ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

Scuola di Scienze
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Symplectic Reduction, with an application to Calogero-Moser systems

Relatore:
Prof. Luca Migliorini

Presentata da:
Carlo Heissenberg

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Sommario

Lo scopo del presente lavoro è di illustrare alcuni temi di geometria simplettica, i cui risultati possono essere applicati con successo al problema dell'integrazione dei sistemi dinamici.
Nella prima parte si formalizza il teorema di Noether generalizzato, introducendo il concetto dell'applicazione momento, e si dà una descrizione dettagliata del processo di riduzione simplettica, che consiste nello sfruttare le simmetrie di un sistema fisico, ovvero l'invarianza sotto l'azione di un gruppo dato, al fine di eliminarne i gradi di libertà ridondanti.
Nella seconda parte, in quanto risultato notevole reso possibile dalla teoria suesposta, si fornisce una panoramica dei sistemi di tipo Calogero-Moser: sistemi totalmente integrabili che possono essere introdotti e risolti usando la tecnica della riduzione simplettica.

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Symplectic Geometry

1 Preliminary Notions

1.1 Derivative of a mapping

Let M and N be differentiable manifolds and let f be a mapping

$$f : M \to N$$

which is differentiable, in the sense that f gives rise, when expressed in local coordinates of M and N, to differentiable functions.

Definition 1.1. The **derivative** of the differentiable function $f : M \to N$ in $\boldsymbol{x} \in M$ is the linear map between tangent spaces

$$f_{*\boldsymbol{x}} : TM_{\boldsymbol{x}} \to TN_{f(\boldsymbol{x})}$$

defined as follows. Given a curve $\varphi : \mathbb{R} \to M$, $\varphi(0) = \boldsymbol{x}$ with velocity vector $\frac{d\varphi}{dt}\big|_{t=0} = \boldsymbol{v}$, then $f_{*\boldsymbol{x}}\boldsymbol{v}$ is the velocity vector of the curve $f \circ \varphi : \mathbb{R} \to N$, thus

$$f_{*\boldsymbol{x}}\boldsymbol{v} \doteq \frac{d}{dt}\Big|_{t=0} f(\varphi(t)). \tag{1.1}$$

We will often use the equivalent notation : $f_* \equiv df$.

Remark 1.1. $f_{*\boldsymbol{x}}\boldsymbol{v}$ does not depend on the curve $\varphi(t)$ but only on the vector $\boldsymbol{v}$, and indeed, given any curve γ, with $\frac{d\gamma}{ds}\big|_{s=0} = \boldsymbol{v}$, we obtain, in local coordinates x^i on M:

$$f_{*\boldsymbol{x}}\boldsymbol{v} = \frac{d}{ds}\Big|_{s=0} f(\gamma(s)) = \frac{\partial f}{\partial x^i}\frac{dx^i}{ds} = \frac{\partial f}{\partial x^i}v^i;$$

$f_{*\boldsymbol{x}} : TM_{\boldsymbol{x}} \to TN_{f(\boldsymbol{x})}$ is manifestly linear, since d/dt is a linear operator.

1.2 Pullback defined by a Mapping

The dual mapping to the one we just introduced is called *pullback* and is described as follows.

Definition 1.2. Let M and N be differentiable manifolds and let f be a differentiable mapping between them

$$f : M \to N.$$

Given a one-form α, defining for any $\boldsymbol{p} \in N$ a linear map of the tangent space to N at $\boldsymbol{p}$

$$\alpha_{\boldsymbol{p}} : T_{\boldsymbol{p}}N \to \mathbb{R},$$

we define the **pullback** of α through f onto M as the map acting, at a given $\boldsymbol{x} \in M$, as:

$$(f^*\alpha)_{\boldsymbol{x}}(\boldsymbol{v}) \doteq \alpha_{f(\boldsymbol{x})}(df_{\boldsymbol{x}}(\boldsymbol{v})). \tag{1.2}$$

The generalization to a $k-$form β is straightforward:

$$(f^*\alpha)_{\boldsymbol{x}}(\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k) \doteq \alpha_{f(\boldsymbol{x})}(df_{\boldsymbol{x}}(\boldsymbol{v}_1), df_{\boldsymbol{x}}(\boldsymbol{v}_2), \dots, df_{\boldsymbol{x}}(\boldsymbol{v}_k)). \tag{1.3}$$

2 Symplectic Manifolds

2.1 Symplectic Forms

Let V be an m-dimensional vector space over $\mathbb{R}$, and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map. The map Ω is **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u)$, for all u, $v \in V$.

Theorem 2.1. *Let Ω be a skew-symmetric bilinear map on V. Then there is a basis $u_1, \dots, u_k, e_1, \dots, e_n, f_1, \dots, f_n$ of V such that*

$$\begin{aligned}
&\Omega(u_i, v) = 0, \text{ for all } v \in V, \\
&\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \\
&\Omega(e_i, f_j) = \delta_{ij}.
\end{aligned}$$

Such base, even though not unique, is often called a "canonical" basis.

Proof. Let $U \doteq \{u \in V | \Omega(u,v) = 0 \text{ for all } v \in V\}$.Choose a basis $u_1, \dots, u_k$ of U and choose a complementary space W to U in V:

$$V = U \oplus W.$$

Take any nonzero $e_1 \in W$. Then there is f_1 such that $\Omega(e_1, f_1) \neq 0$ and by rescaling we can obtain $\Omega(e_1, f_1) = 1$. Let

$$\begin{aligned} W_1 &= \mathrm{Span}\{e_1, f_1\} \\ W_1^{\Omega} &= \{w \in W | \Omega(w,v) = 0 \text{ for all } v \in W_1 \}. \end{aligned}$$

First we observe that $W_1 \cap W_1^{\Omega} = \{0\}$.

Indeed, suppose that $v = ae_1 + bf_1 \in W_1 \cap W_1^{\Omega}$, then $0 = \Omega(v, e_1) = -b$ and also $0 = \Omega(v, f_1) = a$.

Furthermore $W = W_1 \oplus W_1^{\Omega}$.

This holds because, given $v \in W$, with $\Omega(v, e_1) = h$, $\Omega(v, f_1) = k$, we can expand it as

$$v = \underbrace{(-hf_1 + ke_1)}_{\in W_1} + \underbrace{(v + hf_1 - ke_1)}_{\in W_1^{\Omega}}.$$

To move one step forward, consider $e_2 \in W_1^{\Omega}$, $e_2 \neq 0$. We can find $f_2 \in W_1^{\Omega}$ such that $\Omega(e_2, f_2) = 1$, then we consider $W_2 = \mathrm{span}\{e_2, f_2\}$ and so forth.

This process eventually comes to an end since $\dim V$ is finite. □

Consider now the map $\tilde{\Omega}$ defined by

$$\tilde{\Omega} : V \to V^*, \qquad \tilde{\Omega}(v)(\ ) = \Omega(v,\ ). \tag{2.1}$$

Its kernel is the set where Ω is "degenerate" since

$$\mathrm{Ker}\tilde{\Omega} = \{v \in V | \text{ for all } u \in V,\ \Omega(v,u) = 0\}.$$

Remark 2.1. $k \doteq \dim \mathrm{Ker}\Omega$ is an invariant of (V, Ω); thus, since $k + 2n = \dim V$, also n is an invariant of (V, Ω), called sometimes *rank* of Ω.

Definition 2.1. A skew-symmetric bilinear map Ω is **symplectic** or **non degenerate** if $\tilde{\Omega}$ is bijective, i.e. $\mathrm{Ker}\tilde{\Omega} = \{0\}$. The map Ω is then called a **linear symplectic structure** on V, and (V, Ω) is called a **symplectic vector space.**

Here we list some immediate properties of a linear symplectic structure Ω:

- Duality: the map $\tilde{\Omega} : V \xrightarrow{\simeq} V^*$ is a bijection.
- $k = 0$, so dimV $= 2n$ is **even**.
- By Theorem 2.1 a symplectic vector space (V, Ω) has a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ satisfying
$$\Omega(e_i, f_j) = \delta_{ij} \quad \text{and} \quad \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0.$$
  Such basis is called a **symplectic basis** of (V, Ω).

The last property tells us that any symplectic vector field is isomorphic to $(\mathbb{R}^{2n}, \Omega_0)$ where $e_i = (\ldots \delta_{ij} \ldots)$, $f_i = (\ldots \delta_{i(j-n)} \ldots)$ is a symplectic basis. This is the *prototype* of a symplectic vector space.

Definition 2.2. A **symplectomorphism** φ between symplectic vector spaces (V, Ω) and (V', Ω') is a linear isomorphism $\varphi : V \xrightarrow{\simeq} V'$ such that

$$\varphi^* \Omega' = \Omega,$$

where $(\varphi^* \Omega')(u, v) = \Omega'(\varphi(u), \varphi(v))$. If V and V' are linked by a symplectomorphism, they are said to be **symplectomorphic**.

The relation of being symplectomorphic is an equivalence relation among all even-dimensional **symplectic** vector spaces.
Moreover, by Theorem 2.1 every $2n$-dimensional symplectic vector space is symplectomorphic to the *prototype* $(\mathbb{R}^{2n}, \Omega_0)$: finding the suitable symplectomorphism amounts to finding a symplectic basis.

2.2 Subspaces

Let (V, Ω) be a symplectic vector space.

- A subspace $W \subset V$ is called **symplectic** if the restriction $\Omega|_W$ is nondegenerate.
- A subspace $W \subset V$ is called **isotropic** if $\Omega|_W \equiv 0$.

We can also look at these properties by introducing the notion of symplectic orthogonal.

Given a linear subspace Y of (V,Ω), its **symplectic orthogonal** Y^Ω is the linear subspace defined by

$$Y^\Omega = \{v \in V | \Omega(v,u) = 0, \text{ for all } u \in Y\}.$$

Some properties:

1. $\dim Y + \dim Y^\Omega = \dim V$;
2. $(Y^\Omega)^\Omega = Y$;
3. If Y and W are subspaces, then $Y \subseteq W \iff W^\Omega \subseteq Y^\Omega$;

Proof. Property 1 is immediate, once we notice that the linear map

$$\begin{aligned} \tilde{\Omega}|_Y : V &\to Y^* \\ v &\mapsto \Omega(v,\ )|_Y \end{aligned}$$

satisfies: $\text{Ker}\tilde{\Omega}|_Y = Y^\Omega$ and $\text{Im}\tilde{\Omega}|_Y = Y^*$.

Property 2: $v \in (Y^\Omega)^\Omega$ means $\Omega(u,v) = 0$ for all u such that $\Omega(u,v') = 0$ for all $v' \in Y$; so, clearly if $v \in Y$ then $v \in (Y^\Omega)^\Omega$. However $\dim(Y^\Omega)^\Omega = \dim V - \dim Y^\Omega = \dim V - \dim V + \dim Y = \dim Y$. Therefore we have $Y = (Y^\Omega)^\Omega$.

Property 3: Consider $y \in Y \subseteq W$, and let $u \in W^\Omega$, i.e. $\Omega(u,v') = 0$ for all $v' \in W$; then $u \in Y^\Omega$: given $y \in Y$, $\Omega(u,y) = 0$ since $y \in W$ as well.
Viceversa assume $W^\Omega \subseteq Y^\Omega$; then for the previous argument $(Y^\Omega)^\Omega \subseteq (W^\Omega)^\Omega$, which for property 2 means $Y \subseteq W$. □

We call the subspace Y:

- a **symplectic** subspace if

$$\Omega|_{Y\times Y} \text{ is nondegenerate} \iff Y \cap Y^\Omega = \{0\} \iff Y \oplus Y^\Omega = V;$$

- an **isotropic** subspace if

$$\Omega|_{Y\times Y} \equiv 0 \iff Y \subseteq Y^\Omega;$$

- a **coisotropic** subspace if

$$Y^\Omega \subseteq Y;$$

- a **lagrangian** subspace if

$$Y \text{ is isotropic and } \dim Y = \frac{1}{2}\dim V \iff Y \text{ is isotropic and coisotropic } \iff Y = Y^\Omega.$$

2.3 From symplectic spaces to symplectic manifolds

Let ω be a two-form on a manifold M: for each $p \in M$, $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is a skew-symmetric, bilinear form, and ω depends smoothly on the point p.
ω is closed if it satisfies the differential equation $d\omega = 0$, d denoting the exterior derivative.

Definition 2.3. The 2-form ω is **symplectic** if ω is closed and ω_p is symplectic for all $p \in M$.

If ω is symplectic, then $\dim T_pM = \dim M$ must be even.

Definition 2.4. A **symplectic manifold** is a manifold M, equipped with a symplectic form ω and denoted (M, ω).

Example 2.2. $(\mathbb{R}^{2n}, \omega_0)$, where in linear coordinates $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$, is symplectic and its symplectic basis is

$$\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \dots, \left(\frac{\partial}{\partial y_n}\right)_p \right\}.$$

Example 2.3. (S^2, ω) where S^2 is identified with the set of unit vectors in $\mathbb{R}^3$ and

$$\omega_p(u, v) \doteq \langle p, u \times v \rangle, \qquad u, v \in T_pS^2 = \{p\}^\perp.$$

is symplectic.

