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Taylor formula for Kolmogorov equations

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Relatore:
Prof.
Andrea Pascucci

Correlatore:
Prof.
Sergio Polidoro

Presentata da:
Michele Pignotti

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Sommario

Dopo una breve presentazione nel primo capitolo della struttura di gruppo di Lie omogeneo indotta da un'equazione di Kolmogorov, nel secondo capitolo definiamo dei polinomi di Taylor e degli spazi di Holder intrinseci e compariamo la nostra definizione con altre note in letteratura. Nel terzo capitolo dimostriamo l'analogo della formula di Taylor cioè una stima del resto in termini della metrica omogenea.

Abstract

After briefly discuss the natural homogeneous Lie group structure induced by Kolmogorov equations in chapter one, we define an intrinsic version of Taylor polynomials and Holder spaces in chapter two. We also compare our definition with others know in literature. In chapter three we prove an analogue of Taylor formula, that is an estimate of the remainder in terms of the homogeneous metric.

Introduction

Since their introduction in 1934 to formalize the evolution of a probability density Kolmogorov equations have been used in a variety of different fields from diffusion theory to kinetic models to mathematical finance in particular pricing Asian options.

The main aim of this dissertation is to prove a Taylor formula for function regular with respect to a geometry induced by the particular equation. We will essentially require an Hölder type regularity along integral curves of a set of vector fields given by the equation that generate a Lie algebra of full dimension. For function with such regularity we will define a Taylor polynomial different from the Euclidean one and prove an estimate of the remainder.

The structure of this dissertation is quite simple. In the first chapter we introduce some basic terminology and state our main assumption. First of all we report the definition of Kolmogorov operators in great generality, their use and some assunction as the Hörmander condition and homogeneity with respect to a particular type of anisotropic dilatations. We also introduce a non commutative group law with respect to the constant coefficients operators are left-translations invariant. Togheter the law and the dilatations give the space the structure of an homogeneous Lie group. Its property and geometry are discussed in the second section, in particular a quasi metric left invariant wrt the law is introduced.

In the second chapter we define suitable Hölder spaces for the underlying homogeneous structure and prove some inclusions between them. Both local and global versions of the spaces are provided. In section two we define the intrinsic Taylor polynomials and state our main results and its corollaries. In section three we use the stated propositions to compare our definition of intrinsic Hölder regularity to others know in literature.

Chapter three is dedicated to the proof of the main theorem. The demonstration is carried by induction but is divided into four steps in order to deal with the different difficulties that arises as the order increases. Essentially, in the first steps we have to prove that some of the derivatives appearing in the Taylor polynomials exist.

Chapter 1

The intrisic geometry

In this chapter we study the structure of $\mathbb{R}^{1+d}$ given by a *Kolmogorov Operator*. Precisely, in the first section we introduce such operators and state our main assumptions. In section two we define a Lie group structure on $\mathbb{R}^{1+d}$ as well as a relative quasi-metric and study their properties.

1.1 Kolmogorov equations

The term *Kolmogorov operators (KOs).* refers to a large class of second order differential ultra parabolic operators $\mathscr{L}$ of the form

$$\mathscr{L} = \sum_{i,j=1}^{p_0} a_{ij}(t,x)\partial_{x_i x_j} + \sum_{i=1}^{p_0} a_i(t,x)\partial_{x_i} + \langle Bx, \nabla_x \rangle + \partial_t, \tag{1.1}$$

with $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $t \in \mathbb{R}$ and $1 \le p_0 \le d$. Furthermore, $A = (a_{ij}(t,x))_{1 \le i,j \le p_0}$ is a symmetric matrix with variable real entries, positive semidefinite for any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, and $B = (b_{ij})_{1 \le i,j \le d}$ is a matrix with constant real entries.

The simpliest (forward version) of such operators is

$$\sum_{i=1}^{n} \partial_{x_i}^2 + \sum_{i=1}^{n} x_i \partial_{x_{n+i}} - \partial_t \tag{1.2}$$

which was introduced by Kolmogorov in [K] in 1934 in order to describe the probability density of a system with $2n = d$ degree of freedom. The $2n$-dimensional space is the phase space, $(x_1, \dots, x_n)$ is the velocity and $(x_{n+1}, \dots, x_{2n})$ the position of the system. By choosing

$$A = I_n \qquad B = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}$$

where I_n and 0 denote respectively the identity and the null $n \times n$ matrices, operator (1.2) can be written in form 1.1. We also recall that (1.2) is a prototype for a family of evolution equations arising in the kinetic theory of gases that take the following general form

$$Yu = \mathcal{J}(u).$$

Here $\mathbb{R}^{2n} \ni x \mapsto u(x,t) \in R$ is the density of particles which have velocity $(x_1, \dots, x_n)$ and position $(x_{n+1}, \dots, x_{2n})$ at time t,

$$Yu = -\sum_{i=1}^{n} x_i \partial_{x_{n+i}} + \partial_t$$

is the so called *total derivative* of u and $\mathcal{J}(u)$ describes some kind of collisions. This last term can take different form, either linear or non linear. For instance, in the usual Fokker-Planck equation, we have

$$\mathcal{J}(u) = \sum_{i,j=1}^{n} a_{i,j} \partial_{x_i,x_j} u + \sum_{i=1}^{n} a_i \partial_{x_i} u + au.$$

where a_{ij}, a_i and a are functions of (t,x).

For the description of wide classes of stochastic processes and kinetic models leading to equations of the previous type, we refer to the classical monographies [C], [DM] and [CC]. In the last decades mathematical models involving linear and non linear Kolmogorov type equations have also appeared in finance [ADK], [Ba], [DHW]. We explicitly mention the equation

$$s^2 \partial_s V + (\log s) \partial_\tau V + \partial_t V, \qquad s > 0, \quad t, \tau \in \mathbb{R}, \tag{1.3}$$

which arises in the problem of pricing geometric average Asian options, typically a Caychy problem with final datum the payoff function. (1.3) can be reduced to the Kolmogorov equation (1.2) with $n = 1$ by means of an elementary change of variables (see [FPP]). Considering the more frequently used arithmetic average Asian options leads to a more general equation than (1.3) with the second order coefficient dependant also on t. In [FPP] the authors gives a numerical method to approximate solutions of such equations using a Taylor series expansion of the coefficients following the ideas in [LPP] but prove no bound on the errors altough by numerical tests the convergence seems to be quite fast.

Error bounds for small times can be found using the Euclidean regularity of the payoff function however, using the intrinsic notion of regularity for the operator (1.1) can lead to higher order of convergence. This thesis, in which we define explicitly the notion of intrinsic regularity, the relative Taylor expansion and prove a conseguent error bound represents the first step to demonstrate a better error bounds than the ones presently know.

We assume the following structural hypothesis to hold:

(H.1) the matrix $(a_{ij}(x))_{1\le i,j\le p_0}$ is uniformly positive definite in $\mathbb{R}^{p_0}$, i.e. there exists a positive constant $M > 0$ such that

$$M^{-1}|\xi|^2 \le \sum_{i,j=1}^{p_0} a_{ij}(t,x)\xi_i\xi_j \le M|\xi|^2, \qquad \xi \in \mathbb{R}^{p_0}, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d.$$

(H.2) the matrix $B = (b_{ij})_{1\le i,j\le d}$ takes the form

$$B = \begin{pmatrix} * & * & \cdots & * & * \\ B_1 & * & \cdots & * & * \\ 0 & B_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_r & * \end{pmatrix}, \tag{1.4}$$

where each B_j is a $p_j \times p_{j-1}$ matrix with rank p_j, with

$$p_0 \ge p_1 \ge \cdots \ge p_r \ge 1, \quad \sum_{j=0}^{r} p_j = d,$$

and the $*$-blocks are arbitrary.

The structural hypothesis (H.1)-(H.2) represent a corner stone in the study of the existence of the fundamental solution of the operator $\mathscr{L}$. It is well known that, in case the of *constant coefficients KOs* ($a_{ij}(t,x) \equiv a_{ij}, a_i(t,x) \equiv a_i, a(x) \equiv a$), such hypothesis are equivalent to the Hormander condition (see [H]):

(H.C) Let $\mathfrak{g}_z := \mathrm{Lie}(X_1, \cdots, X_{p_0}, Y)$ be the Lie algebra generated by the vector fields

$$X_i = \partial_{x_i}, \quad i = 1, \cdots, p_0, \qquad Y \equiv Y(x) = \langle Bx, \nabla_x\rangle + \partial_t, \tag{1.5}$$

at the point $z = (t,x) \in \mathbb{R}\times\mathbb{R}^d$, then

$$\text{rank } \mathfrak{g}_z = d+1, \qquad \forall z \in \mathbb{R}\times\mathbb{R}^d.$$

Moreover, it can be show that

$$\mathfrak{g}_z = \underbrace{\mathrm{span}\{X_1,\dots,X_{p_0}\}}_{=:W_0} \oplus \underbrace{\mathrm{span}\{Y\}}_{=:W_1} \oplus \underbrace{[W_0,W_1]}_{=:W_2} \oplus \cdots \oplus \underbrace{[W_0,W_r]}_{=:W_{r+1}}, \tag{1.6}$$

with is a *gradation* i.e $[W_i, W_j] \subset W_{i+j}$ for every $i, j = 0, \dots, r$ (set $W_i = 0$ if $i > r$) with $\dim W_0 = p_0$, $\dim W_1 = 1$ and $\dim W_i = p_{i-1}$ for $i = 2, \dots, r+1$.

Therefore, we can hope that opportune regularity wrt the vector fields $X_1, \dots, X_{p_0}, Y$ implies regularity wrt the other vector fields in (1.6). This will be proved in chapter three.

1.2 Intrinsic geometry of $\mathbb{R} \times \mathbb{R}^d$

Definition 1.1. We now introduce the non-commutative law $\circ$ on $\mathbb{R} \times \mathbb{R}^d$ given by

$$(t, x) \circ (s, \xi) = (t + s, E(s)x + \xi), \qquad (t, x), (s, \xi) \in \mathbb{R} \times \mathbb{R}^d,$$

where

$$E(t) = e^{tB}, \quad t \in \mathbb{R}.$$

$(\mathbb{R} \times \mathbb{R}^d, \circ)$ is then a Lie group with identity element $\mathrm{Id} = (0, 0)$ and inverse $(t, x)^{-1} = (-t, -E(-t)x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}^d$. The group law first appeared in [GL] and then in [LP].

It is important to observe that *constant coefficients KOs* are invariant to the left translations with respect to $\circ$, i.e.

$$\big(\mathscr{L} u^{(s,\xi)}\big)(t, x) = (\mathscr{L} u)\big((s, \xi) \circ (t, x)\big), \qquad (t, x), (s, \xi) \in \mathbb{R} \times \mathbb{R}^d,$$

where

$$u^{(s,\xi)}(t, x) = u((s, \xi) \circ (t, x))$$

Definition 1.2. We also introduce the family of dilations $(D(\lambda))_{\lambda \geq 0}$ on $\mathbb{R} \times \mathbb{R}^d$ given by

$$D(\lambda)(t, x) = \mathrm{diag}\big(\lambda^2, \lambda I_{p_0}, \lambda^3 I_{p_1}, \cdots, \lambda^{2r+1} I_{p_r}\big)(t, x), \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

where I_{p_j} are $p_j \times p_j$ identity matrixes.

We esplicitely remark that, if the (formal) degree of the vector fields X_i is one for $i = 1, \dots, p_0$ and the formal degree of Y is set 2 as usual when dealing with parabolic equations then the W_i (of dimension p_{i-1}) appearing in the gradation (1.6) are spaces of vector fields of (formal) degree $2(i-1)+1$ (except for $W_1 = \mathrm{span}\{Y\}$), in accordance with the degree of the dilatation of the corresponding block.

It is easy to check that, if (and only if) the $*$-blocks in (1.4) are null as well as the first order coefficients a_i, then the *constant coefficients KO* $\mathscr{L}$ is also homogeneous of degree two with respect the dilations $D(\lambda)$, i.e.

$$\big(Lu^{(\lambda)}\big)(t,x) = \lambda^2 (\mathscr{L}u)\big(D(\lambda)(t,x)\big), \qquad \lambda > 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d,$$

where

$$u^{(\lambda)}(t,x) = u(D(\lambda)(t,x))$$

Hereafter we will always suppose such condition i.e. the matrix B takes the following form:

$$B = \begin{pmatrix} 0_{p_0 \times p_0} & 0_{p_0 \times p_1} & \cdots & 0_{p_0 \times p_{r-1}} & 0_{p_0 \times p_r} \\ B_1 & 0_{p_1 \times p_1} & \cdots & 0_{p_1 \times p_{r-1}} & 0_{p_1 \times p_r} \\ 0_{p_2 \times p_0} & B_2 & \cdots & 0_{p_2 \times p_{r-1}} & 0_{p_2 \times p_r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{p_r \times p_0} & 0_{p_r \times p_1} & \cdots & B_r & 0_{p_r \times p_r} \end{pmatrix}, \tag{1.7}$$

where $0_{p_i \times p_j}$ is a $p_i \times p_j$ null block.

Definition 1.3. We say that $\mathcal{G} \equiv (\mathbb{R}^N, \circ, \delta(\lambda))$ is an *homogeneous Lie group* if the following facts hold: $(\mathbb{R}^N, \circ)$ is a Lie group and $\delta(\lambda)_{\lambda>0}$ is a family of dilatation i.e.

$$\delta(\lambda)(x_1, \dots, x_N) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_N} x_N) \quad \sigma_i \geq 1;$$

witch are also group-homomorphism.

Definition 1.4. We observe that the matrix B in (1.7) univocally identifies an *homogeneous Lie group* $\mathcal{G}_B \equiv (\mathbb{R} \times \mathbb{R}^d, \circ, D(\lambda))$ called *Kolmogorov-type* Lie group.

We now denote by $D_0(\lambda)_{\lambda>0}$ the restriction of the dilations $D(\lambda)$ on $\mathbb{R}^d$:

$$D_0(\lambda)x = \operatorname{diag}\big(\lambda I_{p_0}, \lambda^3 I_{p_1}, \cdots, \lambda^{2r+1} I_{p_r}\big)x, \qquad x \in \mathbb{R}^d.$$

Next we decompose $\mathbb{R}^d$ in a direct sum of vector subspace according to the behaviour of the different variables in the dilatations $D_0(\lambda)$ and show how powers of the matrix B link the previous subspace.

Let

$$\bar{p}_{-1} = 0, \qquad \bar{p}_k = p_0 + p_1 + \cdots + p_k, \quad 0 \leq k \leq r.$$

For any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $n = 0, \dots, r$, we define $x^{[n]} \in \mathbb{R}^d$ as the projection of x on $\{0\}^{\bar{p}_{n-1}} \times \mathbb{R}^{p_n} \times \{0\}^{d-\bar{p}_n}$, i.e.

$$x^{[n]}_k = \begin{cases} x_k & \text{for } \bar{p}_{n-1} < k \leq \bar{p}_n, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

Thus $\mathbb{R}^d$ is the direct sum

$$\mathbb{R}^d = \bigoplus_{n=0}^{r} V_n, \qquad V_n := \{x^{[n]} \mid x \in \mathbb{R}^d\}, \quad n = 0, \dots, r.$$

Definition 1.5. Given an index $i \in \{1, \dots, d\}$ we will say that the variable x_i is of *level* k if $e_i \in V_k$ where e_i is the i-th vector of the canonical basis of $\mathbb{R}^d$. In this case we will also say that the derivative ∂_{x_i} is of *level* k and set its B-order to be $2k + 1$.

Note that for $i = 1, \dots, p_0$ that is, for level zero derivatives, the B-order coincides with the usual (Euclidean) one.

For any $n \le r$ we have

$$B^n = \begin{pmatrix} 0_{\bar{p}_{n-1}\times p_0} & 0_{\bar{p}_{n-1}\times p_1} & \cdots & 0_{\bar{p}_{n-1}\times p_{r-n}} & 0_{\bar{p}_{n-1}\times(\bar{p}_r-\bar{p}_{r-n})} \\ \prod_{j=1}^{n} B_j & 0_{p_n\times p_1} & \cdots & 0_{p_n\times p_{r-n}} & 0_{p_n\times(\bar{p}_r-\bar{p}_{r-n})} \\ 0_{p_{n+1}\times p_0} & \prod_{j=2}^{n+1} B_j & \cdots & 0_{p_{n+1}\times p_{r-n}} & 0_{p_{n+1}\times(\bar{p}_r-\bar{p}_{r-n})} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{p_r\times p_0} & 0_{p_r\times p_1} & \cdots & \prod_{j=r-n+1}^{r} B_j & 0_{p_r\times(\bar{p}_r-\bar{p}_{r-n})} \end{pmatrix}, \tag{1.9}$$

where

$$\prod_{j=1}^{n} B_j = B_n B_{n-1} \cdots B_1.$$

Note that such products are non commutative. Moreover $B^n = 0$ for $n > r$, so that

$$e^{\delta B} = I_d + \sum_{h=1}^{r} \frac{B^h}{h!}\delta^h. \tag{1.10}$$

where I_d is the $d \times d$ identity matrix.

