= \int_t^T \alpha_{s-t}ds\ v_t + \int_t^T \beta_{s-t}ds \\
&= \int_t^T e^{-\bar{k}(s-t)}ds\ v_t + \int_t^T\left(\theta + \frac{\lambda\mu_v}{\bar{k}}\right)\left(1-e^{-\bar{k}(s-t)}\right)ds \\
&= \frac{1}{\bar{k}}v_t\left(e^{-\bar{k}(T-t)}-1\right) + \left(\theta + \frac{\lambda\mu_v}{\bar{k}}\right)(T-t) + \left(e^{-\bar{k}(T-t)}-1\right)\left(\frac{\theta}{\bar{k}} + \frac{\lambda\mu_v}{\bar{k}^2}\right) \\
&= v_t\left(\frac{1-e^{-\bar{k}(T-t)}}{\bar{k}}\right) + \left(\theta + \frac{\lambda\mu_v}{\bar{k}}\right)\left(\tau - \frac{1-e^{-k(T-t)}}{\bar{k}}\right)
\end{aligned}$$

that leads to the following expression

$$E\left[\int_t^T v_s ds\Big|\mathcal{F}_t\right] = \bar{a}_\tau v_t + \bar{b}_\tau$$

where

$$\bar{a}_\tau = \frac{1-e^{-\bar{k}\tau}}{\bar{k}}$$

[2] $e^{\bar{k}t}\sqrt{v_t}$ belongs to $\mathbb{L}^2$, indeed

$$E\left[\int_0^T (e^{\bar{k}s}\sqrt{v_s})^2\,ds\right] \leq e^{2\bar{k}T}E\left[\int_0^T v_s\,ds\right] < \infty$$

as v_t belongs to $\mathbb{L}^2$ by the assumptions on the SVJJ model.

and

$$\bar{b}_\tau = \left(\theta + \frac{\lambda \mu_v}{\bar{k}}\right)\left(\tau - \frac{1 - e^{-k\tau}}{\bar{k}}\right).$$

Finally, the VIX squared is expressed by

$$\mathrm{VIX}_t^2 = \left(\frac{\bar{a}_\tau}{\tau} v_t + \frac{\bar{b}_\tau}{\tau} + c_\tau\right) \times 100^2$$

whence the claim.

□

4
GAMMA-LAGUERRE EXPANSIONS

Under the risk-neutral valuation framework, the risk-neutral probability measure is a crucial ingredient for asset valuation, since the value of a financial derivative is given by the expected value, with respect to the risk-neutral measure, of the future payoff, corresponding to the derivative, discounted at the risk-free interest rate. Therefore the valuation of a European option translates, in mathematical terms, into the computation of the following integral

$$E^Q\left[H(S_T)\right] = \int_0^\infty H(x) f(x)\, dx$$

where f and H denote the density of the risk-neutral measure Q and the payoff function, respectively. The computation of the expectation above for an arbitrary underlying distribution may be possible only by numerical integration techniques because of the analytical intractability of the distribution function. Nevertheless, very often the most popular numerical methods, although theoretically correct, reveal their inefficiency from the practical point of view. An alternative, appealing approach is to approximate the underlying distribution with an alternate, and more tractable, distribution. Jarrow and Rudd (1982) [10] pioneered the density expansion approach to option pricing using an Edgeworth series expansion of the terminal underlying asset price risk-neutral density around the log-normal density. This approach, similar to the familiar Taylor series expansion for an analytic function, has the desirable property that the coefficients in the expansion are simple functions of the moments of the approximating distribution. Subsequently, Corrado and Su (1996) [4] adopted the Jarrow-Rudd framework and derived an option pricing formula using a Gram-Charlier type A series expansion of the underlying asset log-return risk-neutral density around the Gaussian density. Recently, Drimus, Necula and Farkas (2013) [5] developed a new method to retrieve the risk-neutral probability measure and to derive an option pricing formula by employing a modified Gram-Charlier type A series expansion, replacing the "probabilists" Hermite polynomials by the "physicists" Hermite polynomials. In this work we use instead sums of polynomials weighted by a Gamma density function ϕ; more precisely, we choose a family of polynomials $\left(p_k\right)_{k\in\mathbb{N}}$ such that

$$\int_0^\infty p_i(x) p_j(x) \phi^2(x)\, dx = C\delta_{ij}$$

and

$$\int_0^\infty \phi(x) \sum_{k=0}^{n} c_k p_k(x) H(x)\, dx \xrightarrow[n\to+\infty]{} E^Q\left[H(S_T)\right] \tag{4.1}$$

for some constant C and some real sequence $(c_k)_{k\in\mathbb{N}}$.

Now, we choose the space of square integrable functions as the functional space where our expansion takes place; roughly speaking, we will expand the density function f in $L^2(\mathbb{R}^+)$ and show that the quadratic convergence of the expansion will lead to the convergence of the integrals in (4.1). The first assumption is obviously that f belongs to $L^2(\mathbb{R})$; we recall that the condition (2.7) is sufficient, but not necessary, for $\hat{f}$ to belong to L^2, whence for f to belong to L^2 as well, the Fourier transform being a linear isometry from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$.

Notation 4.1. The following notation will be needed throughout the chapter: we denote by ν the Borel regular measure on $\mathbb{R}$ with density (with respect to Lebesgue measure) given by

$$d\nu(x) = \mathbb{1}_{\mathrm{supp}(\phi)}(x)\,dx.$$

THEOREM - 4.0.2. *Let $(p_k)_{k\in\mathbb{N}}$ be a sequence of orthogonal polynomial functions and $(c_k)_{k\in\mathbb{N}}$ a real sequence such that*

$$||f-\phi\sum_{k=0}^{n}c_kp_k||_{2,\nu}\xrightarrow[n\to+\infty]{}0. \tag{4.2}$$

Then, denoting by q_n and q

$$\begin{aligned} q &:= \int_{\mathbb{R}} f(x)H(x)\,dx \\ q_n &:= \int_{\mathbb{R}} \phi(x)\sum_{k=0}^{n}c_kp_k(x)H(x)\,dx \end{aligned}$$

respectively, it follows that, for every payoff function H such that $H\in L^2_\nu(\mathbb{R})$,

$$q_n\xrightarrow[n\to+\infty]{}q.$$

Proof. We have

$$|q-q_n| = \Big|\int_{\mathbb{R}} H(x)\Big(f(x)-\phi(x)\sum_{k=0}^{n}c_kp_k(x)\Big)\,dx\Big|$$

and for every payoff function H such that $H\in L^2_\nu(\mathbb{R})$, by the Cauchy-Schwarz inequality

$$\begin{aligned} &\le ||H||_{2,\nu}\Big|\Big|f-\phi\sum_{k=0}^{n}c_kp_k\Big|\Big|_{2,\nu} \\ &= C\Big|\Big|f-\phi\sum_{k=0}^{n}c_kp_k\Big|\Big|_{2,\nu} \end{aligned}$$

for some constant C. Therefore, in view of (4.2), we may conclude that

$$\Big|\int_{\mathbb{R}} f(x)H(x)\,dx-\int_{\mathbb{R}}\phi(x)\sum_{k=0}^{n}c_kp_k(x)H(x)\,dx\Big|\xrightarrow[n\to+\infty]{}0.$$

□

4.1 The Gamma choice

To adapt the density expansion approach to the context of volatility options, we choose as ϕ a Gamma density function. This leads to the natural choice of Laguerre polynomials as $(p_k)_{k\in\mathbb{N}}$ since they are orthogonal over $[0,+\infty)$ with respect to the measure with weighting function the Gamma distribution.

Definition 4.2. For any $\alpha > 0, \beta > 0$, the distribution with density

$$\gamma(\alpha,\beta;x) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1} \mathbb{1}_{x\geq 0}(x), \qquad x \in \mathbb{R},$$

is called *Gamma distribution* with parameters α and β.

Definition 4.3. We define *Laguerre polynomial* of degree k and parameter α the polynomial $L_k^{(\alpha)} \in C^\infty(\mathbb{R})$ of the following form

$$L_k^{(\alpha)}(x) = \sum_{i=0}^{k} \frac{(-1)^i}{i!} \begin{pmatrix} k+\alpha \\ k-i \end{pmatrix} x^i.$$

Definition 4.4. We define *(generalized) Laguerre polynomial* of degree k and parameters (α,β) the polynomial $L_k^{(\alpha,\beta)} \in C^\infty(\mathbb{R})$ of the following form

$$L_k^{(\alpha,\beta)}(x) = \sum_{i=0}^{k} \frac{(-\beta)^i}{i!} \begin{pmatrix} k+\alpha \\ k-i \end{pmatrix} x^i.$$

Remark 4.5. From the definitions above, it immediately follows that

$$L_k^{(\alpha,\beta)}(x) = L_k^{(\alpha)}(\beta x).$$

Now, choosing the weighting function ϕ^2 as follows

$$\phi^2(\alpha,\beta;x) = C^2_{\alpha,\beta} e^{-\beta x} x^\alpha \mathbb{1}_{x\geq 0}(x)$$

where

$$C^2_{\alpha,\beta} = \frac{\left[\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+1}\right]^2}{\Gamma^2(\frac{\alpha}{2}+1)}$$

it entails ϕ to be a Gamma distribution with parameters $\frac{\alpha}{2}+1$ and $\frac{\beta}{2}$:

$$\begin{aligned}\phi(\alpha,\beta;x) &= C_{\alpha,\beta} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}} \mathbb{1}_{x\geq 0}(x) \\ &= \frac{\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+1}}{\Gamma(\frac{\alpha}{2}+1)} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}} \mathbb{1}_{x\geq 0}(x) = \gamma\left(\frac{\alpha}{2}+1, \frac{\beta}{2}; x\right).\end{aligned}$$

Definition 4.6. We call *Gamma-Laguerre function* of degree k and parameters (α,β) the function $\mathcal{L}_k^{(\alpha,\beta)} \in C^\infty(\mathbb{R})$ given by

$$\mathcal{L}_k^{(\alpha,\beta)}(x) = \phi(\alpha,\beta;x) \sum_{i=0}^{k} \frac{(-\beta)^i}{i!} \begin{pmatrix} k+\alpha \\ k-i \end{pmatrix} x^i.$$

THEOREM - 4.1.1 (Completeness and orthogonality of Laguerre polynomials in L^2)**.**
For every $\alpha > -1$ the family $\left(L_k^{(\alpha)}\right)_{k\in\mathbb{N}}$ is an orthogonal and complete system for the space $L_\nu^2(\mathbb{R})$ with respect to the weight function $e^{-x}x^\alpha$. In particular

$$\int_0^{+\infty} L_i^{(\alpha)}(x)L_j^{(\alpha)}(x)e^{-x}x^\alpha\,dx = \Gamma(\alpha+1)\begin{pmatrix} i+\alpha \\ i \end{pmatrix}\delta_{ij} \tag{4.3}$$

for every $i, j \in \mathbb{N}$.

Proof. First of all we remark that proving that, for every $\alpha > -1$, the family $\left(L_k^{(\alpha)}\right)_{k\in\mathbb{N}}$ is an orthogonal and complete system for the space $L_\nu^2(\mathbb{R})$ with respect to the weight function $e^{-x}x^\alpha$ is equivalent to show that the functions $\left(e^{-\frac{x}{2}}x^{\frac{\alpha}{2}}L_k^{(\alpha)}\right)_{k\in\mathbb{N}}$ are a complete orthogonal system in $L_\nu^2(\mathbb{R})$ which in turn corresponds to demonstrate that the orthogonal functions $f_k(x) = e^{-\frac{x}{2}}x^{\frac{\alpha}{2}}x^k$ span a dense subspace of $L_\nu^2(\mathbb{R})$. In particular, we recall that proving that the system is orthogonal and complete corresponds to showing that the same system represents an orthogonal basis for $L_\nu^2(\mathbb{R})$, where the notion of orthogonal basis from linear algebra has been generalized to the case of Hilbert spaces. Completeness of an orthogonal family of functions is a bit tricky on unbounded intervals, while it is relatively straightforward on bounded intervals: in our case there is a nice trick due to von Neumann that allows the reduction to bounded intervals. Let $e^{-\frac{x}{2}}x^{\frac{\alpha}{2}}f(x)$ be a function in $L_\nu^2(\mathbb{R})$, that is

$$\int_0^{+\infty} e^{-x}x^\alpha f^2(x)\,dx < +\infty.$$

The first idea is to use the change of variable $y = e^{-x}$ to reduce to the case of $L^2(0,1)$, whence we obtain

$$\int_0^1 \ln\left(\frac{1}{y}\right)^\alpha f^2\left(\ln\left(\frac{1}{y}\right)\right)\,dy$$

that is the function $\ln\left(\frac{1}{y}\right)^{\frac{\alpha}{2}} f\left(\ln(\frac{1}{y})\right)$ belongs to $L^2(0,1)$. Now, since the set of continuous functions is dense in L^2 on a bounded interval, every function in $L^2(0,1)$ can be approximated in L^2-norm by a continuous function which in turn can be approximated, in view of the Weierstrass approximation theorem, by a polynomial in the sup norm. That being so, it is legitimate to approximate $\ln\left(\frac{1}{y}\right)^{\frac{\alpha}{2}} f\left(\ln(\frac{1}{y})\right)$ by functions of the form $\ln\left(\frac{1}{y}\right)^{\frac{\alpha}{2}} p(y)$ where p is a polynomial. Transforming back to $(0,+\infty)$ this entails that, for any $\epsilon > 0$, a polynomial $p(y)$ can be determined so that

$$\int_0^{+\infty} e^{-x}x^\alpha\left(f(x) - p(e^{-x})\right)^2\,dx < \epsilon.$$

Hence this reduces the task to prove that, for all non-negative integer m and for any $\delta > 0$, there exists a polynomial q such that

$$\int_0^{+\infty} e^{-x}x^\alpha\left(e^{-mx} - q(x)\right)^2\,dx < \delta. \tag{4.4}$$

For this purpose, von Neumann's trick is to use the generating function of the Laguerre polynomials $L_k^{(\alpha)}$

$$(1-w)^{-\alpha-1}\exp\left(-\frac{xw}{1-w}\right) = \sum_{k=0}^{+\infty} L_k^{(\alpha)}(x)w^k$$

where $w = \frac{k}{k+1}$ so that $\exp\left(-\frac{xw}{1-w}\right) = \exp(-kx)$. Thus a natural choice for q is

$$q_n = (1-w)^{\alpha+1}\sum_{k=0}^{n} L_k^{(\alpha)}(x)w^k$$

with n sufficiently large. Plugging this in (4.4), we get

$$\int_0^{+\infty} e^{-x}x^{\alpha}\left(e^{-mx}-q_n(x)\right)^2\,dx = (1-w)^{2\alpha+2}\int_0^{\infty} e^{-x}x^{\alpha}\left(\sum_{k=n+1}^{+\infty} L_k^{\alpha}(x)w^k\right)^2\,dx$$

(by the orthogonality of the Laguerre polynomials)

$$(1-w)^{2\alpha+2}\int_0^{\infty} e^{-x}x^{\alpha}\sum_{k=n+1}^{+\infty}\left(L_k^{\alpha}(x)w^k\right)^2\,dx$$

(by Fubini's theorem [1])

$$= (1-w)^{2\alpha+2}\sum_{k=n+1}^{+\infty} w^{2k}\int_0^{\infty} e^{-x}x^{\alpha}\left(L_k^{\alpha}(x)\right)^2\,dx$$

(by 4.5)

$$= (1-w)^{2\alpha+2}\Gamma(\alpha+1)\sum_{k=n+1}^{+\infty}\left(\begin{array}{c} k+\alpha \\ k\end{array}\right)w^{2k}.$$

Therefore it remains to observe that the last term becomes arbitrarily small when n is sufficiently large, and this establishes the statement.

□

Corollary 4.1.2. *For every $\alpha > -1, \beta > 0$ the family $\left(L_k^{(\alpha,\beta)}\right)_{k\in\mathbb{N}}$ is an orthogonal and complete system for the space $L^2_\nu(\mathbb{R})$ with respect to the weight function $\phi^2(\alpha,\beta;x)$. In particular*

$$\int_0^{+\infty} L_i^{(\alpha,\beta)}(x)L_j^{(\alpha,\beta)}(x)\phi^2(\alpha,\beta;x)\,dx = \Gamma(\alpha+1)\left(\begin{array}{c} i+\alpha \\ i\end{array}\right)\frac{C_{\alpha,\beta}}{\beta^{\alpha+1}}\delta_{ij} \tag{4.5}$$

for every $i, j \in \mathbb{N}$.

