Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE

Corso di Laurea magistrale in Matematica, Curriculum Generale-Applicativo

Pricing Stochastic-Local Volatility Models with default

Tesi di Laurea in
Equazioni differenziali stocastiche

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I **Sessione**
Anno Accademico 2013/2014

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Introduction

In recent years is becoming increasingly important to handle credit risk. To value and to hedge credit risk in a consistent way, one needs to develop a quantitative model. The main objective of the quantitative models of the credit risk is to provide ways to price financial contracts that are sensitive to credit risk.
This thesis deal with a local stochastic volatility model with default.
Local stochastic volatility model is a continuous market model with non-constant volatility. In particular this model is an extension of the Black-Scholes one: the general idea is to modify the dynamics of the underlying asset, thus obtaining a model in which the volatility is a function of time and of the price of the underlying asset X.
That is $\sigma_t = \sigma(t, X_t)$. This is due to the fact that, consistent with the observed market dynamics, implied volatilities today change as maturity and strike changes. Moreover, implied volatilities for fixed maturities and strikes also change over time and so volatility is non-constant. One advantage of local volatility models is that transition densities of the underlying – as well as European option prices – are often available in closed-form as infinite series of special functions. In order to take into account the possibility of bankruptcy of the counterparty of a contract, the models have to consider probability of default. Default is now permeating the valuation of derivatives in any area.
Whenever a derivative is traded, the default risk of the counterparty should, in principle, enter the valuation. Recently, regulatory institutions insisted on the need to include this kind of risk in the pricing paradigm. Our goal is to develop a payoff depending on the underlying of the basic derivative and on the default of the counterparty.
We will model the defaultable stock price process as a time-changed diffusion process with state-dependent local volatility. The stock price process will have local-stochastic

volatility and default intensity. We will consider the JDCEV model of Carr and Linetsky (2006) and obtain a rich class of analytically tractable models with local-stochastic volatility and default intensity. These models can be used to jointly price equity and credit derivatives.

This work is organized as follows:

Chapter 1: in this chapter we will consider a random time i.e. a random variable whose value is interpreted as the time at which a given stochastic process exhibits a certain behavior of interest. We will analize the problem of the evaluation of conditional expectation when the filtration available **G** is formed by the natural filtration **H** of a random time and an arbitrary filtration **F**. We will introduce the concept of survival process and hazard process. Finally we will show an important lemma which allows to substitute the conditional expectation with respect to $\mathscr{G}_t$ with the conditioning relative to the σ-algebra $\mathscr{F}_t$.

Chapter 2: in this chapter we will present our model architecture. We will prove that assumption made in the previous chapter holds and so we can used the main results in our case. We will introduce the concept of the default time τ, a stopping time which represents the instant in which the counterparty goes bankrupt. We will present a defaultable asset S with risk-neutral dynamics. We will introduce the European option-pricing problem and derive a partial differential equation (PDE) for the price of an option. So we will show that the study of the price of our model can be transferred in the study of a partial differential operator with variables coefficients.

Chapter 3: in this chapter we will provide some interesting examples. We will present the so called JDCEV model with respectively costant and stochastic short rate. As regards the stochastic case we will present two classical time-homogeneous short-rate models, namely the Vasicek (1977) and the Cox, Ingersoll and Ross (1985) models. We will show that both models possess an affine term structure: this argument acquires importance due to the fact that from an analytical and computational point of view the existence of an affine term structure extremely simplifies the content.

Chapter 4: we will introduce two numerical methods in order to approximate different bond prices. The former give an analytical approximation formulas and it is based on the notion of polynomial expansion basis: the idea is to expand the local volatility and drift functions as a Taylor series about an arbitrary point, i.e.

$$f(x) = \sum_{n=0}^{\infty} a_n (x - \overline{x})^n$$

in order to achieve their approximation result. We will derive a formal asymptotic expansion for the function that solves the option pricing PDE. These formulas are implemented in *Wolfram Mathematica*. The latter is based on Monte Carlo method, one of the most popular numerical methods, which allow to calculate approximately the price of the contract through several simulations. This method is implemented in *Matlab*.

Chapter 5: Finally we will provide numerical examples that illustrate the effectiveness and versatility of our methods. We will tests every code and we will summarized in some tables the theoretical results obtained for each scheme.

In the appendix there are some interesting theoretical results which are used in the thesis. Appendix A deals with Probability spaces, distributions and characteristic function. Appendix B treats of Stochastic process, Brownian motion and martingale. Appendix C deals with Stochastic differential equations and presents the Feynman-Kač theorem, which shows the connection between SDEs and PDEs.

It is essential to stress that we make the common standard technical assumptions:

- all filtrations satisfy the *usual conditions*[1] of right-continuity and completeness

[1]Given a probability space $(\Omega, \mathscr{G}, \mathbb{P})$ we say that $\mathscr{G}_t$ satisfies the so-called *usual hypotheses* with respect to $\mathbb{P}$ if

1. $N \subset \mathscr{G}_0$ where N is the family of the events A s.t. $\mathbb{P}(A) = 0$
2. the filtration is right-continuos i.e. for every $t \geq 0$

$$\mathscr{G}_t = \bigcap_{\varepsilon > 0} \mathscr{G}_{t+\varepsilon}$$

- the sample paths of all sthocastic processes are assumed to be *càdlàg*[2]
- all r.v. and s.p. satisfy suitable integrability conditions, which ensure the existence of considered conditional expectations, deterministic or stochastic integrals.

[2]càdlàg is the French shortening for 'right-continuos with finite left limits at all t'

Introduzione

Negli ultimi anni sta diventando sempre più importante gestire il rischio di credito. Per la valutazione e per la copertura del rischio di credito in modo coerente, si ha la necessità di sviluppare un modello quantitativo.
L'obiettivo principale dei modelli quantitativi del rischio di credito è quello di fornire metodi per prezzi di contratti finanziari che sono sensibili al rischio di credito.
Questa tesi tratta di modelli stocastici con volatilità locale.
Un modello locale di volatilità stocastica è un modello continuo di mercato con volatilità non costante. In particolare, questo modello è un'estensione di quello di Black-Scholes: l'idea generale è quella di modificare la dinamica del sottostante, in modo da ottenere un modello in cui la volatilità è una funzione del tempo e del prezzo del sottostante .
In altri termini $\sigma_t = \sigma(t, X_t)$. Ciò è dovuto al fatto che, in linea con le dinamiche di mercato osservate, le volatilità implicite cambiano al cambiare della scadenza e dello strike. Inoltre, fissati la scadenza e lo strike, le volatilità implicite cambiano anche nel tempo e così la volatilità è non costante. Un vantaggio dei modelli a volatilità locale è che le densità di transizione del sottostante - così come i prezzi delle opzioni europee - sono spesso disponibili in forma chiusa come una serie infinita di funzioni speciali. Al fine di tener conto della possibilità di fallimento della controparte di un contratto, i modelli devono considerare la probabilità di default. Il default è ormai diventato parte importante nella valutazione dei derivati.
Ogni volta che un derivato viene scambiato, il rischio di default della controparte dovrebbe, in linea di principio, entrare nella valutazione. Recentemente, istituzioni di regolamentazione hanno insistito sulla necessità di includere questo tipo di rischio nel paradigma dei prezzi. Il nostro obiettivo è quello di sviluppare un payoff che dipenda dal sottostante

del derivato di base e dal default della controparte.
Modelleremo il processo del prezzo delle azioni con una volatilità locale dipendente dallo stato. Il processo di quotazione avrà volatilità locale stocastica e un default predefinito. Prenderemo in considerazione il modello di JDCEV di Carr e Linetsky (2006) ed otterremo una ricca classe di modelli analiticamente trattabili.

Questa tesi è organizzata come segue:

Capitolo 1: in questo capitolo prenderemo in considerazione un tempo casuale cioè una variabile aleatoria il cui valore viene interpretato come il tempo in cui un dato processo stocastico presenta un certo comportamento di interesse. Analizzeremo il problema della valutazione dell'attesa condizionata quando la filtrazione disponibile **G** è formata da una filtrazione naturale **H** di un tempo casuale e da una filtrazione arbitraria **F**. Introdurremo il concetto di processo di sopravvivenza e di processo di pericolo. Infine mostreremo un lemma importante che permette di sostituire l'attesa condizionata rispetto a $\mathscr{G}_t$ con un attesa condizionata relativa alla σ-algebra $\mathscr{F}_t$.

Capitolo 2: in questo capitolo presenteremo il nostro modello oggetto di studio. Dimostreremo che le ipotesi fatte nel capitolo precedente sono verificate e così potremo utilizzare i principali risultati ottennuti nel nostro particolare caso. Introdurremo il concetto di tempo di default τ, un tempo di arresto che rappresenta l'istante in cui la controparte fallisce. Introdurremo il problema dell'option-pricing europea e ricaveremo un'equazione differenziale alle derivate parziali (PDE) per il prezzo di un'opzione. Quindi dimostreremo che lo studio del prezzo del nostro modello può essere trasferito allo studio di un operatore differenziale alle derivate parziali con coefficienti variabili.

Capitolo 3: in questo capitolo formiremo alcuni esempi interessanti. Presenteremo il cosiddetto modello JDCEV rispettivamente con tasso a breve costante e stocastico. Per quanto riguarda il caso stocastico presenteremo due classici modelli di tasso a breve, vale a dire il modello Vasicek (1977) e quello Cox, Ingersoll e Ross (1985). Mostreremo che entrambi i modelli possiedono una struttura affine: tale argomento acquista particolare importanza grazie al fatto che da un punto di vista analitico e

computazionale l'esistenza di una struttura affine semplifica estremamente il contenuto.

Capitolo 4: introdurremo due metodi numerici per approssimare diversi prezzi dei bond. Il primo fornisce formule d'approssimazione analitiche e si basa sulla nozione di espansione di una base polinomiale: l'idea è quella di espandere la volatilità locale e la funzione di drift come una serie di Taylor intorno a un punto arbitrario, vale a dire

$$f(x) = \sum_{n=0}^{\infty} a_n (x - \overline{x})^n$$

al fine di raggiungere un risultato approssimato. Deriveremo una formula di sviluppo asintotico per la funzione che risolve i prezzi dell' opzione. Queste formule sono state poi implementate in *Wolfram Mathematica*. L' ultimo è basato sul metodo Monte Carlo, uno dei metodi numerici più popolari, che permette di calcolare approsimativamente il prezzo del contratto attraverso diverse simulazioni e facendone la media. Questo metodo è stato poi implementato in *Matlab*.

Chapter 5: infine forniremo esempi numerici che illustreranno l'efficacia e la versatilità dei metodi studiati. Testeremo ogni algoritmo implementato e riassumeremo in alcune tabelle i risultati teorici ottenuti.

Nell'appendice sono riportati alcuni risultati teorici interessanti che sono utilizzati nella tesi. L'appendice A si occupa di spazi di probabilità, distribuzioni e funzione caratteristica. L'appendice B tratta di processi stocastici, moto browniano e martingale. L'appendice C invece tratta di equazioni differenziali stocastiche e presenta il teorema di Feynman-Kač, che mostra il legame tra SDE and PDE.

Riportiamo di seguito le ipotesi tecniche che assumiamo valide per tutta la tesi:

- ogni filtrazione soddisfa le *ipotesi usuali*[3] di continuità e completezza

[3]Dato uno spazio di probabilità $(\Omega, \mathscr{G}, \mathbb{P})$ diciamo che $\mathscr{G}_t$ soddisfa le *ipotesi usuali* rispetto a $\mathbb{P}$ se

1. $N \subset \mathscr{G}_0$ dove N è la famiglia degli eventi A t.c. $\mathbb{P}(A) = 0$

- i cammini di tutti i processi stocastici si assumono essere *càdlàg*[4]
- tutte le variabili aleatorie e i processi stocastici soddisfano le condizioni di integrabilità adeguate che garantiscono l'esistenza di attese condizionate, integrali deterministici o stocastici.

2. la filtrazione è continua da destra ovvero per ogni $t \geq 0$

$$\mathscr{G}_t = \bigcap_{\varepsilon > 0} \mathscr{G}_{t+\varepsilon}$$

[4]càdlàg è un' abbreviazione francese che sta per 'continuità da desta con limite da destra finito in t'

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General notations

- $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ is the set of natural numbers
- $\mathbb{R}$ is the set of real numbers
- $\mathbb{R}^+ =]0, +\infty[$
- $\mathbb{I}_H$ is the indicator function of H
- $A^C = \Omega \setminus A$ is the complementary of A (referred to events)
- $E[X] := E_{\mathbb{P}}[X]$ (referred to random variables)

For any $a, b \in \mathbb{R}$

- $a \wedge b = min\{a, b\}$
- $a \vee b = max\{a, b\}$

Function spaces

- B_m = space of B-measurable functions
- L^p = space of functions integrable of order p
- L^p_{loc} = space of functions locally integrable of order p
- C^k = space of functions with continuos derivates up to order $k \in \mathbb{N}$

Process spaces

- $\mathbb{L}^p$ = space of progressively measurable process in $L^p([0, T]$ x $\Omega)$

- $\mathbb{L}^p_{loc}$ = space of progressively measurable process X s.t. $X(\omega) \in L^p([0,T])$ for almost any ω

Abbreviations

- r.v.= random variable
- s.p.= stochastic process
- s.t.= such that
- i.i.d.= independent and identically distribueted (referred to random variables)
- mg= martingale
- B.m.= Brownian motion
- i.e.= id est≡ that is
- PDE= Partial Differential Equation
- SDE= Stochastic Differential Equation

Chapter 1

Preliminary results

Let us start with some general remarks which will be useful for the costruction of our defaultable model.
Let $\tau : \Omega \longrightarrow \mathbb{R}$ be a non-negative r.v. on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$.
For our model τ represents the *random time*. For convenience we assume:

- $\mathbb{P}(\tau = 0) = 0$
- $\mathbb{P}(\tau > t) > 0$ for any $t \in \mathbb{R}^+$

The last condition means that τ is assumed to be unbonded.

Definition 1.1. We define the *jump process* H associated with the random time τ by setting

$$H_t = \mathbb{I}_{\{\tau \leq t\}} \qquad for \quad t \in \mathbb{R}^+$$

The process H has right-continuos sample paths. Specifically each sample paths is equal to 0 before random time τ and it equals 1 for $t \geq \tau$.
Let $\mathbf{H} = (\mathscr{H}_t)_{t \geq 0}$ stand for the filtration generated by the process H where $\mathscr{H}_t$ is the σ-algebra defined by $\mathscr{H}_t = \sigma(H_u : u \leq t)$. Finally we set $\mathscr{H}_\infty = \sigma(H_u : u \in \mathbb{R}^+)$.
The σ-algebra $\mathscr{H}_t$ represents the information generated by the observations on the time interval $[0, t]$. Let $\mathbf{F} = (\mathscr{F}_t)_{t \geq 0}$ be an arbitrary filtration on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$.
All filtration are assumed to satisfy 'usual condition' of right continuity and completeness.
We assume that the filtration $\mathbf{G}$ has the form $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ i.e. $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{H}_t$ for any

$t \in \mathbb{R}^+$. For each $t \in \mathbb{R}^+$, the information available at time t is captured by the σ-algebra $\mathscr{G}_t$. We want to stress that the process H is obviosly **G**-adapted, but it is not necessary **F**-adapted. In other words, the random time τ is a **G**-stopping time[1], but it may fail to be an **F**-stopping time.

Definition 1.2. For any $t \in \mathbb{R}^+$ we denote by G the ***F**-survival process* of τ with respect to the filtration **F**, given as:

$$G_t = \mathbb{P}(\tau > t | \mathscr{F}_t) \qquad t \in \mathbb{R}^+$$

Remark 1.
G_t is $\mathscr{F}_t$-measurable thanks to the properties of conditional expectation. (See A.3).

Remark 2.
Since $\mathscr{F}_0$ contains no information, we have:

$$G_0 = \mathbb{P}(\tau > 0 | \mathscr{F}_0) = \mathbb{P}(\tau > 0) > 0$$

The hazard process of the default time, given the flow of information represented by the filtration **F** is formally introduced through the following definition.

Definition 1.3. Assume that $G_t > 0$ for any $t \in \mathbb{R}^+$. Then:

$$\Gamma_t := -\ln G_t \qquad t \in \mathbb{R}^+$$

is called the ***F**-hazard process* of τ.

Remark 3.
It is important to observe that for any $A \in \mathscr{G}_t$ we have $A \cap \{\tau > t\} = B \cap \{\tau > t\}$ for some event $B \in \mathscr{F}_t$. Indeed we have two possibilities:

- if $A = \{\tau \leq u\}$ for some $u \leq t \Rightarrow \exists B \in \mathscr{F}_t$ s.t. $A \cap \{\tau > t\} = B \cap \{\tau > t\}$. It is enough to take $B = \varnothing$.
- if $A \in \mathscr{F}_t \Rightarrow \exists B \in \mathscr{F}_t$ s.t. $A \cap \{\tau > t\} = B \cap \{\tau > t\}$. It is enough to take $B = A$.

[1] In a filtered space $(\Omega, \mathscr{G}, (\mathscr{G}_t), \mathbb{P})$ a r.v. $\tau : \Omega \longrightarrow \mathbb{R}^+$ is called stopping time with respect to the filtration $(\mathscr{G}_t)$ if $\{\tau \leq t\} \in \mathscr{G}_t \ \forall t \geq 0$
Clearly a deterministic time $\tau \equiv t$ is a stopping time.

