Alma Mater Studiorum · Università di Bologna

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI

Corso di Laurea in Matematica

HILBERT TRANSFORM

Tesi di Laurea in Finanza Matematica

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Seconda Sessione
Anno Accademico 2013/2014

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Introduction

The Hilbert transform is a linear operator defined as follows:

Definition 0.1. The Hilbert transform of a function $f \in L^p(\mathbb{R}),\ 1 \leq p < \infty$ is:

$$\mathcal{H}(f)(x) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f(y)}{x-y} dy$$

The integral is an extension of the Riemann definition of integral, called the Cauchy principal value.
As we will see in Chapter 3, the Hilbert transform arises from the study of the Fourier transform. In particular David Hilbert first noticed that the transform relates the image of the real line of a harmonic conjugate pair of functions. This tool was first introduced by Hilbert to solve a special case of the Riemann-Hilbert problem for holomorphic functions in 1905. Only in 1928 Marcel Riesz proved that the Hilbert transform is well-defined for functions in $L^p(\mathbb{R}),\ 1 \leq p < \infty$.
Aside this, the other main field of application of this tool is signal processing. As the complex notation of the harmonic wave form in electrical engineering is

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$$

the Hilbert transform is the $\frac{\pi}{2}$ phase-shift operator that describes, along with the original function, the so called strong analytic signal. The following example shows this.

Example 0.1.
$$\mathcal{H}\{\sin(x)\} = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\sin(y)}{x-y} dy$$
Applying the change of variables $y = x + t$
$$= -\frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\sin(x+t)}{t} dt = -\frac{\sin(x)}{\pi} PV \int_{-\infty}^{\infty} \frac{\cos(t)}{t} dt - \frac{\cos(x)}{\pi} PV \int_{-\infty}^{\infty} \frac{\sin(t)}{t} dt$$
from the odd property of $\frac{\cos(t)}{t}$ it follows
$$PV \int_{-\infty}^{\infty} \frac{\cos(t)}{t} dt = 0$$
Hence
$$\mathcal{H}\{\sin(x)\} = -\frac{\cos(x)}{\pi} PV \int_{-\infty}^{\infty} \frac{\sin(t)}{t} dt = -\frac{\cos(x)}{\pi} \int_{-\infty}^{\infty} \frac{\sin(t)}{t} dt$$
This is one of the Dirichlet integrals, therefore as the integral value is π it becomes
$$\mathcal{H}\{\sin(x)\} = -\cos(x)$$

Only lately the Hilbert tranform has been used in finance. As a matter of fact in 2008 Feng and Linetsky proposed that the fast Hilbert transform method could be used to describe the price of barrier and Bermudan style options. This method was actually proposed as an improvement to the fast Fourier method. In the latter method we consider pricing a discrete path dependent options. It is possible that the form of the density function may not be readily available. Hence the characteristic function of the process, that is the Fourier transform of the density function, generally admits analytical closed form representation provided the process is a Lévy process. Therefore after the Fourier time-stepping integration across successive monitoring instants, one has to perform Fourier inversion back to the price of the options. This permits to check if the options should be exercised or not. Hence the fast Hilbert transform could solve this computational inconvenience. Indeed in Section 1.6 we will prove that:
$$\mathcal{F}(\text{sgn } \cdot f)(\xi) = -2i\mathcal{H}\hat{f}(\xi)$$

As a consequence of this fact, one can prove a similar equation for the indicator function ([3], Kwok and Zeng, 2013). This help us because multiplying a function by the indicator function is associated with the barrier feature. Thus we can compute a sequence of Hilbert transforms (instead of the Fourier inversion) at all discrete monitoring instants to check for the knock-out or the exercise condition of the options and we have to apply the Fourier inversion only at the last step to obtain the option price.

Chapter 1

Some properties of the Hilbert transform

We used in the definition of Hilbert transform the principal value integral, that is defined as follows.

Definition 1.1 (Cauchy principal value)**.** Consider a function f that has a singularity in the interval over which the integral is evaluated. The Cauchy principal value is hence defined as

$$PV \int_a^b f(x)dx = \lim_{\epsilon \to 0} \left[\int_a^{t-\epsilon} f(x)dx + \int_{t+\epsilon}^b f(x)dx \right]$$

where $f(x)$ has a singularity at $x = t$.

Next theorem will be proved in Chapter 2:

Theorem 1.0.1 (Riesz inequality)**.** *For* $1 < p < \infty$

$$\int_{-\infty}^{\infty} |\mathcal{H}f(x)|^p dx \leq \{\mathfrak{R}_p\}^p \int_{-\infty}^{\infty} |f(x)|^p dx$$

where $\mathfrak{R}_p$ *is a constat depending only on* p*.*

.