Proof. By the change of variable $\beta x = x$ in (4.5), we have

$$\begin{aligned}
\Gamma(\alpha+1)\left(\begin{array}{c} i+\alpha \\ i\end{array}\right)\delta_{ij} &= \int_0^{+\infty} L_i^{(\alpha)}(x)L_j^{(\alpha)}(x)e^{-x}x^{\alpha}\,dx \\
&= \int_0^{+\infty} L_i^{(\alpha)}(\beta x)L_j^{(\alpha)}(\beta x)e^{-\beta x}(\beta x)^{\alpha}\beta\,dx \\
&= \int_0^{+\infty} L_i^{(\alpha,\beta)}(x)L_j^{(\alpha,\beta)}(x)e^{-\beta x}x^{\alpha}\beta^{\alpha+1}\,dx \\
&= \frac{\beta^{\alpha+1}}{C_{\alpha,\beta}}\int_0^{+\infty} L_i^{(\alpha,\beta)}(x)L_j^{(\alpha,\beta)}(x)C_{\alpha,\beta}e^{-\beta x}x^{\alpha}\,dx \\
&= \frac{\beta^{\alpha+1}}{C_{\alpha,\beta}}\int_0^{+\infty} L_i^{(\alpha,\beta)}(x)L_j^{(\alpha,\beta)}(x)\phi^2(\alpha,\beta;x)\,dx
\end{aligned}$$

whence the claim.

□

[1] By the Cauchy-Schwarz inequality:

$$\int_0^{+\infty} e^{-x}x^{\alpha}|L_k^{\alpha}(x)||L_{k'}^{\alpha}(x)|\,dx \le \left(\int_0^{+\infty} e^{-x}x^{\alpha}\left(L_k^{\alpha}(x)\right)^2\,dx\right)^{\frac{1}{2}}\left(\int_0^{+\infty} e^{-x}x^{\alpha}\left(L_{k'}^{\alpha}(x)\right)^2\,dx\right)^{\frac{1}{2}} < \infty.$$

Corollary 4.1.3. *For every $\alpha > -1, \beta > 0$ the family $\left(\mathcal{L}_k^{(\alpha,\beta)}\right)_{k\in\mathbb{N}}$ is an orthogonal and complete system for the space $L^2_\nu(\mathbb{R})$. In particular*

$$\int_0^{+\infty} \mathcal{L}_i^{(\alpha,\beta)}(x)\mathcal{L}_j^{(\alpha,\beta)}(x)\,dx = \Gamma(\alpha+1)\binom{i+\alpha}{i}\frac{C_{\alpha,\beta}}{\beta^{\alpha+1}}\delta_{ij} \tag{4.6}$$

for every $i, j \in \mathbb{N}$.

As a consequence of the previous result, we also have the following useful:

Remark 4.7. For every probability density function f satisfying

(i) $f \in L^2_\nu(\mathbb{R}) = L^2((0,+\infty))$

(ii) $\operatorname{supp}(f) \subseteq \operatorname{supp}(\phi) = (0,+\infty)$

we can find a real sequence $(c_k)_{k\in\mathbb{N}}$ such that

$$||f - \sum_{k=0}^{n} c_k \mathcal{L}_k^{(\alpha,\beta)}||_{2,\nu} \xrightarrow[n\to+\infty]{} 0 \tag{4.7}$$

and, for every payoff function $H \in L^2_\nu(\mathbb{R})$, Theorem 4.0.2 ensures that

$$\Big|\int_{\mathbb{R}} f(x)H(x)\,dx - \int_{\mathbb{R}}\sum_{k=0}^{n} c_k \mathcal{L}_k^{(\alpha,\beta)}(x)H(x)\,dx\Big| \xrightarrow[n\to+\infty]{} 0.$$

Furthermore, we have, for every $k \in \mathbb{N}$

$$c_k = \frac{\langle f, \mathcal{L}_k^{(\alpha,\beta)}\rangle_{2,\nu}}{\langle \mathcal{L}_k^{(\alpha,\beta)}, \mathcal{L}_k^{(\alpha,\beta)}\rangle_{2,\nu}} = \frac{\langle f, \mathcal{L}_k^{(\alpha,\beta)}\rangle_{2,\nu}}{||\mathcal{L}_k^{(\alpha,\beta)}||^2_{2,\nu}}. \tag{4.8}$$

Note. (4.7) can be also expressed by saying that f can be represented as a *Gamma-Laguerre series expansion.* The truncated series

$$f(x) = \sum_{k=0}^{n} c_k L_k^{(\alpha,\beta)}(x)\phi(x)$$

may be viewed as the Gamma probability density function multiplied by some polynomials that account for the effects of departure from a strict Gamma behaviour of the variance risk-neutral density function.

4.1.1 Applications

In order to illustrate the convergence of the Gamma-Laguerre expansion to a target probability density function we present two illustrative examples based on the Inverse Gaussian (IG) distribution and the density of the Heston model.

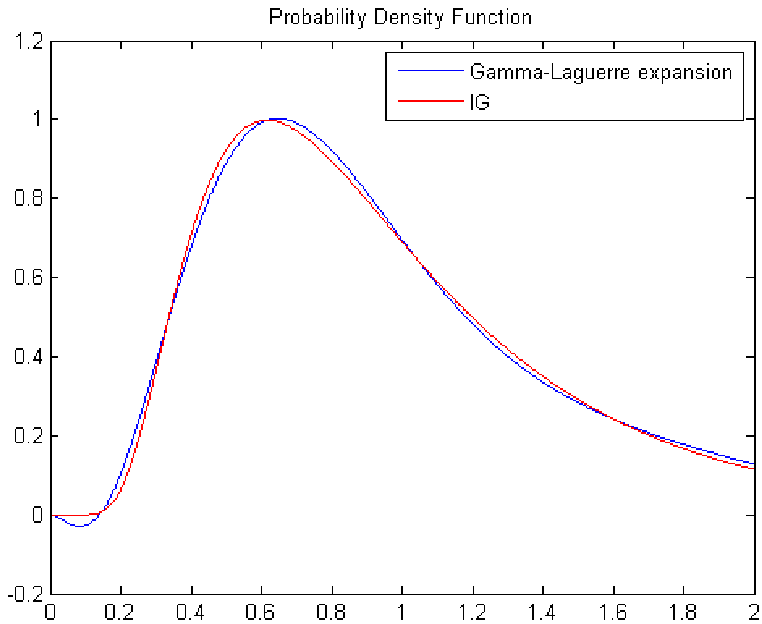
Example 4.8 (Inverse Gaussian distribution)**.**
For any $\mu, \lambda > 0$, the distribution with density

$$f(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} \exp\left(\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right) \mathbb{1}_{x\geq 0}(x)$$

is called *Inverse Gaussian (IG) distribution* with parameters μ, λ and denoted by $IG(\mu, \lambda)$.

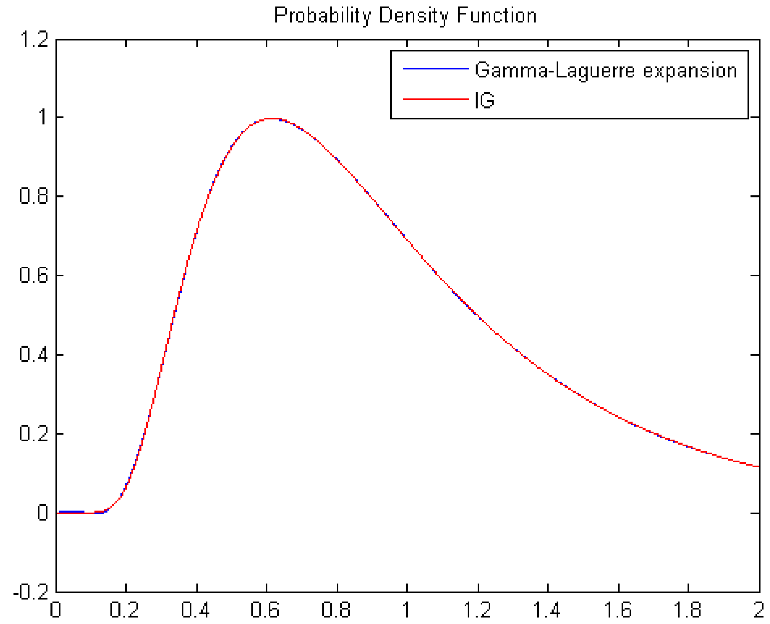
Figures 4.1, 4.2 below depict the probability distribution function of the target distribution (IG) with parameters $\mu = 1$ and $\lambda = 3$ and the Gamma-Laguerre approximation truncated after 5 terms and 25 terms, respectively.

Figure 4.1: The Gamma-Laguerre approximation of the IG distribution after 5 expansion terms

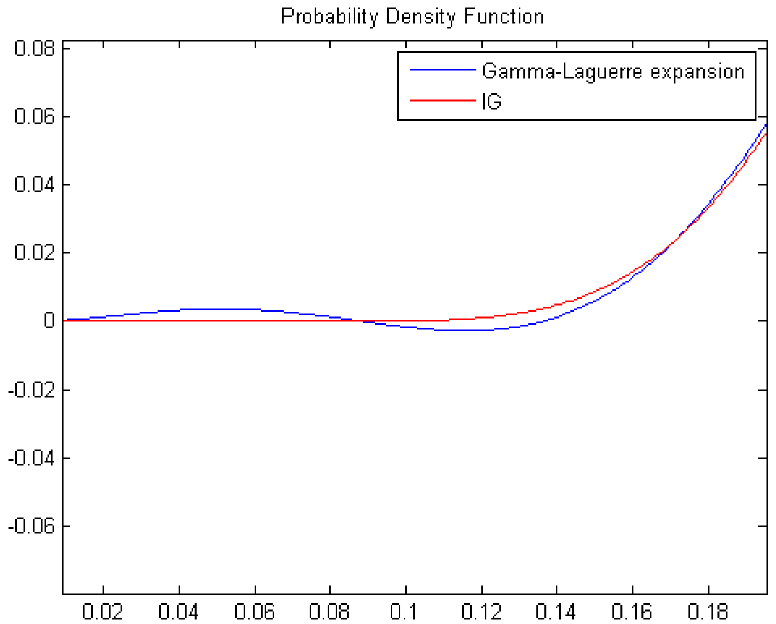


The graph above shows that after 5 terms the Gamma-Laguerre expansion does not represent a good approximation for the target density function, whilst increasing the expansion order to 25 the approximation is notably enhanced, as shown in Figure 4.2 below. Nevertheless, if we confine our attention to a restricted area around zero, some problems pop out: indeed, the Gamma-Laguerre expansion is an excellent method for approximating densities away from zero. As we will see, this behaviour around zero, being inherent of the Gamma-Laguerre expansions, suggests that we could have some problems in using this approximation method in pricing options with the great part of their mass concentrated around zero.

Figure 4.2: The Gamma-Laguerre approximation of the IG distribution after 25 expansion terms



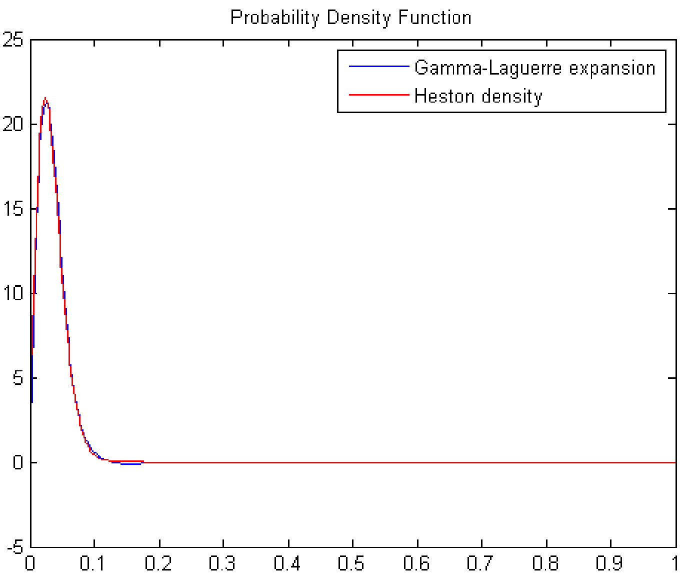
(a)

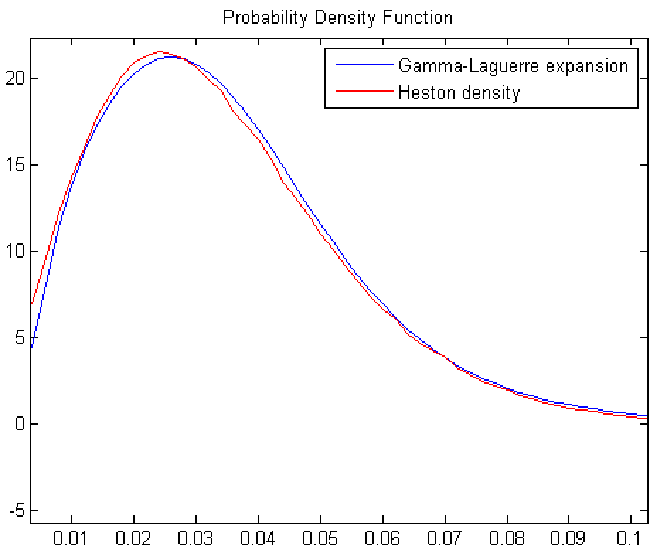


(b) Behaviour around zero

Example 4.9 (Heston density)**.**
We want now to show how well the Gamma-Laguerre expansion approximates the Heston density. Here, the Heston density has been obtained by a simulation based on a Euler scheme with 10^6 realizations of a CIR process and $\frac{1}{10^3}$ month as discretization parameter. Figures 4.3, 4.4 below depict the probability distribution function of the target distribution (Heston) and the Gamma-Laguerre approximation truncated after 5 terms and 25 terms, respectively.

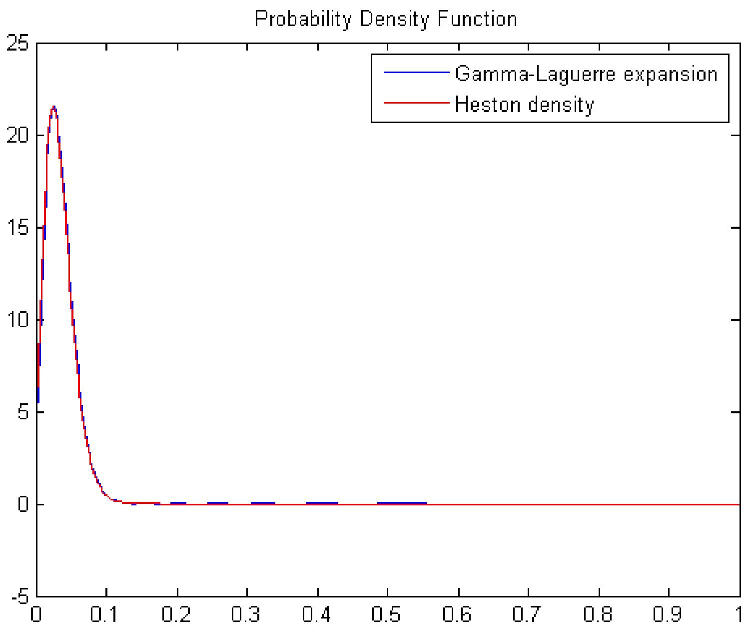
Figure 4.3: The Gamma-Laguerre approximation of the Heston distribution after 5 expansion terms



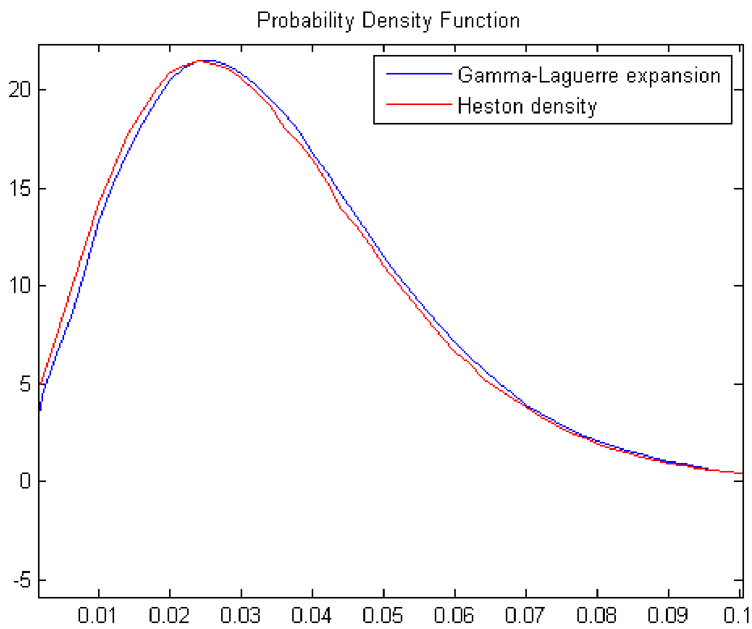


(a) Behaviour around zero

Figure 4.4: The Gamma-Laguerre approximation of the Heston distribution after 25 expansion terms



(a)



(b) Behaviour around zero

Results of convergence for the Heston model are considerably precise even at low orders of expansion: the performance of the approximation of order 25 is, indeed, just slightly better than the one with 5 expansion steps. However, the problematic behaviour around zero is notable and, as shown in the graphs above, by it being an inherent issue of the Gamma-Laguerre expansion, cannot be fixed by increasing the order of approximation to 25. These two considerations lead us to remark that, firstly, the idea of approximating the Heston density by means of a Gamma-Laguerre expansion reveals to be a suitable choice since the expansion, even at low orders, adequately reproduces the dynamics of a CIR process; secondly, as already remarked in the previous example, the behaviour around zero may produce some inaccuracy in the pricing approximation of options whose mass is concentrated around zero, as in the case of Put options. We will later see how this latter empirical fact will have a higher impact to the final results than the theoretical results of convergence of the Gamma-Laguerre expansion method previously studied.

