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Monoidal categories for the Physics of integrable models

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Abstract

Scopo di questa tesi é di evidenziare le connessioni tra le categorie monoidali, l'equazione di Yang-Baxter e l'integrabilitá di alcuni modelli. Oggetto principale del nostro lavoro é stato il monoide di Frobenius e come sia connesso alle C^*algebre. In questo contesto la totalitá delle dimostrazioni sfruttano la strumentazione dell'algebra diagrammatica, nel corso del lavoro di tesi sono state riprodotte tali dimostrazioni tramite il piú familiare linguaggio dell'algebra multilineare allo scopo di rendere piú fruibili questi risultati ad un raggio piú ampio di potenziali lettori.

Introduction

In both Physics and Mathematics the problem of studying integrable models as solvable models has always received a great deal of attention since its very first appearance. In particular, within quantum theories, integrability is strictly connecetd to the Yang-Baxter equation. As a matter of fact, this equation arises in several different models, such as

- 1+1D integrable field theories and conformal field theories
- Quantum spin chains
- Quantum groups

Indeed, theoretical Physics has been interested in 1+1D field theories for several years. Among them a prominent role is undoubtedly played by conformal field theories, *i.e.* theories that are invariant under the action of the Poincaré group and scale transformations. They display interesting properties. Notably, they are integrable, that is to say they have infinitely many conserved quantities.
Furthermore, conformal theories are closely related to quantum spin chains in so far as the critical points are scale invariants. This means that a perturbation of a conformal theory can give more information about the way to approach to the critical points. Generally speaking, symmetries play a vital role in Physics in light of Noether's theorem. As symmetry transformations are straightfowardly seen to form a group under composition, Algebra has always been the natural language to deal with them. Along this line, studing the quantum inverse scattering method, R.J. Baxter, A.B. Zamolodchikov and Al.B. Zamolodchikov discovered a new algebraic structure, later generalized and called quantum group by Drinfield. What's more, it is still possible to define a Yang-Baxter equation in a categorical framework by using the powerful tool provided by braided monoidal categories (abstract categories modeled on the basic example of $\mathfrak{Vec}_{\mathbb{K}}$, the category of vector spaces over a field $\mathbb{K}$).

Although the standard axiomatic presentation of quantum mechanics, essentially due to von Neumann (1932), has long provided the mathematical bedrock of the subject, the advent of quantum information has given rise to new kinds of questions and mathematical exigences. To take but one example, it is enough to consider the changes of perception of quantum entanglement after Samson Abramsky and Bob Coecke developed a categorical formulation of quantum mechanics. A very notable structure has thus arisen in their studies: that of Frobenius monoid.

Through my thesis work I tried to examine the categorical connection between the Yang-Baxter equation, quantum groups and 1+1D topological quantum field theories. I focused my interest mainly on **FdHilb**, the category of finite dimensional Hilbert spaces, because, among other things, B. Coecke, D. Pavlovic and J. Vicary proved that it is possible to turn any Frobenius monoid into an orthogonal basis of a Hilbert space and *viceversa*. One of their most remarkable results, furthermore, states that Frobenius monoids can be endowed with a C^*-algebra structure in a natural way. Because their proofs strongly rely on diagrammatic algebra, a topic that is seldom discussed in basic courses, we have decided to reproduce them by using, instead, the more familiar language of multilinear algebra in order to give a more readable and accessible presentation to the topic as to attract a wider range of potential readers.

Un pensiero permanente a DASY

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Chapter 1

A short introduction to classical integrable systems

1.1 Hamiltonian systems

Let M be a differentiable n-manifold and let m be a point on M of coordinates $(q_1, ..., q_n)$.These identify $\vec{q}$, a contravariant vector on M, *i.e.* the components of $\vec{q}$ transform by inverse change of basis matrix . Let $t \in \mathbb{R}$, the vectors $\vec{q}(t)$ parametrize a regular curve on M, for each point on this curve we identify with $\dot{\vec{q}}$ the tangent vectors to M and with the couple $(\vec{q}(t), \dot{\vec{q}}(t))$ we can parametrize the tangent space $T_m M$ to manifold M. The unions over all the points in M of the tangent space is called *tangent bundle*, TM.

Let $L : TM \to \mathbb{R}$ be a regular function, we fix two point on M with coordinates $\vec{q}(t_i)$ and $\vec{q}(t_f)$ on a regular curve $\vec{q}(t)$, we consider

$$\mathcal{I}(t_i, t_f) = \int_{t_i}^{t_f} \mathcal{L}(\vec{q}(t_i), \vec{q}(t_f))$$

$\mathcal{L}$ is called *lagrangian* and $\mathcal{I}$ is the *action integral.*
The critical points of action integral are the trajectory of a mechanical system, these are given by famous Euler-Lagrange equation :

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = 0\,. \tag{1.1}$$

We set $\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}$, $\vec{p}$ is a covariant vector i.e. it transforms by change of basis matrix.The transformation

$$H(p, q) = p\dot{q} - \mathcal{L}(p, q) \tag{1.2}$$

is called *Legendre transformation* and the fuctional H is said *hamiltonian* of the system. The couple (p, q) parametrize the cotangent space $T^*_m M$, the union over all the points of M is the cotanget bundle.

From the hamiltonian we can extract the equations of motion

$$\frac{\partial H}{\partial p} = \dot{q}; \tag{1.3}$$

$$\frac{\partial H}{\partial q} = -\dot{p}; \tag{1.4}$$

A dynamical system satisfying these equations is called hamiltonian system. For this kind of systems there exist a geometrical description based on vector field called Poisson bracket.

1.2 Vector fields and simplectic structure

Definition 1.1. A vector field X is a first order differential homogeneus operator defined by setting

$$X = \sum_{k=1}^{n} X_k \frac{\partial}{\partial x_k} \tag{1.5}$$

where the $X_k = X_k(x_1, ..., x_n)$ are functions of dynamical variables. The set of vector fields on a manifold M will be denoted by $\nu^1(M)$.
A vector field is a derivation *i.e.* it fulfills

$$i) X(f + g) = X(f) + X(g) \tag{1.6}$$

$$ii) X(fg) = X(f)g + fX(g) \tag{1.7}$$

for all $f, g : \mathbb{R}^n \to \mathbb{R}$ or more generally from a smooth manifold M locally isomorphic to $\mathbb{R}^n$.
It follows that the action of the vector fields on scalars is :

$$X(c) = 0, \quad \forall c \in \mathbb{R}^n. \tag{1.8}$$

The most important operation between vector fields is the Lie bracket.

Definition 1.2. Let X, Y be vector fields . Their Lie Bracket is defined by setting :

$$[X, Y] = X \circ Y - Y \circ X. \tag{1.9}$$

It is easy to prove that $[X, Y]$ is a vector field, namely

$$Z = \sum_{k=1}^{n} Z_k \frac{\partial}{\partial x_k} \tag{1.10}$$

where

$$Z_k = X(Y_k) - Y(X_k). \tag{1.11}$$

Some important properties of Lie brackets:

- Bilinear
- skew-symmetric
- Jacobi identity .

A very important tool in classical mechanics are the Poisson brackets .

Definition 1.3. Given two functions $f, g \in C^\infty(T^*M)$, the Poisson bracket of (f, g) is the function defined by setting

$$\{f, g\} = \sum_{i,j}^{2n} \frac{\partial f}{\partial x_i} (J_0)_{ij} \frac{\partial g}{\partial x_j} \tag{1.12}$$

where

$$J_0 = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}. \tag{1.13}$$

The coordinates $x_1, ..., x_{2n}$ are usually written in the form $p_1, ..., p_n$, $q_1, ..., q_n$ and therefore the Poisson bracket takes the form

$$\{f, g\} = \sum_{i}^{n} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_j} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_j}. \tag{1.14}$$

Definition 1.4. Let f be regular function. We call *hamiltonian vector field* of f the vector field

$$X_f = \sum_{i,j}^{2n} \frac{\partial f}{\partial x_i} (J)_{ij} \frac{\partial}{\partial x_j}. \tag{1.15}$$

This definition means that

$$X_f(g) = \{f, g\} \tag{1.16}$$

A sympletic square matrix M is a matrix of order $2n$ such that

$$M^T J M = M. \tag{1.17}$$

Now suppose to change the coordinates p, q

$$(p', q') = S(p, q) \tag{1.18}$$

where S is a regular function. If the matrix associated to the transformation is a sympletic matrix we say that S is a canonical transformation . A *canonical transformation* preserves the Poisson bracket therefore we can define a new matrix $J = MJ_0$ with $M \in \mathbf{Sp}(2n, \mathbb{R})$, the vector space of symplectic matrices $2n \times 2n$. Now the Poisson brackets are skew-symmetric, fulfill the Jacobi identity and the Leibniz rule. It is also possible to define Poisson brackets by means of sympletic matrix or more generally in the following form :

Definition 1.5. A sympletic manifold is a couple (M, ω) where M is a smooth manifold and ω is a 2-closed non-degenerate form

$$\omega = \omega_{ij}(x) dx_i \wedge dy_j \tag{1.19}$$

where $det(\omega(x))_{ij} \neq 0 \quad \forall x \in M$.

1.3 Hamiltonian systems and Liouville integrability

Definition 1.6. A Hamiltonian system is a dynamical system for which the equations of motion are in this form

$$\dot{x} = \{x, H\} \tag{1.20}$$

where $x = (x_1, ..., x_{2n})$ and the function H is the hamiltonian of the system.

Definition 1.7. A function F on a sympletic manifold M is a called constant of motion if only if

$$\partial_t F + \{F, H\} = 0. \tag{1.21}$$

Definition 1.8. If $\{f, g\} = 0$ we say that f, g are in **involution**.

Definition 1.9. (Liouville integrability) A Hamiltonian system on a $2n$ sympletic manifold M is completely integrable if it has n constants of motion functionally independent and in involution.

1.3.1 Isospectral deformation method

One of the most powerful tool in the study of integrability for dynamical systems is the isospectral deformation method, introduced for the first time in 1968 from Lax. The term " isospectral deformation" was suggested later by Moser, in 1975.
The basic idea of this method, for the finite dimensional case, is to find two operators, i.e. two square matrices, L and M such that the equations of motion $\dot{x} = \{x, H\}$ take the form

$$\dot{L} = [L, M] \tag{1.22}$$

where $[,]$ is the usual commutator between square matrices. L is called Lax matrix.

$$L(t) = U^{-1}(t)L(0)U(t) \tag{1.23}$$

where $U(t)$ is the solution of this equation

$$\dot{U}(t) = M(t)U(t). \tag{1.24}$$

The most important property is the temporal independence of the eigenvalues of the matrix $L(t)$. It is said, hence the matrix L undergoes an *iso-spectral deformation* .

1.4 Lie-Poisson Groups, Lie bialgebras and Yang-Baxter equation

A class of examples of Poisson manifolds is given by Poisson-Lie groups introduced by Drinfel'd . The motivations of the introduction of these facilities are located mainly in the study of quantum groups. For more details about geometry of Poisson brackets see [3] .

Definition 1.10. A PoissonLie group is a Lie group G equipped with a Poisson bracket for which the group multiplication

$$\mu : G \times G \to G \tag{1.25}$$

is a Poisson map, $\mu \in P^{\infty}(G \times G, G)$.

Definition 1.11. Let G be a Lie group and V a representation of G, let $\mathfrak{g}$ be its corresponding Lie algebra and $\mathfrak{g}^*$ its dual . A map $\gamma : G \to V$ such that

$$\gamma(gh) = g \cdot \gamma(h) + \gamma(g) \tag{1.26}$$

is called 1-cocycle.

Drinfeld proved the following theorem in [8],[10].
Theorem 1.12 Let G be a Lie-Poisson group. Let π be a tensor such that the Schouten-Nijenhuis bracket of π is 0, $[\pi, \pi]_s = 0$, and $\gamma = D\pi$ the derived of tensor. Then the dual map

$$\gamma^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^* \tag{1.27}$$

defines a Lie bracket on $\mathfrak{g}^*$.

Definition 1.12. A Lie bialgebra is a couple $(\mathfrak{g}^*, \gamma)$ where $\mathfrak{g}^*$ is a Lie algebra, $\gamma : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$ is a 1-cocycle respect to adjoint representation which makes the dual $\mathfrak{g}^*$ a Lie algebra.
This means

$$\gamma[X, Y] = [\gamma X, Y] + [X, \gamma Y]. \tag{1.28}$$

So the Drinfeld's theorem says the the Lie algebra of a Poisson-Lie is a Lie bialgebra : the structure of manifold is reflected in brackets on $\mathfrak{g}$, the Poisson bracket is reflected in the bracket on $\mathfrak{g}^*$ and the compatibility between Poisson structure and Lie product is codified in the cocycle condition.

More generally :

Theorem 1.14(Drinfel'd) There exists a functor between the category of Poisson-Lie groups and the cateogory of Lie bialgebras. If we ristict to the connected or simply connected groups , the categories are equivalent.
A complete discussion about this fact can be found in [21].

A particularly interesting class of these bialgebras can be found by considering the theory of Yang-Baxter equation, which originates in statistical mechanics . A general formulation of the theory of Yang-Baxter can be given presented as follows :
Let $\mathfrak{g}$ be a Lie algebra and R a linear map

$$R : \mathfrak{g} \to \mathfrak{g}. \tag{1.29}$$

Then we can define a bilinear skew-symmetric application by setting

$$[X, Y]_R = [RX, Y] + [X, RY]. \tag{1.30}$$

Definition 1.13. A classical R-matrix over a Lie algebra is a linear operator such that $[,]_R$ is a Lie bracket .

When $R : \mathfrak{g} \to \mathfrak{g}$ is a R-matrix we can consider the Lie-Poisson structure on $\mathfrak{g}^*$ induced by Lie brackets $[,]_R$, that has relevance in the theory of diffusion: in particular it can be proved that the Casimir's functions,with respect to the usual Lie brackets on $\mathfrak{g}$ forms a commutatative subalgebra respect the $[,]_R$ and the equations of motion respect to the Lie-Poisson brackets induced by $[,]_R$ are in the Lax form, (cfr. [17],[19]) .

Now introduce the new notation

$$B_R(X, Y) = [RX, RY] - R[RX, Y] - R[X, RY] \tag{1.31}$$

then the Jacobi identity for the $[,]_R$ becomes

$$[B_R(X,Y),Z]+[B_R(Y,Z),X]+[B_R(Z,X),Y]=0. \tag{1.32}$$

this is called the *classical Yang-Baxter equation*(cYBE).

We note that if R is an R-matrix such that

$$B_R(X,Y)=0 \tag{1.33}$$

we have

$$[RX,RY]=R[X,Y]_R \tag{1.34}$$

i.e. R is a homomorphism between Lie algebras and this is a natural constraint to fulfill, called *classical Yang-Baxter equation.*
Another important condition follows from the YBE

$$B_R(X,Y)=\alpha[X,Y] \quad \alpha\in\mathbb{K}. \tag{1.35}$$

In this way the cYBE becomes the Jacobi identity for $[,]$ and the latter is the *modified Yang-Baxter equation.*

Chapter 2

Quantum integrability and Yang-Baxter equation

In the previous chapter we saw that a Hamiltonian system with a integral of motion is integrable. In this chapter we will see two examples of quantum integrable models : 1+1D field theory and quantum spin chains. Above all our target is to show how the Yang-Baxter equation arises in thes models.
The general S-matrices of a field theories are complicated objects, even in 1+1 dimension . In the years 1975-1980 some papers by Polyakov, Parke, Zamolodchikov and Zamolodchikov, [16],[23] [24], proved that, for 1+1 dimensional models, the existence of conserved charges implies no production of particles in the scattering process. This means that matrix S can be factorized in two-body interactions.
A model for spin chain was proposed for the first time by Heisemberg in 1930 and it is connected to magnon quasiparticles introduced by Bloch to explain the reduction of the spontaneous magnetization in a ferromagnet. We will see two models, XXX and XXZ and how they are stryctly related to the definition of classical integrability and the Yang-Baxter equation. An important structure to connect to the quantum spin chains and to 1+1D field theory is the Yangian. We will dedicate space to its analysis.

2.1 Introduction to quantum spin chains

This part is largely inspired by de Leeuw M. : Introduction to integrability, lecture notes corse in ETH .

Quantum spin-chains are particular examples of exactly solvable or "quantum integrable" systems in 1+1 spacetime dimensions. Consider a ring of atoms which periodic boundary conditions. Each of which possesses a quantum "degree of freedom", called a "spin", which can point in two directions, up or down. "Quantum" means that we allow for all positions of the different possible spin configurations of the ring, this set forms the physical state space.

A much studied model is the Heisenberg spin-chain. Historically, Bethe's 1931 work on the isotropic case known as the XXX model, had a major impact and was the starting point for many of the subsequent developments in this area. He made an "ansatz" for the stationary states of the XXX spin-chain to be a superposition of plane waves whose momenta/wave vectors have to satisfy an intricate set of non-linear equations, called Bethe's equations. In the literature his approach is nowadays referred to as "coordinate Bethe ansatz" and has been applied to numerous other quantum integrable systems. It is the combinatorics and the algebraic aspects behind Bethe's ansatz which are of mathematical importance.

Many-particle systems, quantum or classical, are usually quite difficult to solve, and except for a few cases, one faces often formidable difficulties in the computations of physically relevant quantities.

The Heisemberg spin-chain nowadays can be experimentally realized in condensed matter systems *e.g.* Mott insulators and the correlation functions can be measured in the laboratory.

The Heisenberg spin chain

This is a one-dimensional model of magnetism or simply of spin-$\frac{1}{2}$ particles that have a spin-spin interaction. In certain metals where there is a one-dimensional isotropy these spin chain appear and describe the dominant physical behaviour .
The spin chain simply consists of N sites, where on each site we consider a spin-$\frac{1}{2}$ particle (for example an electron). This electron can have spin up or down and therefore any electron is in a linear state

$$a\left|\uparrow\right\rangle + b\left|\downarrow\right\rangle \quad : |a|^2+|b|^2=1$$

in a two-dimensional Hilbert space. In a system with N electrons the total Hilbert space where the physical states live in is

$$H=\bigotimes_N \mathbb{C}^2. \tag{2.1}$$

The spin operators $S_i^{x,y,z}$ act on each site i and they satisfy local commutation relations in the sense that

$$[S_h^a, S_k^b]=i\delta_{hk}\varepsilon^{abc}S_h^c \quad . \tag{2.2}$$

The Hamiltonian describes a nearest neighbor spin-spin interaction. More precisely, we have

$$\mathcal{H}=\frac{JN}{4}-J\sum_i \vec{S}_i\cdot\vec{S}_{i+1}, \quad \vec{S}_{N+1}=\vec{S}_1 \quad . \tag{2.3}$$

Let us introduce the usual raising and lowering operators

$$S^{\pm}=S^x\pm iS^y \tag{2.4}$$

such that

$$S^+\left|\uparrow\right\rangle=0, \qquad S^-\left|\uparrow\right\rangle=\left|\downarrow\right\rangle, \qquad S^z\left|\uparrow\right\rangle=\frac{1}{2}\left|\uparrow\right\rangle \tag{2.5}$$

$$S^+\left|\downarrow\right\rangle=\left|\uparrow\right\rangle, \qquad S^-\left|\downarrow\right\rangle=0, \qquad S^z\left|\downarrow\right\rangle=-\frac{1}{2}\left|\downarrow\right\rangle \quad . \tag{2.6}$$

Then we can rewrite the Hamiltonian as

$$\mathcal{H}=\frac{JN}{4}-J\sum_i\left[\frac{1}{2}\left(S_i^+S_{i+1}^-+S_i^-S_{i+1}^+\right)+S_i^zS_{i+1}^z\right] \quad . \tag{2.7}$$

Let us look at the different terms. The terms involving $S^{\pm} = S^x \pm iS^y$ are called hopping terms since they move a spin up or spin down to a neighboring site. There is a constant term proportional to N added for convenience. This is a rudimentary model of (ferro)magnetism. It is an overall shift of the energy levels depending on the sign of J. Written out in components, the Hamiltonian is a special case of a more general Hamiltonian which takes the form

$$\mathcal{H} = \sum_i (J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z) \quad . \tag{2.8}$$

This model is usually called the XYZ spin chain. In the case $J_x = J_y$ it is called the XXZ spin chain and our model is $J_x = J_y = J_z = J$ referred to as the XXX spin chain .

Symmetries

To look the symmetries of the system is one way to reducing the size of Hamiltonian. Consider the operator

$$S^z = \sum_i S_i^z \tag{2.9}$$

which measures the total number of up or down spins. It is easy to check that it commutes with the Hamiltonian. This implies that the Hibert space decomposes to in subspaces of fixed numbers of spin up or down. The spin operators form an $\mathfrak{su}(2)$ algebra and consequently this spin chain has $\mathfrak{su}(2)$ as a symmetry of algebra and this means that the eigenstates of the Hamiltonian will arrange themselves in multiplets of $\mathfrak{su}(2)$.

2.1.1 The XXX spin chain

The XXX spin chain is exactly solvable i.e. the spectrum of the Hamiltonian is known. This is possible via coordinate Bethe Ansatz, a Ansatz for the eigenstates of the Hamiltonian. It was used by Bethe in 1931 and after, this technique has been applied to more general models. The idea behind Bethe Ansatz is to consider a reference state which is an eigenstate of the Hamiltonian where all the spins are up and then flip some spins. These spins will behave like quasi-particles called magnons.

Consider first the ferromagnetic case.
Ground state . The total spin is conserved, this implies that the state with all spins aligned is a eigenstate of the Hamiltonian. This is the ferromagnetic vacuum, let us define the vacuum to be

$$|0\rangle = |\uparrow\uparrow ... \uparrow\uparrow\rangle. \tag{2.10}$$

and the energy is $\mathcal{H}|0\rangle = 0$.
Excited states, called **magnons**, are obtained by the action of S_n^-. In general we write

$$|n_1, ..., n_k\rangle = S_{n_1}^- ... S_{n_k}^- |0\rangle. \tag{2.11}$$

Every eigenstate with k flipped spins is a linear combination of $|n_1, ..., n_k\rangle$

$$|\psi\rangle = \sum_{1\leq n_1 < ... n_k \leq N} a(n_1, ..., n_k))|n_1, ..., n_k\rangle \tag{2.12}$$

with some unknown coefficients $a(n_1, ..., n_k)$. The periodicity can be formulated as

$$a(n_2, ..., n_k, n_1 + N) = a(n_1, ..., n_k). \tag{2.13}$$

The **Bethe Ansatz** postulates the form of these coefficient to be

$$a(n_1, ..., n_k) = \sum_{\sigma\in S_k} A_\sigma e^{ip_{\sigma_i} n_i} \tag{2.14}$$

this is just a plain-wave type Ansatz.

Bethe equations . Now using the eigenfunctions, we write the Bethe equations, that arise from periodicity conditions

$$e^{ip_i N} = \prod A(p_j, p_i). \tag{2.15}$$

Again the interpretation is rather simple and corresponds to moving the i-th particle around the spin chain. Written out in terms of the rapidity, $u_i = 2\cot\frac{p_i}{2}$ it simply becomes

$$\left[\frac{u_i + \frac{i}{2}}{u_i - \frac{i}{2}}\right]^N = \prod_{i \neq j} \frac{u_i - u_j + i}{u_i - u_j - i} \tag{2.16}$$

The energy in terms of the rapidity is given by

$$E = \frac{2J}{4 + u^2} \tag{2.17}$$

We can find the spectrum by solving the Bethe equations and summing the energies of the different magnons.

Monodromy and R-matrix

In this part we'll show the connection between the YBE, Lax pair and the quantum spin chain.
Take again a chain with N sites and corresponding Hilbert space

$$\mathcal{H} = \bigotimes_i H_i.$$

In our case $H_i = \mathbb{C}^2$. A Lax operator is an endomorphism

$$L : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$$

and in our model takes the form

$$L_{nm}(u) = u \otimes \mathbb{I} + iS_i^m \otimes \sigma_i^m \tag{2.18}$$

where σ_i^m are the Pauli matrices acting on site m. For spin-$\frac{1}{2}$ they are connected to the spin operator as $S^j = \frac{1}{2}\sigma^j$. Using the permutation operator we write

$$L_{nm}(u) = (u - \frac{i}{2}) \otimes \mathbb{I} + i\mathcal{P}_{nm} \quad . \tag{2.19}$$

Remembering the commutation relation(CR) for Spins operator

$$[S^l, S^m] = i\varepsilon^{lmk} S^k \,, \tag{2.20}$$

it is possible to write the CR of the Lax matrix

$$R_{mj}(u_1 - u_2)L_{nm}(u_1)L_{nj}(u_2) = L_{nj}(u_2)L_{nm}(u_1)R_{mj}(u_1 - u_2) \tag{2.21}$$

where R is the quantum R-matrix and has the form

$$R_{mj} = \lambda \otimes \mathbb{I} + i\mathcal{P}_{mj} \quad . \tag{2.22}$$

Any R-matrix which fulfills this CR has to satisfy the quantum YBE :

$$L_1 L_2 L_3 = R_{12}^{-1} R_{13}^{-1} L_2 L_3 L_1 R_{12} R_{13} = R_{12}^{-1} R_{13}^{-1} R_{23}^{-1} L_3 L_2 L_1 R_{23} R_{13} R_{12} \tag{2.23}$$

and

$$L_1 L_2 L_3 = R_{23}^{-1} R_{13}^{-1} L_2 L_3 L_1 R_{13} R_{23} = R_{23}^{-1} R_{13}^{-1} R_{12}^{-1} L_3 L_2 L_1 R_{12} R_{13} R_{23} \tag{2.24}$$

hence both relations coincide and the R-matrix must satisfy the

$$\boxed{R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.}$$

Using Lax we define the monodromy matrix

$$T_n = L_{N,n}(u) \ldots L_{1,n}(u). \tag{2.25}$$

It can be seen as a matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{2.26}$$

Where $A(u), B(u), C(u)$ and $D(u)$ are operators acting on a Hilbert space. From T it is possible derive a set of conserved charges that characterize integrable systems.

We need first the CR between $A(u), B(u), C(u)$ and $D(u)$. These can be found using the fundamental CR for the Lax operator. We define a transfer matrix

$$t(u) = tr_n T_n = A(u) + D(u) \tag{2.27}$$

and by cyclicity of the trace we find

$$[t(u_1), t(u_2)] = 0. \tag{2.28}$$

Now we expand $t(u)$ around the point $u = \frac{i}{2}$. Using various properties of permutation it is possible to find the important expression

$$\frac{d}{du} \ln t(u) \Big|_{u=\frac{i}{2}} = -i \sum_n \mathcal{P}_{n,n+1}. \tag{2.29}$$

Since we can express P in terms of Pauli matrices it is possible write the Hamiltonian

$$H = -\frac{J}{2} \sum_n \mathcal{P}_{n,n+1} \tag{2.30}$$

and we find that the **transfer matrix generates a set of conserved quantities** .

Algebraic Bethe Ansatz - The transfer matrix generates a set of commuting conserved quantities so we can diagonalize them simultaneously. Now we can find, in addition to the spectrum of Hamiltonian, the spectrum of all the conserved quantities by a different kind of the Bethe Ansatz called algebraic BA based on a different use of the monodromy matrix. The fundamental ingredient of the algebraic Bethe Ansatz approach were the FCR, which are completely described in terms of the R-matrix. Any R-matrix that satisfies the Yang-Baxter equation is associated to a integrable spin chain.

Relation between R and L

Proof. We suppose that $R_{ij}(u_i - u_j)$ is such that

$$\boxed{R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.}$$

and for some λ

$$R(\lambda) = \mathcal{P} \tag{2.31}$$

We define the Lax operator L

$$L_{nm}(u) = R_{nm}(u - \mu) \quad \mu \in \mathbb{K} \tag{2.32}$$

then using hypothesis L satisfies the fundamental CR and now we can define the monodromy matrix and t-matrix in the usual way . By the fundamental commutation relations the transfer matrix defines a family of commuting quantities. Since $R(\lambda)$ is the special point where the R-matrix becomes zero. The t-matrix at $x = \lambda - \mu$ becomes

$$t(x) = e^{iP}. \tag{2.33}$$

Now consider the derivative of monodromy matrix and use $\mathrm{tr}_j \mathcal{P}_{N,j} = 1 \quad \mathcal{P}_{ij}^2 = 1$ to find

$$\mathcal{H} \equiv \frac{dt}{du}(x)t^{-1}(x) = \sum_n \frac{dL_{n,n+1}}{du}(x)\mathcal{P}_{n,n+1}. \tag{2.34}$$

Recalling the relation between the Lax operator and the R-matrix and by switching the permutation with the Lax operator we arrive at the desired result.

2.1.2 The XXZ Spin chain

We recall briefly the results regarding this model without explicitly repeat the calculations, which are quite similar to the XXX case.

The Hamiltonian

We expose briefly the XXZ model for N spin-$\frac{1}{2}$ particles .The Hamiltonian is

$$\mathcal{H} = \Delta\frac{JN}{4} - J\sum_i \frac{1}{2}S_i^+S_{i+1}^- + S_i^-S_{i+1}^+ + \Delta S_i^zS_{i+1}^z \tag{2.35}$$

where $\Delta \in \mathbb{R}$ and $S_{N+1}^a = S_1^a$. This model is completely integrable by using both the Ansatz, algebraic and coordinate. It easy to prove that

$$[\mathcal{H}, S^z] = 0. \tag{2.36}$$

As consequence the total 3rd component of the spin is conserved and the states organize in sectors of given S^z.

The coordinate Bethe Ansatz

The $|0\rangle$ is defined in the same way for the model XXX

$$|0\rangle = |\uparrow\uparrow ... \uparrow\uparrow\rangle. \tag{2.37}$$

where the energy $E_0 = 0$. Next step is to find the eigenstates of the Hamiltonian in the case one spin is flipped.
Take the state

$$|k\rangle = \sum_n e^{ikn}S_n^-|0\rangle = e^{ik}|\downarrow\uparrow\uparrow ...\rangle + e^{2ik}|\uparrow\downarrow\uparrow ...\rangle + e^{3ik}|\downarrow\uparrow\uparrow ...\rangle + ... \tag{2.38}$$

$|k\rangle$ is an eigenstate of the Hamiltonian with eigenvalue

$$E(k) = \frac{1}{2} = J(2\Delta - e^{ik} + e^{-ik}) \tag{2.39}$$

this because $e^{iN} = 1$ as consequence of *p.b.c.* .
To determine the scattering phase we use this state

$$|k_1, k_2\rangle = \sum_{j_1<j_2} \left[e^{ik_1j_1+k_2j_2} + Ae^{ik_2j_1+k_1j_2}\right] S_{j_1}^-S_{j_2}^-|0\rangle. \tag{2.40}$$

The Ansatz for $|k_1, k_2\rangle$ implies that

$$E(k_1) + E(k_2) \tag{2.41}$$

is the eigenvalue of the $|k_1, k_2\rangle$.
Considering $|... \downarrow_j \downarrow_{j+1}\rangle$ in the Ansatz for the wave function it is possible to prove that

$$A = -\frac{e^{(k_1+k_2)} + 1 - 2\Delta e^{ik_2}}{e^{(k_1+k_2)} + 1 - 2\Delta e^{ik_1}}. \tag{2.42}$$

As s consequence of *p.b.c.*, the Bethe eq. as for the XXX chain hold

$$e^{ik_j N} = \prod_{j \neq l} A(k_l, k_j). \tag{2.43}$$

Introducing $\Delta = \cos \hbar \quad e^{ik} = \frac{\sinh \hbar(u+\frac{i}{2})}{\sinh \hbar(u-\frac{i}{2})}$ the Bethe eq. takes the form

$$\left[\frac{\sinh \hbar(u_j + \frac{i}{2})}{\sinh \hbar(u_j - \frac{i}{2})}\right]^N = \prod_{l \neq j} \frac{\sinh \hbar(u_j - u_l + i}{\sinh \hbar(u_j - u_l - i)} \tag{2.44}$$

and now if we send $\hbar \to 0$ we recover the XXX Bethe eq.

