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HYPOELLIPTIC DIFFERENTIAL OPERATORS IN HEISENBERG GROUPS

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Introduction

The Heisenberg group $\mathbb{H}_n$ and its Lie algebra $\mathfrak{h}$ originally arose in the mathematical formalizations of quantum mechanics (see [3]). Today they appear in many research fields such as several complex variables, Fourier analysis and partial differential equations of subelliptic type.

This thesis presents a characterization of hypoellipticity for homogeneous left-invariant differential operators on $\mathbb{H}_n$, which was proved by C. Rockland in [9].

By definiton, a differential operator P on a Lie group $\mathbb{G}$ is called *hypoelliptic* if for any distribution $u \in \mathcal{D}'(\mathbb{G})$ and any open set $\Omega \subset \mathbb{G}$, the condition $Pu \in C^\infty(\Omega)$ implies that $u \in C^\infty(\Omega)$.

This characterization is given in terms of the unitary irreducible representations of the group.

In particular, the main result is the following.

Theorem 0.0.1. *Let P be a left-invariant homogeneous differential operator on the Heisenberg group $\mathbb{H}_n$. Then the following are equivalent:*

1. *P and P^t are both hypoelliptic;*

2. *for every unitary irreducible representation π of $\mathbb{H}_n$ (except the 1-dimensional identity representation), $\pi(P)$ has a bounded two-sided inverse;*

3. *for every unitary irreducible representation π of $\mathbb{H}_n$ (except the 1-dimensional identity representation), $\pi(P)v \neq 0$ and $\pi(P)^*v \neq 0$ for every nonzero C^∞-vector v of π.*

This is the analogue for $\mathbb{H}_n$ of the following result for $(\mathbb{R}^n, +)$ with dilations $x \mapsto rx$: a differential operator with constant coefficients is hypoelliptic if and only if it is elliptic.

We now show that the Theorem holds for a particular operator.

Example 0.1. As we will see, the standard basis of $\mathfrak{h}$ is given, for $i = 1, \ldots, n$, by

$$X_i := \partial_{x_i} - \frac{1}{2} y_i \partial_z, \quad Y_i := \partial_{y_i} + \frac{1}{2} x_i \partial_z, \quad Z := \partial_z.$$

The only non-trivial commutation relations are $[X_j, Y_j] = Z$, for $j = 1, \ldots, n$.

We now consider the sub-Laplacian on $\mathbb{H}_n$, which is given by

$$P = \sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

This is a left-invariant differential operator homogeneous of order 2.

Since

$$\text{rank}(\text{Lie}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}) = 2n + 1$$

at any point of $\mathbb{R}^{2n+1}$, it follows by Hörmander's Theorem (see [1]) that P is hypoelliptic. Since $P^t = P$, the first condition of the Theorem is satisfied.

The two families of unitary irreducible representations of $\mathbb{H}_n$ are given by

$$\pi_{(\xi,\eta)}(X_j) = i\ \xi_j, \quad \pi_{(\xi,\eta)}(Y_j) = i\ \eta_j, \quad \pi_{(\xi,\eta)}(Z) = 0,$$

for $(\xi, \eta) \in \mathbb{R}^{2n}$ and

$$\tilde{\pi}_\lambda(X_j) = |\lambda|^{\frac{1}{2}} \frac{d}{dt_j}, \quad \tilde{\pi}_\lambda(Y_j) = i(sgn\ \lambda)|\lambda|^{\frac{1}{2}} t_j, \quad \tilde{\pi}_\lambda(Z) = i\lambda,$$

for $\lambda \in \mathbb{R} \setminus \{0\}$. Since if P is homogeneous of order m, then

$$\tilde{\pi}_\lambda(P) = |\lambda|^{\frac{m}{2}} \tilde{\pi}_1(P), \qquad \lambda > 0,$$
$$\tilde{\pi}_\lambda(P) = |\lambda|^{\frac{m}{2}} \tilde{\pi}_{-1}(P), \qquad \lambda < 0,$$

we can consider only $\tilde{\pi}_1(P)$ and $\tilde{\pi}_{-1}(P)$. We have:

$$\pi_{(\xi,\eta)}(P) = -\sum_{j=1}^{n} (\xi_j^2 + \eta_j^2) \neq 0, \qquad (\xi, \eta) \neq (0, 0),$$

hence $\pi_{(\xi,\eta)}(P)$ is invertible if $\pi_{(\xi,\eta)}$ is not trivial. Moreover

$$\tilde{\pi}_1(P) = \tilde{\pi}_{-1}(P) = \sum_{j=1}^{n} \left(\frac{d^2}{dt_j^2} - t_j^2 \right) = -\sum_{j=1}^{n} (D_{t_j}^2 + t_j^2),$$

where $D_{t_j} = \frac{1}{i}\frac{d}{dt_j}$. Since $L^2(\mathbb{R})$ has a complete orthonormal basis consisting of eigenfunctions $\{v_j\}_{j=0}^{\infty}$ for the harmonic oscillator $D_t^2 + t^2$, we find a bounded two-sided inverse T of $\tilde{\pi}_1(P)$, which is given by:

$$Tv_k = \frac{1}{\sum_{j=1}^{n}(2k_j+1)} v_k.$$

Hence, also the second condition of the Theorem is satisfied.

Given now $v \in \mathcal{S}(\mathbb{R}^n)$, if we consider for $(\xi, \eta) \neq (0,0)$

$$\pi_{(\xi,\eta)}(P)v = -\sum_{j=1}^{n}(\xi_j^2 + \eta_j^2)v = 0,$$

this implies $v = 0$. Similarly, the equation $\tilde{\pi}_1(P)v = 0$ implies

$$\sum_{j=1}^{n}\int_{\mathbb{R}}(|D_{t_j}v|^2 + t_j^2|v|^2)dt_j = 0.$$

Hence, for any $j = 1, \ldots, n$

$$\int |D_{t_j}v|^2 dt_j = 0, \quad \int t_j^2|v|^2 dt_j = 0,$$

so that $v = 0$.

Thus, the third condition is satisfied, so the Theorem holds in case P is the sub-Laplacian.

The proof of the Theorem relies on representation theory and in particular on Plancherel Theorem, that gives a parametrization of unitary irreducible representations of a simply connected nilpotent Lie group through a Zarisky-open subset of $\mathbb{R}^q$ (for a certain q). The version we present is a compilation of its L^2 formulation (see [2]) and a geometrical exposition of the distributional version provided by Kirillov in [6].

Plancherel Theorem is used in the construction of a parametrix for the operator P. In Plancherel Theorem, however, only the generic representations of $\mathbb{H}_n$ occur and if we try to find a fundamental solution for P using them, a convergence problem arises. It is the hypothesis that $\pi(P)$ is invertible for degenerate π (i.e. π maps the vertical field to 0, see $\pi_{(\xi,\eta)}$ in the previous example) that allows us to solve the problem, finding W in the center of the enveloping algebra such that $P + W$ is elliptic and constructing a distribution u such that $P(\delta + u) = (P + W)\delta$. Taking the convolution of u with a compactly supported parametrix for $P + W$, we find the parametrix for P.

The methods that are used are quite general, in the sense that they are not crucially tied to $\mathbb{H}_n$, so they could be extended to simply-connected nilpotent Lie groups with dilations, as Rockland suggests in his paper [9] (for a generalization see [4]).

We proceed to an outline of the thesis.

In Chapter 1 we present basic notions on Lie groups and algebras, left-invariant and hypoelliptic differential operators. We recall some basic properties of special classes of operators on Hilbert spaces (unitary, trace-class and Hilbert-Schmidt operators) and the polar decomposition of a bounded operator.

Chapter 2 contains some results on unitary irreducible representations of nilpotent Lie groups. We introduce the concepts of C^∞-vectors and weak C^∞-vectors and representation of differential operators and distributions. The main result is Plancherel Theorem, which we present along with some consequences.

In Chapter 3 we consider the special case of $(\mathbb{R}^n, +)$. In particular, we prove the result for a homogeneous operator because a similar approach will be used to prove a necessary condition for hypoellipticity on $\mathbb{H}_n$.

In Chapter 4 we present the proof of the main Theorem. After some preliminaries on Heisenberg group, we classify the two families of unitary irreducible representations of $\mathbb{H}_n$, according to Stone-von Neumann Theorem.

Under the hypothesis that condition 2 of the Theorem holds, we then construct a parametrix for P and show that it is C^∞ away from the origin, which implies that P^t is hypoelliptic.

Using a similar approach as the one used in chapter 3 for $\mathbb{R}^n$, we show that if P and P^t are both hypoelliptic then for every unitary irreducible representation π of $\mathbb{H}_n$ (except the 1-dimensional identity representation), $\pi(P)v \neq 0$ and $\pi(P)^*v \neq 0$ for every nonzero C^∞-vector v of π.

Thus the Theorem is proved (modulo the proof of the fact that condition 3 implies 1, which is not shown).

Introduzione

Il gruppo di Heisenberg $\mathbb{H}_n$ e la sua algebra di Lie $\mathfrak{h}$ comparvero inizialmente nella formalizzazione matematica della meccanica quantistica (si veda [3]). Ora sono utilizzati in molti campi di ricerca, come funzioni di più variabili complesse, analisi di Fourier ed equazioni alle derivate parziali di tipo subellittico.

La tesi presenta un risultato di caratterizzazione dell'ipoellitticità di operatori differenziali omogenei invarianti a sinistra su $\mathbb{H}_n$, che è stato provato da Rockland ([9]).

Per definizione un operatore differenziale P su un gruppo di Lie $\mathbb{G}$ è detto *ipoellittico* se per ogni distribuzione $u \in \mathcal{D}'(\mathbb{G})$ e per ogni aperto $\Omega \subset \mathbb{G}$, la condizione $Pu \in C^\infty(\Omega)$ implica $u \in C^\infty(\Omega)$.

Questa caratterizzazione dell'ipoellitticità è data in termini delle rappresentazioni unitarie irriducibili del gruppo. In particolare viene provato il seguente teorema.

Theorem 0.0.2. *Sia P un operatore differenziale omogeneo invariante a sinistra sul gruppo di Heisenberg $\mathbb{H}_n$. Le proprietà seguenti sono equivalenti:*

i) sia P che P^t, il trasposto formale di P, sono ipoellittici;

ii) per ogni rappresentazione unitaria irriducibile π di $\mathbb{H}_n$ (a parte quella banale), $\pi(P)$ ha un'inversa limitata;

iii) per ogni rappresentazione unitaria irriducibile π di $\mathbb{H}_n$ (a parte quella banale), $\pi(P)v \neq 0$, $\pi(P)^*v \neq 0$ per ogni vettore C^∞ $v \neq 0$ della rappresentazione π.

Questo è l'analogo per $\mathbb{H}_n$ del seguente risultato valido su $(\mathbb{R}^n, +)$ con le dilatazioni $x \mapsto rx$: un operatore differenziale a coefficienti costanti è ipoellittico se e solo se è ellittico.

Vediamo ora con un esempio che il Teorema precedente vale nel caso di un particolare operatore.

Example 0.2. Come vedremo, la base standard di $\mathfrak{h}$ è data, per $i = 1, \ldots, n$, da

$$X_i := \partial_{x_i} - \frac{1}{2} y_i \partial_z, \quad Y_i := \partial_{y_i} + \frac{1}{2} x_i \partial_z, \quad Z := \partial_z.$$

Le uniche relazioni non nulle sono $[X_j, Y_j] = Z$, for $j = 1, \ldots, n$.

Ora consideriamo il sub-Laplaciano su $\mathbb{H}_n$, che è dato da

$$P = \sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

Esso è un operatore invariante a sinistra omogeneo di ordine 2.

Dato che

$$\text{rank}(\text{Lie}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}) = 2n + 1$$

in ogni punto di $\mathbb{R}^{2n+1}$, segue dal teorema di Hörmander (si veda [1]) che P è ipoellittico. Dato che $P^t = P$, la prima condizione del Teorema è soddisfatta.

Le due famiglie di rappresentazioni unitarie irriducibili di $\mathbb{H}_n$ sono date da

$$\pi_{(\xi,\eta)}(X_j) = i\ \xi_j, \quad \pi_{(\xi,\eta)}(Y_j) = i\ \eta_j, \quad \pi_{(\xi,\eta)}(Z) = 0,$$

per $(\xi, \eta) \in \mathbb{R}^{2n}$ e

$$\tilde{\pi}_\lambda(X_j) = |\lambda|^{\frac{1}{2}} \frac{d}{dt_j}, \quad \tilde{\pi}_\lambda(Y_j) = i(sgn\ \lambda)|\lambda|^{\frac{1}{2}} t_j, \quad \tilde{\pi}_\lambda(Z) = i\lambda,$$

per $\lambda \in \mathbb{R} \setminus \{0\}$. Dato che se P è omogeneo di ordine m, allora

$$\tilde{\pi}_\lambda(P) = |\lambda|^{\frac{m}{2}} \tilde{\pi}_1(P), \qquad \lambda > 0,$$
$$\tilde{\pi}_\lambda(P) = |\lambda|^{\frac{m}{2}} \tilde{\pi}_{-1}(P), \qquad \lambda < 0,$$

possiamo considerare solo $\tilde{\pi}_1(P)$ and $\tilde{\pi}_{-1}(P)$. Si ha:

$$\pi_{(\xi,\eta)}(P) = -\sum_{j=1}^{n} (\xi_j^2 + \eta_j^2) \neq 0, \qquad (\xi, \eta) \neq (0, 0),$$

quindi $\pi_{(\xi,\eta)}(P)$ è invertibile se $\pi_{(\xi,\eta)}$ non è banale. Inoltre

$$\tilde{\pi}_1(P) = \tilde{\pi}_{-1}(P) = \sum_{j=1}^{n} \left(\frac{d^2}{dt_j^2} - t_j^2 \right) = -\sum_{j=1}^{n} (D_{t_j}^2 + t_j^2),$$

dove $D_{t_j} = \frac{1}{i}\frac{d}{dt_j}$. Siccome $L^2(\mathbb{R})$ ha una base ortonormale completa di autofunzioni $\{v_j\}_{j=0}^{\infty}$ dell'oscillatore armonico $D_t^2 + t^2$, si può trovare un'inversa limitata T di $\tilde{\pi}_1(P)$, che è data da:

$$Tv_k = \frac{1}{\sum_{j=1}^{n}(2k_j+1)} v_k.$$

Dunque anche la seconda condizione del Teorema è soddisfatta.

Sia ora $v \in \mathcal{S}(\mathbb{R}^n)$. Se consideriamo, per $(\xi, \eta) \neq (0,0)$

$$\pi_{(\xi,\eta)}(P)v = -\sum_{j=1}^{n}(\xi_j^2 + \eta_j^2)v = 0,$$

questo implica $v = 0$. Analogamente, l'equazione $\tilde{\pi}_1(P)v = 0$ implica

$$\sum_{j=1}^{n}\int_{\mathbb{R}}(|D_{t_j}v|^2 + t_j^2|v|^2)dt_j = 0.$$

Quindi, per ogni $j = 1, \ldots, n$

$$\int |D_{t_j}v|^2 dt_j = 0, \quad \int t_j^2|v|^2 dt_j = 0,$$

cioè $v = 0$.

In conclusione vale anche la terza condizione, cioè il Teorema è valido nel caso in cui P sia il sub-Laplaciano.

La prova del Teorema 0.0.2 si basa sulla teoria delle rappresentazioni ed in particolare sul Teorema di Plancherel, che permette di parametrizzare mediante un aperto di $\mathbb{R}^q$ (per q opportuno) nella topologia di Zarisky le rappresentazioni unitarie irriducibili di un gruppo di Lie nilpotente semplicemente connesso. La versione presentata riunisce la formulazione L^2 del teorema (si veda [2]) e un'esposizione geometrica della versione distribuzionale proposta da Kirillov ([6]).

Il Teorema di Plancherel viene utilizzato nella costruzione di una parametrice per l'operatore P. Tuttavia, nel teorema compaiono solo le rappresentazioni generiche di $\mathbb{H}_n$ e se si cerca di scrivere una soluzione fondamentale utilizzandole si incorre in un problema di convergenza. L'ipotesi che $\pi(P)$ sia invertibile anche per π degenere (cioè π manda il campo verticale in 0) permette risolvere il problema, trovando un campo W nel centro dell'algebra inviluppante tale che $P + W$ sia ellittico e costruendo una distribuzione u tale che $P(\delta + u) = (P + W)\delta$. Prendendo la convoluzione di u con una parametrice a supporto compatto di $P + W$, si ottiene la parametrice per P.

I metodi utilizzati sono piuttosto generali, nel senso che non sono essenzialmente legati a $\mathbb{H}_n$, quindi possono essere estesi a gruppi di Lie nilpotenti semplicemente connessi, come suggerisce Rockland nel suo lavoro [9] (per una generalizzazione si veda [4]).

Vediamo ora come è strutturata la tesi.

Nel Capitolo 1 vengono introdotte le nozioni di base su gruppi e algebre di Lie, operatori invarianti a sinistra ed ipoellittici. Inoltre sono richiamate alcune proprietà fondamentali di alcune classi di operatori su spazi di Hilbert (unitari, di traccia e di Hilbert-Schmidt) e la decomposizione polare di operatori limitati.

Il Capitolo 2 contiene alcuni risultati sulle rappresentazioni unitarie irriducibili di gruppi di Lie nilpotenti. Si introducono i concetti di vettori C^∞ e C^∞ deboli e la rappresentazione di operatori differenziali e distribuzioni. Il risultato fondamentale è il Teorema di Plancherel, di cui vengono presentate anche alcune conseguenze.

Nel Capitolo 3 viene trattato il caso particolare di $(\mathbb{R}^n, +)$. In particolare, si prova il risultato nel caso di un operatore omogeneo poiché un approccio simile viene utilizzato nel seguito per provare una condizione necessaria per l'ipoellitticità sul gruppo $\mathbb{H}_n$.

Il Capitolo 4 presenta la dimostrazione del Teorema 0.0.2. Dopo un'introduzione sul gruppo di Heisenberg, vengono classificate, in base al teorema di Stone-von Neumann, le due famiglie di rappresentazioni unitarie irriducibili dell'algebra di Heisenberg $\mathfrak{h}$.

Per provare che la condizione $ii)$ del Teorema 0.0.2 implica l'ipoellitticità si costruisce una parametrice per l'operatore P e successivamente si prova che essa è C^∞ fuori dall'origine.

Infine si dimostra, utilizzando un procedimento simile a quello del capitolo 3 nel caso di $\mathbb{R}^n$, che se sia P che P^t sono ipoellittici allora per ogni rappresentazione unitaria irriducibile π di $\mathbb{H}_n$ (a parte quella banale), $\pi(P)v \neq 0$, $\pi(P)^*v \neq 0$ per ogni vettore C^∞ $v \neq 0$ della rappresentazione π.

Questo conclude la prova del Teorema (a meno della dimostrazione che $iii)$ implica $i)$, che non viene presentata).

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Chapter 1

Preliminaries on Lie groups and operators in Hilbert spaces

1.1 Lie groups and Lie algebras

Definition 1.1. A **Lie group** $\mathbb{G}$ is a smooth manifold $\mathbb{G}$ which is also a group and such that the map

$$\begin{aligned} \mathbb{G} \times \mathbb{G} &\to \mathbb{G} \\ (x, y) &\mapsto xy^{-1} \end{aligned}$$

is smooth.

Proposition 1.1.1. *The following transformations on a Lie group $\mathbb{G}$ are smooth:*

- *the inversion $x \mapsto x^{-1}$;*
- *the left translations $l_a(x) = ax$ and the right translations $r_a(x) = xa^{-1}$, with $a \in \mathbb{G}$.*

Consequently, the following operators transform smooth functions into smooth functions:

- *the "check" operation $f \mapsto \check{f}$, where $\check{f}(x) = f(x^{-1})$.*
- *the left and right translation operators $L_a f(x) = f(a^{-1}x)$, $R_a f(x) = f(xa)$.*

Definition 1.2. A (real) **Lie algebra** is a real vector space $\mathfrak{g}$ with a bilinear operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called Lie bracket) such that for every $X, Y, Z \in \mathfrak{g}$ we have:

1. anti-commutativity: $[X, Y] = -[Y, X]$;
2. Jacobi identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

Given any Lie group, there exists a certain Lie algebra such that the group properties are reflected into properties of the algebra. We will now see how to find it.

Definition 1.3. Let $\mathbb{G}$ be a Lie group. A smooth vector field X on $\mathbb{G}$ is called **left-invariant** if for every $a \in \mathbb{G}$ and every smooth function f

$$L_a(Xf) = XL_a(f).$$

The commutator of two left-invariant vector fields is also left-invariant.

Definition 1.4. Let $\mathbb{G}$ be a Lie group. Then the **Lie algebra of** $\mathbb{G}$ is the set of the smooth left-invariant vector fields on $\mathbb{G}$ and it will be denoted by $\mathfrak{g}$.

There is an identification between left-invariant vector fields and tangent vectors at the identity element e of $\mathbb{G}$, given by the following proposition.

Proposition 1.1.2. *Given any tangent vector v at e in $\mathbb{G}$, there exists a unique left-invariant vector field X such that $X_e = v$. It is given by*

$$Xf(x) = v(L_{x^{-1}}f). \tag{1.1}$$

Proof. Since $Xf(x) = L_{x^{-1}}(Xf)(e)$, in order X be left-invariant, it must be $Xf(x) = X(L_{x^{-1}}f)(e)$, so the identity (1.1) is forced. We now have to prove that (1.1) actually defines a left-invariant vector field. Given $a \in \mathbb{G}$, we have

$$L_a(Xf)(x) = Xf(a^{-1}x) = v(L_{x^{-1}a}f) = v(L_{x^{-1}}L_af) = X(L_af)(x).$$

□

Thus we have: $\dim \mathfrak{g} = \dim \mathbb{G}$ and we can take $T_e\mathbb{G}$ as the underlying vector space for $\mathfrak{g}$.

1.2 Vector fields, flows, exponential

Let $\mathbb{G}$ be a Lie group and X a real smooth vector field on $\mathbb{G}$. Then X can be expressed in local coordinates (x, Ω) as:

$$X = \sum_{j=1}^{n} a_j(x)\partial_{x_j}. \tag{1.2}$$

We set

$$X_x = (a_1(x), \ldots, a_n(x)).$$

An *integral curve* of X is a curve $\gamma : (-\epsilon, \epsilon) \to \Omega$ such that $\gamma'(t) = X_{\gamma(t)}$. Given a point $x_0 \in \Omega$, we consider the Cauchy problem

$$\begin{cases} \gamma'(t) = X_{\gamma(t)} \\ \gamma(0) = x_0. \end{cases} \tag{1.3}$$

Theorem 1.2.1. *Let X be a smooth vector field on $\mathbb{G}$ and let (x, Ω) be a chart of $\mathbb{G}$. Then:*

i) for every $x_0 \in \Omega$ the problem (1.3) *has a unique solution $\gamma_{x_0}(t)$ defined on a maximal open interval I_{x_0} containing 0;*

ii) given $K \subset \Omega$ compact, there is $\epsilon_K > 0$ such that γ_x is defined for $|t| < \epsilon_K$ for every $x \in K$;

iii) the map $(x, t) \mapsto \gamma_x(t)$ is smooth on its domain;

iv) more generally, if

$$X_y = \sum_{j=1}^{n} a_j(x, y)\partial_{x_j}$$

is a family of vector fields with coefficients depending smoothly on x and y, and $\gamma_{y,x_0}(t)$ is the solution of the Cauchy problem (1.3) *relative to X_y, then the map $(x, y, t) \mapsto \gamma_{y,x}(t)$ is smooth.*

For fixed t, let $\Omega_t \subset \Omega$ consist of the elements x such that $\gamma_x(t)$ is defined and let $\varphi_{X,t} : \Omega_t \to \Omega$ be given by:

$$\varphi_{X,t}(x) = \gamma_x(t).$$

Then Ω_t is open and $\varphi_{X,t}$ is smooth. Moreover, $\varphi_{X,0} = Id$ and $\varphi_{X,t} \circ \varphi_{X,t'} = \varphi_{X,t+t'}$, when defined. The maps $\varphi_{X,t}$ form the *flow* of the vector field X on Ω.

A vector field X is called *complete* when the map $(x, t) \mapsto \varphi_{X,t}(x)$ is defined for every x and every t.

Remark 1. When analyzing the flow of a vector field X locally, that is on a compact subset K of Ω, it is sometimes convenient to replace X by $X' = \eta X$, with $\eta \in C_0^\infty(\Omega)$ and identically equal to 1 on a neighborhood of K. Then the flow of X' coincides with that of X on K but has the advantage that X' is complete.

Proposition 1.2.2. *For each $t \in (-\delta, \delta)$, let $\varphi_t : \Omega_t \to \Omega$ be a smooth map, with $\Omega_t \subset \Omega$, and assume that*

i) for each compact subset K of Ω there is a $\delta(K) > 0$ such that $K \subset \Omega_t$ for $|t| < \delta(K)$;

ii) $\varphi_0 = Id$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ whenever the composition makes sense;

iii) the map $(x, t) \mapsto \varphi_t(x)$ is smooth on its domain.

If

$$Xf = \frac{d}{dt}_{|t=0} f \circ \varphi_t,$$

then X is a smooth vector field on Ω and φ_t coincides with the flow of X restricted to Ω_t.

Proof. See Proposition 1.2 in [8]. □

Definition 1.5. We call *exponential* of tX the operator $\exp(tX)$ acting on a function f on a subdomain of Ω as

$$\exp(tX)f(x) = f(\varphi_{X,t}(x)).$$

Observe that $\exp(tX)f$ is defined only on the domain Ω_t of $\varphi_{X,t}$.

