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ZETA AND L-FUNCTIONS OF ELLIPTIC CURVES

Tesi di Laurea in Teoria dei Numeri

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Introduction

Elliptic curves are extremely rich and fascinating objects, whose study involves techniques borrowed from many different areas of mathematics: from number theory to complex analysis, from algebraic geometry to the theory of Riemann surfaces. Our point of view in this thesis will be mainly algebraic: we will study elliptic curves within the context of plane algebraic curves, and we will be mainly interested in the properties of elliptic curves defined over finite fields and over $\mathbb{Q}$.
However, the seemingly purely algebraic problem of studying elliptic curves defined over finite fields and over $\mathbb{Q}$ will naturally lead us to introduce two types of analytic functions attached to an elliptic curve, the so called Zeta functions and L-functions. Our main aim will be to study some fundamental properties of these functions, investigating how the analytic properties of the Zeta and L-function of an elliptic curve reflect on the arithmetic properties of the curve. The idea of associating analytic functions to algebraic objects, using analysis to shed new light on number theoretic problems, is widespread in modern number theory and very far reaching. We will analyse in this thesis only some basic, yet very significant examples of this general approach.
Many of the most important results that we are going to describe are particular instances of general theorems whose proof requires advanced tools. However, it is often possible to prove special cases of these results using quite elementary means. Thus we will usually give complete proofs of particular cases of many general results (such as the Riemann hypothesis for elliptic curves, which is proved for two particular families of curves using only the properties of Jacobi sums) and then state the corresponding general theorems without proof. This has the advantage of allowing the reader to familiarize with some deep theorems, to see how far in their proof one can get using only elementary tools and to experience "concretely" the need for more abstract and powerful theories to achieve a complete proof. After all, this is how mathematics develops.
Along the way, we sometimes describe additional results which can be easily proved with the techniques developed (e. g. the quadratic reciprocity law and a counterexample to Hasse-Minkowski local-global principle for cubic equations) and hint at more advanced theorems and conjectures closely related to the topics we deal with (such as Birch and Swinnerton-Dyer conjecture, Weil conjectures, Sato-Tate conjecture). This allows the reader to have a look at some further applications of the results described in the thesis, and at the many fascinating open problems which naturally arise from our discussion and are at the hearth of contemporary mathematical research.

The organisation of the thesis is as follows: in Chapter 1 we briefly recall some preliminary results which will be used throughout the thesis; the reader can just skip through this

Chapter, and come back to it later when its results are cited. Chapter 2 deals with the basic theory of algebraic curves, with a great emphasis on the link between the geometric properties of curves and the algebraic properties of the associate coordinate ring. We also state Riemann-Roch theorem, an essential tool in the sequel. In Chapter 3 we introduce Gauss and Jacobi sums and study some of their properties. Their interest for us lies in the fact that they are among the most effective elementary tools which can be used to count the number of points on curves defined over finite fields. In Chapter 4 we study the Zeta function of affine and projective curves. In particular, we prove the rationality of the Zeta function of a projective plane curve. Chapter 5 introduces elliptic curves and their L-functions. We study in detail elliptic curves with affine equation $y^2 = x^3 + D$ and $y^2 = x^3 - Dx$, proving Riemann hypothesis for these curves and then facing the problem of the existence of an analytic continuation of the associated L-function to the whole complex plane. To solve this problem we define Hecke characters and the associated L-functions, and we hint at the theory of elliptic curves with complex multiplication.
The prerequisites for reading this thesis are quite modest. The reader is assumed to have a basic knowledge of abstract algebra, commutative algebra and complex analysis. Some familiarity with projective geometry and algebraic number theory could be useful but is not strictly necessary: all the results that are used are collected in Chapter 1.

I want to express my sincere gratitude to Gabriele Rembado for his constant support, for the long time he spent discussing with me about mathematics and, above all, for conveying to me his enthusiasm and teaching me the importance of hard work and perseverance.

Introduzione

Le curve ellittiche sono oggetti estremamente ricchi e affascinanti, il cui studio coinvolge tecniche provenienti da diverse aree della matematica: dalla teoria dei numeri all'analisi complessa, dalla geometria algebrica alla teoria delle superfici di Riemann. In questa tesi adotteremo un punto di vista prevalentemente algebrico: studieremo le curve ellittiche inserite nel contesto delle curve algebriche piane, con particolare interesse per le proprietà delle curve ellittiche definite su campi finiti e su $\mathbb{Q}$.
Tuttavia il problema, in apparenza puramente algebrico, di studiare le curve ellittiche definite su campi finiti e su $\mathbb{Q}$ ci porterà naturalmente ad introdurre due tipi di funzioni analitiche associate alle curve ellittiche, le cosidette funzioni Zeta e L. Il nostro principale scopo sarà studiare le proprietà fondamentali di queste funzioni, analizzando come le proprietà analitiche delle funzioni Zeta e L di una curva ellittica si riflettano sulle proprietà aritmetiche della curva. L'idea di associare funzioni analitiche ad oggetti algebrici, utilizzando l'analisi per gettare nuova luce su problemi di teoria dei numeri, è assai diffusa nella moderna teoria dei numeri ed è di portata assai ampia. In questa tesi analizzeremo solo alcuni esempi basilari, ma molto significativi, di questo approccio generale.
Molti dei risultati più importanti che descriveremo sono casi particolari di teoremi generali la cui dimostrazione richiede strumenti avanzati. Tuttavia, è spesso possibile dimostrare questi teoremi in alcuni casi più semplici utilizzando mezzi elementari. Daremo quindi dimostrazioni complete di casi particolari di molti risultati generali (ad esempio, l'ipotesi di Riemann per le curve ellittiche è dimostrata per due famiglie specifiche di curve usando solamente le proprietà delle somme di Jacobi); enunceremo poi i corrispondenti teoremi generali senza dimostrarli. Ciò ha il vantaggio di permettere al lettore di familiarizzare con alcuni risultati profondi, di comprendere quanto lontano ci si possa spingere nella loro dimostrazione usando solo strumenti elementari e di sperimentare "concretamente" la necessità di strumenti teorici più astratti e potenti per ottenere una prova completa. Dopotutto, è così che la matematica si sviluppa.
Nel corso della trattazione sono descritti a volte risultati aggiuntivi la cui dimostrazione si ottiene facilmente con le tecniche sviluppate (ad esempio la legge di reciprocità quadratica e un controesempio al principio di Hasse-Minkowski per equazioni di terzo grado) e sono accennati alcuni teoremi e congetture di natura più avanzata strettamente correlati agli argomenti trattati (come la congettura di Birch e Swinnerton-Dyer, le congetture di Weil, la congettura di Sato-Tate). Questo permette al lettore di esplorare ulteriori applicazioni dei risultati descritti nella tesi, e molti problemi aperti che sorgono naturalmente dalla nostra esposizione e sono al centro della moderna ricerca matematica.

La tesi è organizzata come segue: nel Capitolo 1 sono richiamati brevemente alcuni risul-

tati preliminari utilizzati in tutta la tesi; il lettore può sfogliare rapidamente questo Capitolo, per ritornarci più tardi quando i risultati ivi esposti sono citati. Il Capitolo 2 espone la teoria di base delle curve algenriche, con particolare enfasi sul legame tra le proprietà geometriche delle curve e le proprietà algebriche dell'anello di funzioni polinomiali associato. Si enuncia inoltre il teorema di Riemann-Roch, strumento essenziale nel seguito. Nel Capitolo 3 sono introdotte le somme di Gauss e Jacobi, di cui si studiano alcune proprietà. Per noi, la loro importanza risiede nel fatto che sono tra gli strumenti elementari più efficaci per contare il numero di punti sulle curve definite su campi finiti. Nel Capitolo 4 si studia la funzione Zeta delle curve affini e proiettive. In particolare, si dimostra che la funzione Zeta di una curva piana proiettiva è una funzione razionale. Nel Capitolo 5 sono introdotte le curve ellittiche e le funzioni L ad esse associate. Si studiano in dettaglio le curve ellittiche di equazione affine $y^2 = x^3 + D$ e $y^2 = x^3 - Dx$, dimostrando l'ipotesi di Riemann per queste curve e affrontando il problema dell'esistenza di un prolungamento analitico al piano complesso delle funzioni L associate. Per risolvere questo problema si introduce la nozione di carattere di Hecke e di funzione L ad esso associata, e si accenna alla teoria delle curve ellittiche con moltiplicazione complessa.
I prerequisiti necessari per leggere questa tesi sono minimi. Si assume che il lettore abbia una conoscenza di base dell'algebra astratta, dell'algebra commutativa e dell'analisi complessa. Una certa familiarità con la geometria proiettiva e la teoria algebrica dei numeri può essere utile, ma non è essenziale: tutti i risultati utilizzati sono richiamati nel Capitolo 1.

Contents

Chapter 1

Some algebraic tools

In this chapter, after having fixed some notations and conventions, we collect some tools and results that will be used throughout the thesis. We will recall some fundamental definitions and theorems, and give references for most of the proofs.
The reader who is not already familiar with the topics we will deal with in this chapter can read for example [2], [8], [10], [11], [13], [16].

1.1 Notations and conventions

- Unless the contrary is explicitly stated, all rings will be assumed to be commutative with unit.
  If K is a field, $\bar{K}$ will denote the algebraic closure of K (see section 1.2 for a discussion of this concept).

- If R is a ring, $Spec(R)$ denotes the set of all prime ideals of R, while $Max(R)$ denotes the set of its maximal ideals.

- If R is a ring, R^* will denote the multiplicative group of units of R. In particular, if F is a field, $F^* = F \setminus \{0\}$.

- If A, B are rings, $A \subseteq B, b \in B$, $A[b]$ denotes the intersection of all subrings of B containing A and b. Similarly, if K, L are fields, $K \subseteq L$ and $x \in L$, $K(x)$ denotes the intersection of all subfields of L containing K and x. If $P = (a, b) \in \mathbb{A}^2(\bar{K})$, $K(P)$ denotes the field $K(a, b) \subseteq \bar{K}$.

- If R is a ring, and $a \in R$, (a) denotes the principal ideal generated by a. Two elements $a, b \in R$ are *associates* if $(a) = (b)$. We also say that a is associated with b.

- If $r, s \in R$, $I \subseteq R$ is an ideal, $r \equiv s \pmod{I}$ means $r - s \in I$. If $r \equiv s \pmod{(a)}$ we will also write $r \equiv s \pmod{a}$.

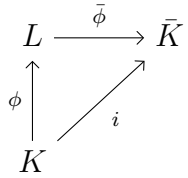
- If A is a set, $|A|$ denotes the cardinality of A.

- If H is a subgroup of a group G, $[G : H]$ denotes the index of H in G.

- $\mathbb{F}_{p^m}$ denotes the field with p^m elements.
- If $f \in \mathbb{F}_p[x,y]$, $N(f=0)$ denotes the number of solutions of the equation $f=0$ in $\mathbb{A}^2(\mathbb{F}_p)$. If $f \in \mathbb{F}_p[x,y,z]$ is homogeneous, $N(f=0)$ denotes the number of solutions to the equation $f=0$ in $\mathbb{P}^2(\mathbb{F}_p)$. (see Chapter 2 for a definition of these spaces).

1.2 Infinite Galois Theory

Good references for the material covered in this section, including proofs of the statements, are [2], [10], [13].
Let K be a field. Recall that there exists an algebraic extension $\bar{K}$ of K, with a field morphism $i : K \to \bar{K}$, characterised by the following universal property: for each algebraic extension L of K and for each field morphism $\phi : K \to L$ there exists a unique morphism $\bar{\phi}$ making the following diagram commute (equivalently, $\bar{\phi}$ is a morphism of $K-$algebras):



Moreover, $\bar{K}$ is unique up to a K-algebra isomorphism. It is called the *algebraic closure* of K. [1]
Now, fix an algebraic closure $\bar{K}$ of K. If K is perfect, the extension $\bar{K}/K$ is separable; clearly, it is also a normal extension. The group

$$Gal(\bar{K}/K) = \{\sigma : \bar{K} \to \bar{K} : \ \sigma(x) = x \ \forall x \in K\}$$

is called the *absolute Galois group* of K. For any intermediate field $\bar{K} \supseteq L \supseteq K$, L is the fixed field of $Gal(\bar{K}/L)$. Moreover, if L/K is a finite extension, $Gal(\bar{K}/L)$ is a subgroup of finite index in $Gal(\bar{K}/K)$, and we have the equality:

$$[Gal(\bar{K}/K) : Gal(\bar{K}/L)] = [L : K]$$

The absolute Galois group $Gal(\bar{K}/K)$ is isomorphic to the inverse limit $\varprojlim(Gal(L/K))$ for L varying in the set of all finite Galois extensions of K, with the morphisms being restrictions.

Example 1.1. $Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p) \cong \varprojlim Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$. The Frobenius automorphism, $\phi : x \mapsto x^p$ generates a cyclic subgroup which is dense in $Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ (with respect to the Krull topology). Let $x \in \bar{\mathbb{F}}_p$. Then $x \in \mathbb{F}_{p^n} \Leftrightarrow \phi^n(x) = x$.

[1] All this can be proved for an arbitrary field using Zorn Lemma. Anyway, in the sequel we will work with the algebraic closure of a finite field $\mathbb{F}_p$, which can also described explicitly: it is $\cup_{n\geq 1}\mathbb{F}_{p^n}$.

1.3 Commutative algebra

Definition 1.1. Let R be a ring. A sequence of prime ideals $P_n \subset P_1 \subset \ldots \subset P_0$, where all inclusions are proper, is called a *chain* of prime ideals. The *height* of a prime ideal P, denoted by $h(P)$, is the supremum of the lengths of all chains of prime ideals with $P_0 = P$. The *Krull dimension* of R is the supremum of the heights of all prime ideals in R. It will be denoted by $dim(R)$.

Example 1.2. If K is a field, $K[x_1, \ldots, x_n]$ is a ring of dimension n. A proof of this fact for the case $n = 2$, which will be enough for us, can be found in [11]. For the general proof, see [15].

Observe that an integral domain R has dimension 1 if and only if every non zero prime ideal of R is maximal. Hence principal ideal domains that are not fields have dimension 1.

1.3.1 Localization

Definition 1.2. Let R be a ring. A subset $S \subset R$ is called *multiplicative* if:

1. $1 \in S$;
2. $a \in S, b \in S \Rightarrow ab \in S$

If S is a multiplicative subset of a ring R, consider the set $R \times S$, and the relation: $(a, s) \equiv (b, t) \Leftrightarrow \exists \sigma \in S : \sigma(at - bs) = 0$. $\equiv$ is an equivalence relation.
Let $S^{-1}R$ be the set of equivalence classes of $R \times S$. Denote by $\frac{a}{s}$ the equivalence class of (a, s).
The following operations are well defined and provide $S^{-1}R$ with a ring structure:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{ts}$$
$$\frac{a}{s}\frac{b}{t} = \frac{ab}{ts}$$

$S^{-1}R$ with this ring structure is called the *localization* of R at S. The map

$$j_S : R \to S^{-1}R,\ j(a) = a/1$$

is a ring homomorphism. It will be denoted simply by j if there is no risk of confusion. Note that j is injective if R is an integral domain. Hence in this case we can regard R as a subring of $S^{-1}R$.

Example 1.3.

1. Let D be an integral domain, $S = D \setminus \{0\}$. Then $S^{-1}D$ is the field of fractions of D.
2. Let R be a ring, $S = D \setminus P$, where P is a fixed prime ideal of R. Then $S^{-1}R$ is called the localization of R at P, and is denoted by R_P.

Proposition 1.1. *Let R be a ring, $S \subseteq R$ a multiplicative subset. Then, the map j induces a bijection:*

$$\begin{aligned} j^* : Spec(S^{-1}R) &\to \{P \in Spec(R) : P \subset R \setminus S\} \\ P &\mapsto j^{-1}(P) \end{aligned}$$

The inverse of j^* is the map sending $P \in Spec(R),\ P \cap S = \varnothing$ to the ideal in $Spec(S^{-1}R)$ generated by $j(P)$.

Proof. See [11, p. 60]. □

Hence, prime ideals in the localization $S^{-1}R$ correspond bijectively to prime ideals in R which do not intersect S. As a consequence, we have the following:

Corollary 1.1. *Let R be a ring, $P \in Spec(R)$. Then R_P has only one maximal ideal, generated by $j(P)$. Moreover, $dim(R_P) = h(P)$. In particular, if $dim(R) = 1$ and $P \in Max(R)$, then $dim(R_P) = 1$.*

Rings with only one maximal ideal are of the uttermost importance, and they deserve a name:

Definition 1.3. A ring with only one maximal ideal is called a *local ring.*

The above corollary states that the localization of a ring at a prime ideal is a local ring.
Let R be a local ring with maximal ideal M. Take $x \in R$. Reminding that any proper ideal in a ring is contained in a maximal ideal, it's easy to show that $x \in M \Leftrightarrow x$ is not a unit.

Definition 1.4. Let A, B be two rings, $A \subseteq B$. An element $b \in B$ is called *integral* over A if it satisfies one of the following equivalent properties: (see [11, p. 12])

1. b is the zero of a *monic* polynomial in $A[x]$;
2. $A[b]$ is a finitely generated $A-$module;
3. $bM \subseteq M$ for a finitely generated $A-$submodule M of B.

The set of all elements in B which are integral over A is a ring, called the *integral closure* of A in B. An integral domain D is called *integrally closed* if it equals its integral closure in its field of fractions.

Proposition 1.2. *Let D be an integral domain. Then the following properties are equivalent:*

1. *D is integrally closed;*
2. *D_P is integrally closed for all $P \in Spec(D)$;*
3. *D_M is integrally closed for all $M \in Max(D)$.*

Proof. See [11, p. 74]. □

Property (2) means that being integrally closed is a *local* property of a domain: in general, a property of a ring is called local when it is satisfied by the ring if and only if it holds for all the localizations of that ring at its prime ideals. Here are other examples of local properties. For the proof, see [11].

Proposition 1.3. *Noetherianity is a local property; being a PID is a local property.*

Proposition 1.4. *Let D be a noetherian local domain of dimension 1. Then D is a PID if and only if it is integrally closed.*

Proof. It is easy to show that factorial domains are always integrally closed. In particular, $PIDs$ are integrally closed.
Conversely, suppose that D is integrally closed. Let M be the maximal ideal of D. Take $x \in M, x \neq 0$. If $(x) = M$, then all prime ideals of D are principal, hence (see [11]) D is a PID.
If $(x) \neq M$, observe first of all that since D is noetherian the ideal (x) contains a power of M.
In fact, any nonzero ideal in a noetherian ring contains a product nonzero of prime ideals (to show it, suppose it's false, and take a maximal element in the family of ideals not containing any product of nonzero prime ideals and obtain a contradiction).
Therefore, there exists $n \in \mathbb{N}$ such that $M^n \subseteq (x)$, and $M^{n-1} \nsubseteq (x)$. Take $y \in M^{n-1} \setminus (x)$
Then, if K denotes the field of fractions of the domain D, $y/x \in K \setminus D$. Since D is integrally closed y/x is not integral over A, so $(y/x)M \nsubseteq M$, as M is a finitely generated $D-$module (D is noetherian).
By construction, we have $yM \subset M^n \subset (x) \Rightarrow (y/x)M \subset D$. Therefore, $(y/x)M$ is an ideal of D not contained in M, so $(y/x)M = D$. Hence D is a PID. □

Definition 1.5. A noetherian local domain whose maximal ideal is principal is called a *discrete valuation ring* (DVR).

Let D be a discrete valuation ring with maximal ideal $M = (\pi)$ (such a π is called a *uniformizing element* of D), $x \in D \setminus \{0\}$. Then, noetherianity of D implies that there exists an integer $n_x \geq 0$ such that $M^{n_x} \subseteq (x)$, and $M^{n_x - 1} \nsubseteq (x)$. Then, $x = \pi^{n_x} u$, where u is a unit in D. The integer n_x does not depend on the choice of the uniformizing parameter π. Hence, we can define:

$$\begin{aligned} v : D &\to \mathbb{Z} \\ x &\mapsto n_x \text{ if } x \neq 0 \\ 0 &\mapsto \infty \end{aligned}$$

This map satisfies the following properties:

1. $v(x) \geq 0$, and $v(x) = \infty \Leftrightarrow x = 0$
2. $v(xy) = v(x) + v(y)$
3. $v(x + y) \geq min\{v(x), v(y)\}$

A map with the above properties is called a *discrete valuation* (sometimes just valuation) on D. It can be extended to the field of fractions K of D by setting: $v(x/y) = v(x) - v(y)$. The valuation defined in this way on a discrete valuation ring D (or on its field of fractions) will be called the *standard valuation* on D.

Example 1.4. Let $p \in \mathbb{Z}$ be a prime. The inverse limit $\mathbb{Z}_p = \varprojlim(\mathbb{Z}/p^n\mathbb{Z})$ (where $n \in \mathbb{N}$ and the morphisms are projections) is called the *ring of p-adic integers*. It is a noetherian local ring, with maximal ideal $p\mathbb{Z}_p$, which is principal. Hence $\mathbb{Z}_p$ is a DVR. Its field of fractions, $\mathbb{Q}_p$, is called the field of *p-adic numbers*. $\mathbb{Q}_p$ has characteristic 0, so $\mathbb{Q} \subseteq \mathbb{Q}_p$. [2]. Let $f(x, y, z) \in \mathbb{Q}[x, y, z]$ be an homogeneous polynomial. If the equation $f(x, y, z) = 0$ has a nontrivial solution $(x_0, y_0, z_0) \neq (0, 0, 0)$ in $\mathbb{Q}^3$, clearly it has a nontrivial solution in $\mathbb{Q}_p^3$ for every prime p, and also a nontrivial solution in $\mathbb{R}^3$. The highly non trivial fact is that in some lucky cases the converse is true:

Theorem 1.1. *(Hasse-Minkowski, local-global principle) Let $f(x, y, z) \in \mathbb{Q}[x, y, z]$ be an homogeneous polynomial of degree 2. If the equation $f(x, y, z) = 0$ has a nontrivial solution in every $\mathbb{Q}_p$ and a nontrivial solution in $\mathbb{R}$, then it has a nontrivial solution in $\mathbb{Q}$.*

A nice account of the proof is given in [17].

Definition 1.6. A *Dedekind domain* is a noetherian, one dimensional, integrally closed domain.

Dedekind domains enjoy the fundamental property of *unique factorization of ideals*: each non trivial ideal I in a Dedekind domain D can be written uniquely (up to order) as a product of prime ideals. This is proved in any introductory book to Algebraic Number Theory, for example [16].

Example 1.5. Let $\mathbb{Q} \subseteq K$ be a finite field extension. Then K is called a *number field*. The set of elements in K which are integral over $\mathbb{Z}$ is a ring, called the *ring of algebraic integers* of K. We will sometimes call it merely the ring of integers of K.
This ring is a Dedekind domain. The study of the properties of rings arising in this way is a major issue of Algebraic Number Theory; we will recall some of its basic results in the following section.

