Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
Corso di Laurea in Fisica

Dispersive Wave Solutions of the Klein-Gordon equation in Cosmology

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Correlatore:
Prof. Roberto Casadio

Presentata da:
Andrea Giusti

Sessione II
Anno Accademico 2012/2013

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Abstract

The Klein-Gordon equation describes a wide variety of physical phenomena such as in wave propagation in Continuum Mechanics and in the theoretical description of spinless particles in Relativistic Quantum Mechanics.
Recently, the dissipative form of this equation turned out to be a fundamental evolution equation in certain cosmological models, particularly in the so called k-Inflation models in the presence of tachyonic fields.
The purpose of this thesis consists in analysing the effects of the dissipative parameter on the dispersion of the wave solutions. Indeed, the resulting dispersion may be normal or anomalous depending on the relative magnitude of the dissipation parameter with respect to the pure dispersive parameter. Typical boundary value problems of cosmological interest are studied with illuminating plots of the corresponding fundamental solutions (Green's functions).

Sommario

L'equazione di Klein-Gordon descrive una ampia varietà di fenomeni fisici come la propagazione delle onde in Meccanica dei Continui ed il comportamento delle particelle spinless in Meccanica Quantistica Relativistica.
Recentemente, la forma dissipativa di questa equazione si è rivelata essere una legge di evoluzione fondamentale in alcuni modelli cosmologici, in particolare nell'ambito dei cosiddetti modelli di k-inflazione in presenza di campi tachionici.
L'obiettivo di questo lavoro consiste nell'analizzare gli effetti del parametro dissipativo sulla dispersione nelle soluzioni dell'equazione d'onda. Saranno inoltre studiati alcuni tipici problemi al controno di particolare interesse cosmologico per mezzo di grafici corrispondenti alle soluzioni fondamentali (Funzioni di Green).

Introduction

Cosmology is in many respects the physics of extremes. On the one hand it handles the largest length scales as it tries to describe the dynamics of the whole Universe. On these scales gravity is the dominant force and (general) relativistic effects become very important. Cosmology also deals with the longest time scales. It describes the evolution of the Universe during the last 13.7 billion years and tries to make predictions for the future of our Universe. Indeed, most cosmologists are working on events that happened in the very early universe with extreme temperatures and densities. These extreme conditions are also present and can be observed today in our Universe in, for example, supernovae, black holes, quasars and highly energetic cosmic rays.
When one try to describe the dynamics of the very early Universe, which is characterized by extremely high energies concentrated in a very tiny region of space, he must deal with both relativistic and quantum effects.
It is an experimental fact that particle physics processes dominated the very early eras of the Universe. A very important consequence of the latter consist in the fact that they leads to a very weird hypothesis: **repulsive gravitational effects** could have dominated the dynamics of the very early Universe. This is exactly the basic idea of the **Theory of the Inflationary Universe**, first proposed by Alan Guth in 1981.
In the early stages of inflationary theory there were hopes to describe the inflation through the Standard Model of particle physics. One of the various proposals consisted in assuming a close relation between the inflaton fields and the GUTs-transition. However, this idea never worked out in a convincing way and, consequently, inflaton fields lived their own life quite detached from the rest of Theoretical Particle Physics.
Theoretical physicists are now trying to describe this era in the context of Unified Theories, of which the most important candidates are the **Superstring Theory** and the **M-theory**.
In this thesis we investigate a $(1 + 1)$-Dimensional Model for Inhomogeneous Fluctuations in the Tachyon Cosmology, which represents a classical example of kinetically driven inflation. Precisely, In our model we assume a specific form

of an Inflaton Field as well as for the Potential Density; these assumptions are completely justified in terms of Superstring Theory. These hypothesis lead to an evolution equation for the time-dependent e inhomogeneous perturbation, in a homogeneous background, which is described by the Dissipative Klein-Gordon Equation. In particular, we find that the Dispersive Wave Solutions of this particular case undergoes an anomalous dispersion.

The **outline** of this thesis is the following. In Chapter 1 we introduce the basic mathematical tool which we will use in the second chapter. In particular, we try to give a formal definition for the concepts of Linear Dispersive Waves and Linear Dispersive Waves with Dissipation.
In Chapter 2 we analyse in very details the Klein-Gordon Equation with Dissipation. Particularly, we will start with studying the Dispersion Relations, then we will solve analytically the Signalling Problem and the Cauchy Problem for this equation using integral transform methods (Laplace and Fourier).
In Chapter 3 we introduce a few basic concepts of Inflationary Physics and String Cosmology. Precisely, we will focus on concept of k-Inflation.
In Chapter 4 we will finally describe in details the Tachyon Matter Cosmology and we will solve a Toy Model for small inhomogeneous perturbation of the tachyon field around time-dependent background solution.

Acknowledgements I would like to take this opportunity to thank the various individuals to whom I am indebted, not only for their help in preparing this thesis, but also for their support and guidance through-out my studies.
The particular choice of topic for this thesis proved to be very rewarding as it allowed me to explore many interrelated areas of Physics and Mathematics that are of great interest to me. Thus I would first like to extend my thanks to my supervisors, Prof. Francesco Mainardi and Prof. Roberto Casadio, for encouraging me to pursue this topic and for providing me with very friendly and insightful guidance when it was needed.
Furthermore, I would like to thank my fellow students for their support and their feedback.
Finally, I would like to thank those who have not directly been part of my academic life yet, but they have been of central importance in the rest of my life. First, and foremost, my parents, Antonella and Maurizio, for their unrelenting support and for teaching me the value of things. Also, to my sisters, Giulia and Giorgia, who have had to deal with my unending rants and other oddities. I would also thank some of my dearest friends, in particular Simone and Martina, for their help and support in preparing this thesis. I also offer my thanks and apologies to my girlfriend, Claudia, for putting up with weekends of me working on my thesis and with my endless, uninteresting updates on the latest traumatic turn of events. You are as kind as you are beautiful.

In conclusion, I would like to dedicate this thesis to my grandparents, Marcellino and Margherita, who are the best people I have ever met.

La matematica è caratterizzata da un lato da una grande libertà,
dall'altro dall'intuizione che il mondo è fatto di cose visibili e invisibili;
essa ha forse una capacità, unica fra le altre scienze,
di passare dall'osservazione delle cose visibili
all'immaginazione delle cose invisibili.
Questo forse è il segreto della forza della matematica.
Ennio De Giorgi

Physics is essentially an intuitive and concrete science.
Mathematics is only a means for expressing the laws that govern phenomena.
Albert Einstein

Per quanto mi riguarda,
mi sembra di essere un ragazzo
che gioca sulla spiaggia e trova
di tanto in tanto
una pietra o una conchiglia
più belle del solito,
mentre il grande oceano della verità
resta sconosciuto davanti a me.
Sir Isaac Newton

Contents

Part I

Linear Dispersive Waves with Dissipation

Chapter 1

Introduction to Partial Differential Equation

A Partial Differential Equation (PDE) is an equation involving an unknown function, its partial derivatives, and the independent variables. PDE's are clearly ubiquitous in science; the unknown function might represent such quantities as temperature, electrostatic potential, concentration of a material, velocity of a fluid, displacement of an elastic material, acoustic pressure, etc. These quantities may depend on many variables, and one would like to find how the unknown quantity depends on these variables.
Typically a PDE can be derived from physical laws and/or modelling assumptions that specify the relationship between the unknown quantity and the variables on which it depends.

So, often we are given a model in the form of a PDE which embodies physical laws and modeling assumptions, and we want to find a solution and study its properties. Some classical examples of PDE in Physics are:

The Heat Equation	$\frac{\partial u}{\partial t} = \Delta u\,, \quad u = u(x,y,z,t)$
Navier-Stokes Equation	$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{f} + \nu\Delta\mathbf{v}$
The Wave Equation	$\frac{\partial^2 u}{\partial t^2} = c^2\Delta u\,, \quad u = u(x,y,z,t)$

In all the previous example we distinguished the role of time from space variables. If we aim to clarifies the formal definition of a PDE for a scalar function u, it is convenient to indicate with $\mathbf{x} \in \mathbb{R}^n$ the vector of independent variables (which may also include the time variable) and with $D^k u$ the collection of k^{th}-order partial derivatives of u.

Then, we have

Definition 1 (PDE of order α)**.** For a given domain $\Omega \subset \mathbb{R}^n$, a function $u : \Omega \to \mathbb{R}$ and a real function $F \in \mathcal{C}^1$ such that

$$\nabla_{q_k} F(q_\alpha, \ldots, q_1, u, \mathbf{x}) \neq 0$$

the equation

$$F(D^\alpha u, \ldots, Du, u, \mathbf{x}) = 0 \tag{1.1}$$

with $\alpha = (\alpha_1, \ldots \alpha_n)$ multi-index such that:

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n$$

is called PDE of order α.

Remark (Notation)*.* We will use

$$u_{x_1}, u_{x_2}, u_{x_2 x_2}, u_{x_1 x_1}, u_{x_1 x_2}, \ldots$$

to denote the partial derivatives of u with respect to independent variables $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.
For much of this paper, we will consider equations involving a variable representing time, which we denote by t. In this case, we will often distinguish the temporal domain from the domain for the other variables, and we generally will use $\Omega \subset \mathbb{R}^n$ to refer to the domain for the other variables only. For instance, if $u = u(\mathbf{x}, t)$, the domain for the function u is a subset of $\mathbb{R}^{n+1}$, perhaps $\Omega \times \mathbb{R}$ or $\Omega \times [0, +\infty)$. In many physical applications, the other variables represent spatial coordinates. $\triangleleft$

We say that a function u solves a PDE if the relevant partial derivatives exist and if the equation holds at every point in the domain when you plug in u and its partial derivatives. This definition of a solution is often called a *classical solution*, however, not every PDE has a solution in this sense, and it is sometimes useful to define a notion of *weak solution*.
The independent variables vary in a domain Ω, which is an open set that may or may not be bounded. A PDE will often be accompanied by *boundary conditions* or *initial conditions* that prescribe the behaviour of the unknown function u at the boundary $\partial\Omega$ of the domain under consideration. There are many boundary conditions, and the type of condition used in an application will depend on modelling assumptions.
Prescribing the value of the solution $u = g$, $\forall \mathbf{x} \in \partial\Omega$, for a certain function g, is called the *Dirichlet boundary condition*.

If $g = 0$, we say that the boundary condition is *homogeneous*.
There are many other types of boundary conditions, depending on the equation and on the application. Some boundary conditions involve derivatives of the solution. For example, instead of $u = g(x, y)$ on the boundary, we might impose $\nu \cdot \nabla u = g(x, y)$, $\forall (x, y) \in \partial\Omega$, for a certain function g and a certain exterior unit normal vector $\nu = \nu(x, y)$. This is called the *Neumann boundary condition*.
There are also applications for which it is interesting to consider terminal conditions imposed at a future time; these are called *terminal value problems*.

Solving a certain *Boundary Value Problem* means finding a function that satisfies both the PDE and the boundary conditions. However, in many cases we are not able to find an explicit representation for the solution, which means that "solving" the problem sometimes means showing that a solution exists or approximating it numerically.

Finally, we have some useful definitions and notations.

Definition 2. A PDE is said to be *linear* if it has the form

$$\sum_{|\alpha| \leq k} A_\alpha(\mathbf{x}) D^\alpha u = f(\mathbf{x}) \tag{1.2}$$

A *semilinear* PDE has the form

$$\sum_{|\alpha| = k} A_\alpha(\mathbf{x}) D^\alpha u + A_0(D^{k-1}u, \dots, Du, u, \mathbf{x}) = f(\mathbf{x}) \tag{1.3}$$

A *quasilinear* PDE has the form

$$\sum_{|\alpha| = k} A_\alpha(D^{k-1}u, \dots, Du, u, \mathbf{x}) D^\alpha u + A_0(D^{k-1}u, \dots, Du, u, \mathbf{x}) = f(\mathbf{x}) \tag{1.4}$$

An equation that depends in a nonlinear way on the highest order derivatives is called fully *nonlinear* PDE.

If an equation is linear and $f \equiv 0$, the equation is called *homogeneous*, otherwise it is called *inhomogeneous*.

1.1 Green's Functions

Consider a set $\Omega \subset \mathbb{R}^n$ and two function $u, f : \Omega \to \mathbb{R}$ for simplicity. Suppose that we want to solve a *linear*, *inhomogeneous* equation of the form

$$\mathcal{L}u(\mathbf{x}) = f(\mathbf{x}) + HBC \tag{1.5}$$

with HBC stands for *homogeneous boundary conditions* and $\mathcal{L}$ is a linear selfadjoint operator on $L^2(\Omega)$. We will assume that equation (1.5) admits a unique solution for every f. This is not the case for all linear operators; for those that do not admit unique solutions, there is an extended notion of *generalized Green's functions* which we do not pursue here.

Definition 3 (Green's Function)**.** If equation (1.5) has a solution of the form

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}\,, \qquad \forall f \tag{1.6}$$

then the function $G(\mathbf{x}, \mathbf{y})$ is called Green's function for the problem (1.5).

The physical interpretation for the previous definition is straightforward. We can think of $u(\mathbf{x})$ as the response at the point $\mathbf{x}$ to the influence given by $f(\mathbf{x})$. For example, if the problem involve the elasticity, u might be the displacement cased by an external force f.

Theorem 1. *If there exists a function G such that G is a Green's function for the problem (1.5) then this function is also unique.*

Proof. Suppose that there is another function G^* such that

$$u(\mathbf{x}) = \int_\Omega G^*(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}\,, \qquad \forall f$$

then

$$u(\mathbf{x}) = \int_\Omega G^*(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y} = \int_\Omega G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}$$
$$\Longrightarrow \quad \int_\Omega \left[G(\mathbf{x}, \mathbf{y}) - G^*(\mathbf{x}, \mathbf{y})\right] f(\mathbf{y}) d\mathbf{y} = 0\,, \qquad \forall f$$

Thus we conclude that $G = G^*$. So the Green's function is unique. □

Relationship to the Dirac δ generalized function

Part of the problem with the definition (1.6) is that it do not tell us how to construct G.
Consider the function f as a point source at $\mathbf{x}_0 \in \Omega$, i.e. $f(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0)$. Plugging into (1.6) we learn that the solution to

$$\mathcal{L}u(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0) + HBC \tag{1.7}$$

is just

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{y})\delta(\mathbf{y} - \mathbf{x}_0)d\mathbf{y} = G(\mathbf{x}, \mathbf{x}_0) \tag{1.8}$$

In other words, we find that the Green's function $G(\mathbf{x}, \mathbf{y})$ formally satisfies

$$\mathcal{L}_\mathbf{x} G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \tag{1.9}$$

where the subscript means that the linear operator acts only on $\mathbf{x}$. This equation physically says that $G(\mathbf{x}, \mathbf{y})$ is the *influence felt at* $\mathbf{x}$ *is due to a source at* $\mathbf{y}$.

The equation (1.9) is clearly a more useful way of defining G since we can, in many cases, solve this *almost homogeneous equation*, either by direct integration or using Fourier techniques.