The following lemma give an important result. It allows to substitute the conditional expectation with respect to $\mathscr{G}_t$ with the conditioning relative to the σ-algebra $\mathscr{F}_t$. Switching from $\mathscr{G}_t$ expectations to $\mathscr{F}_t$ expectations is important because for some variables the $\mathscr{F}_t$ conditional expectations are easier to compute.

Lemma 1. *Let Y be a r.v. on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$. Then for any $t \in \mathbb{R}^+$ we have:*

$$E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{G}_t] = \mathbb{P}(\tau > t|\mathscr{G}_t)\frac{E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{F}_t]}{G_t} \tag{1.1}$$

Proof.
We need to verify that (recall that $\mathscr{F}_t \subseteq \mathscr{G}_t$)

$$E[\mathbb{I}_{\{\tau>t\}}Y\mathbb{P}(\tau > t|\mathscr{F}_t)|\mathscr{G}_t] = E[\mathbb{I}_{\{\tau>t\}}E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{F}_t]|\mathscr{G}_t]$$

We need to show that for any $A \in \mathscr{G}_t$ we have:

$$\int_A \mathbb{I}_{\{\tau>t\}}Y\mathbb{P}(\tau > t|\mathscr{F}_t)d\mathbb{P} = \int_A \mathbb{I}_{\{\tau>t\}}E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{F}_t]d\mathbb{P}$$

In view of the previous remark for any $A \in \mathscr{G}_t$ we have $A \cap \{\tau > t\} = B \cap \{\tau > t\}$ for some event $B \in \mathscr{F}_t$ and so:

$$\begin{aligned}
\int_A \mathbb{I}_{\{\tau>t\}}Y\mathbb{P}(\tau > t|\mathscr{F}_t)d\mathbb{P} &= \int_{A\cap\{\tau>t\}} Y\mathbb{P}(\tau > t|\mathscr{F}_t)d\mathbb{P} = \\
&= \int_{B\cap\{\tau>t\}} Y\mathbb{P}(\tau > t|\mathscr{F}_t)d\mathbb{P} = \\
&= \int_B \mathbb{I}_{\{\tau>t\}}Y\mathbb{P}(\tau > t|\mathscr{F}_t)d\mathbb{P} = \\
&= \int_B E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{F}_t]E[\mathbb{I}_{\{\tau>t\}}|\mathscr{F}_t]d\mathbb{P} = \\
&= \int_B E[\mathbb{I}_{\{\tau>t\}}E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{F}_t]|\mathscr{F}_t]d\mathbb{P} = \\
&= \int_B \mathbb{I}_{\{\tau>t\}}E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{F}_t]d\mathbb{P} = \\
&= \int_{B\cap\{\tau>t\}} E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{F}_t]d\mathbb{P} = \\
&= \int_{A\cap\{\tau>t\}} E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{F}_t]d\mathbb{P} = \\
&= \int_A \mathbb{I}_{\{\tau>t\}}E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{F}_t]d\mathbb{P} =
\end{aligned}$$

Where in those steps the only properties that we have used are simply the properties of conditional expectation (See A.3). □

Corollary 1. *If $\mathscr{H}_t \subseteq \mathscr{G}_t$ then:*

$$E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{G}_t] = \mathbb{I}_{\{\tau>t\}}\frac{E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{F}_t]}{G_t} \tag{1.2}$$

Proof.
It suffices to observe that, since $\mathscr{H}_t \subseteq \mathscr{G}_t$ and H_t is $\mathscr{H}_t$-measurable, we have

$$\mathbb{P}(\tau > t|\mathscr{G}_t) = E[\mathbb{I}_{\{\tau>t\}}|\mathscr{G}_t] = \mathbb{I}_{\{\tau>t\}}$$

In the last equality we have used the properties of conditional expectation (See A.3). □

The formula (1.2) can be rewritten as follows:

$$E[\mathbb{I}_{\{\tau>t\}}Y|\mathscr{G}_t] = \mathbb{I}_{\{\tau>t\}}E[\mathbb{I}_{\{\tau>t\}}e^{\Gamma_t}Y|\mathscr{F}_t] \tag{1.3}$$

Lemma 2. *Let Y be a r.v. on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$ and let $t \leq T$. Then:*

$$E[\mathbb{I}_{\{\tau>T\}}Y|\mathscr{G}_t] = \mathbb{I}_{\{\tau>t\}}E[\mathbb{I}_{\{\tau>T\}}e^{\Gamma_t}Y|\mathscr{F}_t] \tag{1.4}$$

If Y is $\mathscr{F}$-measurable, then

$$E[\mathbb{I}_{\{\tau>T\}}Y|\mathscr{G}_t] = \mathbb{I}_{\{\tau>t\}}E[e^{\Gamma_t-\Gamma_T}Y|\mathscr{F}_t] \tag{1.5}$$

Proof.
In view of (1.3), to show that (1.4) holds, it is enough to observe that $\mathbb{I}_{\{\tau>T\}}\mathbb{I}_{\{\tau>t\}} = \mathbb{I}_{\{\tau>T\}}$. For (1.5), by virtue of (1.4), we obtain

$$\begin{aligned}
E[\mathbb{I}_{\{\tau>T\}}Y|\mathscr{G}_t] &= \mathbb{I}_{\{\tau>t\}}E[\mathbb{I}_{\{\tau>T\}}e^{\Gamma_t}Y|\mathscr{F}_t] = \\
&= \mathbb{I}_{\{\tau>t\}}E[E[\mathbb{I}_{\{\tau>T\}}|\mathscr{F}_T]e^{\Gamma_t}Y|\mathscr{F}_t] \\
&= \mathbb{I}_{\{\tau>t\}}E[\mathbb{P}(\tau > T|\mathscr{F}_T)e^{\Gamma_t}Y|\mathscr{F}_t] \\
&= \mathbb{I}_{\{\tau>t\}}E[G_Te^{\Gamma_t}Y|\mathscr{F}_t] \\
&= \mathbb{I}_{\{\tau>t\}}E[e^{\Gamma_t-\Gamma_T}Y|\mathscr{F}_t]
\end{aligned}$$

□

1.1 Market assumptions

Before proceeding to the description of the models that will be discussed below, it is useful to mention the properties of the financial market that are taken as the assumptions on which these models are built. Some of these issues do not exist in real markets, but they serve to simplify the mathematical treatment of the problem.

- It is assumed that the market is free of arbitrage opportunities. The absence of arbitrage is an important concept in the mathematical theory of contingent claim pricing which imposes constraints on the way instruments are priced in a market. The word absence of arbitrage essentially addresses the opportunity to make a *risk-free profit*: it is equivalent to the impossibility to invest zero today and receive tomorrow a nonnegative amount that is positive with positive probability.
  In other words, two portfolios having the same payoff at a given future date must have the same price today.

- An important concept in the mathematical theory of contingent claim pricing is risk-neutral pricing. It represents the price of any instrument in an arbitrage-free market as its discounted expected payoff under an appropriate probability measure.

- It is assumed the existence of an equivalent martingale measure[2] which leads to absence of arbitrage.

- It is assumed frictionless market i.e. a financial market without transaction costs.

[2]See (B.5)

Chapter 2

Construction of the model

Our goal is, using the previous notations, to costruct a market model with default. Let $(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, \mathbb{P})$ a filtered probability space. All stochastic processes defined below live on this probability space and all expectations are taken with respect to $\mathbb{P}$.

1. Let $W = (W_t)_{t\geq 0}$ be a real B.m.[1]

2. Let ε be a r.v. with exponential distribution with parameter 1 i.e. $\varepsilon \sim exp(1)$[2] and independent of W.

3. Let X be a process which is the solution of the follow SDE[3]:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.1}$$

   where:

   - $\mu = \mu(t, x) : [0, T]$ x $\mathbb{R} \longrightarrow \mathbb{R}$ is a measurable[4] function
   - $\sigma = \sigma(t, x) : [0, T]$ x $\mathbb{R} \longrightarrow \mathbb{R}$ is a non-negative measurable function

4. Let $\gamma : [0, +\infty[$ x $\mathbb{R} \longrightarrow \mathbb{R}$ non-negative measurable function.

[1] See (B.3)
[2] See (A.1.1)
[3] See (C.1)
[4] In a measure space $(\Omega, \mathscr{F}, \mathbb{P})$ we say that a function $f : \Omega \longrightarrow \mathbb{R}$ is $\mathscr{F}$-measurable if $f^{-1}(H) \in \mathscr{F}$ for any $H \in \mathscr{B}$

We define $\tau : \Omega \longrightarrow \mathbb{R}$ be a non-negative r.v. on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$ by setting:

$$\tau = inf\{t \geq 0 : \int_0^t \gamma(s, X_s)ds \geq \varepsilon\}$$

We shall now check that properties listed in the previous chapter are satisfied.
First of all we show that $\mathbb{P}(\tau = 0) = 0$. Since clearly $\{\tau = 0\} = \{\int_0^0 \gamma(s, X_s)ds \geq \varepsilon\}$ we have:

$$\mathbb{P}(\tau = 0) = \mathbb{P}(0 \geq \varepsilon) = 0$$

Now we show that $\mathbb{P}(\tau > t) > 0$ for any $t \in \mathbb{R}^+$. Before to prove this equality we have to introduce this lemma.

Lemma 3. *Let X, Y be a r.v. on $(\Omega, \mathscr{G}, \mathbb{P})$. Let $\mathscr{F} \subseteq \mathscr{G}$ be a σ-algebra s.t.*

- *X is independent of $\mathscr{F}$;*
- *Y is $\mathscr{F}$-measurable.*

Then for every $\mathscr{B}$-measurable bounded function h we have

$$E[h(X, Y)|\mathscr{G}] = g(Y) \qquad where \quad g(y) = E[h(X, y)] \tag{2.2}$$

Proof.
We have to prove that the r.v. $g(Y)$ is a version of the conditional expectation of $h(X, Y)$. i.e. that for any $G \in \mathscr{F}$ we have

$$\int_G h(X, Y)d\mathbb{P} = \int_G g(Y)d\mathbb{P}$$

Using the notation $\mathbb{P}^W$ to denote the distribution of a given r.v. W, we have

$$g(y) = \int_{\mathbb{R}} h(x, y)\mathbb{P}^X(dx)$$

Then, since g is a $\mathscr{B}$-measurable function, by assumption ii) results that $g(Y)$ is a $\mathscr{F}$-

measurable. Further, given $G \in \mathscr{F}$ and putting $Z = \mathbb{I}_G$, we get

$$\begin{aligned}\int_G h(X,Y)d\mathbb{P} &= \int_\Omega h(X,Y)Zd\mathbb{P} = \int\int\int h(x,y)z\mathbb{P}^{(X,Y,Z)}(d(x,y,z)) = \\ &= \int\int\int h(x,y)z\mathbb{P}^{X}(dx)\mathbb{P}^{(Y,Z)}(d(y,z)) = \\ &= \int\int g(y)z\mathbb{P}^{(Y,Z)}(d(y,z)) = \\ &= \int_G g(Y)d\mathbb{P}\end{aligned}$$

Where in third equality we have used the independence assumption i) and in fourth equality Fubini's theorem. □

Now we can show that $\mathbb{P}(\tau > t) > 0$. Since $\{\tau > t\} = \{\int_0^t \gamma(s, X_s)ds < \varepsilon\}$ we have:

$$\begin{aligned}\mathbb{P}(\tau > t) &= \mathbb{P}(\int_0^t \gamma(s,X_s)ds < \varepsilon) \\ &= E[\mathbb{I}_{\{\int_0^t \gamma(s,X_s)ds<\varepsilon\}}] = E[E[\mathbb{I}_{\{\int_0^t \gamma(s,X_s)ds<\varepsilon\}}|\mathscr{F}_t]] = \\ &= E[\mathbb{P}(\int_0^t \gamma(s,X_s)ds < \varepsilon|\mathscr{F}_t)]\end{aligned}$$

Since ε is independent of $\mathbf{F}$ and $\{\int_0^t \gamma(s, X_s)ds < \varepsilon\}$ is $\mathscr{F}_t$-measurable, using the lemma we obtain:

$$\mathbb{P}(\tau > t) = E[h(\int_0^t \gamma(s,X_s)ds)]$$

Where $h(x) = \mathbb{P}(\varepsilon > x)$ and since ε has an exponential distribution we have

$$E[h(\int_0^t \gamma(s,X_s)ds] > 0$$

This ends the proof.

Remark 4. Thanks to lemma we have

$$\mathbb{P}(\tau > t|\mathscr{F}_t) = \mathbb{P}(\int_0^t \gamma(s,X_s)ds < \varepsilon|\mathscr{F}_t) = e^{-\int_0^t \gamma(s,X_s)ds} > 0 \tag{2.3}$$

Therefore the $\mathbf{F}$-survival process of τ is positive i.e.

$$G_t = \mathbb{P}(\tau > t|\mathscr{F}_t) > 0 \qquad t \in \mathbb{R}^+$$

So the $\mathbf{F}$-hazard process of τ is well defined i.e.

$$\Gamma_t := -\ln G_t \qquad t \in \mathbb{R}^+$$

Since $G_t = e^{-\Gamma_t}$, by (2.3) we obtain:

$$\Gamma_t := \int_0^t \gamma(s, X_s)ds \tag{2.4}$$

Therefore the random time τ is seen as the first time when the hazard process Γ is greater or equal to the random level $\varepsilon \sim exp(1)$ i.e.

$$\tau = inf\{t \geq 0 : \Gamma_t \geq \varepsilon\}$$

2.1 Default time

In recent years is becoming increasingly important to handle credit risk. Credit risk is the risk associated with the possibility of bankruptcy. More precisely, if a derivative provides for a payment at cert time T but before that time the counterparty defaults, at maturity the payment cannot be effectively performed, so the owner of the contract loses it entirely or a part of it. It means that the payoff of the derivative, and consequently its price, depends on the underlying of the basic derivative and on the risk of bankruptcy of the counterparty. To value and to hedge credit risk in a consistent way, one needs to develop a quantitative model. The standard way to model the counterparty risk of bankruptcy is to introduce the default time τ : it is a stopping time which represents the instant in which the counterparty goes bankrupt.
Default time is introduced because it gives information which is not contained in the usual filtration $\mathbf{F}$. In order to keep track of the event $\{\tau \leq t\}$ we have introduce the filtration $\mathbf{H} = (\mathscr{H}_t)_{t\geq 0}$ stand for the filtration generated by a default indicator process H defined in the previous chapter i.e.

$$H_t = \mathbb{I}_{\{\tau \leq t\}} \qquad for \quad t \in \mathbb{R}^+$$

Therefore if we want a filtration which provides for the whole flow of information, we should introduce an enlarged filtration $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ i.e. $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{H}_t$ for any $t \in \mathbb{R}^+$.
The filtration $\mathbf{G}$ represents the history of the market. It describes the default-free market variables up to t (by filtration $\mathscr{F}_t$) and tells whether default occurred before t (by filtration $\mathscr{H}_t$).
Now we can defined our model.

2.2 Market model with costant short interest

In this section we will look at default model with deterministic interest rates.
For simplicity, we assume a frictionless market, no arbitrage, zero interest rates and no dividends. We take, as given, an equivalent martingale measure $\mathbb{P}$[5] chosen by the market on a complete filtered probability space $(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual hypothesis of completeness and right continuity. The filtration **G** is defined as above and **F** is the filtration generated by X.

We consider a defaultable asset S whose risk-neutral dynamics are given by

$$\begin{aligned} S_t &= \mathbb{I}_{\{\tau>t\}} e^{X_t} \\ dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ \tau &= inf\{t \geq 0 : \int_0^t \gamma(s, X_s)ds \geq \varepsilon\} \end{aligned} \tag{2.5}$$

- dX_t may be interpreted as the change in X_t over the period $[t, t + dt]$

- $\mu(t, X_t)$ and $\sigma(t, X_t)$ are deterministic functions of X_t and t

- τ represents the default time of S and is the first arrival time of a process with local intensity function $\gamma(t, x) > 0$

We assume that the coefficients are measurable in t and suitably smooth in x to ensure the existence of a solution to (2.5). In the absence of arbitrage, S must be an **G**-martingale[6].
We define $M_t = e^{-rt}S_t$. It represents the discounted value of the asset.
We have to prove that the equality $M_t = [M_T|\mathscr{G}_t]$ holds for $t \leq T$.
Since (2.4) holds and e^{X_T} is $\mathscr{F}_t$-measurable we can apply the lemma 2 of the previous

[5] See (B.5)
[6] See (B.4).

chapter.

$$\begin{aligned}
M_t = E[M_T|\mathscr{G}_t] &\iff e^{-rt}S_t = E[e^{-rT}S_T|\mathscr{G}_t] \\
&\iff S_t = E[e^{r(t-T)}S_T|\mathscr{G}_t] \\
&\iff e^{X_t}\mathbb{I}_{\{\tau>t\}} = e^{r(t-T)}E[e^{X_T}\mathbb{I}_{\{\tau>T\}}|\mathscr{G}_t] \\
&\iff e^{X_t}\mathbb{I}_{\{\tau>t\}} = e^{r(t-T)}\mathbb{I}_{\{\tau>t\}}E[e^{X_T-\int_t^T\gamma(s,X_s)ds}|\mathscr{F}_t] \\
&\iff e^{X_t} = e^{r(t-T)}E[e^{X_T-\int_t^T\gamma(s,X_s)ds}|\mathscr{F}_t] \\
&\iff e^{X_t-rt-\int_0^t\gamma(s,X_s)ds} = E[e^{X_T-rT-\int_0^T\gamma(s,X_s)ds}|\mathscr{F}_t]
\end{aligned}$$

Therefore M_t is a **G**-mg $\iff Y_t := e^{X_t-rt-\int_0^t\gamma(s,X_s)ds}$ is a **F**-mg.
The drift condition follows by applying the Ito's formula to the process $e^{X_t-rt-\int_0^t\gamma(s,X_s)ds}$ and setting the drift term to zero i.e.

$$\begin{aligned}
dY_t &= Y_t(-r-\gamma(t,X_t))dt + Y_t dX_t + \frac{1}{2}Y_t d<X>_t \\
&= Y_t(-r-\gamma(t,X_t))dt + Y_t\mu(t,X_t)dt + Y_t\sigma(t,X_t)dW_t + \frac{1}{2}Y_t\sigma^2(t,X_t)dt \\
&= Y_t(-r-\gamma(t,X_t)+\mu(t,X_t)+\frac{1}{2}\sigma^2(t,X_t))dt + Y_t\sigma(t,X_t)dW_t
\end{aligned}$$

Therefore:

$$\begin{aligned}
&-r-\gamma(t,X_t)+\mu(t,X_t)+\frac{1}{2}\sigma^2(t,X_t) = 0 \\
\implies &\mu(t,X_t) = r+\gamma(t,X_t)-\frac{1}{2}\sigma^2(t,X_t)
\end{aligned}$$

Thus, in order to guarantee that M_t is a mg we have to impose tha additional restrictions on the drift $\mu(t,X_t)$ i.e. $\mu(t,X_t)$ is fixed by $\sigma(t,X_t)$, $\gamma(t,X_t)$ and r.
where

- r is the risk-free rate
- $\sigma^2(t,x)$ is the local volatility
- $\gamma(t,X_t)$ is the intensity function of default

And so:

$$\begin{aligned}
dX_t &= \mu(t,X_t)dt + \sigma(t,X_t)dW_t \\
dX_t &= (r+\gamma(t,X_t)-\frac{1}{2}\sigma^2(t,X_t))dt + \sigma(t,X_t)dW_t
\end{aligned} \tag{2.6}$$

2.3 Option pricing

We consider a European derivative expiring at time T.
Taking care of the credit risk of the counterparty, the price at time t of a derivative with payoff $H(S_T)$ is given by

$$E[H(S_T)|\mathscr{G}_t] \tag{2.7}$$

In order to calculate this, it is necessary to know how to deal with default time and the new filtration. The lemma 2 in the previous chapter shows how to express (2.7) in terms of $\mathscr{F}_t$ instead of $\mathscr{G}_t$.