Definition 2.5. Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds, and let $\varphi : M_1 \to M_2$ be a diffeomorphism. φ is a **symplectomorphism** if

$$\varphi^* \omega_2 = \omega_1.$$

The theorem below, of which we do not provide a proof (see for instance [1]) states that the only local invariant of symplectic manifolds up to symplectomorphisms is their *dimension*.

Theorem 2.2 (Darboux)**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold, let p be any point in M. Then there is a coordinate chart, called a Darboux chart, $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n)$ centered at p such that on $\mathcal{U}$:*

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

The Darboux theorem asserts that locally any symplectic manifold of dimension $2n$ "looks like" $(\mathbb{R}^{2n}, \omega_0)$.

3 Isotopies and Vector Fields

Let M be a manifold and $\rho : M \times \mathbb{R} \to M$ a map, where we set $\rho_t(p) = \rho(p, t)$.

Definition 3.1. The map ρ is an **isotopy** if, for each t, $\rho_t : M \to M$ is a diffeomorphism and $\rho_0 = \mathrm{id}_M$.

Given an isotopy, we obtain a time-dependent vector field $\boldsymbol{v}_t$, that is a family of vector fields $\boldsymbol{v}_t$, $t \in \mathbb{R}$, which at $p \in M$ satisfy:

$$v_t(p) = \frac{d}{ds}\Big|_{s=t} \rho_s(q), \qquad q = \rho_t^{-1}(p), \tag{3.1}$$

or in other words

$$\frac{d\rho_t}{dt} = \boldsymbol{v}_p \circ \rho_t. \tag{3.2}$$

On the other hand, given a time-dependent vector field $\boldsymbol{v}_t$, under the hypothesis that M is *compact* or that the $\boldsymbol{v}_t$'s are compactly supported, we can find a suitable isotopy ρ satisfying the ordinary differential equation (3.1).

Definition 3.2. Given a time-*independent* vector field $\boldsymbol{v}$, the associated isotopy ρ is called **exponential map** or **flow** of $\boldsymbol{v}$ and is denoted $e^{t\boldsymbol{v}}$.
Thus $\{e^{t\boldsymbol{v}} : M \to M, t \in \mathbb{R}\}$ is the unique smooth family of diffeomorphisms satisfying:

$$e^{t\boldsymbol{v}}\Big|_{t=0} = \mathrm{id}_M \qquad \text{and} \qquad \frac{d}{dt}\left(e^{t\boldsymbol{v}}p\right) = v\left(e^{t\boldsymbol{v}}p\right) \tag{3.3}$$

Definition 3.3. The **Lie derivative** is the operator

$$\mathcal{L}_{\boldsymbol{v}} : \Omega^k(M) \to \Omega^k(M)$$

$$\mathcal{L}_{\boldsymbol{v}}\omega \doteq \frac{d}{dt}\Big|_{t=0}((e^{t\boldsymbol{v}})^*\omega). \tag{3.4}$$

In case the vector field $\boldsymbol{v}_t$ does depend on time, we can still define a corresponding isotopy ρ, by Picard's theorem. Therefore, in the neighborhood of any point $p \in M$ and for sufficiently small t there is a one-parameter family of local diffeomorphisms given by:

$$\frac{d\rho_t}{dt} = \boldsymbol{v}_p \circ \rho_t \qquad \rho_0 = \mathrm{id}_M. \tag{3.5}$$

And the Lie derivative with respect to a time-dependent field is therefore:

$$\mathcal{L}_{\boldsymbol{v}_t} : \Omega^k(M) \to \Omega^k(M)$$

$$\mathcal{L}_{\boldsymbol{v}_t}\omega \doteq \frac{d}{dt}\Big|_{t=0}(\rho_t^*\omega). \tag{3.6}$$

The following formulas prove useful in such a variety of cases that they are definitely worth mentioning.

Proposition 3.1. ***Cartan magic formula***

$$\mathcal{L}_{\boldsymbol{v}}\omega = \imath_{\boldsymbol{v}}d\omega + d\imath_{\boldsymbol{v}}\omega. \tag{3.7}$$

For a time-dependent vector field $\boldsymbol{v}_t$ and its local isotopy ρ:

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{\boldsymbol{v}_t}\omega; \tag{3.8}$$

and finally for a smooth family ω_t, $t \in \mathbb{R}$, of d-forms:

$$\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*\left(\mathcal{L}_{\boldsymbol{v}_t}\omega_t + \frac{d\omega_t}{dt}\right). \tag{3.9}$$

Proof. For (3.7) and (3.8) we can observe the following facts. First, they both hold for 0-forms $f \in \Omega^0(M) = C^\infty(M)$:

$$\left(\mathcal{L}_{\boldsymbol{v}}f\right)(p) = \frac{d}{dt}\Big|_{t=0} f(e^{t\boldsymbol{v}}p) = \boldsymbol{v}f|_p = df_p(\boldsymbol{v}) = \imath_{\boldsymbol{v}}df_p;$$

$$\frac{d}{dt}(\rho_t^*f)(p) = \frac{d}{dt}f(\rho_t(p)) = (\mathcal{L}_{\boldsymbol{v}_t}f)(\rho_t(p)) = \rho_t^*\left(\mathcal{L}_{\boldsymbol{v}_t}f\right)(p).$$

Then we notice that both sides of (3.7) and (3.8) commute with the exterior derivative d, essentially because the pullback * commutes with it:

$$d(\mathcal{L}_{\boldsymbol{v}}\omega) = d\left[\frac{d}{dt}(\rho_t^*\omega)\right] = \frac{d}{dt}(\rho_t^*d\omega) = \mathcal{L}_{\boldsymbol{v}}(d\omega); \qquad d\left(\imath_{\boldsymbol{v}}d\omega + d\imath_{\boldsymbol{v}}\omega\right) = d\imath_{\boldsymbol{v}}d\omega = \imath_{\boldsymbol{v}}dd\omega + d\imath_{\boldsymbol{v}}d\omega;$$

$$d\left[\frac{d}{dt}\rho_t^*\omega\right] = \frac{d}{dt}\rho_t^*d\omega; \qquad d\left(\rho_t^*\mathcal{L}_{\boldsymbol{v}_t}\omega\right) = \rho_t^*\mathcal{L}_{\boldsymbol{v}_t}d\omega.$$

Both sides are derivation of the algebra $(\Omega^*(M), \wedge)$, i.e.

$$\mathcal{L}_{\boldsymbol{v}}\left(\omega \wedge \alpha\right) = \left(\mathcal{L}_{\boldsymbol{v}}\omega\right) \wedge \alpha + \omega \wedge \left(\mathcal{L}_{\boldsymbol{v}}\alpha\right)$$

since the Lie derivative operator *is* a derivation itself. This concludes the proof, since for any chart $\mathcal{U}$, $\Omega^\bullet(\mathcal{U})$ is generated by functions and their differentials.

Formula (3.9) is a consequence of (3.8) via chain rule:

$$\frac{d}{dt}\rho_t^*\omega_t = \underbrace{\frac{d}{d\alpha}\Big|_{\alpha=t}\rho_\alpha^*\omega_t}_{\rho_t^*\mathcal{L}_{\boldsymbol{v}_t}\omega_t} + \underbrace{\frac{d}{d\beta}\Big|_{\alpha=t}\rho_t^*\omega_\beta}_{\rho_t^*\frac{d\omega_t}{dt}} = \rho_t^*\left(\mathcal{L}_{\boldsymbol{v}_t}\omega_t + \frac{d\omega_t}{dt}\right).$$

□

4 Hamiltonian Mechanics

4.1 Hamiltonian and Symplectic Vector Fields

Let (M, ω) be a symplectic manifold, and let $H : M \to \mathbb{R}$ be a smooth function. Since ω is non-degenerate, there exists a unique vector field X_H satisfying:

$$\imath_{X_H}\omega = dH. \tag{4.1}$$

Definition 4.1. The vector field X_H is called **hamiltonian vector field** and H is referred to as Hamilton function.

Supposing M is compact, or that X_H is complete, we can define, by integration, a one-parameter family of diffeomorphisms $\rho_t : M \to M$, $t \in \mathbb{R}$, as

$$\begin{cases} \rho_0 = id \\ \\ \dfrac{d\rho_t}{dt} \circ \rho_t^{-1} = X_H. \end{cases} \tag{4.2}$$

Claim. *$\forall t$, ρ_t preserves the symplectic structure ω.*

Proof. By (3.8) we have: $\frac{d}{dt}(\rho_t^*\omega) = \rho_t^*\mathcal{L}_{X_H}\omega = \rho_t^*(d\underbrace{\imath_{X_H}\omega}_{dH} + \imath_{X_H}\underbrace{d\omega}_{=0}) = 0.$ □

Remark 4.1. This shows that, for each t, ρ_t is a *symplectomorphism*; notice however how this proof involved both the non-degeneracy and the closedness of ω.

Remark 4.2 (Energy conservation)**.** H is preserved along the trajectories of X_H:

$$\frac{d}{dt}(\rho_t^*H)(x) = \frac{d}{dt}H(\rho_t x) = \mathcal{L}_{X_H}H = \imath_{X_H}dH = \imath_{X_H}\imath_{X_H}\omega = 0.$$

Which means: $(\rho_t^*H)(x) = H(\rho_t x), \quad \forall t.$

Definition 4.2. A vector field X on (M, ω) is called a **symplectic vector field** whenever $\mathcal{L}_X \omega = 0$.

The following are equivalent:

- X is a symplectic vector field
- $\mathcal{L}_X \omega = 0$
- $\rho_t^* \omega = \omega, \quad \forall t$
- $\imath_X \omega$ is closed

However the condition necessary for X_H to be a symplectic hamiltonian vector field is stronger:

- $\exists H$ such that $\imath_{X_H} \omega = dH$.

Since the form $\imath_X \omega$ is closed, such a function H exists in a neighborhood of every point, by Poincaré lemma. The existence of a global function H requires some supplementary hypothesis of topological type on the variety under consideration. For instance, if the variety is simply connected, then every symplectic field is hamiltonian.

4.2 Classical Mechanics

Let us consider the euclidean space $\mathbb{R}^{2n}$, with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, equipped with the canonical symplectic structure $\omega_0 = \sum_i dq_i \wedge dp_i$. The integral curves of hamiltonian vector field for the Hamilton function H are described by Hamilton Equations. Indeed

$$\imath_{X_H} \omega_0 = dH$$

however

$$X_H = \sum_i \left(\frac{dq_i}{dt} \frac{\partial}{\partial q_i} + \frac{dp_i}{dt} \frac{\partial}{\partial p_i} \right), \qquad dH = \sum_i \left(\frac{dH}{dq_i} dq_i + \frac{dH}{dp_i} dp_i \right);$$

hence:

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q_i}. \end{cases} \tag{4.3}$$

For $n = 3$, these equations describe the motion of a particle of mass m subject to a potential $V(\boldsymbol{q})$ in the three dimensional space $\mathbb{R}^3$ with coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$. Newton's second law

$$m\frac{d^2\boldsymbol{q}}{dt^2} = -\nabla V(\boldsymbol{q})$$

can be rewritten by introducing the **momenta** $p_i = m\frac{dq_i}{dt}$ and the hamilton function (**energy**) $H(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2m}\boldsymbol{p}^2 + V(\boldsymbol{q})$. Considering now $T^*\mathbb{R}^3 = \mathbb{R}^6$ with coordinates $(\boldsymbol{q}, \boldsymbol{p})$, Newton's second law in $\mathbb{R}^3$ is equivalent to the Hamilton equations in $\mathbb{R}^6$.

4.3 Brackets

For a function $f \in C^\infty(M)$ and a vector field X, we define:

$$Xf = df(X) = \mathcal{L}_X f.$$

Given two vector fields X, Y, there exists a unique vector field $W \doteq [X, Y]$, called **Lie Bracket** of X and Y, satisfying the condition:

$$\mathcal{L}_{[X,Y]} f = \mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f = [\mathcal{L}_X, \mathcal{L}_Y] f. \tag{4.4}$$

Lemma 4.1. *For any form* α

$$\imath_{[X,Y]}\alpha = \mathcal{L}_X \imath_Y \alpha - \imath_Y \mathcal{L}_X \alpha = [\mathcal{L}_X, \imath_Y]\alpha. \tag{4.5}$$

Proof. Both sides behave as anti-derivations with respect to the wedge product; indeed let α and β be a p- and q-form respectively, then:

$$\imath_{[X,Y]}(\alpha \wedge \beta) = (\imath_{[X,Y]}\alpha) \wedge \beta + (-1)^q \alpha \wedge (\imath_{[X,Y]}\beta),$$

by definition of $\wedge$,

$$\begin{aligned}
&\mathcal{L}_X \imath_Y(\alpha \wedge \beta) - \imath_Y \mathcal{L}_X(\alpha \wedge \beta) = \\
&\qquad = \mathcal{L}_X\Big(\imath_Y \alpha \wedge \beta + (-1)^q \alpha \wedge \imath_Y \beta\Big) - \imath_Y\Big(\mathcal{L}_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta\Big) = \\
&\quad = (\mathcal{L}_X \imath_Y \alpha) \wedge \beta + (-1)^q \alpha \wedge (\mathcal{L}_X \imath_Y \beta) - (\imath_Y \mathcal{L}_X \alpha) \wedge \beta - (-1)^q \alpha \wedge (\imath_Y \mathcal{L}_X \beta),
\end{aligned}$$

by skew-symmetry of $\wedge$ with respect to contraction $\imath$ and by the Leibnitz rule for $\mathcal{L}$ (the cross terms, therefore, cancel out!).