Some limiting/special cases of the XXZ model

Let us now consider some interesting limiting cases for the anisotropy parameter in the Heisenberg XXZ Hamiltonian.

- $\Delta = 1$: we obtain the XXX Hamiltonian
- $\Delta = 0$ yields the so-called XX model. Via a Jordan-Wigner transformation, one can map this model to free fermions on a lattice.
- $\Delta J \to \infty$ yields the well-known Ising model, of which the ground state is $|\uparrow\uparrow ... \uparrow\uparrow\rangle$ The lowest energy excitations have one spin flipped down, which yields a state of the form $|\uparrow\uparrow ... \uparrow\downarrow\uparrow ... \uparrow\rangle$. Such a state is referred to as a one-magnon state. All the other ones can be generated by a permutation of the one down spin over the lattice sites. Note that the magnon is a boson as the ground state has total spin $\frac{N}{2}$ in the z-direction, whereas the one-magnon state has total spin $\frac{N}{2} - 1$. Thus the magnon has spin $S = 1$ and is a boson.
- $\Delta J \to -\infty$ yields an anti-ferromagnetic Ising model, with two ground states:

  $$|\uparrow\downarrow ... \uparrow\downarrow\uparrow\rangle$$

  and

  $$|\downarrow\uparrow\downarrow\uparrow\downarrow ...\rangle$$

  which are called Néel states. The lowest energy excitations of these ground states are called domain walls, which look like

  $$|\downarrow\uparrow\downarrow ... \uparrow\downarrow\downarrow\uparrow ...\rangle$$

- $J\Delta > 0$ and $|\Delta| > 1$ yields a ferromagnet along the z-direction. We can deduce this as the overall sign of the Hamiltonian is negative, yielding a preference for alignment. Furthermore, the fact that $|\Delta| > 1$ represents a dominance of the z-term as opposed to the x and y terms in the Hamiltonian, so that we may neglect the latter two.
- $J\Delta < 0$ and $|\Delta| > 1$ yields an overall plus sign of the Hamiltonian, thus favoring misalignment. Thus we have an anti-ferromagnet along the z-direction.
- $J\Delta \neq 0$ and $|\Delta| < 1$: now the configurations in the XY-plane energetically dominate those in the z-direction and depending on the overall sign of the Hamiltonian we get (mis)alignment in the XY-plane, also called the planar regime. As consequence the correlation length $\lambda \to \infty$ and we have the conformal invariance .

2.1.3 Yangian and Yang-Baxter equation

To introduce the Yangian of simple Lie algebra we must define the universal enveloping algebra. In this section, we'll see the connection between the Yangian, YBE and R-matrix saw in the preceding sections. Next chapter will introduce the basic definitions of category theory.

Definition 2.3 Let $\mathfrak{g}$ be a Lie algebra and $\mathrm{T}(\mathfrak{g})$ the tensorial algebra on the underlying vector space $\mathfrak{g}$. $\mathrm{T}(\mathfrak{g})_0 = \mathbb{K} \cdot 1$ and $\mathrm{T}(\mathfrak{g})_m$ is the subspace of $\mathrm{T}(\mathfrak{g})$ of all homogeneous tensors of m-degree .

We put

$$u_{\mathbf{v},\mathbf{w}} = \mathbf{v} \otimes \mathbf{w} - \mathbf{w} \otimes \mathbf{v} - [\mathbf{v},\mathbf{w}] \quad \mathbf{v},\mathbf{w} \in \mathfrak{g} \tag{2.45}$$

We denote

$$\mathfrak{L}(\mathfrak{g}) = \sum_{\mathbf{v},\mathbf{w} \in \mathfrak{g}} T(\mathfrak{g}) \otimes u_{\mathbf{v},\mathbf{w}} \otimes T(\mathfrak{g}). \tag{2.46}$$

Since

$$u_{\mathbf{v},\mathbf{w}} \in T(\mathfrak{g})_1 + T(\mathfrak{g})_2 \Rightarrow \mathfrak{L}(\mathfrak{g}) \subseteq \sum_{m \geq 1} T(\mathfrak{g})_m \tag{2.47}$$

$\mathfrak{L}(\mathfrak{g})$ is two-sides ideal in $\mathrm{T}(\mathfrak{g})$.

We define **universal enveloping algebra** $\mathfrak{g}$ as the quotient

$$\mathfrak{U}(\mathfrak{g}) = \frac{T(\mathfrak{g})}{\mathfrak{L}(\mathfrak{g})}. \tag{2.48}$$

Exemple 2.4 Consider $\mathfrak{sl}(2, \mathbb{K})$.

$$\mathfrak{sl}(2,\mathbb{K}) = span_{\mathbb{K}} \left\{ H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \tag{2.49}$$

from this we can find $[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H$.

The Poincaré - Birkhoff - Witt theorem say that a basis for

$$\mathfrak{U}(\mathfrak{sl}(2)) = \frac{T(\mathfrak{sl}(2))}{\begin{bmatrix} H \otimes X - X \otimes H - 2X \\ H \otimes Y - Y \otimes H + 2Y \\ X \otimes Y - Y \otimes X - H \end{bmatrix}} \tag{2.50}$$

is

$$\left\{ H^h, X^x, Y^y, h, x, y \in \mathbb{Z}^+ \right\}. \tag{2.51}$$

To give a formal definition of Yangian we would need many definitions. Such an approach is beyond the scope of this section therefore we'll expose this concept drawing inspiration [14].

Definition 2.5 A coproduct on $\mathfrak{U}(\mathfrak{g})$ is a map $\Delta : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ defined on the generators of $\mathfrak{g}$, $\{I_j\}, j = 1, ..., dim\mathfrak{g}\}$

$$\Delta(I_j) = I_j \otimes 1 + 1 \otimes I_j \tag{2.52}$$

such that

- this diagram

$$\begin{array}{ccc} \mathfrak{U}(\mathfrak{g}) & \xrightarrow{\Delta} & \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) \\ \Delta \downarrow & & \downarrow \mathfrak{U}(\mathfrak{g})\otimes\Delta \\ \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) & \xrightarrow{\Delta\otimes\mathfrak{U}(\mathfrak{g})} & \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) \end{array} \tag{2.53}$$

  is commutative $\forall x \in \mathfrak{g}$. The diagram expresses the coassociativity of Δ, physically we say that the action of x on a 3-particle state is unique.

- homomorphism

$$\Delta([x, y]) = [\Delta(x), \Delta(y)] \quad \forall x, y \in \mathfrak{g} \tag{2.54}$$

  i.e. physically the multiparticle states carry representations of the symmetry algebra.

The Yangian $Y(\mathfrak{g})$ is the eveloping algebra generated by $\{I_j\}_{j=1,...,dim\mathfrak{g}}$ and a second set of generators $\{J_\mu, \mu = 1, ..., dim\mathfrak{g}\}$, in the adjoint representation of $\mathfrak{g}$ so that

$$[I_\nu, J_\mu] = \Gamma_{\nu\mu k} J_k \tag{2.55}$$

equiped with a coproduct

$$\Delta(J_\mu) = J_\mu \otimes 1 + 1 \otimes J_\mu + \frac{\alpha}{2} \Gamma_{\nu\mu k} I_k \otimes I_\mu \tag{2.56}$$

for $\alpha \in \mathbb{C}$.

The Yangian $Y(\mathfrak{g})$ is a Hopf algebra defining a co-unit map

$$\varepsilon : Y(\mathfrak{g}) \to \mathbb{C}, \quad \varepsilon(I_i) = \varepsilon(J_i) = 0 \tag{2.57}$$

physically a one-dimensional vacuum representation and antipode map

$$s : Y(\mathfrak{g}) \to Y(\mathfrak{g}) \quad s(I_i) = -(I_i) \quad s(J_i) = -J_i + \frac{1}{2}\Gamma_{\nu\mu k} I_k I_\mu \tag{2.58}$$

an anti-automorphism and physically a PT-transformation .

It is the moment to spend some words about the $[J_a, J_b]$. Since Δ must be a homomorphism we have a "terrific" constrains

- $$[J_a, [J_b, I_c]] - [I_a, [J_b, J_c]] = \alpha^2 \Lambda_{abcdef} \{I_d, I_c, I_g\} \tag{2.59}$$

  where

  $$\Lambda_{abcdef} = \frac{1}{24} f_{adi} f_{bej} f_{cgk} f_{ijk} \tag{2.60}$$

  and

  $$\{x, y, z\} = \sum_{i \neq j \neq k} x_i y_j z_k \tag{2.61}$$

- $$[[J_a, J_b], [I_l, J_m]] + [[J_l, J_m], [I_a, J_b]] = \alpha^2 \Lambda_{abcdef} f_{lmc} + \Lambda_{lmcdeg} f_{abc} \{I_d, I_e, J_g\}. \tag{2.62}$$

Drinfel'd called those relations "terrific" .

The R-matrix and YBE

A way to see link between YBE and $Y(\mathfrak{g})$ is this definition of monodromy matrix

$$T(\lambda) \equiv exp(-\frac{1}{\lambda} t^a I_a + \frac{1}{\lambda^2} t^a J_a - \frac{1}{\lambda^3} t^a \frac{1}{c_A} f_{abc} [J_c, J_b] + ...) \tag{2.63}$$

where $\lambda \in \mathbb{C}$ is a new, "spectral" parameter .

Now we see that T is a matrix where the elements $\in Y(\mathfrak{g})$. The significance of T lies in the fact that

$$\Delta(T_{ij}(\lambda)) = T_{ik}(\lambda) \otimes T_{kj}(\lambda) \tag{2.64}$$

$Y(\mathfrak{g})$ has an automorphism

$$L_{mu} : I_a \mapsto I_a, J_a \mapsto J_a + \mu I_a \quad (\mu \in \mathbb{C}) \tag{2.65}$$

whose action on T is

$$T(\lambda) \mapsto T(\lambda + \mu). \tag{2.66}$$

Let us consider the intertwiners $\tilde{R}$ which are required to satisfy commutativity with the action on T

$$\tilde{R}(\nu-\mu).\, L_\mu \times L_\nu(\Delta(x)) = L_\nu \times L_\mu(\Delta(x)).\, \tilde{R}(\nu-\mu) \tag{2.67}$$

for any $x \in Y(\mathfrak{g})$ .Their equivalence

$$\tilde{R}(\lambda-\nu)\otimes 1.\, 1\otimes\tilde{R}(\lambda-\mu).\, \tilde{R}(\mu-\nu)\otimes 1 = 1\otimes\tilde{R}(\nu-\mu).\, \tilde{R}(\lambda-\mu)\otimes 1.\, 1\otimes\tilde{R}(\lambda-\nu) \tag{2.68}$$

is the **Yang-Baxter equation** . This is a

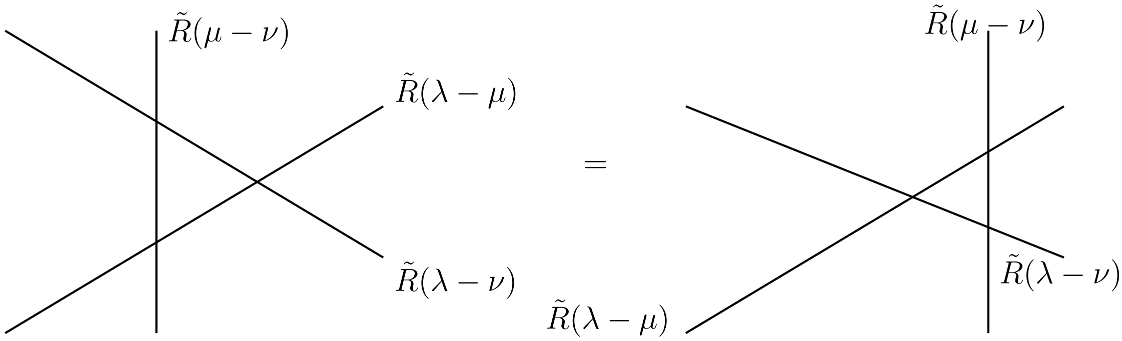


Figura 2.1 Graphic Yang-Baxter equation from Yangian

This is the same equation of 1+1D S-matrix theory stating the condition for factorization of multiparticle S-matrix into 2-particles factors. Here, each line in figure will carry a representation of the Yangian.

2.2 Two-dimensional conformal field theory

Conformal field theory: a brief overview

The conformal maps are present in complex analisis but also in complex geometry, here we give a definition and we prove a very important feature of conformal maps .
Definition 2.6 A holomorphic map $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is **conformal** if

$$f'(z) \neq 0 \quad \forall z \in U. \tag{2.69}$$

The most important consequence for the conformal functions and the theory of fields is :
Theorem 2.7 A conformal map is angle-preserving and sense-preserving.

Proof . Let $f : U \to \mathbb{C}$ be a holomorphic map on an open set U . Let $z_0 \in U$ and let $\gamma_1 : [-1, 1] \to U$ and $\gamma_2 : [-1, 1] \to U$ be two paths which meet at $z_0 = \gamma_1(0) = \gamma_2(0)$. The original curves meet at z_0 in the (signed) angle

$$\theta = \arg\, \gamma_2'(0) - \arg\, \gamma_1'(0) = \arg\, \frac{\gamma_2'(0)}{\gamma_1'(0)} \tag{2.70}$$

The images of the curves $f(\gamma_1)$ and $f(\gamma_2)$ meet at $f(z_0)$ at angle

$$\begin{aligned}
\phi &= \arg\, (f\gamma_2)'(0) - \arg\, (f\gamma_1)'(0) \\
&= \arg\, \frac{(f\gamma_2)'(0)}{(f\gamma_1'(0))} \\
&= \arg\, \frac{f'(\gamma_2)(0)\gamma_2'(0)}{f'(\gamma_1(0))\gamma_1'(0)} \\
&= \arg\, \frac{f'(z_0)\gamma_2'(0)}{f'(z_0)\gamma_1'(0)} \\
&= \arg\, \frac{\gamma_2'(0)}{\gamma_1'(0)} = \theta.
\end{aligned} \tag{2.71}$$

□

Let's talk about Physics. A classical model of statistical mechanics in D spatial dimension on a lattice is equivalent to a euclidean field theory in $D - 1$ spatial dimension and one temporal dimension . The equivalence is obtained in the continuum limit when the lattice spacing $a \to 0$.

The equivalence is essentially summarized by the following points :

- it is possible to represent the partition function as a path integral
- the limit of the fields for $a \to 0$ is finite i.e. the fields are renormalizable .
- the continuum limit of field theory is justified around the critical point where the fluctuations of observables are correlated at macroscopic distance.
- Correlation length λ is connected to the mass m of the theory by $\lambda = \frac{1}{m}$ therefore at the critical point, when λ is macroscopic corrisponds to vanishing of m .

Let consider an example, a scalar field $\phi(x)$ in D euclidean dimension . The action associated to $\phi(x)$ is

$$S = \frac{1}{2}\int d^D x \partial^\mu \phi(x) \partial_\mu \phi(x) \tag{2.72}$$

is invariant under this transformation

$$x \mapsto kx; \quad k \in \mathbb{K} \tag{2.73}$$

if the the field transform

$$\phi(x) \mapsto k^{-\frac{D-2}{2}}\phi(x) \tag{2.74}$$

then for 2-point correlator

$$\langle \phi(x)\phi(y) \rangle \frac{1}{|x-y|^{D-2}} \quad . \tag{2.75}$$

Hence a classical theory massless is invariant under conformal transformation. The conformal invariance arise from a generalization of the (2.70). A infinitesimal transformation of coordinates $x_\mu \mapsto x_\mu + \varepsilon_\mu$ is a conformal transformation if

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \quad \text{where } \Omega(x) \text{ is a scale factor} \tag{2.76}$$

and this is a deformation of euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$. The transformation is

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - \partial_\mu \varepsilon(x)_\nu - \partial_\nu \varepsilon(x)_\mu \tag{2.77}$$

now eqating (2.75) and (2.76) we have

$$\partial_\mu \varepsilon(x)_\nu + \partial_\nu \varepsilon(x)_\mu = (1 - \Omega(x)) g_{\mu\nu}(x) \equiv \Lambda(x) g_{\mu\nu}(x). \tag{2.78}$$

Taking the trace

$$\partial_\mu \varepsilon(x)_\nu + \partial_\nu \varepsilon(x)_\mu = \frac{2}{D} \partial \cdot \varepsilon g_{\mu\nu}. \tag{2.79}$$

It means that the trace of symmetric part is vanishing when we do a conformal change of coordinates .
A key tool in CFT is the stress-energy tensor defined with respect to deformations of coordinates :

$$\delta S = \frac{1}{(2\pi)^{D-1}} \int d^D x T_{\mu\nu} \partial^\mu \varepsilon^\nu. \tag{2.80}$$

The invariance under traslations, rotations and local dilatation implies

$$\partial_\mu T_{\mu\nu} = 0, \quad T_{\mu\nu} = T_{\nu\mu}, \quad T^\mu_\mu = 0. \tag{2.81}$$

i.e. if is hold (2.81) we have

$$\delta S = 0. \tag{2.82}$$

In light of Poliakov's theorem we can see the conformal symmetry as natural extension under dilatations for theories with local interaction that admit well-defined $T_{\mu\nu}$.

Conformal invariance in 1+1 dimensional field theory

For $D = 2$ the condition (2.78) is

$$\begin{cases} \partial_1 \varepsilon_2 = -\partial_2 \varepsilon_1, \\ \partial_1 \varepsilon_1 = -\partial_2 \varepsilon_2. \end{cases} \tag{2.83}$$

Where $z = x + iy$, $\varepsilon(z) = \varepsilon_1 + i\varepsilon_2$ (2.79) are the usual Cauchy-Riemann condition for the analyticity of $\varepsilon(z)$:

$$\partial_{\bar{z}} \varepsilon(z) = \partial_z \bar{\varepsilon}(\bar{z}) = 0. \tag{2.84}$$

In D=2 the conformal transformation are all the functions analytic in z and anti-analytic in $\bar{z}$:

$$\begin{cases} z \mapsto z + \varepsilon(z), \\ \bar{z} \mapsto \bar{z} + \varepsilon(\bar{z}) \end{cases} \tag{2.85}$$

as consequence the conformal group is infinite dimensional. The algebra which generates these transformations is infinite dimensional and the corresponding infinitesimal generators are found to be

$$l_n = -z^{n+1}\partial, \quad \text{and } \bar{l}_n = -\bar{z}^{n+1}\bar{\partial}, \tag{2.86}$$

where

$$\partial := \frac{\partial}{\partial z}, \; \bar{\partial} := \frac{\partial}{\partial \bar{z}} \quad \text{and } n \in \mathbb{Z}. \tag{2.87}$$

These generators satisfy the Witt algebra,

$$[l_n, l_m] = (n-m)l_{n+m}, \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m} \tag{2.88}$$

with

$$[l_n, \bar{l}_n] = 0, \text{ for any } n, m \in \mathbb{Z}. \tag{2.89}$$

Therefore the conformal algebra is the direct sum of two isomorphic subalgebras generated by $l_n, \bar{l}_n$. Not all the conformal transformation are globally defined : only the automorphism of Riemann sphere ($\mathbb{C} \cup \infty$) are the Moebius transformation or global conformal transformation :

$$z \mapsto w(z) = \frac{\alpha z + \beta}{\gamma z + \delta}; \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}; \quad \Delta = \alpha\delta - \beta\gamma \neq 0. \tag{2.90}$$

They correspond to $\varepsilon(z) = \alpha + \beta z + \gamma z^2$. i.e. traslations, dilatations and inversions. In the other cases, the transformation introduce a singularity and do not correspond to symmetries of the states of theory.

A conformal field theory is equipped with particular fields called **primary**, $\phi_{h\bar{h}}(z, \bar{z})$. Primaries fields transform as covariant tensors :

$$\phi_{h\bar{h}}(z, \bar{z}) \mapsto \left(\frac{dw}{dz}\right)^h \left(\frac{d\bar{w}}{dz}\right)^{\bar{h}} \phi_{h\bar{h}}(w, \bar{w}) \tag{2.91}$$

$(h, \bar{h}) \equiv i$ are called **conformal weights** of $\phi_{h\bar{h}} \equiv \phi_i$; $\Delta = h + \bar{h}$ is the scale dimension and $s = h - \bar{h}$ is the conformal spin. The 2-points functions are uniquely determined by the invariance under the global conformal transformation (2.82):

$$\begin{aligned} \langle \phi_i(z_1, \bar{z_1}) \phi_j(z_2, \bar{z_2}) \rangle &= \frac{\delta_{ij}}{(z_1 - z_2)^{2h} (\bar{z_1} - \bar{z_2})^{s\bar{h}}} \\ &= \frac{\delta_{ij}}{|z_1 - z_2|^{2\Delta}} \left(\frac{\bar{z_1} - -\bar{z_2}}{z_1 - z_2} \right)^s . \end{aligned} \tag{2.92}$$

The primary fields **are mapped univocally in to irriducible representation** of the conformal algebra.

Definition 2.8 A integrable field theory is a theory with an infinity of commuting conserved charges.

The conserved charges associated to the QFT in the z-plane are generated by the energy momentum tensor $T_{\mu\nu}$: which is always symmetric and in conformally invariant theories, also traceless . It is usually more convenient to express the components of the energy momentum tensor in terms of the $z, \bar{z}$ coordinates.

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \tag{2.93}$$

$$T_{\bar{z},\bar{z}} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \tag{2.94}$$

$$T_{z,\bar{z}} = T_{\bar{z},z}\frac{1}{4}(T_{00} + T_{11}) = \frac{\Theta}{4}. \tag{2.95}$$

The conservation of the energy momentum tensor amounts to the imposition of the following constraints,

$$\bar{\partial}T_{zz} = \partial T_{\bar{z},\bar{z}} = 0, \tag{2.96}$$

which justify the definitions

$$T(z) := T_{zz}, \quad \bar{T}(\bar{z}) := T_{z,\bar{z}}. \tag{2.97}$$

Consequently, local conformal transformations in the complex z-plane are generated by the holomorphic and antiholomorphic components of the energy momentum tensor defined before. In fact the constraints (2.96) suggests the introduction of the generators

$$L_n, \bar{L}_n$$

which arise as the coefficients of the Laurent expansion of the holomorphic and anti-holomorphic components of the stress-energy tensor :

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-2-n} L_n \Leftrightarrow L_n = \oint d\gamma(\gamma - z)^{n+1}T(\gamma). \tag{2.98}$$

It is possible to define a similar expansion for the component $\bar{T}(\bar{z})$ in terms of $\bar{L}_n$.
To compute the algebra of commutators satisfied by these modes it is required the evaluation of commutators of contour integrals of the type $[\oint dz, \oint d\gamma]$

together with the computation of *operator product expansions* (OPE) of the holomorphic and anti- holomorphic components of the energy momentum tensor. OPE's characterise the behaviour in the limit $z \rightarrow \gamma$. In 1+1 dimensions and in the Euclidean regime give us the following OPE

$$T(z)T(\gamma) = \frac{\frac{c}{2}}{(z-\gamma)^4} + \frac{2T(\gamma)}{(z-\gamma)^2} + \frac{\partial T(\gamma)}{(z-\gamma)}. \tag{2.99}$$

The constant c is called *central charge* of the CFT and depends on the particular theory. By the OPE we can compute the commutator of the generators above introduced

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{2.100}$$

and is known as *Virasoro algebra* .

2.2.1 Scaling near critical point

For the quantum case the scaling invariance is obtained near the critical points where the beta function vanish. Dynamics in the vicinity of second order phase transitions can be described by CFT perturbed by the addition of operators that break the conformal symmetry and introduce a mass scale in the system. The specific values of the parameters for which a statistical system is critical are associated to fixed points of the renormalization group flow. A renormalization group trajectory flowing away from a fixed point is obtained by combinations of the relevant scalar operators Φ_i present in the corresponding CFT. The corresponding off-critical action is given by

$$S_\lambda = S_{CFT} + \sum_n \lambda_n \int \Phi_n(x) d^2x \tag{2.101}$$

where the λ_n are the coupling constants and S_{CFT} is the action of the original unperturbed CFT. The coupling constant has scale dimension $2(1-h)$ and so has conformal dimensions $(1-h, 1-h)$. The CFT is a fixed point of the renormalisation group; so provided $2(1-h) > 0$ a RG transformation moves the model away from the critical point. Thus Φ is called *relevant operator* if $h < 1$ and *irrelevant* if $h > 1$.

The integrability of a PCFT was proved by Zamolodchikov using the counting argument [26] .The quantum integrability of a 1+1-dimensional massive QFT possessing an infinite number of quantum conserved charges was established in light of the results found in [28].

Parke demonstrated the existence of two of these quantities different from the energy momentum tensor and having different spin from each other needs to be proven in order to conclude the quantum integrability of the theory.

2.2.2 S matrix in 1+1 dimensional theories

Factorisability and Parke's theorem

For a scattering process with m massive particles incoming and n outgoing the elements of S matrix are defined by

$$S_{1,...,m}^{1',...,n} = \langle F_1'(p_1')...F_m'(p_m')||F_1(p_1)...F_n(p_n)\rangle. \qquad (2.102)$$

From Coleman, Mandula's theorem we know that a field theory with a conserved charge that transform under the action of Lorentz's group like a tensor of rank 2 has a trivial S matrix. It is obvius that the conserved charge is the energy-momentum vector. For a 1+1 field theory with 2 conserved charges different from impulse, the set of initial impulses are conserved i.e. :

$$\{p_1', ..., p_m'\} = \{p_1, ..., p_m\} \qquad (2.103)$$

and the S matrix is factorized. This is Parke's theorem,(see [16]), and implies that there is no production of particles. These key properties are obtained if we assume the following hypothesis:

- There are two different conserved charges by the impulses, Q^+ and Q^-. They transforms under action of Lorentz group in this way:

  $$Q'^+ = \Lambda^{+q}Q^+ \quad Q'^- = \Lambda^{-n}Q^- \qquad (2.104)$$

  where q, n are odd numbers such that $q \geq n \geq 1$.

- $$Q^{\pm} = \int j_{\pm}^0 dx \qquad (2.105)$$

- $$[Q^+, Q^-] = 0 \qquad (2.106)$$

- Q^+, Q^- on no trivial linear combination of particles in a multiplet is never null.

With the assumptions made until now one can show that the following properties hold :

1. There is no creation of particles.

2. $$\{p_1', ..., p_m'\} = \{p_1, ..., p_m\}$$

3. The S-matrix is factorized in terms of 2 particles interaction

More details on (3), let's consider a diagram with m lines of different inclination like in figure case m=4. Time flows from the top down. At each intersection point corresponds an element $S_{ij}^{kh}(\theta_{ab})$ The given rule has an ambiguity: a certain element $S_{i_1,\dots,i_m}^{j_1,\dots,j_m}$ can be associated to several diagrams. Instead, these diagrams should coincide. This is possible only if S satisfies the following factorization equation :

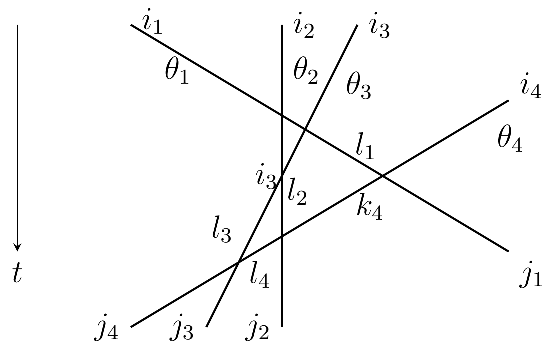


Figura 1 : S-matrix for m=4 and $\theta_1 > \theta_2 > \theta_3 > \theta_4$

$$\boxed{S_{i_1 j_2}^{k_1 k_2}(\theta_{12}) S_{j_1 k_3}^{k_1 i_3}(\theta_{13}) S_{j_1 j_3}^{k_2 k_3}(\theta_{23}) = S_{j_1 j_2}^{k_1 k_2}(\theta_{12}) S_{k_1 j_3}^{i_1 k_3}(\theta_{13}) S_{i_2 i_3}^{k_2 k_3}(\theta_{23})} \tag{2.107}$$

the Yang-Baxter- Zamolodchikov-Faddeev, where $\theta_{13} = \theta_{23} + \theta_{12}$. This equation has the same structure of YB equation found in statistical mechanics for the Boltzman's weights in a vertex model .

2.2.3 Yangian symmetry in 1+1D field theory

Yangian from classical charges

Suppose a 1+1D field theory equipped with a symmetry associated to a Lie algebra $\mathfrak{g}$. Noether theorem tells that there exists a conserved current

$$j_\mu(x,t) \in \mathfrak{g} : \partial^\mu j_\mu = 0. \tag{2.108}$$

Under the further hypothesis that

$$\partial^0 j_1 - \partial^1 j_0 + [j_1, j_0] = 0 \tag{2.109}$$

we can write

$$j_\mu = j_{\mu a} t^a \tag{2.110}$$

where t^a are the generators of $\mathfrak{g}$. Then we have the charges

- local

$$Q_a^{(0)} = \int_{-\infty}^{+\infty} j_{0a} dx \tag{2.111}$$

- and non local

$$Q_a^{(1)} = \int_{-\infty}^{+\infty} j_{1a} dx + \frac{f_{abc}}{2} \int_{-\infty}^{+\infty} j_{0b}(x) \left(\int_{-\infty}^{x} j_{0c}(y) dy \right) dx. \tag{2.112}$$

Using these charges we can define a classical Yangian making the correspondence

$$Q_a^{(0)} = I_a \quad Q_a^{(1)} = J_a \tag{2.113}$$

and replacing the commutators with

$$\{j_{\mu a}, j_{\nu b}\} = f_{abc} j_{\sigma c} \delta(x-y) \quad \sigma = |\mu - \nu|. \tag{2.114}$$

The antipode map is a PT-transformation

$$j_\mu(x,t) \mapsto j_\mu(-x,-t). \tag{2.115}$$

It remains to define a coproduct. A way to interpret this map is by splitting space into two regions (positive and negative x, say), each of which would naturally contain just one of a pair of asymptotically-separate, particle-like lumps.

