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Quantum field theory of $(p, 0)$-forms on Kähler manifolds

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Introduction

In this thesis we want to study the quantum field theory of a special class of differential forms living on a complex manifold endowed with a *Kähler* metric. Higher forms gauge theories have been studied for a long time and find applications in a very broad class of physical theories. They are quite ubiquitous as low energy states in String theories [15] [14]. Various supergravity models made extensive use of antisymmetric tensor fields, and it was in this wide area that the first complete and general quantization schemes for p-forms appeared [24] [25] [22]. They are also useful in the theory of vortex motion in an irrotational incompressible fluid [20].

As we will see, one can separate the coordinates basis of a complex manifold into two pieces: holomorphic and antiholomorphic coordinates, respectively denoted by z and $\bar{z}$. We will deal only with forms of p degree in the holomorphic sector or $(p,0)$-forms. The main variable of our theory is a $(p,0)$-form potential:

$$A = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_p}. \tag{1}$$

As in the Maxwell, real 1-form case, one can define a $(p+1,0)$ field strength for these complex forms, with coordinates:

$$F_{\mu_1 \ldots \mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]} = \partial_{\mu_1} A_{\mu_2 \ldots \mu_{p+1}} \pm \text{cyclic perm}, \tag{2}$$

where $[\ldots]$ denotes weighted antisymmetrization. The equations of motion of our system in terms of the potential A are:

$$\bar{\partial}^{\mu} \partial_{\mu} A_{\mu_1 \ldots \mu_p} + (-1)^p p \bar{\partial}^{\mu} \partial_{[\mu_1} A_{\mu_2 \ldots \mu_p]\mu} = 0 \tag{3}$$

and possess the very important feature of being invariant under gauge transformations:

$$A_{\mu_1 \ldots \mu_p} = p \partial_{[\mu_1} \Lambda^0_{\mu_2 \ldots \mu_p]}, \tag{4}$$

where $\Lambda^0_{\mu_2\cdots\mu_p}$ is a $(p-1,0)$-form gauge parameter. The main feature of those gauge invariances is that they are not all independent, indeed they possess redundancy, or they are completely ineffective if we choose for gauge parameter:

$$\Lambda^0_{\mu_1\cdots\mu_{p-1}} = p\partial_{[\mu_1}\Lambda^1_{\mu_2\cdots\mu_{p-1}]}. \tag{5}$$

It is clear that there is a full tower of redundancy of this kind, until one reaches the last step in which the gauge parameter comes without indices (i.e. it is a 0-forms).

The quantization of systems with (redundant) gauge invariances is a well studied subject. All these gauge invariances are there to make manifest some symmetries like the Lorentz one, but usually the canonical quantization is a hamiltonian procedure that spoils manifest Lorentz covariance to retain manifest unitarity. Several formalisms have been constructed to deal with gauge systems without spoiling neither unitarity or covariance. Quite all of them need the introduction of additional variables of opposite statistics (instead of a reduction of the phase space) named ghosts, to account for the elimination of the gauge degrees of freedom from the physical sector. The first example of such a procedure is the Faddeev-Popov trick for SU(N) non abelian gauge theories [8]. In this path integral approach the integration over all the degrees of freedom yields an over counting because there are entire (infinite) subsets of configuration space that are physically equivalent. To cope with this problem one formally divides by the volume of those subsets. This amounts to a redefinition of the integration measure and the appearance of a functional jacobian (i.e. the Faddeev-Popov determinant). This determinant can be represented as an integration over fermionic variables, the ghosts. In the new action there is no gauge invariance anymore but it appears a new, odd, global, nilpotent symmetry between old fields and new ghosts, called BRST symmetry, named after the four physicists Becchi, Rouet, Stora and Tyutin. The BRST symmetry revealed by the Fadeev-Popov trick is actually a very general structure. Indeed, it can be proved that this symmetry can be realized from scratch for all gauge systems and that the ghosts are exactly what is needed to eliminate the gauge degrees of freedom from the physical sector. This enlarged space of variables could be achieved both in an Hamiltonian or Lagrangian setting. We will mainly review the Lagrangian setting for it is the best suited when redundancies are present, and it is quite powerful in application. The main difference here is that the symplectic structure of the Hamiltonian formalism is replaced by a odd-symplectic one provided by a new kind of brackets called “Antibrackets”. The variables needed for a complete BRST description will be separated in fields and antifields (indeed this is sometimes called the Field-Antifield formalism).

Since the first studies, thirty years ago, mainly due to physicists working on Supergravity, it has been realized that the redundancies in the gauge invariances need a general modification of the BRST structure and require the introduction of a tower of ghost variables (named ghosts for ghosts). Also in the BV formalism we will see that for each stage of reducibility a new set of fields and antifields is required.

The goal of this thesis will be the calculation of an expansion of the 1-loop effective action Γ of the theory that encodes some UV divergences at the quantum level. Usually the starting point is the representation of the 1-loop effective action in terms of the trace of the Heat kernel of the (kinetic) operator D:

$$\Gamma = \int \frac{dt}{t} Tr\left(e^{-tD}\right). \tag{6}$$

For small t, the heat trace has an expansion, whose coefficients are called Seeley-DeWitt coefficients. This is a standard tool to compute the effective action at 1 loop [26] [12] [18] [16] [11]. The coefficients provide all the information about divergences and counterterms needed in the renormalization of the theory.

There are other approaches that could be used to find those coefficients, like the worldline formalism [21] [3] [6] [4]. The starting point is the same eq. (6). The difference is that this approach represents the heat trace of some field theory with kinetic operator D, as a path integral of a particle mechanical model described by its world line coordinates and additional internal degrees of freedom. In particular in [5] it has been found the correct particle model that reproduces the gauge structure of the $(p, 0)$-form gauge theory. In that paper a calculation of the first few Seeley-DeWitt coefficient has been performed. Moreover, a non trivial coupling with the trace of the background connection Γ arose in the quantization of the particle model.

In this thesis we will see how the same Seeley-DeWitt coefficients are computed in the more standard Batalin-Vilkovisky approach following the Heat Kernel expansion, even in the presence of the further coupling with the trace part of the connection. We verify the correctness of the worldline approach reproducing exactly the same coefficients as in [5].

We conclude this introduction with a brief outline of the various chapters.

- In the First Chapter we will give some concepts in complex differential geometry and *Kähler* manifolds in particular. We will introduce the classical theory of $(p, 0)$-forms we want to study. We will derive the equations from an action principle and we will couple them to a curved background geometry.

- In the Second Chapter we will focus on the connections between constrained systems and gauge ones, reviewing the BRST construction and the Fields-Antifields formalism. We will introduce the concepts of ghosts for ghosts and the cohomological structure behind BRST invariance. At the end we will find the master equations of the BV formalism and introduce the problem of the gauge fixing procedure.

- In the Third Chapter we will construct step by step the BV formalism for the $(p, 0)$-form theory. We will start with the simpler case of real p-forms to give a first bite of the formalism, then we will generalize it to the full theory we want to deal with.

- In the Fourth Chapter we will review the connections between the 1-loop effective action and the heat kernel. We will state the general formulas for the first few Seeley-DeWitt coefficients in a form that is best suited for the comparison with other results in literature. Then we will perform explicitly the computation of some coefficients.

Sommario

In questa tesi abbiamo studiato la quantizzazione di una teoria di gauge di forme differenziali su spazi complessi dotati di una metrica di *Kähler* . La particolarità di queste teorie risiede nel fatto che esse presentano invarianze di gauge riducibili, in altre parole non indipendenti tra loro. L'invarianza sotto trasformazioni di gauge rappresenta uno dei pilastri della moderna comprensione del mondo fisico. La caratteristica principale di tali teorie è che non tutte le variabili sono effettivamente presenti nella dinamica e alcune risultano essere ausiliarie. Il motivo per cui si preferisce adottare questo punto di vista è spesso il fatto che tali teorie risultano essere manifestamente covarianti sotto importanti gruppi di simmetria come il gruppo di Lorentz. Non sempre i metodi usati per la quantizzazione di tali teorie preservano la manifesta covarianza sotto trasformazioni di Lorentz o mantengono la manifesta unitarietà nel passaggio alla teoria quantistica. La maggior parte dei metodi che preservano queste importanti proprietà richiede l'introduzione di campi non fisici detti ghosts e di una simmetria globale e fermionica che sostituisce l'iniziale invarianza locale di gauge, la simmetria BRST.

Nella presente tesi abbiamo scelto di utilizzare uno dei più moderni formalismi per il trattamento delle teorie di gauge: il formalismo Lagrangiano di Batalin-Vilkovisky. Questo metodo prevede l'introduzione di ghosts per ogni grado di riducibilità delle trasformazioni di gauge e di opportuni "antifields" associati a ogni campo precedentemente introdotto. Questo formalismo ci ha permesso di arrivare direttamente a una completa formulazione in termini di path integral della teoria quantistica delle $(p, 0)$-forme. In particolare esso permette di dedurre correttamente la struttura dei ghost della teoria e la simmetria BRST associata. Per ottenere questa struttura è richiesta necessariamente una procedura di gauge fixing per eliminare completamente l'invarianza sotto trasformazioni di gauge. Tale procedura prevede l'eliminazione degli antifields in favore dei campi originali e dei

ghosts.

Nell'ultima parte abbiamo presentato un'espansione dell'azione efficace (euclidea) che permette di tenere sotto controllo le divergenze (a 1 loop) della teoria. In particolare abbiamo calcolato i primi coefficienti di tale espansione (coefficienti di Seeley-DeWitt) tramite la tecnica dell'heat kernel. Questo calcolo ha tenuto conto dell'eventuale accoppiamento a una metrica di background cosi come di un possibile ulteriore accoppiamento alla traccia della connessione associata alla metrica di *Kähler* .

Chapter 1

$(p, 0)$-Forms: an introduction

1.1 Complex manifolds

Here we will introduce some general concepts about complex manifolds. We will stress only basic facts that differs from the real case. For a detailed description we refer to Nakahara [10].

We define a complex manifold M in the usual way:

Definition 1.1. A complex manifold M is defined by the following properties:

1. M is a topological space
2. there exist a family of pairs (U_i, ϕ_i) (i.e. an Atlas), where U_i are open sets that cover M and ϕ_i are homeomorphisms from M to an open subset of $\mathbb{C}^d$
3. for each U_i and U_j with non-empty intersection, the map $\psi_{ji} = \phi_j \circ (\phi_i)^{-1}$ from $\phi_i(U_i \cap U_j)$ to $\phi_j(U_i \cap U_j)$ is holomorphic (i.e. it satisfies the Cauchy-Riemann equations).

Some comments are in order: first, M is a differentiable manifold of real dimension $2d$. Second, there could be many atlases, if the union of two atlases is again an atlas, that is, it satisfies the properties listed above, then it is said to belong to the same complex structure, defined as the collection of equivalent classes of atlases.

1.1.1 Holomorphic/Antiholomorphic decomposition

Let z be the coordinates of a point $p \in (U,\phi)$, the tangent space T_pM is spanned by $2d$-vectors: [1]

$$\left\{ \frac{\partial}{\partial z^{\mu_i}}, \frac{\partial}{\partial \bar{z}^{\mu_i}} \right\} \quad i = 1, \dots, d. \tag{1.1}$$

We define the tensor J as:

$$J_p : T_pM \quad \to \quad T_pM \tag{1.2}$$

$$J_p \frac{\partial}{\partial z^\mu} = i \frac{\partial}{\partial z^\mu} \quad ; \quad J_p \frac{\partial}{\partial \bar{z}^\mu} = -i \frac{\partial}{\partial \bar{z}^\mu} \tag{1.3}$$

J_p has the following very important property:

$$J_p^2 = -\mathbb{1} \tag{1.4}$$

M is a complex manifold and the complex structure implies the global existence of this tensor. If we were on a real even-dimensional manifold, even the global existence of J would be in doubt. In general an even-dimensional manifold M' that has a global tensor like J is defined an almost complex manifold. What is missing for M' to become a complex manifold is an integrability condition on the J, which allows to introduce complex coordinates.

The property 1.4 allows one to decompose $T_pM^{\mathbb{C}}$ into two orthogonal subspaces, denoted T_pM^+ and T_pM^- respectively. The vectors belonging to the first one are called *holomorphic*, the ones belonging to the second one are called *antiholomorphic*. The decomposition takes the form

$$T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^- \tag{1.5}$$

$$\text{where: } T_pM^{\pm} = \left\{ Z \in T_pM^{\mathbb{C}} \,\middle|\, J_pZ = \pm iZ \right\} \tag{1.6}$$

Obviously the vectors (1.1) are basis respectively of T_pM^+ and T_pM^-

We denote the space of complexified vector fields $\chi(M)^{\mathbb{C}}$. The same decomposition can be achieved on it with the use of J.

[1] Where the symbol of the coordinate $\bar{z}$ is present we won't put a bar on the index. We will use barred indices only on general tensors or forms.

1.1.2 (r, s)-forms

Let M be a complex differentiable manifold. We define as usual the space of forms $\Omega^r(M)$ (we have not retained the dependency from the point because we denote by "form" an object in the space of sections of the antisymmetric cotangent bundle). If we take two r-forms α and β belonging to $\Omega^r(M)$, it is possible to define a complex r-form as $\gamma = \alpha + i\beta \in \Omega^r(M)^{\mathbb{C}}$.

Now, given the orthogonal decomposition 1.5 it is possible to define a (r, s)-form in the following way:

Definition 1.2. Take a complex manifold M of complex dimension d, a form $\alpha \in \Omega^q(M)^{\mathbb{C}}$, 2 integers $r, s \geq 0$ such that $r + s = q$, and vectors $V_i \in \chi^{\mathbb{C}}$ belonging to χ^+ or χ^-, where $i = 1 \ldots q$.
ω is called a (r, s)-form $\in \Omega^{r,s}(M)$ if $\omega(V_1 \ldots V_q) = 0$ unless there are r of the $V_i \in \chi^+$ and s of the $V_i \in \chi^-$

In the basis (1.1) a (r, s)-form ω is simply

$$\omega = \frac{1}{r!s!}\omega_{\mu_1\ldots\mu_r,\bar{\nu}_1\ldots\bar{\nu}_s}\, dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{\nu_s} \tag{1.7}$$

the real dimension of $\Omega^{r,s}(M)$ is given by

$$Dim_{\mathbb{R}}\, \Omega^{r,s}(M) = \binom{d}{r}\binom{d}{s} \quad 0 \leq r, s \leq d \tag{1.8}$$

1.1.3 Dolbeault differentials

The de Rham differential d has a simple action on (r, s)-forms in the basis (1.1), namely:

$$\begin{aligned} d\omega =& \frac{1}{r!s!}\left\{\frac{\partial}{\partial z^\lambda}\omega_{\mu_1\ldots\mu_r,\bar{\nu}_1\ldots\bar{\nu}_s}\, dz^\lambda + \frac{\partial}{\partial \bar{z}^\lambda}\omega_{\mu_1\ldots\mu_r,\bar{\nu}_1\ldots\bar{\nu}_s}\, d\bar{z}^\lambda\right\} \\ & \wedge dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{\nu_s} \end{aligned} \tag{1.9}$$

This action can be separated into two pieces, called the *Dolbeault operators*:

$$d = \partial + \bar{\partial}, \tag{1.10}$$

$$\text{where:} \quad \partial : \Omega^{r,s}(M) \rightarrow \Omega^{r+1,s}(M), \tag{1.11}$$

$$\bar{\partial} : \Omega^{r,s}(M) \rightarrow \Omega^{r,s+1}(M), \tag{1.12}$$

$$\partial\partial = \bar{\partial}\bar{\partial} = (\partial\bar{\partial} + \bar{\partial}\partial) = 0, \tag{1.13}$$

i.e. they anticommute to each other and they are separately differentials. This last property allows one to define the *Dolbeault complex* in the very same way one defines the de Rham complex. For example in the case of ∂:

$$\Omega^{0,s}(M) \rightarrow \Omega^{1,s}(M) \rightarrow \ldots \rightarrow \Omega^{d,s}(M) \rightarrow 0. \tag{1.14}$$

The cohomology group is defined as the set of equivalence classes of ∂-closed (r, s)-forms denoted by $Z^{r,s}_{\partial}(M)$ ($\partial\omega = 0$) modulo the exact ones (i.e $\{\omega \in Z^{r,s}_{\partial}(M)\, \exists \eta$ s.t. $\omega = \partial\eta)\}$ denoted by $B^{r,s}_{\partial}(M)$.

1.1.4 Hermitian metric and *Kähler* manifolds

Let M be a Riemannian manifold and g_{MN} be the components of a Riemannian metric on it. Note that in holomorphic coordinates (denoted with greek letters) there is a distinction between holomorphic and anti holomorphic indices, the symmetry between all the indices is complete also in those coordinates.

$$g_{\mu\nu} = g_{\nu\mu} \tag{1.15}$$

$$g_{\bar{\mu}\nu} = g_{\nu\bar{\mu}} \tag{1.16}$$

$$g_{\bar{\mu}\bar{\nu}} = g_{\bar{\nu}\bar{\mu}}. \tag{1.17}$$

Definition 1.3. Let J be a complex structure, then g is an Hermitian metric if satisfies

$$g(JX, JY) = g(X, Y) \quad \forall X, Y \in \chi(M). \tag{1.18}$$

There are very important features related with this structure:

- JX is orthogonal to X with respect to the metric g, i.e. $g(JX, X) = 0$. From this it follows that only $g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu} \neq 0$.
- Every complex manifold has an Hermitian metric.
- One can define an inner product on $TM^{\pm}$ by

$$p(X, Y) = g(X, \bar{Y}). \tag{1.19}$$

One can define on M, Hermitian manifold, a 2-form Ω, called the *Kähler* form as:

$$\Omega(X,Y) = g(JX,Y) \tag{1.20}$$

where $X, Y \in T_pM$ and J is an almost complex structure. In holomorphic coordinates this is a $(1,1)$-form so only the mixed components are different from zero.

