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Unusual corrections to scaling in excited states of conformal field theory

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Abstract

In questo lavoro abbiamo studiato la presenza di correzioni, dette unusuali, agli stati eccitati delle teorie conformi. Inizialmente abbiamo brevemente descritto l'approccio di Calabrese e Cardy all'entropia di entanglement nei sistemi unidimensionali al punto critico. Questo approccio permette di ottenere la famosa ed universale divergenza logaritmica di questa quantità. Oltre a questo andamento logaritmico son presenti correzioni, che dipendono dalla geometria su cui si basa l'approccio di Calabrese e Cardy, il cui particolare scaling è noto ed è stato osservato in moltissimi lavori in letteratura. Questo scaling è dovuto alla rottura locale della simmetria conforme, che è una conseguenza della criticità del sistema, intorno a particolari punti detti branch points usati nell'approccio di Calabrese e Cardy.
In questo lavoro abbiamo dimostrato che le correzioni all'entropia di entanglement degli stati eccitati della teoria conforme, che può anch'essa essere calcolata tramite l'approccio di Calabrese e Cardy, hanno lo stesso scaling di quelle osservate negli stati fondamentali.
I nostri risultati teorici sono stati poi perfettamente confermati dei calcoli numerici che abbiamo eseguito sugli stati eccitati del modello XX. Sono stati inoltre usati risultati già noti per lo stato fondamentale del medesimo modello per poter studiare la forma delle correzioni dei suoi stati eccitati. Questo studio ha portato alla conclusione che la forma delle correzioni nei due differenti casi è la medesima a meno di una funzione universale.

Contents

Introduction

In recent years the entanglement has been extensively studied in many very different fields of physics like quantum information theory, quantum many body systems, phase transitions and black hole theory.
From the birth of quantum mechanics the Entanglement between particles has been considered one of the most mysterious and peculiar characteristics of the theory. The presence of this strange "spooky action at distance", as Einstein called it, was described by Schrodinger as "the" nature of quantum physics, see Ref.[1], and actually it is still one of the most rapidly evolving research fields. Entanglement has been intensively studied both from the theoretical and the experimental point of view, where in the last years many results have been confirmed by experiments also on macroscopic scale. For example an experimental group created entanglement between pairs of particles on the scale of 143 Km, see Ref.[2]. In quantum computation, see Ref.[3] and Ref.[4], the use of entangled states can change the complexity class of the algorithms and this is impossible in classical computation theory. Two beautiful examples are the Quantum Fourier Transform, see Ref.[3], and the Shor's algorithm that allows to find prime numbers in a much more easy way with respect to the classical computation theory, for more information see Ref.[5]. Since the entanglement can be used to improve communications and computation protocols, many ways to quantify and describe the entanglement were defined and they are called entanglement measures, for a more complete discussion about this topic see Ref.[6] and Ref.[7]. For the case of bipartite entanglement, the entanglement that takes place between two parts of the same system, the most used ones are the von Neumann entropy and the Renyi entropy. The applications of these entanglement measure to many body physics are useful to have a deepen description of the ground state properties of the many body system and to describe what happens in a phase transition, for a complete review of entanglement in many body systems see Ref.[8], Ref.[9]. In particular the behaviour of 1D many body systems at the critical point is particularly interesting. In Ref.[10], Ref. [11] and Ref.[12] it is demonstrated that the entanglement entropy of the ground state of those systems takes an universal logarithmic behaviour that is one of the most amazing consequences of the concept of universality in physics. The logarithmic divergence can be obtained mapping the system under study on a complicated Riemann surface and then computing its partition function over it, see Ref.[13], this approach is called Replica Trick approach. The model on the Riemann surface can be mapped on the complex plane defining a new model called multi copy model, see Ref.[14], that is defined in a suitable way to preserve the locality of the theory. Furthermore the logarithmic behaviour of entanglement entropy receives geometry dependent corrections due to particular points defined on the Riemann surface, that are called branch points, and these corrections are called unusual corrections, see Ref.[15]. They have a scaling behaviour that is very different from the standard renormalization group one but they were observed in many theoretical and numerical works on lattice models such as spin chains, like Ref.[16],

Ref.[17] and Ref.[18]..
The aim of our work is to investigate the presence of unusual corrections also in the excited states of conformal field theory. In fact only the leading order behaviour of the Renyi entropy of the excited states is known, see Ref.[19], and it can be computed using the same Replica Trick approach valid for the ground state. A possible comment could be that unusual corrections should arise also in the excited state case, due to the geometrical structure of the Riemann surface used in both cases. In this work we will derive how unusual corrections arise in all details. We will analyse also the specific case of the Two dimensional Ising model where, using Ref.[20], it is possible to make explicit computations. The study of the corrections and their scaling properties will be performed using the same theoretical tools employed in the ground state case. At the end of our work we will present some numerical computations to confirm our predictions and to propose some possible extensions.
The structure of the work is the following one:

Chapter 1 In this chapter we will introduce the main concepts of the entanglement and quantum phase transitions. We will follow Ref.[21] and Ref.[22].

Chapter 2 is dedicated to the computation of the entanglement entropy of 1D quantum critical systems that exhibit conformal invariance using the Replica Trick method developed by Calabrese and Cardy in Ref. [10]

Chapter 3 We will extend the Replica Trick approach to the excited states of a generic conformal filed theory following Ref. [19], in the end we will restrict to the specific case of bosonic system that is the continuum limit of the XX spin chain.

Chapter 4 In this chapter we will expose how to extract the corrections to the leading order of entanglement entropy and in particular we will study their scaling as did Calabrese and Cardy in their Ref. [15].

Chapter 5 In this chapter we will extract the unusual correction from the excited state of conformal field theory and we will demonstrate that them can be extracted by the same argument used in the ground state case.

Chapter 6 This is the last chapter where we will present numerical computations that are in perfect agreement with our theoretical predictions.

Chapter 1

Entanglement in many body systems

The aim of this section is to present some important and general results about the entanglement in order to give a global view on the topic. There are many introductory review about entanglement and entanglement entropy, we choose to follow in the first part Ref. [21] and in the second part the book Ref.[22].
In the first part we will describe the general theory of entanglement, studying the situation of a system S divided in two subsystems A and B. We will analyse the situation from the point of view of the density matrix formulation of quantum mechanics introducing the reduced density matrix of the subsystems. We will also describe three methods commonly used to compute it and some explicit computations. For a more complete view about entanglement see also Ref.[9], focused on many body systems, or Ref.[7], concerning the quantum information point of view.
In the second part we will briefly describe the fundamental aspects of phase transitions. The physics of quantum systems at critical points is peculiar because the energy spectrum becomes gapless and the correlation length diverges. We will see in Chapt.2 how these physical facts in 1D quantum systems give us the famous *universal* logarithmic behaviour of entanglement entropy,see Ref.[13], Ref.[11] and Ref.[12].

1.1 The Schmidt decomposition

It is a well known fact that entanglement manifests itself when we divide a system S into two parts A and B where $S = A \cup B$.
We can write the wave function of the whole system S, called $|\Psi\rangle$, as a combination of the local basis $|\Psi_m^A\rangle$ and $|\Psi_n^B\rangle$ of the Hilbert spaces of the subsystems A and B:

$$|\Psi\rangle = \sum_{m,n} A_{m,n} |\Psi_m^A\rangle |\Psi_n^B\rangle . \tag{1.1}$$

The matrix $\mathbb{A}$ is a rectangular matrix, in general $\dim \mathcal{H}_A \neq \dim \mathcal{H}_B$, that is formed by the projections of $|\Psi\rangle$ on the local basis of $\mathcal{H}_A$ and $\mathcal{H}_B$.
It is a well known result from linear algebra that a rectangular matrix can be rewritten using the singular value decomposition as:

$$\mathbb{A} = \mathbb{U}\mathbb{D}\mathbb{V}', \tag{1.2}$$

where $\mathbb{U}$ is an unitary transformation, $\mathbb{D}$ is a diagonal matrix and $\mathbb{V}$ is a matrix with orthonormal rows.
Inserting 1.2 in 1.1 we can find a more suitable basis to write the state $|\Psi\rangle$ of S:

$$|\Phi_l^A\rangle = \sum_m U_{m,l} |\Psi_m^A\rangle \tag{1.3}$$

$$|\Phi_l^A\rangle = \sum_n V'_{l,n} |\Psi_n^B\rangle . \tag{1.4}$$

The state of S written as superposition of $|\Phi_l^A\rangle$ and $|\Phi_l^B\rangle$:

$$|\Psi\rangle = \sum_{l \leq \min\{m,n\}} \lambda_l |\Phi_l^A\rangle |\Phi_l^B\rangle , \tag{1.5}$$

where $\lambda_l \equiv D_{l,l}$. The normalization of the wave function $|\Psi\rangle$ imposes a constraint on the coefficients λ_l:

$$\langle\Psi|\Psi\rangle = \sum_l |\lambda_l|^2 = 1, \tag{1.6}$$

that holds for the property of the matrices $\mathbb{U}$ and $\mathbb{V}'$.
The entanglement properties are all encoded in the values λ_l because they give us a measure of the overlap between the quantum states of the two subsystems, physical effect responsible of entanglement.
In order to have a more clear vision of how the coefficients λ_i affect the entanglement we can analyse two extreme cases.
The first one is:

$$\lambda_l = \delta_{l,l_0}, \tag{1.7}$$

that gives us:

$$|\Psi\rangle = |\Phi_{l_0}^A\rangle |\Phi_{l_0}^B\rangle . \tag{1.8}$$

The state 1.8 is separable and in this case there is no entanglement between the two subsystems; a measure on the subsystem A will not affect a measure on the subsystem B.
The most different case is the following:

$$\lambda_n = \lambda \to \sum_l \lambda^2 = 1 \to \lambda = \frac{1}{\sqrt{N}}, \tag{1.9}$$

that defines the maximally entangled state:

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_l |\Phi_l^A\rangle |\Phi_l^B\rangle . \tag{1.10}$$

Generally it is very difficult to detect entanglement in a generic quantum state, for example in weak entangled states Ref.[7], and it is also difficult to define which states are entangled and which not. This is the reason why we can define the entanglement only using a non definition:
A state $|\Psi\rangle$ of the whole system S is *non entangled* if it could be written as a direct product of the states of the two subsystems:

$$|\Psi\rangle = |\varphi\rangle_A \otimes |\phi\rangle_B . \tag{1.11}$$

Every other state is called entangled and this means that it is possible to influence the state of the subsystem A performing local measurements on the subsystem B.

1.2 Reduced density matrix and entanglement entropy

We have studied the division of the whole system in two parts from the point of view of the wave functions of the system and subsystems. It is a well known fact that the quantum state of systems can be described by their density matrices and it is possible to rewrite the quantum mechanics using density matrices instead of wave functions.
If the whole system S is in a pure state defined by the vector $|\Psi\rangle \in \mathcal{H}_S$ its density matrix is defined as:

$$\rho = |\Psi\rangle \langle\Psi| . \tag{1.12}$$

The reduced density matrix of one of the two subsystems can be obtained simply by tracing out the degrees of freedom of the other one:

$$\rho_A = \mathrm{Tr}_B \rho \qquad \rho_B = \mathrm{Tr}_A \rho. \tag{1.13}$$

We can compute the density matrix of S using its Schmidt decomposition 1.5 and the reduced density matrix of the subsystem α becomes:

$$\rho_\alpha = \sum_l |\lambda_l|^2 |\Phi_l^\alpha\rangle \langle\Phi_l^\alpha| . \tag{1.14}$$

From this relation we can see that the Schmidt basis $|\Phi_l^\alpha\rangle$ is the one that diagonalize the reduced density matrix ρ_α with eigenvalues $w_l = |\lambda_l|^2$.
We can also notice from 1.14 that the reduced density matrix ρ_α describes a mixed state even if we started from a density matrix ρ that was in a pure state. The expectation value of a generic observable restricted to the subsystem A is obtained as usual for a generic mixed state:

$$\langle A_\alpha \rangle = \sum_n |\lambda_n|^2 \langle\Phi_n^\alpha| A_\alpha |\Phi_n^\alpha\rangle . \tag{1.15}$$

From 1.14 we can see that the reduced density matrix ρ_α is a hermitian operator, with positive eigenvalues and we can rewrite it as an exponential operator:

$$\rho_\alpha = \frac{1}{Z} e^{-\mathcal{H}_\alpha}, \tag{1.16}$$

where the $\mathcal{H}_\alpha$ is the entanglement hamiltonian, that is different from the hamiltonian H of the system, and the Z is a normalization factor that ensures the proper normalization of the reduced density matrix:

$$Z = \mathrm{Tr} e^{-\mathcal{H}_\alpha} \to \mathrm{Tr}\rho_\alpha = 1. \tag{1.17}$$

Another important property of the expression 1.14 is that the reduced density matrices ρ_A and ρ_B share the same non zero eigenvalues, it holds because the whole system S was in a pure state. This implies that their entanglement properties are the same and when we will define a measure of entanglement we have to take into account the existence of this particular symmetry.
In the Density Matrix Renormalization Group algorithm the properties of the eigenvalues w_l are used to truncate the Hilbert space selecting only their m higher values and the corresponding states $|\Phi_l^\alpha\rangle$. This method is expected to work only if the total weight of the erased spectrum is sufficiently small.
It is clear that, in order to obtain reasonable results from DMRG, it is necessary to have a knowledge, even only qualitative, of the behaviour of the spectrum w_n of the reduced density matrix.

Lest us assume to have computed in some way, numerical or analytic, the spectrum of reduced density matrix and that now, for example, we want to perform a measure of the entanglement encoded in the subsystem A. Since we know that all the information about the mixing between A and B lies in the coefficients w_n, we want to have a measure that it is only a function of these variables. In classical information theory there is a measure of the information that is the Shannon entropy. We can compute it for the weights w_l and use the result as entanglement measure called *entanglement entropy*:

$$S_A = -\sum_n w_n \ln w_n. \tag{1.18}$$

This expression is a function only of the eigenvalues of the reduced density matrix, that implies $S_A = S_B$ for arbitrary bipartitions of system S in a pure state. Although there are many other entanglement measures, see Ref.[6] for a more complete list of them, but the entanglement entropy is the most used for bipartite entanglement. We have to keep in mind that 1.18 measures a *mutual connection* between subsystems and it is not proportional to their size. We can compute the entanglement entropy for 1.10 and 1.8. In the 1.8 we have the weights $w_n = \delta_{n,n_0}$ and the entanglement entropy vanishes. In the opposite case of 1.10, $w_n = 1/M$ for $n = 1, ..., M$, we have the maximum of the entropy: $S = \ln M$.
The last statement allows us to define an effective number of coupled states M_{eff} that describes the entanglement entropy as:

$$S \equiv \ln M_{\text{eff}}. \tag{1.19}$$

We can rewrite the entropy as a functional of the reduced density matrix using the spectral theorem:

$$S = -\text{Tr}_A \rho_A \ln \rho_A, \tag{1.20}$$

that is formally equal to the definition of thermal entropy in statistical mechanics. This similarity is only apparent since it has a completely different behaviour with the system size.
In fact the thermal entropy is a measure of absence of information due to the fact that many microstates can give the same macrostate of the system.
In general this quantity scales with the number of microstates that can be approximated with the volume of phase space accessible to the system. The phase space is a direct product between the space of momenta and the real space accessible at the mechanical system so it is natural that the thermal entropy scales with the direct volume accessible to the system.
In conclusion we have that the thermal entropy of a system of typical length l in d dimensions scales as:

$$S_T \sim l^d. \tag{1.21}$$

The situation is very different if we look at the entanglement entropy where the quantity that play the key role is the surface that divides the two subsystems: this will lead to an area law behaviour instead of a volume law.
We can rewrite this statement in a more formal way: the system S, in generic dimensions d, is divided in two parts A and B. In general for a *non critical* quantum system described by a local quantum field theory we expect that the entangled degrees of freedom are the ones placed near the surface that separates A and B. This simple analysis implies that entanglement entropy scales with the *area* of the surface that divides the subsystems, this means that in generic dimensions d it has to scale as:

$$S_L \sim c_1 \left(\frac{l}{\epsilon}\right)^{d-1}, \tag{1.22}$$

where l is the size of the subsystem and l^d is its volume and ϵ is a cutoff. For a more complete discussion of area law and its implication on physical systems see Ref.[23].
For $d = 1$ the precedent expression would lead us to a bounded entanglement entropy, but we will see that in some cases there are violations of the previous argument. These violations take place at critical points where the correlation lengths of the theory is divergent and the system is more correlated than the non critical case.

1.3 Reduced Density Matrix for free lattice models

In this section we will expose three methods that are commonly used to compute explicitly the reduced density matrices.
We will focus our attention only on very simple Hamiltonians that are quadratic in the bosonic/fermionic creation/annihilation operators and they can be diagonalized using a Bogoliubov transformation. This is a very large class of models and we will use only the most physically relevant ones:

- Fermionic hopping models with conserved number of particles:

$$H = -\frac{1}{2}\sum_{\langle m,n\rangle} t_{m,n} c_m^\dagger c_n, \tag{1.23}$$

  where the symbol $\langle \dots \rangle$ means that the sum is restricted to the nearest neighbors.

- Coupled oscillators with eigenfrenquency ω_0 and hamiltonian:

$$H = \sum_n \left[-\frac{1}{2}\frac{\partial^2}{\partial x_n^2} + \frac{1}{2}\omega_0^2 x_n^2\right] + \frac{1}{4}\sum_{\langle m,n\rangle} k_{m,n}(x_m - x_n)^2. \tag{1.24}$$

- spin one-half models that can be mapped into free fermionic models using the Jordan-Wigner transformation, as the XY model:

$$H = -\sum_n \left[\frac{1+\gamma}{2}\sigma_n^x\sigma_{n+1}^x + \frac{1-\gamma}{2}\sigma_n^y\sigma_{n+1}^y\right] - h\sum_n \sigma_n^z, \tag{1.25}$$

  where the σ^α are the Pauli matrices.
  If we set $\gamma = 0$ the model becomes the XX spin chain, that will be studied intensively in Chapt. 6, on the other side if we set $\gamma \neq 0$ the model admits the creation and annihilation of pairs. The case of $\gamma = 1$ describes the Ising model in external magnetic field that could be written rescaling the unit of energy as the coupling between nearest neighbors:

$$H = -\sum_n \sigma_n^z - \lambda\sum_n \sigma_n^x\sigma_{n+1}^x. \tag{1.26}$$

In general it is very difficult to compute the reduced density matrix starting from the density matrix of the whole system S. Sometimes there are particular symmetries or properties of the system, like integrability, that can help us in this task.
As we saw in the last section 1.14, we can write the reduced density matrix of the subsystem as an exponential of the entanglement hamiltonian:

$$\mathcal{H}_\alpha = \sum_{i=1}^{l} \epsilon_i f_i^\dagger f_i, \tag{1.27}$$

where l is the length of the subsystem α, that could be A or B, and the operators $f_l^\dagger$ and f_l are fermionic or bosonic and they correspond to single particle states with eigenvalues ϵ_l. The important fact is that the entanglement hamiltonian $\mathcal{H}_\alpha$ and the true hamiltonian of the system H are functions of the same creation/annihilation operators.
Now we can expose three of the most used methods to obtain the reduced density matrix of a subsystem.
The first one is an application of the definition of reduced density matrix: we obtain the reduced density matrix of the subsystem A by tracing out the degrees of freedom of the subsystem B from the density matrix of the whole system S.
This can be done explicitly for example in a system of N coupled harmonic oscillators in their ground state, see Ref.[24] and Ref.[25]. We know from quantum mechanics that the wave function of the ground state of a single harmonic oscillator is a gaussian function. This allows us to write the ground state of the whole system as:

$$\Psi(x_1, x_2, \ldots, x_n) = C \exp\left(-\frac{1}{2}\sum_{m,n}^{N} A_{m,n} x_m x_n\right), \tag{1.28}$$

$A_{m,n}$ is defined as the square root of the matrix associated to the potential energy:

$$\mathbb{A}^2 = \mathbb{V} \rightarrow A_{m,l} A_{l,n} = V_{m,n}. \tag{1.29}$$

From the wave function we can easily derive the density matrix ρ of the system and then we can integrate the degrees of freedom of the variables $x_{l+1}, x_{l+2}, \ldots, x_N$ that will leave the reduced density matrix ρ_A of the subsystem composed by the first l sites. Since we integrate over a great number of gaussian functions we expect that the result would be still a gaussian function in remaining variables.
We can define linear combinations of the remaining variables, namely y_l and y_l', in order to have only combinations of the type y_l^2, $(y_l')^2$ and $(y_l - y_l')^2$. We can write the differences $(y_l - y_l')^2$ as second derivatives with respect to the variables obtaining the reduced density matrix of the subsystem:

$$\rho_A = K \prod_{i=1}^{l} \exp\left(-\frac{1}{4}\omega_i^2 y_i^2\right) \exp\left(\frac{1}{2}\frac{\partial^2}{\partial y_i^2}\right) \exp\left(-\frac{1}{4}\omega_i^2 y_i^2\right). \tag{1.30}$$

The exponent can be recasted in a quadratic expression of bosonic creation/annihilation operators and from its diagonalization can be found that the entanglement hamiltonian is simply a collection of l harmonic oscillators.
The eigenvalues ϵ_i can be obtained from the submatrices of $A_{m;n}$, namely: $a^{AA}, a^{AB}, a^{BA}, a^{BB}$, where the labels A and B describe the position of the site.
We can define the $l \times l$ matrix:

$$a^{AA}\left(a^{AA} - a^{AB}(a^{BB})^{-1} a^{BA}\right)^{-1}, \tag{1.31}$$

that can be diagonalized since it is hermitian. Its eigenvalues are functions of the entanglement eigenvalues: $\coth^2(\epsilon_l/2)$. The last statement gives us all the entanglement properties of the subsystem under study and it ends the demonstration.
The procedure exposed above works also in the case of fermionic systems where we have to replace real valued variables with Grassman ones, see Ref.[26] and Ref.[27] for more details.

In the case of a free fermionic system at zero temperature there is a very easy way to find the eigenvalues of the reduced density matrix from the correlation matrix (see Ref.[28],

Ref.[12] and Ref.[29]) and it avoids the explicit computation of the reduced density matrix starting from the density matrix of the system. This method will be also used in the Chapt.6 to compute the entanglement entropy of the excited states of the XX chain.
The ground state of an N particles Fermi system at zero temperature is a Slater determinant of single particle wave functions. In such state all the many particles correlation functions factorize into products of one particle functions as:

$$\langle a_m^\dagger a_n^\dagger a_k a_l \rangle_0 = \langle a_m^\dagger a_l \rangle_0 \langle a_n^\dagger a_k \rangle_0 - \langle a_m^\dagger a_k \rangle_0 \langle a_n^\dagger a_l \rangle_0, \tag{1.32}$$

where $\langle\rangle_0$ indicates an expectation value taken over the ground state of the system.
If all the sites are in the same subsystem the previous expression can be computed using the reduced density matrix and the definition of expectation value. Wick's theorem forces the reduced density matrix to be in the form:

$$\rho_\alpha = K \exp\left(- \sum_{i,j} H_{ij} c_i^\dagger c_j \right), \tag{1.33}$$

where i and j are sites in the subsystem. The hopping matrix H_{ij} is determined imposing that it gives the correct correlation matrix:

$$C_{ij} = \mathrm{Tr}\left(c_i^\dagger c_j \rho_\alpha \right). \tag{1.34}$$

The matrices $\mathbb{C}$ and $\mathbb{H}$ are diagonalized by the same transformation and they satisfy the formal relation:

$$\mathbb{H} = \ln\left(\frac{\mathbf{1} - \mathbb{C}}{\mathbb{C}} \right). \tag{1.35}$$

From the spectral theorem we have that the same relation holds for the eigenvalues ζ_l of $\mathbb{H}$ and ϵ_l of $\mathbb{C}$. In particular they satisfy:

$$(\mathbf{1} - 2\mathbb{C})\, \varphi_l = \tanh\left(\frac{\epsilon_l}{2} \right) \varphi_l. \tag{1.36}$$

This approach is valid also in presence of creation and annihilation of pairs described by the matrices:

$$\mathbb{F}_{i,j} = \langle c_i^\dagger c_j^\dagger \rangle, \qquad \mathbb{F}^*_{i,j} = \langle c_i c_j \rangle. \tag{1.37}$$

Obviously the terms responsible of generation and annihilation of pairs have to be present also in the entanglement hamiltonian. In the case of a real valued matrix $\mathbb{F}$ we can obtain an equation satisfied by the hopping matrix in the entanglement hamiltonian:

$$(2\mathbb{C} - \mathbf{1} - 2\mathbb{F})\,(2\mathbb{C} - \mathbf{1} + 2\mathbb{F})\, \varphi_l = \tanh^2\left(\frac{\epsilon_l}{2} \right) \varphi_l. \tag{1.38}$$

The eigenvalues of the entanglement hamiltonian, and the entanglement entropy, can be found from the solution of the previous equation.
This method is extremely flexible and it can be use to explore many different physical situations like the presence of defects in the system, random systems and systems at finite temperature.
The main assumptions are:

- The ground state can be written as a Slater determinant

- The Hamiltonian of the system H is quadratic in the creation/annihilation fermionic operators

The third way is to relate the model to some statistical solvable model, Ref.[30] and Ref.[31] for a more complete discussion about this topic. In general a spin system in d dimensions can be related to a classical $d+1$ dimensional statistical model using the transfer matrix.
In a the case of a finite length spin chain we can obtain its quantum state $|\Psi\rangle$ starting from a generic state $|\Psi_s\rangle$ and then applying n times a proper operator T. A suitable choice of this operator T is the row-to-row transfer matrix, that is used in the solution of the classical Ising model. In this situation we create a relation between the quantum state $|\Psi\rangle$ and the partition function of a semi-infinite strip of that classical system, the corresponding density matrix is computed using two strings of operators T.
If we want to compute the reduced density matrix for the first part of the system, for example, we have to trace out the degrees of freedom of the subsystem B, and it is simply done by sewing together its edges and leaving an open cut in presence of the subsystem A, as it is represented in Fig.1.1. In this situation ρ_α is the partition function of a system that is a full strip with a perpendicular cut.
This method works for a very large class of systems, here are some examples:

- the transverse Ising chain is correlated to a two-dimensional Ising model on a rotated square lattice (Ref.[31])
- the chain of coupled oscillator studied above is correlated to a two-dimensional Gaussian model (Ref.[24])
- the XY spin chain is correlated to an Ising model on a triangular lattice (Ref.[32])
- XXZ and XYZ and other higher spin chains are correlated to different vertex models (Ref.[33],Ref.[31] and Ref.[34])

The problem of this approach is the explicit computation of the partition function of the statistical model. This can be done using the corner transfer matrices that are partition functions of the quadrants represented in Fig. 1.1.

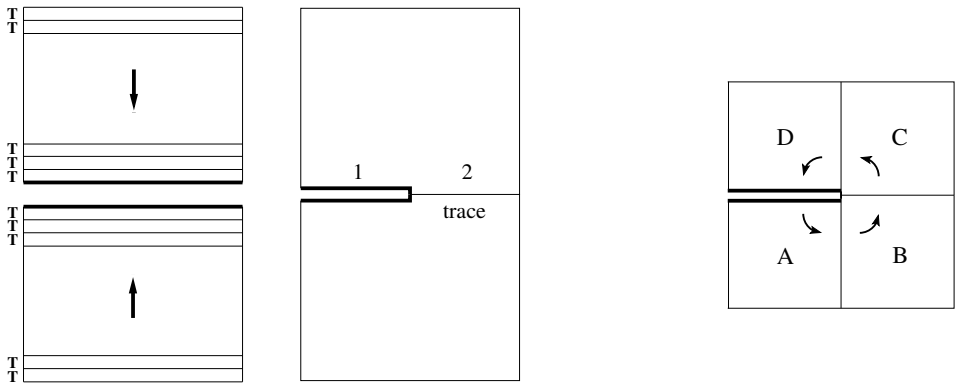

Figure 1.1: *Left: Density matrices for a quantum chain seen as a two dimensional partition function. On the far left we have an expression of the total density matrix ρ as a consecutive application of a large number of operators T. The central figure is the expression of the reduced density matrix ρ_1, it is represented by the open cut. On the right we have a partition function seen as composed by four corner transfer matrices A,B,C,D. This figure has been taken from Ref.[21]*

If the subsystem under study is half of an infinite system, like in Fig.1.1, it is possible to compute explicitly its reduced density matrix as an ordinate product of infinite size corner transfer matrices of quadrants:

$$\rho_\alpha \sim ABCD. \tag{1.39}$$

In particular, for infinite systems the form of the corner transfer matrices is known:

$$A = e^{-u\mathcal{H}_{\text{CTM}}}, \tag{1.40}$$

that is a consequence of the star triangle relation of integrability introduced by Baxter in Ref.[35].
This approach gives the entanglement hamiltonian $\mathcal{H}_\alpha$ and an explicit expression for its eigenvalues ϵ_l. We have described this approach in the case of an infinite system divided in two half-infinite subsystems, but the derivation is still valid in the case of finite systems of length L much larger than the correlation length ξ, we are in the non critical case and this quantity is finite.
In this section we exposed the most used techniques to obtain the entanglement hamiltonian and its eigenvalues. Unfortunately in many cases it is impossible to compute analytically the ϵ_l and then we have to proceed numerically. Even in numerical computations some problems arise: the ϵ_l approach very fast to the values ± 1 when the number of sites in the subsystem increases. This creates many problems in the correct evaluation of physical quantities. These problems can be avoided only using sophisticated numerical techniques, but the computation is possible only if the number of sites is smaller than 100. We will see in Chapt.2 that in the case of a critical 1D quantum system there is another, and very elegant, way to compute the entanglement entropy. This method takes into account that at the critical point the system becomes scale invariant and this will lead us to the famous logarithmic divergence of the entanglement entropy found in Ref.[13], Ref.[11] and Ref.[10].