Moreover this inequality proves that the Hilbert transform of an L^p function is still in L^p for $1 < p < \infty$. Even though this does not hold for $p = 1$, it will be proved in the same chapter that the Hilbert transform of an L^1 function is anyway well defined almost everywhere.
Hence from now on we will consider the Hilbert transform of L^p functions with $1 \leq p < \infty$.

1.1 Inversion property

The following theorem will be proved in Chapter 3.

Theorem 1.1.1 (Hilbert Inversion Theorem). *Given $f \in L^p(\mathbb{R})$, $1 < p < \infty$:*

$$\mathcal{H}(f)(x) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f(y)}{x-y} dy$$

$$f(x) = -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{\mathcal{H}(f)(y)}{x-y} dy$$

It is not possible to consider the case $p = 1$ as in general the Hilbert transform of $f \in L^1(\mathbb{R})$ is not integrable.

Hence the following result arises from the application of Theorem 1.1.1

$$\mathcal{H}^2 f(x) = \mathcal{H}(\mathcal{H}f)(x) = -f(x), \ a.e.$$

This is true given the assumption of $f \in L^p$ for $p > 1$ as the Riesz inequality insures that we can apply the second Hilbert transform.

1.2 Linear scale changes

Let us consider $g(x) = \mathcal{H}f(x)$, then:

- If $a > 0$

$$\mathcal{H}f(ax) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f(at)}{x-t} dt$$

with the change of variables $s = at$

$$\mathcal{H}f(ax) = \frac{1}{a\pi} PV \int_{\mathbb{R}} \frac{f(s)}{x - s/a} ds = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f(s)}{ax - s} ds = g(ax)$$

- If $a > 0$

$$\mathcal{H}f(-ax) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(-at)}{x - t} dt$$

with the change of variables $s = -at$

$$\mathcal{H}f(-ax) = -\frac{1}{a\pi} PV \int_{\infty}^{-\infty} \frac{f(s)}{x + s/a} ds = -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f(s)}{-ax - s} ds = -g(-ax)$$

- If $a, b \in \mathbb{R}$

$$\mathcal{H}f(ax + b) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(at + b)}{x - t} dt$$

with the change of variables $s = at + b$

$$\mathcal{H}f(ax + b) = \begin{cases} \frac{1}{a\pi} PV \int_{-\infty}^{\infty} \frac{f(s)}{x - (s-b)/a} ds & \text{if } a > 0 \\ -\frac{1}{a\pi} PV \int_{-\infty}^{\infty} \frac{f(s)}{x - (s-b)/a} ds & \text{if } a < 0 \end{cases}$$

$$= \pm \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(s)}{ax + b - s} ds = \operatorname{sgn}(a) g(ax + b)$$

1.3 Translation, dilation and reflection

As usual we set $g = \mathcal{H}f$. What follows comes directly from the results above.

- The translation operator is defined as

$$\tau_a f(x) = f(x - a), \quad a \in \mathbb{R}$$

Thus

$$\mathcal{H}(\tau_a f)(x) = \mathcal{H}(f)(x - a) = g(x - a) = \tau_a \mathcal{H}f(x)$$

Hence the Hilbert transform is a translation-invariant operator.

- The dilation operator is defined as

$$S_a f(x) = f(ax), \quad a > 0$$

  Thus

$$\mathcal{H}(S_a f)(x) = \mathcal{H}(f)(ax) = g(ax) = S_a \mathcal{H} f(x)$$

  Hence also the dilation operator commutes with the transform.

- The reflection operator is defined as

$$Rf(x) = f(-x)$$

  Thus

$$\mathcal{H}(Rf)(x) = \mathcal{H}(f)(-x) = -g(-x) = -R\mathcal{H} f(x)$$

  This operator anti-commutes with the Hilbert transform.

1.4 Derivatives

The Hilbert transform commutes with the differential operator. To prove this we first need the following theorem that is a consequence of the dominated convergence theorem.

Theorem 1.4.1. *Given $f : A \times I \longrightarrow \mathbb{R}$, with $A \subset \mathbb{R}^N$ measurable and $I = (a, b) \subset \mathbb{R}$ real interval. Suppose then*

(i) $f(t, x)$ is integrable as a function of t in A for every $x \in I$;

(ii) $f(t, x)$ is differentiable with respect to x in I for almost every $t \in A$;

(iii) exists g integrable in A such that

$$\left| \frac{\partial f}{\partial x}(t, x) \right| \leq g(t)$$

for almost every $t \in A$, for every $x \in I$.