4.2 Laguerre-Gamma expansion coefficients

In order for this approximation to be useful one needs to know, in closed form, the factors c_k. By the following computation, the expansion coefficients c_k can be carried out explicitly in a final closed form. Indeed we have

$$c_k = \frac{1}{||\mathcal{L}_k^{(\alpha,\beta)}||_{2,\nu}^2}\int_{\mathbb{R}} f(x)\mathcal{L}_k^{(\alpha,\beta)}(x)\,dx$$

(by (4.6))

$$= \frac{\beta^{\alpha+1}}{C_{\alpha,\beta}\Gamma(\alpha+1)\begin{pmatrix} k+\alpha \\ k\end{pmatrix}}\int_0^{+\infty} f(x)\sqrt{C_{\alpha,\beta}}e^{-\frac{\beta}{2}x}x^{\frac{\alpha}{2}}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j\end{pmatrix}x^j\,dx$$

$$= \frac{\beta^{\alpha+1}}{\sqrt{C_{\alpha,\beta}}\begin{pmatrix} k+\alpha \\ k\end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j\end{pmatrix}\int_0^{+\infty} f(x)e^{-\frac{\beta}{2}x}x^{\frac{\alpha}{2}}x^j\,dx$$

(if $\hat{f}\in L^1(\mathbb{R})$ [2], by the Fourier inversion formula)

$$= \frac{\beta^{\alpha+1}}{\sqrt{C_{\alpha,\beta}}\begin{pmatrix} k+\alpha \\ k\end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j\end{pmatrix}\int_{\mathbb{R}}\mathbb{1}_{x\geq 0}(x)\left(\frac{1}{2\pi}\int_{\mathbb{R}}e^{-ix\xi}\hat{f}(\xi)\,d\xi\right)e^{-\frac{\beta}{2}x}x^{\frac{\alpha}{2}}x^j\,dx$$

$$= \frac{\beta^{\alpha+1}}{2\pi\sqrt{C_{\alpha,\beta}}\begin{pmatrix} k+\alpha \\ k\end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j\end{pmatrix}\int_{\mathbb{R}}\mathbb{1}_{x\geq 0}(x)\int_{\mathbb{R}}e^{-ix\xi}\hat{f}(\xi)\,d\xi\,e^{-\frac{\beta}{2}x}x^{\frac{\alpha}{2}}x^j\,dx$$

(changing the order of integration, by Fubini's Theorem [3])

$$= \frac{\beta^{\alpha+1}}{2\pi\sqrt{C_{\alpha,\beta}}\begin{pmatrix} k+\alpha \\ k\end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j\end{pmatrix}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{-ix\xi}e^{-\frac{\beta}{2}x}x^{\frac{\alpha}{2}}x^j\mathbb{1}_{x\geq 0}(x)\,dx\,\hat{f}(\xi)d\xi$$

$$= \frac{\beta^{\alpha+1}}{2\pi\sqrt{C_{\alpha,\beta}}\begin{pmatrix} k+\alpha \\ k\end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j\end{pmatrix}\int_{\mathbb{R}}\mathcal{F}\left(e^{-\frac{\beta}{2}x}x^{\frac{\alpha}{2}+j}\mathbb{1}_{x\geq 0}(x)\right)(-\xi)\hat{f}(\xi)d\xi$$

$$= \frac{\beta^{\alpha+1}}{2\pi\sqrt{C_{\alpha,\beta}}\begin{pmatrix} k+\alpha \\ k\end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j\end{pmatrix}\frac{\Gamma\left(\frac{\alpha}{2}+j+1\right)}{\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+j+1}}\int_{\mathbb{R}}\mathcal{F}\left(\gamma\left(\frac{\alpha}{2}+j+1,\frac{\beta}{2}\right)\right)(-\xi)\hat{f}(\xi)d\xi$$

$$= \frac{\beta^{\alpha+1}}{2\pi\sqrt{C_{\alpha,\beta}}\begin{pmatrix} k+\alpha \\ k\end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j\end{pmatrix}\frac{\Gamma\left(\frac{\alpha}{2}+j+1\right)}{\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+j+1}}\int_{\mathbb{R}}\left(1+\frac{2i\xi}{\beta}\right)^{-\frac{\alpha}{2}-j-1}\hat{f}(\xi)d\xi.$$

[2] By Remark 2.2, if the Feller condition holds then $\hat{f}$ belongs to L^1.

[3] If $\hat{f}\in L^1(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}\left|e^{-ix\xi}\hat{f}(\xi)\right|\,d\xi\,e^{-\frac{\beta}{2}x}x^{\frac{\alpha}{2}}x^j\mathbb{1}_{x\geq 0}(x)\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}}\left|\hat{f}(\xi)\right|\,d\xi\,e^{-\frac{\beta}{2}x}x^{\frac{\alpha}{2}}x^j\mathbb{1}_{x\geq 0}(x)\,dx\\ &= \int_{\mathbb{R}}||\hat{f}(\xi)||_1 e^{-\frac{\beta}{2}x}x^{\frac{\alpha}{2}}x^j\mathbb{1}_{x\geq 0}(x)\,dx\\ &= ||\hat{f}(\xi)||_1\int_{\mathbb{R}}e^{-\frac{\beta}{2}x}x^{\frac{\alpha}{2}}x^j\mathbb{1}_{x\geq 0}(x)\,dx<\infty.\end{aligned}$$

4.2.1 Heston model

Under Heston model, in view of Corollary 2.1.2, the expansion coefficients take the following form:

$$c_k = \frac{\beta^{\alpha+1}}{2\pi\sqrt{C_{\alpha,\beta}}\begin{pmatrix} k+\alpha \\ k \end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j \end{pmatrix}\frac{\Gamma\left(\frac{\alpha}{2}+j+1\right)}{\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+j+1}}\int_{\mathbb{R}}\left(1+\frac{2i\xi}{\beta}\right)^{-\frac{\alpha}{2}-j-1} e^{a_1(i\xi,\tau)+a_2(i\xi,\tau)v_t}d\xi$$

where

$$a_1(i\xi,\tau) = \frac{-2\bar{k}\theta}{\epsilon^2}\ln\left(1+\frac{\epsilon^2 i\xi}{2\bar{k}}\left(e^{-\bar{k}\tau}-1\right)\right)$$

$$a_2(i\xi,\tau) = \frac{2\bar{k}i\xi}{\epsilon^2 i\xi+(2\bar{k}-\epsilon^2 i\xi)e^{\bar{k}\tau}}.$$

Finally, we get

$$\begin{aligned} c_k =& \frac{\beta^{\alpha+1}}{2\pi\sqrt{C_{\alpha,\beta}}\begin{pmatrix} k+\alpha \\ k \end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j \end{pmatrix}\frac{\Gamma\left(\frac{\alpha}{2}+j+1\right)}{\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+j+1}}\int_{\mathbb{R}}\left(1+\frac{2i\xi}{\beta}\right)^{-\frac{\alpha}{2}-j-1} \\ &\left(1+\frac{\epsilon^2 i\xi}{2\bar{k}}\left(e^{-\bar{k}\tau}-1\right)\right)^{\frac{-2\bar{k}\theta}{\epsilon^2}} e^{\frac{2\bar{k}i\xi}{\epsilon^2 i\xi+(2\bar{k}-\epsilon^2 i\xi)e^{\bar{k}\tau}}v_t}d\xi. \end{aligned} \tag{4.9}$$

4.2.2 SVJJ model

Under the SVJJ model, in view of Corollary 2.1.4, the expansion coefficients can be written in the following explicit expression:

$$\begin{aligned} c_k =& \frac{\beta^{\alpha+1}}{2\pi\sqrt{C_{\alpha,\beta}}\begin{pmatrix} k+\alpha \\ k \end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j \end{pmatrix}\frac{\Gamma\left(\frac{\alpha}{2}+j+1\right)}{\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+j+1}}\int_{\mathbb{R}}\left(1+\frac{2i\xi}{\beta}\right)^{-\frac{\alpha}{2}-j-1} \\ & e^{a_1(i\xi,\tau)+a_2(i\xi,\tau)v_t+a_3(i\xi,\tau)}d\xi \end{aligned}$$

where

$$a_1(i\xi,\tau) = \frac{-2\bar{k}\theta}{\epsilon^2}\ln\left(1+\frac{\epsilon^2 i\xi}{2\bar{k}}\left(e^{-\bar{k}\tau}-1\right)\right)$$

$$a_2(i\xi,\tau) = \frac{2\bar{k}i\xi}{\epsilon^2 i\xi+(2\bar{k}-\epsilon^2 i\xi)e^{\bar{k}\tau}}$$

$$a_3(i\xi,\tau) = \frac{2\mu_v\lambda}{2\mu_v\bar{k}-\epsilon^2}\ln\left(1+\frac{\left(\epsilon^2-2\mu_v\bar{k}\right)i\xi}{2\bar{k}\left(1-\mu_v i\xi\right)}\left(e^{-\bar{k}\tau}-1\right)\right).$$

Finally, we get

$$\begin{aligned} c_k =& \frac{\beta^{\alpha+1}}{2\pi\sqrt{C_{\alpha,\beta}}\begin{pmatrix} k+\alpha \\ k \end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix} k+\alpha \\ k-j \end{pmatrix}\frac{\Gamma\left(\frac{\alpha}{2}+j+1\right)}{\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+j+1}}\int_{\mathbb{R}}\left(1+\frac{2i\xi}{\beta}\right)^{-\frac{\alpha}{2}-j-1} \\ &\left(1+\frac{\epsilon^2 i\xi}{2\bar{k}}\left(e^{-\bar{k}\tau}-1\right)\right)^{\frac{-2\bar{k}\theta}{\epsilon^2}} e^{\frac{2\bar{k}i\xi}{\epsilon^2 i\xi+(2\bar{k}-\epsilon^2 i\xi)e^{\bar{k}\tau}}v_t}\left(1+\frac{\left(\epsilon^2-2\mu_v\bar{k}\right)i\xi}{2\bar{k}\left(1-\mu_v i\xi\right)}\left(e^{-\bar{k}\tau}-1\right)\right)^{\frac{2\mu_v\lambda}{2\mu_v\bar{k}-\epsilon^2}}d\xi. \end{aligned} \tag{4.10}$$

5

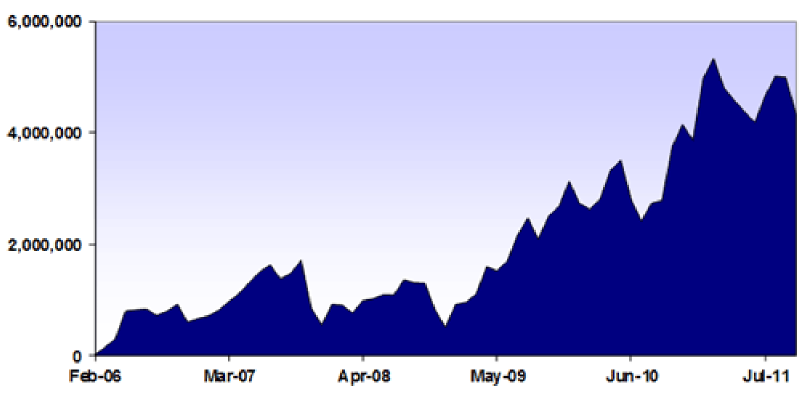
PRICING VIX OPTIONS

In this chapter we indicate how our method may be used to price vanilla options on the VIX Index. Firstly, we dwell on the financial interest of the contracts we are going to price, secondly, we establish the assumptions that should hold to apply the method theoretically and we finally provide accurate pricing formulas under both the Heston model and the more general SVJJ model. The pricing formula can be applied to every model with an analytically tractable characteristic function and represents a valid alternative to the classic Fourier methods, based on the inversion of the characteristic function.

5.1 Option contracts on the VIX

VIX volatility index options were introduced in 2006 and their acceptance as a method of trading an opinion on expected market volatility has been terrific. The chart below illustrates the rapid growth in the open interest of VIX option contracts.

Figure 5.1: Growth in the interest of VIX derivatives



One reason for the increase in popularity of VIX options is their ease of use in hedging equity portfolios. Given the historical inverse relationship between the direction of the $S\&P$ 500 Stock Index (SPX) and the VIX index, VIX Call options can be purchased as a hedge against a declining stock market: when the market drops, the VIX index often rallies and the percentage rise in VIX index is frequently much larger than the percentage decline in SPX. This pricing relationship has made VIX options potentially useful tools for hedging against a forecast move in the overall stock market.

The multiplier for VIX options is 100, just like equity options, that is a VIX option purchased for a price of "3.00", for instance, would cost \$300 plus commissions. Moreover, VIX options are European-style exercise, which means they can only be exercised at expiration. The underlying instrument that determines the cash settlement value of a VIX option at expiration is the VIX index. At expiration, the holder of an in-the-money VIX option will receive a cash payment based on the amount the option is in the money, as the following illustrative example shows:

Example 5.1 (VIX Call option).
Let us consider a Call VIX option with expiration in October and strike price 20 and suppose that the October VIX settlement is 22.10. Thus, we get

$$\begin{aligned}\text{Call settlement } &= \$100 \times (\text{VIX Settlement - Call Strike Price})\\ &= \$100 \times (22.10 - 20.00) = \$210\end{aligned}$$

which entails that a long option holder would receive \$210 and a short option holder would pay \$210.

VIX options offer traders and investors an easy way to trade a forecast of expected market volatility. If a trader forecasts a rise in expected market volatility, then buying a VIX Call option might be an appropriate strategy, whilst it might be convenient to buy a VIX Put option in anticipation of a lower VIX index. Similar to buying a VIX call option, a forecast that justifies purchasing a VIX put should involve three parts, a forecast for the VIX index, a forecast for the time period and an awareness of the price of the relevant VIX futures contract: if VIX futures are already anticipating what we forecast, then a strategy other than buying VIX options may be the best strategy.

5.2 Princing formulas

In view of Remark 4.7, the convergence of our method is guaranteed only if we assume that the payoff of the option belongs to $L^2_\nu(\mathbb{R})$ and this immediately prevents us from dealing with Call options, in favour of Put options, instead. Therefore, let us consider a Put option on the VIX index, which corresponds to a financial derivative that gives the right to its holder to sell an amount of the underlying asset at a future date for a prespecified price, where the underlying asset is assumed to be the VIX index. This entails to choose

$$H(\text{VIX}_T) = (K - \text{VIX}_T)^+$$

as corresponding payoff function, with K being the fixed strike. Therefore the price of the Put V_{PUT} can be performed as follows

$$V_{\text{PUT}} = e^{-rT} E^Q \left[H(\text{VIX}_T)\right].$$

The following result shows that Put options actually fulfill the assumption for their payoffs to belong to $L^2_\nu(\mathbb{R})$.

Proposition 5.2.1. *If H denotes the payoff function of a Put option, then $H \in L^2_\nu(\mathbb{R})$.*

Proof. $H \in L^2(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} H(x)^2 \, dx < \infty.$$

Now, since the underlying is positive

$$\begin{aligned}
\int_{\mathbb{R}} \left[(K-x)^+\right]^2 \, dx &= \int_0^{+\infty} \left[(K-x)^+\right]^2 \, dx \\
&= \int_0^K (K-x)^2 \, dx \\
&= \int_0^K \left(K^2 - 2Kx + x^2\right) dx \\
&= K^2 \int_0^K dx - 2K \int_0^K x \, dx + \int_0^K x^2 \, dx \\
&= K^3 - K^3 + \frac{K^3}{3} \\
&= \frac{K^3}{3}
\end{aligned}$$

whence the claim.

□

Remark 5.2. Although mathematical theory imposes this restriction upon the option payoff, empirical evidence on the Gamma-Laguerre expansions suggests that the choice of a Call option, instead of a Put option, could lead to better results since its mass is concentrated away from zero and the problematic behaviour of the Gamma-Laguerre expansion around zero is not involved. Roughly speaking, there is a tradeoff between having a payoff belonging to $L^2_\nu(\mathbb{R})$, required by a theoretical convergence result, and avoiding the problems deriving from an approximation around zero, required by empirical results. For this reason, we “allow” to consider both Put options and Call options on the VIX Index and we will give the last word to the final results. We recall that the price of a Call option on the VIX, denoted by V_{CALL}, can be performed as follows

$$V_{\text{CALL}} = e^{-rT} E^Q \left[H(\text{VIX}_T)\right]$$

where

$$H(\text{VIX}_T) = (\text{VIX}_T - K)^+$$

is the payoff function.