1.4 Quantum phase transitions

In all the thesis we will study one dimensional quantum systems, even if some arguments can be extended to arbitrary dimensions. In particular we will focus on *critical* 1D quantum systems and now we briefly describe the physical background of this situation.
A physical system is described by a hamiltonian depending on a generic number of parameters that regulates the microscopic behaviour of the quantum system. The values taken by the parameters can put the system in macroscopic states with very different properties and these different macroscopic states are called *Phases*. A system can have a number of phases larger than one and this means that there is at least one combination of the parameters shared by two different phases, called *critical line*.
There are some mathematical objects that can be used to detect and distinguish between different phases of the system: they are called *order parameters*. An order parameter is a quantity that is zero, even statistically, outside its phase and it differs from zero inside it. This particular behaviour of the order parameters is due to a more deeper concept: the spontaneous symmetry breaking. In the spontaneous symmetry breaking we have a microscopic hamiltonian that is invariant under some generic symmetry but it has some low energy states, like the ground state, that explicitly violate the symmetry itself.
A very famous example where the spontaneously symmetry breaking occurs is the Ising model. It is the simplest model developed to study the magnetic properties of matter and it has been studied in many works in the literature. It is described by a global $\mathbb{Z}_2$ invariant

hamiltonian:

$$H = -J \sum_{\langle i,j \rangle} S_i S_j, \tag{1.41}$$

where S_i are classical variables that range in the set $\{\pm 1\}$, they should behave like a classical version of quantum spins and they interact like little magnetic dipoles. The notation $\langle i, j \rangle$ stands for the nearest neighbour sum. We will assume that the coupling constant J is positive and this will support the ferromagnetic phase instead of the paramagnetic.
The hamiltonian 1.41 is trivially invariant under the simultaneous flip of all spins, namely $\{\pm 1\} \to \{\mp 1\}$. In fact the energy of a certain spin configuration is determined by the relative positions of the spins, we can flip them all and we will obtain another configuration that has the same energy of the first one.
We can easily find the ground state of the system since the energy is lower for states with a great number of spins aligned along the same direction: up direction is $\{+1\}$ and down direction is $\{-1\}$. It is obvious that the two lower energy configurations are the two with all the spins aligned along the directions ± 1.
Both these configurations are not invariant under the action of the group $\mathbb{Z}_2$, the one with all up spins goes to the one with all down spins and vice versa: they are connected by the action of the group.
Obviously we can be in one of these configurations only if the temperature is sufficiently low, in the specific case of 1D Ising model only at $T = 0$, because they are two extremely ordered configurations. We know that at high temperature the favourite configurations are the disordered ones.
In the Ising model the order parameter is simply the magnetization along the direction of the spins: in the high temperature limit it is statistically zero, the number of up and down spins are equal and their contributions cancel out. When we are in the ordered phase the macroscopic magnetization has a non zero value along a direction, $\{\pm 1\}$ depending on the case, because there is a macroscopic number of spins aligned along the up or down direction. We saw above these two cases are connected by the action of the invariance group $\mathbb{Z}_2$.
In general, when we are in the high temperature regime, the system is completely described by the classical statistical mechanics and it has classical fluctuations around the mean value of its observables. In this regime we have an energy scale that is $K_B T$ where T is the absolute temperature and a length scale that is λ_T , called thermal wavelength. The intrinsic quantum nature of matter is not evident here but, when we approach the low temperature regime $T \approx 0$, the mean values of observables are affected by a completely different kind of fluctuations that are the *quantum* ones.
In the low temperature regime the spectrum of low energy excitations has typical frequency ω and the typical energy scale $\hbar\omega$. The quantum fluctuations are described by two scales: the correlation length ξ and the correlation time τ. Both of them diverge with a power law behaviour approaching the critical line.
The main parameter for a system near a critical point, occurring at critical temperature T_C, is the reduced temperature defined as:

$$t = \frac{T - T_C}{T_C}, \tag{1.42}$$

we are at the critical point when $t = 0$, in an ordered phase when $t < 0$ and in a disordered phase when $t > 0$.
We can parametrize the divergences of ξ and τ as:

$$\xi \sim |t|^{-\nu} \qquad \tau \sim |t|^{-z\nu}. \tag{1.43}$$

The correlation time defines an energy scale Δ that is called gap energy:

$$\Delta \sim \frac{1}{\tau} \sim |t|^{z\nu}, \tag{1.44}$$

this gap in the spectrum tends to zero approaching the critical point, at the critical point the system is called *gapless*.
Using the definition of the correlation length 1.43 we have the simple relation:

$$\Delta \sim \frac{1}{\xi^z}. \tag{1.45}$$

In general at the critical point the correlation length is divergent and the spectrum is gapless, that has a lot of strong consequences in the physics of critical systems.
The classical fluctuations are totally suppressed only at zero temperature and the only fluctuations present are the quantum ones. Obviously there are some problems with this picture because experimentally we cannot reach the zero temperature limit, fortunately all the previous considerations can be extended to a region where the quantum fluctuations and the thermal energy are comparable, namely:

$$K_B T \sim \Delta, \tag{1.46}$$

that defines the quantum critical region.
In a system at $T = 0$ we have that the correlation length is not dependent on the temperature but on the set of parameters present in the hamiltonian. In the simple case of a single parameter g we can parametrize the gap Δ in this region as:

$$\Delta \sim |g - g_c|^{\nu z}, \tag{1.47}$$

where g_c is the critical values of the parameter g, where the transition occurs. Since we are at $T = 0$ the behaviour of the system is completely described by quantum mechanics and the transition that takes place at g_c is called *quantum phase transition*. We can now define the region called *quantum critical* as the region where the gap is comparable to the thermal energy:

$$K_B T \sim |g - g_c|^{\nu z}. \tag{1.48}$$

In the following chapters we will study principally 1D systems that are very important in condensed matter physics because their behaviour is peculiar compared to the higher dimensions ones. In particular we have that in 1D the Mermin Wagner Hohenberg theorem holds: in $d \leq 2$ spatial dimensions and at finite temperature a system with sufficiently short range interaction cannot spontaneously break any continuous symmetry of its hamiltonian, see Ref.[36], Ref.[**?**]. This means that using a little energy cost, a great number of long-range fluctuations can be created. These fluctuations are favored because they increase the entropy of the system. This creates at low temperature a strange mixture of order and disorder and it is called quasi-long-range order and it is described by a power law decay of correlation functions.
The absences of a length scale and the power law decay of the correlation function are characteristic of critical one dimensional systems that can be described properly using conformal field theory. This has very important consequences for the entanglement entropy of 1D critical systems that will lead to the famous universal behaviour:

$$S \sim \alpha \ln l. \tag{1.49}$$

α is an universal constant that will be fixed in Chapt.2 and its specific geometrical meaning will be clear. This expression was obtained in very different contexts in Ref.[13], Ref.[10], Ref.[11] and Ref.[12]. The divergence of the correlation length at the critical point has an important consequence: the logarithmic scaling of the entanglement entropy. In fact all the degrees of freedom of the system are connected and not only the ones around the surface that separates the two subsystems. The area law argument is still valid away from criticality meaning that we have an entanglement entropy saturation. In dimensions higher than one we have that at critical point the area law is an understimation of the entanglement entropy of the system. In general there are violations of the area law but they depends on the model and it is not always true that corrections are dominant with respect to power law behaviour. For example we could have a violation of the type:

$$S_l \sim (l)^{d-1} \ln l. \tag{1.50}$$

The scaling behaviour is known for some specific models like critical fermions in $d = 2$, Ref.[37], that is:

$$S_l \sim l \ln l. \tag{1.51}$$

All the violations of the area law in $d > 1$ dimensional systems are model dependent. In the 1D case the violation of the area law is universal as we will see in the next chapter and it is one of the beautiful example of the concept of universality that takes place in the 1D world.

Chapter 2

Entanglement entropy in conformal field theory

In this chapter we will focus our attention on critical 1D quantum systems, even if some part of our approach can be extended to quantum systems in generic dimensions.
As we have seen in the previous chapter, in critical systems we have a divergent correlation length that motivates the scale invariance symmetry and, most important, that these systems violates the area law of the entanglement entropy, that holds for non critical systems.
The 1D case is the most beautiful case of violation of area law, the entanglement entropy takes the famous (see Ref.[13], Ref.[29] and Ref.[11]) logarithmic **universal** form :

$$S_l = \frac{c}{3} \ln l + \gamma. \tag{2.1}$$

This logarithmic divergence is completely model independent, the model enters only in the central charge c that multiply the divergence.
We will compute the entanglement entropy for a generic 1D system that exhibit conformal invariance using the Replica Trick, see Ref.[13], approach described below.
The expression 2.1 has become one of the most used expression in the entanglement theory since it is the natural way to find out the central charge of a generic model and that gives us a great number of information about it like the universality class. We will follow Ref.[10] where the conformal field theory approach to 1D critical quantum system is presented together with other interesting results.

2.1 Entanglement and entanglement measures

Let be ρ the density matrix of a system S that is assumed to be in a pure state $|\Psi\rangle$, its density matrix is:

$$\rho = |\Psi\rangle \langle\Psi| . \tag{2.2}$$

We can divide S in two parts A and B in this way:

$$S = A \cup B, \tag{2.3}$$

and the Hilbert space can be written as:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{2.4}$$

The reduced density matrix of the subsystem A is obtained as usual by tracing out the degrees of freedom of B:

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\rho. \tag{2.5}$$

We can define the entanglement entropy as the von Neumann entropy associated to the reduced density matrix ρ_A:

$$S_A = -\mathrm{Tr}_{\mathcal{H}_A}\rho_A \ln \rho_A. \tag{2.6}$$

It is important to notice that, since S is in a pure state, the entanglement entropy of the subsystem A is equal to the entanglement entropy of the subsystem B, as we said before.
It is useful to define a family of functionals of the reduced density matrix called Renyi entropy, see Ref.[38], labelled by an integer number n:

$$S_A^{(n)} = \frac{1}{1-n}\ln \mathrm{Tr}\rho_A^n. \tag{2.7}$$

The reason why we introduced the Renyi entropy is the following: although the von Neumann entropy is a good measure of the entanglement between the two subsystems in many cases it is very difficult to compute. On the other side there is a very elegant way to compute the Renyi entropy for a generic index n as we will see below. Furthermore the Renyi entropy is a very important quantity for quantum information since it takes into the accounts all the correlations, also the long range ones, inside a many body system.
Assuming to have computed the Renyi entropy as function of n, we can easily obtain the von Neumann entropy taking the limit:

$$S_A = \lim_{n\to 1^+} S_A^{(n)}. \tag{2.8}$$

When the system S is in a mixed state, for example S is in a thermal state, the entanglement entropy is no longer a good measure of entanglement because we have classical and quantum correlations in the system. This means that the relation between the Renyi entropy $S_A^{(n)} = S_B^{(n)}$ does not hold anymore.
In this case from the Renyi entropies we can define the *mutual information* of the system:

$$I_{A:B}^{(n)} = S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)}, \tag{2.9}$$

this quantity is, by definition, symmetric in A and B. It has the property of satisfying the area law of entanglement, see Ref.[39] and Ref.[23], also at finite temperature, see Ref.[40]. On the other hand the mutual information is not a good estimator of entanglement as demonstrated in Ref.[8].

2.2 Replica trick approach

As we have seen in the first chapter the eigenvalues of the reduced density matrix ρ_A are very difficult to obtain analytically. In many cases, in order to find its eigenvalues λ_i, we are forced to use numerical methods: the most used one for 1D systems is the Density Matrix Renormalization Group (DMRG), that we briefly described in the Chapt.1.
Once we have obtained the eigenvalues of the reduced density matrix, the von Neumann entropy can be evaluated using the spectral theorem:

$$S_A = -\sum_i \lambda_i \ln \lambda_i. \tag{2.10}$$

The main problem is that the eigenvalues of the reduced density matrix ρ_A are very difficult to compute, also numerically there are some great problems.
In order to avoid these problems we will follow a different way that is reminiscent of the Replica Trick approach, Ref.[13]. This method is described in details in literature, see like Ref.[11], Ref.[41] and Ref.[42], but we will review it another time here because it is crucial for all the results in this thesis.
By definition, all the eigenvalues of the density matrix lie in the interval $[0;1]$ and they sum to 1. This fact implies that the series $\sum_i \lambda_i^n$ is absolutely convergent and analytic in the region $\mathrm{Re}\, n > 1$. We can take also the derivative with respect to n treated as a real variable intead of an integer one:

$$-\lim_{n\to 1^+}\frac{\partial}{\partial n}\mathrm{Tr}\rho_A^n = -\lim_{n\to 1^+}\mathrm{Tr}\rho_A^n \ln\rho_A = \lim_{n\to 1^+} S_A^{(n)} = S_A. \tag{2.11}$$

This result shows that if we are able to compute the $\mathrm{Tr}\rho_A^n$ for every real parameter n we can get the von Neumann entropy simply taking its derivative and then the limit to 1. Unfortunately the computation of $\mathrm{Tr}\rho_A^n$ for a generic index n and for a generic system is an almost unsolvable problem due to its complexity. We will use a method similar to the Replica-trick approach discovered by Holzey in Ref.[13], and we will compute $\mathrm{Tr}\rho_A^n$ as a partition function of a system living on a complicated Riemann surface that we will define soon. Then the main problem will be to show if the expression obtained can be analytically continued to $n = 1^+$.

2.3 The Riemann surface

In this section we will show how to define the Riemann surface in a proper way.
We will describe this method in the case of an 1D critical quantum system but it works anyway in arbitrary dimensions.
Let be x the variable, discrete, that labels the sites of the lattice, with spacing a. The length of the system is called L and it can be finite, infinite or semi-infinite. We will work in continuous time.
Let be $\{\hat{\phi}_x\}$ a set of complete commuting observables with eigenvalues $\{\phi_x\}$ and eigenstates $|\{\phi_x\}\rangle$. A complete basis of the Hilbert space of the system is given by the states:

$$\bigotimes_x |\{\phi_x\}\rangle = |\prod_x \{\phi_x\}\rangle\,. \tag{2.12}$$

The matrix elements of the density matrix in this basis are:

$$\rho\left(\{\phi_x\} \mid \{\phi'_{x'}\}\right) \equiv \langle \prod_x \{\phi_x\}|\,\rho\,|\prod_{x'}\{\phi'_{x'}\}\rangle\,. \tag{2.13}$$

We start our analysis from the density matrix of a system in a thermal state of inverse temperature $\beta = 1/T$:

$$\rho = \frac{e^{-\beta H}}{Z(\beta)}, \tag{2.14}$$

where H is the hamiltonian of the system that describes its time evolution, and $Z(\beta)$ is the partition function that ensures the proper normalization of the density matrix:

$$Z(\beta) = \mathrm{Tr}e^{-\beta H}. \tag{2.15}$$

Equation 2.13 can be rewritten using the path integral formulation of quantum mechanics:

$$\rho\left(\{\phi_x\} \mid \{\phi'_{x'}\}\right) = Z(\beta)^{-1} \int [d\phi(y;\tau)] \prod_{x'} \delta\left(\phi(y;0) - \phi_{x'}\right) \prod_{x} \delta\left(\phi(y;\tau) - \phi_x\right) e^{-S_E}, \quad (2.16)$$

where S_E is the euclidean action defined using the euclidean lagrangian $\mathcal{L}_E$:

$$S_E(\beta) = \int_0^\beta d\tau \mathcal{L}_E. \quad (2.17)$$

In the path integral formulation we can compute the partition function setting $\{\phi_x\} = \{\phi_{x'}\}$, that corresponds to the trace operation, and then integrating over the field. This operation can be seen as sewing together the initial and finial states creating a cylinder of circumference β and integrating over it.
This visual scheme for the computation of the partition function is useful in order to have a picture of what happens every time we evaluate a trace of a generic quantity in the path integral formulation.
A generic subsystem A of an 1D system is composed by all the x coordinates in the disjoint intervals $(u_1; v_1) \dots (u_N; v_N)$. The trace over the degrees of freedom of B is made sewing together the x in B and not the x in A. This procedure leaves open cuts in correspondence of every interval $(u_i; v_i)$ along the line $\tau = 0$ of the cylinder described above for the partition function.
Lets assume now to create n copies of the system on study and gluing them cyclically along the cuts corresponding to $x \in A$ in this way:

$$\phi_j(x; \tau = \beta^-) = \phi_{j+1}(x; \tau = 0^+) \qquad \forall j x \in A, \quad (2.18)$$

$$\phi_1(x; \tau = \beta^-) = \phi_n(x; \tau = 0^+) \qquad \forall x \in A, \quad (2.19)$$

this defines an n-sheeted structure that is represented in Fig.2.1:

Figure 2.1: *Riemann surface of the case $\mathcal{R}_{3,1}$. Figure from Ref.[10]*

The partition function over this surface is called $Z_n(A)$ and it is possible to define $\mathrm{Tr}\rho_A^n$ as:

$$\mathrm{Tr}\rho_A^n = \frac{Z_n(A)}{Z^n(A)}. \quad (2.20)$$

From this expression we can compute the von Neumann entropy as in 2.11:

$$S(A) = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}\rho_A^n = -\lim_{n\to 1}\frac{\partial}{\partial n}\frac{Z_n(A)}{Z^n(A)}. \tag{2.21}$$

Now we have to extend this approach, derived for a lattice system, to a continuous system. We start from a lattice theory and we take the continuum limit defined as $a \to 0$ keeping other lengths fixed. The variable x becomes real valued after the limit and we have to integrate the values of the fields $\phi(x;\tau)$ over a n-sheeted Riemann surface with branch points at u_i and v_i. The Riemann surface is fully defined by two values:

- $2N$: the number of branching points labelling the disjoint intervals that form the subsystem A
- n: the number of copies

and we will denote the most generic Riemann surface by $\mathcal{R}_{n;N}$.
The most interesting case is a quantum field theory described by a Lorentz invariant action, in this case the adimensional free energy of the system in two dimensions is well studied and it takes the form:

$$\ln Z = f_1\mathcal{A}a^{-2} + f_2\mathcal{L}a^{-1} + \dots, \tag{2.22}$$

where $\mathcal{A}$ is the area of the domain and $\mathcal{L}$ is length of its boundary; f_1 and f_2 are non universal bulk and boundary free energy.
In the case of a conformal field theory there are also **universal** terms proportional to $\ln a$, see Ref.[43]. This is due to the non-zero curvature of the Riemann surface around the branch points that breaks locally the conformal invariance.
This local non zero curvature will be responsible of other important physical effects as the unusual corrections to the scaling as we will see in Chapt.4.

2.4 From the Multi copy model to the Twist fields

The calculation of the partition function over the Riemann surface is, in many cases, an unachievable task but there is a way to solve the problem mapping it into a simplest one.
Since the Riemann surface has zero curvature everywhere except around the points v_i and u_i, that are a finite number, we can try to map the problem into the complex plane and we can choose proper boundary conditions around these points with non zero curvature in order to implement the structure of the Riemann surface.
This is possible because the lagrangian density $\mathcal{L}$ is local and it is not affected by the global structure of $\mathcal{R}$.
Let now take the specific case of a single interval $[u_1;v_1]$ which generates the Riemann surface $\mathcal{R}_{n;1}$.
We want to create a theory on the complex plane $z = x + \imath\tau$ and we expect that the partition function of this theory depends on the values of some "fields", that we will define later, around the points $(u_1;0)$ and $(v_1;0)$.
These fields and their correlation functions are defined from the partition function, on the Riemann surface, of the system:

$$Z_{\mathcal{R}} = \int [d\phi]_{\mathcal{R}} \exp\left(-\int_{\mathcal{R}} dx d\tau \phi(x;\tau)\right). \tag{2.23}$$

The main problem of 2.23 is that it makes the theory on the complex plane non local, as it is discussed in Ref.[14].
Since locality is a fundamental hypothesis for all the results of quantum field theory we want to define a new model that preserves it.
Therefore we can think to implement the complicated structure of the Riemann surface on the target field instead of the complex plane and we use the so called multi copy model as it was suggested by Cardy and Doyon in the reference cited above.
We define a new system composed by n copies of the system previously defined and we will call it multi-copy model; it is formally described by the following partition function:

$$Z_{\mathcal{R}} = \int_{\mathcal{C}_{(u_1;v_1)}} [d\phi_1 \dots d\phi_n] \exp\left(- \int_{\mathbb{C}} dx d\tau (\mathcal{L}[\phi_1](x;\tau) + \dots + \mathcal{L}[\phi_n](x;\tau)) \right). \tag{2.24}$$

It is important to notice that the number of the copies n is the same number that labels the Riemann surface $\mathcal{R}_{n;1}$. In 2.24 we used the so called *restricted path integral* $\int_{\mathcal{C}_{(u_1;v_1)}}$ that is a short hand notation to indicate the path integral with boundary conditions:

$$\phi_i(x;0^+) = \phi_{i+1}(x;0^-) \qquad \forall x \in [u_1;v_1] \qquad \forall i = 1,2\dots,n, \tag{2.25}$$

where we have to use the cyclical condition $n+i \equiv i$.
The lagrangian density of the multi copy model is:

$$\mathcal{L}^{(n)}[\phi_1;\dots;\phi_n](x;\tau) \equiv \mathcal{L}[\phi_1](x;\tau) + \dots + \mathcal{L}[\phi_n](x;\tau), \tag{2.26}$$

and it is easy to see, from 2.26, that the multi copy energy density will be the sum of the n energy densities of the copies.
We have that 2.24 and 2.23 define the same model, but the first one gives us a model with local fields in $(u_1;0)$ and $(v_1;0)$ and it allows us to use the full power of local quantum field theory.
The field defined in 2.24 are an example of a very wide class of fields called *twist fields*.
Every time we have a theory that admits an internal (i.e. not involving the space-time degrees of freedom) and rigid (i.e. not depending on the space-time point) symmetry σ, so that:

$$\int dx d\tau \mathcal{L}[\sigma\phi](x;\tau) = \int dx d\tau \mathcal{L}[\phi](x;\tau), \tag{2.27}$$

where the fields $\phi(x;\tau)$ are called *twist fields*.
In our case the internal symmetry σ is the following one:

$$\mathcal{T}_n \equiv \mathcal{T}_\sigma \qquad \sigma : i \mapsto i+1 \qquad \mod n \ \ \forall i = 1,2,\dots,n+1 \equiv 1 \tag{2.28}$$

$$\tilde{\mathcal{T}}_n \equiv \mathcal{T}_{\sigma^{-1}} \quad \sigma^{-1} : i+1 \mapsto i \quad \mod n \ \ \forall i = 1,2,\dots,n+1 \equiv 1, \tag{2.29}$$

and the corresponding twist fields as called *branch-points twist fields*, denoted by $\mathcal{T}_n(x;\tau)$.
The partition function of the model, in the general case of $\mathcal{R}_{n;N}$, can be written as a correlation function of the fields $\mathcal{T}_n$ and $\tilde{\mathcal{T}}_n$:

$$Z_{\mathcal{R}_{n,N}} \propto \langle \mathcal{T}_n(u_1;0)\tilde{\mathcal{T}}_n(v_1;0)\dots\mathcal{T}_n(u_N;0)\tilde{\mathcal{T}}_n(v_N;0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}. \tag{2.30}$$

In fact for all $x \in A$ the fields $\mathcal{T}_n(v_i;0)$ give us the connection at $\tau = 0$ whereas for the points $x \in B$ we have that the presence of both $\tilde{\mathcal{T}}_n$ and $\mathcal{T}_n$ cancels out and these points do not contribute to the partition function.

We are also interested to express the expectation value of a string of generic operators $O(x;\tau;\text{i-th sheet})$ in terms of branch-points twist fields, in particular we are interested in the case of $\mathcal{R}_{n,1}$ where we have:

$$\langle O(x;\tau;\text{i-th sheet})\dots\rangle_{\mathcal{L},\mathcal{R}_n} = \frac{\langle \mathcal{T}_n(u_1;0)\tilde{\mathcal{T}}_n(v_1;0)O_i(x;\tau)\dots\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \mathcal{T}_n(u_1;0)\tilde{\mathcal{T}}_n(v_1;0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}, \tag{2.31}$$

and the proportionality factor cancels out in the ratio.