Hence the function

$$\phi : I \longrightarrow \mathbb{R}, \quad \phi(x) := \int_A f(t,x)dt$$

is differentiable and

$$\phi'(x) = \int_A \frac{\partial f}{\partial x}(t,x)dt$$

for every $x \in I$*.*

Now it is possible to prove that

Theorem 1.4.2. *If* $f \in L^p,\ p > 1$ *and it is differentiable with* $f' \in L^q,\ q \geq 1$*, then*

$$\mathcal{H}\left\{\frac{df(x)}{dx}\right\} = \frac{d}{dx}\mathcal{H}f(x)$$

Proof. We start from Definition 0.1

$$\mathcal{H}(f)(x) = \frac{1}{\pi}PV\int_{\mathbb{R}} \frac{f(y)}{x-y}dy$$

If we subsitute y with $x - t$

$$\mathcal{H}(f)(x) = \frac{1}{\pi}PV\int_{\mathbb{R}} \frac{f(x-t)}{t}dt$$

Thus we apply the derivative of x on both sides

$$\frac{d}{dx}\mathcal{H}(f)(x) = \frac{d}{dx}\left(\frac{1}{\pi}PV\int_{\mathbb{R}} \frac{f(x-t)}{t}dt\right)$$

Now we check that it is possible to apply Theorem 1.4.1 to the right-hand side of the equation.

(i) $\frac{f(x-t)}{t}$ is integrable as a function of t in $\mathbb{R}$ for every $x \in \mathbb{R}$, in fact

$$PV\int_{\mathbb{R}} \frac{f(x-t)}{t}dt = PV\int_{\mathbb{R}} \frac{f(y)}{x-y}dy = \mathcal{H}(f)(x)$$

that is well defined for every $x \in \mathbb{R}$ as a consequence of the Riesz inequality.

(ii) $\frac{f(x-t)}{t}$ is differentiable with respect to the variable x as for almost every t as it is an hypothesis in the theorem.

(iii) As we chose $f' \in L^q,\ q \geq 1$, we can do the same as in (i)

$$PV \int_{\mathbb{R}} \left| \frac{f'(x-t)}{t} \right| dt = PV \int_{\mathbb{R}} \left| \frac{f'(y)}{x-y} \right| dy = \mathcal{H}(f')(x)$$

that is well defined for every $x \in \mathbb{R}$, hence it makes $\left|\frac{f'(x-t)}{t}\right|$ integrable itself.

Now applying Theorem 1.4.1

$$\begin{aligned} \frac{d}{dx}\mathcal{H}(f)(x) &= \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{d}{dx}\left(\frac{f(x-t)}{t} \right) dt \\ &= \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f'(x-t)}{t} dt \end{aligned}$$

In conclusion, applying the proper substitution, we get

$$\mathcal{H}\left\{ \frac{df(x)}{dx} \right\} = \frac{d}{dx}\mathcal{H}f(x)$$

□

The generalization for the n-derivatives is straightforward:

$$\mathcal{H}\left\{ \frac{d^n f(x)}{dx^n} \right\} = \frac{d^n}{dx^n}\mathcal{H}f(x)$$

with the same assumptions on the higher derivatives.

1.5 Convolution property

The convolution property can be stated as follows

Theorem 1.5.1. *Given $f \in L^p$ and $h \in L^q$*

$$\mathcal{H}\{f * h\}(x) = \{\mathcal{H}f * h\}(x) = \{f * \mathcal{H}h\}(x) \tag{1.1}$$

Proof. Thanks to Young's Inequality, the convolution of an L^p function with an L^q function is an L^r function with $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$.
Therefore from the left-hand side of Eq.(1.1)

$$\mathcal{H}\{f*h\}(x)=\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{1}{x-s}\int_{-\infty}^{\infty}f(u)h(s-u)duds$$

From the right-hand side we get

$$\{\mathcal{H}f*h\}(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(PV\int_{-\infty}^{\infty}\frac{f(s)}{u-s}ds\right)h(x-u)du$$

Applying the change of variable $s'=s-u+x,\ u'=u$, it follows that the Jacobian of the matrix of the change of variables has value 1. Hence it becomes

$$\{\mathcal{H}f*h\}(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}PV\int_{-\infty}^{\infty}\frac{f(u'-x+s')}{x-s'}h(x-u')ds'\ du'$$

then changing the order of integration[1]:

$$=\frac{1}{\pi}PV\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(u'-x+s')}{x-s'}h(x-u')du'\ ds'$$

using the change of variables $\mu=u'-x+s',\ \nu=s'$ that has Jacobian 1

$$\{\mathcal{H}f*h\}(x)=\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{1}{x-\nu}\int_{-\infty}^{\infty}f(\mu)h(\nu-\mu)d\mu\ d\nu=\mathcal{H}\{f*h\}(x)$$