Remark. In particular, the equation (1.9) can be rewritten as two conditions:

$$\begin{cases} \mathcal{L}_\mathbf{x} G(\mathbf{x}, \mathbf{y}) = 0\,, \quad \forall \mathbf{x} \in \Omega \,:\, \mathbf{x} \neq \mathbf{y} \\ \displaystyle\int_{B(\mathbf{y},r)} \mathcal{L}_\mathbf{x} G(\mathbf{x}, \mathbf{y}) d\mathbf{x} = 1\,, \quad \forall B(\mathbf{y}, r),\ r > 0 \end{cases} \tag{1.10}$$

where $B(\mathbf{y}, r)$ stand for a ball centred at $\mathbf{y}$ with radius r.
In addition to the conditions (1.10), G must also satisfy the homogeneous boundary conditions as the solution u does in the original problem. $\triangleleft$

Example 1 (One dimensional Poisson equation)**.** Suppose $u, f : \mathbb{R} \to \mathbb{R}$ solve the ordinary differential equation

$$\begin{cases} u_{xx} = f, \\ u(0) = u(L) = 0. \end{cases} \tag{1.11}$$

Obviously, the corresponding Green's function will solve

$$\begin{cases} G_{xx}(x, x_0) = 0, \quad x \neq x_0; \\ G(0, x_0) = G(L, x_0) = 0; \\ \displaystyle\int_{x_0 - r}^{x_0 + r} G_{xx}(x, x_0) dx = 1\,, \quad \forall r > 0 : [x_0 - r, x_0 + r] \subset [0, L] \end{cases} \tag{1.12}$$

One can easily show that G is continuous but that it has a jump in its derivative at $x = x_0$:

$$\lim_{x \to x_0^+} G_x(x, x_0) - \lim_{x \to x_0^-} G_x(x, x_0) = 1$$

Considering the first two equations of (1.12) one can easily conclude that:

$$G(x, x_0) = \begin{cases} c_1 x \,, & x < x_0, \\ c_2(x - L) \,, & x > x_0, \end{cases} \tag{1.13}$$

Imposing continuity at $x = x_0$ and the jump condition gives

$$c_1 x_0 = c_2(x_0 - L) \,, \qquad c_2 - c_1 = 1 \tag{1.14}$$

so that

$$\begin{aligned} u(x) &= \int_0^L G(x, x_0) f(x_0) dx_0 \\ &= \frac{1}{L} \left(\int_0^x x_0(x - L) f(x_0) dx_0 + \int_x^L x(x_0 - L) f(x_0) dx_0 \right) \end{aligned} \tag{1.15}$$

1.2 Linear Dispersive Waves

Many problems in Physics can be described in terms of Nonlinear Hyperbolic PDE. However, the possibility that this kind of equations can develop singularities, i.e. *shock waves*, it is not compatible with the physical meaning associated with the concept of waves amplitude. Nevertheless, it is reasonable to assume that the wave front can be spatially extended on a finite region. We can also suppose that on this region the wave can undergo some *dissipative* and *dispersive effects* which tend to contrast the formation of a shock waves.

Generalities

Differently form hyperbolic waves, which are clearly well defined, it is extremely difficult to find a formal definition for the *dispersive waves.* We can try to define this kind of waves as follows.

Definition 4 (Dispersive Waves)**.** Consider the generic *one dimensional* Linear PDE:

$$\mathcal{D}\varphi = 0\,, \qquad \mathcal{D} = \mathcal{D}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) \tag{1.16}$$

where $\mathcal{D}$ is a formal polynomial with real constant coefficients and $\varphi = \varphi(x,t)$ a generic real function.
The equation (1.16) is called *dispersive* if:

1. It admits solutions of the forms:

$$\begin{aligned} &\varphi(x,t) = Ae^{i\vartheta(x,t)}\,, \\ &\vartheta(x,t) = \kappa x - \omega t\,,\ A = const. \end{aligned} \tag{1.17}$$

   then κ, the wave number, and ω, the angular frequency, are roots of the implicit equation:

$$\mathcal{D}(-i\omega, i\kappa) = 0 \tag{1.18}$$

   defining locally a *dispersion relation* $\omega = \omega(\kappa)$.
   For this equation one can choose a different determinations (also called branches); so we usually define these different determination *modes* (of oscillation).

2. The dispersion relation is *real valued*, which formally means $\omega(\kappa) \in \mathbb{R}$, and
$$\omega''(\kappa) \neq 0\,, \quad \text{almost everywhere} \tag{1.19}$$
The function $\omega''(\kappa)$ is then called *dispersion.*

Consider $n \in \mathbb{Z}$, for $\vartheta(x,t) = 2n\pi$ we have that $Re\varphi$ is maximum, instead, for $\vartheta(x,t) = (2n+1)\pi$ then $Re\varphi$ is minimum. In both cases, the equations describes linear manifolds in spacetime (straight lines in $1+1$ dimensions), which evolves with velocity

$$v_p(\kappa) = \frac{\omega(\kappa)}{\kappa}\,, \qquad \text{(Phase Velocity)} \tag{1.20}$$

in the direction specified by the wave number.
Moreover, we have that:

$$\kappa = \vartheta_x\,, \qquad \omega = -\vartheta_t$$

And then one can easily deduce the following statement

$$\kappa_t + \omega_x = 0 \tag{1.21}$$

that clearly follows immediately form the Schwarz's Lemma (concerning the multiple derivation).
The latter equation represent the conservation of the number of maxima of $\vartheta(x,t)$.

Example 2. Consider the Schrödinger equation

$$i\varphi_t + \gamma\varphi_{xx} = 0\,, \qquad \varphi \in \mathbb{C} \tag{1.22}$$

If we propose a solution of this equation of the form:

$$\varphi(x,t) = A\exp\left[i(\kappa x - \omega t)\right]\,, \qquad A = const.$$

we obtain that (ω, κ) must undergo the following dispersion relation:

$$\omega - \gamma\kappa^2 = 0$$

Thus, the Schrödinger equation, if $\gamma \in \mathbb{R} \setminus \{0\}$, is a dispersive wave equation with dispersion $\omega''(\kappa) = 2\gamma$.

Wave Packets

From the linearity of the equation (1.16), the general solution of the latter can be cast as a superposition of monochromatic plane waves

$$\varphi(x,t) = \int d\kappa\, A(\kappa) \exp\left[i(\kappa x - \omega(\kappa)t)\right] \tag{1.23}$$

where $A(\kappa)$ is an arbitrary function of the wave number. A solution of this form is called *wave packet*.
As we know from the previous section, the quantity $\omega''(\kappa)$ is non vanishing, then the phase velocity of any wave composing the wave packet depends on the wave number κ; $v_p = v_p(\kappa)$. Consequently, the *wave packet spreads out* as t flows.

Although the wave packet tends to spread out with time, the wave packet still carries information that can be detected for $t \gg 1$, in an appropriate time-scale. Given that, in such time the wave packet (more properly, the single waves composing the packet) will have been propagated also in space then, from the condition $t \gg 1$, clearly it follows that $x \gg 1$ in an appropriate space-scale.
Then, it is convenient to study the behaviour of the wave packet in the limit such that

$$t \to \infty,\ x \to \infty, \qquad \frac{x}{t} \to O(1)$$

This case is realized for an observer which, in some way, travels along the wave, as we will clarifier later.
Let us therefore study the wave packet in this limit.
We can recast the equation (1.23) as follows:

$$\begin{aligned} \varphi(x,t) &= \int d\kappa\, A(\kappa) \exp\left(-it\, \chi(\kappa, x/t)\right) \\ \chi(\kappa, x/t) &= -\kappa\, \frac{x}{t} + \omega(\kappa) \end{aligned} \tag{1.24}$$

which clearly shows that, per unit of κ, the exponential oscillate rapidly.
Using the *saddle point approximation* one can prove that, in this limit, we can obtain the following asymptotic behaviour of the wave packet:

$$\varphi(x,t) \sim A(x,t) \exp\left[i\Theta(x,t)\right] \tag{1.25}$$

with

$$A(x,t) = A_0 \sqrt{\frac{2\pi}{|\omega''(\kappa_0)|t}}, \quad \text{and} \quad \Theta(x,t) = \kappa_0 x - \omega(\kappa_0)t \tag{1.26}$$

where κ_0 is such that $\chi'(\kappa_0, x/t) \equiv \partial_\kappa \chi(\kappa_0, x/t) = 0$.
The asymptotic behaviour of the wave packet (1.25) is then called *non uniform wave packet*, because the distance from two successive wave crests is no more constant as well as the temporal separation.
The degree of dis-uniformity is shown by the derivations:

$$\frac{1}{\kappa_0}\frac{\partial \kappa_0}{\partial x} = O\left(\frac{1}{x}\right), \qquad \frac{1}{\kappa_0}\frac{\partial \kappa_0}{\partial t} = O\left(\frac{1}{t}\right)$$

The wave form which is obtained for a wave packet then consists in a *carrier wave* modulated by an *envelope wave*.

Phase Velocity and Group Velocity

Consider the condition of *stationary phase* (κ_0 such that $\chi'(\kappa_0, x/t) = 0$) and omits, for simplicity, the index 0. Thus we have:

$$\begin{aligned} \Theta_x &= \kappa + [x - \omega'(\kappa)t]\kappa_x = \kappa(x,t) \\ \Theta_t &= \omega(\kappa) - [x - \omega'(\kappa)t]\kappa_t = \omega(\kappa; x, t) \end{aligned} \tag{1.27}$$

where we used the fact that $\kappa_0 = \kappa_0(x/t)$.
Then, we can clearly conclude:

$$\kappa_t + \omega_x = 0 \tag{1.28}$$

as we have already seen.
Moreover,

$$\kappa = \kappa(x/t) \quad \Longrightarrow \quad \partial_x \omega(\kappa) = \omega'(\kappa)\frac{\partial \kappa}{\partial x} = \omega'(\kappa)\,\kappa_x$$

and then, from the equation (1.28), we can easily conclude that

$$\kappa_t + \omega'(\kappa)\,\kappa_x = 0 \tag{1.29}$$

which clearly shows that the wave number propagates with a velocity $\omega'(\kappa)$ following an hyperbolic law of propagation.
The wave crests does not travel with the same speed any more, however, in places (x,t) such that $\kappa = \kappa(x/t)$ then we have

$$x - \omega'(\kappa)\,t = 0 \tag{1.30}$$

An observer who travels with speed (for a certain κ)

$$v_g = \frac{x}{t} = \omega'(\kappa) \equiv \frac{\partial \omega}{\partial \kappa} \tag{1.31}$$

can only see a certain value of the wave number (fixed), and then the same frequency, but he do not see always the same amplitude for the wave crests.
The velocity

$$v_g = \left.\frac{\partial \omega}{\partial \kappa}\right|_{\kappa_0} \tag{1.32}$$

is then called *group velocity*. We can finally observe that if an energy density is associated with the magnitude of a dispersive wave, it is clear that the transport of energy occurs with the group velocity.

1.3 Linear Dispersive Waves with Dissipation

We note that the dispersion law

$$\mathcal{D}(\omega, \kappa) = 0$$

is an implicit relationship that can generally be satisfied by complex values of κ and ω.
Let us assume that it can be solved explicitly with respect to a real variable (κ or ω) by means of complex valued branches

$$\begin{aligned} &\overline{\omega}_l(\kappa)\,, \quad \kappa \in \mathbb{R}\,, \quad l \in \mathbb{N} \\ &\overline{\kappa}_m(\omega)\,, \quad \omega \in \mathbb{R}\,, \quad m \in \mathbb{N} \end{aligned} \tag{1.33}$$

where the overlines expresses the possible complex nature of the dependent variables and l, m are the *mode indices*.
These branches provide the so-called *Normal Mode Solutions*:

$$\begin{aligned} \varphi_l(x,t;\kappa) &= Re\left\{A_l(\kappa)\exp\left[+i(\kappa x - \overline{\omega}_l(\kappa)t)\right]\right\} \\ \varphi_m(x,t;\omega) &= Re\left\{A_m(\omega)\exp\left[-i(\omega t - \overline{\kappa}_m(\omega)x)\right]\right\} \end{aligned} \tag{1.34}$$

Henceforth we will denote a normal mode simply by $\varphi_l(\kappa)$ or $\varphi_m(\omega)$, so dropping the dependence on x, t; furthermore we will use the notation $\overline{\varphi}_l(\kappa)$ or $\overline{\varphi}^c_m(\omega)$ to denote the *complex modes*, i.e. the complex solutions whose real part provides $\varphi_l(\kappa)$ or $\varphi_m(\omega)$, respectively.

One can easily note that the normal mode solutions represent *pseudo-monochromatic waves* since generally they are not sinusoidal in both space and time. Omitting the mode index, the following straightforward considerations on solutions (1.34) are in order.

1. The first solution of (1.34) is sinusoidal in space with *wavelength* $\lambda = 2\pi/\kappa$, but it is not necessarily sinusoidal in time, since:

$$\overline{\omega}(\kappa) = \omega_r(\kappa) + i\omega_i(\kappa) \tag{1.35}$$

then, only if $\overline{\omega} \in \mathbb{R}$ the solution effectively represents a monochromatic wave both in space and time. In general, this wave exhibits a *pseudo-period* $T = 2\pi/\omega_r$ and it propagates with speed (the *phase velocity*)

$$v_p(\kappa) := \frac{\omega_r(\kappa)}{\kappa} \tag{1.36}$$

exhibiting, if $\omega_i(\kappa) \leq 0$, an *exponential time-decay in amplitude* provided by the *time-damping factor*:

$$\gamma(\kappa) := -\omega_i(\kappa) \geq 0 \tag{1.37}$$

The last three equations allow us to write a given normal mode as:

$$\varphi(\kappa) = \exp(-\gamma(\kappa)t)\, Re\,\{A(\kappa)\exp\left[+i\kappa(x - v_p(\kappa)t)\right]\} \tag{1.38}$$

2. The second solution of (1.34) is sinusoidal in time with period $T = 2\pi/\omega$, but it is not necessarily sinusoidal in space, since $\overline{\kappa}$ may be complex, say

$$\overline{\kappa}(\omega) = \kappa_r(\omega) + i\kappa_i(\omega) \tag{1.39}$$

then, only if $\overline{\kappa} \in \mathbb{R}$ the solution effectively represents a monochromatic wave both in space and time. In general, this wave exhibits a *pseudo-wavelength* $\lambda = 2\pi/\kappa_r$ and it propagates with speed (the *phase velocity*)

$$v_p(\omega) := \frac{\omega}{\kappa_r(\omega)} \tag{1.40}$$

exhibiting, if $\kappa_i(\omega) \geq 0$, an *exponential space-decay in amplitude* provided by the *space-damping factor*:

$$\delta(\omega) := \kappa_i(\omega) \geq 0 \tag{1.41}$$

The last three equations allow us to write a given normal mode as:

$$\varphi(\omega) = \exp(-\delta(\omega)x)\, Re\,\{A(\omega)\exp\left[-i\omega(t - x/v_p(\omega))\right]\} \tag{1.42}$$

The (real) solution of a given problem, of type (1) and (2), is assumed to be uniquely determined with complex normal modes by a suitable *superposition* *of Fourier integrals.* Hence, we can write:

$$
\begin{aligned}
\varphi^{(1)}(x,t) &= \sum_l \int_{C_l} \hat{\alpha}_l(\kappa)\, A_l(\kappa) \exp\left[+i(\kappa x - \overline{\omega}_l(\kappa) t)\right] d\kappa = \\
&= \sum_l \int_{C_l} \hat{\alpha}_l(\kappa)\, \overline{\varphi}_l(\kappa) d\kappa \\
\varphi^{(2)}(x,t) &= \sum_m \int_{C_m} \hat{\beta}_m(\omega)\, A_m(\omega) \exp\left[-i(\omega t - \overline{\kappa}_m(\omega) x)\right] d\omega = \\
&= \sum_m \int_{C_m} \hat{\beta}_m(\omega)\, \overline{\varphi}_m(\omega) d\omega
\end{aligned}
\tag{1.43}
$$

where the path of integration C denotes either the real axis $\mathbb{R}$ or, when singular points occur on it, a suitable parallel line (in the complex plane) which ensure the convergence and $\hat{\alpha}_l(\kappa)$, $\hat{\beta}_m(\omega)$ are complex functions to be determined to satisfy the initial or boundary conditions.