The two components of the coproduct correspond to the integrals over the two regions, and the non-triviality of coproduct is connected to the non-locality of $Q_a^{(1)}$.

More specifically :

$$Q_a^{(0)} = \int_{-\infty}^{0} j_{0a}(x) dx + \int_{0}^{+\infty} j_{0a}(x) dx = Q_{a-}^{(0)} + Q_{a+}^{(0)} \tag{2.116}$$

and

$$Q_a^{(1)} = Q_{a-}^{(1)} + Q_{a+}^{(1)} + \frac{f_{abc}}{2} Q_a^{(0)} Q_{c-}^{(0)} Q_{b+}^{(0)} \tag{2.117}$$

therefore by correspondence and (2.52), (2.56) it is easy define the coproduct.

Now an important question will take us to find again the YBE : how to incorporate the boundary conditions into field theories with $Y(\mathfrak{g})$ symmetry without losing integrability?

We take as our starting point the boundary equation of motion for the model on $-\infty < x \leq 0$, written in terms of the currents :

$$j_{+a}(0) j_{-a}(0) = 0 \tag{2.118}$$

and we solve this with

$$j_+(0) = \alpha(j_-(0)) \tag{2.119}$$

where

$$\begin{aligned} \alpha : \mathfrak{g} &\to \mathfrak{g} \\ t^a &\longmapsto \alpha_{ab} t^b \end{aligned} \tag{2.120}$$

in light-cone coordinates (2.90) lead to local charges connected to symmetrized trace, an invariant tensor of $\mathfrak{g}$. For more details about $d_{a_1 a_2 \ldots a_m}$ and the Casimir operator associated see [18].
The local charges important for our expositions are :

$$q_{\pm s} = \int_{-\infty}^{+\infty} d_{a_1 a_2 \ldots a_m} j_\pm^{a_1} j_\pm^{a_2} \ldots j_\pm^{a_m} dx \tag{2.121}$$

Now let us require that α be such as to leave precisely one of each pair $q_s + q_{-s}$ of local charges conserved, this is so if α in involution .

$$\alpha(\mathfrak{g}) = \mathfrak{h} \oplus \mathfrak{m} \tag{2.122}$$

α decomposes $\mathfrak{g}$ in two subalgebra, $\mathfrak{h}$ is the subalgebra with eigenvalue + 1 and $\mathfrak{m}$ is the -1 eigenspace .
By some calculation, on half-line the $Q_p^{(1)}$, $\mathfrak{m}$ components are not conserved but the modified charges

$$\tilde{Q}_p^{(1)} \equiv Q_p^{(1)} + \frac{f_{piq}}{4}(Q_i^{(0)} Q_q^{(0)} + Q_q^{(0)} Q_i^{(0)}) \tag{2.123}$$

are conserved.
We denote as $Y(\mathfrak{g}, \mathfrak{h})$ he subalgebra of Y($\mathfrak{g}$) generated by $Q_a^{(0)}$, $\tilde{Q}_a^{(1)}$. The key algebraic property of $Y(\mathfrak{g}, \mathfrak{h})$, which fixes the special form of the $\tilde{Q}_a^{(1)}$, is that

$$\Delta(Y(\mathfrak{g}, \mathfrak{h})) \subset Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h}) \tag{2.124}$$

This property makes $Y(\mathfrak{g}, \mathfrak{h})$ a coideal subalgebra. Its significance is that boundary states form representations of $Y(\mathfrak{g}, \mathfrak{h})$ and, just as the usual coproducts being a homomorphism (2.54) enables two-particle states to represent the correct symmetry algebra, so this property allows a state consisting of a bulk particle and a boundary to represent $Y(\mathfrak{g}, \mathfrak{h})$.

The analogue of $\tilde{R}$ and its relation (2.68) is the "reflection"- or K-matrix, which satisfies

$$K(\mu) L_\mu(x) = L_{-\mu} K(\mu) \quad \forall \in Y(\mathfrak{g}, \mathfrak{h}). \tag{2.125}$$

The analogue of the Yang-Baxter equation is the reflection equation or **boundary Yang-Baxter equation**

$$\boxed{\tilde{R}(\nu-\mu).\,1\otimes K(\nu).\,\tilde{R}(\mu+\nu).\,1\otimes K(\mu)=1\otimes K(\mu).\,\tilde{R}(\mu+\nu).\,1\otimes K(\nu).\,\tilde{R}(\nu-\mu)} \tag{2.126}$$

We conclude this by saying that there are many other models where the Yang-Baxter equation emerges . Thery are not cited for reasons of space. This chapter was intended to show how this equation is related to integrability. Next step will be the introduction to category theory and how its wonderful logic allows to connect and unify different models.

Chapter 3

Prolegomenon of category theory to the practing physicist

The Category theory arises in the context of algebraic topology . It was developed by Saunders Mac Lane and Samuel Eilenberg in 1945. The first concept on which they worked was the natural transformation around which they developed the idea of category. This theory allows the unification of many aspects of science. Is it not the unification what it seeks to do theoretical physics? The Category theory it's not just a set tools to codify algebraically the connection between different aspects of mathematics or physics, it's poetry, the logic flow and describes, makes light in the kingdom of caos elegantly. Is the correct enviroment where to work for whom try to solve the foundational problems of quantum field theory .

3.1 Basical definition

Definition 3.1 A category $\mathcal{C}$ is

- a class of objects denoted by Ob($\mathcal{C}$)

- $\forall$ C_1,$C_2 \in$ Ob($\mathcal{C}$) a set $Hom_{\mathcal{C}}(C_1, C_2)$ called the set of morphisms from C_1 to C_2

- for every $C_1, C_2, C_3 \in Ob(\mathcal{C})$ there is a map :

$$\circ : Hom_{\mathcal{C}}(C_1, C_2) \times Hom_{\mathcal{C}}(C_2, C_3) \longrightarrow Hom_{\mathcal{C}}(C_1, C_3) \quad (3.1)$$

$$(f, g) \longmapsto g \circ f \quad (3.2)$$

  called the composite of g and f satisfying the following conditions :

  - if $(C_1, C_2) \neq (C_3, C_4), Hom_{\mathcal{C}}(C_1, C_2) \cap Hom_{\mathcal{C}}(C_3, C_4) = \emptyset$

  - if $h \in Hom_{\mathcal{C}}(C_1, C_2)$, $h \circ (g \circ f) = (h \circ g) \circ f$;

  - for every $C \in Ob(\mathcal{C})$ there exists $1_{\mathcal{C}} \in Hom_{\mathcal{C}}(C, C)$ such that for every $f \in Hom_{\mathcal{C}}(C, C), f \circ 1_C = f = 1_{C'} \circ f$.

Example 3.1. Sets, together with functions between sets, form the category $\mathfrak{Sets}$. For every algebraic structure you can consider its category: take sets endowed with that algebraic structure as objects and take morphisms between two objects as mor- phisms. In this way, you obtain the category of groups, of rings, of right R-modules and so on.

Definition 3.2. A category is called **small** if the class of its objects is a set; **discrete** if, given two objects C_1, C_2 such that $C_1 = C_2$ implies that $Hom_{\mathcal{C}}(C_1, C_2) = \{1_{C_1}\}$ if $C_1 \neq C_2$ then $Hom_{\mathcal{C}}(C_1, C_2) = \emptyset$.

The opposite category of a category $\mathcal{C}$ is the category $\mathcal{C}^{op}$ where Ob($\mathcal{C}^{op}$) = $Ob(\mathcal{C})$ and $Hom_{\mathcal{C}^{op}}(C_1, C_2) = Hom_{\mathcal{C}}(C_2, C_1)$.

Definition 3.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories . A **covariant** functor $F : \mathcal{C} \to \mathcal{D}$ consists of assigning to each object $C \in \mathcal{C}$ an object $F(C) \in \mathcal{D}$ and to each morphism $f : C_1 \to C_2$ a morphism $F(f) : F(C_1) \to F(C_2)$ such that

$$F(1_C) = 1_{F(C)}\,,\; F(g \circ f) = F(g) \circ F(f) \qquad (3.3)$$

F is **contravariant** functor if $F(f) \in Hom_{\mathcal{D}}(F(C_2), F(C_1))$ and

$$F(g \circ f) = F(f) \circ F(g). \qquad (3.4)$$

Definition 3.4. Consider the map

$$F^{C1}_{C_2} : Hom_{\mathcal{C}}(C_1, C_2) \to Hom_{\mathcal{C}}(F(C_1), F(C_2)) \qquad (3.5)$$
$$f \longmapsto F(f)$$

- F is faithful if $F^{C1}_{C_2}$ is injective for every $C_1, C_2 \in \mathcal{C}$

- F is full if $F^{C1}_{C_2}$ is surjective for every $C_1, C_2 \in \mathcal{C}$

Example 3.5. Let $\mathcal{C}$ a category and $C_1 \in \mathcal{C}$, we define a functor

$$h^{C_1} = Hom_{\mathcal{C}}(C_1, \bullet) : \mathcal{C} \to \mathfrak{Set} \qquad (3.6)$$

that allows to embed each category in the category $\mathfrak{Set}$.
h^{C_1} on the objects :

$$h^{C_1}(C_2) = Hom_{\mathcal{C}}(C_1, C_2) \in \mathfrak{Set} \qquad (3.7)$$

h^{C_1} on $f : C_3 \to C_4$

$$h^{C_1}(f) = Hom_{\mathcal{C}}(C_1, f) : Hom_{\mathcal{C}}(C_1, C_3) \to Hom_{\mathcal{C}}(C_1, C_4) \qquad (3.8)$$
$$(g : C_1 \to C_3) \mapsto (f \circ g : C_1 \to C_4).$$

h^{C_1} is covariant :

- $$h^{C_1}(1_C)(g) = 1_C \circ g = g \Rightarrow h^{C_1}(1_C) = 1_{h^{C_1}(C)} \qquad (3.9)$$

- $$\begin{aligned} h^{C_1}(k \circ f)(g) &= k \circ f \circ g \\ &= h^{C_1}(k)(f \circ g) \\ &= (h^{C_1}(k) \circ h^{C_1}(f)) \circ g \end{aligned} \qquad (3.10)$$

□

Similarly, we can define a contravariant functor

$$h_{C_1} = Hom_{\mathcal{C}}(\bullet, C_1) : \mathcal{C} \to \mathfrak{Set}.$$

Notation 3.6 From now on, if not otherwise specified, the world functor will mean covariant functor.

Definition 3.7. Given two functors $\mathcal{C} \xrightarrow{F,G} \mathcal{D}$ a functorial morphism (or natural transformation) $\varphi : F \to G$ is a collection of morphisms in $\mathcal{D}$,

$$(\varphi_C : F(C) \to G(C))_{C \in \mathcal{C}}$$

such that, for every $f : C_1 \to C_2$,

$$\varphi_{C_2} \circ F(f) = G(f) \circ \varphi_{C_2} \tag{3.11}$$

i.e. the following diagram

$$\begin{array}{ccc} F(C_1) & \xrightarrow{\varphi_{C_1}} & G(C_1) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C_2) & \xrightarrow{\varphi_{C_2}} & G(C_2) \end{array} \tag{3.12}$$

is commutative $\forall f : C_1 \to C_2$ .F, G are isomorph if $\forall C \in \mathcal{C}$, φ_C is an isomophism and we write $F \cong G$.

Definition 3.8. Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be functor . We say that

- F is an equivalence of categories if there is a functor $\mathcal{D} \xrightarrow{G} \mathcal{C}$ such that
$$FG \cong 1_{\mathcal{D}}, \qquad GF \cong 1_{\mathcal{C}}.$$
- F is an isomorphism of categories if there is a functor Let $\mathcal{D} \xrightarrow{G} \mathcal{C}$ such that
$$FG = 1_{\mathcal{D}}, \qquad GF = 1_{\mathcal{C}}.$$

Theorem 3.9. Let $\mathcal{C} \xrightarrow{T} \mathcal{D}$ be functor. Then T is an equivalence of categories if and only if T is full, faithful and, for every $D \in \mathcal{D}$, there exist $C \in \mathcal{C}$ and an isomorphism $T(C) \xrightarrow{\lambda_D} \mathcal{D}$.

Proof. Assume first that T is an equivalence, then there exist a functor $\mathcal{D} \xrightarrow{S} \mathcal{C}$ and functorial isomorphisms $\alpha : ST \to 1_{\mathcal{C}}, \beta : TS \to 1_{\mathcal{D}}$.

T is **faithful**.

Let $f, f' \in Hom_{\mathcal{C}}(C_1, C_2)$with $T(f) = T(f')$, then $ST(f) = ST(f')$. Since α is a functorial morphism we have the following commutative diagram

$$\begin{array}{ccc} ST(C_1) & \xrightarrow{\alpha_{C_1}} & C_1 \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C_2) & \xrightarrow{\alpha_{C_2}} & G(C_2) \end{array} \tag{3.13}$$

$$\alpha_{C_2} \circ ST(f) = f \circ \alpha_{C_1} \quad \alpha_{C_2} \circ ST(f') = f' \circ \alpha_{C_1}$$

since α is an isomorphis, is invertible :

$$\alpha_{C_2} \circ ST(f) \circ \alpha_{C_1}^{-1} = f \quad \alpha_{C_2} \circ ST(f) \circ \alpha_{C_1}^{-1} = f' \tag{3.14}$$

but $ST(f) = ST(f') \Rightarrow f = f'$.

T is **full**.

We put $h : T(C_1) \to T(C_2)$ and

$$g = \alpha_{C_2} \circ S(h) \circ \alpha_{C_2}^{-1} \in Hom_{\mathcal{C}}(C_1, C_2)$$

. Simce α is an isomorphism we have

$$ST(g) = \alpha_{C_2}^{-1} \circ g \circ \alpha_{C_1} = S(h) \tag{3.15}$$

by definition of g, but S is an equivalence therefore is faithful $\Rightarrow h = T(g)$:

$$\begin{array}{ccc} ST(C_1) & \xrightarrow{\alpha_{C_1}} & C_1 \\ {\scriptstyle S(f)=ST(g)}\downarrow & & \downarrow{\scriptstyle g} \\ F(C_2) & \xrightarrow{\alpha_{C_2}} & G(C_2) \end{array} \tag{3.16}$$

For all $D \in \mathcal{D}$ if we put $C = T(D)$, the isomorphism being looked for is $\beta_D : ST(D) \to D$.

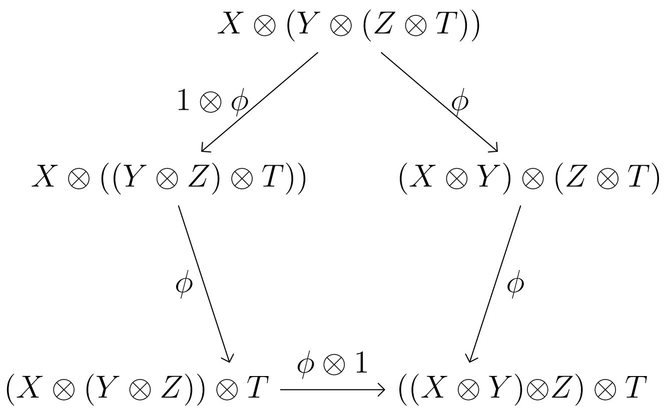
□

3.2 Monoidal category

Definition 3.10 A **monoidal** category$(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ is a category endowed

- with an object $\mathbf{1} \in \mathcal{C}$
- a functor
$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \tag{3.17}$$
  called **tensor product**.
- An associative constraint for $\otimes$ is a functorial isomorphism
$$a_{X,Y,Z} : (X \otimes Y \otimes)Z \to X \otimes (Y \otimes Z) \tag{3.18}$$
  such that the diagram
$$\begin{array}{ccc} (X \otimes Y \otimes)Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) \\ {\scriptstyle (f\otimes g)\otimes h}\downarrow & & \downarrow{\scriptstyle f\otimes(g\otimes h)} \\ (X' \otimes Y' \otimes)Z' & \xrightarrow{a_{X',Y',Z'}} & X' \otimes (Y' \otimes Z') \end{array} \tag{3.19}$$
  is commutative for every $f, g, h \in \mathcal{C}$.

The associativity constraint a satisfies the **Pentagon** Axiom if the pentagonal diagram



commutes for all objects $X, Y, Z, T \in \mathcal{C}$.

- Fix an object 1 in $\mathcal{C}$, a **left unit constraint** and right unit constraint with respect to 1 are a natural isomorphisms

$$l_X : \mathbf{1} \otimes X \to X \tag{3.20}$$

$$r_X : X \otimes \mathbf{1} \to X \tag{3.21}$$

such that

$$\begin{array}{ccc} \mathbf{1} \otimes X & \xrightarrow{l_X} & X \\ {\scriptstyle \mathbf{1}\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{1} \otimes X' & \xrightarrow{l_{X'}} & X' \end{array} \tag{3.22}$$

$$\begin{array}{ccc} X \otimes \mathbf{1} & \xrightarrow{r_X} & X \\ {\scriptstyle f\otimes \mathbf{1}}\downarrow & & \downarrow{\scriptstyle f} \\ X' \otimes \mathbf{1} & \xrightarrow{r_{X'}} & X' \end{array} \tag{3.23}$$

commutes for every f .

In a monoidal category l_X, r_X satisfy the Triangle Axiom :

$$\begin{array}{ccc} (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{a_{X,1,Y}} & X \otimes (\mathbf{1} \otimes Y) \\ & {\scriptstyle r_X \otimes Y} \searrow \quad \swarrow {\scriptstyle X \otimes l_Y} & \\ & X \otimes Y & \end{array}$$

commutes for all objects $X, Y \in \mathcal{C}$.

Proposition 3.11 For any object $X \in \mathcal{C}$ one has the equalities

$$l_{\mathbf{1}\otimes X} = X \otimes l_X \text{ and } r_{X\otimes \mathbf{1}} = r_X \otimes X. \tag{3.24}$$

Proof. It follows from the functoriality of l that the following diagram commutes

$$\begin{array}{ccc} \mathbf{1}\otimes(\mathbf{1}\otimes X) & \xrightarrow{X\otimes l_X} & \mathbf{1}\otimes X \\ {\scriptstyle l_{\mathbf{1}\otimes X}}\downarrow & & \downarrow{\scriptstyle l_X} \\ \mathbf{1}\otimes X & \xrightarrow{l_X} & X \end{array} \tag{3.25}$$

Since l_X is an isomorphism, the first identity follows. The second identity follows similarly from the functoriality of r.

□

Proposition 3.12 The unit object in a monoidal category is unique up to a unique isomorphism.

Proof. If $X = \mathbf{1} \Rightarrow l_X = r_X = \iota$. Let $\mathbf{1}, \mathbf{1}'$ be two unit objects. Let $(r, l), (r', l')$ be the corresponding unit constraints. Then we have the isomorphism

$$\eta := l_{\mathbf{1}'} \circ r'_{\mathbf{1}} : \mathbf{1} \to \mathbf{1}'. \tag{3.26}$$

It is easy to show using commutativity of the above triangle diagrams that η maps ι to ι'. It remains to show that η is the only isomorphism with this property. To do so, it suffices to show that if $f : \mathbf{1} \to \mathbf{1}$ is an isomorphism such that the diagram

$$\begin{array}{ccc} \mathbf{1}\otimes\mathbf{1} & \xrightarrow{f\otimes f} & \mathbf{1}\otimes\mathbf{1} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ \mathbf{1} & \xrightarrow{f} & \mathbf{1} \end{array} \tag{3.27}$$

is commutative, then $f = 1_{\mathbf{1}}$. To see this, it suffices to note that for any morphism $g : \mathbf{1} \to \mathbf{1}$ the diagram

$$\begin{array}{ccc} \mathbf{1}\otimes\mathbf{1} & \xrightarrow{g\otimes 1} & \mathbf{1}\otimes\mathbf{1} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ \mathbf{1} & \xrightarrow{g} & \mathbf{1} \end{array} \tag{3.28}$$

is commutative, so $f \otimes f = f \otimes 1$ and hence $f = 1$.

□

Examples of monoidal category

- $\mathfrak{Sets}$ the category of sets is a monoidal category, where the tensor product is the Cartesian product and the unit object is a one element set; the structure morphisms a,, l, r are obvious.
- Let G be a group . The category $\mathfrak{Rep}(\mathfrak{G})$ of all representations of G over a vector space V, where $\otimes$ is the tensorial product of representation : if for a representation V we denote by ρ_V the corresponding : $G \to GL(V)$ then

  $$\rho_{V\otimes W}(g) := \rho_V(g) \otimes \rho_W(g).$$

  The unit object in this category is the trivial representation .
- Similarly if $\mathfrak{g}$ is a Lie algebra, the category of its representations $\mathfrak{Rep}(\mathfrak{g})$ where the tensor product is defined by

  $$\rho_{V\otimes W}(a) := \rho_V(g) \otimes Id_W + Id_V \otimes \rho_W(a).$$

  where $\rho_Y : \mathfrak{g} \to \mathfrak{gl}(Y)$ is the homomorphism associated to a represen tation Y of $\mathfrak{g}$, and 1 is the 1-dimensional representation with the zero action of $\mathfrak{g}$.

3.2.1 Monoidal category of vector space

The most important example of a monoidal category is given by the category $\mathfrak{Vec}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$. The latter is equipped with a monoidal structure for which $\otimes$ is the usual tensor product defined by

$$V \otimes W = Ll(V \times W)\big/_{U} \tag{3.29}$$

as quotient vector space of $Ll(V \times W)$ the free vector space on $V \times W$. $U = Span(N_1 \cup N_2 \cup N_3 \cup N_4)$

$$N_1 = \{\delta_{(v,w)} - \delta_{(\lambda v,w)}, v \in V, w \in W, \lambda \in \mathbb{K}\} \tag{3.30}$$

the homogeneity on the first component

$$N_2 = \{\delta_{(v,w)} - \delta_{(v,\lambda w)}, v \in V, w \in W, \lambda \in \mathbb{K}\} \tag{3.31}$$

the homogeneity on the second component and the additivity on first and second component

$$N_3 = \{\delta_{(v,w'+w)} - \delta_{(v,w)} - \delta_{(v,w')}, v \in V, w, w' \in W, \lambda \in \mathbb{K}\} \tag{3.32}$$

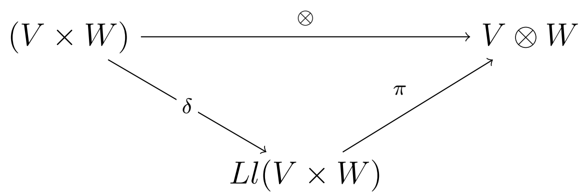
$$N_4 = \{\delta_{(v+v',w)} - \delta_{(v,w)} - \delta_{(v',w)}, v, v' \in V, w \in W, \lambda \in \mathbb{K}\} \tag{3.33}$$

where

$$\delta : V \times W \to Ll(V \times W)$$

acting like Kroneker's δ on the couple $(v, w) \in V \times W$.

After the quotient we have $\pi : Ll(V \times W) \to V \otimes W$ and the following diagram is commutative



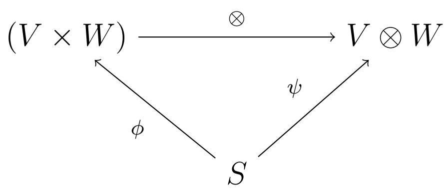
.

$\otimes$ satisfie the universal property : For each S vector space and for each bilinear map

$$\phi : V \times W \to S$$

$$\exists\,!\,\psi \text{ (bilinear) } : S \to V \otimes W$$

such that



.

this diagram is commutative :

$$\psi = \otimes \circ \phi. \tag{3.34}$$

The unit object 1 is $\mathbb{K}$ and the associativity and unit constraints are the natural isomorphisms :

$$a_{V_1,V_2,V_3} : V_1 \otimes (V_2 \otimes V_3) \to (V_1 \otimes V_2) \otimes V_3$$

$$(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$$

$$l_V : V \to \mathbb{K} \otimes V$$

$$v \mapsto 1 \otimes v$$

$$r_V : V \to V \otimes \mathbb{K}$$

$$v \mapsto v \otimes 1.$$

Note that the inverse to l_V is

$$l_V^{-1} : \mathbb{K} \otimes V \to V$$

$$k \otimes v \mapsto k \cdot v.$$

The scalars are provided by $\mathbb{K}$ itself, since it is in bijective correspondence with the linear maps from $\mathbb{K}$ to itself.

3.2.2 Braided categories

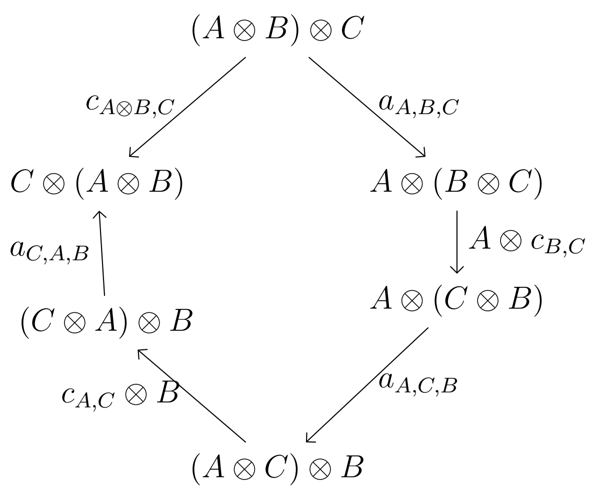
These categories are the fundamental instrument to introduce the categorical notion of Yang-Baxter equation.[19]

Definition 3.11 A functorial system of isomorphisms $c_{A,B} : A \otimes B \to B \otimes A$ in a monoidal category $(\mathcal{C}, \otimes, 1, a, l, r)$ is called a *commutativity constraint* if it satisfes the *hexagon identities*

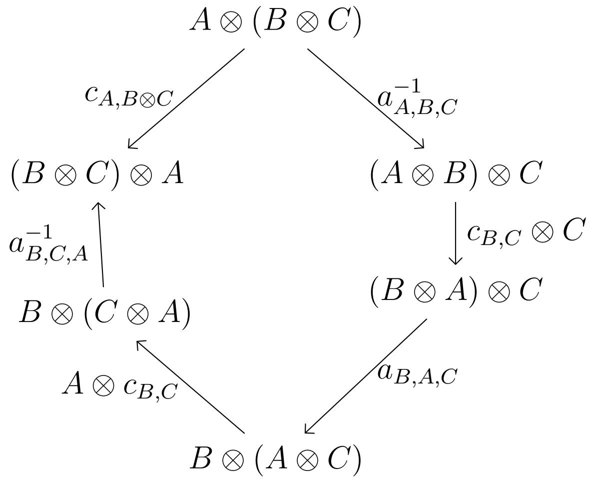
$$c_{A\otimes B,C} = a_{C,A,B} \circ (c_{A,C} \otimes B) \circ a^{-1}_{A,C,B} \circ (A \otimes c_{B,C}) \circ a_{A,B,C} \qquad (3.35)$$

$$c_{A,B\otimes C} = a^{-1}_{B,C,A} \circ (A \otimes c_{B,C}) \circ a_{B,A,C} \circ (c_{B,C} \otimes C) \circ a^{-1}_{A,B,C} \qquad (3.36)$$

or the commutativity of the hexagon diagrams:



(3.37)



(3.38)

The functoriality means that it commutes with morphisms in $\mathcal{C}$, i.e. the diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{c_{A,B}} & B \otimes A \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ C \otimes D & \xrightarrow{c_{C,D}} & D \otimes C \end{array} \tag{3.39}$$

is commutative for all $A, B, C, D \in Ob(C)$, and all $f : A \to B$ and $g : C \to D$.

Definition 3.12 A monoidal category with a commutativity constraint is called a *braided monoidal category* .

Definition 3.13 A category which has a commutativity constraint satisfying

$$c_{A,B} \circ c_{B,A} = 1_{A\otimes B} \tag{3.40}$$

symmetric monoidal category .

3.2.3 Dagger category

Definition 3.14 A dagger category is a category $\mathcal{C}$ with an involutive, identity-on-objects, contravariant functor

$$\dagger : \mathcal{C}^{op} \to \mathcal{C}$$

this means that to every morphism $f : A \to B$ one associates a morphism $\dagger(f) = f^\dagger : B \to A$ called the *adjoint* of f, such that for all $f : A \to B$, $g : B \to C$

- $1_A^\dagger = 1_A$
- $(g \circ f)^\dagger = f^\dagger \circ g^\dagger : C \to A$
- $f^{\dagger\dagger} = f$

Definition 3.15 (Unitary map, self-adjoint map) In a dagger category, a morphism $f : A \to B$ is called *unitary* if it is an isomorphism and

$$f^{-1} = f^\dagger$$

and is called *self-adjoint* or hermitian if

$$f = f^\dagger.$$

Definition 3.16 (Dagger symmetric monoidal category) A *dagger symmetric monoidal category* is a symmetric monoidal category with a dagger structure, such that the contravariant functor

$$\dagger : \mathcal{C}^{op} \to \mathcal{C}$$

coherently preserves the symmetric monoidal structure:

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger : B \otimes D \to A \otimes C \tag{3.41}$$

$$\alpha^\dagger_{A,B,C} = \alpha^{-1}_{A,B,C} : (A \otimes B \otimes)C \to A \otimes (B \otimes C) \tag{3.42}$$

$$\lambda^\dagger_A = \lambda^{-1}_A : 1 \otimes A \to A \tag{3.43}$$

$$c^\dagger_{A,B} = c^{-1}_{A,B} : A \otimes B \to B \otimes A \tag{3.44}$$

Definition 3.17 In a †-category a morphism $f : A \to B$ is an *isometry* if

$$f \circ f^\dagger = 1_A \tag{3.45}$$

and *normal* if

$$f \circ f^\dagger = f^\dagger \circ f \tag{3.46}$$

Definition 3.18 An object A in a monoidal category has a *left dual* if there exists an object A^{*L} and *left-duality morphisms*

$$\varepsilon_A^L : 1 \to A^{*L} \otimes A \tag{3.47}$$

$$\eta_A^L : A \otimes A^{*L} \to 1 \tag{3.48}$$

satisfying the triangle equations:

$$\begin{array}{ccc} A & \xrightarrow{Id_A} & A \\ & {\scriptstyle A \otimes \varepsilon_A^L} \searrow \quad \nearrow {\scriptstyle \eta_A^L \otimes A} & \\ & A \otimes A^{*L} \otimes A & \end{array} . \tag{3.49}$$

$$\begin{array}{ccc} A^{*L} & \xrightarrow{Id_{A^{*L}}} & A^{*L} \\ & {\scriptstyle \varepsilon_A^L \otimes A^*} \searrow \quad \nearrow {\scriptstyle A^* \otimes \eta_A^L} & \\ & A^{*L} \otimes A \otimes A^{*L} & \end{array} . \tag{3.50}$$

Analogously, an object A has a right dual if there exists an object A^{*R} and right-duality morphisms

$$\varepsilon_A^R : 1 \to A \otimes A^{*R} \tag{3.51}$$

$$\eta_A^R : A \otimes A^{*R} \to 1 \tag{3.52}$$

satisfying similar equations to those given above.