2.5 The single interval case in Conformal Field Theory

We conclude our analysis with the particular case of a conformal field theory with the explicit calculation of the entanglement entropy of a single interval, $N = 1$, in the case of an infinite 1D quantum system at zero temperature. This particular case was also considered in Ref.[13] and in Ref.[11] where the same result is obtained in a slightly different way.
The subsystem A starts from the point u and ends in v and its length is $l = |v - u|$. We will see that the entanglement entropy will exhibit the famous universal logarithmic divergence multiplied by an amplitude that will depend only on the central charge of the conformal field theory.
This behaviour is completely different from the one predicted by the area law because we are dealing with a critical system. In the case of non critical 1D quantum systems the area law holds and we have an upper bound to the entanglement entropy as expected by our argument in Chapt.1.
Let's start with mapping the Riemann surface into the complex plane. It can be done with the conformal transformation:

$$z = \left(\frac{w-u}{w-v}\right)^{\frac{1}{n}} \equiv (\zeta)^{\frac{1}{n}}, \tag{2.32}$$

where:

$$z, \zeta \in \mathbb{C}$$
$$w \in \mathcal{R}_{n;1} \equiv \mathcal{R}_n.$$

A visual representation of the transformation is presented in Fig.2.2:

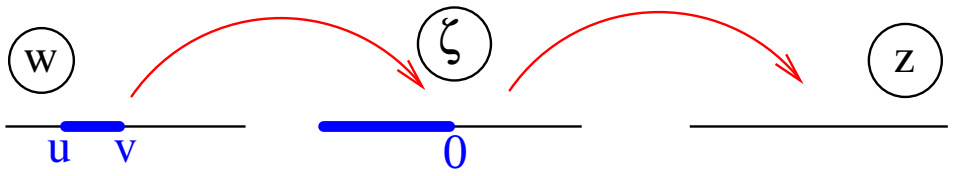

Figure 2.2: *the effect of the conformal mapping 2.32. Figure from Ref.[10]*

The transformation of the holomrphic stress energy tensor $T(w)$ was found in Ref.[44] and it is:

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z;w\}, \tag{2.33}$$

where we used the Schwartzian derivative:

$$\{z;w\} \equiv \frac{z'''z' - \frac{3}{2}(z'')^2}{(z')^2}. \tag{2.34}$$

The c parameter is the central charge of the theory and tells us how the free energy behaves when the conformal invariance is broken, also locally. For a more complete discussion about the role of the central charge see Ref.[45].
Taking now the expectation value of 2.33 and using $\langle T(z)\rangle_{\mathbb{C}} = 0$ for the rotational and traslational invariance of the theory on the complex plane we have:

$$\langle T(w)\rangle_{\mathcal{R}_n} = \frac{c}{24}\left(1 - \frac{1}{n^2}\right)\frac{(u-v)^2}{(w-u)^2(w-v)^2}. \tag{2.35}$$

We know from our previous arguments that the expectation value of a generic operator on the surface $\mathcal{R}_n$ can be computed using 2.31:

$$\frac{\langle \mathcal{T}_n(u_1;0)\tilde{\mathcal{T}}_n(v_1;0)T_j(z)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \mathcal{T}_n(u_1;0)\tilde{\mathcal{T}}_n(v_1;0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}, \tag{2.36}$$

where j labels the copies in the multi-copy model.
From the definition of $\mathcal{L}^{(n)}$ we have that the energy density of the multi-copy model is the sum of the energy densities of the copies and, since the right hand side of 2.35 does not depend on j, we can conclude that all the copies have the the same stress energy tensor.
This gives us the following result:

$$\begin{aligned}\frac{\langle \mathcal{T}_n(u_1;0)\tilde{\mathcal{T}}_n(v_1;0)T^{(n)}(z)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \mathcal{T}_n(u_1;0)\tilde{\mathcal{T}}_n(v_1;0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}} &= n\left(\frac{\langle \mathcal{T}_n(u_1;0)\tilde{\mathcal{T}}_n(v_1;0)T_j(z)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \mathcal{T}_n(u_1;0)\tilde{\mathcal{T}}_n(v_1;0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}\right) = \\ &= \frac{c}{24}\left(n - \frac{1}{n}\right)\frac{(u-v)^2}{(z-u)^2(z-v)^2}.\end{aligned} \tag{2.37}$$

We can compare this expression with the holomorphic Ward identity for the energy momentum tensor, see Ref.[44]:

$$\begin{aligned}\langle \mathcal{T}_n(u_1;0)\tilde{\mathcal{T}}_n(v_1;0)T(w)\rangle_{\mathcal{L}^{(n)},\mathbb{C}} = &\left(\frac{1}{w-u}\frac{\partial}{\partial u} + \frac{h_{\mathcal{T}_n}}{(w-u)^2} + \frac{1}{w-v}\frac{\partial}{\partial v} + \frac{h_{\tilde{\mathcal{T}}_n}}{(w-v)^2}\right)\times \\ &\times \langle \mathcal{T}_n(u_1;0)\tilde{\mathcal{T}}_n(v_1;0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}},\end{aligned} \tag{2.38}$$

and we obtain the scaling dimensions d_n and $\tilde{d}_n$ of the fields $\mathcal{T}_n$ and $\tilde{\mathcal{T}}_n$.
This can be done simply rewriting 2.37 as:

$$\begin{aligned}&\frac{c(n^2-1)}{24n}\frac{(u-v)^2}{(w-u)^2(w-v)^2} = \frac{c(n^2-1)}{24n}\frac{(w-v-(w-u))^2}{(w-u)^2(w-v)^2} = \\ &= \frac{c(n^2-1)}{24n}\left(\frac{1}{(w-u)^2} + \frac{1}{(w-v)^2} - \frac{2}{(w-v)(w-u)}\right),\end{aligned}$$

and by comparison with 2.38 we obtain:

$$h_{\tilde{\mathcal{T}}_n} = h_{\mathcal{T}_n} = \frac{c}{24}\left(n - \frac{1}{n}\right) = \frac{d_n}{2} \to d_n = \frac{c}{12}\left(n - \frac{1}{n}\right). \tag{2.39}$$

The same result was also obtained by Knizhnik Ref.[46] in a completely different context.
In conformal field theory, see Ref.[45], the holomorphic and antiholomorphic dimensions fix the two point function:

$$\langle \mathcal{T}_n(u;0)\tilde{\mathcal{T}}_n(v;0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}} \sim \frac{1}{|u-v|^{4d_n}}. \tag{2.40}$$

Let now focus our attention on the trace of the n-th power of the reduced density matrix, that is the most important quantity of our study.
In the multi copy model defined on the Riemann surface it is defined as:

$$\mathrm{Tr}\rho_A^n \propto \frac{Z_n(A)}{Z^n}, \tag{2.41}$$

and it has to behave, apart from an overall constant, identically to the correlation function 2.40:

$$\mathrm{Tr}\rho_A^n = c_n \left(\frac{v-u}{a}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}, \tag{2.42}$$

where the multiplicative factors c_n are model-dependent and they cannot be determined with this approach. They are known just for few integrable models, see Ref.[14], Ref.[47] and Ref.[48] for example.
The parameter a has been inserted in order to make the final result dimensionless as it should be.
The coefficient c_1 is the only one fixed as consequence of the normalization of the reduced density matrix:

$$\lim_{n\to 1}\mathrm{Tr}\rho_A^n = \lim_{n\to 1} c_n \left(\frac{v-u}{a}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} = c_1 = 1. \tag{2.43}$$

From 2.42 we can easily obtain the Renyi entropy for a generic real valued index n:

$$S^{(n)}(A) = \frac{1}{1-n}\ln \mathrm{Tr}\rho_A^n = \frac{1}{1-n}\ln c_n \left(\frac{v-u}{a}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} = \frac{c}{6}\left(1+\frac{1}{n}\right)\ln\frac{l}{a} + c_n', \tag{2.44}$$

where we used the definitions $c_n' \equiv \ln(c_n)/(1-n)$ and $l \equiv v-u$ that is the length of the subsystem. It is important to notice that this expression in true also for real value of the index n, not only for integer ones.
The von Neumann entropy is now computed simply by taking the limit $n \to 1^+$ of 2.44:

$$S(A) = \lim_{n\to 1} S^{(n)}(A) = \frac{c}{3}\ln\left(\frac{l}{a}\right) + c_1'. \tag{2.45}$$

This expression shows the well known logarithmic divergence, that violates the area laws for 1D systems, with an universal prefactor defined by the central charge c.
This result was also confirmed by random matrices computations, see Ref.[49], and it is an important result because it creates a connection between two different universality classes: the conformal field theory and the random matrices ensambles.
The constants c_n and c_n', as we said before, are model dependent and they are known only for few simple and integrable models, as the XX spin chain for example.

2.6 Generalization to finite temperature and finite size

In the precedent section we studied an infinite 1D quantum system at zero temperature, now we show how to extend the 2.44 and 2.45 to infinite systems at non-zero temperature and to finite systems at zero temperature. The last case will be important for us since in the Chapt.s 5, 4 and 3 we will work with a finite size systems at zero temperature.
in the last section we mapped the surface $\mathcal{R}_n$ (generated by n copies of an infinite system at zero temperature) into the complex plane $\mathbb{C}$ using the conformal mapping:

$$w = \left(\frac{z-u}{z-v}\right)^{\frac{1}{n}}. \tag{2.46}$$

Physically, we know that the axis τ of the complex plane is related to the inverse temperature β. In the previous case we have no bound on the imaginary axis and it means that $T = 0$, now we want to identify the points at a fixed distance β in order to have a bound:

$$z_1 \sim z_2 \Longleftrightarrow \tau_1 = \tau_2 + k\beta \qquad k \in \mathbb{Z}. \tag{2.47}$$

This operation corresponds to the compactification of the complex plane into a cyinder, and analytically can be done with the transformation:

$$w = \mathrm{e}^{2\pi z/\beta}. \tag{2.48}$$

It is easy to see that this transformation is also conformal. The two point function transforms as usual in conformal field theory (see as usual Ref.[45]):

$$\langle \mathcal{T}_n(z_1;\bar{z}_1)\tilde{\mathcal{T}}_n(z_2;\bar{z}_2) = |w_1'(z_1)w_2'(z_2)|^{d_n} \langle \mathcal{T}_n(w_1;\bar{w}_1)\tilde{\mathcal{T}}_n(w_2;\bar{w}_2)\rangle. \tag{2.49}$$

This transformation affects the trace of ρ_A^n in this way:

$$\mathrm{Tr}\rho_A^n \propto \langle \mathcal{T}_n(z_1)\tilde{\mathcal{T}}_n(z_2)\rangle_{cyl} = \left(\left(\frac{2\pi}{\beta}\right)^2 e^{\frac{2\pi}{\beta}(z_1+z_2)}\right)^{d_n} \frac{1}{\left(e^{\frac{2\pi}{\beta}z_1} - e^{\frac{2\pi}{\beta}z_2}\right)^{2d_n}} \tag{2.50}$$

$$= \left(\frac{\beta}{\pi}\sinh\left(\frac{\pi l}{\beta}\right)\right)^{-2d_n}. \tag{2.51}$$

The Renyi entropy and von Neumann entropy are:

$$S^{(n)}(A) = \frac{c}{6}\left(1+\frac{1}{n}\right)\ln\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right) + c_n', \tag{2.52}$$

$$S(A) = \frac{c}{3}\ln\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right) + c_1', \tag{2.53}$$

these results were found also in Ref.[11] and Ref.[50].
It is usefull to study the last expressions in the high and low temperature regimes in order to recover in the first case the well known thermal entropy and in the second one the infinite system entropy found in last section.
The limit of zero temperature is defined by $l \ll \beta$, formally the cylinder has a so small curvature that it is almost flat, the von Neumann entropy in this regime is:

$$S(A) \approx \frac{c}{3}\ln\left(\frac{\beta}{\pi a}\frac{\pi l}{\beta}\right) + c_1' = \frac{c}{3}\ln\left(\frac{l}{a}\right) + c_1', \tag{2.54}$$

and we recover 2.45 that was obtained in the case of infinite system at zero temperature. The high temperature limit is defined by $\beta \ll l$:

$$S(A) \approx \frac{\pi c}{3\beta} l + c_1', \tag{2.55}$$

in this regime the correlations due to entanglement is negligible compared to the thermal ones and we recover the classical thermal form of the Von Neumann entropy. The expression 2.55 was found in Ref.[51] and Ref.[52] using the free energy for a standard CFT.
In the finite size and zero temperature case we want to identify the points separated by a distance of L along the real axis of the complex plane:

$$z_1 \sim z_2 \Longleftrightarrow x_1 = x_2 + kL \qquad k \in \mathbb{Z}, \tag{2.56}$$

and the transformation is:

$$w = e^{2\pi \imath z/L}. \tag{2.57}$$

Following the same steps of the non zero temperature case we obtain the Renyi entropy and the von Neumann entropy:

$$S^{(n)}(A) = \frac{c}{6}\left(1 + \frac{1}{n}\right) \ln\left(\frac{L}{\pi a} \sin\left(\frac{\pi l}{L}\right)\right) + c_n', \tag{2.58}$$

$$S(A) = \frac{c}{3} \ln\left(\frac{L}{\pi a} \sin\left(\frac{\pi l}{L}\right)\right) + c_1'. \tag{2.59}$$

The infinite size result can be found taking the limit $l \ll L$:

$$S(A) \approx \frac{c}{3} \ln\left(\frac{l}{a}\right) + c_1'. \tag{2.60}$$

These expressions are very important since they show us a very simple rule to move from the infinite size case to the finite size one, it can be simply done using the replacement:

$$l \to \left(\frac{L \sin(\pi l/L)}{\pi}\right). \tag{2.61}$$

This correspondence will be intensively used in the following chapters to compare our results, that will be computed for finite systems, with other results that were obtained in the infinite size case.
We can think to the case of non zero temperature and finite size system. Here we have to identify the points on real and imaginary axes:

$$z_1 \sim z_2 \Longleftrightarrow \tau_1 = \tau_2 + k\beta \qquad k \in \mathbb{Z}, \tag{2.62}$$

$$z_1 \sim z_2 \Longleftrightarrow x_1 = x_2 + jL \qquad j \in \mathbb{Z}, \tag{2.63}$$

and these two conditions identify a torus.
The topological structure of the torus is completely different from the structure of the complex plane (they have a different genus) and they cannot be mapped one into the other with an analytic conformal map. Anyway there are some cases when it is possible to perform analytic computation like in Ref.[53] for massless Dirac Fermions.

Chapter 3

Entropy of the excited states of conformal field theory

In the previous chapter we studied the entanglement entropy of an 1D system with conformal invariance, now we want to study the entanglement entropy of the excited states of this system and, in particular, if there are differences in the entanglement entropy of the different excited states.
G.Sierra, M.I.Berganza and F.C. Alcaraz in Ref.[19] solved this problem in the case of excitations due to primary fields, in particular they found a way to compute, using also the replica trick approach exposed in 2, the following quantity:

$$F_{\Upsilon}^{(n)} = \frac{\mathrm{Tr}\rho_{\Upsilon}^{n}}{\mathrm{Tr}\rho^{n}}, \tag{3.1}$$

where ρ_{Υ} is the reduced density matrix of the excited state.
At the end of this chapter we will also show the explicit calculation of the function F_{Υ}^{n} for the spinless massless bosonic theory and we will study the changes in the entanglement entropy due to its different primary operators.

3.1 Primary fields and their entanglement entropy

We want to study the effects of an excitation produced by a primary field Υ. Since in the Cardy Calabrese approach exposed in Chapt.2 we evaluated all the quantities in the limit $\beta \to \infty$ now we need to evaluate the perturbation field in the in and out states. This can be done using the radial quantization where we have that the infinite past is mapped in the origin of the complex plane and the infinite future is mapped at long radial distances from the origin.
The in state of a perturbation Υ is defined as:

$$|\Upsilon\rangle = \lim_{(z;\bar{z})\to 0} \Upsilon(z;\bar{z})\,|0\rangle\,, \tag{3.2}$$

where $|0\rangle$ is the vacuum state.
The out state is mapped to the "radial" infinity and the out state can be written as:

$$\langle\Upsilon| = \lim_{(z;\bar{z})\to 0} z^{-2h}\bar{z}^{-2\bar{h}}\,\langle 0|\,\Upsilon\left(\frac{1}{z};\frac{1}{\bar{z}}\right), \tag{3.3}$$

for a technical description of radial quantization see Ref.[45].
Now we can use in and out states of the excitation Υ in the Cardy-Calabrese approach exposed in Chapt.2, and we can compute $Tr\rho_A^n$ as a path integral over a Riemann surface $\mathcal{R}_n$, defined in the same way as in 2.3.
We use the same set of bosonic degrees of freedom $\{\phi(x)\}$ and we compute the matrix elements of the ground state density matrix:

$$\langle\{\phi''(x'')\}|\,\rho_{\mathbf{I}}\,|\{\phi'(x')\}\rangle = \frac{1}{Z}\int[d\phi(x;\tau)]\prod_x \delta(\phi(x;-\infty)-\phi'(x'))\prod_x \delta(\phi(x;\infty)-\phi'(x''))e^{-S_E}, \tag{3.4}$$

for sake of clarity here we use the canonical coordinates $z = x + \imath\tau$ instead of the radial quantization.
The matrix elements of the excited density matrix can be easily computed in the path integral formulation:

$$\begin{aligned}&\langle\{\phi''(x'')\}|\,\rho_{\mathbf{\Upsilon}}\,|\{\phi'(x')\}\rangle = \\ &= \frac{1}{Z}\int[d\phi]\prod_x \delta(\phi(x;-\infty)-\phi'(x'))\prod_x \delta(\phi(x;\infty)-\phi'(x''))\quad \Upsilon(0;\infty)\Upsilon^*(0;-\infty)e^{-S_E}.\end{aligned} \tag{3.5}$$

The approach of Calabrese and Cardy, Chapt.2, allows us to compute the $Tr\rho_{\mathbf{\Upsilon}^{\mathbf{n}}}$ as a correlation function of a theory defined over the Riemann surface $\mathcal{R}_n$:

$$Tr\rho_{\mathbf{\Upsilon}^{\mathbf{n}}} = C^n Z_n(A)\langle\Upsilon_0(0;-\infty)\Upsilon_0^\dagger(0;+\infty)\dots\Upsilon_n(0;-\infty)\Upsilon_n^\dagger(0;\infty)\rangle_{\mathcal{R}_n}, \tag{3.6}$$

where $Z_n(A)$ is the partition function of the multi-copy model on the surface $\mathcal{R}_n$. C is a numerical constant fixed by the normalization of the density matrix:

$$Tr\rho_{\mathbf{\Upsilon}} = CZ(A)\langle\Upsilon_0(0;-\infty)\Upsilon_0^\dagger(0;+\infty)\rangle_{\mathcal{R}_1} = 1, \tag{3.7}$$

that fixes the normalization constant to:

$$C = \frac{1}{Z_1(A)\langle\Upsilon_0(0;-\infty)\Upsilon_0^\dagger(0;+\infty)\rangle_{Ru}}. \tag{3.8}$$

This allows us to write a closed expression for $\mathrm{Tr}\rho_{\mathbf{\Upsilon}}^n$:

$$Tr\rho_{\mathbf{\Upsilon}}^n = \frac{Z_n(A)}{Z^n(A)}\frac{\langle\prod_{i=0}^{n-1}\Upsilon_i(0;\infty)\Upsilon_i^\dagger(0;-\infty)\rangle_{\mathcal{R}_n}}{[\langle\Upsilon_0(0;-\infty)\Upsilon_0^\dagger(0;+\infty)\rangle_{\mathcal{R}_1}]^n}. \tag{3.9}$$

From equation 2.41 we know that:

$$\mathrm{Tr}\rho_{\mathbf{I}}^n = \frac{Z_n(A)}{Z^n}, \tag{3.10}$$

and this can be used to rewrite 3.9 as:

$$F_{\mathbf{\Upsilon}}^n(l/L) = \frac{Tr\rho_{\mathbf{\Upsilon}}^n}{Tr\rho_{\mathbf{I}}^n} \equiv \frac{\langle\prod_{i=0}^{n-1}\Upsilon_i(0;\infty)\Upsilon_i^\dagger(0;-\infty)\rangle_{\mathcal{R}_n}}{[\langle\Upsilon_0(0;-\infty)\Upsilon_0^\dagger(0;+\infty)\rangle_{\mathcal{R}_1}]^n}. \tag{3.11}$$

This expression is connected to the Renyi entropies of perturbed and unperturbed states:

$$F_{\mathbf{\Upsilon}}^{(n)} = \frac{\mathrm{Tr}\rho_{\mathbf{\Upsilon}}^n}{\mathrm{Tr}\rho_{\mathbf{I}}^n} = \frac{e^{(1-n)S_{\mathbf{\Upsilon}}^{(n)}}}{e^{(1-n)S_{\mathbf{I}}^{(n)}}} = e^{(1-n)(S_{\mathbf{\Upsilon}}^{(n)}-S_{\mathbf{I}}^{(n)})}. \tag{3.12}$$

In the previous analysis the shape of the subsystem A did not play any role and all the previous expressions hold for a generic shape of the subsystem A.

3.2 The single interval case

Let now focus our attention on the well studied single interval case. In order to compute the correlation function involving the primary operators we have to obtain a transformation that maps the Riemann surface $\mathcal{R}_n$ composed by n copies of the system into the complex plane.
In the precedent section we used the conformal transfrmation:

$$z = \left(\frac{\zeta - u}{\zeta - v}\right)^{\frac{1}{n}}, \tag{3.13}$$

that allows us to map the Riemann surface $\mathcal{R}_n$ into the complex plane $\mathbb{C}$. In that case we were studying an infinite 1D quantum system at zero temperature and the Riemann surface was composed by a series of planes sewed consequently one to the others.
Now we want to study a **finite** size system at zero temperature and we want to find the conformal transformation that maps its Riemann surface, generated sewing consequently n *cylinders* parallel to the imaginary axis, modifying properly the transformation 3.13.
As we saw in the previous chapter the transformation that maps the complex plane to the cylinder is:

$$\zeta = e^{\frac{2\pi\imath}{L}w}, \tag{3.14}$$

that is a compactification of the complex plane into an infinite cylinder.
The composition of the two mappings $w \mapsto \zeta \mapsto z$ is still conformal and it is:

$$z = \left(\frac{e^{\frac{2\pi\imath}{L}W} - e^{\frac{2\pi\imath}{L}u}}{e^{\frac{2\pi\imath}{L}W} - e^{\frac{2\pi\imath}{L}v}}\right)^{\frac{1}{n}}, \tag{3.15}$$

and rewriting it in terms of sin functions:

$$z^n = e^{-\imath\pi x} \qquad \left(\frac{\sin\left(\frac{\pi(w-u)}{L}\right)}{\sin\left(\frac{\pi(w-v)}{L}\right)}\right),$$

reabsorbing now the phase in z we obtain the final result:

$$z = \zeta^{\frac{1}{n}}, \qquad \zeta = \frac{\sin\left(\frac{\pi(w-u)}{L}\right)}{\sin\left(\frac{\pi(w-v)}{L}\right)}. \tag{3.16}$$

In Fig. 3.1 we illustrate the transformation in the case of $n = 3$ and there are also plotted the in and out states of the mapping, they will be defined rigorously in 3.24:

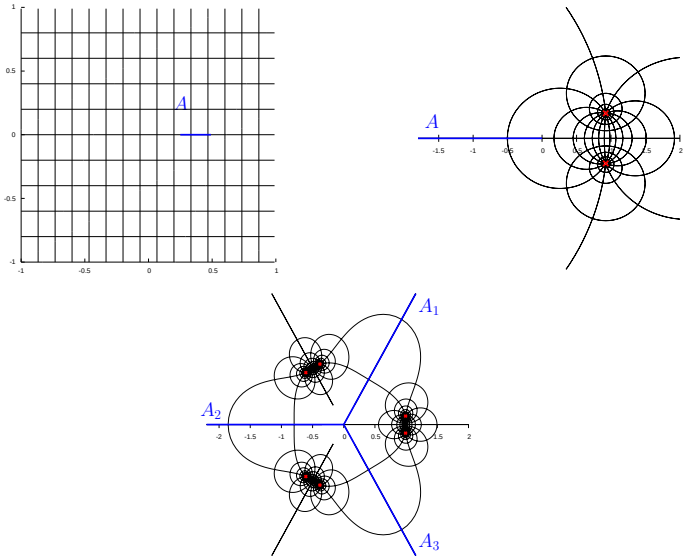

Figure 3.1: The cylinder w with the cut A is mapped *via* (3.16) to the center panel (we took $x = |u-v|/L = 1/8$). The transformation (3.24) with $n = 3$ has the effect shown in the right panel. The red crosses indicate the images ζ_∞, ζ'_∞, z_k, z'_k of the points w_∞, w'_∞. Figure from Ref.[19].

For sake of simplicity we will choose the subsystem that lies between $(u = 0; 0)$ and $(v = l; 0)$:

$$z = \left(\frac{\sin\left(\frac{\pi w}{L}\right)}{\sin\left(\frac{\pi(w-l)}{L}\right)} \right)^{\frac{1}{n}}, \tag{3.17}$$

where we can introduce the ratio:

$$x = \frac{l}{L}, \tag{3.18}$$

that indicates the fraction of the whole system S is in the subsystem A, the system B has a fraction $1 - x$.
We can invert the previous relation and find the inverse mapping $\mathbb{C} \to \mathcal{R}_n$:

$$\begin{aligned}
z^n &= \frac{\sin\left(\frac{\pi w}{L}\right)}{\sin\left(\frac{\pi(w-l)}{L}\right)} = \frac{e^{\frac{\imath\pi w}{L}} - e^{-\frac{\imath\pi w}{L}}}{e^{\frac{\imath\pi(w-l)}{L}} - e^{-\frac{\imath\pi(w-l)}{L}}} \\
z^n &\left(\exp\left(\frac{\imath\pi(w-l)}{L}\right) - \exp\left(-\frac{\imath\pi(w-l)}{L}\right)\right) = \exp\left(\frac{\imath\pi w}{L}\right) - \exp\left(-\frac{\imath\pi w}{L}\right) \\
z^n &\left(e^{(-\imath\pi x)} - e^{(\imath\pi x)} \exp\left(-\frac{2\pi\imath w}{L}\right)\right) = 1 - \exp\left(-\frac{2\pi\imath w}{L}\right) \\
&\exp\left(-\frac{2\pi\imath w}{L}\right)(1 - z^n e^{\imath\pi x}) = 1 - z^n e^{-\imath\pi x} \\
&\exp\left(-\frac{2\pi\imath w}{L}\right) = \frac{1 - z^n e^{-\imath\pi x}}{1 - z^n e^{\imath\pi x}} \to w = \left(\frac{\imath L}{2\pi}\right) \ln\left(\frac{1 - z^n e^{-\imath\pi x}}{1 - z^n e^{\imath\pi x}}\right).
\end{aligned} \tag{3.19}$$

In the Chapt.s 4 and 5 it will be useful the derivative of the inverse conformal mapping:

$$\begin{aligned}
\frac{dw}{dz} &= \left(\frac{\imath L}{2\pi}\right)\left(\frac{1-z^n e^{\imath\pi x}}{1-z^n e^{-\imath\pi x}}\right)\left(\frac{d}{dz}\right)\left(\frac{1-z^n e^{-\imath\pi x}}{1-z^n e^{\imath\pi x}}\right) = \\
&= \left(\frac{\imath L}{2\pi}\right)\left(\frac{\left(-nz^{n-1}e^{-\imath\pi x}\right)\left(1-z^n e^{\imath\pi x}\right)-\left(-nz^{n-1}e^{\imath\pi x}\right)\left(1-z^n e^{-\imath\pi x}\right)}{\left(1-z^n e^{\imath\pi x}\right)^2}\right) = \\
&= \left(\frac{\imath L}{2\pi}\right)\frac{+nz^{2n-1}-nz^{n-1}e^{-\imath\pi x}-nz^{2n-1}+nz^{n-1}e^{\imath\pi x}}{\left(1-z^n e^{\imath\pi x}\right)\left(1-z^n e^{-\imath\pi x}\right)} = \\
&= \left(-\frac{nL\sin(\pi x)}{\pi}\right)\frac{z^{n-1}}{\left(e^{\imath\pi x}-z^n\right)\left(e^{-\imath\pi x}-z^n\right)}.
\end{aligned}$$

Obviously all these results could be translated into the infinite size case simply taking the limit $x \to 0$.
Now we can use the mapping 3.17 to map the primary fields from the Riemann surface to the complex plane as we did in Chapt.2:

$$\Upsilon(w;\bar{w}) = \left(\frac{dz}{dw}\right)^h \left(\frac{d\bar{z}}{d\bar{w}}\right)^{\bar{h}} \Upsilon(z;\bar{z}). \tag{3.20}$$

As we said before we will evaluate these fields and their adjoints in the in and out states and we have to compute the derivative of the mapping $\mathcal{R}_n \to \mathbb{C}$ and then we will evaluate it in the in and out states defined by $w \to \pm\imath\infty$. The derivative of the mapping is:

$$\frac{dz}{dw} = -\frac{\pi\sin(\pi x)}{nL}\frac{z}{\sin\left(\frac{\pi(w-u)}{L}\right)\sin\left(\frac{\pi(w-v)}{L}\right)}. \tag{3.21}$$

We can now take the limits $w \to \pm\imath\infty$:

$$\lim_{w\to-\imath\infty}\frac{dz}{dw} = \frac{z_k}{n}\left(\frac{4\pi\sin(\pi x)e^{\frac{\imath\pi(u+v)}{L}}e^{-\frac{2\pi}{L}|w|}}{L}\right), \tag{3.22}$$

$$\lim_{w\to\imath\infty}\frac{dz}{dw} = \frac{z'_k}{n}\left(\frac{4\pi\sin(\pi x)e^{-\frac{\imath\pi(u+v)}{L}}e^{-\frac{2\pi}{L}|w|}}{L}\right). \tag{3.23}$$

We notice that these two derivatives are exponentially suppressed and the result of this transformation would be zero, but the function $F^{(n)}_{\mathbf{\Upsilon}}$ is defined as a ratio between an equal number of primary field and it will give us a finite result.
The in and out states on the complex plane are simply defined by taking the limits $w \to \pm\imath\infty$ of the mapping 3.17:

$$z_\infty = \lim_{w\to-\imath\infty}\left(\frac{\sin\left(\frac{\pi w}{L}\right)}{\sin\left(\frac{\pi(w-l)}{L}\right)}\right)^{\frac{1}{n}} = \left(e^{\imath\pi x}\right)^{\frac{1}{n}} = z_j = e^{\frac{\imath\pi}{n}(x+2j)},$$

$$z'_\infty = \lim_{w\to\imath\infty}\left(\frac{\sin\left(\frac{\pi w}{L}\right)}{\sin\left(\frac{\pi(w-l)}{L}\right)}\right)^{\frac{1}{n}} = \left(e^{-\imath\pi x}\right)^{\frac{1}{n}} = z'_j = e^{\frac{\imath\pi}{n}(-x+2j)},$$

where the index j takes values $j = 0, 1, \dots n-1$.
Using the mapping and its derivative we can obtain the function F^n_{Υ}:

$$F^n_{\Upsilon}(x) = n^{-2(h+\bar{h})} \frac{\prod_{k=0}^{n-1} z_k^h \bar{z}_k^{\bar{h}} z_k'^h \bar{z}_k'^{\bar{h}}}{(z_0 \bar{z}_0 z_0' \bar{z}_0')^n} \frac{\langle \prod_{i=0}^{n-1} \Upsilon_i(z_i;\bar{z}_i) \Upsilon_i^\dagger(z_i';\bar{z}_i') \rangle_{\mathbb{C}}}{[\langle \Upsilon_0(z_0;\bar{z}_0) \Upsilon_0^\dagger(z_0';\bar{z}_0') \rangle_{\mathbb{C}}]^n}. \tag{3.24}$$

This expression can be further simplified using the mapping between the complex plane and the cylindrical geometry:

$$z = e^{\imath t} \to \Upsilon(t;\bar{t}) = e^{\imath\pi(h-\bar{h})} z^h \bar{z}^{\bar{h}} \Upsilon(z;\bar{z}), \tag{3.25}$$

and the precedent expression becomes:

$$F^n_{\Upsilon}(x) = n^{-2(h+\bar{h})} \frac{\langle \prod_{i=0}^{n-1} \Upsilon\left(\frac{\pi}{n}(x+2k)\right) \Upsilon^\dagger(\frac{\pi}{n}(-x+2k)) \rangle_{cyl}}{[\langle \Upsilon(\pi x) \Upsilon^\dagger(-\pi x) \rangle_{cyl}]^n}. \tag{3.26}$$

All the results developed in this section are obtained in the specific case of a single interval.