It is possible to introduce also covariant derivatives ∇ and show that the only non vanishing components of the connection are $\Gamma^{\lambda}_{\mu\nu}$ and $\Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}}$. For example in complex coordinates the covariant derivatives acts on a tensor $T^{\bar{\lambda}}_{\mu\nu}$ as:

$$(\nabla_\eta T)^{\bar{\lambda}}_{\mu\nu} = \partial_\eta T^{\bar{\lambda}}_{\mu\nu} - \Gamma^{\xi}_{\eta\mu} T^{\bar{\lambda}}_{\xi\nu} - \Gamma^{\xi}_{\eta\nu} T^{\bar{\lambda}}_{\mu\xi} \tag{1.21}$$

$$(\nabla_{\bar{\eta}} T)^{\bar{\lambda}}_{\mu\nu} = \partial_\eta T^{\bar{\lambda}}_{\mu\nu} + \Gamma^{\bar{\lambda}}_{\bar{\eta}\bar{\xi}} T^{\bar{\xi}}_{\mu\nu} \tag{1.22}$$

In this thesis we are interested in a special class of complex manifolds called *Kähler* manifold.

Definition 1.4. A *Kähler* manifold is an Hermitian manifold M with metric g whose *kähler* form is d-closed. The metric g is called now a *Kähler* metric.

If one requires the metric compatibility (that is the metric is covariantly constant) then the connection coefficients are:

$$\Gamma^{\lambda}_{\mu\nu} = g^{\bar{\eta}\lambda}\partial_\mu g_{\nu\bar{\eta}}, \qquad \Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}} = g^{\eta\bar{\lambda}}\partial_{\bar{\mu}} g_{\bar{\nu}\eta}. \tag{1.23}$$

One can also prove directly that the complex structure J is covariantly constant with respect to this connection.

$$(\nabla_A J)^B_C = 0. \tag{1.24}$$

where A, B, C are either holomorphic or anti-holomorphic indices.

The *Kähler* nature of g has some consequences on the differential geometry of M. The connection Γ is torsionless, that is it is symmetric in the covariant indices

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$$
$$\Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}} = \Gamma^{\bar{\lambda}}_{\bar{\nu}\bar{\mu}}. \tag{1.25}$$

The Riemann tensor has only two kind of non vanishing components, namely:

$$R^{\rho}{}_{\sigma\bar{\mu}\nu} = \partial_{\bar{\mu}}\Gamma^{\rho}_{\sigma\nu}$$
$$R^{\bar{\rho}}{}_{\bar{\sigma}\mu\bar{\nu}} = \partial_{\mu}\Gamma^{\bar{\rho}}_{\bar{\sigma}\bar{\nu}}. \qquad (1.26)$$

Moreover the Riemann tensor has all the usual symmetries, but for a *kähler* manifold there is one extra symmetry as one can easily see from (1.25) (1.26).

$$R^{\rho}{}_{\sigma\bar{\mu}\nu} = R^{\rho}{}_{\nu\bar{\mu}\sigma}, \qquad (1.27)$$

along with all the other symmetries derived from this last one and known symmetries of the Riemann tensor.

1.1.5 Adjoint Dolbeault operators

If α is a (r,s)-form, then the Hodge-* map is defined as:

$$\bar{*} : \Omega^{r,s}(m) \rightarrow \Omega^{d-r,d-s}(M), \qquad (1.28)$$
$$\bar{*}\alpha =^{*} \bar{\alpha}. \qquad (1.29)$$

For example in holomorphic coordinates, * is:

$$*dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{\nu_s} \propto$$
$$\epsilon^{\mu_1,\ldots,\mu_r}{}_{\bar{\mu}_{r+1},\ldots,\bar{\mu}_d}\epsilon^{\nu_1,\ldots,\nu_s}{}_{\bar{\nu}_{s+1},\ldots,\bar{\nu}_d} d\bar{z}^{\mu_{r+1}} \wedge \ldots \wedge d\bar{z}^{\mu_d} \wedge d\bar{z}^{\nu_{s+1}} \wedge \ldots \wedge d\bar{z}^{\nu_d}. \qquad (1.30)$$

With the help of the Hodge-* operator we can construct an inner product between forms also in the case of complex manifolds:

$$(\alpha,\beta) = \int_M \alpha \wedge \bar{*}\beta \qquad (1.31)$$

and define adjoints with respect to this product:

$$(\alpha,\partial\beta) = (\partial^{\dagger}\alpha,\beta) \qquad (1.32)$$
$$(\alpha,\bar{\partial}\beta) = (\bar{\partial}^{\dagger}\alpha,\beta) \qquad (1.33)$$
$$\partial^{\dagger} : \Omega^{r,s}(M) \rightarrow \Omega^{r-1,s}(M) \qquad (1.34)$$
$$\bar{\partial}^{\dagger} : \Omega^{r,s}(M) \rightarrow \Omega^{r,s-1}(M) \qquad (1.35)$$

Since M is always even dimensional as a differentiable manifold, the adjoint of the de Rham operator can always be written as:

$$d^\dagger = -{}^*\bar{d}\,{}^*, \tag{1.36}$$

and the two adjoint Dolbeault operators have the following form

$$\partial^\dagger = -{}^*\bar{\partial}\,{}^* \tag{1.37}$$

$$\bar{\partial}^\dagger = -{}^*\partial\,{}^*. \tag{1.38}$$

1.2 Equations of motion for the $(p,0)$-form gauge theory

We are interested in this thesis in the quantum gauge theory of $(p,0)$-forms. The gauge invariance has a prominent role in the whole discussion and it implies that in the Lagrangian there is no room for quadratic (mass) terms. The starting point are the Maxwell-like equations written in the geometric form.

Let $F \in \Omega^{(p+1,0)}(M)$ on a complex manifold M of complex dimension d. The equations of motion we want to study are:

$$\partial F = 0 \tag{1.39}$$

$$\partial^\dagger F = 0 \tag{1.40}$$

$F = \partial A$ solves identically the Bianchi equation (1.39) and then the second equation (1.40) becomes:

$$\partial^\dagger \partial A = 0 \tag{1.41}$$

these equations have a simple and well known form when written in components, in flat space:

$$\partial_{[\mu} F_{\mu_1 \dots \mu_{p+1}]} = 0 \tag{1.42}$$

$$\bar{\partial}^{\mu_1} F_{\mu_1 \dots, \mu_{p+1}} = 0 \tag{1.43}$$

where the square brackets stand for weighted antisymmetrization.

The introduction of a gauge potential $(p,0)$-form yields

$$F_{\mu_1 \dots \mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]} = \partial_{\mu_1} A_{\mu_2 \dots \mu_{p+1}} \pm \text{cyclic perm}, \tag{1.44}$$

$$\bar{\partial}^{\mu} \partial_{\mu} A_{\mu_1 \dots \mu_p} + (-1)^p p \bar{\partial}^{\mu} \partial_{[\mu_1} A_{\mu_2 \dots \mu_p]\mu} = 0 \tag{1.45}$$

1.2.1 Redundant gauge invariances

This system has one special feature with respect to the usual 1-form system. As in the 1-form case, one has the possibility to add a (∂)-exact form to the gauge potential without affecting the equations of motion. Indeed if $A_{(p)} \to A_{(p)} + \partial\lambda_{(p-1)}$ then $F_{(p+1)} = \partial A_{(p)} + \partial\partial\lambda_{(p-1)} = \partial A_{(p)}$, since $(\partial)^2 = 0$, where we have indicated the form degree in brackets. Moreover it is clear that also $\lambda_{(p-1)}$ shares the same gauge invariance as $A_{(p)}$ and that there is a full tower of gauge invariances until one reaches $\delta\lambda_{(1)} = \partial\lambda_{(0)}$.

$$\begin{aligned} \delta A_{(p)} &= \partial A_{(p-1)} \\ &\vdots \\ \delta\lambda_{(1)} &= \partial\lambda_{(0)}. \end{aligned} \tag{1.46}$$

As we will see, this crucial difference has profound implications and yields many complications in the quantization procedure.

1.2.2 Classical Action

The construction of a classical action that reproduce the equations (1.40) as Euler Lagrange equation is analogous to the case of a real Riemannian manifold M, substituting the usual scalar product by (1.31).

$$S = \int_M F \wedge \bar{*} F, \tag{1.47}$$

where F is a complex $(p+1, 0)$-form.

This action can be written in components as:

$$S = \frac{1}{(p+1)!} \int_M d^d z d^d \bar{z}\, \bar{F}^{\mu_1 \ldots \mu_{p+1}} F_{\mu_1 \ldots \mu_{p+1}}, \tag{1.48}$$

1.3 Coupling with a background *Kähler* metric

1.3.1 Minimal coupling with the *Kähler* metric

The coupling with a background metric is performed substituting all the derivatives with covariant ones. This substitution in particular does not spoil the crucial property of the Dolbeault operators of being nilpotent. In this way we can easily recover the curved form of the equations of motion.

The adjoint Dolbeault operator become:

$$\partial^\dagger = -\frac{\delta}{\delta(dz^\mu)} g^{\mu\bar{\nu}} \partial_{\bar{\nu}}. \tag{1.49}$$

Indeed, the equations of motion (1.41) can be put in a more familiar form[2] extracting the covariant Laplacian:

$$(\partial^\dagger \partial + \partial \partial^\dagger) A_{(p)} - \partial \partial^\dagger A_{(p)} = 0, \tag{1.50}$$

where $A_{(p)} = A_{\mu_1 \dots \mu_p} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p}$.
When written in components this equation has the following form :

$$-\frac{1}{2}\nabla^2 A_{(p)} + \frac{p}{2}\mathbf{Ric} \cdot A_{(p)} - \partial \partial^\dagger A_{(p)} = 0, \tag{1.51}$$

where $\mathbf{Ric} \cdot A_{(p)} = R^{\lambda}_{\mu_1} A_{\lambda,\mu_2,\dots,\mu_p} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p}$. Note that the last term contains a divergence of A, i.e. it can be set to zero or ignored with an appropriate gauge fixing condition. Actually if we were not on a *Kähler* manifold it would have appeared also a term proportional to the Riemann tensor contracted with two indices of the form A (for $p > 0$), this term is:

$$p(p-1) R^{\sigma}{}_{\mu_1}{}^{\rho}{}_{\mu_2} A_{\sigma,\rho,\mu_3 \dots,\mu_p} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p}, \tag{1.52}$$

but this term is zero due to the fact that A is antisymmetric in the exchange of (ρ, σ), while the Riemann tensor in the *Kähler* case is symmetric in the first and third components (as well as in the second and fourth ones) as we saw in (1.27).

1.3.2 Coupling with the $U(1)$ part of the holonomy

In [5] emerged a possible coupling of a $(p, 0)$-form with the trace of the connection Γ (that is the U(1) part of the holonomy group of the *Kähler* manifold M). In that paper this possibility come from an ordering ambiguity in the quantization of the relativistic particle model used in the worldline representation of the effective

[2] this form will be very important when we will compute the heat kernel coefficients for a $(p, 0)$-form

action of our theory. In our case this coupling has to be put in by hand to perform a comparison with the results of that approach.

The minimal coupling with the trace of the connection amounts in a twisting of the Dolbeault operator ∂:

$$\partial_q = (\partial + q\Gamma), \tag{1.53}$$

where q measures the strength of the coupling and Γ is the 1-form $\Gamma = \Gamma^{\lambda}_{\mu\lambda} dz^{\mu}$.

This operator has the very important feature of being nilpotent, i.e. $(\partial_q)^2 = 0$. This property allows one to follow the same arguments that lead to the equation of motion (1.51) in the previous subsection, and to rephrase all the gauge invariances in terms of this new differential.

In particular one finds that the adjoint operator acts like a twisted divergence:

$$\partial_q^{\dagger} = -\frac{\delta}{\delta(dz^{\mu})} g^{\mu\bar{\nu}} (\partial_{\bar{\nu}} - q\Gamma_{\bar{\nu}}). \tag{1.54}$$

Moreover the equation of motion can be cast in the form $(\partial_q^{\dagger}\partial_q + \partial_q\partial_q^{\dagger})A_{(p)} - \partial_q\partial_q^{\dagger}A_{(p)} = 0$, this yields in components two new parts, the first one adds a coupling to the Ricci scalar R while the other one shifts the contribution to the coupling with the Ricci tensor:

$$\left(-\frac{1}{2}\nabla_q^2 + qR\right) A_{(p)} + \frac{p}{2}(1-4q)\mathbf{Ric}\cdot A_{(p)} - \partial\partial^{\dagger} A_{(p)} = 0, \tag{1.55}$$

where the covariant Laplacian now takes the form:

$$\nabla_q^2 = g^{\mu\bar{\nu}} \left\{ (\nabla_{\mu} + q\Gamma_{\mu})(\nabla_{\bar{\nu}} - q\Gamma_{\bar{\nu}}) + (\nabla_{\bar{\nu}} - q\Gamma_{\bar{\nu}})(\nabla_{\mu} + q\Gamma_{\mu}) \right\} \tag{1.56}$$

Note that there is no contribution from the Riemann tensor as in (1.52). For later purposes is better to rewrite the equation (1.55) in the following form:

$$D^{(p)} A_{(p)} - 2\partial\partial^{\dagger} A_{(p)} = 0. \tag{1.57}$$

The operator $D^{(p)}$ acts on the space of $(p,0)$-forms and is:

$$D^{(p)} = -(\nabla_q^2 + E) \quad \text{where} \tag{1.58}$$

$$E = -2qR\mathbb{1}_{\Omega^{(p,0)}(M)} - p(1-4q)\left\{ R^{\nu_1}_{[\mu_1} \delta^{\nu_2}_{\mu_2} \dots \delta^{\nu_p}_{\mu_p]} \right\}, \tag{1.59}$$

where the square brackets stand, as usual, for weighted antisymmetrization of the indices.

Chapter 2

Gauge systems and the BV-BRST formalism

In this thesis we want to deal with a system with gauge invariances. Those systems are tightly connected to constrained dynamical ones. Indeed under very general conditions one can state that quite every gauge system is a constrained one. Even classically it could be quite tricky to fully understand the general structure of those systems but it is certainly on the quantum side that the real difficulties arise. A gauge system has the particular feature of being under determined: not all variables are physical and in the dynamics some functions with arbitrary dependence on time appear. In the quantization procedure this could be a problem. One can reduce the (phase) space before quantization and try to find a representation of the Poisson algebra of the reduced phase space functions in terms of commutators or retain all the variables and reduce the physical Hilbert space after quantization. In each of these formalisms there are problems to deal with. As a general feature the gauge symmetry is there to preserve manifest symmetries like manifest Lorentz covariance, while retaining the gauge variables also after the quantization procedure can spoil manifest unitarity. Moreover the process of reduction and quantization not always commute and the systems described could be different (see [19] for a simple example).

There is a general scheme to reformulate a gauge theory, at classical and quantum level, in a way that encodes the whole gauge structure without manifestly showing it, and that, at the same time, does not spoil covariance nor unitarity.

The theory is rewritten in an enlarged space of variables. This space is no more a commuting space but it becomes a graded algebra, with Grassmann variables next to the usual ones. The original gauge symmetry is substituted by a rigid (global), odd (anticommuting), nilpotent symmetry called BRST symmetry between all the variables. Instead of reducing the phase space to a physical one, one is forced to enlarge it. The new variables are there to (in a certain sense) "kill" the gauge degrees of freedom. This formalism has a beautiful algebraic structure that we will try to briefly show in this chapter. Indeed the gauge invariant functions will be the functions belonging to the zeroth order cohomology group of the BRST symmetry operator. These ideas have been formulated both in a Hamiltonian and Lagrangian setting. After a general introduction to the topic we will focus mainly on the construction of the BRST symmetry in the Lagrangian case. This is called Field-Antifield or Batalin-Vilkovisky formalism. There are two main features of this formalism. First of all the lack of a Poisson structure that will be substituted by another bracket structure. Second the need for a final gauge fixing procedure to be sure that the BRST procedure cancels out all the remnant gauge invariances. This topic is way to wide too be completely covered in a thesis so we will sketch only the main ideas and concepts useful for a direct application to the theory of $(p, 0)$-forms and refer the reader to the very complete books and reviews in literature [17] [7] [13] [2].

2.1 Constrained dynamical systems

We postulate that our theory is defined by a Lagrangian action principle with Lagrangian function denoted by $\mathcal{L}$. The Euler-Lagrange equations of motion read:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) - \frac{\partial \mathcal{L}}{\partial q^i}. \tag{2.1}$$

They can be written as a second order partial differential equation as:

$$A_{ij}\ddot{q}^j + B_{ij}\dot{q}^j - C_i = 0. \tag{2.2}$$

where $A_{ij} = \frac{\delta^2 \mathcal{L}}{\delta \dot{q}^i \delta \dot{q}^j}, B_{ij} = \frac{\delta^2 \mathcal{L}}{\delta \dot{q}^i \delta q^j}, C_i = \frac{\delta \mathcal{L}}{\delta q^i}$.

A constrained system is characterized by the non invertibility of the A_{ij} matrix. This means that not all the accelerations can be determined by $(q^i, \dot{q}^i)$. In the dynamics arbitrary functions of time eventually appear, so the motion is underdetermined.

Now, we switch to the Hamiltonian formalism. The conjugate momenta are:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}. \tag{2.3}$$

The non invertibility of A_{ij} now means that one cannot explicit all the velocities in terms of momenta and coordinates. So, there is a certain number N of constraints:

$$\phi_m(q,p) = 0 \quad \text{with} m = 1 \dots N. \tag{2.4}$$

The last equations define a surface on the phase space Φ , called the constraint surface Σ. One can prove that, under certain conditions, a function, vanishing on Σ equals a linear combination of constraints. This fact is encoded in the symbol $\approx$, which means "weakly vanishing" or better "vanishing on the constraint surface."

One could expect that the constraints have to satisfy the consistency condition of being preserved in time at least on the constraint surface Σ. Using the canonical Poisson Brackets (PB), denoted by $\{,\}$ this consistency condition can be rephrased as:

$$\{\phi_m, H\} = \sum_k \alpha_k \phi_k \qquad \text{or} \qquad \{\phi_m, H\} \approx 0. \tag{2.5}$$

There are two main classifications of constraints. Primary/secondary constraints and first class/second class constraints. The first classification derive from an algorithmic procedure to extract from (2.4) and the consistency condition (2.5) all possible independent constraints. This is not so useful as a classification. It is far better to consider the notion of first class and second class functions. A phase function $f \in \mathcal{F}(\Phi)$ is called first class with respect to the system of constraints (ϕ_m) if, for all m:

$$\{f, \phi_m\} \approx 0. \tag{2.6}$$

In this thesis we will deal only with first class constraints, i.e. constraints that satisfy the conditions:

$$\{\phi_m, \phi_n\} = f_{mnl}\phi^l \tag{2.7}$$

$$\{\phi_m, H\} \approx 0. \tag{2.8}$$

Where f_{mnl} are called structure constants. The last equations mean that the constraints are compatible with each other (i.e. they form a closed Poisson algebra) and they are preserved in time.