Chapter 2

Klein-Gordon Equation with Dissipation

Examples of linear hyperbolic systems, which are of physical interest for their dispersive properties and energy propagation, are provided by the following 1-dimensional wave equation.

$$\varphi_{tt} + 2\alpha\varphi_t + \beta^2\varphi = c^2\varphi_{xx} \tag{2.1}$$

where $\varphi = \varphi(x,t)$ and $\alpha, \beta, c \in \mathbb{R}_0^+$.
In particular, α , β denote two non negative parameters which have dimension of a frequency $[T]^{-1}$ and c denotes the characteristic velocity. When $\alpha\beta = 0$ one can recognize some well known equations:

1. *D'Alembert Equation* when $\alpha = \beta = 0$;
2. *Klein-Gordon Equation* when $\beta > \alpha = 0$;
3. *Viscoelastic Maxwell Equation* when $0 = \beta < \alpha$;

Then, in modern Mathematical Physics, the equation (2.1) is known as the *Klein-Gordon Equation with Dissipation* (KGD) or *Dissipative Klein-Gordon Equation.*

The KGD is quite interesting since it provides instructive examples both of *normal dispersion* (usually met in the absence of dissipation) and *anomalous dispersion* (always present in viscoelastic waves). Parameters α and β actually play a fundamental role in the characterization of the dispersion properties of the solution and so does the non dimensional parameter $m = \alpha/\beta$ as well, as we will see in the next section.

2.1 Dispersion Relations

The dispersive feature of the KGD are independent from the physical phenomenon to which the equation refers since they are obtained formally from the associated dispersion relations. Nevertheless, we must distinguish two cases:

(i) Complex frequency $\overline{\omega} \in \mathbb{C}$ and real wave number $\kappa \in \mathbb{R}$;

(ii) Real frequency $\omega \in \mathbb{R}$ and complex wave number $\overline{\kappa} \in \mathbb{C}$.

Thus, we agree to write for plane waves:

$$\begin{aligned} \varphi(x,t) &= A\exp(-\gamma t)\cos(\kappa x - \omega t)\,, \quad \overline{\omega} = \omega - i\gamma \\ \varphi(x,t) &= A\exp(-\delta x)\cos(\kappa x - \omega t)\,, \quad \overline{\kappa} = \kappa + i\delta \end{aligned} \tag{2.2}$$

where $\gamma > 0$ and $\delta > 0$ are usually called time and space attenuation factors, respectively.
If we introduce the (2.2) in the KGD we clearly obtain:

$$\begin{aligned} -\overline{\omega}^2 + c^2\kappa^2 - 2i\alpha\overline{\omega} + \beta^2 = 0 \\ -\omega^2 + c^2\overline{\kappa}^2 - 2i\alpha\omega + \beta^2 = 0 \end{aligned} \tag{2.3}$$

which represent the *dispersion relations for the KGD.*
If we consider, for example, the first relation of the (2.3), recalling the definition $\overline{\omega} = \omega - i\gamma$, we immediately get:

$$-(\omega - i\gamma)^2 + c^2\kappa^2 - 2i\alpha(\omega - i\gamma) + \beta^2 = 0 \tag{2.4}$$

and then

$$(-\omega^2 + \gamma^2 + c^2\kappa^2 - 2\alpha\gamma + \beta^2) + i(2\omega\gamma - 2\omega\alpha) = 0 \tag{2.5}$$

thus

$$\begin{cases} -\omega^2 + \gamma^2 + c^2\kappa^2 - 2\alpha\gamma + \beta^2 = 0 \\ 2\omega\gamma - 2\omega\alpha = 0 \end{cases} \tag{2.6}$$

and we finally obtain the dispersion relation in terms of $\omega = \omega(\kappa)$:

$$\omega(\kappa) = c\kappa\sqrt{1 + \frac{\beta^2 - \alpha^2}{c^2\kappa^2}} \tag{2.7}$$

If we now recall the definitions of phase and group velocity, we obtain:

$$\begin{aligned} v_p(\kappa) &= \frac{\omega(\kappa)}{\kappa} = \sqrt{c^2 + \frac{\beta^2 - \alpha^2}{\kappa^2}} \\ v_g(\kappa) &= \frac{\partial\omega(\kappa)}{\partial\kappa} = \frac{c^2\kappa}{\sqrt{c^2\kappa^2 + \beta^2 - \alpha^2}} \end{aligned} \tag{2.8}$$

If we quote the plots of the (2.7) and (2.8) we can point out the following consideration:

1. For $0 \leq \alpha < \beta$: $v_p \geq v_g \geq 0$ - *normal dispersion*;
2. For $0 \leq \beta < \alpha$: $v_g \geq v_p \geq 0$ - *anomalous dispersion*;
3. For $0 < \alpha = \beta$: $v_g = v_p = c$ with constant attenuation $\gamma = \alpha$ and $\delta = \alpha/c$ - *no dispersion/distortionless.*

Now, if we consider the dissipative properties which arise from the KGD we must again take in account the equations (2.3). If we consider the two cases of dispersion ($(\overline{\omega}, \kappa)$ and $(\omega, \overline{\kappa})$), as shown in (2.2), we can immediately conclude that:

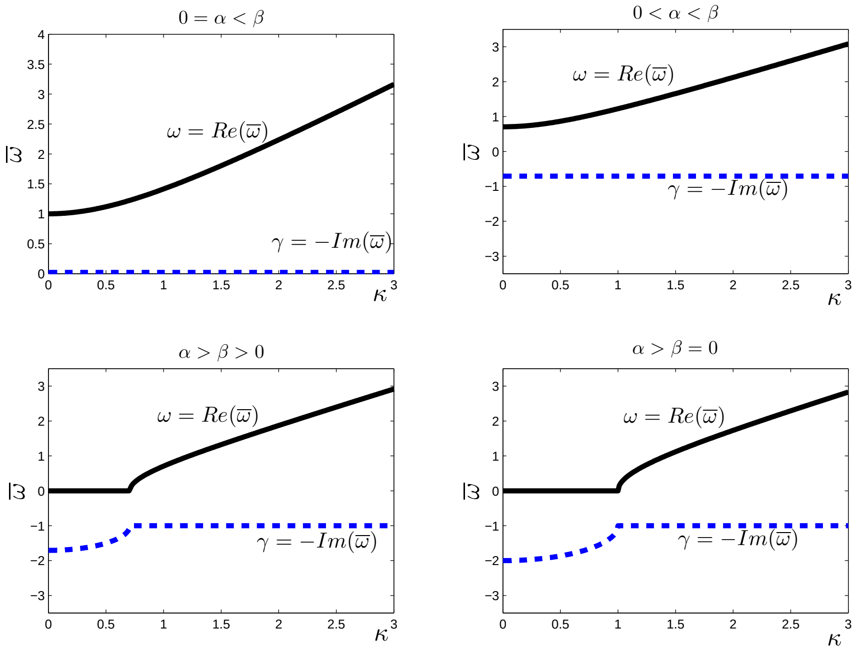
$$\begin{cases} \omega = \omega(\kappa) = Re(\overline{\omega}) \\ \gamma = \gamma(\kappa) = -Im(\overline{\omega}) \end{cases} \quad \text{and} \quad \begin{cases} \kappa = \kappa(\omega) = Re(\overline{\kappa}) \\ \delta = \delta(\omega) = Im(\overline{\kappa}) \end{cases} \tag{2.9}$$

Thus, considering the (2.3) we conclude that:

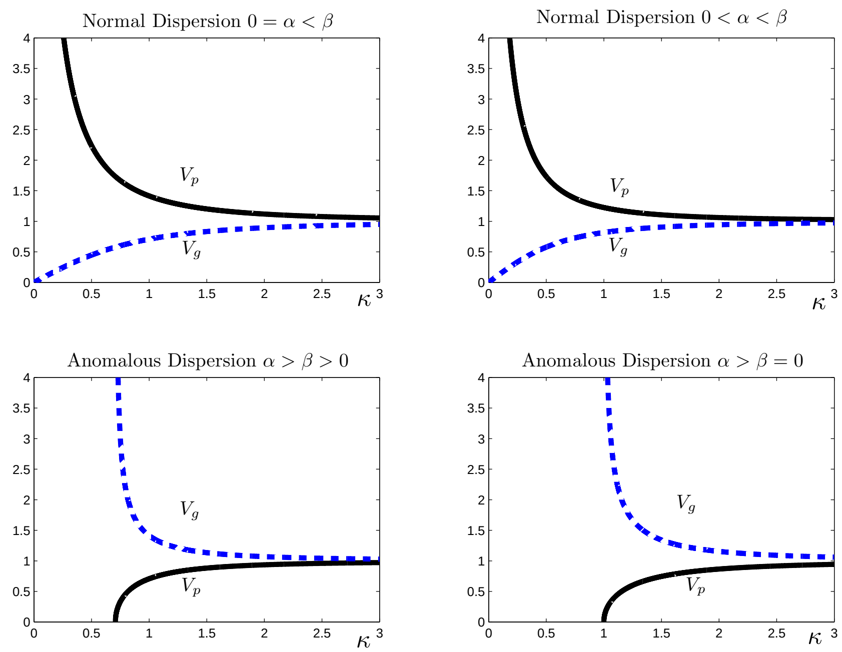
$$\begin{aligned} \overline{\omega} &= i\alpha \pm \sqrt{c^2\kappa^2 + (\beta^2 - \alpha^2)} \\ |\kappa(\omega)| &= \frac{1}{c\sqrt{2}} \sqrt{(\omega^2 - \beta^2) + \sqrt{(\omega^2 - \beta^2)^2 + (2\alpha\omega)^2}} \\ |\delta(\omega)| &= \frac{1}{c\sqrt{2}} \sqrt{\sqrt{(\omega^2 - \beta^2)^2 + (2\alpha\omega)^2} - (\omega^2 - \beta^2)} \end{aligned} \tag{2.10}$$

which gives us an expression for $Re(\overline{\omega})$, $Im(\overline{\omega})$, $Re(\overline{\kappa})$ and $Im(\overline{\kappa})$ depending basically on κ, ω and $sgn(\beta^2 - \alpha^2)$.
If we quote the plots of the (2.9) we can point out that in the κ-representation one can notice a *cut-off* in κ if $0 \leq \beta < \alpha$, while in the ω-representation the *cut-off* arise only in absence of dissipation, i.e. $\alpha = 0$.

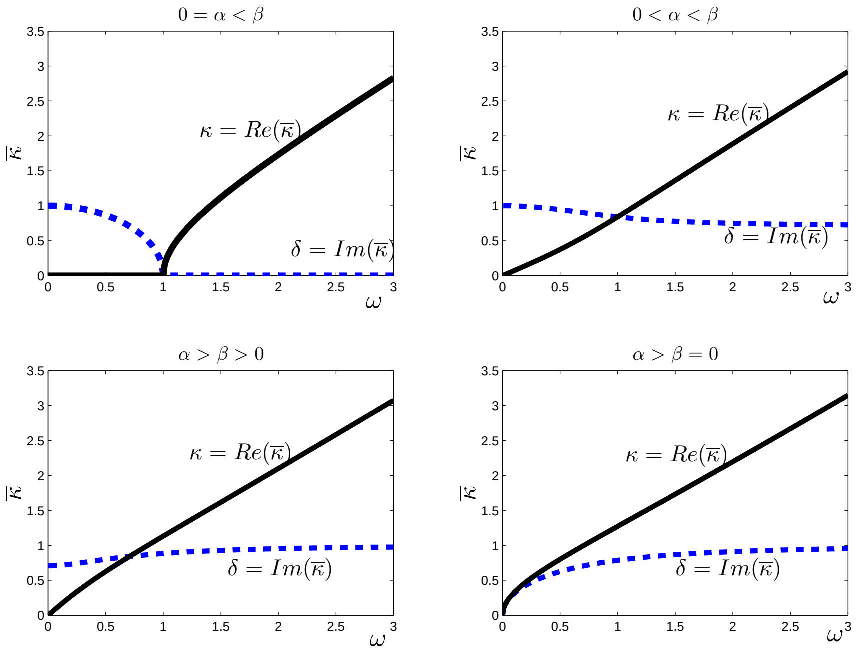
$\overline{\omega} = \overline{\omega}(\kappa)$ Representation



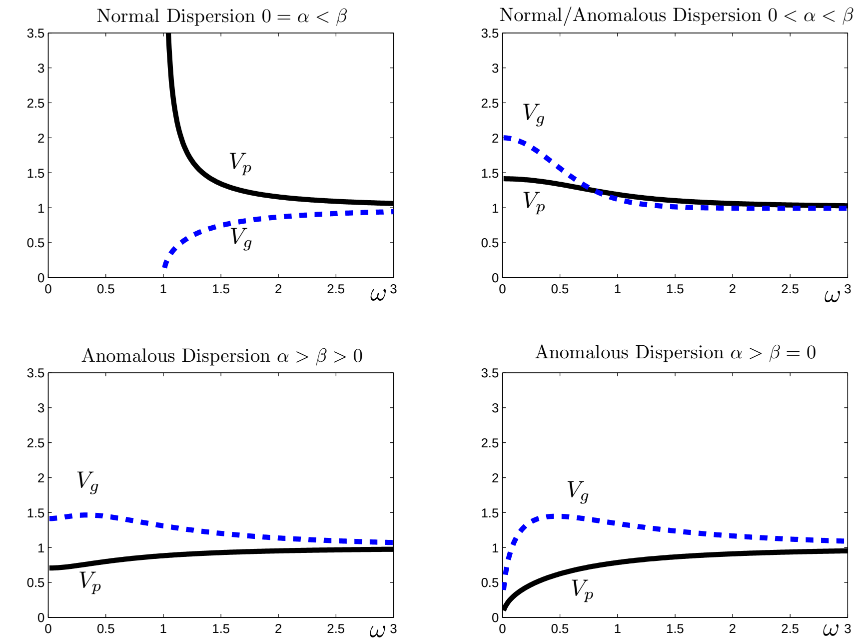
$V_p = V_p(\kappa)$ and $V_g = V_g(\kappa)$



$\overline{\kappa} = \overline{\kappa}(\omega)$ Representation



$V_p = V_p(\omega)$ and $V_g = V_g(\omega)$



2.2 The Signalling Problem

As usual for any PDE occurring in mathematical physics, we must specify some boundary conditions in order to guarantee the existence, the uniqueness and, in some lucky cases, the determination of a solution of physical interest to the problem.
The two basic problems for PDE's are usually referred to as the Signalling problem and the Cauchy problem. The former consist in a mixed initial and boundary value problem where the solution is to be found for $x > 0$ and $t > 0$, from the data prescribed at $t = 0$ and $x = 0$.