Since the solution $\gamma_{sX,x_0}(t)$ of (1.3), with X replaced by its scalar multiple sX, equals $\gamma_{X,x_0}(st)$, it follows that

$$\exp((ts)X) = \exp(t(sX)).$$

Lemma 1.2.3. *Given K compact in Ω, there is $\delta_K > 0$ such that* $\exp(tX)f$ *is defined for $|t| < \delta_K$ and $f \in C_0^\infty(K)$. The exponential of X satisfies the following properties:*

i) $\exp(0X)f = f$*;*

ii) $\exp(-tX) = \exp(tX)^{-1}$*;*

iii) $\exp((t+s)X) = \exp(tX)\exp(sX)$*;*

iv) if f is C^1 then $\frac{d}{dt}\exp(tX)f = X\exp(tX)f = \exp(tX)Xf$*;*

v) if $f \in C_0^\infty(K)$ and $k \in \mathbb{N}$ then

$$\exp(tX)f(x) = \sum_{j=0}^{k} \frac{t^j}{j!} X^j f(x) + O(t^{k+1}),$$

where the remainder term depends continuously on f and X.

Proof. By Remark 1 we can assume that X is complete.

Then statements $i)$ and $iii)$ follow directly from the corresponding identities for $\varphi_{X.t}$ and $ii)$ is a direct consequence.

We now prove $iv)$. If X is in local coordinates as in (1.2) then for every smooth function f

$$Xf(x) = \sum_{j=1}^{n} a_j(x) \frac{\partial f}{\partial x_j}$$

and for every x in the domain of $\varphi_{X,t}$

$$\frac{d}{ds}_{|s=0} f(\varphi_{X,s}(x)) = \frac{df}{dx}(x) \cdot X_{\gamma(0)} = \sum_{j=1}^{n} a_j(x) \frac{\partial f}{\partial x_j}.$$

Thus

$$Xf(x) = \frac{d}{ds}_{|s=0} f(\varphi_{X,s}(x)).$$

Hence

$$\frac{d}{dt} \exp(tX)f(x) = \frac{d}{ds}_{|s=0} \exp((s+t)X)f(x) = \frac{d}{ds}_{|s=0} \exp(sX)\exp(tX)f(x) =$$
$$= \frac{d}{ds}_{|s=0} (\exp(tX)f)(\varphi_{X,s}(x)) = X(\exp(tX)f)(x)$$

but also

$$\frac{d}{dt} \exp(tX)f(x) = \frac{d}{ds}_{|s=0} \exp(tX)\exp(sX)f(x) = \exp(tX)Xf(x).$$

Finally $v)$ follows from $iv)$. □

1.3 The exponential map on a Lie group

Let $\mathbb{G}$ be a Lie group.

Definition 1.6. A one-parameter group in $\mathbb{G}$ is a smooth map $\gamma : \mathbb{R} \to \mathbb{G}$ such that $\gamma(s+t) = \gamma(s)\gamma(t)$ for every $s, t \in \mathbb{R}$.

Theorem 1.3.1. *Let $\{\varphi_t\}$ be the flow on $\mathbb{G}$ generated by a left-invariant vector field X. Then φ_t is defined on all $\mathbb{G}$ for every $t \in \mathbb{R}$. Moreover $\gamma(t) = \varphi_t(e)$ is a one-parameter group and*

$$\varphi_t(x) = x\gamma(t) \tag{1.4}$$

for every $x \in \mathbb{G}$ and $t \in \mathbb{R}$.

Conversely, given any one-parameter group $\gamma(t)$ in $\mathbb{G}$, there is a left-invariant vector field X whose flow is given by (1.4).

Proof. By Theorem 1.2.1, $\gamma(t)$ is defined at least on some interval $(-\delta, \delta)$. For $x \in \mathbb{G}$, we have since X is left-invariant:

$$\begin{aligned}\frac{d}{dt} f(x\gamma(t)) = \frac{d}{dt}(L_{x^{-1}} f)(\gamma(t)) = X(L_{x^{-1}} f)(\gamma(t)) = \\ = L_{x^{-1}}(Xf)(\gamma(t)) = Xf(x\gamma(t)).\end{aligned}$$

Thus it follows:

$$\begin{aligned}&\frac{d}{dt}(\exp(-tX) f(x\gamma(t))) = \\ &= -\exp(-tX)(Xf)(x\gamma(t)) + \exp(-tX)(Xf)(x\gamma(t)) = 0,\end{aligned}$$

which implies that $\exp(-tX) f(x\gamma(t))$ is constant. Since at $t = 0$, it equals $f(x)$, it follows that $\exp(-tX) f(x\gamma(t)) = f(x)$, hence:

$$\exp(tX) f(x) = f(x\gamma(t)).$$

This proves (1.4) for $|t| < \delta$. In particular, for $|t| < \delta$, φ_t is defined on the whole $\mathbb{G}$.

The definition of φ_t can be extended uniquely to $t \in \mathbb{R}$ in order to preserve the property: $\varphi_{s+t} = \varphi_s \circ \varphi_t$. It follows from Proposition 1.2.2 that this extension is the flow generated by X. Moreover,

$$\gamma(s+t) = \varphi_{s+t}(e) = \varphi_s(\varphi_t(e)) = \varphi_s(\gamma(t)) = \gamma(t)\gamma(s),$$

i.e. γ is a one-parameter group.

Conversely, let γ be a one-parameter group. Then the maps $\varphi_t(x) = x\gamma(t)$ satisfy the assumptions of Proposition 1.2.2. Therefore they give the flow generated by a vector field X. We have to show that X is left-invariant. This is true because for every $a \in \mathbb{G}$:

$$\begin{aligned}X(L_a f)(x) = \frac{d}{dt}_{|t=0} L_a f(x\gamma(t)) = \frac{d}{dt}_{|t=0} f(a^{-1}x\gamma(t)) = \\ = Xf(a^{-1}x) = L_a(Xf)(x).\end{aligned}$$

□

Corollary 1.3.2. *There is a one-to-one correspondence between left-invariant vector fields on $\mathbb{G}$ and one-parameter groups in $\mathbb{G}$. It assigns to every one-parameter group $\gamma(t)$ the vector field X generating the flow $\varphi_t(x) = x\gamma(t)$.*

Definition 1.7. Let $v \in T_e\mathbb{G}$, let X be the left-invariant vector field such that $X_e = v$ and $\gamma_v(t)$ the corresponding one-parameter group, according to Corollary 1.3.2. The **exponential map** $\exp_{\mathbb{G}} : \mathfrak{g} \to \mathbb{G}$ is given by

$$\exp_{\mathbb{G}}(v) = \gamma_v(1).$$

The following identities hold (for a smooth function f on $\mathbb{G}$):

$$\exp(tX)f(x) = f(x\ \exp_{\mathbb{G}}(tv)),$$
$$Xf(x) = \frac{d}{dt}_{|t=0} f(x\ \exp_{\mathbb{G}}(tv)).$$

Since $\exp_{\mathbb{G}}$ is defined on the "abstract" Lie algebra, it is correct to write $\exp_{\mathbb{G}}(X)$ instead of $\exp_{\mathbb{G}}(X_e)$. Moreover, we will use the notation $\exp(X)$ instead of $\exp_{\mathbb{G}}(X)$.

Proposition 1.3.3. *The exponential map is smooth and it is a diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ onto a neighborhood of e in $\mathbb{G}$.*

Proof. If we consider the map

$$(x, v, t) \mapsto \gamma_{v,x}(t) = x\ \exp_{\mathbb{G}}(tv)$$

it follows by Theorem 1.2.1 *iv*) that it is smooth because the left-invariant vector fields X_v depend linearly on v. When we restrict the map to $x = e$ and $t = 1$, we obtain that the exponential map $\exp_{\mathbb{G}}$ is smooth.

Let $d\ \exp_{\mathbb{G}}(0) : \mathfrak{g} \to T_e\mathbb{G} \sim \mathfrak{g}$ be the differential of $\exp_{\mathbb{G}}$ at 0. Then for $u \in \mathfrak{g}$ and f smooth on $\mathbb{G}$, we have

$$(d\ \exp_{\mathbb{G}}(0)u)(f) = \frac{d}{ds}_{|s=0} f(\exp_{\mathbb{G}}(su)) = u(f).$$

Hence $d\ \exp_{\mathbb{G}}(0) = Id$, which implies by the inverse mapping theorem that $\exp_{\mathbb{G}}$ is a diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ onto a neighborhood of e in $\mathbb{G}$. □

It can be proven that every (abstract) finite dimensional real Lie algebra is isomorphic to the Lie algebra of a Lie group.

1.4 Nilpotent Lie algebras and groups

Definition 1.8. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if there is a k such that every iterated Lie bracket of order k:

$$[\dots[[x_1, x_2], x_3]\dots, x_k]$$

is zero.

In this case it is called *step k* if k is the smallest integer for which all Lie brackets of order $k + 1$ are zero.

A Lie group $\mathbb{G}$ is called nilpotent if its Lie algebra is nilpotent.

Definition 1.9. The *center* of a Lie algebra $\mathfrak{g}$ consists of all elements $x \in \mathfrak{g}$ such that $[x, y] = 0$ for every $y \in \mathfrak{g}$.

1.5 Homogeneous Lie groups

Let $\mathbb{G}_1$, $\mathbb{G}_2$ be Lie groups and $\psi : \mathbb{G}_1 \to \mathbb{G}_2$ be a smooth group homomorphism. If v is in the Lie algebra $\mathfrak{g}_1$ of G_1, then $\psi(\exp_{\mathbb{G}_1}(tv))$ is a one-parameter group in $\mathbb{G}_2$. By Corollary 1.3.2 there is a unique $v' = \psi_*(v) \in \mathfrak{g}_2$ such that $\psi(\exp_{\mathbb{G}_1}(tv)) = \exp_{\mathbb{G}_2}(tv')$.

Lemma 1.5.1. *The map $\psi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, i.e. it is linear and*

$$\psi_*([u, v]) = [\psi_*(u), \psi_*(v)]$$

for every $u, v \in \mathfrak{g}_1$. If $\mathbb{G}_1$ is connected then ψ_* uniquely determines ψ.

Proof. See proof of Theorem 2.1.50 in [1]. □

Remark 2. We say that the Lie algebra homomorphism ψ_* is induced by the group homomorphism ψ. Observe that ψ_* is nothing but the differential of ψ at the identity of $\mathbb{G}_1$.

Let V be a real vector space. A family $\{\delta_t\}_{t>0}$ of linear maps of V to itself is called a set of *dilations* on V if there are real numbers $\lambda_j > 0$ and subspaces W_{λ_j} of V such that V is the direct sum of the W_{λ_j} and

$$(\delta_t)|_{W_{\lambda_j}} = t^{\lambda_j} Id$$

for every j.

Definition 1.10. Let $\mathfrak{g}$ be a Lie algebra and $\{\delta_t\}_{t>0}$ be a set of dilations on its underlying vector space. If each δ_t is an automorphism of $\mathfrak{g}$, then the pair $(\mathfrak{g}, \{\delta_t\})$ is called a **homogeneous Lie algebra**.

A **homogeneous Lie group** is a connected Lie group $\mathbb{G}$ endowed with a family $\{D_t\}_{t>0}$ of automorphisms such that its Lie algebra $\mathfrak{g}$ is homogeneous under the $\delta_t = (D_t)_*$.

The same Lie algebra can have different homogeneous structures.

Lemma 1.5.2. *A homogeneous Lie group is nilpotent and simply connected.*

Proof. For a proof of nilpotence of homogeneous Lie groups on $\mathbb{R}^N$ see Proposition 1.3.12 in [1]. □

1.6 Graded and stratified Lie algebras

Definition 1.11. A *gradation* on a Lie algebra $\mathfrak{g}$ is a decomposition of $\mathfrak{g}$ as the direct sum of linear subspaces $\{W_j\}_{1\leq j\leq m}$ such that $[W_j, W_k] \subset W_{j+k}$ (or $[W_j, W_k] = \{0\}$ if $j+k > m$). A Lie algebra endowed with a gradation is called a *graded* Lie algebra and the associated connected and simply connected Lie group a *graded* Lie group.

If $\{W_j\}$ is a gradation of $\mathfrak{g}$, the dilations

$$\delta_t(x) = t^j x,$$

with $x \in W_j$ are automorphism, so that $\mathfrak{g}$ canonically inherits a homogeneous structure.

Conversely, if the dilations δ_t on a homogeneous Lie algebra have eigenvalues t^j (i.e. with integer exponents), the eigenspaces W_j relative to the eigenvalues t^j form a gradation of $\mathfrak{g}$.

Definition 1.12. A *stratified* Lie algebra is a graded Lie algebra $\mathfrak{g}$ such that W_1 generates $\mathfrak{g}$. W_1 is called the *horizontal subspace*.

1.7 Universal enveloping algebra

Definition 1.13. Let $\mathfrak{g}$ be a Lie algebra and let T be the tensor algebra of $\mathfrak{g}$, that is:

$$T = T^0 \oplus T^1 \oplus \cdots \oplus T^n \oplus \ldots,$$

where $T^n = \mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (n times).

Let J be the two-sided ideal of T generated by the tensors

$$x \otimes y - y \otimes x - [x, y],$$

with $x, y \in \mathfrak{g}$.

The associative algebra T/J is called the *universal enveloping algebra* of $\mathfrak{g}$ and is denoted by $\mathcal{U}(\mathfrak{g})$.

The composite mapping σ of the canonical mappings $\mathfrak{g} \to T \to \mathcal{U}(\mathfrak{g})$ is called the canonical mapping of $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})$; for every $x, y \in \mathfrak{g}$ we have:

$$\sigma(x)\sigma(y) - \sigma(y)\sigma(x) = \sigma([x, y]).$$

The center of $\mathcal{U}(\mathfrak{g})$ is denoted by $\mathcal{Z}(\mathfrak{g})$.

1.8 Integration and convolution on Nilpotent groups

Every locally compact group (in particular any Lie group) admits a positive, regular Borel measure that is invariant under left translations, called the *left Haar measure*, which is unique up to scalar multiples; similarly such a group admits a unique, up to scalar multiples, *right Haar measure*. In general they do not coincide.

The existence of a Haar measure on $\mathbb{G}$ which is both left- and right-invariant is expressed by saying that $\mathbb{G}$ is *unimodular*.

Let $\mathbb{G}$ be a nilpotent simply connected Lie group and let $\mathfrak{g}$ be its Lie algebra. It can be proven (see Theorem 6.1 in [8]) that the Lebesgue measure on $\mathfrak{g}$ is invariant under both left and right translations of $\mathbb{G}$.

Theorem 1.8.1. *The Lebesgue measure dx on $\mathfrak{g}$ satisfies the following property: if f is an integrable function on $\mathbb{G}$ and $a \in \mathbb{G}$, then*

$$\int_{\mathbb{G}} f(xa)dx = \int_{\mathbb{G}} f(ax)dx = \int_{\mathbb{G}} f(x)dx.$$

Let $L^1(\mathbb{G})$ be the space of summable functions on $\mathbb{G}$. We can define the convolution by

$$(f * g)(x) = \int_{\mathbb{G}} f(xy^{-1})g(y)dy = \int_{\mathbb{G}} f(y)g(y^{-1}x)dy$$

with dy that denotes the Haar measure. Then $L^1(\mathbb{G})$ with the convolution is an algebra.

In general, if $\mathbb{G}$ is non-commutative, $f * g \neq g * f$.

We denote by $\mathcal{D}_k(\mathbb{G})$ the space of C^k functions compactly supported on $\mathbb{G}$, with the C^k norm, and by

$$\mathcal{D}(\mathbb{G}) = \bigcap_{k \in \mathbb{N}} \mathcal{D}_k(\mathbb{G})$$

the space on C^∞ functions compactly supported on $\mathbb{G}$.

We denote by $\mathcal{D}'(\mathbb{G})$ the space of distributions on $\mathbb{G}$ and by $\mathcal{S}'(\mathbb{G})$ the space of tempered distributions. The space of C^k distributions is denoted by $\mathcal{D}'_k(\mathbb{G})$.

For $u \in \mathcal{D}'(\mathbb{G})$ and $f \in \mathcal{D}(\mathbb{G})$, we can define:

$$u * f(x) = (u, L_x \check{f}),$$

$$f * u(x) = (u, R_{x^{-1}}\check{f}),$$

where $\check{f}(x) = f(x^{-1})$.

The same definition makes sense for $u \in \mathcal{S}'(\mathbb{G})$ and $f \in \mathcal{S}(\mathbb{G})$.

Theorem 1.8.2. *If $u \in \mathcal{D}'(\mathbb{G})$ and $f \in \mathcal{D}(\mathbb{G})$, then $u * f$ and $f * u$ are C^∞ functions on $\mathbb{G}$.*

The rule

$$supp(f * g) \subset (supp\ f)(supp\ g)$$

is respected also by convolution between a function and a distribution.

Note that:

$$\delta_a * f = L_a f \tag{1.5}$$

and

$$f * \delta_a = R_{a^{-1}} f, \tag{1.6}$$

where δ_a is the Dirac delta distribution supported on a, i.e. $(\delta_a, f) = f(a)$. In fact:

$$\delta_a * f(x) = (\delta_a, L_x\check{f}) = L_x\check{f}(a) = f(a^{-1}x) = L_a f(x)$$

and similarly for the other identity.

Moreover, if $k \in \mathfrak{g}(= T_0\mathbb{G})$ and X_k is the corresponding left-invariant vector field, then:

$$f * (\partial_k \delta_0) = X_k f. \tag{1.7}$$

In fact:

$$\begin{aligned} f * (\partial_k\delta_0)(x) &= (\partial_k\delta_0, R_{x^{-1}\check{f}} = -\partial_k(R_{x^{-1}\check{f}})(0) = \\ &= -\partial_k(L_{x^{-1}}f)\check{}(0) = \partial_k(L_{x^{-1}}f)(0) = X_k f(x). \end{aligned}$$

The convolution of two distributions is not always defined. However, if $u, v \in \mathcal{D}'(\mathbb{G})$ and one of them, say u, has compact support, then we can set for $f \in \mathcal{D}(\mathbb{G})$,

$$(u * v, f) = (v, \check{u} * f),$$
$$(v * u, f) = (v, f * \check{u}),$$

where $\check{u}$ is the distribution such that

$$(\check{u}, f) = (u, \check{f}).$$

Then $u * v$ and $v * u$ are in $\mathcal{D}'(\mathbb{G})$. If $v \in \mathcal{S}'(\mathbb{G})$ then they are also in $\mathcal{S}'(\mathbb{G})$.

If any two of the distributions u, v, w have compact support, then the associative property

$$(u * v) * w = u * (v * w)$$

holds.

For $u \in \mathcal{D}'(\mathbb{G})$ we define $L_a u$, $R_a u$ by

$$(L_a u, f) = (u, L_{a^{-1}} f), \qquad (R_a u, f) = (u, R_{a^{-1}} f).$$

This definition is motivated by the fact that, by invariance properties of the Lebesgue measure, for any test functions f, g we have:

$$(L_a f, g) = \int_{\mathbb{G}} f(a^{-1}x) g(x) dx = \int_{\mathbb{G}} f(x) g(ax) dx = (f, L_{a^{-1}} g),$$

and similarly for right translations.

We have that: $L_a u = \delta_a * u$ and $R_a u = u * \delta_{a^{-1}}$. Indeed by (1.5):

$$(L_a u, f) = (u, L_{a^{-1}} f) = (u, \delta_{a^{-1}} * f) = (u, \check{\delta}_a * f) = (\delta_a * u, f).$$

Proposition 1.8.3. *If X is a left-invariant vector field and $k = X_0$ then $Xu = u * (\partial_k \delta_0)$ and the following identities hold:*

$$(Xu, f) = -(u, Xf), \tag{1.8}$$

$$X(u * v) = u * (Xv) \tag{1.9}$$

whenever the convolution is defined.

Proof. By invariance of the Lebesgue measure, if $f, g \in \mathcal{D}(\mathbb{G})$, the integral

$$\int_{\mathbb{G}} f(xa) g(xa) dx$$

does not depend on a. Taking $a = \exp_{\mathbb{G}}(tk)$ and differentiating at $t = 0$, we obtain that

$$\int_{\mathbb{G}} (Xf(x)g(x) + f(x)Xg(x)) dx = 0,$$

so that

$$\langle Xf, g \rangle = -\langle f, Xg \rangle,$$

i.e.

$$X^t = -X. \tag{1.10}$$

By definition,

$$(Xu, f) = (u, X^t f) = -(u, Xf).$$

Hence by (1.7) and the definition of convolution:

$$(Xu, f) = -(u, f * (\partial_k \delta_0)) = -(u * (\partial_k \delta_0)\check{}, f) = (u * (\partial_k \delta_0), f).$$

Then (1.9) follows. □

1.9 Left-invariant differential operators

Let $\mathbb{G}$ be a simply connected nilpotent Lie group.

If $X_1, \dots, X_k$ are left-invariant vector fields on $\mathbb{G}$, then $P = X_1 X_2 \cdots X_k$ is a left-invariant differential operator on $\mathbb{G}$, and such is any linear combination of compositions of this kind.

We can identify the set of left-invariant differential operators on $\mathbb{G}$ with the enveloping algebra $\mathcal{U}(\mathfrak{g})$.

We fix a basis $(e_1, \dots, e_n)$ of $\mathfrak{g}$ and denote by X_j the left invariant vector field such that $(X_j)_0 = \partial_{e_j}$.

If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we set:

$$X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}.$$

Theorem 1.9.1. ***(Poincaré-Birkhoff-Witt)*** *Let P be a left-invariant differential operator on $\mathbb{G}$. Then P can be written in one and only one way as*

$$P = \sum_{|\alpha| \le m} c_\alpha X^\alpha = \sum_{|\alpha| \le m} c_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n} \tag{1.11}$$

where $c_\alpha \in \mathbb{C}$ and $|\alpha| = \alpha_1 + \dots \alpha_n$.

Proof. Let $(x_1, \dots, x_n)$ be the coordinates on $\mathfrak{g}$ induced by the fixed basis. Then

$$Pf(0) = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha f(0)$$

for some coefficients $a_\alpha \in \mathbb{C}$.

The proof goes by induction on m. If $m = 0$, then $Pf(0) = af(0)$ (thus a is determined) and

$$Pf(x) = L_{x^{-1}}(Pf)(0) = P(L_{x^{-1}}f)(0) = aL_{x^{-1}}f(0) = af(x).$$

If $m = 1$ then

$$Pf(0) = \sum_{j=1}^n a_j \partial_{x_j} f(0) + a_0 f(0).$$

If $P' = P - \sum_{j=1}^n a_j X_j$, then $P'f(0) = a_0 f(0)$. Since P' is also left-invariant, it follows that

$$P = \sum_{j=1}^n a_j X_j + a_0.$$

Assume now that the statement is true for $m-1$ and for P of order m set

$$P' = P - \sum_{|\alpha| = m} a_\alpha X^\alpha.$$

Observe that for each j,

$$X_j = \partial_{x_j} + \sum_{k=1}^{n} b_{j,k}(x)\partial_{x_k}$$

where every $b_{j,k}$ vanishes at the origin. Therefore

$$X^\alpha f(x) = \partial^\alpha f(x) + \dots$$

where the other terms either vanish at 0 or are lower-order terms. Hence $X^\alpha f(0) = \partial^\alpha f(0) +$ lower-order derivatives of f at 0.

It follows that $P' f(0)$ is a combination of derivatives of f at 0 of order not exceeding $m-1$, so we can use the inductive assumption.

The proof shows that the coefficients c_α in (1.11) coincide with a_α if $|\alpha| = m$. By induction, the representation (1.11) is unique. □

The uniqueness part depends heavily on the fact that the vector fields X_j have been ordered and that this ordering is respected when composing the monomials X^α. If this restriction is removed then the same operator P can have more than one representation as in (1.11). If, for instance, $X_3 = [X_1, X_2]$ and $P = X_2 X_1$, its correct expression according to (1.11) is $P = X_1 X_2 - X_3$.

If P is a differential operator on $\mathbb{G}$, we denote by P^t its formal transpose with respect to Haar measure, i.e. for every pair of test functions f, g:

$$\int_{\mathbb{G}} Pf(x)g(x)dx = \int_{\mathbb{G}} f(x)P^t g(x)dx$$

and by P^* its formal adjoint:

$$\int_{\mathbb{G}} Pf(x)g(x)dx = \int_{\mathbb{G}} f(x)\overline{P^* g(x)}dx.$$

It follows that $P^* = (\bar{P})^t = \overline{P^t}$, where $\bar{P}$ is defined by $\bar{P}(f) = \overline{P(\bar{f})}$.

If P is the left-invariant differential operator given by (1.11), then:

$$\bar{P} = \sum_{|\alpha| \le m} \bar{c}_\alpha X^\alpha. \tag{1.12}$$

Moreover, it follows by (1.10) that:

$$P^t = \sum_{|\alpha| \le m} c_\alpha (-X_n)^{\alpha_n} \cdots (-X_1)^{\alpha_1} \tag{1.13}$$

and

$$P^* = \sum_{|\alpha| \le m} \bar{c}_\alpha (-X_n)^{\alpha_n} \cdots (-X_1)^{\alpha_1}. \tag{1.14}$$

Definition 1.14. If $\mathbb{G}$ is a homogeneous Lie group with dilations $\{D_t\}$ and P is a left-invariant differential operator on $\mathbb{G}$, we say that P is *homogeneous of order* k if:

$$P(f \circ D_t) = t^k (Pf) \circ D_t$$

for every $f \in C^\infty(\mathbb{G})$.

Let $\{X_1, \dots, X_n\}$ be a basis of $\mathfrak{g}$ consisting of homogeneous vector fields with orders $\lambda_1, \dots, \lambda_n$.

Then the operator

$$X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$$

is homogeneous of order

$$d(\alpha) = \sum_{j=1}^{n} \alpha_j \lambda_j.$$

The following theorem is a consequence of the Poincaré-Birkhoff-Witt theorem.

Theorem 1.9.2. *A left-invariant differential operator P is homogeneous of degree k if and only if*

$$P = \sum_{d(\alpha)=k} c_\alpha X^\alpha.$$

1.10 Hypoelliptic operators

Definition 1.15. Let P be a differential operator on a Lie group $\mathbb{G}$. Then P is called **hypoelliptic** if for any distribution $u \in \mathcal{D}'(\mathbb{G})$ and any open set $\Omega \subset \mathbb{G}$, the condition $Pu \in C^\infty(\Omega)$ implies that $u \in C^\infty(\Omega)$.