1.4 Algebraic Number Theory

Let $K/\mathbb{Q}$ be a finite field extension of degree n. The integral closure of $\mathbb{Z}$ in K:

$$\mathcal{O}_K = \{x \in K : \exists\, a_0, \ldots, a_{n-1} \in \mathbb{Z} : x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0\}$$

called the *ring of algebraic integers* of K enjoys the following properties (see [16]):

1. $\mathcal{O}_K$ is a free $\mathbb{Z}-$module of rank $n = [K : \mathbb{Q}]$;
2. $\mathcal{O}_K$ is a Dedekind domain.

[2] See [16] for a detailed study of *p-adic* fields.

Let $\mathcal{O}_K$ be the ring of algebraic integers of a number field K. Take $p \in \mathbb{Z}$ prime. The (non trivial) ideal $P = p\mathcal{O}_K \subseteq \mathcal{O}_K$ can be factored as:

$$P = \prod_i P_i^{e_i}$$

with $P_i \in Spec(\mathcal{O}_K)$. We say that the primes P_i are *over* p. The integer e_i is called the *ramification index* of P_i over p. Clearly $P_i \supseteq P \ \forall i$ and $P_i \cap \mathbb{Z} = p$. Therefore, $\mathcal{O}_K/P_i$ is a finite field extension of $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$. The degree of the extension is denoted by f_i and is called the *residual degree* of P_i over p.
The integers e_i, f_i and n are related by the following fundamental relation:

$$\sum_i e_i f_i = n$$

An easy proof of this equality can be found in [8].
Notice that, as a consequence of the above equality, a fixed prime $p \in \mathbb{Z}$ has at most n prime ideals of $\mathcal{O}_K$ above it.
Let $I \subseteq \mathcal{O}_K$ be a non zero ideal, $a \in I$. Then $a\mathcal{O}_K \subseteq I \subseteq \mathcal{O}_K$. As $a\mathcal{O}_K$ and $\mathcal{O}_K$ are free $\mathbb{Z}-$modules of rank n, I is also a free $\mathbb{Z}-$module of rank n. Therefore, there is a $\mathbb{Z}-$basis $e_1, \dots, e_n$ of $\mathcal{O}_K$ and there are integers $1 < d_1 \mid d_2 \dots \mid d_n$ such that $d_1 e_1, \dots, d_n e_n$ is a $\mathbb{Z}-$basis of I. Hence $\mathcal{O}_K/I$ is finite (of cardinality $d_1 \dots d_n$). We define:

$$N(I) = |\mathcal{O}_K/I|$$

$N(I)$ is called the norm of the ideal I. It can be shown that, if I, J are non zero ideals of $\mathcal{O}_K$, $N(IJ) = N(I)N(J)$. We say that the norm is *multiplicative*.
This relation remains true if we replace $\mathcal{O}_K$ by an arbitrary Dedekind domain D such that the quotient D/I is a finite set for each non zero ideal I of D, so that the definition of the norm of an ideal still makes sense. We call it a Dedekind domain with *finite quotients*. See [11] for a proof of these facts.
If $a \in \mathcal{O}_K, a \neq 0$, we define $N(a) = N((a))$.
Let us study in more detail two rings of algebraic integers which will be very useful later.

1.4.1 The ring $\mathbb{Z}[i]$

The ring $\mathbb{Z}[i] = \{a + bi, a, b \in \mathbb{Z}\}$ is the ring of algebraic integers of the field $\mathbb{Q}(i)$. Besides being a Dedekind domain, it is a PID (actually, it's also a euclidean domain; see [8] for a proof of these facts).[3] Let $x = a + ib \in \mathbb{Z}[i]$. Then $N(x) = x\bar{x} = a^2 + b^2$. In fact, with respect to the basis $(1, i)$ of $\mathbb{Z}[i]$, a basis of (x) is: $(a, b), (-b, a)$. It follows that $|\mathbb{Z}[i]/x\mathbb{Z}[i]| = det\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) = a^2 + b^2$.
The fact that the norm is multiplicative implies that units in $\mathbb{Z}[i]$ must have norm 1. The only elements with this property in $\mathbb{Z}[i]$ are $1, -1, i, -i$, which are obviously units.

[3] Warning: this is absolutely far from true in general: for example, the only quadratic imaginary fields, that is, fields of the form $\mathbb{Q}[\sqrt{d}]$, $d < 0$, whose ring of algebraic integers is a PID are those corresponding to $d = -163, -67, -43, -19, -11, -7, -3, -2, -1$. We also mention, *en passant*, that for $d = -19$ we obtain an example of a ring which is a PID but not a euclidean domain.

Let us determine explicitly the prime ideals in $\mathbb{Z}[i]$ (which are ideals generated by prime elements, as $\mathbb{Z}[i]$ is a PID).
There is only one prime of norm 2 up to associates, namely $1+i$.
Assume now that $\pi \in \mathbb{Z}[i]$ is prime with norm different from 2. Then $\pi \mid \pi\bar{\pi} = N(\pi) \in \mathbb{Z}$, hence, as π is prime, π divides one of the prime factors of $N(\pi)$, call it p. Hence, $N(\pi) \mid p^2$, so $N(\pi) = p$ or p^2.
If $p \equiv 3 \pmod 4$ then the equation $a^2 + b^2 = p$ has no integer solutions, hence $N(\pi) = p^2$ and p is associated with π.
If $p \equiv 1 \pmod 4$, by Proposition 3.5 there exist $a, b \in \mathbb{Z}$ such that $a^2 + b^2 = 1 \Rightarrow (a+bi)(a-bi) = p = \pi\bar{\pi}$. Now, $\pi' = a + bi$ is prime in $\mathbb{Z}[i]$. In fact, suppose $\pi' = \alpha\beta$. Then $p = N(\pi') = N(\alpha)N(\beta) \Rightarrow N(\alpha) = 1$ or $N(\beta) = 1$. So either α or β is a unit, hence π' is prime. It follows that $\pi' \mid \pi$ or $\pi' \mid \bar{\pi}$. Hence π is associated with π' or to $\bar{\pi}'$. We've obtained the following:

Proposition 1.5. *Up to associates, primes $\pi \in \mathbb{Z}[i]$ are of the form:*

1. $\pi = 1 + i$
2. $\pi = p$, *with p prime in $\mathbb{Z}$,* $p \equiv 3 \pmod 4$
3. $\pi = a + ib$, *with $a^2 + b^2 = p$, p prime,* $p \equiv 1 \pmod 4$.

Each prime in $\mathbb{Z}$ is associated with its opposite; anyway, each set of associated primes contains one and only one *positive* element.
In $\mathbb{Z}[i]$ each prime is associated with three other elements. We want to find a way to choose a "canonical" element among the set of associates of a fixed prime.

Definition 1.7. A prime $\pi \in \mathbb{Z}[i]$ is called *primary* if $\pi \equiv 1 \pmod{2+2i}$

An easy computation shows that $\pi = a+ib$ is primary if and only if $a \equiv 1 \pmod 4$, $b \equiv 0 \pmod 4$ or $a \equiv 3 \pmod 4$, $b \equiv 2 \pmod 4$.
More generally, we will say that a nonunit $\alpha \in \mathbb{Z}[i]$ is primary if $\alpha \equiv 1 \pmod{2+2i}$.

Proposition 1.6. *Let $\pi = a + bi \in \mathbb{Z}[i]$ prime, $(\pi) \neq (1+i)$. Then there exists a unique π' primary associated with π.*

Proof. As a and b cannot be both even, there is a unit ϵ such that $\epsilon\pi = a' + ib'$ with a' odd, b' even. Then either $\epsilon\pi$ or $-\epsilon\pi$ is primary.
If ϵ, ϵ' are units such that $\epsilon\pi$ and $\epsilon'\pi$ are primary, then $2(1+i) \mid (\epsilon - \epsilon')$ (as $(1+i) \nmid \pi$). This forces $\epsilon = \epsilon'$. □

1.4.2 The ring $\mathbb{Z}[\omega]$

Let $\omega = (-1 + \sqrt{-3})/2$. Then $\mathbb{Z}[\omega] = \{a + b\omega, a, b \in \mathbb{Z}\}$ is the ring of algebraic integers of the field $\mathbb{Q}(\omega)$. It shares many properties with the ring $\mathbb{Z}[i]$, which can be proved in a very similar way. We will list the fundamental properties of $\mathbb{Z}[\omega]$ without proof; the interested reader can find them in [8], or adapt the previous proofs.

1. $\mathbb{Z}[\omega]$ is a euclidean domain;

2. if $x = a + b\omega \in \mathbb{Z}[\omega]$, $N(x) = x\bar{x} = a^2 - ab + b^2$;
3. the units in D are $\pm 1, \pm\omega, \pm\omega^2$;
4. up to associates, primes $\pi \in \mathbb{Z}[i]$ are of the form:
   (a) $\pi = 1 - \omega$
   (b) $\pi = p$, with p prime in $\mathbb{Z}$, $p \equiv 2 \pmod 3$
   (c) $\pi = a + \omega b$, with $a^2 - ab + b^2 = p$, p prime, $p \equiv 1 \pmod 3$.

Definition 1.8. A prime $\pi \in \mathbb{Z}[\omega]$ is called *primary* if $\pi \equiv 2 \pmod 3$

As in the case of $\mathbb{Z}[i]$, we find that primes in $\mathbb{Z}[\omega]$ which are not associated with $1 - \omega$ have a unique primary associate.

1.4.3 The Ideal Class Group of a Number Field

Proofs of the statements in this section can be found in [16].
Let K be a number field with ring of algebraic integers $\mathcal{O}_K$. If I, J are non zero ideals of $\mathcal{O}_K$, we say that I and J are equivalent, and we write $I \sim J$, if there exist two principal ideals $(\alpha), (\beta)$ such that $(\alpha)I = (\beta)J$. This is easily seen to be an equivalence relation. The set of equivalence classes of ideals of $\mathcal{O}_K$ is denoted by $Cl(K)$.
Let $\bar{I}$ denote the equivalence class of an ideal I. The composition law: $\overline{I} \cdot \overline{J} = \overline{IJ}$ is well defined, and turns $Cl(K)$ into an abelian group, called the *ideal class group* of K. The identity element of the group is the equivalence class of all principal ideals of $\mathcal{O}_K$.
Clearly, the ideal class group of K is trivial if and only if $\mathcal{O}_K$ is a PID. We can consider the size of the ideal class group as a measure of the extent to which $\mathcal{O}_K$ fails to be a PID. The rings of algebraic integers of number fields enjoy the following fundamental property (which is not true for arbitrary Dedekind domains):

Theorem 1.2. *The ideal class group of a number field is finite.*

This important result can be proven using geometric ideas due to Minkowski.
We will also need the following more general result, whose proof needs more advanced tools:

Proposition 1.7. *Let $\mathcal{O}_K$ be the ring of algebraic integers of a number field K. Let M be a non zero ideal of $\mathcal{O}_K$, and $C_M = \{I : I \text{ ideal of } \mathcal{O}_K,\ (I, M) = (1)\}/\sim$ where $I \sim J$ if there exist two principal ideals $(\alpha), (\beta)$ with $\alpha \equiv 1 \pmod M$, $\beta \equiv 1 \pmod M$, such that $(\alpha)I = (\beta)J$. Then the composition law $\overline{I} \cdot \overline{J} = \overline{IJ}$ is well defined and turns C_M into a group; moreover, this group is finite.*

Observe that for $M = \mathcal{O}_K$ we obtain the finiteness of the class number.

Chapter 2

Algebraic curves

2.1 Affine curves

Let K be a field. The *affine space* of dimension n over K is the set:

$$\mathbb{A}^n(K) = \{(x_1, \ldots, x_n) : \ x_i \in K \ \forall i = 1, \ldots, n\}$$

In the sequel, we will be mainly interested in the case $n = 2$. In this case, the affine space is called *affine plane*.

Definition 2.1. Let $f(x, y) \in K[x, y]$. The set $C(K) = \{(x, y) \in \mathbb{A}^2(K) : f(x, y) = 0\}$ is called an *affine (plane) curve*.
$C(K)$ is said *geometrically irreducible* if f is an irreducible polynomial in $\bar{K}[x, y]$.
The degree of the polynomial $f(x, y)$ is called the *degree* of the curve.

Remark 2.1.

1. Let $C(\bar{K}) = \{(x, y) \in \mathbb{A}^2(\bar{K}) : f(x, y) = 0\}$, with $f(x, y) \in \bar{K}[x, y]$. Of course, there are many other polynomials whose zeros are exactly the points of $C(\bar{K})$: it is enough to consider $g(x, y) = \lambda f(x, y), \ \lambda \in \bar{K}^*$. We will say that $C(\bar{K})$ is defined over K if there exists a $\lambda \in \bar{K}^*$ such that $g(x, y) = \lambda f(x, y) \in K[x, y]$. In this case, we will denote: $C(K) = \{(x, y) \in \mathbb{A}^2(K) : g(x, y) = 0\}$.
   If C is defined over K, the Galois group $Gal(\bar{K}/K)$ acts on $C(\bar{K})$ in a natural way: for $\sigma \in Gal(\bar{K}/K), \ P = (a, b) \in C(\bar{K}), \ P^\sigma = (\sigma(a), \sigma(b))$. The points of $C(K)$ are exactly the fixed points of this action.
   More generally, let $P = (a, b) \in C(\bar{K})$. We say that P is *defined over* a field $L \supseteq K$ if $K(P) = K(a, b) \subset L$. Clearly P is defined over L if and only if it is fixed by the action of $Gal(\bar{K}/L)$.

2. We will often denote an affine curve simply by C, without mentioning explicitly the field K.

Definition 2.2. Let $f(x, y) \in K[x, y]$ and $C(K) = \{(x, y) \in \mathbb{A}^2(K) : f(x, y) = 0\}$ be an affine plane curve. The ring $K[C] = K[x, y]/(f)$ is the *affine coordinate ring* of C.
If C is a geometrically irreducible curve then its affine coordinate ring is an integral domain. Its field of fractions is called the *field of (rational) functions* of C, and is denoted by $K(C)$.

Remark 2.2. Since we will only deal with geometrically irreducible curves, from now on the word affine curve will indicate, unless the contrary is explicitly stated, a *geometrically irreducible curve*. The expression *irreducible curve* will also be used to refer to a geometrically irreducible curve. However, this terminology is not standard: usually a curve $C(K) = \{(x, y) \in \mathbb{A}^2(K) : f(x, y) = 0\}$ is called irreducible if $f(x, y)$ is irreducible in $K[x, y]$.

Given an affine curve C, our aim is to study its geometric properties looking at algebraic properties of the associated coordinate ring.
The following theorem is the fundamental example of the close correspondence between the structure of geometric objects and properties of algebraic structures naturally associated with them.

Theorem 2.1. *(Hilbert Nullstellensatz) Let $f(x, y) \in K[x, y]$ and $C(K) = \{(x, y) \in \mathbb{A}^2(K) : f(x, y) = 0\}$ be an affine plane curve with coordinate ring $K[C] = K[x, y]/(f)$.*

1. *Every maximal ideal of $K[C]$ is of the form $\psi_P = ker(ev_P)/(f)$ for some $P = (a, b) \in C(\bar{K})$, where $ev_P : K[x, y] \to \bar{K}$, $g(x, y) \mapsto g(a, b)$ is the* evaluation map *at the point P.*

2. *If $K = \bar{K}$ is algebraically closed, then the map $P \mapsto \psi_P$ is a bijection between $C(K)$ and $Max(K[C])$.*

Proof. See [15] (in which a proof of a more general result is given). □

This Theorem is the first "bridge" between algebra and geometry. It states that, over an algebraically closed field, points on the geometric object $C(K)$ correspond bijectively to maximal ideals in the associated coordinate ring $K[C]$.
If the field K is not algebraically closed, things are not so simple. For example, the ideal $(x^2 + 1)$ in $\mathbb{R}[x]$ is maximal, but doesn't correspond to any point in $\mathbb{R}$. Anyway, the above result tells us that maximal ideals in $K[C]$ are kernels of evaluations at points on the curve defined over algebraic field extensions of K.
In general, evaluations at different points can correspond to the same maximal ideal in $K[C]$. In fact, let $\sigma \in Gal(\bar{K}/K)$, $P, Q \in C(\bar{K})$ such that $P = \sigma(Q)$. Then for each $g(x, y) \in K[x, y]$ we have: $ev_P(g) = \sigma \circ ev_Q(g) = 0 \Leftrightarrow ev_Q(g) = 0$. Hence $\psi_P = \psi_Q$.
Conversely, if $\psi_P = \psi_Q$ then $K(P) \cong K[C]/\psi_P = K[C]/\psi_Q \cong K(Q)$. The isomorphism between the two $K-$algebras $K(P)$ and $K(Q)$, which sends P to Q, can be extended to an isomorphism $\sigma \in Gal(\bar{K}/K)$ (because of the universal property of the algebraic closure). Thus P and Q are in the same orbit with respect to the action of $Gal(\bar{K}/K)$.
To sum up: maximal ideals in $K[C]$ correspond bijectively to orbits of points in $C(\bar{K})$ with respect to the action of $Gal(\bar{K}/K)$.
Observe also that, if $P \in C[\bar{K}]$, the orbit of P contains $[K(P) : K]$ different points, since the stabilizer of P is $Gal(\bar{K}/K(P))$, which has index $[K(P) : K]$ in $Gal(\bar{K}/K)$. (Section 1.2)

Example 2.1. Let $C(K)$ be a curve defined on $K = \mathbb{F}_p$. Then, if N_m denotes the number of points on $C(\bar{K})$ defined over $\mathbb{F}_{p^m}$ and $b_d = |\{M \in Max(K[C]) : [K[C]/M : \mathbb{F}_p] = d\}|$, we have:

$$N_m = \sum_{d|m} db_d$$

In fact, $P = (a, b) \in C(\bar{K})$ is defined over $\mathbb{F}_{p^m}$ if and only if ψ_P is such that $d = [K[C]/\psi_P : \mathbb{F}_p] = [K(P) : \mathbb{F}_p] \mid m$. Moreover, each maximal ideal ψ_P corresponds to an orbit containing $[K(P) : \mathbb{F}_p] = [K[C]/\psi_P : \mathbb{F}_p] = d$ different points defined over $\mathbb{F}_{p^d} \subseteq \mathbb{F}_{p^m}$. Hence the total number of points on the curve defined on $\mathbb{F}_{p^m}$ is exactly $\sum_{d|m} db_d$.

Definition 2.3. The *dimension* of an affine curve $C(K)$ is the Krull dimension of the ring $\bar{K}[C]$.

Recall that $K[x_1, \ldots, x_n]$ is a ring of dimension n (Example 1.2); in particular, $dim\ \bar{K}[x, y] = 2$. As a consequence we obtain the following

Lemma 2.1. *Affine curves have dimension 1.*

Proof. Recall that curve is always, for us, an irreducible curve. Its coordinate ring is $\bar{K}[C] = \bar{K}[x, y]/(f)$, where f is an irreducible polynomial. Let I be a non zero prime ideal in $\bar{K}[C]$. Then $I = \pi(J)$, where $J \in Spec(\bar{K}[x, y])$, $J \supset (f)$. Hence $0 \subset (f) \subset J$ is a chain of length 2 in $\bar{K}[x, y]$; it follows that $J \in Max(\bar{K}[x, y])$ and $\pi(J) = I \in Max(\bar{K}[C])$. □

We're now going to define the notion of smooth point and tangent line to a point of a curve. We will first begin with a geometric definition, borrowed from differential geometry; after that, according to our philosophy, we will look for a characterisation of smoothness in terms of the algebraic properties of the coordinate ring of the curve.

Definition 2.4. Let $f(x, y) \in K[x, y]$. A point $P = (a, b) \in C(\bar{K}) = \{(x, y) \in \mathbb{A}^2(\bar{K}),\ f(x, y) = 0\}$ is called a *singular point* if $\partial f/\partial y(P) = \partial f/\partial x(P) = 0$.
A point which is not singular is called *smooth*. A curve is smooth if all of its points in $\bar{K}$ are smooth.
The affine space $T_P = \{(x, y) \in \mathbb{A}^2(\bar{K}),\ \partial f/\partial y(P)(x - a) + \partial f/\partial y(P)(y - b) = 0\}$ is called *tangent line* to the curve at P.

It is well known that the geometric notion of smooth point on a curve is a *local* notion, which depends only on the structure of the curve in an arbitrarily small neighbourhood of the point. The ring-theoretic translation of the idea of "neighbourhood of a point" is that of *localisation* at a *prime ideal*.
Let $P = (a, b)$ be a point on a curve C. Let $\bar{K}[C]_P$ denote the localisation of $\bar{K}[C]$ at the maximal ideal $(x - a, y - b)$. This is called the *local ring* of C at P.