Definition 3.19 A monoidal category has left duals (or has right duals) if every object A has an assigned left dual or a right dual along with assigned duality morphisms, such that

$$I^{*L} = I \tag{3.53}$$

$$(A \otimes B)^{*L} = B^{*L} \otimes A^{*L} \tag{3.54}$$

or the equivalent with L replaced with R .

Definition 3.20 In a monoidal category with left or right duals, with an assigned left dual for each object or a chosen right dual for each object, the left duality functor$(-)^{*L}$ is a contravariant functor that take an abject A to their assigned duals, and act on morphisms

$$f^{*L} := (A^* \otimes \eta_B^L) \circ (A^* \otimes f \otimes B^*) \circ (\varepsilon_A^L \otimes B^*) \tag{3.55}$$

and right duality functor

$$f^{*R} := (\eta_B^R \otimes A*) \circ (B^* \otimes f \otimes A^*) \circ (A^* \otimes \varepsilon_A^L). \tag{3.56}$$

Definition 3.21. A *monoidal* † - *category* is a monoidal category equipped with a -functor, such that the associativity and unit natural isomorphisms are unitary. If the monoidal category is equipped with natural braiding isomorphisms, then these must also be unitary.
We will not assume that our monoidal categories are strict. A good reference for the essentials of monoidal category theory is[13] .

Definition 3.22. In a monoidal category, the scalars are the monoid $Hom(I, I)$. In a *monoidal* † - *category* the scalars form a monoid with involution.

Definition 3.23 . In a *monoidal* † - *category* a *state* of an object X is a morphism

$$\psi : \mathbf{1} \to X. \tag{3.57}$$

Definition 3.24 . In a *monoidal* † - *category* the *squared norm* of a state $\psi : \mathbf{1} \to X$. is the scalar

$$\psi^\dagger \circ \psi : \mathbf{1} \to \mathbf{1}. \tag{3.58}$$

Lemma 3.25 In a monoidal †-category, left-dual objects are also right-dual objects.
Proof. Give an object A with left dualA^{*L} witnessed by left-duality morphisms

$$\varepsilon_A^L : \mathbf{1} \to A^{*L} \otimes A$$

$$\eta_A^L : A \otimes A^{*L} \to \mathbf{1}$$

we can define

$$\varepsilon_A^R := \eta_A^{L\dagger} \quad \eta_A^L := \varepsilon_A^{L\dagger} \tag{3.59}$$

which witness that A^{*L} is a right dual for A.

□

Since left or right duals are always unique up to isomorphism, left duals must be iso morphic to right duals in a monoidal †-category. We will exploit this isomorphism to write A^* instead of A^{*L} or A^{*R}, and it follows that $A^{**} \cong A$.

Definition 3.26 A *monoidal †-category with duals* is a monoidal †-category such that each object A has an assigned dual object A^* with this assignment satisfying $(A^*)^* = A$, and assigned left and right duality morphisms for each object, such that these assignments are compatible with †-functor in the following way :

$$\varepsilon_A^L = \eta_A^{R\dagger} = \eta_{A^*}^{L\dagger} = \varepsilon_{A^*}^R, \quad \eta_A^L = \varepsilon_A^R = \varepsilon_{A^*}^{L\dagger} = \eta_{A^*}^R, \quad ((-)^{*L})^\dagger = ((-)^\dagger)^{*L} \tag{3.60}$$

Since the left and right duality morphisms can be obtained from each other using the $\dagger - functor$, from now on we will only refer directly to the left-duality morphisms, definig

$$\varepsilon_A := \varepsilon_A^L, \quad \eta_A := \eta_A^L.$$

Definition 3.27 In a monoidal †-category with duals, the conjugation functor $(-)_*$ is defined on all morphisms f by

$$f_* = (f^*)^\dagger = (f^\dagger)^* \tag{3.61}$$

Since the †-functor is the identity on objects, we have $A^* = A_*$ for all objects. To make this equality clear we will write A^* exclusively, and the A_*form will not be used. For any morphism $f : A \to B$ we can use these functors to construct

$$f_* : A^* \to B^*$$
$$f^* : B^* \to A^*$$
$$f^\dagger : B \to A$$

Definition 3.28 In a †-category, a morphism $f : X \to Y$ is an *isometry* if

$$f^\dagger \circ f = 1_X. \tag{3.62}$$

Definition 3.29 In a †-category, a morphism $f : X \to Y$ is *unitary* if

$$f^\dagger \circ f = 1_X. \tag{3.63}$$

and

$$f \circ f^\dagger = 1_Y; \tag{3.64}$$

in other words, if f is an isomorphism and $f^{-1} = f^\dagger$.
Definition 3.30 In a †-category, a morphism $f : X \to X$ is *self-adjoint* if

$$f = f^\dagger. \tag{3.65}$$

Definition 3.31 . In a †-category, a morphism $f : X \to X$ is *normal* if

$$f^\dagger \circ f = f \circ f^\dagger. \tag{3.66}$$

Involution monoids

An important tool in functional analysis is the $* - algebra$: a complex, associative, unital algebra equipped with an antilinear involutive homomorphism from the algebra to itself which reverses the order of multiplication. Category-theoretically, such a homomorphism is not very convenient to work with, since morphisms in a category of vector spaces are usually chosen to be the linear maps. However, if the vector space has an inner product, this induces a canonical antilinear isomorphism from the vector space to its dual. Composing this with the antilinear self- involution, we obtain a linear isomorphism from the vector space to its dual. This style of isomorphism is much more useful from a categorical perspective, and we use it to define the concept of an *involution monoid.* [21]

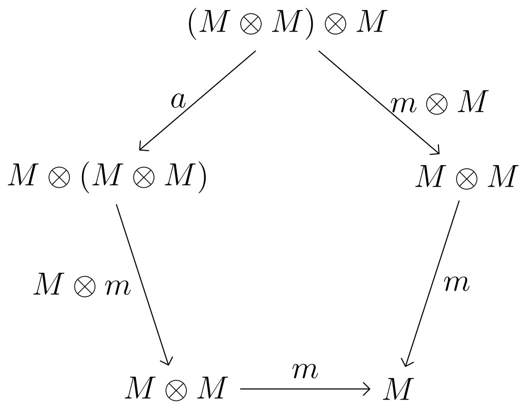
Definition 3.32 In the context of monoidal categories a **monoid** is an ordered triple (H, m, u) consisting of an object H, a multiplication morphism

$$m : H \otimes H \to H$$

and a unit morphism

$$u : \mathbf{1} \to H$$

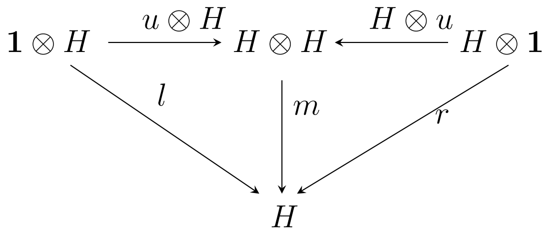
which satisfy associtativity condition



(3.67)

$$a \circ M \otimes m \circ m = m \otimes M \circ m$$

and unit condition :

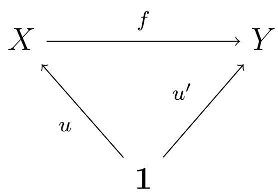


(3.68)

Definition 3.33 In a symmetric monoidal category a morphism $f : X \to Y$ is a monoid homomorphism for monoids (X, m, u) (Y, m', u') if the following diagrams are commutative

$$\begin{array}{ccc} X \otimes X & \xrightarrow{m} & X \\ {\scriptstyle f\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ Y \otimes Y & \xrightarrow{m'} & Y \end{array} \tag{3.69}$$

i.e. $f \circ m = m' \circ (f \otimes f)$ and



.

$f \circ u = u'$.

Definition 3.34 In a monoidal †-category with duals, an **involution monoid** $(A, m, u; s)$ is a monoid equipped with a morphims $s : A \to A^*$ called **linear involution**, which is a morphism of monoids with respect to monoid structure (A^*, m_*, u_*) on A^*, and wich satisfies the **involution condition**

$$s_* \circ s = 1_A. \tag{3.70}$$

It follows from this definition that s and s_* are mutually inverse morphisms, since applying the conjugation functor to the involution condiction gives $s \circ s_* = 1_{A^*}$.
We also note that for any such involution monoid $s : A \to A^*$ and $s^* : A \to A^*$ are parallel morphism, but they are not necessarily the same.

Definition 3.35 In a monoidal †-category with duals, given involution monoids (A, m, u, s_A),(B, m, v, s_B) a morphism $f : A \to B$ is a **homomorphism of involution monoids** if it is a morphism of monoids, and if it satisfies the **involution preservation condition**

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle s_A}\downarrow & & \downarrow{\scriptstyle s_B} \\ A^* & \xrightarrow{f_*} & B^* \end{array} \tag{3.71}$$

i.e.

$$s_B \circ f = f \circ s_A.$$

If an object B is self-dual, it is possible for the involution $s_B : B \to B$ to be the identity. Let $(B, m, v, 1_B)$ be such an involution monoid . In the case, it is sometimes possible to find an embedding : $(A, m, u, s_A) \hookrightarrow (B, m, v, 1_B)$ of involution monoids event whe the s_A is not trivial .
Definition 3.36 In a monoidal category, for an object H with a dual H^* , the endomorphism monoid $End(H)$ is defined by

$$End(H) := (H^* \otimes H, H^* \otimes \eta_H \otimes H, \varepsilon_H). \tag{3.72}$$

The following lemma describes a well-known connection between categorical duality and Frobenius structures.

3.3 The Monoidal Category of FdHilb

Proposition 3.37 Let $\mathcal{H}, \mathcal{K}$ be two finite-dimensional Hilbert spaces on $\mathbb{C}$, the vector tensor product $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{K}$ is an Hilbert space.

Proof. Let $n = dim(\mathcal{H})$ and $m = dim(\mathcal{K})$. Denote by $e_i \otimes f_i$ the i-th orthonormal basis vector of $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{K}$ and let $\{\sum_i a_i^n e_i \otimes f_i \}_{n\in\mathbb{N}}$ be a Cauchy sequence . Then, by Parseval theorem we have

$$|| \sum_i a_i^n e_i \otimes f_i - \sum_j a_j^m e_j \otimes f_j||^2 = \sum_i |a_i^{(n)} - a_i^{(n)}|^2 \tag{3.73}$$

This show that $\{a_i^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence for each $1 \leq i \leq nm$. Hence for each i we have $a_i := lim_{n\to\infty} a_i^{(n)}$. Now thanks to the finiteness and the linearity of the limit we have :

$$lim_{n\to\infty} \left\{ \sum_i a_i^n e_i \otimes f_i \right\} = \sum_i \left(lim_{n\to\infty} a_i^n \right) e_i \otimes f_i = \sum_i a_i e_i \otimes f_i \tag{3.74}$$

thus $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{K}$ is complete .

□

This proposition makes working with the tensor a lot more simple. This is because elements in $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{K}$ are in general converging sequences of elements in the vector tensor product, this means we are dealing with objects of the form

$$|\psi\rangle \otimes |\phi\rangle = \sum_{ij} \psi_i |i\rangle \otimes \phi_j |j\rangle, \quad i = 1, ..., n \quad j = 1, ..., m$$

where we have used the Dirac notation .

Definition 3.38 Given two morphisms $f : \mathcal{H} \to \mathcal{H}'$ and $g : \mathcal{K} \to \mathcal{K}'$ we define

$$\otimes(f, g) = f \otimes g : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}' \tag{3.75}$$

$$\otimes(f, g)(h \otimes k) = f|\psi\rangle \otimes g|\phi\rangle. \tag{3.76}$$

Composition is also defined component wise and therefore we have a functor. To complete the structure we need a unit. Because we used the complex tensor, it will come as no surprise that $\mathbb{C}$ satisfies the necessary properties. The remaining details on the natural isomorphisms are given in the following proposition.

Proposition 3.39 (**FdHilb**. $\otimes, \mathbb{C}$) is a symmetric monoidal category .

Proof. We have already seen that

$$\otimes : \mathbf{FdHilb} \times \mathbf{FdHilb} \to \mathbf{FdHilb}$$

is a functor . Thus Proof. We have already seen that $\otimes$ is a functor, thus we only need to prove the existence of the four natural isomorphisms . Let $\mathcal{H}, \mathcal{K}, \mathcal{L} \in \mathbf{FdHilb}$:
the morphism

$$a : \mathcal{H} \otimes (\mathcal{K} \otimes \mathcal{L}) \to (\mathcal{H} \otimes \mathcal{K}) \otimes \mathcal{L}$$

is defined by

$$|h\rangle \otimes (|k\rangle \otimes |l\rangle) \mapsto (|h\rangle \otimes |k\rangle) \otimes |l\rangle \tag{3.77}$$

It is easy to see that this is well defined and linear. By definition of the inner product we have

$$\begin{aligned} ||a(h \otimes (k \otimes l))||^2 &= \langle |h\rangle \otimes (|k\rangle \otimes |l\rangle)|(|h\rangle \otimes |k\rangle) \otimes |l\rangle\rangle^2 \\ &= \langle h|h\rangle \langle k|k\rangle \langle l|l\rangle \\ &= ||h||^2 ||k||^2 ||l||^2 \end{aligned} \tag{3.78}$$

Therefore it is bounded and hence a morphism in Hilb. Moreover because $\otimes$ is defined component wise on functions it is natural. The inverse is obvious so a is a natural isomorphism. Next we define the transformation

$$\lambda : \otimes(\mathbb{C}, \bullet) \to 1_{\mathbf{FdHilb}} \tag{3.79}$$

$$\lambda(c \otimes |h\rangle) = c|h\rangle.$$

This is also well defined, linear, bounded and natural. It has an inverse defined by

$$\lambda^{-1}(h) = 1 \otimes h$$

for

$$\lambda^{-1}\lambda(c \otimes h) = \lambda^{-1}(ch) = 1 \otimes ch = c \otimes h$$
$$\lambda^{-1}\lambda(h) = \lambda(1 \otimes h) = h$$

Hence, λ is a natural isomorphism. The definition of the transformation ρ is given by

$$h \otimes c \mapsto ch.$$

While the commutative transformation γ has the obvius definition

$$h \otimes k \mapsto k \otimes h.$$

The commutativity of diagrams is straightforward.

□

3.3.1 A dagger on FdHilb

Being monoidal is a start but we need **FdHilb** to be a symmetric dagger monoidal category. This means we have to define a dagger. We use Riesz Representation Theorem which states that for each bounded linear functional

$$f : \mathcal{H} \to \mathbb{C}$$

there exists a unique vector $|h_0\rangle \in \mathcal{H}$ such that

$$f|h\rangle = \langle h|h_0\rangle \tag{3.80}$$

for all $|h\rangle \in \mathcal{H}$ and $||\,|h_0\rangle|| = ||f||$.

Now let

$$f : \mathcal{H} \to \mathcal{K} \quad \text{fix } |k\rangle \in \mathcal{K}.$$

Consider the function

$$F_k : \mathcal{K} \to \mathbb{C} \tag{3.81}$$

$$F_k|h\rangle := \langle (f|h\rangle)|k\rangle.$$

This is clearly linear and bounded , $||F_k|h\rangle||^2 = |\langle (f|h\rangle)|k\rangle|^2$

$$|\langle (f|h\rangle)|k\rangle|^2 \leq \langle (f|h\rangle)|(f|h\rangle)\rangle\langle k|k\rangle \leq ||(f|h\rangle)||^2||\,|k\rangle||^2 \leq ||f||^2||\,|k\rangle||^2||\,|h\rangle||^2 \tag{3.82}$$

Hence by the Riesz Representation Theorem there is a unique $|h_k\rangle$ such that

$$\langle (f|h\rangle)|k\rangle = \langle h|h_k\rangle. \tag{3.83}$$

Now define

$$f^\dagger : \mathcal{H} \to \mathcal{K}$$

by

$$f^\dagger|k\rangle = |h_k\rangle \tag{3.84}$$

then we have the following :

Lemma 3.40 Let $f : \mathcal{H} \to \mathcal{K} \in$ **FdHilb** then

- i) $f^\dagger$ is linear
- ii)$f^\dagger$ is bounded

Proof.i) Let $|k_1\rangle, |k_2\rangle \in \mathcal{K}$, $|h\rangle \in \mathcal{H}$ and $c_1, c_2 \in \mathbb{C}$ then

$$\begin{aligned} \langle h|(|c_1 f^\dagger|k_1\rangle + c_2 f^\dagger|k_2\rangle)\rangle &= c_1\langle h|f^\dagger|k_1\rangle + c_2\langle h|f^\dagger|k_2\rangle \\ &= \langle\overline{c_1}h|f^\dagger|k_1\rangle + \langle\overline{c_2}h|f^\dagger|k_2\rangle \\ &= \langle f\overline{c_1}h|k_1\rangle + \langle f\overline{c_2}h|k_2\rangle \\ &= \langle(f|h\rangle)|(c_1|k_1\rangle + c_2|k_2\rangle)\rangle \end{aligned} \tag{3.85}$$

Hence by uniqueness $f^\dagger(c_1|k_1\rangle + c_2|k_2\rangle) = c_1 f^\dagger|k_1\rangle + c_2 f^\dagger|k_2\rangle$.
ii) Let $|k\rangle \in \mathcal{K}, |h\rangle \in \mathcal{H}$, then by calculation we did earlier

$$||F_k|| \leq ||f|| \, || \, |k\rangle|| \tag{3.86}$$

and hence by Riesz Rapresentation Theorem

$$||f^\dagger|k_1\rangle|| = ||F_k|| \leq ||f|| \, || \, |k\rangle|| \tag{3.87}$$

so $f^\dagger$ is bounded.

□

This lemma proves that $f^\dagger$ is a morphism in **FdHilb**. Next we show that it satisfies the conditions of dagger .

Lemma 3.33 Let $f : \mathcal{H} \to \mathcal{K}$ and $g : \mathcal{K} \to \mathcal{L}$ be morphisms in **FdHilb** then :

- i)$1^\dagger = 1$
- $(f^\dagger)^\dagger = f$
- $(gf)^\dagger = f^\dagger g^\dagger$

Notation $f|h\rangle = |fh\rangle$

Proof. Recall that by lemma 3.32 $f^\dagger, g^\dagger$ are morphisms in **FdHilb** .

i) Is trivial .

ii) Take $|h\rangle \in \mathcal{H}, |k\rangle \in \mathcal{K}$

$$\langle k|fh\rangle = \overline{\langle fh|k\rangle} = \overline{\langle h|f^\dagger k\rangle} = \langle f^\dagger k|h\rangle \tag{3.88}$$

hence by uniqueness $(f^\dagger)^\dagger = f$.

iii)Take $|h\rangle \in \mathcal{H}, |l\rangle \in \mathcal{L}$ then

$$\langle h|f^\dagger g^\dagger l\rangle = \langle fh|g^\dagger l\rangle = \langle gfh|l\rangle \tag{3.89}$$

so again by uniqueness $(gf)^\dagger = f^\dagger g^\dagger$

□

The above considerations and lemmas are summarized in the following definition:

Definition 3.41 . Define the functor

$$\dagger : \mathbf{FdHilb}^{op} \to \mathbf{FdHilb}$$

as the identity on and on morphisms $f : \mathcal{H} \to \mathcal{K}$ by defining $f^\dagger|k\rangle$ to be unique element in $\mathcal{H}$, such that

$$\langle h|f^\dagger k\rangle = \langle fh|k\rangle$$

for all $|h\rangle \in \mathcal{H}$.

Lemma 3.32 together with lemma 3.33 say that † is well defined and a dagger on **FdHilb**, remains to show that it preserve the monoidal structure but we need a lemma :

Lemma 3.42. Let $f \in Hom_{\mathbf{FdHilb}}(\mathcal{H}, \mathcal{K})$ then

i) $f^\dagger \circ f = 1_{\mathcal{H}} \Longleftrightarrow \langle h|h'\rangle = \langle fh|fh'\rangle$ for all $|h\rangle, |h'\rangle \in \mathcal{H}$

ii) $f \circ f^\dagger = 1_{\mathcal{K}} \Longleftrightarrow \langle k|k'\rangle = \langle f^\dagger k|f^\dagger k'\rangle$ for all $|k\rangle, |k'\rangle \in \mathcal{K}$

Proof.
Suppose $f^\dagger \circ f = 1_{\mathcal{H}}$ then

$$\langle h|h'\rangle = \langle h|f^\dagger fh'\rangle = \langle fh|fh'\rangle.$$

Now suppose

$$\langle h|h'\rangle = \langle h|f^\dagger fh'\rangle = \langle fh|fh'\rangle.$$

for all $|h\rangle, |h'\rangle \in \mathcal{H}$.
Then

$$\langle h|h'\rangle = \langle fh|fh'\rangle = \langle h|f^\dagger fh'\rangle$$

and because of uniqueness it follows that

$$f^\dagger f|h'\rangle = |h'\rangle \tag{3.90}$$

for all $|h'\rangle \in \mathcal{H}$.

ii).
Suppose $f \circ f^\dagger = 1_\mathcal{K}$ then

$$\langle k|k'\rangle = \langle k|ff^\dagger k'\rangle = \langle f^\dagger k|f^\dagger k'\rangle.$$

Now suppose

$$\langle k|k'\rangle = \langle f^\dagger k|f^\dagger k'\rangle$$

for all $|k\rangle, |k'\rangle \in \mathcal{K}$. Then

$$\langle k|k'\rangle = \langle f^\dagger k|f^\dagger k'\rangle = \langle k|ff^\dagger k'\rangle$$

and because of uniqueness it follows that

$$ff^\dagger|k'\rangle = |k'\rangle \tag{3.91}$$

for all $|k'\rangle \in \mathcal{K}$.

□

Proposition 3.43 FdHilb is a symmetric dagger monoidal category .

Proof. We have already shown that **FdHilb** is a symmetric monoidal category and that is has a dagger. What remains to prove is that the dagger and the tensor commute and the four structure morphisms a, l, r and c are unitary.
Given two morphisms $f : \mathcal{H} \to \mathcal{H}'$ and $g : \mathcal{K} \to \mathcal{K}'$ in **FdHilb**, then

$$\begin{aligned}(\langle h| \otimes \langle k|)|f^\dagger \otimes g^\dagger(h' \otimes k')\rangle &= \langle h|f^\dagger h'\rangle\langle k, g^\dagger k'\rangle \\ &= \langle fh|h'\rangle\langle gk, k'\rangle \\ &= \langle f \otimes g(h \otimes k)|(|h'\rangle \otimes |k'\rangle) \end{aligned} \tag{3.92}$$

so by uniqueness

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$$

which proves that

$$\dagger\otimes = \otimes\dagger. \tag{3.93}$$

We'll prove that l_X is unitary, the prove for the other transformations are similar :

$$\langle c \otimes h|\lambda^{-1}h\rangle = \langle c \otimes h|1 \otimes h\rangle = \langle ch|h\rangle = \langle ch|h\rangle = \langle (c \otimes h)|h\rangle \tag{3.94}$$

hence

$$\lambda^{-1} = \lambda^\dagger. \tag{3.95}$$

□

Two key maps in FdHilb

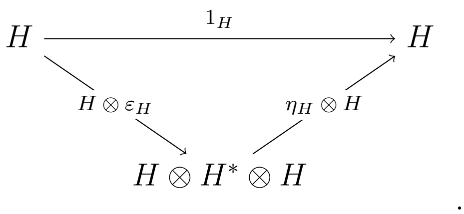
In **FdHilb** we define

$$\eta_H \otimes H : H \otimes H^* \otimes H \to \mathbb{C} \otimes H \cong H$$

by setting

$$|\psi\rangle \otimes \langle w| \otimes |\varphi\rangle \mapsto \langle \psi|w\rangle|\varphi\rangle. \tag{3.96}$$

Just by imposing that this diagram



(3.97)

is commutative *i.e.*

$$1_H = (\eta_H \otimes H) \circ (H \otimes \varepsilon_H). \tag{3.98}$$

It is possible to proof that $\varepsilon_H : \mathbb{C} \to H^* \otimes H$ is

$$1 \mapsto \sum_{i=1}^{n} \langle i| \otimes |i\rangle \tag{3.99}$$

Proof. The most general $\varepsilon_H : \mathbb{C} \to H^* \otimes H$ is

$$|\psi\rangle \otimes \langle w| \otimes |\varphi\rangle = \sum_{i,j,k=1}^{n} \psi_k w_i^* \varphi_j |k\rangle \otimes \langle i| \otimes |j\rangle \tag{3.100}$$

$$\Downarrow$$

$$(\eta_H \otimes H)(\sum_{i,j,k=1}^{n} \psi_k w_i^* \varphi_j |k\rangle \otimes \langle i| \otimes |j\rangle) = \sum_{i,j,k=1}^{n} \psi_k w_i^* \varphi_j \delta_{ik} |j\rangle \tag{3.101}$$

$$\Downarrow$$

$$(\eta_H \otimes H)(\sum_{i,j,k=1}^{n} \psi_k w_i^* \varphi_j |k\rangle \otimes \langle i| \otimes |j\rangle) = \sum_{i,j=1}^{n} \psi_i w_i^* \varphi_j |j\rangle. \tag{3.102}$$

Now we impose the commutativity

$$|\psi\rangle = \sum_{t=1}^{n} \psi_t |t\rangle = \sum_{i,j=1}^{n} \psi_i w_i^* \varphi_j |j\rangle. \tag{3.103}$$

Then we have

$$w_i^* \varphi_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{3.104}$$

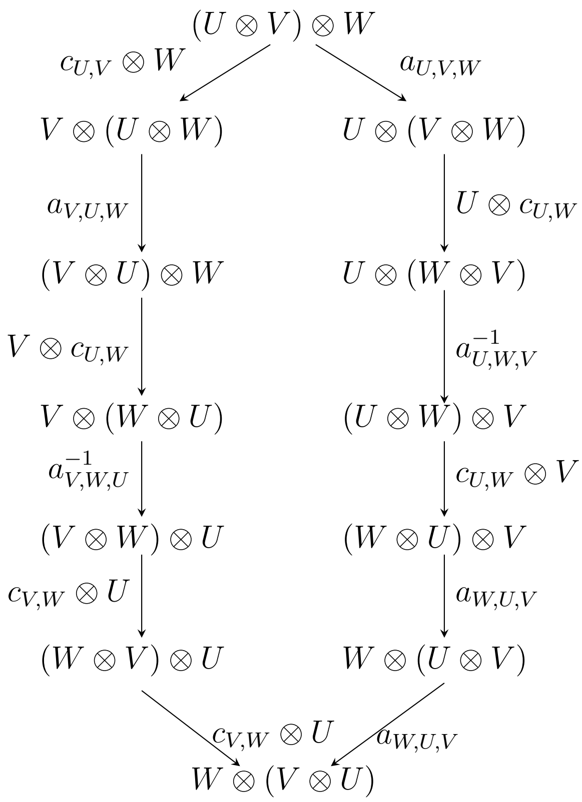
hence

$$\begin{aligned} \varepsilon_H : \mathbb{C} &\to H^* \otimes H \\ 1 &\mapsto \sum_{i=1}^{n} \langle i| \otimes |i\rangle. \end{aligned} \tag{3.105}$$

3.4 Yang-Baxter equation and braided categories

One of the main properties of a braided monoidal category is stated in the following theorem which may be considered as the categorical version of Yang-Baxter equation.[19]

Theorem 3.44 Let $U, V, W \in \mathcal{C}$ be objects in a braided monoidal category, then the dodecagon



commutes .

Proof. We cut the dodecagon in two hexagon and a square : the clockwise composition of the morphisms in the dodecagon starting from $(U \otimes V) \otimes W$ and ending at $W \otimes (U \otimes V)$ is equal to $c_{U \otimes V,W}$. Similarly the counterclockwise composition of the morphisms from $(V \otimes U) \otimes$ to $W \otimes (V \otimes U)$ is equal to $c_{V \otimes U,W}$.

It remains to check the commutativity of the square

$$\begin{array}{ccc} (U \otimes V) \otimes W) & \xrightarrow{c_{U\otimes V,W}} & W \otimes (U \otimes V) \\ {\scriptstyle c_{U,V}\otimes W}\downarrow & & \downarrow{\scriptstyle W\otimes c_{U,V}} \\ (V \otimes U) \otimes W & \xrightarrow{c_{V\otimes U,W}} & W \otimes (V \otimes U) \end{array} \tag{3.106}$$

but this is a special case of the commutative square (3.28) expressing the functoriality of the braiding where f is replaced by $c_{U,V}$ and g by 1_W .

This theorem implies that if the category is strict :

$$X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z \tag{3.107}$$

and

$$X \otimes \mathbf{1} = X = \mathbf{1} \otimes X \tag{3.108}$$

the commutativity of dodecagon diagram is

$$(c_{V,W} \otimes U) \circ (V \otimes c_{U,W}) \circ (c_{U,V} \otimes W) = (W \otimes c_{U,V}) \circ (c_{U,V} \otimes V) \circ (U \otimes c_{V,W}) \tag{3.109}$$

the **Yang-Baxter equation**.

Resuming

In this chapter we have seen some tools of category theory , which the most important in our work are the monoidal categories(e.g. FdHilb) and braided categories. The latter give us the possibility to define a categorical notion of YBE. In next chapter we will see the quantum groups and how are connected to the Hopf algebras.

Chapter 4

Hopf algebras, quantum groups and algebraic quantum field theory

The notion of deformation is very familiar to the physicist. In this connection, quantum mechanics may be considered as a deformation (the deformation parameter being $\hbar$) of classical mechanics and relativistic mechanics is, to a certain extent, another deformation (with 1/c as deformation parameter) of classical mechanics. Although a sharp distinction should be established between deformations and quantized universal enveloping algebras or quantum algebras, the concept of a quantum algebra is more easily introduced in the parlance of deformations. The concept of a quantum algebra (or quantum group) goes back to the end of the seventies. It was introduced, under different names, by Kulish, Reshetikhin, Sklyanin, Drinfeld (from the Faddeev school) and Jimbo in terms of a quantized universal enveloping algebra or an Hopf bi-algebra and, independently, by Woronowicz in terms of a compact matrix pseudo-group. Among the various motivations that led to the concept of a quantum group, we have to mention the quantum inverse scattering technique, the solution of the quantum Yang-Baxter equation and, more generally, the study of exactly solvable models in statistical mechanics. Some applications of quantum algebras concern : 1+1 conformal field theories ; quantum dynamical systems ; quantum optics ; nuclear spectroscopies ; condensed matter physics ; knot theory, theory of link invariants and braid groups ;The concept of a quantum group is a basic tool in non-commutative geometry.