Theorem 2.3.1. *The no-arbitrage price V_t is given by*

$$V_t = K + \mathbb{I}_{\{\tau>t\}} E[e^{-\int_t^T \gamma(s,X_s)ds}(h(X_T) - K)|\mathscr{F}_t] \tag{2.8}$$

Proof.
For convenience, we introduce

$$h(x) := H(e^x) \qquad and \quad H(0) = K$$

So we have two possibilities:

1. if $\{\tau > T\}$

$$\Longrightarrow S_t = e^{X_t} \Longrightarrow H(S_t) = H(e^{X_t}) = h(X_t) \tag{2.9}$$

2. if $\{\tau \leq T\}$

$$\Longrightarrow S_t = 0 \Longrightarrow H(S_t) = H(0) = K \tag{2.10}$$

Using risk-neutral pricing, the value V_t of the derivative at time t is given by the conditional expectation of the option payoff

$$\begin{aligned}
V_t &= E[H(S_T)|\mathscr{G}_t] = \\
&= E[\mathbb{I}_{\{\tau>T\}} h(X_T)|\mathscr{G}_t] + KE[\mathbb{I}_{\{\tau\leq T\}}|\mathscr{G}_t] = \\
&= E[\mathbb{I}_{\{\tau>T\}} h(X_T)|\mathscr{G}_t] + K - KE[\mathbb{I}_{\{\tau>T\}}|\mathscr{G}_t] = \\
&= K + E[\mathbb{I}_{\{\tau>T\}}(h(X_T) - K)|\mathscr{G}_t] = \\
&= K + \mathbb{I}_{\{\tau>t\}} E[e^{-\int_t^T \gamma(s,X_s)ds}(h(X_T) - K)|\mathscr{F}_t]
\end{aligned}$$

In second equality we have used respectively (2.9) and (2.10).
In fifth equality, since $h(X_t) - K$ is a $\mathscr{F}_t$-measurable r.v. (indeed X_t is $\mathscr{F}_t$-measurable and h is a continuos function) and (2.4) holds, we have used the formula of the lemma 2. □

One notes that the pricing formula (2.8) consists of two parts: the value of payoff conditional on no default and the value of the cash payment K in the event of default.

From (2.8), we see that, in order to compute the price of an option, we must evaluate functions of the form

$$u(t,x) := E[e^{-\int_t^T \gamma(s,X_s)ds} h(X_T)|X_t = x] \tag{2.11}$$

By a direct application of the Feynman-Kač representation theorem[7], the function $u(t,x)$ defined in (2.11) is the classical solution (when it exists) of the following Cauchy problem

$$\begin{cases} (\mathcal{A} + \partial_t)u(t,x) = 0 & t \in [0,T[, x \in \mathbb{R} \\ u(T,x) = h(x) & x \in \mathbb{R} \end{cases} \tag{2.12}$$

where $\mathcal{A}$ is the second order elliptic differential operator with variable coefficients

$$\mathcal{A}(t,x)f(x) = \frac{1}{2}\sigma^2(t,x)\partial_{xx}f(x) + \mu(t,x)\partial_x f(x) - \gamma(t,x)f(x) \tag{2.13}$$

We say that $\mathcal{A}$ is the characteristic operator of X_t.
If the operator $\mathcal{A} + \partial_t$ has a fundamental solution $p(t,x,T,y)$ then, for every integrable datum h, the Cauchy problem (2.12) has a classical solution that can be represented as

$$u(t,x) = e^{-\int_t^T \gamma(s,X_s)ds} \int_{\mathbb{R}} p(t,x,T,y)h(y)dy \tag{2.14}$$

Therefore, by Feynman-Kač formula, we have that

$$E[e^{-\int_t^T \gamma(s,X_s)ds} h(X_T)|X_t = x] = e^{-\int_t^T \gamma(s,X_s)ds} \int_{\mathbb{R}} p(t,x,T,y)h(y)dy$$

i.e.

$$E[e^{-\int_t^T \gamma(s,X_s^{t,x})ds} h(X_T^{t,x})] = e^{-\int_t^T \gamma(s,X_s)ds} \int_{\mathbb{R}} p(t,x,T,y)h(y)dy \tag{2.15}$$

where $X_T^{t,x}$ represents the process that at the istant t is in x.

[7] See C.2.

Remark 5. The equation (2.15) means that, for fixed $x \in \mathbb{R}$ and $t < T$, the function

$$y \mapsto p(t, x, T, y)$$

is the density of the r.v. $X_T^{t,x}$. We express this fact by saying that p is the transition density of the SDE given in the model (2.5). This fundamental result unveils the deep connection between PDEs and SDEs.

Therefore, in order to compute the price of an option, we must solve PDE in (2.12) with coefficients in function of two variables: t and x.
The solution of variables coefficients partial differential equations is an important engineering problem. In the literature several methods have been proposed to overcome this problem. In chapter 4 we will show our methodology in order to solve PDE with variabiles coefficients.

2.4 Market model with stochastic short interest

In this section we will look at general local-stochastic volatility models with default with stochastic short interest.
As above, we assume a frictionless market, no arbitrage, zero interest rates and no dividends. We take, as given, an equivalent martingale measure $\mathbb{P}$[8] chosen by the market on a complete filtered probability space $(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual hypothesis of completeness and right continuity. The filtration $\mathbf{G}$ is defined as above and $\mathbf{F}$ is the filtration generated by X.

We consider a defaultable asset S whose risk-neutral dynamics are given by

$$\begin{aligned} S_t &= \mathbb{I}_{\{\tau > t\}} e^{X_t} \\ dX_t &= \mu(t, X_t, r_t)dt + \sigma(t, X_t, r_t)dW_t^1 \\ dr_t &= \alpha(t, X_t, r_t)dt + \beta(t, X_t, r_t)dW_t^2 \\ d < W^1, W^2 >_t &= \rho(t, X_t, r_t) \\ \tau &= inf\{t \geq 0 : \int_0^t \gamma(s, X_s)ds \geq \varepsilon\} \end{aligned} \tag{2.16}$$

[8] See (B.5)

with the correlation $-1 < \rho < 1$.

As the asset price S must be a martingale, by applying the Ito's formula to the process $e^{X_t - \int_0^t r_s ds - \int_0^t \gamma(s,X_s)ds}$ and setting the drift term to zero we obtain that the drift function μ must be given by

$$\mu(t, X_t, r_t) = r_t + \gamma(t, X_t) - \frac{1}{2}\sigma^2(t, X_t, r_t)$$

Making the same steps of the previous section we denote by V the no-arbitrage price of European derivative expiring at time T with payoff $H(S_T)$. It is well known, extended the formula of (2.8) that

$$V_t = K + \mathbb{I}_{\{\tau > t\}} E[e^{-\int_t^T r_s + \gamma(s,X_s)ds}(h(X_T) - K)|\mathscr{F}_t] \tag{2.17}$$

Then, to value a European-style option, one must compute functions of the form

$$u(t, x, r) := E[e^{-\int_t^T r_s + \gamma(s,X_s)ds} h(X_T)|X_t = x, r_t = r] \tag{2.18}$$

By a direct application of the Feynman-Kač representation theorem, the function $u(t, x, r)$ defined in (2.18) is the classical solution (when it exists) of the following Cauchy problem

$$\begin{cases} (\mathcal{A} + \partial_t)u = 0 \\ u(T, x, y) = h(x, y) \end{cases} \tag{2.19}$$

where $\mathcal{A}$ the operator is given explicitly by

$$\begin{aligned} \mathcal{A}(t,x)f(x,r) = \frac{1}{2}\sigma^2(t,x,r)\partial_{xx}f(x,r) + \frac{1}{2}\beta^2(t,x,r)\partial_{rr}f(x,r) + \mu(t,x)\partial_x f(x,r) + \\ + \alpha(t,x,r)\partial_r f(x,r) + \rho\sigma(t,x,r)\beta(t,x,r)\partial_{xr}f(x,r) - (r + \gamma(t,x))f(x,r) \end{aligned} \tag{2.20}$$

As we can see, the results of the previous section can be extended in a straightforward fashion to include models with stochastic interest rates.

In the next chapter we will show some finance models.

Chapter 3

Examples

3.1 JDCEV Model

In mathematical finance, the JDCEV or Jump to Default Constant Elasticity of Variance model is a stochastic volatility model, which attempts to capture stochastic volatility and the leverage effect. The model is widely used in the financial industry, especially for modelling equities and commodities.
This work develops an analytical solution to the pricing problem under the Jump to Default Constant Elasticity of Variance. Consider a defaultable bond, written on S, that pays one dollar at time $T > t$ if no default occurs prior to maturity and pays zero dollars otherwise. Indicating with τ the default time variable, this means that $S_T > 0$ if $\tau > T$. Then by theorem (2.3.1) with $K = 0$ and $h(x) = 1$ we have that the time t value of the bond is given by

$$V_t = \mathbb{I}_{\{\tau>t\}} E[e^{-\int_t^T \gamma(s,X_s)ds} | \mathscr{F}_t] \tag{3.1}$$

We set

$$u(t, X_t, T) = E[e^{-\int_t^T \gamma(s,X_s)ds} | \mathscr{F}_t]$$

thus

$$V_t = \mathbb{I}_{\{\tau>t\}} u(t, X_t, T) \tag{3.2}$$

In practice the price takes care of the risk of default of the issuer. Note that $u(t,x,T)$ is both the price of a bond and the conditional survival probability i.e.

$$\mathbb{P}(\tau > T|\mathscr{G}_t) = E[\mathbb{I}_{\{\tau>T\}}|\mathscr{G}_t] = E[e^{-\int_t^T \gamma(s,X_s)ds}|\mathscr{F}_t]$$

The yield $Y(t,x,T)$ of such a bond, on the set $\tau > T$ is defined as

$$Y(t,x,T) = \frac{-\log u(t,x,T)}{T-t}$$

The credit spread is defined as the yield minus the risk-free rate of interest i.e.

$$CSpread(t,x,T) = Y(t,x,T) - r = \frac{-\log u(t,x,T)}{T-t} - r$$

Obviously, in the case of zero interest rates, we have: yield = credit spread.
In the time-homogeneous case, the underlying S is described by (2.6) with

$$\begin{aligned}
\sigma(x) &= \delta e^{(\beta-1)x} \\
\gamma(x) &= b + c\sigma^2(x) = b + c\delta^2 e^{2(\beta-1)x} \\
\mu(x) &= r + \gamma(x) - \frac{1}{2}\sigma^2(x) = (c - \frac{1}{2})\delta^2 e^{2(\beta-1)x} + r + b
\end{aligned}$$

where $\delta > 0$, $b \geq 0$,$c \geq 0$.
Therefore, in this model, the instantaneous volatility is specified as a power function where

- $0 \leq \beta \leq 1$ is the volatility elasticity parameter
- $\delta > 0$ is the volatility scale parameter

The limiting case with $\beta = 0$ corresponds to the constant volatility assumption in the BSM model.

Remark 6. We will restrict our attention to cases in which $\beta < 1$. From a financial perspective, this restriction makes sense, as it results in volatility and default intensity increasing as $S \to 0^+$ which is consistent with the leverage effect.

To be consistent with the empirical evidence linking corporate bond yields to equity volatility, Carr and Linetsky propose to specify the default intensity as a function $\gamma(x)$ of the instantaneous stock variance where

- b is a constant parameter governing the state-independent part of the jumpto-default intensity
- c is a constant parameter governing the sensitivity of the intensity to the local volatility σ^2

The model is:

$$\begin{aligned} S_t &= \mathbb{I}_{\{\tau>t\}} e^{X_t} \\ dX_t &= ((c-\frac{1}{2})\delta^2 e^{2(\beta-1)X_t} + r + b)dt + \delta e^{(\beta-1)X_t} dW_t \\ \tau &= inf\{t \geq 0 : \int_0^t b + c\delta^2 e^{2(\beta-1)X_t} \geq \varepsilon\} \end{aligned} \tag{3.3}$$

The characteristic operator of X_t by (2.13)

$$\mathcal{A}(t,x)f(x) = \frac{1}{2}\delta^2 e^{2(\beta-1)x}\partial_{xx}f(x) + \big[(c-\frac{1}{2})\delta^2 e^{2(\beta-1)x} + r + b\big]\partial_x f(x) - \gamma(t,x)f(x)$$

The exact price i.e. the survival probability $u(t,x;T)$ in this setting , which requires a Kummer confluent hypergeometric function, is computed in Mendoza-Arriaga et al. (2010) .

$$\begin{aligned} u(t,x;T) = \sum_{n=0}^{\infty} e^{-(b+\omega n)T} \frac{\Gamma(1+\frac{c}{|\beta|})\Gamma(\frac{n+1}{2|\beta|})}{\Gamma(\nu+1)\Gamma(\frac{1}{2|\beta|})n!} \cdot \\ \cdot A^{\frac{1}{2|\beta|}} e^{x} e^{-Ae^{-2\beta x}} F_1(1-n+\frac{c}{|\beta|}; \nu+1; Ae^{-2\beta x}) \end{aligned} \tag{3.4}$$

where

- F_1 is the Kummer confluent hypergeometric function i.e. a generalized hypergeometric series introduced in (Kummer 1837), given by

$$F_1(a,b,z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}$$

where

$$\begin{aligned} (a)_0 &= 1 \\ (a)_n &= a(a+1)(a+2)\dots(a+n-1) \end{aligned}$$

is the rising factorial.

- $\Gamma(x)$ is a Gamma function i.e.

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$$

- ν, A, ω are defined as follow

$$\nu = \frac{1+2c}{2|\beta|}$$
$$A = \frac{r+b}{\delta^2|\beta|}$$
$$\omega = 2|\beta|(r+b)$$

In this case we have make the assumption of deterministic rates.
When pricing a long-maturity option, however, the stochastic feature of interest rates has a stronger impact on the option price. Therefore the short rate process have to enter in the valuation. The general dynamic of the stochastic-local volatility model is given in (2.16).
In the literature have been proposed numerous way on how to specify the dynamic of the short rate under the equivalent martingale measure. We start the section by introducing two popular short rate models: the Vasicek (1977) model and the Cox, Ingersoll and Ross (1985) model. These are all endogenous term-structure models, meaning that the current term structure of rates is an output rather than an input of the model. The success of models like that of Vasicek (1977) and that of Cox, Ingersoll and Ross (1985) was mainly due to their possibility of pricing analytically bonds and bond options. Before define those models we have to introduced the notion of affine term structure.

3.2 Affine term structure

This argument acquires importance due to the fact that from an analytical and computational point of view the existence of an affine term structure extremely simplifies the content.

Definition 3.1. A model is said to possess an affine term structure if the zero coupon bond can be written in the form

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)} \tag{3.5}$$

where A and B are deterministic functions and with zero coupon bond we mean a contract which guarantees the holder 1 dollar to be paid on the date T i.e.

$$p(t,T) = E[e^{-\int_0^T r_s ds}]$$

It turns out that the existence of an affine term structure is extremely pleasing from an analytical and computational point of view, so it is considerable interest to understand when such a structure appears. Both the Vasicek and CIR models we will see are affine models, since the bond price has an expression of the above form in both cases. Assume we have a general risk-neutral dynamics for the short rate,

$$dr_t = b(t,r_t)dt + \sigma(t,r_t)dW_t$$

We may wonder whether there exist conditions on b and σ such that the resulting model displays an affine term structure. The answer is simply that the coefficients b and σ^2 need be affine functions themselves. More precisely if the coefficients b and σ^2 are of the form

$$\begin{aligned} b(t,x) &= \lambda(t)x + \eta(t) \\ \sigma^2(t,x) &= \gamma(t)x + \delta(t) \end{aligned} \tag{3.6}$$

for suitable deterministic time functions $\lambda, \eta, \gamma, \delta$. The functions A and B can be obtained from the coefficients $\lambda, \eta, \gamma, \delta$ by solving an opportune differential equations indeed we have the following theorem.