In a similar way we get the equality with $\{f*\mathcal{H}h\}(x)$. □

1.6 Hilbert transform of a Fourier transform

The following statement can be proved and it relates Hilbert and Fourier transforms. Given $f\in L^p(\mathbb{R})$

$$\mathcal{F}(\text{sgn }\cdot f)(\xi)=-2i\mathcal{H}\hat{f}(\xi) \tag{1.2}$$

[1]Changing the order of integration is not obvious as one of the integral is a principal value integral. Anyway it can be proved using the dominated convergence and Fubini's theorems that this holds.

where

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$$

is the Fourier transform of f. One can prove that the Fourier transform is well-defined for functions in $L^p(\mathbb{R})$ for $1 \leq p \leq 2$, although it is not in general well-defined for other values of p. Hence we consider the definition of Fourier transform in generalized sense (tempered distributions).
To prove Eq. (1.2) we need few results.

Theorem 1.6.1 (Fourier Convolution theorem)**.** *Given* $f \in L^p,\ g \in L^{p'}$*:*

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}$$

Proof. As in Theorem 1.5.1 the convolution is well defined given the condition on p and p'. Now we apply the definition of convolution, and then the definition of Fourier transform to the left-hand side to get

$$\mathcal{F}\{f(x) * g(x)\} = \mathcal{F}\left\{ \int_{-\infty}^{\infty} f(t)g(x-t)dt \right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(x-t)dt\ e^{-isx}dx$$

Hence we subsitute a new variable u for $x - t$, while we do not make any change on the variable t. Hence the Jacobian of the matrix of the change of variables is 1 and we can write:

$$\mathcal{F}\{f(x) * g(x)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(u)e^{-is(u+t)}dt\ du =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-ist}g(u)e^{-isu}dt\ du = \int_{-\infty}^{\infty} f(t)e^{-ist}dt \int_{-\infty}^{\infty} g(u)e^{-isu}du =$$

$$= \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\}$$

renaming t and u as x. □

Proposition 1.6.2. *Consider* f, g *as before, then*

$$\mathcal{F}\{f \cdot g\} = \mathcal{F}\{f\} * \mathcal{F}\{g\}$$

Proof. We first define the following operator as the inverse Fourier transform:

$$\mathcal{F}^{-1}\{g\}(\xi) = \int_{\mathbb{R}} g(x)e^{ix\xi}dx$$

It follows that

$$\mathcal{F}^{-1}\{g\}(\xi) = \mathcal{F}\{g\}(-\xi) \tag{1.3}$$

Hence from Theorem 1.6.1

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}$$

Now we write

$$F = \mathcal{F}f,\ G = \mathcal{F}g \implies f = \mathcal{F}^{-1}(F),\ g = \mathcal{F}^{-1}(G)$$

Therefore the convolution theorem equation can be re-written as

$$F \cdot G = \mathcal{F}\{\mathcal{F}^{-1}(F) * \mathcal{F}^{-1}(G)\}$$

Finally if we apply the inverse operator on both sides we get

$$\mathcal{F}^{-1}\{F \cdot G\} = \mathcal{F}^{-1}(F) * \mathcal{F}^{-1}(G)$$

Using Eq.(1.3)

$$\mathcal{F}^{-1}\{F{\cdot}G\}(\xi) = \{\mathcal{F}^{-1}(F){*}\mathcal{F}^{-1}(G)\}(\xi) \implies \mathcal{F}\{F{\cdot}G\}(-\xi) = \{\mathcal{F}(F){*}\mathcal{F}(G)\}(-\xi)$$

That actually proves the initial statement. $\square$

Observation 1. We also need the following fact

$$\mathcal{F}\{\mathrm{sgn}\}(\xi) = \frac{2}{i\xi}$$

Proof. Consider the following function:

$$f_\alpha(t) = \begin{cases} e^{-\alpha t} & t > 0 \\ -e^{\alpha t} & t < 0 \end{cases}$$

for $\alpha > 0$. Hence

$$\lim_{\alpha \to 0} f_\alpha(t) = \text{sgn } (t)$$

Computing the Fourier transform of f_α gives

$$\mathcal{F}\{f_\alpha\}(\xi) = \int_{-\infty}^{\infty} f_\alpha(t) e^{-it\xi} dt =$$

$$= \int_{0}^{\infty} e^{-t(i\xi+\alpha)} dt - \int_{-\infty}^{0} e^{-t(i\xi-\alpha)} dt =$$

$$\left[-\frac{1}{i\xi+\alpha} e^{-t(i\xi+\alpha)} \right]_{t=0}^{\to\infty} - \left[-\frac{1}{i\xi-\alpha} e^{-t(i\xi-\alpha)} \right]_{\to-\infty}^{t=0} =$$

$$= \left(\frac{1}{i\xi+\alpha} + \frac{1}{i\xi-\alpha} \right) = -\frac{2i\xi}{\xi^2+\alpha^2}$$