4.1 Hopf algebras

4.1.1 Algebras and Coalgebras

Definition 4.1 Let $\mathbb{K}$ be a field. A $\mathbb{K}$-*algebra* is a $\mathbb{K}$-vector space with two linear maps

$$m : A \otimes_{\mathbb{K}} A \to A$$

$$u : \mathbb{K} \to A$$

satisfying the associativity and the unit axioms .

Definition 4.2 A $\mathbb{K}$-*coalgebra* is a $\mathbb{K}$-vector space C with two linear maps

$$\Delta : C \to C \otimes_{\mathbb{K}} C$$

$$u : C \to \mathbb{K}$$

such that the coassociativity and the counit axioms are verified.

Definition 4.3 Let A and B be $\mathbb{K}$-algebras. A linear map $\psi : A \to B$ is a algebra homomorphism if the diagrams

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\psi\otimes\psi} & B \otimes B \\ {\scriptstyle m_A}\downarrow & & \downarrow{\scriptstyle m_B} \\ A & \xrightarrow{\psi} & B \end{array} \tag{4.1}$$

and

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{1_{\mathbb{K}}} & \mathbb{K} \\ {\scriptstyle u_A}\downarrow & & \downarrow{\scriptstyle u_B} \\ A & \xrightarrow{\psi} & B \end{array} \tag{4.2}$$

commute.

Definition 4.4 Let C and D be $\mathbb{K}$-coalgebras. A linear map $\phi : C \to D$ is a coalgebra homomorphism if the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\phi} & C \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_D} \\ C \otimes C & \xrightarrow{\phi\otimes\phi} & D \otimes D \end{array} \tag{4.3}$$

and

$$\begin{array}{ccc} C & \xrightarrow{\phi} & D \\ \varepsilon_C \downarrow & & \downarrow \varepsilon_D \\ \mathbb{K} & \xrightarrow{1_{\mathbb{K}}} & \mathbb{K} \end{array} \tag{4.4}$$

commute.

Definition 4.5 Let A be an $\mathbb{K}$-algebra. The *opposite algebra* A^{op} is the same vector space with a multiplication

$$m_{op} : A \otimes A \to A :: m_{op}((v, w)) = m((w, v)) \tag{4.5}$$

where m is the multiplication map on A.
Similarly, if C is a coalgebra the *opposite algebra* C^{op} is the same vector space with a comultiplication defined by

$$\Delta_{op}(v) := \sigma \circ \Delta(v) \tag{4.6}$$

where σ is the permutation map.
Example 4.6 A $\mathbb{K}$-vector space V with basis $\mathcal{B}$ is a coalgebra if we set

$$\Delta(v) = v \otimes v, \quad \forall v \in \mathcal{B}$$

$$\varepsilon(v) = 1, \quad \forall v \in \mathcal{B}.$$

Example 4.7 Now consider the polynomial algebra

$$A(X) = \mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}].$$

As a vector space, it's basis is

$$\{x_{11}^i, x_{12}^j, x_{21}^k, x_{22}^l : i, j, k, l \in \mathbb{Z}^+\},$$

and examples of elements of $A(X)$

$$x_{11}x_{22} - 3x_{11}x_{22} + x_{11}$$

$$x_{11}x_{12} - x_{22}x_{21}.$$

If we think of

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

then we can think of the polynomials of $A(X)$ as functions from $M(2, \mathbb{C})$ to $\mathbb{C}$. They are in fact often called the **regular functions** of $M(2, \mathbb{C})$.

We define the comultiplication and counit maps on A(X)

$$\Delta(x_{ij}) = x_{i1}x_{j1} + x_{i2}x_{j2}$$
$$\varepsilon(x_{ij}) = \delta_{ij}.$$

We extend the action of Δ to the rest of $A(X)$ by defining it to be a algebra homomorphism

$$\Delta(x_{ij}x_{kl}) = \Delta(x_{ij})\Delta(x_{kl})$$

then $A(X)$ is a coalgebra.

Definition 4.8 A **bialgebra** is a quintuple $(B, \Delta, \varepsilon, \mu, \eta)$ where (B, Δ, ε) is a coalgebra, (B, μ, η) is an algebra and either of the following equivalent conditions is true:

- Δ and ε are algebra morphisms,
- μ and η are coalgebra morphisms.

Example 4.9 Consider the $\mathbb{K}$-vector space $M_n(\mathbb{K})$ of $n \times n$ matrices with coefficient in $\mathbb{K}$. It has a monoid structure with respect to the multiplication, since not all elements are invertible. Let $\mathcal{O}(M_n(\mathbb{K}))$ be the commutative algebra over $\mathbb{K}$ generated by the elements

$$\{X_{ij} : 1 \leq i, j \leq n\}. \tag{4.7}$$

As algebra, it is simply the commutative ring of polynomials in n^2 variables

$$\mathcal{O}(M_n(\mathbb{K})) = \mathbb{K}[X_{ij} : 1 \leq i, j \leq n]. \tag{4.8}$$

Moreover $\mathcal{O}(M_n(\mathbb{K}))$ is a subalgebra of the algebra of functions

$$\{f : M_n(\mathbb{K}) \to \mathbb{K}\}$$

on $M_n(\mathbb{K})$ where X_{ij} is the function defined by matrix coefficient

$$X_{ij}(A) = a_{ij} \,\forall\, A = (a_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{K}. \tag{4.9}$$

If we denote by E_{ij} the matrix with a 1 in the entry (i, j) and 0 in all others position, the set $\{E_{ij}\}_{1\leq i,j\leq n}$ is a linear basis of $M_n(\mathbb{K})$ and the set $\{X_{ij}\}_{1\leq i,j\leq n}$ is the corresponding dual basis with

$$\langle X_{ij}, E_{kl}\rangle = \delta_{ik}\delta_{jl}.$$

Therefore, $\mathcal{O}(M_n(\mathbb{K}))$ is the algebra of regular functions on $M_n(\mathbb{K})$.

$\mathcal{O}(M_n(\mathbb{K}))$ is a bialgebra with the coalgebra determined by

$$\Delta(X_{ij}) = \sum_{i=1}^{n} X_{ik} \otimes X_{kj}, \quad \varepsilon(X_{ij}) = \delta_{ij} \,\forall 1 \leq i, j \leq n. \tag{4.10}$$

Indeed, since $\mathcal{O}(M_n(\mathbb{K}))$ is generated as a free algebra by the elements $\{X_{ij} : 1 \leq i, j \leq n\}$, to define the algebra maps Δ and ε, it suffices to define them on the generators. Moreover, since both maps are uniquely determined by their values on the generators, it is enough to check the coassociativity and the counit axioms on them.
For the coassociativity we have

$$\begin{aligned}
[\Delta \otimes 1_{\mathcal{O}(M_n(\mathbb{K}))}]\Delta(X_{ij}) &= [\Delta \otimes 1_{\mathcal{O}(M_n(\mathbb{K}))}] \left(\sum_{i=1}^{n} X_{ik} \otimes X_{kj} \right) \\
&= \sum_{i=1}^{n} \Delta(X_{ik}) \otimes X_{kj} \\
&= \sum_{i=1}^{n} X_{il} \otimes X_{lk} \otimes X_{kj}
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
[1_{\mathcal{O}(M_n(\mathbb{K}))} \otimes \Delta]\Delta(X_{ij}) &= [1_{\mathcal{O}(M_n(\mathbb{K}))} \otimes \Delta] \left(\sum_{i=1}^{n} X_{il} \otimes X_{lj} \right) \\
&= \sum_{i=1}^{n} X_{il} \otimes \Delta(X_{lj}) \\
&= \sum_{i=1}^{n} X_{il} \otimes X_{lk} \otimes X_{kj},
\end{aligned} \tag{4.12}$$

$\forall 1 \leq i, j \leq n$. Thus, Δ is coassociative. For the counit we have

$$\begin{aligned}
m(\varepsilon \otimes 1_{\mathcal{O}(M_n(\mathbb{K}))})\Delta(X_{ij}) &= m(\varepsilon \otimes 1_{\mathcal{O}(M_n(\mathbb{K}))}) \left(\sum_{i=1}^{n} X_{il} \otimes X_{lj} \right) \\
&= m \left(\sum_{i=1}^{n} \varepsilon(X_{ik}) \otimes X_{kj} \right) \\
&= m \left(\sum_{i=1}^{n} \delta_{ik} \otimes X_{kj} \right) \\
&= m(1 \otimes X_{ij}) = X_{ij}
\end{aligned} \tag{4.13}$$

$$
\begin{aligned}
m(1_{\mathcal{O}(M_n(\mathbb{K}))} \otimes \varepsilon)\Delta(X_{ij}) &= m(\varepsilon \otimes 1_{\mathcal{O}(M_n(\mathbb{K}))}) \left(\sum_{i=1}^{n} X_{ik} \otimes X_{kj} \right) \\
&= m \left(\sum_{i=1}^{n} X_{ik} \otimes \varepsilon(X_{kj}) \right) \\
&= m \left(\sum_{i=1}^{n} X_{ik} \otimes \delta_{kj} \right) \\
&= m(X_{ij} \otimes 1) = X_{ij}, \qquad (4.14)
\end{aligned}
$$

$\forall 1 \leq i, j \leq n$; which proves that ε is a counit and thus $\mathcal{O}(M_n(\mathbb{K}))$ is a bialgebra.

Definition 4.10.(Convolution) Given an algebra (A, μ, η), a coalgebra(C, Δ, ε) and two linear maps $f, g : C \to A$ then the *convolution* of f and g is the linear map

$$f * g : C \to A \qquad (4.15)$$

defined by

$$(f * g)(c) = \mu \circ (f \otimes g) \circ \Delta(c), c \in C. \qquad (4.16)$$

Definition 4.11 Let $(H, \Delta, \varepsilon, \mu, \eta)$ be a bialgebra . An endomorphism S of H is called an antipode for the bialgebra H if

$$1_H * S = S * 1_H = \eta \circ \varepsilon. \qquad (4.17)$$

A **Hopf algebra** is a bialgebra with an antipode.

Example 4.12. Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$.

$$\Delta(g) = g \otimes 1 + 1 \otimes g, \forall g \in \mathfrak{g} \qquad (4.18)$$

$$\varepsilon(g) = 0, \forall g \in \mathfrak{g} \qquad (4.19)$$

$$S(g) = g^{-1} \qquad (4.20)$$

$U(\mathfrak{g})$ is an Hopf algebra. It is remarkable that under the same definition the tensor algebra on a vector space is a Hopf algebra.

Example 4.13. Recall from Example 2.11 that for $n = 2$ the algebra $\mathcal{O}(M_2(\mathbb{K}))$ has a bialgebra structure. $\mathcal{O}(SL_2)$ is the subalgebra generated by $X_{11}, X_{12}, X_{21}, X_{22}$ satisfying the relation

$$X_{11}X_{22} - X_{12}X_{21} = 1.$$

It is possible that $\mathcal{O}(SL_2)$ inherits the bialgebra structure of $\mathcal{O}(M_2(\mathbb{K}))$. $\mathcal{O}(SL_2)$ is a Hopf algebra with the antipode map given by

$$\mathcal{S}(X_{11}) = X_{22},\ \mathcal{S}(X_{12}) = -X_{12},\ \mathcal{S}(X_{21}) = -X_{21},\ \mathcal{S}(X_{22}) = -X_{22}. \quad (4.21)$$

to define an antipode on a bialgebra it is enough to define $\mathcal{S}$ on the generators such that

$$\mathcal{S} : B \to B^{op}$$

is an algebra homomorphims and (4.17) holds for all the elements of the basis. Since $\mathcal{S}(1) = 1$ and

$$\begin{aligned} \mathcal{S}(X_{11}X_{22} - X_{12}X_{21}) &= \mathcal{S}(X_{22})\mathcal{S}(X_{11}) - \mathcal{S}(X_{21})\mathcal{S}(X_{12}) \\ &= X_{11}X_{22} - (-X_{12})(-X_{21}) \\ &= X_{11}X_{22} - X_{12}X_{21} \end{aligned} \quad (4.22)$$

it follows that $\mathcal{S}$ is well-defined algebra homorphims. To check equation (4.17) for the generators is equivalent to prove the following matrix equality

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} \mathcal{S}(X_{11}) & \mathcal{S}(X_{12}) \\ \mathcal{S}(X_{21}) & \mathcal{S}(X_{22}) \end{pmatrix} = \quad (4.23)$$

$$\begin{pmatrix} \mathcal{S}(X_{11}) & \mathcal{S}(X_{12}) \\ \mathcal{S}(X_{21}) & \mathcal{S}(X_{22}) \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} =$$

$$\begin{pmatrix} \varepsilon X_{11} & \varepsilon X_{12} \\ \varepsilon X_{21} & \varepsilon X_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.24)$$

which follows from the equality $X_{11}X_{22} - X_{12}X_{21} = 1$.

There is no universally accepted definition for the term *quantum group*. I would prefer to use the term for **quasi-triangular Hopf algebras**. Some authors use it as a synonym for Hopf algebras, some for certain subclasses of quasi-triangular Hopf algebras.

Definition 4.14. Let $\mathcal{H}$ be a Hopf algebra. $\mathcal{H}$ is *quasi-cocommutative* if there exists an invertible element $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$ such that

$$\Delta^{op}(v) = \mathcal{R}\Delta(v)\mathcal{R}^{-1},\ \forall v \in \mathcal{H}. \quad (4.25)$$

Definition 4.15. Let $\mathcal{H}$ be a quasi-cocommutative Hopf algebra. $\mathcal{H}$ is *quasi-triangular Hopf algebra* if

$$(\mathcal{H}\otimes\Delta)(\mathcal{R} = \sum_i h_i\otimes k_i) = (\sum_i h_i\otimes 1\otimes k_i)(\sum_i h_i\otimes k_i\otimes 1) = \mathcal{R}_{13}\mathcal{R}_{12} \quad (4.26)$$

$$(\Delta\otimes\mathcal{H})(\mathcal{R} = \sum_i h_i\otimes k_i) = (\sum_i h_i\otimes 1\otimes k_i)(\sum_i 1\otimes h_i\otimes k_i) = \mathcal{R}_{13}\mathcal{R}_{23}. \quad (4.27)$$

Theorem 4.16. Let $\mathcal{H}$ be a quasi-triangular Hopf algebra, then $\mathcal{R}$ satisfies the Yang-Baxter equation:

$$\mathcal{R}_{13}\mathcal{R}_{12}\mathcal{R}_{23} \overset{\text{(TH)}}{=} \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \quad (4.28)$$

Proof.

$$\begin{aligned}
[(\sigma\circ\Delta)\otimes\mathcal{H}]\mathcal{R} &= (\Delta^{op}\otimes\mathcal{H})\mathcal{R} \\
&= \sum_i \Delta^{op}\otimes\mathcal{H}h_i\otimes k_i \\
&= \sum_i \Delta^{op}(h_i)\otimes k_i \\
&= \sum_i \mathcal{R}_{12}\Delta(h_i)\mathcal{R}_{12}^{-1}\otimes k_i \\
&= \mathcal{R}_{12}\left(\sum_i \Delta(h_i)\otimes k_i\right)\mathcal{R}_{12}^{-1} \\
&= \mathcal{R}_{12}[(\Delta\otimes\mathcal{H})\mathcal{R}]\mathcal{R}_{12}^{-1} \\
&= \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{12}^{-1}. \quad (4.29)
\end{aligned}$$

$$\begin{aligned}
[(\sigma\circ\Delta)\otimes\mathcal{H}]\mathcal{R} &= (\Delta^{op}\otimes\mathcal{H})\mathcal{R} \\
&= \sigma_{12}(\Delta\otimes\mathcal{H})\mathcal{R} \\
&= \sigma_{12}\mathcal{R}_{13}\mathcal{R}_{23} \\
&= \mathcal{R}_{23}\mathcal{R}_{13}. \quad (4.30)
\end{aligned}$$

.

$$\begin{aligned}
\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{12}^{-1} &= \mathcal{R}_{23}\mathcal{R}_{13} \\
\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{12}^{-1}\mathcal{R}_{12} &= \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \\
\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} &= \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \quad (4.31)
\end{aligned}$$

.

□

4.2 Quantum groups

Definition 4.17. Let $\mathbb{K}$ a algebraically closed field of characteristic zero and let $q \in \mathbb{K} : q^2 \neq 1$. $\mathcal{O}_q(M_2(\mathbb{K}))$ is the algebra generated by elements $X_{11}, X_{12}, X_{21}, X_{22}$ satisfying the relations

$$qX_{11}X_{12} = X_{22}X_{21} \quad X_{22}X_{12} = qX_{12}X_{22} \quad X_{21}X_{11} = qX_{11}X_{21} \tag{4.32}$$

and

$$X_{22}X_{21} = qX_{21}X_{22} \quad X_{12}X_{21} = X_{21}X_{12} \quad X_{11}X_{22} - X_{22}X_{11} = (q^{-1} - q)X_{12}X_{21}. \tag{4.33}$$

To make the notation not so heavy we write from now

$$x = X_{11},\ y = X_{12},\ z = X_{21},\ k = X_{22}$$

Theorem 4.18. The algebra homomorphisms

$$\Delta : \mathcal{O}_q(M_2(\mathbb{K})) \to \mathcal{O}_q(M_2(\mathbb{K})) \otimes \mathcal{O}_q(M_2(\mathbb{K})) \tag{4.34}$$

$$\varepsilon : \mathcal{O}_q(M_2(\mathbb{K})) \to \mathbb{K} \tag{4.35}$$

are uniquely determined by

$$\Delta(x) = x \otimes x + y \otimes z,\ \Delta(y) = x \otimes y + y \otimes k \tag{4.36}$$

$$\Delta(z) = z \otimes x + k \otimes z,\ \Delta(k) = z \otimes y + k \otimes k \tag{4.37}$$

$$\varepsilon(x) = \varepsilon(k) = 1,\ \varepsilon(y) = \varepsilon(z) = 0. \tag{4.38}$$

In order to prove that Δ and ε are well-defined algebra maps, it is enough to show that the relations (4.32) and (4.33) hold under Δ and ε, e.g.

$$\Delta(yx) \overset{\text{(TH)}}{=} q\Delta(x)\Delta(y) \tag{4.39}$$

Proof.

$$\Delta(yx) = \Delta(y)\Delta(y) = x^2 \otimes yx + xy \otimes yz + yx \otimes kx + y^2 \otimes kz \tag{4.40}$$

$$\begin{aligned} q\Delta(xy) &= qx^2 \otimes xy + qxy \otimes xk + qyx \otimes zy + qy^2 \otimes zk \\ &= x^2 \otimes qxy + yx \otimes (kx + (q^{-1} - q)yz) + qxy \otimes yz + y^2 \otimes qzk \\ &= x^2 \otimes yx + yx \otimes kx + q^{-1}yx \otimes yz - qyx \otimes yz + qyx \otimes yz + y^2 \otimes kz \\ &= x^2 \otimes yx + xy \otimes yz + yx \otimes kx + y^2 \otimes kz. \end{aligned} \tag{4.41}$$

analogously, one can prove that $\Delta(ky) = q\Delta(yk), \Delta(zx) = q\Delta(xz), \Delta(kz) = q\Delta(zk), \Delta(yz) = q\Delta(zy)$ and $\Delta(xk - kx) = (q^{-1} - q)\Delta(yz)$. For ε it is completely analogus,

$$\varepsilon(yx) = \varepsilon(y)\varepsilon(x) = 0 = q\varepsilon(xy) = q\varepsilon(x)\varepsilon(y)$$

$$\varepsilon(ky) = \varepsilon(k)\varepsilon(y) = 0 = q\varepsilon(yk) = q\varepsilon(y)\varepsilon(k)$$

$$\varepsilon(yz) = \varepsilon(y)\varepsilon(z) = 0 = \varepsilon(zy) = \varepsilon(z)\varepsilon(y)$$

$$\varepsilon(kz) = \varepsilon(k)\varepsilon(z) = 0 = q\varepsilon(zy) = q\varepsilon(z)\varepsilon(k)$$

$$\varepsilon(zx) = \varepsilon(z)\varepsilon(x) = 0 = q\varepsilon(xz) = \varepsilon(x)\varepsilon(z)$$

$$\varepsilon(xk - kx) = \varepsilon(x)\varepsilon(k) - \varepsilon(k)\varepsilon(x) = 0 = (q^{-1} - q)\varepsilon(yz) = (q^{-1} - q)\varepsilon(y)\varepsilon(z).$$

□

corollary 4.19 $(\mathcal{O}_q(M_2(\mathbb{K})), \Delta, \varepsilon)$ is a bialgebra .

Proof. Since the coalgebra structure defined on $(\mathcal{O}_q(M_2(\mathbb{K})), \Delta, \varepsilon)$ is the same as the one defined on $(\mathcal{O}(M_2(\mathbb{K})), \Delta, \varepsilon)$, it follows that $(\mathcal{O}_q(M_2(\mathbb{K})), \Delta, \varepsilon)$ is a coalgebra, that is, ε is a counit and Δ is associative. Since both maps are algebra maps, it follows that $(\mathcal{O}_q(M_2(\mathbb{K})), \Delta, \varepsilon)$ is indeed a bialgebra. It is not commutative if $q \neq 1$ and it not cocommutative since

$$\Delta(x) = x \otimes x + y \otimes z \neq x \otimes x + z \otimes y = \sigma \circ \Delta(x). \tag{4.42}$$

□

Definition 4.20. Let z be a coalgebra and let $z \in z$. We say that z is a *group-like element* if

$$\Delta(z) = z \otimes z,\ \varepsilon(z) = 1. \tag{4.43}$$

We denote the set of group-like elements by $G(z)$. If c has a bialgebra structure, then $G(z)$ is a group under the multiplication.

Theorem 4.21. If $\det_q = xk - q^{-1}yz = kx - qyz$, then $\Delta(det_q) = det_q \otimes det_q$ and $\varepsilon(det_q) = 1$, that is, $\det_q$ is a group-like element in $\mathcal{O}_q(M_2(\mathbb{K}))$.

Proof.

$$\begin{aligned}\Delta(det_q) &= \Delta(x)\Delta(k) - q^{-1}\Delta(y)\Delta(z)\\ &= (x \otimes x + y \otimes z)(z \otimes y + k \otimes k) - q^{-1}(x \otimes y + y \otimes k)(z \otimes x + k \otimes z)\\ &= xz \otimes xy + xk \otimes xk + yz \otimes zy + yk \otimes zk - q^{-1}yz \otimes kx - q^{-1}yk \otimes kz -\\ & q^{-1}xz \otimes yxq^{-1}xk \otimes yz\end{aligned} \tag{4.44}$$

$$
\begin{aligned}
&= xz \otimes xy + xk \otimes (xk - q^{-1}yz) + yz \otimes zy + yk \otimes zk - q^{-1}yz \otimes kx - yk \otimes q^{-1}kz \\
&- xz \otimes q^{-1}yx \\
&= xz \otimes xy + xk \otimes (xk - q^{-1}yz) + yz \otimes zy + yk \otimes zk - q^{-1}yz \otimes kx - yk \otimes zk - xz \otimes xy \\
&= xk \otimes (xk - q^{-1}yz) + yz \otimes zy - q^{-1}yz \otimes kx \\
&= xk \otimes (xk - q^{-1}yz) + yz \otimes zy - q^{-1}yz \otimes (xk - (q^{-1} - q)yz) \\
&= xk \otimes (xk - q^{-1}yz) + yz \otimes zy - q^{-1}yz \otimes xk + q^{-2}yz \otimes yz - yz \otimes yz \\
&= xk \otimes (xk - q^{-1}yz) - q^{-1}yz \otimes (xk - q^{-1}yz) \\
&= (xk - q^{-1}yz) \otimes (xk - q^{-1}yz) = det_q \otimes det_q. \qquad (4.45)
\end{aligned}
$$

□

Corollary 4.22. $\det_q = xk - q^{-1}yz = kx - qyz$ beyond to the center of algebra $\mathcal{O}_q(M_2(\mathbb{K}))$.

Proof. To prove the thesis it is enough to verify it on the generators:

$$
\begin{aligned}
det_q x &= (xk - q^{-1}yz)x = xkx - q^{-1}yzx \\
&= x(xk - (q^{-1} - q)yz) - q^{-1}q^2 xyz \\
&= x(xk - q^{-1}yz) + qxyz - qxyz = xdet_q, \\
det_q y &= (xk - q^{-1}yz)y = xky - q^{-1}yzy \\
&= q^{-1}qyxk - byz = y(xk - q^{-1}yz) \\
&= ydet_q, \\
det_q z &= (xk - q^{-1}yz)z = xkz - q^{-1}yzz \\
&= q^{-1}qyxk - zyz = z(xk - q^{-1}yz) = zdet_q, \\
det_q k &= (xk - q^{-1}yz)k = xkk - q^{-1}yzk \\
&= (kx + (q^{-1} - q)yz)k - q^{-1}yzk \\
&= kxk + q^{-1}yzk - qyzk - q^{-1}yzk \\
&= kxk - qq^{-2}kyz = kdet_q
\end{aligned}
\qquad (4.46)
$$

□

Definition 4.23. We define $\mathcal{O}_q(SL_2(\mathbb{K}))$ as the $\mathbb{K}$ algebra given by the quotient

$$\mathcal{O}_q(SL_2(\mathbb{K})) = \mathcal{O}_q(M_2(\mathbb{K}))/(det_q - 1) \qquad (4.47)$$

where $(det_q - 1)$ is the two-sided ideal of $\mathcal{O}_q(M_2(\mathbb{K}))$ generated by the element $(det_q - 1)$.

In other words, the algebra $\mathcal{O}_q(SL_2(\mathbb{K}))$ can be presented as the $\mathbb{K}$ algebra generated by the elements x, y, z, ksatisfying the relations (4.33).
Since $det_q$is a central group-like element, the ideal $(det_q - 1)$ of $\mathcal{O}_q(M_2(\mathbb{K}))$ is indeed a bi-ideal and thus $\mathcal{O}_q(SL_2(\mathbb{K}))$ is a bialgebra with the comultiplication and counit defined on the generators as in $\mathcal{O}_q(M_2(\mathbb{K}))$.

Theorem 4.24.$\mathcal{O}_q(SL_2(\mathbb{K}))$ is a Hopf algebra with the antipode determined by

$$\begin{pmatrix} \mathcal{S}(x) & \mathcal{S}(y) \\ \mathcal{S}(z) & \mathcal{S}(k) \end{pmatrix} = \begin{pmatrix} k & -qy \\ -q^{-1}z & x \end{pmatrix}. \tag{4.48}$$

Proof.
First we have to prove that$\mathcal{S} : \mathcal{O}_q(SL_2(\mathbb{K})) \to \mathcal{O}_q(SL_2(\mathbb{K}))^{op}$ is a well-defined algebra map:

$$\begin{aligned} \mathcal{S}(yx) &= \mathcal{S}(x)\mathcal{S}(y) \\ &= k(-qy) = -qky = -q^2yk \\ &= q\mathcal{S}(y)\mathcal{S}(x) \\ &= q\mathcal{S}(xy) \end{aligned} \tag{4.49}$$

$$\begin{aligned} \mathcal{S}(ky) &= \mathcal{S}(y)\mathcal{S}(k) \\ &= (-qy)x = -q^2xy \\ &= q\mathcal{S}(k)\mathcal{S}(y) \\ &= q\mathcal{S}(yk) \end{aligned} \tag{4.50}$$

$$\begin{aligned} \mathcal{S}(zx) &= \mathcal{S}(x)\mathcal{S}(z) \\ &= k(-q^{-1}z) = -q^{-1}kz = -zk \\ &= q\mathcal{S}(z)\mathcal{S}(x) = q\mathcal{S}(xz) \end{aligned} \tag{4.51}$$

$$\begin{aligned} \mathcal{S}(kz) &= \mathcal{S}(z)\mathcal{S}(k) \\ &= (-q^{-1}z)x = -xz \\ &= q\mathcal{S}(k)\mathcal{S}(z) \\ &= q\mathcal{S}(zk) \end{aligned} \tag{4.52}$$

$$\begin{aligned}\mathcal{S}(yz) &= \mathcal{S}(z)\mathcal{S}(y) = (-q^{-1}z)(-qy) = zy \\ &= yz\mathcal{S}(y)\mathcal{S}(z) \\ &= \mathcal{S}(zy)\end{aligned} \tag{4.53}$$

$$\begin{aligned}\mathcal{S}(xk - kx) &= \mathcal{S}(xk) - \mathcal{S}(kx) = \mathcal{S}(k)\mathcal{S}(x) - \mathcal{S}(x)\mathcal{S}(k) \\ &= xk - kx = (q^{-1} - q)yz \\ &= (q^{-1} - q)zy \\ &= (q^{-1} - q)\mathcal{S}(z)\mathcal{S}(y) \\ &= (q^{-1} - q)\mathcal{S}(yz)\end{aligned} \tag{4.54}$$

$$\begin{aligned}\mathcal{S}(xk - q^{-1}yz) &= \mathcal{S}(xk) - q^{-1}\mathcal{S}(yz) \\ &= \mathcal{S}(k)\mathcal{S}(x) - q^{-1}\mathcal{S}(z)\mathcal{S}(y) \\ &= xk - q^{-1}q^{-1}qzy = xk - q^{-1}zy \\ &= xk - q^{-1}yz = 1 = \mathcal{S}(1).\end{aligned} \tag{4.55}$$

To prove that $\mathcal{S}$ defines an antipode for $\mathcal{O}_q(SL_2(\mathbb{K})$, we have to check equation(4.17) for the generators. as for the case of $\mathcal{O}(SL_2(\mathbb{K})$, this is equivalent to prove the following matrix equality

$$\begin{pmatrix} x & y \\ z & k \end{pmatrix} \begin{pmatrix} \mathcal{S}(x) & \mathcal{S}(y) \\ \mathcal{S}(z) & \mathcal{S}(k) \end{pmatrix} = \tag{4.56}$$

$$\begin{pmatrix} \mathcal{S}(x) & \mathcal{S}(y) \\ \mathcal{S}(z) & \mathcal{S}(k) \end{pmatrix} \begin{pmatrix} x & y \\ z & k \end{pmatrix} =$$

$$\begin{pmatrix} \varepsilon(x) & \varepsilon(y) \\ \varepsilon(z) & \varepsilon(k) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.57}$$

which follows from the defining relations of $\mathcal{O}_q(SL_2(\mathbb{K}))$.

□

4.2.1 $U_q(\mathfrak{sl}_2)$

The quantum group we introduce in this section corresponds to the deformation in one parameter of the enveloping algebra $\mathfrak{U}(\mathfrak{sl})_2$ of $\mathfrak{sl}_2$. The deformation uses the classification of semisimple Lie algebras over an algebraically closed field of characteristic zero, done by Cartan and Killing. Thus, the field $\mathbb{K}$ is an arbitrary field with these properties. The origins of the subject of quantum groups lie in mathematical physics, where the term quantum comes from. The starting point of the study of this subject lies in the Quantum Inverse Scattering Method, with the aim of solving certain integrable quantum systems. A key ingredient in this method is the Quantum Yang-Baxter Equation (QYBE).

