Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE

Corso di Laurea Magistrale in Matematica

HODGE CYCLES AND DELIGNE'S PRINCIPLE B

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I Sessione
Anno Accademico 2013-2014

If people do not believe that mathematics is simple,
it is only because they do not realize how complicated life is.
(J. Von Neumann)

Introduzione

Sia X una varietà quasiproiettiva su $\mathbb{C}$ e sia X_{an} la sua varietà complessa associata. Per il lemma di Poincaré olomorfo, il complesso di de Rham olomorfo $(\Omega^{\bullet}_{X_{an}}, d)$ è una risoluzione del fascio costante $\mathbb{C}$, pertanto abbiamo che

$$H^k(X_{an}, \mathbb{C}) = \mathbb{H}^k(X, \Omega^{\bullet}_{X_{an}}),$$

dove $\mathbb{H}^k$ denota il k-esimo gruppo di ipercoomologia. La scelta di un'immersione proiettiva definisce su X una struttura di varietà Kähler , dunque, per la teoria di Hodge, i suoi gruppi di coomologia $H^k(X_{an}, \mathbb{C}) \cong H^k(X_{an}, \mathbb{Q}) \otimes \mathbb{C}$ ammettono la decomposizione di Hodge

$$H^k(X_{an}, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}.$$

Dall' isomorfismo di Dolbeault

$$H^{p,q} \cong H^q(X_{an}, \Omega^p_{X_{an}}),$$

e dalla corrispondenza GAGA di Serre otteniamo l'isomorfismo canonico

$$H^q(X_{an}, \Omega^p_{X_{an}}) = H^q(X, \Omega^p_X),$$

dove il termine di destra indica la coomologia del fascio coerente delle p forme regolari sulla varietà algebrica X. Da ciò otteniamo una versione relativamente semplice del più profondo teorema di de Rham algebrico, dovuto a Grothendieck, che afferma l'esistenza di isomorfismi

$$H^k(X_{an}, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^{\bullet}_X) \cong \bigoplus_{p+q=k} H^q(X, \Omega^p_X)$$

per ogni varietà algebrica complessa. Il teorema afferma quindi la possibilità di definire in modo puramente algebrico l'invariante trascendente $H^k(X_{an}, \mathbb{C})$. In particolare, per ogni automorfismo di campo $\sigma \in \mathbb{C}$, detta X^σ la varietà algebrica complessa ottenuta applicando σ ai coefficienti delle equazioni che definiscono X, abbiamo in modo naturale un isomorfismo $\sigma(\mathbb{C})$-lineare σ^* tra i gruppi di coomologia $H^*(X_{an}, \mathbb{C})$ e $H^*(X^\sigma_{an}, \mathbb{C})$, compatibile con la filtrazione di Hodge F^* associata alla decomposizione di Hodge. Tale isomorfismo esiste nonostante X^σ_{an} e X_{an} possano essere completamente diverse: ad esempio, gli spazi vettoriali $H^*(X_{an}, \mathbb{C})$ and $H^*(X^\sigma_{an}, \mathbb{C})$ hanno una struttura naturale su $\mathbb{Q}$ proveniente dal teorema dei coefficienti universali $H^*(-, \mathbb{C}) = H^*(-, \mathbb{Q}) \otimes \mathbb{C}$ e, in generale, σ^* non è compatibile con queste strutture razionali.
In un certo senso, la teoria delle classi di Hodge assolute di Deligne verte esattamente su tale (mancanza di) compatibilità.

Se Z è una sottovarietà algebrica di X di codimensione p, esiste un modo per associare ad essa una classe di coomologia

$$[Z] \in H^{2p}(X_{an}, \mathbb{Q}) \cap H^{p,p}.$$

La celebre congettura di Hodge afferma che è vero anche il viceversa:

Congettura 0.0.1 (Congettura di Hodge). *Sia X una varietà proiettiva liscia su $\mathbb{C}$. Per ciascun intero non negativo p, il sottospazio di grado p delle classi di Hodge razionali*

$$H^{2p}(X_{an}, \mathbb{Q}) \cap H^{p,p}$$

è generato su $\mathbb{Q}$ dalle classi di coomologia di sottovarietà algebriche di X di codimensione p.

Chiamiamo cicli di Hodge gli elementi di $H^{2p}(X_{an}, \mathbb{Q}) \cap H^{p,p}$.

É importante osservare che la definizione di classe di Hodge ha due aspetti di natura diversa:

1. La razionalità delle classi di coomologia, fatto che di per sé non ha senso considerando X come varietà algebrica astratta,

2. il fatto di essere di tipo (p, p), o più precisamente di appartenere a $F^p\mathbb{H}^{2p}(X, \Omega^p_X)$, che è una nozione prettamente algebrica .

Nell'articolo [08] Deligne considera una classe piú ristretta di cicli di Hodge, alla quale a priori devono appartenere le classi dei cicli algerbici, definendo i cicli di Hodge assoluti come quelle classi di coomologia che rimangono Hodge applicando un isomorfismo σ^* associato ad un automorfismo di $\mathbb{C}$. Come detto sopra, la condizione forte che caratterizza le classi di Hodge assolute sta nel fatto che queste debbano rimanere classi razionali.

Definizione 0.1 (**Classe di Hodge assoluta**). *Sia X una varietà complessa proiettiva liscia e sia p un intero non negativo. Sia poi α un elemento di $H^{2p}(X/\mathbb{C}(p))$. Diciamo che α è una classe di Hodge assoluta se per ogni automorfismo σ di $\mathbb{C}$ la classe di coomologia $\alpha^\sigma \in H^{2p}(X^\sigma_{an}, \mathbb{C}) \cong H^{2p}(X^\sigma/\mathbb{C})$ è una classe di Hodge.*

Non è difficile provare che le classi di varietà algebriche sono classi di Hodge assolute. Questo fatto permette di spezzare la congettura di Hodge nelle seguenti sottocongetture.

Congettura 0.0.2. *Le classi di Hodge su una varietà complessa proiettiva liscia sono classi di Hodge assolute.*

Congettura 0.0.3. *Sia X una varietà complessa proiettiva liscia. Allora le classi di Hodge assolute sono generate su $\mathbb{Q}$ dalle classi di sottovarietà algebriche.*

La congettura 0.0.2 è stata risolta affermativamente da Deligne per le classi di Hodge su varietà abeliane. La dimostrazione si basa sui due seguenti principi:

A) Siano $t_1, \dots, t_N$ cicli di Hodge assoluti su una varietà complessa proiettiva liscia X e sia G il sottogruppo algebrico massimale di $GL(H^*(X, \mathbb{Q})) \times GL(\mathbb{Q})$ che fissa i t_i; allora ogni classe di coomologia t su X fissata da G è un ciclo di Hodge assoluto.

B) Se $(X_b)_{b \in B}$ è una famiglia algebrica di varietà proiettive lisce con B liscia connessa e $(t_b)_{b \in B}$ è una famiglia di cicli razionali tale che t_b è un ciclo di Hodge assoluto per un certo $b \in B$, allora t_b è un ciclo di Hodge assoluto per ogni $b \in B$.

L'obiettivo di questo lavoro è provare il principio B. Tale principio fornisce uno strumento essenziale per dimostrare che alcune classi di Hodge sono assolute. In particolare, nel caso delle varietà abeliane, permette di ridurre la dimostrazione al caso di varietà abeliane con moltiplicazione complessa.

Introduction

Let X be a smooth projective variety over $\mathbb{C}$, and X_{an} its associated complex manifold. The holomorphic de Rham complex $(\Omega^\bullet_{X_{an}}, d)$ is, by the holomorphic Poincaré Lemma, a resolution of the constant sheaf $\mathbb{C}$, hence

$$H^k(X_{an}, \mathbb{C}) = \mathbb{H}^k(X, \Omega^\bullet_{X_{an}}),$$

where $\mathbb{H}^k$ denotes the k-th hypercohomology group. The choice of a projective embedding endows X with a Kähler structure, hence, by Hodge theory, its cohomology groups $H^k(X_{an}, \mathbb{C}) \cong H^k(X_{an}, \mathbb{Q}) \otimes \mathbb{C}$ admit the Hodge decomposition

$$H^k(X_{an}, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}.$$

By the Dolbeault isomorphism

$$H^{p,q} \cong H^q(X_{an}, \Omega^p_{X_{an}}),$$

and by the GAGA principle of Serre we have a canonical isomorphism

$$H^q(X_{an}, \Omega^p_{X_{an}}) = H^q(X, \Omega^p_X),$$

where the last term indicates the cohomology of the coherent sheaf of regular p forms on the algebraic variety X. Thus we have isomorphisms (a relatively easy version of the deep algebraic de Rham theorem of Grothendieck)

$$H^k(X_{an}, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^\bullet_X) \cong \bigoplus_{p+q=k} H^q(X, \Omega^p_X).$$

The highly non-trivial point of this is that the transcendental invariant $H^k(X_{an}, \mathbb{C})$ affords an algebraic characterization. In particular, for any field automorphism $\sigma \in \mathbb{C}$, letting X^σ be the complex algebraic variety obtained by applying σ to the coefficients of the defining equations of X, we have a natural $\sigma(\mathbb{C})$-linear isomorphism σ^* between the cohomology groups $H^*(X_{an}, \mathbb{C})$ and $H^*(X^\sigma_{an}, \mathbb{C})$, which is compatible with the Hodge filtration F^* associated to the Hodge decomposition. This isomorphism holds despite the fact that X^σ_{an} and X_{an} are completely different: for instance, the vector spaces $H^*(X_{an}, \mathbb{C})$ and $H^*(X^\sigma_{an}, \mathbb{C})$ are endowed with a natural $\mathbb{Q}$ structure, coming from the universal coefficient theorem $H^*(-, \mathbb{C}) = H^*(-, \mathbb{Q}) \otimes \mathbb{C}$ and there is in general no compatibility of the isomorphism σ^* with these rational structures. In a sense, Deligne's theory of absolute Hodge classes deals precisely with this (lack of) compatibility:

If Z is an algebraic subvariety of X of codimension p, we have a way to associate a cohomology class

$$[Z] \in H^{2p}(X_{an}, \mathbb{Q}) \cap H^{p,p}.$$

The famous Hodge conjectures states that the viceversa holds:

Conjecture 0.0.4 (Hodge conjecture)**.** *Let X be a smooth projective variety over $\mathbb{C}$. For any nonnegative integer p, the subspace of degree p rational Hodge classes*

$$H^{2p}(X_{an}, \mathbb{Q}) \cap H^{p,p}$$

is generated over $\mathbb{Q}$ by the cohomology classes of codimension p subvarieties of X.

We call the elements of $H^{2p}(X_{an}, \mathbb{Q}) \cap H^{p,p}$ Hodge cycles.

It is important to note that the definition of Hodge class contains two aspects:

1. Rationality of the cohomology class, clearly a transcendental issue, which makes no sense for X considered as an abstract algebraic variety.
2. Being of type (p, p), or more precisely in $F^p\mathbb{H}^{2p}(X, \Omega^p_X)$, is definitely an algebraic notion.

In [08] Deligne puts a stronger condition on Hodge cycles, defining absolute Hodge cycles as those classes which remain Hodge after applying any isomorphism σ^* associated with a field automorphism of $\mathbb{C}$. In view of the preceding discussion the strong requirement is that these classes remain rational. We then have the following definition of absolute Hodge class.

Definition 0.1 (Absolute Hodge class). *Let X be a smooth complex projective variety. Let p be a nonnegative integer and let α be an element of $H^{2p}(X/\mathbb{C}(p))$. We say that the cohomology class α is an absolute Hodge class if for every automorphism σ of $\mathbb{C}$ the cohomology class $\alpha^\sigma \in H^{2p}(X^\sigma_{an}, \mathbb{C}) \cong H^{2p}(X^\sigma/\mathbb{C})$ is a Hodge class.*

It is not hard to prove that the class of an algebraic subvariety is absolute Hodge. This allows to split the Hodge conjecture in the two following subconjectures.

Conjecture 0.0.5. *Hodge classes on smooth complex projective varieties are absolute Hodge*

Conjecture 0.0.6. *Let X be a smooth complex projective variety. Absolute Hodge classes are generated over $\mathbb{Q}$ by algebraic cycles classes.*

Conjecture 0.0.5 has been solved affermatively by Deligne for Hodge classes on abelian varieties. The proof is based on the following two principles.

A) Let $t_1, \ldots, t_N$ be absolute Hodge cycles on a smooth projective variety over X and let G be the largest algebraic subgroup of $GL(H^*(X, \mathbb{Q})) \times GL(\mathbb{Q})$ fixing the t_i; then every cohomology class t on X fixed by G is an absolute Hodge cycles.

B) If $(X_b)_{b \in B}$ is an algebraic family of smooth projective varieties with B smooth connected and $(t_b)_{b \in B}$ is a family of rational cycles such that t_b is an absolute Hodge cycle for one b, then t_b is an absolute Hodge cycle for all b.

The goal of this work is to prove principle B. This principle provides a great tool for proving that some Hodge classes are absolute. In particular, in the case of abelian varities considered by Deligne, it allows for a reduction to the proof of the conjecture in the case of abelian varieties with complex multiplication.

Contents

Chapter 1

Cohomology and Hodge decomposition

We now do a short review on cohomology. Since this is not the main topic of this work, we will not give proofs of many of the result we are presenting. Precise details about the following section can be found in [14] and [25]

1.1 A review of sheaf cohomology

Given a topological space X and a sheaf $\mathcal{F}$ of abelian groups, we have the natural functor Γ of global section, which to $\mathcal{F}$ associates $\Gamma(X, \mathcal{F}) := \mathcal{F}(X)$, with values in the category of abelian groups. This functor is left exact but not right exact, i.e. a surjective morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ does not necessarily induce a morphism at the level of global sections. Thus it is interesting to compute this defect in exactness via the use of invariants, namely the images under the right derived functors $R^i\Gamma$, written $H^i(X, \cdot)$, of the sheaves $\mathcal{F}, \mathcal{G}, ker(\phi)$. These functors are defined as follows: consider two abelian cathegories $\mathcal{C}$, $\mathcal{C}'$ and assume that $\mathcal{C}$ has enough injectives. If $F : \mathcal{C} \to \mathcal{C}'$ is a left exact functor and A is an object of $\mathcal{C}$, we take an injective resolution $I^\bullet$ of A and define the i-th right derived functor as the i-th cohomology group of the complex $F(I^\bullet)$:

$$R^iF(A) := H^i(F(I^\bullet)) \qquad \forall i \geq 0.$$

It can be proved that if we choose another injective resolution, we obtain canonically isomorphic objects. Also, the functors R^iF satisfy the following properties for all $A, B, C \in Ob(\mathcal{C})$:

(i) $R^0F(A) = F(A)$;

(ii) for any exact sequence

$$0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

we can construct a long exact sequence (i.e. an exact complex)

$$0 \to F(A) \xrightarrow{F(\phi)} F(B) \xrightarrow{F(\psi)} F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \dots;$$

(iii) For every injective object $I \in Ob(\mathcal{C})$: $R^iF(I) = 0, \quad \forall i > 0$.

As both the category of sheaves of abelian groups and the functor Γ satisfy the conditions we have given above, it is possible to construct the right derived functors $R^i\Gamma$. We will use these functors to define the cohomology of a sheaf $\mathcal{F}$ of abelian groups over X

$$H^i(X, \mathcal{F}) := R^i\Gamma(\mathcal{F})$$

and prove comparison theorems.
However, injective resolutions are difficult to manipulate. Thus we would like to find a weaker and easier condition for our pourposes. Luckily, such a condition exists and it is acyclicity.

Definition 1.1. *We say that an object A of $\mathcal{C}$ is acyclic for a functor F (or F-acyclic) if $R^iF(A) = 0$ for all $i \geq 0$. A resolution is said acyclic for F if its objects are F-acyclic.*

From the properties of derived functors we immediately notice that injective objects are acyclic for any functor. As a consequence, any injective resolution is acyclic. Anyway, it is still possible to compute the right derived functors from the cohomology of an acyclic resolution.

Theorem 1.1.1. *Let $M^\bullet$ be a F-acyclic resolution of an object A of $\mathcal{C}$. Then*

$$R^iF(A) = H^iF(M^\bullet)$$

An important category of Γ-acyclic sheaves is that of flasque sheaves.

Definition 1.2. *Let $\mathcal{F}$ be a sheaf of abelian groups over a topological space X. We say that $\mathcal{F}$ is flasque (or flabby) if for every open set $U \subset X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.*

Theorem 1.1.2. *Flasque sheaves are acyclic for the functor* Γ.

The previous theorem allow us to use Godement resolutions, which have the advantage of being canonical and functorial, to compute the cohomology of a sheaf. Given a sheaf $\mathcal{F}$, the Godement resolution of $\mathcal{F}$ is computed by considering the inclusion of $\mathcal{F}$ in the sheaf $\mathcal{C}^0\mathcal{F}$ which to any open set U associates the direct product of the stalks of $\mathcal{F}_x$ of its elements:

$$U \to \mathcal{C}^0\mathcal{F}(U) = \prod_{x\in U} \mathcal{F}_x.$$

As the restriction map is surjective by definition, $\mathcal{C}^0\mathcal{F}$ is flasque. Then we can inject $\mathcal{Q}_1 := \mathcal{C}^0\mathcal{F}/\mathcal{F}$ into $\mathcal{C}^1\mathcal{F} := \mathcal{C}^0\mathcal{Q}_1$. We repeat this procedure by setting

$$\begin{aligned} \mathcal{Q}_k &= \mathcal{C}^0\mathcal{Q}_{k-1}/\mathcal{Q}_{k-1} \\ \mathcal{C}^k\mathcal{F} &= \mathcal{C}^0\mathcal{Q}_k. \end{aligned}$$

In this way we construct a long exact sequence

$$0 \to \mathcal{F} \to \mathcal{C}^0\mathcal{F} \to \mathcal{C}^1\mathcal{F} \to \dots$$

called the Godement canonical resolution of $\mathcal{F}$ and its terms $\mathcal{C}^k\mathcal{F}$ are the Godement sheaves of $\mathcal{F}$. One can show (see [10]) that any sheaf morphism $\phi : \mathcal{F} \to \mathcal{G}$ induces a morphism $\mathcal{C}^k\phi : \mathcal{C}^k\mathcal{F} \to \mathcal{C}^k\mathcal{G}$ and that each $\mathcal{C}^k()$ is an exact functor from sheaves to sheaves, called the k-th Godement functor.

We have the following definition.

Definition 1.3. *A sheaf $\mathcal{F}$ is called fine if for any open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X there exists a family of endomorphisms $\phi_i : \mathcal{F} \to \mathcal{F}$ such that*

i) $\sum_i \phi_i(x) = id_{\mathcal{F}_x}$ *where the sum is required to be locally finite;*

ii) $Supp(\phi) \subset U_i$.

Such a family is called a partition of the unity associated to $\mathcal{F}$ and $\mathcal{U}$.

Fine sheaves have an interesting property which will be useful when we prove the de Rham theorem.

Theorem 1.1.3. *Every fine sheaf $\mathcal{F}$ over a topological space X is Γ-acyclic, i.e. $H^i(X, \mathcal{F}) = 0, \quad \forall i > 0$.*

Example 1.1. The sheaf $\mathcal{A}^k$ of smooth k-forms on a manifold X is a fine sheaf on X. Indeed we know that, given a locally finite open cover $\{U_\alpha\}$ of X, there is a $\mathcal{C}^\infty$ partition on unity ρ_α associated to it. However ρ_α is a collection of smooth real valued functions, not sheaf maps. Thus, for any open $U \subset X$ we define

$$\eta_{\alpha,U} : \mathcal{A}^k(U) \to \mathcal{A}^k(U), \qquad \eta_{\alpha,U}(\omega) := \rho_\alpha(\omega)$$

and prove that it is a partition of unity associated to $\mathcal{A}^k$.
If $x \notin U_\alpha$, then x has a neighbourhood U disjoint from supp ρ_α. Hence ρ_α vanishes identically on U and $\eta_{\alpha,U} = 0$, so that the stalk map $\eta_{\alpha,x} : \mathcal{A}^k_x \to \mathcal{A}^k_x$ is he zero map. This proves that supp$\eta_\alpha \subset U_\alpha$. For any $x \in X$, the stalk map $\eta_{\alpha,x}$ is multiplication by the germ of ρ_α, so $\sum_\alpha eta_{\alpha,x}$ is the identity map on the stalk $\mathcal{A}^k_x$. Hence, $\{\eta_\alpha\}$ is a partition of unity of the sheaf $\mathcal{A}^k$ subordinate to $\{U_\alpha\}$.

1.1.1 Cohomology groups on a manifold and comparison theorems

Betti Cohomology

The degree i Betti cohomology group $H^i(X, R)$ of a topological manifold X with value in any abelian group R (usually $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$) is the i^{th} cohomology group of the constant sheaf R. It can be computed in several ways, which correspond to various choices of acyclic resolutions of the constant sheaf R on X. In the case of a complex algebraic variety, we will refer to this cohomology groups, which take into account only the topological space (or differentiable manifold) associated with X, as the Betti cohomology.

De Rham cohomology

Let X be a differentiable manifold X and let T_X denote its tangent bundle. Set $A_X^k := \Lambda^k T_X^*$ and let $\mathcal{A}_X^k$ be the associated sheaf of sections. We denote by $\mathcal{A}^k(X) := \Gamma(X, \mathcal{A}^k)$ the space of $\mathcal{C}^\infty$ k-forms on X. Given the complex $(\mathcal{A}^\bullet, d)$, Poincaré's lemma tells us that $d^2 = 0$ and thus we can define the k-th de Rham cohomology group:

$$H_{dR}^k(X, \mathbb{R}) := \frac{Ker\{d : \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X)\}}{Im\{d : \mathcal{A}^{k-1}(X) \to \mathcal{A}^k(X)\}}$$

The same construction works if we take the $\mathcal{C}^\infty$-forms with complex coefficients and we have an identification

$$H_{dR}^k(X, \mathbb{R}) \otimes \mathbb{C} = H_{dR}^k(X, \mathbb{C}).$$

Singular cohomology

To define singular cohomology one considers the standard k-simplex

$$\Delta_k := \{(x_0, \dots, x_n) \in \mathbb{R}^{k+1} : \sum x_i = 1, \quad x_i \geq 0\}.$$

The faces of Δ_k are the subsets $\Delta_q^{k-1} = \Delta_k \cap \{x_q = 0\}$ with inclusion maps $j^q : \Delta_q^{k-1} \to \Delta_k$. We denote with $S_k(X)$ the group generated by continuous maps $\sigma : \Delta_k \to X$ and call its elements k-singular chains. If R is a ring, we set $S_k(X, R) := S_k(X) \otimes R$ and $S^k(X, R) := Hom(S_k(X, R), R)$. There exists a boundary map

$$\partial : S_k(X, R) \to S_{k-1}(X, R), \qquad \sigma \mapsto \sum_q (-1)^q \sigma \circ j^q.$$

The coboundary map $\delta^k : S^k(X, R) \to S^{k+1}(X, R)$ is defined as the transpose of ∂_k. The singular homology and cohomology groups are defined by

$$H_k^{sing}(X, R) := H_k(S_\bullet(X, R)) = \frac{Ker\{\partial : \mathcal{S}_k(X) \to \mathcal{S}_{k-1}(X)\}}{Im\{\partial : \mathcal{S}_{k+1}(X) \to \mathcal{S}_k(X)\}}$$

$$H_{sing}^k(X, R) := H^k(S^\bullet(X, R)) = \frac{Ker\{\delta : \mathcal{S}^k(X) \to \mathcal{S}^{k+1}(X)\}}{Im\{\delta : \mathcal{S}^{k-1}(X) \to \mathcal{S}^k(X)\}}$$

We now state the first comparison theorem:

Theorem 1.1.4. *Let X be a locally contractible topological space, and R a commutative ring. Then we have a canonical isomorphism*

$$H^k_{sing}(X,\mathbb{Z}) \cong H^k(X,\mathbb{Z})$$

between singular and Betti cohomology.