Thus, it is sufficient to check the formula on local generators of the exterior algebra of forms. For a function f, both sides vanish identically.
For an exact one form df:

$$\imath_{[X,Y]} df = X(Yf) - Y(Xf) = \mathcal{L}_X \imath_Y df - \imath_Y \mathcal{L}_X df$$

since $df(X) \equiv Xf \equiv \mathcal{L}_X f$. $\square$

Proposition 4.2. *Given two symplectic vector fields, X, Y, on a symplectic manifold (M,ω), then $[X,Y]$ is hamiltonian and its hamilton function is $\omega(Y,X)$:*

$$H_{[X,Y]} = \omega(Y,X). \tag{4.6}$$

Proof. Using Cartan magic formula (3.7):

$$\begin{aligned} \imath_{[X,Y]}\omega =& \mathcal{L}_X \imath_Y \omega - \imath_Y \mathcal{L}_X \omega \\ =& d\imath_X \imath_Y \omega + \imath_X \underbrace{d\imath_Y \omega}_{0} - \imath_Y \underbrace{d\imath_X \omega}_{0} - \imath_Y \imath_X \underbrace{d\omega}_{0}. \\ =& d\left(\omega(Y,X)\right). \end{aligned}$$

□

Definition 4.3. A **Lie algebra** is a vector space $\mathfrak{g}$ together with a Lie bracket $[\,,]$, i.e. a bilinear map

$$[\,,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying:

$$[x,y] = -[y,x], \quad \forall x,y \in \mathfrak{g}; \qquad \textbf{(antisymmetry)}$$

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]], \quad \forall x,y,z \in \mathfrak{g}. \qquad \textbf{(Jacobi Identity)}$$

Definition 4.4. A **Poisson algebra** is a commutative algebra A together with a Poisson bracket $\{\,,\}$, defining a map

$$\{\,,\} : A \times A \to A$$

which is a derivation:$\{f,gh\} = \{f,g\}h + g\{f,h\}$.

On a symplectic manifold (M,ω), we can actually give a Poisson bracket and define the Poisson structure $(C^\infty(M),\{\,,\})$ by introducing the formula:

$$\{f,g\} \doteq \omega(X_f,X_g), \tag{4.7}$$

where X_f and X_g denote the hamiltonian vector fields with f, respectively g as hamilton functions.

Claim. *Our $\{\,,\}$ is indeed a Poisson bracket since*

$$\{f,gh\} = \{f,g\}h + g\{f,h\}$$

Proof. He have

$$\{f, gh\} = \omega(X_f, X_{gh});$$

however X_{gh} is defined by

$$d(gh) = dg\ h + g\ dh = \imath_{X_{gh}}\omega$$

therefore

$$\omega(X_f, X_{gh}) = dg(X_f)\ h + g\ dh(X_f) = \omega(X_f, X_g)h + g\omega(X_f, X_h) = \{f, g\}h + g\{f, h\}. \quad \square$$

Remark 4.3. We have $X_{\{f,g\}} = -[X_f, X_g]$ since $X_{\{f,g\}} = X_{\omega(X_f,X_g)} = [X_g, X_f]$, where in the last passage we used equation (4.6).

Proposition 4.3. $\{\,,\}$ *satisfies Jacobi Identity:*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Proof. By the last Remark we have $\{f, \{g, h\}\} = \omega(X_f, X_{\{g,h\}}) = -\omega(X_f, [X_g, X_h])$, thus our identity follows from Jacobi Identity for the Lie bracket of vectors. $\square$

To sum up, we have shown that on a symplectic manifold (M, ω) we have a Poisson algebra $(C^\infty(M), \{\ ,\})$; and moreover we have a Lie algebra anti-homomorphism between the Lie algebra of vector fields $(\chi(M), [\,,])$ and the Lie algebra of functions $(C^\infty(M), \{\,,\})$

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & \chi(M) \\ H & \longmapsto & X_H \\ \{\,,\} & \rightsquigarrow & -[\,,]. \end{array}$$

One can also define, in parallel with the notion of symplectic manifold, the weaker notion of Poisson manifold.

Definition 4.5. A smooth manifold M is a **Poisson manifold** if $(C^\infty(M), \{\,,\})$ is a Poisson algebra.

And of course, similarly to symplectomorphisms for the former, for the latter we have:

Definition 4.6. A **Poisson map** is a regular map $\varphi : M \to N$ between two Poisson manifolds $(M, \{\,,\}_M)$, $(N, \{\,,\}_N)$ which defines a homomorphism of Poisson algebras, i.e. given $f, g \in C^\infty(N)$:

$$\varphi^*\{f, g\}_N = \{\varphi^* f, \varphi^* g\}_M \in C^\infty(M).$$

From a general point of view, one can proceed with the notion of Poisson structure on a manifold M only, and *define*:

$$\{f, g\} \doteq (df \otimes dg)(\Pi), \qquad f, g \in C^\infty(M). \tag{4.8}$$

Then, we can define the corresponding of a symplectic structure:

Definition 4.7. A **Poisson structure** on a smooth manifold M is defined by a **Poisson bivector** i.e. a bivector on M $\Pi \in \Gamma(M, \Lambda^2 TM)$ such that its Shouten bracket with itself is zero: $[\Pi, \Pi] = 0$.

The Shouten bracket mentioned above is just the extension to p-vectors (in this case bivectors) of the usual Lie bracket of vector fields $[\,,]$.

Following this more abstract path, leads to a straightforward definition of Hamiltonian vector field of a given function $H \in C^\infty(M)$ that exploits the following homomorphism between the Lie algebra of functions and the Lie algebra of vector fields $\chi_\Pi(M)$ preserving the Poisson structure:

$$v : C^\infty(M) \longrightarrow \chi_\Pi(M) \tag{4.9}$$

$$H \longmapsto v_H = \{H,\ \}. \tag{4.10}$$

4.4 Integrable Systems

Definition 4.8. A **hamiltonian system** is a triple (M, ω, H), where (M, ω) is a symplectic manifold and $H \in C^\infty(M; \mathbb{R})$ is the **hamiltonian function** of the system.

Here we have our first version of Noether theorem.

Theorem 4.4. *We have $\{f, H\} = 0$ if and only if f is constant along the integral curves of the hamiltonian field X_H.*

Proof. Let ρ_t be the flow of X_H. Then

$$\begin{aligned} \frac{d}{dt}(\rho_t^* f) &= \rho_t^* \mathcal{L}_{X_H} f = \rho_t^* \imath_{X_H} df = \rho_t^* \imath_{X_H} \imath_{X_f} \omega \\ &= \rho_t^* \omega(X_f, X_H) = \rho_t^* \{f, H\}. \end{aligned}$$

□

A function which is constant along the trajectories of motion is called an **integral of motion** (or a **first integral**). Given n functions $f_1, \ldots, f_n$ they are said to be **independent** if their differentials $(df_1)_p, \ldots, (df_n)_p$ are linearly independent at all points p in some open dense subset of M.

In a loose sense, a hamiltonian system is integrable if it has *enough* commuting integrals of motion, where commutativity is with respect to the Poisson bracket. This means that:

$$\omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0,$$

so we are requesting that the set of hamiltonian fields $(X_{f_1})_p, \dots, (X_{f_n})_p$ generates an isotropic subspace of T_pM for each p. By symplectic linear algebra, then, n can be *at most* half the dimension of M.

Theorem 4.5. *(Arnold-Liouville) Let (M, ω, H) be an integrable system of dimension $2n$ with integrals of motion $f_1 = H, f_2, \dots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of $f \doteq (f_1, \dots, f_n)$. The corresponding level $f^{-1}(c)$ is a lagrangian submanifold of M.*

*(a) If the flows of $X_{f_1}, \dots, X_{f_n}$ starting at a point $p \in f^{-1}(c)$ are complete, then the connected component of $f^{-1}(c)$ containing p is a homogeneous space for the group $\mathbb{R}^n$. There are affine coordinates $\varphi_1, \dots, \varphi_n$ on this component, known as **angle coordinates**, with respect to which the flows of the vector fields $X_{f_1}, \dots X_{f_n}$ are linear, where by affine coordinates we mean that the action of the group $\mathbb{R}^n$ is by translations.*

*(b) There are coordinates $I_1, \dots, I_n$ known as **action coordinates**, complementary to the angle coordinates such that the I_i's are integrals of motion and $\phi_1, \dots, \phi_n, I_1, \dots, I_n$ give a Darboux chart.*

Moment Maps

The concept of moment map is a generalization of that of Hamilton function. The notion of a moment map associated to a group action on a symplectic manifold formalizes the Noether principle, which associates to every symmetry in a mechanical system a conserved quantity.

5 Actions

5.1 One-parameter subgroups

Let M be a manifold and let X be a complete vector field on M. Consider now the one-parameter family of diffeomorphisms $\rho_t : M \to M,\ t \in \mathbb{R}$, generated by X, which satisfies:

$$\begin{cases} \rho_0(p) = p \\ \\ \dfrac{d\rho_t(p)}{dt} = X(\rho_t(p)). \end{cases}$$

Claim. *The family ρ_t can be regarded as a* ***one-parameter group of diffeomorphisms****, denoted $\rho_t = e^{tX}$, since it meets the requirements:*

- $\rho_t \circ \rho_s = \rho_{t+s}$;
- $\rho_0 = \mathrm{id}$;
- $\rho_{-t} = \rho_t^{-1}$.

So the map $\mathbb{R} \to \mathrm{Diff}(M),\ t \mapsto \rho_t$ is a group homomorphism.

Proof. Let $\rho_s(q) = p$ and reparametrize as $\tilde{\rho}_t(q) = \rho_{t+s}(q)$. Then we have:

$$\begin{cases} \tilde{\rho}_0(q) = \rho_s(q) = p \\ \\ \dfrac{d\tilde{\rho}_t(q)}{dt} = \dfrac{d\rho_{t+s}(q)}{dt} = X(\rho_{t+s}(q)) = X(\tilde{\rho}_t(q)), \end{cases}$$

i.e. $\tilde{\rho}_t$ is an integral curve of X through p. By uniqueness: $\tilde{\rho}_t(q) = \rho_t(p)$, hence $\rho_{t+s}(q) = \rho_t(\rho_s(q))$. □

5.2 Lie Groups

Definition 5.1. A **Lie group** is a manifold G equipped with a group structure where the group operations

$$\begin{aligned} G \times G \to G \qquad & (a, b) \mapsto a \cdot b \\ G \to G \qquad & a \mapsto a^{-1} \end{aligned}$$

are smooth maps.

Definition 5.2. A representation of a Lie group G on a vector space V is a group homomorphism $G \to GL(V)$, which is also a map of differentiable manifolds.

5.3 Smooth Actions

Definition 5.3. An **action** of the Lie group G on the manifold M is a group homomorphism:

$$\begin{aligned} \psi : G &\to \mathrm{Diff}(M) \\ g &\mapsto \psi_g. \end{aligned}$$

The **evaluation map** associated with an action $\psi : G \to \mathrm{Diff}(M)$ is

$$\begin{aligned} \mathrm{ev}_\psi : M \times G &\to M \\ (p, g) &\mapsto \psi_g(p). \end{aligned}$$

The action is said to be **smooth** if its evaluation map is smooth.

It follows from our previous considerations that there is a one-to-one correspondence between smooth actions of $\mathbb{R}$ on M and complete vector fields on M given by:

$$X \mapsto e^{tX}$$

$$X_p = \left.\frac{d\psi_t(p)}{dt}\right|_{t=0} \leftarrow \psi.$$

5.4 Orbit Spaces

Let $\psi : G \to \text{Diff}(M)$ be any action.

Definition 5.4. The **orbit** of G through $p \in M$ is $\{\psi_g(p) | g \in G\}$.
The **stabilizer** of $p \in M$ is the subgroup $G_p \doteq \{g \in G | \psi_g(p) = p\}$.

Furthermore we can describe an action by the characterizing the properties of its orbits.

Definition 5.5. The action of G on M is:

- **transitive** if there is just one orbit,
- **free** if all stabilizers are trivial,
- **locally free** if all stabilizers are discrete.