Considering the KGD equation (2.1), we propose the following conditions for the *Signalling Problem*:

$$\begin{cases} \varphi(x,0) = \varphi_t(x,0) = 0; \\ \varphi(0,t) = \Psi(t); \qquad\qquad \text{for } x \geq 0,\, t \geq 0 \\ \varphi(+\infty,t) = 0 \end{cases} \tag{2.11}$$

Then, if we consider the (2.1), the solution of this equation can be easily obtained using the *Laplace transform technique.*
Indeed, if we recall the definition of the Laplace Transform with respect to the time variable t:

$$\widetilde{f}(s) \equiv \mathcal{L}[f(t)] := \int_0^\infty e^{-st} f(t)dt \,, \qquad f(t) \div \widetilde{f}(s) \tag{2.12}$$

we can clearly conclude that

$$\varphi_t(x,t) \div s\widetilde{\varphi}(x,s) - \varphi(x,0)\,, \qquad \varphi_{tt}(x,t) \div s^2\widetilde{\varphi}(x,s) - s\varphi(x,0) - \varphi_t(x,0)$$

and then, applying these results to the KGD and considering the boundary/initial conditions furnished by the Signalling Problem (2.11), one can conclude that:

$$\varphi_{tt} + 2\alpha\varphi_t + \beta^2\varphi = c^2\varphi_{xx} \quad \stackrel{\mathcal{L}}{\Longrightarrow} \quad s^2\widetilde{\varphi} + 2\alpha s\widetilde{\varphi} + \beta^2\widetilde{\varphi} = c^2\widetilde{\varphi}_{xx} \tag{2.13}$$

thus

$$\widetilde{\varphi}_{xx} - \delta\,\widetilde{\varphi} = 0\,, \qquad \delta = \frac{s^2 + 2\alpha s + \beta^2}{c^2} \tag{2.14}$$

with the boundary/initial conditions:

$$\begin{cases} \widetilde{\varphi}(0,s) = \widetilde{\Psi}(s); \\ \widetilde{\varphi}(+\infty,s) = 0 \end{cases} \tag{2.15}$$

The equation (2.14) is an ordinary differential equation in the spatial coordinate x which is extremely easy to solve. Indeed, the general solution for the (2.14) is given by:

$$\widetilde{\varphi}(x,s) = c_1(s)\exp(\sqrt{\delta}\,x) + c_2(s)\exp(-\sqrt{\delta}\,x) \tag{2.16}$$

Now, considering the conditions (2.15) one can easily conclude that:

$$c_1(s) = 0\,, \qquad c_2(s) = \widetilde{\Psi}(s)$$

thus

$$\widetilde{\varphi}(x,s) = \widetilde{\Psi}(s)\,\exp(-\sqrt{\delta}\,x)\,, \qquad \delta = \frac{s^2 + 2\alpha s + \beta^2}{c^2} \tag{2.17}$$

or, more explicitly

$$\widetilde{\varphi}(x,s) = \widetilde{\Psi}(s)\,\exp\left(-\frac{x}{c}\,\sqrt{(s+\alpha)^2 + \beta^2 - \alpha^2}\right) \tag{2.18}$$

Now, if we define

$$\tau = t - \frac{x}{c}, \quad \xi = \frac{x}{c}, \quad \chi = \sqrt{|\beta^2 - \alpha^2|} \tag{2.19}$$

and

$$\begin{aligned} F^{+}(\xi,\tau) &:= \frac{J_1\left(\chi\sqrt{\tau(\tau+2\xi)}\right)}{\sqrt{\tau(\tau+2\xi)}}\,, \qquad \text{if } \beta^2 - \alpha^2 > 0 \\ F^{-}(\xi,\tau) &:= \frac{I_1\left(\chi\sqrt{\tau(\tau+2\xi)}\right)}{\sqrt{\tau(\tau+2\xi)}}\,, \qquad \text{if } \beta^2 - \alpha^2 < 0 \end{aligned} \tag{2.20}$$

then we have:

$$\begin{aligned} \widetilde{\varphi}(x,s) \div \varphi(x,t) &= \Psi \,*\, \mathcal{L}^{-1}\left[\exp\left(-\frac{x}{c}\,\sqrt{(s+\alpha)^2 \pm \chi^2}\right)\right] = \\ &= \Psi \,*\, \left[e^{-\alpha\xi}\delta(\tau) \mp \chi\xi e^{-\alpha\xi}e^{-\alpha\tau}\,F^{\pm}(\xi,\tau)\,\Theta(\tau)\right] \end{aligned} \tag{2.21}$$

thus

$$\boxed{\varphi(\xi,\tau) = \exp(-\alpha\xi)\left[\Psi(\tau) \mp \chi\xi\exp(-\alpha\tau)F^{\pm}(\xi,\tau) \,*\, \Psi(\tau)\right]} \tag{2.22}$$

where J_1 and I_1, which appear in $F^{\pm}$, denote the ordinary and the modified Bessel function of the first order, respectively.
Clearly, the first term of the equation (2.22) represents the input signal, propagating at velocity c and exponentially attenuated in space; the second one is

responsible for the distortion of the signal that depends on the position and on the time elapsed from the wave front. The amount of distortion can be measured by the parameter χ that indeed vanishes for $\alpha = \beta$, which represents the *distortion-less case*.

Remark. The *convolution* is to be intended *in Laplace sense*:

$$(f * g)(t) := \int_0^t f(t-\tau)g(\tau)d\tau$$

Moreover, $\Theta(\tau)$ is the *Heaviside step function*.

2.3 The Cauchy Problem

Let us now consider the *Cauchy problem* (or initial value problem), where the solution is to be found for $x \in \mathbb{R}$ and $t > 0$ from the data prescribed at $t = 0$. Consider the KGD problem; if we rename the function $\varphi = \varphi(x,t)$ as:

$$\varphi(x,t) = e^{-\alpha t} v(x,t) \tag{2.23}$$

the equation (2.1) immediately become:

$$v_{tt} - c^2 v_{xx} \pm \chi^2 v = 0 \tag{2.24}$$

which clearly is the one dimensional Klein-Gordon equation with $m^2 = \pm\chi^2$. If we want to study the initial value problem for the latter equation then we have to solve the following Cauchy problem:

$$\begin{cases} v_{tt} - c^2 v_{xx} \pm \chi^2 v = S_0 \delta(x)\delta(t) \\ v(x,0) = U_0 \delta(x) \\ v_t(x,0) = V_0 \delta(x) \end{cases} \qquad x \in \mathbb{R},\ t > 0 \tag{2.25}$$

where $v = v(x,t)$, $\alpha, \beta, c \in \mathbb{R}_0^+$ and $S_0, U_0, V_0 \in \mathbb{R}$ (with initial conditions suitable for a PDE of the second order in time).
If we now apply to (2.40) both Laplace (with respect to t) and Fourier Transforms (with respect to x) we get:

$$\widehat{\widetilde{v}}(\kappa, s) = \frac{S_0 + V_0 + U_0 s}{s^2 \pm \chi^2 + c^2\kappa^2} \tag{2.26}$$

thus we have that the initial condition $V_0 \neq 0$ provides a solution similar to that provided by the source $S_0 \neq 0$.
Therefore, the problem is divided into three cases.

Case 1 - $U_0 = V_0 = 0$
In this particular case we have that the equation (2.26) becomes:

$$\widehat{\widetilde{v}}(\kappa, s) \equiv \widehat{\widetilde{G_S}}(\kappa, s) = \frac{S_0}{s^2 \pm \chi^2 + c^2\kappa^2} \tag{2.27}$$

where $G_S = G_S(x,t)$ represents the Green's function for the *Source Problem*. If we reverse the Fourier Transform we immediately get:

$$\widetilde{G_S}(x,s) = \frac{S_0}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\kappa x}}{s^2 + c^2\kappa^2 \pm \chi^2}\, d\kappa = \frac{S_0}{\pi} \int_0^{+\infty} \frac{\cos(\kappa x)}{s^2 + c^2\kappa^2 \pm \chi^2}\, d\kappa \tag{2.28}$$

If we now reverse the Laplace Transform using the following result:

$$\sin \alpha t \div \frac{\alpha}{s^2 + \alpha^2}$$

we obtain:

$$G_S(x,t) = \frac{S_0}{\pi} \int_0^\infty \frac{\cos(\kappa x) \sin\left(t\sqrt{c^2\kappa^2 \pm \chi^2}\right)}{\sqrt{c^2\kappa^2 \pm \chi^2}} \, d\kappa \tag{2.29}$$

and, recalling the following results:

$$J_0\left(\alpha\sqrt{t^2 - A^2}\right)\Theta(t - A) \div \frac{\exp\left(-A\sqrt{s^2+\alpha^2}\right)}{\sqrt{s^2+\alpha^2}}$$
$$I_0\left(\alpha\sqrt{t^2 - A^2}\right)\Theta(t - A) \div \frac{\exp\left(-A\sqrt{s^2-\alpha^2}\right)}{\sqrt{s^2-\alpha^2}}$$

we finally conclude that:

$$\begin{aligned} G_S^+(x,t) &= \frac{S_0}{2c} J_0\left(\chi\sqrt{t^2 - (x/c)^2}\right)\Theta(ct - |x|)\,, \qquad \text{if } \beta^2 - \alpha^2 > 0 \\ G_S^-(x,t) &= \frac{S_0}{2c} I_0\left(\chi\sqrt{t^2 - (x/c)^2}\right)\Theta(ct - |x|)\,, \qquad \text{if } \beta^2 - \alpha^2 < 0 \end{aligned} \tag{2.30}$$

Case 2 - $S_0 = U_0 = 0$
Now we have that the equation (2.26) becomes:

$$\widehat{\widetilde{v}}(\kappa, s) \equiv \widehat{\widetilde{G_{2C}}}(\kappa, s) = \frac{V_0}{s^2 \pm \chi^2 + c^2\kappa^2} \tag{2.31}$$

where $G_{2C} = G_S(x,t)$ represents the Green's function for the *Second Cauchy Problem.*
As we said before, this case provide a Green's Function such that

$$G_{2C} \propto G_S$$

indeed we can conclude that:

$$\begin{aligned} G_{2C}^+(x,t) &= \frac{V_0}{2c} J_0\left(\chi\sqrt{t^2 - (x/c)^2}\right)\Theta(ct - |x|)\,, \qquad \text{if } \beta^2 - \alpha^2 > 0 \\ G_{2C}^-(x,t) &= \frac{V_0}{2c} I_0\left(\chi\sqrt{t^2 - (x/c)^2}\right)\Theta(ct - |x|)\,, \qquad \text{if } \beta^2 - \alpha^2 < 0 \end{aligned} \tag{2.32}$$

Case 3 - $S_0 = V_0 = 0$
In this case we have:

$$\widehat{\widetilde{v}}(\kappa, s) \equiv \widehat{\widetilde{G_{1C}}}(\kappa, s) = \frac{U_0 s}{s^2 \pm \chi^2 + c^2\kappa^2} = \frac{U_0 s}{c^2} \frac{1}{\frac{s^2 \pm \chi^2}{c^2} + \kappa^2} \tag{2.33}$$

If we now reverse the Fourier Transform using the following result:

$$\mathcal{F}^{-1}\left[\frac{2a}{a^2 + \kappa^2}\right] = \exp(-a|x|)$$

then we conclude:

$$\widetilde{G_{1C}}(x, s) = \frac{1}{2c} \frac{U_0 s}{\sqrt{s^2 \pm \chi^2}} \exp\left(-\frac{|x|}{c}\sqrt{s^2 \pm \chi^2}\right) \tag{2.34}$$

thus,

$$\widetilde{G_{1C}}(x, s) = \frac{U_0}{2c}\, s\, \frac{\exp\left(-\frac{|x|}{c}\sqrt{s^2 \pm \chi^2}\right)}{\sqrt{s^2 \pm \chi^2}} \tag{2.35}$$

Then, if one consider the following results:

$$e^{-A\beta}\delta(t - A) + \frac{d}{dt}\left[e^{-\beta t} I_0\left(\alpha\sqrt{t^2 - A^2}\right)\Theta(t - A)\right] \div s\, \frac{\exp\left(-A\sqrt{(s+\beta)^2 - \alpha^2}\right)}{\sqrt{(s+\beta)^2 - \alpha^2}}$$
$$e^{-A\beta}\delta(t - A) + \frac{d}{dt}\left[e^{-\beta t} J_0\left(\alpha\sqrt{t^2 - A^2}\right)\Theta(t - A)\right] \div s\, \frac{\exp\left(-A\sqrt{(s+\beta)^2 + \alpha^2}\right)}{\sqrt{(s+\beta)^2 + \alpha^2}}$$

one can immediately conclude that:

$$\begin{aligned} G_{1C}^{+}(x, t) &= \frac{U_0}{2c}\left\{\delta\left(t - \frac{|x|}{c}\right) + \frac{d}{dt}\left[J_0\left(\chi\sqrt{t^2 - (|x|/c)^2}\right)\Theta\left(t - \frac{|x|}{c}\right)\right]\right\} \\ &\quad \text{if } \beta^2 - \alpha^2 > 0 \\ G_{1C}^{-}(x, t) &= \frac{U_0}{2c}\left\{\delta\left(t - \frac{|x|}{c}\right) + \frac{d}{dt}\left[I_0\left(\chi\sqrt{t^2 - (|x|/c)^2}\right)\Theta\left(t - \frac{|x|}{c}\right)\right]\right\} \\ &\quad \text{if } \beta^2 - \alpha^2 < 0 \end{aligned} \tag{2.36}$$

Now, if we define:

$$\begin{aligned} \Lambda^+(x,t) &:= J_0\left(\chi\sqrt{t^2-(|x|/c)^2}\right), \qquad \text{if } \beta^2-\alpha^2>0 \\ \Lambda^-(x,t) &:= I_0\left(\chi\sqrt{t^2-(|x|/c)^2}\right), \qquad \text{if } \beta^2-\alpha^2<0 \end{aligned} \tag{2.37}$$

then we can rewrite the Green's Functions for the equation (2.40) (PDE in $v = v(x,t)$) as follows:

$$\begin{aligned} G_S^{\pm}(x,t) &= \frac{S_0}{2c}\,\Lambda^{\pm}(x,t)\Theta(ct-|x|) \\ G_{1C}^{\pm}(x,t) &= \frac{U_0}{2c}\left\{\delta\left(t-\frac{|x|}{c}\right)+\frac{d}{dt}\left[\Lambda^{\pm}(x,t)\Theta\left(t-\frac{|x|}{c}\right)\right]\right\} \\ G_{2C}^{\pm}(x,t) &= \frac{V_0}{2c}\,\Lambda^{\pm}(x,t)\Theta(ct-|x|) \end{aligned} \tag{2.38}$$

and then follows the Green's Functions for the equation (2.1)

$$\begin{aligned} \mathcal{G}_S^{\pm}(x,t) &= \frac{S_0}{2c}\,e^{-\alpha t}\Lambda^{\pm}(x,t)\Theta(ct-|x|) \\ \mathcal{G}_{1C}^{\pm}(x,t) &= \frac{U_0}{2c}\,e^{-\alpha t}\left\{\delta\left(t-\frac{|x|}{c}\right)+\frac{d}{dt}\left[\Lambda^{\pm}(x,t)\Theta\left(t-\frac{|x|}{c}\right)\right]\right\} \\ \mathcal{G}_{2C}^{\pm}(x,t) &= \frac{V_0}{2c}\,e^{-\alpha t}\Lambda^{\pm}(x,t)\Theta(ct-|x|) \end{aligned} \tag{2.39}$$

Now, if we consider the General Cauchy Problem for the KGD:

$$\begin{cases} \varphi_{tt}-c^2\varphi_{xx}+2\alpha\varphi_t+\beta^2\varphi=0 \\ \varphi(x,0)=\Phi(x) & x\in\mathbb{R},\ t>0 \\ \varphi_t(x,0)=\Psi(x) \end{cases} \tag{2.40}$$

where $\varphi = \varphi(x,t)$ and $\alpha, \beta, c \in \mathbb{R}_0^+$; the solution is clearly is given by:

$$\varphi(x,t) = \mathcal{G}_{1C}^{\pm}(x,t) * \Phi(x) + \mathcal{G}_{2C}^{\pm}(x,t) * \Psi(x) \tag{2.41}$$

fact which conclude this analysis.

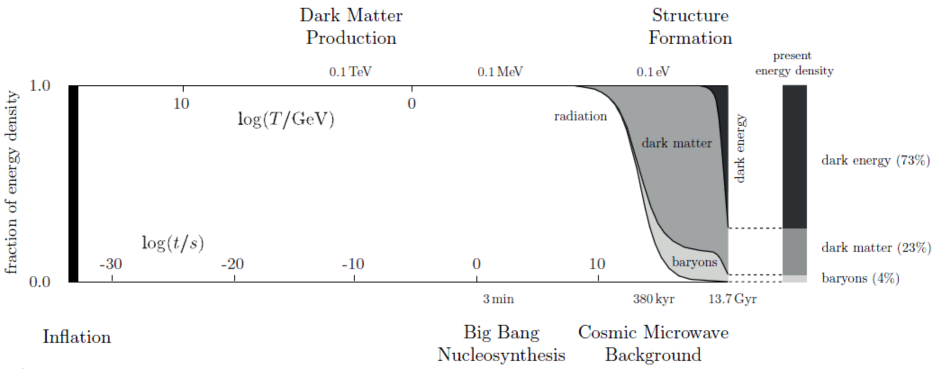
Part II

Inflationary Cosmology

Chapter 3

The Physics of Inflation

At early times, the universe was extremely hot and dense. Interactions between particles were frequent and energetic. Matter was in the form of free electrons and nuclei with light bouncing between them. As the primordial plasma cooled, the light elements formed. At some point, the energy had dropped enough for the first stable atoms to exist. At that moment, photons started to stream freely. Today, billions of years later, we observe this afterglow of the Big Bang as *microwave radiation*. This radiation is found to be almost completely uniform, with a uniform temperature about $2.7\,K$ almost in all directions. A remarkable fact is that the *Cosmic Microwave Background* (CMB) contains small variations in temperature at a level of 1 part in 10^4. Modern Cosmological theories assume that these anisotropies reflect some small variations in the primordial density of matter. These inhomogeneities probably started as quantum fluctuations in the so called *inflationary epoch* which were stretched to cosmic sizes by the *inflationary expansion* but remain constant in amplitude when larger than the contemporary Hubble scale.