Remark 1. *The previous results holds for any commutative ring R.*

Proof. Consider the sheaf S^k of the singular cochains associated to the presheaf

$$U \mapsto S^k(U,\mathbb{Z}).$$

The differential δ on each section $S^k(U,\mathbb{Z})$ gives a differential at the level of sheaves

$$\delta : S^k \to S^{k+1}.$$

The complex of singular cochains gives a resolution of the constant sheaf $\mathbb{Z}$. In fact, since the cohomology of a constant sheaf on every contractible open U is 0 in positive degree, $(S^\bullet, \delta)$ exact in positive degree and the Kernel of $\delta_0 : S^0 \to S^1$ is the constant sheaf $\mathbb{Z}$, as X is locally pathways connected. Also, such a resolution is Γ-acyclic because S^k is flasque. □

Theorem 1.1.5 (de Rham). *Let X be a $\mathcal{C}^\infty$ differentiable manifold. Then*

$$H^k_{dR}(X,\mathbb{R}) \cong H^k(X,\mathbb{R}) \cong H^k_{sing}(X,\mathbb{R})$$

Proof. First we observe that the constant sheaf of stalk $\mathbb{R}$ is naturally included in the sheaf $\mathcal{C}^\infty(X)$ of $\mathcal{C}^\infty$ functions. Poincaré lemma tells us that every closed form of degree $k > 0$ is locally exact, which means that the sequence

$$\ldots \to \mathcal{A}^{k-1} \xrightarrow{d} \mathcal{A}^k \xrightarrow{d} \mathcal{A}^{k+1} \xrightarrow{d} \ldots$$

is exact in the middle for $k \geq 1$. Furthermore, if we look at the kernel of $d_0 : \mathcal{C}^\infty = \mathcal{A}^0 \to \mathcal{A}^1$, we notice that $df = 0$ if and only if f is locally constant, i.e. $\ker d_0 = \mathbb{R}$. We have thus constructed a resolution with fine sheaves of the constant sheaf $\mathbb{R}$, called the de Rham resolution, such that:

$$H^k(\mathcal{A}^\bullet) = \begin{cases} \mathbb{R} & \text{for k=0} \\ 0 & \text{otherwise.} \end{cases}$$

As the de Rham resolution is Γ-acyclic by theorem 1.1.3, the sheaf $H^k(X,\mathbb{R})$ is equal to the cohomology of the complex of the global sections of $\mathcal{A}^\bullet$, which is precisely $H^k_{dR}(X,\mathbb{R})$. From theorem 1.1.4, we also have an isomorphism

$$H^k(X,\mathbb{R}) \cong H^k_{sing}(X,\mathbb{R}).$$

□

Remark 2. *The direct isomorphism*

$$H^k_{dR}(X,\mathbb{R}) \cong H^k_{sing}(X,\mathbb{R})$$

is given by the following pairing: choose a closed form ω *and a singular chain* $\gamma \in Ker\{\partial : \mathcal{S}_k(X) \to \mathcal{S}_{k-1}(X)\}$. *Define:*

$$I(\omega,\gamma) := \int_\gamma \omega.$$

The numbers $I(\omega,\gamma)$ *are called periods of* ω. *By Stokes' theorem,* I *defines a pairing:*

$$H^k_{dR}(X,\mathbb{R}) \times H_k^{sing}(X,\mathbb{R}) \to \mathbb{R}$$

and thus, for every $\omega \in H^k_{dR}(X,\mathbb{R})$ *a functional in* $H^k_{sing}(X,\mathbb{R}) = Hom(H_k^{sing}(X,\mathbb{R}),\mathbb{R})$ *which maps* γ *into* $\int_\gamma \omega$. *The map* $\omega \mapsto \int \omega$ *is an isomorphism between* $H^k_{dR}(X,\mathbb{R})$ *and* $H^k_{sing}(X,\mathbb{R})$.

Remark 3. *De Rham theorem also holds in the case of a complex manifold. Choosing the resolution of the* $\mathcal{C}^\infty$ *k-forms* $\mathcal{A}^k(X)$ *with complex coefficients, we have*

$$H^k_{dR}(X,\mathbb{C}) \cong H^k(X,\mathbb{C}) \cong H^k_{sing}(X,\mathbb{C})$$

and the isomorphism between de Rham and singular cohomology is given as above.

Dolbeault cohomology

Let X be a complex manifold of dimension n. Consider the sheaf Ω^p of holomorphic p-forms. If U is an open set of X we have that $\Omega^p = \ker\{\mathcal{A}^{p,0}(U) \to \mathcal{A}^{p,1}(U)\}$. It follows, from the $\bar\partial$-Poincaré lemma, that for each $0 \le p \le n$ we get a complex

$$0 \to \Omega^p(U) \hookrightarrow \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \mathcal{A}^{p,n} \xrightarrow{\bar\partial} 0$$

called the Dolbeault complex. Its cohomology spaces

$$H^{p,q}_{\bar{\partial}} := \frac{\ker\{\bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}\}}{Im\{\bar{\partial} : \mathcal{A}^{p,q-1} \to \mathcal{A}^{p,q}\}}$$

are called Dolbeault cohomology groups. Since the sheaves $\mathcal{A}^{p,q}$ are fine, we also have that the Dolbeault complex is an acyclic resolution of the sheaf Ω^p and so there is an isomorphism:

$$H^{p,q}_{\bar{\partial}} = H^q(X, \Omega^p).$$

1.2 The Hodge Decomposition

De Rham theorem tells us two things:

(i) if all the periods of a differential form vanish, the form is exact;

(ii) Every cohomology class $a \in H^k(X, \mathbb{R})$ can be represented by a closed differential fomr.

The representative in (ii) is of course not unique, since it is defined up to adding an exact form. The aim of Hodge theorem will be to identify $H^k(X, \mathbb{R})$ with a subspace, rather than a subquotient, of $\mathcal{A}^k(X)$ by finding a unique representative for a cohomology class. This representative is provided by an isomorphism of $H^k(X, \mathbb{R})$ with the harmonic k-forms.

1.2.1 Harmonic forms and laplacians

The L^2 metric

Let (X, g) be a compact oriented riemannian manifold of dimension n. Choose an orthonormal basis for the tangent space $T_{X,x}$ with respect to the inner product g_x. We can extend the inner product g to $T^*_{X,x} =: A^1_{X,x}$ and so to each vector bundle $\bigwedge^k A^1_{X,x} =: A^k_{X,x}$ by requiring that the corresponding bases of these spaces are orthonormal with respect to g. Given $\alpha, \beta \in \mathcal{A}^k(X)$, we have $(\alpha, \beta)_x := g_x(\alpha_x, \beta_x)$ and we define the L^2 metric (or Hodge metric) on $\mathcal{A}^k(X)$ by

$$(\alpha, \beta)_{L^2} := \int_X (\alpha, \beta)_x Vol$$

where Vol is the volume form associated to (X, g).

To construct a formal adjoint for d with respect to this metric, we introduce the Hodge star operator. We recall that the volume form $Vol \in A^n_{X,x}$ determines an isomorphism between $\mathcal{A}^n$ and $\mathbb{R}$ given by integration. Thus, the wedge product induces a bilinear form

$$I : \mathcal{A}^k(X) \times \mathcal{A}^{n-k}(X) \to \mathbb{R}$$

which gives an isomorphism of vector spaces $p : \mathcal{A}^{n-k}(X) \xrightarrow{\simeq} Hom(\mathcal{A}^k(X), \mathbb{R})$.

Also, we have an isomorphism m given by the L^2 metric between $\mathcal{A}^k(X)$ and $Hom(\mathcal{A}^k(X), \mathbb{R})$. We define the Hodge operator $*$ as the unique isomorphism for which the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}^k(X) & \xrightarrow{m} & Hom(\mathcal{A}^k, \mathbb{R}) \\ \downarrow * & & \| \\ \mathcal{A}^{n-k}(X) & \xrightarrow{p} & Hom(\mathcal{A}^k, \mathbb{R}) \end{array}$$

where

$$m : \beta \mapsto (\alpha \mapsto \int_X (\alpha, \beta) Vol)$$
$$p : \gamma \mapsto (\alpha \mapsto \int_X \alpha \wedge \gamma).$$

Given $\beta \in \mathcal{A}^k(X)$, $*\beta$ will be the only element of $\mathcal{A}^{n-k}(X)$ such that

$$\int_x (\alpha, \beta) Vol = \int_X (\alpha \wedge *\beta).$$

One can easily verify from the definition that the Hodge operator satisfies the following properties:

(i) $* \circ * = (-1)^{k(n-k)}$

(ii) $(*\alpha, *\beta) = (\alpha, \beta)$

Proposition 1.2.1. *The operator*

$$d^* = (-1)^{nk+1} * d* : \mathcal{A}^{k+1}(X) \to \mathcal{A}^k(X)$$

is the formal adjoint of d with respect to the L^2 metric.

Proof. Let $\alpha \in \mathcal{A}^k(X), \beta \in \mathcal{A}^{k+1}(X)$. Using Stokes theorem we have:

$$\begin{aligned}(d\alpha, \beta)_{L^2} &= \int_X d\alpha \wedge *\beta \\ &= (-1)^{k+1} \int_X \alpha \wedge d * \beta \\ &= (-1)^{k+1} \int_X \alpha \wedge *(*^{-1} d * \beta) \\ &= (-1)^{k(n-k)+k+1} \int_X \alpha \wedge *(*d * \beta) \\ &= (-1)^{nk+1} (\alpha, *d * \beta).\end{aligned}$$

□

Laplacians

Definition 1.4. *Let X be a compact Riemannian manifold. We define the d-Laplacian operator*

$$\Delta_d := dd^* + d^*d : \mathcal{A}^k(X) \to \mathcal{A}^k(X)$$

Lemma 1.2.2. *In the prevoius hypotheses we have that* $\ker \Delta_d = \ker d \cap \ker d^*$.

Proof. Take $\omega \in \mathcal{A}^k(X)$.

$$(\omega, \Delta_d \omega)_{L^2} = (\omega, dd^*\omega)_{L^2} + (\omega, d^*d\omega)_{L^2} = (d^*\omega, d^*\omega)_{L^2} + (d\omega, d\omega)_{L^2}. \tag{1.1}$$

Clearly $d\omega = d^*\omega = 0$ if and only if $\Delta_d \omega = 0$. □

Corollary 1.2.3. *From the formula 1.1, we notice that the laplacian has the followwing properties:*

(i) $\Delta = \Delta^*$

(ii) Δ *commutes with* $*, d, d^*$.

Definition 1.5. *Let $\omega \in \mathcal{A}^k(X)$. We say that ω is harmonic if $\Delta_d \omega = 0$. We will denote the vector space of k-harmonic forms with $\mathcal{H}^k$.*

In particular we have that every harmonic form is closed. So we have a map i: $\mathcal{H}^k \to H^k_{dR}(X, \mathbb{R})$ which to a form α associates its de Rham cohomology class. In order to prove that this is actually an isomorphism, we need the famous Hodge theorem:

Theorem 1.2.4 (Hodge theorem). *Let X be a compact oriented riemannian manifold. For all k we have:*

(i) $dim\mathcal{H}^k < \infty$

(ii) $\mathcal{A}^k(X) = \mathcal{H}^k \oplus \Delta\mathcal{A}^k(X) = \mathcal{H}^k \oplus d\mathcal{A}^{k-1}(X) \oplus d^*\mathcal{A}^{k+1}(X)$

where the direct sum in (ii) is rispect to the L^2 metric.

We are now able to prove the following theorem:

Theorem 1.2.5. *There map*

$$\mathcal{H}^k \to H^k_{dR}(X, \mathbb{R})$$
$$\alpha \mapsto [\alpha]_{dR}$$

is an isomorphism.

Proof. By Hodge theorem we have the decomposition

$$\mathcal{A}^k(X) = \mathcal{H}^k \oplus \Delta\mathcal{A}^k(X) = \mathcal{H}^k \oplus d\mathcal{A}^{k-1}(X) \oplus d^*\mathcal{A}^{k+1}(X).$$

Let $\alpha \in \mathcal{A}^k(X)$ be a closed form, and write $\alpha = \omega + \Delta\beta$ with ω harmonic. As α and ω are closed, we then have

$$0 = d\alpha = d\omega + d^2d^*\beta + dd^*d\beta = dd^*d\alpha \Rightarrow d^*d\beta \in Ker d \cap Im d^* \Rightarrow d^*d\beta = 0.$$

So $\alpha = \omega + dd^*\beta$ (i.e. $[\alpha]_{dR} = [\omega]_{dR}$) and thus i is surjective. To see that i is also injective we take a harmonic exact form and we show that it is necessarily 0. Let $\alpha \in \mathcal{H}^k$ be exact. Then by lemma 1.2.2 $\alpha \in \ker d^*$. However α is also in $Im d$ so, by Hodge theorem, α must be zero. $\square$

The complex case

In the case of a complex manifold we can extend the definition of * by $\mathbb{C}$-linearity to complex valued forms, which we will again denote by $\mathcal{A}^k(X)$. We extend the metrics $(,)_x$ to Hermitian metrics on the complexified bundles $A^k_{X,\mathbb{C}}$. On the subject of the Hermitian metric induced on the complexified bundles, let us remark the following fact. If V is a complex vector space, $W = Hom(V, \mathbb{R})$ and h is an Hermitian metric on V, we have a decomposition:

$$W_{\mathbb{C}} := W \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}, \qquad \Lambda^l W_{\mathbb{C}} = \bigoplus_{p+q=k} W^{p,q}.$$

Each component $W^{p,q} := \bigwedge^p W^{1,0} \oplus \bigwedge^q W^{0,1}$ has a hermitian metric $h^{p,q}$ induced by h on $W^{1,0} \cong Hom_{\mathbb{C}}(V, \mathbb{C})$, $W^{0,1} \cong Hom_{\overline{\mathbb{C}}}(V, \mathbb{C})$ and their tensor products. Furthermore, we have the Hermitian metric h^k induced by g on $\bigwedge^k W_{\mathbb{C}} \cong \bigwedge^k W \otimes \mathbb{C}$.

Lemma 1.2.6. *$2^k h_k = \sum h^{p,q}$ on $\bigwedge^k W_{\mathbb{C}}$, where $\sum$ denotes the direct sum of the metrics $h^{p,q}$.*

On a complex manifold with hermitian metric h we define

$$* : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{n-q,n-p}(X)$$

such that $\alpha \wedge \overline{*\beta}. = h(\alpha, \beta) Vol$ for all $\alpha, \beta \in \mathcal{A}^{p,q}(X)$. The L^2 inner product will be given by

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge \overline{*\beta}.$$

The decomposition

$$\mathcal{A}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$

is orthogonal with respect to this inner product.

As in the riemannian case, we define the formal adjoint of the differential d and also the adjoint operators of ∂ and $\bar{\partial}$. Using the same techniques of the riemannian case, one can show that

$$\partial^* = - * \bar{\partial} * \qquad \bar{\partial}^* = - * \partial * .$$

We define the laplacian operators

$$\Delta_d = dd^* + d^*d$$

$$\Delta_\partial = \partial\partial^* + \partial^*\partial$$

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

and the space of (p, q)-harmonic forms as

$$\mathcal{H}^{p,q} := \{\omega \in \mathcal{A}^{p,q} \mid \Delta_{\bar\partial}\omega = 0\}$$

By elliptic operator theory one obtains an analogue of Hodge theorem for $\Delta_{\bar\partial}$.

Theorem 1.2.7. *Let X be a compact complex manifold. For all k we have:*

(i) $dim_{\mathbb{C}}\mathcal{H}^{p,q} < \infty$

(ii) $\mathcal{A}^{p,q}(X) = \mathcal{H}^{p,q} \oplus \Delta_{\bar\partial}\mathcal{A}^k(X) = \mathcal{H}^k \oplus Im\bar\partial \oplus Im\bar\partial^*$

where the direct sum in (ii) is rispect to the L^2 metric.

Corollary 1.2.8. *The natural map from the space of complex valued (p, q)-harmonic forms to the Dolbeault cohomology group $H^{p,q}_{\bar\partial}$ is an isomorphism.*

$$\mathcal{H}^{p,q} \cong H^{p,q}_{\bar\partial}$$

Proof. The proof is the same as theorem 1.2.5, with $\bar\partial$ instead of d. □

1.2.2 Hodge theory on Kähler manifolds

Let us first recall the definition of Kähler manifold. Let X be a complex manifold of dimension n with an hermitian metric h. We can define a differential 2-form $\omega \in \mathcal{A}^{1,1}(X)$ by:

$$\omega = -\Im(h)$$

where $\Im$ denotes the imaginary part of h. In local coordinates $\{z_1, \ldots, z_n\}$ this form can be written as

$$\omega = -\frac{i}{2}\sum_{j,k=1}^{n} h_{jk} dz_j \wedge d\bar z_k, \qquad h_{jk} = \overline{h_{kj}}$$

and one can show (see [25]) that $\omega = \bar{\omega}$, i.e. that ω is real. If ω is closed we call h the Kähler metric and we say that X is Kähler .

We now give some examples of Kähler manifolds

Example 1.2.
- $\mathbb{C}^n$ with the standard hermitian metric is a Kähler manifold.
- Consider $\mathbb{P}^n(\mathbb{C})$, with the usual holomorphic atlas (U_j, ψ_j), $\jmath = 0, \ldots, n$ where $U_j = \{[z_0, \ldots, z_n] \in \mathbb{P}^n(\mathbb{C}) \mid z_j \neq 0\} \cong \mathbb{C}^n$ by $\psi_j : [z_0, \ldots, z_n] \mapsto (z_0/z_j, \ldots, \hat{1}, \ldots z_n/z_j)$. Let also $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n(\mathbb{C})$ be the canonical projection $\pi(z_0, \ldots, z_n) = [z_0, \ldots, z_n]$. It also induces maps

$$\phi_j : \mathbb{P}^n(\mathbb{C}) \to \mathbb{C}^{n+1}, \quad [z_0, \ldots, z_n] \mapsto (z_0/z_j, \ldots, 1, \ldots z_n/z_j)$$

such that on $U_j \cap U_k$ we have $\phi_j = ([z_j/z_k \phi_k)$. We have thus constructed a line bundle L which has the ϕ_j as sections.

Definition 1.6. *Let us denote the dual of L by $\mathcal{O}_{\mathbb{P}^n}(1)$*

Now let h be the standard hermitian metric on $\mathbb{C}^{n+1}$. By restriction, the inclusion of vector bundles $L \subset \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1}$ gives a hermitian metric on L, as well as on its dual $\mathcal{O}_{\mathbb{P}^n}(1)$. The real closed (1,1) form associated to h^* is, by definition, equal to

$$\frac{1}{2\pi i}\partial\bar{\partial} log h^*(\phi_i^*), \qquad \text{on } U_i,$$

where ϕ_i^* is the section dual to ϕ_i in U_i. We have $h^*(\phi_i^*) = \frac{1}{h(\phi_i)}$. Finally, the identification $U_i \cong \mathbb{C}^n$, the sections ϕ of L, which we can consider as holomorphic $\mathbb{C}^{n+1}$-valued map, is given by $\phi_i(z_0, \ldots, z_n) = (z_0, \ldots, 1, \ldots z_n)$, where 1 is in the $i - th$ position. We thus obtain

$$h(\phi_i) = 1 + \sum_i \mid z_i \mid^2,$$

and

$$\omega_i := \frac{1}{2\pi i}\partial\bar{\partial} log \frac{1}{1 + \sum_i \mid z_i \mid^2}.$$

By gluing the ω_i we obtain a form ω on $\mathbb{P}^n(\mathbb{C})$ which is positive. The associated Kähler metric is called the Fubini-Study metric.

- As every submanifold of a Kähler manifold is Kähler , every complex projective manifold is Kähler

Kähler identities and proportionality

Definition 1.7. *Let (X, ω) be a n-dimensional, compact Kähler manifold. We define the Lefschetz operator as*

$$L : \mathcal{A}^k(X) \to \mathcal{A}^{k+2}(X)$$

$$\alpha \mapsto \omega \wedge \alpha$$

We also denote with $\Lambda : \mathcal{A}^k(X) \to \mathcal{A}^{k-2}(X)$ its formal adjoint relative to the L^2 inner product and one verifies that

$$\Lambda = (-1)^k * L_\omega *$$

We have the following commutation relations of L and Λ with d, ∂ and $\bar{\partial}$: the first two are known as Kähler identities. See [25] for the proof.

Proposition 1.2.9. ?? *Let (X, ω) be a n-dimensional, compact Kähler manifold. Then the following identities hold:*

(i) $[\partial, L] = [\bar{\partial}, L] = [\partial^*, \Lambda] = [\bar{\partial}^*, \Lambda] = 0.$

(ii) $[\bar{\partial}^*, L] = i\partial, \quad [\partial^*, L] = -i\bar{\partial}, \quad [\bar{\partial}, \Lambda] = i\partial^*, \quad [\partial, \Lambda] = -i\bar{\partial}^*.$

(iii) L, Λ *commutes with* Δ_d.

(iv) $[L, \Lambda] = (k - n)Id$ *on* $\mathcal{A}^k(X)$.