Proposition 2.1. *P is nonsingular if and only if $\bar{K}[C]_P$ is a discrete valuation ring.*

Proof. As $\bar{K}[C] = \bar{K}[x, y]/(f)$ is noetherian, the localisation $\bar{K}[C]_P$ is noetherian (by Proposition 1.3). Moreover we know that the localisation at a maximal ideal of a dimension 1 ring has dimension 1 (Corollary 1.1).
Let m_P be the only maximal ideal of $\bar{K}[C]_P$. By definition, $\bar{K}[C]_P$ is a discrete valuation ring if and only if m_P is principal. Hence, we need to show that $P = (a, b)$ is

smooth if and only if m_p is principal. We may assume, after a linear change of variables, that $P = (a, b) = (0, 0)$. Let us also assume, without loss of generality, that $\partial f/\partial y(0, 0) = \delta \neq 0$. Then the Taylor expansion of $f(x, y)$ at $(0, 0)$ can be written in the form:

$$\begin{aligned} f(x, y) &= \partial f/\partial x(0, 0)x + \delta y + (\text{higher order terms}) = \\ &= \sum_{i=1}^{n} b_i x^i + y(\delta + g(x, y)) \end{aligned}$$

where $b_i \in \bar{K}$, $g(x, y) \in \bar{K}[x, y]$, $g(0, 0) = 0$. In $\bar{K}[C]_P$, the above equality becomes:

$$y(\delta + g(x, y)) = -\sum_{i=1}^{n} b_i x^i$$

(by an abuse of notation, we still denote by x, y the images in $\bar{K}[C]_P$ of $x, y \in \bar{K}[x, y]$). Now $g(0, 0) = 0 \Rightarrow g(x, y) \in m_p$. As δ is a unit, $g(x, y) + \delta \notin m_p \Rightarrow g(x, y) + \delta$ is a unit. Hence, in $\bar{K}[C]_P$, we find that y belongs to the ideal generated by x. Since x and y generate m_p, we conclude that m_p is principal.
Conversely, assume that $\bar{K}[C]_P = (z)$. As $\bar{K}[C]_P = (x, y)$ we have:

$$\begin{aligned} ux + vy = z &\text{ for } u, v \in m_p \\ x = zs &\text{ for } s \in m_p \\ y = zr &\text{ for } r \in m_p \end{aligned}$$

Since $\bar{K}[C]_P$ is a domain, $us + vr = 1$. Then either s or r is a unit in $\bar{K}[C]_P$. Assume that s is a unit. Since $rx - sy = 0$ in $\bar{K}[C]_P$, we can find polynomials $\tilde{r}, \tilde{s}, g \in \bar{K}[x, y]$ such that $f(x, y)g(x, y) = \tilde{r}(x, y)x - \tilde{s}(x, y)y$ and $\tilde{s}(x, y)$ has non-trivial constant term. By comparing the coefficients of the monomial y on the two sides we conclude that $\partial f/\partial y(0, 0) \neq 0$ □

What determines whether a point P on a curve C is smooth or singular is thus the local behaviour of the coordinate ring $\bar{K}[C]$ at the point P. By collecting local information at each point of the curve we obtain the following global characterisation of smooth curves in terms of their coordinate ring:

Proposition 2.2. *An affine curve $C(\bar{K})$ is smooth if and only if its coordinate ring $\bar{K}[C]$ is a Dedekind domain.*

Proof. The ring $\bar{K}[C]$ is noetherian and 1-dimensional (Lemma 2.1). Thus it is a Dedekind domain if and only if it is integrally closed. Proposition 1.2 states that $\bar{K}[C]$ is integrally closed if and only if $\bar{K}[C]_P$ is integrally closed for every $P \in C(\bar{K})$ (maximal ideals correspond to points on the curve, by Theorem 2.1). By Proposition 1.4 $\bar{K}[C]_P$ is integrally closed if and only it is a PID. We just proved that this is equivalent to the fact that P is a smooth point on C. Therefore, $\bar{K}[C]$ is a Dedekind domain if and only if all points of $C(\bar{K})$ are smooth, that is, if C is a smooth curve. □

Corollary 2.1. *Let K be a field, $C(K)$ an irreducible curve. If $\bar{K}[C]$ is a Dedekind domain, then $K[C]$ is also a Dedekind domain.*

Proof. $K[C]$ has dimension 1 and is noetherian. Let us show that every localisation of $K[C]$ at a maximal ideal is a PID. This will prove the theorem.
By the Theorem 2.1 we know that each maximal ideal M of $K[C]$ is of the form $M = ker(ev_P)/(f),\ P = (a,b) \in C(\bar{K})$. Let $g(x)$ denote the minimal polynomial of a over K. Then there exists $h \in K[x,y]$ such that $ker(ev_P) = (g(x), h(x,y))$. In fact, let $I = g(x)K[x,y]$. Then:

$$K[x,y]/I \cong (K[x]/g(x))[y] \cong L[y]$$

where $L = K(a)$. The ideal M/I is maximal in $K[x,y]/I$, which is a PID; hence, M/I is generated by a $\pi(h(x,y)),\ h \in k[x,y]$, and $M = (g(x), h(x,y))$.
Now, since $f(x,y) \in ker(ev_P)$ there exist $\alpha(x,y),\ \beta(x,y) \in K[x,y]$ such that

$$f(x,y) = \alpha(x,y)g(x) + \beta(x,y)h(x,y)$$

Therefore in $K[C]_P$ $\bar{g}$ divides $\bar{h}\bar{\beta}$ (where $\bar{g}$ denotes the class of g in $K[C]_P$). Since by hypothesis $\bar{K}[C]$ is a Dedekind domain, which is equivalent to the fact that the curve C is smooth, we may assume that $\partial f/\partial y(P) \neq 0$. We have

$$\partial f/\partial y(x,y) = \partial\alpha/\partial y(x,y)g(x) + \partial\beta/\partial y(x,y)h(x,y) + \beta(x,y)\partial h/\partial y(x,y)$$

and so: $\partial f/\partial y(P) = \beta(P)\partial h/\partial y(P)$. Hence $\beta(P) \neq 0 \Rightarrow \beta \notin ker(ev(P)) \Rightarrow \bar{\beta}$ is a unit in $K[C]_P$. As $\bar{g}$ divides $\bar{h}\bar{\beta}$, we deduce that $\bar{g}$ and $\bar{h}$ are associates, and so each of them generates $K[C]_P$, which is therefore principal. □

2.2 Projective curves

The *projective space* of dimension n over a field K is the set:

$$\mathbb{P}^n(K) = \{[x_0, \ldots, x_n] :\ x_i \in K\ \forall i = 1, \ldots, n\}$$

Recall that coordinates in the projective space are defined only up to a non zero multiplicative constant: $[x_0, \ldots, x_n] = [y_0, \ldots, y_n]$ if and only if there exists $\lambda \in K^*$ such that $x_i = \lambda y_i\ \forall i = 0, \ldots, n$. The projective space of dimension 2 is called *projective plane*.
Algebraic curves in the projective plane are defined in the same way as in the affine case:

Definition 2.5. Let $f(x,y,z) \in K[x,y,z]$ be an homogeneous polynomial. The set $C(K) = \{[x,y,z] \in \mathbb{P}^2(K) : f(x,y,z) = 0\}$ is called a *projective (plane) curve*. We say that $C(K)$ is *(geometrically) irreducible* if f is an irreducible polynomial in $\bar{K}[x,y,z]$.
The *degree* of the curve $C(K)$ is the degree of the polynomial f.

Remark 2.3. 1. Everything we said in Remark 2.1 can be repeated for projective curves. Terminology and notations introduced there will be also used for projective curves.

2. Note that the definition makes sense, as the polynomial is required to be homogeneous.

3. As in the affine case, we will always assume that projective curves are (geometrically) irreducible.

4. Let $H_0 = \{[x, y, z] \in \mathbb{P}^2(K),\ x = 0\}$.
The bijection

$$\begin{aligned} \phi_0 : \mathbb{P}^2(K) \setminus H_0 &\to \mathbb{A}^2(K) \\ [x, y, z] &\mapsto \left(\frac{y}{x}, \frac{z}{x}\right) \end{aligned}$$

whose inverse is the function $(a, b) \mapsto [1, a, b]$, identifies the affine plane with the projective plane deprived of a "line at infinity".
If $C(K) = \{[x, y, z] \in \mathbb{P}^2(K) : f(x, y, z) = 0\}$ is a projective curve, then $C(K) \cap (\mathbb{P}^2(K) \setminus H_0)$ corresponds, with the above bijection, to the affine curve: $C^*(K) = \{[a, b] \in \mathbb{A}^2(K) : f^*(a, b) = 0\}$, where $f^*(a, b) = f(1, a, b)$ is the dehomogenization of f with respect to x.
Conversely, to an affine curve $C^*(K) = \{[a, b] \in \mathbb{A}^2(K) : f^*(a, b) = 0\}$ we can associate its *projective closure* $C(K) = \{[x, y, z] \in \mathbb{P}^2(K) : f(x, y, z) = 0\}$, where $f(x, y, z) = x^{deg(f^*)} f^*(y/x, z/x)$ is the homogenization of f^*.
The same process can be carried out with respect to any other variable, substituting $H_1 = \{[x, y, z] \in \mathbb{P}^2(K),\ y = 0\}$ or $H_2 = \{[x, y, z] \in \mathbb{P}^2(K),\ z = 0\}$ to H_0. The corresponding maps will be denoted by ϕ_1, ϕ_2.
Roughly speaking, the projective plane allows us to look at our curve from a global point of view; on the contrary, we can always work in the affine setting when we are concerned with local properties of the curve. This motivates the following definitions (the doubtful reader can check that they're all good definitions):

Definition 2.6. Let C be a projective curve, H_i such that $C \nsubseteq H_i$. The *function field* of C, denoted by $K(C)$, is the function field of the affine curve $\phi_i(C \cap (\mathbb{P}^2 \setminus H_i))$.
The *dimension* of C is the dimension of $\phi_i(C \cap (\mathbb{P}^2 \setminus H_i))$.
Let $P \in C \cap (\mathbb{P}^2 \setminus H_i)$. The *local ring* of C at P, denoted by $K[C]_P$, is the local ring of $\phi_i(C \cap (\mathbb{P}^2 \setminus H_i))$ at $\phi_i(P)$.
We say that P is *smooth* if the point $\phi_i(P)$ on the affine curve $\phi_i(C \cap (\mathbb{P}^2 \setminus H_i))$ is a smooth point. A curve is smooth if all its points are smooth.

Remark 2.4. From now on, all curves will be supposed to be smooth.

Now, take a projective curve $C(\bar{K})$ defined over K. We want to state what it means for a point $P = [x, y, z]$ on C to be defined over a field $\bar{K} \supseteq L \supseteq K$. We cannot just say, as in the affine case, that $K(P) = K(x, y, z) \subseteq L$ as x, y, z are defined only up to a multiplicative constant $\lambda \in \bar{K}^*$.
Instead, we're going to exploit the action of the Galois group $Gal(\bar{K}/K)$ on the points of the curve, which we already studied in the affine case. The action is defined in the same way: for $\sigma \in Gal(\bar{K}\ K),\ P = (x, y, z) \in C(\bar{K}),\ P^\sigma = (\sigma(x), \sigma(y), \sigma(z))$ (it's easy to verify that this is well defined).

Definition 2.7. Let $C(\bar{K})$ be a projective curve, $P \in C$. We say that P is *defined over a field* $\bar{K} \supseteq L \supseteq K$ if $\sigma(P) = P \; \forall \sigma \in Gal(\bar{K}/L)$, that is, if P is fixed by $Gal(\bar{K}/L)$.

One of the reasons why the projective plane is a good place to do algebraic geometry is that two irreducible curves $C(\bar{K})$ and $D(\bar{K})$ in the projective plane $\mathbb{P}^2(\bar{K})$ meet as many times as possible. This is the content of the following fundamental

Theorem 2.2. *(Bezout) Let $C(\bar{K})$ and $D(\bar{K})$ be two curves of degree m and n. Then C and D intersect in exactly mn points, counted with multiplicities.*

For a precise explanation of the concept of intersection multiplicity of two curves at a given point and an elementary proof of this result see [3].

2.2.1 Divisors and Riemann-Roch Theorem

Concepts and results presented in this section are of fundamental importance for the study of curves, and have a lot of extraordinary consequences (one of them is described in Chapter 4). Unfortunately, the proof of most statements requires some rather advanced machinery, and will not be given here. The standard reference for these topics is [6]. The same results can also be stated and proved within the (maybe) simpler theory of Riemann Surfaces (see [14]).
Anyway, the reader can just trust the main theorems in this section (expecially Riemann-Roch) and see them at work in the following chapters.

Definition 2.8. Let $C(\bar{K}) = \{[x, y, z] \in \mathbb{P}^2(\bar{K}) : p(x, y, z) = 0\}$ be a projective curve defined over K. The *divisor group* of C, denoted by $Div(C)$, is the free abelian group generated by the points of $C(\bar{K})$. A divisor on C is therefore a formal sum:

$$D = \sum_{P \in C(\bar{K})} n_P P$$

where the coefficients n_P are integers, and only finitely many of them are non zero. The *degree* of the divisor is by definition:

$$deg(D) = \sum_{P \in C(\bar{K})} n_P$$

If $D = \sum_{P \in C(\bar{K})} n_P P$, and $D' = \sum_{P \in C(\bar{K})} n'_P P$, we say that $D \leq D' \Leftrightarrow n_P \leq n'_P \; \forall P \in C(\bar{K})$.
A divisor $D \geq 0$ is called a *positive* (or effective) divisor.
The action of an element $\sigma \in Gal(\bar{K}/K)$ on a divisor $D = \sum_{P \in C(\bar{K})} n_P P$ is given by:

$$(\sigma, D) \mapsto D^\sigma = \sum_{P \in C(\bar{K})} n_P P^\sigma$$

As usual, a divisor D is said to be *defined over a field* $\bar{K} \supseteq L \supseteq K$ if it is fixed by the action of every $\sigma \in Gal(\bar{K}/L)$. The set of divisors defined over L is denoted by $Div_L(C)$.

Recall that if C is a smooth (affine or projective) curve and $P \in C$, the local ring $\bar{K}[C]_P$ is a discrete valuation ring with field of fractions $\bar{K}(C)$ (Proposition 2.1). Denote by v_P the standard valuation on $\bar{K}[C]_P$, extended to its field of fractions.
We say that $f \in \bar{K}(C)^*$ has a *zero* (respectively *pole*) at P if $v_P(f) > 0$ (respectively $v_P(f) < 0$).

Definition 2.9. Let $f \in \bar{K}(C)^*$; the divisor:

$$div(f) = \sum_{P \in C(\bar{K})} v_P(f)P$$

is called the divisor *associated with* f. Two divisors D and D' are called *equivalent* if $D - D' = div(f)$ for some $f \in \bar{K}(C)^*$. If D is equivalent to D', we write $D \sim D'$.

Remark 2.5. To show that $div(f)$ is actually a divisor, it is necessary to prove that a function $f \in \bar{K}(C)^*$ has only a finite number of zeros or poles on a curve C. This is a consequence of Bezout theorem (or it can be verified by more elementary means, with the aid of discriminants of polynomials). By the same means one can also show that if $f \neq 0$ then $deg(div(f)) = 0$. Moreover $div(f) = 0 \Leftrightarrow f \in \bar{K}^*$.

Notation 2.1. Let $D \in Div(C)$. We define:

$$L(D) = \{f \in \bar{K}(C)^* : div(f) + D \geq 0\} \cup \{0\}$$

If $D \in Div_K(C)$, let

$$L_K(D) = \{f \in K(C)^* : div(f) + D \geq 0\} \cup \{0\}$$

Then $L(D)$ (resp. $L_K(D)$) is a $\bar{K}$-vector space (resp. K-vector space), whose dimension is denoted by $l(D)$ (resp. $l_K(D)$).

Remark 2.6. It can be proved that $L(D)$ is always a finite dimensional vector space; moreover, if $deg(D) < 0$, then $L(D) = 0$. In fact, let $f \in L(D),\ f \neq 0$; then:

$$0 = deg(div(f)) \geq deg(-D) = -deg(D) > 0$$

which is absurd.

The dimension of the space $L(D)$ (or $L_K(D)$) tells us, roughly speaking, how many functions we can find on a curve, with poles (zeros) in a fixed set of points not exceeding a prescribed order (having at least a prescribed order). It is extremely useful, in order to study the properties of a curve, to have precise information about the dimension of $L(D)$ or $L_K(D)$. This is the content of the following theorem, of crucial importance in algebraic geometry:

Theorem 2.3. *(Riemann-Roch) Let $C(\bar{K})$ be a (smooth) projective curve defined over K. There exists a divisor $K_C \in Div_K(C)$ and an integer $g \geq 0$, called the* genus *of the curve, such that:*

$$l(D) - l(K_C - D) = deg(D) + 1 - g$$
$$l_K(D) - l_K(K_C - D) = deg(D) + 1 - g$$

Proof. A proof via sheaf theory is given in [6]. A more elementary treatment can be found in [3]. □

Corollary 2.2. *With the same hypotheses as above, we have:*

1. $l(K_C) = g$
2. $deg(K_C) = 2g - 2$
3. *if* $deg(D) > 2g - 2$, *then* $l(D) = l_K(D) = deg(D) + 1 - g$

Proof.
1. Use the previous theorem with $D = 0$, and Remark 2.5, which implies that $l(0) = 1$.
2. Use the previous point and Riemann-Roch, with $D = K_C$.
3. From (2) we obtain $deg(K_C - D) < 0 \Rightarrow l(K_C - D) = 0$, from remark 2.6. Now use Riemann-Roch.

□

The last thing we need to know in order to be able to use Riemann-Roch theorem is how to compute the genus of a curve. For smooth projective curves, we have the following

Proposition 2.3. *If C is a smooth projective curve of degree d, then:*

$$g = (d - 1)(d - 2)/2$$

Chapter 3

Gauss and Jacobi sums

3.1 Multiplicative characters

Definition 3.1. Let G be a finite group. A *character* on G is a morphism of groups

$$\chi : G \to \mathbb{C}^*$$

The set of characters on a group G with the composition law $\chi\psi(g) = \chi(g)\psi(g)$ is a group, called the *dual group* of G and denoted by $\hat{G}$. Its identity element will be denoted by ϵ and will be called the *trivial character*.
Let $\mathbb{F}_p$ denote the field with p elements. A multiplicative character on $\mathbb{F}_p$ is a character defined on the multiplicative group $\mathbb{F}_p^*$ of $\mathbb{F}_p$.
We extend the domain of definition (as well as the codomain) of characters χ on $\mathbb{F}_p$ as follows: $\chi(0) = 0$ if $\chi \neq \epsilon$; $\epsilon(0) = 1$.

Remark 3.1. Properties: Let $\chi \in \hat{G}, g \in G$; let e be the identity of G. The following facts are easy to verify:

1. $\chi(e) = 1$
2. $\chi(g)^{|G|} = 1$
3. $\chi(g^{-1}) = \chi(g)^{-1} = \overline{\chi(g)}$

Recall that $\mathbb{F}_p^*$ is a cyclic group. Let a be a generator of $\mathbb{F}_p^*$. Then $\chi \in \hat{\mathbb{F}}_p^*$ is determined by its value $\chi(a) = \zeta_{p-1}^k$, where ζ_{p-1} is a fixed primitive $(p-1)$-th root of unity. For each $k \in \{0, \ldots, p-1\}$, the formula

$$\chi(a^n) = \zeta_{p-1}^{kn}$$

defines a character of $\mathbb{F}_p$, and all characters have this form. it follows that $\hat{\mathbb{F}}_p^*$ is cyclic, and is isomorphic to $\mathbb{F}_p^*$.
Generators of $\hat{\mathbb{F}}_p^*$ correspond to the values of k which are coprime with $p-1$.

Proposition 3.1. *Let $\chi \in \hat{\mathbb{F}}_p^*, a \in \mathbb{F}_p^*$*

1. *if $\chi \neq \epsilon$, there exists $b \in \mathbb{F}_p^*$ such that $\chi(b) \neq 1$;*

2. *if* $a \neq 1$, *there exists* $\psi \in \hat{\mathbb{F}}_p^*$ *such that* $\psi(a) \neq 1$*;*

3. *if* $\chi \neq \epsilon$, $\sum_{t\in\mathbb{F}_p} \chi(t) = 0$*;*
   $\sum_{t\in\mathbb{F}_p} \epsilon(t) = p$*;*

4. *if* $a \neq 1$, $\sum_{\chi\in\hat{\mathbb{F}}_p^*} \chi(a) = 0$.

Proof. 1. This is clear.

2. Let b be a generator of $\mathbb{F}_p^*$, $a = b^k$. By hypothesis, $p-1$ doesn't divide k. Define $\psi(b^m) = \zeta_{p-1}^m$. Then ψ has the required properties.

3. Suppose $\chi \neq \epsilon$ (otherwise the claim is obvious). From (1), we can take $b : \chi(b) \neq 1$. Then we obtain:

$$\chi(b)\sum_{t\in\mathbb{F}_p}\chi(t) = \sum_{t\in\mathbb{F}_p}\chi(tb) = \sum_{t\in\mathbb{F}_p}\chi(t) \Rightarrow (\chi(b)-1)\sum_{t\in\mathbb{F}_p}\chi(t) = 0$$

   from which the claim follows.

4. take $\psi \in \hat{\mathbb{F}}_p^*$ such that $\psi(a) \neq 1$ (point (2)), and use the same trick as in the previous point.

□

We're going to use characters in order to determine the number of solutions of equations in $\mathbb{F}_p$. In particular, characters turn out to be very useful in the study of equations of the form:

$$x^n = a$$

where $a \in \mathbb{F}_p^*,\ n \mid p-1$.
We make the following preliminary remark: fix a generator b of $\mathbb{F}_p^*$. Then $a = b^k$ is an $n-th$ power (equivalently, $x^n = a$ has a solution) if and only if $n \mid k$.

Proposition 3.2. *Let* $a \in \mathbb{F}_p^*,\ n \mid p-1$*;*

1. *if* $x^n = a$ *has a solution in* $\mathbb{F}_p$ *and* χ *is a character such that* $\chi^n = \epsilon$*, then* $\chi(a) = 1$

2. *if* $x^n = a$ *has no solutions in* $\mathbb{F}_p$*, then there is a character* χ *such that* $\chi^n = \epsilon$ *and* $\chi(a) \neq 1$

Proof. 1. fix a generator b of $\mathbb{F}_p^*$. From the above observation we have: $a = b^{kn} \Rightarrow \chi(a) = \chi(b^{kn}) = (\chi(b^k))^n = 1$

2. by he above remark $a = b^l$ and $n \nmid l$. Write $l = kn + m$, with $0 < m < n$. Define $\chi(b^t) = \zeta_{p-1}^{(p-1)t/n}$. Then χ is a character of order n, and $\chi(a) = \chi(b^{kn})\chi(b^m) = \zeta_{p-1}^{(p-1)m/n} \neq 1$.

□

Let $n \mid p-1$ and $a \in \mathbb{F}_p^*$. Notice that if the equation $x^n = a$ has a solution α, then it has n solutions. In fact, $\beta^n = a \Leftrightarrow (\alpha\beta^{-1})^n = \delta^n = 1$. Hence each solution of $x^n = a$ is of the form $\alpha\delta$, where δ is an element of order dividing n in $\mathbb{F}_p^*$. But $\mathbb{F}_p^*$ is cyclic, and it is known that if G is a cyclic group of finite order and $d \mid |G|$, G has d elements of order dividing d.
Since we know that $\hat{\mathbb{F}}_p^*$ is cyclic of order $p-1$, the same reasoning proves that there are n characters in $\hat{\mathbb{F}}_p^*$ of order dividing n. Collecting this information we obtain the following fundamental:

Proposition 3.3. *Let $a \in \mathbb{F}_p, n \mid p-1$. Then:*

$$N(x^n = a) = \sum_{\chi^n = \epsilon} \chi(a)$$

Proof. For $a = 0$, $x^n = 0$ has one solution ($x = 0$) and $\sum_{\chi^n=\epsilon} \chi(0) = \epsilon(0) = 1$ (this is why we defined $\chi(0) = 0$ if $\chi \neq \epsilon$ and $\epsilon(0) = 1$).
Suppose now that $a \neq 0$. Then, as we observed above, $x^n = a$ has either 0 or n solutions. In the latter case the previous proposition tells that, for every character χ such that $\chi^n = \epsilon$, $\chi(a) = 1$. It follows that $\sum_{\chi^n=\epsilon} \chi(a) = n$.
If a is not a $n-th$ power, from the same proposition we know that there is a character ψ such that $\psi^n = \epsilon$ and $\psi(a) \neq 1$. With a familiar trick, we obtain:

$$\psi(a) \sum_{\chi^n=\epsilon} \chi(a) = \sum_{\chi^n=\epsilon} \chi\psi(a) = \sum_{\chi^n=\epsilon} \chi(a) \Rightarrow (\psi(a) - 1) \sum_{\chi^n=\epsilon} \chi(a) = 0$$

(as characters of order $\leq n$ form a group). It follows that $\sum_{\chi^n=\epsilon} \chi(a) = 0$. □

3.2 Gauss sums

Definition 3.2. Let χ be a character on $\mathbb{F}_p$, $a \in \mathbb{F}_p$, $\zeta = e^{2\pi i/p}$.

$$g_a(\chi) = \sum_{t \in \mathbb{F}_p} \chi(t)\zeta^{at}$$

is called a *Gauss sum* on $\mathbb{F}_p$ belonging to the character χ.