As $\alpha \longrightarrow 0$

$$\mathcal{F}\{f_\alpha\}(\xi) = -\frac{2i\xi}{\alpha^2+\xi^2} \longrightarrow -\frac{2i}{\xi} = \frac{2}{i\xi}$$

Thus it is possible to prove that[2]

$$\lim_{\alpha \to 0} \mathcal{F}\{f_\alpha\}(\xi) = \mathcal{F}\{\lim_{\alpha \to 0} f_\alpha\}(\xi)$$

□

Now the proof of Eq.(1.1) follows directly as we get

$$\mathcal{F}(\text{sgn } \cdot f)(\xi) = (\mathcal{F}\{\text{sgn}\} * \mathcal{F}\{f\})(\xi)$$

from Proposition 1.6.2, hence

$$\mathcal{F}(\text{sgn } \cdot f)(\xi) = PV \int_{\mathbb{R}} \frac{2\hat{f}(t)}{i(\xi - t)} dt$$

from the Observation. In conclusion, Eq.(1.2) arises from the definition of Hilbert transform:

$$\mathcal{F}(\text{sgn } \cdot f)(\xi) = -2i \; PV \int_{\mathbb{R}} \frac{\hat{f}(t)}{\xi - t} dt = -2i\mathcal{H}\hat{f}(\xi)$$

[2]The proof of this concerns some properties of the tempered distributions, that is not proved here.

Chapter 2

Domain of the Hilbert transform

It is important to verify for which set of functions the Hilbert transform is well defined. It is possible to prove that the Hilbert transform is well defined on $L^p(\mathbb{R})$ for $p \geq 1$.
Let us recall Theorem 1.0.1.

Theorem 2.0.3 (Riesz's Inequality)**.** *Given* $f \in L^p(\mathbb{R}),\ 1 < p < \infty$

$$\int_{-\infty}^{\infty} |\mathcal{H}f(x)|^p dx \leq \{\mathfrak{K}_p\}^p \int_{-\infty}^{\infty} |f(x)|^p dx$$

where $\mathfrak{K}_p$ *is a constat depending only on* p*.*
Moreover this inequality proves that the Hilbert transform of an L^p *function is still in* L^p*.*

Proof. The proof is divided into two parts. For $1 < p \leq 2$ we are going to consider $f \in L^p(\mathbb{R})$. Without loss of generality it can be considered a.e. positive as in the inequality the function f is with absolute value. Furthermore observing that

$$\Phi(z) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{f(t)}{z-t} dt \qquad (\Im(z) > 0)$$

comes from an application to the $\Phi(z)$ in Theorem 3.0.9 (written as integral of $a(t)$ and $b(t)$) of Parseval's relation

$$\int_{-\infty}^{\infty} \mathcal{F}\{f(-t)\}\mathcal{F}\{h(t)\}dt = \int_{-\infty}^{\infty} f(t)h(t)dt$$

it shows that f is the limit as $y \longrightarrow 0$ of the real part of Φ, that is holomorphic in the upper-half of the complex plane. Let $\Phi(z) = u(x,y) + iv(x,y)$, thus the following inequality holds:

$$|v|^p \leq A_p u^p - B_p \Re(w^p) \ , \qquad A_p, B_p > 0$$

To prove this fact, let us write $w = Re^{it}$, then the inequality yields to

$$|\sin\theta|^p \leq A_p \cos^p\theta - B_p \cos p\theta$$

For $-\pi/2 \leq \theta \leq \pi/2$ and $1 < p \leq 2$ can be directly checked that the inequality holds (for appropriate constants) as at least one of the two terms in the right hand-side is positive. These values of θ correspond to a positive u, matching what we supposed just above.
Furthermore $w(z)$ is analytic for $y > 0$ and $w(z) \sim \frac{1}{z}$ as $z \longrightarrow \infty$. Hence

$$|v(x,y)|^p \leq A_p u(x,y)^p - B_p \Re(w(x+iy)^p)$$

and integrating over the real axis

$$\int_{-\infty}^{\infty} |v(x,y)|^p dx \leq A_p \int_{-\infty}^{\infty} u(x,y)^p dx - B_p \int_{-\infty}^{\infty} \Re(w(x+iy)^p)dx \qquad (2.1)$$