A remarkable consequence of the Kähler identities is the fact that on a complex Kähler manifold , the laplacians $\Delta_d, \Delta_{\bar{\partial}}$ and Δ_∂ are multipliples of each other:

Theorem 1.2.10 (Proportionality)**.** *Let X be a compact Kähler manifold. Then:*

$$\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_\partial$$

Proof. We have

$$\begin{aligned}\Delta_d &= (\partial+\bar\partial)(\partial^*+\bar\partial^*)+(\partial^*+\bar\partial^*) = \\ &= (\partial+\bar\partial) = \Delta_\partial+\Delta_{\bar\partial}+\bar\partial^*\partial+\partial\bar\partial^*+\bar\partial\partial^*+\partial^*\bar\partial\end{aligned}$$

It suffices to show that

(i) $\bar\partial^*\partial+\partial\bar\partial^*+\bar\partial\partial^*+\partial^*\bar\partial$

(ii) $\Delta_\partial=\Delta_{\bar\partial}$

To show (i) we use the identity $[\Lambda,\partial]=i\bar\partial^*$ to obtain

$$i(\bar\partial^*\partial+\partial\bar\partial^*)=\partial[\Lambda,\partial]+[\Lambda,\partial]\partial=\partial\Lambda\partial-\partial\Lambda\partial=0.$$

By complex conjugation we find $\bar\partial\partial^*+\partial^*\bar\partial=0$ as well, so (i) is proved. Now we have to show (ii):

$$\begin{aligned}\Delta_\partial &= \partial\partial^*+\partial^*\partial-i\partial[\Lambda,\bar\partial]+i\partial[\Lambda,\bar\partial] \\ &= i(\partial\Lambda\bar\partial-\partial\bar\partial\Lambda+\Lambda\bar\partial\partial-\bar\partial\Lambda\partial) \\ &= i(\Lambda(\partial\bar\partial+\bar\partial\partial)+(\partial\bar\partial+\bar\partial\partial)\Lambda-i(\bar\partial\bar\partial^*+\bar\partial^*\bar\partial)) \\ &= \bar\partial\bar\partial^*+\bar\partial^*\bar\partial=\Delta_{\bar\partial}\end{aligned}$$

□

Hodge decomposition theorem

Proportionality leads to one of the main results of Hodge theory: the Hodge decomposition.

Theorem 1.2.11. *Let X be a compact Kähler manifold. We have a decomposition*

$$H^k(X,\mathbb{C})\cong\bigoplus_{p+q=k}H^{p,q}(X)$$

such that $\overline{H^{p,q}}=H^{q,p}$, where the complex conjugation comes from the isomorphism $H^k(X,\mathbb{C})\cong H^k(X,\mathbb{R})\otimes\mathbb{C}$.

Proof. Put $H^{p,q} := \mathcal{H}^{p,q}$. We have seen that

(i) $H^k(X, \mathbb{C}) \cong \mathcal{H}^k$

(ii) $H^{p,q}_{\bar{\partial}} \cong \mathcal{H}^{p,q}$

Hence it suffices to show that $\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$. Given $\alpha \in \mathcal{H}^k \subset \mathcal{A}^k(X)$, write $\alpha = \sum_{p,q} \alpha_{p,q}$ with $\alpha_{p,q} \in \mathcal{A}^{p,q}(X)$. As $\Delta_d(\mathcal{A}^{p,q}(X)) \subset \mathcal{A}^{p,q}(X)$, proportionality tells us that the components $\alpha_{p,q}$ are harmonic. In fact

$$0 = \Delta_d \alpha = 2\Delta_{\bar{\partial}}\alpha = \sum_{p,q} \Delta_{\bar{\partial}}\alpha_{p,q} \Longrightarrow \Delta_{\bar{\partial}}\alpha_{p,q} = 0 \quad \forall p, q.$$

This shows that the map

$$i : \mathcal{A}^k(X) \to \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X)$$
$$\alpha \mapsto (\alpha_{k,0}, \alpha_{k-1,1}, \ldots, \alpha_{0,k})$$

induces an injective map

$$i : \mathcal{H}^k \to \bigoplus_{p+q=k} \mathcal{H}^{p,q}.$$

Given $\beta \in \mathcal{H}^{p,q}$, we have $0 = \Delta_{\bar{\partial}}\beta = \frac{1}{2}\Delta_d\beta$, i.e. $\beta \in \mathcal{H}^k$ and i is surjective.
Furthermore the complex conjugation on $\mathcal{A}^k(X) \cong \mathcal{A}^k_{\mathbb{R}} \otimes \mathbb{C}$ induces a $\overline{\mathbb{C}}$-linear isomorphism $\mathcal{A}^{p,q}(X) \cong \mathcal{A}^{q,p}(X)$. As $\Delta_{\bar{\partial}} = \frac{1}{2}\Delta_d$ commutes with complex conjugation, we obtain an induced $\overline{\mathbb{C}}$-linear isomorphism

$$\mathcal{H}^{p,q} \cong \mathcal{H}^{q,p} \Longrightarrow H^{p,q} \cong \overline{H^{q,p}}.$$

□

Proposition 1.2.12. *The Hodge decomposition does not depend on the choice of a Kähler metric on X.*

Proof. From the proof of theorem 1.2.5 it is clear that the isomorphism $\mathcal{H}^k \cong H^k_{dR}(X, \mathbb{C})$ does not depend on the metric, however it is not so for the isomorphisms $\mathcal{H}^{p,q} \cong H^{p,q}$. We shall identify every space $\mathcal{H}^{p,q}$ with a subspace of $H^{p+q}_{dR}(X, \mathbb{C})$. Set

$$C^{p,q}(X) := \{[\alpha]_{dR} \in H^{p+q}_{dR}(X, \mathbb{C}) \mid \alpha \in \mathcal{A}^{p,q}(X), d\alpha = 0\}.$$

As $\ker \Delta_d = \ker d \cap \ker d^*$, there is a natural injective map:

$$i : \mathcal{H}^{p,q} \to C^{p,q}(X)$$
$$\alpha \mapsto [\alpha]_{dR}$$

To show that i is surjective, we choose $\alpha \in C^{p,q}(X)$ and thanks to Hodge theorem, we can write it as $\alpha = \beta + \Delta_d \gamma$ with β harmonic. As $\Delta_d = 2\Delta_{\bar{\partial}}$ preserves types, we obtain

$$\alpha = \beta_{p,q} + \Delta_d \gamma_{p,q}.$$

By applying d to the equation we get

$$0 = d\alpha = d\Delta_d \gamma_{p,q} = dd^* d\gamma_{p,q}.$$

Again, as $Imd^* \cap \ker d = 0$, we gave $d^* d\gamma_{p,q} = 0$ and so

$$\alpha = \beta + d(d^*\gamma)$$

and $[\alpha]_{dR} = [\beta]_{dR} \in C^{p,q}(X)$. This shows that i is an isomorphism and since $C^{p,q}$ does not depend on the choice of the metric neither does i. □

Corollary 1.2.13. *If a cohomology class $c \in H^k(X, \mathbb{C})$ is represented by closed forms $\alpha_{p,q}$ and $\alpha_{r,s}$ such that $(p, q) \neq (r, s)$ then $c = 0$.*

Proof. As $C^{p,q} \cong \mathcal{H}^{p,q}(X)$, we have $C^{p,q} \cap C^{r,s} = 0$. □

Another useful consequence of Hodge decomposition is the following:

Corollary 1.2.14. *The odd Betti number $b_{2k+1} = dim_{\mathbb{C}} H^{2k+1}$ are even.*

Example 1.3. A Hopf surface is the quotient of $\mathbb{C}^2 - \{(0, 0)\}$ by the equivalence relation $(z_1, z_2) \simeq \frac{1}{2}(z_1, z_2)$. It is diffeomorphic to $S^3 \times S^1$, and its Betti numbers are $1, 1, 0, 1, 1$. By the above corollary, we see that Hopf surfaces are not Kähler .

Corollary 1.2.15. *The cup product $\cup : H^k(X, \mathbb{C}) \otimes H^l(X, \mathbb{C}) \to H^{k+l}(X, \mathbb{C})$ is bigraded for the bigraduation given by the Hodge decomposition*

Proof. If α is a closed form of type (p,q) and β is a closed form of type (p',q') then $\alpha\wedge\beta$ is a closed form of type $(p+p',q+q')$. □

Corollary 1.2.16 ($\partial\bar\partial$-lemma)**.** *Let $\alpha\in\mathcal{A}^{p,q}(X)$ such that $\partial\alpha=\bar\partial\alpha=0$. Then if α is ∂ or $\bar\partial$-exact, there exists a form β such that $\alpha=\partial\bar\partial\beta$.*

Proof. For istance we can take ω $\bar\partial$-exact, $\omega=\bar\partial\alpha$. By Hodge theorem, we can write $\alpha=\beta+\Delta_d\gamma$ with β-harmonic. As $\Delta_d=2\Delta_{\bar\partial}$, we have $\bar\partial\beta=0$. Further, we noticed in the proof of **??** that $\bar\partial\partial^*=-\partial^*\bar\partial$. Thus,

$$\omega=2\bar\partial(\partial\bar\partial^*+\partial\partial^*)\gamma=-2\partial^*(\partial\bar\partial\gamma)+2\bar\partial\partial\partial^*\gamma.$$

As both ω and $2\bar\partial\partial\partial^*\gamma=-2\partial\bar\partial\partial^*$ are ∂-closed, it follows that $\partial^*(\bar\partial\partial\gamma)$ is also. However, $\partial^*(\bar\partial\partial\gamma)\in Im\partial^*$, then it must be 0. So we obtain

$$\omega=\bar\partial\partial(\partial^*\gamma).$$

□

1.2.3 Hard Lefschetz theorem

The commutation relation between L and Λ has the following consequence.

Lemma 1.2.17. *Let X be a compact Kähler manifold of dimension n with Kähler form ω. The morphism*

$$L^k:\mathcal{A}^{n-k}_{\mathbb{R}}(X)\to\mathcal{A}^{n+k}_{\mathbb{R}}(X)$$
$$\alpha\mapsto\omega^k\wedge\alpha$$

is an isomorphism.

Proof. See [25]. □

Definition 1.8. *We say that a form α is primitive if and only if* $\ker(L^{n-k+1})$.

As ω is a closed $(1,1)$ real form, L^k respects bigraduation and also it induces morphisms in cohomology:

$$L^k : H^{n-k}(X,\mathbb{C}) \to H^{n+k}(X,\mathbb{C}) \tag{1.2}$$

$$L^k : H^{n-p,n-q}(X,\mathbb{C}) \to H^{n-p+k,n-q+k}(X,\mathbb{C}) \tag{1.3}$$

Theorem 1.2.18. *In the previous hypotheses*

(i)

$$H^k(X,\mathbb{C}) = \bigoplus_{r\geq max(k-n,0)} L^r H^{k-2r}_{pr}(X,\mathbb{C})$$

$$H^{p,q}(X) = \bigoplus_{r\geq max(p+q-n,0)} L^r H^{n-p+k,n-q+k}_{pr}(X)$$

(ii) (Hard Lefschetz) The morphisms 1.2 *and* 1.3 *are isomorphisms.*

Remark 4. *Even if the result is stated for cohomology with complex coefficients, it also hold for $H^*(X,\mathbb{R})$ because ω is a $(1,1)$-real form. Further if ω is integral, the result is valid for $H^*(X,\mathbb{Q})$ as well.*

An important application of the Lefschetz theorems are the so called Hodge-Riemann bilinear relations. We will meet them again when introduce the concept of polarization of Hodge structure. Define a bilinear form

$$Q : H^k(X,\mathbb{C}) \times H^k(X,\mathbb{C}) \to \mathbb{C}, \qquad Q(u,v) = \int_X u \wedge v \wedge \omega^{n-k} \tag{1.4}$$

By the same argument as in remark 4, Q is real valued on $H^k(X,\mathbb{R})$ and if the the Kähler class is integral, Q takes integral values on $H^k(X,\mathbb{Z})$. Also, Hard Lefschetz theorem implies that it is non-degenerate.

Proposition 1.2.19. *The bilinear form Q has the following properties, which are called the Hodge Riemann bilinear relations:*

(i) Q is symmetric when k- is even and antisymmetric when k is odd;

(ii) The Hodge decomposition is orthogonal respect to Q, i.e. if $u \in H^{p,q}(X)$ and $v \in H^{r,s}(X)$ then $Q(u,v) = 0$ if $(q,p) \neq (r,s)$;

(iii) If $u \in H^{p,q}_{pr}(X)$ *and* $u \neq 0$*, then*

$$(-1)^{k(k-1)/2} i^{p-q} Q(u, \bar{u}) > 0$$

1.3 Cohomology of an algebraic variety

A complex algebraic variety X carries two natural topologies: the Euclidean (or classical), given by the embedding in $\mathbb{C}^n$ (if affine) or $\mathbb{P}^n(\mathbb{C})$ (if quasi-projective), and the Zariski topology, defined by the property that the closed subsets are the algebraic subsets of X, that is, subsets defined by the vanishing of polinomial functions restricted to X. These sets are closed for the classical topology, which so happens to be stronger. Indeed, Zariski topology is very weak. For example, if X is irreducible, any two open Zariski sets have no empty intersection by analytic continuation. Thus the only continuous sections of the constant sheaf $\mathbb{C}$ are the constant function and so $\mathbb{C}$ is flasque. Hence, by theorem 1.1.2, $H^k(X, \mathbb{C}) = 0$ for all $k > 0$. However, the Zariski topology behaves much more better in computing cohomology on other sheaves, namely coherent sheaves, and, more essentially, it is defined in completely algebraic terms. For instance, if we conjugate a complex algebraic variety by an automorphism of $\mathbb{C}$, a Zariski open set remain open, while this does not happen for the open sets of the classical topology.

1.3.1 Coherent sheaves

Definition 1.9. *Let* $\mathcal{R}$ *be a sheaf of commutative rings on a topological space* X*. A sheaf* $\mathcal{F}$ *of* $\mathcal{R}$*-modules on* X *is locally free of rank* p *if every point* $x \in X$ *has a neighbourhood* U *on which there is a sheaf isomorphism* $\mathcal{F}_{|U} \cong \mathcal{R}^{\oplus p}_{|U}$*.*

Example 1.4. Let $\mathcal{O}_{X_{an}}$ be the sheaf of holomorphic functions over a complex manifold X of dimension n. A sheaf of $\mathcal{O}_{X_{an}}$-modules is also called an analytic sheaf. The sheaf Ω^k of holomorphic k-forms is an analytic sheaf. Indeed, it is locally free of rank $\binom{n}{k}$ with local frame $\{dz_{i_1} \wedge \ldots \wedge dz_{i_k}\}$ with $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Example 1.5. The sheaf $\mathcal{O}^*_{X_{an}}$ of nowhere-vanishing holomorphic functions with pointwise multiplication is not an analytic sheaf, since multiplying a nowwhere vanishing function by the zero function $0 \in \mathcal{O}_{X_{an}}$ will result in a function not in $\mathcal{O}^*_{X_{an}}$.

Let $\mathcal{R}$ and $\mathcal{F}$ as above and let $s_1, \ldots, s_p$ be sections of $\mathcal{F}$ over an open set U in X . For any $r_1 \ldots r_p \in \mathcal{R}(U)$, the map

$$\mathcal{R}^{\oplus p}(U) \to \mathcal{F}(U),$$
$$(r_1 \ldots r_p) \mapsto \sum_i r_i s_i$$

defines a sheaf map $\phi : \mathcal{R}^{\oplus p}_{|U} \to \mathcal{F}_{|U}$ over U. The kernel of ϕ is a subsheaf of $\mathcal{R}^{\oplus p}$ called the sheaf of relations among $s_1, \ldots, s_p$, denoted by $\mathcal{S}(s_1, \ldots, s_p)$. We say that $\mathcal{F}_{|U}$ is generated by $s_1, \ldots, s_p$ if ϕ is surjective over U.

Definition 1.10. *A sheaf $\mathcal{F}$ of $\mathcal{R}$ modules over a topological space X is of finite type if for all $x \in X$ there exists an open neighbourhood U of x and $s_1, \ldots, s_p \in \Gamma(U, \mathcal{F})$ such that for any $y \in U$ the stalk $\mathcal{F}_y$ is generated by $s_1(y), \ldots, s_p(y)$.*

Definition 1.11. *A sheaf $\mathcal{F}$ of $\mathcal{R}$ modules on a topological space X is coherent if*

(i) $\mathcal{F}$ is of finite type,

(ii) for any open set $U \subset X$ and any collection of section $s_1, \ldots, s_p \in \mathcal{F}(U)$, the sheaf of relation $\mathcal{S}(s_1, \ldots, s_p)$ is of finite type over U.

Example 1.6. The sheaf of holomorphic functions $\mathcal{O}_{X_{an}}$ on a complex manifold X is a coherent sheaf of rings, by a classical theorem of Oka.

Example 1.7. The sheaf of regular functions $\mathcal{O}_{X_{alg}}$ over an algebraic variety X is a coherent sheaf of rings[1] We will call algebraic sheaves the sheaves of $\mathcal{O}_{X_{alg}}$ modules. Let $X = \mathbb{A}^n$ (i.e. $\mathbb{C}^n$ endowed with the Zariski topology). Then $\mathcal{O} = \mathcal{O}_{X_{alg}}$ is a coherent sheaf of rings. Given $x \in X$, let U be an open neighbourhood of x and $s_1, \ldots, s_p \in \Gamma(U, \mathcal{O}_{X_{alg}})$. Up to restricting U to a smaller neighbourhood V of x, we can write $s_i = \dfrac{P_i}{Q_i}$ with $P, Q \in \mathbb{C}[x_1, \ldots, x_n]$, $\quad Q \neq 0$

[1]If $\mathcal{R}$ is a sheaf of rings, we can see it as a sheaf of $\mathcal{R}$-modules and ask whether it is coherent or not. By Hilbert's Nullstellensatz, $\mathcal{R}$ is of finite type, so it will be coherent if and only if any sheaf of relation $\mathcal{S}(s_1, \ldots, s_p)$ over an open set U is of finite type.

in any point of U. Let $y \in U$ and $r_i \in \mathcal{O}$ such that $\sum_{i=1}^{p} r_i s_i = 0$ in a neighbourhood of y. We can also write r_i in the form $r_i = \dfrac{R_i}{T_i}$, $T \neq 0$. Thus, in a neighbourhood of y the relation $\sum_{i=1}^{p} r_i s_i = 0$ is equivalent to $\sum_{i=1}^{p} R_i P_i = 0$. Since $\mathbb{C}[x_1, \dots, x_n]$ is noetherian, $S(P_1, \dots, P_n)$ is finitely generated; then $\mathcal{S}(s_1, \dots, s_p)$ is of finite type.

Example 1.8. If X is a smooth algebraic variety the sheaf Ω^k_{alg} of algebraic k-forms is an algebraic sheaf, locally free of rank $\binom{n}{k}$, where $n = dimX$

Remark 5. *We know that given a complex manifold every Zariski-open set is also Euclidean-open. Even if we consider $\mathcal{O}_{X_{alg}}$ and $\mathcal{O}_{X_{an}}$ only on Zariski-open sets, these sheaves are far from being equal: for example if we consider $\mathbb{A}^1$, we notice that the function $e^z \in \mathcal{O}_{X_{an}}$ but it is not in $\mathcal{O}_{X_{alg}}$.*

In the following sections we will work with complex algebraic varieties. We will write $(X_{an}, \mathcal{O}_{X_{an}})$ when we consider X as a complex manifold with the classical topology, $(X_{alg}, \mathcal{O}_{X_{alg}})$ when we consider X as an algebraic variety with the Zariski topology. We would like to find a connection between the cohomology of these two structures. This leads to use a more general notion of cohomology, that is hypercohomology. Hypercohomology is just the cohomology of a complex instead of the cohomology of a sheaf. Cohomology is the special case of hypercohomology in which a sheaf is considered as a complex concentrated only in degree 0. Before stating some important theorems on hypercohomology and give some examples, we introduce the notion on spectral sequence, which will be useful in proving the algebraic de Rham theorem and other crucial results in Hodge theory. We refer to [10] and [25] for proofs and further details.

1.3.2 Spectral sequences of a filtered complex

Let $\mathcal{C}$ be an abelian category and $(K^\bullet, d)$ a complex in $\mathcal{C}$. A filtration $F^\cdot$ of the complex is defined by a family of subobjects $F^i \subset K^j$ satisfying $d_j(F^i K^j) \subset F^i K^{j+1}$. A complex $K^\bullet$ endowed with a filtration $F^\bullet$ is called a filtered complex. We consider decreasing filtrations $F^{i+1} \subset F^i$. In the case of an increasing filtration W_i, we obtain the terms of the spectral sequence from the decreasing case by a change of the sign of the indices of the filtration which transforms the increasing filtration into a decreasing one F with $F^i = W_{-i}$.

Definition 1.12. *Let $K^\bullet$ be a complex of objects of an abelian category $\mathcal{C}$, with a decreasing filtration by subcomplexes $F^\bullet$. It induces a filtration $F^\bullet$ on the cohomology $H^*(K^\bullet)$, defined by:*

$$F^iH^j(K^\bullet) := Im\{H^j(F^i(K^\bullet)) \to H^j(K^\bullet)\}, \qquad i, j \in \mathbb{Z}$$

Let F^iK/F^jK for $i < j$ denote the complex $(F^iK^r/F^jK^r, d_r)_{r\in\mathbb{Z}}$ with induced filtration; in particular we set $Gr_F^p(K) := F^pK/F^{p+1}K$. The associated graded object is $Gr_F^* := \bigoplus_{p\in\mathbb{Z}} Gr_F^p(K)$. Similarly, we define $Gr_F^iH^j(K)$ and:

$$Gr_F^*H^j(K) := \bigoplus_{i\in\mathbb{Z}} Gr_F H^j(K) = \bigoplus_{j\in\mathbb{Z}} F^iH^j(K)/F^{i+1}H^j(K)$$

Though at a first glance spectral sequences might seem unpleasant, they give a method to compute the graded object $Gr_F H^*(K)$ out of the cohomology $H^*(F^iK/F^jK)$ for various indices $i > j$ of the filtration. A spectral sequence of a filtered complex $(K^\bullet, d, F^\bullet)$ consists of :

1. indexed objects of $\mathcal{C}$ $E_r^{p,q}$, $r > 0$, $p, q \in \mathbb{Z}$ such that $E_0^{p,q} = Gr_F^p(K^{p+q})$;

2. differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, such that $d_r \circ d_{r+1} = 0$, and d_0 is induced by d;

3. isomorphisms:
$$E_{r+1}^{p,q} \cong \frac{Ker\{d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}\}}{Im\{d_r : E^{p-r,q+r-1} \to E_r^{p,2}\}}$$

4. We also require that for $p + q$ and r sufficiently large, we have:
$$E_r^{p,q} =: E_\infty^{p,q} = Gr_F^p H^{p+q}(K).$$

The aim of the spectral sequence is to compute the term $E_\infty^{p,q}$, which is called the limit of the spectral sequence.

Definition 1.13. *We say that a spectral sequence degenerates at E_{r_0} if the differential d_r of $E_r^{p,q}$ vanish for all $r \geq r_0$.*

The existence of such a sequence is of course not obvious. We briefly describe the construction of the terms and refer to [25] for the proof of the theorem. Set

$$Z_r^{p,q} = \{x \in F^p K^{p+q} \mid dx \in F^{p+r} K^{p+q+1}\}.$$

$Z_r^{p,q}$ naturally contains $Z_{r-1}^{p+1,q-1}$ and $dZ_{r-1}^{p-r+1,q+r-2}$. Let

$$B_r^{p,q} := Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2} \subset Z_r^{p,q}.$$

We set

$$E_r^{p,q} := Z_r^{p,q} / B_r^{p,q}$$

As d sends $Z_r^{p,q}$ to $Z_r^{p+r,q-r+1}$, and $B_r^{p,q}$ to $B_r^{p+r,q-r+1}$, we have a differential

$$d_r : E_r^{p,q} \to E^{p+r,q-r+1}$$

which clearly satisfies $d_r^2 = 0$ since it is induced by d.