The Lie algebra of matrices 2×2 traceless is

$$\mathfrak{sl}(2,\mathbb{K}) = span_{\mathbb{K}} \left\{ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \tag{4.58}$$

from this we can find $[H,E] = 2E, [H,F] = -2F[E,F] = H$.

The Poincaré - Birkhoff - Witt theorem say that a basis for

$$\mathfrak{U}(\mathfrak{sl}(2)) = \frac{T(\mathfrak{sl}(2))}{\begin{bmatrix} H \otimes E - E \otimes H - 2E \\ H \otimes F - F \otimes H + 2F \\ E \otimes F - F \otimes E - H \end{bmatrix}} \tag{4.59}$$

is

$$\left\{ H^h, E^x, F^y, h, x, y \in \mathbb{Z}^+ \right\}. \tag{4.60}$$

Definition 4.25. Let $\mathbb{K}$ a algebraically closed field of characteristic zero and let $q \in \mathbb{K} : q^2 \neq 1$. We define $U_q(\mathfrak{sl}_2)$as the algebra generated by the elements E, F, K, K^{-1}satisfying the relations

$$KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F, \tag{4.61}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{4.62}$$

Theorem 4.26. There exist algebra maps

$$\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \tag{4.63}$$

$$\varepsilon : U_q(\mathfrak{sl}_2) \to \mathbb{K} \tag{4.64}$$

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \tag{4.65}$$

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \tag{4.66}$$

$$\varepsilon(K) = \varepsilon(K^{-1}) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0. \tag{4.67}$$

Proof. We first show that Δ defines an algebra map. For this it is enough to check that the ideal of relations is a coideal, or equivalently, that the following equalities hold

$$\Delta(KK^{-1}) = \Delta(K^{-1}K) = 1 \otimes 1 = \Delta(1) \tag{4.68}$$

$$\Delta(KFK^{-1}) = q^-2\Delta(F) \tag{4.69}$$

$$\Delta(KEK^{-1}) = q^2\Delta(E) \tag{4.70}$$

$$\Delta(EF - FE) = \Delta\left(\frac{K - K^{-1}}{q - q^{-1}}\right) \tag{4.71}$$

The first relations are clear since

$$\begin{aligned} \Delta(KK^{-1}) &= \Delta(K)\Delta(K^{-1}) \\ &= (K \otimes K)(K^{-1} \otimes K^{-1}) \\ &= KK^{-1} \otimes KK^{-1} = 1 \otimes 1. \end{aligned} \tag{4.72}$$

For the others we have

$$\begin{aligned} \Delta(KEK^{-1}) &= (K \otimes K)(1 \otimes E + E \otimes K)(K^{-1} \otimes K^{-1}) \\ &= (K \otimes KE + KE \otimes K2)(K^{-1} \otimes K^{-1}) \\ &= 1 \otimes KEK^{-1} + KEK^{-1} \otimes K \\ &= 1 \otimes q^2E + q^2E \otimes K = q^2\Delta(E). \end{aligned} \tag{4.73}$$

The relation for F is completely analogous and we leave it as exercise for the reader. For the last relation we have $\Delta(EF-FE) = \Delta(E)\Delta(F)-\Delta(F)\Delta(E)$

$$\Delta(E)\Delta(F) - \Delta(F)\Delta(E) =$$

$$
\begin{aligned}
&= (1 \otimes E + E \otimes K)(K^{-1} \otimes F + F \otimes 1) - (K^{-1} \otimes F + F \otimes 1)(1 \otimes E + E \otimes K) \\
&= K^{-1} \otimes EF + F \otimes E + EK^{-1} \otimes KF + EF \otimes K - K^{-1} \otimes FE - K^{-1}E \otimes FE - \\
&F \otimes E - FE \otimes K \\
&= K^{-1} \otimes (EFdfdf - FE) + (EF - FE) \otimes K + EK^{-1} \otimes KF - K^{-1}E \otimes FK \\
&= K^{-1} \otimes (EF - FE) + (EF - FE) \otimes K + q^2 q^{-2} K^{-1}E \otimes FK - K^{-1}E \otimes FK \\
&= K^{-1} \otimes (EF - FE) + (EF - FE) \otimes K \\
&= K^{-1} \otimes \left(\frac{K - K^{-1}}{q - q^{-1}}\right) + \left(\frac{K - K^{-1}}{q - q^{-1}}\right) \otimes K \\
&= \frac{1}{q - q^{-1}}(K^{-1} \otimes K - K \otimes K^{-1} + K \otimes K - K \otimes K^{-1}) \\
&= \frac{1}{q - q^{-1}}(K \otimes K - K^{-1} \otimes K^{-1}) \\
&= \Delta \left(\frac{K - K^{-1}}{q - q^{-1}}\right). \qquad (4.74)
\end{aligned}
$$

Now we check that ε is a well-defined algebra map by showing that the equalities in the relations hold after applying ε:

$$\varepsilon(KK^{-1}) = \varepsilon(K)\varepsilon(K^{-1}) = \varepsilon(1) = \varepsilon(K^{-1})\varepsilon(K) = \varepsilon(K^{-1}K) \qquad (4.75)$$

$$\varepsilon(KEK^{-1}) = \varepsilon(K)\varepsilon(E)\varepsilon(K^{-1}) = 0 = q^2\varepsilon(E) \qquad (4.76)$$

$$\varepsilon(KFK^{-1}) = \varepsilon(K)\varepsilon(F)\varepsilon(K^{-1}) = 0 = q^-2\varepsilon(F) \qquad (4.77)$$

$$\varepsilon(EF - FE) =$$

$$\varepsilon(E)\varepsilon(F) - \varepsilon(F)\varepsilon(E) = 0 = \varepsilon\left(\frac{K - K^{-1}}{q - q^{-1}}\right) = \frac{\varepsilon(K) - \varepsilon(K^{-1})}{q - q^{-1}}. \qquad (4.78)$$

□

Corollary 4.27. With these morphisms, $U_q(\mathfrak{sl}_2)$ is a bialgebra which is non-commutative and non-cocommutative.
Proof.
To prove that $U_q(\mathfrak{sl}_2)$ is a bialgebra, we need to show that $(U_q(\mathfrak{sl}_2), \Delta, \varepsilon)$ is a coalgebra, since by theorem 4.26 , we know that Δ and ε are algebra maps. We prove that ε is a counit and Δ is coassociative by checking the equalities

$$m \circ (\varepsilon \otimes U_q(\mathfrak{sl}_2)) \circ \Delta = m \circ (U_q(\mathfrak{sl}_2 \otimes \varepsilon) \circ \Delta = 1_{U_q(\mathfrak{sl}_2)} \qquad (4.79)$$

and

$$(\Delta \otimes U_q(\mathfrak{sl}_2)) \circ \Delta = (U_q(\mathfrak{sl}_2) \otimes \Delta) \circ \Delta \qquad (4.80)$$

on the generators.

We begin by the counit:

$$\begin{aligned} m(\varepsilon \otimes U_q(\mathfrak{sl}_2))\Delta(K) &= m(\varepsilon \otimes U_q(\mathfrak{sl}_2)(K \otimes K) \\ &= m(\varepsilon(K) \otimes K) = m(1 \otimes K) = K \end{aligned} \tag{4.81}$$

and

$$\begin{aligned} m(U_q(\mathfrak{sl}_2) \otimes \varepsilon)\Delta(K) &= m(U_q(\mathfrak{sl}_2 \otimes \varepsilon)(K \otimes K) \\ &= m(K \otimes \varepsilon(K)) = m(K \otimes 1) = K \end{aligned} \tag{4.82}$$

$$\begin{aligned} m(U_q(\mathfrak{sl}_2) \otimes \varepsilon)\Delta(E) &= m(U_q(\mathfrak{sl}_2) \otimes \varepsilon)(1 \otimes E + E \otimes K) \\ &= m(1 \otimes \varepsilon(E) + E \otimes \varepsilon(K)) \\ &= m(E \otimes 1) \\ &= E \end{aligned} \tag{4.83}$$

$$\begin{aligned} m(\varepsilon \otimes U_q(\mathfrak{sl}_2)\Delta(E) &= m(\varepsilon \otimes U_q(\mathfrak{sl}_2)(1 \otimes E + E \otimes K) \\ &= m(\varepsilon(1) \otimes E + \varepsilon(E) \otimes K) \\ &= m(1 \otimes E) = E \end{aligned} \tag{4.84}$$

$$\begin{aligned} m(\varepsilon \otimes U_q(\mathfrak{sl}_2)\Delta(F) &= m(\varepsilon \otimes U_q(\mathfrak{sl}_2)(K^{-1} \otimes F + F \otimes 1) \\ &= m(\varepsilon(K^{-1}) \otimes F + \varepsilon(F) \otimes 1) \\ &= m(1 \otimes F) \\ &= F \end{aligned} \tag{4.85}$$

$$\begin{aligned} m(U_q(\mathfrak{sl}_2 \otimes)\varepsilon)\Delta(F) &= m(U_q(\mathfrak{sl}_2) \otimes \varepsilon)(K^{-1} \otimes F + F \otimes 1) \\ &= m(K^{-1} \otimes \varepsilon(F) + F \otimes \varepsilon(1)) \\ &= m(F \otimes 1) \qquad = F \end{aligned} \tag{4.86}$$

$$\begin{aligned} m(U_q(\mathfrak{sl}_2 \otimes \varepsilon)\Delta(K^{-1}) &= m(U_q(\mathfrak{sl}_2 \otimes \varepsilon)(K^{-1} \otimes K^{-1}) \\ &= m(K^{-1} \otimes \varepsilon(K^{-1})) \\ &= m(K^{-1} \otimes 1) = K^{-1} \end{aligned} \tag{4.87}$$

$$
\begin{aligned}
m(\varepsilon \otimes U_q(\mathfrak{sl}_2))\Delta(K^{-1}) &= m(\varepsilon \otimes U_q(\mathfrak{sl}_2))(K^{-1} \otimes K^{-1}) \\
&= m(\varepsilon(K^{-1}) \otimes K^{-1}) \qquad = m(1 \otimes K^{-1}) = K^{-1}.
\end{aligned} \tag{4.88}
$$

For the coassociativity we have

$$
(\Delta \otimes U_q(\mathfrak{sl}_2))\Delta(K) = (\Delta \otimes U_q(\mathfrak{sl}_2))(K \otimes K) = \Delta(K) \otimes K = K \otimes K \otimes K, \tag{4.89}
$$

$$
(U_q(\mathfrak{sl}_2) \otimes \Delta)\Delta(K) = (U_q(\mathfrak{sl}_2) \otimes \Delta)(K \otimes K) = K \otimes \Delta(K) = K \otimes K \otimes K. \tag{4.90}
$$

$$
\begin{aligned}
(\Delta \otimes U_q(\mathfrak{sl}_2))\Delta(K^{-1}) &= (\Delta \otimes U_q(\mathfrak{sl}_2))(K^{-1} \otimes K^{-1}1) \\
&= \Delta(K^{-1}1) \otimes K^{-1} \\
&= K^{-1} \otimes K^{-1} \otimes K^{-1}
\end{aligned} \tag{4.91}
$$

$$
\begin{aligned}
(U_q(\mathfrak{sl}_2) \otimes \Delta)\Delta(K^{-1}) &= (U_q(\mathfrak{sl}_2) \otimes \Delta)(K^{-1} \otimes K^{-1}) \\
&= K^{-1} \otimes \Delta(K^{-1}) \\
&= K^{-1} \otimes K^{-1} \otimes K^{-1}
\end{aligned} \tag{4.92}
$$

$$
\begin{aligned}
(\Delta \otimes U_q(\mathfrak{sl}_2))\Delta(E) &= (\Delta \otimes U_q(\mathfrak{sl}_2))(1 \otimes E + E \otimes K) \\
&= \Delta(1) \otimes E + \Delta(E) \otimes K \\
&= 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K
\end{aligned} \tag{4.93}
$$

$$
\begin{aligned}
(U_q(\mathfrak{sl}_2) \otimes \Delta)\Delta(E) &= (U_q(\mathfrak{sl}_2) \otimes \Delta)(1 \otimes E + E \otimes K) \\
&= 1 \otimes \Delta(E) + E \otimes \Delta(K) \\
&= 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K
\end{aligned} \tag{4.94}
$$

$$
\begin{aligned}
(\Delta \otimes U_q(\mathfrak{sl}_2))\Delta(F) &= (\Delta \otimes U_q(\mathfrak{sl}_2))(K^{-1}1 \otimes F + F \otimes 1) \\
&= \Delta(K^{-1}1) \otimes F + \Delta(F) \otimes 1 \\
&= K^{-1}1 \otimes K^{-1}1 \otimes F + K^{-1}1 \otimes F \otimes 1 + F \otimes 1 \otimes 1
\end{aligned} \tag{4.95}
$$

$$
\begin{aligned}
(U_q(\mathfrak{sl}_2) \otimes \Delta)\Delta(F) &= (U_q(\mathfrak{sl}_2) \otimes \Delta)(K^{-1}1 \otimes F + F \otimes 1) \\
&= K^{-1}1 \otimes \Delta(F) + F \otimes \Delta(1) \\
&= K^{-1}1 \otimes K^{-1}1 \otimes F + K^{-1}1 \otimes F \otimes 1 + F \otimes 1 \otimes 1.
\end{aligned} \tag{4.96}
$$

Thus Δ is coassociative and clearly $U_q(\mathfrak{sl}_2)$ is not cocommutative since $\sigma \circ \Delta \neq \Delta$ because

$$
\Delta(E) = 1 \otimes E \otimes F + E \otimes K \neq E \otimes 1 + K \otimes E = \sigma \circ \Delta. \tag{4.97}
$$

□

Lemma 4.28.

$$
\mathcal{S} : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{op}
$$

determined by

$$
\mathcal{S}(E) = -EK^{-1}, \quad \mathcal{S}(F) = -KF, \quad \mathcal{S}(K) = K^{-1} \quad \mathcal{S}(K^{-1}) = K \tag{4.98}
$$

is a algebra homomorphism.

Proof. To show that $\mathcal{S}$ defines an algebra map, we have to verify that the equalities of the relations hold when appying $\mathcal{S}$, but using the opposite multiplication, for example

$$
\mathcal{S}(KEK^{-1}) = \mathcal{S}(K^{-1})\mathcal{S}(E)S(K) = q^2\mathcal{S}(E) \tag{4.99}
$$

but

$$
\begin{aligned}
\mathcal{S}(KEK^{-1}) &= \mathcal{S}(K^{-1})\mathcal{S}(E)\mathcal{S}(K) \\
&= K(-EK^{-1})K^{-1} \\
&= -KEK^{-1}K^{-1} \\
&= -q^2EK^{-1} \\
&= q^2\mathcal{S}(E).
\end{aligned} \tag{4.100}
$$

Clearly it holds for$K$and $K^{-1}$and the computation for F is completely analogous to the computation above. For the last relation we have

$$\begin{aligned}\mathcal{S}(EF-FE) &= \mathcal{S}(F)\mathcal{S}(E)-\mathcal{S}(E)\mathcal{S}(F)\\ &= (-KF)(-EK^{-1})-(-EK^{-1})(-KF)\\ &= KFEK^{-1}-EF = KFq^2K^{-1}E-EF\\ &= q^{-2}q^2KK^{-1}FE-EF\\ &= FE-EF\\ &= -\frac{K-K^{-1}}{q-q^{-1}} = -\frac{\mathcal{S}(K)-\mathcal{S}(K^{-1})}{q-q^{-1}} = \mathcal{S}\left(-\frac{K-K^{-1}}{q-q^{-1}}\right)\end{aligned} \tag{4.101}$$

Theorem 4.29. In light lemma 4.28 $\mathcal{S} : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{op}$ is a well-defined algebra map then :

$$m \circ (1_{U_q(\mathfrak{sl}_2)} \otimes S) \circ \Delta \overset{(\mathrm{TH})}{=} u \circ \varepsilon \overset{(\mathrm{TH})}{=} m \circ (S \otimes 1_{U_q(\mathfrak{sl}_2)}) \circ \Delta \tag{4.102}$$

Proof.

$$\begin{aligned}m(U_q(\mathfrak{sl}_2)\otimes\mathcal{S})\Delta(K) &= m(U_q(\mathfrak{sl}_2)\otimes\mathcal{S})(K\otimes K)\\ &= m(K\otimes\mathcal{S}(K)) = m(K\otimes K^{-1}) = 1\end{aligned} \tag{4.103}$$

$$\begin{aligned}m(\mathcal{S}\otimes U_q(\mathfrak{sl}_2))\Delta(K) &= m(\mathcal{S}\otimes U_q(\mathfrak{sl}_2))(K\otimes K)\\ &= m(\mathcal{S}(K)\otimes K) = m(K^{-1}\otimes K)\\ &= 1\end{aligned} \tag{4.104}$$

$$\begin{aligned}m(U_q(\mathfrak{sl}_2)\otimes\mathcal{S})\Delta(F) &= m(U_q(\mathfrak{sl}_2)\otimes\mathcal{S})(K^{-1}\otimes K)+F\otimes 1)\\ &= m(K^{-1}\otimes\mathcal{S}(F)+F\otimes\mathcal{S}(1))\\ &= m(K^{-1}\otimes(-KF)+F\otimes 1)\\ &= K^{-1}(-KF)+F=0\end{aligned} \tag{4.105}$$

$$\begin{aligned}m(\mathcal{S}\otimes U_q(\mathfrak{sl}_2))\Delta(F) &= m(\mathcal{S}\otimes U_q(\mathfrak{sl}_2))K^{-1}\otimes\mathcal{S}(F)+F\otimes\mathcal{S}(1))\\ &= m(\mathcal{S}(K^{-1}\otimes F+\mathcal{S}\otimes 1)\\ &= m(K\otimes\mathcal{S}F+(-KF)\otimes\mathcal{S}1)\\ &= KF-KF=0\end{aligned} \tag{4.106}$$

The equalities for K^{-1} and E are again completely analogous .

□

4.3 Integrable Systems and Quantum Groups

An important direction of research opened by the introduction of the tetrahedron Zamolodchikov algebra is the investigation of the three dimensional integrable structures in the context of the AdS/CFT correspondence. Specifically, one finds a natural object in Shastry's construction, referred to as $\mathbb{S}$ in this section, which obeys the tetrahedron Zamolodchikov equation,[23]. The AdS/CFT correspondence was first proposed by Juan Maldacena in 1997 provide a powerful tool.

4.3.1 The free fermion model

We start our journey by writing down the free fermion model using oscillators and by describing the tetrahedron Zamolodchikov algebra. We define the fermionic creation operator $c_j^\dagger$ as well as the the annihilation operators c_j where $j \in \mathbb{Z}$ labels the lattice site. The operators obey to the canonica anti-commutation relations

$$\{\hat{c}_j, \hat{c}_i^\dagger\} = \delta_{ij},\ i, j \in \mathbb{Z}. \tag{4.107}$$

We define two compound operators also

$$\hat{n}_j = \hat{c}_j^\dagger \hat{c}_j\ \hat{m}_i = \hat{c}_j \hat{c}_j^\dagger. \tag{4.108}$$

By these definitions the R-matrix for XXZ model become

$$R_{jk}(A) = -a\hat{n}_j\hat{n}_k - ib\hat{n}_j\hat{m}_k - ic\hat{m}_j\hat{m}_k + \hat{c}_j^\dagger\hat{c}_k + \hat{c}_k^\dagger\hat{c}_j \tag{4.109}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{B}). \tag{4.110}$$

This choice is known as the *free fermionic condition.* The free fermionic model is quantum integrable, as its R-matrix satisfies the Yang-Baxter equation. Without using operators in the construction, the representation of R_{12} as a 4×4 matrix would look like the following:

$$R_{12}(A) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & ib & 1 & 0 \\ 0 & 0 & ic & 0 \\ 0 & 0 & 0 & d \end{pmatrix}. \tag{4.111}$$

By choosing a curve in $SL(2, \mathbb{K})$ we can obtain the hamiltonian density of spin chain in other words making A depend on a parameter $u \in \mathbb{C}$ such that for $u = u_0$ the coefficient $a = d = 1$ and $b = c = 0$ implying the relation

$$R_{jk}(A(u_0)) = \mathcal{P}_{jk}. \tag{4.112}$$

The map $\mathcal{P}_{jk}$ sends $s_j \otimes s_k$ in $s_k \otimes s_j$ where s_i is a sping operator. For the sake of simplicity, we can represent s_i in a two dimensional vector space. This would be useful in connecting $\mathcal{P}_{jk}$ to the matricial form of R_{jk}. Since $\mathcal{P}_{jk}(s_j \otimes s_k) = s_k \otimes s_j$ we can obtain the form of $\mathcal{P}_{jk}$

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.113}$$

We can recognise this matrix as R_{12} when $a = b = 1$, $b = c = 0$. So in terms of fermionic operators

$$\mathcal{P}_{jk} = -\hat{n}_j \hat{n}_k + \hat{m}_j \hat{m}_k + \hat{c}_j^{\dagger} \hat{c}_k + \hat{c}_k^{\dagger} \hat{c}_j. \tag{4.114}$$

The monodromy matrix is constructed in terms of the R-matrix by taking the product of $R_{jk}(u)$ in all possible index k

$$T_j(u) = R_{jN}(u) R_{jN-1}(u) ... R_{j1}(u). \tag{4.115}$$

We have used the R-matrix instead of the Lax operator because they are both representation of the same algebra. Now we can compute the hamiltonian density by

$$H = \frac{d}{du} ln\, Tr_j T_j((u)) \Big|_{u=u_0}. \tag{4.116}$$

A natural choice for the curve is $a = d = cosu$,$b = c = sinu$. This way we obtain the XX model

$$H = \sum_{j=1}^{n} \hat{c}_j^{\dagger} \hat{c}_{j+1} + \hat{c}_{j+1}^{\dagger} \hat{c}_j. \tag{4.117}$$

4.3.2 The quantum affine $U_q(\mathfrak{sl}_2)$

The affine extension of any algebra is the vector space spanned by several copies of the generators of the algebra. In the case of $U_i(\mathfrak{sl}_2)$ we will denote this affine extension by $\mathfrak{U}_\mathfrak{i}(\mathfrak{sl}_2)$ which is the algebra generated by

$$k_r, f_r, k_r \text{ and } k_r^{-1} \text{ for } r = 0, 1.$$

that for $q = i$ obey the relations

$$[k_r, k_s] = 0,\ \{k_r, e_s\} = 0,\ \{k_r f_s\} = 0,\ [e_r, f_s] = \delta_{rs} \frac{k_r - k^{-1}}{r} \tag{4.118}$$

together with the Serre relations that we have omitted. Impressively, we can use the elements of the free fermion model R-matrix to write an expression for its generators. Studying the commutation and anticommutation relations of the generators of $\mathfrak{U}_{\mathfrak{i}}(\mathfrak{sl}_2)$ we realize k_r can be interpreted as bosonic generators and e_r, f_r as fermionic generators. This gives us a hint on how to construct the a family two dimensional representations: k_r must be a linear combination of $\hat{m}, \hat{n}$ and the fermionic ones must be proportional to $\hat{c}, \hat{c}^\dagger$. We shall ignore the lattice index for simplicity.

Then it is quite easy to check that

$$k_0 = \lambda^{-1}(\hat{m} - \hat{n}), \quad e_0 = \varphi x^{-1}\hat{c}^\dagger, f_0 = \varphi x\hat{c} \quad h_0 = \mu - \hat{m} - \hat{n} \tag{4.119}$$

$$k_1 = \lambda(\hat{m} - \hat{n}), \quad e_0 = \varphi x^{-1}\hat{c}^\dagger, f_0 = \varphi x\hat{c} \quad h_0 = \mu - \hat{m} - \hat{n} \tag{4.120}$$

where λ, μ, x and y are complex parameters. We have introduced the element φ through the equation

$$\varphi^2 = \frac{\lambda - \lambda^{-1}}{2i}.$$

By $k_r = q^{h_r}$ the parameter λ is fixed by $\lambda = i^{-\mu-1}$. We may name this family of 2-dimensional representations $\mathcal{V}_{\mu;x,y}$

An intertwiner is a map between two representations of the same algebra which is invariant under the action of the algebra itself. S-matrices are themselves a type of intertwiners. We need to find a S-matrix for a $\mathfrak{U}_{\mathfrak{i}}(\mathfrak{sl}_2)$-invariant theory. After we must to check if is possible to obtain R_{jk} by similarity transformations. In order to do so we define the coproduct for each element in $\mathfrak{U}_{\mathfrak{i}}(\mathfrak{sl}_2)$.

As done in [23] :

$$\Delta(k_r) = k_r \otimes k_r, \quad \Delta(Z) = Z \otimes Z, \quad \Delta(F) = F \otimes F, \quad \Delta(h_r) = h_r \otimes \mathbb{I} + \mathbb{I} \otimes h_r. \tag{4.121}$$

Where are introduced two operators

$$F = \hat{m} - \hat{n}, \text{ a grading operator and } Z \text{ a central element .} \tag{4.122}$$

While for the non-diagonal element

$$\Delta(e_0) = e_0 \otimes Z + k_0 F \otimes e_0, \quad \Delta(e_1) = e_1 \otimes \mathbb{I} + Z k_1 F \otimes e_1, \tag{4.123}$$

$$\Delta(f_0) = f_0 \otimes k^{-1} Z^{-1} + F \otimes f_0, \quad 0\Delta(f_1) = f_1 \otimes k^{-1} + Z^{-1} F \otimes f_1. \tag{4.124}$$

The intertwiner map $\mathfrak{R}_{12}$ is acting on $\mathcal{V}_{\mu_1;x_1,y_1} \otimes \mathcal{V}_{\mu_2;x_2,y_2}$ because if the system is integrable, the S- matrix of any process can be decomposed into 2-body S-matrix.

This kind of intertwined map must fulfills

$$\mathfrak{R}_{12}\Delta(X)\mathfrak{R}_{12}^{-1} = \Delta^{op}(X),\ X \in \mathfrak{U}_{\mathfrak{i}}(\mathfrak{sl}_2). \tag{4.125}$$

The solution to the previous symmetry constraints is

$$\begin{aligned}\mathfrak{R}_{12} = & (x_1y_1\lambda_1\lambda_2x_2y_2)\hat{n}_1\hat{n}_2 + z^{-1}(x_2y_2\lambda_1x_1y_1\lambda_2)\hat{n}_1\hat{m}_2 + \\ & +z(x-2y_2\lambda_2x_1y_1\lambda_1)\hat{m}_1\hat{n}_2 + (x_1y_1x_2y_2\lambda_1\lambda_2)\hat{m}_1\hat{m}_2 + \\ & -\sqrt{(\lambda_1-\lambda_1^{-1})(\lambda_2-\lambda_2^{-1})}(x_1y_2\lambda_2c_2^\dagger c_1 + x_2y_1\lambda_1c_1^\dagger c_2).\end{aligned} \tag{4.126}$$

where z is the eigenvalue of the operator Z and physically represents the value of a conserved charge.
As noted in [23] the solution contain $(\hat{n}_1+\hat{m}_1)(\hat{n}_2+\hat{m}_2)$ factor with different coefficients in each compound operator combination, and the $c_2^\dagger c_1 + c_1^\dagger c_2$ factor.
If there exists an R-matrix which is invariant under the action of a algebra A, then it must be related by a similarity transformation to an intertwiner such algebra which acts on spaces of representations , provided the central charges of the representations are conserved through such map. Although it is not obvious at first sight, the operator $\mathfrak{R}_{12}$ fulfills the YBE. This will be clear if we study the relation between $\mathfrak{R}_{12}$ and R_{12} . To find this connection, as well as is done in [23], we define an operator by settings

$$K_j = \hat{m}_j + \sqrt{\frac{y_j}{x_j}\lambda_j\hat{n}_j} \tag{4.127}$$

and we obtain

$$R_{12} = -\frac{K_1^{-1}K_2^{-1}\mathfrak{R}_{12}K_1K_2}{\sqrt{(\lambda_1-\lambda_1^{-1})(\lambda_2-\lambda_2^{-1})x_1y_1x_2y-2\lambda_1\lambda_2}}. \tag{4.128}$$

The relation is correct if the parameters a, b, c and d of A has been adjusted as function of x, y, λ and z, [23]. This means that R_{12} intertwiner of and hence it has a $\mathfrak{U}_{\mathfrak{i}}(\mathfrak{sl}_2)$ symmetry and in conclusion it is an appropriate R-matrix for a theory $\mathfrak{U}_{\mathfrak{i}}(\mathfrak{sl}_2)$-invariant.

4.3.3 Tetrahedron Zamolodchikov equation

Let us consider the quantum group $\mathfrak{W}$ that is generated by the k_0, e_0, f_0 and h . We denote a representation given by restricting the generators as V_{μ_r} . In this case the space of solution to the equation

$$\Delta^{op}(W)\mathfrak{R}_{12} = \mathfrak{R}_{12}\Delta(W), \quad W \in \mathfrak{W} \tag{4.129}$$

is two-dimensional since

$$V_{\mu_1} \otimes V_{\mu_2} \cong V_{\mu_1+\mu_2+1} \oplus V_{\mu_1+\mu_2-1} \tag{4.130}$$

decomposes into two irreducible subspace . A basis for this space is given by the set $\{\mathfrak{R}_{12}, \mathfrak{R}'_{12}\}$ where $\mathfrak{R}_{12}$ is the operator solution to the symmetry constraints (4.125) . $\mathfrak{R}'_{12}$ it is also solution of (4.125) but for $\mathcal{V}_{\mu_1;x_1,x_1} \otimes \mathcal{V}_{\mu_2;x_2,-x_2}$.

The following tensor product

$$\mathcal{V}_{\mu_1} \otimes \mathcal{V}_{\mu_2} \otimes \mathcal{V}_{\mu_3} \tag{4.131}$$

decompose generically as

$$\mathcal{V}_{\mu_1+\mu_2+\mu_3+2} \oplus 2\mathcal{V}_{\mu_1+\mu_2+\mu_3} \oplus \mathcal{V}_{\mu_1+\mu_2+\mu_3-2}. \tag{4.132}$$

Now we want to describe the space of $\mathfrak{W}$-intertwiners of the tensor product $\mathcal{V}_{\mu_1} \otimes \mathcal{V}_{\mu_2} \otimes \mathcal{V}_{\mu_3}$. To do this we define a new basis, in [23] it is possible to find the transformations leads to the basis operators $\mathfrak{W}$-invariant . $\mathcal{R}_{12}, \mathcal{R}'_{12}$ with

$$\mathcal{R}_{12}(A_1, A_2) = R_{12}(A_2 A_1^{-1}), \quad \mathcal{R}'_{12}(A_1, A_2) = R_{12}(A_2\sigma_3 A_1^{-1}\sigma_3) \tag{4.133}$$

where $\sigma_3 = diag(1, 1)$ and the A_i are elements of $SL(2, \mathbb{C})$. We want to use these two operators to describe the space of $\mathfrak{W}$-invariant intertwiners on the the tensor product of three $\mathcal{V}_{\mu_r}$. The 16 operators $\mathcal{R}_{12}^{\alpha}\mathcal{R}_{13}^{\beta}\mathcal{R}_{23}^{\gamma}$ and $\mathcal{R}_{23}^{\alpha}\mathcal{R}_{13}^{\beta}\mathcal{R}_{12}^{\gamma}$ for $\alpha, \beta, \gamma \in \{0,1\}$ are $\mathfrak{W}$-invariant thanks to the invariance properties of $\mathcal{R}_{12}, \mathcal{R}'_{12}$.
The dimension of the space of such invariant intertwiners is 6 therefore at most six of them can be linearly independent. The relationships between the various intertwiners is described by the

- *Tetrahedron Zamolodchikov algebra*
- the linear dependence equations.