One can impose the constraints to the action modifying the structure of the Lagrangian with the introduction of Lagrange multipliers λ_i:

$$S = \int dt \left\{ p_i \dot{q}^i - H(q,p) - \lambda_i \phi^i(q,p) \right\} . \tag{2.9}$$

Note that the equation of motion for λ enforces the constraints surface equations. The other equation of motion now display arbitrary time dependent functions $\lambda_m(t)$.

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + \lambda_m(t) \frac{\partial \phi^m}{\partial p_i} \tag{2.10}$$

$$\dot{p}^i = -\frac{\partial H}{\partial q_i} - \lambda_m(t) \frac{\partial \phi^m}{\partial q_i} \tag{2.11}$$

2.1.1 Gauge invariance

The Poisson structure is very useful to see how gauge invariances arise from the constraints algebra. Indeed a Poisson (symplectic) structure always allows to associate to every phase function f a vector field over the phase space such as:

$$X_f = \{f, \cdot\} \tag{2.12}$$

This vector field defines a one-parameter group of transformation on phase space, i.e. the flux or orbits of the vector field. In particular we can interpret the consistency condition also as a crucial property of the algebra of constraints: ϕ_m are the generators of local symmetries. Indeed (2.12) and (2.5) mean also that on Σ the dynamics (encoded by the Hamiltonian function) is left unchanged along the orbits of X_{ϕ_m}. In the very same way we define the infinitesimal transformation of a generic phase function f as:

$$\delta_\epsilon f = \{\epsilon(t)\phi, f\} \tag{2.13}$$

where ϵ is an infinitesimal parameter of the transformation. Note that this transformation is local (i.e. it depends on time t in a classical mechanics setting, it will depend on x^μ in a field theoretic context). These transformations are symmetries of the action S, i.e.:

$$\delta_\epsilon S = 0. \tag{2.14}$$

2.1.2 Gauge fixing and observables

It should be clear now that not all the degrees of freedom (phase space points) and not all the phase functions are physically acceptable in a constrained system. Now we want to characterize better these concepts. First of all we saw that the dynamics lies on Σ, a subset of Φ, defined by (2.4). This is not the only reduction. Indeed we saw that the phase space is foliated by the orbits of the vector fields X_{ϕ_m}. On each orbit the Hamiltonian is the same, i.e. each point of an orbit is physically equivalent. So Σ is a collection of equivalence classes of states. We can choose for each class a representative. This procedure is called gauge fixing and amounts for example to choose an appropriate phase function F that defines another phase space reduction via the equation $F(q,p) = 0$. The gauge fixing surface intersect Σ, selecting the class representative from the orbits of X_ϕ[1].

It can be proved that (if the gauge condition is properly chosen) this double reduction defines the true physical phase space with a symplectic structure inherited from the one of Φ. So the reduction from the whole set of variables to the physical ones is a two step process (or, as it has been said: "*the gauge strikes twice*"):

1. reduction to the constraints surface Σ,

2. gauge fixing, that is selection of one representative from each class of gauge orbits.

Let now move to the notion of observables. It is not so simple to characterize this notion completely, as it would need the comparison between a certain theory with some experimental apparatus, but what we can do now is to give a mathematical necessary condition. An observable is a phase function f that is also gauge invariant. Gauge invariant means, generally, constant along gauge orbits. Actually, the notion of a gauge invariant function involves a two step process like

[1] F has to satisfy a number of conditions to be an appropriate gauge fixing, the most natural is that it selects one and only one representative from each class. If this were not the case the gauge fixing procedure suffers from the so called Gribov ambiguities.

the one described before. To be more precise, we set f to be in the algebra of smooth phase functions, i.e. $\mathcal{C}^\infty(\Phi)$. We are dealing only with a subset Σ of phase spase, so we denote the smooth functions on Σ as $\mathcal{C}^\infty(\Sigma)$. Define now $\mathcal{N} \subset \mathcal{C}^\infty(\Phi)$ as the set of phase functions that vanish on Σ [2]. This set form a double ideal of the algebra $\mathcal{C}^\infty(\Phi)$, indeed each phase function multiplied (to the left or to the right) to a function vanishing on Σ belongs to $\mathcal{N}$. Now we can say that:

$$\mathcal{C}^\infty(\Sigma) = \mathcal{C}^\infty(\Phi)/\mathcal{N}. \tag{2.15}$$

What we said before is sufficient to prove that $\mathcal{C}^\infty(\Sigma)$ is itself an algebra. Equation (2.15) means simply that the reduction to Σ for phase functions is performed identifying the phase functions that are equal on Σ (whose difference on Σ is zero). As before a two step process is necessary to fully characterize a physical quantity:

1. reduce to Σ as in (2.15),
2. take only those functions $f \in \mathcal{C}^\infty(\Sigma)$ that are constant along gauge orbits, i.e.:

$$f \quad \text{is a gauge invariant function} \quad \Longleftrightarrow \quad \{\phi_m, f\} \approx 0, \quad \forall i. \tag{2.16}$$

2.1.3 Longitudinal derivatives

We have defined the vector fields X_{ϕ_m} associated to constraints in the previous section. The space of those fields is called the space of longitudinal vectorfields. Longitudinal here means that the fields point in the direction of gauge transformation, i.e. in the direction on which physics does not change. Along with this space one can construct in the usual manner the associated (tensor product) dual space, that is the space of longitudinal p-forms. Those forms acts on the tensor product of longitudinal fields. Obviously this construction allows also the definition of a longitudinal (de Rham) differential acting on this space of longitudinal forms. The main reason to introduce the formalism of differential forms on the

[2] Note that $\mathcal{N}$ is the space of functions proportional to a combination of constraints, so it is exactly the algebra generated by the constraints

space of longitudinal vector fields is that if we pick up a 0-form, i.e. a function $f \in \mathcal{C}^\infty(\Sigma)$ the formula:

$$df = 0. \tag{2.17}$$

denotes exactly the vanishing of the directional derivative of f in the direction of the gauge orbits, so the last equation rephrase the fact that f is a gauge invariant function. At this point we should observe the crucial fact that the gauge invariant functions arise as d-closed 0-forms, or in a more elegant way as the zeroth cohomological space of d: $H^0(d)$. Note that we are considering functions on the constraints surface Σ, so if we want to pursue a formalism that use this geometrical view, we have to characterize also in the very same way the reduction to Σ. This is precisely what BRST symmetry does, providing a beautiful and uniform formalism to represent the two step process described so far.

2.2 BRST formalism

The BRST formalism is based on the idea of substituting the original gauge invariance with a rigid odd symmetry s, defined on an extended phase space containing now anticommuting (i.e. odd) variables named ghosts. We will construct two different operator d, δ such that one will implement the restriction to Σ and the other will implement the cohomology described before. Those two operators are combined to form the so called BRST differential s in order to preserve its nilpotency ($s^2 = 0$). The zeroth cohomology group of s will give exactly the gauge invariant functions. The very fact that s defines a symmetry means that it is possible to construct from the beginning an action principle on the extended phase space, that includes fields and ghosts, such that the application of s leave the action invariant. Let us sketch briefly what is the procedure in the Hamiltonian case, in the Lagrangian case we will follow the same strategy but on different functional spaces.

2.2.1 The general construction

Let A be a graded algebra and s a differential on it. According to the grading of A, s could be expanded in a series of operators. Since s is a differential its degree is $dg(s) = 1$. If we start the series with a differential δ we will have:

$$s = \delta + d + \text{“}more\text{”}. \tag{2.18}$$

The higher order terms are relevant in the discussion only because they guarantee that s preserves the nilpotency property. Using this property one can see that:

$$0 = s^2 = \overbrace{\delta^2}^{dg=2} + \overbrace{\{\delta, d\}}^{dg=3} + \overbrace{d^2 + \{\delta, s_1\}}^{dg=4} + \text{“higher degree terms”}. \tag{2.19}$$

This implies that:

$$\delta^2 = 0, \tag{2.20}$$
$$\{d, \delta\} = 0, \tag{2.21}$$
$$d^2 = -[\delta, s_1]. \tag{2.22}$$

The first property does not add anything, δ is a differential. The last two properties say that d is a so called “differential modulo δ”. (2.21) shows that d is still a derivation on the homology of δ, $H_*(\delta)$, i.e. on $x \in A$ such that $\delta x = 0$. (2.22) shows that d is a differential (it is also nilpotent) if we restrict to $H_*(\delta)$. For the BRST construction it will be very useful to study the cohomology of d when we restrict the domain of d to the homology space of δ. This is denoted by $H^*(d|H_*(\delta))$. Generally the problem to solve is the opposite if we have δ and d with the previous properties, we want to construct (under a certain grading k of A) a differential s with the expansion (2.18) and such that

$$H^k(s) = H^k(d|_k(\delta)) \tag{2.23}$$

This is possible thanks to a property that is always true in the BRST case, i.e. δ provides a homological resolution.

Definition 2.1. Let A be a graded algebra, a homological resolution of A is a graded algebra $\bar{A}$ with differential δ of degree -1 such that

$$H_k(\delta) = 0 \quad k \neq 0,$$
$$H_0(\delta) = A \tag{2.24}$$

The grading of $\bar{A}$ is called resolution degree (rdg). Now we can state the central theorem of homological perturbation theory, which establishes that a differential s like the aforementioned one can actually be constructed from d and δ. Let δ provide a resolution of $H_0(\delta)$. Let d be a derivation modulo δ of resolution degree 0. If we denote the degree associated to d as dg_d with:

$$dg_d(\delta) = 0, \qquad dg_d(d) = 1. \tag{2.25}$$

We can define a total grading known as "total ghost number":

$$gh(x) = dg_d(x) - rdg(x). \tag{2.26}$$

Note that $gh(\delta) = gh(d) = 1$. Now we can state the very important

Theorem 2.2.1. *If $H_k(\delta) = 0 \, \forall k \neq 0$, there exist a differential s that combines d and δ with the properties:*

1. $s(AB) = As(B) - s(A)B$.
2. $s = \delta + d + s^1 + s^2 + \ldots,$
3. $rdg(s^k) = k,$
4. $gh(s) = 1,$
5. $s^2 = 0,$
6. $H^k(s) = H^k(d|H_k(\delta))$.

Obviously the most important property is the last one. Indeed it states that at the cohomology level only d and δ are relevant, and s^k are there only to preserve the nilpotency of s.

2.2.2 BRST differential in gauge theories

Let us briefly sketch what is the idea behind the BRST symmetry and how to construct an extended phase space on which s acts representing however all the feature of the gauge invariant theory we start from. δ and d implement the two part process of reduction of phase space (we saw that "the gauge strikes twice"). In particular the δ operator is called "Koszul-Tate" differential. It is constructed to provide a resolution of the algebra of functions $\mathcal{C}^\infty(\Sigma)$. The algebra that resolves it is:

$$\mathcal{C}^\infty(\Phi) \otimes Pol[P_\alpha], \tag{2.27}$$

that is the polynomials with coefficients in $\mathcal{C}^\infty(\Phi)$ and with generators P_α. The generators are there to "kill", on $\mathcal{C}^\infty(\Phi)$, those functions that are vanishing on Σ. This is the first enlargement of the space of variables needed in the BRST formalism.

The second step is the selection of the gauge invariant functions on Σ. We have already encountered a differential that implements that request. It is the longitudinal differential d. One extend d on the space of longitudinal forms with generators η_α with coefficients in the previously extended space of variables, so that one obtains at the end the total extended space:

$$\mathcal{C}^\infty(\Phi) \otimes Pol[P_\alpha] \otimes Pol[\eta_\beta]. \tag{2.28}$$

Actually one has to extend d such that it becomes a differential modulo δ and in order to guarantee the construction of the operator s through the Homological perturbation theory. We have already noted that the zeroth cohomological space of d (on the homology of δ that is $\mathcal{C}^\infty(\Sigma)$) is actually equal to $H^0(s)$ so we arrive at the very important result:

$$H^0(s) = \{\text{gauge invariant functions on}\,\Sigma\} \tag{2.29}$$

2.2.3 Canonical action for s

By now s is undetermined and so is the extended space of variables. The most striking fact about Hamiltonian BRST is that one can construct s such that the number of P_α and η_β is equal and that the new space of variables can be endowed with a canonical symplectic structure. Moreover s can be viewed as a symmetry of the system that has a canonical action:

$$sx = \{x, \Omega\epsilon\}\,, \tag{2.30}$$

where x belongs to the extended phase space, ϵ is an infinitesimal parameter and Ω is the BRST canonical generator. The nilpotency of s yields to the fundamental equation of BRST symmetry:

$$\{\Omega, \Omega\} = 0. \tag{2.31}$$

This equation allows one to construct recursively Ω.

2.3 Action formalism

2.3.1 Equation of motion and Noether identities

To reformulate the BRST construction in Lagrangian terms we have to state the gauge invariance problem directly on the action. We collect all the variables in the notation $y^i(t)$, and write the action as:

$$S(y^i(t)) = \int_{t_1}^{t_2} \mathcal{L}\, dt. \tag{2.32}$$

The equations of motion (e.o.m from now on) read:

$$\frac{\delta S}{\delta y^i(t)} = 0 \tag{2.33}$$

the δ derivation on the LHS is a functional derivative defined by:

$$\delta S = \int \delta y^i(t) \frac{\delta S}{\delta y^i(t)}\, dt, \tag{2.34}$$

note that the variation on the variables $y^i(t)$ vanishes at the boundaries, set for the variational principle (2.32). The gauge transformations are denoted briefly with:

$$\delta_\epsilon y^i(t) = R^i_{(0)\alpha}\epsilon^\alpha + R^i_{(1)\alpha}\dot{\epsilon}^\alpha + \dots \tag{2.35}$$

where ϵ are arbitrary gauge parameters. Under these variation the Lagrangian transforms as a total derivative, so they do not affect the e.o.m. It is more convenient to rewrite the last gauge invariance equations as:

$$\delta_\epsilon y^i(t) = R^i_\alpha \epsilon^\alpha \iff \delta_\epsilon y^i(t) = \int dt' R^i_\alpha(t, t')\epsilon^\alpha(t')$$
$$\text{where} \quad R^i_\alpha = R^i_{(0)\alpha}\delta(t - t') + R^i_{(1)\alpha}\delta'(t - t') + \dots. \tag{2.36}$$

In this notation the variation of the action is:

$$\delta_\epsilon S = \frac{\delta S}{\delta y^i}\delta_\epsilon y^i = \frac{\delta S}{\delta y^i} R^i_\alpha \epsilon^\alpha = 0. \tag{2.37}$$

This has to be true $\forall\epsilon^\alpha$, so we arrive at the celebrated *Noether identities*:

$$\frac{\delta S}{\delta y^i} R^i_\alpha = 0. \tag{2.38}$$

Those identities are very important, they show for example that not all e.o.m. are independent, due to gauge invariances.

We are working with a great number of gauge transformation, but not all of them are really relevant. for example we can safely factor out those that are proportional to the e.o.m. These are called trivial gauge transformations and they form an ideal of the gauge transformations algebra. So we pick up only the quotient space of gauge transformations modulo trivial ones. At the end we restrict only to a generating set of gauge transformations, i.e. the minimal set of gauge transformations that contains all the information about Noether identities.

The commutator of to elements of a generating set is also a gauge transformation so we can this means that:

$$R^j_\alpha \frac{\delta R^i_\beta}{\delta y^j} - R^j_\beta \frac{\delta R^i_\alpha}{\delta y^j} = C^\gamma_{\alpha\beta} R^i_\gamma + M^{ij}_{\alpha\beta} \frac{\delta S}{\delta y^j}, \tag{2.39}$$

where $M^{ij}_{\alpha\beta}$ are antisymmetric in i, j.

2.3.2 Reducible case

The theory we want to develop in this thesis has the fundamental property of being reducible. With that we mean that not all gauge transformations are independent. The gauge transformations are all independent if:

$$\delta_\mu F = \{F, \mu^\alpha \phi_\alpha\} \approx 0 \quad \forall F \quad \Rightarrow \quad \mu \approx 0. \tag{2.40}$$

In a reducible theory instead, we have that certain combination of constraints vanish:

$$Z^\alpha_A \phi_\alpha = 0 \tag{2.41}$$

For example if we take as gauge parameters:

$$\mu^\alpha = \mu^A Z^\alpha_A \tag{2.42}$$

that are not vanishing on-shell, then we obtain:

$$\delta_\mu F \approx 0. \tag{2.43}$$

2.3.3 Covariant phase space

The BRST construction in a Lagrangian setting (Batalin-Vilkoviski formalism), proceed along the same lines as in the Hamiltonian one. What is needed is a redefinition of the relevant functional spaces on which a (BV)-BRST differential will act.