Spectral sequences of the simple complex associated to a double complex

In the case of the filtration of the simple complex $(K^\bullet, d)$ associated to a double complex $(K^{\cdot,\cdot}, \partial, \bar\partial)$ spectral sequences come in a particularly simple form.

Proposition 1.3.1. *Let $K^\bullet$ be the simple complex associated to a double complex $(K^{\cdot,\cdot}, \partial, \bar\partial)$. There exists two natural decreasing filtrations $F'^\bullet, F''^\bullet$ defined by:*

$$F'^p(K^n) = \bigoplus_{r \geq p, r+s=n} K^{r,s}$$

$$F''^p(K^n) = \bigoplus_{r \geq p, r+s=n} K^{r,s}$$

and the spectral sequences associated to them have first terms given by:

- ${}'E_0^{p,q} = {}'' E_0^{p,q} = K^{p,q}$, $d_0' = (-1)^p \bar\partial$ *and* $d_0'' = \partial$
- ${}'E_1^{p,q} = H^q(K^{p,\bullet}, \bar\partial)$, ${}''E_1^{p,q} = H^q(K^{p,\bullet}, \partial)$.
  The differential $d_1' : H^q(K^{p,\bullet}) \to H^q(K^{p+1,\bullet})$ is induced by the morphism of complexes

$$\partial : K^{p,\bullet} \to K^{p+1,\bullet},$$

while $d_1'': H^q(K^{\bullet,p}) \to H^q(K^{\bullet,p+1})$ *is induced by the morphism of complexes*

$$\bar\partial : K^{\bullet,p} \to K^{\bullet,p+1},$$

- $'E_2^{p,q} := H^p[H^q(K^{\bullet,\bullet}, \bar\partial), \partial], \quad ''E_2^{p,q} := H^p[H^q(K^{\bullet,\bullet}, \partial), \bar\partial]$

Example 1.9. Let X be a complex manifold of dimension n. Now, consider the double complex given by complex differential forms $(\mathcal{A}^{p,q}, \partial, \bar\partial)$. We define the de Rham complex as the simple complex associated to it. On this complex we have the filtration:

$$F^p\mathcal{A}^k = \bigoplus_{r\geq p, r+s=k} \mathcal{A}^{r,s}.$$

The spectral sequence associated to this complex is called the Frölicher spectral sequence. The first term is quite easy to compute: in fact, if we apply proposition 1.3.1, we deduce

$$E_1^{p,q} = H^q(\mathcal{A}^{p,\bullet}, \bar\partial) = H_{\bar\partial}^{p,q}.$$

This spectral sequence satisfies a crucial property in the case of compact Kähler manifolds.

Theorem 1.3.2. *Let X be a compact Kähler manifold. Then the Frölicher spectral sequence degenerates at E_1*

1.3.3 Hypercohomology

To define hypercohomology of a complex $\mathcal{M}^\bullet$ of sheaves of abelian groups over a topological space X, we first construct the double complex of global sections of the Godement resolution of the sheaves $\mathcal{M}^q$:

$$K = \bigoplus_{p+q=k} K^{p,q} := \bigoplus_{p+q=k} \Gamma(X, \mathcal{C}^p\mathcal{M}^q). \tag{1.5}$$

This double complex comes with two differentials:

horizontal $\partial : K^{p,q} \to K^{p+1,q}$, coming from the Godement resolution

vertical $\bar\partial : K^{p,q} \to K^{p,q+1}$, coming from the complex

Since the differential $d : \mathcal{M}^q \to \mathcal{M}^{q+1}$ induces a morphism of complexes $\mathcal{C}^\bullet\mathcal{M}^q \to \mathcal{C}^\bullet\mathcal{M}^{q+1}$ where $\mathcal{C}$ is the Godement resolution, ∂ and $\bar\partial$ commutes.

Definition 1.14. *We define the i^{th} hypercohomology group as the i^{th} cohomology group of the simple complex $(K^\bullet, \partial + (-1)^p\bar{\partial})$ associated to the double complex $K^{p,q}$ (we set $K^l = \bigoplus_{p+q=l} K^{p,q}$).*

$$\mathbb{H}^i(X, \mathcal{M}^\bullet) := H^i(K^\bullet)$$

Remark 6. *The cohomology of a single sheaf $\mathcal{F}$ can be seen as a special case of hypercohomology. Indeed, we can consider $\mathcal{F}$ as a complex $\mathcal{F}^\bullet$ which is $\mathcal{F}$ in degree 0 and 0 elsewhere. Hence, the double complex $K^\bullet = \bigoplus \Gamma(X, \mathcal{C}^p\mathcal{F}^q)$ has nonzero entries only in degree 0. Thus*

$$\mathbb{H}^i(X, \mathcal{F}^\bullet) = H^i(\Gamma(X, \mathcal{C}^\bullet\mathcal{F})) = H^k(X, \mathcal{F})$$

Spectral sequences of hypercohomology

Given a double complex $(K^{\bullet,\bullet}, \bar{\partial}, \partial)$ with commuting differential $\bar{\partial}, \partial$, we have the two spectral sequences defined in 1.3.1. Fix a nonnegative integer p and let $T = \Gamma(X, \mathcal{C}^p(\cdot))$ be the Godement sections functor that associates to a sheaf $\mathcal{F}$ on a topological space X the group of sections $\Gamma(X, \mathcal{C}^p\mathcal{F})$ of the Godement sheaf $\mathcal{C}^p\mathcal{F}$. Since T is an exact functor, it commutes with cohomology:

$$H^q(T(\mathcal{M}^\bullet)) = T(H^q(M^\bullet)).$$

For the double complex defined in 1.5, the $'E_1$ term of the first spectral sequence is, by proposition 1.3.1, the cohomology of K with respect to the vertical differential $\bar{\partial}$. Thus, $'E_1^{p,q} = H_{\bar{\partial}}^{p,q}$ is the q-th cohomology of the p-th column $K^{p,\bullet} = \Gamma(X, \mathcal{C}^p\mathcal{M}^\bullet)$ of K:

$$\begin{aligned} 'E_1^{p,q} = H_{\bar{\partial}}^{p,q} &= H^q(\Gamma(X, \mathcal{C}^p\mathcal{M}^\bullet)) \\ &= H^q(T(\mathcal{M}^\bullet)) && \text{(definition of T)} \\ &= T(H^q(\mathcal{M}^\bullet)) && \text{(T is exact)} \\ &= \Gamma(X, \mathcal{C}^p H^q(\mathcal{M}^\bullet)) && \text{(definition of T).} \end{aligned}$$

Hence, the term $'E_2$ is

$$'E_2^{p,q} = H^p(H^q(K^{\bullet,\bullet}, \bar{\partial}), \partial) = H^p((H^{\bullet,q}, \bar{\partial}), \partial) = H^p((\Gamma(X, \mathcal{C}^\bullet H^q)) = H^p(X, H^q). \tag{1.6}$$

We then notice that the q^{th} row of the double complex $\bigoplus K^{p,q} = \bigoplus \Gamma(X, \mathcal{C}^p\mathcal{M}^q)$ calculates the sheaf cohomology of $\mathcal{M}^q$ on X. Thus

$$''E_1^{p,q} = H_{\partial}^{p,q} = H^p(K^{\bullet,q}) = H^p(\Gamma(X, \mathcal{C}^\bullet\mathcal{M}^q)) = H^p(X, \mathcal{M}^q)$$

and the second term is

$$''E_2^{p,q} = H^q(H^p(K^{\bullet,\bullet}, \partial), \bar{\partial}) = H^q((H^{p,\bullet}, \partial), \bar{\partial}) = H^q(H^p(X, \mathcal{M}^\bullet)). \tag{1.7}$$

We now give some important results on hypercohomology.

Theorem 1.3.3. *Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes of sheaves of abelian groups over a topological space X. If $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are quasi-isomorphic (i.e. there is a morphism $\Phi : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of complexes of sheaves which induces isomorphisms on the cohomology sheaves) then Φ induces an isomorphism on the hypercohomology sheaves and we have*

$$\mathbb{H}^*(X, \mathcal{F}^\bullet) \cong \mathbb{H}^*(X, \mathcal{G}^\bullet).$$

Theorem 1.3.4. *If $\mathcal{M}^\bullet$ is a complex of acyclic sheaves of abelian groups on a topological space X, then the hypercohomology of $\mathcal{M}^\bullet$ is isomorphic to the cohomology of the complex of global sections of $\mathcal{M}^\bullet$:*

$$\mathbb{H}^i(X, \mathcal{M}^\bullet) \cong H^i(X, \mathcal{M}^\bullet)$$

Theorem 1.3.5. *If $0 \to \mathcal{F} \to \mathcal{M}^\bullet$ is an acyclic resolution of $\mathcal{F}$ (i.e. $\mathcal{M}^q$ is Γ-acyclic for any q),then the cohomology of $\mathcal{F}$ can be computed from the complex of global sections of $\mathcal{M}^\bullet$:*

$$H^k(X, \mathcal{F}) = H^k(X, \mathcal{M}^\bullet).$$

This theorem tells us that when we compute the cohomology of a sheaf $\mathcal{F}$, we can take any acyclic resolution of $\mathcal{F}$ instead of the Godement resolution.

1.3.4 Analytic and algebraic de Rham theorems

The analytic and algebraic de Rham theorems are the analogues of the classical de Rham theorem for respectively complex manifolds and algebraic varieties. The analytic de Rham theorem states that the cohomology of the constant sheaf $\mathbb{C}$ can be computed from the holomorphic forms. Thanks to the holomorphic Poincaré Lemma, this theorem is far easier to prove than the algebraic counterpart.

Lemma 1.3.6 (Holomorphic Poincaré Lemma). *Let X be a complex manifold of dimension n and set $\Omega^k_{an} := \mathcal{A}^k \cap \ker \bar{\partial}$ the sheaf of holomorphic k-forms on X. Then the sequence*

$$0 \xrightarrow{i} \mathbb{C} \xrightarrow{\partial} \mathcal{O}_{X_{an}} \to \Omega^1_{an} \xrightarrow{\partial} \dots \xrightarrow{\partial} \Omega^n_{an} \to 0$$

is exact. In other words the complex $\Omega^\bullet_{an}$ is, via i, a resolution of the constant sheaf $\mathbb{C}$.

Proof. We want to show that the sheaves of cohomology $H^k = H^k(\Omega^\bullet_{an})$ satisfy $H^0 = i(\mathbb{C})$ and $H^k = 0$ for $k > 0$. Now, we have the inclusion of the holomorphic de Rham complex into the de Rham complex of $\mathbb{C}$-valued differential forms:

$$(\Omega^k_{an}, \partial) \to (\mathcal{A}^k, d)$$

since d and ∂ coincide on holomorphic forms. Moreover we can see the usual de Rham complex as the simple complex associated to the double complex $(\mathcal{A}^{p,q}, \partial, (-1)^p\bar{\partial})$. By Poincaré lemma, we now that

$$H^k(\mathcal{A}^\bullet) = \begin{cases} \mathbb{C} & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$

By $\bar{\partial}$-Poincaré lemma, each column $(\mathcal{A}^{p,\bullet}, (-1)^p\bar{\partial})$ gives a resolution of Ω^p_{an} and so the $'E_1$ term of the spectral sequence of $(\mathcal{A}^{p,q}, \partial, (-1)^p\bar{\partial})$ is given by:

$$'E_1^{0,0} = \mathcal{O}_{X_{an}} = \Omega^0_{an} \qquad 'E_1^{p,0} = \Omega^p_{an} \qquad 'E_1^{p,q} = 0 \text{ otherwise.}$$

Hence, the E_2 term is given by

$$'E_2^{p,q} = \begin{cases} H^p(\Omega^\bullet_{an}) & \text{for } q = 0 \\ 0 & \text{for } q > 0 \end{cases}$$

Since the spectral sequence degenerates at $'E_2$,

$$H^k(\Omega^\bullet_{an}) =' E_2 =' E_\infty \cong H^k(\mathcal{A}^\bullet) = \begin{cases} \mathbb{C} & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$

□

Corollary 1.3.7 (Analytic de Rham theorem)**.** *Let X be a complex manifold. Then*

$$H^k(X, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^\bullet_{an}).$$

Proof. By the holomorphic Poincaré lemma, we have a quasi-isomorphism between the $\mathbb{C}^\bullet$ ($\mathbb{C}$ in degree 0, 0 elsewhere) and $\Omega^\bullet_{an}$. This induces, by theorem 1.3.3, an isomorphism in hypercohomology:

$$\mathbb{H}^k(X, \mathbb{C}) \cong \mathbb{H}(X, \Omega^\bullet_{an}).$$

Also, by remark 6, we have $\mathbb{H}^k(X, \mathbb{C}) = H^k(X, \mathbb{C})$ and so we have finished. □

Let now X be a nonsingular quasi-projective variety defined over a field K of characteristic zero. One has the morphism $X \to Spec(\mathbb{C})$ and the sheaf of Kähler (or algebraic) differentials $\Omega_{X/K}$ which is a locally free algebraic coherent sheaf on X, locally generated by the differentials df_i where the $f_i a$ are algebraic functions on X defined in a neibourghood of $x \in X$. The relations are given by $da = 0$ for $a \in K$ and the Leibniz rule $d(fg) = fdg + gdf$.
We can form the locally free sheaves $\Omega^l_{X/K} := \bigwedge^l \Omega_{X/K}$ and, by the definition of $\Omega_{X/K}$ and using Leibniz rule, we get the differentials $d : \mathcal{O}_{X_{alg}} \to \Omega_{X/K}$, $d : \Omega^l_{X/K} \to \Omega^{l+1}_{X/K}$. The complex

$$0 \to \mathcal{O}_{X_{alg}} \to \Omega_{X/K} \to \ldots \to \Omega^n_{X/K} \to 0$$

is called the *algebraic de Rham complex* and we denote it by $\Omega^\bullet_{X/K}$. When $K = \mathbb{C}$ we will denote it by $\Omega^\bullet_{alg}$.

The painful aspect of the algebraic de Rham complex is that there is no Poincaré lemma, i.e. the complex

$$0 \to \mathbb{C} \to \mathcal{O}_{X_{alg}} \to \Omega^1_{alg} \to \Omega^2_{alg} \to \ldots$$

is in general not exact. Also, as we told earlier, the sheaf cohomology groups computed regarding Zariski topology are trivial. Fortunately we can still use hypercohomology.

Definition 1.15. *The algebraic de Rham cohomology of X is defined as the hypercohomology of the algebraic de Rham complex :* $H^l(X/K) := \mathbb{H}^l(X, \Omega^\bullet_{X/K})$

Remark 7. *This definition is compatible with field extensions. In fact given a field extension $K \subseteq L$, we let $X_L = X \times_{Spec(K)} Spec(L)$ denote the variety obtained from X by estensions of scalars. Since $\Omega_{X_L/L} \cong \Omega_{X/K} \otimes_K L$, we obtain $H^i(X_L/L) \cong H^i(X/K) \otimes_K L$.*

Theorem 1.3.8 (Algebraic de Rham theorem)**.** *Let X_{alg} be a nonsingular projective variety over $\mathbb{C}$ and let X_{an} denote the associated complex manifold. Then there is a canonical isomorphism*

$$H^k(X_{alg}/\mathbb{C}) \cong H^k(X_{an}, \mathbb{C})$$

and under this isomorphism, $F^p H^k(X_{alg}/\mathbb{C}) \cong F^p H^k(X_{an}, \mathbb{C})$ gives the Hodge filtration on singular cohomology.

Proof. By the analytic de Rham theorem, we have an isomorphism:

$$H^k(X_{an}, \mathbb{C}) \cong \mathbb{H}^k(X_{an}, \Omega^\bullet_{an}).$$

Also, in the second spectral sequence converging to $\mathbb{H}^*(X_{an}, \Omega^\bullet_{an})$ the $''E_1$ term is, by equation 1.7,

$$''E^{p,q}_{1,an} = H^p(X_{an}, \Omega^q_{an}).$$

Similarly, in the second spectral sequence converging to the hypercohomology $\mathbb{H}^*(X_{alg}, \Omega^\bullet_{alg}) = H^*(X_{alg}/\mathbb{C})$ the $''E_1$ term is

$$''E^{p,q}_{1,alg} = H^p(X_{alg}, \Omega^q_{alg}).$$

By Serre's GAGA principle, we have an isomorphism

$$H^p(X_{an}, \Omega^q_{an}) \cong H^p(X_{alg}, \Omega^q_{alg}).$$

The isomorphism $E_{1,an} \cong E_{1,alg}$ induces an isomorphism in E_∞. Hence,

$$H^*(X_{alg}/\mathbb{C}) = \mathbb{H}^*(X_{alg}, \Omega^\bullet_{alg}) \cong \mathbb{H}^*(X_{an}, \Omega^\bullet_{an}) = H^*(X_{an}, \mathbb{C}).$$

Since the Hodge filtration on $H^i(X_{an}, \mathbb{C})$ is induced by the filtration on the complex $\Omega^\bullet_{an}$

$$F^p\Omega^\bullet_{an} := \Omega^{\geq p}_{an},$$

the second assertion follows by the same argument. □

Chapter 2

Hodge structures

2.1 Hodge structures

Let R be $\mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$.

Definition 2.1 (**Hodge structure**). *A R-Hodge structure of weight k is the datum of*

- *A free abelian module V_R of finite type over R;*
- *a complex vector space V with a decomposition of complex vector spaces*

$$V = \bigoplus_{p+q=k} V^{p,q} \qquad \text{with } V^{p,q} = \overline{V^{q,p}}; \tag{2.1}$$

- *an isomorphism α*

$$V \cong V_{\mathbb{C}} := V_R \otimes \mathbb{C}. \tag{2.2}$$

Given such a decomposition, we define the associated Hodge filtration $F^\bullet V$ by

$$F^p V := \bigoplus_{r \geq p} V^{r,k-r}.$$

It is a decreasing filtration on $V_{\mathbb{C}}$ which satisfies

$$V = F^p V \oplus \overline{F^{k-p+1} V}.$$

The Hodge filtration determines the Hodge decomposition by

$$V^{p,q} = F^pV \cap \overline{F^qV}, \text{ for } p+q=k.$$

This shows that we can equivalently define a Hodge structure either from the Hodge filtration or the decomposition.

Remark 8. *This definition can be simplified by setting $V = V_{\mathbb{C}} := V_R \otimes \mathbb{C}$ and forgetting about the comparison isomorphism. However in practice, the filtration $F^\bullet$ is often not defined on $V_R \otimes \mathbb{C}$ itself, but on an isomorphic vector space. Keeping track of the isomorphism can be useful. Anyway, unless specify otherwise, the isomorphism in* 2.2 *will be the identity.*

Definition 2.2 (**Polarised Hodge structure**)**.** *We say that a R-Hodge structure (V_R, F, Q) is polarised if there exists a bilinear form $Q : V_R \times V_R \to R$ that satisfies the Hodge-Riemann bilinear relations of proposition 1.2.19.*

Definition 2.3. *A morphism of R-Hodge structures $\phi : (V_R, F^\bullet) \to (W_R, F'^\bullet)$ of weight respectively k is a homomorphism of abelian groups $\phi : V_R \to W_R$ such that $\phi_{\mathbb{C}} \otimes \mathbb{C} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ satisfies*

$$\phi_{\mathbb{C}}(F^pV_{\mathbb{C}}) \subseteq F'^pW_{\mathbb{C}}.$$

By a slight abuse of notation, we will also consider morphisms of Hodge structures of type (r, r) between a Hodge structure V_R of weight k and one W_R of weight $k + 2r$, requiring that $\phi_{\mathbb{C}}(F^pV_{\mathbb{C}}) \subseteq F'^{p+r}W_{\mathbb{C}}$.

Definition 2.4. *Let $(V_R, F^\bullet)$ be a Hodge structure of weight k. The dual Hodge structure $(V_R^*, F^\bullet)$ is the Hodge structure of weight $-k$ defined by*

$$V_R^* = Hom_R(V_R, R) \qquad F^pV_{\mathbb{C}}^* = Hom_{\mathbb{C}}(F^pV_{\mathbb{C}}, \mathbb{C}).$$

Let us now give some examples of Hodge structures. A crucial one is given by the Hodge decomposition of the cohomology of a compact Kähler manifold.

When X is a compact Kähler manifold we have the Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

with$H^{p,q} = \overline{H^{q,p}}$. If we set $V = H^k(X,\mathbb{Z})$ (resp. $V = H^k(X,\mathbb{Q})$) we have an integral (resp. rational) Hodge structure of weight k. The associated Hodge filtration has the following property.

Proposition 2.1.1. *Let $F^p\mathcal{A}^k(X)$ be the set of complex differential forms of type $(r, k-r)$ with $r \geq p$ at every point. Then we have*

$$F^pH^k(X,\mathbb{C}) = \frac{Ker\{d : F^p\mathcal{A}^k(X) \to F^p\mathcal{A}^{k+1}(X)\}}{Im\{d : F^p\mathcal{A}^{k-1}(X) \to F^p\mathcal{A}^k(X)\}}.$$

Proof. Consider the projection $Ker\{d : F^p\mathcal{A}^k(X) \to F^p\mathcal{A}^{k+1}(X)\} \to H^k(X,\mathbb{C})$, which to a closed form in $F^p\mathcal{A}^k(X)$ associates its class. The image of this map contains $F^pH^k(X,\mathbb{C})$, which is generated by a closed form of type $(r, k-r)$ with $r \geq p$, since such a form lies in $Ker\{d : F^p\mathcal{A}^k(X) \to F^p\mathcal{A}^{k+1}(X)\}$. Conversely, consider $[\alpha]$ with α closed form $F^p\mathcal{A}^k(X)$. Given a Kähler metric on X we can write $\alpha = \beta + \Delta\gamma$, with β harmonic. Now, as Δ respects bigraduation and the expression above is unique, β belongs to $F^p\mathcal{A}^k(X)$ and we can assume that $\gamma \in F^p\mathcal{A}^k(X)$ as well. As α and β are closed, $d^*d\gamma$ belongs both to $kerd$ amd Imd^*, so it must be 0 and $\alpha = \beta + dd^*\gamma$. Thus $[\alpha] = [\beta]$, but as β is harmonic, also its components of type $(r, k-r)$ for $r \geq p$ are, and the others are 0. Thus

$$[\beta^{r,k-r}] \in H^{r,k-r}(X) \subset F^p(X,\mathbb{C}) \quad \forall r$$

and $\alpha \in F^pH^k(X,\mathbb{C})$.
Now we show that the kernel of this map is exactly

$$Im\{d : F^p\mathcal{A}^{k-1}(X) \to F^p\mathcal{A}^k(X)\}.$$

We use decreasing induction on p. If $p = k$, a closed form of type (k,0) is holomorphic and both $\partial, \bar{\partial}$-closed. If it is exact, by $\partial\bar{\partial}$-lemma it is equal to $\partial\bar{\partial}\eta$, then it is 0 for reasons of type. Assume now that the property is satisfied for $p+1$ and let $\alpha \in \mathcal{A}^k(X)$ be a closed form of class 0. Then its harmonic representative β is zero, thus we have $\alpha = \Delta\gamma$ and also $\alpha^{p,q} = \Delta\gamma^{p,q} = 2\Delta_{\bar{\partial}}\gamma^{p,q}$. As the components $\alpha^{p,q}$ are $\bar{\partial}$-closed and α has no component of type (m, l) with $l > q$, we deduce that $\bar{\partial}^*\bar{\partial} = 0$ and thus that

$$\alpha^{p,q} = 2d\bar{\partial}^*\gamma^{p,q}.$$

Then the form $\alpha' = \alpha - 2\bar{\partial}\bar{\partial}^*\gamma \in F^{p+1}\mathcal{A}^k(X)$ and of class 0. By the induction hypothesis $\alpha' = d\beta', \beta' \in F^{p+1}\mathcal{A}^{k-1}(X)$. As $\bar{\partial}\gamma^{p,q} \in F^p\mathcal{A}^{k-1}$ and $\alpha^{p,q} = 2d\bar{\partial}^*\gamma^{p,q} + d\beta'$ the result is shown also for p. □

Example 2.1 (Hodge structure on $\mathbb{P}^1(\mathbb{C})$). We consider the Hodge structure on $\mathbb{P}^1$. The Hodge decomposition tells us that

$$H^2(\mathbb{P}^1, \mathbb{C}) = H^{2,0}(\mathbb{P}^1) \oplus H^{1,1}(\mathbb{P}^1) \oplus H^{0,2}(\mathbb{P}^1)$$

with $H^{2,0} = \overline{H^{0,2}}$. As $H^2(\mathbb{P}^1, \mathbb{C}) = \mathbb{C}$, $H^{2,0} = H^{0,2} = 0$ because they should have the same dimension. Thus

$$H^2(\mathbb{P}^1, \mathbb{C}) = H^{1,1}(\mathbb{P}^1)$$

and it is generated by the Kähler form $\omega = \frac{1}{2\pi i}\partial\bar{\partial}log\frac{1}{(1+\mid z \mid^2)^2}$, which is an integral class.