This picture of the universe is an experimental fact.
The majority of the actual universe consists of strange forms of matter and energy which we have never seen in High Energy Physics experiments. Dark matter is required to explain the stability of galaxies and the rate of formation of large-scale structures. Dark energy is required to rationalise the striking fact that the expansion of the universe started to accelerate recently (a few billion years ago). Unfortunately, what dark matter and dark energy are is still unknown.

Another experimental fact is that particle physics processes dominated the very early eras of the Universe. Quantum field theory effects were predominant in this epoch then this leads to an important possibility: *scalar fields producing repulsive gravitational effects could have dominated the dynamics of the very early universe.* This leads to the *Theory of the Inflationary Universe*, proposed by Alan Guth in 1981.
This particular repulsive effect can happen if a scalar field dominates the dynamics of the early universe and then an extremely short period of accelerating expansion will precede the *Hot Big Bang era.* This produces a very cold and smooth vacuum-dominated state and ends in "reheating" which consists in the conversion of the scalar field to radiation, initiating the hot big bang epoch. This inflationary process is claimed to explain the puzzles related to the Causal limitations and the peculiarities of our universe (homogeneity, isotropy, spatial curvature). Inflationary expansion explains these features because particle horizons in inflationary Friedmann-Lemaitre-Robertson-Walker (FLRW) models would be much larger than in the standard models with ordinary matter, *allowing causal connection of matter on scales larger than the visual horizon.*

3.1 Introduction to Standard Cosmology

Let start to consider the basic theoretical framework which enables to interpret the cosmological observations. The starting point is the *Cosmological Principle* which states that it is possible to define in space-time a family of space-like sections (space-like foliation of spacetime) such that on top of all of them the Universe has the same physical properties in each point and in every direction (homogeneity and isotropy).
We consider a 4-dimensional Lorentzian manifold (a spacetime) compatible with the cosmological principle: the most general metric for such a space having homogeneous and isotropic spatial sections (i.e. which has a *maximally symmetric three-dimensional subspace*) can be parametrized by the following

line element:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3.1}$$

where $a(t)$ is known as the *cosmic scale factor*, the parameter $k = 1, 0, -1$ is the *curvature constant* (corresponding respectively to spherical, euclidean and hyperbolic spatial sections) and (t, r, θ, ϕ) are known as *comoving coordinates* (they are related to physical coordinates x_p^i by the scale factor $a(t)$, i.e. $x_p = a(t)x$). An observer whose 4-velocity in these coordinates is $U^\mu = (1, 0, 0, 0)$ is called a *comoving observer* and t is the physical time experienced by a comoving observer.
Another fundamental coordinate choice is obtained by substituting in the previous FRW metric:

$$dt = a(t(\eta))d\eta \tag{3.2}$$

where η is often called *conformal time*, whereas t is the *cosmic time.*

The Einstein Field equations can be derived from the action

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_m. \tag{3.3}$$

where

$$\begin{aligned}\mathcal{S}_{EH} &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R - 2\Lambda)\,, \quad \text{Einstein-Hilbert Action} \\ \mathcal{S}_m &= \sum_{fields}\int d^4x\sqrt{-g}\mathcal{L}_{field}\,, \quad \text{"Matter" Action}\end{aligned} \tag{3.4}$$

and where we included the *Cosmological Constant* Λ. Then we have the field equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \qquad \text{with } c = 1 \tag{3.5}$$

where we introduced the Einstein tensor

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

and we used the definition

$$T_{\mu\nu} := \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_m}{\delta g^{\mu\nu}}$$

By using the Bianchi identity $\nabla^\nu G_{\mu\nu} = 0$ one can recover the continuity equation for the sources:

$$\nabla^\nu T_{\mu\nu} = 0 \tag{3.6}$$

Now, inserting the FRW metric (3.1) into (3.5) we get the so called *Friedmann equations*:

$$\begin{aligned} H^2 + \frac{k}{a^2} &= \frac{8\pi G_N}{3}\,\rho + \frac{\Lambda}{3} \\ \dot{H} + H^2 \equiv \frac{\ddot{a}}{a} &= -\frac{4\pi G_N}{3}\,\rho\left(1 + \frac{3p}{\rho}\right) + \frac{\Lambda}{3} \end{aligned} \tag{3.7}$$

where p and ρ are respectively the pressure and the energy density of "matter" and $H(t) = \dot{a}/a$ is the *Hubble Parameter*.
The cosmological principle fixes the form of the stress-energy tensor to

$$T^\mu_\nu = a^2(t)\,diag(-\rho, p, p, p)$$

so the continuity equation for the sources can be rewritten in a non covariant form as

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{3.8}$$

where an overdot stands for derivative with respect to the cosmic time t.
Among possible sources, we may also include the so-called *vacuum* or *dark energy*, whose equation of state is

$$\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G_N} \tag{3.9}$$

The previous relation (3.8) implies that assuming a barotropic equation of state for the cosmological fluid, $p = w\rho$ we obtain

$$\rho \propto a^{-3(w+1)} \tag{3.10}$$

To keep contact with standard notation we introduce

$$\Omega_m := \frac{8\pi G_N}{3H^2}\,\rho\,, \quad \Omega_\Lambda := \frac{\Lambda}{3H^2}\,, \quad \Omega_k := -\frac{k}{(aH)^2} \tag{3.11}$$

The critical energy density ρ_c corresponds to:

$$\rho_c := \frac{3H_0^2}{8\pi G_N} = 8.10h^2 \times 10^{-47}\,GeV^4\,, \qquad h \simeq (0.71 \pm 0.03) \tag{3.12}$$

Finally, if we define the deceleration factor q_0 as:

$$q_0 := -\frac{\ddot{a}a}{\dot{a}^2}\bigg|_{t_0} \tag{3.13}$$

then we get the two fundamental equations of Standard Cosmology:

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1\,, \qquad q_0 = \frac{\Omega_{m0}}{2}\left(1 + \frac{3p}{\rho}\right) - \Omega_{\Lambda 0} \tag{3.14}$$

We can solve the equations (3.7), or alternatively any of the equation (3.7) and equation (3.8), to find the qualitative evolution of the cosmic scale factor once $p = p(\rho)$ is known.
Any stable energy component with negative pressure is of little importance in the early stage of the cosmological evolution: at early times it must have been a minor fraction of the cosmological source because of (3.10).
One can immediately derive the evolution of the scale factor for $w = 1/3$, but the time coordinate cannot, for this case, be extended beyond a critical value in the past, conventionally set to $t = 0$, where a *singularity* is met almost unavoidably if the dominant energy source has $p/\rho > 0$.
Clearly this analysis is based on Classical General Relativity and breaks down at $t \sim t_P$[1]. The singularity is not a formal artifact of our characteristic solution, but within the framework of General Relativity it has been demonstrated by Hawking and Penrose under well defined, but reasonable, hypotheses.

Theorem 2 (Penrose-Hawking Singularity Theorem)**.** *Let $\mathcal{M}$ be a globally hyperbolic space-time with non-compact Cauchy surfaces satisfying the null energy condition. If $\mathcal{M}$ contains a trapped surface Σ then $\mathcal{M}$ is future null geodesically incomplete.*

Or, more simply: a spacetime $\mathcal{M}$ necessarily contains incomplete, inextendible timelike or null geodesics (i.e. a *singularity*) under the following hypotheses:

1. $\mathcal{M}$ contains no closed timelike curves (unavoidable causality requirement);

2. At each point in $\mathcal{M}$ and for each unit timelike vector with component u^α the energy momentum tensor satisfies (null energy condition):

$$\left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} T^\alpha_\alpha\right) u^\mu u^\nu \geq 0$$

3. The manifold is not too highly symmetric so that for at least one point the curvature is not lined up with the tangent through the point;

[1]Planck time: $t_P = \sqrt{\frac{\hbar G_N}{c^5}} \simeq 5.39106 \times 10^{-44}\, s$

4. $\mathcal{M}$ does not contain any trapped surface.

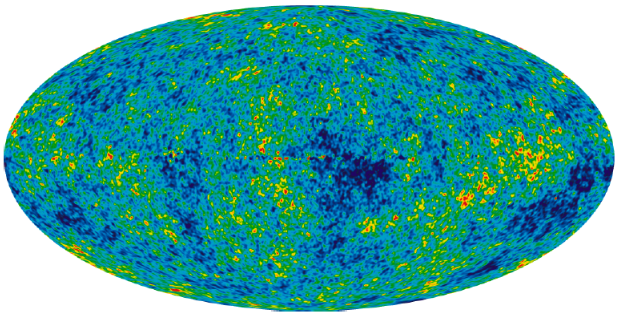
All conditions except the last one, which is rather technical, are completely reasonable for a realistic spacetime.

From the previous model we know that the early universe was extremely hot and dense. The interactions between particles were therefore very frequent. As long as the interaction rate Γ for a certain particle species exceeded the expansion rate H, these particles were in equilibrium. The particle densities were then given by the standard results from equilibrium statistical mechanics. Relativistic particles ($m < T$) dominated the energy and particle number densities of the universe. The abundances of non-relativistic particles ($m > T$) were Boltzmann-suppressed, $n \propto \exp(-m/T)$. Massive particles therefore became less important as the universe cooled. If equilibrium persisted until today, the universe would be mostly composed by photons.
Fortunately, equilibrium did not last forever. As the universe cooled, the interaction rates for many particle species dropped below the expansion rate, $\Gamma < H$. This led to deviations from equilibrium, resulting in the freeze-out of massive particles. Non-equilibrium processes were crucial for the production of dark matter, the synthesis of the light elements during *Big Bang nucleosynthesis* and the *recombination* of electrons and protons into neutral hydrogen. The following table summarize the *Thermal History of the Universe.*

Event	**Time** t	**Redshift** z	**Temperature** T
EW phase transition	$20\,ps$	10^{15}	$100\,GeV$
QCD phase transition	$20\,\mu s$	10^{12}	$150\,GeV$
Neutrino decoupling	$1\,s$	6×10^{9}	$1\,MeV$
Electron-positron annihilation	$6\,s$	2×10^{9}	$500\,keV$
Big Bang nucleosynthesis	$3\,min$	4×10^{8}	$100\,keV$
Matter-radiation equality	$60\,kyr$	3200	$0.75\,keV$
Recombination	$260 - 380\,kyr$	$1100 - 1400$	$0.26 - 0.33\,eV$
CMB decoupling	$380\,kyr$	$1000 - 1200$	$0.23 - 0.28\,eV$
Reionization	$100 - 400\,Myr$	$11 - 30$	$2.6 - 7.0\,meV$
Dark energy-matter equality	$9\,Gyr$	0.4	$0.33\,meV$
Present	$13.7\,Gyr$	0	$0.24\,meV$

3.2 Inflation

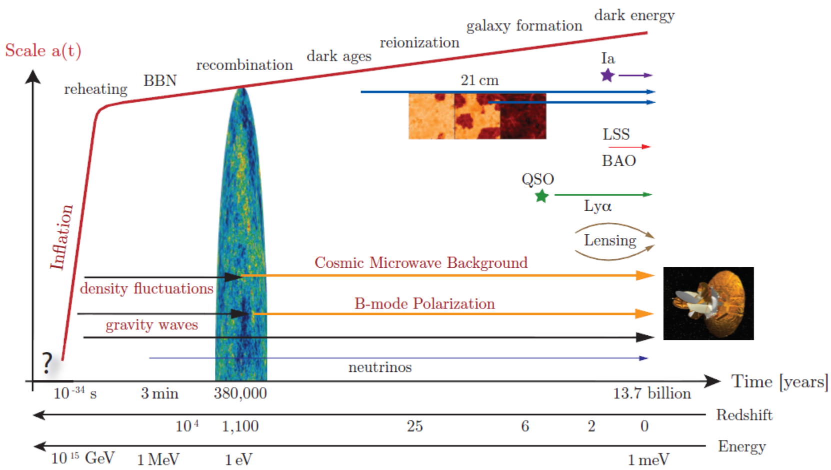
Running back in time to the very beginning of the expansion of the Universe, the uniformity of the CMB becomes mysterious. It is a famous fact that in the conventional Big Bang cosmology the CMB at the time of decoupling consisted of about 10^4 causally independent patches. Two points on the sky with an angular separation exceeding $2°$ should never have been in causal contact; however they are observed to have the same temperature with an extremely high precision. This fact is called the *Horizon Problem*.
The horizon problem in the form stated above assumes that no new physics becomes relevant for the dynamics of the universe at early times. In this chapter, we will see how a specific form of new physics may lead to a negative pressure component and to a quasi-exponential expansion (de Sitter expansion). This period of *inflation* produces the apparently acausal correlations in the CMB and hence solves the horizon problem.
Moreover, inflation also explains why the CMB has small inhomogeneities as shown in the figure.



Quantum mechanical *zero-point fluctuations* during inflation are promoted to cosmic significance as they are stretched outside of the horizon. When the perturbations re-enter the horizon at later times, they seed the fluctuations in the CMB. Through explicit calculation one finds that the primordial fluctuations from inflation are just of the *right type* (Gaussian, scale-invariant and adiabatic) to explain the observed spectrum of CMB fluctuations.

Basics of Inflationary Cosmology

Let us consider the schematic representation of the History of the Universe shown in the figure.



The main topic of the next few sections will be just the first part of the graph which goes from the beginning of inflation to the reheating.

Most of current models of inflation are based on Einstein's theory of General Relativity with a matter source given by a scalar field φ.
The idea of inflation is very simple. We assume there is a time interval beginning at $t_i = 0$ and ending at t_R (the *reheating time*) during which the Universe is exponentially expanding, i.e.,

$$a(t) \propto e^{Ht}, \qquad t \in [t_i, t_R] \tag{3.15}$$

with a constant Hubble parameter H. Such a period is called *de Sitter expansion* or *inflation*.
The success of Big Bang nucleosynthesis sets an upper limit to the time t_R of reheating, $t_R \ll t_{NS}$, t_{NS} being the *time of nucleosynthesis*.
During the inflationary phase, the number density of any particles initially present at $t = t_i = 0$ decays exponentially. At $t = t_R$, all of the energy which is responsible for inflation is released as thermal energy. This is a non-adiabatic process during which the entropy increases by a large factor.
A period of inflation can solve the *Homogeneity Problem* of standard cosmology, the reason being that during inflation the physical size of the forward light cone exponentially expands and thus can easily become larger than the physical size of the past light cone at t_{rec}, the time of last scattering, thus explaining

the near isotropy of the cosmic microwave background (CMB). Inflation also solves the *flatness problem*.
Most importantly, inflation provides a mechanism which in a causal way generates the primordial perturbations required for galaxies, clusters and even larger objects. In inflationary Universe models, the Hubble radius (apparent horizon) and the (actual) horizon (the forward light cone) do not coincide at late times. Provided that the duration of inflation is sufficiently long, then all scales within our present apparent horizon were inside the horizon since t_i. Thus, it is in principle possible to have a causal generation mechanism for perturbations.
The density perturbations produced during inflation, as already said, are due to quantum fluctuations in the matter and gravitational fields. The amplitude of these inhomogeneities corresponds to a temperature $T_H \sim H$, the *Hawking temperature* of the de Sitter phase. This leads one to expect that at all times t during inflation, perturbations with a fixed physical wavelength $\lambda \sim H^{-1}$ will be produced. Consequently, the length of the waves is stretched with the expansion of space, and soon becomes much larger than the Hubble radius $l_H(t) = H^{-1}(t)$. The phases of the inhomogeneities are random. Thus, the inflationary Universe scenario predicts perturbations on all scales ranging from the comoving Hubble radius at the beginning of inflation to the corresponding quantity at the time of reheating. In particular, provided that inflation lasts sufficiently long, perturbations on scales of galaxies and beyond will be generated. Note, however, that it is very dangerous to interpret de Sitter Hawking radiation as thermal radiation. In fact, the equation of state of this "radiation" is not thermal.