5.2.1 Heston model

In this section we derive interesting pricing formulas under the Heston model.

Proposition 5.2.2 (Pricing formula for Put options on VIX).
For every positive strike K, the price of a Put option on the VIX index given by

$$e^{-rT}E^Q\left[H(100\times\sqrt{a_\tau v_T+b_\tau})\right]$$

can be approximated by the following formula

$$100\times e^{-rT}\int_0^{\frac{(K^*)^2-b_\tau}{a_\tau}}\left(K^*-\sqrt{a_\tau x+b_\tau}\right)\sum_{k=0}^{n}c_k\phi(\alpha,\beta;x)L_k^{(\alpha,\beta)}(x)\,dx \tag{5.1}$$

where

$$K^*=\frac{K}{100}$$
$$a_\tau=\frac{1-e^{-\bar{k}\tau}}{\bar{k}\tau}$$
$$b_\tau=\theta\,(1-a_\tau)$$
$$\tau=\frac{30}{365}$$
$$c_k=\frac{\beta^{\alpha+1}}{2\pi\sqrt{C_{\alpha,\beta}}\begin{pmatrix}k+\alpha\\k\end{pmatrix}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix}k+\alpha\\k-j\end{pmatrix}\frac{\Gamma\left(\frac{\alpha}{2}+j+1\right)}{\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+j+1}}\int_{\mathbb{R}}\left(1+\frac{2i\xi}{\beta}\right)^{-\frac{\alpha}{2}-j-1}$$
$$\left(1+\frac{\epsilon^2 i\xi}{2\bar{k}}\left(e^{-\bar{k}\tau}-1\right)\right)^{\frac{-2\bar{k}\theta}{\epsilon^2}}e^{\frac{2\bar{k}i\xi}{\epsilon^2 i\xi+(2\bar{k}-\epsilon^2 i\xi)e^{\bar{k}\tau}}v_t}\,d\xi$$
$$\sqrt{C_{\alpha,\beta}}=\frac{\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+1}}{\Gamma(\frac{\alpha}{2}+1)}$$
$$\phi(\alpha,\beta;x)=\gamma\left(\frac{\alpha}{2}+1,\frac{\beta}{2}\right)$$
$$L_k^{(\alpha,\beta)}(x)=\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\begin{pmatrix}k+\alpha\\k-j\end{pmatrix}x^j.$$

Proof. A direct computation shows that

$$\begin{aligned}E\left[H(\mathrm{VIX}_T)\right]&=E\left[(K-\mathrm{VIX}_T)^+\right]\\&=E\left[\left(K-100\times\sqrt{a_\tau v_T+b_\tau}\right)^+\right]\\&=100\times E\left[\left(K^*-\sqrt{a_\tau v_T+b_\tau}\right)^+\right]\\&=100\times\int_0^{\infty}\left(K^*-\sqrt{a_\tau x+b_\tau}\right)^+f(x)\,dx\end{aligned}$$

$(K^* - \sqrt{a_\tau x + b_\tau} \geq 0$ if and only if $x \leq \frac{(K^*)^2 - b_\tau}{a_\tau})$

$$= 100 \times \int_0^{\frac{(K^*)^2-b_\tau}{a_\tau}} \left(K^* - \sqrt{a_\tau x + b_\tau}\right) f(x)\, dx$$
$$= 100 \times \int_0^{\frac{(K^*)^2-b_\tau}{a_\tau}} \left(K^* - \sqrt{a_\tau x + b_\tau}\right) \sum_{k=0}^{n} c_k \phi(\alpha, \beta; x) L_k^{(\alpha,\beta)}(x)\, dx$$

whence the claim.

□

Analogously, for Call options the following pricing formula holds:

Proposition 5.2.3 (Pricing formula for Call options on VIX)**.**
For every positive strike K, the price of a Call option on the VIX index given by

$$e^{-rT} E^Q \left[H(100 \times \sqrt{a_\tau v_T + b_\tau}) \right]$$

can be approximated by the following formula

$$100 \times e^{-rT} \int_{\frac{(K^*)^2-b_\tau}{a_\tau}}^{\infty} \left(\sqrt{a_\tau x + b_\tau} - K^*\right) \sum_{k=0}^{n} c_k \phi(\alpha, \beta; x) L_k^{(\alpha,\beta)}(x)\, dx \tag{5.2}$$

where the usual notations have been used.

Remark 5.3. We are now interested in finding the proper expression for (5.1) and (5.2) that makes the calculations on MATLAB more efficient. In this way, our approximation method turns out to be a concrete alternative to the usual Monte Carlo method, since it shows much more effectiveness in terms of computational speed.

By means of direct computations we obtain the final working formulas which may be directly transcribed on our MATLAB codes:

(1) For Put options:

$$100 \times \sqrt{C_{\alpha,\beta}} \sum_{k=0}^{n} \sum_{j=0}^{k} c_k \frac{(-\beta)^j}{j!} \binom{k+\alpha}{k-j} \left[K^* \int_0^{\frac{(K^*)^2-b_\tau}{a_\tau}} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx - \int_0^{\frac{(K^*)^2-b_\tau}{a_\tau}} \sqrt{a_\tau x + b_\tau} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx \right] \tag{5.3}$$

(2) For Call options:

$$100 \times \sqrt{C_{\alpha,\beta}} \sum_{k=0}^{n} \sum_{j=0}^{k} c_k \frac{(-\beta)^j}{j!} \binom{k+\alpha}{k-j} \left[\int_{\frac{(K^*)^2-b_\tau}{a_\tau}}^{\infty} \sqrt{a_\tau x + b_\tau} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx - K^* \int_{\frac{(K^*)^2-b_\tau}{a_\tau}}^{\infty} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx \right] \tag{5.4}$$

where the integrals

$$\int_0^{\frac{(K^*)^2-b_\tau}{a_\tau}} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx, \quad \int_{\frac{(K^*)^2-b_\tau}{a_\tau}}^{\infty} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx$$

can be easily computed by remarking that they are incomplete Gamma functions.

5.2.2 SVJJ model

The explicit pricing formulas for vanilla options on the VIX Index under the SVJJ model are stated in the following results.

Proposition 5.2.4 (Pricing formula for Put options on VIX).
For every positive strike K, the price of a Put option on the VIX index given by

$$e^{-rT}E^{Q}\left[H(100\times\sqrt{a_\tau v_T+b_\tau+c_\tau})\right]$$

can be approximated by the following formula

$$100\times e^{-rT}\int_0^{\frac{(K^*)^2-b_\tau-c_\tau}{a_\tau}}\left(K^*-\sqrt{a_\tau x+b_\tau+c_\tau}\right)\sum_{k=0}^{n}c_k\phi(\alpha,\beta;x)L_k^{(\alpha,\beta)}(x)\,dx \tag{5.5}$$

where

$$K^*=\frac{K}{100}$$
$$a_\tau=\frac{1-e^{-\bar{k}\tau}}{\bar{k}\tau}$$
$$b_\tau=\left(\theta+\frac{\lambda\mu_v}{\bar{k}}\right)(1-a_\tau)$$
$$c_\tau=2\lambda(c-\mu_S-\rho_Y\mu_v)$$
$$c=\frac{e^{\mu_S+\frac{1}{2}\sigma_S^2}}{1-\rho_Y\mu_v}-1$$
$$\tau=\frac{30}{365}$$
$$c_k=\frac{\beta^{\alpha+1}}{2\pi\sqrt{C_{\alpha,\beta}}\binom{k+\alpha}{k}\Gamma(\alpha+1)}\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\binom{k+\alpha}{k-j}\frac{\Gamma\left(\frac{\alpha}{2}+j+1\right)}{\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+j+1}}\int_{\mathbb{R}}\left(1+\frac{2i\xi}{\beta}\right)^{-\frac{\alpha}{2}-j-1}$$
$$\left(1+\frac{\epsilon^2 i\xi}{2\bar{k}}\left(e^{-\bar{k}\tau}-1\right)\right)^{\frac{-2\bar{k}\theta}{\epsilon^2}}e^{\frac{2\bar{k}i\xi}{\epsilon^2 i\xi+(2\bar{k}-\epsilon^2 i\xi)e^{\bar{k}\tau}}v_t}\left(1+\frac{\left(\epsilon^2-2\mu_v\bar{k}\right)i\xi}{2\bar{k}\left(1-\mu_v i\xi\right)}\left(e^{-\bar{k}\tau}-1\right)\right)^{\frac{2\mu_v\lambda}{2\mu_v\bar{k}-\epsilon^2}}d\xi$$
$$\sqrt{C_{\alpha,\beta}}=\frac{\left(\frac{\beta}{2}\right)^{\frac{\alpha}{2}+1}}{\Gamma(\frac{\alpha}{2}+1)}$$
$$\phi(\alpha,\beta;x)=\frac{\beta^{\frac{\alpha+1}{2}}}{\Gamma^{\frac{1}{2}}(\alpha+1)}e^{-\frac{\beta}{2}}x^{\frac{\alpha}{2}}\mathbb{1}_{x\geq 0}(x)$$
$$L_k^{(\alpha,\beta)}(x)=\sum_{j=0}^{k}\frac{(-\beta)^j}{j!}\binom{k+\alpha}{k-j}x^j.$$

Proof. A direct computation shows that

$$\begin{aligned}
E\left[H(\mathrm{VIX}_T)\right]&=E\left[(K-\mathrm{VIX}_T)^+\right]\\
&=E\left[\left(K-100\times\sqrt{a_\tau v_T+b_\tau+c_\tau}\right)^+\right]\\
&=100\times E\left[\left(K^*-\sqrt{a_\tau v_T+b_\tau+c_\tau}\right)^+\right]\\
&=100\times\int_0^{\infty}\left(K^*-\sqrt{a_\tau x+b_\tau+c_\tau}\right)^+f(x)\,dx
\end{aligned}$$

$(K^* - \sqrt{a_\tau x + b_\tau + c_\tau} \geq 0$ if and only if $x \leq \frac{(K^*)^2 - b_\tau - c_\tau}{a_\tau})$

$$= 100 \times \int_0^{\frac{(K^*)^2 - b_\tau - c_\tau}{a_\tau}} \left(K^* - \sqrt{a_\tau x + b_\tau + c_\tau}\right) f(x)\, dx$$

$$= 100 \times \int_0^{\frac{(K^*)^2 - b_\tau - c_\tau}{a_\tau}} \left(K^* - \sqrt{a_\tau x + b_\tau + c_\tau}\right) \sum_{k=0}^{n} c_k \phi(\alpha, \beta; x) L_k^{(\alpha,\beta)}(x)\, dx$$

whence the claim.

□

Proposition 5.2.5 (Pricing formula for Call options on VIX)**.**
For every positive strike K, the price of a Put option on the VIX index given by

$$e^{-rT} E^Q \left[H(100 \times \sqrt{a_\tau v_T + b_\tau + c_\tau}) \right]$$

can be approximated by the following formula

$$100 \times e^{-rT} \int_{\frac{(K^*)^2 - b_\tau - c_\tau}{a_\tau}}^{\infty} \left(\sqrt{a_\tau x + b_\tau + c_\tau} - K^*\right) \sum_{k=0}^{n} c_k \phi(\alpha, \beta; x) L_k^{(\alpha,\beta)}(x)\, dx \tag{5.6}$$

where the usual notations have been used.

Remark 5.4. Similarly to the Heston case, we provide the exact formulas we used in our MATLAB codes to guarantee more computational efficiency:

(1) For Put options:

$$100 \times \sqrt{C_{\alpha,\beta}} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_k \frac{(-\beta)^j}{j!} \left(\begin{array}{c} k+\alpha \\ k-j \end{array} \right) \left[K^* \int_0^{\frac{(K^*)^2 - b_\tau - c_\tau}{a_\tau}} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx \right.$$
$$\left. - \int_0^{\frac{(K^*)^2 - b_\tau - c_\tau}{a_\tau}} \sqrt{a_\tau x + b_\tau + c_\tau} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx \right]$$

(2) For Call options:

$$100 \times \sqrt{C_{\alpha,\beta}} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_k \frac{(-\beta)^j}{j!} \left(\begin{array}{c} k+\alpha \\ k-j \end{array} \right) \left[\int_{\frac{(K^*)^2 - b_\tau - c_\tau}{a_\tau}}^{\infty} \sqrt{a_\tau x + b_\tau + c_\tau} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx \right.$$
$$\left. - K^* \int_{\frac{(K^*)^2 - b_\tau - c_\tau}{a_\tau}}^{\infty} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx \right]$$

where the integrals

$$\int_0^{\frac{(K^*)^2 - b_\tau - c_\tau}{a_\tau}} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx, \quad \int_{\frac{(K^*)^2 - b_\tau - c_\tau}{a_\tau}}^{\infty} e^{-\frac{\beta}{2}x} x^{\frac{\alpha}{2}+j}\, dx$$

can be easily computed by remarking that they are two incomplete Gamma functions.

6
Numerical results

In this chapter we assess the performance of the method described above. Before presenting our numerical results it is worth mentioning the broader context our pricing formula fits in. Indeed, our technique can be included in the class of methods based on the knowledge in closed form of the characteristic function and represents an alternative to the inverse Fourier transform methodology. To the best of our knowledge, in an influential paper in the option-pricing literature [8], Heston showed that the risk-neutral probabilities appearing in the Call option-pricing formulas for bonds, currencies and equities can be computed by Fourier inversion of the conditional characteristic function which he showed is known in his particular affine stochastic volatility model.

For the purpose of examing the efficiency of our pricing formula, the numerical study presented here is based on the Heston stochastic volatility model, which is a special case covered by the general SVJJ model. We have adopted the parameters from calibrations reported in Bakshi et al. (1997) and Jacquier et al. (2012) which we recall in Table 6.1 below.

	B. et al.	**J. et al.**
$\bar{k}$	1.15	1.15
θ	0.0348	0.04
ϵ	0.39	0.2
$\sqrt{v0}$	0.1865	0.2

Table 6.1: Parameters for Heston model

In our numerical computations we have assessed how well results obtained by our pricing formula match with those obtained from the Monte Carlo simulations. In particular, we have implemented a Monte Carlo algorithm based on an Euler simulation scheme with 10^6 simulations and $\frac{1\text{ month}}{10^3}$ as discretization parameter. We have chosen one month as option maturity and we have considered a vector of strikes such that the "moneyness" $\frac{K}{\text{VIX}_0}$, varies between 0 and 3. Finally, the interest rate r has been assumed to be zero.

Once we have set the parameters of the model, some further indication is needed for the choice of the parameters α, β corresponding to the Gamma-Laguerre expansion. An intuitive choice for α and β could be a calibration of them by making a matching between the moments of the model density f and the moments of the approximating density $\phi(\alpha,\beta)$. This approach is based on an optimization technique called "moment matching". The characteristic function of the volatility process, by it being known in closed-form, allows us to easily compute the first two moments, by recalling that for any $n \in \mathbb{N}$, if $(x^n f) \in L^1(\mathbb{R})$

$$\begin{aligned} m_n = E\left[v^n\right] &= \int_{\mathbb{R}} x^n f(x)\,dx \\ &= \int_{\mathbb{R}} e^{ix\xi} x^n f(x)\,dx \Big|_{\xi=0} \\ &= \mathcal{F}(x^n f(x))(0) \\ &= (-1)^n \frac{d}{dx^n} \mathcal{F}(f(x))(0). \end{aligned}$$

Thus, we compute the first two moments of $\phi(\alpha,\beta)$ which we recall being a Gamma distribution with parameters $\alpha^* = \frac{\alpha}{2}+1$ and $\beta^* = \frac{\beta}{2}$. We have:

$$\begin{cases} m_1 = \text{Mean} = \frac{\alpha^*}{\beta^*} \\ m2 = \text{Variance} + m_1^2 = \frac{\alpha^*}{(\beta^*)^2} + m_1^2 \end{cases}$$

that is

$$\begin{cases} \beta^* = \frac{\alpha^*}{m_1} \\ m2 = \alpha^* \frac{m_1^2}{(\alpha^*)^2} + m_1^2 = \frac{m_1^2}{\alpha^*} + m_1^2 \end{cases} \qquad \begin{cases} \beta^* = \frac{m_1^2}{m_2 - m_1^2}\frac{1}{m_1} = \frac{m_1}{m_2 - m_1^2} \\ \alpha^* = \frac{m_1^2}{m_2 - m_1^2} \end{cases}$$

whence

$$\begin{cases} \frac{\beta}{2} = \frac{m_1}{m_2 - m_1^2} \\ \frac{\alpha}{2} + 1 = \frac{m_1^2}{m_2 - m_1^2} \end{cases}$$

and finally

$$\begin{cases} \beta = 2\frac{m_1}{m_2 - m_1^2} \\ \alpha = 2\left(\frac{m_1^2}{m_2 - m_1^2} - 1\right). \end{cases}$$

Roughly speaking, making a Gamma-Laguerre expansion of the model density means that we are approximating f by means of a Gamma distribution with the addition of some corrective terms, i.e.

$$\begin{aligned} f(x) &= \sum_{k=0}^{n} \phi^{\frac{1}{2}}(\alpha,\beta,x) L_k^{(\alpha,\beta)}(x) \\ &= c_0 \phi^{\frac{1}{2}}(\alpha,\beta,x) + \sum_{k=1}^{n} \phi^{\frac{1}{2}}(\alpha,\beta,x) L_k^{(\alpha,\beta)}(x) \quad \text{as } L_0^{(\alpha,\beta)}(x) = 1. \end{aligned}$$

The moment matching technique, by choosing α and β as above, requires that, in absence of corrective terms, the Gamma distribution perfectly approximates the target density, and their moments consequently: this is equivalent to require the coefficient c_0 to be equal to one and the remaining other $(c_k)_{k\geq 1}$ equal to zero. This calibration for the expansion parameters reveals to be efficient by empirical stability tests, as suggested by the following Example.