Example 2.2. Let R be again $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$.

- **(Trivial Hodge structure)** The trivial Hodge structure $R(0)$ is the Hodge strucure of weight 0 given by

  $$R(0) = R \quad R(0)_{\mathbb{C}} = \mathbb{C}, \qquad F^0 = \mathbb{C} \qquad F^1 = \{0\}.$$

  The only non-trivial Hodge subspace is $R(0)^{0,0} = \mathbb{C}$.

- **(Tate structures)** The Tate R-Hodge structure is defined as follows:

  $$R(1) = 2\pi i R, \quad R(1)_{\mathbb{C}} = 2\pi i\mathbb{C}, \qquad F^{-1} = 2\pi i\mathbb{C}, \quad F^0 = \{0\}$$

  It is a Hodge structure of weight -2 with $R(1)^{-1,-1} = \mathbb{C}$ as unique non-trivial subspace. We then define for any $n \in \mathbb{Z}$

  $$R(n) := R(1)^{\otimes n}$$

  $R(n)$ is a Hodge structure of weight $-2n$ and it has

  $$R(n) = (2\pi i)^n R \quad R(n)_{\mathbb{C}} = (2\pi i)^n\mathbb{C}, \qquad F^{-n} = \mathbb{C} \quad F^{-n+1} = \{0\}$$

  Again, the only non trivial subspace is $R(n)^{-n,-n} = \mathbb{C}$.

- Let X be a compact Kähler manifold. Set $V_{\mathbb{Q}} = H^k(X, \mathbb{Q})_{prim}$. We have seen in the previous example that it carries a polarized Hodge structure of weight k.

- Set $V_{\mathbb{Z}} = H^2(\mathbb{P}^1, \mathbb{Z})$ and $V_{\mathbb{C}} = H^2(\mathbb{P}^1, \mathbb{C})$, $F^1V_{\mathbb{C}} = V_{\mathbb{C}}$, $F^2V_{\mathbb{C}} = \{0\}$. Let α be the inverse of the isomorphism between de Rham and singular cohomology

$$H^2(\mathbb{P}^1, \mathbb{C}) \to H^2(\mathbb{P}^1, \mathbb{Z}) \otimes \mathbb{C} = Hom_{\mathbb{C}}(H_2(\mathbb{P}^1, \mathbb{Z}), \mathbb{C}), \qquad [\omega] \mapsto (\sigma \mapsto \int_\sigma \omega)$$

  This Hodge structure is isomorphic to the so-called Lefschetz structure $\mathbb{Z}(-1)$ and the isomorphism sends the fundamental class $[\mathbb{P}^1]$ to $(2\pi i)^{-1}$. Thus it is of weight 2 and type $(1, 1)$.

- Let $X \subset \mathbb{P}^N$ be a compact projective manifold. The Lefschetz operator $L : H^k(X, \mathbb{C}) \to H^{k+2}(X, \mathbb{C})$ defines a morphism of Hodge structures of type $(1, 1)$. As the Kähler form ω can be written as $\frac{1}{2\pi i}\partial\bar{\partial} logh$ we have that $[\omega] \in H^2(X, \mathbb{Z})(-1)$ and so it is more natural to think of L as a morphism oh Hodge structures between $H^k(X, \mathbb{C}) \to H^{k+2}(X, \mathbb{C})(-1)$.

Remark 9 (Compatibility of Hodge structure with Poincaré duality)**.** *Hodge Tate structures deserve a particular attention. Let X be a compact oriented manifold. In the first chapter we have given an isomorphism between* $H^k_{dR}(X, \mathbb{C}) \cong H^k(X, \mathbb{C})$ *and now we want to examine the Hodge structures we have over them. We define*

$$H^k(X, \mathbb{Z})(n) := H^k(X, \mathbb{Z}) \otimes \mathbb{Z}(n)$$

and thus

$$H^k(X, \mathbb{Z})(n)_{\mathbb{C}} = (2\pi i)^n H^k(X, \mathbb{C})$$

If $dimX = n$*, we have a map in cohomology, the so called the trace map*

$$Tr : H^n(X, \mathbb{C}) \to \mathbb{C}, \qquad [\alpha] \mapsto \int_X \alpha.$$

If we compose this map with the cup product

$$H^k(X, \mathbb{C}) \otimes H^{n-k}(X, \mathbb{C}) \xrightarrow{\cup} H^n(X, \mathbb{C}) \xrightarrow{Tr} \mathbb{C}$$

we obtain the Poincaré duality isomorphism between $H^k(X, \mathbb{C})$ *and* $Hom(H^{n-k}(X, \mathbb{C}), \mathbb{C})$. *Then we define the trace map of Hodge structures:*

$$Tr : H^{2n}(X, \mathbb{C}) \xrightarrow{\cong} \mathbb{C}(-n), \qquad \alpha \mapsto \frac{1}{(2\pi i)^n} \int_X \alpha$$

such that Poincaré duality is compatible with the Hodge structure:

$$H^{n-k}(X, \mathbb{C}) \cong Hom(H^{n+k}(X, \mathbb{C}), \mathbb{C}(-n))$$

where the duality between $H^{p,q}(X)$ *and* $H^{n-p,n-q}$ *corresponds to Serre duality. The Hodge structure on homology is defined by duality:*

$$(H_k(X, \mathbb{C}), F) \cong Hom((H^k(X, \mathbb{C}), F), \mathbb{C})$$

where $\mathbb{C}$ *is considered as the trivial Hodge structure of weight 0, hence* $H_k(X, \mathbb{Z})$ *is of weight* $-k$. *Then Poincaré duality becomes an isomorphism of Hodge structures:*

$$H^{n+k}(X, \mathbb{C}) \cong H_{n-k}(X, \mathbb{C})(-n)$$

.

Chapter 3

Variations of Hodge structures

In the preceding chapter we showed the existence of a Hodge structure on the cohomology of a Kähler manifold, depending only on its complex structure. Now, we wish to describe how this Hodge structure varies with the complex structure. First we introduce the notion of family of compact complex manifolds, then a theorem by Ehresmann allows us to see families as deformations of the complex structures of fixed manifolds, at least locally. In particular, the cohomology groups of the fibres X_t of this family can be considered locally as a variable Hodge structure on a constant lattice. We refer to [25] for the proofs of some theorems and further details.

3.1 Smooth families

Definition 3.1. *Let X and B be complex manifold and $\pi : X \to B$ be a holomorphic map. We say that $X \xrightarrow{\pi} B$ is a smooth family if π is a proper holomorphic submersion, i.e. π is surjective and, for every $x \in X$, the differential*

$$\pi_{*,x} : T_{X,x} \to T_{B,\pi(x)}$$

is surjective. It follows from the submersion theorem that for each $t \in B$, the fibre $X_t := \pi^{-1}(t)$ is a complex submanifold of X of codimension equal to the dimension of B.

If B is connected and $0 \in B$ is a reference point, we say that X is a family of deformations of the fibre X_0 and each fibre X_t, $t \in B$, is called a deformation.

Example 3.1. Consider $B = \mathbb{C} - 0, 1$ and put

$$X = \{([x_0, x_1, x_2], \lambda) \in \mathbb{P}^2 \times B \mid x_2^2 x_0 = x_1(x_1 - x_0)(x_1 - \lambda x_0)\},$$

and $\pi : X \to B$ the projection. Then $X \xrightarrow{\pi} B$ is a family and the fibre X_λ is a smooth plane cubic with j-invariant

$$J(X_\lambda) = \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

Note that the fibres are diffeomorphic, but in general not biholomorphic since we know that two elliplitic curves are biholomorhic if and only if they have the same j-invariant.

Theorem 3.1.1 (Ehresmann). *Let $X \xrightarrow{\pi} B$ be a smooth family of differentiable manifold and suppose B is contractible with base point 0. Then there exists a diffeomorphism*

$$T : X \cong X_0 \times B$$

such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\pi} & B \\ {\scriptstyle T_0}\downarrow & \overset{T}{\searrow} & \uparrow{\scriptstyle pr_2} \\ X_0 & \xleftarrow[pr_1]{} & X_0 \times B \end{array}$$

Remark 10. *As $pr_2 \circ T = \pi$, the trivialisation will be determined by its value on the first component $T_0 = pr_1 \circ T$, which induces a diffeomorphism $X_t \cong X_0$ for any $t \in B$. Indeed, the diffeomorphism is given by considering*

$$T^{-1}_{|X_0 \times \{t\}} : X_0 \to X_t$$

and its inverse is $(T_0)_{|X_t}$. Up to composing T with $(T_0)^{-1}_{|X_0}$, we may assume that $(T_0)_{|X_0} = Id$, i.e. that T_0 is a retraction of X onto X_0.

In the complex case, we cannot in general choose the trivialisation T to be holomorphic. However, via each isomorphism $T_0 : X_t \to X_0$, a $\mathcal{C}^\infty$ trivialisation enables us to consider the complex structure on X_t as a complex structure on X_0 which varies with t. Further, a more precise statement holds.

Proposition 3.1.2. *Let $X \xrightarrow{\pi} B$ be a family of complex manifolds and let $0 \in B$ be a point of B. Then, up to replacing B by a contractible neighbourhood of 0, there exists a $\mathcal{C}^\infty$ trivialisation $T = (T_0, \pi) : X \to X_0 \times B$ such that the fibres of T_0 are complex submanifolds of X.*

These fibres are submanifold of X which are diffeomorphic to B. The fact that they are complex implies that the family of complex structures on X_0 parametrised by B, which to $t \in B$ associates the complex structure of $X_t \cong^{(T_0)_{|X_t}} X_0$, varies holomorphically with t.

3.2 Local systems and Gauss Manin connection

Let B be a differentiable manifold.

Definition 3.2. *A local system over B is a sheaf of finitely generated abelian groups H, locally isomorphic to the constant sheaf of stalk G, where G is a fixed abelian group.*

Given an open cover $\{U_i\}$ of B, H can be trivialised in the U_i and give rise to transition isomorphism $M_{ij} \in Aut(G)$.
Given a local system H of abelian groups over B, we can consider the associated sheaf of free $\mathcal{C}^\infty(B)$-modules (or $\mathcal{O}_B$-modules, in case B is complex)

$$\mathcal{H} = H \otimes_{\mathbb{R}} \mathcal{C}^\infty(B)$$

The $\mathcal{C}^\infty$ holomorphic vector bundles obtained in this manner are equipped with an additional structure: a flat connection. On $\mathcal{H}$ we define the following connection

$$\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega_B$$
$$\sum_i \alpha_i \sigma_i \mapsto \sum_i \sigma_i \otimes d\alpha_i$$

where $\{\sigma_i\}$ is a basis of a local trivialisation of H. In the $\mathcal{C}^\infty$ case, this construction gives a $\mathcal{C}^\infty$ vector bundle equipped with a $\mathcal{C}^\infty$ connection ∇ such that $\nabla\sigma \in \mathcal{H} \otimes_{\mathcal{C}^\infty(B)} \Omega_B$, where Ω_B denotes the bundle of $\mathcal{C}^\infty$ differential 1-forms. In the holomorphic case, we obtain a holomorphic vector bundle equipped with a holomorphic connection, i.e. $\nabla\sigma \in \mathcal{H} \otimes_{\mathcal{O}_B} \Omega_B$ where Ω_B denotes the bundle of holomorphic differential 1-forms.

We notice that the expression of ∇ does not depend on the choice of the trivialisation , since another local trivialisation is obtained from the first one by a transition matrix with constant coefficients, which commutes with the derivations. Given a connection ∇ we define its curvature

$$\Theta : \mathcal{H} \to \bigwedge^2 \Omega_B$$

as follows: ∇ gives a map

$$\nabla : \mathcal{H} \otimes \Omega_B \to \mathcal{H} \otimes \bigwedge^2 \Omega_B$$
$$\sigma \otimes \omega \mapsto \nabla\sigma \wedge \omega + \sigma \wedge d\omega.$$

Definition 3.3. *The curvature of ∇ is then defined by $\Theta = \nabla \circ \nabla$. We say that ∇ is flat if* $\Theta = 0$

This map is $\mathcal{O}_B$ linear, i.e. $\Theta(f\sigma) = f\Theta(\sigma)$, so Θ is a section of $End(\mathcal{H}) \otimes \bigwedge^2 \Omega_B$. We have the same in the differentiable framework. The connection associated to a local system is flat. In fact, in a local trivialisation we have

$$\Theta(\sigma) = \nabla(\sum_i \sigma_i \otimes d\alpha_i) = \sum_i (\nabla(\sigma_i) \otimes d\alpha_i + \sigma_i \otimes d^2\alpha_i) = 0.$$

Proposition 3.2.1. *The previous correspondence between $\mathcal{C}^\infty$ (or holomorphic in case B is complex) vector bundles with a flat connection and isomorphism classes of local systems of vector spaces is bijective.*

Remark 11. *In the second case vector spaces are complex, in first one they can be real if we consider real bundles equipped with a real connection.*

Proof. (Idea)
The inverse map associates to the vector bundle $(\mathcal{H}, \nabla)$ the local system H of the flat sections, i.e. those annihilated by ∇. We need to see that H is a local system and that we have $\mathcal{H} = H \otimes \mathcal{C}^\infty(B)$. We refer to [25] for this check. □

Now let $X \xrightarrow{\pi} B$ be a smooth family of complex manifolds. By Ehresmann's theorem, in the neighbourhood of X_0 X is isomorphic to $X_0 \times B_0$, where B_0 is a contractible neighbourhood of 0 in B. We can consider the direct image sheaf on B

$$\pi_* A : U \mapsto A(\pi^{-1}(U)), \qquad A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$$

and compute the higher direct image $R^k \pi_* A$, which is the sheaf associated to the presheaf

$$U \mapsto H^k(\pi^{-1}(U), A).$$

As B_0 is contractible, we have an isomorphism

$$H^k(X_0 \times B_0, A) \cong H^k(X_0, A).$$

Considering a fundamental system of neighbourhood of 0, we deduce that $R^k \pi_* A$ is a local system because it is locally isomorphic to constant sheaf of stalk $H^k(X_0, A)$. We denote it by H_A^K. Of course, as the fibres are diffeomorphic, we have

$$H^k(X_t, A) \cong H^k(X_0, A)$$

for all $t \in B_0$.

Definition 3.4. *The flat connection*

$$\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_B$$

on the vector bundle $\mathcal{H}^k = H_A^K \otimes_{\mathbb{C}} \mathcal{O}_B$ *associated to the local system* H_A^K *is called the Gauss-Manin connection.*

3.3 The Kodaira-Spencer Map

Let $\pi : X \to B$ be a family of complex manifold as above. Let T_{X_0},T_X and T_B denote the holomorphic tangent bundles of X_0, X and B, respectively. Recalling that for each $x \in X_0$,

$$T_{X_0,x} = \ker\{\pi_{*,x} : T_{X,x} \to T_{B,0}\}$$

we have an exact sequence of holomorphic vector bundles over X_0:

$$0 \to T_{X_0} \hookrightarrow TX_{|X_0} \xrightarrow{\pi_*} X_0 \times T_{B,0} \to 0.$$

On the other hand, the fact that π is a submersion means that we also have an exact sequence of bundles over X:

$$0 \to T_{X/B} \to T_X \xrightarrow{\pi_*} \pi^*(T_B) \to 0,$$

where $\pi^*(T_B)$ is the pull-back bundle and $T_{X/B}$ is the relative bundle defined as $\ker \pi_*$. We notice that we obtain the first sequence by restriction of the second to X_0 and $\pi^*(T_B)_{|X_0}$ is the trivial holomorphic vector bundle of fibre $T_{B,0}$. The first exact sequence gives rise to a long exact sequence in cohomology. In particular we have a map:

$$\rho : T_{B,0} = H^0(X_0, \pi^*(T_B)_{|X_0}) \to H^1(X_0, T_{X_0}).$$

Definition 3.5. *The map ρ so defined, is called the Kodaira-Spencer map at 0 of the family $X \xrightarrow{\pi} B$.*

The Kodaira-Spencer map can be seen as the differential of the map which associates to each $t \in B$ the complex structure of the fibre $X_t \cong X_0$.

3.4 The Kähler case

Let $\pi : X \to B$ be a family of complex manifold and assume that $X_0 = \pi^{-1}(0)$ with $0 \in B$, is Kähler . Now we want to show that if do not get too far from 0, the fibres X_t are still Kähler manifold. First we show that the Hodge numbers are constant for t near 0.

Proposition 3.4.1. *Let $t \in B$ be near 0. Then we have $h^{p,q}(X_t) = h^{p,q}(X_0)$. Moreover the Frölicher spectral sequence degenerates at E_1.*

Proof. Let $\Omega^p_{X_t}$ denote the vector bundle of the holomorphic p-forms on X_t. By a foundational theorem, which we do not prove here, the function which to $t \in b$ associates $h^{p,q}(X_t) = dim H^q(X_t, \Omega^p_{X_t})$ is upper semicontinuous, i.e. $dim H^q(X_t, \Omega^p_{X_t}) \leq H^q(X_0, \Omega^p_{X_0})$ (see [25] for the complete proof). Thus, we have that

$$h^{p,q}(X_t) \leq h^{p,q}(X_0).$$

Now we recall that

$$H^q(X_t, \Omega^p_{X_t}) = E_1^{p,q}(X_t)$$

where $E_1^{p,q}(X_t)$ denotes the first term of the Frölicher spectral sequence defined in example **??**.

As we observed, we have

$$E_\infty^{p,q}(X_t) = F^p H^{p+q}(X_t)/F^{p+1}H^{p+q}(X_t) \text{ and } dim E_\infty^{p,q}(X_t) \leq E_1^{p,q}(X_t).$$

The first identity gives

$$dim H^k(X_t, \mathbb{C}) = \sum dim E_\infty^{p,q}(X_t).$$

However $X_t \cong X_0$ by Ehresmann's theorem (maybe up to shrinking the neighbourhood of 0), so $dim H^k(X_t, \mathbb{C}) = dim H^k(X_0, \mathbb{C}) =: b_k$. We then construct the following chain of inequalities, all involving positive numbers,

$$b_k = \sum dim E_\infty^{p,q}(X_t) \leq \sum dim E_1^{p,q}(X_t) = \sum h^{p,q}(X_t) \leq \sum h^{p,q}(X_0) = b_k.$$

Thus all inequalities are actually equalities and we have

$$h^{p,q}(X_t) = h^{p,q}(X_0), \qquad E_\infty^{p,q}(X_t) = E_1^{p,q}(X_t).$$

□

Actually, for t near 0, a much stronger condition holds: X_t admits the Hodge decomposition.

Proposition 3.4.2. *For t near 0, we have*

$$H^k(X_t, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X_t)$$

with $H^{p,q}(X_t) = \overline{H^{q,p}}(X_t)$ and $H^{p,q}(X_t) \cong H^q(X_t, \Omega^p_{X_t})$.

Proof. The proof is based on a theorem by Kodaira ([17]) which states the following

Theorem 3.4.3. *Let $\Delta = (\Delta_t)_{t\in B}$ be a relative differential operator acting on a vector bundle $F \to X$, such that each induced operator Δ_t on F_t is elliptic of fixed order. Then if $dim \ker \Delta_t$ is independent of t, the subspace $\ker \Delta_t \subset \mathcal{C}^\infty(F_t)$ varies in a $\mathcal{C}^\infty$ way with t.*

Now, we notice that the dimension of the subspace $F^pH^k(X_t,\mathbb{C}) \subset H^k(X_t,\mathbb{C}) \cong H^k(X_0,\mathbb{C})$ are constant by proposition 3.4.1. If we take $F = \Omega_X$ and Δ as the laplacian, we can apply Kodaira's theorem and so we have that $F^pH^k(X_t,\mathbb{C})$ varies in a $\mathcal{C}^\infty$ way. For $t = 0$, X_0 is Kähler , so we have:

$$H^k(X_0,\mathbb{C}) = F^pH^k(X_0,\mathbb{C}) \oplus \overline{F^{q+1}H^k(X_0,\mathbb{C})}, \qquad p+q=k. \tag{3.1}$$

By continuity, this also holds for t near 0. Set $H^{p,q}(X_t) := F^pH^k(X_0,\mathbb{C}) \cap \overline{F^qH^k(X_0,\mathbb{C})}, \qquad p+q=k$. As these two spaces generate $H^k(X_t,\mathbb{C})$, the dimension of $H^{p,q}(X_t)$ is equal to the dimension of $H^{p,q}(X_0)$. Now by 3.1 we have

$$H^{p,q}(X_t) \hookrightarrow F^pH^k(X_t,\mathbb{C}) \to F^pH^k(X_t,\mathbb{C})/F^{p+1}H^k(X_t,\mathbb{C}) \cong H^q(X_t,\Omega^p_{X_t})$$

is an isomoprhism. Finally we have

$$H^k(X_t,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X_t)$$

with $H^{p,q}(X_t) \cong H^q(X_t,\Omega^p_{X_t})$. The complex conjugation property $H^{p,q}(X_t) = \overline{H^{q,p}}(X_t)$ is a straightforward consequence of 3.1. □

A striking consequence of the previous proposition is the following, which we do not prove (again we refer to [25] for the proof).