This condition is equivalent to

$$\textit{sing supp } u \subset \textit{sing supp}(Pu),$$

where *sing supp* u is the complement of the largest open set where u is C^∞.

Definition 1.16. Let P be a differential operator on $\mathbb{G}$. We say that P is *locally solvable* at $x \in \mathbb{G}$ if, for every k, x has a neighborhood V_k such that for every distribution $\psi \in \mathcal{D}'_k(\mathbb{G})$ there is a distribution $u \in \mathcal{D}'(\mathbb{G})$ such that $Pu = \psi$ on V_k.

Theorem 1.10.1. *Let P be hypoelliptic on $\mathbb{G}$. Then P^t is locally solvable at every point of $\mathbb{G}$.*

Proof. See Theorem 1.5 in [8]. □

Definition 1.17. Let P be a differential operator on $\mathbb{G}$. A *parametrix* for P is a distribution $u \in \mathcal{D}'(\mathbb{G})$ such that:

$$Pu = \delta + f, \tag{1.15}$$

where $f \in C_0^\infty(\mathbb{G})$.

Theorem 1.10.2. *If P has a parametrix which is C^∞ away from the origin, then P^t is hypoelliptic.*

Proof. See Theorem 52.1 in [10]. □

1.11 Unitary, trace-class and Hilbert-Schmidt operators

When studying the representations of Lie groups, we will need some definitions about operators on a Hilbert space.

Definition 1.18. A bounded linear operator T on a Hilbert space H into H is called *unitary* if it is surjective and isometric, that is:

$$\langle Tx, Ty \rangle = \langle x, y \rangle$$

for every $x, y \in H$.

Proposition 1.11.1. *A bounded linear operator T on a Hilbert space H into H is unitary if and only if $T^* = T^{-1}$.*

We denote by $U(H)$ the group of unitary operators on H.

We will now introduce trace-class and Hilbert-Schmidt operators (see chapter VI.6 in [7] for details and proofs of the statements).

Theorem 1.11.2. *Let H be a separable Hilbert space and $\{e_k\}_k$ be an orthonormal basis of H.Then for any positive bounded operator A on H we define:*

$$trA = \sum_k \langle Ae_k, e_k \rangle.$$

The number trA is called the trace of A and it is independent of the choice of the orthonormal basis.

The trace has the following properties:

i)

$$tr(A+B) = tr(A) + tr(B).$$

ii) For all $\lambda \geq 0$,

$$tr(\lambda A) = \lambda \; trA.$$

iii) For any unitary operator U,

$$tr(UAU^{-1}) = trA.$$

iv) If $0 \leq A \leq B$ *then*

$$trA \leq trB.$$

Definition 1.19. A bounded operator A on H is called *trace-class* if and only if

$$tr|A| < \infty.$$

Theorem 1.11.3. *The family of trace-class operators is an ideal, that is:*

i) it is a vector space;

ii) if A *is trace-class and* B *is bounded, then both* AB *and* BA *are trace-class;*

ii) if A *is trace-class, then so is* A^*.

Theorem 1.11.4. *Let* $|| \cdot ||_1$ *be defined on trace-class operators by* $||A||_1 = tr|A|$. *Then the family of trace-class operators is a Banach space with this norm and* $||A|| \leq ||A||_1$.

Theorem 1.11.5. *A trace-class operator is compact. A compact operator* A *is trace-class if and only if:*

$$\sum_{k=1}^{\infty} \lambda_k < \infty, \tag{1.16}$$

where $\{\lambda_k\}_{k=1}^{\infty}$ *are the singular values of* A.

Theorem 1.11.6. *If* A *is trace-class and* $\{e_n\}_{n=1}^{\infty}$ *is any orthonormal basis, then:*

$$\sum_n \langle Ae_n, e_n \rangle$$

converges absolutely and the limit is independent of the choice of the basis.

Theorem 1.11.7. *The trace satisfies the following properties on trace-class operators:*

i) tr *is linear;*

ii) $trA^* = \overline{trA}$*;*

iii) $trAB = trBA$ *if* A *is trace-class and* B *is bounded.*

Definition 1.20. Let H be a Hilbert space. A bounded operator T on H is called *Hilbert-Schmidt* if and only if

$$tr(T^*T) < \infty.$$

Theorem 1.11.8. *The following properties hold:*

a) The family of Hilbert-Schmidt operators is an ideal.

b) If A, B *are Hilbert-Schmidt then for any orthonormal basis* $\{e_n\}$*,*

$$\sum_n \langle A^* B e_n, e_n \rangle$$

is absolutely summable and its limit, denoted by $\langle A, B\rangle_2$*, is independent of the orthonormal basis chosen.*

c) The family of Hilbert-Schmidt operators with inner product $\langle \cdot, \cdot \rangle_2$ *is a Hilbert space.*

d) If $||A||_2 = \sqrt{\langle A, A\rangle_2} = (tr(A^*A))^{\frac{1}{2}}$*, then:*

$$||A|| \leq ||A||_2 \leq ||A||_1, \qquad ||A||_2 = ||A^*||_2.$$

e) Every Hilbert-Schmidt operator is compact and a compact operator is Hilbert-Schmidt if and only if $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$*, where* $\{\lambda_k\}_{k=1}^{\infty}$ *are its the singular values.*

f) A *is trace-class if and only if* $A = BC$*, with* B, C *Hilbert-Schmidt operators.*

Remark 3. By Cauchy-Schwartz inequality, if S, T are Hilbert-Schmidt operators, then:

$$|\langle S, T\rangle_2| \leq ||S||_2 ||T||_2,$$

which means:

$$|tr(S^*T)| \leq (tr(S^*S))^{\frac{1}{2}} (tr(T^*T))^{\frac{1}{2}}. \tag{1.17}$$

1.12 Polar decomposition

Definition 1.21. Let U be an operator on a Hilbert space H with domain $D(U) = H$ and let $R(U) = \{Uf : f \in D(U)\}$ be its range. We say that U is a *partial isometry* if there exists a closed subspace M of H such that U is an isometry on M and $Uf = 0$ for every $f \in M^{\perp}$. M is called *initial domain* of U and $R(U)$ *final domain*.

The following theorem gives the *polar decomposition* of a bounded operator (see Theorem 7.20 in [11] for a more general statement and a proof).

In the same Theorem 7.20 in [11] is defined the nth root $A^{1/n}$ of a non-negative self-adjoint operator A, using the spectral family of A. If A is compact, then $A^{1/n}$ is also compact.

Theorem 1.12.1. *Let A be a bounded operator on a Hilbert space H. Then A can be uniquely represented in the form $A = UT$, where U is a partial isometry with initial domain $\overline{R(T)}$ and final domain $\overline{R(A)}$ and $T = (A^*A)^{1/2}$.*

Chapter 2

Unitary irreducible representations of nilpotent Lie groups

2.1 Unitary irreducible representations and C^∞-vectors

Let $\mathbb{G}$ be a simply-connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and (complexified) universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Definition 2.1. A **unitary representation** of $\mathbb{G}$ on a Hilbert space H is a group homomorphism $\mathbb{G} \to U(H)$, where $U(H)$ is the group of unitary operators on H.

Definition 2.2. Let π be a unitary representation of $\mathbb{G}$ on H. A vector $v \in H$ is called a $C^\infty - vector$ for π if the map $x \mapsto \pi(x)v$ from G to H is C^∞.
The space of C^∞-vectors from a vector subspace of H, which we denote by H_∞.

Given a unitary representation π of $\mathbb{G}$ on H, this determines a Lie algebra representation π of $\mathfrak{g}$ as linear maps $H_\infty \to H_\infty$ defined by

$$\pi(X)v = \frac{d}{dt}\bigg|_{t=0} \pi(\exp\ tX)v, \qquad X \in \mathfrak{g}, \quad v \in H_\infty. \tag{2.1}$$

This extends uniquely to a representation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ as linear maps $H_\infty \to H_\infty$. Indeed if $P \in \mathcal{U}(\mathfrak{g})$ is given by (1.11) then

$$\pi(P) = \sum_{|\alpha| \le m} c_\alpha \pi(X_1)^{\alpha_1} \cdots \pi(X_n)^{\alpha_n}.$$

Moreover π determines a representation of the algebra $L^1(\mathbb{G})$.

For any function $f \in L^1(\mathbb{G})$ we define $\pi(f)$ as the bounded linear operator on H given by

$$\pi(f)v = \int_{\mathbb{G}} f(y)\pi(y)v dy, \qquad v \in H. \tag{2.2}$$

This is actually a representation because:

$$\pi(f * g) = \pi(f)\pi(g). \tag{2.3}$$

Indeed:

$$\begin{aligned}
\pi(f * g) &= \int_{\mathbb{G}} (f * g)(y)\pi(y)dy = \\
&= \int_{\mathbb{G}} \left(\int_G f(yz^{-1})g(z)dz \right) \pi(y)dy = \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} f(yz^{-1})\pi(y)dy \right) g(z)dz = \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} f(w)\pi(wz)dw \right) g(z)dz = \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} f(w)\pi(w)\pi(z)dw \right) g(z)dz = \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} f(w)\pi(w)dw \right) g(z)\pi(z)dz = \\
&= \pi(f)\pi(g).
\end{aligned}$$

From (2.2) follows an estimate of the bounded operator norm $||\cdot||$ of $\pi(f)$:

$$\begin{aligned}
||\pi(f)|| &= \sup_{||v||=1} |\pi(f)v| = \\
&= \sup_{||v||=1} \left| \int_{\mathbb{G}} f(y)\pi(y)v dy \right| \le \\
&\le \sup_{||v||=1} \int_{\mathbb{G}} |f(y)||\pi(y)v| dy \le \\
&\le \int_{\mathbb{G}} |f(y)| dy = ||f||_{L^1(\mathbb{G})}.
\end{aligned}$$

Thus:

$$||\pi(f)|| \le ||f||_{L^1(\mathbb{G})}. \tag{2.4}$$

Definition 2.3. A representation of $\mathbb{G}$ is called **irreducible** if it has no nontrivial invariant subspaces.

We recall that given a representation π of $\mathbb{G}$ on H a subspace $H' \subset H$ is invariant if $\pi(g)H' \subset H'$ for any $g \in \mathbb{G}$.

Definition 2.4. Given two representations π_1, π_2 of $\mathbb{G}$ on Hilbert spaces H_1, H_2, they are called **unitarily-equivalent** if there exists a unitary operator $U : H_1 \to H_2$ such that $U\pi_1(x) = \pi_2(x)U$ for any $x \in \mathbb{G}$.

If π is an irreducible representation of $\mathbb{G}$ on H, then there is a unitary equivalence taking H to $L^2(\mathbb{R}^n)$ for some n and H_∞ to $\mathcal{S}(\mathbb{R}^n)$.

Kirillov has shown in [6] that if π is irreducible then:

- $\pi(f)$ is a compact operator for $f \in L^1(\mathbb{G})$;
- $\pi(\phi)$ is of trace-class for $\phi \in C_0^\infty(\mathbb{G})$.

For a unitary representation π, we also have that $\pi(\phi)$ maps H into H_∞ for $\phi \in C_0^\infty(\mathbb{G})$.

The Gårding subspace $\{\pi(\phi)v | \phi \in C_0^\infty(\mathbb{G}), v \in H\}$ is dense in H, and hence so is H_∞.

Lemma 2.1.1. *Let $\phi \in C_0^\infty(\mathbb{G})$ and $X \in \mathfrak{g}$. Then for any unitary representation π of $\mathbb{G}$ and any $v \in H_\infty$,*

$$\pi(X\phi)v = \pi(\phi)\pi(-X)v. \tag{2.5}$$

Proof.

$$\begin{aligned}
\pi(X\phi)v &= \int_{\mathbb{G}} (X\phi)(x)\pi(x)v dx = \\
&= \int_{\mathbb{G}} \left.\frac{d}{dt}\right|_{t=0} \phi(x\ \exp tX)\pi(x)v dx = \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_{\mathbb{G}} \phi(x\ \exp tX)\pi(x)v dx = \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_{\mathbb{G}} \phi(x)\pi(x\ \exp(-tX))v dx = \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_{\mathbb{G}} \phi(x)\pi(x)\pi(\exp(-tX))v dx = \\
&= \int_{\mathbb{G}} \phi(x)\pi(x) \left(\left.\frac{d}{dt}\right|_{t=0} \pi(\exp(-tX))v \right) dx = \\
&= \int_{\mathbb{G}} \phi(x)\pi(x)\pi(-X)v dx = \pi(\phi)\pi(-X)v.
\end{aligned}$$

□

Lemma 2.1.2. *For any unitary representation π of $\mathbb{G}$ and for any $X \in \mathfrak{g}$,*

$$\pi(-X) = \pi(X)^*$$

that is $\langle \pi(X)v, w\rangle = \langle v, \pi(-X)w\rangle$ for every $v, w \in H_\infty$.

Proof. π is a unitary representation of $\mathbb{G}$ so it's skew-adjoint on $\mathfrak{g}$. Thus: $\pi(X)^* = -\pi(X) = \pi(-X)$. □

Since $X \in \mathfrak{g}$ is real, it follows that $X^* = X^t = -X$. Thus by Lemma 2.1.2:

$$\pi(X^*) = \pi(X)^*. \tag{2.6}$$

Then it follows that for any $P \in \mathcal{U}(\mathfrak{g})$:

$$\begin{aligned}\pi(P^*) &= \pi\left(\sum_{|\alpha|\le m} \bar{c}_\alpha(-X_n)^{\alpha_n}\cdots(-X_1)^{\alpha_1}\right) = \\ &= \sum_{|\alpha|\le m} \bar{c}_\alpha \pi(-X_n)^{\alpha_n}\cdots\pi(-X_1)^{\alpha_1} = \\ &= \sum_{|\alpha|\le m} \bar{c}_\alpha (\pi(X_n)^*)^{\alpha_n}\cdots(\pi(X_1)^*)^{\alpha_1} = \pi(P)^*.\end{aligned} \tag{2.7}$$

An argument based on Lemma 2.1.1 shows that for any $P \in \mathcal{U}(\mathfrak{g})$, for any $\phi \in C_0^\infty(\mathbb{G})$ and for any $v \in H_\infty$:

$$\pi(P\phi)v = \pi(\phi)\pi(P^t)v. \tag{2.8}$$

From this follows immediately:

$$\pi(P^t\phi)v = \pi(\phi)\pi(P)v, \tag{2.9}$$

$$\pi(P^*\phi)v = \pi(\phi)\pi(\bar{P})v, \tag{2.10}$$

$$\pi(\bar{P}\phi)v = \pi(\phi)\pi(P)^*v. \tag{2.11}$$

We now define for any $\phi \in C^\infty(\mathbb{G})$:

$$\check{\phi}(x) = \phi(x^{-1}); \qquad \phi^\#(x) = \overline{\phi(x^{-1})}. \tag{2.12}$$

Lemma 2.1.3. *For any $\phi \in C_0^\infty(\mathbb{G})$ and any unitary representation π of $\mathbb{G}$:*

$$\pi(\phi^\#) = \pi(\phi)^*.$$

Proof.

$$\begin{aligned}\pi(\phi^\#) &= \int_{\mathbb{G}} \overline{\phi(x^{-1})}\pi(x)dx = \\ &= \int_{\mathbb{G}} \overline{\phi(x)}\pi(x^{-1})dx = \\ &= \int_{\mathbb{G}} \overline{\phi(x)}\pi(x)^*dx = \\ &= \left(\int_{\mathbb{G}} \phi(x)\pi(x)dx\right)^* = \pi(\phi)^*.\end{aligned}$$

□

From this Lemma and (2.3) it follows that for any $\phi \in C_0^\infty(\mathbb{G})$:

$$\pi(\phi * (\phi^\#)) = \pi(\phi)\pi(\phi)^*. \tag{2.13}$$

Lemma 2.1.4. *For any $\phi \in C_0^\infty(\mathbb{G})$ and for any unitary representation π of $\mathbb{G}$:*

$$\phi * \phi^\#(e) = ||\phi||^2_{L^2(\mathbb{G})} = ||\phi^\#||^2_{L^2(\mathbb{G})},$$

where e denotes the identity element of $\mathbb{G}$.

Proof. From the definition of convolution we have that:

$$\phi * \phi^\#(e) = \int_{\mathbb{G}} \phi(y)\phi^\#(y^{-1})dy = \int_{\mathbb{G}} \phi(y)\overline{\phi(y)}dy = ||\phi||^2_{L^2(\mathbb{G})}.$$

But since $\mathbb{G}$ is unimodular, we also have:

$$\int_{\mathbb{G}} \phi(y)\overline{\phi(y)}dy = \int_{\mathbb{G}} \phi(y^{-1})\overline{\phi(y^{-1})}dy = \int_{\mathbb{G}} \overline{\phi^\#(y)}\phi^\#(y)dy = ||\phi^\#||^2_{L^2(\mathbb{G})}.$$

□

2.2 Representation of distributions

We now want to apply unitary representations of $\mathbb{G}$ to compactly-supported distributions on $\mathbb{G}$, $\mathcal{E}'(\mathbb{G})$. We thus need to introduce the notion of weak C^∞-vectors.

Definition 2.5. Let π be a unitary representation of $\mathbb{G}$ on H. A *weak $C^\infty-$ vector* for π is a vector $v \in H$ such that:

- For every $w \in H$ the map

$$\begin{aligned} \phi_{v,w} :& \mathbb{G} \to \mathbb{C} \\ & x \mapsto \langle \pi(x)v, w\rangle \end{aligned}$$

  is C^∞.

- The map

$$\begin{aligned} & H \to C^\infty(\mathbb{G}) \\ & w \mapsto \phi_{v,w} \end{aligned}$$

  is continuous (with respect to the norm topology on H and the topology of uniform convergence on compact subsets of all partial derivatives on $C^\infty(\mathbb{G})$).

The weak C^∞-vectors form a linear subspace, H_∞^w, of H, such that $H_\infty \subseteq H_\infty^w$. We will see that when π is irreducible the two spaces coincide.

Definition 2.6. Let π be a unitary representation of $\mathbb{G}$ on H and let $u \in \mathcal{E}'(\mathbb{G})$. We define $\pi(u)$ as a possibly unbounded linear operator on H with domain H_∞^w as follows. For any $v \in H_\infty^w$, $\pi(u)v$ is the unique vector in H satisfying

$$\langle \pi(u)v, w\rangle = (u, \phi_{v,w}) \tag{2.14}$$

for every $w \in H$.

Remark 4. This definition makes sense because:

- $\phi_{v,w} \in C^\infty(\mathbb{G})$ so $(u, \phi_{v,w})$ is a well defined complex number;
- For fixed $u \in \mathcal{E}'(\mathbb{G})$ and $v \in H_\infty^w$ the map $w \mapsto (u, \phi_{v,w})$ from H to $\mathbb{C}$ is conjugate-linear and continuous (by definition 2.5) and hence (by Riesz theorem) there exists a unique vector $\pi(u)v \in H$ such that $\langle \pi(u)v, w\rangle = (u, \phi_{v,w})$.

We will now see that many of the results valid for $\pi(\phi)$, $\phi \in C_0^\infty(\mathbb{G})$, continue to hold for $\pi(u)$.

We define $\check{u}$, $u^\# \in \mathcal{E}'(\mathbb{G})$ as follows:

$$(\check{u}, \phi) = (u, \check{\phi}), \qquad (u^\#, \phi) = \overline{(u, \phi^\#)}$$

for every $\phi \in C^\infty(\mathbb{G})$.

Remark 5. If $u \in C_0^\infty(\mathbb{G})$ then this definition agrees with (2.12). Indeed:

$$(\check{u}, \phi) = (u, \check{\phi}) = \int_{\mathbb{G}} u(x)\phi(x^{-1})dx = \int_{\mathbb{G}} u(x^{-1})\phi(x)dx$$

so $\check{u}(x) = u(x^{-1})$. And similarly:

$$(u^\#, \phi) = \overline{(u, \phi^\#)} = \overline{\int_{\mathbb{G}} u(x)\overline{\phi(x^{-1})}dx} = \int_{\mathbb{G}} \overline{u(x)}\phi(x^{-1})dx = \int_{\mathbb{G}} \overline{u(x^{-1})}\phi(x)dx$$

so that $u^\#(x) = \overline{u(x^{-1})}$.

Lemma 2.2.1. *For every u_1, $u_2 \in \mathcal{E}'(\mathbb{G})$ we have:*

$$(u_1 * u_2)^\# = u_2^\# * u_1^\#. \tag{2.15}$$

Proof. For every $\phi \in C^\infty(\mathbb{G})$ we have by definition of convolution:

$$((u_1 * u_2)^\#, \phi) = \overline{(u_1 * u_2, \phi^\#)} = \overline{(u_2, \check{u}_1 * \phi^\#)}$$

and

$$(u_2^\# * u_1^\#, \phi) = (u_2^\#, \phi * \check{u}_1^\#) = \overline{(u_2, (\phi * \check{u}_1^\#)^\#)}$$

so in order to prove (2.15) it suffices to show:

$$\check{u}_1 * \phi^\# = (\phi * \check{u}_1^\#)^\#.$$

But again by definition of convolution:

$$\check{u}_1 * \phi^\#(x) = (\check{u}_1, L_x \check{\phi}^\#)$$

and

$$(\phi * \check{u}_1^\#)^\#(x) = \overline{(\phi * \check{u}_1^\#)(x^{-1})} = \overline{(\check{u}_1^\#, R_x \check{\phi})} = (\check{u}_1, R_x \check{\phi}^\#)$$

so the equality holds. □

Proposition 2.2.2. *Let π be a unitary representation of $\mathbb{G}$ on H.*

1. *If $f \in L^1(\mathbb{G}) \cap \mathcal{E}'(\mathbb{G})$ then $\pi(f)$, defined on the dense subspace H_∞^w of H, extends uniquely to a bounded operator on H which equals the original $\pi(f)$.*

2. *For any $u \in \mathcal{E}'(\mathbb{G})$ we have:*

$$\pi(u^\#) = \pi(u)^*. \tag{2.16}$$

   More precisely, for any $v, w \in H_\infty^w$, $\langle \pi(u)v, w\rangle = \langle v, \pi(u^\#)w\rangle$.

3. *For any $u \in \mathcal{E}'(\mathbb{G})$, $\pi(u)v \in H_\infty^w$ for any $v \in H_\infty^w$.*

4. *If $u \in \mathcal{E}'(\mathbb{G})$ and $\phi \in C_0^\infty(\mathbb{G})$ then:*

$$\pi(u * \phi) = \pi(u)\pi(\phi), \tag{2.17}$$

   where both sides are viewed as bounded operators on H.

5. *For any $u_1, u_2 \in \mathcal{E}'(\mathbb{G})$ we have:*

$$\pi(u_1 * u_2) = \pi(u_1)\pi(u_2), \tag{2.18}$$

   where both sides are viewed as (unbounded) operators defined on H_∞^w.

6. *For any* $P \in \mathcal{U}(\mathfrak{g})$ *and any* $u \in \mathcal{E}'(\mathbb{G})$ *we have:*

$$\pi(Pu) = \pi(u)\pi(P^t), \tag{2.19}$$

viewed as (unbounded) operators on H_∞.

7. $\pi(\delta) = I$, *where* δ *is the Dirac delta function supported at the identity element and* I *is the identity operator on* H.

Remark 6.

- We have $u * \phi \in C_0^\infty(\mathbb{G})$ and hence $\pi(u * \phi)$ is a bounded operator on H. Thus $\pi(u*\phi)$ maps H into H_∞. But by (2.17), $\pi(u*\phi) = \pi(u)\pi(\phi)$ so $\pi(u)\pi(\phi)$ is defined on all H and $\pi(u)\pi(\phi)v \in H_\infty$ for any $v \in H$. It follows that $\pi(u)$ maps the Gårding subspace into itself.

- Since by (3) $\pi(u)v \in H_\infty^w$ for any $v \in H_\infty^w$, the right-hand side of (2.18) makes sense.

- From (2.19) and (7) follows that:

$$\pi(P) = \pi(P^t\delta). \tag{2.20}$$

- If π is irreducible then $H_\infty^w = H_\infty$. Indeed:

  It suffices to prove this for any unitarily equivalent representation. Since π is irreducible we can assume without loss of generality that $H = L^2(\mathbb{R}^n)$ for some n, $H_\infty = \mathcal{S}(\mathbb{R}^n)$ and $\{\pi(P) | P \in \mathcal{U}(\mathfrak{g})\} = A_n(\mathbb{C})$, the algebra of all differential operators on $\mathbb{R}^n$ with polynomial coefficients (the Weyl algebra).
  We want to show that $H_\infty^w \subset H_\infty = \mathcal{S}(\mathbb{R}^n)$. For every $P \in \mathcal{U}(\mathfrak{g})$, $v \in H_\infty^w$, $w \in H_\infty$ we have by (2.20) that $\langle \pi(P)v, w\rangle = \langle \pi(P^t\delta)v, w\rangle = \langle v, \pi(P^t\delta)^* w\rangle$, which by (2.16) equals $\langle v, \pi((P^t\delta)^\#)w\rangle$. Since for $w \in H_\infty$ we have $\pi((P^t\delta)^\#)w = \pi(P)^* w = \pi(P^*)w$, it follows that : $\langle \pi(P)v, w\rangle = \langle v, \pi(P^*)w\rangle$ for any $v \in H_\infty^w$, $v \in H_\infty$.
  This equation says that $\pi(P)v \in L^2(\mathbb{R}^n)$, defined by (2.14), coincides with the tempered distribution obtained by applying the differential operator with polynomial coefficients $\pi(P)$ to $v \in L^2(\mathbb{R}^n)$ viewed as an element of $\mathcal{S}'(\mathbb{R}^n)$. Thus, in particular, $Qv \in L^2(\mathbb{R}^n)$ for every $Q \in A_n(\mathbb{C})$, which implies by Sobolev lemma: $Qv \in C^k(\mathbb{R}^n)$ for every $Q \in A_n(\mathbb{C})$ and every positive integer k. It follows that $v \in \mathcal{S}(\mathbb{R}^n)$.