Let (q_0, p_0) be the initial conditions at $t = t_0$ that determine completely every $(q(t), p(t))$ at later times through the Hamiltonian evolution. The phase space is then the space of all solutions to the e.o.m. If one switches to a Lagrangian setting and eliminates the p's the e.o.m. are now second order differential equations (2.2) with solution $q(t)$. So one can view the"covariant" phase space as the functional space (infinite dimensional!) of the solutions to the e.o.m. in Lagrangian form. One identifies each point in the phase space with the entire classical trajectory determined by it. one can construct an entire functional space, denoted by I that contains all possible histories of a theory.

$$I = \left\{\text{space of all histories}\, q^i(t)\right\} . \tag{2.44}$$

This includes the possibility to apply the same arguments to a configuration history of a field theory. A point ϕ^i in this functional space could not satisfy the e.o.m. so to recover the concept of "covariant phase space" one selects a submanifold of I, denoted by Σ. It is the space of the field configurations that solve the e.o.m., namely

$$\frac{\delta S}{\delta \phi}(\phi^i) = 0. \tag{2.45}$$

This submanifold is called the "Stationary surface". Functionals on the stationary surface are then smooth functionals on I modulo the ones that vanish on Σ, i.e.:

$$\mathcal{C}^\infty(\Sigma) = {}^{\mathcal{C}^\infty(I)}\!\big/_{\mathcal{N}} \tag{2.46}$$

In the presence of gauge invariances the Noether identities (2.38) are such that a solution to the e.o.m is mapped by a gauge transformation to a solution of the e.o.m., so there are well-defined orbits, denoted by G on Σ, due to the gauge invariances. The "gauge strikes twice argument" is the same here as in the Hamiltonian formalism. The gauge invariant functions are constant along the gauge orbits that lie on the stationary surface. The gauge invariant functions are then functions on the quotient space Σ/G, i.e. $f \in \mathcal{C}^\infty(\Sigma/G)$. this space is the new "covariant

	Hamiltonian	Lagrangian
phase space	Φ	I
reduced surface	Constraint Surface Σ	Stationary Surface Σ
gauge orbits (transf.)	$\delta_\epsilon F = \{F, \epsilon^\alpha \phi_\alpha\}$	$\delta_\epsilon F = \frac{\delta F}{\delta \phi^i} R^i_\alpha \epsilon^\alpha$
gauge invariant functions	$f \in {}^{\mathcal{C}^\infty(\Phi)}/_{\mathcal{N}(\Sigma)}$	${}^{\mathcal{C}^\infty(I)}/_{\mathcal{N}(\Sigma)}$

Figure 2.1: Hamiltonian setting vs Lagrangian setting

phase space" where only the true physical solutions to the e.o.m live. Fig. (2.1) summarize and translate the relevant functional spaces.

2.4 Batalin-Vilkovisky formalism

In this final section we will be able to explore the BRST formalisms (as sketched in sect. (2.2)) but in a Lagrangian setting. We will refer to the notation exposed in the last section and in the Lagrangian part of Fig.(2.1).

We want to construct explicitly a BRST differential s with the usual properties (2.29), (2.18). As in the previous case, δ is the *Koszul-Tate* differential that implements the restriction to Σ and d is a differential modulo δ, namely the longitudinal differential that extracts the gauge invariant functions defined on Σ.

2.4.1 The Koszul-Tate differential δ

The differential δ have to implement, in the covariant phase space I, the restriction to the functionals that satisfy the e.o.m. To do that, we first enlarge the space of variables including, for each field ϕ^i a correspondent "antifield" ϕ^*_i. Next we impose that all the fields ϕ^i are δ-closed and the antifields become what is needed to make the equations of motion δ-exact, namely:

$$\delta\phi^i = 0 \tag{2.47}$$

$$\delta\phi^*_i = -\frac{\delta S_0}{\delta \phi^i}. \tag{2.48}$$

We denote the degree induced by δ as the antighost number or $antigh(.)$ and with $\epsilon(s)$ the parity, such that a real or complex variable has $\epsilon = 0$ and a Grassmann variable has $\epsilon = 1$. Note that:

$$antigh(\phi^*_i) = 1, \tag{2.49}$$

$$antigh(\phi^i) = 0, \tag{2.50}$$

$$\epsilon(\phi^*_i) = 1, \tag{2.51}$$

$$\epsilon(\phi^i) = 0. \tag{2.52}$$

This procedure is quite effective. It is clear that there is no cohomology of δ of degree $k \neq 0$ because only the antighosts could provide such a cohomology, but they are all not even closed, by definition (if there are no gauge invariances). So all the cohomology is at zero antighost number. To see what is $H^0(\delta)$ note that from (2.47):

$$(\mathrm{Ker}\,\delta)_0 = \mathcal{C}^\infty(I) \tag{2.53}$$

and from (2.48):

$$(\mathrm{Im}\,\delta)_0 = \mathcal{N}. \tag{2.54}$$

To understand why the last equation is true remember that all the functionals vanishing on the stationary surface Σ could be written as a linear combination of the e.o.m. The extension of the space of variables and the need of a BRST differential even in the case without gauge invariances is a peculiar feature of the Lagrangian formalism that is not encountered in the Hamiltonian counterpart.

But we are dealing with gauge invariant theories and we saw that gauge invariances are encoded in the Noether identities (2.38). They say that the e.o.m are not all independent. This fact produces an interesting difference from the previous case. It is not true anymore that there is no cohomology at antighost number different from zero. Indeed there is one particular combination of the antifields that is δ-closed:

$$\delta(R^i_\alpha \phi^*_i) = -R^i_\alpha \frac{\delta S_0}{\delta \phi^i} = 0, \tag{2.55}$$

due to the Noether identities. This is not good because to fulfill the hypothesis of homological perturbation theory δ has to provide a resolution. In particular all the higher degree cohomology spaces should be trivial. The only way to bypass this obstacle is to further enlarge the space of variables to include objects that make those combinations δ-exact. Those new variables are such that:

$$\begin{aligned} &antigh(\phi^*_\alpha) = 2 \\ &\delta\phi^*_\alpha = R^i_\alpha \phi^*_i \end{aligned} \tag{2.56}$$

this produces the result of cancelling all the cohomology of degree 1 (the others are still zero),

$$H_1(\delta) = 0. \tag{2.57}$$

The power of the formalism introduced in the last section reside in the fact that the reduction to Σ, the Noether identities and the reducibility condition have the same form. Indeed if there are reducibility conditions like (2.41) we have also that:

$$Z^\alpha_A R^i_\alpha = C^{ij}_A \frac{\delta S_0}{\delta \phi^i} \quad \text{with} \quad C^{ij}_A = -C^{ji}_A. \tag{2.58}$$

Due to this last equation the combination:

$$-Z^\alpha_A \phi^*_\alpha - \frac{1}{2} C^{ij}_A \phi^*_i \phi^*_j, \tag{2.59}$$

is δ-closed without being δ-exact. This means that the cohomology at degree 2 is not zero. As before the only way to cope with this problem is to further enlarge the antifields sector introducing ϕ^*_A:

$$\begin{aligned} &antigh(\phi^*_A) = 3, \\ &\delta\phi^*_A = -Z^\alpha_A \phi^*_\alpha - \frac{1}{2} C^{ij}_A \phi^*_i \phi^*_j. \end{aligned} \tag{2.60}$$

this implies that:

$$H_2(\delta) = 0. \tag{2.61}$$

It should be clear that for further reducibility conditions the procedure is quite the same, and one has to introduce antifields of increasing antighost number to make the higher degree cohomologies trivial.

2.4.2 Longitudinal derivative

The differential d modulo δ needed for the construction of the BRST differential s is exactly the longitudinal exterior differential acting on longitudinal p-forms. The order of the forms is also the grading induced by d and it is called "*pure ghost* *number*". To account for the gauge invariances in the irreducible case one as to fix the gradings of the longitudinal 1-forms denoted now by C^α:

$$puregh(C^\alpha) = 1, \qquad \epsilon(C^\alpha) = 1 \tag{2.62}$$

These forms are called "Ghosts" and allows us to eliminates all the functionals $F \in \mathcal{C}^\infty(\Sigma)$ that are not gauge invariant from the cohomology of d simply by:

$$dF = \frac{\delta F}{\delta \phi^i} R^i_\alpha C^\alpha. \tag{2.63}$$

Indeed F is exact (and so it is removed from cohomology) unless F is gauge invariant, i.e.:

$$\frac{\delta F}{\delta \phi^i} R^i_\alpha = 0 \quad \Leftrightarrow \quad \delta_\epsilon F = \frac{\delta F}{\delta \phi^i} R^i_\alpha \epsilon = 0 \quad \forall \epsilon. \tag{2.64}$$

d has to be a differential so we define the action of d on 1-forms as:

$$dC^\alpha = \frac{1}{2} C^\alpha_{\beta\gamma} C^\beta C^\gamma, \tag{2.65}$$

Where $C^\alpha_{\beta\gamma}$ has been introduced in (2.39). The reducible case needs the introduction of a "model" D for the differential d with the same cohomology. In this case D acts in the same way d does with respect to s. To deal with the redundancy in the gauge transformations we introduce other forms of degree 2, C^A called "Ghosts for ghosts":

$$puregh(C^A) = 2, \qquad \epsilon(C^A) = 0. \tag{2.66}$$

such that:

$$DF = \frac{\delta F}{\delta \phi^i} R^i_\alpha C^\alpha \tag{2.67}$$

$$dC^\alpha = \frac{1}{2} C^\alpha_{\beta\gamma} C^\beta C^\gamma + Z^\alpha_A C^A \tag{2.68}$$

2.4.3 Antibrackets

The two operators d and δ are what we need for the main theorem of homological perturbation theory. we define a total degree called "ghost number":

$$gh = puregh - antigh. \tag{2.69}$$

Reducibility	Gauge	Σ reduction	fields	Gauge	Reducibility
ϕ^*_A	$\phi^(*)_\alpha$	ϕ^*_i	ϕ^i	C^α	C^A
(-3)	(-2)	(-1)	(0)	(1)	(2)

Figure 2.2: Extended space of variables. The number in the third line is the ghost number.

and the cohomology of ghost number zero of s is equal to the cohomology of the same degree of d, and correspond to the gauge invariant functionals on Σ. This formalism has by now an important difference with respect to the Hamiltonian one. The BRST differential does not act canonically on the extended space of variables, because the Poisson brackets simply does not exist. This can be seen by inspecting the grading property of the full set of variables as in Fig.(2.2). However there is a symmetry between fields and antifields. This led to the introduction of a different kind of brackets called "Antibrackets". Those brackets are an odd derivation and carry a ghost number.

Definition 2.2. if A, B are functions of the fields and antifields defined above, the Antibracket is defined by:

$$\begin{aligned}(A,B) = & \frac{\delta^R A \delta^L B}{\delta\phi^i\delta\phi^*_i} - \frac{\delta^R A \delta^L B}{\delta\phi^*_i\delta\phi^i} \\ & + \frac{\delta^R A \delta^L B}{\delta C^\alpha\delta\phi^*_\alpha} - \frac{\delta^R A \delta^L B}{\delta\phi^*_\alpha\delta C^\alpha} \\ & + \frac{\delta^R A \delta^L B}{\delta C^A\delta\phi^*_A} - \frac{\delta^R A \delta^L B}{\delta\phi^*_A\delta C^A}\end{aligned} \tag{2.70}$$

Since we are dealing also with fermionic (Grassmann) variables, R, L denotes right and left derivation. The Antibracket has the following properties:

1. it is odd, i.e. $\epsilon A, B = 1 + \epsilon(A) + \epsilon(B)$,

2. $gh(A, B) = 1 + gh(A) + gh(B)$,

3. it satisfy the usual Jacobi identity, i.e. $(A,(B,C))$ + cyclic permutations of $A, B, C = 0$,

4. $(A, B) = -(-)^{(\epsilon_A+1)(\epsilon_B+1)}(B, A)$.

With this bracket structure we declare that:

$$(\phi^i, \phi^*_j) = \delta^i_j, \quad (C^\alpha, \phi^*_\beta) = \delta^\alpha_\beta, \quad (C^A, \phi^*_B) = \delta^A_B. \tag{2.71}$$

So we can treat fields, ghosts and ghosts for ghosts as "coordinates" and the antifields as "momenta".

2.4.4 The master equation

As in the Hamiltonian case we can define a "canonical" action for the s differential, if A is a dynamical variable:

$$sA = (A, S). \tag{2.72}$$

S is the generator of the differential s and has the following properties:

$$\epsilon(S) = 0, \tag{2.73}$$

$$gh(S) = 0, \tag{2.74}$$

$$(S, S) = 0. \tag{2.75}$$

The very last property is due to the nilpotency of s and it is the fundamental equation of the Batalin-Vilkoviski formalism, it is called the Master Equation. This formula is very important because one can easily see that S is nothing but the extension to the whole set of variables of the classical action. Indeed if we separate the contributions to S from different antighost numbers (i.e. how many antifields are present) we obtain, as solution to the master equation:

$$S^0 = S_0, \tag{2.76}$$

$$S^1 = \phi^*_i R^i_\alpha C^\alpha + \dots \tag{2.77}$$

$$S^2 = \phi^*_\alpha Z^\alpha_A C^A + \dots \tag{2.78}$$

as one can easily prove using the definition of the antibracket and the formulas defining the action of d and δ on all the variables, as presented in the last sections. If there are other reducibility conditions we would have other terms like the last one. S_0 is exactly the gauge invariant action we started from. Moreover those pieces are all truncated to the first order, the following orders would contain the gauge structure functions [3].

As we will see a gauge fixing procedure is needed. To implement some very important derivative gauge conditions it is necessary to further enlarge the space of variables. This is actually possible if we add cohomologically trivial pairs, i.e. pairs that are excluded in the cohomology of s (otherwise they would alter the main result (2.29)). Those "canonical" pairs are (C, C^*), (b, b^*) such that:

$$\left\{ \begin{array}{c} sC = b \\ sb = 0 \end{array} \right\} \qquad \left\{ \begin{array}{c} sb^* = C^* \\ sb^* = 0 \end{array} \right\} \tag{2.79}$$

Note that any of those fields are present in cohomology because they are either exact or not even closed. Lets state what are the ghost degrees of those fields.

$$gh(C) = gh(b) - 1, \tag{2.80}$$

$$gh(C^*) = -gh(C) - 1, \tag{2.81}$$

$$gh(b^*) = -gh(b) - 1. \tag{2.82}$$

The solution to the master equation (2.75) is now extended to include the fields in the non minimal sector:

$$S = S_{\text{minimal}} + \sum C^* b. \tag{2.83}$$

where the sum runs over all non minimal sectors one has the need to add.

2.4.5 Residual gauge invariance and the gauge fixing fermion

Differently to what happens in the Hamiltonian formalism, the master equation gives a solution that is BRST invariant but that does not eliminate all the gauge invariances. In particular if we collect all the $2N$ variables (fields, ghosts, ghosts for ghosts, and antifields) in the notation:

$$z^\alpha = \left\{ \phi^A, \phi^*_A \right\} \quad \text{with} A = 1, \ldots, N, \tag{2.84}$$

one can prove that the matrix:

$$B^a_b = \sigma^{ac} \frac{\delta^L \delta^R S}{\delta z^c \delta z^b}, \tag{2.85}$$

[3]The theory we are dealing with in this thesis is abelian and free, so there is no need to include such terms.

with

$$\sigma^{ab} = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^A_B & 0 \end{pmatrix} \tag{2.86}$$

define a gauge transformation:

$$\delta_\epsilon z^a = B^a_b \epsilon^b \tag{2.87}$$

or in a form that resemble the Noether identities:

$$\frac{\delta^R S}{\delta z^a} B^a_b = 0. \tag{2.88}$$

The Lagrangian formalism for the BRST invariance has been introduced to work directly with a path integral quantization, without dealing with Hamiltonians. The main concern in using path integrals for gauge systems is that one has to integrate over entire histories that are physically equivalent, producing an (infinite!) over counting. The Hamiltonian BRST formalism kills all gauge invariance and substitute them with a rigid symmetry providing the right setting to deal with path integrals. Now the gauge invariances are still present, but this is only a feature that is introduced by the antifields. What we have to do is to correctly implement the functional measure of the path integral and choose an appropriate gauge fixing to eliminate the antifields from the action. To do this, the simple choice of setting all the antifields to zero is useless because we would return to the original gauge invariant action S_0. A more appropriate choice is to set:

$$\phi^*_a = \frac{\delta\psi}{\delta\phi^a} \tag{2.89}$$

where we have introduced the functional ψ, that depends only on fields and ghosts. To match the statistics and the ghost numbers we have to impose:

$$gh(\psi) = -1, \quad \epsilon(\psi) = 1. \tag{2.90}$$

For this reason ψ is called the "gauge fixing fermion" and is appropriately chosen to fix all the gauge invariance such that in the final action the propagators exist. At the path integral (quantum) level we can impose this gauge fixing as:

$$\int \mathcal{D}\phi\mathcal{D}\phi^*\, \delta\left(\phi^*_a - \frac{\delta\psi}{\delta\phi^a}\right) \exp\left(\frac{i}{\hbar} W[\phi, \phi^*]\right). \tag{2.91}$$

Here δ stands for the Dirac delta functional. What we have to require is that the path integral does not depend on the gauge fixing fermion choice. If we perform an infinitesimal change in ψ, namely $\delta\psi$, the difference between the previous and the transformed path integral is proportional, at the lowest order in $\delta\psi$ to:

$$\Delta \exp\left(\frac{i}{\hbar}W\right) \tag{2.92}$$

where the Δ operator is a kind of odd Laplacian:

$$\Delta \equiv (-)^{\epsilon_A+1}\frac{\delta^R}{\delta\phi^A}\frac{\delta^R}{\delta\phi^*_A}, \tag{2.93}$$

$$\epsilon(\Delta) = 1, \tag{2.94}$$

$$\Delta^2 = 0. \tag{2.95}$$

The requirement of path integral invariance under variations of ψ amounts to the vanishing of (2.92). This is equivalent to the so called “Quantum master equation”:

$$i\hbar\Delta W - \frac{1}{2}(W,W) = 0. \tag{2.96}$$

Note that this equation reduces to the classical master equation (2.75) in the limit $\hbar \to 0$.

As we will see, the theory of p-forms provides a general example of a redundant gauge theory, and we will be able to show in that context an example of a gauge fixing procedure.

Chapter 3

p-forms in the Batalin-Vilkoviski approach

The BV formalism, reviewed in the last chapter is certainly the best suited approach for the search of a BRST gauge fixed Lagrangian of p-form theory. Indeed this theory is free (non interacting), abelian and redundant in the gauge invariances. The very last feature is the most complicated one to deal with in the hamiltonian BRST formalism, but it is quite simple to treat in the Lagrangian setting. In what follows we considered only the euclidean version of the theory.

It is convenient to illustrate the BV formalism with the simplest non trivial example of a real 2-form on a real manifold. This example encodes all the most important properties of the BV formalisms and we will be able to generalize it to a p-form theory simply repeating the same arguments for each reducibility stage. At the end of the chapter we will arrive at the full theory on complex manifolds. Before starting it is important to note that the redundant gauge transformations (1.46) have the same form at each stage. In the notation used for example in [17] at the s-th stage the gauge generator is the differential $\partial_{(p-1-s)}$ acting on the space of $(p-1-s)$-forms:

$$
\begin{aligned}
R_0 &= \partial_{(p-1)}, \\
Z_1 &= \partial_{(p-2)}, \\
&\vdots \\
Z_{p-1} &= \partial_{(0)}.
\end{aligned} \tag{3.1}
$$

The procedure to recover the whole ghost structure and the gauge fixed action is for this reason quite recursive.