Remark 1.6 *We have*

$$v \in V_0 \quad \Longrightarrow \quad B^n v \in V_n, \quad n = 0, \dots, r. \tag{1.11}$$

In particular, we have

$$B^n v = \bar{B}_n v,$$

with

$$\bar{B}_n = \begin{pmatrix} 0_{\bar{p}_{n-1}\times p_0} & 0_{\bar{p}_{n-1}\times(r-p_0)} \\ \prod_{j=1}^{n} B_j & 0_{p_n\times(r-p_0)} \\ 0_{(\bar{p}_r-\bar{p}_n)\times p_0} & 0_{(\bar{p}_r-\bar{p}_n)\times(r-p_0)} \end{pmatrix},$$

where, by Hypothesis **(H.2)**, $\prod_{j=1}^{n} B_j$ *is a* $p_n \times p_0$ *matrix with (rows) full rank. Therefore, the linear applications* $\bar{B}_n : V_0 \to V_n$ *are surjective, but not necessary injective. Nevertheless, it is possible to define, for any* $n = 0, \cdots, r$, *the subspaces* $V_{0,n} \subset V_0$ *as*

$$V_{0,n} := \{x \in V_0 | x_j = 0 \ \forall j \notin \Pi_{B,n}\},$$

with $\Pi_{B,n}$ *being the set of the indexes corresponding to the first* p_n *linear independent columns of* $\prod_{j=1}^{n} B_j$. *It is now trivial that the linear maps* $\bar{B}_n : V_{0,n} \to V_n$ *are also injective, and thus bijective. By trivial linear algebra arguments it is possible to show that*

$$V_{0,r} \subset V_{0,r-1} \subset \cdots \subset V_{0,1} \subset V_{0,0} = V_0. \tag{1.12}$$

Example 1.7. Denote the points of $\mathbb{R} \times \mathbb{R}^2$ by $z = (t, x_1, x_2)$ and consider the simpliest *KO*

$$\mathcal{K} = \partial_{x_1,x_1} + x_1 \partial_{x_2} + \partial_t = \partial_{x_1,x_1} + \langle B(x_1, x_2)^t, \nabla \rangle + \partial_t$$

where

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In this case we have

$$\mathbb{R}^2 = V_0 \oplus V_1 = \text{span}\{e_1\} \oplus \text{span}\{e_2\}, \qquad V_{0,0} = V_{0,1} = \text{span}\{e_1\}.$$

The dilatations $D(\lambda)$, $D_0(\lambda)$ take the following explicit form

$$D(\lambda)(t, x_1, x_2) = (\lambda^2 t, \lambda x_1, \lambda^3 x_2) \qquad D_0(\lambda)(x_1, x_2) = (\lambda x_1, \lambda^3 x_2);$$

and, as

$$e^{tB} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (I + tB) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + t x_1 \end{pmatrix},$$

the group law $\circ$ and its inverse become

$$\zeta \circ z = (s, \xi_1, \xi_2) \circ (t, x_1, x_2) = (s + t, x_1 + \xi_1, x_2 + \xi_2 + t\xi_1)$$
$$z^{-1} = (-t, -x_1, -x_2 + t x_1).$$

The group law $\circ$ together with the dilations $D(\lambda)$ induce a geometry different from the Euclidean one. It is natural to define new norms on $\mathbb{R}^d$ and in $\mathbb{R} \times \mathbb{R}^d$ which are homogeneous of degree one with respect to the dilations $D_0(\lambda)$ and $D(\lambda)$.

Definition 1.8.

$$|x|_B = \sum_{j=1}^{N} |x_j|^{1/q_j}, \qquad \|(t,x)\|_B = |t|^{1/2} + |x|_B \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d$$

where $(q_j)_{1\le j\le d}$ are integers such that

$$D(\lambda) = \operatorname{diag}\big(\lambda^2, \lambda^{q_1}, \cdots, \lambda^{q_d}\big).$$

The proprieties of the norm $\|\cdot\|_B$ are listed in the next proposition. For its proof we refer to [BCM].

Proposition 1.9 *The following statements hold for all $z, \zeta \in \mathbb{R}\times\mathbb{R}^d$ and $\lambda > 0$:*

i. $||D(\lambda)z||_B = \lambda ||z||_B$;

ii. $||z+\zeta||_B \le ||z||_B + ||\zeta||_B$;

iii. $||z\circ\zeta||_B \le c_B\,(||z||_B + ||\zeta||_B)$;

iv. $\frac{1}{c_B}||z||_B \le ||z^{-1}||_B \le c_B\,||z||_B$.

Here $c_B \ge 1$ is a constant that depends only *on B.*

Definition 1.10. Combining the norm $\|\cdot\|_B$ together with the group law $\circ$ we can define a functional d_B as

$$d_B(z,\zeta) := \left\|\zeta^{-1}\circ z\right\|_B.$$

For the Example 1.7 we have

$$|(x_1,x_2)|_B = |x_1| + |x_2|^{\frac{1}{3}}, \quad \|(t,x_1,x_2)\|_B = |t|^{\frac{1}{2}} + |x_1| + |x_2|^{\frac{1}{3}};$$
$$d_B((t,x_1,x_2),(t,\xi_1,\xi_2)) = |t-s|^{\frac{1}{2}} + |x_1-\xi_1| + |x_2-\xi_2-(t-s)\xi_1|^{\frac{1}{3}}.$$

Lemma 1.11 *The functional d_B is a quasimetric on $\mathbb{R}\times\mathbb{R}^d$ i.e. for all $z,\zeta\,\omega \in \mathbb{R}\times\mathbb{R}^d$ it holds*

1. $d_B(z,\zeta) \ge 0$;
2. $d_B(z,\zeta) = 0$ *iff* $z=\zeta$;
3. $d_B(z,\zeta) \le c_B\,(d_B(z,\omega) + d_B(\omega,\zeta))$;
4. $d_B(z,\zeta) \le c_B\, d_B(\zeta,z)$.

Here c_B is the same constant that appears in Proposition 1.9. Moreover for every bounded subset Ω of $\mathbb{R}\times\mathbb{R}^d$ there exists a constant $C_\Omega > 0$ such that

$$d_B(z,\zeta) \le C_\Omega |z-\zeta|^{\frac{1}{2r+1}}, \qquad z,\zeta \in \Omega. \tag{1.13}$$

Proof. Properties 1 and 2 follow directily from the definition. Regarding 3, using (iii) of Proposition 1.9 we get

$$d_B(z,\zeta) = \left\|\zeta^{-1}\circ z\right\|_B = \left\|(\zeta^{-1}\circ\omega)\circ(\omega^{-1}\circ z)\right\|_B \le c_B(d_B(\omega,\zeta)+d_B(z,\omega)).$$

In order to obtain 4 just use the right side of (iv). Finally, to prove (1.13) we suppose $z=(t,x)$, $\zeta=(s,\xi)$ and so

$$\begin{aligned} d_B(z,\zeta) = &\left\|(s,\xi)^{-1}\circ(,x)\right\|_B = |t-s|^{\frac{1}{2}} + |x-e^{(t-s)B}\xi|_B \\ =&|t-s|^{\frac{1}{2}} + \sum_{i=1}^{d}|(x-e^{(t-s)B}\xi)_i|^{\frac{1}{q_i}}. \end{aligned}$$

Since Ω is bounded we get $|t-s|^{\frac{1}{2}} \le C|t-s|^{\frac{1}{2r+1}} \le C|z-\zeta|^{\frac{1}{2r+1}}$ and, if i is an index of level say j by (1.9), (1.10)

$$(x-e^{(t-s)B}\xi)_i = x_i-\xi_i-(t-s)l.c.\{\xi_1,\dots,\xi_{p_0}\}+\dots-(t-s)^{j-1}l.c.\{\xi_1,\dots,\xi_{\bar{p}_{j-1}}\},$$

where $l.c\{\dots\}$ stands for linear combination. Then as $q_j \le 2r+1$ we obtain

$$|(x-e^{(t-s)B}\xi)_i|^{\frac{1}{q_i}} \le C(|\zeta|,B)|z-\zeta|^{\frac{1}{2r+1}} \le C_\Omega|z-\zeta|^{\frac{1}{2r+1}}.$$

□

Next we study how higher level derivatives can be obtained commutating the vector fields Y and $X_1,\dots,X_{p_0}$.

We use the following notations:

$$\nabla = (\partial_{x_1},\dots,\partial_{x_d}), \qquad \nabla\cdot v = \langle\nabla, v\rangle = \sum_{i=1}^{d} v_i\partial_{x_i}, \qquad v\in\mathbb{R}^d.$$

Next we set

$$Y_v^{(k)} := [\cdots[[\nabla\cdot v,\underbrace{Y],Y]\cdots,Y]}_{k}, \qquad k=0,\dots,r \qquad v\in V_0. \tag{1.14}$$

Lemma 1.12 *Let $u\in C^\infty(\mathbb{R}\times\mathbb{R}^d)$, $v\in V_0$. Then we have $Y_v^{(0)}=\nabla\cdot v$ and*

$$Y_v^{(k)}u = \langle B^k v,\nabla\rangle u \qquad k\ge 1. \tag{1.15}$$

Proof. The statement can be directly verified for $k=0$ and $k=1$. Let us notice that, by definition, $Y_v^{(k)} = [Y_v^{(k-1)},Y]$ for $k\ge1$. Now, we assume (1.15) to hold for $k\ge1$ and prove it for $k+1$. Expanding the commutator and using the inductive hypothesis we get

$$\begin{aligned} Y_v^{(k+1)}u =& Y_v^{(k)}Yu - Y\,Y_v^{(k)}u = \langle B^k v, \nabla\rangle Yu - Y\langle B^k v, \nabla\rangle u \\ =& \langle B^k v, \nabla\rangle\langle x, B^\top\nabla\rangle u - \langle x, B^\top\nabla\rangle\langle B^k v, \nabla\rangle u. \end{aligned}$$

The statement then stems from the identity:

$$\langle a, \nabla\rangle\langle x, C\nabla\rangle = \langle a, C\nabla\rangle + \langle x, C\nabla\rangle\langle a, \nabla\rangle, \qquad x \in \mathbb{R}^d,$$

for any $a \in \mathbb{R}^d, C \in \mathbb{R}^{d\times d}$, and by setting $a = B^k v$, $C = B^\top$. □

By (1.11) $Y_v^{(k)}$ is a linear combination of derivatives of level k and B-order $2k+1$ i.e. each time we commute $\nabla \cdot v$ with the vector field Y we obtain an operator of one level greater and whit B-order increased by two, coerently whit the fact that the B-order of Y equals two.

In particular if $v_i^{(n)} \in V_{0,n}$ is the (unique) vector such that $B^n v^{(n)_i} = e_i^{(n)}$, the i-th vector of the canonical basis of V_n (see Remark 1.6), we have

$$\partial_{x_{\bar{p}_{n-1}+i}} u \;=\; Y^{(n)}_{v^{(n)_i}} u.$$

In chapter three we will establish an intrinsic Taylor expansion for a class of functions regular only wrt $X_1, \dots, X_{p_0}, Y$ but with none a priori regularity wrt other vector fields. The above formula is then of particular interest since indicate how to recover the lacking regularity.

Next we show how to approximate integral curves of the commutators $Y_v^{(k)}$ using a rather classical technique from control theory. We use the notation $e^{sX}(z)$ to indicate the integral curve starting at z of the vector field X evaluated at the time s i.e. the solution of

$$\begin{cases} \dot\gamma(s) = (X\gamma)(s); \\ \gamma(0) = z. \end{cases}$$

Note that solving such systems for the vector fields X_i and Y in (1.5) we obtain

$$e^{\delta X_i}(t,x) = (t, x + \delta e_i) \qquad e^{\delta Y}(t,x) = (t+\delta, e^{\delta B}x),$$

for any $(t,x) \in \mathbb{R}\times\mathbb{R}^d$ and $\delta \in \mathbb{R}$.

In the Example 1.7 we obtain

$$\begin{aligned} e^{\delta X_1}(t, x_1, x_2) &= (t, x_1 + \delta, x_2), \\ e^{\delta Y}(t, x_1, x_2) &= (t+\delta, x_1, x_2 + \delta x_1). \end{aligned}$$

For any $z\in\mathbb{R}\times\mathbb{R}^d$, $\delta\in\mathbb{R}$ and $v\in\mathbb{R}^d$, we define iteratively the family of trajectories $\left(\gamma^{(k)}_{v,\delta}(z)\right)_{k=0,\dots,r}$ as

$$\gamma^{(0)}_{v,\delta}(z)=e^{\delta\nabla\cdot v}(z)=(t,x+\delta v), \tag{1.16}$$

$$\gamma^{(k+1)}_{v,\delta}(z)=e^{-\delta^2 Y}\left(\gamma^{(k)}_{v,-\delta}\left(e^{\delta^2 Y}\left(\gamma^{(k)}_{v,\delta}(z)\right)\right)\right). \tag{1.17}$$

Lemma 1.13 *We have*

$$\gamma^{(k)}_{v,\delta}(t,x)=(t,x+S_k(\delta)v), \qquad k=0,\dots,r, \tag{1.18}$$

where

$$S_0(\delta)=\delta I_d, \quad \text{and} \quad S_k(\delta):=(-1)^k\sum_{\substack{h\in\mathbb{N}^k\\ |h|\le r}}\frac{(-B)^{|h|}}{h!}\delta^{2|h|+1}, \qquad k=1,\dots,r.$$

with $|h|=h_1+\cdots+h_k$ *and* $h!=h_1!\cdots h_k!$.

Proof of Lemma 1.13. We proceed by induction on k. The case $k=0$ is trivial. Now, assuming (1.18) as inductive hypothesis and noting that $S_k(-\delta)=-S_k(\delta)$, we have

$$\begin{aligned}
\gamma^{(k+1)}_{v,\delta}(t,x)&=e^{-\delta^2 Y}\left(\gamma^{(k)}_{v,-\delta}\left(e^{\delta^2 Y}\left(\gamma^{(k)}_{v,\delta}(t,x)\right)\right)\right)\\
&=e^{-\delta^2 Y}\left(\gamma^{(k)}_{v,-\delta}\left(e^{\delta^2 Y}\left(t,x+S_k(\delta)v\right)\right)\right)\\
&=e^{-\delta^2 Y}\left(\gamma^{(k)}_{v,-\delta}\left(t+\delta^2,e^{\delta^2 B}\left(x+S_k(\delta)v\right)\right)\right)\\
&=e^{-\delta^2 Y}\left(t+\delta^2,e^{\delta^2 B}\left(x+S_k(\delta)v\right)-S_k(\delta)v\right)\\
&=\left(t,e^{-\delta^2 B}\left(e^{\delta^2 B}\left(x+S_k(\delta)v\right)-S_k(\delta)v\right)\right)\\
&=\left(t,x+S_k(\delta)v-e^{-\delta^2 B}S_k(\delta)v\right).
\end{aligned}$$

On the other hand, by (1.10) we have

$$x+S_k(\delta)v-e^{-\delta^2 B}S_k(\delta)v=x+S_k(\delta)v-\left(I_d+\sum_{h=1}^{r}\frac{(-B)^h}{h!}\delta^{2h}\right)S_k(\delta)=$$
$$x+S_{k+1}(\delta)v,$$

and this concludes the proof. □

In general, for $n\in 0,\cdots,r$, $z\in\mathbb{R}\times\mathbb{R}^d$, $\delta\in\mathbb{R}$ and $v\in\mathbb{R}^d$, we define iteratively the family of trajectories $\left(\gamma^{(n,k)}_{v,\delta}(z)\right)_{k=n,\dots,r}$ as

$$\gamma^{(n,n)}_{v,\delta}(z)=e^{\delta^{2n+1}Y^{(n)}_v}(z)=\left(t,x+\delta^{2n+1}B^n v\right),$$
$$\gamma^{(n,k+1)}_{v,\delta}(z)=e^{-\delta^2 Y}\left(\gamma^{(n,k)}_{v,-\delta}\left(e^{\delta^2 Y}\left(\gamma^{(n,k)}_{v,\delta}(z)\right)\right)\right), \qquad n\le k\le r-1.$$

We have the analogous of Lemma 1.13 whose proof is identical to the one just showed and is then left.

Lemma 1.14 *For any $n \in \{0, \cdots, r\}$, we have*

$$\gamma_{v,\delta}^{(n,k)}(t,x) = (t, x + S_{n,k}(\delta)v), \qquad k = n, \ldots, r,$$

where

$$S_{n,n}(\delta) = \delta^{2n+1} B^n v,$$

and

$$S_{n,k}(\delta) := (-1)^{k-n} \delta^{2n+1} B^n \sum_{\substack{h \in \mathbb{N}^{k-n} \\ |h| \le r-n}} \frac{(-B)^{|h|}}{h!} \delta^{2|h|}, \qquad k = n+1, \ldots, r.$$

with $|h| = h_1 + \cdots + h_k$ and $h! = h_1! \cdots h_k!$.