Proof. (of Proposition 2.2.2)

1. $\pi(f)$ is the operator on H^w_∞ defined by (2.14). Since H^w_∞ is dense in H, by Hahn-Banach theorem this extends (uniquely) to an operator on H, which has the same norm of the original one (in particular it's bounded).

2. Generalization of proof of Lemma 2.1.3.

3. We have to prove that for every $v \in H^w_\infty$:

$$\forall w \in H \quad \textit{the} \quad \textit{map} \quad \phi_{\pi(u)v,w} : \mathbb{G} \to \mathbb{C} \quad \textit{is} \quad C^\infty; \tag{2.21}$$

$$\begin{array}{c} \textit{the} \quad \textit{map} \quad H \to C^\infty(\mathbb{G}) \quad \textit{is} \quad \textit{continuous.} \\ w \mapsto \phi_{\pi(u)v,w} \end{array} \tag{2.22}$$

But

$$\phi_{\pi(u)v,w}(x) = \langle \pi(x)\pi(u)v, w\rangle = \langle \pi(u)v, \pi(x)^* w\rangle = (u, \phi_{v,\pi(x)^* w})$$

and

$$\phi_{v,\pi(x)^*w}(y) = \langle \pi(y)v, \pi(x)^*w\rangle = \langle \pi(x)\pi(y)v, w\rangle = \langle \pi(xy)v, w\rangle = \phi_{v,w}(xy).$$

Thus

$$\phi_{\pi(u)v,w}(x) = (u_y, \phi_{v,w}(xy)) = u * \check{\phi}_{v,w}(x^{-1})$$

which is C^∞. So (2.21) holds.
Moreover, for any $u \in \mathcal{E}'(\mathbb{G})$ the map $\phi \mapsto u * \phi$ from $C^\infty(\mathbb{G})$ to $C^\infty(\mathbb{G})$ is continuous as well as the map $\phi \mapsto \check{\phi}$. But also the map $w \mapsto \phi_{v,w}$ from H to $C^\infty(\mathbb{G})$ is continuous because $v \in H^w_\infty$. Since $\phi_{\pi(u)v,w} = (u * \check{\phi}_{v,w})\check{}$, it follows that (2.22) holds.

4. We have to prove that for any $v, w \in H$:

$$\langle \pi(u * \phi)v, w\rangle = \langle \pi(u)\pi(\phi)v, w\rangle.$$

We know that for $v, w \in H$ the function $\phi_{v,w}(x) = \langle \pi(x)v, w\rangle$ is continuous, so in particular it's a distribution. Since

$$\begin{aligned}\langle \pi(u * \phi)v, w\rangle = \int_{\mathbb{G}} (u * \phi)(x)\phi_{v,w}(x)dx = (u * \phi) * \check{\phi}_{v,w}(e) = \\ = u * (\phi * \check{\phi}_{v,w})(e) = (u, (\phi * \check{\phi}_{v,w})\check{})\end{aligned}$$

and by definition

$$\langle \pi(u)\pi(\phi)v, w\rangle = (u, \phi_{\pi(\phi)v,w}),$$

it suffices to prove that for any $v, w \in H$

$$\phi_{\pi(\phi)v,w} = (\phi * \check{\phi}_{v,w})\check{}.$$

But we showed this in the previous point so the proof is completed.

5. Let $u_1, u_2 \in \mathcal{E}'(\mathbb{G})$, $\phi \in C_0^\infty(\mathbb{G})$. Then we have $(u_1 * u_2) * \phi = u_1 * (u_2 * \phi)$ so that

$$\pi((u_1 * u_2) * \phi) = \pi(u_1 * (u_2 * \phi)).$$

But, since $u_2 * \phi \in C_0^\infty(\mathbb{G})$, from the previous point we have that this is equivalent to:

$$\pi(u_1 * u_2)\pi(\phi) = \pi(u_1)\pi(u_2 * \phi),$$

where the right-hand side is (again by the previous point) equal to $\pi(u_1)\pi(u_2)\pi(\phi)$.
In particular, for every $v \in H, w \in H_\infty^w$:

$$\langle \pi(u_1 * u_2)\pi(\phi)v, w\rangle = \langle \pi(u_1)\pi(u_2)\pi(\phi)v, w\rangle$$

which by (2.16) is equivalent to:

$$\langle \pi(\phi)v, \pi((u_1 * u_2)^\#)w\rangle = \langle \pi(\phi)v, \pi(u_2^\#)\pi(u_1^\#)w\rangle.$$

But by (2.15), $(u_1 * u_2)^\# = u_2^\# * u_1^\#$ so for every $v \in H, w \in H_\infty^w$:

$$\langle \pi(\phi)v, \pi(u_2^\# * u_1^\#)w\rangle = \langle \pi(\phi)v, \pi(u_2^\#)\pi(u_1^\#)w\rangle.$$

Since $\{\pi(\phi)v | \phi \in C_0^\infty(\mathbb{G}), v \in H\}$ is dense in H, it follows that for every $w \in H_\infty^w$:

$$\pi(u_2^\# * u_1^\#)w = \pi(u_2^\#)\pi(u_1^\#)w.$$

Replacing $u_1^\#, u_2^\#$ by u_1, u_2 we complete the proof.

6. Let $u \in \mathcal{E}'(G)$, $P \in \mathcal{U}(\mathfrak{g})$. We have:

$$Pu = P(u * \delta) = u * P\delta,$$

where the second equality follows from (1.9).
Thus by (2.18) it follows:

$$\pi(Pu) = \pi(u)\pi(P\delta).$$

Let $v \in H_\infty$, $w \in H$. Then:

$$\langle \pi(P\delta)v, w\rangle = (P\delta, \phi_{v,w}) = (\delta, P^t\phi_{v,w}) = P^t(\langle \pi(x)v, w\rangle)|_{x=e},$$

which equals $\langle \pi(P^t)v, w\rangle$. It follows that for any $v \in H_\infty$ $\pi(P\delta)v = \pi(P^t)v$, so we have proved that $\pi(Pu) = \pi(u)\pi(P^t)$.

7. For any $v \in H_\infty$, $w \in H$:

$$\langle \pi(\delta)v, w\rangle = (\delta, \phi_{v,w}) = \phi_{v,w}(e) = \langle \pi(e)v, w\rangle = \langle v, w\rangle.$$

Thus $\pi(\delta) = I$.

□

2.3 Plancherel Theorem and consequences

We will now give a parametrization of unitary irreducible representations of a simply connected nilpotent Lie group through Plancherel Theorem, without proving it (see [6] and [2] for details). We first need a definition.

Definition 2.7. Given a Lie group $\mathbb{G}$ with Lie algebra $\mathfrak{g}$, we define the coadjoint orbit of $x \in \mathfrak{g}^*$ as the orbit $\{gx^{-1}, g \in \mathbb{G}\}$ inside $\mathfrak{g}^*$.

It can be shown that every coadjoint orbit is a symplectic manifold.

Moreover, we recall that the closed sets in the Zarisky topology on $\mathbb{R}^n$ are the algebraic sets, which are the zeros of polynomials.

The Weyl algebra $A_n(\mathbb{C})$ is the algebra of all differential operators on $\mathbb{R}^n$ with polynomial coefficients.

Theorem 2.3.1. ***(Plancherel Theorem)***
Let $\mathbb{G}$ be a simply connected nilpotent Lie group of dimension N. Let $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g}^*$ the dual of the Lie algebra, $\mathcal{U}(\mathfrak{g})$ the complexified enveloping algebra and $\mathcal{Z}(\mathfrak{g})$ the center of the enveloping algebra. Every coadjoint orbit in $\mathfrak{g}^*$ is even dimensional. Let $2n$ be the maximal dimension which occurs and let $q = N - 2n$. Let $W_1, \ldots, W_q$ be selfadjoint, algebraically independent elements of $\mathcal{Z}(\mathfrak{g})$ which generate the field of fractions of $\mathcal{Z}(\mathfrak{g})$. Then there exists a nonempty Zarisky-open subset Γ of $\mathbb{R}^q$ and for any $\lambda = (\lambda_1, \ldots, \lambda_q) \in \Gamma$ a unitary irreducible representation π_λ of $\mathbb{G}$ in $H = L^2(\mathbb{R}^n)$ so that the following properties hold:

1. *For every $\lambda \in \Gamma$ and any $i = 1, \ldots, q$, $\pi_\lambda(W_i) = \lambda_i I$.*
   Moreover any unitary irreducible representation of $\mathbb{G}$ satisfying this property is unitarily equivalent to π_λ.

2. *For every $\lambda \in \Gamma$, the space of C^∞-vectors for π_λ is $\mathcal{S}(\mathbb{R}^n)$.*

3. *For every $\lambda \in \Gamma$ the algebra of operators $\pi_\lambda(Q)$, $Q \in \mathcal{U}(\mathfrak{g})$, is the Weyl algebra $A_n(\mathbb{C})$.*

4. *For every fixed* $Q \in \mathcal{U}(\mathfrak{g})$, $\pi_\lambda(Q)$ *is a finite linear combination of elements in* $A_n(\mathbb{C})$, *independent of* λ, *whose coefficients are rational functions of* λ, *regular on* Γ.

5. *For every* $f \in L^1(\mathbb{G})$ *and every* $v \in H$, *the map*

$$\begin{aligned} \Gamma &\to H \\ \lambda &\mapsto \pi_\lambda(f)v \end{aligned}$$

is continuous. Moreover the function $\lambda \mapsto ||\pi_\lambda(f)||$ *tends to 0 as* λ *tends to* ∞.

6. *For every* $f \in L^1(\mathbb{G})$ *the nonnegative function*

$$\begin{aligned} \Gamma &\to [0,\infty) \\ \lambda &\mapsto tr(\pi_\lambda(f)\pi_\lambda(f)^*), \end{aligned}$$

where tr is the trace, is lower semicontinuous.
There exists a real-valued rational function R, regular on Γ, *and unique up to multiplication by* -1, *such that for every* $f \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$

$$\int_{\mathbb{G}} |f(x)|^2 dx = \int_\Gamma tr(\pi_\lambda(f)\pi_\lambda(f)^*) d\mu(\lambda) \tag{2.23}$$

where $d\mu(\lambda) = |R(\lambda_1,\dots,\lambda_q)| d\lambda_1 \dots d\lambda_q$.
In particular this implies that $\pi_\lambda(f)$ *is a Hilbert-Schmidt operator* $d\mu$*-almost everywhere.*

7. *Let K be the space of Hilbert-Schmidt operators on H and let* $L^2(\Gamma; K)$ *denote the* L^2*-functions on* Γ *with values in K, with respect to the measure* $d\mu$. *Then there exists a unique bijective isometry* $\Phi : L^2(\mathbb{G}) \to L^2(\Gamma; K)$ *such that for every* $f \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, $\Phi(f)$ *is the function* $\lambda \mapsto \pi_\lambda(f)$.

8. *For every* $\phi \in C_0^\infty(\mathbb{G})$ *and every* $\lambda \in \Gamma$, $\pi_\lambda(\phi)$ *is of trace-class and the function* $\lambda \mapsto tr\pi_\lambda(\phi)$ *is in* $L^1(\Gamma; d\mu)$. *Moreover:*

$$(\delta, \phi) = \phi(e) = \int_\Gamma tr\pi_\lambda(\phi) d\mu(\lambda). \tag{2.24}$$

We will now discuss some consequences of Plancherel Theorem.

Lemma 2.3.2. *Let* $\mathbb{G}$ *be a simply connected nilpotent Lie group. Then for any* $u \in \mathcal{E}'(\mathbb{G})$ *and any* $\phi \in C_0^\infty(\mathbb{G})$:

$$(u, \bar{\phi}) = \int_\Gamma tr(\pi_\lambda(u)\pi_\lambda(\phi)^*) d\mu(\lambda). \tag{2.25}$$

Proof. Since for any $\psi \in C_0^\infty(\mathbb{G})$ holds: $(u,\psi) = (u * \check{\psi})(e)$, we have in this case:

$$(u,\bar{\phi}) = (u * \check{\bar{\phi}})(e) = (u * \phi^\#)(e)$$

and $u * \phi^\# \in C_0^\infty(\mathbb{G})$. Thus by (2.24):

$$(u * \phi^\#)(e) = \int_\Gamma tr(\pi_\lambda(u * \phi^\#))d\mu(\lambda).$$

But by (2.17): $\pi_\lambda(u * \phi^\#) = \pi_\lambda(u)\pi_\lambda(\phi^\#)$ and by Lemma 2.1.3 $\pi_\lambda(\phi^\#) = \pi_\lambda(\phi)^*$, so we have the claim. □

Lemma 2.3.3. *Let $\mathbb{G}$ be a simply connected nilpotent Lie group. Then for any $f, g \in L^2(\mathbb{G})$:*

$$\int_\mathbb{G} f(x)\overline{g(x)}dx = \langle \Phi(f), \Phi(g)\rangle \tag{2.26}$$

where $\langle \cdot,\cdot\rangle$ is the inner product of $L^2(\Gamma; K)$.

Proof. The left-hand side of (2.26) is the inner product of f and g in $L^2(\mathbb{G})$, so the result holds because Φ is an isometry of Hilbert spaces. □

Proposition 2.3.4. *Let $\mathbb{G}$ be a simply connected nilpotent Lie group and let $u \in \mathcal{E}'(\mathbb{G})$. If the function $\lambda \mapsto \pi_\lambda(u)$ from Γ into the space of unbounded linear operators on $H = L^2(\mathbb{R}^n)$ (with domain $\mathcal{S}(\mathbb{R}^n)$) is in $L^2(\Gamma; K)$, then $u \in L^2(\mathbb{G}) \cap \mathcal{E}'(\mathbb{G})$.*

Proof. Since $\Phi : L^2(\mathbb{G}) \to L^2(\Gamma; K)$ is bijective, there exists $f \in L^2(\mathbb{G})$ such that $\Phi(f)(\lambda) = \pi_\lambda(u)$. Let $\phi \in C_0^\infty(\mathbb{G})$. By (2.25):

$$(u,\bar{\phi}) = \int_\Gamma tr(\pi_\lambda(u)\pi_\lambda(\phi)^*)d\mu(\lambda).$$

But $C_0^\infty(\mathbb{G}) \subset L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ so $\Phi(\phi)(\lambda) = \pi_\lambda(\phi)$ (by Plancherel Theorem (7)). Thus by (2.26):

$$\int_\mathbb{G} f(x)\overline{\phi(x)}dx = \int_\Gamma tr(\pi_\lambda(u)\pi_\lambda(\phi)^*)d\mu(\lambda),$$

which implies $(u,\bar{\phi}) = (f,\bar{\phi})$. Since $\phi \in C_0^\infty(\mathbb{G})$ is arbitrary, it follows $u = f$. □

Remark 7. Let $\mathbb{G}$ be a simply connected nilpotent Lie group and let $\{\delta_r\}$ be a group of dilations on $\mathbb{G}$. Suppose that each W_i, $i = 1,\ldots,q$ in Plancherel Theorem is homogeneous of degree m_i, i.e. $\delta_r(W_i) = r^{m_i}W_i$.

1. Define $\delta_r : \mathbb{R}^q \to \mathbb{R}^q$, $\delta_r(\lambda_1, \ldots, \lambda_q) = (r^{m_1}\lambda_1, \ldots, r^{m_q}\lambda_q)$ and suppose that δ_r maps Γ into Γ for any r and $0 \notin \Gamma$. For every $\lambda \in \mathbb{R}^q$, $\lambda \neq 0$, there exists a unique $r \in \mathbb{R}^+$ such that the Euclidean norm $||\delta_{r^{-1}}\lambda|| = 1$. We call this r the norm of λ and we denote it by $|\lambda|$.

2. Since δ_r is an automorphism of $\mathbb{G}$, $\pi_\lambda \circ \delta_r$ is an irreducible unitary representation of $\mathbb{G}$ for every λ. Thus $\pi_\lambda \circ \delta_r(W_i) = \pi_\lambda(r^{m_i}W_i) = r^{m_i}\lambda_i I$, which implies by Plancherel Theorem (1) that $\pi_\lambda \circ \delta_r$ is unitarily equivalent to $\pi_{\delta_r\lambda}$.
   This means that for every $\lambda \in \Gamma$, $r \in \mathbb{R}^+$, there exists a unitary operator $U_{\lambda,r}$ on H such that

$$\pi_\lambda \circ \delta_r = U_{\lambda,r}\pi_{\delta_r\lambda}U_{\lambda,r}^{-1}. \tag{2.27}$$

We will now renormalize the choice of π_λ to get rid of the factor $U_{\lambda,r}$.

Lemma 2.3.5. *For every $\lambda \in \Gamma$ there is a unitary operator V_λ on H such that the representations $\tilde{\pi}_\lambda = V_\lambda\pi_\lambda V_\lambda^{-1}$ satisfy the same properties as π_λ in Plancherel Theorem (only 4 is partially modified) and in addition:*

$$\tilde{\pi}_\lambda \circ \delta_r = \tilde{\pi}_{\delta_r\lambda} \tag{2.28}$$

for every $\lambda \in \Gamma$ and $r \in \mathbb{R}^+$.

Proof. If we have:

$$V_{\delta_r\lambda} \cong V_\lambda U_{\lambda,r} \tag{2.29}$$

where $\cong$ denotes equality up to a scalar multiple of modulus 1, then by (2.27):

$$\begin{aligned}\tilde{\pi}_\lambda \circ \delta_r = (V_\lambda\pi_\lambda V_\lambda^{-1}) \circ \delta_r = V_\lambda(\pi_\lambda \circ \delta_r)V_\lambda^{-1} = \\ = V_\lambda U_{\lambda,r}\pi_{\delta_r\lambda}U_{\lambda,r}^{-1}V_\lambda^{-1} = V_{\delta_r\lambda}\pi_{\delta_r\lambda}V_{\delta_r\lambda}^{-1} = \tilde{\pi}_{\delta_r\lambda}\end{aligned}$$

so (2.28) holds. So it suffices to prove (2.29).

On the other hand, $U_{\lambda,r}$ satisfies the "cocycle" condition:

$$U_{\lambda,r_1} \circ U_{\delta_{r_1}\lambda,r_2} \cong U_{\lambda,r_1r_2}. \tag{2.30}$$

Indeed we know that:

$$\pi_\lambda \circ \delta_{r_1r_2} = U_{\lambda,r_1r_2}\pi_{\delta_{r_1r_2}\lambda}U_{\lambda,r_1r_2}^{-1} \tag{2.31}$$

and

$$\pi_\lambda \circ \delta_{r_1} = U_{\lambda,r_1}\pi_{\delta_{r_1}\lambda}U_{\lambda,r_1}^{-1}$$

which implies:

$$\pi_{\delta_{r_1}\lambda} = U_{\lambda,r_1}^{-1}(\pi_\lambda \circ \delta_{r_1})U_{\lambda,r_1}. \tag{2.32}$$

Moreover from:

$$\pi_{\delta_{r_1}\lambda} \circ \delta_{r_2} = U_{\delta_{r_1}\lambda,r_2}\pi_{\delta_{r_1 r_2}\lambda}U_{\delta_{r_1}\lambda,r_2}^{-1}$$

and (2.32) follows:

$$(U_{\lambda,r_1}^{-1}(\pi_\lambda \circ \delta_{r_1})U_{\lambda,r_1}) \circ \delta_{r_2} = U_{\delta_{r_1}\lambda,r_2}\pi_{\delta_{r_1 r_2}\lambda}U_{\delta_{r_1}\lambda}^{-1}. \tag{2.33}$$

Now since $\delta_{r_1} \circ \delta_{r_2} = \delta_{r_1 r_2}$, (2.31) and (2.33) imply (2.30).

We now define V_λ by:

$$V_\lambda = U_{\delta_{|\lambda|^{-1}}\lambda,|\lambda|}. \tag{2.34}$$

By (2.30) and the definition of $|\lambda|$ it follows:

$$\begin{aligned} V_{\delta_r\lambda} &= U_{\delta_{|\delta_r\lambda|^{-1}}(\delta_r\lambda),|\delta_r\lambda|} \cong U_{\delta_{r|\delta_r\lambda|^{-1}}\lambda,|\delta_r\lambda|r^{-1}} \circ U_{\lambda,r} \cong \\ &\cong U_{\delta_{|\lambda|^{-1}}\lambda,|\lambda|} \circ U_{\lambda,r} \cong V_\lambda \circ U_{\lambda,r}. \end{aligned}$$

Hence (2.29) holds, i.e. the "cocycle" $U_{\lambda,r}$ is a "coboundary".

We now have to verify that $\tilde{\pi}_\lambda$ satisfies the same properties as π_λ in Plancherel Theorem.

First we show that property 2 holds, that is the space of C^∞-vectors for $\tilde{\pi}_\lambda$ is $\mathcal{S}(\mathbb{R}^n)$. It follows from (2.27) that for any $v \in H = L^2(\mathbb{R}^n)$, v is a C^∞-vector for $\pi_{\delta_r\lambda}$ if and only if $U_{\lambda,r}v$ is a C^∞-vector for $\pi_\lambda \circ \delta_r$, which means: for any λ, r and any $v \in L^2(\mathbb{R}^n)$, $U_{\lambda,r}v \in \mathcal{S}(\mathbb{R}^n)$ if and only if $v \in \mathcal{S}(\mathbb{R}^n)$. Hence, by definition (2.34) of V_λ it follows that the space of C^∞-vectors for $\tilde{\pi}_\lambda$ is $\mathcal{S}(\mathbb{R}^n)$.

Moreover, property 3 holds because: for every $\lambda \in \Gamma$ by the same property for π_λ we know that $\{\pi_\lambda(Q)|Q \in \mathcal{U}(\mathfrak{g})\} = A_n(\mathbb{C})$, so by (2.27):

$$A_n(\mathbb{C}) = \{U_{\lambda,r}TU_{\lambda,r}^{-1}|T \in A_n(\mathbb{C})\}.$$

We will now see how property 4 of Plancherel Theorem is slightly modified for $\tilde{\pi}_\lambda$. Let $Q \in \mathcal{U}(\mathfrak{g})$. Then by Plancherel Theorem (4) there exist $T_1, \dots, T_k \in A_n(\mathbb{C})$ and rational functions $R_1(\lambda), \dots, R_k(\lambda)$ such that

$$\pi_\lambda(Q) = \sum_{j=1}^{k} R_j(\lambda)T_j. \tag{2.35}$$

If Q is homogeneous of degree m then:

$$\pi_\lambda \circ \delta_r(Q) = r^m \sum_{j=1}^{k} R_j(\lambda)T_j. \tag{2.36}$$

But by (2.27):

$$\pi_\lambda \circ \delta_r(Q) = U_{\lambda,r}\left(\sum_{j=1}^{k} R_j(\delta_r\lambda)T_j\right)U_{\lambda,r}^{-1}. \tag{2.37}$$

It follows that for any λ, r the right-hand sides of (2.36) and (2.37) are equal.

If we now replace λ by $\delta_{|\lambda|^{-1}}\lambda$ and r by $|\lambda|$ (then $\delta_r\lambda$ is replaced by λ) this equality becomes:

$$V_\lambda\left(\sum_{j=1}^{k} R_j(\lambda)T_j\right)V_\lambda^{-1} = |\lambda|^m \sum_{j=1}^{k} R_j(\delta_{|\lambda|^{-1}}\lambda)T_j,$$

where the left-hand side equals $\tilde{\pi}_\lambda(Q)$. Hence we have that if $Q \in \mathcal{U}(\mathfrak{g})$ satisfies (2.35) and is homogeneous of degree m then:

$$\tilde{\pi}_\lambda(Q) = |\lambda|^m \sum_{j=1}^{k} R_j(\delta_{|\lambda|^{-1}}\lambda)T_j. \tag{2.38}$$

Since every $Q \in \mathcal{U}(\mathfrak{g})$ is uniquely expressible as a finite sum of homogeneous elements of $\mathcal{U}(\mathfrak{g})$, applying (2.38) to each of these we obtain the analogue of property 4.

Properties 1,6,8 and the second part of 5 are invariant under unitary equivalence. Indeed:

1. For any $\lambda \in \Gamma$ and any $i = 1, \dots, q$

$$\tilde{\pi}_\lambda(W_i) = V_\lambda \pi_\lambda(W_i)V_\lambda^{-1} = V_\lambda \lambda_i I V_\lambda^{-1} = \lambda_i I.$$

6. For any $f \in L^1(\mathbb{G})$ and any $\lambda \in \Gamma$

$$tr(\tilde{\pi}_\lambda(f)\tilde{\pi}_\lambda(f)^*) = tr(V_\lambda \pi_\lambda(f)\pi_\lambda(f)^* V_\lambda^{-1}) = tr(\pi_\lambda(f)\pi_\lambda(f)^*).$$

8. For any $\phi \in C_0^\infty(\mathbb{G})$ and any $\lambda \in \Gamma$, since $\tilde{\pi}_\lambda(\phi) = V_\lambda \pi_\lambda(\phi)V_\lambda^{-1}$, it follows that $\tilde{\pi}_\lambda(\phi)$ is trace-class and $tr(\tilde{\pi}_\lambda(\phi)) = tr(\pi_\lambda(\phi))$.

5. (second part) For any $f \in L^1(\mathbb{G})$ and any $\lambda \in \Gamma$, $||\tilde{\pi}_\lambda(f)|| = ||\pi_\lambda(f)$.

Property 7 for $\tilde{\pi}_\lambda$ can be proven in the same way as for π_λ.

For the first part of property 5 see [2] (Lemma 33) and [9] (proof of Lemma 2.13). □

Corollary 2.3.6. *If $P \in \mathcal{U}(\mathfrak{g})$ is homogeneous of degree m, then for any $\lambda \in \Gamma$:*

$$\tilde{\pi}_\lambda(P) = |\lambda|^m \tilde{\pi}_{\delta_{|\lambda|^{-1}}\lambda}(P) = |\lambda|^m \pi_{\delta_{|\lambda|^{-1}}\lambda}(P). \tag{2.39}$$

Proof. The first equality follows from (2.28):

$$\tilde{\pi}_{\delta_{|\lambda|^{-1}}\lambda}(P) = \tilde{\pi}_\lambda \circ \delta_{|\lambda|^{-1}}(P) = |\lambda|^{-m}\tilde{\pi}_\lambda(P).$$

Since δ_1 is the identity map, it follows from (2.27) that $U_{\lambda,1} \cong I$ for any $\lambda \in \Gamma$. Thus by (2.34) $V_\lambda \cong I$ if $|\lambda| = 1$. Hence $\tilde{\pi}_\lambda = \pi_\lambda$ if $|\lambda| = 1$ and so the second equality holds. □

We will need to work interchangeably with P and $\bar{P}$ (resp., P^* and P^t), thus we observe the following.