3.1 Real p-forms

3.1.1 The 2-form case

This theory is defined in n (real) dimensions. The fundamental variable is the 2-form $A_{\mu\nu}$. We introduce the totally antisymmetric field strength:

$$F_{\mu\nu\lambda} = \partial_\mu A_{\nu\lambda} + \partial_\lambda A_{\mu\nu} + \partial_\nu A_{\lambda\mu} \tag{3.2}$$

The theory we want to deal with is defined by the Lagrangian:

$$\mathcal{L} = \frac{1}{12} F^{\mu\nu\lambda} F_{\mu\nu\lambda}, \tag{3.3}$$

The Lagrangian is invariant under the following gauge transformations:

$$\delta A_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \tag{3.4}$$

where Λ is a 1-form gauge parameter. Those gauge transformations are not all independent, since they vanish if we choose a function Γ such that:

$$\Lambda_\mu = \partial_\mu \Gamma \tag{3.5}$$

In order to construct a BV-BRST invariant action we have to enlarge the configuration space of our theory to include a set of fields: a vector C_μ, and a scalar C_1, and antifields: an antisymmetric tensor $A^{*\,\mu\nu}$, a vector $C^{*\,\mu}$, and a scalar C^*. Those fields and antifields define our BRST complex on which the

antifields			fields			
C_1^*	$C^{*\,\mu}$	$A^{*\,\mu\nu}$	$A_{\mu\nu}$	C_μ	C_1	
-3	-2	-1	0	1	2	gh
1	0	1	0	1	0	ϵ

Figure 3.1: Fields-antifields structure.

BRST variation s acts. They are arranged as in Fig. 3.1 with respect to their gradings, defined on the BRST complex, namely the ghost number and parity.

The Master equation (2.75) has a simple solution, since the theory is free and the constraints form an abelian algebra. This solution is the BV-BRST action:

$$S = S^{(0)} + S^{(1)} + S^{(2)}. \tag{3.6}$$

The correspondent Lagrangian terms are

$$\mathcal{L}^{(0)} = \mathcal{L}_0(A_{\mu\nu}), \tag{3.7}$$

$$\mathcal{L}^{(1)} = A^{*\,\mu\nu} R_0(C)_{\mu\nu}, \tag{3.8}$$

$$\mathcal{L}^{(2)} = C^{*\,\mu}(Z_1)_\mu C_1. \tag{3.9}$$

where R_0, Z_1 are the gauge generators (3.1) and $\mathcal{L}_0(A_{\mu\nu})$ is the classical action. The explicit form of these pieces read:

$$\mathcal{L}^{(0)} = -\frac{1}{12} F^{\mu\nu\lambda} F_{\mu\nu\lambda}, \tag{3.10}$$

$$\mathcal{L}^{(1)} = A^{*\,\mu\nu}(\partial_\mu C_\nu - \partial_\nu C_\mu), \tag{3.11}$$

$$\mathcal{L}^{(2)} = C^{*\,\mu}(\partial_\mu C_1). \tag{3.12}$$

This action has some gauge invariances that have to be fixed through an appropriate gauge fixing procedure. We want to implement a Feynman-Lorentz (FL) gauge condition to eliminate from the action all the divergences of the fields. This condition is explicitly:

$$\partial^\mu A_{\mu\nu} = 0. \tag{3.13}$$

To do this we proceed step by step. The FL condition is of derivative type so in order to correctly implement it we enlarge the configuration space to include other fields that disappear in the cohomology of the BRST differential. In this way we can construct a non minimal solution to the master equation that is BRST invariant but does not affect the gauge invariant degrees of freedom Explicitly this non minimal sector is $(\bar{C}_\mu, \bar{C}^*_\mu, b_\mu, b^*_\mu)$[1]. Starred and non-starred variable with the same name are conjugate to each other with respect to the antibrackets (2.70). The BRST structure of the non-minimal sector is, following (2.79):

$$\begin{aligned} sb_\mu &= 0, \\ s\bar{C}_\mu &= b_\mu, \\ sb^{*\,\mu} &= \bar{C}^{*\,\mu}. \end{aligned} \tag{3.14}$$

We saw in the last chapter that the corresponding solution to the master equation is (2.83). In the case at hand we have:

$$S_{NM} = \int d^n x\, \bar{C}^{*\,\mu} b_\mu. \tag{3.15}$$

and it has to be of zero ghost number and parity. This solution is obviously BRST invariant due to (3.14) and the nilpotency of s. We introduce the gauge fixing fermion ψ, with the properties: $\mathrm{gh}(\psi) = -1$ and $\epsilon = 1$ in order to implement the FL gauge fixing condition. An appropriate choice seems to be:

$$\psi = \int d^n x\, \bar{C}_\mu (\partial_\nu A^{\nu\mu}) \tag{3.16}$$

but it turns out to be incomplete as we will see. This gauge fixing fermion has the structure encountered in Chapter 2 (see [13] for a more general treatment). It is used to eliminate the antifields and to fix the remaining gauge invariances of the action, in order to safely work with a path integral approach. Such a gauge fixing fermion is called "properly chosen". In fact the elimination of the antifields is performed by the conditions

$$A^*_{\mu\nu} = \frac{\delta\psi}{\delta A^{\mu\nu}} = \partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu, \tag{3.17}$$

$$\bar{C}^{*\,\mu} = \frac{\delta\psi}{\delta \bar{C}_\mu} = \partial_\nu A^{\nu\mu}. \tag{3.18}$$

$$C^{*\,\mu} = 0. \tag{3.19}$$

[1] $\bar{C}_\mu$ is usually "called antighost", because in the 1-form theory it is the same as the Fadeev-Popov antighost, while b_μ is usually called the auxiliary field

This leads to the action:

$$S_\psi = \int_M d^n x \left(-\frac{1}{12} F^{\mu\nu\lambda} F_{\mu\nu\lambda} - \partial_\nu \bar{C}_\mu (\partial^\mu C^\nu - \partial^\nu C^\mu) + (\partial_\mu A^{\mu\nu}) b_\nu \right). \qquad (3.20)$$

The ghost numbers of the non minimal sector fields have to be consistent with the ghost numbers of the other fields and the previous equations. In particular one finds that:

$$\begin{aligned} gh\,\bar{C}^\mu &= -1, \\ gh\,b^\mu &= 0, \\ gh\,\bar{C}^{*\mu} &= 0, \\ gh\,b^{*\mu} &= -1. \end{aligned} \qquad (3.21)$$

The FL gauge condition on the 2-form is implemented in the path integral:

$$Z = \int DA_{\mu\nu} D\bar{C}_\mu DC_\mu Db_\mu DC_1\, e^{-iS_\psi}, \qquad (3.22)$$

by the integration over b_μ. This yields the so called "delta function gauge fixing" $\delta(\partial_\mu A^{\mu\nu})$. But those gauge conditions are not all independent. This is the first way to see that this gauge fixing procedure is incomplete and produces singular terms. Indeed the degeneracy in the FL conditions leads to a $\delta(0)$ in the path integral. Morover C_1 is not present in the action. The degeneracy shows up also through the appearance in the action (3.20) of further gauge invariances in the ghost C_μ and the antighost $\bar{C}_\mu$, namely

$$\delta C_\mu = \partial_\mu \lambda, \qquad (3.23a)$$

$$\delta \bar{C}_\mu = \partial_\mu \lambda'.. \qquad (3.23b)$$

Those arguments show that the gauge fixing fermion ψ is not properly chosen and we have to amend it. Fortunately it is not a difficult task. The last gauge invariances are of the same type as the original one. So we can use the same strategy to fix the gauges with the two LF conditions:

$$\partial^\mu C_\mu = 0, \qquad (3.24a)$$

$$\partial^\mu \bar{C}_\mu = 0, \qquad (3.24b)$$

by means of a further enlarged non minimal sector. This actually amounts to introducing two new sectors, one for (3.23a) and the other for (3.23b).

$$\begin{pmatrix} \bar{C}_1 & \bar{C}_1^* \\ b_1 & b_1^* \end{pmatrix} \quad \text{for } C_\mu, \tag{3.25}$$

$$\begin{pmatrix} \bar{C}_2 & \bar{C}_2^* \\ b_2 & b_2^* \end{pmatrix} \quad \text{for } \bar{C}_\mu, \tag{3.26}$$

with BRST variation:

$$s\bar{C}_1 = b_1, \tag{3.27}$$

$$sb_1^* = \bar{C}_1^*, \tag{3.28}$$

and

$$s\bar{C}_2 = b_2, \tag{3.29}$$

$$sb_2^* = \bar{C}_2^*. \tag{3.30}$$

The total non minimal action is then:

$$S_{NM} = \int d^n x \left(\bar{C}^{*\,\mu} b_\mu + \bar{C}_1^* b_1 + \bar{C}_2^* b_2 \right). \tag{3.31}$$

In the same manner as we did before, we can add terms in the gauge fixing fermion ψ to implement a δ function gauge fixing for (3.24a) and (3.24b).

$$\psi = \int d^n x \left(\bar{C}_\mu (\partial_\nu A^{\nu\mu}) + \bar{C}_1 (\partial^\mu C_\mu) + \bar{C}_2 (\partial^\mu \bar{C}_\mu) \right). \tag{3.32}$$

The ψ allows us to eliminate the antifields from the action:

$$A^{*\,\mu\nu} = \partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu, \tag{3.33}$$

$$\bar{C}^{*\,\mu} = \partial_\nu A^{\nu\mu} - \partial^\mu \bar{C}_2, \tag{3.34}$$

$$\bar{C}_1^* = \partial^\mu C_\mu, \tag{3.35}$$

$$C^{*\,\mu} = -\partial^\mu \bar{C}_1, \tag{3.36}$$

$$\bar{C}_2^* = \partial^\mu \bar{C}_\mu. \tag{3.37}$$

To complete this analysis let us state what are the ghost numbers and parities of the non minimal variables and what is the form that the gauge fixed action assumes at the end:

	$\bar{C}_\mu$	b_μ	$\bar{C}^{*\mu}$	$b^{*\mu}$	b_1	$\bar{C}_1$	b_1^*	$\bar{C}_1^*$	b_2	$\bar{C}_2$	b_2^*	$\bar{C}_2^*$
gh	-1	0	0	1	-1	-2	0	1	1	0	-2	-1
ϵ	1	0	0	1	1	0	0	1	1	0	0	1

$$S = \int d^n x \Big\{ -\frac{1}{12} F^{\mu\nu\lambda} F_{\mu\nu\lambda} - \partial_{[\nu}\bar{C}_{\mu]}(\partial^\mu C^\nu - \partial^\nu C^\mu) - \partial^\mu \bar{C}_1 \partial_\mu C_1$$
$$+ (\partial_\mu A^{\mu\nu} - \partial^\nu C_2) b_\nu + (\partial^\mu C_\mu) b_1 + (\partial^\mu \bar{C}_\mu) b_2 \Big\} \tag{3.38}$$

where we have suppressed the bar on C_2 because it is a simple real scalar field. A few comments are in order. There are no gauge invariances in this action, so the gauge fixing procedure has been properly chosen. We have introduced the minimal and non minimal sectors and solved the master equation in order to have a BRST symmetric action. Indeed this symmetry is present as one can easily verify. To show that, it is important to remember that along with the transformation rules (3.30) ,(3.28) and (3.14) there are the transformation rules for the minimal sector:

$$sA_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu, \tag{3.39}$$
$$sC_\mu = \partial_\mu C_1. \tag{3.40}$$

Moreover, s is a graded differential of parity $\epsilon_s = 1$, i.e. if A has parity ϵ_A then whatever the parity of B, $s(AB) = A(sB)(-)^{\epsilon_B}(sA)B$.

The fact that ψ does not depend on the antifields allows us to always recover the action in a simple way and to see the BRST invariance immediately:

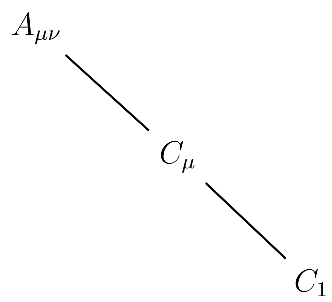
$$S = S_0 + s\psi. \tag{3.41}$$

where S_0 is the usual classical action. Indeed, the BRST invariance of the gauge fixed action S follows directly from the nilpotency of s and the fact that S_0 is BRST closed.

3.1.2 The Ghost-Antighost Tree Diagram

In order to generalize this process to an arbitrary p-form, we will represent the steps that led us to the gauge fixed action (3.38) with a diagram that encodes all the properties that are relevant in the situation when ghosts of ghosts are present.

This diagram can summarize the important information of the gauge fixing process. This diagram will contain only the fields and the antighosts. There is a correspondent diagram for the auxiliary fields. Let us start with the 2-form. We picture the fields and the ghosts in this way:



this part of the diagram corresponds to the minimal solution of the master equation, so to each of the fields displayed corresponds a term in the action as in (3.12). In the diagram, the form degree decreases at each step until it reaches zero. The statistics (grassmann parity) instead has an alternating pattern.

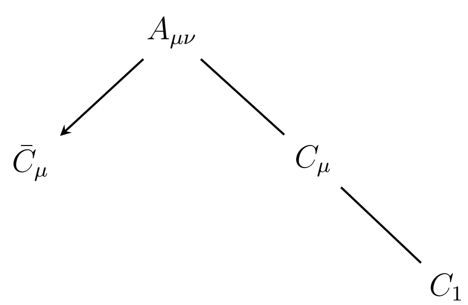
In order to fix the gauge invariance of $A_{\mu\nu}$ we introduced the first non minimal sector (3.14). We added to the Lagrangian a term of the form:

(antifield of the antighost) x (auxiliary field),

and we use the gauge fixing fermion ψ of the form:

(antighost) x (FL gauge condition),

to reach a covariant delta function gauge fixing. We represent this procedure with an arrow:



Note that each row continue to respect the gradings described above. We have already noted that this is not the end of the story because of further gauge invariances that show up for C_μ and $\bar{C}_\mu$ (3.23a), (3.23b). To cope with this problem we will follow the same procedure as for the first step. The introduction of the second non minimal sectors (3.26) can be represented with two other arrows, meaning similar insertions in the action and in the gauge fixing fermion as in the first step. At the end we have the diagram:

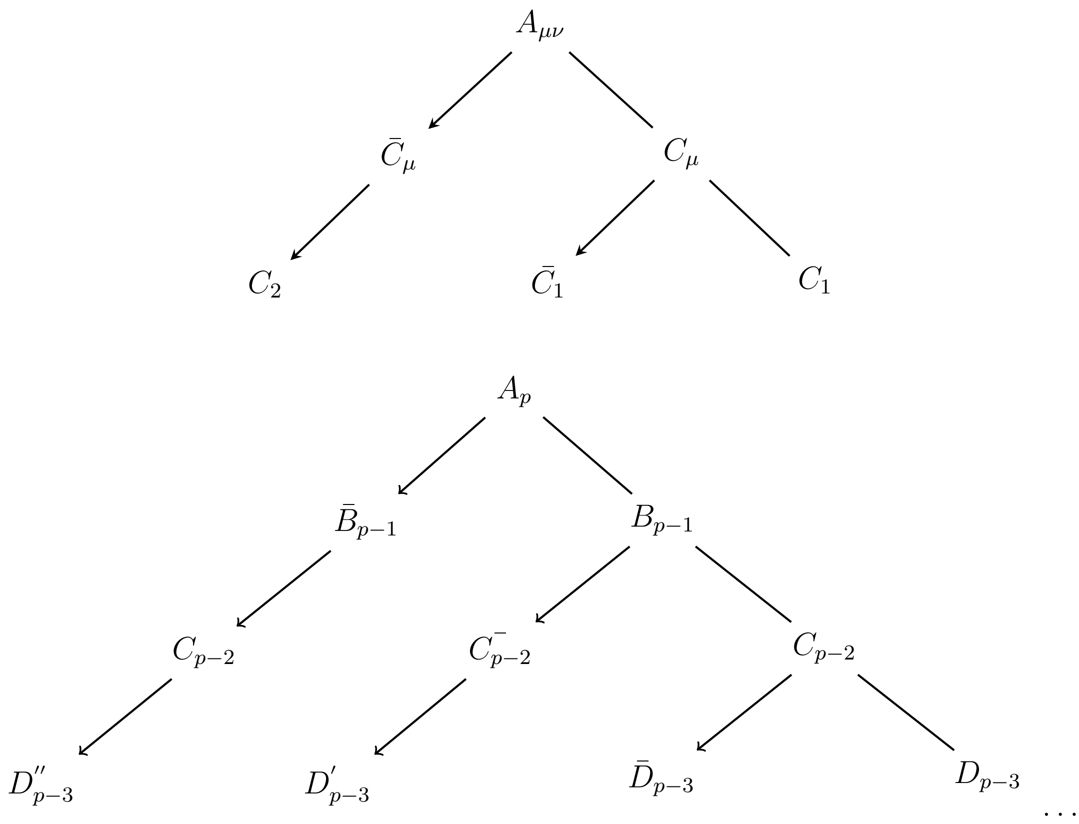


Figure 3.2: Ghost structure tree diagram.

3.1.3 The p-form case

It would be clear that this procedure is the same for theories with higher order reducibilities. On the right we have the fields and ghosts (and ghosts for ghosts, etc...). At each step we introduce a non minimal sector for each gauge invariance present in the action at that step. We fix the residual invariances with an appropriate gauge fixing fermion. The diagram is pictured in Fig.(3.1.2)

For each A, B in an arrow $A \to B$, $gh(A) + gh(B) = -1$ and we construct the gauge fixing fermion as:

$$\psi = \int d^n x \, B f(A) \tag{3.42}$$

where f is the gauge condition. In our case we choose always: $f(A) = \partial \cdot A$.