3.3 The bosonic theory

In this section we will apply the theory developed in the previous section to the case of a massless spinless boson that is the simplest example of conformal field theory.
This particular case is interesting because some of the most used spin chains, the XX model and many other spin chains, are described in their continuum limit by that quantum field theory. The action of a massless bosonic field in 2 dimensions is:

$$S = \frac{1}{8\pi} \int dz d\bar{z} \partial_z \varphi \partial_{\bar{z}} \varphi. \tag{3.27}$$

The φ can be decomposed in two chiral parts:

$$\varphi(z;\bar{z}) = \phi(z) + \bar{\phi}(\bar{z}). \tag{3.28}$$

One of the primary operators of the theory, see Ref.[45], is the vertex operator defined as:

$$V_{(\alpha;\bar{\alpha})} \equiv : e^{\imath(\alpha\phi + \bar{\alpha}\bar{\phi})} : , \tag{3.29}$$

and from the OPE with the energy momentum tensor we know that its holomorphic and antiholomorphic dimensions are:

$$h_\alpha = \bar{h}_\alpha = \frac{\alpha^2}{2}. \tag{3.30}$$

The holomorphic part of the generic correlator between vertex operators is well known, Ref.[45]:

$$\langle V_{(\alpha_1)}(z_1) \dots V_{(\alpha_n)}(z_n) \rangle_{\mathbb{C}} = \prod_{i<j} (z_i - z_j)^{\alpha_i \alpha_j}, \tag{3.31}$$

if the α_i parameters satisfy the *neutrality condition*: $\sum_i \alpha_i = 0$. In all other cases the correlation function is 0. Obviously the antiholomorphic part of the correlation function 3.31 takes the same form but it is a function of $\bar{z}_i$ instead of z_i.
We can map 3.31 on a cylindrical geometry using the transformation 3.25, and it becomes:

$$\langle V_{\alpha_1}(z_1) \dots V_{\alpha_n}(z_n) \rangle_{cyl} = \prod_{i<j} \left[2 \sin\left(\frac{w_{ij}}{2} \right) \right]^{\alpha_i \alpha_j}. \tag{3.32}$$

In order to elucidate how equation 3.32 can be obtained from 3.31 we will perform the computation explicitly in the simple case of a two point function:

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\rangle_{\mathbb{C}} = (z_1 - z_2)^{\alpha_1\alpha_2}\,. \tag{3.33}$$

The neutrality condition is:

$$\alpha_1 + \alpha_2 = 0 \to \alpha_1 = -\alpha_2. \tag{3.34}$$

The holomorphic dimension of the vertex operator is:

$$h_i = \frac{\alpha_i^2}{2}, \tag{3.35}$$

and the neutrality condition implies:

$$h_1 = \frac{\alpha_1^2}{2} = \frac{(-\alpha_2)^2}{2} = \frac{\alpha_2^2}{2} = h_2. \tag{3.36}$$

The vertex operators transforms as usual, Ref.[45], and here we are interested in their holomorphic part:

$$V_{\alpha_i}(z_i) = \left(\frac{dz_i}{dw_i}\right)^{-h_i} V_{\alpha_i}(w_i). \tag{3.37}$$

The correlation function on cylindrical geometry is:

$$\langle V_{\alpha_1}(w_1)V_{\alpha_2}(w_2)\rangle_{cyl} = \left(\frac{dz_1}{dw_1}\right)^{h_1}\left(\frac{dz_2}{dw_2}\right)^{h_2}[e^{\imath w_1} - e^{\imath w_2}]^{\alpha_1\alpha_2} = \tag{3.38}$$

$$\imath^{h_1+h_2}(z_1)^{h_1}(z_2)^{h_2}\left[2\imath e^{\frac{\imath}{2}(w_1+w_2)}\sin\left(\frac{w_1-w_2}{2}\right)\right]^{\alpha_1\alpha_2},$$

where we used the previous results and the derivative of the mapping 3.25.
Making use of the neutrality condition 3.34 we obtain:

$$\langle V_{\alpha_1}(w_1)V_{\alpha_2}(w_2)\rangle_{cyl} = \imath^{2h}\,(z_1z_2)^h\left[2\imath\sqrt{z_1z_2}\sin\left(\frac{w_1-w_2}{2}\right)\right]^{-2h} = \tag{3.39}$$

$$= \left[2\sin\left(\frac{w_1-w_2}{2}\right)\right]^{\alpha_1\alpha_2}. \tag{3.40}$$

Obviously the same approach works for arbitrary long string of vertex operators and it leads to the expression 3.32.
The $F^{(n)}_{\mathbf{V}_\alpha}$ function is defined using the general expression 3.26:

$$F^n_{\mathbf{V}_\alpha}(x) = n^{-2(h+\bar{h})}\frac{\langle\prod_{i=0}^{n-1}V_\alpha\left(\frac{\pi}{n}(x+2k)\right)V^\dagger_\alpha(\frac{\pi}{n}(-x+2k))\rangle_{cyl}}{[\langle V_\alpha(\pi x)V^\dagger_\alpha(-\pi x)\rangle_{cyl}]^n}. \tag{3.41}$$

The α parameter of the adjoint vertex operator can be found from definition 3.29:

$$\alpha_{V^\dagger} = -\alpha_V \to h_{V^\dagger} = h_V. \tag{3.42}$$

The expression 3.41 can be computed explicitly for a generic index n, also real valued, and it takes the form:

$$F^n_{V_\alpha}(x) = \left(n^{-n}\left[\frac{\sin(\pi x)}{\sin\left(\frac{\pi x}{n}\right)}\right]^n\prod_{m=1}^{n-1}\left[\frac{\sin^2(\pi m/n)}{\sin\left(\frac{\pi(m-x)}{n}\right)\sin\left(\frac{\pi(m+x)}{n}\right)}\right]^{n-m}\right)^{\alpha^2}. \tag{3.43}$$

We can evaluate equation 3.43 for the value $n = 2$, it takes the form:

$$F_{V_\alpha}^{n=2} = \left(2^{-2} \left[\frac{\sin(\pi x)}{\sin\left(\frac{\pi x}{2}\right)} \right]^2 \left[\frac{\sin^2(\pi/2)}{\sin\left(\frac{\pi(1-x)}{2}\right) \sin\left(\frac{\pi(1+x)}{2}\right)} \right]^{-1} \right)^{\alpha^2} = \tag{3.44}$$

$$= \left(\cos^2\left(\frac{\pi x}{2}\right) \left[\frac{1}{\cos\left(\frac{\pi x}{2}\right)} \right]^2 \right)^{\alpha^2} = 1. \tag{3.45}$$

This result is valid for every value of n and the interested reader can check by hand that $F_{V_\alpha}^{n=3}$ is still 1 and so on for every value of n.
To convince the reader of this statement we notice that the expression 3.43 is analytic (as function of x) for every value of n. We can also notice that the numerator and the denominator share the same simple poles (all rational), so we can say that the numerator and the denominator are proportional because they are both analytic functions. The proportionality constant is fixed to 1 simply taking the limit $x \to 0$ of the expression 3.43.
This argument demonstrates that, in the case of vertex operators, the function $F_{V_\alpha}^{(n)}$ is:

$$F_{V_\alpha}^{n} = 1 \quad \forall n, \alpha. \tag{3.46}$$

This result shows that the entropy of the ground state and the entropy of the states $|V_\alpha\rangle$ are the same, this will be clear in section 6.2 where we will analyse the different types of excitations of the XX model and their continuum limit.
There is another primary operator defined in the bosonic theory and it is:

$$\imath\partial\phi, \tag{3.47}$$

that has holomorphic and antiholomorphic dimensions equal to $(h; \bar{h}) = (1; 0)$, these fix the correlation function to be:

$$\langle \imath\partial\phi(z_1)\imath\partial\phi(z_2)\rangle_{\mathbb{C}} = \frac{1}{z_{12}^2}. \tag{3.48}$$

Since the action 3.27 describes a free field theory, we compute the $2n$ correlation function using the Wick's theorem:

$$\langle \prod_{j=1}^{2n} \imath\partial\phi(z_j)\rangle_{\mathbb{C}} = \mathrm{Hf}\left(\frac{1}{(z_i - z_j)^2} \right), \tag{3.49}$$

where we used the Haffian operator, see Ref.[45], that is defined as:

$$\mathrm{Hf}(\mathbb{A}) = \frac{1}{2^n n!} \sum_{\pi \in S_{2n}} \prod_{i=1}^{2n} A_{\pi(2i-1),\pi(2i)}, \tag{3.50}$$

where $\mathbb{A}$ is a generic matrix, S_{2n} is the group of permutations of $2n$ indices, and π is one of its elements.
In Ref.[45] it is demonstrated that:

$$\mathrm{Hf}\left(\frac{1}{(z_i - z_j)^2} \right) = \det\left(\frac{1}{z_i - z_j} \right). \tag{3.51}$$

This allows us to express the $2n$ points function as a determinant:

$$\langle \prod_{j=1}^{2n} \imath\partial\phi(z_j) \rangle_{\mathbb{C}} = \det\left(\frac{1}{z_i - z_j}\right). \tag{3.52}$$

Now we can perform the change of coordinates from the complex plane to the cylindrical geometry:

$$F^n_{\imath\partial\phi} = n^{-2n}[\sin(\pi x)]^{2n} \det\left(\frac{1}{2\sin\left(w_i - w_j\right)/2}\right), \tag{3.53}$$

where w_i are the same in and out states on the cylinder that we used in equation 3.39 in the case of vertex operators.
The expression 3.53 can be evaluated exactly for different integer values of n because it involves a determinant that is not defined for generic real n. Computing by hand the determinant in 3.53 for the value $n = 2$ we obtain the explicit expression for $F^2_{\imath\partial\phi}$:

$$F^2_{\imath\partial\phi}(x) = 1 - 2(\sin(\pi x/2))^2 + 3(\sin(\pi x/2))^4 - 2(\sin(\pi x/2))^6 + (\sin(\pi x/2))^8. \tag{3.54}$$

This expression is related to the Renyi entropy and it is important that it exhibits the symmetry:

$$F^n_{\imath\partial\phi}(x) = F^{(n)}_{\imath\partial\phi}(1 - x), \tag{3.55}$$

that reflects the well-known fact:

$$S^n(A) = S^n(B), \tag{3.56}$$

that holds since the system $S = A \cup B$ is in a pure state, that was our initial assumption on the state of S. It is clear that in this case the entanglement entropy of the excited state is totally different from the ground state one. The expression 3.53 has only one problem: it involves a determinant. This is a very big problem if we want to evaluate our $F^{(n)}_{\Upsilon}$ function with real values of n, not only integer, and maybe performing the analytical continuation around the point $n = 1^+$ in order to obtain the entanglement entropy of the excited state. This problem was solved, in Ref. [54], by Essler, Läuchli and Calabrese where they found an expression for 3.53 that can be easily continued to $n \to 1^+$ because it does not involve a determinant. The expression is:

$$F^n_{\imath\partial\phi}(x) = \prod_{p=1}^{n}\left[1 - \frac{(n-2p+1)^2}{n^2}\sin(\pi x)\right] = \left[\left(\frac{2\sin(\pi x)}{n}\right)^n \frac{\Gamma\left(\frac{1+n(\csc(\pi x)+1)}{2}\right)}{\Gamma\left(\frac{1+n(\csc(\pi x)-1)}{2}\right)}\right]^2. \tag{3.57}$$

If we take the limit $n \to 1^+$ we get:

$$\lim_{n\to 1^+} \frac{\ln F^n(x)}{1-n} = \ln|2\sin(\pi x)|\psi\left(\frac{1}{2\sin(\pi x)}\right) + \sin(\pi x), \tag{3.58}$$

where the function $\Psi(z)$ is the polygamma function defined as:

$$\Psi(z) = \frac{d}{dz}\ln(\Gamma(z)). \tag{3.59}$$

The physical difference between the perturbation of the vertex operator and the perturbation of the $\imath\partial\phi$ will be clear when we will analyse the XX model and its different excitations in section 6.2.

Chapter 4

Unusual corrections to scaling

In Chapt.2 we analysed the asymptotic behaviour for small a of the von Neumann and Renyi entropy, in this section we want to analyse the corrections to the leading order behaviour and in particular their scaling.
The corrections to scaling have been computed for many different systems both numerically and theoretically and it has been discovered the presence of corrections that depend on the n index of the Renyi entropy. J.Cardy and P.Calabrese showed, in Ref. [15], how unusual corrections can be due to the presence in the action of an irrelevant bulk perturbation operator that, for particular values of n, gives corrections of the form:

$$\left(\frac{L\sin(\pi x)}{\pi}\right)^{-\frac{2\Delta}{n}}, \tag{4.1}$$

where $\Delta > 0$ is the scaling dimension of the locally perturbing operator.
In all the other cases the leading correction is the one computed using the finite size scaling behaviour of the renormalization group, see Ref.[55] and Ref.[45]:

$$\left(\frac{L\sin(\pi x)}{\pi}\right)^{-2(\Delta-2)}, \tag{4.2}$$

where $\Delta > 2$. These corrections are subleading with respect to the unusual corrections.

4.1 From quantum field theory to unusual corrections to scaling

In Chapt.2 we showed that $\mathrm{Tr}\rho_A^n$ is proportional to Z_n/Z_1^n (see also Ref.[10], Ref.[13] and Ref.[14]) where Z_n is the partition function computed on the Riemann surface.
We can take the logarithm of $\mathrm{Tr}\rho_A^n$ obtaining:

$$(1-n)S^{(n)} = \ln \mathrm{Tr}\rho_A^n = \ln \frac{Z_n}{Z_1^n} = -(F_n - nF_1), \tag{4.3}$$

where we used the definition of adimensional free energy:

$$F_n \equiv -\ln Z_n. \tag{4.4}$$

We start our analysis considering a lattice system, that has a natural short distance cutoff called ϵ, and, as we did in section 2.3, we can take its continuum limit to describe the system using a quantum field theory.
In this procedure the effect of the lattice is taken into account by imposing that all the correlation functions takes the usual form only for distances greater than the cutoff. This approximation could be seem a bit too coarse but we are interested only in the *scaling* of the corrections and not in their amplitudes.
The changes in the parameter ϵ affects the free energy through the integrated trace of the energy momentum tensor $\Theta(z)$, see Ref.[43]:

$$-\epsilon\frac{\partial F}{\partial \epsilon} = \frac{1}{2\pi}\int_{\mathcal{R}_n}\langle\Theta(w)\rangle d^2z. \tag{4.5}$$

In the case of a flat space time CFT we have that $\langle\Theta\rangle$ is 0 for rotational and traslational invariance and F is scale invariant, after the subtraction of the bulk free energy.
In the case of the Riemann surface $\mathcal{R}_n$ described in section 2.3, the presence of branch points gives several contributions to the free energy. These contributions are due to the local breaking of the scale invariance near conical singularities, see Ref.[43].
Assuming to have, in addition to the conformal invariant action S_0, an irrelevant bulk operator $\Phi(w)$ of scaling dimension $\Delta > 2$, the action S takes the form:

$$S = S_0 + \lambda\int_{\mathcal{R}_n}\Phi(w)d^2w, \tag{4.6}$$

where λ is the dimensional coupling constant that we assume small enough to use power expansions.
We can rewrite λ as a function of the adimensional coupling constant g and of the cutoff ϵ:

$$\lambda = \frac{g}{\epsilon^{2-\Delta}}. \tag{4.7}$$

The perturbed adimensional free energy can be expressed as a power series in λ:

$$-\delta F_n = \sum_{N=1}^{\infty}\frac{(-\lambda)^N}{N!}\int\cdots\int_{\mathcal{R}_n}\langle\Phi(w_1)\dots\Phi(w_N)\rangle_{\mathcal{R}_n}d^2w_1\dots d^2w_N, \tag{4.8}$$

where the sum involves only the connected Feynman diagrams of the theory defined the Riemann surface $\mathcal{R}_n$.
We can now map the correlation functions from $\mathcal{R}_n$ into $\mathbb{C}$ using the transformation:

$$w = \left(\frac{\imath L}{2\pi}\right)\ln\left(\frac{1-z^n e^{-\imath\pi x}}{1-z^n e^{\imath\pi x}}\right). \tag{4.9}$$

As usual the generic correlation function transforms as, Ref.[45]:

$$\langle\Phi(w_1)\dots\Phi(w_N)\rangle_{\mathcal{R}_n} = \prod_{i=1}^{N}\left|\frac{dw_i}{dz_i}\right|^{-\Delta}\langle\Phi(z_1)\dots\Phi(z_N)\rangle_{\mathbb{C}}. \tag{4.10}$$

We can evaluate the power series 4.8 term by term in the coupling constant. The first order term $\langle\Phi(z_1)\rangle_{\mathbb{C}}$ is trivially zero for translational and rotational invariance on the complex plane. Therefore we will focus on the second order of 4.8, namely:

$$-\delta F_n^2 = \frac{1}{2}\frac{g^2}{\epsilon^{2(2-\Delta)}}\iint_{\mathcal{R}_n}\langle\Phi(w_1)\Phi(w_2)\rangle_{\mathcal{R}_n}d^2w_1d^2w_2. \tag{4.11}$$

We can map it to an integral over the complex plane:

$$\delta F_n^2 = -\frac{1}{2}\frac{g^2}{\epsilon^{2(2-\Delta)}} \iint_{\mathbb{C}} \left|\frac{dw_1}{dz_1}\right|^{2-\Delta} \left|\frac{dw_2}{dz_2}\right|^{2-\Delta} \langle \Phi(z_1)\Phi(z_2)\rangle_{\mathbb{C}} = \tag{4.12}$$

$$= -\frac{g^2}{2}\left(\frac{nL\sin(\pi x)}{\pi\epsilon}\right)^{4-2\Delta} \times \tag{4.13}$$

$$\times \iint_{\mathbb{C}} \frac{|z_1 z_2|^{(2-\Delta)(n-1)}}{|(z_1^n - e^{\imath\pi x})(z_1^n - e^{-\imath\pi x})(z_2^n - e^{\imath\pi x})(z_2^n - e^{-\imath\pi x})|^{2-\Delta}|z_1 - z_2|^{2\Delta}},$$

where we used the definition $x \equiv l/L$.
It is important to notice that the variables ϵ and L appear everywhere in the combination:

$$\frac{L\sin(\pi x)}{\pi\epsilon}, \tag{4.14}$$

we can move to the infinite 1D system case, studied in Ref.[15], simply taking the limit $x \to 0$:

$$\frac{L\sin(\pi x)}{\pi\epsilon} \to \frac{L\pi l}{L\pi\epsilon} = \frac{l}{\epsilon}. \tag{4.15}$$

The integral 4.12 could be a source of many divergences and it could be necessary to use additional cut-offs. They have to be of order $O(\epsilon)$ in the w-plane. These further cut-offs will give us, by scaling considerations, further powers of the combination $L\sin(\pi x)/\pi$ that generate deviation from the finite size scaling behaviour of the renormalization group.
First of all we analyse the case $n - 1 = 0^+$, where the integral is:

$$\iint_{\mathbb{C}} \frac{1}{|(z_1 - e^{\imath\pi x})(z_1 - e^{-\imath\pi x})(z_2 - e^{\imath\pi x})(z_2 - e^{-\imath\pi x})|^{2-\Delta}|z_1 - z_2|^{2\Delta}}, \tag{4.16}$$

this expression is regular in $z_i \approx 0$ as long as $z_1 - z_2$ is different from zero.
Lets take a look to the asymptotic behaviour of the integrals in 4.12 when, for example, z_1 becomes large:

$$\int_{\mathbb{C}} \frac{1}{|z_1|^4} d^2 z_1, \tag{4.17}$$

that is well behaved and it does not need to be regulated.
For larger values of Δ the region $z_1 \to z_2$ is a source of divergences and we have to introduce a cut-off $|w_1 - w_2| < \epsilon$, in the w-plane, to regulate them. The divergence is $O(\epsilon^{2-2\Delta})$ and the contribution to the quantity δF_n^2 is proportional to:

$$\frac{1}{\epsilon^2}\text{Area}(\mathcal{R}_n), \tag{4.18}$$

and this divergence cancels out in the subtraction $F_n - nF_1$.
Furthermore there are other divergences around the points:

$$z_i^n = e^{\pm\imath\pi x}, \tag{4.19}$$

and they will also cancel out in the subtraction $F_n - nF_1$, that will leave us with a well behaved.
We have subtracted off the divergences previously described and we have a regular expression for the values $1 < \Delta < 3$. The cutoff ϵ appears only in the pre-factor $\epsilon^{2(2-\Delta)}$ that arises

from the finite size scaling of the renormalization group.
The scaling of the corrections in L can be obtained by the scaling of the ones in ϵ:

$$\left(\frac{L\sin(\pi x)}{\pi}\right)^{-2(\Delta-2)}. \tag{4.20}$$

As we did before we can move to the case of system of infinite size, the case considered in Ref.[15], simply taking the limit$x \to 0$ and we find that the scaling of the corrections is:

$$l^{-2(\Delta-2)}, \tag{4.21}$$

where l is the size of the subsystem, that is the only length scale present in the system.
In case of a non vanishing value of $\langle\Phi(z_1)\rangle_{\mathbb{C}}$ we would have corrections that scale as $(L\sin(\pi x)/\pi)^{2-\Delta}$.
Let now take into account the fact that some divergences can arise for high values of n near a branch point. These divergences are due to the local breaking of scale invariance and they are genuine divergence that will not cancel out in the subraction $F_n - nF_1$.
Let now study the behaviour of our integral near the branch point $z_1 \approx 0$:

$$\int_{\mathbb{C}} d^2 z_1 \left|\frac{dw_1}{dz_1}\right|^2 |z_1|^{(1-n)\Delta}. \tag{4.22}$$

We can map this integral onto the Riemann surface $\mathcal{R}_n$, where we have our natural cutoff of order $O(\epsilon)$. We are studying the behaviour of the integral around the point $w_1 \approx 0$, where the conformal mapping takes the form:

$$z(w \approx 0) = \left(-\frac{\pi w}{L\sin(\pi x)}\right)^{\frac{1}{n}}. \tag{4.23}$$

Inserting this expression inside the integral 4.22 we have:

$$\int_{\mathcal{R}_n} d^2 w_1 |w_1|^{\Delta\left(\frac{1}{n}-1\right)}. \tag{4.24}$$

This integral diverges for the following values of n:

$$\Delta\left(1-\frac{1}{n}\right) > 2 \to n > \frac{\Delta}{\Delta-2}, \tag{4.25}$$

this condition defines a critical vale of the index n, namely: $n_c = \Delta/(\Delta-2)$. For $n > n_c$ the integral 4.25 diverges and it has to be further regulated using an additional cutoff $|w_1| < \epsilon$. This procedure gives us an extra ϵ-dependent factor:

$$\epsilon^{2+\Delta\left(\frac{1}{n}-1\right)}. \tag{4.26}$$

The integral approximated around the other branch point, $z_2 \to \infty$, takes the form:

$$\int_{\mathbb{C}} d^2 z_2 \left|\frac{dw_2}{dz_2}\right|^2 \frac{|z_2|^{\Delta(1-n)}}{|z_2|^{-2n\Delta}|z_2|^{2\Delta}} = \int_{\mathbb{C}} d^2 z_2 \left|\frac{dw_2}{dz_2}\right|^2 |z_2|^{\Delta(n-1)}. \tag{4.27}$$

The limit of $z_2 \to \infty$ is mapped in $w_2 \approx l$ and the conformal mapping approximated around this point is:

$$z(w \approx l) = \left(\frac{L\sin(\pi x)}{\pi(w-l)}\right)^{\frac{1}{n}}. \tag{4.28}$$

It is easy to see that the integrals around different branch points take the same form:

$$\int_{\mathcal{R}_n} d^2 w_2 |w_2 - l|^{\Delta\left(\frac{1}{n}-1\right)}. \tag{4.29}$$

As we did before, we have to introduce a further cutoff if $n > n_c$ and it leads us to the same ϵ dependent result seen in the precedent case:

$$\epsilon^{2+\Delta\left(\frac{1}{n}-1\right)}. \tag{4.30}$$

Looking at the expression $\delta F_n^2 - nF_1^{(2)}$ as function of the variables z_1 and z_2, we have two possibilities:

- z_1 and z_2 approach the same branch point: in this case the divergences disappear in the subtraction $F_n - nF_1$ and the quantity is finite.
- z_1 and z_2 approach *different* branch points: in this case the divergences do not cancel in the subtraction $F_n - nF_1$ and we have a finite size scaling behaviour that is different from the renormalization group one.

In the analysis of the scaling behaviour of the corrections we have two contributions: the first is the one that we computed above that is due to the local breaking of conformal invariance:

$$\epsilon^{4+2\Delta\left(\frac{1}{n}-1\right)}, \tag{4.31}$$

and the other one comes from the finite size scaling behaviour of the renormalization group:

$$\epsilon^{-2(\Delta-2)}. \tag{4.32}$$

The total dependence of $\delta F_n^{(2)}$ is:

$$\frac{\epsilon^{4+2\Delta\left(\frac{1}{n}-1\right)}}{\epsilon^{4-2\Delta}} = \epsilon^{\frac{2\Delta}{n}}. \tag{4.33}$$

The corrections in powers of L are obtained by scaling arguments:

$$\left(\frac{L\sin(\pi x)}{\pi}\right)^{-\frac{2\Delta}{n}}, \tag{4.34}$$

these corrections are called *unusual* because of the explicit presence of the index n at the exponent.

4.2 Relevant operators at conical singularities

In section 4.1 we saw how an irrelevant bulk operator can be responsible of corrections of the form $(L\sin(\pi x)/\pi)^{-2\Delta/n}$. These are different from the finite size scaling of the Renormalization group but they had been observed in many numerical and theoretical works about spin chains.
Now we want analyse how a *relevant* operator can give the same type of corrections.
Obviously it is not possible to have a bulk relevant operator because it would drive the system away from criticality and the conformal invariance of the system would be destroyed.

Furthermore it is possible to have a relevant perturbation located around the conical singularities of the Riemann surface.
Let consider the case of a quantum spin chain, where the degrees of freedom are located on sites, we could have a relevant perturbation operator localised near one of the branch points that define the Riemann surface $\mathcal{R}_n$. The scaling dimensions of this operator inside the bulk are $\Delta < 2$, by definition of relevant operator. On the other hand it is located near a branch point and its scaling dimension are enhanced to Δ/n. This means that it does not drive the system away from criticality anymore, and the conformal invariance is preserved.
Lets focus on the 2d Ising case, that will be analysed in section 5.5, the computation of $\mathrm{Tr}\rho_A^n$ involves a partition function over a Riemann surface defined by n branch point of degree n, as we saw in section 2.3. The degrees of freedom of the system are localised inside the sites of the lattice. If we use a model with discrete time and discrete space, every branch point can be placed in the middle of a plaquette. It is clear that for every $n > 1$ all the degrees of freedom inside the plaquette have a larger number of neighbors than the degrees of freedom localised inside the bulk. In the simplest case of nearest neighbor interaction we have that the order variable at the centre of the plaquette interacts exactly with 4 other spins, but the dual spin situated on the top of the plaquette interacts with $4n$ other variables. This situation drives the system away from criticality. Lets assume now to perform the continuum limit, the system will be described by a field theory and the energy operator located around the conical singularity will give us corrections of the form: $(L\sin(\pi x)/\pi)^{-2/n}$, the energy operator of the Ising model has $\Delta = 1$ (see Ref.[45]).
The previous arguments can suggest us that the right action of the model on the Riemann surface $\mathcal{R}_n$ has to take the form:

$$S = S_{CFT} + \sum_j \lambda_j \int_{\mathcal{R}_n} d^2w \Phi_j(w) + \sum_P \sum_k \lambda_k \Phi_k^{(n)}(P), \tag{4.35}$$

where the second term is the sum over all the possible irrelevant operators with $\Delta_j > 2$ and the third term takes into account all the possible operators localised at branch points P with scaling dimensions Δ_k/n with all the allowed Δ_k including $\Delta_k < 2$. In the power expansion in λ_k every operator appears at least one time since we can use the OPE to simplify the expressions involving an higher number of operators. In the case of an infinite system $\langle \Phi_k^{(n)}(P)\rangle = 0$ and the corrections are of the form $(L\sin(\pi x)/\pi)^{-2\Delta_k/n}$. On the other hand if $\langle \Phi(z)\rangle_{\mathbb{C}} \neq 0$, corrections of the type:$(L\sin(\pi x)/\pi)^{-\Delta_k/n}$ appear.