Let now $\sim$ be the equivalence relation that identifies two points in the same orbit: $p, q \in M$,

$$p \sim q \iff p, q \text{ are in the same orbit.}$$

The space of orbits $M/\sim = M/G$ is called the **orbit space**.
The natural projection π is the one associating a point to the G-orbit through it:

$$\begin{aligned} \pi : M &\longrightarrow M/G \\ p &\longmapsto \text{orbit through } p. \end{aligned}$$

5.5 Symplectic and Hamiltonian actions

Let (M, ω) be a symplectic manifold, and G a Lie group acting on it through a smooth action $\psi : G \to \text{Diff}(M)$.

Definition 5.6. The action ψ is a **symplectic action** if

$$\psi : G \to \text{Sympl}(M, \omega) \subset \text{Diff}(M).$$

This means that ψ gives rise to symplectomorphisms.

The general definition of *hamiltonian* action, which we are going to inspect in a moment, requires the notion of moment map.

5.6 Lie algebra of a Lie group

Let G be a Lie group. Given $g \in G$ let

$$\begin{aligned} L_g : G &\to G \\ a &\to g \cdot a \end{aligned}$$

be the **left multiplication** by g.
Consider now a vector field X on G. Such field is called **left-invariant** if

$$(L_g)_* X = X$$

for every $g \in G$. Since the Lie bracket of two left-invariant vector fields is again left invariant we may give the following:

Definition 5.7. Let $\mathfrak{g}$ be the vector space of all left-invariant vector fields on G. Together with the Lie bracket $[\,,]$ of vector fields, $\mathfrak{g}$ forms a Lie algebra, called the **Lie algebra of the Lie group** G.

Claim. *The map*

$$\begin{aligned} \mathfrak{g} &\longrightarrow T_e G \\ X &\longmapsto X_e \end{aligned}$$

where e denotes the identity element of the Lie group G, is an isomorphism of vector spaces.

Proof. Given a vector field $X \in \mathfrak{g}$, we associate to X its corresponding vector in the identity element X_e; clearly we have, for two distinct fields X, Y:

$$(\alpha X + \beta Y)_e = \alpha X_e + \beta Y_e$$

where α, β are *constant* coefficients for the linear combinations.

Conversely it is clear that any element $X_e \in T_e G$ defines a unique left-invariant vector field through the relation $X_g \doteq (L_g)_{*e} X_e$: indeed given another $h \in G$,

$$\begin{aligned} (L_h)_{*g} X_g = (L_h)_{*g} \circ (L_g)_{*e} X_e = (L_h)_{*g} \frac{d}{dt}\Big|_{t=0} (g \cdot e^{tX_e}) = \\ = \frac{d}{dt}\Big|_{t=0} (h \cdot g \cdot e^{tX_e}) = (L_{h \cdot g})_{*e} X_e = X_{h \cdot g}. \end{aligned}$$

Finally $[X, Y]$ in $\mathfrak{g}$ corresponds to $[X, Y]_e \equiv= [X_e, Y_e]$ in $T_e G$, while we have:

$$(L_g)_{*e}[X, Y]_e = (L_g)_{*e} \frac{d}{dt}\Big|_{t=0} (e^{tX} e^{tY} - e^{tY} e^{tX}) = \frac{d}{dt}\Big|_{t=0} g(e^{tX} e^{tY} - e^{tY} e^{tX}) = [X, Y]_g. \quad \square$$

We observe that the last claim means we can plainly identify the Lie algebra of a Lie group with its tangent space in the identity element, which is just the way many authors actually *define* it.

6 Adjoint and Coadjoint representations

Let G be a Lie Group and $\mathfrak{g}$ its Lie Algebra, i.e. its tangent space at the identity element, provided with a commutator operation $[\,,]$.

Given $g \in G$, a group element, we have the **left** and **right translations**, which are diffeomorphisms through which the group acts on itself:

$$L_g : G \to G, \quad h \in G \mapsto L_g h = gh \in G \tag{6.1}$$
$$R_g : G \to G, \quad h \in G \mapsto R_g h = hg \in G. \tag{6.2}$$

6.1 Adjoint Representation

The derivatives (def.1.1) of the diffeomorphisms described above are:

$$L_{g*} : TG_h \to TG_{gh} \tag{6.3}$$
$$R_{g*} : TG_h \to TG_{hg} \tag{6.4}$$

The *conjugation* $R_{g^{-1}} \circ L_g$ acts as follows:

$$R_{g^{-1}} \circ L_g : \; G \to G, \quad h \in G \mapsto ghg^{-1}. \tag{6.5}$$

It is an inner automorphism of the group G. In particular, it fixes the unit element e, since $R_{g^{-1}} \circ L_g(e) = g\ e\ g^{-1} = g\ g^{-1} = e$.

Its derivative in the unit element, which we shall denote Ad_g, constitutes a linear map of the Lie algebra $\mathfrak{g}$ to itself:

$$Ad_g = (R_{g^{-1}} \circ L_g)_{*e}, \quad Ad_g : \mathfrak{g} \to \mathfrak{g}. \tag{6.6}$$

Therefore, interpreting such procedure as a map

$$Ad : G \to GL(\mathfrak{g}), \quad g \mapsto Ad_g \tag{6.7}$$

we define the **adjoint representation** of the group G on its Lie algebra $\mathfrak{g}$.

We can show that Ad_g is a homomorphism for the algebra, i.e. it preserves the structure of the Lie bracket

$$Ad_g[X, Y] = [Ad_g X, Ad_g Y], \quad X, Y \in \mathfrak{g}, \tag{6.8}$$

and furthermore that Ad_g has indeed the properties of a representation

$$Ad_{gh} = Ad_g Ad_h, \quad g, h \in G. \tag{6.9}$$

Proof. Let $Ad_g X = \boldsymbol{v}$, $Ad_g Y = \boldsymbol{u}$ we have:

$$[Ad_g X, Ad_g Y] = [\boldsymbol{v}, \boldsymbol{u}] = \frac{d}{dt}\Big|_{t=0} \left(e^{t\boldsymbol{v}}e^{t\boldsymbol{u}} - e^{t\boldsymbol{u}}e^{t\boldsymbol{v}}\right) = (\boldsymbol{vu} - \boldsymbol{uv}) =$$
$$= \frac{d}{ds}\Big|_{s=0}(ge^{sX}g^{-1})\frac{d}{ds}\Big|_{s=0}(ge^{sY}g^{-1}) - \frac{d}{ds}\Big|_{s=0}(ge^{sY}g^{-1})\frac{d}{ds}\Big|_{s=0}(ge^{sX}g^{-1}) =$$
$$= \frac{d}{d\tau}\Big|_{\tau=0} g(e^{\tau Y}e^{\tau X} - e^{\tau X}e^{\tau X})g^{-1} = Ad_g[X, Y].$$

Given $X \in \mathfrak{g}$,

$$Ad_{gh}X = Ad_{gh}\frac{d}{dt}\Big|_{t=0} e^{tX} = \frac{d}{dt}\Big|_{t=0} \left((gh)e^{tX}(gh)^{-1}\right) =$$
$$= \frac{d}{dt}\left(ghe^{tX}h^{-1}g^{-1}\right)\Big|_{t=0} = Ad_g\left(he^{tX}h^{-1}\right)\Big|_{t=0} = Ad_g Ad_h X. \quad \square$$

Let us once more consider the map from the group into the space of linear operators on the algebra given by Ad:

$$Ad : G \to GL(\mathfrak{g}), \quad g \mapsto Ad_g \in GL(\mathfrak{g}) \tag{6.10}$$

its derivative in the unit element of G is a linear map from the Lie algebra $\mathfrak{g}$ into the space of linear operators on $\mathfrak{g}$ itself; we shall denote this by ad in the following way:

$$Ad_{*e} = ad, \quad ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad X \in \mathfrak{g} \to ad_g \in \mathrm{End}(\mathfrak{g}). \tag{6.11}$$

Given a 1-parameter subgroup e^{tX}, $X \in \mathfrak{g}$, then, by definition we get:

$$ad_X = \frac{d}{dt}\Big|_{t=0} Ad_{e^{tX}}; \tag{6.12}$$

and we can finally express ad in terms of the Lie algebra only:

$$ad_X Y = [X, Y]. \tag{6.13}$$

Proof.

$$ad_X Y = \frac{d}{dt}\Big|_{t=0} Ad_{e^{tX}}Y = \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\left(e^{tX}e^{sY}e^{-tX}\right) =$$
$$= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}(1 + sY + tX + tsXY - tsYX + \ldots) = [X, Y].$$

$\square$

Example 6.1. For matrix groups G (i.e. subgroups of $GL(n;\mathbb{R})$, where we have

$$Ad_g(Y) = gYg^{-1}, \qquad \forall g \in G,\ \forall Y \in \mathfrak{g}$$

and

$$[X, Y] = XY - YX \qquad \forall X, Y \in \mathfrak{g},$$

we can check directly that (6.13) holds.

$$ad_X Y = [X, Y] \qquad \forall X, Y \in \mathfrak{g}.$$

6.2 Coadjoint Representation

Now we are about to express some properties of the action of the Lie Group onto itself which are *dual* to those presented in the previous paragraph.

Let $\mathfrak{g}^*$ be the dual to the Lie algebra $\mathfrak{g}$, namely the vector space T^*G_e of linear functions on $\mathfrak{g}$. We will express the contraction between a vector and a one-form with angle brackets $\langle\,,\rangle$: $\langle \omega, X\rangle \in \mathbb{R}$ where $\omega \in \mathfrak{g}^*,\ X \in \mathfrak{g}$.

Left and right translations induce, beside the mappings between tangent spaces defined above, also mappings between cotangent spaces:

$$L_g^* : TG_h^* \to TG_{gh}^* \tag{6.14}$$

$$R_g^* : TG_h^* \to TG_{hg}^* \tag{6.15}$$

defined as dual maps of L_{g*} and R_{g*}, namely $\left\langle L_g^*\omega, X\right\rangle = \langle \omega, L_{g*}X\rangle$ for a vector $X \in \mathfrak{g}$ and analogously for R_g^*.

Again we have the dual operator to Ad_g, which is denoted Ad_g^*:

$$Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^*, \quad \left\langle Ad_g^*\omega, X\right\rangle = \langle \omega, Ad_{g^{-1}}X\rangle\,. \tag{6.16}$$

Again we can consider:

$$Ad^* : G \to GL(\mathfrak{g}^*), \quad g \mapsto Ad_g^* \tag{6.17}$$

as a representation, called **coadjoint representation**. Indeed:

$$Ad_{gh}^* = Ad_g^* Ad_h^*. \tag{6.18}$$

Proof.

$$\left\langle Ad^*_{gh}\omega, X\right\rangle = \left\langle \omega, Ad_{(gh)^{-1}}X\right\rangle = \langle \omega, Ad_{h^{-1}g^{-1}}X\rangle =$$
$$= \langle \omega, Ad_{h^{-1}}Ad_{g^{-1}}X\rangle = \langle Ad^*_h\omega, Ad_{g^{-1}}X\rangle = \left\langle Ad^*_g Ad^*_h\omega, X\right\rangle. \quad \square$$

The derivative of Ad^*_g with respect to g in the identity element is a linear map of the Lie algebra $\mathfrak{g}^*$ onto the space of linear operators acting *upon* the dual space $\mathfrak{g}^*$, and we will denote it ad^* in the following manner:

$$ad^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*), \quad X \in \mathfrak{g} \to ad^*_X \in \mathrm{End}(\mathfrak{g}^*) \tag{6.19}$$

$$ad^*_X : \mathfrak{g}^* \to \mathfrak{g}^*, \quad \omega \in \mathfrak{g} \to ad^*_X\omega \in \mathfrak{g}^*. \tag{6.20}$$

ad^*_X turns out to be dual to ad_X as well, i.e. for any $\omega \in \mathfrak{g}^*$ and $Y \in \mathfrak{g}$:

$$\langle ad^*_X\omega, Y\rangle = \langle \omega, ad_X Y\rangle. \tag{6.21}$$

If we make use of the expression of ad in terms of the Lie bracket $[\,,]$ given above we obtain:

$$\langle ad^*_X\omega, Y\rangle = \langle \omega, [Y, X]\rangle, \quad X, Y \in \mathfrak{g},\ \omega \in \mathfrak{g}^*. \tag{6.22}$$

Proof. $\langle ad^*_X\omega, Y\rangle = \langle \frac{d}{dt}\Big|_{t=0} Ad^*_{e^{tX}}\omega, Y\rangle = \frac{d}{dt}\Big|_{t=0}\langle \omega, Ad_{e^{-tX}}Y\rangle =$
$= -\langle \omega, ad_X Y\rangle = \langle \omega, [Y, X]\rangle$ $\square$

7 Hamiltonian Actions

7.1 Moment and Comoment Maps

Let (M, ω) be a symplectic manifold, G a Lie group and ψ a smooth symplectic action of G on M(See definition 5.6). As usual we denote the Lie algebra of G as $\mathfrak{g}$ and its dual as $\mathfrak{g}^*$.

Definition 7.1. The action ψ is a **hamiltonian action** if there exists a map

$$\mu : M \to \mathfrak{g}^*$$

satisfying:

1. For each $X \in \mathfrak{g}$, let
   - $\mu^X : M \to \mathbb{R}$, $\mu^X(p) \doteq \langle \mu(p), X \rangle$ be the component of μ along X,
   - $X^\sharp$ be the vector field on M generated by the one-parameter subgroup $e^{tX} \subseteq G$.