Theorem 3.2.1. *Assume that $b(t,x)$ and $\sigma(t,x)$ are of the form in (3.6), then the model admits an affine term structure of the form (3.5) where A and B satisfy the system*

$$\begin{cases} \partial_t B(t,T) + \lambda(t)B(t,T) - \dfrac{1}{2}\gamma(t)B^2(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{3.7}$$

$$\begin{cases} \partial_t A(t,T) = \eta(t)B(t,T) - \dfrac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0 \end{cases} \tag{3.8}$$

The first equation is a Riccati differential equation that, in general, needs to be solved numerically. However, in the particular cases of Vasicek where

$$\begin{aligned} \lambda(t) &= -K \\ \eta(t) &= K\theta \\ \gamma(t) &= 0 \\ \delta(t) &= \sigma^2 \end{aligned} \tag{3.9}$$

or CIR where

$$\begin{aligned} \lambda(t) &= -K \\ \eta(t) &= K\theta \\ \gamma(t) &= \sigma^2 \\ \delta(t) &= 0 \end{aligned} \tag{3.10}$$

we have that the equations are explicitly solvable.

Remark 7. We note that equation (3.7) is an equation for the determination of B which does not involve A. Having solved (3.7) we may then insert the solution B into (3.8) and simply integrate in order to obtain A.

3.2.1 Vasicek model

In finance, the Vasicek model is a mathematical model describing the evolution of interest rates (1977). The model specifies that the instantaneous interest rate follows the stochastic differential equation:

$$dr_t = K(\theta - r_t)dt + \sigma dW_t \qquad r_0, K, \theta, \sigma > 0 \tag{3.11}$$

where

- r_0 is the initial condition
- The drift factor $K(\theta - r_t)$ represents the expected instantaneous change in the interest rate at time t.
- σ determines the volatility of the interest rate

This dynamics has some peculiarities that make the model attractive. The equation is linear and can be solved explicitly and the distribution of the short rate is Gaussian. Indeed integrating equation[1] (3.11), we obtain, for each $s \leq t$

$$\begin{aligned} r_t &= e^{-tK}(r_0 + K\theta \int_0^t e^{sK} b ds + \int_0^t e^{sK} \sigma dW_s) \\ &= r_0 e^{-tK} + \theta e^{-tK}(e^{tK} - 1) + \int_0^t e^{-K(t-s)} \sigma dW_s \\ &= r_0 e^{-tK} + \theta(1 - e^{-tK}) + \int_0^t e^{-K(t-s)} \sigma dW_s \end{aligned}$$

$r_t \sim N(m_t, V_t)^2$ where

$$\begin{aligned} m_t &:= E[r_t] = r_0 e^{-Kt} + \theta(1 - e^{-Kt}) \\ V_t &:= Var[r_t] = \frac{\sigma^2}{2K}(1 - e^{-2Kt}) \end{aligned} \tag{3.12}$$

This implies that, for each time t, the rate r_t can be negative with positive probability. The possibility of negative rates is indeed a major drawback of the Vasicek model.
As a consequence of (3.12), the short rate r is mean reverting, since the expected rate tends, for t going to infinity, to the value θ.
The simulation of the paths are plotted in figure (3.1).

As we say in previous section Vasicek possesses an affine term structure. Equations (3.7) and (3.8) became

$$\begin{cases} \partial_t B(t,T) - KB(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{3.13}$$

$$\begin{cases} \partial_t A(t,T) = K\theta B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) = 0 \end{cases} \tag{3.14}$$

The first equation is a simple ODE in the t-variable. It can easy be solved as

$$B(t,T) = \frac{1}{K}(1 - e^{-K(T-t)})$$

[1] We have used Theorem(C.1.1) with $b := K\theta, B := -K, \sigma := \sigma$
[2] We have used Remark (15)

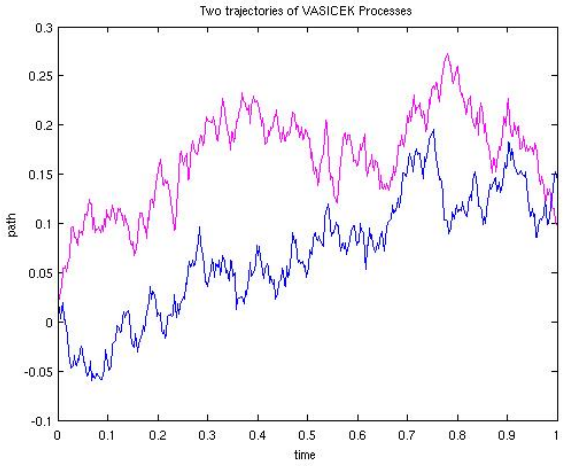

Figure 3.1: Path of Vasicek processes

where from the second system we obtain

$$A(t,T) = (\theta - \frac{\sigma^2}{2K^2})(B(t,T) - T + t) - \frac{\sigma^2}{4K}B^2(t,T)$$

Therefore the zero coupon bond, using (3.5), is

$$p(t,T) = e^{(\theta - \frac{\sigma^2}{2K^2})(B(t,T)-T+t) - \frac{\sigma^2}{4K}B^2(t,T) - \frac{1}{K}(1-e^{-K(T-t)})t_t} \tag{3.15}$$

See function bond_VASICEK.m which compare the exact formula with the approximation result of the bond.
In light of expression (2.16), the process that describes the evolution of the price of the underlying is

$$\begin{aligned} S_t &= \mathbb{I}_{\{\tau > t\}} e^{X_t} \\ dX_t &= ((c - \frac{1}{2})\delta^2 e^{2(\beta-1)X_t} + r_t + b)dt + \delta e^{(\beta-1)X_t} dW_t^1 \\ dr_t &= K(\theta - r_t)dt + \widehat{\sigma} dW_t^2 \\ d < W_t^1, W_t^2 >_t &= \rho \end{aligned} \tag{3.16}$$

Therefore in term of the istantaneos correlation coefficient we set

$$W_t^2 = \rho W_t^1 + \sqrt{1-\rho^2} \cdot dZ_t \tag{3.17}$$

So we obtain

$$dX_t = ((c - \frac{1}{2})\delta^2 e^{2(\beta-1)X_t} + r_t + b)dt + \delta e^{(\beta-1)X_t} dW_t^1$$
$$dr_t = K(\theta - r_t)dt + \widehat{\sigma}\rho dW_t^1 + \widehat{\sigma}\sqrt{1-\rho^2}dZ_t$$

We defiy $Y_t = (X_t, r_t)$ and $W_t = (W_t^1, Z_t)$, so we have:

$$dY_t = b(t, X_t, r_t)dt + C(t, X_t, r_t)dW_t \tag{3.18}$$

with

$$b(t,x,r) = \begin{pmatrix} \mu(t,x) \\ K(\theta - r) \end{pmatrix}$$

$$C(t,x,r) = \begin{pmatrix} \sigma(t,x) & 0 \\ \widehat{\sigma}\rho & \widehat{\sigma}\sqrt{1-\rho^2} \end{pmatrix}$$

In order to write the characteristic operator of the SDE we have to compute $C \cdot C^*$ where $C := C(t,x,r)$ so we have

$$C \cdot C^* = \begin{pmatrix} \sigma^2(t,x) & \widehat{\sigma}\rho\sigma(t,x) \\ \widehat{\sigma}\rho\sigma(t,x) & \widehat{\sigma}^2 \end{pmatrix}$$

Finally the characteristic operator of (3.27) is

$$A = \frac{1}{2}\sigma^2(t,x)\partial_{xx} + \widehat{\sigma}\rho\sigma(t,x)\partial_{xr} + \frac{1}{2}\widehat{\sigma}^2\partial_{rr} + \mu(t,x)\partial_x + K(\theta - r)\partial_r - (r + \gamma(t,x))$$

Our goal is to compute a representation formula for the classical solution u of the Cauchy problem

$$\begin{cases} (A + \partial_t)u = 0 \qquad in \quad [0,T[x\mathbb{R} \\ u(T,x) = 1 \qquad x \in \mathbb{R} \end{cases} \tag{3.19}$$

Thanks to Feynman-Kač formula the solution can be written in terms of expectation[3] i.e.

$$u(t,x) = E[e^{-\int_t^T (r_s + \gamma(s, X_s))ds}] \tag{3.20}$$

See function VASICEK_simulaz.m.

[3] See (C.2)

3.2.2 CIR model

Another important model which is more studied in the financial literature is the Cox–Ingersoll–Ross model (1985). The resulting model has been a benchmark for many years because of its analytical tractability and the fact that, contrary to the Vasicek model, the instantaneous short rate is always positive.
The model specifies that the instantaneous interest rate follows the stochastic differential equation:

$$dr_t = K(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t \qquad r_0, K, \theta, \sigma > 0 \tag{3.21}$$

The drift factor $K(\theta - r_t)$ is exactly the same as in the Vasicek model. It ensures mean reversion of the interest rate towards the long run value θ.
The condition

$$2K\theta > \sigma^2$$

has to be imposed to ensure that the origin is inaccessible to the process (3.21), so that we can grant that r remains positive. The mean and the variance of r_t are given by

$$\begin{aligned} E[r_t] &= r_0 e^{-Kt} + \theta(1 - e^{-Kt}) \\ Var[r_t] &= r_0 \frac{\sigma^2}{K}(e^{-Kt} - e^{-2Kt}) + \theta\frac{\sigma^2}{2K}(1 - e^{-Kt})^2 \end{aligned} \tag{3.22}$$

The simulation of the paths are plotted in figure (3.2).

As we say in previous section CIR possesses an affine term structure. Equations (3.7) and (3.8) became

$$\begin{cases} \partial_t B(t,T) - KB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{3.23}$$

$$\begin{cases} \partial_t A(t,T) = K\theta B(t,T) \\ A(T,T) = 0 \end{cases} \tag{3.24}$$

The price at time t of a zero-coupon bond with maturity T is

$$p(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{3.25}$$

where

$$A(t,T) = \left(\frac{2he^{\frac{1}{2}(K+h)(T-t)}}{2h + (K+h)(e^{(T-t)h} - 1)}\right)^{\frac{2h\theta}{\sigma^2}}$$

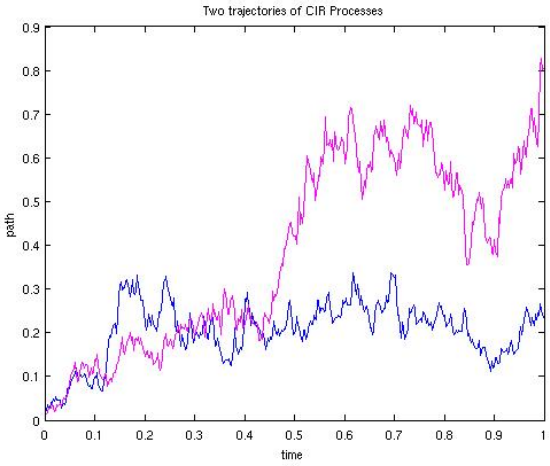

Figure 3.2: Path of CIR processes

$$B(t,T) = \frac{2(^{(T-t)h}-1)}{2h + (K+h)(e^{(T-t)h}-1)}$$
$$h = \sqrt{K^2 + 2\sigma^2}$$

See function bond_CIR.m which compare the exact formula with the approximation result of the bond.

In light of expression (2.16), the process that describes the evolution of the price of the underlying is

$$\begin{aligned} S_t &= \mathbb{I}_{\{\tau>t\}} e^{X_t} \\ dX_t &= ((c-\frac{1}{2})\delta^2 e^{2(\beta-1)X_t} + r_t + b)dt + \delta e^{(\beta-1)X_t} dW_t^1 \\ dr_t &= K(\theta - r_t)dt + \widehat{\sigma}\sqrt{r_t} dW_t^2 \\ d < W_t^1, W_t^2 >_t &= \rho \end{aligned} \tag{3.26}$$

Therefore writting the linear combination we have shown in (3.17) we obtain

$$\begin{aligned} dX_t &= ((c-\frac{1}{2})\delta^2 e^{2(\beta-1)X_t} + r_t + b)dt + \delta e^{(\beta-1)X_t} dW_t^1 \\ dr_t &= K(\theta - r_t)dt + \widehat{\sigma}\rho\sqrt{r_t} dW_t^1 + \widehat{\sigma}\sqrt{1-\rho^2}\sqrt{r_t} dZ_t \end{aligned}$$

We defiy $Y_t = (X_t, r_t)$ and $W_t = (W_t^1, Z_t)$, so we have:

$$dY_t = b(t, X_t, r_t)dt + C(t, X_t, r_t)dW_t \tag{3.27}$$

where $b(t,x,r)$ is the same in Vasicek and

$$C(t,x,r) = \begin{pmatrix} \sigma(t,x) & 0 \\ \widehat{\sigma}\rho\sqrt{r} & \widehat{\sigma}\sqrt{1-\rho^2}\sqrt{r} \end{pmatrix}$$

In order to write the characteristic operator of the SDE we have to compute $C \cdot C^*$ where $C := C(t,x,r)$ so we have

$$C \cdot C^* = \begin{pmatrix} \sigma^2(t,x) & \widehat{\sigma}\rho\sigma(t,x)\sqrt{r} \\ \widehat{\sigma}\rho\sigma(t,x)\sqrt{r} & \widehat{\sigma}^2 r \end{pmatrix}$$

Finally the characteristic operator of (3.27) is

$$A = \frac{1}{2}\sigma^2(t,x)\partial_{xx} + \widehat{\sigma}\rho\sqrt{r}\sigma(t,x)\partial_{xr} + \frac{1}{2}r\widehat{\sigma}^2\partial_{rr} + \mu(t,x)\partial_x + K(\theta - r)\partial_r - (r + \gamma(t,x))$$

The classical solution u of the Cauchy problem (3.19) is (3.20).
See function CIR_simulaz.m.

Chapter 4

Numerical method

In this section we describe the approximation methodology. The goal of this work is to solve the partial differential equations with variables cofficients and give a numerical approximation of the solution. Our Cauchy problem is given by

$$\begin{cases} (\mathcal{A} + \partial_t)u(t,x) = 0 & t \in [0,T[, x \in \mathbb{R} \\ u(T,x) = h(x) & x \in \mathbb{R} \end{cases} \tag{4.1}$$

where

$$\mathcal{A}(t,x)f(x) = a(t,x)\partial_{xx}f(x) + \mu(t,x)\partial_x f(x) - \gamma(t,x)f(x) \tag{4.2}$$

and

$$a(t,x) := \frac{1}{2}\sigma^2(t,x)$$

4.1 Taylor series expansion

This kind of method, which is proposed by Pagliarani, Pascucci, and Riga (2013), consist of approximating the drift and diffusion coefficients of $\mathcal{A}$ with a polynomial expansion.

Definition 4.1. We say that (a_n, γ_n) is a *polynomial expansion basis* for $\mathcal{A}$ if the following are satisfied:

- the sequences $(a_n(t,x))_{n\geq 0}$ and $(\gamma_n(t,x))_{n\geq 0}$ are sequences of continuous functions that depend polynomially on x with $a_0(t,x) \equiv a_0(t)$ and $\gamma_0(t,x) \equiv \gamma_0(t)$

- We have convergence

$$a(t,x) = \sum_{n=0}^{\infty} a_n(t,x) \qquad \gamma(t,x) = \sum_{n=0}^{\infty} \gamma_n(t,x)$$

  in some sense (pointwise or in norm).