Remark 1.15 *Since*

$$S_{n,k}(\delta) = \delta^{2k+1} B^k + \widetilde{S}_{n,k}(\delta), \qquad n \le k \le r$$

with

$$\widetilde{S}_{n,n}(\delta) := 0$$

and

$$\widetilde{S}_{n,k}(\delta) := (-1)^{k-n} \delta^{2n+1} B^n \sum_{\substack{h \in \mathbb{N}^{k-n} \\ k-n < |h| \le r-n}} \frac{(-B)^{|h|}}{h!} \delta^{2|h|}, \qquad k = n+1, \ldots, r,$$

(note that $\widetilde{S}_{n,r}(\delta) = 0$), then we deduce from (1.18) that

$$\gamma_{v,\delta}^{(n,k)}(z) = \big(t, x + \delta^{2k+1} B^k v\big) + \big(0, \widetilde{S}_{n,k}(\delta) v\big), \qquad n \le k \le n. \tag{1.19}$$

If $v \in V_0$ then the remainders $\widetilde{S}_{n,k}(\delta)v$ have the following important properties: first of all, as a remarkable consequence of (1.11), we have that

$$\widetilde{S}_{n,k}(\delta)v \in \bigoplus_{j=k+1}^{r} V_j, \qquad k = n, \ldots, r. \tag{1.20}$$

Moreover, using notation (1.8), for any $k = n, \ldots, r$ we have

$$\left| \big(\widetilde{S}_{n,k}(\delta)v\big)^{[j]} \right| \le c_B |\delta|^{2j+1} |v|, \qquad j = k+1, \ldots, r, \tag{1.21}$$

where the constant c_B depends only on the matrix B. If $|v| = 1$ this also implies

$$\begin{aligned} \left\| \big(\gamma_{v,\delta}^{(n,k)}(z)\big)^{-1} \circ z \right\|_B &= \left\| \big((t, x + \delta^{2k+1} B^k v) + (0, \widetilde{S}_{n,k}(\delta)v)\big)^{-1} \circ (t,x) \right\|_B \\ &= \left\| \big(0, -\delta^{2k+1} B^k v - \widetilde{S}_{n,k}(\delta)v\big) \right\|_B \\ &= | - \delta^{2k+1} B^k v - \widetilde{S}_{n,k}(\delta) v|_B \le c_B |\delta|. \end{aligned} \tag{1.22}$$

We also set:

$$\gamma_{v,\delta}^{(-1,k)}(z) \equiv \gamma_{v,\delta}^{(k)}(z), \qquad 0 \le k \le r.$$

If $v \in V_0$ equation (1.19) togheter with (1.20) say that the function $\gamma_{v,\delta}^{(n,k)}$ modifies the spatial components of level greater or equal to k but left untouched the others. Moreover since the maps

$$B^k : V_{0,k} \longrightarrow V_k$$

are bijective, choosign conveniently the vector v we can make the increment in the level k arbitrary. Exploiting such features of the trajectories $\gamma_{v,\delta}^{(n,k)}$ we can connect two arbitrary point in $\mathbb{R} \times \mathbb{R}^d$.

Proposition 1.16 *Given any two points z, $\zeta \in \mathbb{R}\times\mathbb{R}^d$ there exists a continuos path*

$$\gamma_{\zeta,z} : [0,1] \longrightarrow \mathbb{R}^{1+d}; \qquad \gamma_{\zeta,z}(0) = \zeta,\ \ \gamma_{\zeta,z}(1) = z,$$

such that $\gamma_{\zeta,z}$ is a concatenation of integral curves of either Y or $\nabla \cdot v$ for suitable vectors $v \in V_0$.

Moreover there exists a constat $C > 0$ depending only on B such that, setting $R = d_B(\zeta, z)$, the support of the curve $\gamma_{z,\zeta}$ is contained in the set $\bar{\mathscr{B}}(\zeta, CR) = \{\omega \in \mathbb{R}\times\mathbb{R}^d \,|\, d_B(\zeta,\omega) \le CR\}$, the closed d_B-ball of centre ζ and radious CR.

Proof. Let $z = (t, x)$, $\zeta = (s, \xi)$. The first step consist in adjusting the temporal component moving along $e^{\tau Y}$. Set

$$z_{-1} = (t, x_{-1}) = e^{(t-s)Y}(\zeta) = (t, e^{(t-s)B}\xi)$$

Next we adjust the spatial components. We define recursively the sequence of points $\big(z_k = (t, x_k)\big)_{k=0,\cdots,r}$ as follows. For $k = 0$ we set

$$v_0 = \frac{(x - e^{(t-s)B}\xi)^{[0]}}{\big|(x - e^{(t-s)B}\xi)^{[0]}\big|}, \qquad \delta_0 = \big|(x - e^{(t-s)B}\xi)^{[0]}\big|, \tag{1.23}$$

and

$$z_0 := \gamma_{v_0,\delta_0}^{(0)}(z_{-1}) = (t, x_{-1}^{[0]} + \delta_0 v_0, x_{-1}^{[1]}, \ldots, x_{-1}^{[r]}) = (t, x^{[0]}, x_{-1}^{[1]}, \ldots, x_{-1}^{[r]}).$$

For $k = 1, \ldots, r$ we consider the unique (see Remark 1.6) unitary vector $v_k \in V_{0,k} \subset V_0$ defined as $v_k = \frac{w_k}{|w_k|}$ where $w_k \in V_{0,k}$ is such that

$$B^k w_k = (e^{(t-s)B}\xi)^{[k]} - x_{k-1}^{[k]}.$$

We also set

$$z_k = \gamma_{v_k,\delta_k}^{(k)}(z_{k-1}), \qquad \delta_k = |w_k|^{\frac{1}{2k+1}}.$$

and $\gamma_{\zeta,z}$ the concatenation of $[0,t-s] \ni \tau \to e^{\tau Y}(\zeta)$ with the trajectories

$$[0,\delta_k] \ni \tau \longrightarrow \gamma^{(k)}_{v_k,\tau}(z_{k-1}) \qquad k = 0,\dots,r.$$

Here we suppose $t \geq s$, if not just take the interval $[t-s,0]$. We can reparametrize such path and therefore suppose that it is defined in $[0,1]$. By construction $z_r = z$ and, since by definitions the trajectories $\gamma^{(k)}_{v_k,\delta_k}$, are themselves composition of integral curves, also $\gamma_{\zeta,z}$ it is. This prove the first part of the proposition.

In order to prove the second part we prove a bound for the δ_k in terms of $\left\|\zeta^{-1}\circ z\right\|_B$. In the following c_B will denote any constant greater than zero depending only on B.

We begin with the first piece of the curve.

$$d_B(\zeta, e^{\tau Y}(\zeta)) = \left\|(e^{\tau Y}(\zeta))^{-1}\circ\zeta\right\|_B = \|(-\tau,0)\|_B = |\tau|^{\frac{1}{2}}.$$

In particular $d_B(\zeta, z_{-1}) = |t-s|^{\frac{1}{2}} \leq d_B(z,\zeta)$ so that

$$e^{\tau Y}(\zeta) \in \bar{\mathscr{B}}(\zeta, R), \qquad \tau \in [0,t-s]. \tag{1.24}$$

By Remark 1.6, it is easy to prove that

$$\delta_k \leq c_B |(e^{(t-s)B}\xi)^{[k]} - x^{[k]}_{k-1}|^{\frac{1}{2k+1}}. \tag{1.25}$$

Moreover, by (1.21) we get

$$|x^{[j]}_k - x^{[j]}_{k-1}| \leq c_B |\delta_k|^{2j+1}, \qquad j = k+1,\dots,r. \tag{1.26}$$

Now we prove by induction that, for any $k = 0,\dots,r$, we have

$$\delta_k \leq c_B|x - e^{(t-s)B}\xi|_B \leq c_B(|t-s|^{\frac{1}{2}} + |x - e^{(t-s)B}\xi|_B) = c_B d_B(z,\zeta). \tag{1.27}$$

For $k=0$, the thesis follows immediately from the definition of δ_0 in (1.23). Assuming that the estimate holds for any $h \leq k$, by (1.25) we have

$$\begin{aligned}\delta_{k+1} &\leq c_B |(e^{(t-s)B}\xi)^{[k+1]} - x^{[k+1]}_k|^{\frac{1}{2(k+1)+1}} \\ &\leq c_B |e^{(t-s)B}\xi)^{[k+1]} - x^{[k+1]}|^{\frac{1}{2(k+1)+1}} + c_B \sum_{h=1}^{k} |x^{[k+1]}_h - x^{[k+1]}_{h-1}|^{\frac{1}{2(k+1)+1}}\end{aligned}$$

(by (1.26))

$$\leq c_B |(e^{(t-s)B}\xi)^{[k+1]} - x^{[k+1]}_k|^{\frac{1}{2(k+1)+1}} + c_B \sum_{h=1}^{k} \delta_h,$$

and the thesis follows assuming the inductive hypothesis

$$\delta_h \le c_B |x - e^{(t-s)B}\xi|_B, \qquad h = 0, \dots, k.$$

Equation (1.24) proves that the first piece of $\gamma_{\zeta,z}$ is contained in the closed d_B-ball $\overline{\mathscr{B}}(\zeta, R)$ while equations (1.22),(1.27) prove that

$$\gamma^{(k)}_{v_k,\tau}(z_{k-1}) \in \bar{\mathscr{B}}(z_{k-1}, c_B\delta_k) \subset \bar{\mathscr{B}}(z_{k-1}, c_B R), \qquad \tau \in [0, \delta_k].$$

Setting C_0 the maximum between the constants c_B in the above formula, the constant in Lemma 1.11 and 1 we have

$$\begin{aligned}
d_B(\zeta, e^{\tau Y}(\zeta)) &\le |\tau|^{\frac{1}{2}} \le C_0 R && \tau \in [0, t-s]\\
d_B(\zeta, \gamma^{(0)}_{v_0,\tau}(z_{-1})) &\le C_0(d_B(\zeta, z_{-1}) + d_B(z_{-1}, \gamma^{(0)}_{v_0,\tau}(z_{-1})))\\
&\le C_0^2 R + C_0^2\delta_0 \le (C_0^2 + C_0^3)R && \tau \in [0, \delta_0]\\
d_B(\zeta, \gamma^{(1)}_{v_1,\tau}(z_0)) &\le C_0(d_B(\zeta, z_0) + d_B(z_0, \gamma^{(1)}_{v_1,\tau}(z_0)))\\
&\le C_0((C_0^2 + C_0^3)R + C_0\delta_0) \le (C_0^4 + 2C_0^3)R && \tau \in [0, \delta_1]\\
&\dots\\
d_B(\zeta, \gamma^{(r)}_{v_r,\tau}(z_{r-1})) &\le C_0(d_B(\zeta, z_{r-1}) + d_B(z_0, \gamma^{(1)}_{v_r,\tau}(z_{r-1})))\\
&\le (C_0^{r+2} + 2C_0^{r+1} + C_0^r + \dots + C_0^3)R && \tau \in [0, \delta_r]
\end{aligned}$$

and the thesis follows setting $C = C_0^{r+2} + 2C_0^{r+1} + C_0^r + \dots + C_0^3$. □

Chapter 2

Intrinsic Hölder spaces and Taylor polynomials

In this chapter we define some intrinsic Hölder spaces for the study of Kolmogorov Operators. For function in such spaces we state a Taylor-type formula whose proof is posponed in chapter three. We also compare our definitions with others note in literature.

2.1 Hölder spaces

In order to specify the notions of B-intrinsic regularity for a function f, we need to make use of the *integral (or characteristic) curves* of the vector fields $X_1, \cdots, X_{p_0}, Y$ as defined in (1.5). Precisely, for a given vector field on $\mathbb{R} \times \mathbb{R}^d$

$$X \equiv X(z) = a_0(z)\partial_t + \sum_{i=1}^{d} a_i(z)\partial_{x_i}, \qquad z \in \mathbb{R} \times \mathbb{R}^d \tag{2.1}$$

with Lipshitz continous coefficients $(a_i)_{i=1,\ldots,d}$ the characteristic curve $\gamma_{X,z} : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^d$ is defined as the unique solution of

$$\begin{cases} \dot{\gamma}_{X,z}(\delta) = X\big(\gamma_{X,z}(\delta)\big), \\ \gamma_{X,z}(0) = z. \end{cases}$$

To simplify the notation, in the sequel we will often write $e^{\delta X}(z)$ to indicate $\gamma_{X,z}(\delta)$. By solving the system of linear ODEs related to the fields $X_1, \cdots, X_{p_0}, Y$ we easily obtain

$$\gamma_{X_i,z}(\delta) = e^{\delta X_t}(z) = z + \delta e_i, \quad i = 1, \cdots, p_0, \quad \gamma_{Y,z}(\delta) = e^{\delta Y}(z) = (t+\delta, e^{\delta B}x),$$

for any $z = (t, x) \in \mathbb{R} \times \mathbb{R}^d$.

Definition 2.1. Let X be a Lipschitz vector field on $\mathbb{R} \times \mathbb{R}^d$ and f a real valued function defined in a neighborhood of $z \in \mathbb{R} \times \mathbb{R}^d$. We say that f is *X-differentiable* in z if the function $\delta \mapsto f\left(e^{\delta X}(z)\right)$ is differentiable in 0. We call

$$(Xf)(z) := \lim_{\delta \to 0} \frac{f\left(e^{\delta X}(z)\right) - f(z)}{\delta},$$

the Lie derivative of f wrt the vector field X in z.

Remark 2.2 *Note that if $f \in C^1$ i.e. f has continuos Euclidean derivatives and X is as in* (2.1) *then*

$$(Xf)(z) = a_0(z)\partial_t f(z) + \sum_{i=1}^{d} a_i(z)\partial_{x_i} f(z), \quad z \in \mathbb{R} \times \mathbb{R}^d.$$

However there exist functions with continuos Lie derivative wrt a Lipschitz vector field which have no Euclidean partial derivatives. For example consider the homogeneous structure introduced in Example 1.7 and the related vector field $Y(t, x_1, x_2) = x_1\partial_{x_2} - \partial_t$. Its integral curve act as $e^{\delta Y}(t, x_1, x_2) = (t + \delta, x_1, x_2 + \delta x_1)$ and therefore the function

$$u(t, x_1, x_2) = |tx_1 - x_2|$$

is constant along such integral curves. this in turn implies that has Lie derivatives identically zero despite having no partial derivatives. More explicitely for any of the derivatives ∂_{x_1}, ∂_{x_2}, ∂_t there exist a point (in fact infinitely many) in $\mathbb{R} \times \mathbb{R}^d$ in which the function u is not derivable. Moreover they can be choosen arbitrary close to $(0, 0, 0)$.

Definition 2.3. Let X be a Lipschitz vector field on $\mathbb{R} \times \mathbb{R}^d$, $m > 0$ a formal degree associated to X and Ω an open subset of $\mathbb{R} \times \mathbb{R}^d$. Then, for $\alpha \in]0, m]$, we say that $f \in C^\alpha_X(\Omega)$ if

$$\|f\|_{C^\alpha_X(\Omega)} := \sup \frac{\left|f\left(e^{\delta X}(z)\right) - f(z)\right|}{|\delta|^{\frac{\alpha}{m}}} < \infty.$$

Here the sup is taken over all the $z \in \Omega$ and the $\delta \in \mathbb{R} \setminus \{0\}$ such that the integral curve $e^{sX}(z)$ lies in Ω for $|s| \leq |\delta|$. Note that $\|\cdot\|_{C^\alpha_X(\Omega)}$ is only a seminorm.

We also say that $f \in C^\alpha_{X,\text{loc}}(\Omega)$ if, for any $z \in \Omega$, there exists $\delta_z > 0$ such that the integral curve $e^{sX}(z)$ lies in Ω for $|s| \leq |\delta_z|$ and

$$\sup_{|\delta| < \delta_z,\, \delta \neq 0} \frac{\left|f\left(e^{\delta X}(z)\right) - f(z)\right|}{|\delta|^{\frac{\alpha}{m}}} < \infty.$$

Coerently with the discussion in chapter one we set the formal degree of the vector field Y to be two and the formal degree of X_i to be one,$i = 1, \dots, p_0$.

Definition 2.4. Let $\alpha \in]0, 1]$ and $k \in \mathbb{N}$ with $k \geq 2$. Then

i) $u \in C_B^{0,\alpha}(\Omega)$ if u is bounded in Ω, $u \in C_Y^{\alpha}(\Omega)$ and $u \in C_{\partial_{x_i}}^{\alpha}(\Omega)$ for any $i = 1, \dots, p_0$. For any $u \in C_B^{0,\alpha}(\Omega)$ we define

$$\|u\|_{C_B^{0,\alpha}(\Omega)} := \sup_{z\in\Omega} |u(z)| + \|u\|_{C_Y^{\alpha}(\Omega)} + \sum_{i=1}^{p_0} \|u\|_{C_{\partial_{x_i}}^{\alpha}(\Omega)} .$$

ii) $u \in C_B^{1,\alpha}(\Omega)$ if u is bounded in Ω, $u \in C_Y^{1+\alpha}(\Omega)$ and $\partial_{x_i} u \in C_B^{0,\alpha}(\Omega)$ for any $i = 1, \dots, p_0$. For any $u \in C_B^{1,\alpha}(\Omega)$ we define

$$\|u\|_{C_B^{1,\alpha}(\Omega)} := \sup_{z\in\Omega} |u(z)| + \|u\|_{C_Y^{1+\alpha}(\Omega)} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{C_B^{0,\alpha}(\Omega)} .$$

iii) $u \in C_B^{k,\alpha}(\Omega)$ if u is bounded in Ω, $Yu \in C_B^{k-2,\alpha}(\Omega)$ and $\partial_{x_i} u \in C_B^{k-1,\alpha}(\Omega)$ for any $i = 1, \dots, p_0$. For any $u \in C_B^{k,\alpha}$ we define

$$\|u\|_{C_B^{k,\alpha}(\Omega)} := \sup_{z\in\Omega} |u(z)| + \|Yu\|_{C_B^{k-2,\alpha}(\Omega)} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{C_B^{k-1,\alpha}(\Omega)} .$$

Definition 2.5. Let $\alpha \in]0, 1]$ and $k \in \mathbb{N}$ with $k \geq 2$. Then

i) $u \in C_{B,\mathrm{loc}}^{0,\alpha}(\Omega)$ if $u \in C_{Y,\mathrm{loc}}^{\alpha}(\Omega)$ and $u \in C_{\partial_{x_i},\mathrm{loc}}^{\alpha}(\Omega)$ for any $i = 1, \dots, p_0$;

ii) $u \in C_{B,\mathrm{loc}}^{1,\alpha}(\Omega)$ if $u \in C_{Y,\mathrm{loc}}^{1+\alpha}(\Omega)$ and $\partial_{x_i} u \in C_{B,\mathrm{loc}}^{0,\alpha}(\Omega)$ for any $i = 1, \dots, p_0$;

iii) $u \in C_{B,\mathrm{loc}}^{k,\alpha}(\Omega)$ if $Yu \in C_{B,\mathrm{loc}}^{k-2,\alpha}(\Omega)$ and $\partial_{x_i} u \in C_{B,\mathrm{loc}}^{k-1,\alpha}(\Omega)$ for any $i = 1, \dots, p_0$.