4.3 The marginal case

In previous sections we have studied the case of relevant and irrelevant operators, now we want to analyse the case of a marginal irrelevant perturbation, that is defined by the limit $\Delta \to 2^+$.
We start our analysis rewriting expression 4.12 in a more suitable way to perform the limit to the marginal case. The easier way to do this is to perform explicitly the subtraction: $\delta F_n - n\delta F_1$.
We have to take care of the cutoff $|w_1 - w_2| > \epsilon$ that is due to the presence of the lattice, as we said in section 4.1.
In order to express this cutoff in a more suitable way for our purpose we rewrite the conformal transformation as:

$$w = \frac{\imath L}{2\pi} \ln\left(\frac{1 - e^{-\imath\pi x} z^n}{1 - e^{\imath\pi x} z^n}\right) \equiv \frac{\imath L}{2\pi} f(z), \tag{4.36}$$

and its derivative as:

$$\frac{dw}{dz} = -\frac{nL\sin(\pi x)}{\pi}\frac{z^{n-1}}{(z^n - e^{\imath\pi x})(z^n - e^{-\imath\pi x})} \equiv -\frac{L\sin(\pi x)}{\pi}g(z). \tag{4.37}$$

Now we can express the cutoff induced by the lattice as function of $f(z_i)$:

$$|w_1 - w_2| = \frac{L}{2\pi}|f(z_1) - f(z_2)|. \tag{4.38}$$

This affects the integral as:

$$\delta F_n^2 = -\frac{g^2}{2}\epsilon^{-2(2-\Delta)} \iint_{|f(z_1)-f(z_2)|>(2\pi\epsilon/L)} \left|\frac{dw_1}{dz_1}\right|^{2-\Delta} \left|\frac{dw_2}{dz_2}\right|^{2-\Delta} \frac{1}{|z_1 - z_2|^{2\Delta}} d^2z_1 d^2z_2. \tag{4.39}$$

The term involved in the subtraction is:

$$-\frac{g^2}{2}\left(\frac{L\sin(\pi x)}{\epsilon}\right)^{4-2\Delta}\int_{\mathcal{R}_n}\frac{2\pi\epsilon^{2-2\Delta}}{2-2\Delta}dw_1 = -\frac{g^2}{2}\left(\frac{L\sin(\pi x)}{\pi\epsilon}\right)^{4-2\Delta}\int |g(z_1)|^2\frac{2\pi\epsilon^{2-2\Delta}}{2-2\Delta}d^2z_1 =$$
$$= -\frac{g^2}{2}\left(\frac{L\sin(\pi x)}{\pi\epsilon}\right)^{4-2\Delta}\int_{|z_1-z_2|>2\pi\epsilon/L|g(z_1)|}\frac{|g(z_1)|^{4-2\Delta}}{|z_1 - z_2|^{2\Delta}}d^2z_1. \tag{4.40}$$

The cutoffs in the expressions 4.39 and 4.40 are different but this difference disappears in the limit $\Delta \to 2^+$. This means that we can replace the cutoff of the integral 4.40 with:

$$|z_1 - z_2| > (2\pi\epsilon/L). \tag{4.41}$$

After the substitution of the cutoff in 4.40, the integrals 4.48 and 4.40 share the same integration region and it is possible to perform the subtraction:

$$\delta F_n - n\delta F_1 =$$
$$-\frac{g^2}{2}\left(\frac{L\sin(\pi x)}{\pi}\right)^{4-2\Delta}\iint_{|f(z_1)-f(z_2)|>2\pi\epsilon/L}\frac{|g(z_1)|^{2-\Delta}|g(z_2)|^{2-\Delta} - |g(z_1)|^{4-2\Delta}}{|z_1 - z_2|^{2\Delta}}d^2z_1 d^2z_2, \tag{4.42}$$

we can put the previous expression in a form that is symmetric in the variables z_1 and z_2:

$$\delta F_n - n\delta F_1 = \frac{g^2}{4}\left(\frac{L\sin(\pi x)}{\pi\epsilon}\right)^{4-2\Delta}\iint_{|f(z_1)-f(z_2)|>2\pi\epsilon/L}\frac{\left(|g(z_1)|^{2-\Delta} - |g(z_2)|^{2-\Delta}\right)^2}{|z_1 - z_2|^{-2\Delta}}d^2z_1 d^2z_2. \tag{4.43}$$

It is clear that in the region $\Delta \approx 2$ this integral is finite also if $\epsilon \approx 0$.
We can also notice that both $g(z_i)$ are regular and consequently the limit $\Delta \to 2$ would return 0 inside the bulk.
The situation changes drastically when both the variables z_1 and z_2 approach a branch point. Lets take the case of the branch point $z = 0$, we have that $|g(z)|^{2-\Delta} \approx |z|^{(n-1)(2-\Delta)}$ and it is clear that the limits $z \to 0$ and $\Delta \to 2^+$ do not commute. The same argument is obviously valid for the other branch point $z \to \infty$.
The previous considerations allow us to restrict the integration region to the set:

$$\{|z_1| < \rho, |z_2| < \rho\} \cup \{|z_1| > \rho^{-1}, |z_2| > \rho^{-1}\} \equiv \mathcal{I}_1 \cup \mathcal{I}_2 \quad 0 < \rho < 1, \tag{4.44}$$

without changing the result of the final result.
We can choose ρ arbitrarily small in order to use the asymptotic expressions for $g(z)$ near the branch points:

$$\iint_{\mathcal{I}_1} \frac{\left(|nz_1^{n-1}|^{2-\Delta} - |nz_2^{n-1}|^{2-\Delta}\right)^2}{|z_1 - z_2|^{2\Delta}} = \tag{4.45}$$
$$= \iint_{\mathcal{I}_1} \frac{|nz_1^2|^{(n-1)(2-\Delta)} + |nz_2^2|^{(n-1)(2-\Delta)} - 2|n^2 z_1 z_2|^{(n-1)(2-\Delta)}}{|z_1 - z_2|^{2\Delta}}.$$

Since we are interested in the behaviour of the integral near $z_i \approx 0$ we can approximate the integrand as:

$$2\iint_{\mathcal{I}_1} \frac{|nz_1|^{(n-1)(2-\Delta)}|nz_2|^{(n-1)(2-\Delta)}}{|z_1 - z_2|^{2\Delta}}, \tag{4.46}$$

where we have neglected the first two terms because they were much smaller than the third. The integral 4.46 is exactly twice what we would obtain for a single branch point and we can replace the expression 4.40 with this one.
In order to obtain an explicit result from 4.46 we use a simple rescaling of the second variable $z_2 = kz_1$, that gives us:

$$2n^{4-2\Delta}\iint \frac{|z_1|^{(n-1)(2-\Delta)}|kz_1|^{(n-1)(2-\Delta)}}{|z_1 - wz_1|^{2\Delta}}|z_1|^2 d^2k d^2z_1 =$$
$$2n^{4-2\Delta}\int |k|^{(n-1)(2-\Delta)}|k-1|^{-2\Delta}d^2k \int |z_1|^{-2-2n(\Delta-2)}d^2z_1, \tag{4.47}$$

using the Mellin transform in the k integral we have:

$$I(n;\Delta) = \int |k|^{(n-1)(2-\Delta)}|k-1|^{-2\Delta}d^2k =$$
$$= \pi \frac{\Gamma\left(1 + \frac{(n+1)(\Delta-2)}{2}\right)\Gamma\left(1 - \frac{(n-1)(\Delta-2)}{2}\right)\Gamma(1-\Delta)}{\Gamma\left(-\frac{(n+1)(\Delta-2)}{2}\right)\Gamma\left(\frac{(n-1)(\Delta-2)}{2}\right)\Gamma(\Delta)}. \tag{4.48}$$

Although the integrals 4.39 and 4.48 are equal only in the region $\Delta \approx 2$, the expression 4.48 exhibits the genuine poles in $\Delta = 1$ and $\Delta = 2n/(n-1)$, that corresponds to n_C.
We can now take the limit $\Delta \to 2^+$ of $I(n;\Delta)$:

$$I(n;\Delta) \approx -\frac{\pi}{4}(n^2-1)(\Delta-2) + O((\Delta-2)^2). \tag{4.49}$$

The integral over z_1 has to be further regulated with another short distance cutoff of the type $\epsilon^{1/n}$ that gives us the result:

$$\int |z_1|^{2n(2-\Delta)-2}d^2z_1 = \frac{2\pi\left(\epsilon^{\frac{1}{n}}\right)^{2n(\Delta-2)}}{2n(\Delta-2)} = \frac{2\pi\epsilon^{2n(\Delta-2)}}{2(\Delta-2)} \approx \frac{\pi}{n(\Delta-2)}. \tag{4.50}$$

The final result is obtained multiplying the two previous expressions:

$$\delta F_n - n\delta F_1 = g^2\left(\frac{L\sin(\pi x)}{\pi\epsilon}\right)^{4-2\Delta}\left(\frac{n^2-1}{n}\right)\left(\frac{\pi^2}{4} + O(\Delta-2)^2\right). \tag{4.51}$$

The limit $\Delta \to 2^+$ gives us an uninteresting constant, namely:

$$\lim_{\Delta\to 2^+} (\delta F_n - n\delta F_1) = \frac{\pi^2 g^2}{4}\left(\frac{n^2-1}{n}\right). \tag{4.52}$$

This result forces us to take into account the third order in the coupling constant g that is a triple integral of the type like:

$$\frac{g^3}{6} b \left(\frac{L\pi \sin(\pi x)}{\pi\epsilon}\right)^{6-3\Delta} \iiint_{\mathbb{C}} \frac{|g(z_1)|^{2-\Delta}|g(z_2)|^{2-\Delta}|g(z_3)|^{2-\Delta}}{|z_1-z_2|^{\Delta}|z_1-z_3|^{\Delta}|z_2-z_3|^{\Delta}} d^2z_1 d^2z_2 d^2z_3, \tag{4.53}$$

where b is the universal coefficient of an OPE.
Again we have that the integral measure is concentrated near the branch points, the short distance divergences are eliminated in the subtraction and the limit $\Delta \to 2$ gives us again a finite result. A similar calculation was performed in Ref.[56] and Ref.[57] for the corrections to the free energy of a cylinder that could be seen as the limit $n \to 0$ of our previous computations.
A more direct way to compute the $O(g^3)$ is the one that involves the Zamolodchikov's c-theorem exposed in Ref. [58]. In order to use it we need the logarithmic derivative of the free energy with respect to the cutoff.
We start from the adimensional free energy up to g^2:

$$F_n - nF_1 = -\frac{c}{6}\left(n-\frac{1}{n}\right)\ln\left(\frac{L}{\pi\epsilon}\sin(\pi x)\right) + \left(n-\frac{1}{n}\right)\frac{g^2\pi^2}{4}\left(\frac{L\sin(\pi x)}{\pi\epsilon}\right)^{4-2\Delta}(1+O(\Delta-2)^2). \tag{4.54}$$

The logarithmic derivative of the previous expression in the region $\Delta \approx 2$ is:

$$-\epsilon\frac{\partial(F_n - nF_1)}{\partial\epsilon} = -\frac{c}{6}\left(n-\frac{1}{n}\right) - \frac{g^2\pi^2}{2}\left(n-\frac{1}{n}\right)(\Delta-2) \equiv -\frac{c_{\text{eff}}}{6}\left(n-\frac{1}{n}\right), \tag{4.55}$$

where we used the effective central charge defined as:

$$c_{\text{eff}} = c - 3g^2\pi^2(2-\Delta). \tag{4.56}$$

The C-theorem states that there exists a function $C(g)$ that decreases along the renormalization group flow and it is stationary at fixed points where it is equal to the conformal anomaly of the corresponding CFT.
Zamolodchikov also demonstrated that:

$$C'(g) \propto (1+O(g^2))\beta(g), \tag{4.57}$$

where β is the beta function:

$$\beta(g) = -\epsilon\frac{\partial g}{\partial\epsilon}, \tag{4.58}$$

where the derivative is performed at fixed coupling.
In a perturbed CFT like 4.6 the first two terms of the β function are known to be universal, Ref.[58] and Ref.[59], and they are equal to:

$$\beta(g) = (\Delta-2)g + \pi b g^2 + O(g^3), \tag{4.59}$$

and up to this order we have that:

$$c'_{\text{eff}} \propto \beta(g). \tag{4.60}$$

We can now find the coefficient of the term g^3 applying the C-Theorem 4.56:

$$c_{\text{eff}} = c+3\pi^2(\Delta-2)g^2+\gamma g^3 \rightarrow c'_{\text{eff}} = 6\pi^2(\Delta-2)g+3\gamma g^2 = 6\pi^2\left((\Delta-2)g+\frac{\gamma}{2\pi^2}g^2\right). \quad (4.61)$$

Comparing this expression with the β function 4.59 we get:

$$\gamma = 2\pi^3 b. \quad (4.62)$$

The central charge up to the third order in the adimensional coupling constant is:

$$c_{\text{eff}} = c + 3\pi^2(\Delta-2)g^2 + 2\pi^3 b g^3. \quad (4.63)$$

The problem of the previous method is that there could be logarithmic corrections in $L\sin(\pi x)/\pi$ due to the pole around $\Delta = 2$, and our previous analysis is not able to reproduce them. The solution is the introduction of the "RG-improving" that can reproduce the logarithmic divergences.
In the following we will perform all calculations in an infinite system where the main parameter is the length of the subsystem l but all results that we will obtain can be easily translated in the case of a finite system with the usual replacement:

$$l \rightarrow \frac{L\sin(\pi x)}{\pi}. \quad (4.64)$$

We can start our analysis from the RG-improving flow equation:

$$l\frac{dg(l)}{dl} = -\beta(g(l)). \quad (4.65)$$

In the case of $\Delta > 2$ the perturbation is irrelevant and the previous equation becomes, at the leading order in g:

$$l\frac{\partial g}{\partial l} = (2-\Delta)g, \quad (4.66)$$

that it is solved by:

$$\frac{g}{g_0} = \left(\frac{l}{\epsilon}\right)^{-(\Delta-2)}, \quad (4.67)$$

and the effective central charge tends to c when g tends to 0:

$$c_{\text{eff}} = c - 3\pi^2(2-\Delta)g^2 \rightarrow c. \quad (4.68)$$

In the case $\Delta < 2$ the perturbation is relevant and there are two possibilities. We could be in the case $g(l) \rightarrow g^*$ where:

$$\beta(g^*) = 0 \rightarrow g^* = \frac{2-\Delta}{\pi b}, \quad (4.69)$$

and the effective central charge is:

$$c_{\text{eff}} = c - \frac{3(2-\Delta)^3}{b^2}. \quad (4.70)$$

On the other hand we could be in the case where $g(l)$ cannot be more described by the perturbation theory, for a complete description of these two situations see Ref.[57].
The interesting case for us is the marginal one defined by $\Delta = 2$ where the β function takes the form:

$$\beta = \pi b g^2. \quad (4.71)$$

The differential equation that defines the coupling $g(l)$ is:

$$l\frac{\partial g}{\partial l} = -\pi b g^2, \tag{4.72}$$

and its solution is:

$$g = \frac{g_0}{1 + g_0 \pi b \ln(l/\epsilon)} \approx \frac{1}{b\pi \ln(l/\epsilon)}. \tag{4.73}$$

The behaviour of the effective central charge is:

$$c_{\text{eff}} = c + \frac{2}{b^2 \left(\ln(l/\epsilon)\right)^3}. \tag{4.74}$$

After the integration with respect to the variable $\ln \epsilon$ we get the result final:

$$c - \frac{1}{b^2(\ln(l))^3}. \tag{4.75}$$

It is important to notice that the central charge is reached from below, and it is an apparent violation of the C-theorem but this happens because we are studying the adimensional free energy instead of the entanglement entropy. In fact the entanglement entropy is a quantity ultraviolet finite at the fixed points, see Ref.[60], and it reaches the value of the central charge c from above in perfect agreement with the C-theorem.

Chapter 5

Unusual correction to scaling in excited states of conformal field theory

In the previous chapters we showed how the replica trick approach can be used to compute the entanglement entropy of an 1D quantum system described in the continuum limit by a conformal field theory and in particular we analysed the unusual corrections to scaling. We demonstrated how an irrelevant operator in the bulk hamiltonian can be responsable of unusual correction to scaling of the type:

$$\left(\frac{L\sin(\pi x)}{\pi}\right)^{-\frac{2\Delta}{n}}. \tag{5.1}$$

In this chapter we will demonstrate that the same arguments used in Chapt.4 hold also in the case of excited states of the conformal field theory. In conclusion we can say that the unusual corrections to scaling are present also in the excited states of conformal field theory and they are due to the same mechanism that is valid in the ground state case.

5.1 Perturbation of the F function

In Chapt.3 we computed the quantity:

$$F^n_\Upsilon(x) \equiv \frac{Tr\rho^n_\Upsilon}{Tr\rho^n_I} = \exp\left((1-n)(S_\Upsilon - S_{gs})\right), \tag{5.2}$$

using the Cardy-Calabrese approach described in Chapt.2.
We demonstrated how it can be rewritten as a ratio involving correlation functions evaluated over the Riemann surface $\mathcal{R}_n$, see Chapt.3:

$$F^n_\Upsilon(x) = \lim_{w\to -\imath\infty} \frac{\langle\prod_{j=0}^{n-1}\Upsilon_j(w)\Upsilon_j^\dagger(-w)\rangle_{\mathcal{R}_n}}{\left[\langle\Upsilon_0(w)\Upsilon_0^\dagger(-w)\rangle_{\mathcal{R}_1}\right]^n}. \tag{5.3}$$

As we did in Chapt.4, we want to study the scaling behaviour of the corrections to entanglement entropy. They are due to the presence in the action of an irrelevant bulk operator

$\Phi(w)$, that changes the action in:

$$S = S_0 + \lambda \int_{\mathcal{R}_n} d^2 w \Phi(w), \tag{5.4}$$

where $\Phi(w)$ is a bulk irrelevant scalar operator, with conformal dimension $\Delta > 2$ and the action S_0 is the conformal invariant action. Obviously we assume that $\lambda \ll 1$ in order to allow us to evaluate a meaningful perturbative expansion.
The function $F^{(n)}_{\Upsilon}$ of 5.3 computed on the perturbed action 5.4 will be denoted by $\tilde{F}^{(n)}_{\boldsymbol{\Upsilon}}$ and it is formally given by the expression:

$$\tilde{F}^n_{\Upsilon}(x) = \lim_{w \to -\imath\infty} \frac{\langle \prod_{j=0}^{n-1} \Upsilon_j(w) \Upsilon_j^\dagger(-w) e^{-\lambda \int_{\mathcal{R}_n} d^2 w \, \Phi(w)} \rangle_{\mathcal{R}_n}}{\left[\langle \Upsilon_0(w) \Upsilon_0^\dagger(-w) e^{-\lambda \int_{\mathcal{R}_1} d^2 w \Phi(w)} \rangle_{\mathcal{R}_1} \right]^n}, \tag{5.5}$$

where the correlation functions are computed using the conformal invariant action S_0.
We can rewrite this expression in terms of the $F^n_{\Upsilon}(x)$, described in Chapt.3:

$$\begin{aligned}
\tilde{F}^n_{\Upsilon}(x) &= \lim_{w \to -\imath\infty} \frac{\langle \prod_{j=0}^{n-1} \Upsilon_j(w) \Upsilon_j^\dagger(-w) \exp\left(-\lambda \int_{\mathcal{R}_n} d^2 w \, \Phi(w)\right) \rangle_{\mathcal{R}_n}}{\left[\langle \Upsilon_0(w) \Upsilon_0^\dagger(-w) \exp\left(-\lambda \int_{\mathcal{R}_1} d^2 w \Phi(w)\right) \rangle_{\mathcal{R}_1} \right]^n} = \\
&= F^n_{\Upsilon}(x) \lim_{w \to -\imath\infty} \left[\left(\frac{\langle \Upsilon_0(w) \Upsilon_0^\dagger(-w) \exp\left(-\lambda \int_{\mathcal{R}_1} d^2 w \Phi(w)\right) \rangle_{\mathcal{R}_1}}{\langle \Upsilon_0(w) \Upsilon_0^\dagger(-w) \rangle_{\mathcal{R}_1}} \right)^{-n} \times \right. \\
&\left. \times \left(\frac{\langle \prod_{j=0}^{n-1} \Upsilon_j(w) \Upsilon_j^\dagger(-w) \exp\left(-g \int_{\mathcal{R}_n} d^2 w \, \Phi(w)\right) \rangle_{\mathcal{R}_n}}{\langle \prod_{j=0}^{n-1} \Upsilon_j(w) \Upsilon_j^\dagger(-w) \rangle_{\mathcal{R}_n}} \right) \right].
\end{aligned}$$

The exponential functions can be evaluated as power expansions, this is possible because we are assuming $\lambda \ll 1$. We can evaluate the function $\tilde{F}^{(n)}_{\boldsymbol{\Upsilon}}$ order by order in the perturbation theory computing all the Feynman diagrams of the theory defined on the Riemann surface $\mathcal{R}_n$, as we did in section 4.1.
The diagrams involved in the evaluation of the expression are both connected and disconnected as usual in quantum field theory.
In order to evaluate only quantities defined by sums of connected Feynman diagrams we take the logarithm of the $\tilde{F}$ and we expand in power series of the coupling λ:

$$\begin{aligned}
&\ln\left(\tilde{F}^n_{\Upsilon}(x)\right) = \ln\left(F^n_{\Upsilon}(x)\right) + \\
&+ \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{k!} \int \cdots \int_{\mathcal{R}_n} \frac{\langle \prod_j \Upsilon_j(w) \Upsilon_j^\dagger(-w) \Phi(w_1) \Phi(w_2) \ldots \Phi(w_k) \rangle_{\mathcal{R}_n}}{\langle \prod_{j=0}^{n-1} \Upsilon_j(w) \Upsilon_j^\dagger(-w) \rangle_{\mathcal{R}_n}} d^2 w_1 \ldots d^2 w_k \\
&- n \left(\sum_{k=1}^{\infty} \frac{(-\lambda)^k}{k!} \int \cdots \int_{\mathcal{R}_1} \frac{\langle \Upsilon_0(w) \Upsilon_0^\dagger(-w) \Phi(w_1) \Phi(w_2) \ldots \Phi(w_k) \rangle_{\mathcal{R}_1}}{\langle \Upsilon_0(w) \Upsilon_0^\dagger(-w) \rangle_{\mathcal{R}_1}} d^2 w_1 \ldots d^2 w_k \right).
\end{aligned} \tag{5.6}$$

From Ref.[45] we know how to rewrite the dimensional coupling constant λ in terms of the adimensional coupling constant g:

$$\lambda = \frac{g}{\epsilon^{\Delta - 2}}. \tag{5.7}$$

The power series written int 5.6 is related to the Renyi entropy using the definition of the $F_{\Upsilon}^{(n)}$:

$$\ln F_{\Upsilon}^{(n)} = (1-n)\,(S_{\Upsilon}^{n} - S_{\mathbf{I}}^{n})\,. \tag{5.8}$$

The unusual corrections to scaling arise from the divergences due to the local breaking of the conformal invariance near the conical singularities. We will study expression 5.6 order by order in λ looking for divergences as we did for the ground state in section 4.1.
We expect from the ground state case studied in Chapt.4 that the unusual corrections to scaling would come from divergences in the λ^2 term of 5.6 and the order λ will not give further contribution at the scaling behaviour of the Renormalization Group.
This means that we will focus our attention on the λ term:

$$\int_{\mathcal{R}_n} \frac{\langle \prod_{j=0}^{n-1} \Upsilon_j(w)\Upsilon_j^{\dagger}(-w)\Phi(w_1)\rangle_{\mathcal{R}_n}}{\langle \prod_{j=0}^{n-1} \Upsilon_j(w)\Upsilon_j^{\dagger}(-w)\rangle_{\mathcal{R}_n}} d^2w_1 - \int_{\mathcal{R}_1} \frac{\langle \Upsilon_0(w)\Upsilon_0^{\dagger}(-w)\Phi(w_1)\rangle_{\mathcal{R}_1}}{\langle \Upsilon_0(w)\Upsilon_0^{\dagger}(-w)\rangle_{\mathcal{R}_1}} d^2w_1, \tag{5.9}$$

and on the most important λ^2 term:

$$\int_{\mathcal{R}_n} \frac{\langle \prod_{j=0}^{n-1} \Upsilon_j(w)\Upsilon_j^{\dagger}(-w)\Phi(w_1)\Phi(w_2)\rangle_{\mathcal{R}_n}}{\langle \prod_{j=0}^{n-1} \Upsilon_j(w)\Upsilon_j^{\dagger}(-w)\rangle_{\mathcal{R}_n}} - \int_{\mathcal{R}_1} \frac{\langle \Upsilon_0(w)\Upsilon_0^{\dagger}(-w)\Phi(w_1)\Phi(w_2)\rangle_{\mathcal{R}_1}}{\langle \Upsilon_0(w)\Upsilon_0^{\dagger}(-w)\rangle_{\mathcal{R}_1}}. \tag{5.10}$$

5.2 The λ term living on $\mathcal{R}_1$

We start our analysis from the simplest term in the expansion 5.6, that is the order λ term onto the Riemann surface $\mathcal{R}_1$, namely:

$$\lim_{w\to -\imath\infty} \frac{\langle \Upsilon_0(w)\Upsilon_0^{\dagger}(-w)\Phi(w_1)\rangle_{\mathcal{R}_1}}{\langle \Upsilon_0(w)\Upsilon_0^{\dagger}(-w)\rangle_{\mathcal{R}_1}}, \tag{5.11}$$

where the limit fixes the perturbation fields in the "in" and "out" states mapped on the complex plane, as we did in Chapt.4.
We can map the correlation functions from $\mathcal{R}_1$ to $\mathbb{C}$ using the conformal transformation:

$$\frac{dw}{dz} = \left(-\frac{L\sin(\pi x)}{\pi}\right) \frac{1}{(e^{\imath\pi x}-z)\,(e^{-\imath\pi x}-z)},$$

that is the mapping 3.17 where we fixed $n=1$ because now we are on the Riemann surface $\mathcal{R}_1$. The quantity 5.11 transforms as usual in CFT, see Ref.[45]:

$$\lim_{w\to -\imath\infty} \frac{\langle \Upsilon_0(w)\Upsilon_0^{\dagger}(-w)\Phi(w_1)\rangle_{\mathcal{R}_1}}{\langle \Upsilon_0(w)\Upsilon_0^{\dagger}(-w)\rangle_{\mathcal{R}_1}} = \tag{5.12}$$
$$\left|\left(\frac{L\sin(\pi x)}{\pi}\right) \frac{1}{(e^{\imath\pi x}-z)\,(e^{-\imath\pi x}-z)}\right|^{-2h_\Phi} \frac{\langle \Upsilon(z_{out})\Upsilon^{\dagger}(z_{in})\Phi(z_1)\rangle_{\mathbb{C}}}{\langle \Upsilon(z_{out})\Upsilon^{\dagger}(z_{in})\rangle_{\mathbb{C}}},$$

where h_Φ is the holomorphic dimension of the spinless field Φ.
In the mapping from the Riemann surface to the complex plane the transformation factors due to perturbation operators cancel out in the ratio.
It is important because if we would evaluate separately the denominator and the numerator we would obtain a vanishing correlation functions in the limit $w \to \pm\imath\infty$.

The three point functions on the complex plane are fixed by conformal invariance, see Ref.[45], and we use their explicit expression:

$$\frac{\langle\Upsilon(z_0')\Upsilon^\dagger(z_0)\Phi(z_1)\rangle_{\mathbb{C}}}{\langle\Upsilon(z_0')\Upsilon^\dagger(z_0)\rangle_{\mathbb{C}}} = \left(\frac{1}{(z_0'-z_0)^{2h-h_\Phi}(z_0'-z_1)^{h_\Phi}(z_0-z_1)^{h_\Phi}}\right)\times$$
$$\times\left(\frac{1}{(\bar{z}_0'-\bar{z}_0)^{-h_\Phi}(\bar{z}_0'-\bar{z}_1)^{h_\Phi}(\bar{z}_0-\bar{z}_1)^{h_\Phi}}\right)\left(\frac{1}{(z_0'-z_0)^{2h}}\right)^{-1} = \left(\frac{|z_0'-z_0|}{|z_0-z_1|\,|z_0'-z_1|}\right)^{2h_\Phi}.$$

We will consider operators Υ with holomorphic dimension $h_\Upsilon = h$ and antiholomorphic dimension $\bar{h}_\Upsilon = 0$, that is the particular case of the operator $\imath\partial\phi$ (the most interesting one in bosonic theory).
Taking now the limit of 5.13 to the in and out states onto $\mathbb{C}$:

$$z = e^{\imath\pi x}$$
$$z' = e^{-\imath\pi x},$$

we obtain the ratio 5.11 as a function only of the variable z_1:

$$\frac{\langle\Upsilon_0(z')\Upsilon_0^\dagger(z)\Phi(z_1)\rangle_{\mathbb{C}}}{\langle\Upsilon_0(z')\Upsilon_0^\dagger(z)\rangle_{\mathbb{C}}} = \left(\frac{|2\sin(\pi x)|}{|e^{\imath\pi x}-z_1|\,|e^{-\imath\pi x}-z_1|}\right)^{2h_\Phi}.$$

The correlation function on the Riemann surface $\mathcal{R}_1$ can be easily computed using the expression 5.12:

$$\lim_{w\to-\imath\infty}\frac{\langle\Upsilon_0(w)\Upsilon_0^\dagger(-w)\Phi(w_1)\rangle_{\mathcal{R}_1}}{\langle\Upsilon_0(w)\Upsilon_0^\dagger(-w)\rangle_{\mathcal{R}_1}} =$$
$$\left|\left(\frac{L\sin(\pi x)}{\pi}\right)\frac{1}{(e^{\imath\pi x}-z_1)\,(e^{-\imath\pi x}-z_1)}\right|^{-2h_\Phi}\left(\frac{|2\sin(\pi x)|}{|e^{\imath\pi x}-z_1|\,|e^{-\imath\pi x}-z_1|}\right)^{2h_\Phi} =$$
$$= \left(\frac{2\pi}{L}\right)^{2h_\Phi}. \tag{5.13}$$

The order λ in the power expansion 5.6 is:

$$\frac{1}{\epsilon^{\Delta-2}}\int_{\mathcal{R}_1} d^2w_1\left(\frac{2\pi}{L}\right)^{2h_\Phi} = \left(\frac{2\pi}{L}\right)^{2h_\Phi}\frac{\mathcal{A}(\mathcal{R}_1)}{\epsilon^{\Delta-2}}, \tag{5.14}$$

where $\mathcal{A}(\mathcal{R}_1)$ is the area of the surface $\mathcal{R}_1$.
This integral has not to be further regulated and it exhibits the same dependence on the parameter ϵ of the finite size scaling behaviour of the renormalization group. We know from Chapt.4 that this behaviour is always negligible compared to the one of the unusual corrections to scaling.