For a fixed polynomial expansion basis (a_n, γ_n) the operator $\mathcal{A}$ can be formally written as

$$\mathcal{A}(t,x) = \sum_{n=0}^{\infty} \mathcal{A}_n(t,x) \tag{4.3}$$

where the operators $\mathcal{A}_n = \mathcal{A}_n(t,x)$ act as

$$\mathcal{A}_n(t,x)f(x) = a_n(t,x)\partial_{xx}f(x) + \mu_n(t,x)\partial_x f(x) - \gamma_n(t,x)f(x) \tag{4.4}$$

Now we approximate the drift and diffusion coefficients of $\mathcal{A}$ as a Taylor series about an arbitrary point $\overline{x} \in \mathbb{R}$. This corresponds to setting

$$\begin{aligned} a_n(t,x) &= \frac{(x-\overline{x})^n}{n!}\partial_x^n a(t,\overline{x}) \\ \gamma_n(t,x) &= \frac{(x-\overline{x})^n}{n!}\partial_x^n \gamma(t,\overline{x}) \end{aligned} \tag{4.5}$$

The choice of $\overline{x}$ is somewhat arbitrary. However, a convenient choice that seems to work well in most applications is to choose $\overline{x}$ near X_t , the level of the process X at time t. Following the classical perturbation approach, we expand the solution u as an infinite sum

$$u = \sum_{n=0}^{\infty} u_n(t,x) \tag{4.6}$$

Inserting (4.6) and (4.3) into (2.12) we have:

$n = 0$

$$(\mathcal{A}_0 + \partial_t)u_0(t,x) = 0$$

$n = 1$

$$\begin{aligned} &(\mathcal{A}_0 + \mathcal{A}_1 + \partial_t)(u_0(t,x) + u_1(t,x)) = 0 \\ &(\mathcal{A}_0 + \partial_t)u_0(t,x) + \mathcal{A}_1 u_0(t,x) + (\mathcal{A}_0 + \partial_t)u_1(t,x) + \mathcal{A}_1 u_1(t,x) = 0 \\ &\mathcal{A}_1 u_0(t,x) + (\mathcal{A}_0 + \partial_t)u_1(t,x) + \mathcal{A}_1 u_1(t,x) = 0 \\ &(\mathcal{A}_0 + \partial_t)u_1(t,x) = -\mathcal{A}_1 u_0(t,x) - \mathcal{A}_1 u_1(t,x) \end{aligned}$$

So we find that the functions $(u_n)_{n\geq 0}$ satisfy the following sequence of nested Cauchy problems

$$\begin{cases} (\mathcal{A}_0 + \partial_t)u_0(t,x) = 0 \qquad t \in [0,T[, x \in \mathbb{R} \\ u_0(T,x) = h(x) \qquad x \in \mathbb{R} \end{cases} \tag{4.7}$$

and

$$\begin{cases} (\mathcal{A}_0 + \partial_t)u_n(t,x) = -\sum_{k=1}^{n} \mathcal{A}_k(t,x)u_{n-k}(t,x) \qquad t \in [0,T[, x \in \mathbb{R} \\ u_n(T,x) = 0 \qquad x \in \mathbb{R} \end{cases} \tag{4.8}$$

Notice that

$$\mathcal{A}_0(t,x)f(x) = a_0(t,x)\partial_{xx}f(x) + \mu_0(t,x)\partial_x f(x) - \gamma_0(t,x)f(x)$$

is the characteristic operator of the following additive process

$$dX_t^0 = (r + \gamma_0(t) - a_0(t))dt + \sqrt{2a_0(t)}dW_t \tag{4.9}$$

whose characteristic function $\hat{p}_0(t,x,T,\xi)$ is given explicitly by

$$\hat{p}_0(t,x,T,\xi) = E[e^{i\xi X_T^0}|X_t = x] = \int_{\mathbb{R}} e^{i\xi y}p_0(t,x,T,\xi)dy \tag{4.10}$$

where $p_0(t,x,T,\xi)$ is the density of X_T^0.
By applying the Ito's formula to the process $e^{i\xi X_T^0}$ we obtain:

$$\begin{aligned} d(e^{i\xi X_t^0}) &= i\xi e^{i\xi X_t^0}dX_t^0 - \frac{\xi^2}{2}e^{i\xi X_t^0}d < X^0 >_t = \\ &= i\xi e^{i\xi X_t^0}((r + \gamma_0(t) - a_0(t))dt + \sqrt{2a_0(t)}dW_t) - \frac{\xi^2}{2}e^{i\xi X_t^0}d < X^0 >_t = \\ &= e^{i\xi X_t^0}[i\xi(r + \gamma_0(t) - a_0(t)) - \frac{\xi^2}{2} \cdot (2a_0(t))]dt + \cdots dW_t \end{aligned}$$

Then

$$\begin{aligned} e^{i\xi X_T^0} &= e^{i\xi X_t^0} + \int_t^T e^{i\xi X_s^0}[i\xi(r + \gamma_0(s) - a_0(s)) - \frac{\xi^2}{2} \cdot (2a_0(s))]ds + \int_t^T \cdots dW_s \\ &= e^{i\xi x} + \int_t^T e^{i\xi X_s^0}[i\xi(r + \gamma_0(s) - a_0(s)) - \frac{\xi^2}{2} \cdot (2a_0(s))]ds + \int_t^T \cdots dW_s \end{aligned}$$

Where in the second equality, since $X_t^0 = x$, we have replaced $e^{i\xi X_t^0}$ with $e^{i\xi x}$.
Therefore

$$\begin{aligned} E[e^{i\xi X_T^0}] &= E[e^{i\xi x}] + \int_t^T E[e^{i\xi X_s^0}][i\xi(r+\gamma_0(s)-a_0(s)) - \frac{\xi^2}{2}\cdot(2a_0(s))]ds \\ &= e^{i\xi(x+\int_t^T (r+\gamma_0(s)-a_0(s))ds) - \frac{\xi^2}{2}\int_t^T 2a_0(s)ds} \\ &= e^{i\xi(x+M(t,T)) - \frac{\xi^2}{2}C(t,T)} \end{aligned}$$

where $M(t,T)$ and $C(t,T)$ are defined as

$$\begin{aligned} M(t,T) &:= \int_t^T (r+\gamma_0(s)-a_0(s))ds \\ C(t,T) &:= \int_t^T 2a_0(s)ds \end{aligned}$$

And so

$$\hat{p}_0(t,x,T,\xi) = e^{i\xi(x+M(t,T)) - \frac{\xi^2}{2}C(t,T)} \tag{4.11}$$

By (A.2.1) it means that X_T^0 has Gaussian distribution with parameter $M(t,T), C(t,T)$ i.e. $X \sim N(M(t,T), C(t,T))$. Therefore the 1-dimensional Gaussian density of X_T^0 is

$$p_0(t,x,T,y) = \frac{1}{\sqrt{2\pi C(t,T)}} e^{-\frac{(x-y-M(t,T))^2}{2C(t,T)}}$$

Our goal is to provide an alternative representation of the price expansion. We need an important result: the following theorem provides an explicit expansion for the characteristic function $\hat{p}(t,x;T,\xi)$ of X_t , expressed in terms of integro-differential operators applied to $\hat{p}_0(t,x,T,\xi)$.

Theorem 4.1.1. *For any $n \geq 1$ we have:*

$$\hat{p}_n(t,x,T,\xi) = \mathscr{L}_n^x(t,T)\hat{p}_0(t,x,T,\xi) \qquad t < T, \quad x,\xi \in \mathbb{R}$$

with $\hat{p}_0$ as in (4.11) and

$$\mathscr{L}_n^x(s_0,T) = \sum_{h=1}^{n}\sum_{i=1}^{3n}\sum_{k=1}^{n}(x-\overline{x})^k F_{i,k}^{(n,h)}(s_0,T)\partial_x^i$$

with

$$F_{i,k}^{(n,h)}(s_0,T) = \int_{s_0}^T \int_{s_1}^T \int_{s_2}^T \cdots \int_{s_{h-1}}^T f_{i,k}^{(n,h)} ds_1 ds_2 \cdots ds_h$$

The coefficients $f_{i,k}^{(n,h)}$ are presented in the works of Lorig, Pagliarani and Pascucci and they have already been computed and saved in a model independent fashion.
(See *coefficients_prices_order3_JDCEV.txt*)

In particular the solution u_n of (4.8), if it exists, is given by

$$u_n(t,x) = \mathscr{L}_n(t,T)u_0(t,x) \tag{4.12}$$

where u_0 is explicitly given by

$$u_0(t,x) = e^{-\int_t^T \gamma_0(s)ds} \int_{\mathbb{R}} p_0(t,x,T,y)h(y)dy$$

In our case, if we want to compute the bond price, (i.e. $h(y) = 1$) since

$$\int_{\mathbb{R}} p(t,x,T,y)dy = 1$$

we have

$$\begin{aligned} u_0(t,x) &= e^{-\int_t^T \gamma_0(s)ds} \\ &= e^{-\int_t^T b+c\delta^2 e^{2(\beta-1)\overline{x}}ds} \\ &= e^{-\int_t^T b+c\delta^2 S_0^{2(\beta-1)}ds} \\ &= e^{-(T-t)(b+c\delta^2 S_0^{2(\beta-1)})} \end{aligned} \tag{4.13}$$

where in the third equality we have put $\overline{x} = \log(S_0)$. Instead, if we want to compute the European option prices, (i.e. $h(y) = e^y - K_0$) we have

$$\begin{aligned} M(t,T) &= \int_t^T ((c-\frac{1}{2})\delta^2 S_0^{2(\beta-1)} + r + b)ds \\ &= (T-t)[(c-\frac{1}{2})\delta^2 S_0^{2(\beta-1)} + r + b] \\ C(t,T) &= \int_t^T 2S_0^{2(\beta-1)}ds \\ &= 2(T-t)S_0^{2(\beta-1)} \end{aligned} \tag{4.14}$$

Therefore

$$u(t,x) = e^{-(T-t)(b+c\delta^2 S_0^{2(\beta-1)})} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi C(t,T)}} e^{-\frac{(x-y-M(t,T))^2}{2C(t,T)}} (e^y - K_0)dy$$

Remark 8 (Accuracy of the pricing approximation). Asymptotic convergence results were proved in Pagliarani et al.(2013); Lorig et al. (2013a). Precisely, assume that the functions $a = a(t, x)$ and $\gamma = \gamma(t, x)$ are differentiable up to order n with bounded and Lipschitz continuous derivatives. Assume also that the covariance matrix is uniformly positive definite and bounded. Let $\overline{x} = \mathrm{x}$. Then for any $N \in \mathbb{N}$ we have

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x) + O(T-t)^{\frac{N+1}{2}} \qquad as \quad t \to T^- \tag{4.15}$$

Remark 9 (Practical implementation). Notice that after a few terms the expression for $\mathscr{L}_n$ becomes very long. In practice, the formulas are feasible only for $n \leq 4$. However, in light of (4.15), it is sufficient to get very accurate results with $n = 2$ or $n = 3$.

In the following section we will present another criteria of approximation which is widely used in finance.

4.2 Monte Carlo Method

A fundamental implication of asset pricing theory is that, under certain circumstances, the price of a derivative can be usefully represented as an expected value.
Valuing derivatives thus reduces to computing expectations. But not every contract can be priced with an explicit formula. Most models in fact can have complicated payoffs so that the expectation of their discounted values cannot be easily calculated. In those cases the derivatives pricing can be simulated only approximately and a good way to determine the price of the contract can be to perform Monte Carlo simulation.
Monte Carlo method is one of the most popular numerical methods and it is used in many circumstances in mathematical finance and in particular in the pricing problem. Monte Carlo method allows to calculate the expected value of a random variable whose distribution is known. It is based on the strong law of large numbers which ensures that this estimate converges to the correct value as the number of draws increases.

Theorem 4.2.1. *Let (X_n) be a sequence of i.i.d. r.v. with $E[X_1] < \infty$.*
Let $\mu = E[X_1]$ and

$$M_n = \frac{X_1 + \cdots + X_n}{n}$$

then we have

$$M_n \longrightarrow \mu \qquad a.s.$$

This theorem tells that if we are able to generate many realizations X_1 , X_2 ,$\cdots$, X_n of the random variable X in an independent way, then we can almost surely use their mean M_n as an approximation of $E[X]$.
In order to give a first estimate of the error of the Monte Carlo method we have to introduce Markov's inequality.

Theorem 4.2.2 (Markov inequality)**.** *Let X be a r.v., $\lambda \in \mathbb{R}$,$\lambda > 0$, $1 \leq p < \infty$.*
Then:

$$\mathbb{P}(|X - E[X]| \geq \lambda) \leq \frac{var(X)}{\lambda^2} \tag{4.16}$$

Proof.

$$\begin{aligned} E[(X - E[X])^p] &= \int_\Omega (X - E[X])^p d\mathbb{P} \\ &\geq \int_{\{X - E[X] \geq \lambda\}} (X - E[X])^p d\mathbb{P} \\ &\geq \lambda^p \int_{\{X - E[X] \geq \lambda\}} d\mathbb{P} \\ &= \lambda^p \mathbb{P}(X - E[X] \geq \lambda) \\ \Longrightarrow \mathbb{P}(X - E[X] \geq \lambda) &\leq \frac{E[(X - E[X])^p]}{\lambda^p} \end{aligned} \tag{4.17}$$

We have (4.16) putting $p = 2$. Indeed

$$\mathbb{P}(X - E[X] \geq \lambda) \leq \frac{E[(X - E[X])^2]}{\lambda^2} = \frac{var(X)}{\lambda^2}$$

□

If we apply this proposition with $X = M_n$ for every $\varepsilon > 0$ w obtain:

$$\begin{aligned}\mathbb{P}(|M_n - \mu| \geq \varepsilon) &\leq \frac{var(M_n)}{\varepsilon^2} = \\ &= \frac{var(\frac{X_1+\cdots+X_n}{n})}{\varepsilon^2} = \\ &= \frac{\frac{1}{n^2}var(X_1 + \cdots + X_n)}{\varepsilon^2} = \\ &= \frac{\frac{1}{n}var(X_1)}{\varepsilon^2} = \\ &= \frac{\sigma^2}{n\varepsilon^2}\end{aligned}$$

where in third equality we have used the properties of the variance, in fourth equality the independence of the generations X_k , $k = 1, \cdots, n$ (thus the variance of their sum is the sum of their variances), and the fact that they are identically distributed (so the variances are all equal to $var(X_1)$).
Therefore

$$\mathbb{P}(|M_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \tag{4.18}$$

We set $p = \frac{\sigma^2}{n\varepsilon^2}$.
Formula (4.18) gives an estimate of the error in terms of three parameters:

1. n: the number of samples i.e. how many random numbers we have generated
2. ε: the maximum approximation error
3. p: the probability that the approximated value M_n not belongs to the confidence interval $[\mu - \varepsilon, \mu + \varepsilon]$

By (4.18) what is interesting to note is that with n tending to infinity the probability of obtaining a bad result (where the 'badness' is fixed by ε) tends to zero and so when n tends to infinity the probability that M_n not belongs to the confidence interval became zero. This fact can be written as follow:

$$M_n \longrightarrow \mu \qquad \text{with probability 1 as} \qquad n \to \infty$$

Remark 10. It is important to stress that the result and the error of the Monte Carlo method are r.v.

Now we introduce the central limit theorem which provides information about the likely magnitude of the error in the estimate after a finite number of draws.

Theorem 4.2.3. *Let (X_n) be a sequence of real i.i.d. r.v.with $\sigma^2 = var(X_1) < \infty$. We put, as usual,*

$$M_n = \frac{X_1 + \cdots + X_n}{n}$$

$$\mu = E[X_1]$$

and we consider the sequence defined by

$$G_n = \sqrt{n}(\frac{M_n - \mu}{\sigma}) \qquad n \in \mathbb{N}$$

Then

$$G_n \sim N(0, 1)$$

meaning that the error on the left has approximately the distribution on the right. Equivalently

$$\sqrt{n}(M_n - \mu) \sim N(0, \sigma^2) \Longrightarrow M_n - \mu \sim N(0, \frac{\sigma^2}{\sqrt{n}})$$

So, the central limit theorem provides an estimate of the speed of the convergence and the error distribution. It asserts that, as the number of replications n increases, the standardized estimator $\sqrt{n}(\frac{M_n - \mu}{\sigma})$ converges in distribution to the standard normal.
The same result holds if σ is replaced with the sample standard deviation s_C. This is important because σ is rarely known in practice but s_C is easily calculated from simulation output. For a more detailed analysis we refer to the next section.

Remark 11. One of the few drawbacks of the Monte Carlo method is that it is almost impossible to have completely independent generations of the random variable: since we need a great number of realizations, we must use a calculator, which can only work following algorithms, thus giving results which are only pseudo-random. The error due to the non-independence of the generations cannot be easily estimated, but if the generator works well the method can be applied and gives good results anyway.

4.2.1 Confidence interval

Thanks to central limit theorem we can introduce the notion of confidence interval. Very often, the estimate of a parameter does not provide sufficient guarantees of accuracy, and it is more reasonable to provide a set of values that can be considered a reasonable estimate of the parameter. Therefore, one way to describe concisely our approximation is to introduce confidence intervals. The confidence interval is a range of values within which is believed to have comprised the parameter in question with a certain degree of confidence. We indicate with δ the level of confidence of the confidence interval and with $p = 1 - \delta$ the probability that our estimate belongs to the interval of confidence. More precisely, for finite but at least moderately large n, we can supplement the point estimate M_n with a confidence interval.
Let

$$s_C = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - M_n)^2}$$

denote the sample standard deviation of X_1 , X_2 ,$\cdots$, X_n and let z_δ denote the $1 - \delta$ quantile of the standard normal distribution. Then

$$[M_n - z_{\frac{\delta}{2}}\frac{s_C}{\sqrt{n}}, M_n + z_{\frac{\delta}{2}}\frac{s_C}{\sqrt{n}}]$$

is an asymptotically valid $1 - \delta$ confidence interval for μ.
This means that the true value of μ is inside the 'window' with probability $p = 1 - \delta$. Here we report the values of z_δ correspondent to some of the most used confidence levels:

δ	Interval confidence	z_δ
0.01	99%	2.58
0.02	98%	2.33
0.05	95%	1.96
0.10	90%	1.65

Table 4.1: Confidence levels

In our approximation we will fix $\delta = 0.01$ therefore the true result will stay in the window

$$[M_n - 2.58\frac{s_C}{\sqrt{n}}, M_n + 2.58\frac{s_C}{\sqrt{n}}]$$

with probability $p = 99\%$.

Remark 12. We can see that, as n increases, the window shrinks as $\frac{1}{\sqrt{n}}$, which is worse than the $\frac{1}{n}$ rate one would typically wish. This means that if we need to reduce the window size to one tenth, we have to increase the number of scenarios by a factor 100. This implies that sometimes, to reach a chosen accuracy (a small enough window), we need to take a huge number of scenarios n.