Theorem 3.4.4. *Let $\pi : X \to B$ be a family of complex manifolds and let $0 \in B$. If the fibre X_0 is Kähler , then so is X_t for all t sufficiently near 0.*

3.5 Period maps and period domains

Let now X be a Kähler manifold and let $\pi : X \to B$ a family of deformations of X. Up to restricting B, we may assume that the fibres X_t satisfies the degeneracy at E_1 of the Frölicher spectral sequence and also that $dimF^pH^k(X_t,\mathbb{C}) = dimF^pH^k(X_0,\mathbb{C}) =: b^{p,k}$ (by theorem 3.4.4 we could also assume that X_t is Kähler but we do not need that). Moreover, up to restricting B, again we can assume that B is contractible and apply Ehresmann's theorem to have diffeomorphisms $X_t \cong X_0$ for $t \in B$. These diffeomorphism leads to isomorphisms in cohomology, namely

$$H^k(X_t, A) \cong H^k(X_0, A), \quad \text{for } A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}.$$

Definition 3.6. *We define the period map*

$$\mathcal{P}^{p,k} : B \to Grass(b^{p,k}, H^k(X_0, \mathbb{C}))$$

where $Grass(b^{p,k}, H^k(X, \mathbb{C}))$ *denotes the grassmannian of* $b^{p,k}$*-dimensional subspaces of* $H^k(X_0, \mathbb{C})$*, as the map which to* $t \in B$ *associates the subspace*

$$F^p H^k(X_t, \mathbb{C}) \subset H^k(X_t, \mathbb{C}) \cong H^k(X_0, \mathbb{C})$$

As we noticed in the proof of proposition 3.4.2, the map

$$t \mapsto F^p H^k(X_t, \mathbb{C})$$

varies in a $\mathcal{C}^\infty$ way, that is $\mathcal{P}^{p,k}$ is $\mathcal{C}^\infty$. Let us recall the fact that given a point $[W] \in Grass(l, V)$, there is a canonical isomorphism

$$T_{[W]} Grass(l, V) \simeq Hom(W, V/W)$$

Griffiths showed that in fact the differential of the period maps lands in a subspace which is smaller than expected.

Theorem 3.5.1. *The period map has the following properties:*

(i) $\mathcal{P}^{p,k}$ *is holomorphic for all* p, k *such that* $p \leq k$.

*(ii) (**Griffiths transversality**) The differential*

$$d\mathcal{P}^{p,k} : T_{B,t} \to Hom(F^p H^k(X_t, \mathbb{C}), H^k(X_t, \mathbb{C})/F^p H^k(X_t, \mathbb{C}))$$

takes values in $Hom(F^p H^k(X_t, \mathbb{C}), F^{p-1} H^k(X_t, \mathbb{C})/F^p H^k(X_t, \mathbb{C}))$

Proof. We refer to [04] for the proof. □

Since we are assuming that X_0 is Kähler , for each integer $k \leq dim_{\mathbb{C}} X_0$ the Hodge decomposition theorem gives a Hodge structure of weight k one the cohomology group $H^k(X_0, \mathbb{C})$ with Hodge numbers $h^{p,q}$. Also, we have the Hodge filtration

$$0 = F^{k+1}H^k(X_0) \subset \dots F^p H^k(X_0) \subset F^{p-1}H^k(X_0, \mathbb{C}) \subset \dots \subset F^0 H^k(X_0) = H^k(X_0, \mathbb{C})$$

by complex subspaces of dimension $b^{p,k} := \sum_{i \geq p} h^{i,k-i}$. We call $b := (b^{1,k}, \dots, b^{k,k})$ and define $F_{b,k}(H^k(X_0, \mathbb{C})$ as the set of the decreasing filtrations on $H^k(X_0, \mathbb{C})$ by complex subspace of dimension $b^{p,k}$ for $0 < p \leq k$.
$F_{b,k}(H^k(X_0, \mathbb{C})$ is a complex submanifold of $\prod_{0<p\leq k} Grass(b^{p,k}, H^k(X_0, \mathbb{C}))$ with the map

$$F^k H^k(X_0) \subset \dots F^p H^k(X_0) \subset \dots \subset F^0 H^k(X_0) \mapsto (F^k H^k(X_0), \dots, F^p H^k(X_0), \dots, F^0 H^k(X_0)).$$

Consider now a family $X \to B$ of deformations of X_0. After restricting to a neighbourhood of 0 such that all the fibres X_t are Kähler , we can construct the period map defined as

$$\begin{aligned} \mathcal{P}^k : B &\to F_{b,k}(H^k(X_t, \mathbb{C})) \cong F_{b,k}(H^k(X_0, \mathbb{C})) \\ t &\mapsto (\mathcal{P}^{1,k}(t), \dots, \mathcal{P}^{k,k}(t)) \end{aligned}$$

which, by the preceding results, it is holomorphic. Also, the Hodge filtration must satisfy the condition

$$F^p H^k(X_t, \mathbb{C}) \oplus \overline{F^{k-p+1} H^k(X_t, \mathbb{C})}$$

and this condition defines an open set $\mathcal{D}$ of $F_{b,k}(H^k(X_t, \mathbb{C}))$, which is called period domain.

3.6 Hodge bundles

Assume now $\pi : X \to B$ is a family of compact Kähler manifolds and let

$$\mathcal{H}^k = R^k \pi_* \mathbb{C} \otimes \mathcal{O}_B$$

be the holomorphic vector bundle defined in section 3.2. We have seen that this bundle is equipped with the Gauss-Manin connection

$$\nabla^k_{\mathcal{H}} \to \mathcal{H}^k \otimes_{\mathcal{O}_B} \Omega_B$$

which is flat and holomorphic. Also $\mathcal{H}^k$ admits local ∇-flat trivialisations

$$\mathcal{H}^k_{|B_0} \cong H^k(X_0, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_B,$$

where B_0 is an open contractible neighbourhood of 0 in B. On this neighbourhood we have the holomorphic period map

$$\begin{aligned} \mathcal{P}^{p,k} : B_0 &\to Grass(b^{p,k}, H^k(X_0, \mathbb{C})) \\ t &\mapsto F^p H^k(X_t, \mathbb{C}) \subset H^k(X_t, \mathbb{C}) \cong H^k(X_0, \mathbb{C}). \end{aligned}$$

This implies that there exists a holomorphic vector subbundle

$$F^p \mathcal{H}^k \subset \mathcal{H}^k$$

defined by the condition:

$$F^p \mathcal{H}^k_t \subset \mathcal{H}^k_t \text{ can be identified with } F^p H^k(X_t, \mathbb{C}) \subset H^k(X_t, \mathbb{C}) \text{ for all } t \in B \tag{3.2}$$

The identification is given by

$$\mathcal{H}^k = (R^k \pi_* \mathbb{C})_t \otimes (\mathcal{O}_B / \mathcal{M}_t \mathcal{O}_{\mathcal{B}})$$

where $\mathcal{M}_t$ denotes the maximal ideal of the functions that vanish at t. One can show ([25]) that this subbundles are equal to

$$R^k \pi_* \Omega^{\geq p}_{X/B} \subset R^k \pi_* (\Omega^\bullet_{X/B}).$$

The bundles $\mathcal{H}^k$ are called Hodge bundles and $F^p \mathcal{H}^k$ are the Hodge subbundles. Their successive quotients satisfy

$$\mathcal{H}^{p,q}_t = (F^p \mathcal{H}^k / F^{p+1} \mathcal{H}^k)_t = F^p H^k(X_t) / F^{p+1} H^k(X_t) = H^q(X_t, \Omega^p_{X_t}), \quad p + q = k.$$

3.7 Algebraic approach

Let $\pi : X \to B$ be a family of smooth complex projective varieties. We have seen that, for each i, we have a variation of Hodge structure whose underlying vector bundle is

$$\mathcal{H}^i = R^i \pi_* \mathbb{Q} \otimes_{\mathbb{Q}} \mathcal{O}_{B_{an}} \cong \mathbb{R}^i \pi_*^{an} \Omega^\bullet_{X_{an}/B_{an}} \cong (\mathbb{R}^i \pi_* \Omega^\bullet_{X/B})_{an} \tag{3.3}$$

where $\Omega_{X/B}$ is the sheaf of differentials of X over B [1] and $\mathbb{R}^i\pi_*\Omega^\bullet_{X/B}$ denotes the hyperdirect image [2]. The first isomorphism comes from the fact that middle term is a resolution of the first, the second is due to GAGA. A relative version of Grothendieck's theorem, tells us that the Hodge bundles are given by $F^p\mathcal{H}^i \cong (\mathbb{R}^i\pi_*\Omega^{\bullet\geq p}_{X/B})_{an}$. Moreover, in [16], Katz and Oda have shown that the Gauss-Manin connection can also be constructed algebraically. Starting from the exact sequence

$$0 \to \pi^*\Omega^1_{B/\mathbb{C}} \to \Omega^1_{X,\mathbb{C}} \to \Omega^1_{X/B} \to 0$$

we set $L^r\Omega^i_{X/\mathbb{C}} = \pi^*\Omega^r_{B/\mathbb{C}} \wedge \Omega^{i-r}_{X/\mathbb{C}}$. We get a short exact sequence of complexes

$$0 \to \pi^*\Omega^1_{B/\mathbb{C}} \otimes \Omega^{\bullet-1}_{X/B} \to \Omega^\bullet_{X,\mathbb{C}}/L^2\Omega_{X/\mathbb{C}} \to \Omega^\bullet_{X/B} \to 0$$

and hence a connecting morphism

$$\mathbb{R}^i\pi_*\Omega^\bullet_{X/B} \to \mathbb{R}^{i+1}\pi_*(\pi^*\Omega^1_{B/\mathbb{C}} \otimes \Omega^{\bullet-1}_{X/B}) \cong \Omega^1_{B/\mathbb{C}} \otimes \mathbb{R}^i\pi_*\Omega^\bullet_{X/B}.$$

The theorem of Katz and Oda shows that the associated morphism between vector bundles is precisely the Gauss-Manin connection. For our purposes the most interesting fact is that if π, X, B are all defined over a subfield K of $\mathbb{C}$, then the same is true for the Hodge bundles and the Gauss-Manin connection and their construction is algebraic.

[1] For a full definition of $\Omega_{X/B}$ we refer to [18]

[2] Given a complex of sheaves $\mathcal{F}^\bullet$ on X, there exists a double complex $I^{\bullet,\bullet}$ such that, for all p, $I^{p,\bullet}$ is a resolution of $\mathcal{F}^p$. We define

$$\mathbb{R}\pi_*(\mathcal{F}^\bullet) := H^k(\pi_*I^\bullet)$$

where $I^\bullet$ is the simple complex associated to $I^{\bullet,\bullet}$

Chapter 4

Absolute Hodge Classes

4.1 Cycle classes

Let X be a nonsingular projective variety over $\mathbb{C}$ of dimension n. We now want to define the cycle class associated to an algebraic subvariety $Z \subseteq X$ of codimension p. We will give two different constructions: the first one is due to Grothendieck and associates to Z an element of $2\pi i H^{2p}(X_{an}, \mathbb{Q})$; the second, by Bloch in [02], is purely algebraic and associates to Z an element of $F^p H^{2p}(X/\mathbb{C})$. There is a comparison theorem between the classes obtained from the two constructions:

Theorem 4.1.1. *Under the isomorphism $H^{2p}(X/\mathbb{C}) \cong H^{2p}(X_{an}, \mathbb{C})$, we have*

$$[Z] = [Z_{an}].$$

4.1.1 Grothendieck's construction

Firstly, we recall that integration of differential forms gives an isomorphism

$$H^{2n}(X_{an}, \mathbb{Q}(n)) \to \mathbb{Q}, \qquad \alpha \mapsto \frac{1}{(2\pi i)^n} \int_{X_{an}} \alpha$$

Now we construct the cycle class

$$[Z_{an}] \in H^{2p}(X_{an}, \mathbb{Q}(p))$$

as follows . By Hironaka's theorem, there exists a resolution $\tilde{Z}$ of the singularities of Z and let $\mu : \tilde{Z} \xrightarrow{\tilde{\mu}} Z \xrightarrow{i} X$ denote the induced morphism. We may consider

$$H^{2n-2p}(X, \mathbb{Q}(n-p)) \xrightarrow{\mu^*} H^{2n-2p}(\tilde{Z}, \mathbb{Q}(n-p)) \xrightarrow{\cong} \mathbb{Q}, \qquad \alpha \mapsto \frac{1}{(2\pi i)^{n-p}} \int_{\tilde{Z}_{an}} \mu^* \alpha.$$

By Poincaré duality, the linear function in the equation is represented by a unique class $\zeta \in H^{2p}(X_{an}, \mathbb{Q}(p))$ with the property that

$$\frac{1}{(2\pi i)^{n-p}} \int_{\tilde{Z}_{an}} \mu^* \alpha = \frac{1}{(2\pi i)^n} \int_{X_{an}} \zeta \cup \alpha.$$

Observe that the class $\zeta \in H^{2p}(X_{an}, \mathbb{Q}(p))$ which is endowed with a weight zero Hodge structure. The class ζ is a Hodge class. Indeed, if $\alpha \in H^{2n-2p}(X, \mathbb{Q}(n-p))$ is of type $(n-i, n-j)$ with $i \neq j$, than either i or j is greater than p, and $\int_{\tilde{Z}_{an}} \mu^* \alpha = 0$. This implies $\int_{X_{an}} \zeta \cup \alpha = 0$ and that, as a consequence, ζ is of type (0,0) in $H^{2p}(X_{an}, \mathbb{Q}(p))$. In fact, one can prove that it actually comes from a class in $H^{2p}(X_{an}, \mathbb{Z}(p))$.

4.1.2 Bloch's construction

Let X be a projective variety and Z a subvariety of codimension p, both defined over a field $K \subseteq \mathbb{C}$ of characteristic 0.

We want to use Bloch's construction, which works for an arbitrary locally complete intersection Z in X. We first need some background on local cohomology, and we will focus our discussion on coherent sheaves. Given a sheaf $\mathcal{F}$ we have the local cohomology groups which fit in the exact sequence

$$\ldots \to H^k_Z(X, \mathcal{F}) \to H^k(X, \mathcal{F}) \to H^k(X - Z, \mathcal{F}) \to\to H^{k+1}_Z(X, \mathcal{F}) \to \ldots$$

We also define the sheaves

$$\mathcal{H}^k_Z(X, \mathcal{F}) := \lim_{\to k} Ext^k_{\mathcal{O}_X}(\mathcal{O}_X / I^k, \mathcal{F})$$

where $I \subset \mathcal{O}_X$ is the ideal of Z. One can show that H_Z and $\mathcal{H}^k_Z$ are left exact functors for the category of abelian sheaves, and that they are related by the so called local-to-global spectral sequence which has the property that

$$E_2^{p,q} = H^p(X, \mathcal{H}_Z^q(X, \mathcal{F})) \cong H_Z^{p+q}(X, \mathcal{F}). \tag{4.1}$$

Finally these notions can be extended to a complex of sheaves $\mathcal{F}^\bullet$ to define the analogues of these functors in hypercohomology. To construct the cohomology class $[Z]$ in algebraic de Rham cohomology, we start with the subsheaf $\Omega_{X/K}^{k,cl}$ of closed forms of degree k over the field K. This maps naturally to the truncated complex $\Omega_{X/K}^{\bullet \geq k}$

$$0 \to \Omega_{X/K}^k \to \Omega_{X/K}^{k+1} \to \ldots \to \Omega_{X/K}^n \to 0,$$

which is itself a subcomplex of the full de Rham complex. In this way we get

$$H^l(X, \Omega_{X/K}^{k,cl}) \to \mathbb{H}^{l+k}(X, \Omega_{X/K}^{\bullet \geq k}) = F^k\mathbb{H}^{l+k}(X, \Omega_{X/K}^\bullet) \hookrightarrow \mathbb{H}^{l+k}(X, \Omega_{alg}^\bullet) = H^{k+l}(X/K)$$

Choose an open set $U \subset X$ such that the subset $U \cap Z$ of U is defined by p equations $f_1, \ldots, f_p$. Then $W = U - (U \cap Z)$ is covered by the open sets $U_1 \ldots U_p$ where U_i is the subset for which $f_i \neq 0$. Consider the closed differential form

$$\omega_U = \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_p}{f_p} \in \Omega_{alg}^{p,cl}(U_1 \cap \ldots \cap U_p).$$

This determines a class in Čech cohomology of degree $p-1$ for the sheaf $\Omega_{X/K}^{p,cl}$ restricted to W, and so a class in

$$H^{p-1}(W, \Omega_{X/K}^{k,cl}) \to \mathbb{H}^{2p-1}(W, \Omega_{X/K}^{\bullet \geq p}) \to \mathbb{H}_{Z \cap U}^{2p}(U, \Omega_{X/K}^{\bullet \geq p}).$$

We can glue these locally defined classes to get a global section in

$$\Gamma(X, \mathcal{H}_Z^{2p}(X, \Omega_{X/K}^{\bullet \geq p})) \cong H_Z^{2p}(X, \Omega_{X/K}^{\bullet \geq p}).$$

where the isomorphism comes from 4.1 and the fact that $\mathcal{H}_Z^i(X, \Omega_{X/K}^{\bullet \geq p}) = 0$ for $i \leq 2p-1$. Using natural maps

$$\mathbb{H}_Z^{2p}(X, \Omega_{X/K}^{\bullet \geq p}) \to \mathbb{H}^{2p}(X, \Omega_{X/K}^{\bullet \geq p}) \to \mathbb{H}^{2p}(X, \Omega_{X/K}^\bullet) = H^{2k}(X/K)$$

we end up with a class in $H^{2k}(X/K)$, which is called the algebraic de Rham representative of Z and is denoted by $[Z]$.

Remark 12. *The construction shows that in particular,* $[Z] \in \mathbb{H}^{2p}(X, \Omega^{\bullet \geq p}_{X/K})$. *Recall that* $\mathbb{H}^{2p}(X, \Omega^{\bullet \geq p}_{X/K}) = F^p H^{2p}(X/K)$. *Thus* $[Z] \in F^p H^{2p}(X/K)$.

In order to give a more general construction, we need to introduce the Chern classes. Also, they will be useful because they provide an example of Hodge class.

4.1.3 Chern classes

Let $\mathcal{E}$ a locally free sheaf of rank r over X. We want to define the Chern classes $c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Z})$, $1 \leq i \leq r$. By convention, we set $c_0(\mathcal{E}) = 1$ and $c_i(\mathcal{E}) = 0$ for $i > r$. First we construct the first Chern class of an algebraic line bundle $\mathcal{L}$. Taking the corresponding holomorphic line bundle $\mathcal{L}_{an}$, we consider the short exact sequence

$$0 \to \mathbb{Z}(1) \to \mathcal{O}_{X_{an}} \xrightarrow{\exp} \mathcal{O}^*_{X_{an}} \to 0.$$

This induces a long exact sequence in cohomology, in particular there is a map:

$$c_1 : H^1(X_{an}, \mathcal{O}^*_{X_{an}}) \to H^2(X_{an}, \mathbb{Z}(1)).$$

As isomorphism class of $\mathcal{L}_{an}$ belongs to $H^1(X, \mathcal{O}^*_{X_{an}})$, we define the first Chern class to be the image $c_1(\mathcal{L}_{an}) \in H^2(X_{an}, \mathbb{Z})$. To relate this to differential forms, we cover X by open simply connected subset $\{U_i\}$ on which $\mathcal{L}_{an}$ is trivial, and let $g_{ij} \in \mathcal{O}_{X_{an}}(U_i \cap U_j)$ denote the holomorphic transition function for this cover. On each U_i we can write $g_{ij} = exp(f_{ij})$, then the cocycle condition tells us that

$$f_{jk} - f_{ik} + f_{ij} \in \mathbb{Z}(1)$$

forms a 2-cocycle that represents $c_1(\mathcal{L}_{an})$. Its image in $H^2(X_{an}, \mathbb{C}) \cong \mathbb{H}^2(X_{an}, \Omega^\bullet_{an})$ is a (1,1) form cohomologous to the class of the 1-cocycle $df_{ij} \in H^1(X_{an}, \Omega^1_{an})$. However, $df_{ij} = dg_{ij}/g_{ij}$, so we find the case of the Bloch's construction.

To define the first Chern class of $\mathcal{L}$ in the algebraic de Rham cohomology, we use the fact that a line bundle is also locally trivial in the Zariski topology. As above, we denote by $\{U_i\}$ the Zariski-open sets on which $\mathcal{L}$ is trivial and by g_ij the corresponding transition function.

We define $c_1(\mathcal{L}) \in F^1H^2(X, \mathbb{C})$ to be the hypercohomology class determined by the cocycle dg_{ij}/g_{ij}. Under the isomorphism of Grothendieck's theorem we have $c_1(\mathcal{L}) = c_1(\mathcal{L}_{an})$.

Now suppose $\mathcal{E}$ is a locally free sheaf of rank r over X. On the associated projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to X$, we have the universal line bundle $\mathcal{O}_{\mathcal{E}}(1)$ toghether with a surjection from $\pi^*\mathcal{E}$. In the Betti cohomology we have a decomposition (cf [25], §7)

$$H^{2r}(\mathbb{P}(\mathcal{E}_{an}), \mathbb{Z}(r)) = \bigoplus_{i=0}^{r-1} \zeta^i \pi^* H^{2r-2i}(X_{an}, \mathbb{Z}(r-i)), \tag{4.2}$$

where $\zeta = c_1(\mathcal{O}_{\mathcal{E}}(1)) \in H^2(\mathbb{P}(\mathcal{E}_{an}), \mathbb{Z}(1)$. Consequently, there are unique classes $c_k \in H^{2k}(\mathbb{P}(\mathcal{E}_{an}), \mathbb{Z}(k))$ such that

$$\zeta^r - \pi^*(c_1)\zeta^{r-1} + \pi^*(c_2)\zeta^{r-2} + \ldots + (-1)^r \pi^*(c_r) = 0.$$

We define

$$c_k(\mathcal{E}_{an}) := c_k.$$

We can do the same construction with the algebraic de Rham cohomology, obtaining Chern classes $c_k(\mathcal{E}) \in F^kH^{2k}(X/\mathbb{C})$. As in the case of line bundles $c_k(\mathcal{E}) = c_k(\mathcal{E}_{an})$ under Grothendieck isomorphism.

Now, coherent sheaves on regular schemes admit finite resolution by locally free sheaves, so it is possible to define Chern classes. A consequence of the Riemann-Roch theorem gives us the formula

$$[Z_{an}] = \frac{(-1)^{p-1}}{(p-1)!} c_p(\mathcal{O}_{Z_{an}}) \in H^{2p}(X_{an}, \mathbb{Q}(p)).$$

Thus we define

$$[Z] := \frac{(-1)^{p-1}}{(p-1)!} c_p(\mathcal{O}_Z) \in F^pH^{2p}(X/\mathbb{C})$$

and we have $[Z_{an}] = [Z]$ as desired.