Example 2.6. There exist functions that exhibit a more regular behaviour under an homogeneous structure that under the Euclidean one. For example consider the structure introduced in 1.7 and the function

$$u : \Omega \to \mathbb{R}, \qquad u(t, x_1, x_2) = |x_2|.$$

Here Ω is a bounded neighbourhood of zero. The function is only Lipschitz in the euclidean sense but, in fact, is $C_B^{1,1}(\Omega)$. To see that just note that $\partial_{x_1} u \equiv 0$ and

$$\begin{aligned} |u(e^{\delta Y}(t, x_1, x_2)) - u(t, x_1, x_2)| &= |u(t + \delta, x_1, x_2 + \delta x_1) - u(t, x_1, x_2)| \\ &= ||x_2 + \delta x_1| - |x_2|| \leq C|\delta|. \end{aligned}$$

Therefore $u \in C_Y^2(\Omega)$.

Proposition 2.7 *For any $k \geq 0$, the following relations holds:*

$$
\begin{aligned}
C^{k,\alpha}_{B,loc}(\Omega) \subset C^{k-1,\alpha}_{B,loc}(\Omega) \subset \cdots \subset C^{0,\alpha}_{B,loc}(\Omega) \\
C^{k,\alpha}_{B}(\Omega) \subset C^{k-1,\alpha}_{B}(\Omega) \subset \cdots \subset C^{0,\alpha}_{B}(\Omega)
\end{aligned} \tag{2.2}
$$

Moreover, we have

$$
C^{k,\alpha}_{B}(\Omega) \subset C^{k,\alpha}_{B,loc}(\Omega), \qquad k \geq 0. \tag{2.3}
$$

Proof. We prove the first claim by induction on k. Let $u \in C^{1,\alpha}_{B,\mathrm{loc}}(\Omega)$. By Definition 2.5 we have $u \in C^{1+\alpha}_{Y,\mathrm{loc}}(\Omega)$ so that for any $z \in \Omega$ there exist a constant $C(z)$ and a $\delta_z > 0$ such that

$$
\left|u\left(e^{\delta Y}(z)\right) - u(z)\right| \leq C(z)|\delta|^{\frac{1+\alpha}{2}} \leq C(z)|\delta_z|^{\frac{1}{2}}|\delta|^{\frac{\alpha}{2}}, \qquad |\delta| \leq \delta_z.
$$

Since δ_z depends only on z it follows that $u \in C^{\alpha}_{Y,\mathrm{loc}}(\Omega)$. For the same reason, for a suitable $\bar{\delta} \leq \delta$,

$$
|u(t, x + \delta e_i) - u(t, x)| = \left|\partial_{x_i} u(t, x + \bar{\delta} e_i)\right| |\delta|^{\alpha} |\delta|^{1-\alpha} \leq C(z)|\delta|^{\alpha}, \tag{2.4}
$$

for $i \in \{1, \dots, p_0\}$. Therefore

$$
C^{1,\alpha}_{B,\mathrm{loc}}(\Omega) \subset C^{\alpha}_{Y,\mathrm{loc}}(\Omega) \cap C^{\alpha}_{X_1,\mathrm{loc}}(\Omega) \cap \cdots \cap C^{\alpha}_{X_{p_0},\mathrm{loc}}(\Omega) = C^{0,\alpha}_{B,\mathrm{loc}}(\Omega). \tag{2.5}
$$

Next suppose $u \in C^{2,\alpha}_{B,\mathrm{loc}}(\Omega)$. Then by definition Yu exists and it is in $C^{0,\alpha}_{B,\mathrm{loc}}(\Omega)$ and we can proceed as in (2.4). Precisely

$$
\left|u\left(e^{\delta Y}(z)\right) - u(z)\right| = \left|Yu(e^{\bar{\delta} Y}(z))\right| |\delta|^{\frac{1+\alpha}{2}} |\delta|^{1-\frac{1+\alpha}{2}} \leq C(z)|\delta|^{\frac{1+\alpha}{2}},
$$

and so $u \in C^{1+\alpha}_{Y,\mathrm{loc}}(\Omega)$. By Definition 2.5 and equation (2.5) we get

$$
\partial_{x_i} u \in C^{1,\alpha}_{B,\mathrm{loc}}(\Omega) \subset C^{0,\alpha}_{B,\mathrm{loc}}(\Omega), \qquad i = 1, \dots, p_0.
$$

Therefore $u \in C^{1,\alpha}_{B,\mathrm{loc}}(\Omega)$.

Now we suppose the thesis true for $k \geq 2$ and we prove it for $k + 1$.

$$
u \in C^{k+1,\alpha}_{B,\mathrm{loc}}(\Omega) \Longrightarrow \begin{cases} Yu \in C^{k-1,\alpha}_{B,\mathrm{loc}}(\Omega) \subset C^{k-2,\alpha}_{B,\mathrm{loc}}(\Omega) \\ \partial_{x_i} u \in C^{k,\alpha}_{B,\mathrm{loc}}(\Omega) \subset C^{k-1,\alpha}_{B,\mathrm{loc}}(\Omega) \quad i = 1, \dots, p_0. \end{cases}
$$

i.e. $u \in C^{k,\alpha}_{B,\mathrm{loc}}(\Omega)$.

Next we prove (2.2). As in the previuos case we prove the thesis directly for $k = 1, 2$ and then use induction to conclude. The key ingredient is the boundness of the function and its derivatives.

Let $u \in C_B^{1,\alpha}(\Omega)$. We have

$$|u(t, x + \delta e_i) - u(t, x)| \leq \begin{cases} \sup |\partial_{x_i} u||\delta|^\alpha, & \text{if } \delta < 1, \\ 2 \sup |u||\delta|^\alpha, & \text{if } \delta \geq 1. \end{cases}$$

Hence $u \in C_{X_i}^\alpha(\Omega)$. Similarly,

$$\left|u\left(e^{\delta Y}(z)\right) - u(z)\right| \leq \begin{cases} \|u\|_{C_Y^{1+\alpha}(\Omega)} |\delta|^{\frac{\alpha}{2}}, & \text{if } \delta < 1, \\ 2 \sup |u||\delta|^{\frac{\alpha}{2}}, & \text{if } \delta \geq 1, \end{cases}$$

and u is also in $u \in C_Y^\alpha(\Omega)$. Being bounded it follows $u \in C_B^{0,\alpha}(\Omega)$. Now, if $u \in C_B^{2,\alpha}(\Omega)$ by the inclusion just proved we get $\partial_{x_i} u \in C_B^{0,\alpha}(\Omega)$. What is left is to prove that $u \in C_Y^{1+\alpha}(\Omega)$. This is done arguing just in the previous case, in fact

$$\left|u\left(e^{\delta Y}(z)\right) - u(z)\right| \leq \begin{cases} \sup |Yu||\delta|^{\frac{1+\alpha}{2}}, & \text{if } \delta < 1, \\ 2 \sup |u||\delta|^{\frac{1+\alpha}{2}}, & \text{if } \delta \geq 1. \end{cases}$$

The induction step can be carried as in the proof for the local version of the spaces. Eventually, the inclusions in (2.3) are a straightforward conseguence of the definitions. □

2.2 The main Theorem

In the very general setting of an homogeneous Lie group $\mathcal{G} \equiv (\mathbb{R}^N, \circ, \delta(\lambda))$ (see Definition 1.3) we say that a function f at the point x_0 has Taylor polynomial $T_n f(x_0, \cdot)$ if $T_n f(x_0, \cdot)$ is a polynomial and

$$f(x) = T_n f(x_0, x) + O(|x_0^{-1} \circ x|_{\mathcal{G}}^{n+\varepsilon}) \qquad \text{as } |x_0^{-1} \circ x|_{\mathcal{G}} \to 0.$$

for some $\varepsilon > 0$ and some $|\cdot|_{\mathcal{G}}$ δ_λ-homogeneous norm.

Existence and uniqueness of such polynomials were proven in [FS] and an (exact) integral expression of the remainder was given in [Bo]. However, in both works the authors assume a global Euclidean regularity, (functions continuosly differentiable up to order $n + 1$) which in our case seems excessive since, as we shall see, not all the derivatives are needed to define the Taylor polynomials.

In order to state our main result, we need to introduce some further notation.

Definition 2.8. For any multi-index $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{N}_0^d$ we define:

- the length $|\beta| := \sum_{j=1}^d \beta_j$, and the factorial $\beta! := \prod_{j=1}^d \beta_j!$;

- for any $i = 0, \cdots, r$, the multi-index $\beta^{[i]} \in \mathbb{N}_0^d$ as

$$\beta_k^{[i]} = \begin{cases} \beta_k & \text{for } \bar{p}_{n-1} < k \le \bar{p}_n, \\ 0 & \text{otherwise;} \end{cases}$$

- the *B-length* $|\beta|_B := \sum_{i=0}^{r} (2i+1) \big|\beta^{[i]}\big|$;

- for any $x \in \mathbb{R} \times \mathbb{R}^d$, the product $x^\beta := x_1^{\beta_1} \cdots x_d^{\beta_d}$;

- the operator $\partial^\beta = \partial_x^\beta := \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d}$.

We are now ready to state our main result and two simple corollaries. All the statements will be proven in chapter three.

Theorem 2.9 *Let $u \in C^{n,\alpha}_{B,loc}(\mathbb{R} \times \mathbb{R}^d)$ with $\alpha \in]0,1]$ and $n \in \mathbb{N}_0$. Then, we have:*

1. *there exist the derivatives*

$$Y^k \partial_x^\beta u \in C^{n-2k-|\beta|_B,\alpha}_{B,loc}(\mathbb{R} \times \mathbb{R}^d), \qquad 0 \le k \le \left\lfloor \frac{n}{2} \right\rfloor, \quad |\beta|_B \le n - 2k;$$

2. *for any $\zeta = (s,\xi) \in \mathbb{R} \times \mathbb{R}^d$, it is well defined the n-*th order B-Taylor polynomial of u around ζ*:*

$$T_n u(\zeta, z) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{|\beta|_B \le n-2k} \frac{1}{k!\,\beta!} Y^k \partial_x^\beta u(\zeta)(t-s)^k \big(x - e^{(t-s)B}\xi\big)^\beta,$$

$z = (t,x) \in \mathbb{R} \times \mathbb{R}^d$, for which the following estimate holds:

$$u(z) = T_n u(\zeta, z) + O\big(\|\zeta^{-1} \circ z\|_B^{n+\alpha}\big), \quad \text{as } \|\zeta^{-1} \circ z\|_B \to 0. \tag{2.6}$$

3. *if $u \in C^{n,\alpha}_B(\mathbb{R} \times \mathbb{R}^d)$, then we have*

$$Y^k \partial_x^\beta u \in C^{n-2k-|\beta|_B,\alpha}_B(\mathbb{R} \times \mathbb{R}^d) \qquad 0 \le k \le \left\lfloor \frac{n}{2} \right\rfloor, |\beta|_B \le n - 2k,$$

and

$$\big|u(z) - T_n u(\zeta, z)\big| \le c_B \|u\|_{C^{n,\alpha}_B} \|\zeta^{-1} \circ z\|_B^{n+\alpha}, \qquad z, \zeta \in \mathbb{R} \times \mathbb{R}^d, \tag{2.7}$$

where c_B is a positive constant that depends on B and n.

Remark 2.10 *If $\mathbb{R} \times \mathbb{R}^d$ is replaced with a generic open subset Ω the bound in* (2.7) *does not holds, in general, for all $z, \zeta \in \Omega$ but it do holds if the points are sufficiently close each others.*

In fact, in order to obtain the estimaties we have to connect z and ζ with integral curves of Y and $X_1, \dots, X_{p_0}$ in a similar way of that used in 1.16. Such curves must lie in Ω and this is, generally speaking, false an example being a disconnected set Ω.

Definition 2.11. We say that $\Omega' \subset \Omega$ is *well contained* in Ω if for every $z, \zeta \in \Omega'$ the integral curves above connecting them lies in Ω.

Arguing as in Proposition 1.16 it can be show that the support of the curves is contained in an open ball whose radius depends only on the distance $\|\zeta \circ z\|_B$. Therefore if Ω' is a subset of Ω whose diameter is small enough then Ω' is well contained in Ω.

A straightforward corollary of the theorem is the following:

Corollary 2.12 *If $u \in C^{0,\alpha}_{B,loc}(\Omega)$ then u is in fact locally Hölder continuos of order α wrt the intrinsic distance d_B of $\mathbb{R} \times \mathbb{R}^d$, in particular u is continuos. More explicitly*

$$|u(z) - u(\zeta)| \leq C\|\zeta^{-1} \circ z\|_B^\alpha$$

for every z, ζ in a well contained subset of Ω. If $u \in C^{0,\alpha}_B(\mathbb{R} \times \mathbb{R}^d)$ then u is globally Hölder continuos of order α.

A less obvious corollary regards the existance of the time derivative for functions in $C^{n,\alpha}_{B,\mathrm{loc}}(\Omega)$

Corollary 2.13 *If $u \in C^{2r+1,\alpha}_{B,loc}(\Omega)$ then there exists $\partial_t u(z)$ for every $z \in \Omega$, $\partial_t u \in C^{0,\alpha}_{B,loc}(\Omega)$ and*

$$\partial_t u(t, x) = Yu(t, x) - \langle \nabla u(t, x), Bx \rangle.$$

In particular if $u \in C^{2r+1,\alpha}_B(\Omega)$ then $\partial_t u \in C^{0,\alpha}_B(\Omega)$.

For the homogeneous structure presented in Example 1.7 the first B-Taylor polynomials are

$$\begin{aligned}
T_0 u(\zeta, z) =& u(\zeta);\\
T_1 u(\zeta, z) =& T_0 u(\zeta, z) + \partial_{x_1} u(\zeta)(x_1 - \xi_1);\\
T_2 u(\zeta, z) =& T_1 u(\zeta, z) + \frac{1}{2!}\partial_{x_1,x_1} u(\zeta)(x_1 - \xi_1)^2 + Yu(\zeta)(t - s)u\\
T_3 u(\zeta, z) =& T_2 u(\zeta, z) + \frac{1}{3!}\partial^3_{x_1} u(\zeta)(x_1 - \xi_1)^3 + Y\partial_{x_1} u(\zeta)(x_1 - \xi_1)^2(t - s)\\
&+ \partial_{x_2} u(\zeta)(x_2 - \xi_2 - (t - s)\xi_1);\\
T_4 u(\zeta, z) =& T_3 u(\zeta, z) + \frac{1}{4!}\partial^4_{x_1} u(\zeta)(x_1 - \xi_1)^4 + \frac{1}{2!}Y\partial^2_{x_1} u(\zeta)(x_1 - \xi_1)^2(t - s)\\
&+ \frac{1}{2!}Y^2 u(\zeta)(t - s)^2 + \partial_{x_2}\partial_{x_1} u(\zeta)(x_1 - \xi_1)(x_2 - \xi_2 - (t - s)\xi_1);
\end{aligned}$$

$$\begin{aligned} T_5u(\zeta,z) =& T_4u(\zeta,z) + \frac{1}{5!}\partial^5_{x_1}u(\zeta)(x_1-\xi_1)^5 + \frac{1}{3!}Y\partial^3_{x_1}u(\zeta)(x_1-\xi_1)^3(t-s) \\ &+ \frac{1}{2!}Y^2\partial_{x_1}u(\zeta)(x_1-\xi_1)(t-s)^2 \\ &+ \frac{1}{2!}\partial_{x_2}\partial^2_{x_1}u(\zeta)(x_1-\xi_1)^2(x_2-\xi_2-(t-s)\xi_1) \\ &+ Y\partial_{x_2}u(\zeta)(x_2-\xi_2-(t-s)\xi_1)(t-s); \\ T_6u(\zeta,z) =& \dots \end{aligned}$$

The proof of Theorem 2.9 is postponed to chapter three.

2.3 Other Hölder spaces

Suitable Hölder spaces for the operator 1.1 were used in the works of Manfredini [M], Di Francesco and Polidoro [DP] and Frentz and others [Fo] to obtain Shauder-type estimates. In the first two papers only the analogue of our spaces $C^{0,\alpha}_B$ and $C^{2,\alpha}_B$ were defined while in [Fo] also an analogue of the space $C^{1,\alpha}_B$ was used.

All the various definitions of the authors coincide for the space of order zero that we will call $C^\alpha(\Omega,B)$ following the notations in [M]. For functions defined in an open subset Ω of $\mathbb{R}\times\mathbb{R}^d$ they require the boundness and Hölder condition of order α wrt the underlying homogeneous structure. Equivalently the norm $|\cdot|_{\alpha,B,\Omega}$ defined as

$$|u|_{\alpha,B,\Omega} := \sup_{z\in\Omega}|u(z)| + \sup_{\substack{z,\zeta\in\Omega\\ z\neq\zeta}}\frac{|u(z)-u(\zeta)|}{\|\zeta^{-1}\circ z\|^\alpha_B}, \tag{2.8}$$

must be finite.

Proposition 2.14 *We have the following inclusions:*

1. $C^\alpha(\Omega,B)\subset C^{0,\alpha}_B(\Omega)$;
2. $C^{0,\alpha}_B(\Omega)\subset C^\alpha(\Omega',B)$.

Here Ω' is any well contained subset of Ω. In particular $C^{0,\alpha}_B(\mathbb{R}\times\mathbb{R}^d) = C^\alpha(\mathbb{R}\times\mathbb{R}^d,B)$.