4.2.2 Euler scheme

Valuing a derivative security by Monte Carlo typically involves simulating paths of stochastic processes used to describe the evolution of underlying asset prices, interest rates and model parameters. The simplest method for approximate simulation of stochastic differential equations is the Euler scheme. Now we see how it can be used in conjunction with the Monte Carlo method in order to improve accuracy.
We consider a process X satisfying a SDE of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

In this case the distribution of X_T is not known explicitly. In order to obtain some realizations of X_T we use a Euler-type scheme. It is clear that, in this way, the discretization error of the SDE must be added to the error of the Monte Carlo method.
Our goal is to compute a numerical approximation of the bond which is represented as an expected value i.e.

$$Bond(T) := E[e^{-\int_0^T r+\gamma(s,X_s)ds}]$$

and, since the risk-free rate of interest r is costant, we have

$$Bond(T) := e^{-rT}E[e^{-\int_0^T \gamma(s,X_s)ds}]$$

We set

$$f(X_T) = e^{-\int_0^T r+\gamma(s,X_s)ds}$$

Thanks to Monte Carlo method, the bond price approximation is given by

$$E[f(X_T)] \approx \frac{1}{n}\sum_{j=1}^{s} f(X_T^{(j)})$$

where $X_T^{(1)}, X_T^{(2)}, \cdots, X_T^{(s)}$ are independent realizations.
Before to illustrate the main steps, in order to guarantee consistence in our work, we introduce our notation. For a fixed discretization step h we divide the time horizont $[0, T]$ into small increments of length h and we consider a time grid $0 = t_0 < t_1 < \cdots < t_n = T$. We set:

- n = number of increments
- s = number of realizations
- $X(i, j)$ = matrix of the process X where i stay for t_i and j for j-th realization
- $t(i)$ = vector of n increments

The steps are follows:

Step 1: First of all we have to simulate paths of stochastic process X_t used to describe the evolution of underlying asset price. We have to produce s independent realizations Z_i of the normal standard distribution $N(0, 1)$ and, using an iterative formula, we have to determine the corresponding realizations of the final value of the underlying asset $X_T^{(1)}, X_T^{(2)}, \cdots, X_T^{(s)}$.
We use a simple recursive procedure for simulating values of X at $t_0 < t_1 < \cdots < t_n$:
For $j = 1, \cdots, s$ for $i = 1, \cdots, n-1$

$$\begin{aligned} X_{t_{i+1}}^{(j)} &= \mu(t_i, X_{t_i}^{(j)})(t_{i+1} - t_i) + \sigma(t_i, X_{t_i}^{(j)})(W_{t_{i+1}} - W_{t_i}) \\ &\text{Since } W_{t_{i+1}} - W_{t_i} = \sqrt{t_{i+1} - t_i} \cdot Z_{t_i} \text{with } Z_{t_i} \sim N(0,1) \\ \Longrightarrow X_{t_{i+1}}^{(j)} &= \mu(t_i, X_{t_i}^{(j)})(t_{i+1} - t_i) + \sigma(t_i, X_{t_i}^{(j)})\sqrt{t_{i+1} - t_i} Z_{t_i} \end{aligned} \tag{4.19}$$

Step 2: Then, for each realization (i.e. for each column), we have to compute

$$\text{IRiemann}(j) = \int_0^T \gamma(t, X_t^{(j)}) dt$$

In order to obtain a good approximation of the integral we have used Composite Simpson's rule which is a method for numerical integration. What give Composite Simpson's rule competitive with alternative method is that if the interval of integration $[0, T]$ is in some sense "small", then Composite Simpson's rule will provide

an adequate approximation to the exact integral. Suppose that the interval $[0, T]$ is split up in n subintervals as above. Then, the composite Simpson's rule is given by

$$\int_0^T f(t)dt \approx \frac{h}{3}(f(t_0) + 4f(t_1) + 2f(t_2) + 4f(t_3) + 2f(t_5) + \cdots + 4f(t_{n-1}) + f(t_n))$$

Step 3: Finally we have to compute the approximation of the price of the derivative. Then we computed

$$f(j) = exp(-r * t(n) - Iriemann(j));$$

Therefore the approximation of the price is given by

$$\text{BondT}_s = \frac{1}{n}\sum_{j=1}^{s} f(j)$$

In order to consider the 99% Interval confidence we have to compute the sample standard deviation. Using the definition we have introduced in section (4.2.1), we obtain

$$s_C = \sqrt{\frac{1}{s-1}\sum_{j=1}^{s}(f(j) - \text{BondT}_s)^2}$$

therefore the true result μ will stay in the window

$$[\text{BondT}_s - 2.58\frac{s_C}{\sqrt{n}}, \text{BondT}_s + 2.58\frac{s_C}{\sqrt{n}}]$$

with probability $p = 99\%$.

$$\frac{1}{n}\sum_{j=1}^{s} f(X_T^{(j)}) = \frac{e^{-rT}}{n}\sum_{j=1}^{s} e^{-\text{IRiemann}(j)} \approx e^{-rT}E[e^{-\int_0^T \gamma(s,X_s)ds}]$$

Remark 13. We want to stress that the error of this code is composed by

1. The error committed by the composite Simpson's rule
2. The discretization error of the SDE committed by the Euler schemes
3. The error of the Monte Carlo method

We give above the different *Matlab* code which we have implemented for our financial problems. Precisely:

function MMC_simulaz.m.: compute $Bond(T) := e^{-rT}E[e^{-\int_0^T \gamma(s,X_s)ds}]$

function Payoff_rcost.m.: compute $Call(T) := e^{-rT}E[e^{-\int_0^T \gamma(s,X_s)ds}(e^{X_T} - K_0)]$

function VASICEK_simulaz.m.: compute $Bond(T) := E[e^{-\int_0^T r_s+\gamma(s,X_s)ds}]$ with

$$dr_t = K(\theta - r_t)dt + \sigma dW_t$$

function CIR_simulaz.m.: compute $Bond(T) := E[e^{-\int_0^T r_s+\gamma(s,X_s)ds}]$ with

$$dr_t = K(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

function bond_VASICEK.m.: compute $Bond(T) := E[e^{-\int_0^T r_s ds}]$

function bond_CIR.m.: compute $Bond(T) := E[e^{-\int_0^T r_s ds}]$

```
function MMC_simulaz.m.
%MMC PER CALCOLARE IL BOND ALL' ISTANTE FINALE (A SCADENZA) DELL INTERVALLO
%DI DISCRETIZZAZIONE PER IL MODELLO JDCEV
%input
sigma = 0.3; beta = 2/3; b = 0.01; c = 2; r = 0.05;

Tmax = 20;
Tmin = 0;
N = 2000; %numero di intervalli in cui discretizzo
iter = 100000; %numero di simulazioni
t = linspace(Tmin, Tmax, N);
delta = t(2) - t(1);

sig = @(x)sigma * exp((beta - 1) * x);
mu = @(x)(c - 0.5) * sigma^2 * exp((2 * beta - 2) * x) + r + b;
gamma = @(x)b + c * sigma^2 * exp((2 * beta - 2) * x);

%inizializzazione
X = zeros(N, iter);%risolve la PDE(ogni colonna mi da una traiettoria nel tempo(da 0 a T))
num = zeros(N, iter);%gamma calcolato nelle traiettorie
num(1, :) = gamma(0);
f = zeros(1, iter);%funzione di cui voglio calcolare la media
Iriemann = zeros(1, iter);%integrale in gamma

%righe=tempo e colonne=simulazioni
for k = 1 : iter
for i = 1 : N - 1
Z = randn(1); %schema di Eulero per calcolare i diversi X
X(i + 1, k) = X(i, k) + mu(X(i, k)) * delta + sig(X(i, k)) * sqrt(delta) * Z;
perm = X(i + 1, k);
num(i + 1, k) = gamma(perm);
end
%metodo di Simpson per calcolare l'integrale
s1 = sum(num(2 : 2 : N - 2, k));
s2 = sum(num(3 : 2 : N - 1, k));
Iriemann(k) = delta * (num(1, k) + num(N, k) + 2 * s1 + 4 * s2)/3;
f(k) = exp(-Iriemann(k));
end


%prezzo bond scontato(da confrontare con user_JDCEV_yelds con r)

bondT = exp(-r * Tmax)/iter * sum(f);
```

```
function Payoff_rcost.m.
%MMC PER CALCOLARE LA CALL ALL' ISTANTE FINALE (A SCADENZA) DELL INTERVALLO
%DI DISCRETIZZAZIONE PER IL MODELLO JDCEV
%input
sigma = 0.3; beta = 2/3; b = 0.01; c = 2; r = 0.05; cappa = 2;

Tmax = 5;
Tmin = 0;
N = 500; %numero di intervalli in cui discretizzo
iter = 100000; %numero di simulazioni
t = linspace(Tmin, Tmax, N);
delta = t(2) - t(1);

sig = @(x)sigma * exp((beta - 1) * x);
mu = @(x)(c - 0.5) * sigma^2 * exp((2 * beta - 2) * x) + r + b;
gamma = @(x)b + c * sigma^2 * exp((2 * beta - 2) * x);

%inizializzazione
X = zeros(N, iter);%risolve la PDE(ogni colonna mi da una traiettoria nel tempo(da 0 a T))
num = zeros(N, iter);%gamma calcolato nelle traiettorie
num(1, :) = gamma(0);
f = zeros(1, iter);%funzione di cui voglio calcolare la media
Iriemann = zeros(1, iter);%integrale in gamma

%righe=tempo e colonne=simulazioni
for k = 1 : iter
for i = 1 : N - 1
Z = randn(1); %schema di Eulero per calcolare i diversi X
X(i + 1, k) = X(i, k) + mu(X(i, k)) * delta + sig(X(i, k)) * sqrt(delta) * Z;
perm = X(i + 1, k);
num(i + 1, k) = gamma(perm);
end
%metodo di Simpson per calcolare l'integrale
s1 = sum(num(2 : 2 : N - 2, k));
s2 = sum(num(3 : 2 : N - 1, k));
Iriemann(k) = delta * (num(1, k) + num(N, k) + 2 * s1 + 4 * s2)/3;
premium(k) = max(exp(X(N, k)) - cappa, 0);
f(k) = premium(k) * exp(-r * Tmax - Iriemann(k));
end


%prezzo bond scontato(da confrontare con user_JDCEV_yelds con r)

bondT = 1/iter * sum(f);
```

Now we consider JDCEV model which we have described above with stochastic interest rate. First of all we describes the short rate through the Vasicek model, then through the CIR model which is more attractive.

```
function VASICEK_simulaz.m.
%MMC PER CALCOLARE IL BOND ALL' ISTANTE FINALE (A SCADENZA) DELL INTERVALLO
%DI DISCRETIZZAZIONE PER IL MODELLO JDCEV CON TASSO CHE SEGUE IL MODELLO DI
%VASICEK
%input
sigma = 0.3; beta = 2/3; b = 0.01; c = 2;
%input per il tasso r
K = 1; teta = 0.03; m = 0.2; rho = -0.3;

Tmax = 20;
Tmin = 0;
N = 2000; %numero di intervalli in cui discretizzo
iter = 100000; %numero di simulazioni
t = linspace(Tmin, Tmax, N);
delta = t(2) - t(1);

sig = @(x)sigma * exp((beta - 1) * x);
mu = @(x)(c - 0.5) * sigma^2 * exp((2 * beta - 2) * x) + b;
gamma = @(x)b + c * sigma^2 * exp((2 * beta - 2) * x);

%inizializzazione
X = zeros(N, iter);%risolve la PDE(ogni colonna mi da una traiettoria nel tempo(da 0 a T))
num = zeros(N, iter);%gamma calcolato nelle traiettorie
num(1, :) = gamma(0);
r = zeros(N, iter);
r(1, :) = 0.02;
f = zeros(1, iter);%funzione di cui voglio calcolare la media
Iriemann = zeros(1, iter);%integrale in gamma
Iriemannr = zeros(1, iter);%integrale in r

%righe=tempo e colonne=simulazioni
for k = 1 : iter
for i = 1 : N - 1
Z1 = randn(1);
W1 = sqrt(delta) * Z1;
Z2 = randn(1);
W2 = rho * W1 + sqrt(1 - (rho * rho)) * Z2 * sqrt(delta);
r(i + 1, k) = r(i, k) + K * (teta - r(i, k)) * delta + m * W2;
%schema di Eulero per calcolare i diversi X
X(i + 1, k) = X(i, k) + mu(X(i, k) + r(i, k)) * delta + sig(X(i, k)) * sqrt(delta) * Z;
perm = X(i + 1, k);
num(i + 1, k) = gamma(perm);
end
%metodo di Simpson per calcolare l'integrale
s1 = sum(num(2 : 2 : N - 2, k));
s2 = sum(num(3 : 2 : N - 1, k));
s1r = sum(r(2 : 2 : N - 2, k));
s2r = sum(r(3 : 2 : N - 1, k));
Iriemann(k) = delta * (num(1, k) + num(N, k) + 2 * s1 + 4 * s2)/3;
Iriemannr(k) = delta * (r(1, k) + r(N, k) + 2 * s1r + 4 * s2r)/3;
f(k) = exp(-Iriemannr(k) - Iriemann(k));
end


bondT = 1/iter * sum(f);%prezzo bond
```

```
function CIR_simulaz.m.
%MMC PER CALCOLARE IL BOND ALL' ISTANTE FINALE (A SCADENZA) DELL INTERVALLO
%DI DISCRETIZZAZIONE X IL MODELLO JDCEV CON TASSO CHE SEGUE IL MODELLO CIR
%input
sigma = 0.3; beta = 2/3; b = 0.01; c = 2;
%input per il tasso r
K = 1; teta = 0.03; m = 0.2; rho = -0.3;

Tmax = 20;
Tmin = 0;
N = 2000; %numero di intervalli in cui discretizzo
iter = 100000; %numero di simulazioni
t = linspace(Tmin, Tmax, N);
delta = t(2) - t(1);

sig = @(x)sigma * exp((beta - 1) * x);
mu = @(x)(c - 0.5) * sigma^2 * exp((2 * beta - 2) * x) + b;
gamma = @(x)b + c * sigma^2 * exp((2 * beta - 2) * x);

%inizializzazione
X = zeros(N, iter);%risolve la PDE(ogni colonna mi da una traiettoria nel tempo(da 0 a T))
num = zeros(N, iter);%gamma calcolato nelle traiettorie
num(1, :) = gamma(0);
r = zeros(N, iter);
r(1, :) = 0.02;
f = zeros(1, iter);%funzione di cui voglio calcolare la media
Iriemann = zeros(1, iter);%integrale in gamma
Iriemannr = zeros(1, iter);%integrale in r

%righe=tempo e colonne=simulazioni
for k = 1 : iter
for i = 1 : N - 1
Z1 = randn(1);
W1 = sqrt(delta) * Z1; Z2 = randn(1);
W2 = rho * W1 + sqrt(1 - (rho * rho)) * Z2 * sqrt(delta);
r(i+1, k) = r(i, k)+K*(teta-r(i, k))*delta+m*sqrt(r(i, k))*W2+0.25*(m^2)*W2*W2-0.25*(m^2)*delta;
if(r(i + 1, k) < 0)
r(i + 1, k) = -r(i + 1, k);
end
%schema di Eulero per calcolare i diversi X
X(i + 1, k) = X(i, k) + mu(X(i, k) + r(i, k)) * delta + sig(X(i, k)) * sqrt(delta) * Z;
perm = X(i + 1, k);
num(i + 1, k) = gamma(perm);
end
%metodo di Simpson per calcolare l'integrale
s1 = sum(num(2 : 2 : N - 2, k));
s2 = sum(num(3 : 2 : N - 1, k));
s1r = sum(r(2 : 2 : N - 2, k));
s2r = sum(r(3 : 2 : N - 1, k));
Iriemann(k) = delta * (num(1, k) + num(N, k) + 2 * s1 + 4 * s2)/3;
Iriemannr(k) = delta * (r(1, k) + r(N, k) + 2 * s1r + 4 * s2r)/3;
f(k) = exp(-Iriemannr(k) - Iriemann(k));
end


bondT = 1/iter * sum(f);%prezzo bond
```

Now we set default $= 0$ i.e. $\gamma(t, X_t) = 0$. Our goal is to compute a numerical approximation of the zero coupon bond which is represented as an expected value i.e.