Example 6.1. Let us consider the simulated probability density function of the Heston model. Figures 6.1, 6.2 below depict the results obtained in the approximation of the density by means of the moment matching technique and by freely choosing the parameters α and β, respectively. The moment matching scheme performs the following values for the expansion parameters

$$\alpha = 3.6928$$
$$\beta = 168.6273$$

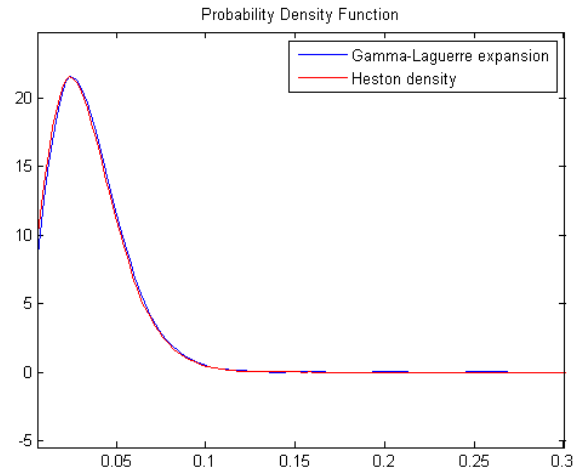
which will be used in Figure 6.1, whereas Figure 6.2 shows numerical outcomes by employing the following free combinations of values:

(a) $\alpha = 1$ and $\beta = 5$

(b) $\alpha = 0.1$ and $\beta = 10$

(c) $\alpha = 3$ and $\beta = 60$.

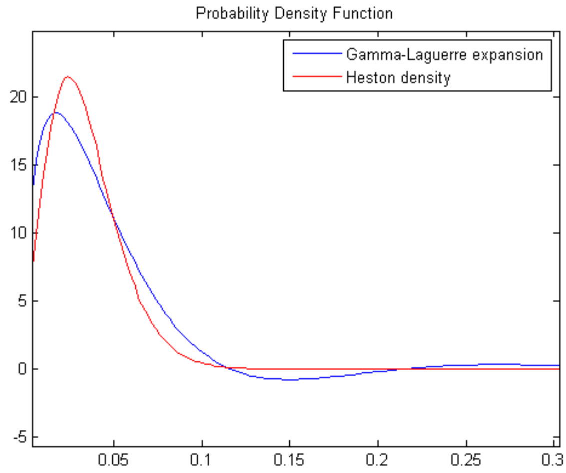
Figure 6.1: The Gamma-Laguerre approximation of the Heston distribution by means of the moment matching technique



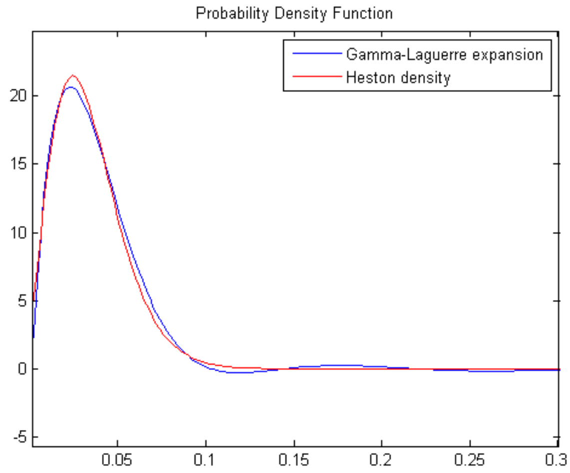
As already mentioned, the moment matching technique imposes that the expansion coefficient of order 0 tends to be equal to 1 and the remaining coefficients equal to zero, as confirmed in our particular example where the expansion has been truncated after 25 steps. The coefficients vector is indeed given by:

$$c = (0.9591, -0.0385, -0.0010, 0.0065, 0.0038, 0.0020, 0.0013, 0.0009, 0.0006, 0.0005, 0.0004, 0.0003, 0.0002, 0.0002, 0.0002, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0000, 0.0000, 0.0000, 0.0000).$$

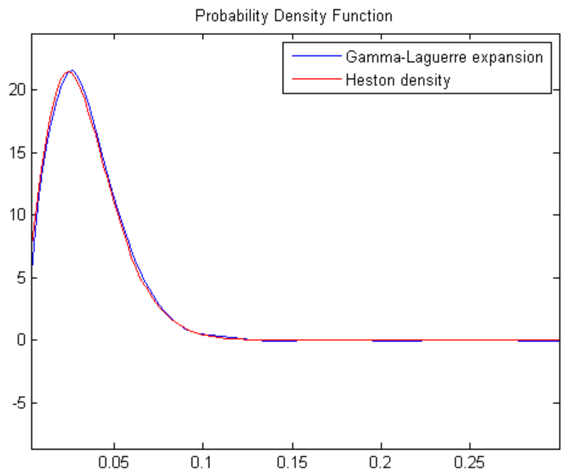
Figure 6.2: The Gamma-Laguerre approximation of the Heston distribution without moment matching



(a) $\alpha = 1$ and $\beta = 5$



(b) $\alpha = 0.1$ and $\beta = 10$



(c) $\alpha = 3$ and $\beta = 60$

It is straightforward to remark that the results are basically in accord with the intuition which stands behind the moment matching technique and show that the latter represents a valid rule for the determination of the expansion parameters.

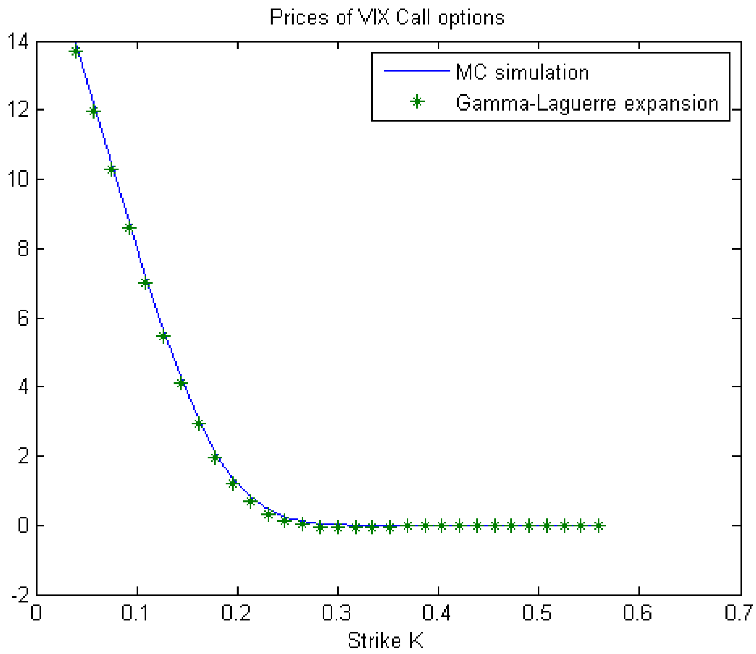
The first immediate advantage of our technique is that, compared to a Monte Carlo method which is assumed to be parallelized on the system CPU cores, is very few time-consuming: the differences in time between the two methods are extremely considerable. It is worth mentioning that the time component of an option pricing methodology is very relevant if it is thought to be used in a calibration process, that is the procedure of selecting model parameters in such a way that the value of a set of benchmark instruments, computed in the model, correspond to their market prices. A calibration algorithm is an optimization scheme where, once a “goodness-of-fit” measure has been chosen, the objective is to find the model parameters so that this goodness-of-fit measure is minimized; the numerical procedures involved in a calibration process are very time-consuming, requiring, at each step of minimization of the error, the computation of several option prices for different strikes and maturities.

As already remarked, the accuracy of our pricing technique depends on both a convergence result which is applicable only to Put options and a problematic behaviour around zero, typical of the Gamma-Laguerre expansion, which manifests for Put options, but can be averted by considering Call options instead.

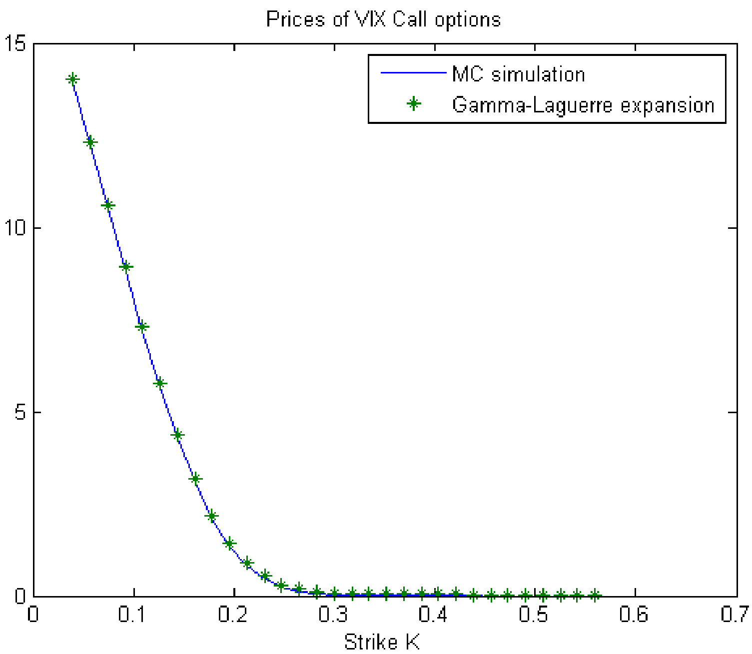
Figure 6.3 depicts a comparison between the VIX Call prices obtained by the numerical implementation of the Gamma-Laguerre approximation (5.4) truncated after $n = 5$ terms (panel (a)) and $n = 20$ terms (panel (b)) and the outcomes from the Monte Carlo simulation, based on the Heston model with parameters by Bakshi et al. The same comparison is also shown in Figure 6.4 where the parameters used are the ones from Jaquier et al. We can clearly observe that our results perfectly match with the outcomes from the Monte Carlo simulation. After 20 terms of the expansion the approximation is enhanced but, recalling that we can not rely on a convergence result, after 25 terms the expansion which employs the parameters by Bakshi et al. still converges to the Monte Carlo simulation, whilst the expansion that makes use of the parameters by Jaquier et al. does not, as depicted in Figure 6.5.

For the sake of completeness, we also provide the results obtained by our formula for VIX Put options (5.3), comparing the Gamma-Laguerre approximation truncated after $n = 5$ terms and $n = 20$ terms to the outcomes from the Monte Carlo simulation. The comparison is based on the Heston model with parameters by Bakshi et al. (Figure 6.6) and by Jacquier et al. (Figure 6.7). The results are quite in line with our expectations, they do not perfectly match with the Monte Carlo simulation and, even increasing the order of the expansion, the convergence being ensured in this case, the results do not improve. This is the effect of the problem that the Gamma-Laguerre expansion has in approximating around zero which cannot be repaired either by a high number of terms in the series expansion.

Figure 6.3: The VIX Call prices via Gamma-Laguerre expansions, B. et al.

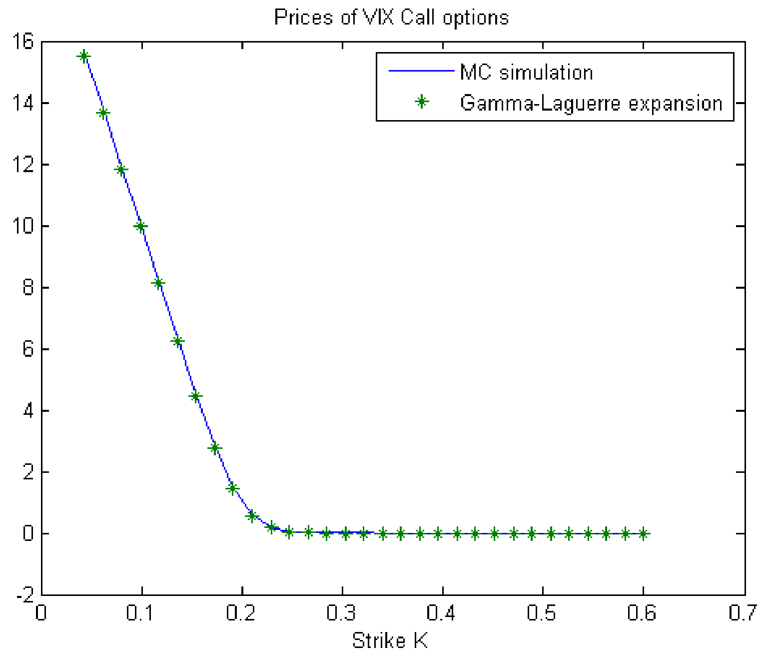


(a) $n = 5$

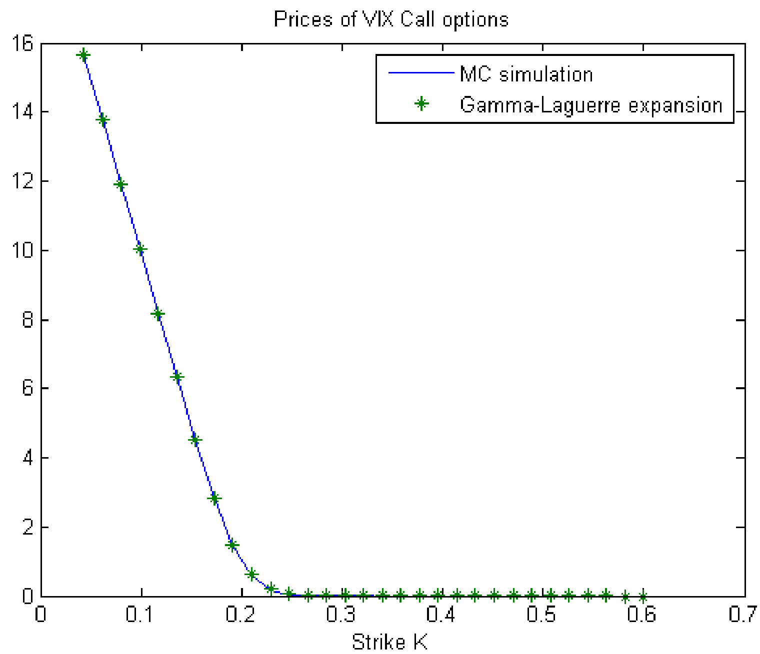


(b) $n = 20$

Figure 6.4: The VIX Call prices via Gamma-Laguerre expansions, J. et al.