Proof. Suppose $u\in C^\alpha(\Omega,B)$ i.e. u is bounded and Hölderian in Ω. If in (2.8) we choose $z=e^{\delta X_i}(\zeta)$ and $z=e^{\delta Y}(\zeta)$ we immediatly see that $u\in C^\alpha_{X_i}(\Omega)$ and $u\in C^\alpha_Y(\Omega)$ and therefore it is in $C^{0,\alpha}_B(\Omega)$. This follows from

$$\|(e^{\delta X_i}(\zeta))^{-1}\circ z\|^\alpha_B = |\delta|^\alpha, \quad \|(e^{\delta Y}(\zeta))^{-1}\circ z\|^\alpha_B = |\delta|^{\frac{\alpha}{2}}. \tag{2.9}$$

The content of the second claim is part of Corollary 2.12. □

We point out that by equations (2.9) the requirements $u \in C^{\alpha}_{Y}(\Omega)$ and $C^{\alpha}_{X_i}(\Omega)$ are in fact an Hölder condition along the integral curves of Y, X_i, $i = 1, \dots, p_0$. The main theorem then say that regularity along such integral curves is sufficient to recover full (local) Hölderianity.

We explicitly remark that, for functions in $C^{\alpha}(\Omega, B)$ Theorem 2.9 is a straightforward consequence of the definition (2.8). The estimate in the main theorem for intrinsic Taylor polynomials of order one is also built-in in the definition of the space $C^{1+\alpha}(\Omega, B)$ in [Fo].

A function u is in $C^{1+\alpha}(\Omega, B)$ if the norm

$$|u|_{1+\alpha,B,\Omega} := |u|_{\alpha,B,\Omega} + \sum_{i=1}^{p_0} |\partial_{x_i} u|_{\alpha,B,\Omega} + \sup_{\substack{z,\zeta\in\Omega \\ z\neq\zeta}} \frac{|u(z) - T_1 u(\zeta, z)|}{\|\zeta^{-1} \circ z\|_B^{1+\alpha}} \tag{2.10}$$

is finite.

Proposition 2.15 *We have the following inclusions:*

1. $C^{1+\alpha}(\Omega, B) \subset C^{1,\alpha}_{B}(\Omega)$*;*

2. $C^{1,\alpha}_{B}(\Omega) \subset C^{1+\alpha}(\Omega', B)$ *if* Ω' *is a well contained subset of* Ω.

Proof. We first prove part 1. By the definition of (2.10) it follows that functions in $C^{1+\alpha}(\Omega, B)$ are bounded and their derivatives wrt the first p_0 spatial variables are in $C^{\alpha}(\Omega, B)$ which, by Proposition 2.14 is contained in $C^{0,\alpha}_{B}(\Omega)$. We are left to prove that such functions are also in $C^{1+\alpha}_{Y}(\Omega)$.

If in (2.10) we choose $z = e^{\delta Y}(\zeta)$ by (2.9) we get

$$|u(e^{\delta Y}(\zeta)) - T_1 u(\zeta, e^{\delta Y}(\zeta))| \le |u|_{1+\alpha,B,\Omega} |\delta|^{\frac{1+a}{2}}$$

and because the integral curves of Y do not act on the first p_0 spatial variables we have $T_1 u(\zeta, e^{\delta Y}(\zeta)) = u(\zeta)$ and the thesis follows. To prove part 2 let Ω' be an open well contained subset of Ω. By Definition 2.4 and Propositions 2.14, 2.7 we get

$$\partial_{x_i} u \in C^{0,\alpha}_{B}(\Omega) \subset C^{\alpha}(\Omega', B), \qquad u \in C^{1,\alpha}_{B}(\Omega) \subset C^{0,\alpha}_{B}(\Omega) \subset C^{\alpha}(\Omega', B).$$

This say that the first two terms in the definition of $|\cdot|_{1+\alpha,B,\Omega'}$ are bounded. Finally, by Theorem 2.9, also the third term is bounded since Ω' is well contained in Ω. $\square$

Various analogues of the space $C^{2,\alpha}_{B}(\Omega)$ are used in the literature. In [M] Manfredini essentially requires bounded Hölder continuos second order derivatives while Di Francesco and Polidoro in [DP] and Frentz and others in [Fo] requires also the function and its first p_0 spatial derivatives to be Hölder continuos. Precisely the norms used were

$$|u|^{(M)}_{2+\alpha,B,\Omega} := \sup_\Omega |u| + \sum_{i=1}^{p_0} \sup_\Omega |\partial_{x_i} u| + \sum_{i,j=1}^{p_0} |\partial_{x_i,x_j} u|_{\alpha,B,\Omega} + |Yu|_{\alpha,B,\Omega} \quad (2.11)$$

(here M stands for Manfredini) and

$$|u|_{2+\alpha,B,\Omega} := |u|_{\alpha,B,\Omega} + \sum_{i=1}^{p_0} |\partial_{x_i} u|_{\alpha,B,\Omega} + \sum_{i,j=1}^{p_0} |\partial_{x_i,x_j} u|_{\alpha,B,\Omega} + |Yu|_{\alpha,B,\Omega}. \quad (2.12)$$

The norm $|\cdot|^{(M)}_{2+\alpha,B,\Omega}$ is more aderent to the classical Euclidean definition of the Hölder space $C^{2,a}$.

While our spaces $C^{0,\alpha}_B(\Omega)$, $C^{1,\alpha}_B(\Omega)$ were greater than the analogue spaces used for $C^{2,\alpha}_B(\Omega)$ this inclusion is reversed. Essentially the problem relies in the function $\partial_{x_i} u$. If the norm $|u|^{(M)}_{2+\alpha,B,\Omega}$ is finite we deduce that $\partial_{x_i} u$ is bounded and has partial derivatives wrt to first p_0 variables Hölder continuos but we can not deduce Hölderianity along the integral curves of Y which act on the others directions. In the case of the norm $|u|_{2+\alpha,B,\Omega}$ instead the problem is more deceitful. We do know that $\partial_{x_i} u$ is Hölder but the order is wrong. We have $\partial_{x_i} u \in C^\alpha_Y$ while we would need $\partial_{x_i} u \in C^{1+\alpha}_Y$.

Precisely we have

$$|\partial_{x_i} u(e^{\delta Y}(z)) - \partial_{x_i} u(z)| \le C_1 |\delta|^{\frac{a}{2}} \nleq C_2 |\delta|^{\frac{1+a}{2}} \qquad \text{as } \delta \to 0.$$

However the other inclusion still stand.

Proposition 2.16 *We have the following inclusions:*

1. $C^{2,\alpha}_B(\Omega) \subset C^{2+\alpha}(\Omega', B)^{(M)}$;
2. $C^{2,\alpha}_B(\Omega) \subset C^{2+\alpha}(\Omega', B)$.

Here Ω' is any well contained subset of Ω, $C^{2+\alpha}(\Omega', B)$ is the space defined by the norm in (2.12) *and $C^{2+\alpha}(\Omega', B)^{(M)}$ the one defined by* (2.11).

Proof. By definition all the derivatives that appear in (2.11) and (2.12) are bounded in Ω. Moreover since

$$\partial_{x_i,x_j} u,\ Yu \in C^{0,\alpha}_B(\Omega) \qquad i, j = 1, \dots, p_0$$

by the first inclusion in Propositon 2.14 we get

$$\sum_{i,j=1}^{p_0} |\partial_{x_i,x_j} u|_{\alpha,B,\Omega'} + |Yu|_{\alpha,B,\Omega'} < \infty$$

for any well containde subset Ω' of Ω and this conclude part 1. Since $C^{1,\alpha}_B(\Omega) \subset C^{0,\alpha}_B(\Omega)$, part 2 follows. □

Chapter 3

Proof of the main Theorem

In this chapter we prove the main theorem 2.9 and its corollaries stated in chapter two. In order to avoid to confine ourselves with points of a well-contained subset Ω' of Ω we deal only whit the case $\Omega = \mathbb{R} \times \mathbb{R}^d$ and hereafter omit it in the spaces $C^{n,\alpha}_{B,\mathrm{loc}}$, $C^{n,\alpha}_{B}$, see Remark 2.10. The proof given still works for general Ω.

Theorem 2.9 will be proved by induction on n, through the following steps:

- **Step 1**: Proof for $n = 0$;
- **Step 2**: Induction from $2n$ to $2n + 1$ for any $0 \le n \le r$;
- **Step 3**: Induction from $2n + 1$ to $2(n + 1)$ for any $0 \le n \le r - 1$;
- **Step 4**: Induction from n to $n + 1$ for any $n \ge 2r + 1$.

In order to prove the main theorem we need to state two complementary results which will be proved according to the steps above along with Theorem 2.9.

Proposition 3.1 *Let $u \in C^{n,\alpha}_{B,loc}$ with $\alpha \in]0, 1]$ and $n \in \mathbb{N}_0$, $n \le 2r + 1$ and set $s = \max\{\lfloor \frac{n}{2} \rfloor - 1, 0\}$. Then, for any $s \le k \le r$ and $v \in V_{0,k}$ with $|v| = 1$, we have:*

$$u\big(\gamma^{(s,k)}_{v,\delta}(z)\big) = T_n u\big(z, \gamma^{(s,k)}_{v,\delta}(z)\big) + O\big(|\delta|^{n+\alpha}\big), \quad \text{as } \delta \to 0. \tag{3.1}$$

In particular, if $u \in C^{n,\alpha}_{B}$, then for all $z = (t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $\delta \in \mathbb{R}$ we also have:

$$\Big| u\big(\gamma^{(s,k)}_{v,\delta}(z)\big) - T_n u\big(z, \gamma^{(s,k)}_{v,\delta}(z)\big) \Big| \le c_B \|u\|_{C^{n,\alpha}_{B}} |\delta|^{n+\alpha}, \tag{3.2}$$

where c_B is a positive constant that only depends on B.

Proposition 3.2 *Let $u \in C^{n,\alpha}_{B,loc}$ with $\alpha \in]0,1]$ and $n \in \mathbb{N}_0$. Then, for any $0 \le k < \lfloor \frac{n}{2} \rfloor$ we have:*

$$u(t,x) = T_n u\big((t,\xi),(t,x)\big) + O\big(|x-\xi|_B^{n+\alpha}\big), \quad \text{as } |x-\xi|_B \to 0,$$

for any $x, \xi \in \mathbb{R}^d$ such that $\xi^{[i]} = x^{[i]}$ for any $i > k$. In particular, if $u \in C^{n,\alpha}_B$, then we also have:

$$\big|u(t,x) - T_n u\big((t,\xi),(t,x)\big)\big| \le c_B \|u\|_{C^{n,\alpha}_B} |x-\xi|_B^{n+\alpha}, \qquad t \in \mathbb{R}, \quad x, \xi \in \mathbb{R}^d,$$

where c_B is a positive constant that only depends on B.

N.B. Hereafter, in the following proofs, we will denote by c_B any positive constant that *only depends on B*.

A brief explanation is needed: the proof of Theorem 2.9 can not be carried in a single induction because of the qualitative differences in the polynomials for different orders. For example suppose the theorem true for $n = 2$ and take a function $u \in C^{3,\alpha}_{B,\mathrm{loc}}$. By the inclusion

$$C^{3,\alpha}_{B,\mathrm{loc}} \subset C^{2,\alpha}_{B,\mathrm{loc}}$$

we get that

$$Y^k \partial_x^\beta u \in C^{2-2k-|\beta|_B,\alpha}_{B,\mathrm{loc}}, \qquad 2k + |\beta|_B \le 2,$$

but $T_3 u$ contains also derivatives of B-order 3. These are exactly

$$\partial_{x_i,x_j,x_k} u, \quad Y\partial_{x_i} u, \quad \partial_{x_l} u, \quad 1 \le i,j,k \le p_0, \; p_0 < l \le \bar{p}_1.$$

While the first two kinds exists by definitions of $C^{3,\alpha}_{B,\mathrm{loc}}$ we have no a priori information on ∂_{x_l} and must prove its existence. Such problem arises for all orders of the type $2k+1$, $k = 1, \dots, r$ i.e. when derivatives wrt a higher level variables appear and this explain the difference between Step 2 and Step 3.

If $n \ge 2r+1$ we have derivatives wrt all the variables and no further problems. This explains step 4.

Propositions 3.1 and 3.2 are particular cases of the main theorem and are preparatory to its proof. Precisely, in the case $n = 0$, having estimates only along the integral curves of the vector fields Y and ∂_{x_i} for $i = 1, \dots, p_0$ to estimate $u(z) - u(\zeta)$ we must connect the points z, ζ using such integral curves: this lead to the functions $\gamma^{(k)}_{v,\delta}$ and to Propositions 3.1.

As n increase derivatives wrt higher level variables become avaible. Therefore we can estimate directly $u(t,x) - T_n u(t,\xi)$ if the only non-zero increments are those relative to such derivatives: this is the content of Proposition 3.2. Otherwise we rely on the functions $\gamma^{(n,k)}_{v,\delta}$ and Proposition 3.1.

We will prove Proposition 3.1 and Theorem 2.9 directily for $n = 0$ in step 1. Then, assuming the main theorem true for $0 \le i \le 2n$ (respectively $0 \le i \le 2n+1$), we will prove the complementary propositions for $2n+1$ (resp. $2n+2$) and use them to prove the theorem for the same order in step 2 (resp. 3). Accordingly with the previous discussion the proof of the main theorem in step 4 (order greater than $2r+1$) will have no need of Proposition 3.1.

3.1 Step 1

Here we prove Proposition 3.1 and Theorem 2.9 for $n = 0$. Note that:

$$T_0 u\big(z, \zeta\big) = u(z), \qquad z, \zeta \in \mathbb{R} \times \mathbb{R}^d.$$

Proof of Propostion 3.1 for $n = 0$.

We prove the thesis by induction on k. For $k = 0$ the estimate (3.1) (and (3.2)) trivially follows from (1.16), along with the assumptions $v \in V_0$, $|v| = 1$, and since $u \in C^{\alpha}_{\partial_{x_i},\mathrm{loc}}$ (or respectively $u \in C^{\alpha}_{\partial_{x_i}}$) for any $i = 1, \dots, p_0$.

Assume now the thesis to hold for $k \ge 0$, and we prove it for $k+1$. We recall (1.17) and set

$$z_0 = z, \quad z_1 = \gamma^{(k)}_{v,\delta}(z_0), \quad z_2 = e^{\delta^2 Y}\left(z_1\right),$$
$$z_3 = \gamma^{(k)}_{v,-\delta}(z_2), \quad z_4 = e^{-\delta^2 Y}\left(z_3\right) = \gamma^{(k+1)}_{v,\delta}(z).$$

Now, by the triangular inequality we get

$$\left|u\big(\gamma^{(k+1)}_{v,\delta}(z)\big) - u(z)\right| \le \sum_{i=1}^{4} \left|u(z_i) - u(z_{i-1})\right|,$$

and thus, (3.1) and (3.2) for $k+1$ follow from the inductive hypothesis and from the assumptions $u \in C^{\alpha}_{Y,\mathrm{loc}}$ and $u \in C^{\alpha}_{Y}$ respectively. □

We are now ready to prove Theorem 2.9 for $n = 0$.

Proof of Theorem 2.9 for $n = 0$.

We only need to prove Part 2 and Part 3. We first consider the particular case $z = (t, x), \zeta = (t, \xi)$, with $x, \xi \in \mathbb{R}^d$. Precisely, we show that, if $u \in C^{0,\alpha}_{B,\mathrm{loc}}$ we have

$$u(t, x) = u(t, \xi) + O\big(|x - \xi|^{\alpha}_{B}\big), \quad \text{as } |x - \xi|_B \to 0, \tag{3.3}$$

and that, in particular, if $u \in C^{0,\alpha}_{B}$ we have

$$|u(t, x) - u(t, \xi)| \le c_B \|u\|_{C^{0,\alpha}_{B}} |x - \xi|^{\alpha}_{B}, \qquad t \in \mathbb{R}, \quad x, \xi \in \mathbb{R}^d. \tag{3.4}$$

We define the sequence of points $\big(z_k = (t, x_k)\big)_{k=0,\cdots,r}$ as in the proof of Proposition 1.16. The proof of (3.3) (and (3.4)) can be easily concluded by using (3.1) (and (3.2)) for $n = 0$

We now prove the general case. For any $z = (t, x), \zeta = (s, \xi) \in \mathbb{R} \times \mathbb{R}^d$, by triangular inequality we get

$$\begin{aligned} |u(z) - u(\zeta)| \leq & |u(z) - u(e^{(t-s)Y}(\zeta))| + |u(e^{(t-s)Y}(\zeta)) - u(\zeta)| \\ = & |u(t, x) - u(t, e^{(t-s)B}\xi)| + |u(e^{(t-s)Y}(\zeta)) - u(\zeta)|. \end{aligned} \tag{3.5}$$

Now, to prove (2.6), we use (3.3) to bound the first term in (3.5), $u \in C^{\alpha}_{Y,\text{loc}}$ to bound the second one, and we obtain

$$|u(z) - u(\zeta)| \leq O\big(|e^{(t-s)B}\xi|^{\alpha}_B\big) + O\big(|t - s|^{\frac{\alpha}{2}}\big) = O\big(\|\zeta^{-1} \circ z\|^{\alpha}_B\big),$$

as $\|\zeta^{-1} \circ z\|_B \to 0$.