   Then μ^X is the hamiltonian function for the vector field $X^\sharp$

   $$d\mu^X = \imath_{X^\sharp}\omega.$$

2. μ is **equivariant** with respect to the given action ψ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$:

   $$\mu \circ \psi_g = Ad^*_g \mu \ , \qquad \text{for all } g \in G$$

   i.e. the following diagram

   $$\begin{array}{ccc} M & \xrightarrow{\psi_g} & M \\ \Big\downarrow \mu & & \Big\downarrow \mu \\ \mathfrak{g}^* & \xrightarrow{Ad^*_g} & \mathfrak{g}^* \end{array}$$

   commutes.

In such case (M, ω, G, μ) is called a **hamiltonian G-space** and μ is called a **moment map**.

For connected Lie groups, hamiltonian actions can be defined using the equivalent concept of **comoment map**:

$$\mu^* : \mathfrak{g} \longrightarrow C^\infty(M) \ , \tag{7.1}$$

where

1. $\mu^*(X) \doteq \mu^X$ is a hamiltonian function for the vector field $X^\sharp$,

2. μ^* is a Lie algebra homomorphism

   $$\mu^*[X, Y] = \{\mu^*(X), \mu^*(Y)\}$$

   where $\{\,,\}$ is the Poisson bracket on $C^\infty(M)$. Or, using the homomorphism of Lie algebras defined in eq. (4.9)

   $$v_{\mu^*(X)} = X^\sharp, \qquad X \in \mathfrak{g}.$$

Remark 7.1. In cases $G = \mathbb{R},\ S^1, \mathbb{T}^n$, which are abelian groups, the coadjoint action is trivial and therefore equivariance becomes mere invariance.

7.2 Some classical examples

Translation:
Consider $\mathbb{R}^6$ with canonical positions and momenta $x_1, x_2, x_3, y_1, y_2, y_3$ as coordinates, equipped with the canonical symplectic form $\omega = \sum_i dx_i \wedge dy_i$. Let $G = \mathbb{R}^3$ act on $M = \mathbb{R}^6$ by translation:

$$\begin{aligned} \vec{a} \in \mathbb{R}^3 \longmapsto \psi_{\vec{a}} &\in \mathrm{Sympl}(\mathbb{R}^6, \omega) \\ \psi_{\vec{a}}(\vec{x}, \vec{y}) &= (\vec{x} + \vec{a}, \vec{y}). \end{aligned}$$

Here $\mathfrak{g} \simeq \mathbb{R}^3$, therefore letting $X = \vec{a} \in \mathfrak{g} \simeq \mathbb{R}^3$, we have $X^\sharp = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3}$ and

$$\mu : \mathbb{R}^6 \longrightarrow \mathbb{R}^3, \qquad \mu(\vec{x}, \vec{y}) = \vec{y}$$

is a moment map; the hamiltonian function for $X^\sharp = \vec{a}$ is:

$$\mu^{\vec{a}}(\vec{x}, \vec{y}) = \langle \mu(\vec{x}, \vec{y}), \vec{a} \rangle = \vec{y} \cdot \vec{a},$$

since

$$\imath_{X^\sharp}\omega = \left(\sum_i dx_i \wedge dy_i \right) \left(\sum_i a_i \frac{\partial}{\partial x_i} \right) = \sum_i a_i dy_i = d\mu^{\vec{a}}(\vec{x}, \vec{y}).$$

It is now clear that the **momentum vector** $\vec{y}$ is the generator of translation in euclidean 3-space.

Rotation: Let $G = SO(3)$, i.e. the set of $A \in GL(3; \mathbb{R})$ satisfying $A^t A = \mathrm{Id}$ and $\det A = 1$. We know that $\mathfrak{g} = so(3; \mathbb{R})$ is the set of 3×3 skew-symmetric matrices, which can be identified with $\mathbb{R}^3$ as follows:

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \longmapsto \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$[A, B] = AB - BA \longmapsto \vec{a} \times \vec{b}.$$

We can now check that the adjoint and coadjoint actions are, under the identifications $\mathfrak{g}, \mathfrak{g}^* \simeq \mathbb{R}^3$, the usual $SO(3)$-action on $\mathbb{R}^3$ by rotations. Indeed, consider the matrix O of rotation about the z axis by angle θ and let ω be in $\mathfrak{g}$, i.e. a skew-symmetric matrix:

$$O = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix};$$

We can now compute

$$Ad_O\omega = O\omega O_1^t = \begin{pmatrix} 0 & -\omega 3 & \cos\theta\omega_2 + \sin\theta\omega 1 \\ \omega_3 & 0 & \sin\theta\omega_2 - \cos\theta\omega_1 \\ -\cos\theta\omega_2 - \sin\theta\omega 1 & -\sin\theta\omega_2 + \cos\theta\omega_1 & 0 \end{pmatrix}.$$

We see that in our dictionary this corresponds to a usual rotation of the vector $\vec{\omega}$ about the zeta axis by angle θ:

$$\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \longmapsto O\vec{\omega} = \begin{pmatrix} -\sin\theta\omega_2 + \cos\theta\omega_1 \\ \cos\theta\omega_2 + \sin\theta\omega 1 \\ \omega_3 \end{pmatrix}.$$

We can proceed with analogous calculations for rotations about the x and y axis as well to complete the argument, but it is now clear that the coadjoint action really induces ordinary rotations of vectors.

We can actually lift the $SO(3)$-action on $\mathbb{R}^3$ to a symplectic action on the cotangent bundle $\mathbb{R}^6$ as follows. The infinitesimal version of this action is:

$$\begin{aligned} \vec{a} \in \mathbb{R}^3 \longmapsto d\psi(\vec{a}) &\in \chi^{\text{sympl}}(\mathbb{R}^6) \\ d\psi(\vec{a})(\vec{x},\vec{y}) &= (\vec{a}\times\vec{x}, \vec{a}\times\vec{y}); \end{aligned}$$

then

$$\mu : \mathbb{R}^6 \to \mathbb{R}^3, \qquad \mu(\vec{x},\vec{y}) = \vec{x}\times\vec{y}$$

is a moment map, whereas the hamiltonian function for $X^\sharp$, $X = \vec{a}$ is

$$\mu^{\vec{a}}(\vec{x},\vec{y}) = \langle \mu(\vec{x},\vec{y}), \vec{a}\rangle = (\vec{x}\times\vec{y})\cdot\vec{a}.$$

Indeed $X^\sharp = (\vec{a}\times\vec{x}, \vec{a}\times\vec{y})$,

$$\imath_{X^\sharp}\omega = \left(\sum_i dx_i \wedge dy_i\right)(\vec{a}\times\vec{x}, \vec{a}\times\vec{y}) = \left(\sum_i dx_i \wedge dy_i\right)\left(\sum_{ijk} \epsilon_{ijk} a_i x_j \frac{\partial}{\partial x_k} + \sum_{ijk} \epsilon_{ijk} a_i y_j \frac{\partial}{\partial y_k}\right) =$$
$$\sum_{ijk} \epsilon_{ijk} a_i x_j dy_k - \sum_{ijk} \epsilon_{ijk} a_i y_j dx_k = \vec{x}\times\vec{dy}\cdot\vec{a} + \vec{dx}\times\vec{y}\cdot\vec{a} = d\left(\vec{x}\times\vec{y}\cdot\vec{a}\right) = d\mu^{\vec{a}}(\vec{x},\vec{y}).$$

We have given a precise sense to the common claim that **angular momentum** is the generator of rotations.

7.3 The Noether principle

Theorem 7.1 (Nöther)**.** *Let (M, ω, G, μ) be a hamiltonian G-space; if the function*

$$f : M \to \mathbb{R}$$

is G-invariant, then μ is constant on the trajectories of the hamiltonian vector field with hamilton function f.

Proof. Denote by v_f the hamiltonian vector field of f: $df = \imath_{v_f}\omega$. Letting $X \in \mathfrak{g}$ and $\mu^X = \langle \mu, X \rangle : M \to \mathbb{R}$ we have:

$$\mathcal{L}_{v_f}\mu^X = \imath_{v_f} d\mu^X = \imath_{v_f}\imath_{X^\sharp}\omega = -\imath_{X^\sharp}\imath_{v_f}\omega = -\imath_{X^\sharp} df = -\mathcal{L}_{X^\sharp} f$$

where the last term is zero since f is G-invariant. □

In order to give a precise sense to the Nöther principle, we can set the following definitions.

Definition 7.2. A G-invariant function $f : M \to \mathbb{R}$ is called an **integral of motion** of (M, ω, G, μ).
If μ is constant on the trajectories of a hamiltonian vector field v_f, then the corresponding one-parameter group of diffeomorphisms e^{tv_f} is called a **symmetry** of (M, ω, G, μ).

The **Nöther principle** asserts that there is a one-to-one correspondence between symmetries and integrals of motion.

8 The Kostant-Kirillov and Lie-Poisson structures

8.1 Coadjoint Orbits and Symplectic Structure

In this paragraph we explain how the orbits of the coadjoint representation of G on $\mathfrak{g}^*$ can be endowed with a *natural* symplectic structure.

Definition 8.1. Let's define ω_ξ, where $\xi \in \mathfrak{g}^*$ as follows:

$$\omega_\xi(X, Y) = \langle \xi, [X, Y] \rangle, \quad X, Y \in \mathfrak{g} \tag{8.1}$$

We are now going to see how this leads indeed to the symplectic structure we are looking for.

Lemma 8.1. *The set* $Ker\omega_\xi = \{X \in \mathfrak{g} | \omega_\xi(X,Y) = 0 \ \forall Y \in \mathfrak{g}\}$ *is precisely the Lie algebra* $\mathfrak{g}_\xi$ *of the stabilizer of* ξ *for the coadjoint representation of* G *on* $\mathfrak{g}^*$.

Proof. Let $X \in \mathrm{Ker}\omega_\xi$, then $0 = \langle \xi, [X,Y] \rangle \ \forall Y \in \mathfrak{g}$; however, for the properties of ad^*, we have:

$$0 = \langle \xi, [X,Y] \rangle = \langle ad^*_Y \xi, X \rangle, \quad \forall Y \in \mathfrak{g},$$

since $\langle \, , \rangle$ is non-degenerate this can only mean: $ad^*_Y \xi = 0$, i.e.

$$\begin{aligned} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Ad^*_{e^{\varepsilon Y}} \xi = 0 \Rightarrow \frac{d}{dt}\Big|_{t=t_0} Ad^*_{e^{tY}} \xi = \lim_{t \to t_0} \frac{Ad^*_{e^{tY}} \xi - Ad^*_{e^{t_0 Y}} \xi}{t - t_0} = \\ = \lim_{t \to t_0} \frac{Ad^*_{e^{t_0 Y} e^{(t-t_0)Y}} \xi - Ad^*_{e^{t_0 Y}} \xi}{t - t_0} = Ad^*_{e^{t_0 Y}} \lim_{\Delta t \to 0} \frac{Ad^*_{e^{\Delta t Y}} \xi - \xi}{\Delta t} = \\ = Ad^*_{e^{t_0 Y}} \left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Ad^*_{e^{\varepsilon Y}} \xi \right) = 0. \end{aligned}$$

Therefore $Ad^*_{e^{tY}} \xi = \xi$, $\forall t \in \mathbb{R}$, which means $e^{tY} \in G_\xi$ and finally $\xi \in \mathfrak{g}_\xi$.

Viceversa supposing $X \in \mathfrak{g}_\xi$, then, for any $Y \in \mathfrak{g}$:

$$\omega_\xi(X,Y) = \langle \xi, [X,Y] \rangle = \langle ad^*_Y \xi, X \rangle$$

where the left term of the contraction is zero by definition.
Hence: $\omega_\xi(X,Y) = 0, \ \forall Y \in \mathfrak{g}$. □

Let ξ be an element of $\mathfrak{g}^*$ and let $\mathcal{O}_\xi \subseteq \mathfrak{g}^*$ be its coadjoint orbit. We have the "evaluation at ξ"map:

$$G \to \mathcal{O}_\xi \subseteq \mathfrak{g}^*$$

sending $g \in G$ to $Ad^*_g(\xi)$. It is surjective by definition, and so is its derivative at ξ

$$T_e G (\simeq \mathfrak{g}) \to T_\xi \mathcal{O}_\xi \subseteq T_\xi \mathfrak{g}^*.$$

Thus, every tangent vector $\boldsymbol{v} \in T_\xi \mathcal{O}_\xi$ can be represented as the velocity vector at the point ξ of some curve, which is the coadjoint representation of a 1-parameter subgroup e^{tX} of G, where $X \in \mathfrak{g}$, therefore:

$$\boldsymbol{v} = \frac{d}{d\tau}\Big|_{\tau=0} Ad^*_{e^{\tau X}} \xi = ad^*_X \xi, \quad X \in \mathfrak{g}, \ \xi \in \mathfrak{g}^*. \tag{8.2}$$

Notice that the element $X \in G$ is determined only up to a vector which is tangent to the stabilizer of ξ.