5.3 The the λ^2 term living on $\mathcal{R}_1$

As we expected from the discussion on the ground state, the order λ does not give further contributions to the finite size scaling behaviour of the renormalization group flow and this is an important check because we have demonstrated that this holds for excited states as well.

Now we want to move to the more interesting order λ^2 term because we expect that it will be the one that has to be regulated and it will lead to the unusual corrections.
We look at the simplest part of the λ^2 term that is the ratio between correlation functions computed on the Riemann surface $\mathcal{R}_1$ of 5.6, namely:

$$\lim_{w\to-\imath\infty}\frac{\langle\Upsilon_0(w')\Upsilon_0^\dagger(w)\Phi(w_1)\Phi(w_2)\rangle_{\mathcal{R}_1}}{\langle\Upsilon_0(w')\Upsilon_0^\dagger(w)\rangle_{\mathcal{R}_1}}, \tag{5.15}$$

as usual we can map it from $\mathcal{R}_1$ to the complex plane $\mathbb{C}$:

$$\frac{\langle\Upsilon_0(w')\Upsilon_0^\dagger(w)\Phi(w_1)\Phi(w_2)\rangle_{\mathcal{R}_1}}{\langle\Upsilon_0(w')\Upsilon_0^\dagger(w)\rangle_{\mathcal{R}_1}}=\left|\frac{dw_1}{dz_1}\right|^{-2h_\Phi}\left|\frac{dw_2}{dz_2}\right|^{-2h_\Phi}\frac{\langle\Upsilon(z_0')\Upsilon^\dagger(z_0)\Phi(z_1)\Phi(z_2)\rangle_{\mathbb{C}}}{\langle\Upsilon(z_0')\Upsilon^\dagger(z_0)\rangle_{\mathbb{C}}},$$

where the Υ operators are evaluated in the in and out points mapped onto $\mathbb{C}$.
The four points function is not totally fixed by conformal invariance since we can always perform a global conformal transformation that maps 4 points (z_1,z_2,z_3,z_4) into $(0,1,\infty,\eta)$. This means that the four point function is known apart from an invariant function of η that is defined as:

$$\eta=\frac{(z_0'-z_1)(z_0-z_2)}{(z_0'-z_2)(z_0-z_1)}. \tag{5.16}$$

The four point function in 5.15 is:

$$\begin{aligned}&\langle\Upsilon(z_0')\Upsilon(z_o)\Phi(z_1)\Phi(z_2)\rangle_{\mathbb{C}}=\\&=f(\eta;\bar\eta)(z_0'-z_0)^{-2h}|z_1-z_2|^{-4h_\Phi}\left(\frac{(\eta-1)^2}{\eta}\right)^{\frac{h}{3}}\left|\frac{(\eta-1)^2}{\eta}\right|^{\frac{2}{3}h_\Phi},\end{aligned} \tag{5.17}$$

In order to demonstrate how we obtained the previous result we will compute separately the holomorphic and antiholomorphic parts, they are different because the fields Υ have $\bar h_\Upsilon=0$. The holomorphic part of the four points correlation function is, see Ref.[45]:

$$\begin{aligned}&(z_0'-z_0)^{\frac{2}{3}h+\frac{2}{3}h_\Phi-2h}(z_0'-z_1)^{\frac{2}{3}h+\frac{2}{3}h_\Phi-h-h_\Phi}(z_0-z_1)^{\frac{2}{3}h+\frac{2}{3}h_\Phi-h-h_\Phi}\\&(z_0'-z_2)^{\frac{2}{3}h+\frac{2}{3}h_\Phi-h-h_\Phi}(z-z_2)^{\frac{2}{3}h+\frac{2}{3}h_\Phi-h-h_\Phi}(z_1-z_2)^{\frac{2}{3}h+\frac{2}{3}h_\Phi-2h_\Phi}=\\&=(z_0'-z_0)^{-2h}\left(\frac{(z_0'-z_0)^2(z_1-z_2)^2}{(z_0'-z_1)(z_0-z_1)(z_0'-z_2)(z_0-z_2)}\right)^{\frac{h}{3}}\times\\&\times\left(\frac{(z_0'-z_0)^2(z_1-z_2)^2}{(z_0'-z_1)(z_0-z_1)(z_0'-z_2)(z_0-z_2)}\right)^{\frac{h_\Phi}{3}}.\end{aligned}$$

The antiholomorphic part can be compute in the same way taking into the account that the excitation operators Υ have zero antiholomorphic dimension:

$$(\bar z_1-\bar z_2)^{-2h_\Phi}\left(\frac{(\bar z_0'-z_0)^2(z_1-\bar z_2)^2}{(\bar z_0'-\bar z_1)(\bar z_0-\bar z_1)(\bar z_0'-\bar z_2)(\bar z_0-\bar z_2)}\right)^{\frac{h_\Phi}{3}}. \tag{5.18}$$

We obtain the algebraic part of the correlation function simply multiplying the holomorphic and anti holomorphic parts previously obtained:

$$\begin{aligned}(z_0'-z_0)^{-2h}|z_1-z_2|^{-4h_\Phi}&\left(\frac{(z_0'-z_0)^2(z_1-z_2)^2}{(z_0'-z_1)(z_0-z_1)(z_0'-z_2)(z_0-z_2)}\right)^{\frac{h}{3}}\times\\&\times\left|\frac{(z_0'-z_0)^2(z_1-z_2)^2}{(z_0'-z_1)(z_0-z_1)(z_0'-z_2)(z_0-z_2)}\right|^{\frac{2}{3}h_\Phi}.\end{aligned}$$

We can express several combinations of the variables as functions of the anharmonic ratio η:

$$\frac{\eta-1}{\eta}=\frac{(z_0'-z_0)(z_1-z_2)}{(z_0'-z_1)(z_0-z_2)}, \tag{5.19}$$

$$\eta-1=\frac{(z_0'-z_0)(z_1-z_2)}{(z_0'-z_2)(z_0-z_1)}. \tag{5.20}$$

Now we have the ratio between the four and two points functions on the Riemann surface $\mathcal{R}_1$ as:

$$\begin{aligned}&\frac{\langle\Upsilon_0(w')\Upsilon_0^\dagger(w)\Phi(w_1)\Phi(w_2)\rangle_{\mathcal{R}_1}}{\langle\Upsilon_0(w')\Upsilon_0^\dagger(w)\rangle_{\mathcal{R}_1}}=\\&=\left|\frac{dw_1}{dz_1}\right|^{-2h_\Phi}\left|\frac{dw_2}{dz_2}\right|^{-2h_\Phi}f(\eta;\bar\eta)|z_1-z_2|^{-4h_\Phi}\left(\frac{(\eta-1)^2}{\eta}\right)^{\frac{h}{3}}\left|\frac{(\eta-1)^2}{\eta}\right|^{\frac{2}{3}h_\Phi}.\end{aligned} \tag{5.21}$$

It is important to notice that the coordinates z_1 and z_2 appear in the same combination as in the ground state case, see 4.12, but multiplied by an invariant function of the parameter η, namely:

$$F(\eta;\bar\eta)=f(\eta;\bar\eta)\left(\frac{(\eta-1)^2}{\eta}\right)^{\frac{h}{3}}\left|\frac{(\eta-1)^2}{\eta}\right|^{\frac{2}{3}h_\Phi}. \tag{5.22}$$

We know from the ground state discussion that the measure is concentrated around the conical singularities and this means that we have to look at the behaviour of the correlation functions near those points.
We look at the behaviour of the function $F(\eta)$ around the conical singularities. Its argument η tends to:

$$\eta=\frac{(z_0'-z_1)(z_0-z_2)}{(z_0'-z_2)(z_0-z_1)}\to\frac{z_0'}{z_0}, \tag{5.23}$$

and the amplitude $F(\eta)$ does not have poles in these points. This means that the divergences eventually will come from the algebraic part of the correlation functions, in analogy with the case studied in section 4.1. The function $F(\eta)$ tends to a constant:

$$F(\eta;\bar\eta)\equiv f(\eta;\bar\eta)\left(\frac{(\eta-1)^2}{\eta}\right)^{\frac{1}{3}h}\left|\frac{(\eta-1)^2}{\eta}\right|^{\frac{2}{3}h_\Phi}\to F\left(\frac{z_0'}{z_0};\frac{\bar z_0'}{\bar z_0}\right). \tag{5.24}$$

This constant is different from zero from the general properties of the $F(\eta)$ function. This constant will not affect the behaviour of the integrals around the conical singularities.
As we did in Chapt.4 we linearize the mapping $\mathcal{R}_1\to\mathbb{C}$ around the conical singularities. Around $z_1\approx 0$ it takes the form:

$$z_1\approx-\frac{\pi}{L\sin(\pi x)}w_1\to\frac{dz_1}{dw_1}=-\frac{\pi}{L\sin(\pi x)},$$

and the integral in z_1 in this region behaves like:

$$\int_{\mathbb{C}}d^2z_1\left|\frac{dw_1}{dz_1}\right|^{2-2h_\Phi}\approx\int_{\mathcal{R}_1}d^2w_1\left|\frac{\pi}{L\sin(\pi x)}\right|^{2h_\Phi}=2\pi\left|\frac{L\sin(\pi x)}{\pi}\right|^{-2h_\Phi}\epsilon^2. \tag{5.25}$$

The mapping 3.17 for $z_2 \to \infty$ is:

$$z_2 = \left(\frac{L\sin(\pi x)}{\pi(w_2 - l)}\right) = \left(\frac{L\sin(\pi x)}{\pi}\right)(w_2 - l)^{-1} \to \frac{dz_2}{dw_2} = -\left(\frac{L\sin(\pi x)}{\pi}\right)(w_2 - l)^{-2},$$

and the integral approximated for large values of the variable z_2 is:

$$\int_{\mathbb{C}} d^2 z_2 \left|\frac{dw_2}{dz_2}\right|^2 \left|\frac{dz_2}{dw_2}\right|^{2h_\Phi} |z_2|^{-4h_\Phi} = \left|\frac{L\sin(\pi x)}{\pi}\right|^{-2h_\Phi} \int_{\mathcal{R}_1} d^2 w_2 = 2\pi \left|\frac{L\sin(\pi x)}{\pi}\right|^{-2h_\Phi} \epsilon^2. \tag{5.26}$$

The previous expressions have the same divergences of the integral studied in 4.1 and it has to be further regulated with additional cutoffs.
We have to take into account the finite size scaling of the renormalization group flow, as we did in section 4.1. This is simply done expressing the dimensional coupling constant λ in terms of the cutoff and the adimensional coupling constant g.
The total ϵ dependence of the term λ^2 living onto $\mathcal{R}_1$ is:

$$\frac{\left(\epsilon^2\right)^2}{\epsilon^{4-4h_\Phi}} = \epsilon^{4h_\Phi}. \tag{5.27}$$

As usual we compute the finite size scaling corrections in the combination $L\sin(\pi x)/\pi$ by scaling:

$$\epsilon^{4h_\Phi} \to \left(\frac{L\sin(\pi x)}{\pi}\right)^{-4h_\Phi} = \left(\frac{L\sin(\pi x)}{\pi}\right)^{-2\Delta}, \tag{5.28}$$

where we used the definition of scaling dimension for a spinless field like the Φ field:

$$2h_\Phi = \Delta. \tag{5.29}$$

The form of the correction is the same of unusual corrections to scaling found in the ground state case, section 4.1, with $n = 1$:

$$\left(\frac{L\sin(\pi x)}{\pi}\right)^{-\frac{2\Delta}{n}} \to \left(\frac{L\sin(\pi x)}{\pi}\right)^{-2\Delta}. \tag{5.30}$$

5.4 The λ^2 term living on $\mathcal{R}_n$

In previous sections we computed the two terms of the expansion 5.6 living on the surface $\mathcal{R}_1$, in both of them we have found the same scaling behaviour of the ground state case. Now we compute the other term of order λ^2 that is living on the Riemann surface $\mathcal{R}_n$. We expect that the following computation will confirm the result found in previous sections and it will lead us again to the unusual corrections.
The λ^2 term living on the surface $\mathcal{R}_n$ is:

$$\frac{g^2}{2}\int_{\mathcal{R}_n} d^2 w_1 \int_{\mathcal{R}_n} d^2 w_2 \frac{\langle \prod_{j=0}^{n-1} \Upsilon_j(w)\Upsilon_j^\dagger(-w)\Phi(w_1)\Phi(w_2)\rangle_{\mathcal{R}_n}}{\langle \prod_{j=0}^{n-1} \Upsilon_j(w)\Upsilon_j^\dagger(-w)\rangle_{\mathcal{R}_n}}, \tag{5.31}$$

and the difficult is to compute the arbitrarily long correlation functions in the numerator and denominator.
A very simple method to do this is to compute in primis the correlation functions in the

case of vertex operators, assuming the neutrality condition fulfilled and then we will move back to the initial case restoring the scaling dimension of fields.
The simplification comes from the fact that correlation functions between an arbitrary large number of vertex operators is well known, see Ref.[45]:

$$\langle \prod_{i=0}^{n-1} \mathcal{V}_{\alpha_i}(z_i) \rangle_{\mathbb{C}} = \prod_{i<j} |z_i - z_j|^{4\alpha_i \alpha_j} . \tag{5.32}$$

The neutrality condition that provides a non zero result is:

$$\sum_i \alpha_i = 0. \tag{5.33}$$

The vertex operator is defined in a bosonic free theory as:

$$\mathcal{V}_\alpha \equiv : e^{\sqrt{2}\imath\alpha\Phi(z)} : . \tag{5.34}$$

Using the OPE between the vertex operator and the normal ordered energy momentum tensor of bosonic theory we can obtain the holomorphic and antiholomorphic dimension of the vertex operator:

$$h = \bar{h} = \alpha^2. \tag{5.35}$$

The last equation provides a relation between the α and $\bar{\alpha}$ parameters and the dimensions h, $\bar{h}$ of a generic operator.
In the case of the spinless field Φ we have the usual relation $h_\Phi = \bar{h}_\Phi$ and we have the same relation valid for vertex operators:

$$\alpha_\Phi = \pm\sqrt{h_\Phi}. \tag{5.36}$$

The field Υ is different, since it has only holomorphic dimension h, this means that we have to assign it only the holomorphic α and antiholomorphic $\bar{\alpha}$ is zero, this choice make the antiholomorphic neutrality condition automatically satisfied.
A possible choice of the parameters for the operators Υ and $\Upsilon^\dagger$ is:

$$(\alpha_\Upsilon; \bar{\alpha}_\Upsilon) = (\sqrt{h}; 0), \tag{5.37}$$

$$(\alpha_{\Upsilon^\dagger}; \bar{\alpha}_{\Upsilon^\dagger}) = (-\sqrt{h}; 0). \tag{5.38}$$

We have a correlation function with n Υ operators and n $\Upsilon^\dagger$, n for the in and n for the out states, this means that the choice 5.37 automatically fulfils the neutrality condition.
A possible choice of parameters of Φ fields is:

$$(\alpha_{\Phi_1}; \bar{\alpha}_{\Phi_1}) = (\sqrt{h_\Phi}; -\sqrt{h_\Phi}), \tag{5.39}$$

$$(\alpha_{\Phi_2}; \bar{\alpha}_{\Phi_2}) = (-\sqrt{h_\Phi}; \sqrt{h_\Phi}). \tag{5.40}$$

Using the definition of correlation function between vertex operators we can compute the ratio:

$$\frac{\langle \prod_{i=0}^{n-1} \mathcal{V}_{\alpha_i}(z_i') \mathcal{V}^\dagger_{\alpha_i}(z_i) \mathcal{V}_{\alpha_{\Phi_1}}(z_1) \mathcal{V}_{\alpha_{\Phi_2}}(z_2) \rangle_{\mathbb{C}}}{\langle \prod_{i=0}^{n-1} \mathcal{V}_{\alpha_i}(z_i') \mathcal{V}^\dagger_{\alpha_i}(z_i) \rangle_{\mathbb{C}}}, \tag{5.41}$$

that is different from zero since we chose the signs of the α and $\bar{\alpha}$ parameters in order to satisfy the neutrality condition for the numerator and the denominator. The ratio 5.41 takes the form:

$$\left(\prod_{i=0}^{n-1}(z_i-z_1)^{2\alpha_{\Upsilon^\dagger}\alpha_{\Phi_1}}(z_i'-z_1)^{2\alpha_\Upsilon\alpha_{\Phi_1}}(z_i-z_2)^{2\alpha_{\Upsilon^\dagger}\alpha_{\Phi_2}}(z_i'-z_2)^{2\alpha_\Upsilon\alpha_{\Phi_2}}\right)|z_1-z_2|^{4\alpha_{\Phi_1}\alpha_{\Phi_2}}.$$

Restoring the holomorphic and antiholomorphic dimensions of the initial operators involved in the correlation function:

$$\left(\prod_{i=0}^{n-1}\frac{(z_i'-z_1)(z_i'-z_2)}{(z_i-z_1)(z_i-z_2)}\right)^{2\sqrt{hh_\Phi}}\frac{1}{|z_1-z_2|^{4h_\Phi}}. \tag{5.42}$$

The initial ratio between correlation functions 5.31 can be computed simply multiplying the previous result by an invariant function of all the possible anharmonic ratios $F(\eta_i)$:

$$\begin{aligned}&\frac{\langle\prod_{j=0}^{n-1}\Upsilon_j(w)\Upsilon_j^\dagger(-w)\Phi(w_1)\Phi(w_2)\rangle_{\mathcal{R}_n}}{\langle\prod_{j=0}^{n-1}\Upsilon_j(w)\Upsilon_j^\dagger(-w)\rangle_{\mathcal{R}_n}}=\\&=\left|\frac{dw_1}{dz_1}\right|^{-2h_\Phi}\left|\frac{dw_2}{dz_2}\right|^{-2h_\Phi}F(\eta_i)\left(\prod_{i=0}^{n-1}\frac{(z_j'-z_1)(z_j'-z_2)}{(z_j-z_1)(z_j-z_2)}\right)^{2\sqrt{hh_\Phi}}\frac{1}{|z_1-z_2|^{4h_\Phi}}.\end{aligned} \tag{5.43}$$

where the anharmonic ratios are defined as usual:

$$\eta_j=\frac{(z_j'-z_1)(z_j-z_2)}{(z_j'-z_2)(z_j-z_1)}. \tag{5.44}$$

Obviously the $\eta_{i,j}$ parameters involving only the in and out states are fixed by the limit $w\to\pm\imath\infty$.
The expression 5.43 shows exactly the same algebraic behaviour of the two points functions in the ground state case but multiplied by an universal function:

$$F(\eta_i)\left(\prod_{i=0}^{n-1}\frac{(z_j'-z_1)(z_j'-z_2)}{(z_j-z_1)(z_j-z_2)}\right)^{2\sqrt{hh_\Phi}}. \tag{5.45}$$

If we take the limits of z_1 going to 0 and z_2 going to ∞, we approach the branch points, the universal amplitude becomes constant:

$$\eta_j=\frac{(z_j'-z_1)(z_j-z_2)}{(z_j'-z_2)(z_j-z_1)}\to\frac{z_j'}{z_j}, \tag{5.46}$$

$$F(\eta_i)\left(\prod_{i=0}^{n-1}\frac{(z_j'-z_1)(z_j'-z_2)}{(z_j-z_1)(z_j-z_2)}\right)^{2\sqrt{hh_\Phi}}\to F\left(\frac{z_j'}{z_j}\right)\prod_{j=0}^{n-1}\left(\frac{z_j'}{z_j}\right)^{2\sqrt{hh_\Phi}}. \tag{5.47}$$

We can ignore this numerical amplitude since it is not a source of divergences, we will restore it at the end of our analysis.
The computation of unusual corrections from 5.43 now can be done in the same way as the ground state case presented in section 4.1. We know that the corrections come from the conical singularities and we will approximate the integral around these points. In case

of divergences we will regulate the integral and we will obtain a scaling behaviour that is different from the one of the renormalization group.
First of all we have to see how the conformal transformation from $\mathcal{R}_n$ to $\mathbb{C}$ behaves near the two branch points and it can be simply done linearizing it:

$$z_1 = \left(\frac{\sin\left(\frac{\pi w_1}{L}\right)}{\sin\left(\frac{\pi(w_1 - l)}{L}\right)} \right)^{\frac{1}{n}} \approx \left(-\frac{\pi w_1}{L \sin(\pi x)} \right)^{\frac{1}{n}} \tag{5.48}$$

$$z_2 = \left(\frac{\sin\left(\frac{\pi w_2}{L}\right)}{\sin\left(\frac{\pi(w_2 - l)}{L}\right)} \right)^{\frac{1}{n}} \approx \left(\frac{L \sin(\pi x)}{\pi(w_2 - l)} \right)^{\frac{1}{n}} . \tag{5.49}$$

The derivatives of the mapping near the branch points are:

$$\frac{dz_1}{dw_1} = \left(-\frac{\pi}{L \sin(\pi x)} \right)^{\frac{1}{n}} \frac{w_1^{\frac{1}{n}-1}}{n}, \tag{5.50}$$

$$\frac{dz_2}{dw_2} = -\left(\frac{L \sin(\pi x)}{L} \right)^{\frac{1}{n}} \frac{(w_2 - l)^{-\frac{1}{n}-1}}{n}. \tag{5.51}$$

These expressions are used to compute the integral 5.31 around the singularities:

$$\begin{aligned}
&\int_{\mathbb{C}} d^2 z_1 \int_{\mathbb{C}} d^2 z_2 \left| \frac{dw_1}{dz_1} \right|^{2-2h_\Phi} \left| \frac{dw_2}{dz_2} \right|^{2-2h_\Phi} \frac{1}{|z_1 - z_2|^{4h_\Phi}} = \\
&= \int_{\mathbb{C}} d^2 z_1 \left| \frac{dw_1}{dz_1} \right|^{2-2h_\Phi} \int_{\mathbb{C}} d^2 z_2 \left| \frac{dw_2}{dz_2} \right|^{2-2h_\Phi} |z_2|^{-4h_\Phi} = \\
&= \left| \frac{nL \sin(\pi x)}{\pi} \right|^{-\frac{4h_\Phi}{n}} \int_{\mathcal{R}_n} d^2 w_1 \, |w_1|^{(\frac{1}{n}-1)2h_\Phi} \int_{\mathcal{R}_n} d^2 w_2 \, |w_2 - l|^{(\frac{1}{n}-1)2h_\Phi} .
\end{aligned} \tag{5.52}$$

The integrals are divergent for $n > n_c$, as in 4.1, and they need a further cutoff, that is in the w space as usual.
The regulated integrals take the form:

$$\left| \frac{nL \sin(\pi x)}{\pi} \right|^{-\frac{4h_\Phi}{n}} (2\pi)^2 \epsilon^{4h_\Phi(\frac{1}{n}-1)}. \tag{5.53}$$

Including now the ϵ dependence of the dimensional coupling constant λ we get:

$$\frac{1}{\epsilon^{4-4h_\Phi}} \left| \frac{nL \sin(\pi x)}{\pi} \right|^{-4h_\Phi} (2\pi)^2 \epsilon^{4h_\Phi(\frac{1}{n}-1)} = (2\pi)^2 \left| \frac{\epsilon \pi}{Ln \sin(\pi x)} \right|^{\frac{4h_\Phi}{n}} . \tag{5.54}$$

Now we can restore the numerical amplitude fixed by the limit to in and out states and to conical singularities and we have the final result:

$$F\left(\frac{z_j'}{z_j}\right) \prod_{j=0}^{n-1} \left(\frac{z_j'}{z_j}\right) (2\pi)^2 \left| \frac{\epsilon \pi}{Ln \sin(\pi x)} \right|^{\frac{4h_\Phi}{n}} . \tag{5.55}$$

This expression shows the same unusual corrections to scaling found in the ground state case:

$$\left(\frac{L\sin(\pi x)}{\pi}\right)^{-\frac{4h_\Phi}{n}}, \tag{5.56}$$

but multiplied by an universal amplitude that does not change their scaling behaviour.

5.5 The λ term living on $\mathcal{R}_n$

The last term in the perturbative expansion 5.6 is the order g term on the surface $\mathcal{R}_n$. As we saw or the case 5.11 we want to demonstrate that this term does not need to be further regulated.
First of all we notice that the approach used in 5.4 is useless here because the neutrality condition would force us to:

$$\alpha_\Phi = \pm\sqrt{h_\Phi} = 0. \tag{5.57}$$

This means that the only operator allowed as perturbation is the one with vanishing scaling dimensions: the identity, and this case is not interesting.
We can argue that this term does not need to be further regulated and it has the scaling behaviour of the renormalization group, like the other λ term in 5.11. In this case the leading corrections to scaling would be the unusual correction found in the last section.
In order to check our statement, we will compute an odd correlation function for a very well studied model as the bidimensional Ising model using bosonization theory.
In particular we want to compute the correlation function:

$$\frac{\langle\sigma(v_1)\sigma(v_2)\sigma(v_3)\sigma(v_4)\epsilon(z_1)\rangle_{\mathbb{C}}}{\langle\sigma(v_1)\sigma(v_2)\sigma(v_3)\sigma(v_4)\rangle_{\mathbb{C}}}, \tag{5.58}$$

that corresponds to the λ term on the Riemann surface $\mathcal{R}_2$. The reader should argue that we are using as perturbation operator the energy operator that has $\Delta_\epsilon = 1$ and it is not an irrelevant operator: it would drive away our quantum system from criticality. We have studied in 4.2 that also relevant operators could give unusual corrections to scaling but they have to be located around conical singularities in order to not drive away the system from criticality.
Here we changed a bit the notation in order to make it match with the one used by Ardonne and Sierra in Ref. [20], where the write expressions for all the correlation functions of the Bidimensional Ising model.
We will focus on the numerator of the precedent relation since it has the dependence on z_1. It can be written as:

$$\langle\sigma(v_1)\sigma(v_2)\sigma(v_3)\sigma_(v_4)\epsilon(z_1)\rangle = \sum_{\mathbf{m}=(m_1;m_2)} \mathcal{F}_{(2;1)}\bar{\mathcal{F}}_{(2;1)}, \tag{5.59}$$

where:

$$\mathcal{F}_{(2;1)} = \langle\sigma(v_1)\sigma(v_2)\sigma(v_3)\sigma(v_4)\psi(z_1)\rangle,$$
$$\mathbf{m} = (m_1; m_2).$$

The entries of the vector $\mathbf{m}$ are 0 or 1, that represent two different fusion channel. The vector $\mathbf{m}$ should have an even number of 1 in it, since the 1 channel represents the fusion of two σ operators to the operator $\Psi(z)$.