Notice that, if $\zeta = e^{2\pi i/p}$ and $t \in \mathbb{F}_p$, ζ^t is well defined.
The following lemma follows easily from the definition:

Lemma 3.1. *If $a \neq 0$ and $\chi \neq \epsilon$ then $g_a(\chi) = \chi(a^{-1})g_1(\chi)$; if $a = 0$ and $\chi \neq \epsilon$ we have $g_0(\chi) = 0$. If $a \neq 0$ and $\chi = \epsilon$, $g_a(\epsilon) = 0$. If $a = 0$ and $\chi = \epsilon$, $g_0(\epsilon) = p$.*

From now on we shall denote $g_1(\chi)$ by $g(\chi)$.

Proposition 3.4. *If $\chi \neq \epsilon$, then $|g(\chi)| = \sqrt{p}$*

Proof. We're going to evaluate $A = \sum_a g_a(\chi)\overline{g_a(\chi)}$ in two different ways:
if $a \neq 0$, then by the previous proposition $g_a(\chi) = \chi(a^{-1})g(\chi)$, and $\overline{g_a(\chi)} = \chi(a)\overline{g(\chi)}$. Thus $g_a(\chi)\overline{g_a(\chi)} = |g(\chi)|^2$. Since $g_0(\chi) = 0$ we obtain $A = (p-1)|g(\chi)^2|$.
On the other hand, we also have:

$$g_a(\chi)\overline{g_a(\chi)} = \sum_{x,y} \chi(x)\overline{\chi(y)}\zeta^{ax-ay}$$

Observe that, for $t \in \mathbb{F}_p$ fixed, $\sum_{a\in\mathbb{F}_p} \zeta^{at} = p$ if $t \equiv 0 \pmod p$, otherwise this sum equals 0.
Hence we have:

$$A = \sum_{x,y} \chi(x)\overline{\chi(y)}\delta(x,y)p = (p-1)p$$

Comparing the two values of A, we get the desired result. □

Remark 3.2. Let $\overline{\chi}$ denote the character whose value at a is $\overline{\chi(a)}$ (which coincides with χ^{-1}). Then we have:

$$\overline{g(\chi)} = \sum_t \overline{\chi(t)}\zeta^{-t} = \chi(-1)\sum_t \overline{\chi(-t)}\zeta^{-t} = \chi(-1)g(\bar{\chi})$$

Thus, the equality $|g(\chi)|^2 = p$ can be written as: $g(\chi)g(\bar{\chi}) = \chi(-1)p$

Remark 3.3. Let $\chi(a) = \left(\frac{a}{p}\right) = a^{(p-1)/2}$ be the quadratic residue character on $\mathbb{F}_p$. Evaluating the sum $\sum_a g_a g_{-a}$ in two ways, in the same way as in the previous proof, we find that $g^2 = (-1)^{(p-1)/2}p$. This is the starting point of an elegant proof of the law of quadratic reciprocity, whose conciseness and beauty make it worth describing.
Let p, q two different odd primes. Let ζ be a primitive $q-$th root of unity in $\bar{\mathbb{F}}_p$. Let χ be the quadratic residue character on $\mathbb{F}_q$ and $g = \sum_{t\in\mathbb{F}_q} \chi(t)\zeta^t \in \bar{\mathbb{F}}_p$ be the corresponding Gauss sum. Then, as $g^2 = (-1)^{\frac{q-1}{2}}q$, $\left(\frac{(-1)^{\frac{q-1}{2}}q}{p}\right) = 1$ if and only if $g \in \mathbb{F}_p \Leftrightarrow g^p = g$.
Now

$$g^p = \sum_{t\in\mathbb{F}_q} \chi(t)^p\zeta^{pt} = \sum_{t\in\mathbb{F}_q} \chi(t)\zeta^{pt} = \chi(p)\sum_{t\in\mathbb{F}_q} \chi(pt)\zeta^{pt} = \chi(p)g$$

So $g^p = g \Leftrightarrow \chi(p) = \left(\frac{p}{q}\right) = 1$.
Hence we obtain:

$$\left(\frac{(-1)^{\frac{q-1}{2}}q}{p}\right) = 1 \Leftrightarrow \left(\frac{p}{q}\right) = 1$$

hence

$$(-1)^{\frac{p-1}{2}\frac{q-1}{2}}\left(\frac{q}{p}\right) = 1 \Leftrightarrow \left(\frac{p}{q}\right) = 1$$

which is the law of quadratic reciprocity.

Remark 3.4. As $g \in \mathbb{Q}[\zeta]$, we obtain that $\sqrt{\pm p} \in \mathbb{Q}[e^{2\pi i/p}]$. With a little extra work, it can be deduced from this fact that each quadratic extension of $\mathbb{Q}$ is contained in a cyclotomic extension. This is a very special case of the Kronecker-Weber theorem, which asserts that

the same is true for every *abelian* extension of $\mathbb{Q}$ (i.e. a finite Galois extension of $\mathbb{Q}$ whose Galois group is abelian).[1]

3.3 Jacobi sums

Definition 3.3. Let χ and λ be characters on $\mathbb{F}_p$. The expression:

$$J(\chi, \lambda) = \sum_{a+b=1} \chi(a)\lambda(b)$$

is called a *Jacobi sum.*

The proof of the following elementary properties of Jacobi sums is left to the reader (see [8] if you need help):

Lemma 3.2. *Let χ, λ be nontrivial characters. Then*

1. $J(\chi, \chi) = p$
2. $J(\epsilon, \chi) = 0$
3. $J(\chi, \chi^{-1}) = -\chi(-1)$
4. *If $\chi\lambda \neq \epsilon$, then*

$$J(\chi, \lambda) = \frac{g(\chi)g(\lambda)}{g(\chi\lambda)}$$

Corollary 3.1. *If $\chi, \lambda, \chi\lambda$ are nontrivial, then $|J(\chi, \lambda)| = \sqrt{p}$. In particular, if χ, χ^2 are nontrivial, $|J(\chi, \chi)| = \sqrt{p}$*

This seemingly innocuous statement actually has many useful consequences, as we will soon discover. It allows us to obtain good estimates for the number of solutions of certain types of equations over finite fields.
The following proposition is another non trivial consequence of the previous corollary:

Proposition 3.5. *Let p be a prime number.*

1. $p \equiv 1 \pmod 4 \Leftrightarrow \exists\, a, b \in \mathbb{Z} : a^2 + b^2 = p$
2. $p \equiv 1 \pmod 3 \Leftrightarrow \exists\, a, b \in \mathbb{Z} : a^2 - ab + b^2 = p$

Proof. If $p \equiv 1 \pmod 4$ then $4 \mid (p-1) = |\mathbb{F}_p^*| \Rightarrow \hat{\mathbb{F}}_p^*$ has an element of order 4. Let us call it λ. Then λ takes its values in the set $\{1, -1, i, -i\}$, so $J(\lambda) \in \mathbb{Z}[i]$, and $|J(\lambda, \lambda)|^2 = |(a + ib)|^2 = a^2 + b^2 = p$, by Corollary 3.1.
If $p \equiv 1 \pmod 3$, take a character λ on $\mathbb{F}_p$ of order 3. Then $J(\lambda) \in \mathbb{Z}[\omega]$, and $p = |J(\lambda, \lambda)|^2 = |(a + \omega b)|^2 = a^2 - ab + b^2$.
The opposite implications are easy (look at the congruence class modulo 4 and 3). □

[1] Actually, the first correct proof of this theorem is due to Hilbert. To prove it one has to explore much deeper waters: it is one of the main results of the so called *class field theory*; see [16].

Proposition 3.6. *Let $p \equiv 1 \pmod{n}$ be a prime, χ a character of order n in $\mathbb{F}_p$. Then*

$$g(\chi)^n = \chi(-1)pJ(\chi,\chi)J(\chi,\chi^2)\dots J(\chi,\chi^{n-2})$$

In particular, if $n = 3$, $g(\chi)^3 = pJ(\chi,\chi)$.

Proof. Using Lemma 3.2 we obtain: $g(\chi)^2 = J(\chi,\chi)g(\chi^2)$. Multiplication by $g(\chi)$ and Lemma 3.2 again give $g(\chi)^3 = J(\chi,\chi)J(\chi,\chi^2)g(\chi^3)$. Continuing in this way, we get:

$$g(\chi)^{n-1} = J(\chi,\chi)J(\chi,\chi^2)\dots J(\chi,\chi^{n-2})g(\chi^{n-1})$$

Now, $\chi^{n-1} = \overline{\chi} \Rightarrow g(\chi^{n-1})g(\chi) = \chi(-1)p$ (Remark 3.2). Multiplication of the above equality by $g(\chi)$ gives the result. □

Proposition 3.7. *Let $p \equiv 1 \pmod{3}$ be a prime, χ a character of order 3 on $\mathbb{F}_p$. Set $J(\chi,\chi) = a + b\omega$. Then $a \equiv -1 \pmod{3}, b \equiv 0 \pmod{3}$.*

Proof. We have the following congruences in the ring of algebraic integers $\mathbb{Z}[\omega]$:

$$g(\chi)^3 = (\sum_t \chi(t)\zeta^t)^3 \equiv \sum_t \chi(t)^3\zeta^{3t} \equiv \sum_{t\neq 0} \zeta^{3t} \equiv -1 \pmod{3}$$

Thus:

$$g(\chi)^3 = pJ(\chi,\chi) \equiv a + b\omega \equiv -1 \pmod{3}$$

Working with $\overline{\chi}$ instead of χ, and observing that $g(\overline{\chi}) = \overline{g(\chi)}$ (as χ is a cubic character, hence $\chi(-1)^3 = 1 \Rightarrow \chi(-1) = 1$) we get:

$$g(\overline{\chi})^3 = pJ(\overline{\chi},\overline{\chi}) \equiv a + b\overline{\omega} \equiv -1 \pmod{3}$$

Subtraction yields $b(\omega - \overline{\omega}) = b\sqrt{-3} \equiv 0 \pmod{3} \Rightarrow 9 \mid -3b^2 \Rightarrow 3 \mid b$. Now, since $a + b\omega \equiv -1 \pmod{3}$, we obtain $a \equiv -1 \pmod{3}$. □

We can now prove a beautiful result due to Gauss; this is a first example of how to use Jacobi sums and their properties in order to obtain information about the number of solutions of a polynomial equation in $\mathbb{F}_p$.

Theorem 3.1. *(Gauss) Let $N(x^3+y^3 = 1)$ denote the number of solutions of the equation $x^3 + y^3 = 1$ in $\mathbb{F}_p$.*

1. *If $p \equiv 2 \pmod{3}$, then $N(x^3 + y^3 = 1) = p$.*
2. *If $p \equiv 1 \pmod{3}$, then there are integers A, B such that $4p = A^2 + 27B^2$; A is uniquely determined by the condition $A \equiv 1 \pmod{3}$, and $N(x^3+y^3 = 1) = p-2+A$*

Proof. 1. As $p \equiv 2 \pmod{3}$, $x \mapsto x^3$ is an automorphism of $\mathbb{F}_p^*$. Then each element in $\mathbb{F}_p$ is the cube of a unique element in $\mathbb{F}_p$, hence $N(x^3+y^3 = 1) = N(x+y = 1) = p$. Of course, we did not need Gauss to prove this.

2. This is going to be more interesting.
   Take a character χ on $\mathbb{F}_p$ of order 3. Then ϵ, χ, χ^2 are all the characters on $\mathbb{F}_p$ of order dividing 3 and, by Proposition 3.3, we have:

$$N(x^3+y^3=1) = \sum_{a+b=1} N(x^3=a)N(y^3=b) = \sum_{a+b=1}\left(\sum_{i=0}^{2}\chi^i(a)\sum_{j=0}^{2}\chi^j(b)\right) =$$
$$= \sum_{i,j=0}^{2} J(\chi^i,\chi^j) = J(\epsilon,\epsilon)+J(\epsilon,\chi)+J(\chi,\epsilon)+J(\chi,\chi^2)$$
$$+ J(\chi^2,\chi)+J(\chi,\chi)+J(\chi^2,\chi^2)$$

   Let $J(\chi,\chi) = a+b\omega$. By Lemma 3.2, we obtain:

$$\begin{aligned} N(x^3+y^3=1) =& p+0+0-\chi(-1)-\chi^2(-1)+J(\chi,\chi)+J(\bar{\chi},\bar{\chi}) = \\ &= p-2+2Re(J(\chi,\chi)) = p-2+2Re(a+b\omega) = \\ &= p-2+(2a-b) = p-2+A \end{aligned}$$

   where $A = 2a-b \equiv 1 \pmod 3$, $b \equiv 0 \pmod 3$ and $a^2-ab+b^2 = p$ (Proposition 3.5 and Proposition 3.7), hence $4p = (2a-b)^2+3b^2 = A^2+27B^2$, where $B = b/3$. It remains only to show that such an A is unique.
   Suppose that $4p = A'^2+27B'^2$, with $A' \equiv 1 \pmod 3$. Then, the equations:

$$\begin{aligned} 3B' =& b' \\ (2a'-b') =& A' \end{aligned}$$

   determine uniquely two integers a', b' such that $4p = (2a'-b')^2+3b'^2 \Rightarrow p = a'^2-a'b'+b'^2$. Moreover, $3|b'$ and $2a'-b' \equiv 1 \pmod 3$.
   In $\mathbb{Z}[\omega]$ we have $N(a+\omega b) = N(a'+\omega b') = p$, so $a+\omega b$ is associated with $a'+\omega b'$. This means that $a'+\omega b' = u(a+\omega b)$ where $u = \pm 1, \pm\omega$ or $\pm\omega^2$. Examination of each case shows that only for $u = 1$ we have $3|b'$ and $2a'-b' \equiv 1 \pmod 3$. Therefore $A' = A,\ B' = B$.

□

Example 3.1. $p = 97$. Then $4p = 388 = 19^2+27$. So $A = 19, B = 1$. Hence, the curve of affine equation $x^3+y^3 = 1$ has $97-2+19 = 114$ points in $\mathbb{A}^2(\mathbb{F}_{97})$. It would have been much harder to obtain the same conclusion by brute force.
Notice that this curve has 3 points at infinity, corresponding to the projective solutions of the equation

$$x^3+y^3 = 0$$

in $\mathbb{P}^1(\mathbb{F}_{97})$. Let $a = y/x$ (observe that in the above equation we must have $x \neq 0, y \neq 0$). Then we obtain the equivalent equation $a^3 = -1$ in $\mathbb{F}_{97}^*$, which has exactly 3 solutions, since $3 \mid (97-1)$. Hence our curve has 117 points in $\mathbb{P}^2(\mathbb{F}_{97})$.

Remark 3.5. If we consider the projective closure of the affine curve $x^3+y^3 = 1$, namely $x^3+y^3-z^3 = 0$, it is easy to show, using the previous result, that the number of points on this curve defined on $\mathbb{F}_p$ is $p+1$ if $p \equiv 2 \pmod 3$, $p+1-A$ if $p \equiv 1 \pmod 3$ (it

suffices to add to affine points the points at infinity of the curve, solutions of the equation $x^3 + y^3 = 0$; these can be determined as in the previous example). In both cases, we observe that the number of projective points on the curve $x^3 + y^3 - z^3 = 0$ satisfies the inequality: $|N(x^3+y^3-z^3=0)-(p+1)| \leq 2\sqrt{p}$. This is a special case of Hasse theorem: we will prove it for other smooth projective curves of degree 3 in Chapter 5.

We conclude this section with a technical result that will be used in the sequel:

Lemma 3.3. *Let p be an odd prime, ρ a character of order 2 and χ any non trivial character of $\mathbb{F}_p$. Then $J(\rho, \chi) = \chi(4)J(\chi, \chi)$.*

Proof.

$$\begin{aligned} J(\rho,\chi) &= \sum_{u+v=1} \rho(u)\chi(v) = \sum_{u+v=1} (1+\rho(u))\chi(v) = \\ &= \sum_{u+v=1} N(t^2 = u)\chi(v) = \sum_t \chi(1-t^2) = \\ &= \chi(4)\sum_t \chi\Big(\frac{1-t}{2}\Big)\chi\Big(\frac{1+t}{2}\Big) = \chi(4)J(\chi,\chi) \end{aligned}$$

□

3.4 Cubic residue character

Let $\pi \in \mathbb{Z}[\omega]$, be a prime, with $N(\pi) \neq 3$. Equivalently, π is not associate with $1-\omega$. Then it's immediate to see that the residue classes of $1, \omega, \omega^2$ are distinct in $\mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega]$. Thus $(\mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega])^*$ contains a subgroup of order 3, and so $3 \mid |(\mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega])^*| = N(\pi)-1$. Now, take $\alpha \in \mathbb{Z}[\omega]$. If π does not divide α, then $\alpha^{(N(\pi)-1)/3} \equiv 1, \omega$ or $\omega^2 \pmod{\pi}$. In fact, let $\alpha^{(N(\pi)-1)/3} \equiv A \pmod{\pi}$. Then A is a zero of the polynomial $(x-1)(x-\omega)(x-\omega^2) \in (\mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega])[x]$, as $A^3 - 1 \equiv \alpha^{N(\pi)-1} \equiv \alpha^{|(\mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega])^*|} \equiv 1 \pmod{\pi}$.
Therefore we can give the following definition:

Definition 3.4. If $N(\pi) \neq 3$, the cubic residue character of α modulo π is defined by:

1. $(\alpha/\pi)_3 = 0$ if $\pi \mid \alpha$;
2. $(\alpha/\pi)_3 \equiv \alpha^{(N(\pi)-1)/3} \pmod{\pi}$, with $(\alpha/\pi)_3 \in \{1, \omega, \omega^2\}$, if $(\pi, \alpha) = 1$.

In this section, will also denote $(\alpha/\pi)_3$ by $\chi_\pi(\alpha)$. The following properties of $\chi_\pi(\alpha)$ are clear:

Lemma 3.4. *1. $\chi_\pi(\alpha\beta) = \chi_\pi(\alpha)\chi_\pi(\beta)$*

2. if $\alpha \equiv \beta \pmod{\pi}$, then $\chi_\pi(\alpha) = \chi_\pi(\beta)$

As a consequence of the previous Lemma, χ_π gives rise to a character defined on the group $(\mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega])^*$, which we will still denote with the same symbol.
Let $\pi \in \mathbb{Z}[\omega], N(\pi) = p \equiv 1 \pmod 3$, p prime in $\mathbb{Z}$. Then π is prime in $\mathbb{Z}[\omega]$ and $(\mathbb{Z}[\omega]/\pi\mathbb{Z}[\omega])$ is a field of cardinality $N(\pi) = p$, thus it may be identified with $\mathbb{F}_p$. Thus, χ_π can be seen as a character of order 3 on $\mathbb{F}_p$. We can determine explicitly the value $J(\chi_\pi, \chi_\pi)$ if π is primary:

Proposition 3.8. *If $\pi \in \mathbb{Z}[\omega]$ is primary and $N(\pi) = p$, then:*

$$J(\chi_\pi, \chi_\pi) = \pi$$

Proof. By corollary 3.1 $J(\chi_\pi, \chi_\pi) = \pi'$ is prime in $\mathbb{Z}[\omega]$ (as its norm is p). By Proposition 3.7 we know that π' is primary. Thus, we need only to show that $\pi \mid \pi'$, as this will imply that π and π' are associates and both primary, hence equal. Now, we have by definition:

$$J(\chi_\pi, \chi_\pi) = \sum_x \chi_\pi(x)\chi_\pi(1-x) \equiv \sum_x x^{(p-1)/3}(1-x)^{(p-1)/3} \pmod{\pi}$$

As the polynomial $x^{(p-1)/3}(1-x)^{(p-1)/3}$ is of degree $< (p-1)$, we conclude applying the following proposition to each monomial in $x^{(p-1)/3}(1-x)^{(p-1)/3}$. □

Proposition 3.9. *1. If $(p-1) \mid n$, $\sum_{x\in\mathbb{F}_p} x^n = p-1$;*

2. If $(p-1) \nmid n$, $\sum_{x\in\mathbb{F}_p} x^n = 0$.

Proof. (1) is clear. Let us prove (2): as $(p-1) \nmid n$, there exists $y \in \mathbb{F}_p^*$ such that $y^n \neq 1$. It suffices to take a generator of the multiplicative group $\mathbb{F}_p^*$. Hence we have, as usual:

$$\sum_{x\in\mathbb{F}_p} x^n = \sum_{x\in\mathbb{F}_p} (xy)^n = y^n \sum_{x\in\mathbb{F}_p} x^n \Rightarrow (y^n - 1)\sum_{x\in\mathbb{F}_p} x^n = 0 \Rightarrow \sum_{x\in\mathbb{F}_p} x^n = 0$$

□

As a consequence of this simple fact, we obtain the following important:

Theorem 3.2. *(Chevalley-Warning) Let $f(x,y,z) \in \mathbb{F}_p[x,y,z]$ be a homogeneous polynomial of degree 2. For each prime p there is always at least a non zero solution of the equation $f(x,y,z) = 0$ in $\mathbb{A}^3(\mathbb{F}_p)$.*
Equivalently, there is always at least a point on the projective curve $C(\mathbb{F}_p) = \{[x,y,z] \in \mathbb{P}^2(\mathbb{F}_p) : f(x,y,z) = 0\}$.

Proof. Fix a prime p. Let N be the number of solutions of the equation $f(x,y,z) = 0$ with $x, y, z \in \mathbb{F}_p$. We're going to show that $N \equiv 0 \pmod{p}$. This will imply the result. The key observation is that $f(x,y,z) \neq 0$ if and only if $f(x,y,z)^{p-1} \equiv 1 \pmod{p}$. Hence

$$N \equiv \sum_{x,y,z\in\mathbb{F}_p} 1 - f(x,y,z)^{p-1} \pmod{p}$$

Now notice that the each monomial $m(x,y,z) = x^i y^j z^k$ in the polynomial $1 - f(x,y,z)^{p-1}$ has degree at most $2(p-1)$. Hence one of x, y, z, say x, appears in m with an exponent which is less than $p-1$. By the above Proposition: $\sum_{x,y,z\in\mathbb{F}_p} m(x,y,z) = \left(\sum_{y,z} y^j z^k\right)\sum_x x^i \equiv 0 \pmod{p}$. The result follows. □

Remark 3.6. A careful examination of the above proof shows that the Theorem is true for any homogeneous polynomial whose degree is strictly inferior to the number of variables. Moreover, a very similar proof works for an arbitrary finite field $\mathbb{F}_{p^m}$.

Remark 3.7. Cubic residue characters allow us to state in a very simple way the law of cubic reciprocity:

Theorem 3.3. *If $\lambda, \pi \in \mathbb{Z}[\omega]$ are primary, then $(\lambda/\pi)_3 = (\pi/\lambda)_3$.*

See [8] for a proof of this result.