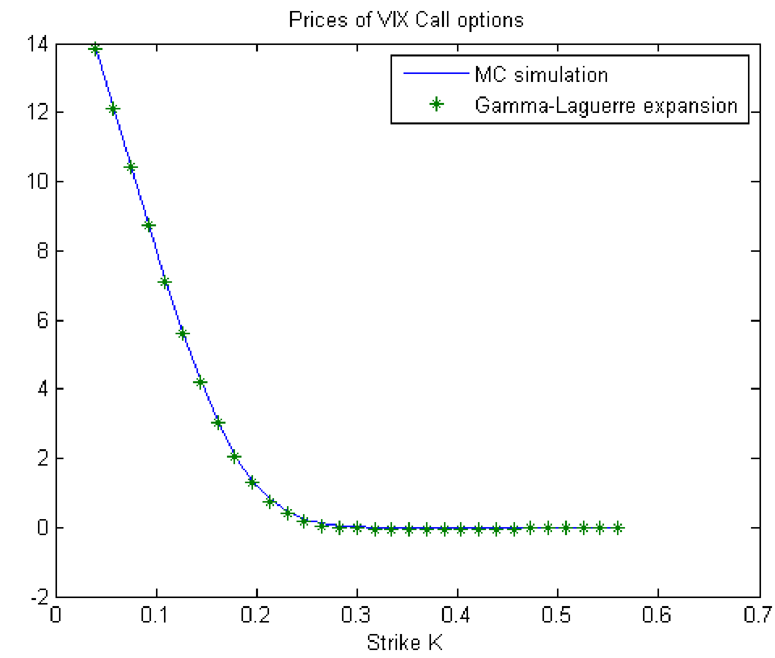


(a) $n = 5$

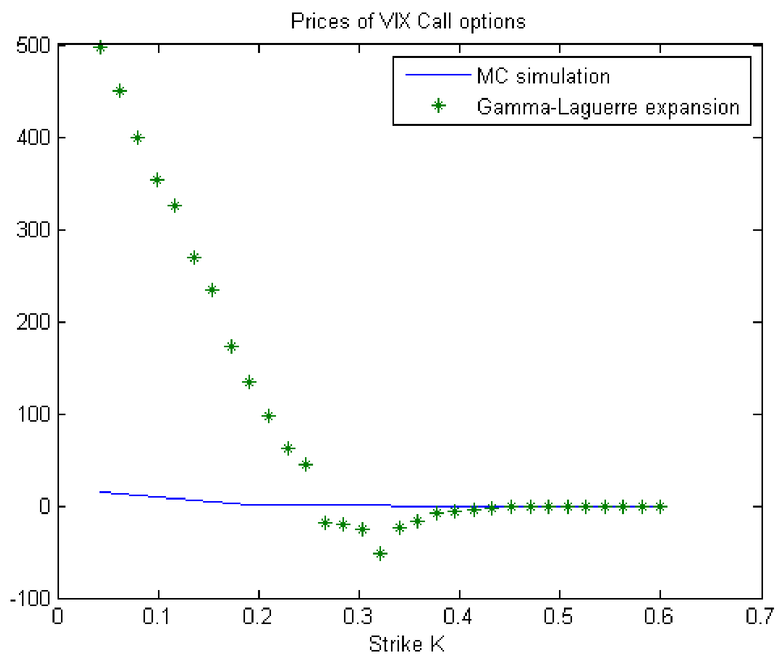


(b) $n = 20$

Figure 6.5: Non convergence for Call prices at high expansion orders

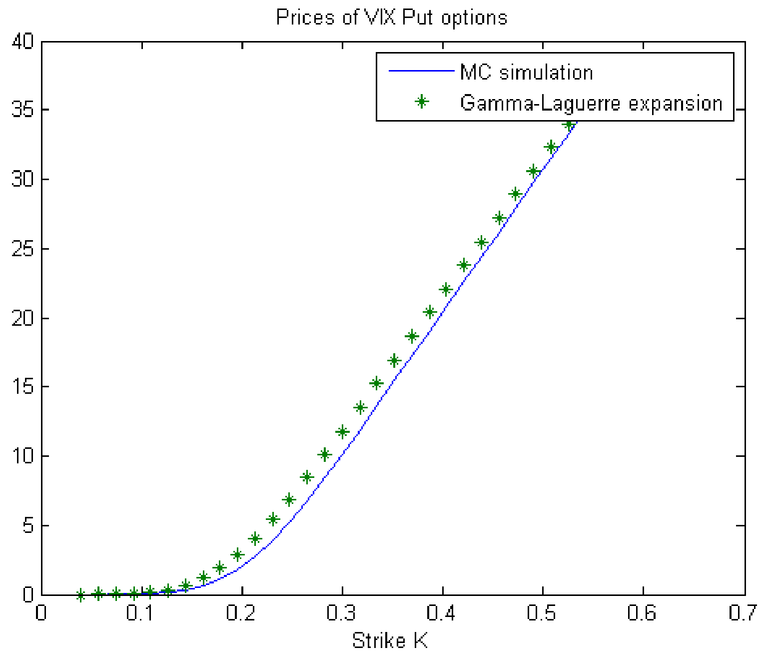


(a) $n = 25$, B. et al.

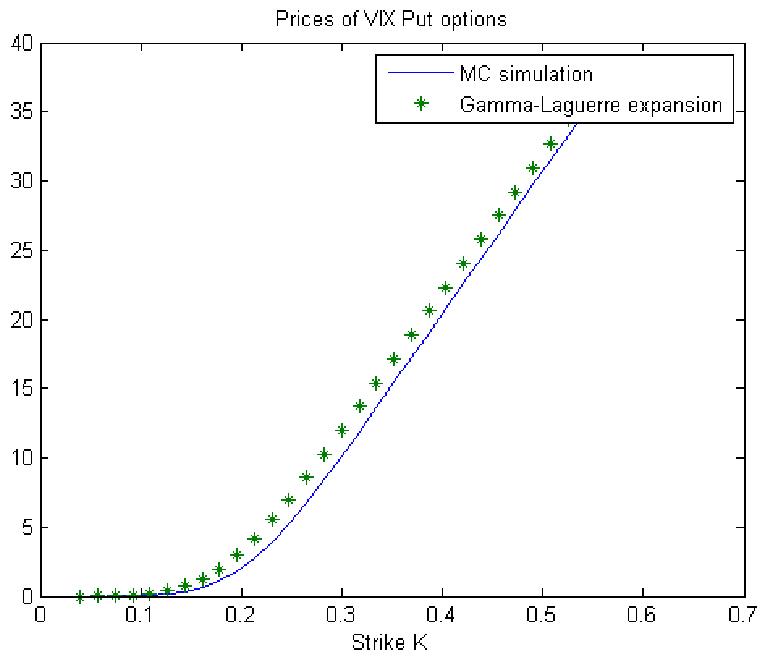


(b) $n = 25$, J. et al.

Figure 6.6: The VIX Put prices via Gamma-Laguerre expansions, B. et al.

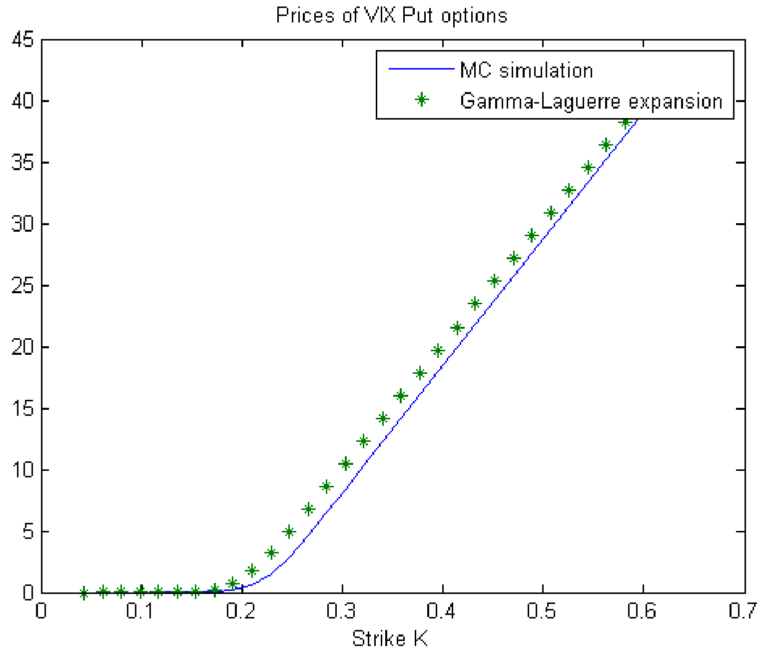


(a) $n = 5$

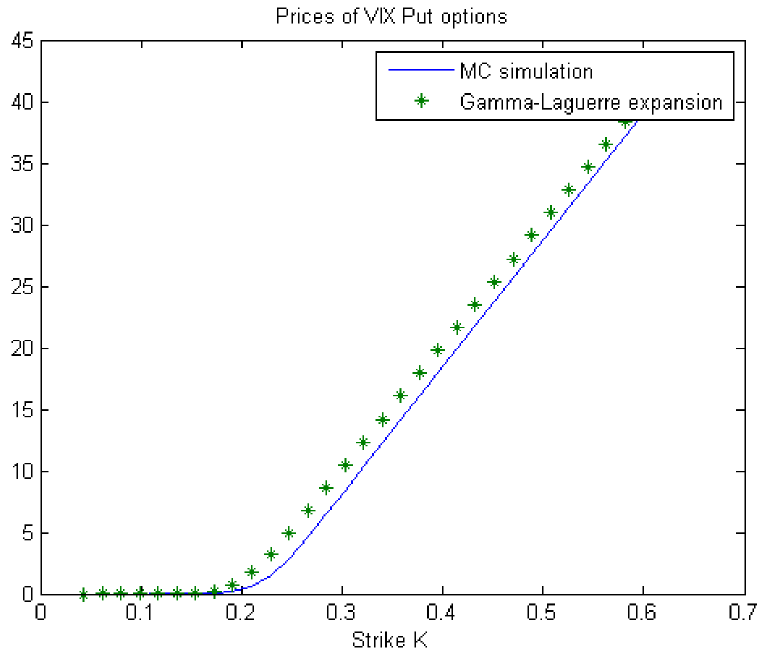


(b) $n = 20$

Figure 6.7: The VIX Put prices via Gamma-Laguerre expansions, J. et al.



(a) $n = 5$



(b) $n = 20$

A

APPENDIX

A.1 ...regarding stochastic calculus

In what follows W is a real Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ where the following hypotheses hold:

(i) $\mathcal{F}_0$ (and so also $\mathcal{F}_t$, for every $t > 0$) contains $\mathcal{N} := \{F \in \mathcal{F} | P(F) = 0\}$;

(ii) the filtration is right-continuous, i.e. for every $t \geq 0$

$$\mathcal{F}_t = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}.$$

Definition A.1. A stochastic process X is called *progressively measurable with respect to the filtration* $(\mathcal{F}_t)$ if, for every t, $X|_{[0,t]\times\Omega}$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t-$ measurable, i.e.

$$\{(s,\omega) \in [0,t] \times \Omega | X_s(\omega) \in H\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t, \quad H \in \mathcal{B}.$$

Definition A.2. The stochastic process u belongs to the class $\mathbb{L}^2$ if

(i) u is progressively measurable with respect to the filtration $(\mathcal{F}_t)$

(ii) $u \in L^2([0,T] \times \Omega)$, i.e.

$$E\left[\int_0^T u_t^2 \, dt\right] < \infty.$$

THEOREM - A.1.1. *For any process $u \in \mathbb{L}^2$ and $0 \leq a < b \leq T$, the following properties hold:*

(i) null expectation:

$$E\left[\int_a^b u_t dW_t \middle| \mathcal{F}_a\right] = 0$$

(ii) the stochastic process

$$X_t = \int_0^t u_s dW_s, \quad t \in [0,T]$$

is a continuous $\mathcal{F}_t-$martingale.

Proof. We present the proof of the theorem only in the particular case of a simple stichastic process: the general theorem can be proved by taking the limit in the analogous relation that holds for the integral of simple stochastic processes. We consider a simple process in $\mathbb{L}^2$

$$u_t = \sum_{k=1}^{N} e_k \mathbb{1}_{]t_{k-1},t_k]}(t), \quad t \in [0,T]$$

where $0 \le t_0 < t_1 < \cdots < t_N \le T$ and e_k are random variables on $(\Omega, \mathcal{F}, P)$. Since we always assume the right-continuity of the filtration and u is progressively measurable, it follows that e_k is $\mathcal{F}_{t_{k-1}}$-measurable for every $k = 1, \cdots, N$.

(i) We have

$$E\left[\int_a^b u_t dW_t \middle| \mathcal{F}_a\right] = \sum_{k=1}^{N} E\left[e_k(W_{t_k} - W_{t_{k-1}})\middle|\mathcal{F}_a\right]$$

(since $t_0 \ge a$, e_k is $\mathcal{F}_{t_{k-1}}$−measurable and so independent of $W_{t_k} - W_{t_{k-1}}$)

$$= \sum_{k=1}^{N} E\left[e_k|\mathcal{F}_a\right] E\left[W_{t_k} - W_{t_{k-1}}\right] = 0.$$

(ii) For $0 \le s < t$, we have

$$E\left[X_t|\mathcal{F}_s\right] = E\left[X_s|\mathcal{F}_s\right] + E\left[\int_s^t u_\tau dW_\tau|\mathcal{F}_s\right] = X_s$$

since X_s is $\mathcal{F}_s$-measurable and (i) holds.

□

Definition A.3. The stochastic process u belongs to the class $\mathbb{L}^2_{\text{loc}}$ if

(i) u is progressively measurable with respect to the filtration $(\mathcal{F}_t)$

(ii) $\int_0^T u_t^2 \, dt < \infty \quad a.s.$

THEOREM - A.1.2. *If $u \in \mathbb{L}^2_{loc}$, then its stochastic integral*

$$X_t = \int_0^t u_s dW_s$$

is a continuous local martingale.

A.1.1 Correlated Brownian motion

For simplicity, we only consider the case of constant correlation matrix even if all the following results can be extended to the more general case of stochastic correlation (cf. Remark 10.23 in [15]). Thus, we assume that

$$W_t = A\bar{W}_t$$

where $\bar{W}_t$ is a standard d-dimensional Brownian motion and $A = \left(A^{ij}\right)_{i,j=1,\cdots,d}$ is a non-singular $d \times d$ constant matrix. We then denote by $\rho = AA^T$ the correlation matrix assuming that, for any $i = 1, \cdots, d$

$$\rho^{ii} = \left|A^i\right|^2 = \sum_{j=1}^{d} \left(A^{ij}\right)^2 = 1, \quad t \in [0,T] \quad \text{a.s.}$$

whence we get that

$$W_t^i = \sum_{j=1}^{d} A^{ij} \bar{W}_t^j \quad i = 1, \cdots, d$$

is a standard real Brownian motion and the covariance processes are given by

$$d\left\langle W^i, W^j \right\rangle_t = \rho^{ij} dt, \quad i, j = 1, \cdots, d.$$

Example A.4 (Particular case $d = 2$)**.** We typically assume

$$A = \begin{pmatrix} 1 & 0 \\ \bar{\rho} & \sqrt{1 - \bar{\rho}^2} \end{pmatrix}$$

where $\bar{\rho} \in]-1, 1[$. Then $W_t = A\bar{W}_t$ is a correlated Brownian motion with non-singular correlation matrix

$$\rho = \begin{pmatrix} 1 & \bar{\rho} \\ \bar{\rho} & 1 \end{pmatrix}.$$

A.1.2 Itô calculus

Here, we provide the definition of Itô process and we present the formula for the "change of variable" extended to the stochastic integration theory, the so-called Itô formula.

Definition A.5. An *Itô process* is a stochastic process X of the form

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s, \quad t \in [0, T] \tag{A.1}$$

where X_0 is a $\mathcal{F}_0$-measurable random variable, $\mu \in \mathbb{L}^1_{\text{loc}}$ and $\sigma \in \mathbb{L}^2_{\text{loc}}$ are the *drift* and *diffusion* coefficients, respectively. Formula (A.1) is usually written in the "differential form"

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

The Itô process X is the sum of the continuous process with bounded variation

$$X_0 + \int_0^t \mu_s \, ds$$

with the continuous local martingale

$$\int_0^t \sigma_s \, dW_s.$$

Remark A.6. An Itô process is a local martingale if and only if it has null drift, namely $\mu = 0$ $m \otimes P)$−a.e. Indeed, by assumption the process

$$\int_0^t \mu_s \, ds = X_t - X_0 - \int_0^t \sigma_s \, dW_s$$

would be both a continuous local martingale and a bounded variation process, it being a Lebesgue integral. However, it can be proved that if a (local) martingale has bounded variation, then it is indistinguishable from the null process. Whence the claim.

THEOREM - A.1.3 (Itô formula)**.**
Let X be an Itô process and $f = f(t,x)$ a function belonging to $C^{1,2}(\mathbb{R}^2)$. Then the stochastic process

$$Y_t = f(t, X_t)$$

is an Itô process as well and we have

$$df(t,X_t) = \partial_t f(t,X_t)dt + \partial_x f(t,X_t)dX_t + \frac{1}{2}\partial_{xx} f(t,X_t)d\langle X\rangle_t. \tag{A.2}$$

where $d\langle X\rangle_t = \sigma_t^2 dt$.

Formula (A.2) can be explicity written as follows

$$df = \left(\partial_t f + \mu_t \partial_x f + \frac{1}{2}\sigma_t^2 \partial_{xx} f\right) dt + \sigma_t \partial_x f dW_t$$

where $f = f(t, X_t)$.