3.1.4 Gaussian gauge fixing

The delta function gauge condition has an important drawback: it imposes in the path integral the condition $f(A) = \partial \cdot A = 0$ on all the fields which we still have to integrate over. This poses a problem because this constrained space of variables is not always easy to handle. It is possible to overcome this problem through the very known trick of averaging over a family of gauge conditions. To do that we need to simply add to the gauge fixing fermion a (consistent) term linear in the auxiliary field. To explain the procedure it is convenient to go back to the 2-form example. The terms needed in this case are:

$$\int d^n x \, \alpha \bar{C}^\mu b_\mu, \int d^n x \, \beta \bar{C}_1 b_2, \int d^n x \, \gamma C_2 b_1. \tag{3.43}$$

where α, β, γ are generic real numbers. note that the condition $gh(\psi) = -1$ is satisfied. The elimination of the antifields with this new gauge fixing fermion produces the following action:

$$\begin{aligned} S = \int d^n x \Big\{ & -\frac{1}{12} F^{\mu\nu\lambda} F_{\mu\nu\lambda} - \partial_{[\nu} \bar{C}_{\mu]} (\partial^\mu C^\nu - \partial^\nu C^\mu) - \partial^\mu \bar{C}_1 \partial_\mu C_1 \\ & + (\partial_\mu A^{\mu\nu} - \partial^\nu C_2 + \alpha b^\nu) b_\nu + (\partial^\mu C_\mu + \beta b_2) b_1 + (\partial^\mu \bar{C}_\mu + \gamma b_1) b_2 \Big\} \end{aligned} \tag{3.44}$$

The functional integrations over the auxiliary fields are then all gaussians. This procedure allows us to eliminate in the action all the terms like $\partial \cdot A$ without constraining the space of variables, but simply cancelling them out. We integrate first over b_ν, that is a real and bosonic variable. Up to a constant the result is:

$$exp \left\{ \int d^n x \, \frac{(\partial_\mu A^{\mu\nu} - \partial^\nu C_2)^2}{\alpha} \right\} \tag{3.45}$$

The terms $\partial_\mu A^{\mu\nu}\partial^\nu C_2$ are zero, upon integration by parts, due to the anti-symmetry of $A_{\mu\nu}$. The term $(\partial^\nu C_2)^2$ is a ghost Klein-Gordon kinetic term (free $0-form$). The quadratic term in the divergence of $A_{\mu\nu}$ with the appropriate choice of α is canceled with the opposite term one can extract from the Maxwell like kinetic term of $A_{\mu\nu}$, after an integration by parts. Finally we are left with an invertible kinetic operator, i.e. the laplacian acting on the full space of 2-indices antisymmetric tensors. The same can be done with the integration over b_1 and b_2. We integrate them out as a multivariable gaussian integral. We define

$$\mathbf{A} = \begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix} \tag{3.46}$$

$$\mathbf{x} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{3.47}$$

$$\mathbf{B} = \begin{pmatrix} \partial C^\mu \\ \partial \bar{C}^\mu \end{pmatrix} \tag{3.48}$$

The relevant terms in the (euclidean) functional integral are then (we suppress the integration symbol in the first line):

$$\begin{aligned} &\int D\mathbf{x} \exp(-\mathbf{x}^t\mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{x}) \propto \exp(\mathbf{B}^t\mathbf{A}^{-1}\mathbf{B}) \\ &= \exp\left(\int d^n x\, (-\frac{1}{\gamma} + \frac{1}{\beta})\partial^\mu \bar{C}_\mu \partial^\nu C_\nu\right), \end{aligned} \tag{3.49}$$

where in the last line we used the fact that C_ν and $\bar{C}_\mu$ are Grassmann variables. This term has to be equal to the one that can be extracted from the Maxwell like complex kinetic term in the action 3.44, i.e. $-\partial^\mu \bar{C}_\mu \partial^\nu C_\nu$. So we can fix the parameters to be, for example: $\beta = 2$ and $\gamma = -2$. After the integrations the action in the path integral as the very suggestive form:

$$S = \int d^n x \left(\frac{1}{4} A^{\mu\nu}\Delta_2 A_{\mu\nu} - \bar{C}^\mu \Delta_1 C_\mu + \bar{C}_1 \Delta_0 C_1 + C_2 \Delta_0 C_2\right) \tag{3.50}$$

where $\Delta_{(p)}$ is the laplacian acting on the full space of p-forms. Now the triangular diagram is very useful, because one can extend the procedure described in this section to a generic p-form. At the end the diagram tells us exactly what are the pieces of the action, once the integration over the auxiliary fields has been performed. Each field in the diagram gives a quadratic term with kinetic operator $\Delta_{(p)}$. The fields to the right of the diagram that comes in pairs (for example C_μ, $\bar{C}^\mu$) give only one term of the same type (as a complex field).

Had we used a delta function gauge fixing procedure, the form of the action would be very similar, except for a crucial difference: the kinetic operator would act not on the full space of p-forms but on the constrained space of forms that satisfy the FL condition. We are interested in the first kind of operators because all the computations are obviously simpler.

Another comment is important. The gauge invariances and the initial action are not affected by the introduction of a minimal coupling to a background metric. This is a consequence of the antisymmetry both of $F_{\mu\nu\lambda}$ and of gauge conditions. Indeed we choose to work with a simple gravitational theory for the background in which the connections is the Levi-Civita one (symmetric and metric compatible). This eliminates the dependence of the gauge conditions from the background. On the other hand it is important to stress that we end up with a gauge fixed action, and we use this as the Lagrangian that produces the correct equation of motion. When the elimination of the divergence of the fields is performed, for example with (3.49), the terms containing the coupling to the metric appear as a modification of the kinetic operator, due to the integration by parts as we pointed out in the first chapter (see eq. (1.55)).

3.2 Complex $(p, 0)$-form

We have sketched the procedure to arrive at a completely gauge fixed and BRST invariant Lagrangian in the real case. What we want to do now is to extend those arguments to the complex case. In the following Chapter we are interested only in the final structure like (3.50). This structure is very important because it is totally splitted in quadratic terms. In each term it appears only a Laplacian type operator $\Delta^{(s)} = -(\nabla + E)^{(s)}$.

In the complex case the starting Lagrangian is:

$$S = \frac{1}{(p+1)!} \int_M d^d z d^d \bar{z} g \, \bar{F}^{\mu_1 \dots \mu_{p+1}} F_{\mu_1 \dots \mu_{p+1}}, \tag{3.51}$$

where indices are raised with the inverse metric $g^{\mu\bar{\nu}}$. We have to deal with the gauge invariances of the field $A_{\mu_1 \dots \mu_p}$ as well as those of its complex conjugate.

The gauge invariances are one the complex conjugate of the other so they are not really independent. However each gauge invariance possesses now its own ghosts and antifield set of the same type as in Fig. 3.1. Repeating the same arguments as in the previous section, one can show that the gauge fixed action has a similar tree structure like in Fig. 3.1.2. Now this structure is repeated twice, one for $A_{\mu_1\ldots\mu_p}$ and the other for its complex conjugate. It is obvious that this diagram couldn't be read as the previous one. For example the very first row of each part of the diagram is not a separate term in the Lagrangian. However each term continue to follow the structure:

$$\phi_{(s)}D^{(s)}\chi_{(s)}, \tag{3.52}$$

where $D^{(s)}$ in our case, with a coupling with the background metric and the trace of the connection is:

$$D^{(s)} = -(\nabla_q^2 + E) \quad \text{where} \tag{3.53}$$

$$E = -2qR\mathbb{1}_{\Omega^{(s,0)}(M)} - s(1-4q)\left\{R^{\nu_1}_{[\mu_1}\delta^{\nu_2}_{\mu_2}\ldots\delta^{\nu_s}_{\mu_s]}\right\}. \tag{3.54}$$

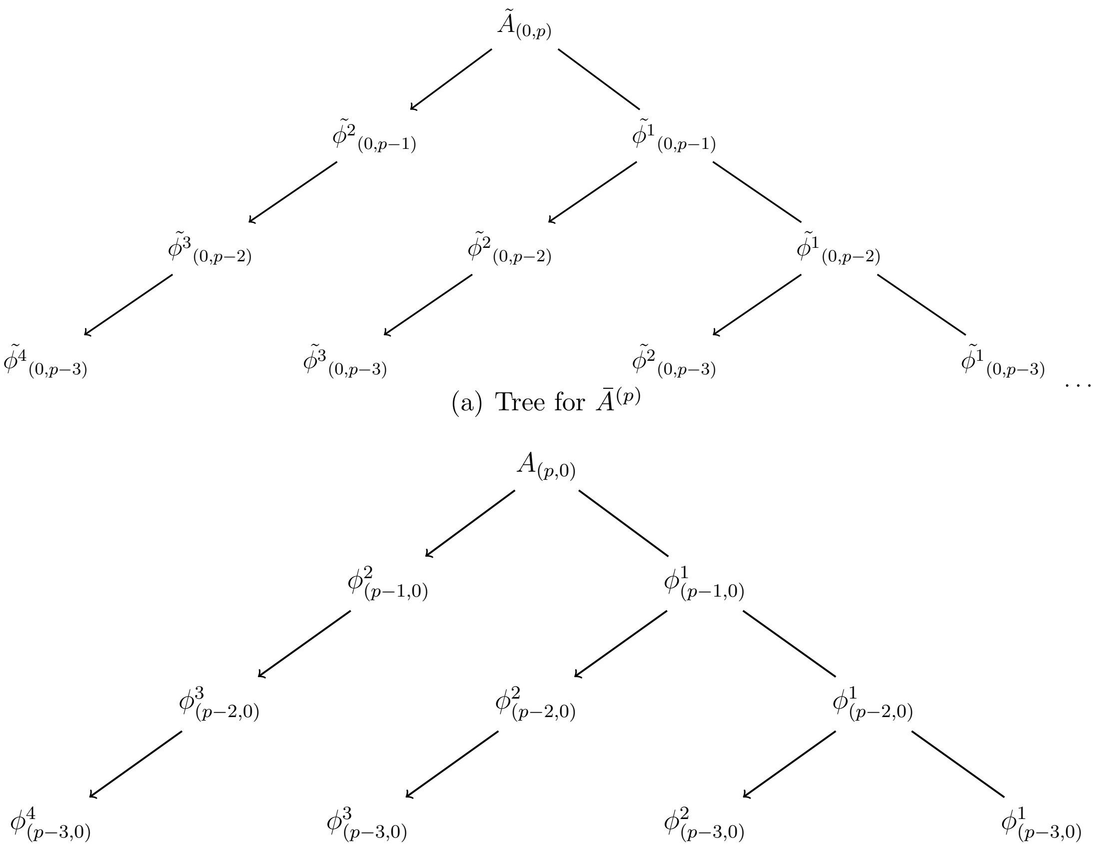
$\phi_{(s)}$ and $\chi_{(s)}$ are fields belonging to the row containing $(s,0)$ and $(0,s)$-forms. Each term then splits the path integral in a product of gaussian integrals. The gaussian integration can be performd easily, at least formally yielding the power of a functional determinant of $\Delta^{(s)}$. The exponent is determined by the nature of the fields involved in the gaussian functional integration, i.e.:

$$(\mathrm{Det}(D_{(s)}))^{(-)^{\epsilon+1}\frac{r}{2}}, \tag{3.55}$$

where ϵ is the parity of the fields (they belong to the same row of the ghost diagram, so they have the same parity) and $r=2$ if there were 2 fields involved in the integration and $r=1$ if there was only 1 field involved.

3.2.1 Example: complex $(1,0)$-forms

To illustrate what is the meaning of the construction sketched in the previous section we present here the case of a $(1,0)$-form in the Batalin-Vilkoviski approach along with the whole gauge fixing procedure. This could be extended as we previously described.



(a) Tree for $\bar{A}^{(p)}$

(b) Tree for $A^{(p)}$

Figure 3.3: Tree structure. Complex case

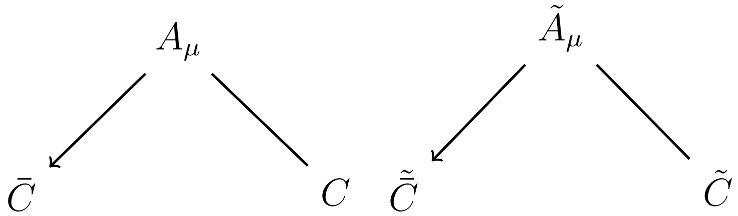
We obtained before a general feature of the Batalin Vilkoviski approach. In the case of a gauge fixing fermion ψ, that is independent from the antifields the BV action takes the form:

$$S = S_0 + s\psi. \tag{3.56}$$

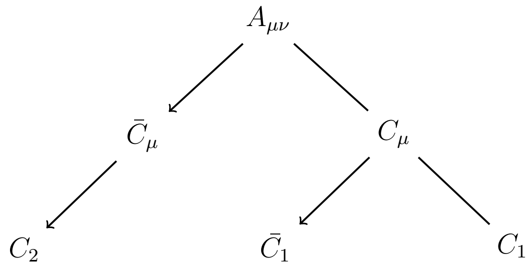
In the case at hand we have denoted the ghosts belonging to the complex conjugate part of the tree with a tilde. Now we have:

$$S_0 = \frac{1}{2}\int_M d^d z d^d \bar{z} g \, \bar{F}^{\mu\nu} F_{\mu\nu} \tag{3.57}$$

$$s\psi = s\left(\bar{C}(\partial_\mu A^\mu - \alpha\tilde{b}) + \bar{\tilde{C}}(\bar{\partial}_\mu \tilde{A}^\mu - \tilde{\alpha} b)\right) \tag{3.58}$$



(a) $(1, 0)$-form tree diagram



(b) Real 2-form

Figure 3.4: Tree structure. Complex case

We show in figure (3.4) the diagram for the ghost structure of the $(1, 0)$-form and the real 2-form. We can see from it why we choose such a gauge fixing fermion. Indeed, compare the complex case we are dealing with to the second and third row of the real 2-form case discussed before, in particular the arrow pointing to the left-downwards. We have two copies of the gauge fixing procedure, but the most important thing to realize is the gaussian average. This is clearly the same as the real 2-form case, at the second stage, when we tried to fix the gauges for $(\bar{C}^\mu, C_\mu)$. So, we enlarge the gauge fixing fermion as in the last equation, with a term linear in the auxiliary fields b and $\tilde{b}$. The functional gaussian integration is exactly the same as in (3.49) with α and $\tilde{\alpha}$ taking the place of β and γ. After a suitable choice of those constants this integration yields a term in the argument of the exponential of the path integral like:

$$- \bar{\partial}^\mu A_\mu \partial^{\bar{\nu}} \tilde{A}_{\bar{\nu}}, \tag{3.59}$$

that cancels out the (opposite) term one can extract from the S_0 part after an integration by parts. As usual we are left only with Laplace type kinetic terms.

Chapter 4

Effective Action and Seeley DeWitt coefficients

4.1 Effective action for $(p, 0)$-forms

In the previous chapter, we found that the path integral splitted in a products of integrals. Each term of the product was a gaussian integral over fields belonging to the same row of the tree diagram 3.3. Each term in the path integral contributes as:

$$(\text{Det}\, D_{(s)})^{(-)^{\epsilon+1}\,\frac{r}{2}}. \tag{4.1}$$

In the last equation ϵ is the grassmann parity of the fields and $r = 2$ if there were two real fields involved in the integration and $r = 1$ if there was only 1 field involved. We label each row of the tree diagram 3.3 with an index k, starting from $k = 0$ for the first row on top. We stress that going from a row to the next, the parity changes. So in the formula (4.1) we can change $\epsilon + 1$ with k. What we want to study here is the so called Euclidean Effective Action W, defined by the relation:

$$\begin{aligned} Z &= e^{-W}, \\ \text{or} \quad W &= -\ln Z \end{aligned} \tag{4.2}$$

In terms of the functional determinants (4.1) each term of the Effective Action reads:

$$W = (-)^k\,\frac{r}{2}\,\ln(\det(D_{(s)})), \tag{4.3}$$

but this time the terms are summed because of the formula $ln(AB) = ln(A) + ln(B)$. Moreover for the operators we are considering in this thesis it is possible to say that:

$$\ln(\mathrm{Det}(D_{(s)})) = \mathrm{Tr}(\ln(D_{(s)})). \tag{4.4}$$

Now we are able to write down the total Effective Action by simple inspection of the tree diagram. The number r simply counts the number of fields in the trees. If we separate the contribution from each row we arrive at the very important formula:

$$W = -\sum_{k=0}^{p}(-1)^k(1+k)(\mathrm{Tr}(\ln\ \mathrm{Det}_{p-k})). \tag{4.5}$$

4.2 Effective action expansion: heat kernel method

4.2.1 Heat kernel

The evaluation of the trace that appeared in the last formula is usally performed by means of the heat kernel of the operator $D_{(s)}$. We review here briefly that method following mainly Vassilievich and Gilkey.([26], [12], [18]). So we define formally the heat kernel of a self adjoint operator D:

$$K(t,x,y;D) = < x|e^{-tD}|y >, \tag{4.6}$$

where the bra-ket notations denotes the usual scalar product on functions belonging to $\mathcal{L}^2$. The heat kernel is formally a solution of the heat equation:

$$(\partial_t + D_x)K(t,x,y,D) = 0 \tag{4.7}$$

with initial condition:

$$K(0,x,y,D) = \delta(x,y) \tag{4.8}$$

To be more precise, the operator D is a self adjoint elliptic operator of Laplace type acting on sections of a vector bundle V. In our case this vector bundle is the $(s,0)$-form bundle, namely $\Omega^{(s,0)}(M)$. We define the Heat Trace as:

$$K(t,f,D) = Tr_{\mathcal{L}^2(V)}(f exp(-tD)) \tag{4.9}$$

where f is a test function belonging to $\mathcal{L}^2(V)$ and t is a positive number. We can relate the Heat Trace (4.9) and the formal heat kernel (4.6) through the integral of the $x \to y$ limit:

$$K(t, f, D) = \int_{\mathcal{M}} d^n x \sqrt{g} K(t, x, x; D) f(x) \tag{4.10}$$

It is possible to prove that the Heat Trace has an expansion for $t \to 0$:

$$Tr_{\mathcal{L}^2}(f exp(-tD)) \simeq \sum_{k \geq 0} t^{\frac{(k-n)}{2}} a_k(f, D) \tag{4.11}$$

$$. \tag{4.12}$$

It is not so simple to prove the existence of this expansion, for a proof one can see for example.[16]. The coefficients $a_k(f, D)$ are usually called "Seeley-DeWitt Coefficients". and their computation in the case of a $(p, 0)$-form theory will be the main goal of this Chapter. Note that the computation of those coefficients has been performed by other means, in particular the worldline formalism by F. Bastianelli and R. Bonezzi in [5]. We rescale the parameter that regulates the expansion as $t = \frac{\beta}{2}$, in order to compare our results with those contained in that paper.