Definition 8.2. We define the form Ω as follows: given two vectors $\boldsymbol{v}_1,\ \boldsymbol{v}_2 \in T_\xi\,\mathcal{O}_\xi$, expressed in terms of the corresponding $X_1,\ X_2 \in \mathfrak{g}$;
then

$$\Omega_\xi(\boldsymbol{v}_1, \boldsymbol{v}_2) = \omega_\xi(X_1, X_2) = \langle \xi, [X_1, X_2] \rangle, \qquad \xi \in \mathfrak{g}^*,\ X_i \in \mathfrak{g}. \tag{8.3}$$

Remark 8.1. We shall examine in detail why $\Omega_\xi(\boldsymbol{v}_1, \boldsymbol{v}_2)$ indeed does not depend on the choice of the representative elements $X_1,\ X_2$, which are of course not unique, in theorem 8.4.

Lemma 8.2. *Ω_ξ defines a non-degenerate two-form on the tangent space at ξ to the coadjoint orbit through ξ, denoted $T_\xi\,\mathcal{O}_\xi$.*

Proof. By contradiction, assume there is $\boldsymbol{v} \in T_\xi\,\mathcal{O}_\xi$ such that $\Omega_\xi(\boldsymbol{v}, \boldsymbol{u}) = 0$, $\forall \boldsymbol{u} \in T_\xi\,\mathcal{O}_\xi$. Looking at the definition of Ω, this leads to a corresponding relation between the representative elements in $\mathfrak{g}$: $\omega_\xi(X, Y) = 0$, $\forall Y \in \mathfrak{g}$, thus $X \in \mathrm{Ker}\omega_\xi$. However, from Lemma 8.1, $X \in \mathfrak{g}_\xi$, the Lie algebra of the *stabilizer* of ξ which means $Ad_{e^{tX}}\xi = \xi$ for all $t \in \mathbb{R}$ and finally $\boldsymbol{v} = \frac{d}{dt}\big|_{t=0} Ad_{e^{tX}}\xi = \frac{d}{dt}\big|_{t=0}\xi = 0$, which shows the non-degeneracy of Ω_ξ. □

Lemma 8.3. *Ω_ξ defines a closed 2-form on the orbit of ξ in $\mathfrak{g}^*$.*

Proof. Clearly from Definition 8.2 Ω_ξ *is* a 2-form: it is bilinear and skew-symmetric since ad^* and $[\,,]$ possess these properties.
To show the closedness of Ω_ξ we can consider a general property: for a generic two-form ω, its exterior derivative $d\omega$ satisfies

$$\begin{aligned} d\omega(X, Y, Z) = X\omega(Y, Z) + Y\omega(Z, X) + Z\omega(X, Y) + \\ - \omega([X, Y], Z) - \omega([Y, Z], X) - \omega([Z, X], Y). \end{aligned} \tag{8.4}$$

In our case we can apply this identity to the generators of $T_\xi\,\mathcal{O}_\xi$, which are of course vectors of the form $X_\xi^\sharp = \frac{d}{dt}\big|_{t=0} Ad^*_{e^{tX}}\xi$, $X \in \mathfrak{g}$; for arbitrary $X_\xi^\sharp, Y_\xi^\sharp, Z_\xi^\sharp$ get

$$\begin{aligned} d\Omega_\xi(X_\xi^\sharp, Y_\xi^\sharp, Z_\xi^\sharp) = d\omega_\xi(X, Y, Z) = X\omega_\xi(Y, Z) + Y\omega_\xi(Z, X) + Z\omega_\xi(X, Y) + \\ - \omega_\xi([X, Y], Z) - \omega_\xi([Y, Z], X) - \omega_\xi([Z, X], Y). \end{aligned}$$

The first three terms vanish:

$$\begin{aligned} &\langle ad^*_{e^{tX}}\xi, [Y, Z]\rangle + \langle ad^*_{e^{tY}}\xi, [Z, X]\rangle + \langle ad^*_{e^{tZ}}\xi, [X, Y]\rangle = \\ &\quad = \langle \xi, ad_{e^{-tX}}[Y, Z]\rangle + \langle \xi, ad_{e^{-tY}}[Z, X]\rangle + \langle \xi, ad_{e^{-tZ}}[X, Y]\rangle = \\ &\qquad = -\langle \xi, \underbrace{[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]}_{=0 \text{ by Jacobi Identity}}\rangle = 0 \end{aligned}$$

The last three terms, on the other hand, can be regrouped as

$$\langle \xi, [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] \rangle$$

and vanish thanks to Jacobi Identity all the same. □

Theorem 8.4.
Ω *constitutes a* well-defined *symplectic structure on the coadjoint orbits* $\mathcal{O}_\xi$, $\xi \in \mathfrak{g}^*$.
This canonical symplectic form is also known as the Lie-Poisson or ***Kostant-Kirillov symplectic structure****.*

Proof. From the previous Lemmas 8.2 and 8.3 we can conclude that Ω_ξ is indeed bilinear, skew-symmetric, non-degenerate and closed.
We are now to verify that $\Omega_\xi(\boldsymbol{v}, \boldsymbol{u})$, $\boldsymbol{v}, \boldsymbol{u} \in T_\xi\, \mathcal{O}_\xi \simeq \mathfrak{g}^*$ does not depend on the choice of the representative elements $X, Y \in \mathfrak{g}$ such that $ad^*_X\xi = \boldsymbol{v}$, $ad^*_Y\xi = \boldsymbol{u}$. Indeed, consider X, X', both satisfying $ad^*_X\xi = ad^*_{X'}\xi = \boldsymbol{v}$; therefore

$$\begin{aligned}\omega_\xi(X,Y) = \langle \xi, [X,Y] \rangle = \langle \xi, ad_X Y \rangle = \\ = \langle ad^*_X\xi, Y \rangle = \langle ad^*_{X'}\xi, Y \rangle = \langle \xi, ad'_X Y \rangle = \langle \xi, [X',Y] \rangle = \omega_\xi(X',Y).\end{aligned}$$ □

8.2 Coadjoint Orbits and Poisson Structure

The Lie Algebra structure of $\mathfrak{g}$ also defines a canonical Poisson structure on $\mathfrak{g}^*$.

Definition 8.3. The Poisson bracket of two functions on $\mathfrak{g}^*$ can be taken to be:

$$\{f,g\}(\xi) \doteq \langle \xi, [df_\xi, dg_\xi] \rangle, \qquad \xi \in \mathfrak{g}^*,\ f,g \in C^\infty(\mathfrak{g}^*). \tag{8.5}$$

The use of the $\langle\, , \rangle$ contraction notation is justified by the fact that $df_\xi : T_\xi \mathfrak{g}^* \simeq \mathfrak{g}^* \to \mathbb{R}$ and it can therefore be identified with an element of $\mathfrak{g} \simeq \mathfrak{g}^{**}$.

Lemma 8.5. $\{\, , \}$ *satisfies the Leibnitz rule:* $\{f, gh\} = \{f,g\}h + g\{f,h\}$.

Proof. $\{f,gh\}(\xi) = \langle \xi, [df_\xi, d(gh)_\xi] \rangle = \langle \xi, [df_\xi, (dg_\xi)h_\xi + g_\xi(dh_\xi)] \rangle = \langle \xi, [df_\xi, dg_\xi] \rangle h_\xi + g_\xi \langle \xi, [df_\xi, dh_\xi] = (\{f,g\}h + g\{f,h\})(\xi)$. □

Lemma 8.6. *The* ***jacobiator***

$$J(f,g,h) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}$$

is a trivector field, that is to say a skew-symmetric, trilinear map

$$C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*)$$

which is a derivation in each argument.

Proof. Skew-symmetry and bilinearity are granted by the fact that $[\,,]$ is skew-symmetric and bilinear, whereas $<\,,>$ is bilinear in its argument.
The property of being a derivation amounts to the Leibnitz rule from Lemma 8.5. ☐

Lemma 8.7. $J \equiv 0$*, i.e.* $\{\,,\}$ *satisfies the Jacobi Identity.*

Proof. This is trivial since the Jacobi identity for $[\,,]$ is indeed valid. ☐

Symplectic Reduction

Dynamical systems that exhibit properties of symmetry allow the introduction of a *reduced* phase space, provided with a natural symplectic structure, as the quotient manifold between the invariant level manifolds of the first integrals defined by those symmetries and the 1-parameter subgroups operating on those manifolds.

9 The Marsden-Weinstein-Meyer Theorem

9.1 Statement

Theorem 9.1 (Marsden-Weinstein-Meyer). *Let (M, ω, G, μ) be a hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on M. Then*

- *the orbit space $M_{\text{red}} = \mu^{-1}(0)/G$ is a manifold,*
- *$\pi : \mu^{-1}(0) \to M_{\text{red}}$ is a principal G-bundle and*
- *there is a symplectic form ω_{red} on M_{red} satisfying*

$$i^*\omega = \pi^*\omega_{\text{red}}.$$

The pair $(M_{\text{red}}), \omega_{\text{red}}$ is often called ***reduction*** *of (M, ω).*

9.2 Ingredients

1. Let $\mathfrak{g}_p$ be the Lie algebra of the stabilizer of $p \in M$. Then we have for $d\mu_p : T_pM \to \mathfrak{g}^*$ the following properties:

$$\text{Ker } d\mu_p = (T_p \mathcal{O}_p)^{\omega_p}$$
$$\text{Im } d\mu_p = \mathfrak{g}_p^0,$$

where $\mathcal{O}_p$ is the G-orbit through p and $\mathfrak{g}_p^0$ is the **annihilator** of $\mathfrak{g}_p$, i.e. the set $\{\xi \in \mathfrak{g}^* \mid \langle \xi, X\rangle = 0, \text{ for all } X \in \mathfrak{g}_p.\}$

Proof. The first fact follows from

$$\begin{aligned} v \in \text{Ker } d\mu_p &\iff d\mu_p(v) = 0 \\ &\iff 0 = \langle d\mu_p(v), X\rangle = \omega_p(X_p^\sharp, v) \ \forall X \in \mathfrak{g} \\ &\iff v \in (T_p \mathcal{O}_P)^{\omega_p}. \end{aligned}$$

As for the second, it is easy to show that Im $d\mu_p \subseteq \mathfrak{g}_p^0$, since letting $\xi = d\mu_p(v)$ for some $v \in T_pM$ one has:

$$\forall X \in \mathfrak{g}_p, \ \langle \xi, X\rangle = \langle d\mu_p(v), X\rangle = \omega_p(X_p^\sharp, v) = \omega_p(0, v) = 0,$$

where we used $X^\sharp = \frac{d}{dt}\big|_{t=0} \psi_{e^{tX}} p \equiv 0$ because $e^{tX} \in G_p$, G_p being the stabilizer of p.
To prove that actually only the equality holds, we can count dimensions as follows: dim M = dim $T_pM = 2n$, dim $T_p \mathcal{O}_p = g - s$ where g = dim G and s = dim G_p. For symplectic linear algebra, a subspace and its symplectic orthogonal have dimensions that add to the dimension of the whole space: dim $(T_p \mathcal{O})^{\omega_p}$ = dim Ker $d\mu_p = 2n-g+s$. Finally dim Ker $d\mu_p$+dim Im $d\mu_p$ = dim T_pM, thus dim Im $d\mu_p = 2n - (2n - g + s) = g - s$.
On the other hand clearly dim $\mathfrak{g}_p^0$ = dim $\mathfrak{g}$ − dim $\mathfrak{g}_p = g - s$. □

The previous properties imply that, if the action is locally free at p, which means $\mathfrak{g}_p = \{0\}$, then $d\mu_p$ is surjective, which means that p is a regular point. In particular if $\mu(p) = 0$, then 0 is a regular value of μ and the level set $\mu^{-1}(0)$ is a closed submanifold of M of codimension equal to dim$G$. Assume furthermore G act freely on $\mu^{-1}(0)$; then $T_p\mu^{-1}(0) = \text{Ker}d\mu_p$ for $p \in \mu^{-1}(0)$ and the previous statement ensures that $T_p\mathcal{O}_p$ and $T_p\mu^{-1}(0)$ are symplectically orthogonal.