A.1.3 Feynman-Kac formula

In this section we state a representation formula for the classical solution of the Cauchy problem

$$\begin{cases} \mathcal{A}u - au + \partial_t u = f, & \text{in } \mathcal{S}_T :=]0,T[\times\mathbb{R}^N \\ u(T,\cdot) = \phi \end{cases} \tag{A.3}$$

where f, a, ϕ are given functions, $(c_{ij}) = \sigma\sigma^T$ and

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{N} c_{ij}\partial_{x_i x_j} + \sum_{j=1}^{N} b_j \partial_{x_j}$$

is the characteristic operator of the SDE

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t. \tag{A.4}$$

We assume that

(i) the coefficients b, σ are measurable and have at most linear growth in x;

(ii) for every $(t,x) \in \mathcal{S}_T$, there exists a solution $X^{t,x}$ of the SDE A.4 relative to a d-dimensional Brownian motion W on the space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$.

Having a probabilistic representation of the solution u in terms of an associated Markovian diffusion process is very interesting for the pricing purpose: the link between option values (defined via risk neutral expectations of terminal payoffs) and solutions of second order parabolic PDEs is established by the well-known Feynman-Kac formula.

THEOREM - A.1.4 (Feynman-Kac formula)**.**
Let $u \in C^2(\mathcal{S}_T) \cup C(\bar{\mathcal{S}}_T)$ be a solution of the Cauchy problem (A.3) where $a \in C(\mathcal{S}_T))$ is such that $a_0 = \inf a > -\infty$. Assume that (i), (ii) and at least one of the following conditions are in force:

1) there exist two positive constants M, p such that

$$|u(t,x)| + |f(t,x)| \le M(1 + |x|^p), \quad (t,x) \in \mathcal{S}_T;$$

2) *the matrix σ is bounded and there exist two positive constants M and α, with α small enough, such that*

$$|u(t,x)| + |f(t,x)| \le M e^{\alpha |x|^2}, \quad (t,x) \in \mathcal{S}_T.$$

Then, for every $(t,x) \in \mathcal{S}_T$, we have the representation formula

$$u(t,x) = E\left[e^{-\int_t^T a(s,X_s)\,ds} \phi(X_T) - \int_t^T e^{-\int_t^s a(r,X_r)\,dr} f(s,X_s)\,ds \right]$$

where, for the sake of simplicity, $X = X^{t,x}$.

A.2 Lévy processes

In this section we merely introduce Lévy processes and discuss some of their general properties. We then give particular stress to the simplest examples of Lévy processes, the compound Poisson processes, which can be considered as Poisson processes with random jump sizes. The class of compound Poisson processes is both simple to study and rich enough to introduce two important theoretical tools of Lévy processes: the Lévy-Khintchine formula that allows to study distributional properties of Lévy processes and the Lévy-Itô decomposition, that describes the structure of their sample paths.

Definition A.7. Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ be a filtrated probability space. A stochastic process $X = (X_t)_{t\ge 0}$ with state space on $\mathbb{R}^d$ is called a *Lévy process* if it possesses the following properties:

(i) $X_0 = 0$;

(ii) Stationary increments: $X_t - X_s \sim X_{t-s}, \quad 0 \le s < t$;

(iii) Independent increments: for every increasing sequence of times $0 \le t_1 < t_2 < \cdots < t_n$ the random variables

$$X_{t_1},\, X_{t_2} - X_{t_1},\, \cdots, X_{t_n} - X_{t_{n-1}}$$

are independent;

(iv) For every $\omega \in \Omega$, the path $t \longrightarrow X_t(\omega)$ is cadlag (i.e. right continuous with finite left limits).

Definition A.8. A measure m on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is called a *Lévy measure* if it satisfies the following properties:

(i) $m(\{0\}) = 0$

(ii) $\int_{\mathbb{R}^d} (1 \wedge ||x||^2)\, m(dx) < \infty$.

Definition A.9. We call *Lévy triplet* the triplet (a, σ, m) where $a \in \mathbb{R}^d$, σ is a $d \times d$ matrix and m is a Lévy measure.

Definition A.10. Let (a, σ, m) be a Lévy triplet. A *Lévy exponent* ψ is a function

$$\psi : \mathbb{R}^d \longrightarrow \mathbb{C}$$

such that

$$\psi(\xi) = i(a\cdot\xi) + \frac{1}{2}||\sigma\xi||^2 + \int_{\mathbb{R}^d} \left[1 - e^{i\xi\cdot x} + i(\xi\cdot x)\mathbb{1}_{(0,1)}(||x||)\right] m(dx).$$

Remark A.11. Let (a, σ, m) be a Lévy triplet. We assume that

$$\int_{\mathbb{R}^d} (1 \wedge ||x||)\ m(dx) < \infty$$

and we put

$$b = a + \int_{||x||<1} x\ m(dx).$$

Then, the Lévy exponential assumes the following form

$$\psi(\xi) = i(b \cdot \xi) + \frac{1}{2}||\sigma\xi||^2 + \int_{\mathbb{R}^d} \left[1 - e^{i\xi \cdot x}\right]\ m(dx).$$

We call (b, σ, m) the *modified Lévy triplet.*

Definition A.12. Let $(S, \mathcal{S}, m)$ denote a $\sigma-$finite measure space with $m(S) > 0$. A *Poisson random measure* with intensity m is a family of random variables $\Pi = \{\Pi(A)\}_{A \in \mathcal{S}}$, with $\Pi(A)$ defined on $(\Omega, \mathcal{F}, P)$, such that

(i) $\Pi(A) \sim \text{Poisson}(m(A))$, $A \in \mathcal{S}$;

(ii) if $A_1, \cdots, A_n \in \mathcal{S}$ are disjoint, then $\Pi(A_1), \cdots, \Pi(A_n)$ are independent;

(iii) For any $\omega \in \Omega$ and $A \in \mathcal{S}$, $A \longrightarrow \Pi(A)(\omega)$ is a measure.

Remark A.13. The existence of a Poisson random measure with intensity m, Π, can be proved and, from the proof it follows that Π can be represented as

$$\Pi = \sum_{j=1}^{N} \delta_{Z_j}$$

where $Z_1, Z_2, \cdots$ are independent S-valued random variables that are also independent of N, and $N \sim \text{Poisson}(m(S))$.

Definition A.14. Let $S^* = \mathbb{R}^+ \times \mathbb{R}^d$, $m^* = \text{LEB} \times m$, $\mathcal{S}^* = \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^d)$, we define the Poisson random measure with respect to the product measure m^* as follows

$$\Pi^* = \sum_{j=1}^{\infty} \delta_{(T_j, Z_j)}, \quad T_j \in \mathbb{R}^+, Z_j \in \mathbb{R}^d.$$

Remark A.15. Let X be a Lévy process with Lévy measure m, then the Poisson random measure Π^* is often referred as *jump measure of X.* Indeed, we have

$$\begin{aligned} \Pi^*((0,t] \times A) &= \sharp\left\{j : T_j \le t, Z_j \in A\right\} \\ &= \sharp\{s \le t, \Delta X_s \in A, \Delta X_s \neq 0\}. \end{aligned}$$

In other terms, $\Pi^*((0,t] \times A)$ counts the number of jumps occurring in the time interval $[0,t]$ and such that their size is in A.

A.2.1 Some examples of Lévy processes

Real Brownian motion

The real Brownian motion $W = (W_t)_{t\geq 0}$ is a particular Lévy process on $\mathbb{R}$: indeed it fulfills the defining properties in A.7 since, by definition

(i) $W_0 = 0$;

(ii) Stationary increments: $W_t - W_s \sim \mathcal{N}_{0,t-s}, \quad 0 \leq s < t$;

(iii) Independent increments: for every $0 \leq s < t$ the random variable $W_t - W_s$ is independent of $\mathcal{F}_s$;

(iv) For every $\omega \in \Omega$, the path $t \longrightarrow X_t(\omega)$ is even continuous, thus in particular cadlag.

Poisson processes

Definition A.16. Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^n \tau_i$. The process $(N_t)_{t\geq 0}$ defined by

$$N_t = \sum_{n\geq 1} \mathbb{1}_{t\geq T_n}$$

is called a *Poisson process* with intensity λ.

The Poisson process is therefore defined as a *counting process*: it counts the number of random times (T_n) which occur between 0 and t, where $(T_n - T_{n-1})_{n\geq 1}$ is an i.i.d. sequence of exponential variables.

Compound Poisson processes

Definition A.17. Let N be a Poisson process with intensity λ and assume that $Y = (Y_i)_{i\geq 1}$ is a sequence of i.i.d. random variables in $\mathbb{R}^d$ with distribution m, i.e. $Y_i \sim m$ for $i \geq 1$, and which are independent of N. The *compound Poisson process* is defined as

$$C_t = \sum_{i=1}^{N_t} Y_i$$

for $t \geq 0$ and where $\sum_{i=1}^0 Y_i := 0$.

The following properties of a compound Poisson process are easily deduced from the definition:

(i) The sample paths of X are càdlàg piecewise constant functions.

(ii) The jump times $(T_i)_{i\geq 1}$ have the same law as the jump times of the Poisson process N_t: they can be expressed as partial sums of independent exponential random variables with parameter λ.

(iii) The jump sizes $(Y_i)_{i\geq 1}$ are independent and identically distributed with law m, while the jumps of N are of fixed size equal to one.

The Poisson process itself can be seen as a compound Poisson process on $\mathbb{R}$ such that $Y_i = 1$. The graph in Figure A.1 depicts a typical trajectory of a compound Poisson process - note the piecewise constant path.

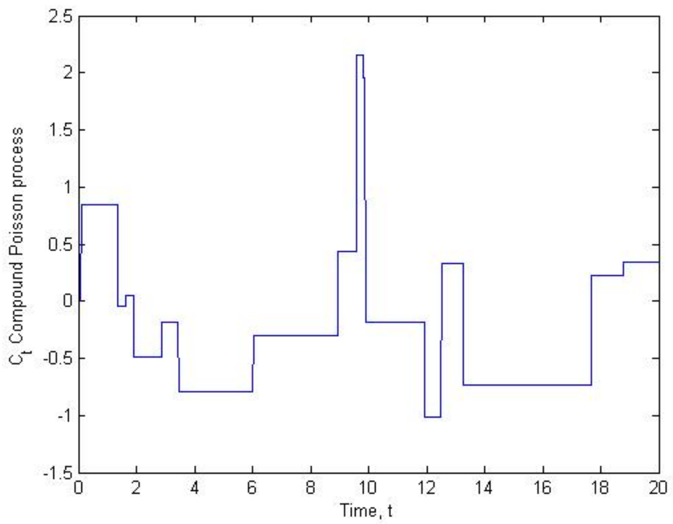


Figure A.1: One path of a compound Poisson process with $\lambda = 1$ and $m = \mathcal{N}(0,1)$

Compound Poisson processes are Lévy processes and they are the only Lévy processes with piecewise constant sample paths, as shown by the following proposition

Proposition A.2.1. *$(X_t)_{t\geq 0}$ is a compound Poisson process if and only if it is a Lévy process and its sample paths are piecewise functions.*

Proposition A.2.2 (Characteristic function of a compound Poisson process)**.**
Let $(C_t)_{t\geq 0}$ be a compound Poisson process on $\mathbb{R}^d$. Its characteristic function has the following representation:

$$E\left[e^{i\xi\cdot C_t}\right] = \exp\left\{t\lambda\int_{\mathbb{R}^d}\left(e^{i\xi\cdot x}-1\right)m(dx)\right\}, \quad \forall\xi\in\mathbb{R}^d, \tag{A.5}$$

where λ denotes the jump intensity and m the jump size distribution.

Introducing a new Borel measure $\tilde{m}(B) = \lambda m(B)$, for any $B \in \mathcal{B}(\mathbb{R}^d)$, we can rewrite Formula (A.5) as follows

$$E\left[e^{i\xi\cdot X_t}\right] = \exp\left\{t\int_{\mathbb{R}^d}\left(e^{i\xi\cdot x}-1\right)\tilde{m}(dx)\right\}, \quad \forall\xi\in\mathbb{R}^d,. \tag{A.6}$$

$\tilde{m}$ is the *Lévy measure* of process $(X_t)_{t\geq 0}$ and Formula (A.6) is a particular case of the Lévy-Khintchine representation A.7.

Proof. For every $t \geq 0$, we have

$$
\begin{aligned}
E\left[e^{i\xi\cdot C_t}\right] &= E\left[\exp\left\{i\xi\cdot\sum_{j=1}^{N_t} Y_j\right\}\right] \\
&= E\left[E\Big[\exp\Big\{i\xi\cdot\sum_{j=1}^{N_t} Y_j\Big\}\Big|N_t\Big]\right] = E\left[E\Big[\prod_{j=1}^{N_t} e^{i\xi\cdot Y_j}\Big|N_t\Big]\right] \\
&= E\left[E\Big[\prod_{j=1}^{N_t} e^{i\xi\cdot Y_j}\Big]\right] \quad \text{by the independence of N and } (Y_j)_{j\geq 1} \\
&= E\left[\prod_{j=1}^{N_t} E\Big[e^{i\xi\cdot Y_j}\Big]\right] \quad \text{by the independence of } (Y_j)_{j\geq 1} \\
&= E\left[\Big(E\Big[e^{i\xi\cdot Y_1}\Big]\Big)^{N_t}\right] \quad \text{as } Y_{j,j\geq 1} \text{ are identically distributed} \\
&= \sum_{n=0}^{\infty}\Big(E\Big[e^{i\xi\cdot Y_1}\Big]\Big)^n P(N_t = n) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\Big(E\Big[e^{i\xi\cdot Y_1}\Big]\Big)^n \quad \text{as N is a Poisson process} \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t E\left[e^{i\xi\cdot Y_1}\right])^n}{n!} = \exp\Big\{-\lambda t\Big\}\exp\Big\{\lambda t E\Big[e^{i\xi\cdot Y_1}\Big]\Big\} \\
&= \exp\Big\{-\lambda t\Big(1 - E\Big[e^{i\xi\cdot Y_1}\Big]\Big)\Big\} \\
&= \exp\Big\{-\lambda t\Big(1 - \int_{\mathbb{R}^d} e^{i\xi\cdot x} m(dx)\Big)\Big\} \quad \text{as } Y_{j,j\geq 1} \text{ have common law } m \\
&= \exp\Big\{\lambda t\Big(\int_{\mathbb{R}^d}(e^{i\xi\cdot x} - 1)m(dx)\Big)\Big\}.
\end{aligned}
$$

□

THEOREM - A.2.3. *Let $\{X_t\}_{t\geq 0}$ be an integrable Lévy process. Then the process*

$$\Big\{X_t - E\left[X_t\right]\Big\}_{t\geq 0}$$

is a martingale.

Proof. For every $0 < s \leq t$, we have

$$
\begin{aligned}
E\Big[X_t - E\left[X_t\right]|\mathcal{F}_s\Big] &= E\Big[X_t|\mathcal{F}_s\Big] - E\Big[E\left[X_t\right]|\mathcal{F}_s\Big] \\
&= E\Big[(X_t - X_s) + X_s|\mathcal{F}_s\Big] - E\left[X_t\right] \\
&= E\Big[X_t - X_s|\mathcal{F}_s\Big] + E\Big[X_s|\mathcal{F}_s\Big] - E\left[X_t\right]
\end{aligned}
$$

(since $X_t - X_s$ is independent of $\mathcal{F}_s$ and X_s is $\mathcal{F}_s$−measurable)

$$
\begin{aligned}
&= E\Big[X_t - X_s\Big] + X_s - E\left[X_t\right] \\
&= E\left[X_t\right] - E\left[X_s\right] + X_s - E\left[X_t\right] \\
&= X_s - E\left[X_s\right].
\end{aligned}
$$

□

Example A.18. The process
$$\tilde{C}_t = C_t - \mu\lambda t,$$
where $\mu = E[Y_1]$, called *compensated compound Poisson process*, is a martingale. Indeed, we have
$$\begin{aligned} E[C_t] &= E[E[C_t|N_t]] \\ &= E\left[E\left[\sum_{i=1}^{N_t} Y_i \middle| N_t\right]\right] \\ &= E[N_t E[Y_1|N_t]] \end{aligned}$$
(since N and Y_1 are independent)
$$\begin{aligned} &= E[N_t E[Y_1]] \\ &= E[N_t]\,E[Y_1] \\ &= \lambda t E[Y_1] \end{aligned}$$
and the claim follows from Theorem A.2.3.

THEOREM - A.2.4. *Let X be a Lévy process with triplet (a,σ,m). Then X has bounded variation if and only if*
$$\sigma = 0$$
and
$$\int_{\mathbb{R}^d} 1 \wedge ||x||\; m(dx) < \infty.$$

Corollary A.2.5. *A compound Poisson process has bounded variation.*

Proof. A Lévy process is a compound Poisson process if and only if it has modified Lévy triplet $(0,0,m)$, where $m(\mathbb{R}^d) < \infty$. Then to prove that a a compound Poisson process is of bounded variation it remains to show that
$$\int_{\mathbb{R}^d} 1 \wedge ||x||\; m(dx) < \infty.$$
Indeed, we have
$$\begin{aligned} \int_{\mathbb{R}^d} 1 \wedge ||x||\; m(dx) &= \int_{||x||\geq 1} 1 \wedge ||x||\; m(dx) + \int_{||x||<1} 1 \wedge ||x||\; m(dx) \\ &= \int_{||x||\geq 1} m(dx) + \int_{||x||<1} ||x||\; m(dx) \\ &\leq \int_{||x||\geq 1} m(dx) + \int_{||x||<1} m(dx) \\ &= \int_{\mathbb{R}^d} m(dx) \\ &= m\left(\mathbb{R}^d\right) < \infty. \end{aligned}$$
□

The starting point of the classification of Lévy processes via their characteristic function is known as the Lévy-Khintchine formula.