4.2.2 Relation between the heat kernel and the Effective Action

The starting point for quite all the computations of the asimptotics of the 1-loop effective actions is the integral representation of the logaritm:

$$ln\lambda_i = -\int_0^\infty \frac{dt}{t} e^{-t\lambda_i} \tag{4.13}$$

This relation is correct modulo an (infinite!) constant that does not depend on λ_i. For this reason we will not consider it in what follows. Consider λ_i as the eigenvalues of our Laplace type operator $D_{(s)}$, now we can extend the last formula to the whole operator and take the functional trace obtaining:

$$\begin{aligned} Tr\, ln(D_{(s)}) &= -\int_0^\infty \frac{dt}{t} Tr_{\mathcal{L}^2(V)}(f exp(-tD(s))) \\ t \to 0 &= -\int_0^\infty \frac{dt}{t} \sum_{k=0}^{\infty} t^{\frac{k-n}{2}} a_k(f, D_{(s)}). \end{aligned} \tag{4.14}$$

Note that all the contributions to the effective action up to $k = n$ for small t are divergent (UV divergences). In particular the most important coefficient is certainly a_n. Indeed it is the coefficient of the logarithmic divergence. As a simple example if we put cutoffs in the previous integral for $k = n$ the logarithmic divergence is:

$$\Gamma^{\log}_{\text{1-loop}} = -a_n \int_{\frac{1}{M^2}}^{\frac{1}{\mu^2}} \frac{dt}{t} = 2a_n \log\frac{M}{\mu}. \tag{4.15}$$

4.2.3 Heat kernel Expansion

For a manifold M[1] without boundaries there exist a method to compute the heat kernel's coefficients due to Gilkey. It is possible to extend the method to manifolds with boundaries but it is not the goal of this thesis to deal with such manifolds. For a more extensive treatment see [11]. For a general self adjoint Laplace type operator D on a smooth manifold without boundaries it is not only possible to prove that an expansion like (4.12) exists but also that the odd k components vanish and that the other even k components are locally computable in terms of geometric invariants of dimension k. So the heat kernel coefficients are:

$$a_k(f, D) = tr_V \int_{\mathcal{M}} d^n x\sqrt{g}\{f(x)a_k(x, D)\} = \sum_I tr_V \int_{\mathcal{M}} d^n x\sqrt{g}\{fA_i(n)\mathcal{I}^i_k(D)\}, \tag{4.16}$$

Here $\mathcal{I}^i_k(D)$ are all the geometric invariants of dimension k. To clarify what will be the geometric invariants that we will be choose, we recall the structure of the elliptic, Laplace type operator $D_{(s)}$. This operator is:

$$D^{(s)} = -(\nabla^2_q + E)_s \quad \text{where} \tag{4.17}$$

$$E = -2qR\mathbb{1}_{\Omega^{(s,0)}} - s(1-4q)\left\{R^{\nu_1}_{[\mu_1}\delta^{\nu_2}_{\mu_2}\dots\delta^{\nu_s}_{\mu_s]}\right\}, \tag{4.18}$$

The s subscript indicates that this operator acts on $(s, 0)$-forms. In particular we have that ∇_q has three terms: the usual Dolbeault operator ∂, the metric connection term (in components $\Gamma_{\text{metric}} = \Gamma^\lambda_{\mu\nu}$) and the further coupling with the trace of the connection and coupling constant q (in components $\Gamma_{\text{trace}} = \Gamma^\lambda_{\lambda\nu}$).

$$\nabla_q = \partial + \Gamma_{\text{metric}} + q\Gamma_{\text{trace}}. \tag{4.19}$$

[1]We will follow the review by Vassilievich in which M is considered an ordinary differentiable manifold. We will specify the differences in the coefficients in the case of a complex *Kähler* manifold later, directly on the coefficients'formulas.

We can define the curvature associated with the full connection term, i.e. $\Gamma_{\text{metric}}+\Gamma_{\text{trace}}$. In components this curvature is the commutator of two twisted covariant derivative $\nabla_{q\,\mu}$

$$\Omega_{\mu\bar{\nu}} = [\nabla_{q\,\mu}, \nabla_{q\,\bar{\nu}}]. \tag{4.20}$$

It is possible to refine (4.16) expliciting the dependence of the factors $A_i(n)$ from the dimension of the manifold n. One can prove that:

$$A_i(n) = \sqrt{4\pi}A_i(n+1) \tag{4.21}$$

from this relation follows that we can extract the true constants α_i:

$$A_i(n) = \frac{\alpha_i}{(4\pi)^{\frac{n}{2}}}. \tag{4.22}$$

The geometric invariants can be chosen in different ways, because of relations between different invariants that makes some of them not independent (like the Bianchi identities). The first few heat kernel coefficients for the operator D are written as:

$$a_0(f,D) = (4\pi)^{-\frac{n}{2}}\int_{\mathcal{M}} d^n x\sqrt{g}\mathrm{Tr}_V\{\alpha_0 f\} \tag{4.23}$$

$$a_2(f,D) = (4\pi)^{-\frac{n}{2}}\frac{1}{6}\int_{\mathcal{M}} d^n x\sqrt{g}\mathrm{Tr}_V\{f(\alpha_1 E + \alpha_2 R(r)\} \tag{4.24}$$

$$\begin{aligned} a_4(f,D) = (4\pi)^{-\frac{n}{2}}\frac{1}{360}\int_{\mathcal{M}} & d^n x\sqrt{g}\mathrm{Tr}_V\{\ f(\alpha_3 E_{;kk} + \alpha_4 R(r)E + \alpha_5 E^2 \\ & \alpha_6 R(r)_{;kk} + \alpha_7 R(r)^2 + \alpha_8 R(r)_{ij}R(r)_{ij} + \alpha_9 R(r)_{ijkl}R(r)_{ijkl} \\ & + \alpha_{10}\Omega(r)_{ij}\Omega(r)_{ij})\}. \end{aligned} \tag{4.25}$$

Here Ω is the curvature operator of the connection acting on the space of $(s,0)$-forms and we denoted the covariant derivative with a ;. We used the same notation as in [26], with a (r) on the invariants to remind that these refers to a real manifold. In particular in that review the geometric invariants are constructed in a local orthogonal frame (the indices i,j,k,. . . are flat). The constants α_i are computed by means of some differential and recursive formulas (see [26] and references within).

They are:

$$\begin{aligned} \alpha_0 &= 1; \quad \alpha_1 = 6; \\ \alpha_2 &= 1; \quad \alpha_3 = 60; \\ \alpha_4 &= 60; \quad \alpha_5 = 180 : \\ \alpha_6 &= 12; \quad \alpha_7 = 5; \\ \alpha_8 &= -2; \quad \alpha_9 = 2; \\ \alpha_{10} &= 30. \end{aligned} \tag{4.26}$$

In order to compare our results with those of [5] we have substituted $t = \frac{\beta}{2}$ so we can define in the expansion (4.14) the rescaled coefficients:

$$\tilde{b}_k = \frac{1}{2^{\frac{k}{2}}} a_k. \tag{4.27}$$

The following geometrical relations allow us to rewrite the coefficients for the case of a complex manifold:

$$\begin{aligned} R(r) &= 2R, \\ R(r)^{MN} R(r)_{MN} &= 2R^{\mu\bar{\nu}} R_{\mu\bar{\nu}} \\ R(r)^{MNPQ} R(r)_{MNPQ} &= 4R^{\mu\bar{\nu}\sigma\bar{\rho}} R_{\mu\bar{\nu}\sigma\bar{\rho}}. \end{aligned} \tag{4.28}$$

$$\Omega_{MN}\Omega^{MN} = 2\Omega_{\mu\bar{\nu}}\Omega^{\mu\bar{\nu}} \tag{4.29}$$

Remembering that $n = 2d$ real dimensions, the new coefficients are:

$$\tilde{b}_0(f, D) = (4\pi)^{-d} \int_{\mathcal{M}} d^d z d^d \bar{z}\, g \,\mathrm{Tr}_V\{f\} \tag{4.30}$$

$$\tilde{b}_2(f, D) = (4\pi)^{-d} \frac{1}{12} \int_{\mathcal{M}} d^d z d^d \bar{z}\, g \,\mathrm{Tr}_V\{f(6E + 2R\} \tag{4.31}$$

$$\begin{aligned} \tilde{b}_4(f, D) = (4\pi)^{-d} \int_{\mathcal{M}} d^d z d^d \bar{z}\, g \,\mathrm{Tr}_V\{\, f\Big(\frac{1}{24}\nabla^2 E + \frac{1}{12} RE + \frac{1}{8} E^2 \\ \frac{1}{60}\nabla^2 R + \frac{1}{72} R^2 - \frac{1}{360}(R_{\mu\bar{\nu}})^2 + \frac{1}{180}(R_{\mu\bar{\nu}\sigma\bar{\rho}})^2 + \frac{1}{24}(\Omega^2_{\mu\bar{\nu}})\Big)\} \end{aligned} \tag{4.32}$$

4.3 Seeley-DeWitt coefficients for the $(p, 0)$-forms

Now we are ready to perform the calculations of the Seeley-DeWitt coefficients of a $(p, 0)$-form and to obtain the first few terms in the expansion of the 1-loop effective action. Let us recall briefly some formulas. First of all the Euclidean Effective Action for a $(p,0)$-form has an expansion:

$$W=\int_0^\infty \frac{d\beta}{\beta}\frac{1}{(\frac{\beta}{2})^2}\sum_{i\,\mathrm{even}}\beta^{\frac{i}{2}}\,b_i. \tag{4.33}$$

Note that with b_i we denote the coefficient of the full theory of $(p,0)$-forms. Only these coefficients are gauge invariant because they have contributions from the whole set of fields and ghosts. From (4.5) we can expand the coefficient b_i in a sum of contributions derived from the full tree diagram of fields and ghosts of Fig.(3.3) .

$$b_i=\sum_{k=0}^{p}(-1)^k(1+k)\tilde{b}_i^{(p-k)} \tag{4.34}$$

where $\tilde{b}_i^{(p-k)}$ is the heat kernel coefficient for the kinetic (Laplacian type) operator acting on the space of $(p-k,0)$-forms. In particular we recall the form of this operator in terms of the form degree $s=p-k$:

$$D^{(s)}=-(\nabla^2+E) \tag{4.35}$$

where E is the endomorphism on $V=\Omega^{(s,0)}(M)$:

$$E=-\frac{s(1-4q)}{(s!)}\left(R^{\nu_1}_{(\mu_1},\delta^{\nu_2}_{\mu_2}\dots\delta^{\nu_s}_{\mu_s)}\right)-2qR\mathbb{1}_{\mathbb{V}}. \tag{4.36}$$

From now on it will be better to explicit the combinatorial factors so the round brackets (...) will denote unweighted antisymmetrization of the indices. Remember that q is a real constant that measure the strength of the coupling with the $U(1)$ part of the holonomy of the background *Kähler* manifold M. The study of the coefficients will be splitted in several parts. From the formulas (4.30)-(4.32), it is clear that the explicit computation of the traces on the space of forms $V=\Omega^{(s,0)}(M)$ will be required.

It is easy to get lost for the great number of pieces involved in the calculations so we will use the following notation that anticipate the final result.

$$W=\int\frac{d\beta}{\beta}\frac{d^dzd^d\bar{z}}{(2\pi\beta)^2}\,g\left\{v_1+v_2\beta+\beta^2[v_3R_{\mu\bar{\nu}\sigma\bar{\rho}}R^{\mu\bar{\nu}\sigma\bar{\rho}}+v_4R_{\mu\bar{\nu}}R^{\mu\bar{\nu}}\right.$$
$$\left.+v_5R^2+v_6\nabla^2R]\right\} \tag{4.37}$$

Note however that in this notation $v_1=b_0$ and $v_1=b_2$, because they yield only one term.

4.3.1 Coefficient b_0

First of all we have to deal with the heat kernel coefficients $\tilde{b}_0^{(s)}$. It explicitly reads:

$$\tilde{b}_0^{(s)} = (4\pi)^{-d} \int_{\mathcal{M}} d^d z d^d \bar{z}\, g\, \mathrm{Tr}_V\{f\} \tag{4.38}$$

It is implicit that we are tracing over the identity on the space $V = \Omega^{(s,0)}(M)$. This trace produce the dimension of this space and it is well known that:

$$dim\Omega^{(s,0)}(M) = \binom{d}{s}, \tag{4.39}$$

where the RHS is the binomial coefficient $\binom{d}{s} = \frac{d!}{s!(d-s)!}$. However, it is instructive to give a proof of this statement, computing directly the trace.

Proof. The trace we have to compute is:

$$\mathrm{Tr}_V(\mathbb{1}_V) = \frac{1}{s!}\mathrm{Tr}_V\left(\delta^{\nu_1}_{(\mu_1}, \delta^{\nu_2}_{\mu_2} \dots \delta^{\nu_s}_{\mu_s)}\right) \tag{4.40}$$

$$\tag{4.41}$$

This trace is performed separately on each space constituting the tensor product, i.e. over the pairs $(\mu_1, \nu_1) \dots (\mu_s, \nu_s)$. The last equation can be simplified by means of (A.5):

$$\begin{aligned} \frac{1}{s!}\mathrm{Tr}_V\left(\delta^{\nu_1}_{(\mu_1}\delta^{\nu_2}_{\mu_2} \dots \delta^{\nu_s}_{\mu_s)}\right) &= \frac{1}{s!}\mathrm{Tr}_V\left(\epsilon^{\nu_1\dots\nu_s}_{\mu_1\dots\mu_s}\right) \\ \frac{1}{s!}\epsilon^{\mu_1\dots\mu_s}_{\mu_1\dots\mu_s} &= \frac{d!}{s!(d-s)!}. \end{aligned} \tag{4.42}$$

where in the last line we used (A.3) □

This result has to be inserted in the sum formula (4.34) to give the b_0 coefficient:

$$\begin{aligned} b_0 &= \sum_{k=0}^{p} (-1)^k (1+k) \tilde{b}_0^{(p-k)} \\ &= \sum_{k=0}^{p} (-1)^k (1+k) \binom{d}{p-k} = \binom{d-2}{p}. \end{aligned} \tag{4.43}$$

where in the last line we use the result (B.7), proved in appendix.

4.3.2 Coefficient b_2

This coefficient is proportional to the Ricci scalar. From (4.31), we need to compute:

$$\frac{1}{12}\mathrm{Tr}_V\left(6E + 2R\right), \tag{4.44}$$

where:

$$E = -\frac{s(1-4q)}{(s!)}\left(R^{\nu_1}_{(\mu_1}\delta^{\nu_2}_{\mu_2}\dots\delta^{\nu_s}_{\mu_s)}\right) - 2qR\mathbb{1}_{\mathbb{V}} \tag{4.45}$$

The unknown part is only:

$$-\frac{s(1-4q)}{(s!)}\mathrm{Tr}_V\left(R^{\nu_1}_{(\mu_1}\delta^{\nu_2}_{\mu_2}\dots\delta^{\nu_s}_{\mu_s)}\right) \tag{4.46}$$

Even if a simple argument can show the value of the last formula, we give also a direct computation that we consider valuable for it shows the basic steps that will lead to other coefficients. The trace of E is:

$$-\frac{s(1-4q)}{d}\binom{d}{s}R - 2qR\binom{d}{s}. \tag{4.47}$$

Proof. Method 1.
We prove only the first part of trace of E, because the second has been shown in the calculations of the b_0 coefficient. We start with the formula:

$$-\frac{s(1-4q)}{s!}\mathrm{Tr}_V\left(R^{\nu_1}_{(\mu_1}\delta^{\nu_2}_{\mu_2}\dots\delta^{\nu_s}_{\mu_s)}\right) \tag{4.48}$$

now, we can separate the Ricci tensor, in each term of the antisymmetrization, in a part proportional to the identity plus a traceless part:

$$R^{\nu}_{\mu} = \frac{R}{d}\mathbb{1} + \tilde{R}^{\nu}_{\mu} \tag{4.49}$$

only the trace part survives and we obtain, then:

$$\begin{aligned} &-\frac{s(1-4q)}{d}\mathrm{Tr}_V\left(\mathbb{1}_{\mathbb{V}}\right)R \\ &= -\frac{s(1-4q)}{d}\binom{d}{s}R \end{aligned} \tag{4.50}$$

Method 2.
With the help of (A.5) we can extract the antisymmetric part of E:

$$
\begin{aligned}
&-\frac{s(1-4q)}{s!}\mathrm{Tr}_V\left(R^{\nu_1}_{(\mu_1}\delta^{\nu_2}_{\mu_2}\dots\delta^{\nu_s}_{\mu_s)}\right)\\
&=-\frac{s(1-4q)}{s!}\mathrm{Tr}_V\left(R^{\nu_1}_{\sigma_1}\delta^{\nu_2}_{\sigma_2}\dots\delta^{\nu_s}_{\sigma_s}\epsilon^{\sigma_1\dots\sigma_s}_{\mu_1\dots\mu_s}\right),
\end{aligned}
\tag{4.51}
$$

then using the deltas we obtain:

$$
-\frac{s(1-4q)}{s!}\mathrm{Tr}_V\left(R^{\nu_1}_{\sigma_1}\epsilon^{\sigma_1\nu_2\dots\nu_s}_{\mu_1\dots\mu_s}\right),
\tag{4.52}
$$

next, performing the trace we obtain:

$$
\begin{aligned}
&-\frac{s(1-4q)}{s!}\left(R^{\mu_1}_{\sigma_1}\epsilon^{\sigma_1\mu_2\dots\mu_s}_{\mu_1\mu_2\dots\mu_s}\right),\\
&=-\frac{s(1-4q)}{s!}\frac{(d-1)!}{(d-s)!}\left(R^{\mu_1}_{\sigma_1}\delta^{\sigma_1}_{\mu_1}\right)=-s(1-4q)\frac{R}{d}\binom{d}{s}
\end{aligned}
\tag{4.53}
$$

where in the last line we have used (A.4). □

The heat kernel coefficient $\tilde{b}_2^{p-k}$ now reads:

$$
\tilde{b}_2^{p-k}=\frac{1}{12}\left\{6\left(-\frac{(p-k)(1-4q)}{d}\right)-2q+2\right\}\binom{d}{p-k}R.
\tag{4.54}
$$

Inserting this result in (4.34), yields:

$$
\begin{aligned}
b_2^p&=\sum_{k=0}^{p}(-1)^k(1+k)\left\{\frac{1}{6}-\frac{(p-k)(1-4q)}{2d}-\frac{1}{6}q\right\}\binom{d}{p-k}R\\
&=\binom{d-2}{p}\left\{\frac{1}{6}-\frac{p}{2(d-2)}-q\frac{d-2-2p}{d-2}\right\}
\end{aligned}
\tag{4.55}
$$

in perfect agreement with [5].