3.5 Biquadratic residue character

In this section we're going to define a biquadratic (or quartic) residue character, in the same way as we defined the cubic residue character in the previous section. We will work in the ring $\mathbb{Z}[i]$ instead of the ring $\mathbb{Z}[\omega]$.
Let π be a prime in $\mathbb{Z}[i]$, with $N(\pi) \neq 2 \Leftrightarrow (\pi) \neq (1+i)$. Then the residues classes of $1, -1, i, -i$ are distinct in $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$, so $(\mathbb{Z}[i]/\pi\mathbb{Z}[i])^*$ contains a subgroup of order 4, hence $4 \mid |(\mathbb{Z}[i]/\pi\mathbb{Z}[i])^*| = N(\pi) - 1$.
Let $\alpha \in \mathbb{Z}[i]$, and suppose that π doesn't divide α. Then in $(\mathbb{Z}[i]/\pi\mathbb{Z}[i])^*$ we have $\alpha^{N(\pi)-1} = \alpha^{|(\mathbb{Z}[i]/\pi\mathbb{Z}[i])^*|} \equiv 1 \pmod{\pi} \Rightarrow \alpha^{(N(\pi)-1)/4} \equiv 1, -1, i \text{ or } -i \pmod{\pi}$.

Definition 3.5. If $N(\pi) \neq 2$, the *biquadratic residue character* of α modulo π is defined by:

1. $(\alpha/\pi)_4 = 0$ if $\pi \mid \alpha$;
2. $(\alpha/\pi)_4 \equiv \alpha^{(N(\pi)-1)/4} \pmod{\pi}$, with $(\alpha/\pi)_4 \in \{1, -1, i, -i\}$, if $(\pi, \alpha) = 1$

In this section we will also denote $(\alpha/\pi)_4$ by $\chi_\pi(\alpha)$.

Lemma 3.5. *1.* $\chi_\pi(\alpha\beta) = \chi_\pi(\alpha)\chi_\pi(\beta)$

2. if $\alpha \equiv \beta \pmod{\pi}$*, then* $\chi_\pi(\alpha) = \chi_\pi(\beta)$

It follows from these properties that χ_π induces a character on the group $(\mathbb{Z}[i]/\pi\mathbb{Z}[i])^*$, which we will still denote with the same symbol.
Suppose that π is prime in $\mathbb{Z}[i]$ and $N(\pi) = p \equiv 1 \pmod{4}$. Then $|\mathbb{Z}[i]/\pi\mathbb{Z}[i]| = N(\pi) = p$, so $\mathbb{Z}[i]/i\mathbb{Z}[i]$ is isomorphic to $\mathbb{F}_p$ and χ_π can be seen as a character of order 4 on $\mathbb{F}_p$.
Suppose that π is primary. We would like to determine, as in the previous section, the value of $J(\chi_\pi, \chi_\pi)$. We need the following Proposition:

Lemma 3.6. $-\chi_\pi(-1)J(\chi_\pi, \chi_\pi)$ *is primary.*

Proof.

$$J(\chi_\pi, \chi_\pi) = 2\sum_{t=2}^{(p-1)/2} \chi_\pi(t)\chi_\pi(1-t) + \chi_\pi\left(\frac{p+1}{2}\right)^2$$

Now, any unit in $\mathbb{Z}[i]$ is congruent to 1 $\pmod{1+i}$. Moreover, $(2+2i) \mid 4 \mid (p-1) \Rightarrow p \equiv 1 \pmod{2+2i}$. Finally, $\chi_\pi(\frac{p+1}{2})^2 = \chi_\pi(2^{-1})^2 = \chi_\pi(2)^{-2} = \chi_\pi(2)^2 = \chi_\pi(-i(1+i)^2)^2 = \chi_\pi(-i)^2 = \chi_\pi(-1)$. Thus:

$$J(\chi_\pi, \chi_\pi) \equiv 2\left(\frac{p-3}{2}\right) + \chi_\pi(-1) \equiv -2 + \chi_\pi(-1) \pmod{2+2i}$$

Thus:

$$-\chi_\pi(-1)J(\chi_\pi, \chi_\pi) \equiv 2\chi_\pi(-1) - 1 \equiv 1 \pmod{2+2i}$$

since $\chi_\pi(-1) = \pm 1$. □

Proposition 3.10. *If* $\pi \in \mathbb{Z}[i]$ *is primary, then:*

$$-\chi_\pi(-1)J(\chi_\pi, \chi_\pi) = \pi$$

Proof. We know that $N(J(\chi,\chi)) = p$ (Corollary 3.1), hence $J(\chi,\chi)$ is prime in $\mathbb{Z}[i]$, and $-\chi_\pi(-1)J(\chi_\pi,\chi_\pi) = \pi'$ is also prime. Moreover, we know by the above Lemma that π' is primary. Thus, it's enough to show that $\pi \mid \pi' \Leftrightarrow \pi \mid J(\chi_\pi,\chi_\pi)$. By definition

$$J(\chi_\pi,\chi_\pi) \equiv \sum_{t=1}^{p-1} t^{(p-1)/4}(1-t)^{(p-1)/4} \pmod{\pi}$$

Now, the polynomial $\sum_{t=1}^{p-1} t^{(p-1)/4}(1-t)^{(p-1)/4}$ has degree $< p-1$, hence we conclude by Proposition 3.9. □

Remark 3.8.

1. We can also define the quartic residue character $(a/p)_4$ with respect to a prime $p \equiv 3 \pmod 4$, which is also prime in $\mathbb{Z}[i]$. This is a character on $\mathbb{Z}[i]/p\mathbb{Z}[i] \cong \mathbb{F}_{p^2}$. For $a \in \mathbb{Z}$ such that $(a,p) = 1$ we obtain: $(a/p)_4 \equiv a^{(N(p)-1)/4} \equiv a^{(p^2-1)/4} \equiv a^{(p-1)(p+1)/4} \equiv 1 \pmod p$ as $a^{p-1} \equiv 1 \pmod p$.

2. Even more generally, we can define, for $a \in \mathbb{Z}[i]$ such that $(1+i) \nmid a$, the biquadratic residue symbol modulo a: $(\alpha/a)_4 = \prod_i (\alpha/\lambda_i)_4$ where $\alpha = \prod_i \lambda_i$ and each λ_i is prime in $\mathbb{Z}[i]$.
   The law of biquadratic reciprocity describes the relation between $(\alpha/a)_4$ and $(a/\alpha)_4$. We will need the following special case of this result:

Theorem 3.4. *If $a \equiv 1 \pmod 4$ and α is primary , with $(\alpha, a) = 1$, then*

$$(\alpha/a)_4 = (a/\alpha)_4$$

We will also need the following "supplement":

Lemma 3.7. *If $\alpha = a + bi$ is primary, then*

$$\Big(\frac{1+i}{\alpha}\Big)_4 = i^{(a-b-b^2-1)/4}$$

For a proof, see [8].

Chapter 4

The Zeta function

In this chapter we're going to define and study some of the fundamental properties of the zeta function of a curve. This function, defined in terms of a power series, provides information about the number of points on a curve over the fields $\mathbb{F}_{p^m}$, where p is a fixed prime and m varies among all natural numbers.
First of all, let us fix some notation. Let $K = \mathbb{F}_p$, $f(x,y) \in \mathbb{F}_p[x,y]$. Let $C(K) = \{(x,y) \in \mathbb{A}^2(K) : f(x,y) = 0\}$ be an irreducible smooth curve. Let $N_m = |\{(x,y) \in C(\bar{K}) : (x,y) \text{ is defined over } \mathbb{F}_{p^m}\}|$.

Definition 4.1. The *zeta function* of the affine curve $C(K)$ is the (formal) series given by:

$$Z_C(t) = exp\left(\sum_{m=1}^{\infty} \frac{N_m t^m}{m}\right)$$

We will denote the zeta function simply by $Z(t)$ when the corresponding curve is clear from the context.
The reader may wonder why we defined the zeta function in such a (seemingly) strange way. Hopefully, the reason should become clear in a few pages.
Let us start by giving a simple example:

Example 4.1. Let $f(x,y) = x$, so that $C(\mathbb{F}_p) = \mathbb{A}^1(\mathbb{F}_p)$. Then, $N_m = p^m$. An easy computation gives:

$$Z_C(t) = exp\left(\sum_{m=1}^{\infty} \frac{p^m t^m}{m}\right) = exp(-log(1-pt)) = (1-pt)^{-1}$$

At least in this case, the zeta function turns out to be much simpler than one could expect from the definition.

Another reasonable question is the following: is there any relation between the zeta function we just defined and the well known Riemann zeta function? Before facing this problem, let us recall briefly some of the basic properties of the Riemann zeta function (for a proof, see [20]):

Proposition 4.1. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges for all $s \in \mathbb{C}$ *such that* $Re\ s > 1$*, and defines an holomorphic function* $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} (1 - p^{-s})^{-1}$ *on this half plane, called the Riemann zeta function, with the following properties:*

1. *(analytic continuation)* $\zeta(s)$ *can be analytically continued to a meromorphic function (still denoted by* $\zeta(s)$*) on the whole plane, with a simple pole at* $s = 1$ *and simple zeros at the points* $-2, -4, \ldots, -2n, \ldots$*, called the* trivial zeros *of* $\zeta(s)$*;*

2. *(functional equation) Let* $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$*, where* $\Gamma(s)$ *denotes Euler's gamma function. Then* $\xi(s)$ *satisfies the following functional equation:*

$$\xi(s) = \xi(1 - s)$$

3. *(Riemann hypothesis) The non trivial zeros of* $\zeta(s)$ *all lie in the strip* $0 < Re\ s < 1$*. Riemann hypothesis conjectures that they all lie on the line* $Re\ s = 1/2$*.*

If K is any number field, we can give a definition of the zeta function associated to the Dedekind domain $\mathcal{O}_K$ which generalizes the Riemann zeta function (corresponding to the case $K = \mathbb{Q}$):

$$Z_K(s) = \sum_I \frac{1}{N(I)^s}$$

where the sum is over all nonzero ideals in $\mathcal{O}_K$. It follows from unique factorization of ideals in $\mathcal{O}_K$ and the fact that the norm is multiplicative that the zeta function can be written in the following form (called an Euler product):

$$Z_K(s) = \prod_{M \in Max(\mathcal{O}_K)} (1 - N(M)^{-s})^{-1}$$

Observe now that the last two formulas make sense for arbitrary Dedekind domains D with finite quotients. Coordinate rings $K[C]$ of affine curves defined over $K = \mathbb{F}_p$ always have this property: in fact, Corollary 2.1 states that $K[C]$ is a Dedekind domain, and Theorem 2.1 tells that if $M \in Max(K[C])$, then $M = \psi_P$, where $P = (a, b) \in \bar{K}$. Hence $(K[C]/M) \cong K(P) = \mathbb{F}_p(P)$, and this is a finite field.
We will now prove that the zeta function of the Dedekind domain $\mathbb{F}_p[C]$, as defined above, coincides with the zeta function of the affine curve defined at the beginning of the paragraph (up to a change of variable).
As in Example 2.1, let $b_d = |\{M \in Max(K[C]) : [K[C]/M : \mathbb{F}_p]\}| = d$. Observe that if $[K[C]/M : \mathbb{F}_p]\}| = d$ then $N(M) = p^d$. Hence we obtain:

$$Z_{K[C]}(s) = \prod_{M \in Max(K[C]} (1 - N(M)^{-s})^{-1} = \prod_{d \in \mathbb{N}} \left(1 - \frac{1}{p^{sd}}\right)^{-b_d}$$

For $t = q^{-s}$, we have (denoting $Z_{K[C]}(t)$ by $Z(t)$):

$$Z(t) = \prod_{d\in\mathbb{N}}(1-t^d)^{-b_d} \Rightarrow log(Z(t)) = -\sum_{d\in\mathbb{N}} b_d log(1-t^d) =$$
$$= \sum_{d\in\mathbb{N}} b_d \left(\sum_{i=1}^{\infty} \frac{t^{di}}{i}\right) =$$
$$= \sum_{n=1}^{\infty}\left(\sum_{d|n} db_d\right)\frac{t^n}{n} = \sum_{n=1}^{\infty} N_n \frac{t^n}{n}$$

since by Example 2.1 we have $\sum_{d|n} db_d = N_n$. So we have: $log(Z(t)) = \sum_{n=1}^{\infty} N_n \frac{t^n}{n}$. Exponentiating both sides we get the desired equality; this also explains why the exponential is used in the definition of the zeta function.
Now we are going to define the zeta function of a projective plane curve. The definition is analogous to the affine case Anyway, as the projective world is better than the affine one, we will see that the zeta function of a projective plane curve enjoys extraordinary properties.
Let $K = \mathbb{F}_p$, $f(x,y,z) \in \mathbb{F}_p[x,y,z]$ homogeneous, $C(K) = \{[x,y,z] \in \mathbb{P}^2(K) : f(x,y,z) = 0\}$ be an irreducible smooth curve.
Let $N_m = |\{P \in C(\bar{K}) : P \text{ is defined over } \mathbb{F}_{p^m}\}|$.

Definition 4.2. The *zeta function* of the projective curve $C(K)$ is the (formal) series given by:

$$Z_C(t) = exp\left(\sum_{m=1}^{\infty} \frac{N_m t^m}{m}\right)$$

Example 4.2. Let $f(x,y,z) = x$, so that $C = \mathbb{P}^1(\mathbb{F}_p)$. Then $N_m = p^m + 1$. Hence we obtain:

$$Z_{\mathbb{P}^1(\mathbb{F}_p)}(t) = exp\left(\sum_{m=1}^{\infty} \frac{(p^m+1)t^m}{m}\right) = \frac{1}{(1-pt)(1-t)}$$

This example allows us to determine with no extra work the zeta function of all projective smooth curves of degree 2. Let C be such a curve.
By Chevalley-Warning Theorem and Remark 3.6 C has always at least one point on each $\mathbb{F}_{p^m}$. Once we have a point P, all the other points on the curve defined over $\mathbb{F}_{p^m}$ are obtained intersecting the curve with a line $ax + by + cz = 0$ passing through P, with $a, b, c \in \mathbb{F}_{p^m}$. Clearly different lines through P correspond to different points on C.
Hence $C(\mathbb{F}_{p^m}) \cong \mathbb{P}^1(\mathbb{F}_{p^m})$, so $N_m = p^m + 1$.
Therefore, the zeta function of an arbitrary smooth projective plane curve of degree 2 is:

$$Z(t) = \frac{1}{(1-pt)(1-t)}$$

Our aim now is to study the fundamental properties of the zeta function of a smooth projective curve. In particular, we want to find out whether this function enjoys similar properties to those of the Riemann zeta function.
First of all, it's easy to see that the zeta function, which we defined just as a formal power

series, can actually be thought as a holomorphic function defined on a small enough disk in the plane centred in the origin. The problem arises of determining whether there is an analytic continuation of this function to a meromorphic function defined on the whole plane. The above example shows that this is true for all curves of degree ≤ 2. In fact, in this case we've seen that much more is true, namely, the zeta function is a *rational* function.
Let us see what happens in the general case.

4.0.1 Rationality of the zeta function

Let C be a projective curve defined over $K = \mathbb{F}_p$.
First of all, we need a clever way to count the number of points on C which are defined over $\mathbb{F}_{p^m}$. Recall that $P \in C(\bar{K})$ is defined over $\mathbb{F}_{p^m}$ if and only if $P^\sigma = P \ \forall \sigma \in Gal(\bar{\mathbb{F}}_p/\mathbb{F}_{p^m}) \Leftrightarrow \phi^m(P) = P$, where $\phi(x) = x^p$ is the Frobenius automorphism.
Let $deg(P)$ denote the least value of d such that $\phi^d(P) = P$. This is called the *degree* of the point P. If P has degree d it is defined over $\mathbb{F}_{p^d}$, and the points $P, \phi(P), \dots \phi^{d-1}(P)$ are distinct points on C defined over $\mathbb{F}_{p^d}$.

Definition 4.3. Let $P \in C(\bar{K})$ be a point of degree d. Then the divisor:

$$\mathcal{P} = P + \phi(P) + \phi^2(P) + \ldots + \phi^{d-1}P$$

is called a *prime divisor*.

Remark 4.1.

1. Observe that $\mathcal{P}^\phi = \mathcal{P}$, so $\mathcal{P}$ is a divisor of degree d defined over $\mathbb{F}_p$.
2. If $D \geq 0$ is a divisor defined over $\mathbb{F}_p$ in which a point P of degree d appears with a non zero coefficient, then the points $\phi^i(P),\ i = 1, \dots, d-1$ must have the same coefficient. This allows to show that each divisor $D \geq 0$ defined over $\mathbb{F}_p$ can be written uniquely in the form: $D = i_1\mathcal{P}_1 + \ldots + i_s\mathcal{P}_s$, with $\mathcal{P}_1, \dots, \mathcal{P}_s$ prime divisors.
3. Let a_d denote the number of prime divisors of degree d. Let $P \in C(\bar{K})$. Then P is defined over $\mathbb{F}_{p^m}$ if and only if its degree d divides m. Moreover, we can divide points of fixed degree d into disjoint sets of d elements, according to the prime divisor in which they appear (with non zero coefficient). Hence we deduce the following equality:

$$N_m = \sum_{d|m} d a_d$$

Let us transform a bit our zeta function exploiting the above remarks. We have:

$$\begin{aligned}
\frac{d}{dt}(log(Z(t))) =& \frac{1}{t}\sum_{m=1}^{\infty} N_m t^m = \\
=& \frac{1}{t}\sum_{m=1}^{\infty}\left(\sum_{d|m} d a_d\right) t^m = \frac{1}{t}\sum_{m=1}^{\infty} m a_m \left(\sum_{i=1}^{\infty} t^{mi}\right) = \\
=& \frac{1}{t}\sum_{m=1}^{\infty}\frac{m a_m t^m}{1-t^m} = \frac{d}{dt}\left(log\prod_{m=1}^{\infty}\left(\frac{1}{1-t^m}\right)^{a_m}\right)
\end{aligned}$$

Now, observe that

$$\prod_{m=1}^{\infty}\left(\frac{1}{1-t^m}\right)^{a_m} = \prod_{\mathcal{P}\text{ prime divisor}}\left(\frac{1}{1-t^{deg(\mathcal{P})}}\right)$$

Remark (2) above implies that the last expression equals

$$\sum_{D\in Div_K(C),\ D\geq 0} t^{deg(D)} = \sum_{m=0}^{\infty} A_m t^m$$

where $A_m = |\{D \in Div_K(C), D \geq 0, deg(D) = m\}|$ (it's easy to see that A_m is finite). Summing up, we have obtained:

$$\frac{d}{dt}(log(Z(t))) = \frac{d}{dt}\left(log\big(\sum_{m=0}^{\infty} A_m t^m\big)\right)$$

From this it easily follows that $Z(t) = \sum_{m=0}^{\infty} A_m t^m$, since $log(Z(t))$ and $log\big(\sum_{m=0}^{\infty} A_m t^m\big)$ have the same derivative and the same value for $t = 0$.
So, we're left with the task of investigating the integers A_m in the above sum. This is where Riemann-Roch is going to help us.

Notation 4.1.

1. Let $\delta\mathbb{Z}$ be the subgroup of $\mathbb{Z}$ image of the map $deg : Div_K(C) \to \mathbb{Z}$.
2. Fix a divisor $D_0 \in Div_K(C)$ of degree δ.
3. Choose $\nu \in \mathbb{N}$ such that $(\nu-1)\delta < g \leq \nu\delta$ where g is the genus of the curve.
4. Let $\{D_1, \ldots, D_h\}$ be a maximal set of positive non equivalent divisors in $Div_K(C)$ of degree $\nu\delta$.
5. Choose a canonical divisor $K \in Div_K(C)$ and let $\mu \in \mathbb{N}$ such that $\mu\delta = 2g - 2 = deg(K)$.

Now, let $D \in Div_K(C)$ of degree $\nu\delta$. By Riemann-Roch:

$$l_K(D) \geq deg(D) + 1 - g \geq 1$$

hence there exists $f \in K(C)^*$ such that $div(f) + D \geq 0$. Then $div(f) + D$ must be equivalent to one of the D_i. Hence $D \sim D_i$ for some i, $1 \leq i \leq h$. Moreover such a D_i is unique, as $D_i \sim D$ and $D_j \sim D$ implies $D_i \sim D_j$, absurd. Hence, each divisor of degree $\nu\delta$ is equivalent to one and only one of the D_i.
Now, if $D \in Div_K(C)$ has degree $n\delta$, then $D - (n-\nu)D_0$ has degree $\nu\delta$, hence there is a unique i such that D is equivalent to $(n-\nu)D_0 + D_i$.
Finally, it follows easily from the definition that the number of positive divisors in $Div_K(C)$ which are equivalent to D is $\frac{p^{l_K(D)}-1}{p-1}$.
To simplify the notation, let us denote $l_K(D)$ simply by $l(D)$. The above observations allow us to write the zeta function in the form:

$$Z(t) = \sum_{n=0}^{\infty}\left(\sum_{i=1}^{h}\frac{p^{l(D_i+(n-\nu)D_0)}-1}{p-1}\right)t^{n\delta}$$

Now, denote

$$\begin{aligned} Z_1(t) =& \sum_{n=0}^{\mu} \left(\sum_{i=1}^{h} \frac{p^{l(D_i+(n-\nu)D_0)}}{p-1} \right) t^{n\delta} \\ Z_2(t) =& Z(t) - Z_1(t) \end{aligned}$$

Then an easy computation gives:

$$Z_2(t) = \sum_{i=1}^{h} \sum_{n=\mu+1}^{\infty} \frac{p^{l(D_i+(n-\nu)D_0)}}{p-1} t^{n\delta} - \frac{h}{p-1} \sum_{n=0}^{\infty} t^{n\delta}$$

The key fact at this point is that for $n > \mu$ we have $deg(D_i+(n-\nu)D_0) = n\delta > \mu\delta = 2g-2$, hence $l(D_i + (n-\nu)D_0) = deg(D_i + (n-\nu)D_0) + 1 - g = n\delta + 1 - g$ by Corollary 2.2. Then

$$\begin{aligned} Z_2(t) =& \sum_{i=1}^{h} \sum_{n=\mu+1}^{\infty} \frac{p^{n\delta+1-g}}{p-1} t^{n\delta} - \frac{h}{p-1} \sum_{n=0}^{\infty} t^{n\delta} = \\ =& \sum_{i=1}^{h} \sum_{n=\mu+1}^{\infty} \frac{p^{1-g}}{p-1} (pt)^{n\delta} - \frac{h}{p-1} \sum_{n=0}^{\infty} t^{n\delta} = \\ =& \frac{hp^{1-g}}{p-1} \sum_{n=\mu+1}^{\infty} (pt)^{n\delta} - \frac{h}{p-1} \sum_{n=0}^{\infty} t^{n\delta} = \\ =& \frac{h}{p-1} \Big(\frac{p^{1-g}(pt)^{(\mu+1)\delta}}{1-(pt)^{\delta}} - \frac{1}{1-t^{\delta}} \Big) \end{aligned}$$

Let us stop and think about what we've done up to now: we had our zeta function, written in the form

$$Z(t) = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{h} \frac{p^{l(D_i+(n-\nu)D_0)} - 1}{p-1} \right) t^{n\delta}$$

We decomposed it into two pieces, Z_1 and Z_2. The first one is a polynomial in t fo degree $\mu\delta = 2g-2$. On the other hand,

$$Z_2(t) = \sum_{i=1}^{h} \sum_{n=\mu+1}^{\infty} \frac{p^{l(D_i+(n-\nu)D_0)}}{p-1} t^{n\delta} - \frac{h}{p-1} \sum_{n=0}^{\infty} t^{n\delta}$$

is *a priori* an infinite series. However, the fact that $n > \mu$ in the sum allows us to apply Riemann-Roch, and we discover that we're actually dealing with a geometric series, whose sum we can compute explicitly, and turns out to be a rational function of t.
Therefore, our computation lead us to show that the zeta function of a smooth projective curve is a *rational function*. Also note that everything we made for a curve $C = C(\mathbb{F}_p)$ in order to obtain this result works in exactly the same way for a curve $C = C(\mathbb{F}_{p^m})$ (just replace p with $q = p^m$).