Remark 8. Suppose that for every $i = 1, \ldots, q$ W_i is either symmetric, i.e. $W_i = W_i^t$, or antisymmetric, i.e. $W_i = -W_i^t$, and let $\tau_i = 1$ or -1 correspondingly. Since each W_i is selfadjoint, i.e. $W_i = W_i^*$, these conditions correspond to $\overline{W}_i = W_i$ and $\overline{W}_i = -W_i$, respectively. For any $\lambda \in \mathbb{R}^q$, let $\tau\lambda = (\tau_1\lambda_1, \ldots, \tau_q\lambda_q)$, and suppose that τ leaves Γ invariant.

For $\lambda \in \Gamma$ define:

$$\hat{\pi}_\lambda(x) = [\tilde{\pi}_\lambda(x^{-1})]^t = \overline{\pi_\lambda(x)}, \tag{2.40}$$

where $\overline{\pi_\lambda(x)}v = \overline{\pi_\lambda(x)\bar{v}}$ for any $v \in L^2(\mathbb{R}^n)$. Then $\hat{\pi}_\lambda$ is again an irreducible unitary representation of $\mathbb{G}$. Moreover, since $\mathcal{S}(\mathbb{R}^n)$ is closed with respect to conjugation, it follows from (2.40) that the space of C^∞-vectors for $\hat{\pi}_\lambda$ is also $\mathcal{S}(\mathbb{R}^n)$.

For any $Q \in \mathcal{U}(\mathfrak{g})$, we have:

$$\hat{\pi}_\lambda(Q) = [\tilde{\pi}_\lambda(Q^t)]^t = \overline{\tilde{\pi}_\lambda(\bar{Q})},$$

where the second equality follows from (2.7) and the fact that $\bar{Q} = Q^{t*}$:

$$\overline{\tilde{\pi}_\lambda(\bar{Q})} = [\tilde{\pi}_\lambda(Q^{t*})]^{t*} = [\tilde{\pi}_\lambda(Q^t)]^{t**} = [\tilde{\pi}_\lambda(Q^t)]^t.$$

Since $\hat{\pi}_\lambda(W_i) = \overline{\tilde{\pi}_\lambda(\overline{W}_i)} = \tau_i\lambda_i I$ for any $i = 1, \ldots, q$, it follows from Plancherel Theorem (1) that $\hat{\pi}_\lambda$ is unitarily equivalent to $\tilde{\pi}_{\tau\lambda}$, i.e. there exists a unitary operator S_λ (unique up to a scalar multiple) such that:

$$\hat{\pi}_\lambda = S_\lambda \tilde{\pi}_{\tau\lambda} S_\lambda^{-1}.$$

It follows that S_λ and S_λ^{-1} map C^∞-vectors to C^∞-vectors, i.e. they map $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Thus for any $Q \in \mathcal{U}(\mathfrak{g})$ and any $\lambda \in \Gamma$, we have:

$$\overline{\tilde{\pi}_\lambda(\bar{Q})} = [\tilde{\pi}_\lambda(Q^t)]^t = S_\lambda \tilde{\pi}_{\tau\lambda}(Q) S_\lambda^{-1}. \tag{2.41}$$

Since S_λ, S_λ^{-1} map $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, it follows that if $\tilde{\pi}_\lambda(P)$ has a bounded left or right inverse (see definition 4.1 in Chapter 4) for every $\lambda \in \Gamma$, then so does $\tilde{\pi}_\lambda(\bar{P})$.

Chapter 3

Special case: $\mathbb{G} = (\mathbb{R}^n, +)$

If we consider the group $\mathbb{G} = (\mathbb{R}^n, +)$ with dilations $x \mapsto rx$, then a basis for its Lie algebra is given by the left-invariant derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$.

The unitary irreducible representations of $\mathbb{R}^n$ are all 1-dimensional and parametrized by $\mathbb{R}^n$ as follows:

$$\pi_\xi(x) = e^{-ix\cdot\xi}, \qquad \xi \in \mathbb{R}^n$$

which means that $\pi_\xi(x)$ acts on the 1-dimensional Hilbert space $\mathbb{C}$ by multiplication by $e^{-ix\cdot\xi}$. The identity representation corresponds to $\xi = 0$.

If we denote by δ_j the vector on $\mathbb{R}^n$ with j-th entry equal to 1 and all other entries 0, we have by (2.1):

$$\begin{aligned}\pi_\xi\left(\frac{\partial}{\partial x_j}\right) &= \frac{d}{dt}_{|t=0}\pi_\xi\left(\exp\ t\frac{\partial}{\partial x_j}\right) = \frac{d}{dt}_{|t=0}\pi_\xi(t\delta_j) = \\ &= \frac{d}{dt}_{|t=0} e^{-it\xi_j} = -i\xi_j.\end{aligned} \tag{3.1}$$

Hence, if $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, then $\pi_\xi(D_{x_j}) = \xi_j$.

For any function $f \in L^1(\mathbb{R}^n)$ it follows by (2.2) that:

$$\pi_\xi(f)v = \int_{\mathbb{R}^n} f(y)e^{-iy\cdot\xi}v dy = \hat{f}(\xi)v,$$

where $\hat{f}(\xi)$ denotes the Fourier transform of f.

Then Lemma 2.1.1 says that for every $\varphi \in C_0^\infty(\mathbb{R}^n)$:

$$\pi_\xi\left(\frac{\partial\varphi}{\partial x_j}\right) = \hat{\varphi}\pi_\xi\left(-\frac{\partial}{\partial x_j}\right)$$

which means

$$\frac{\hat{\partial \varphi}}{\partial x_j} = i\xi_j \varphi.$$

This is the usual formula for the Fourier transform of the derivative of a function.

For any distribution $u \in \mathcal{E}'(\mathbb{R}^n)$ it follows by (2.14) that:

$$\pi_\xi(u) = \hat{u}(\xi). \tag{3.2}$$

A left-invariant (hence bi-invariant) differential operator P on $\mathbb{R}^n$ is a constant coefficient differential operator. Moreover, P is homogeneous of degree m precisely if it has only terms of highest order m in the usual sense, i.e. we can write P as:

$$P = \sum_{|\alpha|=m} a_\alpha D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \qquad a_\alpha \in \mathbb{C}. \tag{3.3}$$

There is a characterization of hypoellipticity for, not necessarily homogeneous, constant-coefficient differential operators on $\mathbb{R}^n$, which is a special case of Hörmander's result.

We will treat the homogeneous case which will give us useful tools for the result on Heisenberg group.

Proposition 3.0.7. *Let P be a constant-coefficient differential operator on $\mathbb{R}^n$, homogeneous of order m. Then the following are equivalent:*

i) $\pi(P)$ is invertible for every unitary irreducible representation π of $\mathbb{R}^n$ except the identity representation (this means that P is elliptic);

ii) P is hypoelliptic;

iii) there exists $u \in \mathcal{E}'(\mathbb{R}^n)$ such that $Pu = \delta + f$ for some $f \in C_0^\infty(\mathbb{R}^n)$.

Proof. Let P be as in (3.3).

First we show that *i)* implies *ii)*. The usual principal symbol of P is given by:

$$\sigma(P)(\xi) = \sum_{|\alpha|=m} a_\alpha \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n},$$

which equals $(-1)^m \pi_\xi(P)$. This shows that *i)* states precisely that P is elliptic, hence implies *ii)*.

We then show that *ii)* implies *iii)*. Every constant-coefficient differential operator on $\mathbb{R}^n$ has a fundamental solution, i.e. there exists $v \in \mathcal{D}'(\mathbb{R}^n)$ such that $Pv = \delta$. If P is hypoelliptic then v is C^∞ away from 0. Hence if we take $u = \varphi v$ with $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi \equiv 1$ in a neighborhood of 0, we have:

$$\begin{aligned} Pu = P(\varphi v) &= \varphi P v + [P, \varphi] v = \\ &= \varphi \delta + [P, \varphi] v = \delta + f, \end{aligned}$$

where $f = [P, \varphi]v$ is C_0^∞ because the derivatives of φ vanish near 0 where v is not C^∞. Thus iii) holds.

Finally we prove that iii) implies i). Since u, and hence Pu, is compactly supported, we can apply unitary representations to $Pu = \delta + f$. It follows by (3.2) that:

$$\hat{u}(\xi)\pi_\xi(P^t) = \pi_\xi(Pu) = \pi_\xi(\delta) + \pi_\xi(f) = 1 + \hat{f}(\xi).$$

But by (3.3) $P^t = (-1)^m P$, hence:

$$\hat{u}(\xi)\sigma(P)(\xi) = 1 + \hat{f}(\xi).$$

Replacing ξ by $r\xi$, $r \in \mathbb{R}^+$, and noting that

$$\sigma(P)(r\xi) = r^m \sigma(P)(\xi),$$

we get:

$$r^m \hat{u}(r\xi)\sigma(P)(\xi) = 1 + \hat{f}(r\xi) \tag{3.4}$$

for every $r \in \mathbb{R}^+$.

Since $f \in C_0^\infty(\mathbb{R}^n)$, we have that $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$. In particular, for fixed $\xi \neq 0$, $|\hat{f}(r\xi)| < \epsilon$ for r sufficiently large. Therefore, for r sufficiently large, the right-hand side of (3.4) does not equal 0, and so $\sigma(P)(\xi) \neq 0$. This means that P is elliptic. □

Chapter 4

Hypoellipticity on the Heisenberg group $\mathbb{H}_n$

The main result of this chapter is to present a representation-theoretic characterization of hypoellipticity for homogeneous left-invariant differential operators on the Heisenberg group $\mathbb{H}_n$.

In particular we will prove the following theorem.

Theorem 4.0.8. *Let P be a left-invariant homogeneous differential operator on the Heisenberg group $\mathbb{H}_n$. Then the following are equivalent:*

1. *P and P^t are both hypoelliptic;*
2. *for every unitary irreducible representation π of $\mathbb{H}_n$ (except the 1-dimensional identity representation), $\pi(P)$ has a bounded two-sided inverse;*
3. *for every unitary irreducible representation π of $\mathbb{H}_n$ (except the 1-dimensional identity representation), $\pi(P)v \neq 0$ and $\pi(P)^*v \neq 0$ for every nonzero C^∞-vector v of π.*

4.1 The Heisenberg group

We denote by $\mathbb{H}_n$ the n-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}_n$ is denoted by $p = (p_1, \dots, p_{2n}, p_{2n+1}) = (p', p_{2n+1})$, with $p' \in \mathbb{R}^{2n}$ and $p_{2n+1} \in \mathbb{R}$, or by $p = (x, y, z)$, with both $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. If p, $q \in \mathbb{H}_n$, the group operation is defined as

$$p \cdot q = \left(p' + q', p_{2n+1} + q_{2n+1} - \frac{1}{2} \sum_{j=1}^{n} (p_j q_{j+n} - p_{j+n} q_j) \right).$$

We denote as $p^{-1} := (-p', -p_{2n+1})$ the inverse of p and as 0 or e the identity of $\mathbb{H}_n$. Sometimes, we write also pq for $p \cdot q$.

For $r > 0$ we define the dilations $\delta_r : \mathbb{H}_n \to \mathbb{H}_n$ as

$$\delta_r(p) := (rp', r^2 p_{2n+1}).$$

We denote by $\mathfrak{h}$ the Lie algebra of the left-invariant vector fields of $\mathbb{H}_n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_{x_i} - \frac{1}{2} y_i \partial_z, \quad Y_i := \partial_{y_i} + \frac{1}{2} x_i \partial_z, \quad Z := \partial_z.$$

The only non-trivial commutation relations are $[X_j, Y_j] = Z$, for $j = 1, \dots, n$.

On the Heisenberg algebra $\mathfrak{h}$ the dilations take the form:

$$\delta_r(X_i) = rX_i, \quad \delta_r(Y_i) = rY_i, \quad \delta_r(Z) = r^2 Z.$$

The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$. We refer to $X_1, \dots, X_n, Y_1, \dots, Y_n$ (identified with first order differential operators) as to the *horizontal derivatives.*

If we set

$$W_i := X_i, \quad W_{i+n} := Y_i, \quad W_{2n+1} := Z, \quad \text{for } i = 1, \cdots, n.$$

and for a multi-index $I = (i_1, \dots, i_{2n+1})$, we set $W^I = W_1^{i_1} \cdots W_{2n}^{i_{2n}} \, Z^{i_{2n+1}}$, then by the Poincaré-Birkhoff-Witt theorem, the differential operators W^I form a basis for the algebra of left-invariant differential operators in $\mathbb{H}_n$. Furthermore, we set $|I| := i_1 + \cdots + i_{2n} + i_{2n+1}$ the order of the differential operator W^I, and $d(I) := i_1 + \cdots + i_{2n} + 2i_{2n+1}$ its degree of homogeneity with respect to group dilations. From the Poincaré-Birkhoff-Witt theorem, it follows, in particular, that any homogeneous linear differential operator in the horizontal derivatives can be expressed as a linear combination of the operators W^I of the special form above. Thus, often we can restrict ourselves to consider only operators of the special form W^I.

Denoting by $\mathfrak{h}_2$ the linear span of Z, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

We will use exponential coordinates on $\mathbb{H}_n$:

$$(a, b, c) \in \mathbb{R}^{2n+1} \mapsto \exp(a \cdot X + b \cdot Y + cZ),$$

where $X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_n)$.

4.2 Unitary irreducible representations of $\mathfrak{h}$

By Stone-von Neumann Theorem (see (1.50) in [3]) there are two classes of unitary irreducible representations of $\mathbb{H}_n$ (see (1.59) in [3]):

1. A "degenerate" family of 1-dimensional representations which map Z to 0.
   To one of this representation $\tau : \mathbb{H}_n \to U(H)$ corresponds a representation $\tilde{\tau} : \mathbb{H}_n/\{\exp\ tZ | t \in \mathbb{R}\} \to U(H)$ defined by $\tilde{\tau}([g]) = \tau(g)$ for any $g \in \mathbb{H}_n$.

   Moreover $\mathbb{H}_n/\{\exp\ tZ | t \in \mathbb{R}\} \cong \mathbb{R}^{2n}$, where the isomorphism is given by $(x, y, z) \mapsto (x, y)$.

   Thus this family of representations is parametrized by $(\xi, \eta) \in \mathbb{R}^{2n}$ and is given by
   $$\pi_{(\xi,\eta)}(x, y, z) = e^{i(x\cdot\xi + y\cdot\eta)}. \tag{4.1}$$
   The trivial 1-dimensional identity representation corresponds to $(\xi, \eta) = (0, 0)$.

   On the Lie algebra $\mathfrak{h}$ these representations are given by
   $$\begin{aligned} \pi_{(\xi,\eta)}(X_j) &= i\ \xi_j, \\ \pi_{(\xi,\eta)}(Y_j) &= i\ \eta_j, \\ \pi_{(\xi,\eta)}(Z) &= 0. \end{aligned} \tag{4.2}$$

2. A "generic" family parametrized by $\lambda \in \mathbb{R} \setminus \{0\}$, acting on $L^2(\mathbb{R}^n)$, which map Z to a nonzero scalar. They are given by
   $$[\pi_\lambda(x, y, z)v](t) = e^{i\lambda(y\cdot t + z + x\cdot y/2)} v(t + x) \quad for \quad v \in L^2(\mathbb{R}^n). \tag{4.3}$$
   The space of C^∞-vectors for each π_λ is $\mathcal{S}(\mathbb{R}^n)$.

   On the Lie algebra $\mathfrak{h}$ these representations take the form
   $$\begin{aligned} \pi_\lambda(X_j) &= \frac{d}{dt_j}, \\ \pi_\lambda(Y_j) &= i\lambda t_j, \\ \pi_\lambda(Z) &= i\lambda. \end{aligned} \tag{4.4}$$

This second family of representations, π_λ, is the one occurring in Plancherel Theorem 2.3.1. Indeed, in this case the center of the enveloping algebra, $\mathcal{Z}(\mathfrak{h})$, is given by the polynomials in Z, so $q = 1$ and we can take $W = \frac{Z}{i}$

(thus we have $\pi_\lambda(W) = \lambda$). Then $\Gamma = \mathbb{R} \setminus \{0\}$ and the Plancherel measure $d\mu(\lambda)$ equals $|\lambda|^n d\lambda$, where $|\quad|$ denotes the absolute value.

We now renormalize as in Lemma 2.3.5.

First we observe that by definition of $|\lambda|$ given in Remark 7 this is the unique $r \in \mathbb{R}^+$ such that $|\delta_{r^{-1}}\lambda| = 1$. Since Z is homogeneous of degree 2, this means that $||\lambda|^{-2}\lambda| = 1$, which implies $|\lambda| = |\lambda|^{\frac{1}{2}}$ for every $\lambda \in \mathbb{R} \setminus \{0\}$.

Remark 9. 1. It follows from (2.39) and the definition of π_λ that for every $j = 1, \ldots, n$:

$$\begin{aligned}
\tilde{\pi}_\lambda(X_j) &= |\lambda| \pi_{\delta_{|\lambda|^{-1}}\lambda}(X_j) = |\lambda| \frac{d}{dt_j} = |\lambda|^{\frac{1}{2}} \frac{d}{dt_j}, \\
\tilde{\pi}_\lambda(Y_j) &= |\lambda| \pi_{\delta_{|\lambda|^{-1}}\lambda}(Y_j) = i|\lambda| \delta_{|\lambda|^{-1}} \lambda t_j = i|\lambda|^{\frac{1}{2}} |\lambda|^{-1} \lambda t_j = i(sgn\ \lambda)|\lambda|^{\frac{1}{2}} t_j, \\
\tilde{\pi}_\lambda(Z) &= |\lambda| \pi_{\delta_{|\lambda|^{-1}}\lambda}(Z) = i|\lambda||\lambda|^{-1}\lambda = i\lambda.
\end{aligned} \tag{4.5}$$

More generally it follows from (2.39) that if $P \in \mathcal{U}(\mathfrak{h})$ is homogeneous of degree m, then:

$$\tilde{\pi}_\lambda(P) = |\lambda|^{\frac{m}{2}} \tilde{\pi}_1(P) = |\lambda|^{\frac{m}{2}} \pi_1(P) \qquad if \quad \lambda > 0 \tag{4.6}$$

and

$$\tilde{\pi}_\lambda(P) = |\lambda|^{\frac{m}{2}} \tilde{\pi}_{-1}(P) = |\lambda|^{\frac{m}{2}} \pi_{-1}(P) \qquad if \quad \lambda < 0.$$

2. It follows from (4.2) that $\pi_{(\xi,\eta)} \circ \delta_r = \pi_{(r\xi, r\eta)}$. Thus if $P \in \mathcal{U}(\mathfrak{h})$ is homogeneous of degree m, then for any $r \in \mathbb{R}^+$ and any $(\xi, \eta) \in \mathbb{R}^{2n}$:

$$\pi_{(r\xi, r\eta)}(P) = r^m \pi_{(\xi,\eta)}(P). \tag{4.7}$$

3. Since $W = -W^t$, we can apply the considerations in Remark 8 and thus we have $\hat{\pi}_\lambda = S_\lambda \tilde{\pi}_{-\lambda} S_\lambda^{-1}$. We now show that $S_\lambda = I$ and hence $\hat{\pi}_\lambda = \tilde{\pi}_{-\lambda}$ for any $\lambda \in \mathbb{R} \setminus \{0\}$. Indeed:

$$\begin{aligned}
\hat{\pi}_\lambda(X_j) &= \overline{\tilde{\pi}_\lambda(\bar{X}_j)} = |\lambda|^{\frac{1}{2}} \frac{d}{dt_j} = \tilde{\pi}_{-\lambda}(X_j), \\
\hat{\pi}_\lambda(Y_j) &= \overline{\tilde{\pi}_\lambda(\bar{Y}_j)} = -i(sgn\ \lambda)|\lambda|^{\frac{1}{2}} t_j = \tilde{\pi}_{-\lambda}(Y_j), \\
\hat{\pi}_\lambda(Z) &= \overline{\tilde{\pi}_\lambda(\bar{Z})} = -i\lambda = \tilde{\pi}_{-\lambda}(Z).
\end{aligned}$$

Hence (2.41) takes the form:

$$\tilde{\pi}_\lambda(Q^t) = [\tilde{\pi}_{-\lambda}(Q)]^t, \qquad \tilde{\pi}_\lambda(\bar{Q}) = \overline{\tilde{\pi}_{-\lambda}(Q)} \tag{4.8}$$

for every $\lambda \in \mathbb{R} \setminus \{0\}$.

Similarly we see:

$$\pi_{(\xi,\eta)}(Q^t) = \pi_{(-\xi,-\eta)}(Q), \qquad \pi_{(\xi,\eta)}(\bar{Q}) = \overline{\pi_{(-\xi,-\eta)}(Q)} \tag{4.9}$$

for every $(\xi, \eta) \in \mathbb{R}^{2n}$.

4.3 Construction of the parametrix

The aim of this section is to construct a parametrix for P if it satisfies certain hypothesis. We will need the following definition:

Definition 4.1. A *bounded right (resp. left) -inverse* for $\pi(P)$ is a bounded linear operator $B : H \to H$ such that:

i) B maps H_∞ into H_∞;

ii) $\pi(P)B = I$ on H_∞ (resp. $B\pi(P) = I$ on H_∞).

We will prove the following theorem.

Theorem 4.3.1. *Let $P \in \mathcal{U}(\mathfrak{h})$ be homogeneous and suppose that $\pi(P)$ has a bounded right-inverse for every irreducible unitary representation π of $\mathbb{H}_n$ (except the trivial, identity representation).*

Then P has a parametrix, that is there exists $u \in \mathcal{D}'(\mathbb{H}_n)$ and $f \in C_0^\infty(\mathbb{H}_n)$ such that:

$$Pu = \delta + f. \tag{4.10}$$

Remark 10. Recall that we observed after (2.41) that if $\tilde{\pi}_\lambda(P)$ (resp. $\tilde{\pi}_\lambda(P^*)$) has a bounded right or left inverse for every $\lambda \in \Gamma$, then so does $\tilde{\pi}_\lambda(\bar{P})$ (resp. $\tilde{\pi}_\lambda(P^t)$).

We first sketch the idea and then proceed with the detailed construction.

Let $P \in \mathcal{U}(\mathfrak{h})$ be homogeneous of degree m and satisfy the hypothesis of theorem 4.3.1. Let B_1, B_{-1} be bounded right inverses for $\tilde{\pi}_1(P)$, $\tilde{\pi}_{-1}(P)$ respectively.

If the integral

$$(u, \varphi) = \int_{\mathbb{R}\setminus\{0\}} tr(\tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(P)^{-1}) d\mu(\lambda)$$

were well defined for $\varphi \in C_0^\infty(\mathbb{H}_n)$ and determined a distribution, then u would satisfy the equation $Pu = \delta$. Indeed, we would have by (2.8) and by (2.24):

$$\begin{aligned}(Pu, \varphi) = (u, P^t\varphi) &= \int_{\mathbb{R}\setminus\{0\}} tr(\tilde{\pi}_\lambda(P^t\varphi)\tilde{\pi}_\lambda(P)^{-1}) d\mu(\lambda) = \\ &= \int_{\mathbb{R}\setminus\{0\}} tr(\tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(P)\tilde{\pi}_\lambda(P)^{-1}) d\mu(\lambda) = \\ &= \int_{\mathbb{R}\setminus\{0\}} tr(\tilde{\pi}_\lambda(\varphi)) d\mu(\lambda) = (\delta, \varphi).\end{aligned}$$

Hence u would be a fundamental solution for P. But by (4.6), we have:

$$\tilde{\pi}_\lambda(P)^{-1} = |\lambda|^{-\frac{m}{2}} B_1$$

if $\lambda > 0$ and

$$\tilde{\pi}_\lambda(P)^{-1} = |\lambda|^{-\frac{m}{2}} B_{-1}$$

if $\lambda < 0$.

Thus, since $d\mu(\lambda) = |\lambda|^n d\lambda$, we have:

$$\int_{\mathbb{R}\setminus\{0\}} tr(\tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(P)^{-1}) d\mu(\lambda) =$$
$$= \int_{\mathbb{R}^+} |\lambda|^{-\frac{m}{2}} tr(\tilde{\pi}_\lambda(\varphi)B_1)|\lambda|^n d\lambda + \int_{\mathbb{R}^-} |\lambda|^{-\frac{m}{2}} tr(\tilde{\pi}_\lambda(\varphi)B_{-1})|\lambda|^n d\lambda.$$

Here both integrands behave well when $|\lambda| \to \infty$ but they may blow up as $|\lambda| \to 0$. Thus u is not well defined. We will now see how to deal with this problem and find a parametrix for P.

We will need the following lemma.

Lemma 4.3.2. *There exists a homogeneous $Q \in \mathcal{U}(\mathfrak{h})$ such that for every irreducible unitary representation π of $\mathbb{H}_n$ except $\pi_{(0,0)}$, $\pi(Q)$ has a bounded two-sided inverse which is, in fact, of trace-class. We can take:*

$$Q = \left(\sum_{i=1}^{n} (X_i^2 + Y_i^2)\right)^N$$

for N a sufficiently large positive integer depending on n.

Proof. Since X_i and Y_i are homogeneous of degree 1, we have that $\sum_{j=1}^n (X_j^2 + Y_j^2)$ is homogeneous of degree 2. By (4.2) it follows:

$$\pi_{(\xi,\eta)}\left(\sum_{j=1}^{n} (X_j^2 + Y_j^2)\right) = -\sum_{j=1}^{n} (\xi_j^2 + \eta_j^2)$$

that is not 0 if $(\xi, \eta) \neq (0,0)$.