This line of thought leads to conclude that $T_p\mathcal{O}_p$, where $p \in \mu^{-1}(0)$ is an isotropic subspace of T_pM, hence **orbits in $\mu^{-1}(0)$ are isotropic**. To convince ourselves of this fact, we can verify it directly; let $X, Y \in \mathfrak{g}$ and $p \in \mu^{-1}(0)$:

$$\omega_p(X_p^\sharp, Y_p^\sharp) = H_{[Y^\sharp, X^\sharp]}(p) = H_{[Y,X]^\sharp}(p) = \mu^{[Y,X]}(p) = 0.$$

2. The existence of such isotropic subspaces hints to the presence of redundant degrees of freedom. To eliminate them, we need the following lemma from linear algebra:

   Lemma 9.2. *Let (V, ω) be a symplectic vector space. Suppose that I is an isotropic subspace of V, i.e. $\omega|_I \equiv 0$. Then ω induces a canonical symplectic form on I^ω/I.*

   Proof. Let $u, v \in I^\omega$ and $[u], [v] \in I^\omega/I$. We set: $\Omega([u],[v]) = \omega(u, v)$.
   First we notice that Ω is well-defined since $\forall i, j \in I$ we have

$$\omega(u+i, v+j) = \omega(u,v) + \underbrace{\omega(u,j)}_{0} + \underbrace{\omega(v,i)}_{0} + \underbrace{\omega(i,j)}_{0}.$$

   Furthermore Ω is nondegenerate; indeed letting $u \in I^\omega$ has $\omega(u, v) = 0$ for all $v \in I^\omega$ implies $u \in I^{\omega\omega} \equiv I$. But then $[u] = 0$. □

3. As a third preliminary fact, we need a statement concerning the regularity of the objects we are dealing with.

   Theorem 9.3. *If a compact Lie group G acts freely on a manifold M, then M/G is a manifold and $\pi : M \to M/G$ is a principal G-bundle.*

   A thorough proof of this fact can be found in both [4] and [2]; a sketch of the reason why it holds goes as follows.
   That the quotient space, endowed with the quotient topology, is Hausdorff and second-countable is a general fact from point-set topology. We just sketch how one can construct a differentiable atlas: Given a point $p \in M$ one can construct a local slice Σ_p for the action, i.e. a subvariety passing through p and transversally intersecting the orbit $\mathcal{O}_p$, with the property that, for every $q \in \Sigma$, we have $\mathcal{O}_q \cap \Sigma_p = \{q\}$ (this statement is sometimes called "Slice" theorem). Then, coordinates on Σ give coordinates for a neighborhood of $\pi(p)$ in M/G. The coordinate-change maps are given by identifying points on different slices by using the action of the group G.

9.3 Proof of the Marsden-Weinstein-Meyer theorem

For the first ingredient, since G acts freely on $\mu^{-1}(0)$, then $d\mu_p$ is surjective for all $p \in \mu^{-1}(0)$, 0 is a regular value and $\mu^{-1}(0)$ is a submanifold of codimension dim$G$. Now the third ingredient, applied to the free action of G on $\mu^{-1}(0)$, yields the first two assertions of the theorem.

The second ingredient (Lemma 9.2) gives a canonical symplectic structure on the quotient $T_p\mu^{-1}/T_p\,\mathcal{O}_p$, $T_p\,\mathcal{O}_p$ being an isotropic vector subspace for $p \in \mu^{-1}(0)$. However, $[p] \in M_{red}$ has tangent space $T_{[p]}M_{red} \simeq T_p\mu^{-1}/T_p\,\mathcal{O}_p$, hence our Lemma defines a nondegenerate 2-form ω_{red} on M_{red} as well. ω_{red} is well defined because ω is G-invariant. By construction $i^*\omega = \pi^*\omega_{red}$ where

$$\begin{array}{ccc} \mu^{-1}(0) & \xrightarrow{\ i\ } & M \\ \big\downarrow{\scriptstyle\pi} & & \\ M_{red}; & & \end{array}$$

hence $\pi^* d\omega_{red} = d\pi^*\omega_{red} = di^*\omega = i^* d\omega = 0$. The closeness of ω_{red} follows from the injectivity of π^*. We notice that the relation also ensures that ω_{red} is smooth, by the construction of an atlas for M/G out of local slices for the action of G on M which was sketched above.

10 Applications

10.1 Elementary theory of reduction

Given the $2n$-dimensional hamiltonian system (M, ω, H), if we can spot a symmetry in this system, i.e. invariance under the action of a group, we can simplify the problem by solving the *reduced* $(2n-2)$-dimensional system $(M_{red}, \omega_{red}, H_{red})$.

To give a more precise idea of this, we can consider local Darboux coordinates for the open set $\mathcal{U}$ of M: $x_1, \ldots, x_n$, $\xi_1, \ldots, \xi_n$, where ξ_n is the integral of motion corresponding to the symmetry mentioned above.
So we have $\xi_n = constant = c$, which means

$$\{\xi_n, H\} = 0 = -\frac{\partial H}{\partial x_n};$$

then $H = H(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c)$. The Hamilton equations read:

$$\begin{aligned}
\frac{dx_1}{dt} &= \frac{\partial H}{\partial \xi_1}(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \\
&\vdots \\
\frac{dx_{n-1}}{dt} &= \frac{\partial H}{\partial \xi_{n-1}}(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \\
\frac{d\xi_1}{dt} &= -\frac{\partial H}{\partial x_1}(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \\
&\vdots \\
\frac{d\xi n_1}{dt} &= -\frac{\partial H}{\partial x_{n-1}}(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c).
\end{aligned}$$

and

$$\begin{aligned}
\frac{dx_n}{dt} &= \frac{\partial H}{\partial \xi_n} \\
\frac{d\xi_n}{dt} &= 0.
\end{aligned}$$

Therefore, switching to the reduced manifold means to set $\xi_n = c$, and to work on this hyperplane. In case we manage to solve the reduced problem, by finding the trajectories $x_1(t), \ldots, x_{n-1}(t), \xi_1(t), \ldots, \xi_{n-1}(t)$, we can also reconstruct the solution to the original problem by direct integration:

$$x_n(t) = x_n(0) + \int_0^t \frac{\partial H}{\partial \xi_n} dt$$

$$\xi_n(t) = c.$$

10.2 Reduction at Other Levels

Let a compact Lie group G act on a symplectic manifold (M, ω) in a hamiltonian way and let $\mu : M \to \mathfrak{g}^*$ be the moment map of such action.
Up to now we have only considered the reduction at the level zero, $\mu^{-1}(0)$; given an arbitrary value $\xi \in \mathfrak{g}^*$, we would like now to reduce at the level ξ of μ; to do so we

need $\mu^{-1}(\xi)$ to be preserved by G.
Since G is equivariant:

$$\begin{aligned} G \text{ preserves } \mu^{-1}(\xi) &\iff G \text{ preserves } \xi \\ &\iff Ad^*_g \xi = \xi, \ \forall g \in G. \end{aligned}$$

Consistently with the above considerations, se see now that 0 is always preserved

10.3 Hamiltonian reduction along an orbit

Let's take $\mathcal{O}$ to be the coadjoint orbit of $\xi \in \mathfrak{g}^*$, equipped with the Konstant-Kirillov symplectic form $\omega_{\mathcal{O}}$ which was discussed before.
Denote $\mathcal{O}^-$ the same orbit equipped with the symplectic structure defined by $-\omega_{\mathcal{O}}$; now the natural product action of G on $M \times \mathcal{O}^-$ is hamiltonian with moment map $\mu_{\mathcal{O}}(p, \xi) = \mu(p) - \xi$, since this shifted map satisfies $\mu_{\mathcal{O}}^{-1}(0) = \mu^{-1}(\xi)$ by construction.

Now, if we can verify the hypothesis of the Marsden-Weinstein-Meyer theorem for $M \times \mathcal{O}^-$, then we obtain the **reduced space with respect to** G **along the coadjoint orbit** $\mathcal{O}$, $\mu(\mathcal{O})/G \equiv R(M, G, \mathcal{O})$.

Lemma 10.1. *If the action of $\mu^{-1}(\mathcal{O})$ is free, then $R(M, G, \mathcal{O})$ is a symplectic manifold of dimension*

$$\dim M - 2\dim G + \dim \mathcal{O}.$$

Proof. We obtain $\mathcal{O}$ by imposing $\dim G - \dim \mathcal{O}$ constraints, therefore we have $\dim\mu^{-1}(\mathcal{O}) = \dim M - (\dim G - \dim \mathcal{O})$. Finally, since G acts freely, thus all the stabilizers are discrete,

$$\dim R(M, G, \mathcal{O}) = \dim\mu^{-1}(\mathcal{O}) - \dim G = \dim M - 2\dim\, G + \dim \mathcal{O}. \quad \square$$

This procedure will prove to be of chief interest in the next section.

Introduction to Calogero-Moser Systems

In this final section we describe and solve the hamiltonian system of Calogero-Moser type, which was originally introduced in the work of F. Calogero.

Here we are going to deal with complex quantities: manifolds are going to be complex manifolds, rather than real manifolds which we dealt with in the previous sections, and also symplectic forms are going to be *holomorphic* symplectic forms.

11 The Calogero-Moser system

11.1 The hamiltonian G-space

In order to build the Calogero-Moser space, we are going to require the following elements:

- Let $M \doteq T^*\mathrm{Mat}_n(\mathbb{C})$.
  Since this is a linear space, we can identify $T_p\mathrm{Mat}_n(\mathbb{C}) \simeq \mathrm{Mat}_n(\mathbb{C})$ and $T\mathrm{Mat}_n(\mathbb{C}) \simeq \mathrm{Mat}_n(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C})$. Thus $\dim M = n^2 + n^2 = 2n^2$.

  Provided we have a nondegenerate bilinear form Γ_p on $T_p\mathrm{Mat}_n(\mathbb{C})$, we can further identify $T_p^*\mathrm{Mat}_n(\mathbb{C}) \simeq T_p\mathrm{Mat}_n(\mathbb{C})$ in the usual way:

  $$X \in T_p\mathrm{Mat}_n(\mathbb{C}) \longmapsto \Gamma_p(X,\ ) \in T_p^*\mathrm{Mat}_n(\mathbb{C}).$$

  So we can regard M as the set of pairs of matrices:

  $$M = T^*\mathrm{Mat}_n(\mathbb{C}) \simeq T\mathrm{Mat}_n(\mathbb{C}) \simeq \mathrm{Mat}_n(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C}) = \{(X,Y) | X, Y \in \mathrm{Mat}_n(\mathbb{C})\}.$$

- Consider also the **trace form**

  $$\omega \doteq tr\,(dY \wedge dX) = tr\,[(dY_{ij}) \wedge (dX_{lm})] = tr\left(\sum_k dY_{ik} \wedge dX_{km}\right) = \sum_{h,k} dY_{hk} \wedge dX_{kh},$$

  which is really nothing more than our usual canonical symplectic form $dp \wedge dq$. The pair (M, ω) constitutes a symplectic manifold.

- Now let $G \doteq PGL_n(\mathbb{C}) = GL_n(\mathbb{C})/\mathbb{C}^*\mathrm{Id}$, where:

  $$GL_n(\mathbb{C}) = \{M \in \mathrm{Mat}_n(\mathbb{C}) | \det M \neq 0\},$$
  $$\mathbb{C}^* = \mathbb{C} \setminus \{0\}.$$

  This is the formal way of saying that we want to consider all nonsingular matrices after identifying those which can be written in the form: A, A', where $A = \lambda A'$, $\lambda \neq 0$.

  Thanks to this degree of freedom, we are able to satisfy the condition $\det \lambda A = 1$ in n different ways:

  $$\det \lambda A = 1 \iff \lambda^n = \frac{1}{\det A}.$$

  This gives the relation connecting $PGL_n(\mathbb{C})$ to $SL_n(\mathbb{C})$, where

  $$SL_n(\mathbb{C}) = \{M \in \mathrm{Mat}_n(\mathbb{C}) | \det M = 1\},$$

  which is just:

  $$PGL_n(\mathbb{C}) = SL_n(\mathbb{C})/\mathbb{C}^*\mathrm{Id} \cap SL_n(\mathbb{C}).$$

  However the matrices in the form $\lambda\mathrm{Id}$ where $\lambda^n = 1$ are precisely n, so the difference between the two groups amounts to a *finite* group. We have shown that:

  $$\dim\, PGL_n(\mathbb{C}) = n^2 - 1; \tag{11.1}$$
  $$\mathfrak{pgl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C}). \tag{11.2}$$

- Let now G act on M by conjugation in the following sense.

  Given $A \in G$

  $$\psi_A : \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C}) \tag{11.3}$$
  $$Q \mapsto \psi_A(M) = A^{-1}QA; \tag{11.4}$$

  this action lifts to an action on $M = T^*\mathrm{Mat}_n(\mathbb{C})$.