Eventually, to prove (2.7), we use (3.4) to bound the first term in (3.5), $u \in C^{\alpha}_Y$ to bound the second one, and we obtain

$$|u(z) - u(\zeta)| \leq c_B \|u\|_{C^{0,\alpha}_B} \|\zeta^{-1} \circ z\|^{\alpha}_B, \qquad z = (t, x), \zeta = (t, \xi) \in \mathbb{R} \times \mathbb{R}^d,$$

which concludes the proof. □

3.2 Step 2

Here we fix $n \in \{0, \cdots, r\}$, we assume Theorem 2.9 and Propositions 3.1, 3.2 to hold for any $0 \leq i \leq 2n$, and we prove them to be true for $2n + 1$. This induction step has to be treated separately because we do not know a priori that the euclidean derivatives wrt the n-th level variables, $\big(\partial_{\bar{p}_{n-1}+i} u\big)_{1 \leq i \leq p_n}$, do exist.

We introduce the following alternative definition of $(2n + 1)$-th order B-Taylor polynomial of u, which will be proved to be equivalent to T_{2n+1}:

$$\begin{aligned} \bar{T}_{2n+1} u(\zeta, z) := & \sum_{k=0}^{\lfloor \frac{2n+1}{2} \rfloor} \sum_{\substack{|\beta|_B \leq 2n+1-2k \\ \beta_{2n+1}=0}} \frac{1}{k!\,\beta!} Y^k \partial^{\beta}_x u(\zeta)(t - s)^k \big(x - e^{(t-s)B}\xi\big)^{\beta} \\ & + \sum_{i=1}^{p_n} Y^{(n)}_{v^{(n)}_i} u(\zeta)\big(x_{\bar{p}_{n-1}+i} - (e^{B(t-s)}\xi)_{\bar{p}_{n-1}+i}\big), \qquad z = (t, x) \in \mathbb{R} \times \mathbb{R}^d, \end{aligned}$$

with $\big(v^{(n)}_i\big)_{1 \leq i \leq p_n}$ being the family of vectors such that $v^{(n)}_i \in V_{0,n}$ with $B^n v^{(n)}_i = e_{\bar{p}_{n-1}+i}$.

More explicitely we will prove that, for a function $u \in C^{2n+1,\alpha}_{B,\text{loc}}$

$$Y^{(n)}_{v^{(n)}_i} u(z) = \partial_{\bar{p}_{n-1}+i} u(z), \qquad z \in \mathbb{R} \times \mathbb{R}^d.$$

Note that the polynomial is well defined i.e. all the derivatives appearing in it exist for $u \in C^{2n+1,\alpha}_{B,\mathrm{loc}}$ and in particular for $u \in C^{2n+1,\alpha}_{B}$. This follows directily from the definition of $C^{2n+1,\alpha}_{B,\mathrm{loc}}$ for the operators $Y^{(n)}_{v^{(n)}_i}$ and by the inductive hypothesis on the main theorem for the derivatives $Y^k\partial^\beta_x$ since by Proposition 2.7 $C^{2n+1,\alpha}_{B,\mathrm{loc}} \subset C^{2n,\alpha}_{B,\mathrm{loc}}$.

Remark 3.3 *We explicitly observe that, by Definition 2.4, $u \in C^{m,\alpha}_{B,loc}$ implies $Y^{\lfloor \frac{m}{2} \rfloor} u \in C^{\alpha}_{Y,loc}$ if m is even and $Y^{\lfloor \frac{m}{2} \rfloor} u \in C^{1+\alpha}_{Y,loc}$ if m is odd. In both cases, by the euclidean mean value theorem along the vector field Y, for any $z = (t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $\delta \in \mathbb{R}$ small enough there exist $\bar\delta$ with $|\bar\delta| \leq |\delta|$ such that*

$$\begin{aligned} u\big(e^{\delta Y}(z)\big) - u(z) - \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \frac{\delta^i}{i!} Y^i u(z) =& \delta^{\lfloor \frac{m}{2} \rfloor} \Big(Y^{\lfloor \frac{m}{2} \rfloor} u\big(e^{\bar\delta Y}(z)\big) - Y^{\lfloor \frac{m}{2} \rfloor} u(z)\Big) \\ =& \begin{cases} O(|\delta|^{\lfloor \frac{m}{2} \rfloor + \frac{1+\alpha}{2}}), & \text{if } m \text{ is odd;} \\ O(|\delta|^{\lfloor \frac{m}{2} \rfloor + \frac{\alpha}{2}}), & \text{if } m \text{ is even,} \end{cases} \end{aligned}$$

where the bounds hold in a neighbourhood of zero. If in particular $u \in C^{m,\alpha}_B$

$$\Big| u\big(e^{\delta Y}(z)\big) - u(z) - \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \frac{\delta^i}{i!} Y^i u(z) \Big| \leq \begin{cases} c_B \|u\|_{C^{m,\alpha}_B} |\delta|^{\lfloor \frac{m}{2} \rfloor + \frac{1+\alpha}{2}}, & \text{if } m \text{ is odd;} \\ c_B \|u\|_{C^{m,\alpha}_B} |\delta|^{\lfloor \frac{m}{2} \rfloor + \frac{\alpha}{2}}, & \text{if } m \text{ is even.} \end{cases}$$

Proof of Proposition 3.1 for $2n+1$.

Here we prove Proposition 3.1 for $\bar T_{2n+1}u$ by induction on k. We begin with the local version, precisely we want to prove that for any $\max\{\lfloor \frac{2n+1}{2} \rfloor - 1, 0\} \leq k \leq r$ and $v \in V_{0,k}$ with $|v| = 1$, we have:

$$u\Big(\gamma^{(\lfloor \frac{2n+1}{2} \rfloor - 1, k)}_{v,\delta}(z)\Big) = T_{2n+1} u\Big(z, \gamma^{(\lfloor \frac{2n+1}{2} \rfloor - 1, k)}_{v,\delta}(z)\Big) + O\big(|\delta|^{2n+1+\alpha}\big), \quad (3.6)$$

as $\delta \to 0$.

Because of the particular definition of $\gamma^{(n,k)}_{v,\delta}$ we have to analyze separately the cases $n = 0$, $n = 1$ and $n > 1$. We begin proving (3.6) directily for $k = \max\{\lfloor \frac{2n+1}{2} \rfloor - 1, 0\}$.

Case $n = 0$

We have $k = 0$ and equation 3.6 rewrites as

$$u(t, x + \delta v) = u(t, x) + \sum_{i=1}^{p_0} \partial_{x_i} u(t, x)\delta v_i + O\big(|\delta|^{1+\alpha}\big), \quad \text{as } \delta \to 0.$$

There exist, by the multidimensional mean value theorem, a family of vectors $(\bar{v}_i)_{i=1,\dots,p_0}$ with $\bar{v}_i \in V_0$ and $|\bar{v}_i| \le 1$ such that

$$\begin{aligned}
u(t,x+\delta v) - u(z) - \sum_{i=1}^{p_0} \partial_{x_i} u(t,x)\delta v_i =& \delta \sum_{i=0}^{p_0} (\partial_{x_i} u(t,x+\delta\bar{v}_i) - \partial_{x_i} u(t,x)) v_i \\
=& \delta O(|\delta v|_B^\alpha) \quad \text{as } \delta \to 0 \qquad (u \in C^\alpha_{\partial x_i}) \\
=& O(|\delta|^{1+\alpha}) \qquad \text{as } \delta \to 0.
\end{aligned}$$

Case $n = 1$

Also in this case we have $k = 0$ but this time we have to prove that

$$\begin{aligned}
u(t,x+\delta v) = u(t,x) + \sum_{i=1}^{p_0} \partial_{x_i} u(t,x)\delta v_i + \frac{\delta^2}{2!} \sum_{i,j=1}^{p_0} \partial_{x_i x_j} u(t,x) v_i v_j + \\
\frac{\delta^3}{3!} \sum_{i,j,l=1}^{p_0} \partial_{x_i x_j x_l} u(t,x) v_i v_j v_l + O\big(|\delta|^{3+\alpha}\big), \quad \text{as } \delta \to 0.
\end{aligned}$$

Again, by the multidimensional euclidean mean-value theorem, there exist $(\bar{v}_{i,j,k})_{1 \le i,j,k \le p_0}$, with $\bar{v}_{i,j,k} \in V_0$ and $|\bar{v}_{i,j,k}| \le 1$, such that

$$\begin{aligned}
& u(t,x+\delta v) - \bar{T}_3 u((t,x),(t,x+\delta v)) \\
&\quad = \frac{\delta^3}{3!} \sum_{i,j,k=1}^{p_0} \big(\partial_{x_i,x_j,x_k} u(t,x+\delta\bar{v}_{i,j,k}) - \partial_{x_i,x_j,x_k} u(z)\big) v_i v_j v_k \\
&\qquad = \delta^3 O(|\delta v|_B^\alpha) \quad \text{as } \delta \to 0 \qquad (\partial_{x_i,x_j,x_k} u \in C_B^{0,\alpha}) \\
&\qquad\qquad = O(|\delta|^{3+\alpha}) \quad \text{as } \delta \to 0.
\end{aligned}$$

Case $n > 1$

This time $k = n - 1 > 0$. We have to prove that for any $v \in V_{0,k}$ with $|v| = 1$,

$$\begin{aligned}
& u(t,x+\delta^{2n-1} B^{n-1} v) = u(t,x) + \\
& \delta^{2n-1} \sum_{i=1}^{p_{n-1}} \partial_{x_{\bar{p}_{n-2}+i}} u(t,x) (B^{n-1}v)_{\bar{p}_{n-2}+i} + O\big(|\delta|^{2n+1+\alpha}\big), \quad \text{as } \delta \to 0. \quad (3.7)
\end{aligned}$$

Note that the derivatives in (3.7) are of level strictly greater than one and exist thanks to the inductive hypothesis on Theorem 2.9 and not because of the definition of the spaces $C^{2n+1,\alpha}_{B,\text{loc}}$ as in the previous cases.

By the multidimensional mean-value theorem, there exist a family of vectors $(\bar{v}_i)_{1\leq i\leq p_{n-1}}$, with $\bar{v}_i \in V_{0,n-1}$ and $|B^{n-1}\bar{v}_i| \leq |B^{n-1}v|$ such that

$$u(t,x+\delta^{2n-1}B^{n-1}v)-u(t,x) = \sum_{i=1}^{p_{n-1}} \partial_{x_{\bar{p}_{n-2}+i}} u(t,x+\delta^{2n-1}B^{n-1}\bar{v}_i)\delta^{n-1}B^{n-1}v_i.$$

Therefore

$$\begin{aligned} u\Big(\gamma_{v,\delta}^{(n-1,n-1)}(t,x)\Big) - T_n u\Big(z,\gamma_{v,\delta}^{(n-1,n-1)}(t,x)\Big) = \\ \sum_{i=1}^{p_{n-1}} \Big(\partial_{x_{\bar{p}_{n-2}+i}} u(t,x+\delta^{2n-1}B^{n-1}\bar{v}_i) - \partial_{x_{\bar{p}_{n-2}+i}} u(t,x)\Big)\,\delta^{2n-1}(B^{n-1}v)_i \\ = O(|\delta|^{2+\alpha})\delta^{2n-1}\sum_{i=1}^{p_{n-1}}(B^{n-1}v)_i \qquad \text{as } \delta\to 0 \\ = O(|\delta|^{2n+1+\alpha}) \quad \text{ as } \delta \to 0. \end{aligned}$$

Where we have used Theorem 2.9 in the second line since
$\partial_{x_{\bar{p}_{n-2}+i}} u(t,x) = T_2\partial_{x_{\bar{p}_{n-2}+i}} u((t,x),(t,x+\delta^{2n-1}B^{n-1}\bar{v}_i))$.

Inductive Step

Now we suppose the thesis true for a fixed $k \geq \max\{\lfloor\frac{2n+1}{2}\rfloor - 1, 0\}$ and we prove it for $k+1$. We set $\widetilde{T}_{2n+1}u(\zeta,z) = \bar{T}_{2n+1}u(\zeta,z) - u(\zeta)$ and

$$z_0 = z, \quad z_1 = \gamma_{v,\delta}^{(\lfloor\frac{2n+1}{2}\rfloor-1,k)}(z_0), \quad z_2 = e^{\delta^2 Y}(z_1),$$

$$z_3 = \gamma_{v,\delta}^{(\lfloor\frac{2n+1}{2}\rfloor-1,k)}(z_2), \quad z_4 = e^{-\delta^2 Y}(z_3) = \gamma_{v,\delta}^{(\lfloor\frac{2n+1}{2}\rfloor-1,k+1)}(z),$$

where $v \in V_{0,k+1}$, $|v| = 1$. Whit this notations we have

$$\begin{aligned} u(z_4) - \bar{T}_{2n+1}u(z_0,z_4) = {} & u(z_4) - u(z_3) - \sum_{i=1}^{n}\frac{(-\delta^2)^i}{i!}Y^i u(z_3) \\ & + u(z_3) - u(z_2) - \widetilde{T}_{2n+1}u(z_2,z_3) \\ & + \sum_{i=1}^{n}\frac{(-\delta^2)^i}{i!}Y^i u(z_2) + u(z_2) - u(z_1) \\ & + \widetilde{T}_{2n+1}u(z_1,z_0) + u(z_1) - u(z_0) \\ & + \sum_{i=1}^{n}\frac{(-\delta^2)^i}{i!}\left(Y^i u(z_3) - Y^i u(z_2)\right) \\ & + \widetilde{T}_{2n+1}u(z_2,z_3) - \widetilde{T}_{2n+1}u(z_1,z_0) - \widetilde{T}_{2n+1}u(z_0,z_4). \end{aligned} \tag{3.8}$$

By Remark 3.3 the first and the third term are $O(|\delta|^{2n+1+\alpha})$ as $\delta \to 0$ and the same bound for the second and the fourth term follows from the inductive hypothesis (note that, by (1.12) $V_{0,k+1} \subset V_{0,k}$). In order to estimate the last terms we need once again to distinguish various cases depending on n and k.

Case $n = 0$

In this case the sums that appear in (3.8) are void and we are left whit the estimate of the last term. Note that, by (1.20), $\widetilde{T}_1 u(z_0, z_4) \equiv 0$ for all k while $\widetilde{T}_1 u(z_2, z_3) - \widetilde{T}_1 u(z_1, z_0) \equiv 0$ only if $k > 0$. If $k = 0$ we have

$$\begin{aligned}\widetilde{T}_1 u(z_2, z_3) - \widetilde{T}_1 u(z_1, z_0) &= -\delta \sum_{i=0}^{p_0} (\partial_{x_i} u(z_2) - \partial_{x_i} u(z_1)) v_i \\ &= O(|\delta|^{1+\alpha}), \quad \text{as } \delta \to 0,\end{aligned}$$

since $\partial_{x_i} u \in C^{\alpha}_{Y,\mathrm{loc}}$.

Case $n = 1$

First suppose $k = 0$. Since $v \in V_{0,1}$ and $Bv_i^{(1)} = e_i^{[1]}$, then $v = \sum_{i=1}^{p_1} \left(Bv\right)_i v_i^{(1)}$ and the sum of the last two terms in (3.8) equals $F_1 + \cdots + F_7$ with each $F_j = O(|\delta|^{3+\alpha})$ as $\delta \to 0$. Precisely,

$$F_1 = (-\delta^2)(Yu(z_3) - T_1 Yu(z_2, z_3)) \qquad (Yu \in C^{1,\alpha}_{B,\mathrm{loc}})$$

$$F_2 = \delta^3 \left((\nabla \cdot v) Yu(z_2) - (\nabla \cdot v) Yu(z_1)\right) \qquad ((\nabla \cdot v) Yu \in C^{\alpha}_{Y,\mathrm{loc}})$$

$$F_3 = -\delta \left((\nabla \cdot v) u(z_2) - (\nabla \cdot v) u(z_1) - \delta^2 Y (\nabla \cdot v) u(z_1)\right) \qquad (Y(\nabla \cdot v) u \in C^{\alpha}_{Y,\mathrm{loc}})$$

$$F_4 = \frac{\delta^2}{2!} \sum_{i,j=1}^{p_0} \left(\partial_{x_i,x_j} u(z_2) - \partial_{x_i,x_j} u(z_1)\right) v_i v_j \qquad (\partial_{x_i,x_j} u \in C^{1+\alpha}_{Y,\mathrm{loc}})$$

$$F_5 = -\frac{\delta^3}{3!} \sum_{i,j,l=1}^{p_0} \left(\partial_{x_i,x_j,x_l} u(z_2) - \partial_{x_i,x_j,x_l} u(z_1)\right) v_i v_j v_l \qquad (\partial_{x_i,x_j,x_l} u \in C^{\alpha}_{Y,\mathrm{loc}})$$

$$F_6 = \delta^3 \left((\nabla \cdot v) Yu(z_1) - (\nabla \cdot v) Yu(z_0)\right) \qquad ((\nabla \cdot v) Yu \in C^{0,\alpha}_{B,\mathrm{loc}})$$

$$F_7 = \delta^3 \left(Y(\nabla \cdot v) u(z_0) - Y(\nabla \cdot v) u(z_1)\right) \qquad (Y(\nabla \cdot v) u \in C^{0,\alpha}_{B,\mathrm{loc}}).$$

Next suppose $k > 0$. Then the fifth term in (3.8) is $O(|\delta|^{3+\alpha})$ as $\delta \to 0$ because

$$-\delta^2 (Yu(z_3) - Yu(z_2)) = -\delta^2 (Yu(z_3) - T_1 Yu(z_2, z_3)).$$

Eventually the last term in (3.8) is identically zero if $k > 1$, otherwise, if $k = 1$, it rewrites as $F_1 + F_2$ with each $F_j = O(|\delta|^{3+\alpha})$ as $\delta \to 0$. Indeed,

$$\begin{aligned} F_1 &= -\delta^3 \left((\nabla \cdot v)Yu(z_2) - (\nabla \cdot v)Yu(z_1) \right) && ((\nabla \cdot v)Yu \in C^{\alpha}_{Y,\mathrm{loc}}) \\ F_2 &= \delta^3 \left(Y(\nabla \cdot v)u(z_2) - Y(\nabla \cdot v)u(z_1) \right) && (Y(\nabla \cdot v)u \in C^{\alpha}_{Y,\mathrm{loc}}). \end{aligned}$$

Case $n > 1$

We distinguish two sub-cases depending on k beign greater or equal of $n-1$.