Now observe that the integral over the semicircular contour with center at iy and diameter parallel to the x-axis of $w(x+iy)$ is null as it is holomorphic on the upper-half plane. Hence:

$$\int_{-\infty}^{\infty} w(x+iy)^p dx = 0 \Longleftrightarrow \int_{-\infty}^{\infty} \Re(w(x+iy)^p) + i\Im(w(x+iy)^p)dx = 0$$

$$\Longleftrightarrow \int_{-\infty}^{\infty} \Re(w(x+iy)^p)dx = 0 \ \text{ and } \int_{-\infty}^{\infty} \Im(w(x+iy)^p)dx = 0$$

Therefore Eq.2.1 simplifies to

$$\int_{-\infty}^{\infty} |v(x,y)|^p dx \leq A_p \int_{-\infty}^{\infty} u(x,y)^p dx$$

As we will prove in Chapter 3

$$\lim_{y \to 0^+} u(x,y) = f(x) \ a.e.$$

and

$$\lim_{y \to 0^+} v(x,y) = \mathcal{H}f(x) \ a.e.$$

Thus this proves the case for $1 < p \leq 2$ as the limit can pass into the integral as a consequence of the dominated convergence theorem.
For $p > 2$ we have to first prove the following result.

Proposition 2.0.4. *If $f \in L^p(a,b),\ 1 \leq p < \infty$, then*

$$\left\{ \int_a^b |f(x)|^p dx \right\}^{p^{-1}} = \sup_g \left| \int_a^b f(x)g(x)dx \right|$$

where all the g are like

$$\left\{ \int_a^b |g(x)|^q dx \right\}^{q^{-1}} \leq 1$$

and p, q are conjugate exponents.

Proof. Let us apply Holder's inequality to the integral of the product of f and g

$$\left| \int_a^b f(x)g(x)dx \right| \leq \left\{ \int_a^b |f(x)|^p dx \right\}^{p^{-1}} \left\{ \int_a^b |g(x)|^q dx \right\}^{q^{-1}} \leq \left\{ \int_a^b |f(x)|^p dx \right\}^{p^{-1}}$$

Choosing

$$g_0(x) = \frac{|f(x)|^{p-1} \mathrm{sgn}\ f(x)}{\left\{ \int_a^b |f(x)|^p dx \right\}^{(p-1)/p}}$$

It is straightforward to show that g_0 has q-norm equals to 1 and that

$$\left| \int_a^b f(x) g_0(x) dx \right| = \left\{ \int_a^b |f(x)|^p dx \right\}^{p^{-1}}$$

and hence is the sup. $\square$

We accept without proof the following result

$$\left| \int_{-\infty}^{\infty} \mathcal{H}f(x) g(x) dx \right| = \left| \int_{-\infty}^{\infty} f(x) \mathcal{H}g(x) dx \right|$$

Now using Holder's inequality

$$\left| \int_{-\infty}^{\infty} f(x) \mathcal{H}g(x) dx \right| \leq \left\{ \int_{-\infty}^{\infty} |f(x)|^q dx \right\}^{q^{-1}} \left\{ \int_{-\infty}^{\infty} |\mathcal{H}g(x)|^p dx \right\}^{p^{-1}} \leq$$

$$\leq A_p \left\{ \int_{-\infty}^{\infty} |f(x)|^q dx \right\}^{q^{-1}} \left\{ \int_{-\infty}^{\infty} |g(x)|^p dx \right\}^{p^{-1}} \leq A_p \left\{ \int_{-\infty}^{\infty} |f(x)|^q dx \right\}^{q^{-1}}$$

Observe that we are allowed to use Holder's inequality as $1 < p \leq 2$ and Riesz inequality is already proved for this case (hence $\mathcal{H}g$ is still in L^p).
Finally this prove what we wanted:

$$\left\{ \int_{-\infty}^{\infty} |\mathcal{H}f(x)|^q dx \right\}^{q^{-1}} = \sup_g \left| \int_{-\infty}^{\infty} \mathcal{H}f(x) g(x) dx \right| = \sup_g \left| \int_{-\infty}^{\infty} f(x) \mathcal{H}g(x) dx \right| \leq$$

$$\leq A_p \left\{ \int_{-\infty}^{\infty} |f(x)|^q dx \right\}^{q^{-1}}$$

Also

$$\frac{1}{p} + \frac{1}{q} = 1 \iff q = \frac{p}{p-1} \geq 2 \quad , \quad 1 < p \leq 2$$

that actually confirms the proof for $q > 2$. $\square$

L^1 **case** In general the Hilbert transform of an L^1 function is not L^1, but it is anyway defined a.e.. To prove this, we need the following two theorems from Titchmarsh (1948) (proofs omitted):

Theorem 2.0.5. *Let $f(x) \in L^1(0,1)$ and $x^{-1}f(x) \in L^1(1,\infty)$. Let $v(x,y)$ be as before, then*

$$\lim_{y\to 0}\left\{ v(x,y) + \frac{1}{\pi}\int_y^\infty \frac{f(x+t)-f(x-t)}{t} \right\} = 0$$

for almost all values of x.