4.4 Coefficient b_4

This coefficient presents some algebraic complications. For this reason we will able to give only the results with $q=0$. Let us rewrite for clarity the coefficient

$\tilde{b}_4(s)$.

$$\tilde{b}_4(s) = (4\pi)^{-d}\int_{\mathcal{M}} d^d z d^d \bar{z}\, g\, \mathrm{Tr}_V\Big\{\ f(\frac{1}{24}\nabla^2 E + \frac{1}{12}RE + \frac{1}{8}E^2$$
$$\frac{1}{60}\nabla^2 R + \frac{1}{72}R^2 - \frac{1}{360}(R_{\mu\bar{\nu}})^2 + \frac{1}{180}(R_{\mu\bar{\nu}\sigma\bar{\rho}})^2 + \frac{1}{48}(\Omega^2_{\mu\bar{\nu}}))\Big\} \tag{4.56}$$

This coefficient will split in a sum of four terms as previously announced in (4.37), Remember that the "tilde" denotes always the contribution of only one row in the ghost diagram. So we will follow the same procedure we used for the first two coefficients. First we compute the coefficient for a single s-form then we insert it in the sum formula (4.34). The first thing to note is that four of the eight pieces of (4.56) are already in the right form, belonging to different four pieces of the result. The trace is performed over the identity as in the case of the b_0 coefficient. We use the compact notation $(v_3; v_4; v_5; v_6)$ for this result. If we insert only these terms in the sum formula we obtain that the coefficients start with:

$$\binom{d-2}{p}\cdot\left(\frac{1}{180}+\dots;-\frac{1}{360}+\dots;\frac{1}{72}+\dots;\frac{1}{60}+\dots\right) \tag{4.57}$$

Now we have to compute the other four traces. We denote the form degree with an s, to stress the fact that this is not the degree of our original theory (which we called p). The first term is:

$$\frac{1}{12}\mathrm{Tr}(RE), \tag{4.58}$$

but this amounts to compute the trace of E. We already know the result, we found it in the last section, see eq. (4.47) (with q=0). So this term gives contributions only to the R^2 piece, i.e. v_5.

$$\frac{1}{12}\mathrm{Tr}(RE) = -s\frac{R^2}{12d}\binom{d}{s} \tag{4.59}$$

The second term is:

$$\frac{1}{24}\mathrm{Tr}(\nabla^2 E), \tag{4.60}$$

but also in this case the only thing to know is the trace of E. The result is:

$$-\frac{1}{24}\frac{s}{d}\binom{d}{s}\nabla^2 R. \tag{4.61}$$

The third term is a bit more difficult:

$$\frac{1}{8}\mathrm{Tr}(E^2). \tag{4.62}$$

If we write it explicitly, it reads:

$$\frac{s^2}{8(s!)^2}\text{Tr}\Big\{R^{\nu_1}_{(\mu_1}\delta^{\nu_2}_{\mu_2}\dots\delta^{\nu_s}_{\mu_s)}\cdot R^{\mu_1}_{(\sigma_1}\delta^{\mu_2}_{\sigma_2}\dots\delta^{\mu_s}_{\sigma_s)}\Big\} \tag{4.63}$$

Remember that $(\dots)$ stand for unweighted antisymmetrizazion. This could be reduced using a similar technique as the one used for b_2, expliciting the antisymmetric part with the help of the Levi-Civita symbol. A very useful formula for this computation is (A.8). At the end we are left with:

$$\frac{s^2}{8(s!)^2}\left\{K_1 R_{\mu\bar{\nu}}R^{\mu\bar{\nu}} + K_2 R^2\right\}, \tag{4.64}$$

where

$$\begin{aligned} K_1 &= \frac{(s!)^2(d-s)}{sd(d-1)}\binom{d}{s} \\ K_2 &= \frac{(s!)^2(p-1)}{sd(d-1)}\binom{d}{s}. \end{aligned} \tag{4.65}$$

So, this term produces two contributions:

$$\frac{s(s-1)}{8d(d-1)}\binom{d}{s} \qquad (v_5) \tag{4.66}$$

$$\frac{s(d-s)}{8d(d-1)}\binom{d}{s} \qquad (v_4). \tag{4.67}$$

Then the last term is:

$$\frac{1}{24}\text{Tr}(\Omega_{\mu\bar{\nu}}\Omega^{\mu\bar{\nu}}). \tag{4.68}$$

We have to explicit what is the curvature operator $\Omega_{\mu\bar{\nu}}$ when it acts on the space of $(s,0)$-forms. One can compute it as the commutator of two covariant derivative acting on a $(p,0)$-form:

$$\begin{aligned} (\Omega_{\mu\bar{\nu}})A_{\sigma_1\dots\sigma_s} &= A_{\sigma_1\dots\sigma_s;(\bar{\nu}\mu)} \\ &\sum_a R^{\lambda}_{\sigma_a\nu\mu}A_{\sigma_1\dots\lambda(a)\dots\sigma_s}, \end{aligned} \tag{4.69}$$

where $\lambda(a)$ means that lambda is in the a-th position . We can extract only the operator part:

$$\frac{1}{s!}\sum_a R^{\lambda}_{(\mu_a}\delta^{\nu_1}_{\mu_1}\dots(\delta^{\nu_a}_{\mu_a})\dots\delta^{\nu_s}_{\mu_s)}, \tag{4.70}$$

where the $\delta^{\nu_a}_{\mu_a}$ in curved brackets is not present in each term of the sum. Inserting this in the trace formula, extracting the antisymmetric parts by means of the Levi-Civita symbol and using (A.8) yields:

$$\frac{s^2}{24(s!)^2}\Big\{-K_1\, R_{\mu\bar{\nu}\sigma\bar{\rho}}R^{\mu\bar{\nu}\sigma\bar{\rho}} - K_2\, R_{\mu\bar{\nu}}R^{\mu\bar{\nu}}\Big\} \tag{4.71}$$

where K_1 and K_2 are the same as in (4.65). At the end this produces two contributions:

$$-\frac{s(s-1)}{24d(d-1)}\binom{d}{s}R_{\mu\bar{\nu}}R^{\mu\bar{\nu}} \qquad (v_4) \tag{4.72}$$

$$-\frac{s(d-s)}{12d(d-1)}\binom{d}{s}R_{\mu\bar{\nu}\sigma\bar{\rho}}R^{\mu\bar{\nu}\sigma\bar{\rho}} \qquad (v_3) \tag{4.73}$$

We have all we need to perform the last step. Each contribution as to be inserted in the sum formula (4.34), paying attention to use $s = p - k$ in each sum, then sum in k. Grouping together the contributions belonging to the same coefficient, we arrive at the final result:

$$v_1 = \binom{d-2}{p} \tag{4.74}$$

$$v_2 = \left\{\frac{1}{6} - \frac{p}{2(d-2)}\right\}\binom{d-2}{p} \tag{4.75}$$

$$v_3 = \left\{\frac{1}{180} - \frac{p(d-p-2)}{24(d-2)(d-3)}\right\}\binom{d-2}{p} \tag{4.76}$$

$$v_4 = \left\{-\frac{1}{360} + \frac{p(3d-4p-5)}{24(d-2)(d-3)}\right\}\binom{d-2}{p} \tag{4.77}$$

$$v_5 = \left\{\frac{1}{72} + \frac{p(3p-2d+3)}{24(d-2)(d-3)}\right\}\binom{d-2}{p} \tag{4.78}$$

$$v_6 = \left\{\frac{1}{60} - \frac{p}{24(d-2)}\right\}\binom{d-2}{p} \tag{4.79}$$

Again this result is in full agreement with the one obtained by means of the worldline formalism in [5].

Conclusions and future prospects

In this thesis we presented a Batalin-Vilkovisky approach to properly quantize the gauge theory of $(p,0)$-forms on *Kähler* manifolds, and we used the heat kernel technique to compute some Seeley-DeWitt coefficients for the 1-loop effective action of the same theory. The reasons to choose the BV formalism were twofold: it is simpler to apply with respect to other approaches when redundancies in the gauge invariance are present and it offers a direct way to the path integral quantization. The redundancies in the gauge invariances required a tower of ghosts for ghosts to reproduce the full gauge structure.

The most important fact we devised is that a complete covariant gauge fixing procedure, performed at each reducibility step, produce a simple and quite recursive way to construct the final Lagrangian as a sum of kinetic terms like:

$$\phi_{(s)}\Delta^{(s)}\chi_{(s)}, \tag{4.80}$$

where $\phi_{(s)}$ and $\chi_{(s)}$ are (Field or Ghosts) forms of degree s that belongs to one row of the simple diagram 3.3, and $\Delta^{(s)}$ is a Laplace type operator acting on $(s,0)$-forms. When the theory is minimally coupled to a *Kähler* background manifold we could add a further coupling to the trace of the metric connection $\Gamma_\mu = \Gamma^\lambda_{\lambda\mu}$ with a coupling constant q. This coupling does not spoil the nilpotency of the Dolbeault operator ∂, so that one can safely perform the usual analysis of the gauge invariances simply substituting it with its twisted version $\partial_q = \partial + q\Gamma$. In this case the Laplace type operator $\Delta^{(s)}$ takes the form:

$$D^{(s)} = -(\nabla_q^2 + E)_s \quad \text{where} \tag{4.81}$$

$$E = -2qR\mathbb{1}_{\Omega^{(s,0)}} - s(1-4q)\left\{R^{\nu_1}_{[\mu_1}\delta^{\nu_2}_{\mu_2}\dots\delta^{\nu_s}_{\mu_s]}\right\}, \tag{4.82}$$

This form is what we needed to perform the study of the 1-loop effective action with the heat kernel technique as we showed in the last chapter. In particular we used a method due to Gilkey in which the coefficients are computed as a trace over a sum of geometric invariants. The computation of the first few Heat kernel's coefficients has been performed in a way that allows a direct comparison with the same results obtained by means of the worldline formalism in [5].

The computation of the coefficients, performed with the method proposed in this thesis, become rapidly very involved for three main reasons. First, one has to compute traces over functions of operators acting on forms of arbitrary order. Second one has to sum the heat kernel coefficients over the full ghosts for ghosts structure to obtain the final gauge invariant coefficients. Third the expansion in geometric invariants start soon to acquire a huge number of terms to deal with. Actually, this is not a great problem when the number n of dimensions is low because all the information about divergences is contained in the coefficients b_k with $k \leq n$.

It is possible to expand the work done so far in different ways. It is possible to compute other heat kernel coefficients, as well as completely determine the form of b_4 in the case of $q \neq 0$. There is also the possibility to substitute the coupling to the trace of the connection with a more general $U(1)$ field B_μ. Actually this coupling would spoil the nilpotency of the Dolbeault operator ∂. However the obstruction to the full nilpotency of ∂ is due simply to the totally holomorphic part of the B_μ field strength, i.e. $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Including the condition of the vanishing of those components of the field strength (for the emergence of those background in string theories see for example [14]), we could perform safely the calculations of the heat kernel coefficients for that coupling.

Appendix A

Totally antisymmetric tensors

When we computed the Seeley DeWitt coefficients with the heat kernel method we faced the problem of the computation of traces over $\Omega^{(p,0)}(M)$. This often requires the knowledge of some formulas concerning totally antisymmetric symbols (known also as Levi-Civita symbols). Albeit they are ubiquitous in mathematics and physics it is not so simple to find in literature listings of those formulas. So we state here some basic facts about them that have been useful in this thesis.

The Levi-Civita Symbol in d dimensions is defined as the totally antisymmetric tensor $\epsilon^{\mu_1 \cdots \mu_d}$ where the indices run from 1 to d. It means that it is zero if two indices are equal and it is conveniently normalized as: $\epsilon^{1,\cdots,d} = 1$. The very basic fact about this tensor is its square, namely:

$$\epsilon^{\mu_1,\ldots,\mu_d}\,\epsilon_{\mu_1,\ldots,\mu_d} = d! \tag{A.1}$$

from the last equation, the total antisymmetry is sufficient to prove that:

$$\begin{aligned}\epsilon^{\lambda_1\ldots\lambda_{d-p},\mu_1,\ldots,\mu_p}\,\epsilon_{\lambda_1\ldots\lambda_{d-p},\nu_1,\ldots,\nu_p} &= \\ &(d-p)!\left\{\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2}\ldots\delta^{\nu_p}_{\mu_p} \pm \text{permutations}\right\}.\end{aligned} \tag{A.2}$$

We denote the part in curly brackets of the RHS of the last equation: $\epsilon^{\nu_1,\ldots,\nu_p}_{\mu_1,\ldots,\mu_p}$. We will call it bi-$\epsilon$ symbol. The trace of this symbol is:

$$\epsilon^{\mu_1,\ldots,\mu_p}_{\mu_1,\ldots,\mu_p} = \frac{d!}{(d-p)!}. \tag{A.3}$$

From this last formula one infer that:

$$\epsilon^{\sigma,\mu_2,\ldots,\mu_p}_{\rho,\mu_2,\ldots,\mu_p} = \frac{(d-1)!}{(d-p)!}\delta^{\sigma}_{\rho}. \tag{A.4}$$

So if an expression contains a total antisymmetry in some indices (denoted by round brackets), one use the symbol above, in the following manner:

$$A_{(\mu_1}B_{\mu_2}\dots Z_{\mu_p)} = A_{\sigma_1}B_{\sigma_2}\dots Z_{\sigma_p}\,\epsilon^{\sigma_1,\dots,\sigma_p}_{\mu_1,\dots,\mu_p} \tag{A.5}$$

Another important formula is the trace of the product of two bi-ϵ symbols:

$$\epsilon^{\nu_1,\dots,\nu_p}_{\mu_1,\dots,\mu_p}\,\epsilon^{\mu_1,\dots,\mu_p}_{\nu_1,\dots,\nu_p} = (p!)^2\binom{d}{p}. \tag{A.6}$$

When we trace only over $(p-1)$ indices in the above formula the result is:

$$\epsilon^{\sigma,\nu_1,\dots,\nu_{p-1}}_{\mu_1,\dots,\mu_p}\,\epsilon^{\mu_1,\dots,\mu_p}_{\rho,\nu_1,\dots,\nu_{p-1}} = (p!)\frac{(d-1)!}{(d-p)!}\delta^\sigma_\rho. \tag{A.7}$$

All those formulas allows one to compute also:

$$\epsilon^{\sigma\nu_1\dots\nu_{p-1}}_{\lambda\mu_1\dots\mu_{p-1}}\,\epsilon^{\rho,\mu_1\dots\mu_{p-1}}_{\gamma\nu_1\dots\nu_{p-1}} = K_1\delta^\sigma_\lambda\delta^\rho_\gamma + K_2\delta^\sigma_\gamma\delta^\rho_\lambda \tag{A.8}$$

where:

$$\begin{aligned} K_1 &= \frac{(p!)^2(d-p)}{pd(d-1)}\binom{d}{p} \\ K_2 &= \frac{(p!)^2(p-1)}{pd(d-1)}\binom{d}{p} \end{aligned} \tag{A.9}$$

Appendix B

Binomial coefficient: useful formulas.

In this appendix we state some useful formulas and two results that are encountered in the main text. First of all the Binomial coefficient is defined by the very well known expression:

$$\binom{d}{s} = \frac{d!}{s!(d-s)!} \tag{B.1}$$

To start we list some important formulas about the binomial coefficient, there are a lot of them. A more complete list could be found in [1].

$$\binom{n-1}{k} = \frac{n-k}{n}\binom{n}{k}, \tag{B.2}$$

$$\binom{n-1}{k} = -\frac{1+k}{n}\binom{n}{k+1}, \tag{B.3}$$

$$\binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}. \tag{B.4}$$

Next we want to prove two important results that are used throughout the thesis:

Proposition B.0.1.

$$\sum_{k=0}^{p}(-1)^k\binom{d}{p-k} = \binom{d-1}{p}. \tag{B.5}$$

Proof. We prove this statement by induction. It is certainly true for $p = 0$ (it yields 1). Now considering that it is true for a certain p, we show that it is also true for $p+1$. Indeed,

$$\begin{aligned}\sum_{k=0}^{p+1}(-1)^k\binom{d}{p+1-k} &= \binom{d}{p+1} + \sum_{k=1}^{p}(-1)^{k+1}\binom{d}{p+1-k}\\ &\binom{d}{p+1} - \sum_{k=0}^{p}(-1)^k\binom{d}{p-k}\\ &\binom{d}{p+1} - \binom{d-1}{p} = \binom{d-1}{p+1},\end{aligned} \tag{B.6}$$

in the second line we shifted by one the index of the sum and in the third we used (B.4).

□

Proposition B.0.2.

$$\sum_{k=0}^{p}(-1)^k(k+1)\binom{d}{p-k} = \binom{d-2}{p}. \tag{B.7}$$

Proof. Again, we prove this statement by induction. It is true for $p = 0$. So as we did for the last proposition,

$$\begin{aligned}\sum_{k=0}^{p+1}(-1)^k(k+1)\binom{d}{p+1-k} &= \binom{d}{p+1} + \sum_{k=1}^{p}(-1)^{k+1}(k+1)\binom{d}{p+1-k}\\ &\binom{d}{p+1} - \sum_{k=0}^{p}(-1)^k(k+1+1)\binom{d}{p-k}\\ &\binom{d}{p+1} - \sum_{k=0}^{p}(-1)^k(k+1)\binom{d}{p-k} - \binom{d-1}{p}\\ &\binom{d}{p+1} - \binom{d-2}{p} - \binom{d-1}{p}\\ &\binom{d-1}{p+1} - \binom{d-2}{p} = \binom{d-2}{p+1}\end{aligned} \tag{B.8}$$

in the second line we shifted the index by one and in the last we used repeatedly (B.4).

□

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