Proposition 4.1.2. *The Chern classes $c_k(\mathcal{E}_{an})$ of an holomorphic vector bundle $\mathcal{E}_{an}$ of rank r over a complex projective variety X are of type (k, k).*

Proof. We have seen that on the associated projective bundle we have a decomposition

$$H^{2r}(\mathbb{P}(\mathcal{E}_{an}), \mathbb{Z}(r)) = \bigoplus_{i=0}^{r-1} \zeta^i \pi^* H^{2r-2i}(X_{an}, \mathbb{Z}(r-i)),$$

where $\zeta = c_1(\mathcal{O}_{\mathcal{E}}(1)) \in H^2(\mathbb{P}(\mathcal{E}_{an}), \mathbb{Z}(1)$. As ζ is represented by a form of type $(1,1)$, we have that the morphism

$$\zeta^i : H^{2r-2l}(X, \mathbb{Z}(r-i)) \to H^{2r}(\mathbb{P}(\mathcal{E}_a n), \mathbb{Z}(r))$$
$$\alpha \mapsto \alpha \cup \zeta^i$$

is a morphism of Hodge structures of bidegree (i,i). Thus, if β is a form of type $(p,q) \in H^{2r}(\mathbb{P}(\mathcal{E}_a n), \mathbb{Z}(r))$, its components α_i in the decomposition 4.2, are of type $(p-i, q-i)$. In particular the Chern classes c_i are the component of ζ^r, thus are of type (r-i,r-i) □

What we have just proved motivates the following defintion.

Definition 4.1. *Let $V = (V_{\mathbb{Z}}, (V_{\mathbb{C}}, F^{\bullet}))$ be an integral Hodge structure of weight $2p$. The Hodge classes of V are the integral classes of type (p,p).*

$$Hdg(V) = V_{\mathbb{Z}} \cap V^{p,p}$$

In the case of a Kähler manifold X and $V_{\mathbb{Z}} = H^2p(X, \mathbb{Z})/torsion$, we write $Hdg^{2p}(X) = Hdg(V)$.

Example 4.1. Let X be a smooth projective variety. We have seen that the classes of algebraic subvarieties of X and the Chern classes are Hodge classes.

Actually an interesting property holds:

Theorem 4.1.3. *Let X be an algebraic variety. Then the subgroup of $Hdg^{2p}(X)$ generated by the algebraic classes and the one generated by Chern classes of vector bundles coincide*

Proof. (Idea) Let us first prove the statement for $p = 1$. If L is a holomorphic line bundle and σ is a holomorphic non-zero section of L, consider the divisor $D \subset X$ od σ. It is the cicle of codimension 1 associated to the subscheme whose sheaf of ideals is locally generated by σ, which we can consider locally as a function by trivialising the bundle L. We have $D = \sum n_i D_i$, and we can show that this decomposition is given locally by the decomposiion into prime elements of the function corresponding to σ in a local trivialisation of L.

Theorem 4.1.4. ***(Lelong)*** *Let L be a holomorphic line bundle and let σ be a non-zero holomorphic section of L. Then the cohomology class of the divisor D of σ and the first Chern class $c_1(L)$ are equal in $H^2(X, \mathbb{Z})$.*

Proof. Each component D_i of D allows one to define a holomorphic line bundle $L_i := \mathcal{O}_X(D_i) := \mathcal{I}^*_{D_i}$, where $\mathcal{I}^*_{D_i}$ is the sheaf of $\mathcal{O}_X$-modules of rank 1 given by the holomorphic functions vanishing on D_i. Indeed, one can show (see [19]) that the sheaf of ideals of D_i is locally generated by an equation f_i. Clearly there exists a section σ_i of L_i whose divisor is equal to D_i: in fact the inclusion

$$j : \mathcal{I}_{D_i} \to \mathcal{O}_X$$

dualises the inclusion

$$j^t : \mathcal{O}_X \to \mathcal{I}_{D_i}$$

which gives the section $\sigma_i = j^t(1)$ of L_i. Now, the equation f_i gives a local generator of $\mathcal{I}_{D_i}$, i.e. a trivialisation

$$\mathcal{I}_{D_i} \cong \mathcal{O}_X$$

and in this trivialisation, the map j is multiplication by f_i. This also holds for j^t, which shows that the divisor of $\sigma_i = J^t(1)$ is equal to D_i.

We then have $L = \bigotimes_i L_i^{\otimes n_i}$, since the section σ of L gives a morphism $\mathcal{O}_X \to L$ whose dual

$$L^* =: L^{-1} \to \mathcal{O}_X$$

identifies L^{-1} with the free $\mathcal{O}_X$-submodule of rank 1

$$\mathcal{I}_D = \prod_i \mathcal{I}_{D_i}^{n_i} \cong \bigotimes_i L_i^{\otimes -n_i}.$$

Since the first Chern class and this cycle class map are both additive, it suffices to prove the result for the section σ_i of L_i, whose divisor D_i can be thought generically smooth (see [25] for details). Now consider the exact sequence of relative cohomology

$$\ldots \to H^2(X, X - D_i, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to H^2(X - D_i, \mathbb{Z}) \to \ldots$$

and the Thom isomorphism

$$H^2(X, X - D_i, \mathbb{Z}) = H^0(D_i, \mathbb{Z}) = \mathbb{Z}.$$

We notice that the Chern class $c_1(L_i)$ vanishes on $X - D_i$, since L_i is trivial on $X - D_i$. Thus it comes from $H^0(D_i, \mathbb{Z})$ and so it is an integral multiple of the class of D_i which can be seen as the image of $1 \in H^0(D_i, \mathbb{Z})$ (see [25],§11.1.2). Actually, something stronger holds: in fact one can show

$$c_1(L_i) = [D_i].$$

Then, as $D = \sum_i n_i D_i$, we can consider the associated line bundle $\mathcal{O}_X(D) := \otimes_i \mathcal{I}_{D_i}^{\otimes - n_i}$ and conclude that $c_1(\mathcal{O}_X(D)) = [D]$. □

As corollaries of Lelong's theorem we have that

Corollary 4.1.5. *If X is a complex projective manifold, the Chern classes of holomorphic line bundles on X are the classes of divisors.*

Corollary 4.1.6. *Let Z be an algebraic cycle of codimension k, and L a line bundle over an algebraic variety X. Then $[Z] \cup c_1(L)$ is the class of an algebraic cycle Z' of codimension $k+1$ of X.*

Next, if E is a holomorphic vector bundle over an algebraic variety equipped with an ample line bundle H, then it is possible to show $E' = E \otimes H^{\otimes N}$ is generated by its global sections for a sufficiently large N (see [20]). The global sections of E' then give a holomorphic map ϕ from X to the Grassmannian $G(N - r, N)$, $r = dim H^0(X, \mathcal{E}')$, where $\mathcal{E}'$ denotes the sheaf of holomorphic sections of E'. This map associates to every $x \in X$ the subspace of the sections of E' which vanish at the point x.
The bundle E' can then be naturally identified with the pullback $\phi^* Q$, where Q is the quotient tautological bundle ober the Grassmannian. One can show (see [11]) that the cohomology of the Grassmannian is generated by classes of smooth algebraic cycles, which we may furthermore assume transverse to ϕ (i.e. smooth cycles Z of codimension k such that the scheme $\phi^{-1}(Z)$ is also smooth of codimension k). We then have

$$\phi^*([Z]) = [\phi^{-1}(Z)],$$

and we deduce that the Chern classes of E are classes of algebraic cycles.
Finally the Chern classes of E can be computed using the Chern classes of $E' = E \otimes H^N$ and their cup-products with powers of $c_1(H)$ (see [25]). Then, by proposition 4.1.6, the Chern classes of E are also classes of algebraic cycles. □

4.2 Absolute Hodge Classes

In this section, we introduce the notion of absolute of absolute Hodge classes in the cohomology of a complex algebraic variety. While Hodge theory applies to general compact Kähler manifolds, absolute Hodge classes are brought in as a way to deal with cohomological properties of a variety coming from its algebraic structure.

4.2.1 Definition of absolute Hodge classes

Here we enter one of the most fascinating aspects of the Hodge conjecture, which seriously involves the fact that the complex manifolds we are considering are algebraic. We now introduce the notion of (de Rham) absolute Hodge class (cf.[08]). Firs of all, let us give a new slightly different definition of Hodge class.

Definition 4.2. *A class $\alpha \in H^{2p}(X, \mathbb{Q}(p))$ is Hodge class of degree $2p$ on X if*

$$\alpha \in H^{2p}(X, \mathbb{Q}(p)) \cap H^{0,0}(X)$$

.

Now let X be a smooth proper complex algebraic variety and Z an algebraic cycle of codimension p in X. As we have shown earlier, Z has a cohomology class

$$[Z] \in H^{2p}(X_{an}, \mathbb{Q}(p))$$

which is a Hodge class, that is, the image of $[Z]$ in $H^{2p}(X_{an}, \mathbb{C}(p)) = H^{2p}(X_{an}, \mathbb{C})(p)$ lies, by remark 12, in

$$F^0 H^{2p}(X, \mathbb{C})(p) = F^p H^{2p}(X, \mathbb{C}).$$

Given any automorphism σ of $\mathbb{C}$, we can form the conjugate variety X^σ defined as $X \times_\sigma Spec(\mathbb{C})$, the unique complex algebraic variety such that the following cartesian diagram commutes

$$\begin{array}{ccc} X^\sigma & \xrightarrow{\sigma^{-1}} & X \\ \downarrow & & \downarrow \\ Spec(\mathbb{C}) & \xrightarrow{\sigma^*} & Spec(\mathbb{C}) \end{array}$$

X^σ is a smooth projective variety. If X is defined by homogeneus polynomials $p_1, \dots, p_r$ in some projective space, then X^σ is defined by the conjugates of the p_i by σ. In this case, the morphism from X^σ to X in the cartesian diagram sends the closed point with coordinates $(x_0 \dots x_n)$ to the closed points $(\sigma^{-1}(x_0) \dots \sigma^{-1}(x_n))$, which allow us to denote it by σ^{-1}. Let us note that X^σ is in general not homotopically equivalent to X. However, the pull-back of Kähler forms still induces an isomorphism between the de Rham complexes of X and X^σ

$$(\sigma^{-1})^* \Omega^\bullet_{X/\mathbb{C}} \xrightarrow{\cong} \Omega^\bullet_{X^\sigma/\mathbb{C}}.$$

Taking hypercohomology and using theorem 4.1.1, we get an isomorphism

$$(\sigma^{-1})^* : H^*(X/\mathbb{C}) \xrightarrow{\cong} H^*(X^\sigma, \mathbb{C}), \quad \alpha \mapsto \alpha^\sigma.$$

This isomorphism is not $\mathbb{C}$-linear, but $\sigma(\mathbb{C})$-linear i.e. for all $\lambda \in \mathbb{C}$ we have $(\lambda\alpha)^\sigma = \sigma(\lambda)\alpha^\sigma$. We thus get an isomorphism of complex vector spaces

$$H^*(X, \mathbb{C}) \otimes_\sigma \mathbb{C} \cong H^*(X^\sigma/\mathbb{C})$$

where the notation $\otimes_\sigma$ means that we are taking tensor product with $\mathbb{C}$ mapping to $\mathbb{C}$ via the morphism σ. Let us note that, since this isomorphism comes from an isomorphism of the de Rham complexes, it preserves Hodge filtration.

We can now apply these considerations to the cycle class of an algebraic cycle $Z \subset X$, of codimension p and form its conjugate $[Z^\sigma] \in H^{2p}(X^\sigma/\mathbb{C})(p)$. The construction of the cycle class map in de Rham cohomology shows that

$$[Z^\sigma] = [Z]^\sigma.$$

Since X^σ is a smooth projective complex variety, its de Rham cohomology group $H^{2p}(X^\sigma/\mathbb{C})(p)$ is canonically isomorphic to the singular cohomology group $H^{2p}(X^\sigma_{an}/\mathbb{C}(p))$. The cohomology class $[Z^\sigma]$ in $H^{2p}(X^\sigma_{an}/\mathbb{C}(p)) \cong H^{2p}(X^\sigma_{an}/\mathbb{C})(p)$ is a Hodge class. This leads to following definition of absolute Hodge class.

Definition 4.3 (**Absolute Hodge class**). *Let X be a smooth complex projective variety. Let p be a nonnegative integer and let* α *be an element of* $H^{2p}(X/\mathbb{C}(p))$. *We say that the cohomology class* α *is an absolute Hodge class if for every automorphism* σ *of* $\mathbb{C}$ *the cohomology class* $\alpha^\sigma \in H^{2p}(X^\sigma_{an}/\mathbb{C}(p)) \cong H^{2p}(X^\sigma_{an}/\mathbb{C})(p)$ *is a Hodge class.*

Using the canonical isomorphism $H^{2p}(X_{an}/\mathbb{C}(p)) \cong H^{2p}(X/\mathbb{C})(p)$ we will say that a class in $H^{2p}(X_{an}/\mathbb{C})$ is absolute Hodge if its image in $H^{2p}(X/\mathbb{C})(p)$ is.

The preceding discussion shows the following fact:

Proposition 4.2.1. *The cohomology class of an algebraic cycle is an absolute Hodge class. In particular, taking* $\sigma = Id_{\mathbb{C}}$, *we see that absolute Hodge classes are Hodge classes.*

4.2.2 Algebraic cycles, absolute Hodge classes and Hodge conjecture

Let X be a smooth projective variety over $\mathbb{C}$, as above. The singular cohomology group of X are endowed with a pure Hodge structure such that for any integer p, $H^{2p}(X;\mathbb{Z}(p))$ has weight 0. We denote by $Hdg^p(X)$ the group of Hodge classes in $H^{2p}(X;\mathbb{Q}(p))$. As we showed earlier, if Z is a subvariety of X of codimension p, its cohomology class $[Z] \in H^{2p}(X;\mathbb{Q}(p))$ is a Hodge class. The Hodge conjecture states that the cohomology classes of subvariety of X span the $\mathbb{Q}$-vector space generated by Hodge classes.

Conjecture 4.2.2 (**Hodge Conjecture**). *Let* X *be a smooth projective variety over* $\mathbb{C}$. *For any nonnegative integer p, the subspace of degree p rational Hodge classes*

$$Hdg^p(X) \otimes \mathbb{Q} \subset H^{2p}(X, \mathbb{Q}(p))$$

is generated over $\mathbb{Q}$ *by the cohomology classes of codimension p subvarieties of* X.

Proposition 4.2.1 allow us to split the Hodge conjecture in two subconjectures:

Conjecture 4.2.3. *Hodge classes on smooth complex projective varities are absolute Hodge*

Conjecture 4.2.4. *Let X be a smooth complex projective variety. Absolute Hodge classes on X are generated over $\mathbb{Q}$ by algebraic cycle classes.*

It is clear that these conjectures together imply the Hodge conjecture. Conjecture 4.2.3 was solved affermatively by Deligne in [08] for Hodge classes on abelian varities.

4.2.3 Generalization to the compact Kähler case

If X is only assumed to be a compact Kähler manifold, the cohomology groups $H^{2p}(X, \mathbb{Z}(p))$ still carry Hodge structures, and analytic subvarities of X still give rise to Hodge classes. However, while a general compact Kähler manifold can have very few analytic subvarieties, Chern classes of coherent sheaves are Hodge classes on the cohomology of X (see [25]). While on a smooth projective complex variety analytic subvarieties are algebraic by GAGA, and Chern classes of coherent sheaves are linear combinations of cohomology classes of algebraic subvarieties (in particular they are classes of divisors), this is no longer true on a general compact Kähler manifold. Indeed, Chern classes of coherent sheaves can generate a strictly larger subspace than that generated by the cohomology classes of analytic subvarieties. One could wonder whether it is possible to generalize the Hodge conjecture to compact Kähler manifolds by asking whether Chern classes of coherent sheaves generate the space of Hodge classes. The answer is still negative, because Voisin proved in [23] that on a Weil torus W the group $Hdg^2(W)$ is torsion free, while for any coherent sheaf $\mathcal{F}$ on W the second Chern classes of $\mathcal{F}$ is 0.
Moreover neither of the two subconjectures make sense in the setting of Kähler manifolds. In fact, automorphisms of $\mathbb{C}$ other than the identity and complex conjugation are very discontinuous, even not measurable. This makes it impossible to define the variety X^σ and absolute Hodge classes.
Even for algebraic varieties, the fact that automorphisms of $\mathbb{C}$ are discontinuous appears: the main problem is that it is not to be expected that the σ-linear isomorphism

$$(\sigma^{-1})^* : H^*(X_{an}, \mathbb{C}) \xrightarrow{\cong} H^*(X_{an}^\sigma, \mathbb{C})$$

maps $H^*(X_{an}, \mathbb{Q})$ to $H^*(X^\sigma_{an}, \mathbb{Q})$ (for example one can see [05]).

4.2.4 Second definition of absolute Hodge class

We can rephrase the definition of absolute Hodge cycles in a slightly more intrinsic way. Let K a fiel of characteristic 0 and let X a smooth projective variety over K. Assume that there exists embeddings of K into $\mathbb{C}$.

Definition 4.4. *Let p be a positive integer and let α be an element of the de Rham cohomology space $H^{2p}(X/K)$. Let τ be an embedding of K into $\mathbb{C}$, and let τX denote the complex variety obtained from X by base change to $\mathbb{C}$. We say that α is a Hodge class relative to τ if the image of α in*

$$H^{2p}(\tau X/\mathbb{C}) = H^{2p}(X/K) \otimes_\tau \mathbb{C}$$

is a Hodge class. We say that α it is absolute if it is a Hodge class relative to every embedding of K into $\mathbb{C}$

Let τ be any embedding of K into $\mathbb{C}$. Since by standard field theory, any two embeddings of K into $\mathbb{C}$ are conjugated by an automorphism of $\mathbb{C}$, it is straightforward to check that such a cohomology class α is absolute Hodge if and only if its image in $H^{2p}(\tau X/\mathbb{C})$ is. The second definition we have given has the advantage of not involving automorphism of $\mathbb{C}$, and allow us to work with absolute Hodge classes in a wider setting by using other cohomology theories such as étale cohomology (see [08] for further details).

4.3 Functoriality properties of absolute Hodge classes

We now want to show some functoriality properties of absolute Hodge classes and give further examples. Firstly, we give a third more general definitions of absolute Hodge class, which allows us to exhibit elementary cases.

Definition 4.5. *Let K be a field of characteristic 0 with cardinality less or equal than the cardinality of $\mathbb{C}$. Let $(X_i)_{i\in I}$ and $(X_j)_{j\in J}$ be smooth projective varieties over $\mathbb{C}$, and let $(p_i)_{i\in I}$,*

$(q_j)_{j\in J}$, n be integers. If α is an element of the tensor product

$$(\bigotimes_{i\in I} H^{p_i}(X_i/K)) \otimes (\bigotimes_{j\in J} H^{q_j}(X_j/K)^*)(n)$$

and τ is an embedding of K into $\mathbb{C}$, we say that α is a Hodge class relative to τ if its image in

$$\begin{aligned}&(\bigotimes_{i\in I} H^{p_i}(X_i/K)) \otimes (\bigotimes_{j\in J} H^{q_j}(X_j/K)^*)(n) \otimes_\tau \mathbb{C}\\ = &(\bigotimes_{i\in I} H^{p_i}(\tau X_i/\mathbb{C})) \otimes (\bigotimes_{j\in J} H^{q_j}(\tau X_j/\mathbb{C})^*)(n)\end{aligned}$$

is a Hodge class. We say that α is absolute Hodge if it is a Hodge class relative to every embedding of K into $\mathbb{C}$.

As before, taking $K = \mathbb{Q}$, we can speak of absolute Hodge classes in the group

$$(\bigotimes_{i\in I} H^{p_i}(X_i/\mathbb{Q})) \otimes (\bigotimes_{j\in J} H^{q_j}(X_j/\mathbb{Q})^*)(n).$$

If X and Y are two smooth projective complex varieties, and if

$$f : H^p(X, \mathbb{Q}(i)) \to H^q(Y, \mathbb{Q}(j))$$

is a morphism of Hodge structures, we will say that f is absolute Hodge - or, more precisely, is given by an absolute Hodge class - if the element corresponding[1] to f in

$$H^q(Y, \mathbb{Q}) \otimes H^p(X, \mathbb{Q})^*(j - i)$$

is an absolute Hodge class. Similarly, we can define what it means for a multilinear form (and so a polarization), to be absolute Hodge. With such a definition we are now able to present some examples.

Example 4.2. Consider the cup product

$$H^p(X, \mathbb{Q}) \otimes H^q(X, \mathbb{Q}) \to H^{p+q}(X, \mathbb{Q}).$$

[1]We have canonical isomorphisms $Hom(H^p(X, \mathbb{Q}(i), H^q(Y, \mathbb{Q}(j))) \cong H^q(Y, \mathbb{Q}(j)) \otimes H^p(X, \mathbb{Q}(i))^* \cong H^q(Y, \mathbb{Q}) \otimes H^p(X, \mathbb{Q})^*(j - i)$

this map is given by an absolute Hodge class.
In fact, given an embedding τ of $\mathbb{Q}$ into $\mathbb{C}$, we have the induced map

$$H^p(\tau X, \mathbb{C}) \otimes H^q(\tau X, \mathbb{C}) \to H^{p+q}(\tau X, \mathbb{C})$$

which is the cup-product on the de Rham cohomology of τX. Since we know that it is compatible with Hodge structures (see [25], chap. 7) we deduce that it is given by a Hodge class.

Morphism given by absolute Hodge classes behave in a functorial way.

Proposition 4.3.1. *Let X, Y, Z be smooth projective varieties of dimension respectively n, m, l and let*

$$f : H^p(X, \mathbb{Q}(i)) \to H^q(Y, \mathbb{Q}(j)), \quad g : H^q(Y, \mathbb{Q}(j)) \to H^r(Z, \mathbb{Q}(k))$$

be morphism of Hodge structures. Then

(i) If f is induced by an algebraic correspondence, then f is absolute Hodge

(ii) If f and g are absolute Hodge, then $g \circ f$ is absolute Hodge.

(iii) Let

$$f^t : H^{2m-q}(Y, \mathbb{Q}(m-j)) \to H^{2n-p}(X, \mathbb{Q}(n-i))$$

be the adjoint operator of f respect to Poincaré duality. Then f is absolute Hodge if and only if f^t is.

(iv) If f is an isomorphism, then is absolute Hodge if and only if f^{-1} is absolute Hodge.

We will need a refinement of the last property of proposition 4.3.1 as follows.

Proposition 4.3.2. *Let X and Y be smooth projective complex varieties, and let*

$$p : H^p(X, \mathbb{Q}(i)) \to H^p(X, \mathbb{Q}(i)) \text{ and } q : H^q(X, \mathbb{Q}(j)) \to H^q(X, \mathbb{Q}(j))$$

be projectors. Assume that p and q are absolute Hodge. Let V (resp. W) be the image of p (resp. q) and let

$$f \circ p : H^p(X, \mathbb{Q}(i)) \to H^q(Y, \mathbb{Q}(j))$$

be absolute Hodge. Assume that $q \circ f \circ p$ induces an isomorphism from V to W. Then the composition

$$p : H^q(Y, \mathbb{Q}(j)) \to W \xrightarrow{(q \circ f \circ p)^{-1}} V \hookrightarrow H^p(X, \mathbb{Q}(i))$$

is absolute Hodge.