$$p(0,T) := E[e^{-\int_0^T r_s ds}]$$

and compare it with exact formula given by Brigo.

```
function bond_VASICEK.m.
%MMC PER CALCOLARE IL PREZZO ZERO COUPON BOND E CONFRONTO CON LA
%FORMULA DEL PREZZO ESATTA DI VASICEK DATA DAL BRIGO
%input per il tasso r
K = 5; teta = 0.1; m = 0.2;

Tmax = 20;
Tmin = 0;
N = 2000; %numero di intervalli in cui discretizzo
iter = 100000; %numero di simulazioni
t = linspace(Tmin, Tmax, N);
delta = t(2) - t(1);

%inizializzazione
r = zeros(N, iter);
r(1, :) = 0.02;
f = zeros(1, iter);%funzione di cui voglio calcolare la media
Iriemannr = zeros(1, iter);%integrale in r

%righe=tempo e colonne=simulazioni
for k = 1 : iter
for i = 1 : N - 1
Z1 = randn(1);
W1 = sqrt(delta) * Z1;
r(i + 1, k) = r(i, k) + K * (teta - r(i, k)) * delta + m * W1;
end
s1r = sum(r(2 : 2 : N - 2, k));
s2r = sum(r(3 : 2 : N - 1, k));
Iriemannr(k) = delta * (r(1, k) + r(N, k) + 2 * s1r + 4 * s2r)/3;
f(k) = exp(-Iriemannr(k));
end


%prezzo zero coupon bond

bondT = 1/iter * sum(f);

B = 1/K * (1 - exp(-K * Tmax));

A = exp((teta - m^2/(2 * K^2)) * (B - Tmax) - m^2/(4 * K) * B^2);

bond_esatta = A * exp(-B * r(1, 1));

errore = norm(bond_esatta - bondT)/norm(bond_esatta); %errore relativo del prezzo
```

```
function bond_CIR.m.
%MMC PER CALCOLARE IL PREZZO ZERO COUPON BOND E CONFRONTO CON LA
%FORMULA DEL PREZZO ESATTA DEL CIR DATA DAL BRIGO
%input per il tasso r
K = 5;
teta = 0.4;
m = 0.8;

Tmax = 20;
Tmin = 0;
N = 2000; %numero di intervalli in cui discretizzo
iter = 100000; %numero di simulazioni
t = linspace(Tmin, Tmax, N);
delta = t(2) - t(1);

%inizializzazione
r = zeros(N, iter);
r(1, :) = 0.02;
f = zeros(1, iter);%funzione di cui voglio calcolare la media
Iriemannr = zeros(1, iter);%integrale in r

%righe=tempo e colonne=simulazioni
for k = 1 : iter
for i = 1 : N - 1
Z1 = randn(1);
W1 = sqrt(delta) * Z1;
r(i+1, k) = r(i, k)+K*(teta-r(i, k))*delta+m*sqrt(r(i, k))*W1+0.25*(m^2)*W1*W1-0.25*(m^2)*delta;
if (r(i + 1, k) < 0)
r(i + 1, k) = -r(i + 1, k);
end
end
s1r = sum(r(2 : 2 : N - 2, k));
s2r = sum(r(3 : 2 : N - 1, k));
Iriemannr(k) = delta * (r(1, k) + r(N, k) + 2 * s1r + 4 * s2r)/3;
f(k) = exp(-Iriemannr(k));
end


%prezzo zero coupon bond

bondT = 1/iter * sum(f);

acca = sqrt(K^2 + 2 * m^2);

B = (2 * (exp(acca * Tmax) - 1))/(2 * acca + (K + acca) * (exp(acca * Tmax) - 1));

esponente = (2 * K * teta)/m^2;

A1 = (2 * acca * exp(0.5 * Tmax * (K + acca)))/(2 * acca + (K + acca) * (exp(acca * Tmax) - 1));

A = A1^(esponente);

bond_esatta = A * exp(-B * r(1, 1));

errore = norm(bond_esatta - bondT)/norm(bond_esatta); %errore relativo del prezzo
```

Chapter 5

Results

In this section we test the performance of the analytical approximation formula presented in the previous sections in the context of one-dimensional local volatility models. In the following experiment we compare the second and third order approximation of the Bond price with an accurate Monte Carlo simulation.
Precisely we show numerical approximation of:

1. JDCEV model with costant interest r

2. JDCEV model with stochastic interest r which follow two different models:

   - Vasicek model
   - Cir model

The true result bond_esatta $:= u(t, x; T)$ is computed in *Wolfram Mathematica* by formula (3.4) truncating the infinite sum at $n = 70$.
Relative error is defined as

$$\text{err} = \frac{||\text{bond\_esatta} - \text{bondT}||}{||\text{bond\_esatta}||}$$

5.0.3 JDCEV model with costant interest

We fix the following parameters:

$$\sigma = 0.3; \qquad \beta = 2/3;$$
$$b = 0.01; \qquad c = 2;$$
$$S0 = 1; \qquad r = 0.05;$$

We have considered 100 discretization points for each year (i.e if $T = 1 \Rightarrow N = 100$,if $T = 2 \Rightarrow N = 200$,eccetera...). Therefore we have fixed $\delta = 0.01$.
Since the Monte Carlo method requires s iterations, if we choose a great number of simulations it turns out to be quite slow. In order to have the result in few minutes we have to keep number not enormous (we mean something like 100000).
First of all we computed the approximate price of a bond i.e.

$$Bond(T) := E[e^{-\int_0^T r+\gamma(s,X_s)ds}]$$

This model is computed in user_JDCEV_yields.nb (*Wolfram Mathematica*) and in MMC_simulaz.m (*Matlab*). With Bondbar[2] and Bondbar[3] we mean respectively the second and the third order approximation. In order to have an high precision we have considered 99% Interval confidence i.e. $z_\delta = 2.58$.
We have summarized in next tables the theoretical results obtained for each scheme.
The Bond price at time $t_0, t_1, \cdots, t_n$ and the simulation of the path are plotted in figure (5.1) and (5.2).

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.794695 ± 0.011210	0.794206	0.06
1000	0.794813 ± 0.003560	0.794206	0.07
10000	0.794869 ± 0.001125	0.794206	0.08
100000	0.794699 ± 0.000356	0.794206	0.06

Table 5.1: Bond price at maturity $T = 1$: The approximate price Bondbar[2]= 0.793964 and Bondbar[3]= 0.794264

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.639907 ± 0.019236	0.642143	0.3
1000	0.642376 ± 0.006139	0.642143	0.03
10000	0.642486 ± 0.001953	0.642143	0.05
100000	0.642507 ± 0.000618	0.642143	0.05

Table 5.2: Bond price at maturity $T = 2$: The approximate price Bondbar[2]= 0.640787 and Bondbar[3]= 0.642532

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.371500 ± 0.031687	0.370376	0.3
1000	0.369358 ± 0.010109	0.370376	0.2
10000	0.370128 ± 0.003212	0.370376	0.06
100000	0.370413 ± 0.001015	0.370376	0.009

Table 5.3: Bond price at maturity $T = 5$: The approximate price Bondbar[2]= 0.364453 and Bondbar[3]= 0.372089

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.181677 ± 0.035609	0.180146	0.8
1000	0.179587 ± 0.011358	0.180146	0.3
10000	0.179500 ± 0.003584	0.180146	0.3
100000	0.180271 ± 0.001137	0.180146	0.06

Table 5.4: Bond price at maturity $T = 10$: The approximate price Bondbar[2]= 0.18438 and Bondbar[3]= 0.169343

Performing the simulations, we note, as expected according to the law of large numbers, that increasing the number of experiments used in the probabilistic Monte Carlo method, the value of the price differs less and less from the exact value obtained by formula of Mendoza (3.4) with an error of 0.01%. This is a test of the robustness of the Monte Carlo method. Similarly we observe that the approximate value given by expanding the drift and diffusion coefficients is quite good. In particular we can note that for maturities between 0.5 and 5 years the third order approximation is better than

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.061044 ± 0.031891	0.0614821	0.7
1000	0.061861 ± 0.010156	0.0614821	0.6
10000	0.061561 ± 0.003194	0.0614821	0.1
100000	0.061406 ± 0.001009	0.0614821	0.1

Table 5.5: Bond price at maturity $T = 20$: The approximate price Bondbar[2]= 0.0886772 and Bondbar[3]= 0.0473193

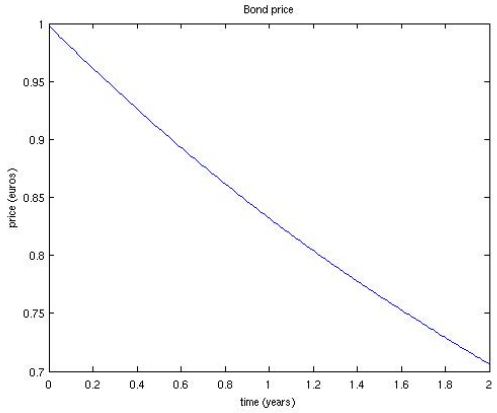

Figure 5.1: Bond price as a function of time

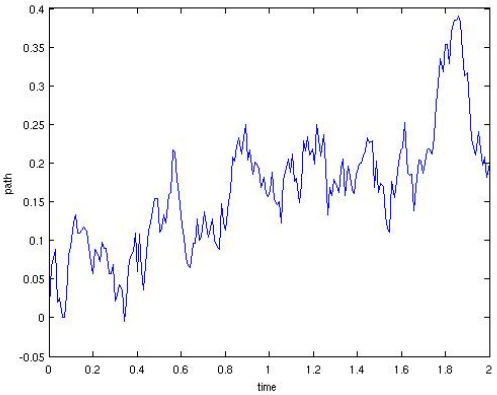

Figure 5.2: Path of X

the second order approximation. While if we consider maturities over 5 years the best result is given by the second order approximation.

Now we considered the case of option pricing i.e.

$$Bond(T) := E[e^{-\int_0^T r+\gamma(s,X_s)ds}(e^{X_T} - K_0)]$$

We fix the following parameters:

$$\begin{aligned} \sigma &= 0.3; \qquad \beta = 2/3; \\ b &= 0.01; \qquad c = 2; \\ S0 &= 1; \qquad r = 0.05; \end{aligned}$$

This model is computed in user_JDCEV_yields_payoff.nb (*Wolfram Mathematica*) and in Payoff_rcost.m (*Matlab*). With Callbar[2] and Callbar[3] we mean respectively the second order and the third approximation.
We have summarized in next tables the theoretical results obtained for each scheme.
In this case we have not the exact price therefore we cannot compute the relative error between the exact value and the approximate price.

Simulations	99% Interval confidence
100	0.521039 ± 0.031891
1000	0.524265 ± 0.011830
10000	0.530646 ± 0.003792
100000	0.530258 ± 0.001208

Table 5.6: Call price at maturity $T = 0.25$ with $K_0 = 0.5$: The approximate price Callbar[2]= 0.528831 and Callbar[3]= 0.528828

Simulations	99% Interval confidence
100	0.133577 ± 0.050686
1000	0.134295 ± 0.015432
10000	0.132254 ± 0.005019
100000	0.131981 ± 0.001600

Table 5.7: Call price at maturity $T = 1$ with $K_0 = 1.2$: The approximate price Callbar[2]= 0.132196 and Callbar[3]= 0.131823

Simulations	99% Interval confidence
100	0.189783 ± 0.090009
1000	0.178767 ± 0.024269
10000	0.181238 ± 0.007888
100000	0.184460 ± 0.002555

Table 5.8: Call price at maturity $T = 3$ with $K_0 = 1.8$: The approximate price Callbar[2]= 0.1884 and Callbar[3]= 0.18112

Simulations	99% Interval confidence
100	0.304975 ± 0.108190
1000	0.322570 ± 0.035480
10000	0.322640 ± 0.011511
100000	0.323634 ± 0.003667

Table 5.9: Call price at maturity $T = 5$ with $K_0 = 2$: The approximate price Callbar[2]= 0.335117 and Callbar[3]= 0.320025

We can seen that the approximate price obtained from the Monte Carlo method is near to the approximate price obtained from Taylor expansion. This mean that the two methods give a similiar results and so they work well. We want to stress that the series approach using in *Wolfram Mathematica* code is better than the simulations approach using in Monte Carlo Method. The latter indeed is more quickly than the former. For this reason we prefer testing only user_JDCEV_yields_payoff.nb with different parameters. Therefore we test the performance of Mathematica code fixing several strike.

We considered the same parameters as above but now we want to use two different values of β i.e.:

- $\beta = 2/3$
- $\beta = 1/3$

First of all we consider $\beta = 2/3$. We have summarized in next tables the theoretical results obtained.

		$\sigma = 0.1$		$\sigma = 0.3$		$\sigma = 0.5$	
T	K_0	Call[2]	Call[3]	Call[2]	Call[3]	Call[2]	Call[3]
	0.5	0.509886	0.509886	0.528831	0.528828	0.563828	0.56378
0.25	0.6580	0.35501	0.35501	0.379985	0.379982	0.426988	0.426919
	1	0.0311581	0.0311581	0.0905966	0.0905883	0.167224	0.167116
	0.3	0.722935	0.722935	0.761811	0.761721	0.81961	0.818629
1	0.7746	0.284674	0.284633	0.389749	0.389507	0.539434	0.536925
	1.2	0.007433	0.007431	0.132196	0.131823	0.322439	0.318584
	0.25	0.802547	0.802538	0.868908	0.867881	0.929499	0.929545
3	0.7071	0.441545	0.441519	0.630842	0.627943	0.800898	0.801087
	2	0.0001318	0.00005497	0.142347	0.13427	0.466416	0.466974
	0.2	0.864487	0.864453	0.92711	0.92558	0.962841	0.972083
5	0.7746	0.475261	0.475131	0.718598	0.712718	0.856188	0.891952
	2.2	0.001908	0.001458	0.288716	0.272119	0.597738	0.699241

Table 5.10: Call price as a function of time to maturity T and strike K_0 with $\beta = 2/3$

Now we fix $\beta = 1/3$.

		$\sigma = 0.1$		$\sigma = 0.3$		$\sigma = 0.5$	
T	K_0	Call[2]	Call[3]	Call[2]	Call[3]	Call[2]	Call[3]
	0.5	0.509873	0.509873	0.528659	0.528636	0.563192	0.562811
0.25	0.6580	0.354993	0.354493	0.379789	0.379763	0.426314	0.425759
	1	0.031112	0.0311117	0.0896343	0.0895746	0.163514	0.162653
	0.3	0.722823	0.72282	0.760731	0.760012	0.817319	0.809473
1	0.7746	0.284352	0.284344	0.386765	0.384824	0.531282	0.511203
	1.2	0.006589	0.006572	0.122499	0.119508	0.303944	0.273103
	0.25	0.801929	0.801854	0.866317	0.858104	0.918568	0.918932
3	0.7071	0.439809	0.439599	0.622695	0.599502	0.76976	0.771274
	1.6	0.003753	0.003173	0.221725	0.169829	0.490558	0.494197
	0.2	0.863491	0.863223	0.924395	0.912173	0.931938	1.00588
5	0.7746	0.471433	0.470395	0.707262	0.660241	0.736797	1.02291
	1.7	0.023194	0.020869	0.387067	0.284418	0.425585	1.05287

Table 5.11: Call price as a function of time to maturity T and strike K_0 with $\beta = 1/3$

5.0.4 JDCEV model with sthocastic interest: Vasicek model

This model is computed in bond_VASICEK.m (*Matlab*). The exact value is given by formulas written in Brigo. The aim of this test is to show the validity of the strong law of large numbers. We want to show that, as the simulations increase, the approximate price begin better and better. We have summarized in next tables the theoretical results obtained for each maturity.

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.923006 ± 0.008385	0.919849	0.3
1000	0.919002 ± 0.002379	0.919849	0.09
10000	0.919769 ± 0.000794	0.919849	0.008
100000	0.919664 ± 0.000254	0.919849	0.02

Table 5.12: Vasicek:Zero coupon bond at maturity $T = 1$

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.835965 ± 0.011809	0.833067	0.3
1000	0.834258 ± 0.003638	0.833067	0.1
10000	0.833359 ± 0.001110	0.833067	0.03
100000	0.832863 ± 0.000356	0.833067	0.02

Table 5.13: Vasicek: Zero coupon bond at maturity $T = 2$

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.615634 ± 0.012960	0.618635	0.4
1000	0.618387 ± 0.004344	0.618635	0.04
10000	0.618226 ± 0.001384	0.618635	0.06
100000	0.618669 ± 0.000440	0.618635	0.005

Table 5.14: Vasicek: Zero coupon bond at maturity $T = 5$

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.377779 ± 0.011944	0.376725	0.2
1000	0.376083 ± 0.003744	0.376725	0.1
10000	0.377061 ± 0.001207	0.376725	0.08
100000	0.376660 ± 0.000386	0.376725	0.01

Table 5.15: Vasicek: Zero coupon bond at maturity $T = 10$

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.139932 ± 0.005882	0.139703	0.01
1000	0.139051 ± 0.001955	0.139703	0.04
10000	0.139666 ± 0.000650	0.139703	0.02
100000	0.139690 ± 0.000204	0.139703	0.009

Table 5.16: Vasicek: Zero coupon bond at maturity $T = 20$

5.0.5 JDCEV model with sthocastic interest: Cir model

This model is computed in bond_CIR.m. The exact value is given by formulas written in Brigo. Also in this case we can see the efficiency and the validity of the strong law of

large numbers.

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.729457 ± 0.013856	0.726221	0.4
1000	0.727159 ± 0.004184	0.726221	0.1
10000	0.725590 ± 0.001348	0.726221	0.08
100000	0.724982 ± 0.000254	0.726221	0.02

Table 5.17: CIR: Zero coupon bond at maturity $T = 1$

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.488751 ± 0.016629	0.489457	0.1
1000	0.488411 ± 0.004869	0.489457	0.2
10000	0.488376 ± 0.001525	0.489457	0.2
100000	0.488576 ± 0.000483	0.489457	0.1

Table 5.18: CIR: Zero coupon bond at maturity $T = 2$

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.148767 ± 0.007593	0.149647	0.5
1000	0.148927 ± 0.002548	0.149647	0.4
10000	0.149330 ± 0.000802	0.149647	0.2
100000	0.149414 ± 0.000256	0.149647	0.1

Table 5.19: CIR: Zero coupon bond at maturity $T = 5$

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.020814 ± 0.001497	0.020764	0.2
1000	0.020719 ± 0.000538	0.020764	0.2
10000	0.020798 ± 0.000166	0.020764	0.1
100000	0.020760 ± 0.000052	0.020764	0.02

Table 5.20: CIR: Zero coupon bond at maturity $T = 10$

Simulations	99% Interval confidence	Exact price	Relative error (%)
100	0.000398 ± 0.000042	0.000400	0.3
1000	0.000399 ± 0.000014	0.000400	0.1
10000	0.000399 ± 0.000005	0.000400	0.2
100000	0.000400 ± 0.000001	0.000400	0.01

Table 5.21: CIR: Zero coupon bond at maturity $T = 20$

The last two tests allow to confirm again the numerical accuracy of our approximations given by Monte Carlo method. Thanks to its usefulness, it is used in many circumstances in mathematical finance and in particular in the pricing problem. Moreover the level of mathematics required is quite basic and so it not difficult to implement. On the other hand a large number of samples is required to reach the desired results therefore calculations can take much time. For this reason it may not always be appropriate and we prefer, if it exist, anoter criteria of approximation.