Lemma 4.1. *The zeta function satisfies the identity:*

$$Z_{C(\mathbb{F}_{p^d})}(t^d) = \prod_{\epsilon^d=1} Z_{C(\mathbb{F}_p)}(\epsilon t)$$

Proof. The right hand side equals:

$$exp\left(\sum_{m=1}^{\infty} N_m \frac{t^m}{m}\Big(\sum_{\epsilon^d=1} \epsilon^m\Big)\right)$$

Now use the fact that $\sum_{\epsilon^d=1} \epsilon^m = 0$ if $d \nmid m$, d if $d \mid m$. □

Theorem 4.1. *The function $Z(t)$ may be written as*

$$Z(t) = \frac{P(t)}{(1-t)(1-pt)}$$

with $P(t) \in \mathbb{Z}[t]$ of degree $2g$.

Proof. The above computations show that

$$\begin{aligned} Z(t) =& Z_1(t) + Z_2(t) = \sum_{n=0}^{\mu} \Big(\sum_{i=1}^{h} \frac{p^{l(D_i+(n-\nu)D_0)}}{p-1}\Big) t^{n\delta} + \frac{h}{p-1}\Big(\frac{p^{1-g}(pt)^{(\mu+1)\delta}}{1-(pt)^\delta} - \frac{1}{1-t^\delta}\Big) = \\ =& \frac{P(t)}{(1-t^\delta)(1-(pt)^\delta)} \end{aligned} \tag{4.1}$$

for some $P(t) \in \mathbb{Z}[t]$. Moreover, Z_1 is a polynomial in t, whereas Z_2 has a simple pole at those t for which $t^\delta = 1$. Hence $Z_{C(\mathbb{F}_{p^\delta})}(t^\delta)$ has a simple pole at each t such that $t^\delta = 1$. Now use Lemma 4.1 with $d = \delta$ to get:

$$Z_{C(\mathbb{F}_{p^\delta})}(t^\delta) = \prod_{\epsilon^\delta=1} \frac{P(\epsilon t)}{(1-(\epsilon t)^\delta)(1-(\epsilon tp)^\delta)} = \frac{\prod_{\epsilon^\delta=1} P(\epsilon t)}{(1-t^\delta)^\delta (1-(tp)^\delta)^\delta}$$

hence $Z_{C(\mathbb{F}_{p^\delta})}(t^\delta)$ has a pole of order δ at each t such that $t^\delta = 1$, so $\delta = 1$. Formula 4.1 with $\delta = 1$ becomes:

$$Z(t) = Z_1(t) + \frac{h}{p-1}\Big(\frac{p^{1-g}(pt)^{(\mu+1)}}{1-pt} - \frac{1}{1-t}\Big)$$

where Z_1 is a polynomial of degree $\mu = 2g-2$ in t. Hence

$$Z(t) = \frac{P(t)}{(1-t)(1-pt)}$$

with $deg(P(t)) = 2g$ □

As clearly $Z(0) = 1$, we obtain the following

Corollary 4.1. $Z(t) = \frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-pt)}$ *where $\alpha_1, \ldots, \alpha_{2g} \in \mathbb{C}$ are the inverses of the zeros of $Z(t)$.*

Corollary 4.2. $N_m = 1 + p^m - \sum_{i=1}^{2g} \alpha_i^m$

Proof. By the previous Corollary, we find:

$$\frac{tZ'(t)}{Z(t)} = \sum_{m=1}^{\infty} \Big(1 + p^m - \sum_{i=1}^{2g} \alpha_i^m\Big) t^m$$

On the other hand

$$Z(t) = exp\Big(\sum_{M=1}^{\infty} \frac{N_m t^m}{m}\Big) \Rightarrow \frac{tZ'(t)}{Z(t)} = \sum_{m=1}^{\infty} N_m t^m$$

Comparing coefficients we get the desired result. □

The fact that the zeta function is rational gives us information about the rate of growth of the number of points on a curve defined over $\mathbb{F}_{p^m}$ for growing m. Note that this number is $p^m + 1 = |\mathbb{P}^1(\mathbb{F}_{p^m})|$ plus an "error term", which depends on the zeros of the zeta function. Thus if we can obtain good bounds for the size of these zeros, we will have an hopefully sharp estimate of the "error term".
In the case $g = 1$ (corresponding to curves of degree 3 by Proposition 2.3) the formula for the zeta function turns out to be very simple: recall that

$$Z(t) = Z_1(t) + Z_2(t) = \sum_{n=0}^{\mu} \Big(\sum_{i=1}^{h} \frac{p^{l(D_i + (n-\nu)D_0)}}{p-1}\Big) t^n + \frac{h}{p-1}\Big(\frac{p^{1-g}(pt)^{(\mu+1)}}{1-pt} - \frac{1}{1-t}\Big)$$

where $\mu = 2g - 2$. So $\mu = 0$ if $g = 1$. Hence the above formula becomes:

$$Z(t) = \Big(\sum_{i=1}^{h} \frac{p^{l(D_i - \nu D_0)}}{p-1}\Big) t^0 + \frac{h}{p-1}\Big(\frac{pt}{1-pt} - \frac{1}{1-t}\Big) = A + \frac{h}{p-1}\Big(\frac{pt}{1-pt} - \frac{1}{1-t}\Big)$$

where $A = \Big(\sum_{i=1}^{h} \frac{p^{l(D_i - \nu D_0)}}{p-1}\Big)$ is a constant not depending on t. Hence we have:

$$\begin{aligned} Z(\frac{1}{pt}) =& A + \frac{h}{p-1}\Big(\frac{(p(\frac{1}{pt}))}{1 - p(\frac{1}{pt})} - \frac{1}{1 - \frac{1}{pt}}\Big) = \\ =& A + \frac{h}{p-1}\Big(\frac{\frac{1}{t}}{1 - \frac{1}{t}} - \frac{pt}{pt-1}\Big) = \\ =& A + \frac{h}{p-1}\Big(\frac{1}{t-1} - \frac{pt}{pt-1}\Big) = Z(t) \end{aligned}$$

Moreover, we know that

$$Z(t) = \frac{P(t)}{(1-t)(1-pt)} = \frac{1 - a_p t + b_p t^2}{(1-t)(1-pt)}$$

The condition $Z(t) = Z(\frac{1}{pt})$ implies:

$$\frac{1 - a_p t + b_p t^2}{(1-t)(1-pt)} = \frac{1 - a_p \frac{1}{pt} + b_p \frac{1}{(pt)^2}}{(1-\frac{1}{pt})(1-p\frac{1}{pt})} =$$
$$= \frac{1}{p}\frac{p^2t^2 - a_p pt + b_p}{(pt-1)(t-1)} = \frac{pt^2 - a_p t + \frac{b_p}{p}}{(pt-1)(t-1)}$$

The last equality holds if and only if $b_p = p$. Hence for curves of degree 3 the zeta function has the form:

$$Z(t) = \frac{1 - a_p t + pt^2}{(1-t)(1-pt)} \tag{4.2}$$

Now, it can be proven that $|a_p| \leq 2\sqrt{p}$, which implies that $1 - a_p t + pt^2$ has two complex conjugate zeros; therefore we obtain:

$$Z(t) = \frac{(1-\pi t)(1-\bar{\pi} t)}{(1-t)(1-pt)}$$

where $\pi, \bar{\pi}$ are the inverses of the zeros of $Z(t)$. Clearly, we also have $\pi\bar{\pi} = p \Rightarrow |\pi| = \sqrt{p}$. A direct proof of these facts for two particular families of elliptic curves (which are curves of degree 3) will be given in Chapter 5.
All the results we obtained in this paragraph are special cases of the following amazing Theorem:

Theorem 4.2. *Let $C(\mathbb{F}_p)$ be a smooth projective curve of genus g. Then the zeta function $Z(t)$ of C satisfies the following properties:*

1. *(analytic continuation) $Z(t)$ is a rational function, and can be written in the form:*

$$Z(t) = \frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-pt)}$$

2. *(functional equation) $Z(\frac{1}{pt}) = (pT^2)^{1-g} Z(t)$*

3. *(Riemann hypothesis) $\alpha_1, \ldots, \alpha_{2g}$ satisfy $|\alpha_i| = \sqrt{p}$*

As a corollary, we obtain the following:

Theorem 4.3. *(Hasse) The number of points on a projective smooth curve $C(\bar{\mathbb{F}}_p)$ of genus g defined over $\mathbb{F}_{p^m}$ satisfy the inequality:*

$$|N_m - p^m - 1| \leq 2g\sqrt{p^m}$$

Proof. The theorem follows immediately from Corollary 4.2 and from the Riemann hypothesis. □

Corollary 4.3. *Let C be a projective curve of degree 3 defined over $\mathbb{F}_p$. Then for each $m \in \mathbb{N}$ there is at least a point on C defined over $\mathbb{F}_{p^m}$.*

Proof. As in this case $g = 1$ we obtain $|N_m - p^m - 1| \leq 2\sqrt{p^m} \Rightarrow N_m = p^m + 1 + \epsilon$, with $|\epsilon| \leq 2\sqrt{p^m} < p^m + 1$. The conclusion follows. □

This result is the analogue, for curves of degree 3, of Chevalley-Warning theorem (which asserts the same thing for curves of degree 2). Anyway, note that in this case the proof requires much more advanced techniques.
Moreover, the above statement is false for projective curves of degree greater than 3. For example, it's easy to check that the projective curve with equation $x^4 + y^4 + z^4 = 0$ has no points in $\mathbb{P}^2(\mathbb{F}_5)$.

Example 4.3. As an application of the previous results, we will now describe a counterexample to the local-global principle (Theorem 1.1) for a curve of degree 3. Precisely, we will show that the projective curve $C(\mathbb{Q}) = \{[x, y, z] \in \mathbb{P}^2(\mathbb{Q}) : 3x^3 + 4y^3 + 5z^3 = 0\}$ has a point in $\mathbb{P}^2(\mathbb{Q}_p)$ for each prime p and a point in $\mathbb{P}^2(\mathbb{R})$. The proof of the fact that it has no points in $\mathbb{P}^2(\mathbb{Q})$ is much more difficult, and will not be given here.
An easy computation shows that the reduced curve $C_p(\mathbb{F}_p) = \{[x, y, z] \in \mathbb{P}^2(\mathbb{F}_p) : \bar{3}x^3 + \bar{4}y^3 + \bar{5}z^3 = 0\}$ is smooth for $p \neq 2, 3$ or 5. Thus for such a p Corollary 4.3 tells us that there is at least a nontrivial solution of the equation $3x^3 + 4y^3 + 5z^3 \equiv 0 \pmod{p}$. By Hensel Lemma [1] this solution lifts to a nontrivial solution in $\mathbb{Z}_p$.
Let $p = 2$. Then the reduced curve $C_2(\mathbb{F}_2) = \{[x, y, z] \in \mathbb{P}^2(\mathbb{F}_2) : x^3 + z^3 = 0\}$ contains the point $[1, 0, 1]$, which lifts to a point on $C(\mathbb{Q}_2)$ thanks to Hensel Lemma.
For the same reason, the point $[1, 2, 0] \in C_5(\mathbb{F}_5) = \{[x, y, z] \in \mathbb{P}^2(\mathbb{F}_2) : 3x^3 + 4y^3 = 0\}$ lifts to a point on $C(\mathbb{Q}_5)$, and the point $[0, 1, 4] \in C_3(\mathbb{F}_3) = \{[x, y, z] \in \mathbb{P}^2(\mathbb{F}_3) : 4y^3 + 5y^3 = 0\}$ lifts to a point on $C(\mathbb{Q}_3)$.
Finally, $[0, \sqrt[3]{5/4}, -1] \in C(\mathbb{R})$.

Let us now explain why point (2) of Theorem 4.2 is called Riemann hypothesis: let $Z(t) = \frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-pt)}$. By the change of variables $t = p^{-s}$ we obtain:

$$Z(s) = \frac{\prod_{i=1}^{2g}(1 - \alpha_i p^{-s})}{(1 - p^{-s})(1 - p^{1-s})}$$

The zeros of $Z(s)$ are: $\beta_i = a_i + ib_i$ such that $p^{\beta_i} = \alpha_i$. Thus $|\alpha_i| = p^{a_i} = p^{Re\,\beta_i}$. Hence property (2) in Theorem 4.2 is equivalent to the fact that the zeros of $Z(s)$ have real part $\frac{1}{2}$.

Remark 4.2. Theorem 4.2 was first proved by André Weil in the 1940*s*. Weil also conjectured that the same results were true for arbitrary algebraic varieties. We're not going to explain here what exactly are algebraic varieties. And in fact, the formulation of this concept in an appropriate language was one of the first problems of algebraic geometers trying to prove Weil conjectures. This gave birth to many of the most important ideas in modern algebraic geometry (for example the concept of scheme), which allowed to prove the rationality of the zeta function for an arbitrary algebraic variety (Dwork, 1960) and the fact that this function satisfies a certain functional equation. The hardest statement to prove was Riemann hypothesis for arbitrary algebraic varieties, which was proved by

[1] See [17, Theorem 1, pag. 15].

Deligne in 1974.
Some material about Weil conjectures can be found in [6, Appendix C].

In the following Chapter we're going to restrict our attention to a particular type of smooth projective curves of degree 3, called elliptic curves. We're going to compute explicitly the zeta function of some families of elliptic curves using the techniques developed so far. This will allow us to verify directly the validity if the Riemann hypothesis for these curves.
After that, we will introduce a new type of function associated to a curve, namely its L-function.

Chapter 5

Elliptic curves

We will now focus on a particular class of projective smooth curves, called *elliptic curves*. Observe that Example 4.2 completely solves the problem of determining the zeta function of a smooth curve of degree $n = 1$ and 2. The following natural case to study is that corresponding to $n = 3$.
Let K be a field.

Definition 5.1. An *elliptic curve* is a smooth projective curve $E = C(K)$ of degree 3 defined over K, together with a point $P \in C(K)$.

Remark 5.1. It follows from Proposition 2.3 that elliptic curves are curves of genus one.

Let $K = \mathbb{Q}$. In this case, it can be proven (see [19]) that with a change of coordinates we can always transform our curve in the form:

$$E = C(\mathbb{Q}) = \{[x_0, x_1, x_2] \in \mathbb{P}^2(\mathbb{Q}) : x_0x_2^2 = x_1^3 - Ax_0^2x_1 + Bx_0^3\}$$

with $A, B \in \mathbb{Q}$. This is called the *Weierstrass form* of the curve. The affine equation of the curve obtained by setting $x = x_1/x_0, y = x_2/x_0$ is:

$$y^2 = x^3 - Ax + B$$

Note that there is only one point at infinity, namely $[0, 0, 1]$.
The transformation $(x, y) \mapsto (c^2x, c^3y)$ transforms the equation into $y^2 = x^3 - c^4Ax + c^6B$. Thus, we may assume that $A, B \in \mathbb{Z}$. We will always suppose this from now on.
The number $\Delta = 16(4A^3 - 27B^2)$ is called the *discriminant* of the curve. An easy calculation shows that E is a smooth curve if and only if its discriminant is not zero.
Reducing A, B modulo a prime p we obtain a curve $E_p = \{(x, y) \in \mathbb{A}^2(\mathbb{F}_p) : y^2 = x^3 - \bar{A}x + \bar{B}\}$. This is a smooth affine curve if and only if $p \nmid \Delta$. If this is the case the projective completion of E_p, with equation $x_0x_2^2 = x_1^3 - \bar{A}x_0^2x_1 + \bar{B}x_0^3$, is also smooth. We will still denote it by E_p.
Primes p for which the reduced curve E_p is smooth are called primes of *good reduction* for E.
Let p be a prime of good reduction for E. By Equation 4.2 the zeta function of E_p, called the *local zeta function* of E at p, is the rational function

$$Z_{E_p}(t) = \frac{P(t)}{(1-t)(1-pt)} = \frac{pt^2 - a_pt + 1}{(1-t)(1-pt)} = \frac{(1-\pi t)(1-\bar{\pi}t)}{(1-t)(1-pt)}$$

Moreover, by Corollary 4.2 we have:

$$N_{p^m} = p^m + 1 - \pi^m - \bar{\pi}^m$$

where N_{p^m} denotes the number of points on E_p defined over $\mathbb{F}_{p^m}$. In particular $N_p = p + 1 - a_p$. Thus, if one calculates N_p this also determines a_p and so π and $\bar{\pi}$, which are the inverses of the roots of the polynomial $pt^2 - a_p t + 1$. Hence the number of points defined over $\mathbb{F}_p$ of the elliptic curve E_p uniquely determines its zeta function and the number of points on E_p defined on each $\mathbb{F}_{p^m}$.

Example 5.1. In Example 3.1 we computed the number of points on the curve E of projective equation $f(x, y, z) = x_0^3 + x_1^3 - x_2^3 = 0$ in $\mathbb{P}^2(\mathbb{F}_{97})$, and found out that they are $N_{97} = 117$.
It is easy to see that this curve is smooth, as $\partial f/\partial x_0(x_0, x_1, x_2) = \partial f/\partial x_1(x_0, x_1, x_2) = \partial f/\partial x_2(x_0, x_1, x_2) = 0 \Leftrightarrow x_0 = x_1 = x_2 = 0$. Thus E is an elliptic curve, and we have: $a_p = 98 - 117 = -19$. From this we obtain $\pi = (-19 + 3\sqrt{-3})/2$. Hence, for example, $N_{9409} = N_{97^2} = 9409 + 1 - \pi^2 - \bar{\pi}^2 = 9243$.

Now let us compute explicitly the zeta functions of two particular classes of elliptic curves.

5.0.2 The curve $y^2 = x^3 + D$

Let $D \in \mathbb{Z}$, $D \neq 0$. Then the curve $E = E(\mathbb{Q})$ defined by the equation $x_0x_2^2 - x_1^3 - Dx_0^3 = 0$ is smooth. The discriminant of E is $\Delta = -2^4 3^3 D^2$, so we will only consider primes different from 2 and 3 and not dividing D, which are primes of good reduction for E.
The affine equation of E_p is $y^2 = x^3 + \bar{D}$. There is only one point at infinity, hence the number of $\mathbb{F}_p$-points on E_p is $N_p = 1 + N(y^2 = x^3 + \bar{D})$ (we will forget the bar in what follows).
If $p \equiv 2 \pmod 3$ then $x \mapsto x^3$ is an automorphism of $\mathbb{F}_p^*$, hence $N(y^2 = x^3 + D) = N(y^2 = x + D) = p$. The last equality follows from Example 4.2 and the observation that $y^2 = x + D$ is the equation of an affine smooth curve of degree 2 with only one point at infinity. Thus, for $p \equiv 2 \pmod 3$, $N_p = p + 1$.
Let $p \equiv 1 \pmod 3$ and let χ be a character of order 3 on $\mathbb{F}_p$, ρ a character of order 2. Then:

$$N(y^2 = x^3 + D) = \sum_{u+v=D} N(y^2 = u)N(x^3 = -v) = \sum_{u+v=D} (1 + \rho(u))(1 + \chi(-v) + \chi^2(-v)) =$$
$$= p + \sum_{u+v=D} \rho(u)\chi(v) + \sum_{u+v=D} \rho(u)\chi^2(v)$$

because of Proposition 3.1 and the fact that $\chi(-1) = 1$.
Setting $u = Du'$, $v = Dv'$ we find:

$$N_p = p + 1 + \rho\chi(D)J(\rho, \chi) + \overline{\rho\chi(D)J(\rho, \chi)}$$

By Lemma 3.3 we obtain

$$N_p = p + 1 + \rho\chi(4D)J(\chi, \chi) + \overline{\rho\chi(4D)J(\chi, \chi)}$$

Let us now specify ρ and χ. Since $p \equiv 1 \pmod 3$, $p = \pi\bar{\pi}$ in $\mathbb{Z}[\omega]$, and we can take $\pi, \bar{\pi}$ primary. Let $(a/\pi)_6$ be the sixth power residue symbol (defined in the same way as the cubic and biquadratic residue symbol) and take $\rho(a) = (a/\pi)_6^3$ and $\chi(a) = (a/\pi)_6^2 = (a/\pi)_3$. Then by Proposition 3.8 we obtain:

$$N_p = p + 1 + \Big(\frac{4D}{\pi}\Big)_6^5 \pi + \Big(\frac{\overline{4D}}{\pi}\Big)_6^5 \bar{\pi} = p + 1 + \Big(\frac{\overline{4D}}{\pi}\Big)_6 \pi + \Big(\frac{4D}{\pi}\Big)_6 \bar{\pi}$$

Let us collect the results we obtained:

Theorem 5.1. *Let $p \neq 2$ or 3 and $p \nmid D$. The number of projective points on the elliptic curve $x_0x_2^2 - x_1^3 - Dx_0^3 = 0$ which are defined over $\mathbb{F}_p$ is:*

1. *If $p \equiv 2 \pmod 3$, $N_p = p + 1$;*
2. *If $p \equiv 1 \pmod 3$, let $p = \pi\bar{\pi}$ in $\mathbb{Z}[\omega]$, with π primary. Then*

$$N_p = p + 1 + \Big(\frac{\overline{4D}}{\pi}\Big)_6 \pi + \Big(\frac{4D}{\pi}\Big)_6 \bar{\pi}$$

Thanks to this result we're able to determine the local zeta function of E for all primes p for which the reduced curve E_p is smooth.
Recall that we have:

$$Z_{E_p}(t) = \frac{1 - a_p t + pt^2}{(1-t)(1-pt)}$$

and $a_p = p + 1 - N_p$. If $p \equiv 2 \pmod 3$ the above formula implies that $a_p = 0$. If $p \equiv 1 \pmod 3$ then:

$$|a_p| = \Big| \Big(\frac{\overline{4D}}{\pi}\Big)_6 \pi + \Big(\frac{4D}{\pi}\Big)_6 \bar{\pi} \Big| \leq 2|\pi| = 2\sqrt{p}$$

Thus in both cases $|a_p| \leq 2\sqrt{p}$. As we already pointed out, this implies that the polynomial $1 - a_p t + pt^2$ has two complex conjugate zeros, hence the local zeta function of E at a prime p of good reduction has the form:

$$Z_{E_p}(t) = \frac{(1 - \pi t)(1 - \bar{\pi} t)}{(1-t)(1-pt)}$$

with $|\pi| = \sqrt{p}$. Hence the Riemann hypothesis for elliptic curves of affine equation $y^2 = x^3 + D$ is proved.