Theorem 2.0.6. *If $\Psi(z)$ is regular and bounded for $y>0$ then $\Psi(z)$ tends to a finite limit as $y \longrightarrow 0$ for almost all x.*

Thus

Theorem 2.0.7. *If $f \in L^1(\mathbb{R})$ then*

$$\mathcal{H}f(x) = \frac{1}{\pi}\int_0^\infty \frac{f(x+t)-f(x-t)}{t}dt$$

is defined almost everywhere.

Proof. As in the proof of Riesz inequality we suppose that $f(x) \geq 0$ and we consider $\Phi(z)$ same as before (defined in the upper-half plane); again $u(x,y) \geq 0$. Now take

$$\Psi(z) = e^{-\Phi(z)} = e^{-u(x,y)-iv(x,y)}$$

From the definition of complex exponential for a fixed z the module of $\Psi(z)$ is given by $e^{-u(x,y)}$. Hence $|\Psi(z)| \leq 1$ and thanks to Theorem 2.0.6 $\Psi(z)$ tends to a finite limit for almost all x. Moreover as $u(x,y)$ tends to the finite limit $f(x)$ a.e., $\Psi(z)$ tends to a finite non-zero limit a.e. Therefore $\Psi(z)$ tends to a finite limit a.e. and consequentially $v(x,y)$ does the same. Hypothesis of Theorem 2.0.5 are verified as

$$f \in L^1(\mathbb{R}) \Longrightarrow f \in L^1(0,1)$$

$$f \in L^1(\mathbb{R}) \Longrightarrow \int_1^\infty \left|\frac{f(x)}{x}\right| dx \leq \int_1^\infty |f(x)| dx < \infty \Longrightarrow x^{-1} f(x) \in L^1(1, \infty)$$

Hence it follows that $v(x, y)$ converges to a finite limit a.e. that is the Hilbert transform of $f(x)$. $\square$

If $f \in L^\infty(\mathbb{R})$ then in general the Hilbert transform is an unbounded operator.

Chapter 3

Derivation of the Hilbert transform

Fourier transform and Hilbert transform are strictly related as it was first noticed by Hilbert. To show how he derived the transform from the Fourier transform we will use i slightly different definition of the latter ([1] Titchmarsh, 1948).
Given $f : \mathbb{R} \longrightarrow \mathbb{C},\ f \in L^p,\ p \geq 1$ as:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx$$

Moreover the Fourier inversion theorem states that:

Theorem 3.0.8 (Fourier inversion theorem). *Given* $f \in L^p(\mathbb{R}),\ 1 \leq p < \infty$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-ix\xi} d\xi$$

The Hilbert-Fourier relation arises in the proof of the inversion theorem for the Hilbert transform.

Theorem 3.0.9 (Hilbert inversion theorem). *Given* $f \in L^p(\mathbb{R}),\ 1 \leq p < \infty$*:*

$$f(x) = -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{H(f)(y)}{x-y} dy$$

Proof. Using the Fourier Inversion theorem, we can re-write f as follows:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-ix\xi} d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\mu) e^{i\xi\mu} d\mu \right) e^{-ix\xi} d\xi =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \big\{ f(\mu)\cos(\xi\mu) + i f(\mu)\sin(\xi\mu) \big\} d\mu \right) \big(\cos(x\xi) - i\sin(x\xi) \big) d\xi$$

If we consider the case of real valued functions we get:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\mu)\cos(\xi\mu) d\mu \right) \cos(x\xi) d\xi$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\mu)\sin(\xi\mu) d\mu \right) \sin(x\xi) d\xi$$

Thus we can finally write:

$$f(x) = \int_{0}^{\infty} \big\{ a(\xi)\cos(x\xi) + b(\xi) sin(x\xi) \big\} d\xi$$

Where:

$$a(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\mu)\cos(\xi\mu) d\mu \quad , \quad b(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\mu)\sin(\xi\mu) d\mu$$

Now we can define

$$u(x,y) = \int_{0}^{\infty} \big\{ a(\xi)\cos(x\xi) + b(\xi) sin(x\xi) \big\} e^{-y\xi} d\xi$$

and it is easy to see that $u(x,y)$ is well defined for $y \geq 0$ and that it is the real part of

$$\Phi(z) = \int_{0}^{\infty} \{ a(\xi) - ib(\xi) \} e^{iz\xi} d\xi$$

where $z = x + iy$. The imaginary part of $\Phi(z)$ is then

$$v(x,y) = -\int_{0}^{\infty} \{ b(\xi)\cos(x\xi) - a(\xi)\sin(x\xi) \} e^{-y\xi} d\xi$$