Moreover, by (4.5):

$$\tilde{\pi}_1\left(\sum_{j=1}^{n} (X_j^2 + Y_j^2)\right) = \tilde{\pi}_{-1}\left(\sum_{j=1}^{n} (X_j^2 + Y_j^2)\right) = \sum_{j=1}^{n} \left(\frac{d^2}{dt_j^2} - t_j^2\right) =$$
$$= -\sum_{j=1}^{n} (D_{t_j}^2 + t_j^2),$$

where $D_{t_j} = \frac{1}{i}\frac{d}{dt_j}$.

We recall that $L^2(\mathbb{R})$ has a complete orthonormal basis consisting of eigenfunctions $\{v_j\}_{j=0}^{\infty}$ for the harmonic oscillator $D_t^2+t^2$. These are called Hermite functions and for every j they satisfy:

$$(D_t^2 + t^2)v_j = (2j+1)v_j, \quad v_j \in \mathcal{S}(\mathbb{R}).$$

Let now $v_k(t) = v_{k_1}(t_1)\cdots v_{k_n}(t_n)$, where $k = (k_1, \ldots, k_n)$ runs through all n-tuples of nonnegative integers. Then v_k form a complete orthonormal basis of $L^2(\mathbb{R}^n)$ and for every k

$$\left[\sum_{j=1}^{n}(D_{t_j}^2 + t_j^2)\right] v_k = \left[\sum_{j=1}^{n}(2k_j+1)\right] v_k.$$

It follows that $\sum_{j=1}^{n}(D_{t_j}^2 + t_j^2)$ has a bounded two-sided inverse T, which is given by:

$$Tv_k = \frac{1}{\sum_{j=1}^{n}(2k_j+1)} v_k.$$

Since all the eigenvalues of T are positive, T^N is of trace-class if and only if:

$$\sum_k \frac{1}{\left[\sum_{j=1}^{n}(2k_j+1)\right]^N} < \infty.$$

This condition is equivalent to:

$$\sum_{k=1}^{\infty} \frac{\alpha(k)}{k^N} < \infty,$$

where $\alpha(k)$ is the number of ways k can be written as a sum of n positive integers, thus $\alpha(k) < k^n$. Hence, the condition holds if $N - n > 1$, i.e. $N > n+1$. □

We next examine the question of measurability of the functions $\lambda \mapsto tr(\tilde{\pi}_\lambda(\varphi)B_1)$ and $\lambda \mapsto tr(\tilde{\pi}_\lambda(\varphi)B_{-1})$. We will work in the general contest of a simply-connected nilpotent Lie group.

Lemma 4.3.3. *Let $\mathbb{G}$ be a simply-connected nilpotent Lie group (with dilations) and let $B : H \to H$ be a bounded linear operator. Then for any $\varphi \in C_0^\infty(\mathbb{G})$ the function:*

$$\begin{aligned} &\Gamma \to \mathbb{C} \\ &\lambda \mapsto tr(\tilde{\pi}_\lambda(\varphi)B) \end{aligned}$$

is measurable. Here Γ is the Zariski-open set (see Plancherel Theorem 2.3.1) that parametrizes the unitary irreducible representations of $\mathbb{G}$.

Moreover, if B is of trace-class then the function is continuous.

Proof. We first suppose only that B is bounded. Since $\tilde{\pi}_\lambda(\varphi)$ is trace-class, so is $\tilde{\pi}_\lambda(\varphi)B$. Let $\{e_i\}_{i=0}^{\infty}$ be any orthonormal basis for H. Then

$$\begin{aligned} tr(\tilde{\pi}_\lambda(\varphi)B) = tr(B\tilde{\pi}_\lambda(\varphi)) = \sum_{i=0}^{\infty}\langle B\tilde{\pi}_\lambda(\varphi)e_i, e_i\rangle = \\ = \sum_{i=0}^{\infty}\langle \tilde{\pi}_\lambda(\varphi)e_i, B^*e_i\rangle. \end{aligned} \tag{4.11}$$

By Plancherel Theorem 2.3.1(5), for every i the function $\lambda \mapsto \langle \tilde{\pi}_\lambda(\varphi)e_i, B^*e_i\rangle$ from Γ to $\mathbb{C}$ is continuous, hence measurable. Since $tr(\tilde{\pi}_\lambda(\varphi)B)$ is the pointwise limit of measurable functions, it is measurable.

Suppose now that B is trace-class. To have that the function $\lambda \mapsto tr(\tilde{\pi}_\lambda(\varphi)B)$ is continuous, we need to show that the series in (4.11) converges uniformly in λ.

Using the polar decomposition given in theorem 1.12.1, we write $B^* = UT$, where $T = (BB^*)^{1/2}$ and U is a partial isometry. We observe that T is positive definite and of trace-class, so in particular compact by Theorem 1.11.5. Hence H has an orthonormal basis $\{e_i\}_{i=0}^{\infty}$ consisting of eigenvectors of T with eigenvalues $\{t_i\}_{i=0}^{\infty}$. The trace norm of B is, in fact, $\sum_{i=0}^{\infty} t_i < \infty$. Then we have:

$$\langle \tilde{\pi}_\lambda(\varphi)e_i, B^*e_i\rangle = \langle U^*\tilde{\pi}_\lambda(\varphi)e_i, Te_i\rangle = t_i\langle U^*\tilde{\pi}_\lambda(\varphi)e_i, e_i\rangle.$$

Thus:

$$\begin{aligned} |\langle \tilde{\pi}_\lambda(\varphi)e_i, B^*e_i\rangle| &= t_i|\langle U^*\tilde{\pi}_\lambda(\varphi)e_i, e_i\rangle| \leq \\ &\leq t_i||U||||\tilde{\pi}_\lambda(\varphi)|| = t_i||\tilde{\pi}_\lambda(\varphi)|| \leq \\ &\leq t_i||\varphi||_{L^1(\mathbb{G})}, \end{aligned}$$

where the last inequality holds by (2.4).

Hence, for every integer $n \geq 0$,

$$\sum_{i=n}^{\infty} |\langle \tilde{\pi}_\lambda(\varphi)e_i, B^*e_i\rangle| \leq \left(\sum_{i=n}^{\infty} t_i\right) ||\varphi||_{L^1(\mathbb{G})}.$$

Since here the right-hand side is independent of λ and since $\sum_{i=0}^{\infty} t_i < \infty$, the series in (4.11) converges uniformly in λ, thus we have the claim. □

We now return to the context $\mathbb{G} = \mathbb{H}_n$.

Lemma 4.3.4. *Let $\epsilon > 0$ and let B be a trace-class operator on $L^2(\mathbb{R}^n)$. If $s \in \mathbb{C}$ with $\operatorname{Re} s > -\frac{n+1}{2}$, then the linear map $u : C_0^\infty(\mathbb{H}_n) \to \mathbb{C}$ given by*

$$(u, \varphi) = \int_{0<\lambda\leq\epsilon} |\lambda|^{s+n} tr(\tilde{\pi}_\lambda(\varphi)B) d\lambda \tag{4.12}$$

is well defined and $u \in \mathcal{D}'(\mathbb{H}_n)$. In fact, there exists a constant C independent of φ such that

$$|(u, \varphi)| \leq \int_{0<\lambda\leq\epsilon} |\lambda|^{\operatorname{Re} s+n} |tr(\tilde{\pi}_\lambda(\varphi)B)| d\lambda \leq C ||\varphi||_{L^2(\mathbb{H}_n)}, \tag{4.13}$$

so $u \in L^2(\mathbb{H}_n)$. The same is true if we replace the domain of integration by $-\epsilon \leq \lambda < 0$.

Proof. By preceding Lemma, the integrand in (4.12) is measurable.We have to show that it is integrable. First we observe that we can write:

$$|\lambda|^{\operatorname{Re} s+n} |tr(\tilde{\pi}_\lambda(\varphi)B)| = |\lambda|^{\operatorname{Re} s+\frac{n}{2}} (|\lambda|^{\frac{n}{2}} |tr(\tilde{\pi}_\lambda(\varphi)B)|).$$

Thus, by the definition of u and Hölder's inequality:

$$\begin{aligned} |(u, \varphi)| &\leq \int_{0<\lambda\leq\epsilon} |\lambda|^{\operatorname{Re} s+n} |tr(\tilde{\pi}_\lambda(\varphi)B)| d\lambda \leq \\ &\leq \left[\int_{0<\lambda\leq\epsilon} |\lambda|^{2\operatorname{Re} s+n} d\lambda \right]^{\frac{1}{2}} \left[\int_{0<\lambda\leq\epsilon} |tr(\tilde{\pi}_\lambda(\varphi)B)|^2 |\lambda|^n d\lambda \right]^{\frac{1}{2}}. \end{aligned}$$

By hypothesis $2\operatorname{Re} s + n > -1$, hence the first integral in this product is finite, call it C'.

Moreover, by (1.17) $|tr(\tilde{\pi}_\lambda(\varphi)B)|^2 \leq tr(BB^*) tr(\tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(\varphi)^*)$. But by Plancherel Theorem (6) and Lemma 2.3.5 the function $\lambda \mapsto tr(\tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(\varphi)^*)$ from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}^+$ is measurable. Hence,

$$\left[\int_{0<\lambda\leq\epsilon} |tr(\tilde{\pi}_\lambda(\varphi)B)|^2 |\lambda|^n d\lambda \right]^{\frac{1}{2}} \leq [tr(BB^*)]^{\frac{1}{2}} \left[\int_{0<\lambda\leq\epsilon} tr(\tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(\varphi)^*) |\lambda|^n d\lambda \right]^{\frac{1}{2}}.$$

Since $tr(\tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(\varphi)^*) \geq 0$ for every $\lambda \in \mathbb{R} \setminus \{0\}$ and since the Plancherel measure is $d\mu(\lambda) = |\lambda|^n d\lambda$, it follows from (2.23) that

$$\left[\int_{0<\lambda\leq\epsilon} tr(\tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(\varphi)^*) |\lambda|^n d\lambda \right]^{\frac{1}{2}} \leq \left[\int_{\mathbb{R}\setminus\{0\}} tr(\tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(\varphi)^*) |\lambda|^n d\lambda \right]^{\frac{1}{2}} = ||\varphi||_{L^2(\mathbb{H}_n)}.$$

If we now let $C = C'[tr(BB^*)]^{\frac{1}{2}}$, we obtain (4.13).

Since the linear map $\varphi \mapsto (u, \varphi)$ is continuous and $L^2(\mathbb{H}_n)$ is its own dual, it follows by Hahn-Banach Theorem that there exists $g \in L^2(\mathbb{H}_n)$ such that for every $\varphi \in C_0^\infty(\mathbb{H}_n)$,

$$(u, \varphi) = \langle \varphi, g \rangle = (\bar{g}, \varphi),$$

i.e. $u = \bar{g} \in L^2(\mathbb{H}_n)$.

The same proof is valid for the domain of integration $-\epsilon \leq \lambda < 0$. □

Lemma 4.3.5. *Let $\epsilon > 0$ and let B be a trace-class operator on $L^2(\mathbb{R}^n)$. For any $s \in \mathbb{C}$ the linear map $u : C_0^\infty(\mathbb{H}_n) \to \mathbb{C}$ given by*

$$(u, \varphi) = \int_{\lambda > \epsilon} |\lambda|^{s+n} tr(\tilde{\pi}_\lambda(\varphi)B) d\lambda \tag{4.14}$$

is well defined and $u \in \mathcal{D}'(\mathbb{H}_n)$. Moreover, for any nonnegative integer N such that $N > \operatorname{Re} s + \frac{n+1}{2}$ there exists a constant C independent of φ such that

$$|(u, \varphi)| \leq \int_{\lambda > \epsilon} |\lambda|^{\operatorname{Re} s+n} |tr(\tilde{\pi}_\lambda(\varphi)B)| d\lambda \leq C ||Z^N \varphi||_{L^2(\mathbb{H}_n)}. \tag{4.15}$$

Thus, there exists $f \in L^2(\mathbb{H}_n)$ such that $u = Z^N f$. The same is true if we replace the domain of integration by $\lambda < -\epsilon$.

Proof. The proof is similar to the one of the preceding lemma, but we now consider the behavior "at ∞" rather than "at 0".

We can write:

$$|\lambda|^{\operatorname{Re} s+n} |tr(\tilde{\pi}_\lambda(\varphi)B)| = |\lambda|^{\operatorname{Re} s+\frac{n}{2}-N} (|\lambda|^{\frac{n}{2}} |\lambda^N tr(\tilde{\pi}_\lambda(\varphi)B)|).$$

Since $Z^t = -Z$, we see that $\tilde{\pi}_\lambda(Z^t) = -\tilde{\pi}_\lambda(Z) = -i\lambda$. Thus by (2.8) $\tilde{\pi}_\lambda(Z^N\varphi) = \tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda((Z^N)^t)) = (-i\lambda)^N \tilde{\pi}_\lambda(\varphi)$. Hence from the previous equality follows:

$$|\lambda|^{\operatorname{Re} s+n} |tr(\tilde{\pi}_\lambda(\varphi)B)| = |\lambda|^{\operatorname{Re} s+\frac{n}{2}-N} (|\lambda|^{\frac{n}{2}} |tr(\tilde{\pi}_\lambda(Z^N\varphi)B)|).$$

Therefore

$$\int_{\lambda > \epsilon} |\lambda|^{\operatorname{Re} s+n} |tr(\tilde{\pi}_\lambda(\varphi)B)| d\lambda \leq \left[\int_{\lambda > \epsilon} |\lambda|^{2(\operatorname{Re} s+\frac{n}{2}-N)} d\lambda\right]^{\frac{1}{2}} \left[\int_{\lambda > \epsilon} |tr(\tilde{\pi}_\lambda(Z^N\varphi)B)|^2 |\lambda|^n d\lambda\right]^{\frac{1}{2}}.$$

By hypothesis N is such that $2(\operatorname{Re} s + \frac{n}{2} - N) < -1$, thus the first integral in this product is finite. Now proceeding exactly as in the previous proof, but with φ replaced by $Z^N\varphi$, we get (4.15).

Let now $V = \{Z^N\varphi | \varphi \in C_0^\infty(\mathbb{H}_n)\}$, which is a subspace of $L^2(\mathbb{H}_n)$. By (4.15) the linear map $Z^N\varphi \mapsto (u,\varphi)$ from V to $\mathbb{C}$ is well defined and continuous. Hence, using Hahn-Banach Theorem as before, there exists $g \in L^2(\mathbb{H}_n)$ such that for every $\varphi \in C_0^\infty(\mathbb{H}_n)$:

$$(u,\varphi) = \langle Z^N\varphi, g\rangle = (\bar{g}, Z^N\varphi) = ((Z^N)^t\bar{g}, \varphi) = (Z^N((-1)^N\bar{g}), \varphi).$$

This means that $u = Z^N f$, where $f = (-1)^N\bar{g} \in L^2(\mathbb{H}_n)$.

The same proof is valid for the domain of integration $\lambda < -\epsilon$. □

Lemma 4.3.6. *Let $P \in \mathcal{U}(\mathfrak{h})$ be homogeneous of degree m and suppose $\pi_{(\xi,\eta)}(P) \neq 0$ for every $(\xi,\eta) \in \mathbb{R}^{2n}$ except $(0,0)$. Then P is of order m as a differential operator.*

Proof. Suppose P is of order k as a differential operator. Then it can be written as

$$P = \sum_{|\alpha|+|\beta|+\gamma\leq k} a_{\alpha\beta\gamma} X_1^{\alpha_1}\cdots X_n^{\alpha_n} Y_1^{\beta_1}\cdots Y_n^{\beta_n} Z^\gamma,$$

with $a_{\alpha\beta\gamma} \in \mathbb{C}$ and $a_{\alpha\beta\gamma} \neq 0$ for some $\alpha\beta\gamma$ such that $|\alpha| + |\beta| + \gamma = k$.

Since P is homogeneous of degree m, then:

$$|\alpha| + |\beta| + 2\gamma = m$$

for every $\alpha\beta\gamma$ such that $a_{\alpha\beta\gamma} \neq 0$. But $|\alpha| + |\beta| + \gamma \leq k$ for every $\alpha\beta\gamma$ so $k \leq m$ for every $\alpha\beta\gamma$ such that $a_{\alpha\beta\gamma} \neq 0$, $\gamma \geq m - k$.

Since $\pi_{(\xi,\eta)}(Z) = 0$ for every (ξ,η) and by hypothesis $\pi_{(\xi,\eta)}(P) \neq 0$ for every $(\xi,\eta) \in \mathbb{R}^{2n}$ except $(0,0)$, it follows from the expression of P that there exists $\alpha\beta$ such that $a_{\alpha\beta 0} \neq 0$. Hence $k = m$. □

Lemma 4.3.7. *Let $P \in \mathcal{U}(\mathfrak{h})$ be homogeneous of degree m and suppose $\pi_{(\xi,\eta)}(P) \neq 0$ for every $(\xi,\eta) \in \mathbb{R}^{2n}$ except $(0,0)$.*

Let $V(P) \subset T^*\mathbb{H}_n \setminus 0$ denote the real characteristic variety of P, i.e. the zero-set of the principal symbol $\sigma_m(P)$. Identifying $T_e^*\mathbb{H}_n$ with $\mathfrak{h}^*$, we have that $V(P)_e$ is the annihilator of the $2n$-dimensional subspace of $\mathfrak{h}$ spanned by $X_i, Y_i, i = 1,\dots,n$.

In terms of the coordinates $\xi_i, \eta_i, \tau, i = 1,\dots,n$ defined on $\mathfrak{h}^*$ by X_i, Y_i, Z this can be expressed as:

$$V(P)_e = \{(\xi,\eta,\tau) | \xi = \eta = 0, \tau \neq 0\}. \tag{4.16}$$

Proof. By previous lemma, P is of order m as a differential operator so it can be written as

$$P = \sum_{|\alpha|+|\beta|+\gamma\leq m} a_{\alpha\beta\gamma} X_1^{\alpha_1}\cdots X_n^{\alpha_n} Y_1^{\beta_1}\cdots Y_n^{\beta_n} Z^\gamma,$$

with $|\alpha| + |\beta| + 2\gamma = m$ for every $\alpha\beta\gamma$ such that $a_{\alpha\beta\gamma} \neq 0$.

We have that:

$$\pi_{(\xi,\eta)}(P) = i^m \sum_{|\alpha|+|\beta|=m} a_{\alpha\beta 0} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \eta_1^{\beta_1} \cdots \eta_n^{\beta_n}.$$

Let now $\sigma_m(P)_e$ be the principal symbol of P at e, which we consider as a polynomial function on $\mathfrak{h}^*$. It is given by

$$\begin{aligned}\sigma_m(P)_e = i^m & \sum_{|\alpha|+|\beta|+\gamma=m} a_{\alpha\beta\gamma} X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_n^{\beta_n} Z^\gamma = \\ = i^m & \sum_{|\alpha|+|\beta|=m} a_{\alpha\beta 0} X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_n^{\beta_n},\end{aligned}$$

where the last equality holds since $|\alpha| + |\beta| + 2\gamma = m$ for every $\alpha\beta\gamma$ such that $a_{\alpha\beta\gamma} \neq 0$.

Evaluating at $(\xi, \eta, \tau) \in \mathfrak{h}^*$, we get:

$$\begin{aligned}\langle \sigma_m(P)_e, (\xi, \eta, \tau) \rangle &= i^m \sum_{|\alpha|+|\beta|=m} a_{\alpha\beta 0} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \eta_1^{\beta_1} \cdots \eta_n^{\beta_n} = \\ &= \pi_{(\xi,\eta)}(P).\end{aligned}$$

Hence by hypothesis, $\langle \sigma_m(P)_e, (\xi, \eta, \tau) \rangle \neq 0$ if $(\xi, \eta) \neq (0, 0)$.

But, clearly, $\langle \sigma_m(P)_e, (0, 0, \tau) \rangle = 0$ for all τ. Thus (4.16) holds. □

Proposition 4.3.8. *Let $P \in \mathcal{U}(\mathfrak{h})$ be homogeneous of degree m and suppose $\pi_{(\xi,\eta)}(P) \neq 0$ for every $(\xi, \eta) \in \mathbb{R}^{2n}$ except $(0, 0)$.*

Then $P\bar{P} + Z^{2m} \in \mathcal{U}(\mathfrak{h})$ is elliptic.

Proof. Since P is of order m, it follows that $P\bar{P}$, and hence $P\bar{P} + Z^{2m}$, is of order $2m$. Thus the principal symbol of $P\bar{P} + Z^{2m}$ at e is given by:

$$\begin{aligned}\langle \sigma_{2m}(P\bar{P} + Z^{2m})_e, (\xi, \eta, \tau) \rangle &= \langle \sigma_m(P)_e \sigma_m(\bar{P})_e, (\xi, \eta, \tau) \rangle + \langle \sigma_{2m}(Z^{2m})_e, (\xi, \eta, \tau) \rangle = \\ &= (-1)^m |\langle \sigma_m(P)_e, (\xi, \eta, \tau) \rangle|^2 + (i\tau)^{2m} = \\ &= (-1)^m |\langle \sigma_m(P)_e, (\xi, \eta, \tau) \rangle|^2 + (-1)^m |\tau|^{2m}.\end{aligned}$$

Here both terms have the same sign so the sum is 0 if and only if they are both 0. Hence, by (4.16), the characteristic variety of $P\bar{P} + Z^{2m}$ is given by

$$V(P\bar{P} + Z^{2m})_e = \{(0, 0, \tau) | \tau \neq 0\} \cap \{(\xi, \eta, 0) | (\xi, \eta) \neq (0, 0)\} = \emptyset.$$

Since $P\bar{P} + Z^{2m}$ is invariant under left translation, this implies that it is elliptic. □

Remark 11. i) If P is homogeneous, then so is $\bar{P}$.

ii) If P satisfies the hypothesis of Theorem 4.3.1, then so does $\bar{P}$. Indeed: by (4.8) for every $\lambda \in \mathbb{R} \setminus \{0\}$ we have

$$\tilde{\pi}_\lambda(\bar{P}) = \overline{\tilde{\pi}_{-\lambda}(P)},$$

thus if $\tilde{\pi}_{-\lambda}(P)$ has a bounded right-inverse, then so does $\tilde{\pi}_\lambda(\bar{P})$. Similarly for $\pi_{(\xi,\eta)}(\bar{P})$ because by (4.9)

$$\pi_{(\xi,\eta)}(\bar{P}) = \overline{\pi_{(-\xi,-\eta)}(P)}.$$

Remark 12. We can, without loss of generality, replace the hypothesis of Theorem 4.3.1 by the following:

i) $P \in \mathcal{U}(\mathfrak{h})$ is homogeneous of degree m;

ii) $\pi_{(\xi,\eta)}(P) \neq 0$ if $(\xi, \eta) \neq (0, 0)$;

iii) $\tilde{\pi}_1(P)$, $\tilde{\pi}_{-1}(P)$ have bounded right-inverses B_1, B_{-1}, respectively, which are both of trace-class;

iv) $P + Z^m$ is elliptic.

Proof. Let Q be as in Lemma 4.3.2.

i) Same hypothesis as in Theorem 4.3.1.

ii) We proceed by replacing P by PQ. If B is the bounded right-inverse of $\pi_{(\xi,\eta)}(P)$ (where $(\xi, \eta) \neq (0, 0)$), then $\pi_{(\xi,\eta)}(Q)^{-1}B$ is a bounded right-inverse for $\pi_{(\xi,\eta)}(PQ)$. Thus, if $\pi_{(\xi,\eta)}(PQ) = 0$ with $(\xi, \eta) \neq (0, 0)$, multiplying both sides by $\pi_{(\xi,\eta)}(Q)^{-1}B$ we get a contradiction.

iii) We proceed by replacing P by PQ as before. We know that the bounded right-inverses $\tilde{\pi}_1(Q)^{-1}$ and $\tilde{\pi}_{-1}(Q)^{-1}$ are of trace-class by Lemma 4.3.2. If B_1, B_{-1} are bounded right-inverses for $\tilde{\pi}_1(P)$, $\tilde{\pi}_{-1}(P)$ respectively, it follows that $\tilde{\pi}_1(Q)^{-1}B_1$, $\tilde{\pi}_{-1}(Q)^{-1}B_{-1}$ are bounded right-inverses for $\tilde{\pi}_1(PQ)$, $\tilde{\pi}_{-1}(PQ)$. Moreover, they are of trace-class because the trace-class operators form an ideal in the space of bounded operators.

iv) If $L = PQ$ is homogeneous of degree μ and $\pi_{(\xi,\eta)}(L) \neq 0$ if $(\xi, \eta) \neq (0, 0)$, then by Proposition 4.3.8 $L\bar{L} + Z^{2\mu}$ is elliptic. Taking here $P = L\bar{L}$ and $2\mu = m$, we have the claim.

$\square$

Fix $\epsilon > 0$. Then for any $s \in \mathbb{C}$ with $\operatorname{Re} s - \frac{m}{2} > -\frac{n+1}{2}$ define distributions u_0^s, u_∞^s, u^s by:

$$\begin{aligned}(u_0^s, \varphi) =& (i)^s \int_{0<\lambda\leq\epsilon} |\lambda|^{s-m/2} tr(\tilde{\pi}_\lambda(\varphi)B_1)|\lambda|^n d\lambda + \\ &+ (-i)^s \int_{-\epsilon\leq\lambda<0} |\lambda|^{s-m/2} tr(\tilde{\pi}_\lambda(\varphi)B_{-1})|\lambda|^n d\lambda,\end{aligned} \tag{4.17}$$

$$\begin{aligned}(u_\infty^s, \varphi) =& (i)^s \int_{\lambda>\epsilon} |\lambda|^{s-m/2} tr(\tilde{\pi}_\lambda(\varphi)B_1)|\lambda|^n d\lambda + \\ &+ (-i)^s \int_{\lambda<-\epsilon} |\lambda|^{s-m/2} tr(\tilde{\pi}_\lambda(\varphi)B_{-1})|\lambda|^n d\lambda,\end{aligned} \tag{4.18}$$

$$u^s = u_0^s + u_\infty^s. \tag{4.19}$$

Remark 13. We know by Lemmas 4.3.4 and 4.3.5 that u_0^s, u_∞^s are distributions. In fact we know:

1. For every s, $u_0^s \in L^2(\mathbb{H}_n)$;
2. For any positive integer $N > \operatorname{Re} s - \frac{m}{2} + \frac{n+1}{2}$ there exists $f \in L^2(\mathbb{H}_n)$ such that $u_\infty^s = Z^N f$.