In the first case we have that the last term in (3.8) is zero. Moreover, for any $i = 1, \dots, n$

$$Y^i u(z_3) - Y^i u(z_2) = Y^i u(z_3) - T_{2(n-i)+1} Y^i u(z_2, z_3) \tag{3.9}$$

and the bound for the fifth term in (3.8) follows form the inductive hypothesis on Theorem 2.9.

In the sub-case $k = n$ equation (3.9) is true only for $i > 1$. Nevertheless, we can write

$$\begin{aligned} Yu(z_3) - Yu(z_2) = Yu(z_3) &- T_{2(n-1)+1} Yu(z_2, z_3) \\ &+ \sum_{i=1}^{p_{n-1}} \partial_{x_{\bar{p}_{n-2}+i}} Yu(z_2)(-\delta)^{2n-1}(B^{n-1}v)_{p_{n-2}+i}, \end{aligned} \tag{3.10}$$

and the difference in the right side can be estimed using Theorem 2.9 on Yu. Now, $\partial_{x_{\bar{p}_{n-2}+i}} u = Y^{(n-1)}_{v_i^{(n-1)}}$ for $i = 1, \dots, p_{n-1}$ by induction hypothesis on Theorem 2.9 and, since $v \in V_{0,n} \subset V_{0,n-1}$

$$v = \sum_{i=1}^{p_{n-1}} (B^{n-1}v)_{\bar{p}_{n-2}+i} v_i^{(n-1)} = \sum_{i=1}^{p_n} (B^n v)_{\bar{p}_{n-1}+i} v_i^{(n)}$$

so that, by definition (1.14) and from $Y_v^{(n)} = [Y_v^{(n-1)}, Y]$, the term in the far right of (3.10) plus the last line in (3.8) can be rewritten as $F_1 + \cdots + F_5$ whit each $F_j = O(|\delta|^{2n+1+\alpha})$ as $\delta \to 0$. Precisely, if we set $q = n - 1$ we have

$$\begin{aligned} F_1 &= -\delta^{2n-1} \left(Y_v^{(q)} u(z_2) - Y_v^{(q)} u(z_1) - \delta^2 Y Y_v^{(q)} u(z_2) \right) && (Y Y_v^{(q)} u \in C^{\alpha}_{Y,\mathrm{loc}}) \\ F_2 &= -\delta^{2n+1} \left(Y Y_v^{(q)} u(z_2) - Y Y_v^{(q)} u(z_1) \right) && (Y Y_v^{(q)} u \in C^{\alpha}_{Y,\mathrm{loc}}) \\ F_3 &= -\delta^{2n+1} \left(Y Y_v^{(q)} u(z_1) - Y Y_v^{(q)} u(z_0) \right) && (Y Y_v^{(q)} u \in C^{0,\alpha}_{B,\mathrm{loc}}) \\ F_4 &= \delta^{2n+1} \left(Y_v^{(q)} Y u(z_2) - Y_v^{(q)} Y u(z_1) \right) && (Y_v^{(q)} Y u \in C^{\alpha}_{Y,\mathrm{loc}}) \\ F_5 &= \delta^{2n+1} \left(Y_v^{(q)} Y u(z_1) - Y_v^{(q)} Y u(z_0) \right) && (Y_v^{(q)} Y u \in C^{0,\alpha}_{B,\mathrm{loc}}). \end{aligned}$$

If in particular $u \in C_B^{2n+1,\alpha}$ the proof still works with minor changes. □

Proof of Proposition 3.2 for $2n+1$.

We prove the thesis by induction on k. For $k=0$ we have to prove that, for any $v \in V_0$

$$u(t,x+v) = T_{2n+1}u\big((t,x),(t,x+v)\big) + O\big(|v|_B^{2n+1+\alpha}\big), \quad \text{as } |v|_B \to 0.$$

We observe $v \in V_0$ implies that $T_{2n+1}u$ contains only increments with respect to the first p_0 variables. By the multidimensional mean-value theorem, there exist a family of vectors $(\bar{v}_I)_{I\in\mathcal{I}}$, with $\mathcal{I} = \{1,\dots,p_0\}^{2n+1}$, $\bar{v}_I \in V_0$ and $|\bar{v}_I| \le |v|$ such that

$$\begin{aligned} u(t,x+v) &- T_{2n+1}u\big((t,x),(t,x+v)\big) \\ &= \frac{1}{(2n+1)!}\sum_{I\in\mathcal{I}}(\partial_x^I u(t,x+\bar{v}_I) - \partial_x^I u(t,x))v^I \\ &= O(|v|^\alpha)\sum_{I\in\mathcal{I}} v^I \ \text{ as } |v|_B \to 0 \quad (\ \partial_x^I u \in C^{0,\alpha}_{B,\text{loc}}) \\ &= O(|v|_B^{2n+1+\alpha}) \quad \text{as } |v|_B \to 0. \end{aligned}$$

Now suppose $k \ge 0$, $\xi \in \bigoplus_{j=0}^k V_j$ and $v \in V_{k+1}$. Then

$$\begin{aligned} u(t,x+\xi+v) - T_{2n+1}u\big((t,x),(t,x+\xi+v)\big) = \\ u(t,x+\xi+v) - T_{2n+1}u\big((t,x+v),(t,x+\xi+v)\big) + \\ T_{2n+1}u\big((t,x+v),(t,x+\xi+v)\big) - T_{2n+1}u\big((t,x),(t,x+\xi+v)\big), \end{aligned}$$

with the first differenze $O(|\xi|_B^{2n+1+\alpha}) = O(|\xi+v|_B^{2n+1+\alpha})$ as $|\xi+v|_B \to 0$ by inductive hypothesis. Recalling notation 2.8 the second difference rewrites as

$$\begin{aligned} &\sum_{\substack{|\beta|_B \le 2n+1 \\ \beta_i = 0 \text{ if } i > \bar{p}_k}} \frac{1}{\beta!}\partial_x^\beta u(t,x+v)\xi^\beta \\ &\quad - \sum_{\substack{|\beta|_B \le 2n+1 \\ \beta_i = 0 \text{ if } i > \bar{p}_k}} \frac{1}{\beta!} \sum_{|\gamma^{[k+1]}|_B = 0}^{2n+1-|\beta|_B} \frac{1}{\gamma^{[k+1]}!}\partial_x^{\gamma^{[k+1]}}\partial_x^\beta u(t,x)\xi^\beta v^{\gamma^{[k+1]}} \\ &= \sum_{\substack{|\beta|_B \le 2n+1 \\ \beta_i = 0 \text{ if } i > \bar{p}_k}} \frac{1}{\beta!}\left(\partial_x^\beta u(t,x+v) - \sum_{|\gamma^{[k+1]}|_B = 0}^{2n+1-|\beta|_B} \frac{1}{\gamma^{[k+1]}!}\partial_x^{\gamma^{[k+1]}}\partial_x^\beta u(t,x) v^{\gamma^{[k+1]}}\right)\xi^\beta \\ &= \sum_{\substack{|\beta|_B \le 2n+1 \\ \beta_i = 0 \text{ if } i > \bar{p}_k}} \frac{1}{\beta!}\big(\partial_x^\beta u(t,x+v) - T_{2n+1-|\beta|_B}\partial_x^\beta u((t,x),(t,x+v))\big)\xi^\beta. \end{aligned}$$

If $|\beta|_B \ge 1$ we can use Theorem 2.9 on $\partial_x^\beta u$. The corresponding term on the summation are then $O(|v|_B^{2n+1-|\beta|_B+\alpha})|\xi|_B^{|\beta|_B} = O(|\xi+v|_B^{2n+1+\alpha})$ as $|\xi+v|_B \to 0$.

Eventually, it remains to estimate the term

$$u(t,x+v) - \sum_{|\gamma^{[k+1]}|_B \le 2n+1} \frac{1}{\gamma^{[k+1]}!} \partial_x^{\gamma^{[k+1]}} u(t,x) v^{\gamma^{[k+1]}}.$$

By definition $|\gamma^{[k+1]}|_B = (2(k+1)+1)|\gamma^{[k+1]}|$ and we set l the maximum integer such that $(2k+3)l \le 2n+1$. Applying the mean-value theorem as in the $k=0$ step we can rewrite the above formula as

$$\frac{1}{l!} \sum_{I \in \mathcal{I}^l_{k+1}} \left(\partial_x^I u(t, x+\bar{v}_I) - \partial_x^I u(t,x) \right) v^I.$$

where $\mathcal{I}^l_{k+1} = \{\bar{p}_k + 1, \dots, \bar{p}_{k+1}\}^l$ and $\bar{v}_I \in V_{k+1}$ with $|\bar{v}_I| \le |v|$. As $\partial_x^I u \in C^{2n+1-(2k+3)l,\alpha}_{B,\mathrm{loc}}$ we can use Theorem 2.9 one more time. The thesis follows now noticing that

$$|v^I| \le c_B |v|^l \le c_B |v|_B^{(2k+3)l}.$$

If in particular $C^{2n+1,\alpha}_B$ the proof still works with minor changes. □

We are now ready to prove step 2 for Theorem 2.9. We only prove the local version of the theorem being the proof for the particular case $u \in C^{2n+1,\alpha}_B$ a straightforward modification of the following one.

proof of Theorem 2.9 for $2n+1$. First of all we consider the case $z = (t,x)$, $\zeta = (t,\xi)$ i.e. there is no increment in the temporal variable. Precisely we show that, if $u \in C^{2n+1,\alpha}_{B,\mathrm{loc}}$ we have

$$u(t,x) = \bar{T}_{2n+1} u((t,\xi),(t,x)) + O(|x-\xi|_B^{2n+1+\alpha}) \quad \text{as } |x-\xi|_B \to 0. \tag{3.11}$$

Define the point $\bar{z} := (t, \bar{x})$ with

$$\bar{x}^{[i]} = \begin{cases} x^{[i]}, & \text{if } i \ge n, \\ \xi^{[i]}, & \text{if } i < n. \end{cases} \tag{3.12}$$

Then,

$$\begin{aligned} u(t,x) - \bar{T}_{2n+1} u((t,\xi),(t,x)) &= u(t,x) - \bar{T}_{2n+1} u((t,\bar{x}),(t,x)) \\ &\quad + \bar{T}_{2n+1} u((t,\bar{x}),(t,x)) - \bar{T}_{2n+1} u((t,\xi),(t,x)). \end{aligned}$$

Applying Proposition 3.2 on the first term we obtain the bound $O(|x - \bar{x}|_B^{2n+1+\alpha}) = O(|x-\xi|_B^{2n+1+\alpha})$ as $|x-\xi|_B \to 0$. Let us notice that, by (3.12), we have

$$(x-\bar{x})^\beta = \begin{cases} (x-\xi)^\beta & \text{if } |\beta|_B \le 2n+1,\ \beta^{[n]}=0, \\ 0, & \text{if } |\beta|_B \le 2n+1,\ \beta^{[n]}\neq 0. \end{cases}$$

Therefore,

$$\bar{T}_{2n+1}u((t,\bar{x}),(t,x)) - \bar{T}_{2n+1}u((t,\xi),(t,x)) =$$

$$\sum_{\substack{|\beta|_B\le 2n+1\\ \beta^{[n]}=0}} \frac{1}{\beta!}\Big(\partial_x^\beta u(t,\bar{x}) - \partial_x^\beta u(t,\xi)\Big)(x-\xi)^\beta - \sum_{i=1}^{p_n} Y^{(n)}_{v_i^{(n)}} u(t,\xi)(x-\xi)_{\bar{p}_{n-1}+i}.$$

Each term of the first sum with $|\beta|_B > 0$ is $O(|\bar{x}-\xi|_B^{2n+1+\alpha-|\beta|_B})|x-\xi|^{|\beta|} = O(|x-\xi|_B^{2n+1+\alpha})$ as $|x-\xi|_B \to 0$ by the inductive hypothesis on Theorem 2.9. What is left to estimate is just

$$u(t,\bar{x}) - u(t,\xi) - \sum_{i=1}^{p_n} Y^{(n)}_{v_i^{(n)}} u(t,\xi)(x-\xi)_{\bar{p}_{n-1}+i}.$$

Define the points

$$z_{n-1} = (t,\xi),\ z_n = \gamma^{(n-1,n)}_{\delta_n,v_n}(z_{n-1}),\ldots, z_r = \gamma^{(n-1,r)}_{\delta_r,v_r}(z_{r-1}) \equiv \bar{z}$$

similarly as in the proof of Theorem 2.9 for $n=0$. To be more precise each time we choose $v_i \in V_{0,i}$, $|v_i| = 1$ and $\delta_i \in \mathbb{R}$ such that the application of $\gamma^{(n-1,i)}_{\delta_i,v_i}$ corrects the i-th level spatial components of z_{i-1} to $\bar{x}^{[i]}$. Moreover, arguing as in Step 1 it can be proven that

$$|\delta_i| \le c_B|\bar{x}-\xi|_B \le c_B|x-\xi|_B \quad i = n,\ldots,r. \tag{3.13}$$

Therefore, we get $\bar{T}_{2n+1}u(z_{i-1},z_i) = u(z_{i-1})$ if $i > n$ and, keeping this in mind, it is clear that

$$\begin{aligned} u(t,\bar{x}) - u(t,\xi) - \sum_{i=1}^{p_n} Y^{(n)}_{v_i^{(n)}} u(t,\xi)(x-\xi)_{\bar{p}_{n-1}+i} &= u(z_r) - \bar{T}_{2n+1}u(z_{n-1},z_n) \\ &= \sum_{i=n+1}^{r} \left(u(z_i) - \bar{T}_{2n+1}u(z_{i-1},z_i)\right) + \big(u(z_n) - \bar{T}_{2n+1}u(z_{n-1},z_n)\big), \end{aligned}$$

and the thesis follows by Proposition 3.2 and (1.22), (3.13).

We are now able to prove part 1 and part 2 of Theorem 2.9.

Choosing $x = \xi + \delta e_i^{[n]}$ in (3.11) ,where $\delta \in \mathbb{R}$ and $e_i^{[n]}$ is the i-th vector of the canonical basis of V_n, we get

$$u(t,\xi+\delta e_i^{[n]}) - u(t,\xi) - \delta Y^{(n)}_{v_i^{(n)}} u(t,\xi) = O(|\delta|^{1+\frac{\alpha}{2n+1}}),$$

which can be rewritten as

$$\frac{\left|u(t,\xi+\delta e_i^{[n]})-u(t,\xi)-\delta Y^{(n)}_{v_i^{(n)}}u(t,\xi)\right|}{|\delta|}\le C|\delta|^{\frac{\alpha}{2n+1}}, \qquad 0<|\delta|<\delta_0,$$

with C and δ_0 two costant greater than zero. This implies that $\partial_{x_{\bar{p}_{n-1}+i}}u(t,\xi)$ exists and

$$\partial_{x_{\bar{p}_{n-1}+i}}u(t,\xi)=Y^{(n)}_{v_i^{(n)}}u(t,\xi) \quad t\in\mathbb{R},\ \xi\in\mathbb{R}^d,\ i=1,\dots,p_n.$$

Thus, as $Y^{(n)}_{v_i^{(n)}}u\in^{0,\alpha}_{B,\mathrm{loc}}$, the proof of part 1 is completed.

Next we prove the general case $z=(t,x)$, $\zeta=(s,\xi)$. By part 1 it is well defined the B-Taylor polynomial $T_{2n+1}u(\zeta,\cdot)$ and it equals $\bar{T}_{2n+1}u(\zeta,\cdot)$. Define the point $\zeta_1:=e^{(t-s)Y}(\zeta)=(t,e^{(t-s)B}\xi)$ and, as usual, write

$$u(z)-T_{2n+1}u(\zeta,z)=u(z)-T_{2n+1}u(\zeta_1,z)+T_{2n+1}u(\zeta_1,z)-T_{2n+1}u(\zeta,z).$$

The first difference is $O(|x-e^{(t-s)B}\xi|_B^{2n+1+\alpha})=O(\|\zeta^{-1}\circ z\|_B^{2n+1+\alpha})$ as $\|\zeta^{-1}\circ z\|_B\to 0$ thanks to the previous case while the second can be rewritten as

$$\begin{aligned}&T_{2n+1}u(\zeta_1,z)-T_{2n+1}u(\zeta,z)=\\&\qquad\sum_{|\beta|_B\le 2n+1}\frac{1}{\beta!}\partial_x^\beta u(e^{(t-s)Y}(\zeta))(x-e^{(t-s)B}\xi)^\beta\\&\quad-\sum_{|\beta|_B\le 2n+1}\sum_{k=0}^{\left\lfloor\frac{2n+1-|\beta|_B}{2}\right\rfloor}\frac{1}{\beta!k!}Y^k\partial_x^\beta u(\zeta)(x-e^{(t-s)B}\xi)^\beta(t-s)^k=\\&\sum_{|\beta|_B=0}^{2n+1}\frac{1}{\beta!}\left(\partial_x^\beta u(e^{(t-s)Y}(\zeta))-\sum_{k=0}^{\left\lfloor\frac{2n+1-|\beta|_B}{2}\right\rfloor}\frac{(t-s)^k}{k!}Y^k\partial_x^\beta u(\zeta)\right)(x-e^{(t-s)B}\xi)^\beta.\end{aligned}$$

Eventually, as $\partial_x^\beta u\in C_{B,\mathrm{loc}}^{2n+1-|\beta|_B,\alpha}$ for any multi-index β such that $|\beta|_B\le 2n+1$, by Remark 3.3, we infer that the corresponding term of the sum is

$$O(|t-s|^{\frac{2n+1+\alpha-|\beta|_B}{2}}|x-e^{(t-s)B}\xi|_B^{|\beta|_B})=O(\|\zeta^{-1}\circ z\|_B^{2n+1+\alpha}) \quad \text{as } \|\zeta^{-1}\circ z\|_B\to 0.$$

□

3.3 Step 3

Next we do Step 3 that is, we suppose the main theorem true for the orders $n=0,\dots,2n+1\le 2r+1$ and we prove it for functions $u\in C_{B,\mathrm{loc}}^{2n+2,\alpha}$. As in the previous step we will use Propositions 3.1 and 3.2 to demostrate the main theorem however, for the sake of brevity we prove only 3.1. The proof of 3.2 is identical (just replace $2n+1$ with $2n+2$) to the one given in Step 2 and the proof of 2.9 is easier since this time we don't have to prove the existence of any derivative.