The basic physical mechanism for producing a period of inflation in the very early universe relies on the existence, at such epochs, of matter in a form that can be described classically in terms of a *scalar field*. Upon quantisation, a scalar field describes a collection of spin-less particles. It may at first seem rather arbitrary to postulate the presence of such scalar fields in the very early universe. However, their existence is suggested by the Standard Model of Fundamental Interactions, which predict that the Universe experienced a succession of *phase transitions*[2] in its early stages as it expanded and cooled. For example, let us model this expansion by assuming that the Universe followed a standard radiation-dominated Friedmann model in its early stages, in

[2]The basic premise of Grand Unification is that the known symmetries of the elementary particles resulted from a larger symmetry group. Whenever a phase transition occurs, part of this symmetry is lost, so the symmetry group changes.

this case:

$$a(t) \propto \sqrt{t} \propto \frac{1}{T(t)} \tag{3.16}$$

where the "temperature" T is related to the typical particle energy by $T \sim E/k_B$.
The basic scenario is as follows.

- $E_P \sim 10^{19}\,GeV > E > E_{GUT} \sim 10^{15}\,GeV$ The earliest point at which the Universe can be modelled as a classical system is the *Planck era*, corresponding to particle energies $E_P \sim 10^{19}\,GeV$ and time scales $t_P \sim 10^{-43}\,s$; prior to this epoch, it is considered that the Universe can be described only in terms of some, as yet unknown, Quantum theory of Gravity.
  At these extremely high energies, Grand Unified Theories (GUTs) predict that the electroweak and strong forces are unified into a single force and that these interactions bring the particles present into thermal equilibrium. Once the universe has cooled enough to $E_{GUT} \sim 10^{14}\,GeV$, there is a spontaneous breaking of the symmetry group characterising the GUT into a product of smaller symmetry groups and the electroweak and strong forces separate. From the equation (3.16), this *GUT phase transition* occurs at $t_{GUT} \sim 10^{-36}\,s$.

- $E_{GUT} \sim 10^{15}\,GeV > E > E_{EW} \sim 100\,GeV$ During this period the electroweak and strong forces are separate and *these interactions sustain thermal equilibrium*. This continues until the Universe has cooled to $E_{EW} \sim 100\,GeV$ when the unified electroweak theory predicts that a *second phase transition* should occur in which the electromagnetic and weak forces separate. Again, the equation (3.16), this *electroweak phase transition* occurs at $t_{EW} \sim 10^{-11}\,s$.

- $E_{EW} \sim 100\,GeV > E > E_{QH} \sim 100\,MeV$ During this period the electromagnetic, weak and strong forces are separate, as they are today.
  It is remarkable, however, that when the universe has cooled to $E_{QH} \sim 100\,MeV$ there is a *final phase transition, according to the theory of Quantum Chromodynamics (QCD)*, in which the strong force increases in strength and leads to the confinement of quarks into hadrons.
  This *Quark-Hadron phase transition* occurs at $t_{QH} \sim 10^{-5}\,s$.

In general, phase transitions occur via a process called *Spontaneous Symmetry Breaking*, which can be characterised by the acquisition of certain non-zero values by scalar parameters known as *Higgs fields.* The symmetry is manifest when the Higgs fields have the value zero; it is spontaneously broken whenever at least one of the Higgs fields becomes non-zero. Thus, the occurrence of phase transitions in the very Early Universe suggests the existence of scalar fields and hence provides the motivation for considering their effect on the expansion of the Universe. *In the context of inflation, we will confine our attention to scalar fields present at, or before, the GUT phase transition.*

For simplicity, let us consider a single scalar field φ present in the very Early Universe. The field φ is traditionally called the *inflaton field* for obvious reasons.
In most current models of inflation, φ is a scalar matter field with standard action:

$$\mathcal{L}_m = -\frac{1}{2}\nabla^\mu\varphi\nabla_\mu\varphi - V(\varphi)\,, \qquad S_m = \int d^4x\sqrt{-g}\,\mathcal{L}_m \tag{3.17}$$

where ∇_μ denotes the covariant derivative, g is the determinant of the metric tensor and the exponential expansion is driven by the potential energy density $V(\varphi)$.
The resulting energy-momentum (comparing the result from the variational approach with the energy-momentum tensor for a perfect fluid, using comoving coordinates) tensor yields the following expressions for the energy density ρ and the pressure p:

$$\begin{aligned} \rho(\varphi) &= \frac{\dot{\varphi}^2}{2} + \frac{(\nabla\varphi)^2}{2a^2} + V(\varphi) \\ p(\varphi) &= \frac{\dot{\varphi}^2}{2} - \frac{(\nabla\varphi)^2}{6a^2} - V(\varphi) \end{aligned} \tag{3.18}$$

It thus follows that if the scalar field is homogeneous and static, but with positive potential energy, then the equation of state $p = -\rho$, which is necessary for exponential inflation, arises.

Most of the current realizations of potential-driven inflation are based on satisfying the conditions:

$$\dot{\varphi}^2 \ll V(\varphi)\,, \qquad a^{-2}(\nabla\varphi)^2 \ll V(\varphi) \tag{3.19}$$

via the idea of *slow rolling*.
If we now consider the equation of motion of the scalar field φ:

$$\ddot{\varphi} + 3H\dot{\varphi} - a^{-2}\Delta\varphi + \frac{dV}{d\varphi} = 0 \tag{3.20}$$

If the scalar field starts out almost homogeneous and at rest, if the Hubble damping term is large and if the potential is quite flat, so the term on the r.h.s. is small, then $\dot{\varphi}$ may remain small compared to $V(\varphi)$. Note that if the spatial gradient terms are initially negligible, they will remain negligible.

Chaotic inflation is a prototype of inflationary scenario. Consider a scalar field φ which is very weakly coupled to itself and other fields. In this case, φ need not to be in thermal equilibrium at the Planck time and most of the phase space for φ will correspond to large values of $|\varphi|$ (typically $|\varphi| \gg M_P$). Consider now a region in space where at the initial time φ is very large, approximately homogeneous and static. In this case, the energy-momentum tensor will be immediately dominated by the large potential energy term and induce an equation of state $p \simeq -\rho$ which leads to inflation. Because of the large Hubble damping term in the scalar field equation of motion, φ will only roll very slowly towards $\varphi = 0$, assuming that $V(\varphi)$ has a global minimum at a finite value of φ which can then be chosen to be $\varphi = 0$. The kinetic energy contribution to ρ and p will remain small, the spatial gradient contribution will be exponentially suppressed due to the expansion of the Universe ($a(t)$ increases rapidly) and thus inflation persists. Note that *the form of $V(\varphi)$ is irrelevant to the mechanism.*
It is difficult to realize chaotic inflation in conventional Supergravity Models since gravitational corrections to the potential of scalar fields typically render the potential steep for values of φ of the order of M_P and larger.
This prevents the slow rolling condition (3.19) from being realizable. Even if this condition can be satisfied, there are constraints from the amplitude of produced density fluctuations which are much harder to satisfy.

Actually, there are many models of potential-driven inflation, but there is no canonical theory. A lot of attention is being devoted to implementing inflation in the context of Unified Theories, the most important candidate being *Superstring Theory* or *M-theory*. String Theory or M-theory live in 10 or 11 space-time dimensions, respectively. When *compactified*[3] to 4 space-time dimensions, there exist many *moduli fields*, scalar fields which describe flat directions in the complicated vacuum manifold of the theory. A lot of attention has recently been devoted to attempts at implementing inflation using moduli fields. Not long ago, it has been suggested that our space-time is a *Brane* in a higher-dimensional space-time. Ways of obtaining inflation on the Brane are also under active investigation.
It should also not be forgotten that inflation can arise from the purely Gravitational sector of the theory, as in the original model of Starobinsky or that *it*

[3]In mathematics, *compactification* is the process or result of making a topological space into a compact space (a space which contains its accumulation points).

may arise from kinetic terms in an effective action as in pre-big-bang cosmology or in k-inflation.
Theories with (almost) exponential inflation generically predicts an (almost) scale-invariant spectrum of density fluctuations. Via the Sachs-Wolfe effect, these density perturbations induce CMB anisotropies with a spectrum which is also scale-invariant on large angular scales.
The heuristic picture is as follows. If the inflationary period which lasts from t_i to t_R is almost exponential, then the physical effects which are independent of the small deviations from exponential expansion are time-translation-invariant. This implies, for example, that quantum fluctuations at all times have the same strength when measured on the same physical length scale.
If the inhomogeneities are small, they can be described by linear theory, which implies that all Fourier modes k evolve independently. The exponential expansion stretches the wavelength of any perturbation. Thus, the wavelength of perturbations generated early in the inflationary phase on length scales smaller than the Hubble radius soon becomes equal to the (existent) Hubble radius and continues to increase exponentially. After inflation, the Hubble radius increases as t while the physical wavelength of a fluctuation increases only as $a(t)$. Thus, eventually the wavelength will cross the Hubble radius again at time $t_f(k)$. Thus, it is possible for inflation to generate fluctuations on cosmological scales by causal physics.

3.3 k-Inflation

Much of Modern Cosmology has dealt with the construction of phenomenologically suitable inflationary models with various potentials and number of inflaton fields. In the early stages of inflationary theory there were hopes of incorporating inflation in more or less standard particle physics. One of the various proposals consisted in assuming a close relation between the inflaton fields and the GUTs-transition. However, this idea never worked out in a convincing way and, consequently, inflation fields lived their own life quite detached from the rest of Theoretical Particle Physics.
Nowadays, the most important theoretical picture, aiming to unify the modern inflationary cosmology with the theory of fundamental interactions, consist in String Theory. In this theory it is well known that parameters describing background geometries and compactifications, the *moduli*, are all promoted into scalar fields. There are, therefore, no lack of potential candidates for the inflaton, even though there are several difficult conditions to be met.
For one thing, the potential of the inflaton must be extremely flat in order to

allow for enough e-foldings[4]. On the other hand, it cannot be completely flat for the idea to work. In Supersymmetric String Theory there are many flat directions in the moduli space of solutions which could serve as useful starting points. The hope would then be that these flat directions are lifted by non-perturbative, supersymmetry breaking terms. Unfortunately, it is difficult to find these non-perturbative corrections explicitly and their expected form is anyway, in many cases, not of the right kind. In addition, there are also other problems to be solved. Apart from the flat, inflationary potential, *one needs potentials that manage to fix dangerous moduli like those controlling the size of the extra dimensions.* It is hard to see how realistic inflationary theories can be obtained without addressing this problem at the same time.
It can be proven that realistic inflationary models can indeed be constructed using string moduli if one introduces Branes, but this is not the purpose of this work. The very basic idea is to use two stacks of Branes separated by a certain distance, corresponding to the inflaton, in a higher dimensional space. As the Branes move, the inflaton rolls, and when the Branes collide inflation stops.
Firstly, I will treat the attempts which go under the name of *String Cosmology.* The idea is to make use of the *dilaton*, i.e. the field corresponding to the way the string coupling varies over space and time, and a variant of the string theoretical *T-duality.* The resulting theory fulfils the condition for inflation, albeit in an unorthodox way.
On second instance, it will be presented a very recent theoretical tool of inflationary cosmology theory, known as *k-inflation*, aiming to help to locate which sector of string theory has inflated a strongly curved initial state into our presently observed large and weakly curved Universe.

3.3.1 Basics of String Cosmology

String cosmology makes use of one of the most basic features of string theory, the concept of dilaton. According to string theory the Einstein-Hilbert action of General Relativity is described by a new dimensionless scalar field, the dilaton φ, and given by

$$\mathcal{S} = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\, e^{-\varphi}\left(R + \partial^\alpha\varphi\partial_\alpha\varphi\right) \tag{3.21}$$

where $\kappa_{10} = (2\pi)^7\alpha'^4/2 \sim l_s^8$ and where the string coupling is related to the dilaton through $g_s^2 = e^\varphi$. The action as given is written in the *string frame.*

[4]E-folding is the time interval in which an exponentially growing quantity increases by a factor of e.

That is, the string length, l_s, is our fundamental unit and what we use as our measuring rod. This means that the Planck mass, the effective coefficient of the scalar curvature R, varies with the dilaton. An alternative way to describe things is to use the *Einstein frame* which in many ways is physically more clear than the string frame. In the Einstein frame it is the Planck length, which is more directly related to macroscopic physics through the strength of gravity, which is used as a fundamental unit. Let me explain the relation between the two frames in more detail.
Immediately, one can note that that the frames are, by definition, related through

$$\int d^D x \sqrt{-g} e^{-\varphi} R = \int d^D x \sqrt{-g_E} (R_E + \cdots) , \qquad g_{\mu\nu} = e^{2\omega\varphi} (g_E)_{\mu\nu} \quad (3.22)$$

where the subscript E indicate the Einstein frame and furthermore

$$\sqrt{-g} = e^{D\omega\varphi} \sqrt{-g_E} \quad (3.23)$$

It follows, from the definition of curvature, that the scalar curvatures are related through

$$R = e^{-2\omega\varphi} \left[R_E - 2\omega(D-1)\Delta\varphi - \omega^2 (D-2)(D-1) \partial^\alpha \varphi \partial_\alpha \varphi \right] \quad (3.24)$$

Hence we have that

$$\sqrt{-g} e^{-\varphi} R = \sqrt{-g_E} e^{(D\omega - 1 - 2\omega)\varphi} \left[R_E - 2\omega(D-1)\Delta\varphi - \omega^2 (D-2)(D-1) \partial^\alpha \varphi \partial_\alpha \varphi \right]$$

and consequently

$$D\omega - 1 - 2\omega = 0 \quad \Longrightarrow \quad \omega = \frac{1}{D-2} \quad (3.25)$$

The action in the Einstein frame finally becomes

$$\mathcal{S} = -\frac{M_D^{D-2}}{2} \int d^D x \sqrt{-g_E} \left(R_E - \frac{1}{D-2} \partial^\alpha \varphi \partial_\alpha \varphi \right) \quad (3.26)$$

also called the D-dimensional Einstein-Hilbert Action, where M_D is the D-dimensional Planck mass. We note that the sign of the kinetic term of the scalar field now is the familiar one.
If we consider a metric of FRW-form (3.1) generalized to D dimensions, we find

$$ds_E^2 = e^{-2\omega\varphi} ds^2 = e^{-2\omega\varphi} \left(-dt^2 + a^2(t) d\mathbf{x}^2 \right) \equiv -dt_E^2 + a_E^2(t) d\mathbf{x}^2 \quad (3.27)$$

where

$$a_E(t) := e^{-\omega\varphi} a(t)\,, \qquad dt_E := e^{-\omega\varphi} dt \tag{3.28}$$

It is important to realize that the two frames are physically equivalent, even if things can, at a first glance, look rather different in the two frames. To fully appreciate string cosmology it is important to keep this in mind.