Chapter 6

Conclusions and future work

Analytical approximation methods in option pricing have attracted an ever increasing interest in the last years. This is due to the demand for more sophisticated pricing models, including local, stochastic volatility that generally cannot be solved in closed-form. In this thesis we have defined a local stochastic volatility model with default i.e. a model which take into account the possibility of bankruptcy of the counterparty of a contract. We have analized our defaultable model and through Feynman-Kač representation theorem we have seen that the price of an option is the classical solution (when it exists) of Cauchy problem therefore we have solved a PDE with variables coefficients. We have illustrated how to obtain fast and accurate pricing approximations by expanding the drift and diffusion as a Taylor series and we have compared the second and third order approximation of the Bond and Call price with an accurate Monte Carlo simulation. Finally we have examined the numerical accuracy of our approximation both with *Mathematica* algorithms and with *Matlab* ones. We have provided several numerical examples illustrating the usefulness and versatility of our methods.

Appendix A

Probability spaces

Modern probability theory has become the natural language for formulating quantitative models of financial markets. In this chapter we presents some of its tools and concepts that will be important to understand every step of the costruction of our model. Our goal is to give some background knowledge on random variables, sthocastic process and Stochastic Differential Equation.

Definition A.1. Let Ω be non-empty set. A *σ-algebra* $\mathscr{F}$ is a collection of subsets of Ω s.t. :

1. contains the empty set: $\varnothing \in \mathscr{F}$
2. contains the complementary of every element: if $A \in \mathscr{F}$ then $A^C \in \mathscr{F}$
3. is stable under unions: for every sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathscr{F}$,

$$\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}$$

An elements of a *σ-algebra* $\mathscr{F}$ is called a *event* therefore *σ-algebra* $\mathscr{F}$ is a family of events. In many applications and especially in mathematical finance, σ-algebras are routinely used to represent the concept of information.

Example The Borel σ-algebra $\mathscr{B}$ is the σ-algebra generated by the Euclidean topology of $\mathbb{R}$ i.e.

$$\mathscr{B} = \sigma(\{A | A \quad open \quad set \quad in \quad \mathbb{R}\})$$

Definition A.2. A measure $\mathbb{P}$ on $\mathscr{F}$ is a map $\mathbb{P}$: $\mathscr{F} \longrightarrow [0,1]$ s.t. :

1. $\mathbb{P}(\varnothing) = 0$
2. for every $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\mathscr{F}$, we have:

$$\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

If $\mathbb{P}(\Omega) < \infty$ we say that $\mathbb{P}$ is a finite measure.
$\mathbb{P}$ is called probability measure. It assigns a probability between 0 and 1 to each event. An event A s.t. $\mathbb{P}(A) = 1$ is said to occur almost surely. If $\mathbb{P}(A) = 0$ this is interpreted by saying the event A is impossible.

Definition A.3. A probability space is a triple $(\Omega, \mathscr{F}, \mathbb{P})$ with:

1. Ω be non-empty set
2. σ-algebra $\mathscr{F}$ on Ω
3. $\mathbb{P}$ a measure on $\mathscr{F}$

Remark 14. In a financial modelling context

Ω : will represent the different scenarios which can occur in the market, each scenario $\omega \in \Omega$ being described in terms of the evolution of prices.

$\mathscr{F}_t$: can be interpreted as the collection of events reflecting informations known up to payment period t

$\mathbb{P}$: is a probability measure that assigns probabilities to the events in the σ-algebra $\mathscr{F}$

A.1 Distributions

Probabilities measure definded on the Euclidean space play an essential role.

Definition A.4. A probability measure on ($\mathbb{R}$, $\mathscr{B}$) is called distribution.

Definition A.5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a non-negative $\mathscr{B}$-measurable function s.t.

$$\int_{\mathbb{R}} f(x)dx = 1$$

Then

$$\mathbb{P}(H) = \int_H f(x)dx \qquad H \in \mathscr{B}$$

is a distribution. We say that f is the density of $\mathbb{P}$ with respect to Lebesgue measure. Therefore there is a natural way to define a distribution that is given a density.

A.1.1 Exponential distribution

Let $\lambda > 0$ and X be a r.v. with exponential distribution with parameter λ i.e. $X \sim exp(\lambda)$. Then X has the following density:

$$f(x) = \lambda e^{-\lambda t}\mathbb{I}_{]0,+\infty[} \qquad t \in \mathbb{R}$$

A.1.2 Gaussian distribution

Let $\mu \in \mathbb{R}$ and $\sigma > 0$ and X be a r.v. with Gaussian distribution with parameter μ, σ i.e. $X \sim N(\mu, \sigma^2)$. Then X has the following density:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu)^2}{2\sigma^2 t}} \qquad x \in \mathbb{R}, t > 0$$

A.2 Fourier transform

We introduce the notion of Fourier transform which play an important role defining each distribution uniquely.

Definition A.6. The *Fourier transform* of a function $f \in L^1(\mathbb{R})$ is defined as follows:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x)dx \tag{A.1}$$

Definition A.7. Let X a r.v. and P^X its distribution. The *characteristic function* of the r.v. X with values in $\mathbb{R}$ is the function $\varphi_X : \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$\varphi_X(\xi) = E[e^{i\xi X}] = \int_{\mathbb{R}} e^{i\xi y} P^X(dy)$$

Therefore φ_X is simply the Fourier transform of the distribution P^X of X.

A.2.1 Gaussian Fourier characteristic function

If $X \sim N(\mu, \sigma^2)$ then we have

$$\begin{aligned}
\varphi_X(\xi) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{i\xi y} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = \\
&= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{i\xi(\mu + x\sqrt{2\sigma^2})} e^{-x^2} dx = \\
&= \frac{e^{i\xi\mu}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{i\xi x\sqrt{2\sigma^2} - x^2} dx = \\
&= \frac{e^{i\xi\mu}}{\sqrt{\pi}} \sqrt{\pi} e^{-\frac{\xi^2\sigma^2}{2}} dx = \\
&= e^{i\xi\mu - \frac{\xi^2\sigma^2}{2}} dx
\end{aligned}$$

A.3 Conditional expectation

In financial applications the price of an asset is generally modeled by a r.v. X and the amount of information available is described by a σ-algebra $\mathscr{G}$. As a consequence it is natural to introduce the notion of conditional expectation of X given $\mathscr{G}$, usually denoted by $E[X|\mathscr{G}]$.

Definition A.8. Let X be a r.v. on $(\Omega, \mathscr{F}, \mathbb{P})$ and $B \in \mathscr{F}$ with $\mathbb{P}(B) > 0$. We define

$$E[X|B] = \frac{1}{\mathbb{P}(B)} \int_B X d\mathbb{P}$$

We denote by $\mathscr{G}$ the σ-algebra generated by B:

$$\sigma(B) = \{\varnothing, B, B^C, \Omega\}$$

The conditional expectation of X given $\mathscr{G}$ is defined by

$$E[X|\mathscr{G}](\omega) = \begin{cases} E[X|B] & if \qquad \omega \in B \\ E[X|B] & if \qquad \omega \in B^C \end{cases}$$

We remark that $E[X|\mathscr{G}]$ is a r.v.

It is easy to prove, as a consequence of the definition and costruction of the conditional expectation, that the following properties hold.

For every X, Y r.v. $\in L^1(\Omega, \mathscr{F}, \mathbb{P})$ and $a, b \in \mathbb{R}$, we have:

1. $E[X|\mathscr{G}]$ is $\mathscr{G}$-measurable
2. $\int_A E[X|\mathscr{G}]d\mathbb{P} = \int_A X d\mathbb{P}$ for every $A \in \mathscr{G}$
3. if X is $\mathscr{G}$-measurable, then $X = E[X|\mathscr{G}]$
4. if X and $\mathscr{G}$ are independent (i.e. $\sigma(X)$ and $\mathscr{G}$ are independent) then $E[X|\mathscr{G}] = E[X]$
5. $E[E[X|\mathscr{G}]] = E[X]$
6. if Y is $\mathscr{G}$-measurable and bounded, then $E[XY|\mathscr{G}] = YE[X|\mathscr{G}]$
7. if $\mathscr{H} \subseteq \mathscr{G}$, then $E[E[X|\mathscr{G}]|\mathscr{H}] = E[X|\mathscr{H}]$
8. LINEARITY: $E[\alpha X + \beta Y|\mathscr{G}] = \alpha E[X|\mathscr{G}] + \beta E[Y|\mathscr{G}]$
9. MONOTONICITY: if $X < Y$ then $E[X|\mathscr{G}] < E[Y|\mathscr{G}]$
10. JENSEN'S INEQUALITY: if φ is a convex function s.t. $\varphi(X) \in L^1(\Omega, F, \mathbb{P})$, then

$$E[\varphi(X)|\mathscr{G}] \geq \varphi(E[X|\mathscr{G}])$$

Appendix B

Stochastic process

Definition B.1. A s.p. is a family $(X_t)_{t\geq 0}$ of r.v. with values in $\mathbb{R}$ s.t. the map

$$X : Ix\Omega \longrightarrow \mathbb{R}, \qquad X(t,\omega) = X_t(\omega) \tag{B.1}$$

is a function of both time t and randomness ω. For each ω, the trajectory

$$X(\omega) : t \mapsto X_t(\omega)$$

defines a function of time, called the sample path of process.

Therefore a s.p. is a family $(X_t)_{t\geq 0}$ of r.v. indexed by time. The time parameter t may be either discrete or continuos but in this work we will consider continuos-time sthocastic processes. So a s.p. can be used to describe a random phenomenon that evolves in time: for istance, we can interpret a positive r.v. X_t as the price of a risky asset at time t. The interpretation of the index t as a time variable introduces a dynamic aspect which needs to be taken into account by properly defining the notion of information in the context of a sthocastic model. So we have to introduce the important concept of filtration which will allow us to define the important notions of past information and to classify processes and random times according to these properties.

Definition B.2. A filtration on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is an increasing family of σ-algebras $(\mathscr{F}_t)_{t\geq 0}$ s.t. $\forall t \geq s \geq 0$ we have $\mathscr{F}_s \subseteq \mathscr{F}_t$

$\mathscr{F}_t$ is then interpreted as the information known at time t, which increases with time. Infact, in a dynamic context, as time goes on, more information is progressively revealed

to the observer. Therefore, to accomodate this additional feature, a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is equipped with a filtration. It is called *filtered probability space*. From an intuitive point of view, the probability of occurrence of a random event will change with time as more information is revealed. An event $A \in \mathscr{F}_t$ in an event s.t. given the information $\mathscr{F}_t$ at time t the observer can decide whether A has occurred or not.

B.1 Brownian Motion

Definition B.3. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. A real *Brownian motion* is a s.p. $W = (W_t)_{t\geq 0}$ in $\mathbb{R}$ s.t.

- $W_0 = 0$ a.s.
- W is $\mathscr{F}$-adapted and continuos
- for $t > s \geq 0$ the r.v. $W_t - W_s$ has normal distribution i.e. $W_t - W_s \sim N_{0,t-s}$ and is independent of $\mathscr{F}_s$

Definition B.4. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. A s.p. M is a *martingale* if:

- $M_t \in \mathbb{L}^1(\Omega)\ \forall t \geq 0$
- $E[M_t|\mathscr{F}_s] = M_s$ for $s \leq t$

B.2 Equivalent Martingale Measure (EMM)

In a given stochastic model, a key result is the established connection between the economic concept of absence of arbitrage and the mathematical property of existence of a probability measure, the equivalent martingale measure (or risk-neutral measure), whose definition is given in the following.

Definition B.5. An *equivalent martingale measure* Q is a probability measure on the space $(\Omega, \mathscr{F})$ s.t.

- Q_0 and Q are equivalent measures i.e.

$$Q_0(A) = 0 \Longleftrightarrow Q(A) = 0 \qquad for \quad every \quad A \in \mathscr{F}$$

- the Radon-Nikodym derivative $\frac{dQ}{dQ_0}$ belongs to $L^2(\Omega, \mathscr{F}, Q_0)$
- the 'discounted asset price' process is a **F**-mg

Appendix C

Stochastic differential equations

Definition C.1. We consider $x_0 \in \mathbb{R}$ and two measurable functions

$$\mu = \mu(t,x) : [0,T]x\mathbb{R} \longrightarrow \mathbb{R}$$
$$\sigma = \sigma(t,x) : [0,T]x\mathbb{R} \longrightarrow \mathbb{R}$$

- μ is called the *drift coefficient*
- σ is called the *diffusion coefficient*

Let W a 1-dimensional B.m. on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ on which the usual hypothesis hold. A solution relative to W of the SDE with coefficients (x_0, μ, σ) is a $\mathscr{F}$-adapted continuos process $(X_t)_{t\in[0,T]}$ s.t.

- $\mu(t,x), \sigma(t,x) \in \mathbb{L}^2_{loc}(\Omega)$
- we have that

$$\begin{aligned} dX_t &= \mu(t,X_t)dt + \sigma(t,X_t)dW_t \\ X_0 &= x_0 \end{aligned} \tag{C.1}$$

C.1 Linear SDE

Definition C.2. We define *Linear Stochastic differential equations* a particular type of SDE in which the coefficients of (C.1) are linear functions of X_t ; it means they are the

ones of the form

$$\begin{aligned} dX_t &= (b + BX_t)dt + \sigma(t)dW_t \\ X_0 &= x_0 \end{aligned} \tag{C.2}$$

with $b, \sigma, B \in C(\mathbb{R}^+)$ costant coefficients.

Theorem C.1.1. *The solution of the SDE (C.2) with initial condition is of the form*

$$X_t = e^{tB}(x_0 + \int_0^t e^{-sB} b ds + \int_0^t e^{-sB}\sigma dW_s) \tag{C.3}$$

Proof.
We define $Y_t := x_0 + \int_0^t e^{-sB} b ds + \int_0^t e^{-sB}\sigma dW_s$.
We have to show that

$$dX_t = d(e^{tB}Y_t)$$

i.e.

$$(b + BX_t)dt + \sigma(t)dW_t = d(e^{tB}Y_t)$$

By applying the Ito's formula to the process $e^{tB}Y_t$ we obtain:

$$\begin{aligned} d(e^{tB}Y_t) &= Be^{tB}Y_t dt + e^{tB}dY_t = \\ &= Be^{tB}Y_t dt + e^{tB}(e^{-tB}bdt + e^{-tB}\sigma dW_t) = \\ &= BX_t dt + bdt + \sigma dW_t = \\ &= (b + BX_t)dt + \sigma(t)dW_t \end{aligned}$$

□

Remark 15. $X_t \sim N(m(t), C(t))$ with

$$\begin{aligned} m(t) &:= E[X_t] = \\ &= E[e^{tB}(x_0 + \int_0^t e^{-sB} b ds + \int_0^t e^{-sB}\sigma dW_s)] = \\ &= E[e^{tB}x_0 + \int_0^t e^{(t-s)B} b ds] \end{aligned} \tag{C.4}$$

$$\begin{aligned} C(t) &:= E[(X_t - m(t))(X_t - m(t))^*] = \\ &= e^{tB} E[(\int_0^t e^{-sB}\sigma dW_s)(\int_0^t e^{-sB}\sigma dW_s)^*]e^{tB^*} = \\ &= e^{tB}\int_0^t (e^{-sB}\sigma)(e^{-sB}\sigma)^* dW_s e^{tB^*} = \\ &= \int_0^t (e^{(t-s)B}\sigma)(e^{(t-s)B}\sigma)^* dW_s \end{aligned} \tag{C.5}$$

C.2 Fundamental solution and transition density

The following theorem establishes an important link between SDEs and PDEs. Our goal is to prove a representation formula for the classical solution u of the Cauchy problem

$$\begin{cases} Au - au + \partial_t u = 0 \qquad in \quad S_T := [0,T[x\mathbb{R} \\ u(T,x) = \varphi(x) \qquad x \in \mathbb{R} \end{cases}$$

where a, φ are given functions and

$$A = \frac{1}{2}\sigma^2(t,x)\partial_{xx} + \mu(t,x)\partial_x$$

is the characteristic operator of the SDE

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t \tag{C.6}$$

We will show that the solution can be written in terms of expectation of $u(t,X_t)$.

Theorem C.2.1 (Feynman-Kač formula)**.** *Let $u \in C^2(S_T) \cap C(\overline{S_T})$ be a solution of the Cauchy problem (C.2). Assume that:*

- *the coefficients $\mu(t,x), \sigma(t,x)$ are measurable and have at most linear growth in x*
- *for every $(t,x) \in S_T$ there exist a solution $X^{t,x}$ of the SDE (C.6) relative to a 1-dimensional B.m. W on the space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$*
- *σ is bounded and there exist two positive costants M and α s.t.*

$$|u(t,x)| \leq Me^{\alpha x^2}$$

Then for every $(t,x) \in S_T$ we have the representation formula

$$u(t,x) = E[e^{-\int_t^T \gamma(s,X_s^{t,x})ds}\varphi(X_T^{x,t})] \tag{C.7}$$

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Acknowledgements

I would like to express my gratitude to Andrea Pascucci for his assistance and relentess supervision. Special thanks then go to my teachers that since I was child make me love maths. I am also grateful to my friends who shared with me these five important years. Above all I would like to express my sincere thanks to Mattia: without him I would never been here. Last, but not least, I wuold like to thank my parents for their patience, encouragement and their support. Thanks to Claudia e Maurizio for everything.
Thanks also to Unipol UGF Banca e Assicurazioni for the data.