Writing $g(x) = -V(x,0)$, we get

$$g(x) = \int_{0}^{\infty} \{ b(\xi)\cos(x\xi) - a(\xi)\sin(x\xi) \} d\xi =$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} \{ f(\mu) \big(\sin(\mu\xi)\cos(x\xi) - \cos(\mu\xi)\sin(x\xi) \big) \} d\mu d\xi =$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \{f(\mu)\sin(\mu - x)\xi\} d\mu d\xi$$

This can be written as

$$g(x) = \lim_{\lambda\to\infty} \frac{1}{\pi}\int_0^\lambda \int_{-\infty}^\infty \{f(\mu)\sin(\mu - x)\xi\} d\mu d\xi$$

Solving for ξ gives

$$g(x) = \lim_{\lambda\to\infty} \frac{1}{\pi}\int_{-\infty}^\infty \left[f(\mu)\left(-\frac{\cos(\mu - x)\xi}{\mu - x}\right)\right]_{\xi=0}^{\xi=\lambda} d\mu =$$

$$= \lim_{\lambda\to\infty} \frac{1}{\pi}\int_{-\infty}^\infty \frac{1-\cos(\mu - x)\lambda}{\mu - x} f(\mu) d\mu =$$

$$= \lim_{\lambda\to\infty} \frac{1}{\pi}\left(\int_0^\infty \frac{1-\cos(\mu - x)\lambda}{\mu - x} f(\mu) d\mu + \int_0^\infty \frac{1-\cos(-\mu' - x)\lambda}{-\mu' - x} f(-\mu') d\mu' \right)$$

Now using the changes of coordinates $\mu - x = t$ and $\mu' + x = t'$ it becomes

$$g(x) = \lim_{\lambda\to\infty} \frac{1}{\pi}\left(\int_0^\infty \frac{1-\cos\lambda t}{t} f(x+t) dt - \int_0^\infty \frac{1-\cos\lambda t'}{t'} f(x-t') dt' \right) =$$

$$= \lim_{\lambda\to\infty} \frac{1}{\pi}\int_0^\infty \frac{1-\cos\lambda t}{t} \big(f(x+t) - f(x-t)\big) dt$$

Under certain conditions on f the part with $\cos\lambda t$ will tend to 0, therefore

$$g(x) = \frac{1}{\pi}\int_0^\infty \frac{f(x+t) - f(x-t)}{t} dt \tag{3.1}$$

similarly for f

$$f(x) = -\frac{1}{\pi}\int_0^\infty \frac{g(x+t) - g(x-t)}{t} dt \tag{3.2}$$

Hilbert first noticed this relationship, given from the Fourier transform, between f and g. It is straight forward the equivalency between the first definition we gave of Hilbert transform and Eq.(3.1). Furthermore Eq.(3.2) leads to the inversion property of the Hilbert transform:

$$f(x) = -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{H(f)(y)}{x-y} dy$$

□

Observation 2. Another relationship arises between the Hilbert and Fourier transform from the calculations above, indeed $a(\xi)$ and $b(\xi)$ can be written as

$$a(\xi) = \frac{1}{\sqrt{2\pi}}\big(\hat{f}(\xi) + \hat{f}(-\xi)\big) \quad , \quad b(\xi) = \frac{1}{i\sqrt{2\pi}}\big(\hat{f}(\xi) - \hat{f}(-\xi)\big)$$

Hence

$$g(x) = \int_0^\infty \{b(\xi)\cos(x\xi) - a(\xi)\sin(x\xi)\}d\xi =$$

$$= \frac{1}{i\sqrt{2\pi}}\int_{0\infty} \big(\hat{f}(\xi) - \hat{f}(-\xi)\big)\cos(x\xi)d\xi - \frac{1}{\sqrt{2\pi}}\int_{0\infty}\big(\hat{f}(\xi) + \hat{f}(-\xi)\big)\sin(x\xi)d\xi =$$

$$= \frac{1}{i\sqrt{2\pi}}\left(\int_0^\infty \hat{f}(\xi)e^{-ix\xi}d\xi - \int_0^\infty \hat{f}(-\xi)e^{ix\xi}d\xi\right)$$

with the change of variable $t = -\xi$ in the second integral it becomes

$$g(x) = \frac{1}{i\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(t)\ \mathrm{sgn}(t)\ e^{-ixt}dt$$

From the Fourier's inversion theorem it follows

$$\hat{g}(t) = -i\hat{f}(t)\ \mathrm{sgn}(t)$$

But $g = \mathcal{H}f$, hence formally

$$\mathcal{F}\mathcal{H}f(t) = -i\ \mathrm{sgn}(t)\ \mathcal{F}f(t)$$

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