Let us now investigate the above action in more detail. I will perform the analysis in the string frame, and, for simplicity, assume a spatially homogenous FRW-metric. One can readily check that the scalar curvature in these coordinates is given by

$$R = -(D-1)(D-2)\frac{\dot{a}}{a} - 2(D-1)\frac{\ddot{a}}{a} \tag{3.29}$$

The action possesses a remarkable symmetry thanks to the presence of the stringy dilaton. The symmetry acts on the scale factor and the dilaton through the transformations

$$\begin{aligned} a(t) &\longrightarrow 1/a(t) \\ \varphi(t) &\longrightarrow \varphi(t) - 2(D-1)\ln a(t) \end{aligned} \tag{3.30}$$

It leaves the action invariant and assures that the solutions of the equations of motion have some very interesting properties that will be important for cosmology. To verify the symmetry, we note that

$$\sqrt{-g}e^{-\varphi}(R+\dot{\varphi}^2) = a^{D+1}e^{-\varphi}\left[-(D-1)\left(\frac{\dot{a}}{a}\right)^2 + \left(\dot{\varphi} - (D-1)\frac{\dot{a}}{a}\right)^2\right] + \text{Total Derivative} \tag{3.31}$$

Since we have (considering the equations (3.30))

$$\begin{aligned} a^{D-1}e^{-\varphi} &\longrightarrow a^{D-1}e^{-\varphi} \\ \frac{\dot{a}}{a} &\longrightarrow -\frac{\dot{a}}{a} \end{aligned} \tag{3.32}$$

we find

$$\begin{aligned} &a^{D+1}e^{-\varphi}\left[-(D-1)\left(\frac{\dot{a}}{a}\right)^2 + \left(\dot{\varphi} - (D-1)\frac{\dot{a}}{a}\right)^2\right] \longrightarrow \\ &\longrightarrow a^{D+1}e^{-\varphi}\left[-(D-1)\left(\frac{\dot{a}}{a}\right)^2 + \left(\dot{\varphi} - (D-1)\frac{\dot{a}}{a}\right)^2\right] \end{aligned} \tag{3.33}$$

and hence an invariance of the action.
In other words, if $a(t)$ and $\varphi(t)$ solves the equations of motion, so does the

transformed functions $1/a(t)$ and $\varphi(t) - 2(D-1)\ln a(t)$.
To fully appreciate what is going on, and to understand the structure of the solutions, we need to note that there is yet another simple symmetry

$$t \longrightarrow -t \tag{3.34}$$

i.e. *time reversal invariance*, which together with (3.30) presents an interesting fact about possible cosmologies.

Combining the two symmetries we can map out how various solutions are related to each other. If we first consider the scale factor $a(t)$, we can easily see how we can construct two new solutions, from a given solution $a(t)$, according to

$$\begin{aligned} a(t) &\longrightarrow 1/a(t)\,, \qquad & H(t) &\longrightarrow -H(t) \\ a(t) &\longrightarrow a(-t)\,, \qquad & H(t) &\longrightarrow -H(-t) \end{aligned} \tag{3.35}$$

The time $t = 0$ is referred to as the Big Bang and it *is natural to allow for two eras, a pre and a post Big Bang.* The basic idea of String Cosmology is that physics can be traced back in time through the Big Bang into an earlier era, the pre Big Bang, where many of the initial conditions for the post Big Bang are determined in a natural and dynamical way.

It should be remarked that the whole previous formulation is in line with the general picture of *T-duality in String Theory.* According to T-duality, it is equivalent to compactify string theory on a small circle (compared with the string scale) and a large circle.
In some sense large and small scales are, therefore, equivalent. Loosely applying this idea to the Big Bang, would suggest that if we trace the expansion far enough back in time, we are better off describing the universe as becoming bigger again, rather than smaller. As can be proven, however, string cosmology suggests that we should take an expanding pre Big Bang theory and match it to an expanding post Big Bang. *But* this is the picture obtained in the string frame. The picture in the Einstein frame is quite different with a contracting rather than an expanding pre Big Bang phase. This is precisely in line with the hand waving argument above.

3.3.2 A brief introduction to k-Inflation

The inflationary paradigm offers the attractive possibility of resolving many of the puzzles of standard hot big bang cosmology. The crucial ingredient of most successful inflationary scenarios is a period of "slow-roll" evolution of the inflaton field, φ, during which the potential energy density $V(\varphi)$, stored in φ, dominates its kinetic energy $\dot{\varphi}^2/2$ and drives a quasi-exponential expansion of the Universe.
As previously mentioned, String Theory provides one with several very weakly coupled scalar fields, *the moduli*, which could be natural inflaton candidates, however, their non-perturbative potentials $V(\varphi)$ do not seem to be able to sustain a slow-roll inflation because, for large values of φ, they tend either to grow, or to tend to zero, too fast. It is therefore *important to explore novel possibilities for implementing an inflationary evolution of the early Universe.*
The main feature of the so known *Kinetically Driven Inflation* (k-inflation) consist in the fact that, *even in absence of any potential energy term*, a general class of non-standard (non-quadratic) kinetic energy terms, for a scalar field φ, can drive an inflationary evolution of the same type as the usually considered potential driven inflation. By "usual type of inflation" we mean here an *accelerated expansion* (in the Einstein conformal frame η) during which the curvature scale starts around a Planckian value and then decreases *monotonically.*
By contrast, the pre big bang scenario uses a standard quadratic kinetic energy term $\dot{\varphi}^2/2$ to drive an accelerated contraction (in the Einstein frame) during which the curvature scale increases. Though we shall motivate below the consideration of non-standard kinetic terms by appealing to the existence, in string theory, of higher-order corrections to the *effective action* for φ. With these statements we do not claim that the structure needed for implementing our kinetically driven inflation arises inevitably in string theory but we pay attention to a new mechanism for implementing inflation.

Let me introduce the basic formalism of the k-Inflation.
As we have already mentioned, slow-roll ination assumes that the kinetic term is canonical, $\mathcal{L}_{s.r.} = X - V(\varphi)$ where $X \equiv -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi$.
As one can easily see, a given FRW background with time-dependent Hubble parameter $H(t)$ corresponds to cosmic acceleration if and only if:

$$\varepsilon := -\frac{\dot{H}}{H^2} < 1 \tag{3.36}$$

For this condition to be sustained for a sufficiently long time, requires

$$|\zeta| := \frac{|\dot{\varepsilon}|}{H\varepsilon} \ll 1 \tag{3.37}$$

i.e. the fractional change of ε per Hubble time is small.
Now, considering the *slow-roll approximation* (3.19) we than obtain:

$$\begin{aligned} \varepsilon &= -\frac{\dot{H}}{H^2} \approx \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 =: \epsilon_v \\ |\zeta| &= \frac{|\dot{\varepsilon}|}{H\varepsilon} \approx M_P^2\frac{|V''|}{V} =: |\zeta_v| \end{aligned} \tag{3.38}$$

where ϵ_v and ζ_v are the so called *potential slow-roll parameters.*
One can immediately understand that when these are small, slow-roll inflation occurs, thus:

$$\epsilon_v = \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1\,, \qquad |\zeta_v| = M_P^2\frac{|V''|}{V} \ll 1 \tag{3.39}$$

Now, if one consider that the kinetic term of the slow roll Lagrangian Density is canonical, this puts strong constraints on the shape of the potential $V(\varphi)$ via the potential slow-roll conditions. However, these conditions for inflation are not absolute, but assume the slow-roll approximations.
In contrast, the Hubble slow-roll conditions, $\varepsilon, \zeta \ll 1$ do not make any approximations and allow for a larger spectrum of inflationary backgrounds. In particular, *the constraints on the inflationary potential can potentially be relaxed if higher-derivative corrections to the kinetic term were dynamically relevant during inflation*, i.e. $|\dot{H}| \ll H^2$ *not because the theory of potential-dominated, but because it allows non-trivial dynamics.*
A useful way to describe these effects is by the following action,

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{M_P^2 R}{2} + P(X,\varphi)\right] \tag{3.40}$$

with

$$P(X,\varphi) = \sum_n c_n(\varphi)\frac{X^n}{\Lambda^{4n-4}} \tag{3.41}$$

For $X \ll \Lambda^4$, the dynamics reduces to that of slow-roll inflation, so we are now interested in the limit $X \sim \Lambda^4$. This is an extremely complicated situation because X/Λ^4 controls the derivative expansion. In particular, in the limit $X \to \Lambda^4$ we have to worry about the appearance of unstable "ghost" states and the stability under radiative corrections. Specifically, in the absence of symmetries, there is no way to protect the coefficients c_n in (3.41) from quantum corrections. The predictions derived from (3.41) then cannot be trusted. However, sometimes the theory is equipped with a symmetry that forbids large renormalizations of these coefficients.

This is the case, for instance in *Dirac-Born-Infeld* (DBI) inflation, where a higher-dimensional boost symmetry protects the special form of the Lagrangian Density

$$P(X,\varphi) = -\Lambda^4(\varphi)\sqrt{1+\frac{X}{\Lambda^4(\varphi)}} - V(\varphi) \tag{3.42}$$

In this case, the boost symmetry forces quantum corrections to involve the two-derivative combination. We remark that, only when they come with protective symmetries are $P(X)$-theories really interesting and predictive theories.
The stress-energy tensor arising from (3.40) has pressure P and energy density

$$\rho = 2XP_{,X} - P \tag{3.43}$$

where $P_{,X}$ denotes a derivative with respect to X. The inflationary parameter ε then becomes:

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{3XP_{,X}}{2XP_{,X} - P} \tag{3.44}$$

The condition for inflation is still $\varepsilon \ll 1$, which now is a condition on the functional form of $P(X)$. However, it should be remembered that unless a protective symmetry is identified, there is no guarantee that the $P(X)$-theory is radiatively stable.
The fluctuations in $P(X)$-theories have a number of interesting features. First, in the limit $X \sim \Lambda^4$, they propagate with a non-trivial speed of sound

$$c_s^2 = \frac{dP}{d\rho} = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}} \tag{3.45}$$

The limit of small sound speed implies enhanced interactions in the cubic and quartic Lagrangian. This leads to large amount of non-Gaussian perturbations.

Chapter 4

Tachyon Matter Cosmology

Let us begin with the question: What are Tachyons? Historically, these were described as particles which travel faster than light. If we use the relativistic equation:

$$v = \frac{pc}{\sqrt{p^2 + m^2c^2}} \tag{4.1}$$

relating the velocity v of a particle to its momentum p, mass m and the velocity of light c, then we see that *tachyons can also be regarded as particles with negative mass squared* (or, equivalently, with an imaginary mass).
Of course, both descriptions sound equally bizarre. On the other hand tachyons have been known to exist in String Theory almost since its birth and hence we need to make sense of them.

Actually, tachyons do appear in conventional Quantum Field Theories (QFT) as well. Consider, for instance, a classical scalar field φ with potential $V(\varphi)$. For simplicity I shall set $\hbar = c = 1$. Consider a $(p+1)$-space, labelled by the time coordinate x^0 and space coordinates x^i, with $1 \leq i \leq p$; then the the Lagrangian Density of the scalar field is:

$$\mathcal{L} = -\frac{1}{2}\,\partial^\mu\varphi\partial_\mu\varphi - V(\varphi) \equiv \frac{1}{2}\,\left(\partial^0\varphi\partial_0\varphi - \partial^i\varphi\partial_i\varphi\right) - V(\varphi) \tag{4.2}$$

Normally, we choose the origin of φ so that the potential $V(\varphi)$ has a minimum at $\varphi = 0$. In this case, the quantization of φ gives a scalar particle such that:

$$m_\varphi^2 = \frac{d^2V}{d\varphi^2}(0) > 0 \tag{4.3}$$

which gives a real mass m_φ. Nevertheless, if we suppose that the potential has a maximum at $\varphi = 0$, then we conclude that:

$$m_\varphi^2 = \frac{d^2V}{d\varphi^2}(0) < 0 \tag{4.4}$$

thus, the quantization of φ gives a tachyon.
In this case, however, we can easily conclude that *this way of thinking is completely wrong*. When we identify $V''(0) = m_\varphi^2$ we are making an approximation. We expanded $V(\varphi)$ in a Taylor series expansion in φ and treated the cubic and higher order terms as small corrections to the quadratic term. This is actually true only if the quantum fluctuations of φ around $\varphi = 0$ are *small*. Hence, we do not have tachyons in QFT of a scalar field.
Thus we see that the existence of a tachyon in a scalar field theory implies that the potential for the scalar field has a local maximum at the origin. In order for the theory to be sensible at least in perturbation theory, the potential must also have a (local) minimum. Typically, whenever the potential in a scalar field theory has more than one extremum we can construct non-trivial classical solutions which depend on one or more spatial directions, but this is not the aim of this work.
The lessons learned from the field theory may be summarized as follows:

1. Existence of tachyons in the spectrum tells us that we are expanding the potential around its maximum rather than its minimum;

2. The correct procedure to deal with such a situation is to find the (global) minimum of the potential and expand the potential around the minimum. The resulting theory has a positive mass^2 scalar particle instead of a tachyon;

3. Associated with the existence of tachyons we often have non-trivial space dependent classical solutions.

We now turn to the discussion of **Tachyons in String Theory**. The conventional description of string theory is based on "first quantized" formalism rather than a field theory. We take a string (closed or open) and quantize it maintaining Lorentz invariance. This gives infinite number of states characterized by momentum $\mathbf{p}$ and other discrete quantum numbers n. It turns out that the energy of the n-th state carrying momentum $\mathbf{p}$ is given by $E_n = \sqrt{p^2 + m_n^2}$, where m_n is some constant. This state clearly has the interpretation of being a particle of mass m_n. Thus string theory contains infinite number of single "particle" states, as if it were a field theory with infinite number of fields.
Quantization of some closed or open strings gives rise to states with negative m_n^2 for some n. This clearly corresponds to a tachyon. For instance, the original bosonic string theory, formulated in $25 + 1$ dimensions, has a tachyon in the spectrum of closed strings. This theory is thought to be inconsistent due to this reason.
Superstring Theories are free from tachyons in the spectrum of closed string,

but, for certain boundary conditions, there can be tachyon in the spectrum of open strings even in Superstring Theories. Thus the question is: Does the existence of tachyons make the theory inconsistent? Or does it simply indicate that we are quantizing the theory around the wrong point?
The problem in studying this question stems from the fact that unlike the example in a scalar field theory, the tachyon in string theory does not originally come from quantization of a scalar field. Hence, in order to understand the tachyon, we have to reconstruct the scalar field and its potential from the known results in string theory, and then analyse if the potential has a minimum.
It turns out that for open string tachyons we now know the answer in many cases. On the other hand, closed string tachyons are only beginning to be explored.

4.1 Introduction

There are many faces of Superstring/Brane Cosmology which come from different sectors of M/String Theories. In particular, many people search for potential candidates to explain Early Universe Inflation. One of the String Theory constructions, Tachyon on D-branes, has been recently proposed for cosmological applications by A. Sen. A relatively simple formulation of the unstable D-brane tachyon dynamics in terms of Effective Field Theory stimulates one to investigate its role in Cosmology.
The rolling tachyon in the string theory may be described in terms of Effective Field Theory for the Tachyon Condensate φ, which in the flat spacetime has a Lagrangian Density:

$$\mathcal{L} = -V(\varphi)\sqrt{1-2X}\,, \qquad X \equiv -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi \tag{4.5}$$

The tachyon potential $V(\varphi)$ has a positive maximum at $\varphi = 0$ and a minimum at φ_0, with $V(\varphi_0) = 0$.
It is noteworthy that the models of type (4.5) already were studied in Cosmology on the phenomenological ground. For certain choices of potentials V and non-minimal kinetic terms one can get kinematically driven inflation, *k-inflation*. However, it remains to be seen how this kind of potentials can be motivated by the String Theory of Tachyon.
In this thesis, we investigate *cosmology with tachyon matter with a string theory motivated potential* of the form $V \sim e^{\varphi/\varphi_0}$. In the next section, we write down the equations for the tachyon matter coupled to gravity. We focus on

self-consistent formulation of the isotropic Friedmann- Robertson-Walker cosmology supported by the tachyon matter. It is described by coupled equations for the time-dependent background tachyon field $\varphi(t)$ and the scale factor of the universe $a(t)$. Finally, we will study in details small fluctuations of the tachyon field $\delta\varphi(\mathbf{x}, t)$ for the previous potential coupled with a suitable inflaton field.