In order to write theme it turns out to be useful to perform a change of basis to light-cone operators

$$\mathcal{R}^{\pm} = \frac{1}{2}(\mathcal{R}^0 \pm \mathcal{R}^1) \tag{4.134}$$

explicitly written using oscillators as

$$R^{+}(A_j, A_k) = (a_k\hat{n}_j + ic_k\hat{m}_j)(d_j\hat{n}_k + ib_j\hat{m}_k) + \hat{c}^{\dagger}\hat{c}_k \tag{4.135}$$

$$R^{-}(A_j, A_k) = (b_k\hat{n}_j + id_k\hat{m}_j)(c_j\hat{n}_k ia_j\hat{m}_k) + +\hat{c}^{\dagger}\hat{c}_k \tag{4.136}$$

where a_k, b_k, c_k, d_k are the free fermion parameters. The tetrahedron Zamolodchikov algebra in this basis is then defined as the set of relations

$$\mathcal{R}^{\alpha}_{23}\mathcal{R}^{\beta}_{13}\mathcal{R}^{\gamma}_{12} = \sum_{\alpha',\beta',\gamma'=\pm} \mathbb{S}^{\alpha\beta\gamma}_{\alpha'\beta'\gamma'}(a_k, b_k, c_k, d_k)\mathcal{R}^{\alpha'}_{12}\mathcal{R}^{\beta'}_{13}\mathcal{R}^{\gamma'}_{23}, \quad k = \{0, 1, 2, 3\} \tag{4.137}$$

where the coefficients $\mathbb{S}^{\alpha\beta\gamma}_{\alpha'\beta'\gamma'}(a_k, b_k, c_k, d_k)$ are given in [23]. Since this shows a relation between 8 generators, and there should only be 6 linearly independent ones, there exist two linear dependence equations:

$$\sum_{\alpha,\beta,\gamma=\pm} \mathbb{K}^{\alpha\beta\gamma}_{\alpha'\beta'\gamma'}(a_k, b_k, c_k, d_k)\mathcal{R}^{\alpha}_{12}\mathcal{R}^{\beta}_{13}\mathcal{R}^{\gamma}_{23} = 0, i = 1, 2. \tag{4.138}$$

The coefficients $\mathbb{K}^{\alpha\beta\gamma}$ are not unique in fatc

$$(\mathbb{S}')^{\alpha\beta\gamma}_{\alpha'\beta'\gamma'} = \mathbb{S}^{\alpha\beta\gamma}_{\alpha'\beta'\gamma'} + \sum_{i=1}^{2} c_i^{\alpha\beta\gamma}\mathbb{K}^{i}_{\alpha\beta\gamma} \tag{4.139}$$

will obey the tetrahedron Zamolodchikov algebra for any $c_i^{\alpha\beta\gamma} \in \mathbb{C}$. If we consider now the product of six R-matrices in lattice order, the *Tetrahedron Zamolodchikov equations* can be obtained by gauging elements of the algebra suitably:

$$\mathbb{S}'_{123}\mathbb{S}'_{124}\mathbb{S}'_{134}\mathbb{S}'_{234} = \mathbb{S}'_{234}\mathbb{S}'_{134}\mathbb{S}'_{124}\mathbb{S}'_{123}. \tag{4.140}$$

This equation should be interpreted as an equation in $End((\mathbb{C}^2)^{\otimes 6})$. Let us introduce 2×2 units e_{ij} and define

$$e^{(12)}_{ij} = e_{ij} \otimes 1^{\otimes 5}, \quad e^{(13)}_{ij} = 1 \otimes e_{ij} \otimes 1^{\otimes 4}, \quad e^{(23)}_{ij} = 1^{\otimes 2} \otimes e_{ij} \otimes 1^{\otimes 3} \tag{4.141}$$

$$e^{(14)}_{ij} = 1^{\otimes 3} \otimes e_{ij} \otimes 1^{\otimes 2}, \quad e^{(24)}_{ij} = 1^{\otimes 4} \otimes e_{ij} \otimes 1, \quad e^{(34)}_{ij} = 1^{\otimes 5} \otimes e_{ij}. \tag{4.142}$$

Then the tensors in tetrahedron Zamolodchikov equations are defined by

$$\mathbb{S}'_{ijk} = \sum_{l,m,n,p,q,r=\pm} (\mathbb{S}'_{ijk})^{lmn}_{pqr} e^{(ij)}_{nr} e^{(ik)}_{mq} e^{(jk)}_{nr}. \tag{4.143}$$

Tetrahedron Zamolodchikov equation is the corresponding of the Yang Baxter equation and its associated algebra for 1+2 dimensional physics: the Yang Baxter equation corresponds to the equality of two scattering matrices of a 3-body process in a bidimesional lattice. The Tetrahedron Zamolodchikov equation corresponds to this equality in a three dimensional lattice and generates integrable three dimensional quantum field theories.

4.4 Categorical approach to quantum field theory

This new approach is due to Romeo Brunetti, Klaus Fredenaghen and Rein Verch . This section is part of [2] .
"The main feature of this new approach is to incorporate in a local sense the principle of general covariance of general relativity, thus giving rise to the concept of a locally covariant quantum field theory. Such locally covariant quantum field theories will be described mathematically in terms of covariant functors between the categories, on one side, of globally hyperbolic spacetimes with isometric embeddings as morphisms and, on the other side, of *-algebras with unital injective * -endomorphisms as morphisms.
Moreover, locally covariant quantum fields can be described in this framework as natural transformations between certain functors. The usual Haag-Kastler framework of nets of operator-algebras over a fixed spacetime background-manifold, together with covariant automorphic actions of the isometry-group of the background spacetime, can be regained from this new approach as a special case. Examples of this new approach are also outlined. In case that a locally covariant quantum field theory obeys the time-slice axiom, one can naturally associate to it certain automorphic actions, called "relative Cauchy-evolutions", which describe the dynamical reaction of the quantum field theory to a local change of spacetime background metrics. The functional derivative of a relative Cauchy-evolution with respect to the spacetime metric is found to be a divergence-free quantity which has, as will be demonstrated in an example, the significance of an energy-momentum tensor for the locally covariant quantum field theory. Furthermore, we discuss the functorial properties of state spaces of locally covariant quantum field theories that entail the validity of the principle of local definiteness. "

Definition 4.30 $\mathfrak{Man}$: This category consists of a class of objects Obj($\mathfrak{Man}$) formed by all four- dimensional, globally hyperbolic spacetimes (M, g) which are oriented and time-oriented. Given any two such objects $(M_1, g_1)$and (M_2, g_2), the morphism

$$\psi : (M_1, g_1) \to (M_2, g_2)$$

an isometric embedding in other words, ψ is a diffeomorphism onto its range $\psi(M_1)$ i.e. the map $\overline{\psi} : M_1 \to \psi(M_1) \subset M_2$ is a diffeomorphism and ψ is an isometry, that is $\psi_* g_1 = g_2 \upharpoonright \psi(M_1)$. With the additional constraints that i)if $\gamma : [a, b] \to M_2$ is any causal curve and $\gamma(a), \gamma(b) \in \psi(M_1)$ then the whole curve must be in the image $\psi(M_1)$, i.e. $\gamma(t) \in \psi(M_1)$, $\forall t \in (a, b)$; ii) the isometric embedding preserves orientation and time-orientation of the embedded spacetime. The composition rule for any $\psi \in Hom_{\mathfrak{Man}}((M_1, g_1)(M_2, g_2))$ and $\psi' \in Hom_{\mathfrak{Man}}((M_2, g_2)(M_3, g_3))$ is to define its composition $\psi' \circ \psi$ as the composition of maps.

$\mathfrak{Alg}$: This is the category whose class of objects Obj($\mathfrak{Alg}$) is formed by all C^*- algebras possessing unit elements, and the morphisms are faithful unit-preserving *-homomorphisms. The composition is again defined as the composition of maps. The unit element for any $A \in Obj(\mathfrak{Alg})$ given by the identical map on A.

Requirement (i) on the morphisms of $\mathfrak{Man}$ is introduced in order that the induced and intrinsic causal structures coincide for the embedded space-time $\psi(M_1) \subset M_2$. Condition (ii) might, in fact, be relaxed; the resulting structure, allowing also isometric embeddings which reverse spatial- and time-orientation.

Definition 4.31. A **locally covariant quantum field theory** is a covariant functor A between the two categories $\mathfrak{Man}$ and $\mathfrak{Alg}$ i.e. writing α_ψ for A_ψ, in diagrammatic form :

$$\begin{array}{ccc} (M, g) & \xrightarrow{\phi} & (M', g') \\ {\scriptstyle A}\big\downarrow & & \big\downarrow{\scriptstyle A} \\ A(M, g) & \xrightarrow{\alpha_\psi} & A(M', g') \end{array} \tag{4.144}$$

together with covariance properties

$$\alpha_\psi \circ \alpha_{\psi'} = \alpha_{\psi \circ \psi'}, \quad \alpha_{1_M} = 1_{A(M)}, \tag{4.145}$$

for all $\psi \in Hom_{\mathfrak{Man}}((M_1, g_1)(M_2, g_2))$ and $\psi' \in Hom_{\mathfrak{Man}}((M_2, g_2)(M_3, g_3))$ and all $(M, g) \in \text{Obj } \mathfrak{Man}$.

Definition 4.32. A locally covariant quantum field theory described by a covariant functor A is called **causal** if the following holds :
whenever there are morphisms $\psi_j \in Hom_{\mathfrak{Man}}((M_j, g_j)(M, g)), j = 1, 2$ so that the sets $\psi(M_1)$ and $\psi(M_2)$ are causally separated in (M, g), the one has

$$[\alpha_{\psi_1}(A(M_1, g_1)), \alpha_{\psi_2}(A(M_2, g_2))] = \{0\}, \tag{4.146}$$

where $[\mathcal{A}.\mathcal{B}] = \{AB - BA : A \in \mathcal{A}, B \in \mathcal{B}$ for any pair of $C^* - algebras\ \mathcal{A}$ and $\mathcal{B}$.

Definition. We say that a locally covariant quantum field theory given by the functor A obeys the time-slice axiom if

$$\alpha_\psi(A(M, g)) = A(M', g') \tag{4.147}$$

holds for all $\psi \in Hom_{\mathfrak{Man}}((M, g)(M', g'))$ such that $\psi(M)$ contains a Cauchy surface for (M', g'). Thus, a locally covariant quantum field theory is an assignment of C^*-algebras to all globally hyperbolic spacetimes so that the algebras are identifyable when the spacetimes are isometric, in the indicated way. Note that we use the term local in the sense of geometrically local in the definition which should not be confused with the meaning of locality in the sense of Einstein causality.
Causality means that the algebras $\alpha_{\psi_1}(A(M_1, g_1))$ and $\alpha_{\psi_2}(A(M_2, g_2))$ commute element wise in the larger algebra $A(M, g)$when the sub-regions $\psi_1(M_1)$and $\psi_2(M_2)$ of$M$ are causally separated (with respect to g). This property is expected to hold generally for observable quantities which can be localized in certain subregions of spacetimes. The time slice axiom (iii), also called strong Einstein causality, or existence of a causal dynamical law, says that an algebra of observables on a globally hyperbolic spacetime is already determined by the algebra of observables localized in any neighbourhood of a Cauchy-surface.

"We consider again the category $\mathfrak{Man}$, and introduce the category $\mathfrak{TAlg}$ consist- ing of topological *-algebras (with unit elements) as objects, and of continuous *-endomorphisms as morphisms (i.e., $\in Hom_{\mathfrak{TAlg}}(A_1, A_2)$ is a morphism of $\mathfrak{TAlg}$ if $: A_1 \to A_2$ is a continuous, unit-preserving, injective *-morphism). In addition, we consider another category $\mathfrak{Test}$ which is the category containing as objects all possible test-function spaces over $\mathfrak{Man}$, that is, the objects consist of all spaces $C_0^\infty(M)$ of smooth, compactly supported test-functions on M , for (M, g) ranging over the objects of $\mathfrak{Man}$, and the morphisms are all possible push-forwards ψ of isometric embeddings $\psi : (M_1, g_1) \to (M_2, g_2)$. The action of any push-forward ψ on an element of a test-function space has been defined above, and it clearly satisfies the requirements for morphisms between test-function spaces. "

Now let a locally covariant quantum field theory A be defined as a functor in the same manner as in Def. 2.1, but with the category $\mathfrak{TAlg}$ in place of the category $\mathfrak{Alg}$, and again following the convention to denote A $(\psi)$by ψ whenever $\psi$is any morphism in $\mathfrak{Man}$. Moreover, let D be the covariant functor between $\mathfrak{Man}$ and $\mathfrak{Test}$ assigning to each $(M, g) \in Obj(\mathfrak{Man})$ the test-function space $D(M, g) = C_0^\infty(M)$, and to each morphism ψ of $\mathfrak{Man}$ its push-forward: $D(\psi) = \psi$. We regard the categories $\mathfrak{Test}$ and $\mathfrak{TAlg}$ as subcategories of the category of all topological spaces Top, and hence we are led to adopt the following :
A **locally covariant quantum field** Φ is a natural transformation between the functors D and A , i.e. for any object (M, g) in $\mathfrak{Man}$ there exists a morphism $\Phi_{(M,g)} : D(M, g) \to A(M, g)$ in $\mathfrak{Top}$ such that for each given morphism $\psi \in Hom_{\mathfrak{Man}}((M_1, g_1)(M_2, g_2))$ the following diagram

$$\begin{array}{ccc} D(M_1, g_1) & \xrightarrow{\Phi_{(M_1,g_1)}} & A(M_1, g_1) \\ \psi_* \Big\downarrow & & \Big\downarrow \alpha_\psi \\ D(M_2, g_2) & \xrightarrow{\Phi_{(M_2,g_2)}} & A(M_2, g_2) \end{array} \tag{4.148}$$

commutes.
The commutativity of the diagram means that

$$\alpha_\psi \circ \Phi_{(M_1,g_1)} = \Phi_{(M_2,g_2)} \circ \psi_* \tag{4.149}$$

i.e., the requirement of covariance for fields.

Chapter 5

New structures for Physics

We show that an orthogonal basis for a finite-dimensional Hilbert space can be equivalently characterised as an abelian †-Frobenius monoid in the category **FdHilb**, which has finite-dimensional Hilbert spaces as objects and bounded linear maps as morphisms, and tensor product for the monoidal structure. The basis is normalised exactly when the corresponding commutative †-Frobenius monoid. Hence orthogonal and orthonormal bases can be formulated in terms of composition of operations and tensor product only, without any explicit reference to the underlying vector spaces.

5.1 A new description of orthogonal bases

We recall an important definition : in a monoidal category, a **monoid** is an ordered triple (A, m, u) consisting

- an object A
- a moltiplication morphism $m : A \otimes A \to A$
- a unit morphism $u : \mathbf{1} \to A$

which satisfy associativity and unit equations :

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C \tag{5.1}$$

$$\mathbf{1} \otimes A \cong A \cong A \otimes \mathbf{1}. \tag{5.2}$$

Now we introduce the key structure of our work : **Frobenius monoid**
Definition 5.1. A Frobenius monoid in a symmetric monoidal category is a quintuple $(H, m, u, \delta, \varepsilon)$ consisting in a internal monoid

$$\mathbf{1} \xrightarrow{u} H \xleftarrow{m} H \otimes H \tag{5.3}$$

and an internal comonoid

$$\mathbf{1} \xleftarrow{\varepsilon} H \xrightarrow{\delta} H \otimes H \tag{5.4}$$

which together satisfy the Frobenius condition : the following diagrams must be commutative

$$\begin{array}{ccc} X \otimes X & \xrightarrow{X \otimes \delta} & X \otimes X \otimes X \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m \otimes X} \\ X & \xrightarrow{\delta} & X \otimes X \end{array} \tag{5.5}$$

$$(m \otimes X) \circ (X \otimes \delta) = \delta \circ m$$

$$\begin{array}{ccc} X \otimes X & \xrightarrow{m} & X \\ {\scriptstyle \delta \otimes X}\downarrow & & \downarrow{\scriptstyle \delta} \\ X \otimes X \otimes X & \xrightarrow{X \otimes m} & X \otimes X \end{array} \tag{5.6}$$

$$(\delta \otimes X) \circ (X \otimes m) = \delta \circ m.$$

Definition 5.2. A Frobenius monoid is commutative if

$$\sigma \circ \delta = \delta \tag{5.7}$$

where σ is the braiding map .

Definition 5.3. A Frobenius monoid is a $\dagger-$Frobenius monoid

$$m^\dagger = \delta, \quad u^\dagger = \varepsilon. \tag{5.8}$$

5.1.1 Turning an orthogonal basis into a commutative $\dagger-$Frobenius monoids

Why $\dagger-$Frobenius monoids? The key property of $\dagger-$Frobenius monoids which makes them so useful is contained in the following observation, due to Coecke, Pavlovic and J. Vicary [5].
Given a finite dimensional Hilbert space H with $dim\, H = n$ and relative orthonormal basis $\{|i\rangle\}_{i=1,\dots,n}$ we can always define the linear maps

$$\delta : H \to H \otimes H \tag{5.9}$$
$$|i\rangle \mapsto |i\rangle \otimes |i\rangle \tag{5.10}$$

$$\varepsilon : H \to \mathbb{C} \tag{5.11}$$
$$|i\rangle \mapsto 1 \tag{5.12}$$

Proposition 5.4.$\delta(|\psi\rangle) = |\psi\rangle \otimes |\psi\rangle \overset{TH}{\Longrightarrow} |\psi\rangle \equiv |i\rangle$
Proof.

$$|\psi\rangle = \sum_{i=1}^{n} c_i|i\rangle 1 \Rightarrow \delta(|\psi\rangle) = \sum_{i=1}^{n} c_i\delta(|i\rangle) = \sum_{i=1}^{n} c_i|i\rangle \otimes |i\rangle \tag{5.13}$$

$$|\psi\rangle \otimes |\psi\rangle = \left(\sum_{i=1}^{n} c_i|i\rangle\right) \otimes \left(\sum_{j=1}^{n} c_j|j\rangle\right) = \sum_{i,j=1}^{n} c_ic_j|i\rangle \otimes |j\rangle \tag{5.14}$$

Now we use the hypothesis (5.13) = (5.14)

$$\sum_{i,j=1}^{n} c_ic_j|i\rangle \otimes |j\rangle - \sum_{i=1}^{n} c_i|i\rangle \otimes |i\rangle = 0 \tag{5.15}$$

this is a linear combination of the $H \otimes H$ hence are linearly independent. This implies the following

$$c_i - c_ic_j = 0,\, i, j = 1, ..., n \tag{5.16}$$

if $i = j \Rightarrow c_i - c_i^2 = 0 \Rightarrow c_i = 1, \forall i = 1, ..., n$
if $i \neq j \Rightarrow c_i - c_i c_j = c_i(1 - c_j) = 0 \Rightarrow c_j = 1, \forall j = 1, ..., n$

□

Then we see that from δ we can recover the basis of H.
Definition 5.5 We define $\delta^\dagger$ by setting

$$\delta^\dagger : H \otimes H \to H \tag{5.17}$$

$$|i\rangle \otimes |j\rangle \mapsto \begin{cases} |i\rangle & i = j \\ 0 & i \neq j \end{cases} \tag{5.18}$$

. On a generic element $|\psi\rangle \otimes |\varphi\rangle = \sum_{i,j=1}^{n} \psi_i \varphi_j |i\rangle \otimes |j\rangle$

$$\begin{aligned} \delta^\dagger(|\psi\rangle \otimes |\varphi\rangle) &= \sum_{i,j=1}^{n} \psi_i \varphi_j \delta^\dagger |i\rangle \otimes |j\rangle \\ &= \sum_{i,j=1}^{n} \psi_i \varphi_j \delta_{ij} |j\rangle \\ &= \sum_{i=1}^{n} \psi_i \varphi_i |i\rangle \end{aligned} \tag{5.19}$$

To see that $\delta^\dagger$ and δ obey the Frobenius condition it suffices to note that

$$|i\rangle \otimes |j\rangle \xmapsto{\delta^\dagger} \begin{cases} |i\rangle & i = j \\ 0 & i \neq j \end{cases} \xmapsto{\delta} \begin{cases} |i\rangle \otimes |i\rangle & i = j \\ 0 & i \neq j \end{cases} \tag{5.20}$$

and

$$|i\rangle \otimes |j\rangle \xmapsto{H \otimes \delta} |i\rangle \otimes |j\rangle \otimes |j\rangle \xmapsto{\delta^\dagger \otimes H} \begin{cases} |i\rangle \otimes |i\rangle & i = j \\ 0 & i \neq j \end{cases} \tag{5.21}$$

As a consequence, by linearity,

$$\delta \circ \delta^\dagger = (\delta^\dagger \otimes H) \circ (H \otimes \delta). \tag{5.22}$$

That (H, δ, ε) is a comonoid is verified.
The unit of the corresponding monoid is defined by setting

$$\varepsilon^\dagger : \mathbb{C} \to H \tag{5.23}$$

$$1 \mapsto \sum_{i=1}^{n} |i\rangle. \quad (5.24)$$

Hence is possible turning an orthogonal basis into a commutative -Frobenius monoid .

5.1.2 Turning a commutative †-Frobenius monoid into an orthogonal basis

We start denoting elements of H as linear maps $Hom(\mathbb{C}, H)$:

$$\alpha : \mathbb{C} \to H \quad (5.25)$$
$$1 \mapsto |\alpha\rangle \quad (5.26)$$

and as kets $= \alpha(1)$. Taking the adjoint of α gives us

$$\alpha^\dagger : H \to \mathbb{C} \quad (5.27)$$
$$|\psi\rangle \mapsto \langle\alpha|\psi\rangle \quad (5.28)$$

and hence $\langle\alpha| = \alpha^\dagger \in H^*$.
Let $(H, m = \delta^\dagger, u)$ be a commutative †-Frobenius monoid.
Given such a commutative †-Frobenius monoid any $\alpha \in H$ induces a linear map :

$$R_{|\alpha\rangle} = m \circ (H \otimes \alpha) : H \to H \quad (5.29)$$

on any element $|\psi\rangle \in H$

$$R_{|\alpha\rangle} : H \otimes \mathbb{C} \cong H \xrightarrow{H\otimes\alpha} H \otimes H \xrightarrow{m} H \quad (5.30)$$

$$|\psi\rangle \otimes 1 \mapsto |\psi\rangle \otimes |\alpha\rangle \mapsto \sum_{i=1}^{n} \psi_i \alpha_i |i\rangle \quad (5.31)$$

$$\Downarrow$$

$$R_{|\alpha\rangle}|i\rangle = \alpha_i |i\rangle. \quad (5.32)$$

Definition 5.6 Let H be a Hilbert space . Let $O : H \to H$ be a bounded linear operator . Then the adjoint of O is a operator $O^\dagger : H \to H$ satisfying

$$\langle O^\dagger\psi|\varphi\rangle = \langle\psi|O\varphi\rangle, \forall|\varphi\rangle, |\psi\rangle \in H \quad (5.33)$$

Existence and uniqueness of this operator follows from the Riesz representation theorem.

Proposition 5.7 If (H, m, u) is a commutative †-Frobenius monoid in a symmetric monoidal †-category then

$$R^{\dagger}_{|\alpha\rangle} = R_{|\alpha'\rangle} \quad for \quad \alpha' = (H \otimes \alpha^{\dagger}) \circ m^{\dagger} \circ u : \mathbb{C} \to H \tag{5.34}$$

Proof.

$$u = \varepsilon^{\dagger} : \mathbb{C} \to H \tag{5.35}$$

$$1 \mapsto \sum_{i=1}^{n} |i\rangle \tag{5.36}$$

$$m^{\dagger} = \delta : H \to H \otimes H \tag{5.37}$$

$$\begin{aligned} m^{\dagger}(\langle\alpha|\psi\rangle|\varphi\rangle) &= \delta(\langle\alpha|\psi\rangle|\varphi\rangle) \\ &= \sum_{i=1}^{n} \alpha_i^* \psi_i \varphi_i |i\rangle \otimes |i\rangle \end{aligned} \tag{5.38}$$

$$H \otimes \alpha^{\dagger} : H \otimes H \to H \otimes \mathbb{C} \cong H \tag{5.39}$$

$$|\psi\rangle \otimes |\varphi\rangle \mapsto \langle\alpha|\psi\rangle|\varphi\rangle = \sum_{i=1}^{n} \alpha_i^* \psi_i \varphi_i |i\rangle \tag{5.40}$$

hence if

$$|\psi\rangle \otimes |\varphi\rangle = |i\rangle \otimes |i\rangle \Rightarrow \psi_i = \varphi_i = 1 \tag{5.41}$$

$$\Downarrow$$

$$\alpha'(1) = \sum_{i=1}^{n} \alpha_i^* |i\rangle = |\alpha'\rangle \tag{5.42}$$

$$\Downarrow$$

$$\begin{aligned} R_{|\alpha'\rangle}|\psi\rangle &= m(|\psi\rangle \otimes |\alpha'\rangle) \\ &= \delta^{\dagger}(|\psi\rangle \otimes |\alpha'\rangle) \\ &= \sum_{i=1}^{n} \psi_i \alpha_i^* |i\rangle \end{aligned} \tag{5.43}$$

$$\langle R_{|\alpha'\rangle}\psi| = \sum_{i=1}^{n} \psi_i^* \alpha_i \langle i| \tag{5.44}$$

$$\Downarrow$$

$$\langle R_{|\alpha'\rangle}\psi|\varphi\rangle = \sum_{i=1}^{n} \psi_i^* \alpha_i \langle i|\varphi\rangle = \sum_{i=1}^{n} \psi_i^* \alpha_i \langle i| \sum_{j=1}^{n} \varphi_j |j\rangle \tag{5.45}$$

$$\Downarrow$$

$$\begin{aligned}
\langle R_{|\alpha'\rangle}\psi|\varphi\rangle &= \sum_{i,j=1}^{n} \psi_i^* \alpha_i \varphi_j \langle i|j\rangle \\
&= \sum_{i,j=1}^{n} \psi_i^* \alpha_i \varphi_j \delta_{ij} \\
&= \sum_{i=1}^{n} \psi_i^* \alpha_i \varphi_i
\end{aligned} \tag{5.46}$$

$$\begin{aligned}
|R_{|\alpha\rangle}\varphi\rangle &= R_{|\alpha\rangle}|\varphi\rangle \\
&= \sum_{i=1}^{n} \varphi_i R_{|\alpha\rangle} \\
&= \sum_{i=1}^{n} \varphi_i \alpha_i |i\rangle
\end{aligned} \tag{5.47}$$

$$\Downarrow$$

$$\langle \psi|R_{|\alpha\rangle}\varphi\rangle = \sum_{i=1}^{n} \psi_i^* \alpha_i \varphi_i \tag{5.48}$$

$$\Downarrow$$

$$\langle \psi|R_{|\alpha\rangle}\varphi\rangle = \langle R_{|\alpha'\rangle}\psi|\varphi\rangle \tag{5.49}$$

hence by uniqueness $R^\dagger_{|\alpha\rangle}$ we conclude that

$$R^\dagger_{|\alpha\rangle} = R_{|\alpha'\rangle} \tag{5.50}$$

□

Proposition 5.8

$$(-)' : Hom(\mathbb{C}, H) \to Hom(\mathbb{C}, H)$$

$$\alpha \quad \mapsto \quad \alpha' = (H \otimes \alpha^\dagger) \circ m^\dagger \circ u$$

then $(-)'$ is an involution :

$$(\alpha')' \overset{\text{(TH)}}{=} \alpha. \tag{5.51}$$

Proof. If

$$\alpha' = (H \otimes \alpha^\dagger) \circ m^\dagger \circ u \Rightarrow (\alpha')' = (H \otimes (\alpha')^\dagger) \circ m^\dagger \circ u \tag{5.52}$$

where

$$(\alpha')^\dagger = u^\dagger \circ m \circ (H \otimes \alpha^\dagger)^\dagger. \tag{5.53}$$

By proposition 3.36 **FdHilb** is a symmetric dagger monoidal category hence

$$(\alpha')^\dagger = u^\dagger \circ m \circ (H \otimes \alpha) \tag{5.54}$$

$$(\alpha')^\dagger : H \otimes \mathbb{C} \cong H \xrightarrow{H\otimes\alpha} H \otimes H \xrightarrow{m} H \xrightarrow{u^\dagger} \mathbb{C}$$

$$|\psi\rangle \mapsto |\psi\rangle \otimes |\alpha\rangle \mapsto \sum_{i=1}^{n} \alpha_i \psi_i |i\rangle \otimes |i\rangle \mapsto \sum_{i=1}^{n} \alpha_i \psi_i \tag{5.55}$$

$$\Downarrow$$

$$H \otimes (\alpha')^\dagger : H \otimes H \otimes \mathbb{C} \cong H \otimes H \to H \otimes H \otimes H \to H \otimes H \to H$$

$$|\psi\rangle \otimes |\varphi\rangle \mapsto |\psi\rangle \otimes |\varphi\rangle \otimes |\alpha\rangle \mapsto |\psi\rangle \otimes \sum_{i=1}^{n} \alpha_i \varphi_i |i\rangle \mapsto \sum_{i=1}^{n} \alpha_i \varphi_i \psi_i |i\rangle \tag{5.56}$$

$$\Downarrow$$

$$(\alpha')' : \mathbb{C} \xrightarrow{u=\varepsilon^\dagger} H \xrightarrow{\delta=m^\dagger} H \otimes H \xrightarrow{H\otimes(\alpha')^\dagger} H$$

$$1 \mapsto \sum_{i=1}^{n} |i\rangle \mapsto \sum_{i=1}^{n} |i\rangle \otimes |i\rangle \mapsto \sum_{i=1}^{n} \alpha_i |i\rangle = |\alpha\rangle \tag{5.57}$$

□

5.1.3 The embedding

In **FdHilb** we define a morphism

$$\begin{aligned}\Lambda : \mathbf{FdHilb}(\mathbb{C}, H) = H \hookrightarrow \mathbf{FdHilb}(H, H) = F(H, H)\\ |\psi\rangle \quad \longmapsto \quad R_{|\psi\rangle}\end{aligned} \tag{5.58}$$

then is an **involution preserving monoid embedding** when endowing **FdHilb**(ℂ,H) and **FdHilb**(H,H) with the monoid structure of the internal monoid .