Proof. We need to check that after conjugating by any automorphism of $\mathbb{C}$, the above composition is given by a Hodge class. Since we know that if q, f and p are absolute Hodge also their inverse are, we only have to check this for the identity automorphism, which is the case. □

4.4 Examples of absolute Hodge classes

The Künneth components

Let X be a smooth projective complex variety of dimension n, and let Δ be the diagonal of $X \times X$. We notice that Δ is an algebraic cycle of codimension n in $X \times X$, hence we can define its cohomology class $[\Delta] \in H^{2n}(X \times X, \mathbb{Q}(n))$. By the Künneth formula for cohomology, there is an isomorphism of Hodge structures

$$H^{2n}(X \times X, \mathbb{Q}) \cong \bigoplus_{i=0}^{2n} H^i(X, \mathbb{Q}) \otimes H^{2n-i}(X, \mathbb{Q})$$

and also projections $H^{2n}(X \times X, \mathbb{Q}) \to H^i(X, \mathbb{Q}) \otimes H^{2n-i}(X, \mathbb{Q})$. We denote by π_i the component of $[\Delta]$ in $H^i(X, \mathbb{Q}) \otimes H^{2n-i}(X, \mathbb{Q})(n) \subset H^{2n}(X \times X, \mathbb{Q})(n)$ and we call them the Künneth components of $[\Delta]$. By definition, the Künneth components are Hodge classes. Now, let σ be an automorphism of $\mathbb{C}$ and denote by Δ^σ the diagonal of $X^\sigma \times X^\sigma = (X \times X)^\sigma$. Let $\pi_{i,dR}$ (resp. $\pi^\sigma_{i,dR}$) denotes the de Rham representative of π_i (π_i^σ). As the Künneth formula holds for de Rham cohomology and it is compatible with the comparison theorem, it follows that

$$(\pi_{i,dR})^\sigma = \pi^\sigma_{i,dR}.$$

Now $\pi^{\sigma}_{i,dR}$ are Hodge classes because they are the Künneth components of Δ^{σ}, thus $(\pi_{i,dR})^{\sigma}$ are Hodge classes as well.

Projections on the primitive components

Fix an embedding of X into a projective space, and let $h \in H^2(X, \mathbb{Q}(1))$ be the cohomology class of a hyperplane section. The hard Lefschetz theorem states that for all $i \leq n = dim_{\mathbb{C}} X$, the morphism

$$L^{n-k} = \cup h^{n-k} : H^k(X, \mathbb{Q}) \to H^{2n-k}(X, \mathbb{Q}(n-k)), \quad x \mapsto x \cup h^{n-k}$$

is an isomorphism. In example 4.2 we have seen that this map is absolute Hodge, thus its inverse

$$f_k : H^{2n-k}(X, \mathbb{Q}(n-k)) \to H^k(X, \mathbb{Q})$$

is absolute Hodge.

As an immediate corollary we obtain the following result.

Proposition 4.4.1. *Let k be an integer such that $2k \leq n$. An element $\alpha \in H^{2k}(X, \mathbb{Q})$ is an absolute Hodge class in and only if $x \cup h^{n-2k} \in H^{2n-2k}(X, \mathbb{Q}(n-2k))$ is an absolute Hodge class.*

Now we can show that the projections on the primitive components of the Lefschetz decomposition are absolute Hodge.

Proposition 4.4.2. *Let X be a smooth projective complex variety of dimension n, and let $h \in H^2(X, \mathbb{Q}(1))$ be the cohomology class of a hyperplane section. Let L denote the operator given by cup-product with h. Consider the Lefschetz decomposition*

$$H^k(X, \mathbb{Q}) = \bigoplus_{j \geq 0} L^i H^{k-2i}(X, \mathbb{Q})_{prim}$$

of the cohomology of X into primitive parts. Then the projection of $H^k(X, \mathbb{Q})$ onto the component $L^i H^{k-2i}(X, \mathbb{Q})_{prim}$ with respect to the Lefschetz decomposition is given by an absolute Hodge class.

Proof. By induction, it is enough to prove that the projection of $H^k(X,\mathbb{Q})$ onto $LH^{k-2}(X,\mathbb{Q})$ is given by an absolute Hodge class. While this could be proved by the same argument as in example 4.2, consider the composition

$$L \circ f_k \circ L^{n-k+1} : H^k(X,\mathbb{Q}) \to H^k(X,\mathbb{Q})$$

where f_k is the inverse of the Lefschetz operator as above. It is the desired projection since $H^k(X,\mathbb{Q})_{prim}$ is the Kernel of L^{n-k+1} in $H^K(X,\mathbb{Q})$. □

Remark 13. *It is currently unknown if the Künneth projectors or the inverse of the Lefschetz operator are classes of algebraic cycles. Roughly speaking this is the content of the set of the so-called "Standard conjectures" of Grothendieck. The lack of progress on these foundational questions led Deligne to introduce the notion of absolute cycles discussed here, which is strong enough to develop some part of the theory of motives. See ([01], §5).*

The proof of proposition 4.4.2 allows for the following result, which shows that the Hodge structures on the cohomology of smooth projective varieties can be polarized by absolute Hodge classes.

Proposition 4.4.3. *Let X be a smooth projective complex variety and k be an integer. There exists an absolute Hodge class giving a pairing*

$$Q : H^k((X,\mathbb{Q}) \otimes H^k(X,\mathbb{Q}) \to \mathbb{Q}(-k)$$

which turns $H^k(X,\mathbb{Q})$ into a polarized Hodge structure.

Proof. Let n be the dimension of X. By the Hard Lefschetz theorem, we can assume $k \leq n$. Let H be an ample line bundle on X with first Chern calss $h \in H^2(X,\mathbb{Q}(1))$, and let L be the endomorphism of the cohomology of X given bu the cup-product with h. Consider the Lefschetz decomposition

$$H^k(X,\mathbb{Q}) = \bigoplus_{i\geq 0} L^i H^{k-2i}(X,\mathbb{Q})_{prim}$$

of $H^k(X,\mathbb{Q})$ into primitive parts. Let s be the linear automorphism of $H^k(X,\mathbb{Q})$ which is given by multiplication by $(-1)^i$ on $L^iH^{k-2i}(X,\mathbb{Q})_{prim}$.

By the Hodge index theorem, the pairing

$$H^k(X, q) \otimes H^k(X, \mathbb{Q}) \to \mathbb{Q}(1), \qquad \alpha \otimes \beta \mapsto \int_X \alpha \cup L^{n-k}(s(\beta))$$

turns $H^k(X, \mathbb{Q})$ into a polarized Hodge structure. By proposition 4.4.2, the projections on $H^k(X, \mathbb{Q})$ onto the factors $L^i H^{k-2i}(X, \mathbb{Q})_{prim}$ are given by absolute Hodge classes. It follows that the morphism s is given by an absolute Hodge class as well.

Since also the cup-product is given by an absolute Hodge class and L is induced by an algebraic correspondence- and so it absolute Hodge-, it follows that the pairing Q is given by an absolute Hodge class, which concludes the proof of the proposition. □

Chapter 5

Deligne's principle B

5.1 Absolute Hodge classes in families

5.1.1 Variational Hodge conjecture

Let B be a smooth connected complex quasi-projective variety, and let $\pi : X \to B$ be a smooth projective morphism. Let 0 be a complex point of B, and let α be a cohomology class in $H^{2p}(X_0, \mathbb{Q}(p))$. Recalling that $X_0 = \pi^{-1}(0)$ and the definition of direct image, assume that α is the cohomology class of some codimension p algebraic cycle Z_0 and that it extends as a section $\tilde{\alpha}$ of the local system $R^{2p}\pi_*\mathbb{Q}(p)$ on B. In [13] Grothendieck makes the following conjecture:

Conjecture 5.1.1 (**Variational Hodge conjecture**). *For any complex point $b \in B$, the class $\tilde{\alpha}_b$ is the cohomology class of an algebraic cycle.*

Using the Gauss-Manin connection and the isomorphism between de Rham and singular cohomology, we can riformulate the conjecture in terms of de Rham cohomology. If we keep notations as above, we have a coherent sheaf $\mathcal{H}^{2p} = \mathbb{R}^{2p}\pi_*\Omega^{\bullet}_{X/B}$ which computes the relative de Rham cohomology of $\mathcal{X}$ over B. We also know that it is endowed with a canonical connection, the Gauss-Manin connection ∇.

Conjecture 5.1.2 (Variational Hodge conjecture for de Rham cohomology). *Let β be a cohomology class in $H^{2p}(X_0, \mathbb{C})$. Assume that β is the cohomology class of some codimension p algebraic cycle Z_0 and that β extends as a section $\tilde{\beta}$ of the coherent sheaf $\mathcal{H}^{2p} = \mathbb{R}^{2p}\pi_*\Omega^\bullet_{X/B}$ with the property that $\tilde{\beta}$ is flat for the Gauss-Manin connection. The variational Hodge conjecture states that for any complex point $b \in B$, the class β_s is the cohomology class of an algebraic cycle.*

The two conjectures above have a different flavor: the first considers the local system $R^{2p}\pi_*\mathbb{Q}(p)$ threfore the rational structure on the cohomology groups: as such, it involves the classical topology. In contrast, the second, given the fact that the Gauss-Manin connection affords a purely algerbaic definition, involves a purely algebraic setting. However we have the following:

Proposition 5.1.3. *The two conjectures are equivalent*

Proof. As we have seen in the preceding chapter, the de Rham comparison isomorphism between singular and de Rham cohomology in a relative context (cfr. equation 3.3) takes the form of a canonical isomorphism

$$\mathbb{R}^{2p}\pi_*\Omega^\bullet_{X/S} \cong R^{2p}\pi_*\mathbb{Q}(p) \otimes_{\mathbb{Q}} \mathcal{O}_B. \tag{5.1}$$

We notice that this formula is not an algebraic geometry one. Indeed, the sheaf $\mathcal{O}_B$ denotes here the sheaf of holomorphic functions on the complex manifold B. The derived functor $\mathbb{R}^{2p}\pi_*$ on the left is a functor between categories of complexes of holomorphic coherent sheaves, while the one on the right is computed for sheaves with the usual complex topology. The Gauss-Manin connection is the connection on $\mathbb{R}^{2p}\pi_*\Omega^\bullet_{X/S}$ for which the local system $R^{2p}\pi_*\mathbb{Q}(p))$ is constant. As we saw in section 3.7, the locally free sheaf $\mathbb{R}^{2p}\pi_*\Omega^\bullet_{X/S}$ is algebraic, i.e. is induced by a locally free sheaf on the algebraic variety B, as well as the Gauss-Manin connection. Given β a cohomology class in the de Rham cohomology group $H^{2p}(X_0/\mathbb{C})$, we know that β belongs to the rational subspace $H^{2p}(X_0, \mathbb{Q}(p))$ because it is the cohomology class of an algebraic cycle. Furthermore, since β is flat for the Gauss-Manin connection and is rational at one point, it corresponds to a section of the local system $R^{2p}\pi_*\mathbb{Q}(p)$ under the comparison isomorphism above. This shows that Conjecture 5.1.1 implies Conjecture 5.1.1.

Conversely, sections of the local system $R^{2p}\pi_*\mathbb{Q}(p)$ induce flat holomorphic sections of the coherent sheaf $\mathbb{R}^{2p}\pi_*\Omega^\bullet_{X/S}$. We have to show that they are algebraic. This is a consequence of the following important result, which is due to Deligne [09].

Theorem 5.1.4 (Global invariant cycle theorem). *Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism of quasi-projective complex varieties, and let $i : \mathcal{X} \hookrightarrow \overline{\mathcal{X}}$ be a smooth compactification of X. Let $0 \in B$ a complex point of B, and $\pi_1(B, 0)$ be the fundamental group of B. For any integer k, the space of monodromy-invariant classes of degree k*

$$H^k(X_0, \mathbb{Q})^{\pi_1(B,0)}$$

is equal to the image of the restriction map

$$i_0^* : H^k(\overline{\mathcal{X}}, \mathbb{Q}) \to H^k(X_0, \mathbb{Q}),$$

where i_0 is the inclusion of X_0 in $\overline{\mathcal{X}}$.

Before concluding the proof, let us make some remarks.

Remark 14. *In the theorem, the monodromy action is the action of the fundamental group $\pi_1(B, 0)$ on the cohomology groups of the fibre X_0. We recall that, given a variation of Hodge structure $(\mathcal{H}, \nabla, F^\bullet)$ over a complex connected manifold B, the monodromy representation acts as follows. Fix $0 \in B$. Given a curve $\mu : [0, 1] \to B$, with $\mu(0) = 0$ and $\mu(1) = b$ we may define a $\mathbb{C}$-linear isomorphism*

$$\mu^* : \mathcal{H}_b \to \mathcal{H}_0$$

by parallel translation relative to the flat connection ∇. These isomorphisms depend only on the homotopy class of μ and we denote by

$$\rho : \pi_1(B, 0) \to GL(\mathcal{H}_0)$$

the resulting representation, which is called the monodromy representation. In our case, $\mathcal{H}$ is $H^{2p}(X_0, \mathbb{Q})$, ∇ is the Gauss-Manin connection and $F^\bullet$ is the Hodge filtration.

Remark 15. *While it is clear that the restriction of a cohomology class in $\mathcal{X}$ to X_0 is monodromy invariant, the converse is non trivial. The global invariant cycle theorem in fact says something stronger, and actually consists of two different statements: first of all, the monodromy invariants in the cohomology of the fibre are precisely the classes obtained by restriction from $\mathcal{X}$. This follows from the fact that the Leray spectral sequence for π degenerates at E_2, so that in particular the map*

$$H^k(\mathcal{X}) \to H^k(X_0, \mathbb{Q})^{\pi_1(B,0)} = H^0(B, R^k\pi_*\mathbb{Q}) = E_2^{0,k}$$

is necessarily surjective. The second statement is much deeper, as it states that in

$$H^k(\overline{\mathcal{X}}) \xrightarrow{a} H^k(\mathcal{X}) \xrightarrow{b} H^k(X_0, \mathbb{Q})^{\pi_1(B,0)}$$

the two maps a and $a \circ b$ have the same image. This stronger, and in fact surprising, statement, is a consequence of the existence on the cohomology groups of X of a functorial Mixed Hodge structure, constructed in [09].

Remark 16. *Note that the theorem also implies that the space $H^k(X_0, \mathbb{Q})^{\pi_1(B,0)}$ is a sub-Hodge structure of $H^k(X_0, \mathbb{Q})$. This despite the fact that the fundamental group of B does not in general act by automorphisms of Hodge structures.*

End of the proof of proposition 5.1.3. The global invariant cycle theorem implies the algebraicity of flat holomorphic sections of the vector bundle $\mathbb{R}^{2p}\pi_*\Omega^\bullet_{X/B}$ as follows. Let $\tilde{\beta}$ be such a section and keep the notation of the theorem. By definition of the Gauss-Manin connection, $\tilde{\beta}$ corresponds to a section of the local system $R^{2p}\pi_*\Omega^\bullet_{X/B}$ under the isomorphism 5.1, that is, to a monodromy-invariant class in $H^{2p}(X_0, \mathbb{C})$. The global invariant cycle theorem shows, using the comparison theorem between singular and de Rham cohomology on X , that $\tilde{\beta}$ comes from a de Rham cohomology class b in $H^{2p}(X, \mathbb{C})$. As such, it is algebraic. The previous remarks readily show the equivalence of the two versions of the variational Hodge conjecture. □

The next proposition shows that the variational Hodge conjecture is implied by the Hodge conjecture.

Proposition 5.1.5. *Let B be a smooth connected quasi projective variety, and let $\pi : \mathcal{X} \to B$ be a smooth projective morphism. Let $0 \in B$ be a complex point of B and let p be an integer.*

(i) Let α be a cohomology class in $H^{2p}(X_0, \mathbb{Q}(p))$. Assume that α is a Hodge class and that it extends as a section $\tilde{\alpha}$ of the local system $R^{2p}\pi_*\mathbb{Q}(p)$ on B. Then for any complex point $t \in B$, the class $\tilde{\alpha}_t$ is a Hodge class.

(ii) Let β be a cohomology class in $H^{2p}(X_0, \mathbb{C})$. Assume that β is a Hodge class and that it extends as a section $\tilde{\beta}$ of the coherent sheaf $R^{2p}\pi_*\Omega^\bullet_{X/B}$ such that β is flat for the Gauss-Manin connection. Then for any complex point $t \in B$, the class $\tilde{\beta}_t$ is a Hodge class.

As an immediate corollary, we have:

Corollary 5.1.6. *The Hodge conjecture implies the variational Hodge conjecture.*

Proof. The two statements are equivalent by the argument of proposition 5.1.3. Let us keep the notations as above. We want to prove that for any complex point $t \in B$, the class $\tilde{\alpha}_t$ is a Hodge class. Let us show how this is a consequence of the global invariant cycle theorem. This is a simple consequence of 4.3.2 in the – easier – context of Hodge classes. Let us prove the result from scratch. As in Proposition 4.4.3, we can find a pairing

$$H^{2p}(X, \mathbb{Q}) \otimes H^{2p}(X, \mathbb{Q}) \to \mathbb{Q}(1)$$

which turns $H^{2p}(X, \mathbb{Q})$ into a polarized Hodge structure. Let $i : X \hookrightarrow \overline{X}$ be a smooth compactification of X, and let i_0 be the inclusion of X_0 in X. By the global invariant cycle theorem, the morphism

$$i_0^* : H^{2p}(X, \mathbb{Q}) \to H^{2p}(X_0, \mathbb{Q})^{\pi_1(B,0)}$$

is surjective. It restricts to an isomorphism of Hodge structures

$$i_0^* : (Ker i_0^*)^\perp \to H^{2p}(X_0, \mathbb{Q})^{\pi_1(B,0)},$$

hence a Hodge class $a \in (Ker i_0^*)^\perp \subset H^{2p}(\overline{X}, \mathbb{Q})$ mapping to α. Indeed, saying that α extends to a global section of the local system $R^{2p}\pi_*\mathbb{Q}(p)$ exactly means that α is monodromy-invariant.

Now let i_s be the inclusion of X_t in $\overline{X}$. Since B is connected, we have $\tilde{\alpha}_t = i_t^*(a)$, which shows that $\tilde{\alpha}_t$ is an Hodge class. □

5.2 Deligne's principle B

Theorem 5.2.1 (Principle B)**.** *Let B be a smooth connected complex quasi-projective variety, and let $\pi : \mathcal{X} \to B$ be a smooth projective morphism. Let $0 \in B$ be a complex point of B and, for some integer p, let α be a cohomology class in $H^{2p}(X_0, \mathbb{Q}(p))$. Assume that α is an absolute Hodge class and that it extends as a section $\tilde{\alpha}$ of the local system $R^{2p}\pi_*\mathbb{Q}(p)$ on B. Then for any complex point $b \in B$, the class $\tilde{\alpha}_b$ is absolute Hodge.*

We can rephrase it for de Rham cohomology in the following way:

Theorem 5.2.2 (Principle B for de Rham cohomology)**.** *Let B be a smooth connected complex quasi-projective variety, and let $\pi : \mathcal{X} \to B$ be a smooth projective morphism. Let $0 \in B$ be a complex point of B and, let β be a cohomology class in $H^{2p}(X_0, \mathbb{C})$. Assume that β is an absolute Hodge class and that it extends as a section $\tilde{\beta}$ of the coherent sheaf $R^{2p}\pi_*\Omega^{\bullet}_{X/B}$ which is flat for the Gauss-Manin connection. Then for any complex point $b \in B$, the class $\tilde{\beta}_b$ is absolute Hodge.*

Proof. We work with de Rham cohomology. Let σ be an automorphism of $\mathbb{C}$. Since $\tilde{\beta}$ is a global section of the locally free sheaf $\mathcal{H}^{2p}$, we can form the conjugate section $\tilde{\beta}^{\sigma}$ of the conjugate sheaf $(\mathcal{H}^{2p})^{\sigma}$ on B^{σ}. Now, this sheaf identifies with the relative de Rham cohomology of X^{σ} over B^{σ}. Fix a complex point $t \in B$. We want to show that the class $\tilde{\beta}_t$ is absolute Hodge. This means that for any automorphism σ of $\mathbb{C}$, the class $\tilde{\beta}^{\sigma}_{\sigma(t)}$ is a Hodge class in the cohomology of $X^{\sigma}_{\sigma(t)}$. Now since $\beta = \tilde{\beta}_0$ is an absolute Hodge class by assumption, $\tilde{\beta}^{\sigma}_{\sigma(0)}$ is a Hodge class. Since the construction of the Gauss-Manin connection commutes with the base change via σ, the Gauss-Manin connection ∇^{σ} on the relative de Rham cohomology of X^{σ} over B^{σ} is the conjugate by σ of the Gauss-Manin connection on $\mathcal{H}^{2p}$. These remarks allow us to write

$$\nabla^{\sigma}\tilde{\beta}^{\sigma} = (\nabla\tilde{\beta})^{\sigma} = 0$$

since $\tilde{\beta}$ is flat. This shows that $\tilde{\beta}^\sigma$ is a flat section of the relative de Rham cohomology of X^σ over B^σ. Since $\tilde{\beta}^\sigma_{\sigma(0)}$ is a Hodge class, proposition 5.1.5 shows that $\tilde{\beta}^\sigma_{\sigma(t)}$ is a Hodge class, which is what we needed to prove. □

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Ringraziamenti

Quando ho cominciato l'Università ero convinta che avrei fatto qualcosa nell'ambito della matematica finanziaria o, quantomeno, che mi sarei sicuramente data all'ambito applicativo. Oggi, in tutta coerenza, presento una tesi in geometria algebrica.
Per tale scelta, più di ogni altra persona devo ringraziare il Professor Luca Migliorini perché, come dice lui, bisogna insegnarmi tutto e da parte sua non sono mai mancate la pazienza e la disponibilità di farlo. Un ringraziamento particolare va anche al Dottor Marco Trozzo, per tutte le ore passate a spiegarmi ciò che non capivo e perché è davvero una bella persona. Grazie ad Alessio, Michele, Matteo, Lia, Giacomo, Lorenzo e Giulio: è merito loro se entro in dipartimento tutte le mattine con il sorriso ed esco (spesso ad orari improbabili) con quello stesso sorriso. Grazie a Caterina, Lisa, Letizia, Stefania, Maria Chiara, Valentina ed Emanuele che non mi fanno mai mancare il loro appoggio e che sono fonte del mio inguaribile buonumore. Infine grazie, grazie e ancora grazie alla mia famiglia che mi sostiene sempre e che rimane il mio punto di riferimento nei momenti più luminosi come in quelli più bui.