Lemma 4.3.9. *The distributions u_0^s, u_∞^s, u^s satisfy the following properties:*

i) For any nonnegative integer k,

$$Z^k u_0^s = u_0^{s+k}, \qquad Z^k u_\infty^s = u_\infty^{s+k}, \qquad Z^k u^s = u^{s+k}. \tag{4.20}$$

ii) For any positive integer $l > \frac{m}{2} - \frac{n+1}{2}$,

$$Pu^l = Z^l \delta. \tag{4.21}$$

Proof. i) We have that for every $\varphi \in C_0^\infty(\mathbb{H}_n)$:

$$(Z^k u_0^s, \varphi) = (u_0^s, (Z^k)^t \varphi).$$

But by (2.8):

$$\tilde{\pi}_\lambda((Z^k)^t\varphi) = \tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(Z^k) = (i\lambda)^k \tilde{\pi}_\lambda(\varphi) = (\pm i)^k |\lambda|^k \tilde{\pi}_\lambda(\varphi),$$

depending on the sign of λ.

Thus by definition of u_0^s:

$$(u_0^s, (Z^k)^t\varphi) =(i)^{s+k}\int_{0<\lambda\leq\epsilon}|\lambda|^{s+k-m/2}tr(\tilde{\pi}_\lambda(\varphi)B_1)|\lambda|^n d\lambda+ \\ +(-i)^{s+k}\int_{-\epsilon\leq\lambda<0}|\lambda|^{s+k-m/2}tr(\tilde{\pi}_\lambda(\varphi)B_{-1})|\lambda|^n d\lambda = (u_0^{s+k},\varphi).$$

It follows that $Z^k u_0^s = u_0^{s+k}$. The same argument shows that $Z^k u_\infty^s = u_\infty^{s+k}$, hence $Z^k u^s = u^{s+k}$.

ii) We have that for every $\varphi \in C_0^\infty(\mathbb{H}_n)$:

$$(Pu_0^s,\varphi) = (u_0^s, P^t\varphi), \qquad (Pu_\infty^s,\varphi) = (u_\infty^s, P^t\varphi).$$

But again by (2.8) for every $v \in \mathcal{S}(\mathbb{R}^n)$:

$$\tilde{\pi}_\lambda(P^t\varphi)v = \tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(P)v.$$

Hence by (4.6) and by definition of B_1 if $\lambda > 0$ and $v \in \mathcal{S}(\mathbb{R}^n)$:

$$\tilde{\pi}_\lambda(P^t\varphi)B_1 v = \tilde{\pi}_\lambda(\varphi)\tilde{\pi}_\lambda(P)B_1 v = \tilde{\pi}_\lambda(\varphi)|\lambda|^{\frac{m}{2}}\tilde{\pi}_1(P)B_1 v = \\ = |\lambda|^{\frac{m}{2}}\tilde{\pi}_\lambda(\varphi)v.$$

Similarly if $\lambda < 0$:

$$\tilde{\pi}_\lambda(P^t\varphi)B_{-1}v = |\lambda|^{\frac{m}{2}}\tilde{\pi}_\lambda(\varphi)v.$$

Since these equations involve bounded, in fact trace-class operators, it follows that:

$$\tilde{\pi}_\lambda(P^t\varphi)B_1 = |\lambda|^{\frac{m}{2}}\tilde{\pi}_\lambda(\varphi), \quad \lambda > 0 \\ \tilde{\pi}_\lambda(P^t\varphi)B_{-1} = |\lambda|^{\frac{m}{2}}\tilde{\pi}_\lambda(\varphi), \quad \lambda < 0.$$

Thus for any l as in the hypothesis:

$$(Pu^l,\varphi) = (Pu_0^l,\varphi) + (Pu_\infty^l,\varphi) = \int_{\mathbb{R}\setminus\{0\}}(i\lambda)^l tr(\tilde{\pi}_\lambda(\varphi))|\lambda|^n d\lambda = \\ = \int_{\mathbb{R}\setminus\{0\}} tr(\tilde{\pi}_\lambda((Z^l)^t\varphi))|\lambda|^n d\lambda = (\delta,(Z^l)^t\varphi) = (Z^l\delta,\varphi).$$

Hence $Pu^l = Z^l\delta$.

□

The following Corollary is an immediate consequence.

Corollary 4.3.10.

$$P(\delta + u^m) = (P + Z^m)\delta.$$

Then we proceed to construct the parametrix. By hypothesis, $P + Z^m$ is elliptic, thus in particular it is locally solvable and hypoelliptic. This means that there exist $v' \in \mathcal{D}'(\mathbb{H}_n)$, C^∞ away from e, and $f' \in C^\infty(\mathbb{H}_n)$ such that:

$$(P + Z^m)v' = \delta + f'. \tag{4.22}$$

Let $\varphi \in C_0^\infty(\mathbb{H}_n)$ such that $\varphi \equiv 1$ in a neighborhood of e and let $v = \varphi v'$. Then:

$$\begin{aligned}(P + Z^m)v &= \varphi(P + Z^m)v' + [P + Z^m, \varphi]v' = \\ &= \varphi\delta + \varphi f' + [P + Z^m, \varphi]v' = \delta + f,\end{aligned}$$

where $f = \varphi f' + [P + Z^m, \varphi]v'$. Since v' is C^∞ away from e and $[P + Z^m, \varphi]$ contains derivatives of φ, that vanish in a neighborhood of e by definition of φ, it follows that f is C^∞. Moreover, both v and f are compactly supported and v is C^∞ away from e.

Since $v \in \mathcal{E}'(\mathbb{H}_n)$, we can define a distribution $u \in \mathcal{D}'(\mathbb{H}_n)$ by

$$u = v * (\delta + u^m). \tag{4.23}$$

This is the parametrix we were looking for, because:

$$\begin{aligned}Pu &= P(v * (\delta + u^m)) = v * P(\delta + u^m) = \\ &= v * (P + Z^m)\delta = (P + Z^m)(v * \delta) = \\ &= (P + Z^m)v = \delta + f.\end{aligned}$$

In order to prove Theorem 4.3.1 it remains to show that this parametrix is C^∞ away from e.

We will do that under the stronger hypothesis that the inverses for $\tilde{\pi}_1(P)$, $\tilde{\pi}_{-1}(P)$ are two-sided.

4.4 Sufficiency for hypoellipticity

We want to show that, at least under the stronger assumption that the inverses for $\tilde{\pi}_1(P)$ and $\tilde{\pi}_{-1}(P)$ are two-sided, the parametrix u for P constructed in the previous section is C^∞ away from e.

We will need to work in the following spaces.

Definition 4.2. For any $\delta > 0$ and any nonnegative integer k we define $H_{(k,\delta)}(\mathbb{R}^n)$ as the set of all functions $v(t)$ such that:

$$(1+|t|)^{(k-|\alpha|)\delta} D_t^\alpha v(t) \in L^2(\mathbb{R}^n)$$

for all multi-indices α with $|\alpha| \le k$.

Remark 14. $H_{(k,\delta)}(\mathbb{R}^n)$ is a Hilbert space and $\mathcal{S}(\mathbb{R}^n) \subset H_{(k,\delta)}(\mathbb{R}^n)$.

Proposition 4.4.1. *$C_0^\infty(\mathbb{R}^n)$ (and so $\mathcal{S}(\mathbb{R}^n)$) is dense in $H_{(k,\delta)}(\mathbb{R}^n)$.*

Proof. For $N \in \mathbb{N} \setminus \{0\}$ let $\psi_N(|x|)$ be a cut-off function on $\mathbb{R}^n$, that is: $\psi_N \in C^\infty(\mathbb{R}^n)$, $\psi_N(|x|) \equiv 1$ for $|x| \le N$ and $\psi_N(|x|) \equiv 0$ for $|x| \ge N+1$.

If $u \in H_{(k,\delta)}(\mathbb{R}^n)$ then $u\psi_N \to u$ in $H_{(k,\delta)}(\mathbb{R}^n)$ as N tends to ∞. Indeed:

$$\begin{aligned}
||u\psi_N - u||^2_{H_{(k,\delta)}(\mathbb{R}^n)} &= \sum_{|\alpha|\le k} ||(1+|t|)^{(k-|\alpha|)\delta} D_t^\alpha (u\psi_N - u)(t)||^2_{L^2(\mathbb{R}^n)} = \\
&= \sum_{|\alpha|\le k} ||(1+|t|)^{(k-|\alpha|)\delta}[(D_t^\alpha u)\psi_N + u(D_t^\alpha \psi_N) - D_t^\alpha u]||^2_{L^2(\mathbb{R}^n)} \le \\
&\le \sum_{|\alpha|\le k} ||(1+|t|)^{(k-|\alpha|)\delta} D_t^\alpha u||_{L^2(\mathbb{R}^n)} ||\psi_N - 1||_{L^2(\mathbb{R}^n)} + \\
&+ \sum_{|\alpha|\le k} ||(1+|t|)^{(k-|\alpha|)\delta} u||_{L^2(\mathbb{R}^n)} ||D_t^\alpha \psi_N||_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

This sum tends to 0 as $N \to \infty$ because $||\psi_N - 1||_{L^2(\mathbb{R})^n} \to 0$, $||D_t^\alpha \psi_N||_{L^2(\mathbb{R}^n)} \to 0$ and the other two norms are bounded by a constant.

Thus if we fix $\epsilon > 0$ we can choose N such that $||u\psi_N - u||_{H_{(k,\delta)}(\mathbb{R}^n)} < \frac{\epsilon}{2}$.

We have that $u\psi_N \in W_0^{k,2}(B(0, N+2))$ because $supp(u\psi_N) = B(0, N+1)$ and in $B(0, N+2)$ we have the estimate:

$$1 \le (1+|t|)^{(k-|\alpha|)\delta} \le (N+3)^{(k-|\alpha|)\delta}.$$

Thus there exists $\varphi \in C_0^\infty(B(0, N+2))$ such that $||\varphi - u\psi_N||_{W^{k,2}} < \sigma\epsilon$, where $\sigma^2 < \frac{1}{4(N+3)^{2k\delta}}$. Then:

$$\begin{aligned}
||\varphi - u\psi_N||^2_{H_{(k,\delta)}} &= \sum_{|\alpha|\le k} \int (1+|t|)^{2(k-|\alpha|)\delta} |D_t^\alpha(\varphi - u\psi_N)|^2 dt \le \\
&\le \sum_{|\alpha|\le k} (N+3)^{2(k-|\alpha|)\delta} \int |D_t^\alpha(\varphi - u\psi_N)|^2 dt \le \\
&\le (N+3)^{2k\delta} ||\varphi - u\psi_N||^2_{W^{k,2}} < \\
&< (N+3)^{2k\delta}\sigma^2\epsilon^2 < \frac{\epsilon^2}{4}.
\end{aligned}$$

Hence:

$$||u-\varphi||_{H_{(k,\delta)}} \leq ||u-u\psi_N||_{H_{(k,\delta)}} + ||u\psi_N-\varphi||_{H_{(k,\delta)}} < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

□

Lemma 4.4.2. *Let $Q \in \mathcal{U}(\mathfrak{h})$ be homogeneous of degree $\leq k$. Then $\tilde{\pi}_1(Q)$ and $\tilde{\pi}_{-1}(Q)$, viewed as operators from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, are bounded if we give the domain the $H_{(k,1)}(\mathbb{R}^n)$ norm and the range the $L^2(\mathbb{R}^n)$ norm. Hence, since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_{(k,1)}(\mathbb{R}^n)$, both $\tilde{\pi}_1(Q)$ and $\tilde{\pi}_{-1}(Q)$ extend uniquely as bounded operators from $H_{(k,1)}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.*

Proof. Let

$$Q = \sum_{|\alpha|+|\beta|+\gamma\leq l} a_{\alpha\beta\gamma} Y_1^{\alpha_1}\cdots Y_n^{\alpha_n} X_1^{\beta_1}\cdots X_n^{\beta_n} Z^{\gamma}$$

be homogeneous of degree $j \leq k$. This means that for every $a_{\alpha\beta\gamma} \neq 0$ we have $|\alpha|+|\beta|+2\gamma = j \leq k$, hence in particular $|\alpha|+|\beta| \leq k$. But by (4.5):

$$\tilde{\pi}_1(Q) = \sum_{|\alpha|+|\beta|+\gamma\leq l} a_{\alpha\beta\gamma}(i)^{|\alpha|+\gamma} t^{\alpha}\frac{\partial^{\beta}}{\partial t},$$
$$\tilde{\pi}_{-1}(Q) = \sum_{|\alpha|+|\beta|+\gamma\leq l} a_{\alpha\beta\gamma}(-i)^{|\alpha|+\gamma} t^{\alpha}\frac{\partial^{\beta}}{\partial t}.$$

Thus it suffices to show that $t^{\alpha}\frac{\partial^{\beta}}{\partial t}$ defines a bounded operator from $H_{(k,1)}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ if $|\alpha|+|\beta| \leq k$. But since by definition $v \in H_{(k,1)}(\mathbb{R}^n)$ if and only if $(1+|t|)^{k-|\sigma|}D^{\sigma}v(t) \in L^2(\mathbb{R}^n)$ for any $|\sigma| \leq k$, we have the claim. □

Remark 15. In the preceding proof it would have been sufficient to assume that the order of Q as a differential operator was $\leq k$.

Actually the same proof shows the following stronger statement: if $Q \in \mathcal{U}(\mathfrak{h})$ is homogeneous of degree $\leq k$, then for any integer $r \geq k$, $\tilde{\pi}_1(Q)$ and $\tilde{\pi}_{-1}(Q)$ define bounded operators from $H_{(k,1)}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

We will need the following Lemma, which we don't prove (see paragraph 6 in [9]).

Lemma 4.4.3. *Let $P \in \mathcal{U}(\mathfrak{h})$ be homogeneous of degree m and suppose that $\pi_{(\xi,\eta)}(P) \neq 0$ for every $(\xi,\eta) \in \mathbb{R}^{2n}\setminus\{0\}$. Fix $\lambda \in \mathbb{R}\setminus\{0\}$. Then the following are equivalent:*

i) neither of the equations $\tilde{\pi}_{\lambda}(P)v = 0$, $\tilde{\pi}_{\lambda}(P^*)v = 0$ has a nontrivial solution $v \in \mathcal{S}(\mathbb{R}^n)$;

ii) $\tilde{\pi}_\lambda(P)$ *has a bounded two-sided inverse* L, *i.e. a bounded operator* $L : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *such that* L *maps* $\mathcal{S}(\mathbb{R}^n)$ *into* $\mathcal{S}(\mathbb{R}^n)$ *and such that* $\tilde{\pi}_\lambda(P)L = I$ *on* $\mathcal{S}(\mathbb{R}^n)$ *and* $L\tilde{\pi}_\lambda(P) = I$ *on* $\mathcal{S}(\mathbb{R}^n)$.

If these equivalent conditions hold, then L *satisfies the following additional properties:*

1. L *maps* $L^2(\mathbb{R}^n)$ *into* $H_{(m,1)}(\mathbb{R}^n)$ *and is bounded as an operator from* $L^2(\mathbb{R}^n)$ *into* $H_{(m,1)}(\mathbb{R}^n)$;
2. $\tilde{\pi}_\lambda(P)L = I$ *on* $L^2(\mathbb{R}^n)$;
3. $L\tilde{\pi}_\lambda(P) = I$ *on* $H_{(m,1)}(\mathbb{R}^n)$;
4. *if* $f \in L^2(\mathbb{R}^n)$ *and* $Lf \in \mathcal{S}(\mathbb{R}^n)$, *then* $f \in \mathcal{S}(\mathbb{R}^n)$.

The following lemmas will be useful in the proof of the crucial Proposition 4.4.7.

Lemma 4.4.4. *Let* $Q \in \mathcal{U}(\mathfrak{h})$ *be homogeneous of degree* k *and let* $\varphi \in C_0^\infty(\mathbb{H}_n)$. *Then there is a finite set of differential operators* $Q_1, \dots, Q_r \in \mathcal{U}(\mathfrak{h})$, *each homogeneous of some degree* $< k$, *and* $\varphi_1, \dots, \varphi_r \in C_0^\infty(\mathbb{H}_n)$ *with supp* $\varphi_j \subset$ *supp* φ, *such that*

$$[Q, \varphi] = \sum_{j=1}^{r} Q_j \varphi_j.$$

Proof. We prove the claim by induction on k.

If $k = 1$ then $Q = X_i$ for some $i = 1, \dots, n$ or $Q = Y_i$ or $Q = Z$. In any case for every $u \in \mathbb{C}_0^\infty(\mathbb{H}_n)$:

$$[Q, \varphi]u = Q(\varphi u) - \varphi(Qu) = (Q\varphi)u + \varphi(Qu) - \varphi(Qu) = (Q\varphi)u,$$

thus we have the claim because $Q\varphi$ has order 0.

Suppose now that we have proved the claim for any operator homogeneous of degree $k-1$ and let Q be homogeneous of degree k. Since Q can be written as sum of terms of degree k, it suffices to prove the claim in case

$$Q = X_{i_1} \cdots X_{i_{k-1}} X_{i_k} = P_{k-1} X_{i_k},$$

where $i_j \in \{1, \dots, n\}$ and $P_{k-1} = X_{i_1} \cdots X_{i_{k-1}}$ is homogeneous of degree $k - 1$. Then:

$$\begin{aligned}
[Q, \varphi]u = [P_{k-1}X_{i_k}, \varphi]u &= (P_{k-1}X_{i_k})(\varphi u) - \varphi((P_{k-1}X_{i_k})u) = \\
&= P_{k-1}(X_{i_k}(\varphi u)) - \varphi(P_{k-1}(X_{i_k}u)) = \\
&= P_{k-1}((X_{i_k}\varphi)u + \varphi(X_{i_k}u)) - \varphi(P_{k-1}(X_{i_k}u)) = \\
&= P_{k-1}((X_{i_k}\varphi)u) + P_{k-1}(\varphi X_{i_k}u) - \varphi P_{k-1}(X_{i_k}u) = \\
&= P_{k-1}((X_{i_k}\varphi)u) + [P_{k-1}, \varphi](X_{i_k}u).
\end{aligned}$$

Using now the inductive hypothesis on $[P_{k-1},\varphi]$ and noting that $P_{k-1}(X_{i_k}\varphi)$ has degree $k-1$, we have the claim. □

Lemma 4.4.5. *Let $u^m = u_0^m + u_\infty^m$ be the distribution solution of $Pu^m = Z^m\delta$ defined by* (4.19)*. Then:*

i) for any integer $k \geq 0$, $Z^k u_0^m \in L^2(\mathbb{H}_n)$;

ii) there exists $f \in L^2(\mathbb{H}_n)$ and an integer $N > 0$ such that:

$$u_\infty^m = Z^N f. \tag{4.24}$$

Proof. We recall that by Remark 13 $u_0^s \in L^2(\mathbb{H}_n)$ for any s and by (4.20) $Z^k u_0^m = u_0^{m+k}$ for any integer $k \geq 0$. Thus we get i).

On the other hand, ii) follows immediately from the same Remark 13. □

Lemma 4.4.6. *Under the hypothesis of the previous lemma, for every $\varphi \in C_0^\infty(\mathbb{H}_n)$ the following hold.*

i) For any integer $k \geq 0$, $Z^k(\varphi u_0^m) \in L^1(\mathbb{H}_n) \cap L^2(\mathbb{H}_n)$.

ii) There exist $\varphi_j \in C_0^\infty(\mathbb{H}_n)$, $j = 0,\ldots,N$ (where N is the integer in (4.24)*), with supp $\varphi_j \subset$ supp φ (and $\varphi_N = \varphi$) such that:*

$$\varphi u_\infty^m = \sum_{j=0}^{N} Z^j(\varphi_j f). \tag{4.25}$$

Note, in particular, that for every j, $\varphi_j f \in L^1(\mathbb{H}_n) \cap L^2(\mathbb{H}_n)$.

Proof. By iterating the product rule for differentiation we get:

$$Z^k(\varphi u_0^m) = \sum_{j=0}^{k} \binom{k}{j} (Z^j\varphi)(Z^{j-k}u_0^m).$$

Since $Z^j\varphi \in C_0^\infty(\mathbb{H}_n)$ and by previous lemma $Z^{j-k}u_0^m \in L^2(\mathbb{H}_n)$, it follows that $(Z^j\varphi)(Z^{j-k}u_0^m) \in L^1(\mathbb{H}_n) \cap L^2(\mathbb{H}_n)$ for any $j = 0,\ldots,k$. Thus i) holds.

The proof of (4.25) is by induction on N.

If $N = 1$, then

$$\varphi u_\infty^m = \varphi Z f = Z(\varphi f) - [Z,\varphi]f.$$

Thus the claim holds with $\varphi_0 = -[Z,\varphi]$.

We now suppose that the claim holds for N and we prove it for $N+1$. We have:

$$\varphi Z^{N+1} f = \varphi Z^N (Zf),$$

which by inductive hypothesis equals to:

$$\sum_{j=0}^{N} Z^j(\varphi_j Z f) = \sum_{j=0}^{N} Z^j(Z(\varphi_j f) - [Z, \varphi_j] f) =$$
$$= \sum_{j=0}^{N} Z^{j+1}(\varphi_j f) - \sum_{j=0}^{N} Z^j([Z, \varphi_j] f).$$

If we set $\psi_0 = -[Z, \varphi_0]$, $\psi_1 = \varphi_0 - [Z, \varphi_1]$, $\dots$,$\psi_N = \varphi_{N-1} - [Z, \varphi_N]$, $\psi_{N+1} = \varphi_N = \varphi$, we get the claim. □

Proposition 4.4.7. *Let $P \in \mathcal{U}(\mathfrak{h})$ be homogeneous of degree m and satisfy:*

i) $\pi_{(\xi,\eta)}(P) \neq 0$ for every $(\xi, \eta) \in \mathbb{R}^{2n} \setminus \{0\}$;

ii) $\tilde{\pi}_1(P)$, $\tilde{\pi}_{-1}(P)$ both have two-sided inverses.

Let u^m be the distribution solution of $Pu^m = Z^m \delta$ defined by (4.19). *Let φ be any function in $C_0^\infty(\mathbb{H}_n)$ such that $e \notin supp\varphi$. Then:*

$$Z^k(\varphi u^m) \in L^2(\mathbb{H}_n) \tag{4.26}$$

for every nonnegative integer k.

Proof. If P satisfies i) and ii), then by (4.8) and (4.9) so does $\bar{P}$. Hence, as before, we can assume without loss of generality that $P + Z^m$ is elliptic.

Moreover, since P satisfies i), so does P^* because $\pi_{(\xi,\eta)}(P^*) = \pi_{(\xi,\eta)}(P)^*$. Also, by Lemma 4.4.3 if P satisfies ii) then so does P^*, and hence, by the above, so does $P^t = \overline{P^*}$. In particular, $\tilde{\pi}_1(P^t)$ and $\tilde{\pi}_{-1}(P^t)$ both have bounded two-sided inverses.

Viewing φu^m as an element of $\mathcal{E}'(\mathbb{H}_n)$, we have:

$$\tilde{\pi}_\lambda(Z^k(\varphi u^m)) = \tilde{\pi}_\lambda(\varphi u^m)\tilde{\pi}_\lambda((Z^k)^t) = (-i)^k \lambda^k \tilde{\pi}_\lambda(\varphi u^m).$$

Thus by Proposition 2.3.4 to prove (4.26) it suffices to show that for any integer $k \geq 0$ and for any $\varphi \in C_0^\infty(\mathbb{H}_n)$ such that $e \notin supp\varphi$, the function:

$$\lambda \mapsto |\lambda|^k \tilde{\pi}_\lambda(\varphi u^m) \tag{4.27}$$

lies in $L^2(\mathbb{R}\setminus\{0\};K)$, where K is the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ and $\mathbb{R}\setminus\{0\}$ has the Plancherel measure $|\lambda|^n d\lambda$.

We now observe that:

$$P(\varphi u^m) = \varphi P u^m + [P,\varphi]u^m = \varphi Z^m\delta + [P,\varphi]u^m.$$

But if $e \notin supp\varphi$, then $\varphi Z^m \delta = 0$. Thus:

$$P(\varphi u^m) = [P,\varphi]u^m.$$

Applying Lemma 4.4.4, we have that there exist finitely many $Q_1,\dots,Q_r \in \mathcal{U}(\mathfrak{h})$ homogeneous of degree $< m$ and $\varphi_1,\dots,\varphi_r \in C_0^\infty(\mathbb{H}_n)$ such that $e \notin supp\varphi_j$ and

$$P(\varphi u^m) = \sum_{j=1}^{r} Q_j(\varphi_j u^m). \tag{4.28}$$

Since $\varphi u_0^m \in \mathcal{E}'(\mathbb{H}_n)$ also lies in $L^1(\mathbb{H}_n)$, by Proposition 2.2.2 (1) we can view $\tilde{\pi}_\lambda(\varphi u_0^m)$ as a bounded operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Moreover, by (4.25) we have:

$$\begin{aligned}\tilde{\pi}_\lambda(\varphi u_\infty^m) &= \sum_{j=0}^{N} \tilde{\pi}_\lambda(Z^j(\varphi_j f)) = \sum_{j=0}^{N} \tilde{\pi}_\lambda(\varphi_j f)\tilde{\pi}_\lambda((Z^j)^t) = \\ &= \sum_{j=0}^{N} (-i)^j \lambda^j \tilde{\pi}_\lambda(\varphi_j f).\end{aligned} \tag{4.29}$$

Since $\varphi_j f \in L^1(\mathbb{H}_n)$, we can again view $\tilde{\pi}_\lambda(\varphi_j f)$, and hence $\tilde{\pi}_\lambda(\varphi u_\infty^m)$, as a bounded operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

We now prove two lemmas that will complete the proof, splitting the claim in two cases.