Lemma 5.9. $H^* \otimes H \cong \mathbf{FdHilb}(H, H)$

Proof. We define two maps

$$\begin{aligned}H^* \otimes H &\stackrel{\varphi}{\to} \mathbf{FdHilb}(H, H)\\ \langle z| \otimes |w\rangle &\longmapsto h : H \to H\\ &\qquad |v\rangle \mapsto \langle z|v\rangle|w\rangle.\end{aligned} \tag{5.59}$$

$$\begin{aligned}\mathbf{FdHilb}(H, H) &\stackrel{\psi}{\to} H^* \otimes H\\ h : H \to H &\mapsto \sum_{i=1}^{n} \langle i| \otimes h|i\rangle\end{aligned} \tag{5.60}$$

$$\begin{aligned}(\psi \circ \varphi)(\langle z| \otimes |w\rangle) &= \psi(|i\rangle \mapsto \langle z|i\rangle|w\rangle)\\ &= \sum_{i=1}^{n} \langle i| \otimes \langle z|i\rangle|w\rangle = \langle z| \otimes |w\rangle\end{aligned} \tag{5.61}$$

$$\begin{aligned}[(\varphi \circ \psi)h]|j\rangle &= [\varphi(\sum_{i=1}^{n} \langle i| \otimes h|i\rangle)]|j\rangle\\ &= \sum_{i=1}^{n} \langle i|j\rangle h|i\rangle\\ &= h|j\rangle\end{aligned} \tag{5.62}$$

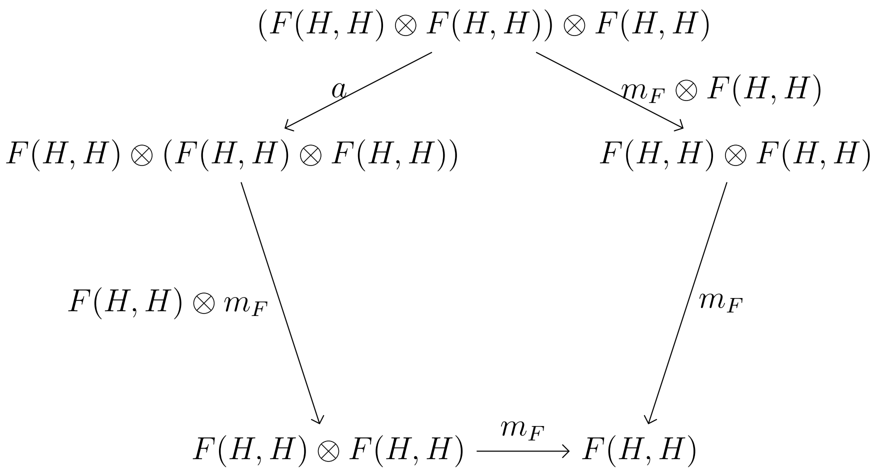
then we have

$$\psi \circ \varphi = 1_{H^* \otimes H} \text{ and } \varphi \circ \psi = 1_{\mathbf{FdHilb}(H,H)} \tag{5.63}$$

□

Theorem 5.10. $(F(H,H), m_F, u_F : \mathbb{C} \to F(H,H) :: 1 \mapsto 1_H)$ is an involution monoid .
Proof. First we prove that $(F(H,H), m_F, u_F : \mathbb{C} \to F(H,H) :: 1 \mapsto 1_H)$ is a monoid.

- We needs to show that this diagram



(5.64)

is commutative in other words

$$m_F \circ (m_F \otimes F(H,H)) \circ a = m_F \circ (F(H,H) \otimes m_F) \tag{5.65}$$

where $m_F = \circ$.

$$a(h \otimes (k \otimes t)) = (h \otimes k) \otimes t. \tag{5.66}$$

$$m_F \otimes F(H,H)((h \otimes k) \otimes t) = (h \circ k) \otimes t \tag{5.67}$$

$$m_F((h \circ k) \otimes t) = (h \circ k) \circ t \tag{5.68}$$

and

$$F(H,H) \otimes m_F(h \otimes (k \otimes t)) = h \otimes (k \circ t) \tag{5.69}$$

$$m_F(h \otimes (k \circ t)) = h \circ (k \circ t) \tag{5.70}$$

$$\circ \text{ is associative } \Rightarrow h \circ (k \circ t) = (h \circ k) \circ t. \tag{5.71}$$

- $l_F \overset{\text{(TH)}}{=} (u_F \otimes F(H,H)) \circ m_F$

$$
\begin{aligned}
l_F : \mathbb{C} \otimes F(H,H) &\to F(H,H) \\
\lambda \otimes h &\mapsto h \qquad (5.72) \\
: \mathbb{C} \otimes F(H,H) \to F(H,H) &\otimes F(H,H) \to F(H,H) \\
\lambda \otimes h \mapsto 1_H \otimes h &\mapsto 1_H \circ h = h \qquad (5.73)
\end{aligned}
$$

- $r_F \overset{\text{(TH)}}{=} (F(H,H) \otimes u_F) \circ m_F$

$$
\begin{aligned}
r_F : F(H,H) \otimes \mathbb{C} &\to F(H,H) \\
h \otimes \lambda &\mapsto h \qquad (5.74) \\
: F(H,H) \otimes \mathbb{C} \to F(H,H) &\otimes F(H,H) \to F(H,H) \\
h \otimes \lambda \mapsto h \otimes 1_H &\mapsto h \circ 1_H = h. \qquad (5.75)
\end{aligned}
$$

An **involution monoid** $(A, m, u; s)$ is a monoid equipped with a morphims $s : A \to A^*$ called **linear involution**, which is a morphism of monoids with respect to monoid structure (A^*, m_*, u_*) on A^*, and which satisfies the **involution condition** $s_* \circ s = 1_A$. Now we have $(A, m, u; s) = (F(H,H), m_F, u_F : \mathbb{C} \to F(H,H) :: 1 \mapsto 1_H; s_F)$ and $(A^*, m_*, u_*) = (F(H,H)^*, m_{F^*}, u_{F^*})$. The question is if $s_F : F(H,H) \to F(H,H)^*$ is morphism of monoids with respect to monoid structure (A^*, m_*, u_*) on A^*. In other words if these diagrams

$$
\begin{array}{ccc}
F(H,H) \otimes F(H,H) & \xrightarrow{m_{F(H,H)} = m_F} & F(H,H) \\
{\scriptstyle s_F \otimes s_F} \downarrow & & \downarrow {\scriptstyle s_F} \\
F(H,H)^* \otimes F(H,H)^* & \xrightarrow{m_{F(H,H)^*} = m_{F^*}} & F(H,H)^*
\end{array} \quad . \qquad (5.76)
$$

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
 & {\scriptstyle u} \nwarrow \quad \nearrow {\scriptstyle u'} & \\
 & \mathbf{1} &
\end{array} \quad . \qquad (5.77)
$$

are commutative otherwise if

$$s_F \circ m_F \overset{\text{(TH)}}{=} s_F \otimes s_F \circ m_{F^*}.$$

and if

$$s_F \circ u_F \overset{(\mathrm{TH})}{=} u_{F^*}.$$

In light Lemma 5.9 $H^* \otimes H \cong \mathbf{FdHilb}(H, H)$ the diagram (5.81) become

$$\begin{array}{ccc} H^* \otimes H \otimes H^* \otimes H & \xrightarrow{m_{F(H,H)}=m_F} & H^* \otimes H \\ {\scriptstyle s_F \otimes s_F}\downarrow & & \downarrow{\scriptstyle s_F} \\ H \otimes H^* \otimes H \otimes H^* & \xrightarrow{m_{F(H,H)^*}=m_{F^*}} & H \otimes H^*. \end{array} \tag{5.78}$$

Where

$$m_F : H^* \otimes H \otimes H^* \otimes H \to H^* \otimes H$$

is defined by

$$\langle\psi| \otimes |w\rangle \otimes \langle\varphi| \otimes |v\rangle \mapsto \langle\psi| \otimes \langle\varphi|w\rangle|v\rangle \tag{5.79}$$

$$m_{F^*} : H \otimes H^* \otimes H \otimes H^* \to H \otimes H^*$$

is defined by

$$|\psi\rangle \otimes \langle w| \otimes |\varphi\rangle \otimes \langle v| \mapsto \langle\psi| \otimes \langle w|\varphi\rangle|v\rangle \tag{5.80}$$

and

$$s_F = s_{H^* \otimes H} : H^* \otimes H \to H \otimes H^*$$

by

$$\langle\psi| \otimes |w\rangle \mapsto |\psi\rangle \otimes \langle w|. \tag{5.81}$$

Now the diagrams (5.77) and (5.78) are straightforwardly commutative.

□

Proposition 5.11 Λ is injective in other words that if

$$\Lambda|\psi\rangle = \Lambda|\varphi\rangle$$

then

$$|\psi\rangle \overset{\text{(TH)}}{=} |\varphi\rangle. \tag{5.82}$$

Proof. We observe that

$$\mathbb{C} \xrightarrow{u=\varepsilon^{\dagger}} H \xrightarrow{R_{|\alpha\rangle}} H$$

$$1 \mapsto \sum_{i=1}^{n} |i\rangle \mapsto R_{|\alpha\rangle} \sum_{i=1}^{n} |i\rangle = \sum_{i=1}^{n} R_{|\alpha\rangle}|i\rangle = |\alpha\rangle \tag{5.83}$$

hence

$$R_{|\alpha\rangle} \circ u = \alpha. \tag{5.84}$$

If

$$\Lambda|\psi\rangle = \Lambda|\varphi\rangle \Rightarrow R_{|\psi\rangle} = R_{|\varphi\rangle} \tag{5.85}$$

$$\Downarrow$$

$$R_{|\psi\rangle} \circ u = R_{|\varphi\rangle} \circ u \tag{5.86}$$

$$\Downarrow \text{ by } (5.85)$$

$$|\psi\rangle = |\varphi\rangle. \tag{5.87}$$

□

Theorem 5.12. Λ is a monoid morphism in other words the following diagrams

$$\begin{array}{ccc} H \otimes H & \xrightarrow{m=\delta^{\dagger}} & H \\ {\scriptstyle \Lambda\otimes\Lambda}\downarrow & & \downarrow{\scriptstyle \Lambda} \\ F(H,H) \otimes F(H,H) & \xrightarrow{m_F} & F(H,H) \end{array} \tag{5.88}$$

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\quad u \quad} & H \\ & {\scriptstyle u_F}\searrow \quad \nearrow{\scriptstyle \Lambda} & \\ & F(H,H) & \end{array}.$$

$$(5.89)$$

are commutative.
Proof. By the first diagram we have

$$[m_F \circ (\Lambda \otimes \Lambda)]|i\rangle \otimes |j\rangle \overset{(\mathrm{TH})}{=} (\Lambda \circ m_F)|i\rangle \otimes |j\rangle. \tag{5.90}$$

The left hand side of (5.91) is

$$[m_F \circ (\Lambda \otimes \Lambda)]|i\rangle \otimes |j\rangle = R_{|i\rangle} \circ R_{|j\rangle} \tag{5.91}$$

$$\Downarrow$$

$$\begin{aligned} R_{|i\rangle} \circ R_{|j\rangle})|t\rangle &= R_{|i\rangle}(R_{|j\rangle})|t\rangle) \\ &= R_{|i\rangle}(\langle t|j\rangle|t\rangle) \\ &= \langle t|j\rangle R_{|i\rangle}|t\rangle \\ &= \langle t|j\rangle\langle t|i\rangle|i\rangle \end{aligned} \tag{5.92}$$

$$\langle t|j\rangle\langle t|i\rangle|i\rangle = \delta_{tj}\delta_{ti}|i\rangle. \tag{5.93}$$

The right hand side

$$(\Lambda \circ m_F)|i\rangle \otimes |j\rangle = \Lambda(\delta_{ij}|i\rangle) = R_{|i\rangle}\delta_{ij} \tag{5.94}$$

$$\Downarrow$$

$$\begin{aligned} R_{|i\rangle}\delta_{ij}|t\rangle &= \langle t|j\rangle\langle t|i\rangle|i\rangle \\ &= \delta_{ti}\delta_{ij}|t\rangle \\ &= \delta_{ti}\delta_{ij}|i\rangle \end{aligned} \tag{5.95}$$

then we have the thesis . By the second diagram we have

$$(\Lambda \circ u)(1_{\mathbb{C}}) \overset{(\mathrm{TH})}{=} 1_H \Rightarrow (\Lambda \circ u)(1_{\mathbb{C}})|i\rangle \overset{(\mathrm{TH})}{=} |i\rangle. \tag{5.96}$$

$$\begin{aligned} (\Lambda \circ u)(1_{\mathbb{C}})|i\rangle &= \Lambda\left(\sum_{j=1}^{n}|j\rangle\right) = \sum_{j=1}^{n} R_{|j\rangle}|i\rangle \\ &= \sum_{j=1}^{n}\langle j|i\rangle|i\rangle = |i\rangle. \end{aligned} \tag{5.97}$$

□

Theorem 5.13. Λ is a involution monoid morphism i.e. the following diagram

$$\begin{array}{ccc} H & \xrightarrow{\Lambda} & F(H,H) \\ {\scriptstyle s_H}\downarrow & & \downarrow{\scriptstyle s_F} \\ H^* & \xrightarrow{\Lambda_*} & [F(H,H)]^* \end{array} \tag{5.98}$$

is commutative.
Proof. We define

$$s_F : F(H,H) \to F(H,H)^*$$

by setting

$$|i,j\rangle \mapsto \langle i,j| \tag{5.99}$$

$|i,j\rangle \in M_n(\mathbb{C})$ is the matrix with 1 in the entry (i,j) and 0 in all others position hence $|i,j\rangle$ is a basis in F(H,H) .

$$(s_F \circ \Lambda)|i\rangle = s_F(R_{|i\rangle} = |i,i\rangle) = \langle i,i|. \tag{5.100}$$

Since

$$\Lambda : H \to F(H,H)$$
$$|i\rangle \mapsto |i,i\rangle \tag{5.101}$$

then

$$\Lambda^* : F^*(H,H) \to H^*$$
$$\langle i,j| \mapsto \delta_{ij}\langle i| \tag{5.102}$$
$$\Downarrow$$
$$(\Lambda^*)^\dagger : H^* \to F^*(H,H)$$
$$|t\rangle \mapsto \sum_{ij}^{n} \langle i,j|\langle i|t\rangle\delta_{ij} = \sum_{ij}^{n} \langle i,j|\delta_{it}\delta_{ij} = \langle t,t| \tag{5.103}$$

□

Theorem 5.15. Any †-Frobenius monoid in **FdHilb** is a C^*-algebra.

Proof. **FdHilb**(H,H) is the C^*-algebra of endomorphism on a Hilbert space, see appendix A . By Lemma 5.9 it is easy to show that **FdHilb**(H,H) is an †-Frobenius monoid.
By the embeddingΛ we know that

$$H \cong \mathbf{FdHilb}(\mathbb{C},H) \cong R_{[\mathbf{FdHilb}(\mathbb{C},H)]} \subseteq \mathbf{FdHilb}(H,H) \tag{5.104}$$

inherits algebra structure from $\mathbf{FdHilb}(H, H)$. Now, since any finite dimensional involution-closed subalgebra of a C^*-algebra is also a C^*-algebra it follows that any †-Frobenius monoid in **FdHilb** is a C^*-algebra, in particular, it can be given a C^*-algebra norm.

□

5.2 The spectral theorem for normal operators

The spectral theorem for normal operators, says that a normal operator on a complex Hilbert space can be diagonalized. For complex Hilbert spaces this follows from the spectral theorem for com- mutative C^*-algebras, since any normal operator generates a commutative C^*-algebra and the spectrum of this algebra performs the diagonalization. This will not necessarily be the case in an arbitrary monoidal †-category, with C^*-algebras replaced by special unitary †-Frobenius monoids. Jamie Vicary in [21] gives a direct categorical description of diagonalization. We proceed as done in [21] by introducing two different categorical properties which capture the geometrical essence of the spectral theorem for normal operators, and then showing that they are equivalent.

Definition 5.16. In a monoidal category, an endomorphism $f : X \to X$ is compatible with a monoid (A, m, u) if the following equations hold:

$$m \circ (f \otimes X) = f \circ m = m \circ (X \otimes f). \tag{5.105}$$

Definition 5.17. In a braided monoidal †-category, an endomorphism $f : X \to X$ is *internally diagonalizable* if it can be written as an action of an element of a commuta- tive †-Frobenius algebra on X; that is, if it can be written as

$$f = m \circ (\phi_f \otimes X), \tag{5.106}$$

where $m : X \otimes X \to X$ is the multiplication of a commutative †-Frobenius algebra on X and $\phi_f : \mathbf{1} \to X$ is a state of X.

Lemma 5.18.$f : X \to X$ is *internally diagonalizable* $\Leftrightarrow$

$$m \circ (f \otimes H) = f \circ m = m \circ (H \otimes f) \tag{5.107}$$

Proof.
($\Rightarrow$)

HP) $f = m \circ (\phi_f \otimes H)$

$$m \circ (f \otimes H) \overset{(\mathrm{TH})}{=} f \circ m \overset{(\mathrm{TH})}{=} m \circ (H \otimes f). \tag{5.108}$$

Proof.
Under the hyphotesis HP) we need to identify

$$\phi_f : \mathbb{C} \to H$$

$$\phi(1) = |\phi_f\rangle = ?.$$

$$\phi_f : \mathbb{C} \to H$$

$$\phi(1) = |\phi_f\rangle = \sum_k^n \varphi_k |k\rangle. \tag{5.109}$$

$$\phi \otimes H : \mathbb{C} \otimes H \cong H \to H \otimes H$$

$$1 \otimes |\psi\rangle \cong |\psi\rangle \mapsto \left(\sum_k^n \varphi_k |k\rangle \right) \otimes \left(\sum_{j=1}^n \psi_j |j\rangle \right) \tag{5.110}$$

$$m \left(\left(\sum_{k=1}^n \varphi_k |k\rangle \right) \otimes \left(\sum_{j=1}^n \psi_j |j\rangle \right) \right) = \sum_{i=1}^n \varphi_i \psi_i |i\rangle \tag{5.111}$$

now we impose the hypothesis

$$f|\psi\rangle = \sum_{i=1}^n \varphi_i \psi_i |i\rangle \tag{5.112}$$

hence

$$\sum_{i=1} \psi_i f|i\rangle - \sum_{i=1}^n \varphi_i \psi_i |i\rangle = 0 \tag{5.113}$$

$$\sum_{i=1} f|i\rangle - \sum_{i=1}^n \varphi_i |i\rangle = 0 \tag{5.114}$$

then

$$f \left(\sum_{i=1} |i\rangle \right) = \sum_{i=1}^n \varphi_i |i\rangle = |\phi_f\rangle. \tag{5.115}$$

Where $f|i\rangle = \varphi_i |i\rangle$, $\forall i = 1, ..., n$.

$$f \otimes H : H \otimes H \to H \otimes H$$

$$|i\rangle \otimes |j\rangle \mapsto \varphi_i |i\rangle \otimes |j\rangle \tag{5.116}$$

$$m(\varphi_i |i\rangle \otimes |j\rangle) = \varphi_i |i\rangle \tag{5.117}$$

$$H \otimes H \xrightarrow{m} H \xrightarrow{f} H$$
$$m(|i\rangle \otimes |j\rangle) = \delta_{ij} |i\rangle \tag{5.118}$$
$$f(\delta_{ij} |i\rangle) = \varphi |i\rangle \tag{5.119}$$

$$H \otimes f : H \otimes H \to H \otimes H$$
$$|i\rangle \otimes |j\rangle \mapsto |i\rangle \otimes \varphi_j |j\rangle \tag{5.120}$$

$$m(|i\rangle \otimes \varphi_j |j\rangle) = \delta_{ij} \varphi_j |j\rangle = \varphi |i\rangle . \tag{5.121}$$

($\Leftarrow$)
HP) $m \circ (f \otimes H) \overset{\text{(TH)}}{=} f \circ m \overset{\text{(TH)}}{=} m \circ (H \otimes f)$.
TH) $f = m \circ (\phi_f \otimes H)$

I choose $\phi_f = f \circ u$ with

$$u : \mathbb{C} \to H :: 1 \mapsto \sum_{i=1}^{n} |i\rangle$$

.

$$\phi_f : \mathbb{C} \to H \xrightarrow{f} H$$
$$1 \mapsto \sum_{i=1}^{n} |i\rangle \mapsto f\left(\sum_{i=1}^{n} |i\rangle\right) = |\phi_f\rangle \tag{5.122}$$

$$\begin{aligned} m \circ ((f \circ u) \otimes H) &= m \circ ((f \otimes H) \circ (u \otimes H)) \\ &= (m \circ (f \otimes H)) \circ (u \otimes H) \\ &= (f \circ m) \circ (u \otimes H) \\ &= f \circ (m \circ u \otimes H) \end{aligned} \tag{5.123}$$

$$\mathbb{C} \otimes H \cong H \xrightarrow{u \otimes H} H \otimes H \xrightarrow{m} H$$
$$1 \otimes |i\rangle \cong |i\rangle \mapsto \sum_{i=1}^{n} |i\rangle \otimes |i\rangle \mapsto |i\rangle \tag{5.124}$$

$$m \circ u \otimes H = 1_H \tag{5.125}$$

$$f \circ (m \circ u \otimes H) = f \tag{5.126}$$

□

Lemma 5.19. Let f be a morphism internally diagonalizable then

$$f \circ f^{\dagger} \overset{(\text{TH})}{=} f^{\dagger} \circ f. \tag{5.127}$$

Proof. If

$$f = m \circ (\phi_f \otimes H) \Rightarrow f^{\dagger} = (\phi_f \otimes H)^{\dagger} \circ m^{\dagger} = (\phi_f^{\dagger} \otimes H) \circ m^{\dagger} \tag{5.128}$$

$$= (\phi_f^{\dagger} \otimes H) \circ \delta \tag{5.129}$$

$$\phi_f^{\dagger} \otimes H : H \otimes H \to \mathbb{C} \otimes H \cong H$$

$$|i\rangle \otimes |j\rangle \mapsto \varphi_i |j\rangle \tag{5.130}$$

$$|i\rangle \otimes |i\rangle \mapsto \varphi_i |i\rangle \tag{5.131}$$

$$f^{\dagger} : H \xrightarrow{\delta} H \otimes H \xrightarrow{\phi_f^{\dagger} \otimes H} H$$

$$|i\rangle \mapsto |i\rangle \otimes |i\rangle \mapsto \varphi_i |i\rangle \tag{5.132}$$

hence

$$f^{\dagger} \circ f = f \circ f^{\dagger} : H \to H$$

$$|i\rangle \mapsto \varphi_i^2 |i\rangle \tag{5.133}$$

□

Theorem 5.19. In **FdHilb** any morphism $f : H \to H$ such that

$$f^{\dagger} \circ f = f \circ f^{\dagger}$$

$$f \overset{(\text{TH})}{=} m \circ (\phi_f \otimes H). \tag{5.134}$$

Proof. As done in [21] we choose a set $\{a_i \in Hom(\mathbb{C}, H), i = 1, ..., n\}$ such that

$$f|a_i\rangle = \lambda_i |a_i\rangle \tag{5.135}$$

and

$$a_i^{\dagger} \circ a_i = \delta_{ij} 1_H. \tag{5.136}$$

This basis set is uniquely determined if and only if f is nondegenerate. We use $\{a_i \in Hom(\mathbb{C}, H), i = 1, ..., n\}$ to construct a monoid(H, m, u) on H as follows

$$m := \sum_{i=1}^{n} a_i \circ (a_i^\dagger \otimes a_i^\dagger) \tag{5.137}$$

$$u := \sum_{i=1}^{n} a_i. \tag{5.138}$$

$$a_i : \mathbb{C} \to H :: 1 \mapsto |a_i\rangle$$

$$\Downarrow$$

$$a_i^\dagger : H \to \mathbb{C} \tag{5.139}$$

such that

$$\langle x|a_i z\rangle = a_i^\dagger|x\rangle \cdot z \tag{5.140}$$

$$|x\rangle = \sum_{i=1}^{n} x_i|i\rangle \Rightarrow \langle x| = \sum_{i=1}^{n} x_i^*\langle i| \tag{5.141}$$

$$\langle x|a_i z\rangle = \sum_{j=1} x_j^* z\langle a_j|a_i\rangle = x_i^* z = a_i^\dagger|x\rangle \cdot z \tag{5.142}$$

$$\Downarrow$$

$$a_i^\dagger|x\rangle = x_i^* \Rightarrow a_i^\dagger|a_i\rangle = \delta_{ij}1_H \tag{5.143}$$

hence

$$(a_i^\dagger \otimes a_i^\dagger)(|a_k\rangle \otimes |a_j\rangle) = \begin{cases} 1 & \text{if } i = j, i = k \\ 0 & \text{in the others cases }. \end{cases} \tag{5.144}$$

It is straightforward to show that this monoid is in fact a †- Frobenius monoid, which copies the chosen basis. Now we will proof the compatibility :

$$m \circ (f \otimes X) \overset{\text{(TH)}}{=} f \circ m \overset{\text{(TH)}}{=} m \circ (X \otimes f). \tag{5.145}$$

$$f \otimes H : H \otimes H \to H \otimes H$$

$$|a_i\rangle \otimes |a_j\rangle \mapsto \lambda_i|a_i\rangle \otimes |a_j\rangle \tag{5.146}$$

$$\Downarrow$$

$$m(\lambda_i|a_i\rangle \otimes |a_j\rangle) = \lambda_i|a_i\rangle \tag{5.147}$$

$$(f \circ m)(|a_i\rangle \otimes |a_j\rangle) = \lambda_i|a_i\rangle \tag{5.148}$$

$$(m \circ f \otimes H)(|a_i\rangle \otimes |a_j\rangle) = m(\lambda_i|a_i\rangle \otimes |a_j\rangle) = \lambda_i|a_i\rangle. \tag{5.149}$$

By lemma 5.18 f is internally diagonalizable.

□

5.3 Conclusion

Statement of the main results
Defining two linear maps by settings

$$\begin{aligned} \delta : H &\to H \otimes H \\ |i\rangle &\mapsto |i\rangle \otimes |i\rangle \end{aligned} \tag{5.150}$$

$$\begin{aligned} \varepsilon : H &\to \mathbb{C} \\ |i\rangle &\mapsto 1 \end{aligned} \tag{5.151}$$

and taking $H \in \mathbf{FdHilb}$ (H, δ, ε) is a commutative †-Frobenius monoid. In proposition 5.4 we have saw that solving

$$\delta(|\psi\rangle) = |\psi\rangle \otimes |\psi\rangle \tag{5.152}$$

it is possible recover the basis in H. A very important property of †-Frobenius monoid is that we can map any element $\alpha \in H$ into the algebra of operators on H by defining a right action

$$R_{|\alpha\rangle} = m \circ (H \otimes \alpha). \tag{5.153}$$

In proposition 5.7 we have seen that the adjoint of $R_{|\alpha\rangle}$ for some α is

$$R^{\dagger}_{|\alpha\rangle} = R_{|\alpha'\rangle} \text{ for } \alpha' = (H \otimes \alpha^{\dagger}) \circ m^{\dagger} \circ u. \tag{5.154}$$

We have seen also the proposition 5.8 which that the map

$$(-)' : Hom_{\mathbf{FdHilb}}(\mathbb{C}, H) :: \alpha \mapsto \alpha' \tag{5.155}$$

is an involution, $(\alpha')' = \alpha$.
In light of these results it is possible to define a powerful involution preserving monoid embedding

$$\begin{aligned} \Lambda : \mathbf{FdHilb}(\mathbb{C}, H) &\hookrightarrow \mathbf{FdHilb}(H, H). \\ \alpha &\mapsto R_{|\alpha\rangle}. \end{aligned} \tag{5.156}$$

Since that we can say :

*Any †-Frobenius monoid in **FdHilb** is in bijective correspondence to a orthogonal basis set for H and viceversa. Every commutative †-Frobenius monoid in **FdHilb** in is embedded in a C^* algebra hence is a C^* algebra.*

Appendix A

Some results in functional analysis

Proposition Let X be a normed space and let Y be a Banach space. Let B(X,Y) the bounded operator vector space between X and Y. then B(X,Y) is a Banach space.
Proof. Let T_n be a Cauchy sequence bounded operators. This implies that

$$\| T_n x - T_m x \| \leq \| T_n - T_m \| \| x \|$$

but T_n is Cauchy sequence hence for$n, m \geq N_0 \in \mathbb{N}$

$$\exists \ \tilde{\varepsilon} > \varepsilon$$

such that

$$\| T_n x - T_m x \| \leq \| T_n - T_m \| \| x \| \leq \varepsilon \| x \| \leq \tilde{\varepsilon}$$

In other words the sequence $T_n x \in Y$ is Cauchy sequence in a Banach space then admits a limit in Y :

$$Tx = \lim T_n x.$$

We see that :

$$\| Tx - T_m x \| = \| \lim T_n x - T_m x \| = \lim \| T_n x - T_m x \| \leq \varepsilon \| x \|$$

but

$$\| Tx \| \leq \| Tx - T_m x \| + \| T_m x \| \leq (\varepsilon + \| T_m \|) \| x \|$$

i.e. T is bounded. Now we needs to proof that T is the the limit for T_n squence. The following is hold

$$\| T_n x - T_m x \| \leq \varepsilon \| x \| \quad \forall x$$

then

$$\| T_n x - Tx \| \leq \varepsilon \| x \| \quad \forall x$$

$$\Downarrow$$

$$\| T_n - T \| \leq \varepsilon$$

□

Proposition$B(X,Y)$ is a C^*-algebra.
Proof.

$$\| F^* F \| \leq \| F^* \| \| F \| = \| F \|^2 .$$

On other hand

$$\| F \|^2 \sup_{\|x\| \leq 1} \| Ax \|^2 = \sup_{\|x\| \leq 1} \langle Fx \| Fx \rangle = \sup_{\|x\| \leq 1} \langle x | F^* Fx \rangle$$

but

$$\sup_{\|x\| \leq 1} \langle x | F^* Fx \rangle \leq \sup_{\|x\| \leq 1} \| x \| \| F^* Fx \| \leq \sup_{\|x\| \leq 1} \| F^* Fx \| = \| F^* F \|$$

$$\Downarrow$$

$$\| F \|^2 \leq \| F^* F \|,$$

both the inequalities are hold then we have :

$$\| F \|^2 = \| F^* F \| .$$

□

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