Claim. *The action of* $G = PGL_n(\mathbb{C})$ *on* $M = T^*Mat_n(\mathbb{C})$ *by conjugation is* ***hamiltonian*** *with moment map*

$$\mu(X,Y) = [X,Y] = XY - YX, \qquad X, Y \in Mat_n(\mathbb{C}).$$

Proof. We first have to compute the action of $PGL_n(\mathbb{C})$ on $T^*\mathrm{Mat}_n(\mathbb{C})$ explicitly. Since $PGL_n(\mathbb{C})$ acts on $\mathrm{Mat}_n(\mathbb{C})$ by conjugation, we have for any $C \in PGL_n(\mathbb{C})$:

$$\psi_C : M \in \mathrm{Mat}_n(\mathbb{C}) \longmapsto C^{-1}MC.$$

The corresponding action of $T\mathrm{Mat}_n(\mathbb{C})$ will be:

$$\psi_C : (M,V) \in T\mathrm{Mat}_n(\mathbb{C}) \longmapsto (C^{-1}MC, C^{-1}VC)$$

since we have $M + \varepsilon V \longmapsto C^{-1}(M + \varepsilon V)C = C^{-1}MC + \varepsilon C^{-1}VC$ or, more formally

$$V = \frac{d(e^{Vt})}{dt}\Big|_{t=0} \implies (\psi_C)_* V = \frac{d}{dt}\Big|_{t=0}(C^{-1}e^{Vt}C) = C^{-1}VC.$$

We identify $T^*\mathrm{Mat}_n(\mathbb{C}) \simeq T\mathrm{Mat}_n(\mathbb{C})$ using a convenient, conjugacy-invariant bilinear form: given $(M,N) \in T^*\mathrm{Mat}_n(\mathbb{C})$ and $V \in T\mathrm{Mat}_n(\mathbb{C})$

$$(M,N)(V) \doteq \mathrm{Tr}(NV);$$

now, as $M \mapsto C^{-1}MC$, we have that $\mathrm{Tr}(NV) \mapsto \mathrm{Tr}(NCVC^{-1})$ since ψ_C is a group homomorphism. Then:

$$\mathrm{Tr}(NCVC^{-1}) = \mathrm{Tr}(C^{-1}NCV)$$

which proves that

$$\psi_C : (M,N) \in T^*\mathrm{Mat}_n(\mathbb{C}) \longmapsto (C^{-1}MC, C^{-1}NC).$$

Our next task is to verify: $d\mu^A = \imath_{A^\sharp}\omega$, where $A \in \mathfrak{pgl}_\mathfrak{n}(\mathbb{C})$, $\mu(M,N) = [M,N]$, $\mu^A = \langle \mu, A \rangle$, $\omega = \mathrm{Tr}(dY \wedge dX)$. Now, using the Liouville one-form $\omega = d\alpha$ and Cartan magic formula 3.7 our condition simplifies to:

$$d\mu^A = \imath_{A^\sharp}\omega = \imath_{A^\sharp} d\alpha = \underbrace{\mathcal{L}_{A^\sharp}\alpha}_{=0} - d\imath_{A^\sharp}\alpha$$

that is to say $-\imath_{A^\sharp}\alpha = \mu^A$. Indeed we have:

$$A^\sharp(M,N) = \frac{d}{dt}\Big|_{t=0}(e^{At}(M,N)) = \frac{d}{dt}\Big|_{t=0}(e^{-At}Me^{At}, e^{-At}Ne^{At}) = ([M,A],[N,A]);$$

$$\begin{aligned} &-\imath_{A^\sharp}\alpha = -\imath_{A^\sharp}\mathrm{Tr}(NdM) = \\ &= -\mathrm{Tr}(N[M,A]) = \mathrm{Tr}(-NMA+NAM) = \mathrm{Tr}(-NMA+MNA) = \mathrm{Tr}([M,N]A). \end{aligned}$$

On the other hand

$$\mu^A(M,N) = \mathrm{Tr}([M,N]A),$$

which proves the statement. □

11.2 The Calogero-Moser Space

Let $\mathcal{O}$ be the orbit of the matrix

$$\begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n-1 \end{pmatrix}$$

The orbit admits a simple parametrization: in order to discuss let us recall that, given a column vector v, and a row vector φ, we can form their tensor product matrix

$$v \otimes \varphi = \Big(\, \Big| \, \Big) (\text{——});$$

this matrix has rank one and $\mathrm{Tr}(v \otimes \varphi) = \varphi(v) = \langle \varphi, V \rangle$. Viceversa, it is easy to see that every rank one matrix is of this form. Furthermore such matrix is diagonalizable if and only if $\varphi(v) \neq 0$, i.e. $\mathrm{Tr}(v \otimes \varphi) \neq 0$.

Lemma 11.1. *The orbit $\mathcal{O}$ is the set of **traceless** matrices T such that $T + Id$ has **rank one**.*

Proof. We have

$$\mathrm{Tr}(C^{-1}TC) = \mathrm{Tr}\, T = 0;$$

$$\mathrm{Rank}(C^{-1}TC + \mathrm{Id}) = \mathrm{Rank}(C^{-1}TC + C^{-1}\mathrm{Id}C) = \mathrm{Rank}(C^{-1}(T + \mathrm{Id})C) = \mathrm{Rank}(T + \mathrm{Id}).$$

On the other hand, if T is traceless, and $T + \mathrm{Id}$ has rank one, then $T + \mathrm{Id} = v \otimes \varphi$, for appropriately chosen v and φ. Since $\mathrm{Tr}(T) = 0$, we have that $\mathrm{Tr}(T + \mathrm{Id}) = n$, hence $v \otimes \varphi$ is diagonalizable and conjugate to

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n \end{pmatrix}$$

and we conclude. □

The Calogero-Moser space is the reduction of (M, ω, G) along the orbit $\mathcal{O}$:

$$\mathcal{C}_n \doteq R(M, G, \mathcal{O}). \tag{11.5}$$

Thus, $\mathcal{C}_n$ is the set space of conjugacy classes of pairs of $n \times n$ matrices (X, Y) such that the matrix $XY - YX + 1$ has rank 1.

From the Marsden-Weinstein-Marsden theorem $\mathcal{C}_n$ is symplectic, and it can also be shown to be *connected*.

From the discussion in the proof of Lemma 11.1, we see that any $T \in \mathcal{O}$ can be written in the form

$$T = v \otimes \varphi - \mathrm{Id}$$

provided that $\langle \varphi, V \rangle = \mathrm{Tr}\ \mathrm{Id} = n$. Since v and φ live in an n-dimensional space, but we have to take both the constraint $\langle \varphi, V \rangle = n$ and the gauge degree of freedom $v \mapsto \lambda v, \varphi \mapsto \varphi / \lambda$ into account, we see that:

$$\dim\ \mathcal{O} = 2n - 2. \tag{11.6}$$

We are now in a position to show the following fact:

Claim. *Tha Calogero-Moser space $\mathcal{C}_n$ is $2n$-dimensional.*

Proof. Applying the formula from lemma 10.1 we have exactly:

$$\dim\ \mathcal{C}_n = \dim\ M - 2\dim\ G + \dim\ \mathcal{O} = 2n^2 - 2(n^2 - 1) + 2n - 2 = 2n. \quad \square$$

11.3 The Calogero-Moser System and Flow

Consider the functions

$$H_i = \mathrm{Tr}(Y^i), \quad i = 1, 2, \dots, n, \tag{11.7}$$

where $(X, Y) \in M = T^*\mathrm{Mat}_n(\mathbb{C})$. These functions satisfy the following properties:

- they are invariant under conjugation, since trace is invariant under change of basis;
- they are trivially in involution with each other, since $\{H_i, H_j\} = 0$ by skew-symmetry of the Lie bracket; indeed they depend only on the Y component of $(X, Y) \in M$, and thus their Poisson bracket can be shown to be zero by exploitation of $\{Y, Y\} = 0$;
- their differential are independent almost everywhere on M.

Let $G = PGL_n(\mathbb{C})$ act on M by conjugation and let $\mathcal{O}$ be the coadjoint orbit, consisting of traceless matrices T such that $T + \mathrm{Id}$ has rank one, considered above. Then the system $H_1, \dots, H_n$ descends to a system of functions in involution on the Calogero-Moser space $R(M, G, \mathcal{O}) \equiv \mathcal{C}_n$. This is called the **Calogero-Moser System**.

The **Calogero-Moser flow** is, by definition, the Hamiltonian flow on $\mathcal{C}_n$ defined by the Hamiltonian $H \doteq H_2 = \mathrm{Tr}(Y^2)$. Thus this flow is integrable by Arnold-Liouville Theorem 4.5, since it can be included in an integrable system: $H_1, \dots, H_n$ provide n independent integrals of motion, whereas we have shown that $\mathcal{C}_n$ is exactly $2n$-dimensional.
Therefore, in principle, its solutions can be found by quadratures using the inductive procedure of reduction of order, according to the elementary theory of reduction.

However one may observe that on the former manifold M the Calogero-Moser flow is just the motion of a free particle in the space of matrices, so it has the form:

$$\rho_t(X, Y) = (X + 2Yt, Y). \tag{11.8}$$

It can be shown that the same formula is valid on $\mathcal{C}_n$ and that the other $n - 1$ flows corresponding to $H_i = \mathrm{Tr}(Y^i)$ have exactly the same form

$$\rho_t^{(i)}(X, Y) = (X + iY^{i-1}t, Y).$$

12 Coordinates on $\mathcal{C}_n$ and the explicit form of the Calogero-Moser system

Let us restrict ourselves to the open dense subset $U_n \subset \mathcal{C}_n$ consisting of conjugacy classes of those pairs (X, Y) for which X is diagonalizable with distinct eigenvalues. Let $P \in U_n$. Clearly, P can be represented by a pair (X, Y) such that

$$\begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{pmatrix}, \qquad x_i \neq x_j, i \neq j.$$

Now,

$$T \doteq XY - YX = \begin{pmatrix} 0 & (x_1 - x_2)y_{12} & \dots & (x_1 - x_n)y_{1n} \\ (x_2 - x_1)y_{21} & 0 & \dots & (x_2 - x_n)y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (x_n - x_1)y_{n1} & (x_n - x_2)y_{n2} & \dots & 0 \end{pmatrix} = ((x_i - x_j)y_{ij}).$$

Now, as we saw $T + \mathrm{Id}$ has rank 1 and its entries κ_{ij} can be written in the form $\kappa_{ij} = v_i \varphi_j$ for some appropriate $v,\ \varphi$:

$$T + 1 = v \otimes \varphi$$

$$\begin{pmatrix} 1 & (x_1 - x_2)y_{12} & \dots & (x_1 - x_n)y_{1n} \\ (x_2 - x_1)y_{21} & 1 & \dots & (x_2 - x_n)y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (x_n - x_1)y_{n1} & (x_n - x_2)y_{n2} & \dots & 1 \end{pmatrix} = \begin{pmatrix} v_1\varphi_1 & v_1\varphi_2 & \dots & v_1\varphi_n \\ v_2\varphi_1 & v_2\varphi_2 & \dots & v_2\varphi_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n\varphi_1 & v_n\varphi_2 & \dots & v_n\varphi_n \end{pmatrix}.$$

By inspecting the last equation we have: $v_j\varphi_j = 1$, so $\varphi_j = v_j^{-1}$ and hence $\kappa_{ij} = v_i v_j^{-1}$. By conjugating (X, Y) by the matrix

$$\begin{pmatrix} v_1 & 0 & \dots & 0 \\ 0 & v_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_n \end{pmatrix}$$

we can reduce to the situation when $v_i = 1$, $\kappa_{ij} = 1$, so:

$$T = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{pmatrix}.$$

This leads to conclude that $(x_i - x_j)y_{ij} = 1$ when $i \neq j$, whereas y_{ii} are unconstrained. We have almost shown the following result.

Theorem 12.1. *Let $\mathbb{C}^n_{reg}$ be the open set of $(x_1, \dots, x_n) \in \mathcal{C}^n$ such that $x_i \neq x_j$ for $i \neq j$. There exists an isomorphism of symplectic manifolds $\xi : T^*(\mathbb{C}^n_{reg}/S_n) \to U_n$, where S_n denotes the symmetric group, given by the formula $(x_1, \dots, x_n, p_1, \dots, p_n) \mapsto (X, Y)$:*

$$X = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{pmatrix}; \qquad Y = \begin{pmatrix} p_1 & \frac{1}{x_1 - x_2} & \dots & \frac{1}{x_1 - x_n} \\ \frac{1}{x_2 - x_1} & p_2 & \dots & \frac{1}{x_2 - x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n - x_1} & \frac{1}{x_n - x_2} & \dots & p_n \end{pmatrix}.$$

Proof. Let $a_k \doteq \mathrm{Tr}(X^k)$ and $b_k \doteq \mathrm{Tr}(X^k Y)$. On M we have:

$$\{b_m, a_k\} = m a_{m+k-1}.$$

On the other hand, by definition of ξ: $\xi^* a_k = \sum x_i^k$, $\xi^* b_k = \sum x_i^k p_i$. Thus we see that

$$\{b_m, a_k\} = \{\xi^* b_m, \xi^* a_k\}.$$

Now we are done, since the functions a_k and b_k form a local coordinate system near a generic point of U_n. □

In such coordinates, the Hamilton function of the Calogero-Moser system reads:

$$H = \mathrm{Tr}(Y^2(x, p)) = \sum_i p_i^2 - \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}. \tag{12.1}$$

Thus the Calogero-Moser Hamiltonian describes the motion of n particles on the line, interacting with potential $-1/x^2$. The procedure we have described ensures that this system is completely integrable, with first integrals

$$H_i = \mathrm{Tr}(Y^i(x, p))$$

and provides an explicit solution:

$$X_t = X_0 + 2tY_0, \qquad P_t = \dot{X}_t,$$

where $(X, Y) = \xi(x, p)$.

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