As usual we prove only the local versions of the statements.

Proof of Proposition 3.1 for $2n+2$.

Note that the condition $\max\{\lfloor \frac{2n+2}{2} \rfloor - 1, 0\} \leq k \leq r$ become $n \leq k \leq r$. We want to prove that for any of such k and $v \in V_{0,k}$ with $|v| = 1$, we have:

$$u\big(\gamma_{v,\delta}^{(n,k)}(z)\big) = T_{2n+2}u\big(z, \gamma_{v,\delta}^{(n,k)}(z)\big) + O\big(|\delta|^{2n+2+\alpha}\big), \quad \text{as } \delta \to 0. \tag{3.14}$$

We have to analyze separately the cases $n = 0$ and $n > 0$. We begin proving (3.14) directily for $k = n$.

Case $n = 0$

We have $k = 0$ and equation 3.14 rewrites as

$$u(t, x + \delta v) = u(t, x) + \sum_{i=1}^{p_0} \partial_{x_i} u(t, x) \delta v_i + \frac{\delta^2}{2!} \sum_{i,j=1}^{p_0} \partial_{x_i x_j} u(t, x) v_i v_j + O(|\delta|^{2n+2+\alpha}) \quad \text{as } \delta \to 0.$$

By the multidimensional euclidean mean-value theorem, there exist $(\bar{v}_{i,j})_{1 \leq i,j \leq p_0}$, with $\bar{v}_{i,j} \in V_0$ and $|\bar{v}_{i,j}| \leq 1$, such that

$$\begin{aligned} u(t, x + \delta v) &- T_2 u((t, x), (t, x + \delta v)) \\ &= \frac{\delta^2}{2!} \sum_{i,j=1}^{p_0} \big(\partial_{x_i, x_j} u(t, x + \delta \bar{v}_{i,j}) - \partial_{x_i, x_j} u(z)\big) v_i v_j \\ &= \delta^2 O(|\delta v|_B^{\alpha}) \quad \text{as } \delta \to 0 \qquad (\partial_{x_i, x_j} u \in C_B^{0,\alpha}) \\ &= O(|\delta|^{2+\alpha}) \quad \text{as } \delta \to 0. \end{aligned}$$

Case $n > 0$

This time $k > 0$. We have to prove that for any $v \in V_{0,k}$ with $|v| = 1$,

$$u(t, x + \delta^{2n+1} B^n v) = u(t, x) + \delta^{2n+1} \sum_{i=1}^{p_n} \partial_{x_{\bar{p}_{n-1}+i}} u(t, x) (B^n v)_{\bar{p}_{n-1}+i} + O\big(|\delta|^{2n+2+\alpha}\big), \quad \text{as } \delta \to 0.$$

This case differs from the precedent because this time we have no derivatives of higher (Euclidean) order. By the multidimensional mean-value theorem, there exist a family of vectors $(\bar{v}_i)_{1\le i\le p_n}$, with $\bar{v}_i \in V_{0,n}$ and $|B^n\bar{v}_i| \le |B^n v|$ such that

$$u(t, x + \delta^{2n+1}B^n v) - u(t, x) = \sum_{i=1}^{p_n} \partial_{x_{\bar{p}_{n-1}+i}} u(t, x + \delta^{2n+1}B^n\bar{v}_i)\delta^{2n+1}B^n v_i.$$

Therefore

$$\begin{aligned} &u\Big(\gamma_{v,\delta}^{(n,n)}(t,x)\Big) - T_n u\Big(z, \gamma_{v,\delta}^{(n,n)}(t,x)\Big) \\ &\quad= \sum_{i=1}^{p_n} \Big(\partial_{x_{\bar{p}_{n-1}+i}} u(t, x + \delta^{2n+1}B^n\bar{v}_i) - \partial_{x_{\bar{p}_{n-1}+i}} u(t,x)\Big)\, \delta^{2n+1}(B^n v)_i \\ &\qquad\qquad= O(|\delta|^{1+\alpha})\delta^{2n+1}\sum_{i=1}^{p_n}(B^n v)_i \quad \text{ as } \delta\to 0 \\ &\qquad\qquad\qquad= O(|\delta|^{2n+2+\alpha}) \quad \text{ as } \delta\to 0. \end{aligned}$$

Where we have used Theorem 2.9 in the second line since $\partial_{x_{\bar{p}_{n-2}+i}} u(t,x) = T_2\partial_{x_{\bar{p}_{n-2}+i}} u((t,x),(t,x+\delta^{2n-1}B^{n-1}\bar{v}_i))$ and $\partial_{x_{\bar{p}_{n-1}+i}} u \in C^{1,\alpha}_{B,\mathrm{loc}})$

Inductive Step

Now we suppose the thesis true for a fixed $k \ge n$ and we prove it for $k+1$. We set $\widetilde{T}_{2n+2}u(\zeta,z) = T_{2n+2}u(\zeta,z) - u(\zeta)$ and

$$z_0 = z, \quad z_1 = \gamma_{v,\delta}^{(n,k)}(z_0), \quad z_2 = e^{\delta^2 Y}(z_1),$$

$$z_3 = \gamma_{v,\delta}^{(n,k)}(z_2), \quad z_4 = e^{-\delta^2 Y}(z_3) = \gamma_{v,\delta}^{(n,k+1)}(z),$$

where $v \in V_{0,k+1}$, $|v| = 1$. Whit this notations we have

$$\begin{aligned} u(z_4) - \bar{T}_{2n+2}u(z_0,z_4) &= u(z_4) - u(z_3) - \sum_{i=1}^{n+1}\frac{(-\delta^2)^i}{i!}Y^i u(z_3) \\ &\quad + u(z_3) - u(z_2) - \widetilde{T}_{2n+2}u(z_2,z_3) \\ &\quad + \sum_{i=1}^{n+1}\frac{(-\delta^2)^i}{i!}Y^i u(z_2) + u(z_2) - u(z_1) \\ &\quad + \widetilde{T}_{2n+2}u(z_1,z_0) + u(z_1) - u(z_0) \\ &\quad + \sum_{i=1}^{n+1}\frac{(-\delta^2)^i}{i!}\left(Y^i u(z_3) - Y^i u(z_2)\right) \\ &\quad + \widetilde{T}_{2n+2}u(z_2,z_3) - \widetilde{T}_{2n+2}u(z_1,z_0) - \widetilde{T}_{2n+2}u(z_0,z_4). \end{aligned} \tag{3.15}$$

By Remark 3.3 the first and the third term are $O(|\delta|^{2n+2+\alpha})$ as $\delta \to 0$ and the same bound for the second and the fourth term follows from the inductive hypothesis (note that, by (1.12) $V_{0,k+1} \subset V_{0,k}$). In order to estimate the last terms we need once again to distinguish various cases depending on n and k.

Case $n = 0$

In this case we have $\widetilde{T}_{2n+2}u(z_0, z_4) \equiv 0$ and

$$Yu(z_3) - Yu(z_2) = O(|\delta|^{\alpha}), \qquad \text{as } \delta \to 0,$$

since $Yu \in C^{0,\alpha}_{B,\text{loc}}$. Regarding the difference $\widetilde{T}_{2n+2}u(z_2, z_3) - \widetilde{T}_{2n+2}u(z_1, z_0)$ we distinguish two cases. If $k > 0$ the two polynomials are identically zero and we are done. If $k = 0$ the above difference rewrites as

$$\begin{aligned}\widetilde{T}_2u(z_2, z_3) - \widetilde{T}_2u(z_1, z_0) = &-\delta \sum_{i=1}^{p_0} (\partial_{x_i}u(z_2) - \partial_{x_i}u(z_1))v_i \\ &+ \frac{(-\delta)^2}{2!} \sum_{i,j=1}^{p_0} (\partial_{x_i,x_j}u(z_2) - \partial_{x_i,x_j}u(z_1))v_iv_j.\end{aligned}$$

Now, since $\partial_{x_i}u \in C^{1,\alpha}_{B,\text{loc}} \subset C^{1+a}_{Y,\text{loc}}$, $\partial_{x_i,x_j}u \in C^{0,\alpha}_{B,\text{loc}} \subset C^{a}_{Y,\text{loc}}$ and $z_2 = e^{\delta^2 Y}(z_1)$ we get

$$\begin{aligned}\partial_{x_i}u(z_2) - \partial_{x_i}u(z_1) &= O(|\delta|^{1+\alpha}), \quad \text{as } \delta \to 0\\ \partial_{x_i,x_j}u(z_2) - \partial_{x_i,x_j}u(z_1) &= O(|\delta|^{\alpha}), \quad \text{as } \delta \to 0.\end{aligned}$$

and this conclude the case $n = 0$.

Case $n > 0$

For any $i = 1, \dots, n+1$

$$\begin{aligned}Y^iu(z_3) - Y^iu(z_2) =& Y^iu(z_3) - T_{2(n-i)+1}Y^iu(z_2, z_3)\\ =& O(|\delta|^{2(n-i+1)+\alpha}) \quad \text{as } \delta \to 0,\end{aligned}$$

and the bound for the fifth term in (3.15) follows form the inductive hypothesis on Theorem 2.9.

Moreover, since the first increment in $\gamma^{(n,k+1)}_{v,d}$ is at least of level $n+1$ we have $T_{2n+2}u(z_0, z_4) \equiv 0$.

Now we analyze the difference $\widetilde{T}_{2n+2}u(z_2, z_3) - \widetilde{T}_{2n+2}u(z_1, z_0)$. If $k > n$ it is identically zero for the same reason above. Otherwise, if $k = n$, it reduces to

$$\sum_{i=\bar{p}_{n-1}+1}^{\bar{p}_n} (\partial_{x_i}u(z_2) - \partial_{x_i}u(z_1))\, \delta^{2n+1}(B^n v)_i.$$

Because $\partial_{x_i}u \in C^{1,\alpha}_{B,\text{loc}} \subset C^{1+a}_{Y,\text{loc}}$ and $z_2 = e^{\delta^2 Y}(z_1)$ we get

$$\widetilde{T}_{2n+2}u(z_2, z_3) - \widetilde{T}_{2n+2}u(z_1, z_0) = O(|\delta|^{1+\alpha}), \quad \text{as } \delta \to 0.$$

and this conclude the proof. □

3.4 Step 4 and corollaries

Here we fix a certain $n \geq 2r+1$, suppose Theorem 2.9 true for any $0 \leq m \leq n$ and we prove it for $n+1$. We will use Proposition 3.2 however its proof is identical to the one given in the step 2 (just replace $2n+1$ with $n+1$ and note that the condition $0 \leq k < \left\lfloor \frac{n+1}{2} \right\rfloor$ become $0 \leq k \leq r$) and is thus omitted.

Proof of Theorem 2.9 for $n+1$.

Let $z = (t, x)$, $\zeta = (s, \xi) \in \mathbb{R} \times \mathbb{R}^d$, define the point $\zeta_1 := e^{(t-s)Y}(\zeta) = (t, e^{(t-s)B}\xi)$ and, as usual, write

$$u(z) - T_{2n+1}u(\zeta, z) = u(z) - T_{2n+1}u(\zeta_1, z) + T_{2n+1}u(\zeta_1, z) - T_{2n+1}u(\zeta, z).$$

The first difference is $O(|x - e^{(t-s)B}\xi|_B^{n+1+\alpha}) = O(\|\zeta^{-1} \circ z\|_B^{n+1+\alpha})$ as $\|\zeta^{-1} \circ z\|_B \to 0$ thanks to Proposition 3.2 while the second can be rewritten as

$$\begin{aligned} T_{n+1}u(\zeta_1, z) - T_{n+1}u(\zeta, z) = \\ \sum_{|\beta|_B \leq n+1} \frac{1}{\beta!} \partial_x^\beta u(e^{(t-s)Y}(\zeta))(x - e^{(t-s)B}\xi)^\beta \\ - \sum_{|\beta|_B \leq n+1} \sum_{k=0}^{\left\lfloor \frac{n+1-|\beta|_B}{2} \right\rfloor} \frac{1}{\beta! k!} Y^k \partial_x^\beta u(\zeta)(x - e^{(t-s)B}\xi)^\beta (t-s)^k = \\ \sum_{|\beta|_B=0}^{n+1} \frac{1}{\beta!} \left(\partial_x^\beta u(e^{(t-s)Y}(\zeta)) - \sum_{k=0}^{\left\lfloor \frac{n+1-|\beta|_B}{2} \right\rfloor} \frac{(t-s)^k}{k!} Y^k \partial_x^\beta u(\zeta) \right) (x - e^{(t-s)B}\xi)^\beta. \end{aligned}$$

Because $\partial_x^\beta u \in C_{B,\text{loc}}^{n+1-|\beta|_B, \alpha}$ for any multi-index β such that $|\beta|_B \leq n+1$, by Remark 3.3, we infer that each of the corresponding term of the sum is

$$O(|t-s|^{\frac{n+1+\alpha-|\beta|_B}{2}} |x - e^{(t-s)B}\xi|_B^{|\beta|_B}) = O(\|\zeta^{-1} \circ z\|_B^{n+1+\alpha}) \quad \text{as } \|\zeta^{-1} \circ z\|_B \to 0.$$

□

Now we prove Corollary 2.13.

Proof. We give two different proofs. the first is based on the euclidean mean value theorem and Theorem 2.9 while the second only on the latter.

Under the hypothesis $u \in C_{B,\text{loc}}^{2r+1,\alpha}(\Omega)$ Theorem 2.9 and Corollary 2.12 assures that all the (Euclidean) spatial derivatives $\partial_{x_1}, \ldots, \partial_{x_d}$ exists and are continuos functions. Therefore for small δ the function

$$[0, \delta] \ni s \longmapsto u(t + \delta, e^{sB}x), \qquad (t, x) \in \Omega$$

is C^1 and we can apply the mean value theorem and infer that

$$u(t+\delta,x)-u(t+\delta,e^{\delta B}x)=-\langle\nabla u(t+\delta,e^{\bar\delta B}x),Be^{\bar\delta B}x\rangle\delta.$$

for a suitable $|\bar\delta|\le|\delta|$. It follows that

$$\begin{aligned}u(t+\delta,x)-u(t,x)=&u(t+\delta,e^{\delta B}x)-u(t,x)+u(t+\delta,x)-u(t+\delta,e^{\delta B}x)\\=&u(t+\delta,e^{\delta B}x)-u(t,x)-\langle\nabla u(t+\delta,e^{\bar\delta B}x),Be^{\bar\delta B}x\rangle\delta.\end{aligned}$$

Therefore, dividing both sides by δ and taking the limit as $\delta\to 0$ we deduce that $\partial_t u$ exists and

$$\partial_t u(t,x)=Yu(t,x)-\langle\nabla u(t,x),Bx\rangle. \tag{3.16}$$

This could also be derived directily from the main theorem taking $\zeta=(t,x)$, $z=(t+\delta,x)$ and noting that the spatial increments would become

$$x-e^{\delta B}x=\sum_{i=1}^{r}\frac{(-\delta B)^i}{i!}x=-\delta Bx+O(\delta^2)\qquad\text{as }\delta\to 0.$$

Thus,

$$u(z)-T_{2r+1}u(\zeta,z)=u(t+\delta,x)-u(t,x)-\delta Yu(t,x)+\delta\sum_{i=1}^{d}(Bx)_i\partial_{x_i}u(t,x)+O(\delta^2)$$

as $\delta\to 0$ and the thesis follows since

$$\|\zeta^{-1}\circ z\|_B^{2r+1+\alpha}=\|(\delta,x-e^{\delta B}x)\|_B^{2r+1+\alpha}=O(|\delta|^{1+\frac{\alpha}{2r+1}})\qquad\text{as }\delta\to 0.$$

That $\delta_t u\in C^{0,\alpha}_{B,\mathrm{loc}}(\Omega)$ easily follows from (3.16) since, by the inclusions in Proposition 2.7 all the derivatives are in $C^{0,\alpha}_{B,\mathrm{loc}}(\Omega)$.

The same reasoning can be applyed for functions in $C^{2r+1,\alpha}_B(\Omega)$ to conclude that the time-derivative is in $C^{0,\alpha}_B(\Omega)$. □

The second proof shows that, generally speaking, the hypothesis $u\in C^{2r+1,\alpha}_{B,\mathrm{loc}}(\Omega)$ can not be weakened to lower orders since for $n<2r+1$ we have

$$\|\zeta^{-1}\circ z\|_B^{n+\alpha}=\|(\delta,x-e^{\delta B}x)\|_B^{n+\alpha}=O(|\delta|^{\frac{n+\alpha}{2r+1}})\qquad\text{as }\delta\to 0.$$

and since $n+\alpha\le 2r+1$ we can not conclude that the time-derivative exists even though the Lie derivative Yu do exists.

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