Lemma 4.4.8. *Let N be the integer appearing in* (4.24)*. Let $k \geq -2N$ be an integer and let $\varphi \in C_0^\infty(\mathbb{H}_n)$ such that $e \notin supp\varphi$. Fix $\epsilon > 0$ and let $\mathbb{R}_\epsilon = \{\lambda \in \mathbb{R} \,|\, |\lambda| > \epsilon\}$. Then the function $\lambda \mapsto |\lambda|^{\frac{k}{2}}\tilde{\pi}_\lambda(\varphi u^m)$ lies in $L^2(\mathbb{R}_\epsilon;K)$, where $\mathbb{R}_\epsilon$ is provided with the Plancherel measure $|\lambda|^n d\lambda$.*

Proof. We prove the statement by induction on k.

First let $k = -2N$. We want to show that the function $\lambda \mapsto |\lambda|^{-N}\tilde{\pi}_\lambda(\varphi u^m)$ lies in $L^2(\mathbb{R}_\epsilon;K)$.

Since $\tilde{\pi}_\lambda(\varphi u^m) = \tilde{\pi}_\lambda(\varphi u_0^m) + \tilde{\pi}_\lambda(\varphi u_\infty^m)$, it suffices to show that both:

$$\lambda \mapsto |\lambda|^{-N}\tilde{\pi}_\lambda(\varphi u_0^m) \tag{4.30}$$

and

$$\lambda \mapsto |\lambda|^{-N}\tilde{\pi}_\lambda(\varphi u_\infty^m) \tag{4.31}$$

lie in $L^2(\mathbb{R}_\epsilon; K)$.

By Lemma 4.4.6 we know in particular that $\varphi u_0^m \in L^1(\mathbb{H}_n) \cap L^2(\mathbb{H}_n)$. Thus by Plancherel Theorem, the function $\lambda \mapsto \tilde{\pi}_\lambda(\varphi u_0^m)$ lies in $L^2(\mathbb{R} \setminus \{0\})$, and so, by restriction, it also lies in $L^2(\mathbb{R}_\epsilon; K)$. But $\lambda \mapsto |\lambda|^{-N}$ is C^∞ and bounded on $\mathbb{R}_\epsilon$, thus (4.30) lies in $L^2(\mathbb{R}_\epsilon; K)$.

Moreover, by (4.29):

$$\tilde{\pi}_\lambda(\varphi u_\infty^m) = \sum_{j=0}^{N} (-i)^j \lambda^j \tilde{\pi}_\lambda(\varphi_j f).$$

Arguing as before, we have that the function $\lambda \mapsto \tilde{\pi}_\lambda(\varphi_j f)$ is in $L^2(\mathbb{R}_\epsilon; K)$ for any $j = 0, \ldots, N$. Since the function $\lambda \mapsto |\lambda|^{-N}\lambda^j$ is C^∞ and bounded on $\mathbb{R}_\epsilon$, it follows that also (4.31) lies in $L^2(\mathbb{R}_\epsilon; K)$.

We now assume that we have shown that for any integer k such that $-2N \leq k \leq l$ and for any $\varphi \in C_0^\infty(\mathbb{H}_n)$ such that $e \notin supp\varphi$, the function $\lambda \mapsto |\lambda|^{\frac{k}{2}}\tilde{\pi}_\lambda(\varphi u^m)$ lies in $L^2(\mathbb{R}_\epsilon; K)$. We want to prove the claim for $k = l+1$.

Since for any integer $r \geq 0$, the function $\lambda \mapsto |\lambda|^{-\frac{r}{2}}$ is C^∞ and bounded on $\mathbb{R}_\epsilon$, it follows that also $\lambda \mapsto |\lambda|^{\frac{k-r}{2}}\tilde{\pi}_\lambda(\varphi u^m)$ lies in $L^2(\mathbb{R}_\epsilon; K)$. Thus in the inductive hypothesis we can take any integer $k \leq l$.

Let $\varphi \in C_0^\infty(\mathbb{H}_n)$ such that $e \notin supp\varphi$. Applying $\tilde{\pi}_\lambda$ to the compactly supported distributions in (4.28), we get:

$$\tilde{\pi}_\lambda(P(\varphi u^m)) = \sum_{j=1}^{r} \tilde{\pi}_\lambda(Q_j(\varphi_j u^m)).$$

Hence by (2.19):

$$\tilde{\pi}_\lambda(\varphi u^m)\tilde{\pi}_\lambda(P^t) = \sum_{j=1}^{r} \tilde{\pi}_\lambda(\varphi_j u^m)\tilde{\pi}_\lambda(Q_j^t).$$

Since P, and hence P^t, is homogeneous of degree m whereas each Q_j, and hence Q_j^t, is homogeneous of degree $s_j < m$, we have by (4.6):

$$\begin{aligned} |\lambda|^{\frac{m}{2}}\tilde{\pi}_\lambda(\varphi u^m)\tilde{\pi}_1(P^t) &= \sum_{j=1}^{r} |\lambda|^{\frac{s_j}{2}}\tilde{\pi}_\lambda(\varphi_j u^m)\tilde{\pi}_1(Q_j^t), \qquad \lambda > 0 \\ |\lambda|^{\frac{m}{2}}\tilde{\pi}_\lambda(\varphi u^m)\tilde{\pi}_{-1}(P^t) &= \sum_{j=1}^{r} |\lambda|^{\frac{s_j}{2}}\tilde{\pi}_\lambda(\varphi_j u^m)\tilde{\pi}_{-1}(Q_j^t), \qquad \lambda < 0. \end{aligned} \tag{4.32}$$

As we have noticed in the beginning of the proof of Proposition 4.4.7, $\tilde{\pi}_1(P^t)$ and $\tilde{\pi}_{-1}(P^t)$ have bounded two-sided inverses, call them L_1 and L_{-1} respectively.

Since P^t is homogeneous of degree m and $\pi_{(\xi,\eta)}(P^t) \neq 0$ for every $(\xi, \eta) \in \mathbb{R}^{2n} \setminus \{0\}$, it follows by Lemma 4.4.3 that L_1 and L_{-1} are bounded operators from $L^2(\mathbb{R}^n)$ into $H_{(m,1)}(\mathbb{R}^n)$.

Moreover, by Lemma 4.4.2 for every $j = 1, \dots, r$, $\tilde{\pi}_1(Q_j^t)$ and $\tilde{\pi}_{-1}(Q_j^t)$ are bounded operators from $H_{(m,1)}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Thus if we set $T_{1,j} = \tilde{\pi}_1(Q_j^t)L_1$ and $T_{-1,j} = \tilde{\pi}_{-1}(Q_j^t)L_{-1}$, we know that these are bounded operators from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ which map $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$.

Applying L_1 to the first equation in (4.32) and L_{-1} to the second one, we get:

$$\tilde{\pi}_\lambda(\varphi u^m) = \sum_{j=1}^{r} |\lambda|^{\frac{s_j - m}{2}} \tilde{\pi}_\lambda(\varphi_j u^m) T_{1,j}, \qquad \lambda > 0$$
$$\tilde{\pi}_\lambda(\varphi u^m) = \sum_{j=1}^{r} |\lambda|^{\frac{s_j - m}{2}} \tilde{\pi}_\lambda(\varphi_j u^m) T_{-1,j}, \qquad \lambda < 0.$$

These equations are initially valid between operators defined on $\mathcal{S}(\mathbb{R}^n)$. But, since $\tilde{\pi}_\lambda(\varphi u^m)$, $\tilde{\pi}_\lambda(\varphi_j u^m)$, $T_{1,j}$ and $T_{-1,j}$ are all bounded operators from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we can view the equations above as equations between bounded operators on $L^2(\mathbb{R}^n)$.

If we now multiply both equations by $|\lambda|^{\frac{l+1}{2}}$, we get:

$$\begin{aligned} |\lambda|^{\frac{l+1}{2}} \tilde{\pi}_\lambda(\varphi u^m) &= \sum_{j=1}^{r} |\lambda|^{\frac{l+1+s_j - m}{2}} \tilde{\pi}_\lambda(\varphi_j u^m) T_{1,j}, \qquad \lambda > 0 \\ |\lambda|^{\frac{l+1}{2}} \tilde{\pi}_\lambda(\varphi u^m) &= \sum_{j=1}^{r} |\lambda|^{\frac{l+1+s_j - m}{2}} \tilde{\pi}_\lambda(\varphi_j u^m) T_{-1,j}, \qquad \lambda < 0. \end{aligned} \tag{4.33}$$

Since $s_j - m < 0$ for any j, we have $l+1+s_j-m \leq l$. Hence, by inductive hypothesis, the function $\lambda \mapsto |\lambda|^{\frac{l+1+s_j-m}{2}} \tilde{\pi}_\lambda(\varphi_j u^m)$ lies in $L^2(\mathbb{R}_\epsilon; K)$ for any j. Letting $\mathbb{R}_\epsilon^+ = \{\lambda \in \mathbb{R} | \lambda > \epsilon\}$ and $\mathbb{R}_\epsilon^- = \{\lambda \in \mathbb{R} | \lambda < -\epsilon\}$, we have by restriction that $\lambda \mapsto |\lambda|^{\frac{l+1+s_j-m}{2}} \tilde{\pi}_\lambda(\varphi_j u^m)$ lies in $L^2(\mathbb{R}_\epsilon^+; K)$ and $L^2(\mathbb{R}_\epsilon^-; K)$.

We now need to observe the following.

Remark 16. It can be shown that if (Ω, μ) is any measure space, H a Hilbert space and T a bounded linear operator $T : H \to H$, then for any function $\lambda \mapsto f(\lambda)$ in $L^2(\Omega; H)$, the function $\lambda \mapsto T(f(\lambda))$ also lies in $L^2(\Omega; H)$.

Indeed, it can be shown that $\lambda \mapsto T(f(\lambda))$ is μ-measurable and

$$\left(\int_\Omega ||T(f(\lambda))||^2 d\mu(\lambda)\right)^{\frac{1}{2}} \leq ||T|| \left(\int_\Omega ||f(\lambda)||^2 d\mu(\lambda)\right)^{\frac{1}{2}}.$$

In our particular case, we recall that every bounded linear operator $T: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ determines a bounded linear operator on the Hilbert space K of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$, defined by $K \ni S \mapsto ST \in K$.

Thus, by the previous Remark the function

$$\lambda \mapsto |\lambda|^{\frac{l+1+s_j-m}{2}} \tilde{\pi}_\lambda(\varphi_j u^m) T_{1,j}$$

lies in $L^2(\mathbb{R}^+_\epsilon; K)$ for any j, and the function

$$\lambda \mapsto |\lambda|^{\frac{l+1+s_j-m}{2}} \tilde{\pi}_\lambda(\varphi_j u^m) T_{-1,j}$$

lies in $L^2(\mathbb{R}^-_\epsilon; K)$ for any j.

Hence, by (4.33) the function $\lambda \mapsto |\lambda|^{\frac{l+1}{2}} \tilde{\pi}_\lambda(\varphi u^m)$ lies in $L^2(\mathbb{R}_\epsilon; K)$ and this completes the proof. □

Lemma 4.4.9. *Let $k \geq 0$ be an integer and let $\varphi \in C_0^\infty(\mathbb{H}_n)$ such that $e \notin supp\varphi$. Then the function $\lambda \mapsto |\lambda|^k \tilde{\pi}_\lambda(\varphi u^m)$ lies in $L^2(\{0 < |\lambda| \leq \epsilon\}; K)$.*

Proof. We argue exactly as we have done before in the case $k = -2N$ (actually we do not need the hypothesis that $e \notin supp\varphi$).

Since $\tilde{\pi}_\lambda(\varphi u^m) = \tilde{\pi}_\lambda(\varphi u_0^m) + \tilde{\pi}_\lambda(\varphi u_\infty^m)$, it suffices to show that both:

$$\lambda \mapsto |\lambda|^k \tilde{\pi}_\lambda(\varphi u_0^m) \tag{4.34}$$

and

$$\lambda \mapsto |\lambda|^k \tilde{\pi}_\lambda(\varphi u_\infty^m) \tag{4.35}$$

lie in $L^2(\{0 < |\lambda| \leq \epsilon\}; K)$.

By Lemma 4.4.6 we know that $\varphi u_0^m \in L^1(\mathbb{H}_n) \cap L^2(\mathbb{H}_n)$. Thus by Plancherel Theorem, the function $\lambda \mapsto \tilde{\pi}_\lambda(\varphi u_0^m)$ lies in $L^2(\mathbb{R} \setminus \{0\})$, and so, by restriction, it also lies in $L^2(\{0 < |\lambda| \leq \epsilon\}; K)$. But $\lambda \mapsto |\lambda|^k$ is C^∞ and bounded on $\{0 < |\lambda| \leq \epsilon\}$, thus (4.34) lies in $L^2(\{0 < |\lambda| \leq \epsilon\}; K)$.

Moreover, by (4.29):

$$|\lambda|^k \tilde{\pi}_\lambda(\varphi u_\infty^m) = \sum_{j=0}^{N} (-i)^j |\lambda|^k \lambda^j \tilde{\pi}_\lambda(\varphi_j f).$$

Arguing as before, we have that the function $\lambda \mapsto \tilde{\pi}_\lambda(\varphi_j f)$ is in $L^2(\{0 < |\lambda| \leq \epsilon\}; K)$ for any $j = 0, \ldots, N$. Since the function $\lambda \mapsto |\lambda|^k \lambda^j$ is C^∞ and bounded on $\{0 < |\lambda| \leq \epsilon\}$, it follows that also (4.35) lies in $L^2(\{0 < |\lambda| \leq \epsilon\}; K)$. □

These two lemmas complete the proof of the proposition because they show that for any integer $k \geq 0$ and for any $\varphi \in C_0^\infty(\mathbb{H}_n)$ such that $e \notin supp\varphi$, the function:

$$\lambda \mapsto |\lambda|^k \tilde{\pi}_\lambda(\varphi u^m)$$

lies in $L^2(\mathbb{R} \setminus \{0\}; K)$. □

Theorem 4.4.10. *Let $P \in \mathcal{U}(\mathfrak{h})$ be homogeneous of degree m and satisfy:*

i) $\pi_{(\xi,\eta)}(P) \neq 0$ for every $(\xi, \eta) \in \mathbb{R}^{2n} \setminus \{0\}$;

ii) $\tilde{\pi}_1(P)$, $\tilde{\pi}_{-1}(P)$ both have bounded two-sided inverses.

Then the parametrix $u = v * (\delta + u^m)$ constructed above for P is C^∞ away from e.

Proof. As we have noted in the beginning of the proof of Proposition 4.4.7, we can assume without loss of generality that $P + Z^m$ is elliptic (in fact we needed this hypothesis to construct the parametrix).

First, we observe that, by construction, $v \in \mathcal{E}'(\mathbb{H}_n)$ is C^∞ away from e. Since convolution by a distribution which is C^∞ away from e does not increase singular support, it suffices to show that u^m is C^∞ away from e (then also $\delta + u^m$ is C^∞ away from e).

Since Z lies in the center of $\mathcal{U}(\mathfrak{h})$, so does Z^m. In particular, Z^m commutes with P. Hence, for every integer $k > 0$, we have:

$$(P + Z^m)^k u^m = \sum_{j=0}^{k} \binom{k}{j} Z^{mj} P^{k-j} u^m.$$

Since $Pu^m = Z^m\delta$, which is supported at e, then $P^{k-j}u^m$ is supported at e when $j < k$. Thus if we multiply $(P+Z^m)^k u^m$ by any function $\varphi \in C_0^\infty(\mathbb{H}_n)$ such that $e \notin supp\varphi$, the only non vanishing term will be the one with $j = k$, i.e.

$$\varphi(P + Z^m)^k u^m = \varphi Z^{mk} u^m. \tag{4.36}$$

We can prove (proceeding in the same way as in the proof of (4.25)) that

$$\varphi Z^{mk} u^m = \sum_{j=0}^{mk} Z^j(\varphi_j u^m),$$

where $\varphi_j \in C_0^\infty(\mathbb{H}_n)$ such that $supp\varphi_j \subset supp\varphi$.

Indeed we proceed by induction on mk.

When $mk = 1$, then:

$$\varphi Z u^m = Z(\varphi u^m) - [Z, \varphi] u^m,$$

hence we have the claim with $\varphi_0 = -[Z, \varphi]$ and $\varphi_1 = \varphi$.

Suppose now that we have proved the claim for mk and we prove it for $mk + 1$. We have:

$$\varphi Z^{mk+1} u^m = \varphi Z^{mk}(Z u^m).$$

Remembering that $Zu^m = u^{m+1}$, we can use the inductive hypothesis so this equals to:

$$\sum_{j=0}^{mk} Z^j(\varphi_j Z u^m) = \sum_{j=0}^{mk} Z^{j+1}(\varphi_j u^m) - \sum_{j=0}^{mk} Z^j([Z, \varphi_j] u^m).$$

Hence, we have the claim if we take $\psi_0 = [Z, \varphi_0]$, $\psi_1 = \varphi_0 - [Z, \varphi_1]$, $\ldots$, $\psi_{mk+1} = \varphi_{mk}$.

Since $e \notin supp\varphi_j$ for any j, it follows by Proposition 4.4.7 that $Z^j(\varphi_j u^m) \in L^2(\mathbb{H}_n)$ for every $j = 0, \ldots, mk$. Hence, $\varphi Z^{mk} u^m \in L^2(\mathbb{H}_n)$.

Thus, by (4.36) $\varphi(P + Z^m)^k u^m \in L^2(\mathbb{H}_n)$ for every integer $k > 0$ and for every $\varphi \in C_0^\infty(\mathbb{H}_n)$ such that $e \notin supp\varphi$. Since $(P + Z^m)^k$ is elliptic of order mk, regularity results for elliptic operators imply that $u^m \in H^{mk}_{loc}(\mathbb{H}_n \setminus \{e\})$, where H^s_{loc} denotes the usual Sobolev space.

Since k is arbitrary, it follows by Sobolev lemma that $u^m \in C^\infty(\mathbb{H}_n \setminus \{e\})$ and this completes the proof. □

Remark 17.

1. From the previous Theorem and 1.10.2, it follows that P^t is hypoelliptic.

2. As we have observed in the beginning of the proof of Proposition 4.4.7, if P satisfies the hypothesis of the previous Theorem, then so do P^*, $\bar{P}$ and P^t. In particular, it follows that also P is hypoelliptic.

3. Hence, the previous Theorem shows that in Theorem 4.0.8 condition 2 implies 1.

4.5 Necessity for hypoellipticity

We will now prove a necessary condition for P and P^t to be hypoelliptic, that will conclude the proof of Theorem 4.0.8.

Lemma 4.5.1. *Let P be a differential operator on $\mathbb{H}_n$ such that both P and P^t are hypoelliptic. Then there exist distributions u_1, $u_2 \in \mathcal{E}'(\mathbb{H}_n)$ both C^∞ away from e and functions f_1, $f_2 \in C_0^\infty(\mathbb{H}_n)$ such that:*

$$Pu_1 = \delta + f_1, \tag{4.37}$$

$$P^t u_2 = \delta + f_2. \tag{4.38}$$

Proof. Since P^t is hypoelliptic, then by Theorem 1.10.1 P is locally solvable, i.e. there exist a neighborhood V of e and a distribution $u \in \mathcal{D}'(\mathbb{H}_n)$ such that:

$$Pu = \delta$$

in V. From the hypoellipticity of P and the fact that δ is C^∞ away from e, it follows that also u is C^∞ away from e. We can choose (eventually restricting V) $V = B(e, r)$ for some $r > 0$.

Let $\phi \in C_0^\infty(B(e, r))$, $\phi \equiv 1$ on $B(e, \frac{r}{2})$. Then $\phi u \in \mathcal{E}'(\mathbb{H}_n)$, it is C^∞ away from e and it satisfies:

$$P(\phi u) = \phi P u + [P, \phi]u = \phi\delta + [P, \phi]u = \delta + [P, \phi]u.$$

But $[P, \phi]$ contains derivatives of ϕ that vanish near e, thus $f_1 = [P, \phi]u \in C_0^\infty(\mathbb{H}_n)$. Hence $u_1 = \phi u$ satisfies (4.37).

Similarly, (4.38) holds. □

Theorem 4.5.2. *Let $P \in \mathcal{U}(\mathfrak{h})$ be homogeneous of degree m and suppose that both P and P^t are hypoelliptic. Then:*

i) $\pi_{(\xi,\eta)}(P) \neq 0$ for every $(\xi, \eta) \in \mathbb{R}^{2n} \setminus \{0\}$;

ii) neither of the equations $\tilde{\pi}_\lambda(P)v = 0$, $\tilde{\pi}_\lambda(P^*)v = 0$ has a nonzero solution $v \in \mathcal{S}(\mathbb{R}^n)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. Applying (4.37) with P replaced by $\bar{P}$, we have $\bar{P}u_1 = \delta + f_1$. Applying now to both sides the representations $\pi_{(\xi,\eta)}$, we get:

$$\pi_{(\xi,\eta)}(\bar{P}u_1) = \pi_{(\xi,\eta)}(\delta) + \pi_{(\xi,\eta)}(f_1).$$

Now we recall that $\pi_{(\xi,\eta)}(\delta) = I$ (because this is valid for every unitary irreducible representation) and by (2.19) $\pi_{(\xi,\eta)}(\bar{P}u_1) = \pi_{(\xi,\eta)}(u_1)\pi_{(\xi,\eta)}(\bar{P}^t)$. Moreover, by (4.9) $\pi_{(\xi,\eta)}(\bar{P}^t) = \pi_{(-\xi,-\eta)}(\bar{P}) = \overline{\pi_{(\xi,\eta)}(P)}$. Thus we get:

$$\pi_{(\xi,\eta)}(u_1)\overline{\pi_{(\xi,\eta)}(P)} = 1 + \pi_{(\xi,\eta)}(f_1). \tag{4.39}$$

When, instead, we apply $\tilde{\pi}_\lambda$ for every $\lambda \in \mathbb{R} \setminus \{0\}$ to both sides of $\bar{P}u_1 = \delta + f_1$, we get for every $v \in \mathcal{S}(\mathbb{R}^n)$:

$$\tilde{\pi}_\lambda(u_1)\tilde{\pi}_\lambda(\bar{P}^t)v = (I + \tilde{\pi}_\lambda(f_1))v.$$

But $\bar{P}^t = P^*$, thus by (2.7) $\tilde{\pi}_\lambda(\bar{P}^t) = \tilde{\pi}_\lambda(P^*) = \tilde{\pi}_\lambda(P)^*$. Hence

$$\tilde{\pi}_\lambda(u_1)\tilde{\pi}_\lambda(P)^*v = (I + \tilde{\pi}_\lambda(f_1))v. \tag{4.40}$$

Applying now $\tilde{\pi}_\lambda$ for every $\lambda \in \mathbb{R} \setminus \{0\}$ to both sides of (4.38), we get for every $v \in \mathcal{S}(\mathbb{R}^n)$:

$$\tilde{\pi}_\lambda(u_2)\tilde{\pi}_\lambda(P)v = (I + \tilde{\pi}_\lambda(f_2))v. \tag{4.41}$$

If we use exponential coordinates on $\mathbb{H}_n$, it follows from (4.1) that for any $\varphi \in C_0^\infty(\mathbb{H}_n)$:

$$\begin{aligned}\pi_{(\xi,\eta)}(\varphi) &= \int_{\mathbb{H}_n} \varphi(x,y,z)\pi_{(\xi,\eta)}(x,y,z)dxdydz = \\ &= \int_{\mathbb{H}_n} \varphi(x,y,z)e^{i(x\cdot\xi + y\cdot\eta)}dxdydz = \hat{\varphi}(-\xi,-\eta,0),\end{aligned}$$

where $\hat{\varphi}$ denotes the Fourier transform of φ.

Applying this together with (4.7) to (4.39), we get for any $r \in \mathbb{R}^+$:

$$r^m\pi_{(r\xi,r\eta)}(u_1)\overline{\pi_{(\xi,\eta)}(P)} = 1 + \hat{f}_1(-r\xi,-r\eta,0). \tag{4.42}$$

We now argue as in the proof of Proposition 3.0.7, that is: since $f_1 \in C_0^\infty(\mathbb{H}_n)$, we have $\hat{f}_1 \in \mathcal{S}(\mathbb{H}_n)$, hence for fixed $(\xi,\eta) \neq (0,0)$, $|\hat{f}_1(-r\xi,-r\eta,0)| < \epsilon$ for r sufficiently large. Therefore, for r sufficiently large the right-hand side of (4.42) does not equal 0, and so $\pi_{(\xi,\eta)}(P) \neq 0$. Hence $i)$ holds.

Applying now (4.6) to (4.40) and (4.41), we see that for any $r \in \mathbb{R}^+$:

$$r^{\frac{m}{2}}\tilde{\pi}_{r\lambda}(u_1)\tilde{\pi}_\lambda(P)^*v = (I + \tilde{\pi}_{r\lambda}(f_1))v. \tag{4.43}$$

$$r^{\frac{m}{2}}\tilde{\pi}_{r\lambda}(u_2)\tilde{\pi}_\lambda(P)v = (I + \tilde{\pi}_{r\lambda}(f_2))v. \tag{4.44}$$

But by Plancherel Theorem (5), we know that $||\tilde{\pi}_{r\lambda}(f_1)|| < \epsilon$ and $||\tilde{\pi}_{r\lambda}(f_2)|| < \epsilon$ for r sufficiently large and fixed λ. Hence by Neumann series, for r sufficiently large, $I + \tilde{\pi}_{r\lambda}(f_1)$ and $I + \tilde{\pi}_{r\lambda}(f_2)$ are invertible bounded linear operators on $L^2(\mathbb{R}^n)$. In particular, the right-hand sides of (4.43) and (4.44) cannot be 0 unless $v = 0$. Thus $ii)$ follows. □

This theorem shows that in Theorem 4.0.8 condition 1 implies 3. Since in the previous section we proved that 2 implies 1 and by Lemma 4.4.3 condition 3 implies 1, we have proved Theorem 4.0.8 (modulo Lemma 4.4.3).

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