Example 5.2. Consider the elliptic curve E of affine equation $y^2 = x^3 + 1$. Let us determine the number of points of its reduction modulo 19. We have $19 = (5+3\omega)(5+3\omega^2)$ in $\mathbb{Z}[\omega]$. Hence:

$$N_{19} = 20 + \overline{\Big(\frac{4}{5+3\omega}\Big)_6}(5 + 3\omega) + \Big(\frac{4}{5+3\omega}\Big)_6 (5 + 3\omega^2)$$

It remains to calculate

$$\Big(\frac{4}{5+3\omega}\Big)_6 = \Big(\frac{2}{5+3\omega}\Big)_3$$

We have

$$2^{(19-1)/3} = 2^6 = 64 \equiv 64 - 19 \times 3 \equiv 7 \pmod{(5+3\omega)}$$

Direct calculation shows that $(5+3\omega) \mid (7-\omega^2)$, hence $(4/(5+3\omega))_6 = \omega^2$. Hence, we obtain:

$$N_{19} = 20 + \omega(5+3\omega) + \omega^2(5+3\omega^2) = 20 + 8(\omega+\omega^2) = 20 + 8(2Re\ \omega) = 20 - 8 = 12$$

The local zeta function of E at $p = 19$ is therefore:

$$Z_{E_{19}}(t) = \frac{19t^2 - a_{19}t + 1}{(1-t)(1-pt)}$$

where $a_{19} = 19 + 1 - N_{19} = 20 - 12 = 8$. So

$$Z_{E_{19}}(t) = \frac{19t^2 - 8t + 1}{(1-t)(1-pt)} = \frac{(1-\pi t)(1-\bar{\pi}t)}{(1-t)(1-pt)}$$

where $\pi = (4+\sqrt{-3})$. Now we can determine with no extra work the number of points on our curve defined over an arbitrary finite extension $\mathbb{F}_{19^m}$ of $\mathbb{F}_{19}$. For example: $N_{130321} = N_{19^4} = 19^4 + 1 - \pi^4 - \bar{\pi}^4 = 130368$.

5.0.3 The curve $y^2 = x^3 - Dx$

Consider the elliptic curve E defined by $x_0x_2^2 - x_1^3 + Dx_1x_0^2 = 0$, where $D \in \mathbb{Z}$, $D \neq 0$. In affine coordinates E has equation $y^2 = x^3 - Dx$, and only one point at infinity. The discriminant of E is $\Delta = 2^6D^3$, hence we will only consider odd primes not dividing D, which are of good reduction for E.
Let p be such a prime. Let us determine the number N_p of points on the reduced curve E_p defined over $\mathbb{F}_p$.
Notice that the polynomial $p(x) = x^3 - Dx$ is odd.
If $p \equiv 3 \pmod 4$ then -1 is not a square in $\mathbb{F}_p$. $p(x)$ has 1 or 3 zeros, depending on whether D is or not a square in $\mathbb{F}_p$.
Take $x \in \mathbb{F}_p$ such that $p(x) \neq 0$. Then either $p(x)$ is a square or $p(-x) = -p(x)$ is a square in $\mathbb{F}_p$. In both cases, there are exactly two points on E_p with first coordinate equal to x or $-x$. So, if D is a square we have $2\frac{p-3}{2} + 3 = p$ affine points on E_p. If D is not a square, we find $2\frac{p-1}{2} + 1$ affine points on E_p. In both cases we have $N_p = p + 1$.
Let $p \equiv 1 \pmod 4$. We want to use again the technique of Jacobi sums in order to count points on E_p. It should be clear from the examples we dealt with so far that this method works for "diagonal" equations such as $x^3 + y^3 - 1 = 0$. Let us transform our equation in this form. If C denotes the curve of equation $u^2 = v^4 + 4D$, it's easy to see that the transformation

$$T(u,v) = \big((u+v^2)/2, v(u+v^2)/2\big)$$

maps bijectively C to $E \setminus (0,0)$ (the inverse map being $(x,y) \mapsto (2x - y^2/x^2, y/x)$). Therefore, we have $N(y^2 = x^3 - Dx) - 1 = N(u^2 = v^4 + 4D)$.

Let λ be a character of order 4 on $\mathbb{F}_p$, $\rho = \lambda^2$. Then we find:

$$\begin{aligned} N(u^2 = v^4 + 4D) &= \sum_{r+s=4D} N(u^2 = r)N(v^4 = -s) = \\ &= \sum_{r+s=4D} (1+\rho(r))(1+\lambda(-s)+\lambda^2(-s)+\lambda^3(-s)) = \\ &= p + J(\rho,\rho) + \rho\lambda(-4D)J(\rho,\lambda) + \rho\lambda^3(-4D)J(\rho,\lambda^3) = \\ &= p - 1 + \overline{\lambda(-4D)}J(\rho,\lambda) + \lambda(-4D)\overline{J(\rho,\lambda)} = \\ &= p - 1 + \overline{\lambda(D)}\lambda(-1)J(\lambda,\lambda) + \lambda(D)\overline{\lambda(-1)J(\lambda,\lambda)} \end{aligned}$$

since by Lemma 3.3 we have $J(\rho,\lambda) = \lambda(4)J(\lambda,\lambda)$.
Now, since $\pi \equiv 1 \pmod 4$ we can write $p = \pi\bar{\pi}$ in $\mathbb{Z}[i]$, where π is primary. Choosing $\lambda(a) = (a/\pi)_4$ (the biquadratic residue on $\mathbb{Z}[i]/\pi\mathbb{Z}[i]$) we have by Proposition 3.10 $-\lambda(-1)J(\lambda,\lambda) = \pi$. Substituting this in the above equation and collecting all the information we've obtained we arrive at:

Theorem 5.2. *Let $p \neq 2$ and $p \nmid D$. The number of projective points on the elliptic curve $x_0x_2^2 - x_1^3 + Dx_1x_0^2 = 0$ defined over $\mathbb{F}_p$ is:*

1. *If $p \equiv 3 \pmod 4$, $N_p = p + 1$;*
2. *If $p \equiv 1 \pmod 4$, let $p = \pi\bar{\pi}$ in $\mathbb{Z}[i]$, with π primary. Then*

$$N_p = p + 1 - \left(\frac{\overline{D}}{\pi}\right)_4 \pi - \left(\frac{D}{\pi}\right)_4 \bar{\pi}$$

Again, this theorem allows us to determine the local zeta function of E for all p for which the reduced curve E_p is smooth. We obtain, as in the previous section, that:

$$Z_{E_p}(t) = \frac{(1-\pi t)(1-\bar{\pi}t)}{(1-t)(1-pt)}$$

with $|\pi| = \sqrt{p}$. This proves the Riemann hypothesis for elliptic curves of affine equation $y^2 = x^3 - Dx$.

Example 5.3. Let us determine the number of points of the reduction E_5 of the elliptic curve E whose affine equation is $y^2 = x^3 - 4x$.
We have: $5 = (2i-1)(-2i-1)$, and $2i-1$ is primary. Hence:

$$N_5 = 5 + 1 - \overline{\left(\frac{4}{-1+2i}\right)_4}(-1+2i) - \left(\frac{4}{-1+2i}\right)_4(-1-2i)$$

Moreover: $(4/(-1+2i))_4 \equiv 4^{\frac{N(-1+2i)-1}{4}} \equiv 4 \equiv -1 \pmod{-1+2i}$.
Hence $N_5 = 6 + (-1+2i) + (-1-2i) = 4$.
In this simple case, one can verify directly that the points on E_5 which are defined over $\mathbb{F}_5$ are: $(0,0), (2,0), (-2,0), \infty$.
The local zeta function of E at $p = 5$ is:

$$Z_{E_5}(t) = \frac{5t^2 - 2t + 1}{(1-t)(1-5t)} = \frac{(1-(1+2i)t)(1-(1-2i)t)}{(1-t)(1-5t)}$$

Hence we have for example: $N_{625} = 625 + 1 + (1+2i)^4 + (1-2i)^4 = 612$.

Remark 5.2. Take an elliptic curve E defined over $\mathbb{Q}$ with affine equation $y^2 = x^3 + D$ or $y^2 = x^3 - Dx$. Let p be a prime of good reduction for E. Let N_p be the number of points on the reduced curve E_p which are defined on $\mathbb{F}_p$. In the previous two paragraphs we verified directly that $N_p = p + 1 + \epsilon_p$, with $|\epsilon_p| \leq 2\sqrt{p}$ (Hasse Theorem). More precisely, when p varies among all primes of good reduction for E, Theorem 5.1 and Theorem 5.2 show that, roughly speaking, $\epsilon_p = 0$ for half of the primes p. Thus the "error term" ϵ_p is not uniformly distributed in the interval $[-2\sqrt{p}, 2\sqrt{p}]$.
The problem arises of determining the distribution of the error terms ϵ_p when p varies among all primes of good reduction for an arbitrary elliptic curve E defined over $\mathbb{Q}$. Let us consider, instead of ϵ_p, the normalised error $\sigma_p = \epsilon_p/2\sqrt{p}$. As $|\sigma_p| \leq 1$, for each p there exist a unique $\theta_p \in [0, \pi]$ such that $\cos\theta_p = \sigma_p$. It has been conjectured that, for "most" elliptic curves, the values of θ_p are "uniformly distributed" in the interval $[0, \pi]$. To formulate exactly the conjecture, we need to clarify what it means for a sequence of real numbers to be "uniformly distributed". This is the content of the following definition:

Definition 5.2. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in the interval $[a, b] \subseteq \mathbb{R}$. Let μ be a measure on $[a, b]$. We say that $\{x_n\}$ is a *μ-equidistributed* sequence if, for every interval $I = [c, d] \subseteq [a, b]$,

$$\lim_{n\to\infty} \frac{|\{k \leq n \in \mathbb{N} : x_k \in I\}|}{n} = \frac{d-c}{b-a}$$

Conjecture 5.1. *(Sato-Tate) Let E be an elliptic curve defined over $\mathbb{Q}$, without complex multiplication.*[1] *Then θ_p is a μ-equidistributed sequence with respect to the probability measure $\mu = \frac{2}{\pi}\sin^2\theta d\theta$ on the interval $[0, \pi]$.*

This is actually no more a conjecture: in fact, it has recently been proven by L. Clozel, M. Harris, N. Shepherd-Barron, R. Taylor.

In the next section we're going to collect local information in order to build the global zeta function of a given elliptic curve, and the closely related L-function. The study of this new object, as we will soon see, originates problems whose importance in Number Theory could hardly be overestimated.

5.1 The L-function

Let $E = \{[x_0, x_1, x_2] \in \mathbb{P}^2(\mathbb{Q}) : x_0x_2^2 = x_1^3 - Ax_0^2x_1 + Bx_0^3\}$ be an elliptic curve. If p is a prime of good reduction for E, we have defined the local zeta function of E at p as the zeta function of the reduced curve E_p:

$$Z_{E_p}(t) = \frac{pt^2 - a_pt + 1}{(1-t)(1-pt)}$$

[1] See Section 5.2.3 for an explanation of what is complex multiplication. In the same paragraph we also notice that elliptic curves of the form $y^2 = x^3 + D$ and $y^2 = x^3 - Dx$ *do* have complex multiplication. As for these curves the claim in the conjecture is clearly false, we see that the hypothesis of not having complex multiplication cannot be dropped.

By the change of variable $t = q^{-s}$ we obtain:

$$Z_{E_p}(s) = \frac{p^{1-2s} - a_p p^{-s} + 1}{(1-p^{-s})(1-p^{1-s})}$$

If p is *not* a prime of good reduction, that is, if $p \mid \Delta$, we define the local zeta function of E at p as[2]:

$$Z_{E_p}(s) = \frac{1}{(1-p^{-s})(1-p^{1-s})}$$

In this way, for each prime p we have defined the local zeta function of E at p. We define the global zeta function of the elliptic curve E as:

$$Z_E(s) = \prod_{p \in Max(\mathbb{Z})} Z_{E_p}(s)$$

Now recall that the Riemann zeta function can be written as: $\zeta(s) = \prod_{p \text{ prime}} (1-p^{-s})^{-1}$ (Proposition 4.1).
Let

$$L_E(s) = \prod_{p \nmid \Delta} (1 - a_p p^{-s} + p^{1-2s})^{-1}$$

Then we obtain the equality:

$$Z_E(s) = \zeta(s)\zeta(1-s)L_E(s)^{-1}$$

The function $L_E(s)$ we just defined is called the L-function of the elliptic curve E. We will denote it simply by $L(s)$ when this will not be too dangerous.

Lemma 5.1. *The product defining the L-function $L(s)$ converges in the half plane $Re\ s > 3/2$, and $L(s)$ is a holomorphic function on this half plane.*

Proof. Observe that, by Hasse theorem, for each $p \nmid \Delta$ the zeros of $1 - a_p p^{-s} + p^{1-2s}$ have real part $1/2$, hence $(1 - a_p p^{-s} + p^{1-2s})^{-1}$ is holomorphic and nonzero on the half plane $Re\ s > 3/2$. Recall that a product of nonzero holomorphic functions f_n is said to converge if the sum $\sum_n log(f_n)$ converges uniformly on compact sets.
We need to check that $-\sum_{p \nmid \Delta} log(1 - a_p p^{-s} + p^{1-2s})$ converges for $Re\ s > 3/2$.
By Hasse theorem we have $1 - a_p p^{-s} + p^{1-2s} = (1 - \pi p^{-s})(1 - \bar{\pi} p^{-s})$ with $|\pi| = \sqrt{p}$. Hence

$$\sum_{p \nmid \Delta} (log(1 - \pi p^{-s}) + log(1 - \bar{\pi} p^{-s}))$$

converges uniformly on compact sets K such that $|\pi p^{-s}| = p^{1/2 - Re\ s} \leq p^{-1-\delta}\ \forall s \in K$, for any fixed $\delta > 0$. It follows that our product converges for all s such that $1/2 - Re\ s < -1 \Leftrightarrow Re\ s > 3/2$. □

[2]Actually, this is not the "official" definition. To state it correctly, it would be necessary to investigate more in depth what happens when we reduce an elliptic curve at a prime which is not of good reduction. For simplicity, we shall not do this here and we will use this simplified form of the definition, which will be enough for us.

Of course, we're not satisfied with this result: we would like to know if it is possible to analytically continue $L(s)$ to a holomorphic function defined on the whole complex plane. Notice that by Proposition 4.1 $\zeta(s)$ can be analytically continued to a meromorphic function on $\mathbb{C}$. As $Z_E(s) = \zeta(s)\zeta(1-s)L_E(s)^{-1}$, our problem is strictly related to the problem of determining if the global zeta function $Z_E(s)$ of an elliptic curve E can be analytically continued to a meromorphic function defined on the whole complex plane. Anyway it turns out that it's easier to work with the L-function, which enjoys some remarkable properties.
In order to explain the importance of the L-function, and to show how its analytic properties are (or could be) linked to fundamental arithmetic properties of the corresponding curve, we're going to briefly discuss one of the most important open problems involving L-functions of elliptic curves.

5.1.1 The Birch and Swynnerton-Dyer conjecture

Let E be an elliptic curve defined over an arbitrary field K, with a distinguished point $O \in E(K)$. Poincaré first discovered that the points of $E(K)$ can be given an abelian group structure, defined as follows.
Let $P, Q \in E(K)$. By Bezout theorem the line through P and Q intersects $E(\bar{K})$ in a third point $R \in \mathbb{P}^2(\bar{K})$; denote this point by $R = P * Q$. It's easy to verify that actually if $P, Q \in E(K)$ then also $P * Q \in E(K)$. Now, for each pair of points $P, Q \in E(K)$ define $P + Q = O * (P * Q)$. Then $(E(K), +)$ is an abelian group.

Remark 5.3. The careful reader will have noticed that we've been quite imprecise in the definition of the group law given above: for example, if $P = Q$ the line through P and Q is by no means unique. In this case we have to consider the tangent line to E at P. The detailed description of the group law, as well as the proof that this law actually enjoys the properties of a group law, can be found for example in [3] or [18]. A very readable description of the group law from the complex analytic point of view is given in [9].

The natural problem at this point is to determine the structure of the group of points of an elliptic curve $E(K)$ for a fixed field K. The following remarkable theorem, proved by Mordell (1923) for $\mathbb{Q}$ and later generalised by Weil, answers this question for number fields.

Theorem 5.3. *(Mordell-Weil) Let $E(K)$ be an elliptic curve defined over a number field K. Then the group $(E(K), +)$ is a finitely generated abelian group.*

Proof. For a proof of the general case see [18]. An elementary proof in a special case, containing all the main ideas that are used in the general proof, can be found in [19]. □

The above theorem and the structure theorem for finitely generated abelian groups imply that, as a group, we have: $E(K) = T \oplus \mathbb{Z}^r$, where T denotes the torsion part of the group. The integer r is called the *rank* of the elliptic curve $E(K)$.
Here is where our L-function plays an unexpected, yet crucial role: suppose for a moment that it is true that the L-function of any elliptic curve E defined over $\mathbb{Q}$ can be analytically continued to a holomorphic function on $\mathbb{C}$. Then it makes sense to speak of the behaviour

of the (extended) L-function at the point $s = 1$. Based on an extensive empirical work with curves of the form $y^2 = x^3 - Dx$ and $y^2 = x^3 + D$ (for which we will see that our supposition is actually true) Birch and Swinnerton-Dyer stated the following

Conjecture 5.2. *(Birch, Swinnerton-Dyer) Let E be an elliptic curve defined over $\mathbb{Q}$. Then the rank of E is equal to the order of vanishing of $L(E, s)$ at $s = 1$.*

The above conjecture, if true, should imply the existence of an intimate relation between analytic properties of the L-function of an elliptic curve and arithmetic properties of that curve. [3]
There is actually a more precise form of this conjecture, but we will not discuss it here. The interested reader can refer to [7].
In a recent paper Manjul Bhargava (Fields Medal 2014), Christopher Skinner and Wei Zhang have shown that for "most" elliptic curves the Birch and Swinnerton-Dyer conjecture is true (See `http://arxiv.org/abs/1407.1826`).
We will now return to our original problem of determining if there exists an analytic continuation of the L-function of a given elliptic curve. The key idea is to look at the L-function from a completely different perspective.

5.2 Hecke L-functions

Hecke L-functions are an important family of functions associated with the so called *Hecke characters*, which are particular characters defined over number fields.
In order to motivate the introduction of this new concept, we shall quickly review the definition of the more classical Dirichlet L-functions. [4]

5.2.1 Dirichlet L-functions

Let ψ be a character on the abelian group $(\mathbb{Z}/m\mathbb{Z})^*$. Extend ψ letting $\psi(a) = 0\ \forall a \in (\mathbb{Z}/m\mathbb{Z}) \setminus (\mathbb{Z}/m\mathbb{Z})^*$. Let $\pi : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ denote the canonical projection. The composition $\chi = \psi \circ \pi : \mathbb{Z} \to \mathbb{C}$ has the following properties:

1. $\chi(n + m) = \chi(n)\ \forall n \in \mathbb{Z}$
2. $\chi(n) \neq 0$ if and only if $(n, m) = 1$
3. $\chi(kn) = \chi(k)\chi(n)\ \forall k, n \in \mathbb{Z}$

A function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying the three properties above is called a *Dirichlet character* modulo m.

[3] The prominent position of this conjecture in modern mathematical research is also proved by the fact that it has been chosen as one of the seven (maybe we should say six, after Perelman) Millenium Prize Problems.

[4] These functions were introduced by Dirichlet who used them to give the first proof of the fact that each arithmetic progression $a + nb$, with $(a, b) = 1$ contains infinitely many prime numbers. See [8] for an account of the proof.

To a given Dirichlet character χ we associate the corresponding *L-function* defined as follows:

$$L_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

As $|\chi(s)| = 1$, we immediately see that $L_\chi(s)$ is an analytic function on the half plane $Re\ s > 1$. Moreover, as $\chi(kn) = \chi(k)\chi(n)$ we obtain the product formula:

$$L_\chi(s) = \prod_{p \text{ prime}} (1 - \chi(p)p^{-s})^{-1}$$

As for the zeta function, it can be shown that Dirichlet L-functions have a meromorphic continuation to $\mathbb{C}$, which satisfies a certain functional equation (see [16]).
Moreover, just like the zeta function, Dirichlet L-functions can be generalized to arbitrary number fields. In what follows, we will study one possible generalisation to *CM* fields, due to Hecke.

5.2.2 Hecke algebraic characters and L-functions

Definition 5.3. Let K be a number field. A morphism $\sigma : K \to \mathbb{C}$ is called *real* if $\sigma(K) \subseteq \mathbb{R}$, otherwise it is called *complex*.
K is called *totally real* if every morphism of K into $\mathbb{C}$ is real; it is called *totally complex* if every morphism of K into $\mathbb{C}$ is complex.
A *CM field* is a totally complex quadratic extension of a totally real subfield K_0.

Example 5.4.
1. If $d \in \mathbb{Z}, d > 0$ is squarefree then $\mathbb{Q}(\sqrt{-d})$ is a CM field. In particular, $\mathbb{Q}(i)$ and $\mathbb{Q}(\omega)$ are CM fields.
2. If $\zeta_m = e^{2\pi i/m}$, $\mathbb{Q}(\zeta_m)$ is a CM field, with totally real subfield $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$.

Remark 5.4. Let K be a Galois extension of $\mathbb{Q}$ which is also a CM field. Let $j \in Gal(K/\mathbb{Q})$ denote the restriction to K of complex conjugation. Let L be the fixed field of j. Then clearly $L \supseteq K_0$, and $[K : L] = 2 = [K : K_0]$. Hence $L = K_0$.
Moreover, j clearly commutes with every $\sigma \in Gal(K_0/\mathbb{Q})$, so it commutes with each element in $Gal(K/\mathbb{Q})$. In this paragraph, we will always assume these hypotheses.

Notation 5.1. Let $G = Gal(K/\mathbb{Q})$. We will denote by $\mathbb{Z}[G]$ the *group ring* of G, whose elements are formal finite sums $\sum_{\sigma \in G} n_\sigma \sigma$ with $n_\sigma \in \mathbb{Z}$. This is a ring with addition and multiplication defined in the only reasonable way. It acts on K as follows: for $\theta = \sum_\sigma n_\sigma \sigma \in \mathbb{Z}[G], \alpha \in K$, $\alpha^\theta = \prod_\sigma \sigma(\alpha)^{n_\sigma}$.