In our simple case we have only two possible choices: $(0;0)$ and $(1;1)$, we have only two couples of σ operators.
The expression:

$$\mathcal{F}_{\mathbf{p}}^{(n;N)} = \langle \sigma(v_1)\sigma(v_2)\dots\sigma(v_{2n})\psi(z_1)\psi(z_2)\dots\psi(z_N)\rangle \tag{5.60}$$

can also be written in another way, see Ref.[20], as function of a binary variable $\mathbf{p}$:

$$\mathcal{F}_{\mathbf{m}}^{(2n;2m-1)} =$$
$$\frac{1}{2}\prod_{a<b}^{2n}(v_a - v_b)^{-\frac{1}{8}}\left(\sum_{q=0}^{2^{n-1}-1}\epsilon_{pq}\sqrt{v_{l_q}v_{l'_q}}\right)^{-\frac{1}{2}}\left[\sum_{q=0}^{2^{n-1}-1}\epsilon_{pq}\sqrt{v_{l_q}v_{l'_q}}\frac{\prod_{k=1}^{n}\sqrt{v_1 - v_{l'_q}}}{\prod_{k=2}^{n}\sqrt{v_1 - v_{l_q}}}\right.$$
$$\left.\sum_{i=1}^{2m-1}(-1)^{i+1}(z_i - v_1)^{-1}\prod_{k=1}^{n}\left(\frac{z_i - v_{l_k}}{z_i - v_{l'_k}}\right)^{\frac{1}{2}}\mathrm{Pf}_{j,k\neq i,2m}\left(\frac{h_{l_q;l'_q}(z_j;z_k)}{z_j - z_k}\right)\right].$$

The function ϵ_{pq} and vectors l_p, l_q restricted to our case ($n = 2$ and $m = 1$) are:

$$\epsilon_{00} = 1$$
$$\epsilon_{01} = 1$$
$$\epsilon_{10} = 1$$
$$\epsilon_{11} = -1$$
$$l_0 = (1;3) \quad l_1 = (1;4)$$
$$l'_0 = (2;4) \quad l'_1 = (2;3).$$

We can perform explicitly the computation in our simple case:

$$\mathcal{F}_{\mathbf{p}}^{(2;1)} = \tag{5.61}$$
$$= 2^{-1}\prod_{a<b}^{4}(v_a - v_b)^{-\frac{1}{8}}\left(\sum_{q=0}^{1}\epsilon_{pq}\sqrt{v_{l_q}v_{l'_q}}\right)^{-\frac{1}{2}}\left[\sum_{q=0}^{1}\epsilon_{pq}\sqrt{v_{l_q}v_{l'_q}}\frac{\prod_{k=1}^{2}\sqrt{v_1 - v_{l'_q}}}{\prod_{k=2}^{2}\sqrt{v_1 - v_{l_q}}}\right.$$
$$\left.\sum_{i=1}^{1}(-1)^{i+1}(z_i - v_1)^{-1}\prod_{k=1}^{2}\left(\frac{z_i - v_{l_k}}{z_i - v_{l'_k}}\right)^{\frac{1}{2}}Pf_{j,k\neq i,2}\left(\frac{h_{l_q;l'_q}(z_j;z_k)}{z_j - z_k}\right)\right] =$$
$$= 2^{-1}\prod_{a<b}^{4}(v_a - v_b)^{-\frac{1}{8}}\left(\sqrt{v_{13}v_{24}} + (-1)^p\sqrt{v_{14}v_{23}}\right)^{-\frac{1}{2}}\left[\sum_{q=0}^{1}\epsilon_{pq}\sqrt{v_{l_q}v_{l'_q}}\frac{\prod_{k=1}^{2}\sqrt{v_1 - v_{l'_q}}}{\prod_{k=2}^{2}\sqrt{v_1 - v_{l_q}}}\right.$$
$$\left.(z_1 - v_1)^{-1}\prod_{k=1}^{2}\left(\frac{z_1 - v_{l_k}}{z_1 - v_{l'_k}}\right)^{\frac{1}{2}}\right] = 2^{-1}\prod_{a<b}^{4}(v_a - v_b)^{-\frac{1}{8}}\left(\sqrt{v_{13}v_{24}} + (-1)^p\sqrt{v_{14}v_{23}}\right)^{-\frac{1}{2}} *$$
$$\left[\sqrt{\frac{(v_1 - v_2)(v_1 - v_4)(z_1 - v_3)(v_2 - v_4)}{(z_1 - v_1)(z_1 - v_2)(z_1 - v_4)}} + (-1)^p\sqrt{\frac{(v_2 - v_3)(v_1 - v_2)(v_1 - v_3)(z_1 - v_4)}{(z_1 - v_2)(z_1 - v_3)(z_1 - v_1)}}\right].$$

The chiral correlator with $p = 0$ is:

$$\mathcal{F}_0^{(2;1)} = \frac{1}{2}\prod_{a<b}^{4}(v_a - v_b)^{-\frac{1}{8}}\left(\sqrt{(v_1-v_3)(v_2-v_4)} - \sqrt{(v_1-v_4)(v_2-v_3)}\right)^{-\frac{1}{2}} \times$$
$$\times\left[\sqrt{\frac{v_1-v_2}{(z_1-v_1)(z_1-v_2)(z_1-v_3)(z_1-v_4)}}\sqrt{(v_1-v_4)(v_2-v_4)}(z_1-v_3) + \right.$$
$$\left. + \sqrt{\frac{v_1-v_2}{(z_1-v_1)(z_1-v_2)(z_1-v_3)(z_1-v_4)}}\sqrt{(v_1-v_3)(v_2-v_3)}(z_1-v_4)\right], \tag{5.62}$$

and the one with $p = 1$ is:

$$\mathcal{F}_1^{(2;1)} = \frac{1}{2}\prod_{a<b}^{4}(v_a - v_b)^{-\frac{1}{8}}\left(\sqrt{(v_1-v_3)(v_2-v_4)} - \sqrt{(v_1-v_4)(v_2-v_3)}\right)^{-\frac{1}{2}} \times$$
$$\times\left[\sqrt{\frac{v_1-v_2}{(z_1-v_1)(z_1-v_2)(z_1-v_3)(z_1-v_4)}}\sqrt{(v_1-v_4)(v_2-v_4)}(z_1-v_3) - \right.$$
$$\left. - \sqrt{\frac{v_1-v_2}{(z_1-v_1)(z_1-v_2)(z_1-v_3)(z_1-v_4)}}\sqrt{(v_1-v_3)(v_2-v_3)}(z_1-v_4)\right]. \tag{5.63}$$

In conformal field theory all the correlation functions have to satisfy a set of differential equations, see Ref.[45] for more informations, in our case the quantities 5.62 and 5.63 have to satisfy:

$$\left(\frac{4}{3}\frac{\partial^2}{\partial v_a^2} - \sum_{v_a\neq v_b}\frac{1}{(v_a-v_b)}\frac{\partial}{\partial v_a} - \frac{1}{(v_a-z_1)}\frac{\partial}{\partial z_1} - \frac{1}{16}\sum_{b\neq a}\frac{1}{(v_a-v_b)^2} - \frac{1}{2}\frac{1}{(v_a-z_1)}\right)\mathcal{F}_{\mathbf{m}}^{(4;1)} = 0,$$
$$\left(\frac{3}{4}\frac{\partial^2}{\partial z_1^2} - \sum_{a}\frac{1}{(z_1-v_a)}\frac{\partial}{\partial v_a} - \frac{1}{16}\sum_{a}\frac{1}{(z_1-v_a)^2}\right)\mathcal{F}_{\mathbf{m}}^{(4;1)} = 0.$$

We checked that all the correlation functions obtained satisfy these differential equations. Taking the limit to the in and out states for the v_i coordinates we get two expressions as function only of the variable z_1:

$$\mathcal{F}_0^{(2;1)} = \frac{(-1)^{\frac{3}{8}}\imath^{\frac{1}{4}}\cos\left(\frac{\pi x}{4}\right)(1+z_1)(\sin(\pi x))^{1/4}}{2^{\frac{1}{2}}\sqrt{\cos\left(\frac{x\pi}{4}\right)^2}\sqrt{-e^{-\frac{1}{2}ix\pi}+z_1}\sqrt{e^{-\frac{1}{2}ix\pi}+z_1}\sqrt{-e^{\frac{ix\pi}{2}}+z_1}\sqrt{e^{\frac{ix\pi}{2}}+z_1}}, \tag{5.64}$$

$$\mathcal{F}_1^{(2;1)} = \frac{(-1)^{\frac{3}{8}}\imath^{\frac{5}{4}}\sin\left(\frac{\pi x}{4}\right)(z_1-1)(\sin(\pi x))^{1/4}}{2^{\frac{1}{2}}\sqrt{\sin\left(\frac{x\pi}{4}\right)^2}\sqrt{-e^{-\frac{1}{2}ix\pi}+z_1}\sqrt{e^{-\frac{1}{2}ix\pi}+z_1}\sqrt{-e^{\frac{ix\pi}{2}}+z_1}.\sqrt{e^{\frac{ix\pi}{2}}+z_1}} \tag{5.65}$$

We can compute the initial correlation function using the expression 5.59:

$$\begin{aligned}
&\langle\sigma(v_1)\sigma(v_2)\sigma(v_3)\sigma(v_4)\epsilon(z_1)\rangle = \\
&= \left|\mathcal{F}_0^{(2;1)}\right|^2 + \left|\mathcal{F}_1^{(2;1)}\right|^2 = \\
&= \left|\frac{(-1)^{\frac{3}{8}}\imath^{\frac{1}{4}}\sin(\pi x)^{\frac{1}{4}}}{2\sqrt{-e^{-\frac{1}{2}ix\pi}+z_1}\sqrt{e^{-\frac{1}{2}ix\pi}+z_1}\sqrt{-e^{\frac{ix\pi}{2}}+z_1}\sqrt{e^{\frac{ix\pi}{2}}+z_1}}\right|^2 \\
&\left(\frac{\left|\cos\left(\frac{\pi x}{4}\right)\right|^2}{\left|\cos\left(\frac{\pi x}{4}\right)\right|^2}|1+z_1|^2 + \frac{\left|\sin\left(\frac{\pi x}{4}\right)\right|^2}{\left|\sin\left(\frac{\pi x}{4}\right)\right|^2}|1-z_1|^2\right) = \\
&\left|\frac{\sin(\pi x)^{\frac{1}{4}}}{\sqrt{-e^{-\frac{1}{2}ix\pi}+z_1}\sqrt{e^{-\frac{1}{2}ix\pi}+z_1}\sqrt{-e^{\frac{ix\pi}{2}}+z_1}\sqrt{e^{\frac{ix\pi}{2}}+z_1}}\right|^2 (1+|z_1|^2).
\end{aligned} \tag{5.66}$$

Finally after a lot of tedious algebra we get the final result:

$$\begin{aligned}
\langle\sigma(v_1)\sigma(v_2)\sigma(v_3)\sigma(v_4)\epsilon(z_1)\rangle_{\mathbb{C}} = \\
= \left|\frac{\sin(\pi x)^{\frac{1}{2}}}{(e^{-\frac{1}{2}ix\pi}-z_1)(e^{-\frac{1}{2}ix\pi}+z_1)(e^{\frac{ix\pi}{2}}-z_1)(e^{\frac{ix\pi}{2}}+z_1)}\right|(1+|z_1|^2).
\end{aligned} \tag{5.67}$$

Now we wanto to use the previous result to compute the ratio:

$$\begin{aligned}
\frac{\langle\sigma(\bar{w}_1)\sigma(\bar{w}_2)\sigma(\bar{w}_3)\sigma(\bar{w}_4)\epsilon(w_1)\rangle_{\mathcal{R}_2}}{\langle\sigma(\bar{w}_1)\sigma(\bar{w}_2)\sigma(\bar{w}_3)\sigma(\bar{w}_4)\rangle_{\mathcal{R}_2}} = &\left|\frac{\pi}{2\sin(\pi x)L}\frac{(e^{\imath\pi x}-z_1^2)(e^{-\imath\pi x}-z_1^2)}{z_1}\right| \times \\
&\times \frac{\langle\sigma(v_1)\sigma(v_2)\sigma(v_3)\sigma(v_4)\epsilon(z_1)\rangle_C}{\langle\sigma(v_1)\sigma(v_2)\sigma(v_3)\sigma(v_4)\rangle_{\mathbb{C}}},
\end{aligned}$$

on the Riemann surface $\mathcal{R}_2$.
We have to compute the correlation function between four spin operators that can be easily found in the literature (Ref.[45] or Ref.[20]):

$$\begin{aligned}
\langle\sigma(v_1)\sigma(v_2)\sigma(v_3)\sigma(v_4)\rangle_{\mathbb{C}} = \\
= \frac{\sqrt{(v_1-v_3)(v_2-v_4)}}{2((v_1-v_2)(v_1-v_3)(v_2-v_3)(v_1-v_4)(v_2-v_4)(v_3-v_4))^{1/4}}.
\end{aligned} \tag{5.68}$$

Taking the limit of v_i to in and out states we get:

$$\langle\sigma(v_1)\sigma(v_2)\sigma(v_3)\sigma(v_4)\rangle_C = \frac{1}{2\sqrt{\sin(\pi x)}}. \tag{5.69}$$

The previous correlation functions were computed on the complex plane $\mathbb{C}$, the same quantities can be computed on $\mathcal{R}_2$ taking into account transformation $\mathcal{R}_2 \mapsto \mathbb{C}$:

$$\frac{\langle\sigma(\bar{w}_1)\sigma(\bar{w}_2)\sigma(\bar{w}_3)\sigma(\bar{w}_4)\epsilon(w_1)\rangle_{\mathcal{R}_2}}{\langle\sigma(\bar{w}_1)\sigma(\bar{w}_2)\sigma(\bar{w}_3)\sigma(\bar{w}_4)\rangle_{\mathcal{R}_2}} = \frac{\pi}{L}\frac{1+|z_1|^2}{|z_1|}. \tag{5.70}$$

We can extract the unusual corrections from this expression as usual restricting the integral in the regions around the branch points and looking for divergences. In this particular case it is easier to compute them starting from the integral on $\mathbb{C}$:

$$\frac{1}{\epsilon^{2-1}}\int_{\mathbb{C}} d^2z_1 \left|\frac{dw_1}{dz_1}\right|^2 \frac{\pi}{L}\frac{1+|z_1|^2}{|z_1|}. \tag{5.71}$$

using the z_1 dependence of the derivative of the mapping we have:

$$\frac{1}{\epsilon^{2-1}}\int_{\mathbb{C}} d^2 z_1 \left|\frac{2L\sin(\pi x)}{\pi}\frac{z_1}{(e^{\imath\pi x}-z_1^2)(e^{-\imath\pi x}-z_1^2)}\right|^2 \frac{\pi}{L}\frac{1+|z_1|^2}{|z_1|} = \tag{5.72}$$

$$= \frac{1}{\epsilon}\int_{\mathbb{C}} d^2 z_1 \left|\frac{2L\sin(\pi x)}{\pi}\frac{z}{(e^{\imath\pi x}-z^2)(e^{-\imath\pi x}-z^2)}\right|^2 \frac{\pi}{L}\frac{1+|z_1|^2}{|z_1|}, \tag{5.73}$$

and the integrand goes to 0 when $z \to 0$ and it goes also to 0 when $z \to \infty$.
Now we can to look for divergences of the integral near the branch points 0 and l.
The conformal transformation from $\mathcal{R}_2$ to $\mathbb{C}$ around the branch points takes the form:

$$z(w \approx 0) = \left(-\frac{L}{\pi\sin(\pi x)}\right)^{\frac{1}{2}} (w)^{\frac{1}{2}}, \tag{5.74}$$

$$z(w \approx l) = \left(\frac{\pi\sin(\pi x)}{L}\right)^{\frac{1}{2}}. \tag{5.75}$$

We can evaluate the integral around $w_1 \approx 0$ for example:

$$\frac{1}{\epsilon}\int d^2 w_1 \left|\frac{\pi\sin(\pi x)}{L}\right|^{\frac{1}{2}} |w_1|^{-\frac{1}{2}}, \tag{5.76}$$

and we can see that this term is not divergent around $w_1 \approx 0$. This means that the order λ term on $\mathcal{R}_2$ does not need to be further regulated introducing new cutoffs.
We can try to generalize our argument to a general n sheeted Riemann surface $\mathcal{R}_n$. In this general case we have to study a correlation function like the following one:

$$\frac{\langle\prod_{i=1}^{2n}\sigma(\bar{w}_i)\epsilon(w_1)\rangle_{\mathcal{R}_n}}{\langle\prod_{i=0}^{2n}\sigma(\bar{w}_i)\rangle_{\mathcal{R}_n}} = \left|\frac{dw_1}{dz_1}\right|^{-1} \frac{\langle\prod_{i=1}^{2n}\sigma(v_i)\epsilon(z_1)\rangle_{\mathbb{C}}}{\langle\prod_{i=1}^{2n}\sigma(v_i)\rangle_{\mathbb{C}}} \tag{5.77}$$

From Ref.[20] we know that there is another equivalent way to write the correlation function 5.59:

$$\langle\prod_{i=1}^{2n}\sigma(v_i)\epsilon(z_1)\rangle_{\mathbb{C}} = \sum_{\mathbf{m}} \mathcal{F}_{\mathbf{m}}^{(2n;N=1)}\bar{\mathcal{F}}_{\mathbf{m}}^{(2n;N=1)}. \tag{5.78}$$

The explicit computation can be done using another version of the expression 5.61, namely:

$$\mathcal{F}_{\mathbf{m}}^{(2n;N=1)} = 2^{-\frac{n}{2}}\prod_{i=1}^{n}(v_{2i-1}-v_{2i})^{-\frac{1}{8}}\prod_{i=1}^{2n}(v_i-z_1)^{-\frac{1}{2}}(A_{2n}^{\mathbf{m}})^{-\frac{1}{2}}\times$$
$$\times\left(\sum_{\mathbf{t}}\left(\prod_{i=1}^{n}t_i^{m_i}\prod_{1<i,j<n}(1-x_{i,j})^{\frac{t_i t_j}{4}}\right)\Psi_{\mathbf{t}}\right),$$

where we used the following definitions:

$$x_{i,j} = \frac{(v_{2i-1}-v_{2i})(v_{2j-1})-v_{2j}}{(v_{2i-1}-v_{2j})(v_{2j-1}-v_{2i})}, \tag{5.79}$$

$$A_{2n}^{\mathbf{m}} = \sum_{\mathbf{t}}\left(\prod_{i=1}^{n}t_i^{m_i}\prod_{1<i,j<n}(1-x_{i,j})^{\frac{t_i t_j}{4}}\right), \tag{5.80}$$

$$\Psi_{\mathbf{t}} = -(v_1-v_2)^{\frac{1}{2}}\left(\prod_{i=2}^{n}\left(\frac{v_1-v_{2i+\frac{t_i-1}{2}}}{v_1-v_{2i-1-\frac{t_i-1}{2}}}\right)^{\frac{1}{2}}(v_{2i-1-\frac{t_i-1}{2}}-z_1)\right). \tag{5.81}$$

We are studying the z_1 dependence of the correlation functions and we will start our analysis from $\Psi_{\mathbf{t}}$ because it is the only one that involves z_1.
From its definition it is clear that we can write $\Psi_{\mathbf{t}}$ as a polynomial of the variable z_1:

$$\Psi_{\mathbf{t}} = \alpha_0 + \alpha_1 z_1 + \cdots + \alpha_{n-1} z_1^{n-1}, \tag{5.82}$$

where the coefficient α_j are functions of the coordinates v_i and they will be fixed after taking the limit to the in and out states.
This fact allows us to write a simple general expression for the chiral correlator where we take care only of the z_1 variable:

$$\mathcal{F}_{\mathbf{m}}^{(2n;N=1)} = \frac{\beta_{0,\mathbf{m}} + \beta_{1,\mathbf{m}} z_1 + \cdots + \beta_{n-1,\mathbf{m}} z_1^{n-1}}{\sqrt{\prod_{i=1}^{2n}(v_i - z_1)}}, \tag{5.83}$$

and we have a different set of coefficients β_i for every vector $\mathbf{m}$, but the important fact is the dependence on the variable z_1, that is the same for every $\mathbf{m}$.
We can write 5.78 as a sum of the generic real polynomials in the variables z_1 and z_1^* of degree $n-1$:

$$\langle \prod_{i=1}^{2n} \sigma(v_i)\epsilon(z_1)\rangle_{\mathbb{C}} = \frac{\gamma_{0;0} + \gamma_{0;1} z_1^* + \gamma_{1;0} z_1^1 + \cdots + \gamma_{n-1;n-1} |z_1|^{2(n-1)}}{\left|\prod_{i=1}^{i=2n}(v_i - z_1)\right|}. \tag{5.84}$$

We can easily compute the correlator on the Riemann surface including the proper transformation factor due to the mapping $\mathcal{R}_n \mapsto \mathbb{C}$:

$$\langle \prod_{i=1}^{2n} \sigma(\bar{w}_i)\epsilon(w_1)\rangle_{\mathcal{R}_n} = \left|\frac{nL\sin(\pi x)}{\pi}\frac{z^{n-1}}{(e^{\imath\pi x} - z^n)(e^{-\imath\pi x} - z^n)}\right|^{-1} \frac{\gamma_{0;0} + \gamma_{0;1} z_1^* + \gamma_{1;0} z_1^1 + \cdots + \gamma_{n-1;n-1} |z_1|^{2(n-1)}}{\left|\prod_{i=1}^{i=2n}(v_i - z_1)\right|}. \tag{5.85}$$

We can simplify the previous expression using the following relation:

$$(e^{\imath\pi x} - z^n)(e^{-\imath\pi x} - z^n) = \prod_{i=1}^{i=2n}(v_i - z_1), \tag{5.86}$$

that is true because of the definition of in and out states on $\mathbb{C}$.
Now we can write a general expression for the z_1 dependence of the correlation function 5.78:

$$\frac{\langle \prod_{i=1}^{2n} \sigma(\bar{w}_i)\epsilon(w_1)\rangle_{\mathcal{R}_n}}{\langle \prod_{i=1}^{2n} \sigma(\bar{w}_i)\rangle_{\mathcal{R}_n}} = \left|\frac{\pi}{nL\sin(\pi x)}\right| \times \times \frac{\tilde{\gamma}_{0;0} + \tilde{\gamma}_{0;1} z_1^* + \tilde{\gamma}_{1;0} z_1^1 + \cdots + \tilde{\gamma}_{n-1;n-1} |z_1|^{2(n-1)}}{|z_1|^{n-1}}. \tag{5.87}$$

The denominator is function only of the v_i and it will become a constant after the limit to the in and out states. We will absorb it in the coefficients $\tilde{\gamma}_{i;j}$ because it does not change the z_1 behaviour of 5.87 that is the important part.

We are interested in the behaviour of expression 5.87 near the branch points and we look at its behaviour for small values of z_1:

$$\frac{\langle \prod_{i=1}^{2n} \sigma(\bar{w}_i)\epsilon(w_1)\rangle_{\mathcal{R}_n}}{\langle \prod_{i=1}^{2n} \sigma(\bar{w}_i)\rangle_{\mathcal{R}_n}} \approx \left|\frac{\pi}{nL\sin(\pi x)}\right| \frac{\tilde{\gamma}_{0;0}}{|z_1|^{n-1}}, \tag{5.88}$$

and for large values of z_1:

$$\frac{\langle \prod_{i=1}^{2n} \sigma(\bar{w}_i)\epsilon(w_1)\rangle_{\mathcal{R}_n}}{\langle \prod_{i=1}^{2n} \sigma(\bar{w}_i)\rangle_{\mathcal{R}_n}} \approx \left|\frac{\pi}{nL\sin(\pi x)}\right| \tilde{\gamma}_{n-1;n-1}\,|z_1|^{n-1}\,. \tag{5.89}$$

We can use these asymptotic expressions to look for divergences around the branch points. If we restrict the integral to the region $z_1 \approx 0$, for example, we have:

$$\frac{1}{\epsilon}\int_{\mathbb{C}} d^2 z_1 \left|\frac{dw_1}{dz_1}\right|^2 \left|\frac{\pi}{nL\sin(\pi x)}\right| \frac{\tilde{\gamma}_{0;0}}{|z_1|^{n-1}}. \tag{5.90}$$

Using the inverse conformal transformation of 3.16 we can transform it in an integral on $\mathcal{R}_n$ around $w_1 \approx 0$:

$$\frac{1}{\epsilon}\int_{\mathcal{R}_n} d^2 w_1 \left|\frac{\pi}{nL\sin(\pi x)}\right|^{1+\frac{1}{n}} \tilde{\gamma}_{0;0}\,|w_1|^{\frac{1}{n}-1}\,. \tag{5.91}$$

The same expression is obtained approximating the integral in the region $w_1 \approx l$:

$$\frac{1}{\epsilon}\int_{\mathcal{R}_n} d^2 w_1 \left|\frac{\pi}{nL\sin(\pi x)}\right|^{1+\frac{1}{n}} \tilde{\gamma}_{n-1;n-1}\,|w_1 - l|^{\frac{1}{n}-1}\,. \tag{5.92}$$

As usual we have used the linearized expressions for the mapping 3.16 around the branch points:

$$z_1(w_1 \approx 0) = \left(-\frac{\pi}{L\sin(\pi x)} w_1\right)^{\frac{1}{n}}, \tag{5.93}$$

$$z_1(w_1 \approx l) = \left(\frac{L\sin(\pi x)}{\pi}\frac{1}{w_1}\right)^{\frac{1}{n}}. \tag{5.94}$$

The integrals 5.91 and 5.5 are finite near the branch points and this means that they has not to be further regulated introducing additional cutoffs. The scaling behaviour of the g term is the same computed for 5.11: the finite size scaling of the renormalization group, namely:

$$\left(\frac{L\sin(\pi x)}{\pi}\right)^{-(\Delta-2)}. \tag{5.95}$$

This scaling behaviour is subleading respect to the unusual corrections:

$$\left(\frac{L\sin(\pi x)}{\pi}\right)^{-\frac{2\Delta}{n}}. \tag{5.96}$$

This means that the unusual corrections to scaling are present also in the excited states of conformal field theory and their form is equal to the one found in the analysis of the ground state situation:

Chapter 6

Numerical results

The results obtained analytically in Chapt. 5 confirm that the unusual corrections to scaling are present also in the excited states of CFT and they can be obtained in the same way as in the ground state case.
In this section we want to show numerical computations in order to check our theoretical predictions. We will start with an analysis of the XX model and of its excited states, then we will show how to extract the entanglement entropy from the correlation matrix in this model and in the end we will show our numerical results.

6.1 The XX model

The XX model is one of the simplest spin chain and it has been studied, in particular in its ground state, in many works in the literature like in Ref.[29]. This model is solvable via the Jordan Wigner transformations and could be mapped into a free fermionic model.
The Hamiltonian operator of the model is the following:

$$H_{XX} = -\frac{1}{2}\sum_{j=1}^{L}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y\right) + \frac{J}{2}\sum_j \sigma_j^z, \tag{6.1}$$

where we used the nearest neighbors coupling as the unit of energy and J is the coupling constant between magnetic field spins.
The σ^i are as usual the Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.2}$$

We assume periodic boundary conditions (PBC) on the chain:

$$\sigma_1^i = \sigma_L^i. \tag{6.3}$$

We can define the raising and lowering operators as:

$$\sigma_\pm = \frac{1}{2}(\sigma^x \pm \imath\sigma^y), \tag{6.4}$$

and in order to rewrite the hamiltonian as function of the ladder operators we have to invert the previous relation:

$$\sigma^x = (\sigma^+ + \sigma^-), \tag{6.5}$$
$$\sigma^y = -\imath(\sigma^+ - \sigma^-). \tag{6.6}$$

The hamiltonian as function of ladder operators is:

$$\begin{aligned} H_{XX} &= -\frac{1}{2}\sum_{i=1}^{L}\left[(\sigma_i^+ + \sigma_i^-)(\sigma_{i+1}^+ + \sigma_{i+1}^-) - (\sigma_i^+ - \sigma_i^-)(\sigma_{i+1}^+ - \sigma_{i+1}^-)\right] - \frac{J}{2}\sum_j \sigma_i^z \\ &= -\sum_{i=1}^{N}[\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+] + \frac{J}{2}\sum_j \sigma_i^z. \end{aligned}$$

The previous equation can be further manipulated using the Jordan Wigner transformation:

$$a_l = \left(\prod_{i=0}^{l-1}\sigma_i^z\right)\sigma_l^-, \tag{6.7}$$

where the product of Pauli matrices makes this transformation non local but it is necessary to have the canonical anticommutation relations for fermionic operators:

$$\{a_m; a_n\} = 0, \tag{6.8}$$
$$\{a_m^\dagger; a_n^\dagger\} = 0, \tag{6.9}$$
$$\{a_m^\dagger; a_n\} = \delta_{mn}. \tag{6.10}$$

The hamiltonian written as function of these new fermionic operators is:

$$H_{XX} = -\sum_{l=1}^{L}\left[a_l^\dagger a_{l+1} + a_l a_{l+1}^\dagger\right] + J\sum_l a_l^\dagger a_l, \tag{6.11}$$

and we can notice that the magnetic field plays the role of a chemical potential.
This hamiltonian is traslationally invariant and we can use the Fourier transform of the creation and annihilation operators to diagonalize it. The Fourier transform for a fermionic operator is:

$$a_l = \frac{1}{\sqrt{L}}\sum_{j=1}^{L} e^{-\imath\frac{2\pi\imath}{L}jl}c_j, \tag{6.12}$$

and for its adjoint:

$$a_l^\dagger = \frac{1}{\sqrt{L}}\sum_{j=1}^{L} e^{\frac{2\pi\imath}{L}jl}c_j^\dagger. \tag{6.13}$$

We can perform the Fourier transform of the hamiltonian term by term.
The first part of the kinetic operator transforms as:

$$\sum_{l=1}^{L} a_l^\dagger a_{l+1} = \sum_l \frac{1}{L}\sum_{j;j'} c_j^\dagger c_{j'} e^{\frac{2\pi\imath}{L}(jl-j'l-j')} = \sum_{j;j'}\delta_{j;j'}c_j^\dagger c_{j'}e^{-\frac{2\pi\imath}{L}j'}, \tag{6.14}$$

where we use the definition of delta function:

$$\delta_{u;v} = \frac{1}{L}\sum_{l} e^{\frac{2\pi\imath}{L}l(u-v)}. \tag{6.15}$$

This expression allows us to eliminate one of the sums and then we remain with:

$$\sum_{l=1}^{L} a_l^\dagger a_{l+1} = \sum_{j} c_j^\dagger c_j e^{-\frac{2\pi\imath}{L}j}. \tag{6.16}$$

The second term in the hamiltonian can be diagonalized simply taking the adjoint of the precedent:

$$\left(\sum_{l=1}^{L} a_l^\dagger a_{l+1}\right)^\dagger \rightarrow \sum_{l=1}^{L} a_{l+1}^\dagger a_l = \sum_{j} c_j^\dagger c_j e^{\frac{2\pi\imath}{L}j}. \tag{6.17}$$