4.2 Cosmology with Rolling Tachyon Matter

Rolling tachyon is associated with unstable D-branes and self-consistent inclusion of gravity may require higher-dimensional Einstein equations with Branes. Still, in the low energy limit, one expects that the Brane Gravity is reduced to the four dimensional Einstein theory.
In this section, we consider tachyon matter coupled with Einstein gravity in four dimensions. Then, the model is given by the action:

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N} - V(\varphi)\sqrt{1 + \nabla^\mu\varphi\nabla_\mu\varphi}\right) \tag{4.6}$$

from which follows the Einstein equations and then the field equation for the tachyon:

$$\nabla_\mu\nabla^\mu\varphi - \frac{\nabla_\mu\nabla_\nu\varphi}{1 + \nabla^\alpha\varphi\nabla_\alpha\varphi}\nabla^\mu\varphi\nabla^\nu\varphi - \frac{V_{,\varphi}}{V} = 0 \tag{4.7}$$

If we apply these equations for the isotropic, homogeneous and spatially flat FRW cosmological model, then one can conclude that the energy-momentum tensor of tachyon matter is reduced to a diagonal form $T^\mu_\nu = diag(-\rho, p, p, p)$ where the energy density ρ is given by:

$$\rho = \frac{V(\varphi)}{\sqrt{1 - \dot{\varphi}^2}} \tag{4.8}$$

the pressure p is

$$p = -V(\varphi)\sqrt{1 - \dot{\varphi}^2} \tag{4.9}$$

and the equation for the evolution of the scale factor follows from

$$H^2(t) \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\frac{V(\varphi)}{\sqrt{1 - \dot{\varphi}^2}} \tag{4.10}$$

The equation for the *time-dependent rolling tachyon in an expanding universe* follows from (4.7),

$$\frac{\ddot{\varphi}}{1 - \dot{\varphi}^2} + 3H\dot{\varphi} + \frac{V_{,\varphi}}{V} = 0 \tag{4.11}$$

4.3 Fluctuations in Rolling Tachyon

The issue of stability of small scalar fluctuations is often essential in scalar field cosmology. In this very short section, we provide a formalism for treating linear inhomogeneous fluctuations in the tachyon field. Let us consider **small inhomogeneous perturbation of the tachyon field** $\delta\varphi(\mathbf{x}, t)$ around **time-dependent background solution** $\varphi(t)$ of equation (4.11)

$$\varphi(\mathbf{x}, t) = \varphi(t) + \delta\varphi(\mathbf{x}, t) \tag{4.12}$$

Now, linearizing the field equation (4.7) with respect to small fluctuations $\delta\varphi$ we obtain evolution equation for the time-dependent for the inhomogeneous perturbation $\delta\varphi(\mathbf{x}, t)$ in a homogeneous background ($\varphi(t)$):

$$\frac{\ddot{\delta\varphi}}{1-\dot{\varphi}^2} + \frac{2\dot{\varphi}\ddot{\varphi}}{(1-\dot{\varphi}^2)^2}\,\dot{\delta\varphi} + \left[-\Delta + (\log V)_{,\varphi\varphi}\right]\delta\varphi = 0 \tag{4.13}$$

that, using the results of the previous section, con be rewritten as follows:

$$\ddot{\delta\varphi} - (1-\dot{\varphi}^2)\,\Delta\delta\varphi + \left[3H(1-3\dot{\varphi}^2) - 2\dot{\varphi}\,\frac{V_{,\varphi}}{V}\right]\dot{\delta\varphi} + \left[\frac{V_{,\varphi\varphi}}{V} - \left(\frac{V_{,\varphi}}{V}\right)^2\right](1-\dot{\varphi}^2)\delta\varphi = 0 \tag{4.14}$$

4.4 A Toy Model for Inhomogeneous Fluctuations in the Tachyon Cosmology

Let us consider a very simple Toy Model for Inhomogeneous Fluctuations in the Tachyon Cosmology.
As we have already seen in the previous section, linearizing the field equation (4.7) with respect to small fluctuations $\delta\varphi$ we obtain the following evolution equation for the time-dependent for the inhomogeneous perturbation $\delta\varphi(\mathbf{x},t)$ in a homogeneous background ($\varphi(t)$):

$$\ddot{\delta\varphi}-(1-\dot{\varphi}^2)\,\Delta\delta\varphi+\left[3H(1-3\dot{\varphi}^2)-2\dot{\varphi}\,\frac{V_{,\varphi}}{V}\right]\dot{\delta\varphi}+\left[\frac{V_{,\varphi\varphi}}{V}-\left(\frac{V_{,\varphi}}{V}\right)^2\right](1-\dot{\varphi}^2)\delta\varphi=0 \tag{4.15}$$

To analyse the very important case of a tachyon potential which has $\varphi_0 \to \infty$ we use exponential asymptotic of the potential

$$V(\varphi) \sim \exp(-\varphi/\varphi_0) \tag{4.16}$$

which is derived from the string theory calculations.
Dimensional parameters of the potential are related to the fundamental length scale, $\varphi \sim l_s$, and V_0 is the Brane Tension. As it was demonstrated by Sen, *the tachyon matter is pressure-less for the potential with the ground state at infinity.* In this case, *tachyon matter may be considered as an interesting candidate to represent Cold Dark Matter.*
Now, if we consider a special case where the Inflaton Field assumes the following form (homogeneous background):

$$\varphi(t) \sim \sqrt{1-c^2}\,t\,, \qquad c \in \mathbb{R} \tag{4.17}$$

then, from (4.16), (4.17) and supposing $H(t) \simeq const.$, we obtain a simpler evolution equation for the time-dependent for the inhomogeneous perturbation $\delta\varphi(\mathbf{x},t)$:

$$\ddot{\delta\varphi}-c^2\Delta\delta\varphi+\left[3H(3c^2-2)+\frac{2\sqrt{1-c^2}}{\varphi_0}\right]\dot{\delta\varphi}=0 \tag{4.18}$$

Just to understand the general behaviour for this kind of inhomogeneous fluctuations in Tachyon Cosmology, let us consider the (1+1)-dimensional problem for the latter equation:

$$(\delta\varphi)_{tt}-c^2(\delta\varphi)_{xx}+\left[3H(3c^2-2)+\frac{2\sqrt{1-c^2}}{\varphi_0}\right](\delta\varphi)_t=0 \tag{4.19}$$

which is exactly the Klein-Gordon equation with Dissipation with

$$2\alpha = 3H(3c^2 - 2) + \frac{2\sqrt{1-c^2}}{\varphi_0}\,, \qquad \text{and } \beta^2 = 0 \tag{4.20}$$

If we consider a simple initial condition as

$$\delta\varphi(x,0) = 0\,, \qquad \text{and } (\delta\varphi)_t(x,0) = \delta(x) \tag{4.21}$$

from the results of the second chapter, one can immediately conclude that:

$$\delta\varphi(x,t) = \mathcal{G}^{-}_{2C}(x,t) = \frac{\exp(-\alpha t)}{2c}\, I_0\left(\chi\sqrt{t^2 - (x/c)^2}\right)\Theta(ct - |x|) \tag{4.22}$$

where:

$$\alpha = \frac{3}{2}\,H(3c^2 - 2) + \frac{\sqrt{1-c^2}}{\varphi_0}\,, \qquad \chi = \sqrt{|\beta^2 - \alpha^2|} = |\alpha| \tag{4.23}$$

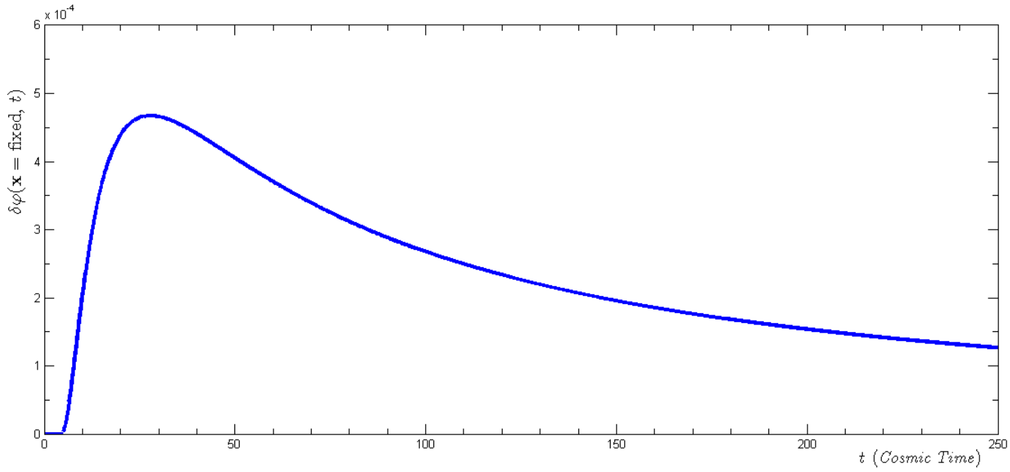


Figure 4.1: The time representation of the Second/Source Green's function $\delta\varphi(x = \text{fixed}, t) = \mathcal{G}_{2C}^{-}(x = \text{fixed}, t)$.

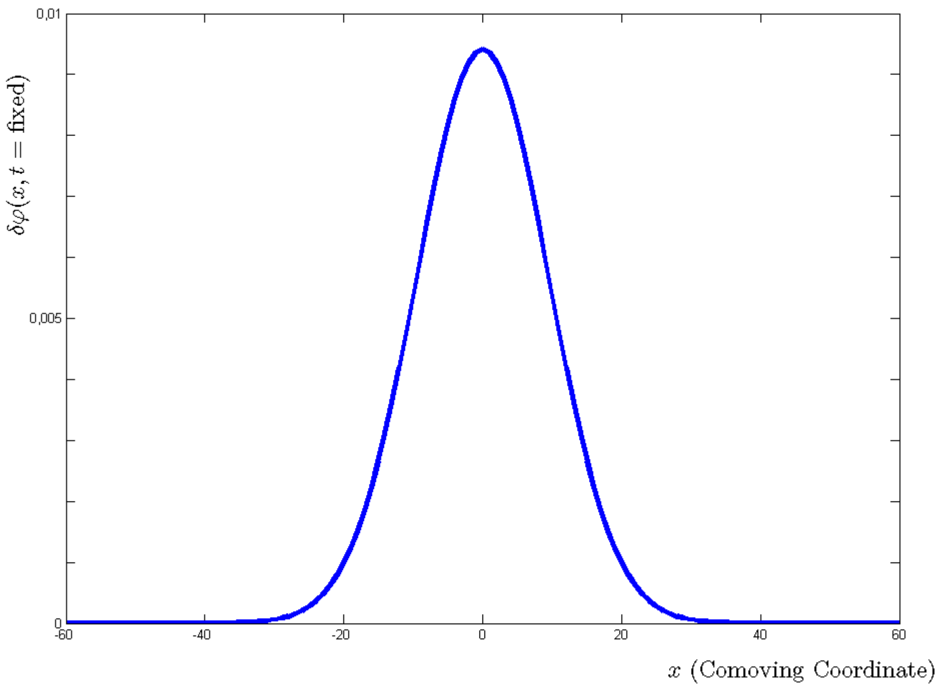


Figure 4.2: The coordinate representation of the Second/Source Green's function $\delta\varphi(x, t) = \mathcal{G}_{2C}^{-}(x, t = \text{fixed})$.

Appendix A

Dispersion Relations - Calculations

Consider the dispersion relation for the KGD:

$$\begin{aligned} -\overline{\omega}^2 + c^2\kappa^2 - 2i\alpha\overline{\omega} + \beta^2 = 0 \\ -\omega^2 + c^2\overline{\kappa}^2 - 2i\alpha\omega + \beta^2 = 0 \end{aligned} \tag{A.1}$$

- **Case 1**: $\overline{\omega} = \omega - i\gamma$.
  From the first equation of (A.1) we get:

  $$-(\omega - i\gamma)^2 + c^2\kappa^2 - 2i\alpha(\omega - i\gamma) + \beta^2 = 0 \tag{A.2}$$

  and then

  $$(-\omega^2 + \gamma^2 + c^2\kappa^2 - 2\alpha\gamma + \beta^2) + i(2\omega\gamma - 2\omega\alpha) = 0 \tag{A.3}$$

  thus

  $$\begin{cases} -\omega^2 + \gamma^2 + c^2\kappa^2 - 2\alpha\gamma + \beta^2 = 0 \\ 2\omega\gamma - 2\omega\alpha = 0 \end{cases} \tag{A.4}$$

  and we finally obtain:

  $$\overline{\omega} = i\alpha \pm \sqrt{c^2\kappa^2 + (\beta^2 - \alpha^2)} \tag{A.5}$$

  thus

  $$\begin{cases} \omega(\kappa) = Re(\overline{\omega}) = \pm Re\left(\sqrt{c^2\kappa^2 + (\beta^2 - \alpha^2)}\right) \\ \gamma(\kappa) = -Im(\overline{\omega}) = -\alpha \mp Im\left(\sqrt{c^2\kappa^2 + (\beta^2 - \alpha^2)}\right) \end{cases} \tag{A.6}$$

- **Case 2**: $\overline{\kappa} = \omega + i\delta$.
From the second equation of (A.1) we get:

$$\overline{\kappa}^2 = \frac{\omega^2 - \beta^2}{c^2} + i\,\frac{2\alpha\omega}{c^2} \tag{A.7}$$

Now, if we consider the equation:

$$z^2 = A^2 + iB^2\,, \qquad z = x + iy, \;\; x, y \in \mathbb{R} \tag{A.8}$$

than we have:

$$x^2 - y^2 + 2ixy = A + iB \;\; \Rightarrow \;\; \begin{cases} A = x^2 - y^2 \\ B = 2xy \end{cases} \tag{A.9}$$

thus

$$\begin{cases} x^2 = y^2 + A \\ y = \dfrac{B}{2x} \end{cases} \quad \Leftrightarrow \quad \begin{cases} x^2 = y^2 + A \\ y^2 = \dfrac{B^2}{4x^2} \end{cases}$$

and then it immediately follows that:

$$4x^4 - 4Ax^2 - B^2 = 0 \tag{A.10}$$

If we now define a new variable $t = x^2$ we then obtain:

$$4t^2 - 4At - B^2 = 0 \tag{A.11}$$

thus

$$x^2 = t_{\pm} = \frac{A \pm \sqrt{A^2 + B^2}}{2} \tag{A.12}$$

Clearly, we can only accept t_+ and then we obtain:

$$x = \pm\frac{1}{\sqrt{2}}\sqrt{A + \sqrt{A^2 + B^2}} \tag{A.13}$$

Now,

$$y^2 = x^2 - A = \frac{A + \sqrt{A^2 + B^2}}{2} - A = \frac{\sqrt{A^2 + B^2} - A}{2} \tag{A.14}$$

thus:

$$\begin{cases} x = \pm\dfrac{1}{\sqrt{2}}\sqrt{A + \sqrt{A^2 + B^2}} \\ y = \pm\dfrac{1}{\sqrt{2}}\sqrt{\sqrt{A^2 + B^2} - A} \end{cases} \tag{A.15}$$

This result can immediately lead us to the following conclusion: if we re-define $\kappa = x$, $\delta = y$, $A = (\omega^2 - \beta^2)/c^2$ and $2\alpha\omega/c^2$ in the equation (A.7), then we obtain the following results:

$$\begin{cases} |\kappa(\omega)| = \dfrac{1}{c\sqrt{2}} \sqrt{(\omega^2 - \beta^2) + \sqrt{(\omega^2 - \beta^2)^2 + (2\alpha\omega)^2}} \\ |\delta(\omega)| = \dfrac{1}{c\sqrt{2}} \sqrt{\sqrt{(\omega^2 - \beta^2)^2 + (2\alpha\omega)^2} - (\omega^2 - \beta^2)} \end{cases} \tag{A.16}$$

fact which conclude our discussion.

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