THEOREM - A.2.6 (Lévy-Khintchine formula)**.**
For every Lévy exponent ψ on $\mathbb{R}^d$ there exists a Lévy process X such that for all $t \geq 0$ and $\xi \in \mathbb{R}^d$
$$E\left[e^{i\xi\cdot X_t}\right] = \exp\left(-t\psi(\xi)\right). \tag{A.7}$$

Outline of the proof. The proof of the Lévy-Khintchine formula follows the treatment of Itô and is divided into two steps:

(i) By direct computation, we know that the characteristic function of a Brownian motion W is given by

$$E\left[e^{i\xi\cdot W_t}\right] = \exp\left(-\frac{1}{2}t||\xi||^2\right)$$

as $W_t \sim \mathcal{N}_{0,t}$. This entails that, given the Lévy triplet $(0,\sigma,0)$, its associated Lévy process is

$$X_t = \sigma\begin{pmatrix} W_t^1 \\ \cdots \\ W_t^d \end{pmatrix}.$$

Therefore it remains to prove that there is a Lévy process X with modified Lévy triplet $(b,0,m)$ such that

$$E\left[e^{i\xi\cdot X_t}\right] = \exp\left(-t\left(ib\cdot\xi + \int_{\mathbb{R}^d}(1-e^{i\xi\cdot x}\, m(dx))\right)\right).$$

(ii) The construction will be based on a Poisson random measure and there will be distinguished three cases:

1) Compound Poisson process case: $m(\mathbb{R}^d) < \infty$;
2) Bounded variation case: $\int_{\mathbb{R}^d}(1\wedge ||x||)\, m(dx) < \infty$;
3) General case: $\int_{\mathbb{R}^d}(1\wedge ||x||^2)\, m(dx) < \infty$.

The Lévy-Itô proof of the Lévy-Khintchine formula shows among other things that a Lévy process admits a process-wise decomposition, called Lévy-Itô decomposition.

THEOREM - A.2.7 (Lévy-Itô decomposition)**.**
If X is a Lévy process with triplet (a,σ,m) we have the following process-wise decomposition

$$X_t = W_t + Z_t + Y_t$$

where:

- $W_t := \sigma B_t - at$, *with* B_t *a Brownian motion on* $\mathbb{R}^d$*;*
- Z_t *is a compound Poisson process with Lévy triplet* $(0,0,m(\cdot\cap\{||x||\geq 1\}))$

$$Z_t = \int_{(0,t]\times\{||x||\geq 1\}} x\Pi^*(d(u,x))$$

and

$$\Delta Z_t = Z_t - Z_{t-} = \Delta X_t \mathbb{1}_{||\Delta X_t||\geq 1},$$

that is Z_t *is the jump component which includes only big jumps.*

- Y_t *is a mean-zero Lévy process with Lévy triplet* $(0,0,m(\cdot\cap\{||x||<1\}))$

$$Y_t = \lim_{n\to\infty}\int_{2^{-n}<||x||<1} x\Pi^*(d(u,x)) - t\int_{2^{-n}<||x||<1} xm(dx)$$

where the convergence is in L^2*. Moreover it is a martingale satisfying*

$$\Delta Y_t = Y_t - Y_{t-} = \Delta X_t \mathbb{1}_{||\Delta X_t||<1}$$

that is Y_t *is the jump component which includes only small jumps.*

A.3 Stochastic calculus for jump-diffusion processes

A.3.1 Itô formula for jump-diffusion processes

In this section we obtain a more general version of the Itô formula by extending its validity to the case of jump-diffusion processes, that is processes consisting of a drift term, a diffusion part and a jump component determined by a compound Poisson process. The more tractability of compound Poisson processes, among Lévy processes, stems from their being, in particular, of bounded variation. Thus it doesn't take a notion of stochastic integral with respect to Lévy processes to deal with them, but it suffices to remind the definition of Riemann-Stieltjes integral and extend the standard Itô formula for continuous bounded variation functions to discontinuous bounded variation functions.

Firstly ,we state a deterministic Itô formula, generalized to the case of discontinuous bounded variation functions, which is basically an extended change of variable formula for piecewise smooth functions.

Consider a function $x : [0,T] \longrightarrow \mathbb{R}$ which has a finite number of discontinuities at $T_1 \le T_2 \le \cdots T_n \le T_{n+1} = T$, but is smooth on each interval $]T_i, T_{i+1}[$. We choose x to be càdlàg at the discontinuity points by defining $x(T_i) := x(T_{i+})$. Such a function may be represented as

$$x(t) = \int_0^t b(s)\, ds + \sum_{i,\, T_i \le t} \Delta x_i$$

where $\Delta x_i = x(T_i) - x(T_{i-}$ and the sum takes into account the discontinuities occuring between 0 and t. Consider now a C^1 function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Since on each interval $]T_i, T_{i+1}[$ x is smooth, $f(x(t))$ is also smooth. Therefore we can apply the change of variable formula for smooth functions and write, for $i = 0, \cdots, n$ and $T_0 = 0$ for convention:

$$f(x(T_{i+1-})) - f(x(T_i)) = \int_{T_i}^{T_{i+1-}} f'(x(t))x'(t)\, dt = \int_{T_i}^{T_{i+1-}} f'(x(t))b(t)\, dt.$$

At each discontinuity point, $f(x(t))$ has jump equal to

$$f(x(T_i)) - f(x(T_i)) = f(x(T_{i-}) + \Delta x_i) - f(x(T_{i-})).$$

Adding these two contributions together, the overall variation of f between 0 and t can be written as:

$$\begin{aligned} f(x(T)) - f(x(0)) &= \sum_{i=0}^{n} \left[f(x(T_{i+1})) - f(x(T_i)) \right] \\ &= \sum_{i=0}^{n} \left[f(x(T_{i+1}) - f(x(T_{i+1-})) + f(x(T_{i+1-})) - f(x(T_i)) \right] \\ &= \sum_{i=1}^{n+1} \left[f(x(T_{i-}) + \Delta x_i) - f(x(T_{i-})) \right] + \sum_{i=0}^{n} \int_{T_i}^{T_{i+1-}} f'(x(t))b(t)\, dt. \end{aligned}$$

Finally we obtain the following

THEOREM - A.3.1 (Deterministic Itô formula)**.**
If x is a piecewise C^1 function given by

$$x(t) = \int_0^t b(s)\, ds + \sum_{i=1,\cdots,n+1,\, T_i \le t} \Delta x_i$$

where $\Delta x_i = x(T_i) - x(T_{i-})$, *then for every* C^1 *function* $f : \mathbb{R} \longrightarrow \mathbb{R}$*:*

$$f(x(T)) - f(x(0)) = \int_0^T b(t) f'(x(t-))\, dt + \sum_{i=1}^{n+1} f(x(T_{i-}) + \Delta x_i) - f(x(T_{i-})).$$

Now, we finally present the version of the extended Itô formula for jump-diffusion processes, which is based on both the contributions of the Brownian Itô formula and the deterministic one.

Consider a jump-diffusion process

$$X_t = \mu t + \sigma W_t + Z_t = X^c(t) + Z_t$$

where Z is a compound Poisson process and X^c is the continuous part of X:

$$Z_t = \sum_{i=1}^{N_t} \Delta X_j$$
$$X_t^c = \mu t + \sigma W_t.$$

Define $Y_t = f(X_t)$ where $f \in C^2(\mathbb{R})$ and denote by T_i, $i = 1, \cdots, N_T$ the jumps times of X.

On $]T_i, T_{i+1}[$, X evolves according to

$$dX_t = dX_t^c = \sigma dW_t + \mu dt$$

hence, by applying the Itô formula in the Brownian case we obtain

$$\begin{aligned} Y_{T_{i+1-}} - T_{T_i} &= \int_{T_i}^{T_{i+1-}} \frac{\sigma^2}{2} f''(X_t)\, dt + \int_{T_i}^{T_{i+1-}} f'(X_t)\, dX_t \\ &= \int_{T_i}^{T_{i+1-}} \left\{ \frac{\sigma^2}{2} f''(X_t)\, dt + f'(X_t)\, dX_t^c \right\} \end{aligned}$$

since $dX_t = dX^c(t)$ on this interval.

If a jump of size ΔX_t occurs, then the resulting change in Y_t is given by $f(X_{t-} + \Delta X_t) - f(X_{t-})$. The total change in Y_t can therefore be written as the sum of these two contributions:

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)\, dX_s^c + \int_0^t \frac{\sigma^2}{2} f''(X_s)\, ds + \sum_{0 \le s \le t,\, \Delta X_s \ne 0} \left[f(X_{s-} + \Delta X_S) - f(X_{s-}) \right]. \quad \text{(A.8)}$$

Remark A.19. Replacing dX_S^c by $dX_s - \Delta X_s$ we obtain an equivalent expression:

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})\, dX_s + \int_0^t \frac{\sigma^2}{2} f''(X_s)\, ds + \sum_{0 \le s \le t,\, \Delta X_s \ne 0} \left[f(X_{s-} + \Delta X_S) - f(X_{s-}) - \Delta X_s f'(X_{s-}) \right]. \quad \text{(A.9)}$$

When the number of jumps is finite, which is the case of compound Poisson processes, this form is equivalent to (A.8). However, the form (A.9) is more general: indeed, for instance, if jumps have infinite variation, the sum in Equation (A.8) may not converge, whereas it can be shown that both the stochastic integral and the sum over the jumps in (A.9) are well-defined for any semimartingale.

Here we have only used the Itô formula for diffusions, which is of course still valid if σ is replaced by a $\mathcal{F}_t$−measurable square-integrable process. Indeed we have the following general

THEOREM - A.3.2 (Itô formula for jump-diffusion processes)**.**
Let X be a diffusion process with jumps, defined as the sum of a drift term, a Brownian stochastic integral and a compound Poisson process:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \sum_{j=1}^{N_t} \Delta X_i$$

where b_t and σ_t are continuous, progressively measurable (with respect to the filtration $\mathcal{F}_t$) processes with

$$E\left[\int_0^T \sigma_t^2\, dt\right] < +\infty.$$

Then, for any $C^{1,2}$ function $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, the process $Y_t = f(t, X_t)$ can be represented as

$$\begin{aligned} f(t,X_t) - f(0,X_0) = &\int_0^t \left[\frac{\partial f}{\partial s}(s,X_s) + \frac{\partial f}{\partial x}(s,X_s) b_s\right] ds \\ &+ \frac{1}{2}\int_0^t \sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s,X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s,X_s)\sigma_s\, dW_s \\ &+ \sum_{\{i\geq 1, T_i \leq t\}} \left[f(X_{T_{i-}} + \Delta X_i) - f(X_{T_{i-}})\right]. \end{aligned}$$

In differential notation:

$$\begin{aligned} dY_t = &\frac{\partial f}{\partial t}(t,X_t)dt + b_t \frac{\partial f}{\partial x}(t,X_t)dt + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)dt \\ &+ \frac{\partial f}{\partial x}(t,X_t)\sigma_t dW_t + \left[f(X_{t-} + \Delta X_t) - f(X_{t-})\right]. \end{aligned}$$

A.3.2 Feynman-Kac representation

Similarly to the diffusion case, in this section, we examine the deep connection between SDEs with jumps and partial integro-differential equations (PIDEs), where the trait d'union is the Itô formula A.3.2.

Let $a : [0,T] \longrightarrow \mathbb{R}$ be a bounded (L^∞) function and $\sigma : [0,T] \longrightarrow \mathbb{R}^+$ be a positive bounded function. Let us denote by Π^* a Poisson random measure on $[0,T] \times \mathbb{R}$ with intensity $m^*(dy, dt) = m(dy)\, dt$ with m a Lévy measure and $\tilde{\Pi}^*$ the compensated version of Π^*, i.e. $\tilde{\Pi}^*(A) = \Pi^*(A) - \int_A dt\, m(dy)$. For a given $t \in [0,T[$, $x \in \mathbb{R}$ we define the jump process $\left(X_s^{t,x}\right)_{s\in[t,T]}$ by

$$X_s^{t,x} = x + \int_t^s a(u)\, du + \int_t^s \sigma(u)\, dW_u + \int_t^s \int_{|y|\geq 1} y\, \Pi^*(du, dy) + \int_t^s \int_{|y|\leq 1} y\, \tilde{\Pi}^*(du, dx). \quad \text{(A.10)}$$

$X_s^{t,x}$ is the position at time $s > t$ of a jump process starting in x at time t and having drift $a(\cdot)$, a time-dependent volatility $\sigma(\cdot)$ and a jump component described by a (pure jump) Lévy process with Lévy measure m.

Remark A.20. If $\sigma(t) = \sigma$ and $a(t) = a$ then $X_S^{t,x} = x + X_{s-t}$ where X is a Lévy process with Lévy triplet (a, σ^2, m) and (A.10) is simply the Lévy-Itô decomposition.

THEOREM - A.3.3 (Feynman-Kac representation)**.**
Consider a bounded function $h \in L^\infty(\mathbb{R})$. If

$$\exists c, \bar{c} > 0,\ \forall t \in [0,T],\ c \geq \sigma(t) \geq \bar{c}$$

then the Cauchy problem

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) + \frac{\sigma^2(t)}{2}\frac{\partial^2 f}{\partial x^2}(t,x) + a(t)\frac{\partial f}{\partial x}(t,x) + \int_{\mathbb{R}}\left[f(t,x+y) - f(t,x) - y\mathbb{1}_{|y|\leq 1}\frac{\partial f}{\partial x}(t,x)\right]\, m(dy) = 0, \quad \forall x \in \mathbb{R} \\ f(T,x) = h(x) \end{cases}$$

has a unique solution given by

$$f(t,x) = E\left[h(X_T^{t,x})\right],$$

where $X_T^{t,x}$ is the process given by (A.10).

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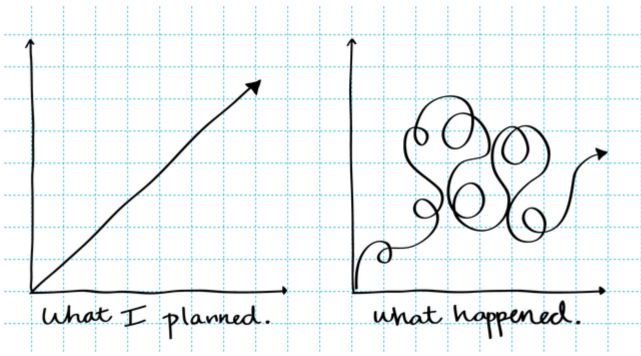
RINGRAZIAMENTI



Vorrei concludere la mia tesi con questo simpatico grafico nel quale vedo raffigurato il mio percorso universitario. Mi sono iscritta all'Università con la convinzione che sarei diventata professoressa di Matematica, mentre ora, dopo cinque anni, presento una tesi in Finanza Quantitativa.

Per questo, desidero ringraziare il Professor Pascucci, Relatore di questa tesi, per aver sempre ricevuto da lui approvazione e sostegno per la realizzazione di questa tesi in collaborazione con l'Università di Aarhus.
Un ringraziamento particolare va alla Professoressa Elisa Nicolato, Co-relatrice di questa tesi, per avermi introdotto, con la sua passione, alla Finanza Quantitativa e, in particolare, all'argomento di cui tratta questa tesi.
Desidero ringraziare inoltre l'Università di Bologna, in particolare la Dott.ssa Alice Barbieri, per avermi autorizzato ed affiancato nel portare a termine il mio percorso universitario in Danimarca, e l'Università di Aarhus per avermi dato la possibilità di seguire validi corsi che hanno arricchito la mia formazione universitaria e contribuito alla stesura di questa tesi.
Un ringraziamento particolare va ad Andrea per il suo continuo incoraggiamento e le ore spese a dirimere i miei dubbi. Grazie per avermi con amore supportata (e anche sopportata) durante la stesura di questo elaborato. E grazie anche per tutto il resto, ma per questo preferisco farlo a voce.
Vorrei ringraziare anche due amici, Stefano e Buket, per aver speso parte del proprio tempo a leggere le bozze del mio lavoro. Grazie per avermelo chiesto, mi ha fatto molto piacere. Sottolineo comunque che la responsabilità per ogni errore contenuto in questa tesi è tutta mia.
Last but not least, un grazie immenso va alla mia famiglia. Mamma e Babbo, grazie per il vostro amore e per il vostro sostegno incondizionato.