Definition 5.4. Let K be a CM field which is also a Galois extension of $\mathbb{Q}$. Let $\mathcal{O}$ be the ring of algebraic integers of K and M a fixed ideal of $\mathcal{O}$.
An *algebraic Hecke character* modulo M is a function $\chi : \{I : I \subseteq \mathcal{O} \text{ ideal}\} \to \mathbb{C}$ that satisfies the following properties:

1. $\chi(\mathcal{O}) = 1$
2. $\chi(A) \neq 0$ if and only if $(A, M) = (1)$

3. $\chi(AB) = \chi(A)\chi(B)$
4. There is an element $\theta = \sum_\sigma n_\sigma \sigma \in \mathbb{Z}[G]$ such that if $\alpha \in \mathcal{O}, \alpha \equiv 1 \pmod{M}$, then $\chi((\alpha)) = \alpha^\theta$
5. There is an integer $m > 0$, called the *weight* of χ, such that $n_\sigma + n_{j\sigma} = m \ \forall \sigma \in G$.

Remark 5.5.
1. Properties (2),(3) of Hecke characters defined above are analogous to properties (2),(3) of Dirichlet characters.
2. It follows from condition (5) above that $(1+j)\sigma = mN$, where $N = \sum \sigma$ is the norm element in $\mathbb{Z}[G]$. The name is motivated by the fact that, for $\alpha \in K$, $\alpha^N = N(\alpha)$ where $N(\alpha)$ is the norm as defined in Section 1.4.

Proposition 5.1. *Let χ be an algebraic Hecke character of weight m. If $(A, M) = 1$, $|\chi(A)| = N(A)^{m/2}$.*

Proof. With the notations of Proposition 1.7, let $h = |C_M|$. Then h is finite by Proposition 1.7, hence there exist $\alpha, \beta \in \mathcal{O}$, $\alpha, \beta \equiv 1 \pmod{M}$ such that $(\alpha)A^h = (\beta)$. Applying χ we get:

$$\alpha^\theta \chi(A)^h = \beta^\theta$$

Taking complex conjugates and multiplying we obtain, recalling that $(1 + j)\sigma = mN$:

$$N(\alpha)^m |\chi(A)|^{2h} = N(\beta)^m$$

On the other hand, we have $(\alpha)A^h = (\beta) \Rightarrow N(\alpha)N(A)^h = N(\beta)$. Comparing the last two equations we obtain:

$$|\chi(A)|^{2h} = N(A)^{mh} \Rightarrow |\chi(A)| = N(A)^{m/2}$$

□

Remark 5.6. The equality $\alpha^\theta \chi(A)^h = \beta^\theta$ shows that the values $\chi(A)$ are algebraic numbers (roots of elements of K). This explains why we call χ an "algebraic" Hecke character.

Now that we've defined a character, we can attack an L-function to it. The definition is the obvious generalisation of the one we gave for Dirichlet L-functions:

Definition 5.5. Let χ be an algebraic Hecke character on a CM field K with ring of algebraic integers $\mathcal{O}$. The *Hecke L-function* associated to χ is:

$$L_\chi(s) = \sum_A \chi(A)N(A)^{-s} = \prod_P (1 - \chi(P)N(P)^{-s})^{-1}$$

where the sum is over all non zero ideals of $\mathcal{O}$, the product over all ideals $P \in Max(\mathcal{O})$ and equality of the two expressions follows from unique factorisation of ideals in $\mathcal{O}$ and the fact that $\chi(AB) = \chi(A)\chi(B)$.

Proposition 5.2. *The product $L_\chi(s) = \prod_P (1 - \chi(P)N(P)^{-s})^{-1}$ converges absolutely in the half plane Re $s > 1 + m/2$, uniformly for Re $s \geq 1 + m/2 + \delta$ for any $\delta > 0$.*

Proof. The product converges absolutely if and only if the series $\sum_P |log(1-\chi(P)N(P)^{-s})|$ converges. This is equivalent to the convergence of the series $\sum_P |\chi(P)N(P)^{-s}| =$
$= \sum_P N(P)^{m/2}N(P)^{-u} = \sum_P N(P)^{(m/2)-u}$, where $u = Re\ s$.
Now $N(P) = |\mathcal{O}/P| \geq p \Rightarrow N(P)^{(m/2)-u} \leq p^{-(u-m/2)}$ where p is the prime such that $P \cap \mathbb{Z} = (p)$. Since we know by Section 1.4 that each prime p as at most $[K : \mathbb{Q}]$ primes in $\mathcal{O}$ above it, we obtain:

$$\sum_P N(P)^{(m/2)-u} \leq [K : \mathbb{Q}] \sum_p p^{-(u-m/2)}$$

and the last sum converges for $Re\ s = u > 1 + m/2$. □

One of the reasons why Hecke L-functions are so important is that they have the property that we would like to be enjoyed by L-functions of elliptic curves:

Theorem 5.4. *Let χ be an algebraic Hecke character and $L_\chi(s)$ the corresponding L-function. If $\chi(A)$ is not equal to 1 for some ideal A, then $L_\chi(s)$ can be analytically continued to a holomorphic function defined on the whole $\mathbb{C}$.*

Proof. The original proof of this (difficult) Theorem is due to Hecke. A different proof has been given by John Tate in his thesis, which is reproduced in [1]. □

With this result in mind, let us return to our problem: we have an elliptic curve E with L-function $L_E(s)$. We want to show that $L_E(s)$ has an analytic continuation to the whole complex plane. We have just discovered that another, seemingly unrelated family of L-functions enjoys this property. There's only one thing we can try to do: show that $L_E(s)$ can be realised as the L-function associated with a certain Hecke character χ defined over the ring of algebraic integers of a well chosen CM field K. Let us try to do this with our two favourite families of elliptic curves.

The L-function of $y^2 = x^3 - Dx$

Let E be the curve whose affine equation is $y^2 = x^3 - Dx$, $D \in \mathbb{Z}$, $D \neq 0$. Since $\Delta = 2^6D^3$ we have:

$$L_E(s) = \prod_{p \nmid 2D} (1 - a_p p^{-s} + p^{1-2s})^{-1}$$

where $a_p = p + 1 - N_p$ and N_p has been determined in Section 5.0.3. Recall that in that Section we worked with the ring $\mathbb{Z}[i]$ in order to count points on E. If we have a chance to find an Hecke character whose associated L-function coincides with that of the curve, there is no better place to look for it than the ring $\mathbb{Z}[i]$.
Precisely, we are going to construct an algebraic Hecke character χ on $\mathbb{Z}[i]$ with respect to the modulus $(8D)$. To define it, it is clearly sufficient to specify the value of χ on prime ideals (P) of $\mathbb{Z}[i]$.

1. If $P \mid 2D$, $\chi(P) = 0$.

2. If $P \nmid 2D$ and $N(P) = p \equiv 1 \pmod 4$, let $(P) = (\pi)$, with $\pi \equiv 1 \pmod{2 + 2i}$. Then $\chi(P) = \overline{(D/p)_4}\pi$. Let $\mathcal{P}_1$ denote the set of prime ideals (P) of this type.

3. If $P \nmid 2D$ and $N(P) = p^2$ then $p \equiv 3 \pmod 4$ and $(P) = (p)$. Then $\chi(P) = -p$. Let $\mathcal{P}_2$ denote the set of prime ideals (P) of this type.

Thanks to Remark 3.8 points (2) and (3) can be joined, and we can define $\chi(P)$ uniformly for primes (P) such that $P \nmid 2D$ as: $\chi(P) = \overline{(D/\pi)_4}\pi$ where $(P) = (\pi)$ and $\pi \equiv 1 \pmod{2+2i}$.
Now suppose for a moment that we knew that χ is an algebraic Hecke character. Then we would obtain:

$$L_E(s) = \prod_{p\nmid 2D, p\equiv 3 \pmod 4} (1 - a_p p^{-s} + p^{1-2s})^{-1} \prod_{p\nmid 2D, p\equiv 1 \pmod 4} (1 - a_p p^{-s} + p^{1-2s})^{-1}$$

By Theorem 5.2 we have: $N_p = p+1 \Rightarrow a_p = 0$ if $p \equiv 3 \pmod 4$; $a_p = \overline{(D/\pi)_4}\pi + (D/\pi)_4\bar{\pi}$ if $p \equiv 1 \pmod 4$, where $p = \pi\bar{\pi}, \pi \equiv 1 \pmod{2+2i}$ Hence we obtain:

$$\begin{aligned} L_E(s) &= \prod_{p\nmid 2D,\ p\equiv 3 \pmod 4} (1 + p^{1-2s})^{-1} \prod_{p\nmid 2D,\ p\equiv 1 \pmod 4} (1 - \overline{(D/\pi)}_4\pi p^{-s})^{-1}(1 - (D/\pi)_4\bar{\pi}p^{-s})^{-1} = \\ &= \prod_{(P)\in\mathcal{P}_2} (1 - \chi(P)N(P)^{-s})^{-1} \prod_{(P)\in\mathcal{P}_1} (1 - \chi(P)N(P)^{-s})^{-1} = \\ &= L_\chi(s) \end{aligned}$$

This is exactly what we wanted. It remains only to show that χ is an algebraic Hecke character.
As $\chi(P) = \overline{(D/\pi)_4}\pi$ for all ideals (P) with $P \nmid 2D$, for any ideal A which is coprime with $(2D)$ we have $\chi(A) = \overline{(D/\alpha)_4}\alpha$, where α is the unique generator of A such that $\alpha \equiv 1 \pmod{2+2i}$ (existence and uniqueness of such an α are checked as in Proposition 1.6).
Thus, to prove that χ is an algebraic Hecke character (of weight one) with respect to the modulus $(8D)$ it will be enough to check that

$$\alpha \equiv 1 \pmod{8D} \Rightarrow (D/\alpha)_4 = 1$$

This is the typical situation in which reciprocity laws can help us. Let us distinguish three cases:

1. $D \equiv 1 \pmod 4$ Then by Proposition 3.4 we have $(D/\alpha)_4 = (\alpha/D)_4 = 1$, since $\alpha \equiv 1 \pmod D$

2. $D \equiv 3 \pmod 4$ In this case, we have: $(D/\alpha)_4 = (i^2\alpha/D)_4 = 1$ as $(i/\alpha)_4 = 1$ for $\alpha \equiv 1 \pmod 8$ (easy computation). Hence, by Proposition 3.4, $(D/\alpha)_4 = (-D/\alpha)_4 = (\alpha/D)_4 = 1$.

3. D is even. Write $D = 2^t D_0$, with D_0 odd. Then by the previous points we have $(D_0/\alpha)_4 = 1$. It remains to show that $(2/\alpha)_4 = 1$.
   Since $\alpha = a + bi \equiv 1 \pmod{8D}$ and D is even ve have $\alpha \equiv 1 \pmod{16}$, hence $a - 1 \equiv 0 \pmod{16}$ and $b \equiv 0 \pmod{16}$. Lemma 3.7 implies that $((1+i)/\alpha)_4 = 1$. Thus:
   $$1 = \left(\frac{1+i}{\alpha}\right)_4^2 = \left(\frac{2i}{\alpha}\right)_4 = \left(\frac{2}{\alpha}\right)_4$$

Hence we have proved the following

Theorem 5.5. *Let $D \in \mathbb{Z}, D \neq 0$. The L-function of the elliptic curve E of affine equation $y^2 = x^3 - Dx$ coincides with the Hecke L-function of the Hecke character modulo $M = (8D)$ defined on $\mathbb{Z}[i]$ by: $\chi(A) = \overline{(D/\alpha)_4}\alpha$ for $A = (\alpha)$ such that $(A, M) = 1$ and α is primary. Hence the L-function of E can be analytically continued to the whole complex plane.*

The L-function of $y^2 = x^3 + D$

We are now going to study the L-function of the elliptic curve E whose affine equation is $y^2 = x^3 + D,\ D \in \mathbb{Z},\ D \neq 0$. The ideas involved are the same as in the previous paragraph. We will therefore describe the main steps without going in depth into the details.
Since in this case $\Delta = -2^4 3^3 D^2$ we have:

$$L_E(s) = \prod_{p \nmid 6D} (1 - a_p p^{-s} + p^{1-2s})^{-1}$$

where $a_p = p + 1 - N_p$ and N_p has been determined in section 5.0.2. By Theorem 5.1 we have: $a_p = 0$ if $p \nmid 6D,\ p \equiv 2 \pmod 3$; $a_p = -\overline{(D/\pi)_6}\pi - (D/\pi)_6\bar{\pi}$ if $p \equiv 1 \pmod 3,\ p \nmid 6D$ and $p = \pi\bar{\pi}, \pi \equiv 2 \pmod 3$. Hence:

$$L_E(s) = \prod_{p \nmid 6D, p \equiv 2 \pmod 3} (1+p^{1-2s})^{-1} \prod_{p \nmid 6D, p \equiv 1 \pmod 3} (1+\overline{(D/\pi)}_6 \pi p^{-s})^{-1}(1+(D/\pi)_6\bar{\pi}p^{-s})^{-1}$$

As in the previous paragraph, we will construct an algebraic Hecke character whose L-function coincides with the L-function of our curve; of course, in this case we will work with the ring $\mathbb{Z}[\omega]$.
We will construct an Hecke character χ of weight 1 on $\mathbb{Z}[\omega]$ with modulus $M = (12D)$. The definition is the following:

1. If $P \mid 6D$, $\chi(P) = 0$.
2. If $P \nmid 6D$ and $N(P) = p \equiv 1 \pmod 3$, let $(P) = (\pi),\ \pi \equiv 2 \pmod 3, \pi\bar{\pi} = p$. Then $\chi(P) = -\overline{(4D/\pi)_6}\pi$.
3. If $P \nmid 6D$ and $N(P) = p^2$ then $p \equiv 2 \pmod 3$ and $(P) = (p)$. Then $\chi(P) = -p$.

As usual, we have used the symbol $(4D/\pi)_6$ to indicate $(4D)^{(N(\pi)-1)/6}$.
Notice that if p is an odd prime and $p \equiv 2 \pmod 3$ we have $(4D/p)_6 = (4D)^{(N(p)-1)/6} = ((4D)^{p-1})^{(p+1)/6} \equiv 1 \pmod p$, so $(4D/p)_6 = 1$.
Hence we can define, for (P) such that $(P, M) = 1$, $\chi(P) = -\overline{(4D/\pi)_6}\pi$ where π is such that $(P) = (\pi)$ and $\pi \equiv 2 \pmod 3$. As a consequence we have, for any ideal A such that $(A, M) = 1$, $\chi(A) = \overline{(4D/\alpha)_6}\alpha$, where α is the generator of A such that $\alpha \equiv 1 \pmod 3$.
Now, if $N(P) = p \equiv 1 \pmod 3$, $(P) = (\pi), \pi \equiv 2 \pmod 3$ we have $\chi(P) = -\overline{(4D/p)_6}\pi$. Hence

$$(1 - \chi(P)N(P)^{-s})(1 - \chi(\bar{P})N(\bar{P})^{-s}) = (1 + \overline{(4D/p)_6}\pi p^{-s})(1 + (4D/p)_6\bar{\pi}p^{-s})$$

If $N(P) = p^2$, $p \equiv 2 \pmod 3$ and $(P) = (p)$ then $\chi(P) = -p$. Hence

$$(1 - \chi(P)N(P)^{-s}) = 1 + p^{1-2s}$$

Therefore we obtain:

$$L_\chi(s) = \prod_{P \in Max(\mathbb{Z}[\omega])} (1 - \chi(P)N(P)^{-s})^{-1} = L_E(s)$$

It remains to show that χ is an algebraic Hecke character. As $\chi(A) = \overline{(4D/\alpha)_6}\alpha$ for any ideal $A = (\alpha)$ such that $(A, M) = 1$, it suffices to show that $\alpha \equiv 1 \pmod{12D} \Rightarrow (4D/\alpha)_6 = 1$. This can be shown using cubic reciprocity. Details can be found in [8].
Summing up, we obtain the following:

Theorem 5.6. *Let $D \in \mathbb{Z}$, $D \neq 0$. The L-function of the elliptic curve E of affine equation $y^2 = x^3 + D$ coincides with the Hecke L-function of the Hecke character modulo $M = (12D)$ defined on $\mathbb{Z}[\omega]$ by: $\chi(A) = \overline{(4D/\alpha)_6}\alpha$ for $A = (\alpha)$ such that $(A, M) = 1$ and $\alpha \equiv 1 \pmod 3$. Hence the L-function of E can be analytically continued to the whole complex plane.*

5.2.3 Why did things work?

Let us try to understand what we've been doing in the last two paragraphs: we were given an elliptic curve with affine equation $y^2 = x^3 + D$ or $y^2 = x^3 - Dx$. We were able to write down the L-function of such an elliptic curve explicitly, and we guessed an algebraic Hecke character defined on the ring of integers of a (reasonable) CM field whose L-function turned out to be exactly equal to the L-function of our elliptic curve.
There are (at least) two natural question which arise at this point:

1. Is there a peculiar property of the elliptic curves we considered that allows to express their L-function as the L-function of an algebraic Hecke character? (notice that, *a priori*, these two kinds of L-functions have nothing in common except the name).

2. Can this method be generalised to other elliptic curves, hence showing that their L-function can be analytically continued to $\mathbb{C}$?

Of course, these questions are closely related.
In this section, we will try to answer them. The key fact, as we shall see, is that elliptic curves of the form $y^2 = x^3 + D$ or $y^2 = x^3 - Dx$ have *complex multiplication*. We will explain what this means, and state the main results which connect the theory of algebraic Hecke characters with that of elliptic curves with complex multiplication. The advanced nature of these topics is far beyond the level of this exposition. Hence we will give no proofs; our aim is only to present concisely some fundamental results, leaving to the interested reader the opportunity to study more in depth these fascinating topics. Some useful material can be found in [7], [16].
Take an elliptic curve E defined over $\mathbb{Q}$, with affine equation: $y^2 = x^3 - Ax + B$, $A, B \in \mathbb{Z}$. Up to now, we've always studied E from the algebraic point of view. Anyway, we can

also look at $E(\mathbb{C})$, the set of complex projective points on E, which is a subset of $\mathbb{P}^2(\mathbb{C})$. This is easily seen to be a Riemann surface. It is of course very reasonable (and in fact it is true) that our algebraic definition of the genus of a curve coincides with the geometric definition of the genus of a Riemann surface (roughly speaking, its number of holes). Hence an elliptic curve is a Riemann surface of genus one, which is a torus.[5] It is well known that a torus can be realised as a quotient space of $\mathbb{C}$ with respect to a lattice L (a discrete subgroup of the plane not contained in a line). Up to an homothety, we can suppose that this lattice is generated by the points $(1, 0)$ and $\tau = (a, b)$ with $b > 0$.
Now consider the ring $\mathcal{O} = \{z \in \mathbb{C} : zL \subseteq L\}$. Clearly, $\mathcal{O} \supseteq \mathbb{Z}$.
It can happen that this inclusion is proper, which means, intuitively, that the lattice L has some nontrivial symmetries. If this is the case, we say that the elliptic curve $\mathbb{C}/L$ has complex multiplication. It is not hard to show that when this happens the ring $\mathcal{O}$ is a subring of the ring of algebraic integers of an imaginary quadratic field $\mathbb{Q}(\tau)$ which generates $\mathbb{Q}(\tau)$ over $\mathbb{Q}$ (such an $\mathcal{O}$ is called an *order* in $\mathbb{Q}(\tau)$).
The algebraic formulation of this concept is the following: to an elliptic curve E we can associate the corresponding *endomorphism ring*, containing those morphisms of the algebraic curve E to itself with also respect the group structure. It's easy to see that for each integer m group multiplication by m endomorphism is an endomorphism of E. These endomorphisms correspond to the trivial symmetries of the lattice in the complex plane. Algebraically speaking, the elliptic curve E has complex multiplication if its endomorphism ring is strictly bigger than $\mathbb{Z}$.
It turns out that having complex multiplication is the key property allowing to realise the L-function of a given elliptic curve as the L-function of an Hecke character:

Theorem 5.7. *(Deuring) Let E be an elliptic curve defined over $\mathbb{Q}$ having complex multiplication. Then there exists an Hecke character χ on a number field K such that:*

$$L_E(s) = L_\chi(s)$$

In particular, $L_E(s)$ has an analytic continuation to a holomorphic function on $\mathbb{C}$.

Proof. See [7] □

Let us return to our old friends, the elliptic curves of affine equation $y^2 = x^3 + D$ or $y^2 = x^3 - Dx$. The corresponding lattices in $\mathbb{C}$ are represented in Figure 5.1. Precisely, a curve of the form $y^2 = x^3 + D$ corresponds to a multiple of the lattice $\mathbb{Z}[\omega]$, while a curve of the form $y^2 = x^3 - Dx$ corresponds to a multiple of the lattice $\mathbb{Z}[i]$ (see [9]). In

Figure 5.1

[5] This can also be shown directly with the aid of the Weierstrass $\wp$ function; see [9] and [20].

the first case, it's easy to see that $\{z \in \mathbb{C} : zL \subseteq L\} = \mathbb{Z}[\omega]$. In the second case, this set equals $\mathbb{Z}[i]$. Therefore our two families of elliptic curves have complex multiplication. That's the scientific reason why our computations were successful.
We eventually answered the questions at the beginning of the paragraph. Anything else?

Conclusion

Deuring's Theorem 5.7 allows to prove that the L-function of an elliptic curve defined over $\mathbb{Q}$ having complex multiplication can be analytically continued to the whole complex plane, by showing that this function can actually be realized as the Hecke L-function of a certain Hecke character. The converse of this result is also true: let E be an elliptic curve defined over $\mathbb{Q}$ such that there exists an Hecke character χ verifying $L_E(s) = L_\chi(s)$. Then E has complex multiplication. Unfortunately, most elliptic curves *do not* have complex multiplication. What can we say about the existence of an analytic continuation of the L-function of an *arbitrary* elliptic curve defined over $\mathbb{Q}$? Hasse and Weil proposed the following conjecture:

Conjecture 5.3. *(Hasse-Weil) The L-function of every elliptic curve defined over* $\mathbb{Q}$ *extends to an analytic function defined on the whole* $\mathbb{C}$.

A proof of this conjecture can be obtained as a consequence of another celebrated conjecture due to Taniyama, Shimura and Weil regarding the theory of *modular* elliptic curves.[6] The Taniyama-Shimura-Weil conjecture was first proved by Andrew Wiles in a special case (which was enough to deduce Fermat's last theorem) and then in its general form by Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor in 2001. Their result is now known as the *modularity theorem*, and is the tip of an enormous iceberg known as the Langlands program, a vast web of conjectures first proposed by Robert Langlands in 1967.[7] Very roughly speaking, these conjectures state that all the L-functions which arise in number theory (of which the L-functions of elliptic curves are a special case) can always be realised as L-functions of the so called *automorphic representations*. The problem is similar in nature to the one we faced in Section 5.2.2: in order to study the L-functions of certain number-theoretic objects (such as elliptic curves) we try to show that these L-functions can actually be obtained within a completely different context.
The amount of machinery needed just to state correctly Langlands's conjectures is very large, and most of these conjectures have not yet been proved. Much hard work still has to be done, but the promised rewards for such an effort are extraordinary. In this thesis we just explored the shore of that mysterious and fascinating sea which is modern number theory. We hope we encouraged the reader to sail towards further, unexplored waters.

[6] See [7], [9] and [12] for details.
[7] See [4] for a very good introduction to the Langlands program.

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