The third piece, the one given by the presence of the magnetic field, is a bit different:

$$\sum_{l} a_l^\dagger a_l = \sum_{l}\frac{1}{L}\sum_{j;j'} c_j^\dagger c_{j'} e^{\frac{2\pi\imath}{L}(j-j')l} = \sum_{j} c_j^\dagger c_j, \tag{6.18}$$

where we used again the delta function representation 6.15.
After the Fourier transformation we have a hamiltonian in diagonal form:

$$H_{XX} = \sum_{j}\left(J - e^{\frac{2\pi\imath j}{L}} - e^{-\frac{2\pi\imath j}{L}}\right) c_j^\dagger c_j = \tag{6.19}$$

$$= \sum_{j}\left(J - 2\cos\left(\frac{2\pi j}{L}\right)\right) c_j^\dagger c_j. \tag{6.20}$$

For what concerns the boundary conditions we have a that the Jordan Wigner transformation has the following property:

$$a_L^\dagger a_1 = -(-1)^{n_\downarrow}\sigma_L^+\sigma_1^-, \tag{6.21}$$

where $n_\downarrow$ is the number of down spins defined as:

$$n_\downarrow = L - \sum_{j=0}^{L-1}\frac{s_j^z - 1}{2}. \tag{6.22}$$

From this we obtain the boundary conditions on the fermionic problem:

$$a_{L+1} = (-1)^{n_\downarrow} a_1, \tag{6.23}$$

this means that periodic boundary conditions on the spin chain are mapped into Antiperiodic Boundary Conditions in the fermionic problem and vice versa in the case of an even value of L.
In order to find the ground state of the system we have to analyse the spectrum of the diagonalized Hamiltonian that has been mapped into a fermionic problem with dispersion relation:

$$\Lambda_k = J - 2\cos\left(\frac{2\pi k}{L}\right). \tag{6.24}$$

In the case of $J > 2$ the eigenvalues are all $\Lambda_k > 0$ because the equation:

$$J - 2\cos\left(\frac{2\pi k}{L}\right) = 0, \tag{6.25}$$

has no solutions. This means that all the spins are aligned along the direction of the magnetic field.
The state of the system here is a product state like the following:

$$\prod_i |\uparrow_i\rangle\,, \tag{6.26}$$

and the entanglement entropy for these states vanishes.
This is a more general characteristic of spin systems: there is a combination of the couplings g_i present in the Hamiltonian $H(g_i)$, called factorizing point, that corresponds to a product state.
For $J > 2$ the ground state of the system is the one annihilated by all annihilation operators:

$$c_k\,|0\rangle = 0 \qquad \forall k, \tag{6.27}$$

that is the low energy state:

$$H\,|0\rangle = 0. \tag{6.28}$$

If the magnetic field is $0 < J < 2$ we have some negative eigenvalues of the Hamiltonian. In fact the equation:

$$J - 2\cos\left(\frac{2\pi k_c}{L}\right) = 0, \tag{6.29}$$

has a solution:

$$k_c = \left[\frac{L}{2\pi}\arccos\left(\frac{J}{2}\right)\right]. \tag{6.30}$$

This is the maximum value of the number k given the magnetic field J. The brackets [] stand for the floor function in order to give a natural number as it should be. For consistency we can go back in the previous case and we found that k_c cannot be defined in the case of $J > 2$.
The Fermi momentum is defined as the value of the momentum corresponding to the number k_c:

$$k_F = \frac{2\pi k_c}{L} = \arccos\left(\frac{J}{2}\right). \tag{6.31}$$

From the Fermi momentum we can define the filling parameter ν:

$$\nu = \frac{k_F}{\pi}. \tag{6.32}$$

If $J = 0$ we have that:

$$k_F = \arccos 0 = \frac{\pi}{2},$$
$$\nu = \frac{1}{2},$$

and this particular case is called *half filling*.
The energy of the ground state is given by:

$$E = \sum_{-k_C}^{+k_C}\left(J - 2\cos\left(\frac{2\pi k}{L}\right)\right). \tag{6.33}$$

In the case of half-filled we have:

$$E_{HF} = -2 \sum_{j \in \Omega_{HF}} \cos\left(\frac{2\pi k}{L}\right). \tag{6.34}$$

Now it is time to choose the range of variables of the variable k.
In general the lower energy configuration will be obtained placing the particles in the lowest accessible momenta and this force us to use a set of k numbers that is symmetric with respect to the 0 momentum.
This means that if the number of particles is odd we have to choose a set of integers that includes the 0:

$$\Omega_{\text{odd}} = \{-\frac{\nu L - 1}{2}; -\frac{\nu L - 3}{2}; \ldots; -1; 0; +1; \ldots; \frac{\nu L - 3}{2}; +\frac{\nu L - 1}{2}\}. \tag{6.35}$$

In the other case, if the number of particles is even, we choose a set of half integers that does not include the 0 momentum:

$$\Omega_{\text{even}} = \{-\frac{\nu L - 1}{2}; -\frac{\nu L - 3}{2}; \ldots; -\frac{1}{2}; +\frac{1}{2}; \ldots; \frac{\nu L - 1}{2}\}. \tag{6.36}$$

Here we assumed an even L in both cases.
The main difference between the two situations is the fact that in the first case every positive number has a symmetric negative mate but in the second one we have an exception that is the $k = 0$. Some numerical examples will be made in following section.

6.2 Excitations of the XX model

It is a well known fact that the XX model is described in the continuum limit by a bosonic field theory and we can assign to some excitation a corresponding operator of the quantum field theory.
For sake of clarity in this chapter we will use the zero magnetic field XX mode, but all following arguments can be easily extended to different values of the magnetic field. The Hamiltonian of the model is:

$$H_{XX}(J = 0) = -\frac{1}{2} \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y, \tag{6.37}$$

and as we did before we can map this hamiltonian in a free fermionic model as:

$$H_{XX} = \sum_{j \in \Omega} \left(-\cos\left(\frac{2\pi j}{L}\right) c_j^\dagger c_j \right). \tag{6.38}$$

The states of the Hamiltonian of the XX model are:

$$\prod_{n_F \leq L} c_{m_j}^\dagger \left|0\right\rangle. \tag{6.39}$$

We will use Periodic Boundary conditions on the chain that are mapped on Antiperiodic Boundary conditions on the Fermionic problem, we are using $L = 2k$ as usual. These conditions define the set Ω of the numbers k:

$$\Omega = \{\pm\frac{1}{2}; \ldots; \pm\frac{L-1}{2}\}, \tag{6.40}$$

in particular the subset m_j used in 6.54 is one of the 2^L subset of Ω.
The ground state of the Hamiltonian can be written as:

$$|n_F\rangle = \prod_{0<j\leq(n_F-1)/2} c_j^\dagger c_{-j}^\dagger |0\rangle , \tag{6.41}$$

Now we can focus on the low energy excitations that we can create from this ground state. Due to conformal invariance we have that the excess of energy of excitations will be of $2\pi\Delta/L$ where Δ is the conformal dimension of the excitation and L is the number of sites in the chain. It is easy to see that in the limit of large L, where the model is critical, we have a vanishing excess of energy.
We could ask if the same thing happens to the entanglement entropy of the excited states of the model. We demonstrated in Chapt.2 that the Renyi entropy of a critical 1D quantum system in its ground state is:

$$S_{gs}^{(n)}(x) = \frac{1}{6}\left(1+\frac{1}{n}\right)\ln\left[\frac{L}{\pi}\sin(\pi x)\right] + c_n', \tag{6.42}$$

where we used the central charge equal to 1 because we are dealing with a free bosonic theory.
We saw in Chapt. 3 that primary operators of CFT can generate excited states that have sometimes the same entropy of the ground state but sometimes they have a very different entanglement entropy. In this section we will see the excited states of the fermionic problem that are represented, in the continuum limit, by the operators studied in 3.
The ground state of the half-filled XX model is formed by a set of $L/2$ particles with moment $2\pi j/L$, here we assume an even $L/2$ and so the set of the numbers j is:

$$\{-\frac{\frac{L}{2}-1}{2};\ldots;-\frac{1}{2};+\frac{1}{2};\ldots;+\frac{\frac{L}{2}-1}{2}\}. \tag{6.43}$$

We have $L/2$ particles that occupies all the states labelled by the momenta from $-\pi(L/2-1)/N$ to $\pi(L/2-1)$, it is easy to see that this set is *compact* since there are no holes inside it.
This characteristic is the most important one since it allows us to make a classification of different excitations of the model:

- *compact* if they do not exhibit holes in momentum space, they are formed by particles of consecutive momenta separated by quantum of momentum $2\pi/L$.
- *non-compact* if they have a hole in momentum space.

We will see that compact ones have the same Renyi entropy as the ground state, a paradigmatic excitation is the vertex operator, and the non compact ones have an entanglement entropy that is different from the ground state, for example the operator $\imath\partial\varphi$.
We can start our study of excitations from the simplest compact excitation obtained from the ground state:

$$c_{-(n_F-1)/2}\,|n_F\rangle , \tag{6.44}$$

that corresponds to the annihilation of the particle with lower momentum. It easy to see that this excitation is compact since the change in the set of occupied numbers j is:

$$\{-\frac{\frac{L}{2}-1}{2};\ldots;-\frac{1}{2};+\frac{1}{2};\ldots;+\frac{\frac{L}{2}-1}{2}\} \to \{-\frac{\frac{L}{2}+1}{2};\ldots;-\frac{1}{2};+\frac{1}{2};\ldots;+\frac{\frac{L}{2}-1}{2}\}. \tag{6.45}$$

Obviously this changes affects the entanglement entropy because it shifts the Fermi momentum. In 6.4 we have the Renyi entropy of $n = 2, 3, 4$ of this excitation as we anticipated above the only difference between the ground state and the excitations, in this case, is due only to the different type of oscillations present.
Another simple excited state that could be created from the ground state of the XX model is:

$$c^{\dagger}_{-(L/2+1)/2} c^{\dagger}_{(L/2+1)/2} \left| L/2 \right\rangle , \tag{6.46}$$

that is still a compact excitation: we just added a particle with $j = -(L/2+1)/2$ and a particle with $j = (L/2+1)/2$. This excitation is obviously compact since the set of numbers j is enlarged by two units but this change does not leave holes in the spectrum. The set of momenta changes in this way:

$$\{-\frac{\frac{L}{2}-1}{2}; \ldots; -\frac{1}{2}; +\frac{1}{2}; \ldots; +\frac{\frac{L}{2}-1}{2}\} \to \{-\frac{\frac{L}{2}+1}{2}; \ldots; -\frac{1}{2}; +\frac{1}{2}; \ldots; +\frac{\frac{L}{2}+1}{2}\}. \tag{6.47}$$

This state is still compact and, as we can see in the Fig. 6.4, it has the same Renyi entropy of the ground state up to oscillations described in Ref.[16]. This type of excitations creates a change in the number of particles presents: $L/2 \to L/2+2$ that moves the Fermi momentum and it affects the entanglement entropy.
There is another compact excitation called *Umklapp* excitation that does not move the Fermi momentum and it is defined as:

$$c^{\dagger}_{(L/2+1)/2} c_{-(L/2-1)/2} \left| L/2 \right\rangle , \tag{6.48}$$

where we move a particle from the higher negative occupied momentum to the higher free positive momentum. This change preserves a compact set of numbers j:

$$\{-\frac{\frac{L}{2}-1}{2}; \ldots; -\frac{1}{2}; +\frac{1}{2}; \ldots; +\frac{\frac{L}{2}-1}{2}\} \to \{-\frac{(\frac{L}{2}-3)}{2}; \ldots; +\frac{\frac{L}{2}-1}{2}; +\frac{\frac{L}{2}+1}{2}\}. \tag{6.49}$$

This excitation corresponds, in the continuum limit, to a vertex operator and we have shown explicitly in 3.46 that the conformal field theory approach leads us to the *right* result:

$$\mathrm{Tr}\rho^{n}_{A,\mathcal{V}_\alpha} = \mathrm{Tr}\rho^{n}_{A,gs}. \tag{6.50}$$

This confirms the fact that the compact excitations have the same Renyi entropy of the ground state for every value of the index n. It can be also demonstrated that the action of the Umklapp excitation corresponds to a shifting of momenta:

$$k \to k + \frac{2\pi}{L}, \tag{6.51}$$

and it is obvious that the transformation 6.51 leaves unchanged the reduced density matrix and the entanglement entropy.

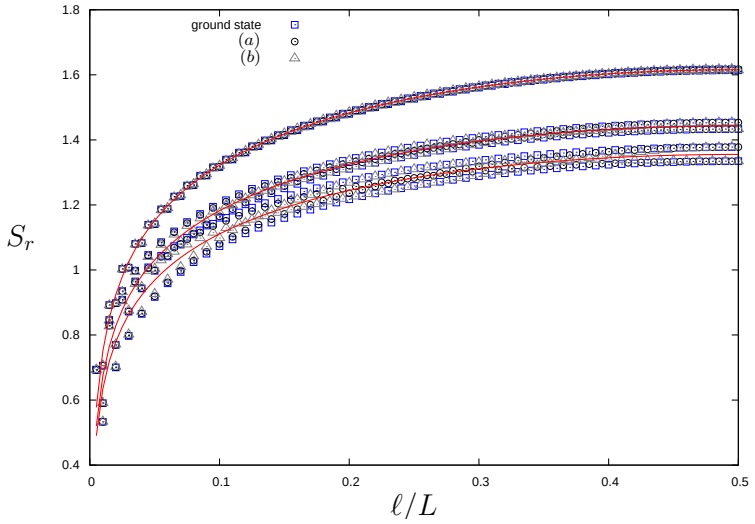

Figure 6.1: *Here are plotted some data from the previously discussed compact excitations from systems. The letters (a) and (b) labels the different types of excitations described in Table 6.1. Figure from Ref.[19]*

Here we have a table with all the different types of excitations and their corresponding operator in continuum limit (table taken from Ref.[19])

name of excitation	field	$(h, \bar{h})$	state ($n_F = L/2$)	$(h:p)$	$L = 8$ example
ground state	$\mathbf{1}$	(0,0)	$\|n_F\rangle$	$(:)$	○ ○ ● ● ● ● ○○
(a)	$e^{-i\phi}$	(1/2,0)	$c_{(n_F-1)/2}\|n_F\rangle$	$(1:)$	○ ○ ● ● ● ○ ○○
(b)	$e^{i\phi+i\bar{\phi}}$	$(1/2, 1/2)$	$c^\dagger_{n_F/2+1/2}\, c^\dagger_{-n_F/2-1/2}\|n_F\rangle$		○ ● ● ● ● ● ●○
Umklapp	$e^{i\phi-i\bar{\phi}}$	$(1/2, 1/2)$	$c_{-(n_F-1)/2}\, c^\dagger_{(n_F+1)/2}\|n_F\rangle$		○ ○ ○ ● ● ● ●○
particle-hole	$i\partial\phi$	$(1, 0)$	$c_{(n_F-1)/2}\, c^\dagger_{(n_F+1)/2}\|n_F\rangle$	$(1:1)$	○ ○ ● ● ● ○ ●○
R-L particle-hole	$\bar{\partial}\bar{\phi}\partial\phi$	$(1, 1)$	$c_{(n_F-1)/2}\, c^\dagger_{(n_F+1)/2}$ $c_{-(n_F-1)/2}\, c^\dagger_{-(n_F+1)/2}\|n_F\rangle$		○ ● ○ ● ● ○ ●○
	-	-	$c_{(n_F-1)/2}\, c^\dagger_{(n_F+3)/2}\|n_F\rangle$	$(1:2)$	○ ○ ● ● ● ○ ○●

Table 6.1: *A summary of the mentioned excitations. The horizontal line separates the compact states from the non-compact ones. The notation (h, p) applies only for chiral excitations, but these parameters are not interesting for our analysis. The corresponding conformal fields are shown for primary states only. Table taken from Ref. [19]*

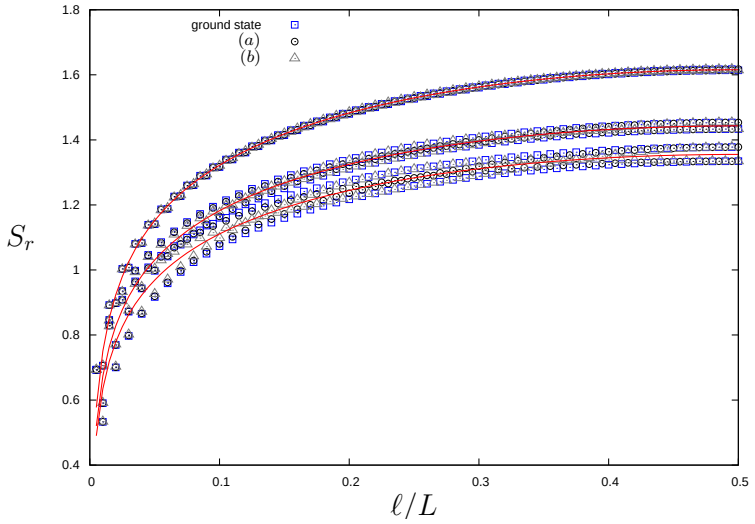

Figure 6.2: *Here are plotted some data from the previously discussed compact excitations from systems. The letters* (a) *and* (b) *labels the different types of excitations described in Table 6.1. Figure from Ref.[19]*

Now we move to the more interesting case of the entanglement entropy of *non compact* excitations. In table 6.1 are shown two type of non compact excitation.
The first one is the excited state corresponding to the operator $\imath\partial\phi$ that we have studied in Chapt.3. It corresponds to the continuum limit of the excitation:

$$c^\dagger_{(N/2+1)/2} c_{(N/2-1)/2} \left|GS\right\rangle . \tag{6.52}$$

This state has an excess of energy $2\pi/L$ and it is called an electron-hole excitation since we create a particle above the Fermi sea and we have an "hole", that can be considered as a particle, below the Fermi sea.
As we demonstrated in Chapt. 3 this excitation, that corresponds to the operator $\imath\partial\varphi$, have a different entanglement entropy from the ground state. From the excess of energy of the excited state we can find out its conformal dimensions that are $\Delta = 1$. We know that the excitation operator under study has only holomorphic dimension and it means $\bar{h} = 0$ and $h = \Delta = 1$.
The excitation $\partial\varphi\bar{\partial}\bar{\varphi}$ has a different Renyi entropy from the ground state, but they can be computed from the Renyi entropy of the excitation $\imath\partial\varphi$ as:

$$F^{(n)}_{\partial\varphi\bar{\partial}\bar{\varphi}} = \left|F^{(n)}_{\imath\partial\varphi}\right|^2 , \tag{6.53}$$

in fact $\bar{\partial}\bar{\varphi}\partial\varphi$ has equal holomorphic and antiholomorphic parts. This means that it can be computed simply taking the square modules of the holomorphic part, that is equal to the one of the operator $\imath\partial\phi$.
We can consider also the general case of an excitation with a "jump" of m momenta:

$$c^\dagger_{(L/2-1)/2+m} c_{(L/2-1)/2} \left|L/2\right\rangle , \tag{6.54}$$

that corresponds to an excitation that has an excess of energy of $2\pi m/L$. The change in the set of numbers j is easy to compute and it is:

$$\{-\frac{\frac{L}{2}-1}{2};\ldots;-\frac{1}{2};+\frac{1}{2};\ldots;+\frac{\frac{L}{2}-1}{2}\}\rightarrow\{-\frac{\frac{L}{2}-3}{2};\ldots;+\frac{\frac{L}{2}-1}{2}+m\},\tag{6.55}$$

and it shows that this excitations are non compact for every value of m.
It is known from other works that the behaviour of the excess of entropy of excitations for $l/L\ll 1$, infinite system length regime,is:

$$\Delta S_{n=1}(l)=S(l)-S^{gs}(l)=\frac{2\pi^2}{3}\left(h+\bar{h}\right)\left(\frac{l}{L}\right)^2+\left[\left(\frac{l}{L}\right)^4\right],\tag{6.56}$$

where we have used the well known result valid in the infinite system regime $S_1^{gs}\sim c/3\ln(l)$ Ref. [10], Ref.[13] and Ref.[29]. Excited states of the type 6.54 have conformal dimension $(h=m;\bar{h}=0)$, that can be found in the same way of the operator $\imath\partial\varphi$. In Fig. 6.3 we plot the low x behaviour of the excess of entanglement entropy computed for some different excitations. In the caption is showed that the quantity $\Delta S_1/m$ has an universal behaviour:

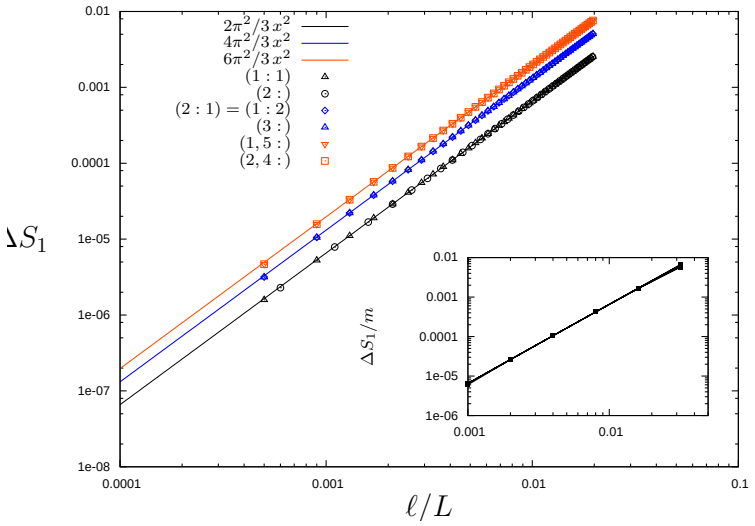

Figure 6.3: Low x regime of some different excitations. In the inset we have the excess of entropy divided by m and we can see that all the curves collapse. Figure from Ref.[19]

Although the small x behaviour is similar and all the figures collapse on the same line when we plot S_1/m, if we look at the behaviour of entanglement entropy for arbitrary values of x it is clear that the curves are similar only near $x\approx 0$:

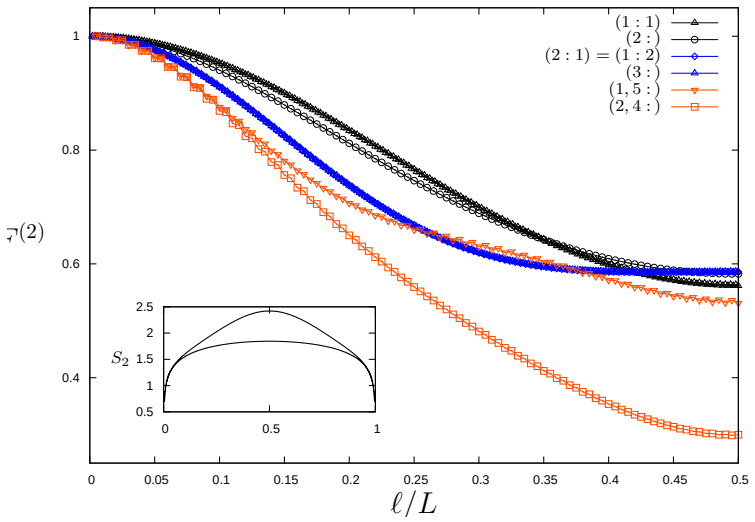

Figure 6.4: *Here we have the full curves of the excitation exposed in Fig.6.3, we can see that they have completely different behaviours outside the region $l \ll L$. Figure taken from Ref.[19]*

6.3 Entanglement entropy from correlation matrix

The XX model is a very simple model and a great number of results are available for it, in particular it is possible to compute the entanglement entropy using the correlation matrix and the exact form of its corrections to scaling is known.
In this section we will show a method, for more information about it Ref. [12], that allows us to extract the entanglement entropy of the subsystem from the correlation matrix. This method can be also extended to the analysis of the entanglement entropy of the excited states.
We are interested in zero temperature finite size XX chain as we did in Chapt. 3. In this situation the XX model is in its ground state that has been analysed in section 6.1. The ground state of the system, in the momentum space, is defined as:

$$c_j^\dagger |GS\rangle = 0 \quad \forall j, \Lambda_j < 0, \tag{6.57}$$

$$c_k |GS\rangle = 0 \quad \forall k, \Lambda_k > 0, \tag{6.58}$$

that is a consequence of the fact that the ground state is a Fermi sea of Fermi momentum k_F.
The correlator between two fermionic creation and annihilation operators in the Fourier transform is:

$$\langle GS| c_j^\dagger c_k |GS\rangle = \delta_{kj} \quad \Lambda_j, \Lambda_k < 0, \tag{6.59}$$

and zero otherwise.
We are interested in computing the correlation functions between two operators in real space

and this can be done using the Fourier transformation:

$$A_{mn} = \langle GS| a_m^\dagger a_n |GS\rangle = \frac{1}{L}\sum_j \sum_k e^{\imath(mk-nj)} \langle GS| c_j^\dagger c_k |GS\rangle = \\ = \frac{1}{L}\sum_j \sum_k e^{\imath(mk-nj)}\delta_{jk} = \frac{1}{L}\sum_k e^{\imath(m-n)k}. \tag{6.60}$$

We restrict ourselves to the case of even L and zero magnetic field, but the method works for every value of ν and L. The values of the variable k that compose the ground state are:

$$\Omega = \{\pm\frac{1}{2};\pm\frac{3}{2};\ldots;\pm\frac{\frac{L}{2}-1}{2}\}. \tag{6.61}$$

The Hamiltonian 6.19 is a free fermionic Hamiltonian and this means that Wick's theorem holds. Any observable involving an arbitrary number of creation/annihilation operators can be computed from 6.60 taking into account the Fermi Dirac statistic:

$$\langle a_i^\dagger a_j^\dagger a_k a_l\rangle_{\mathrm{GS}} = \langle a_i^\dagger a_l\rangle_{\mathrm{GS}}\langle a_j^\dagger a_k\rangle_{\mathrm{GS}} - \langle a_i^\dagger a_k\rangle_{\mathrm{GS}}\langle a_j^\dagger a_l\rangle_{\mathrm{GS}}. \tag{6.62}$$

We can compute the correlation matrix simply using the definition of observable defined by the reduced density matrix of a subsystem of length l:

$$A_{mn} = Tr\left(a_m^\dagger a_n \rho_l\right), \tag{6.63}$$

from its definition it is clear that the correlation matrix is hermitian and we can diagonalize it using a unitary transformation:

$$D_{pq} = \sum_{mn} U_{pm} A_{mn} U_{nq}^* \equiv \langle GS| d_p^\dagger d_q |GS\rangle \delta_{pq}, \tag{6.64}$$

where we defined new fermionic operators:

$$d_p = \sum_m U_{pm} a_m. \tag{6.65}$$

As usual we can write the expression 6.64 using the reduced density matrix of the subsystem:

$$D_{pq} = \mathrm{Tr}(d_p^\dagger d_q \rho_l) = \lambda_p \delta_{mn}, \tag{6.66}$$

this result implies that the ρ_l can be written as:

$$\rho_l = \bigotimes_{i=1}^{l} \rho_i. \tag{6.67}$$

Now we can study the eigenvalues of one of the modes, for example the i-th, where the operators previously defined take the matrix form:

$$d_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{6.68}$$

$$d_i^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{6.69}$$

$$\rho_i = \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i^* & 1-\alpha_i \end{pmatrix}. \tag{6.70}$$

The density matrix satisfies automatically the normalization condition:

$$\mathrm{Tr}\rho_i = 1 \quad \forall i. \tag{6.71}$$

We want to determine the parameters α_i and β_i: the β_i can be easily obtained from the one point function:

$$\langle GS| d_i |GS\rangle = \mathrm{Tr}(d_i\rho_l) = \beta_i = 0. \tag{6.72}$$

and the α can be found from the diagonal matrix elements D_{pp}:

$$\mathrm{Tr}\left(d_i^\dagger d_i \rho_l\right) = \mathrm{Tr}\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha_i & 0\\ 0 & 1-\alpha_i\end{pmatrix}\right] = \alpha_i = \lambda_i. \tag{6.73}$$

This method shows us that there is a connection between the eigenvalues of the correlation matrix computed on the ground state and the elements of the density matrices ρ_i.
The spectral theorem allows us to compute the entanglement entropy as a function of the eigenvalues of the correlation matrix:

$$S(l) = -\sum_i^l \left[(\lambda_i)\ln(\lambda_i) + (1-\lambda_i)\ln(1-\lambda_i)\right], \tag{6.74}$$

and the Renyi entropy is:

$$S^n(l) = \frac{1}{1-n}\sum_i^l \ln\left(\lambda_i^n + (1-\lambda_i)^n\right). \tag{6.75}$$

This approach is totally general and we can extend it to the excited states of XX model simply changing the correlation matrix.
Lets start our analysis of the excited state from the most important one: the electron hole excitation. We have seen in 6.52 that the state corresponding the the operator $\imath\partial\phi$ is:

$$|\text{e-h}\rangle = c^\dagger_{(n_F+1)/2} c_{(n_F-1)/2} |GS\rangle, \tag{6.76}$$

we assume to have $n_F = 2k$, where $n_F = \nu L$ and L is the number of the sites.
The correlation matrix is defined as:

$$\langle e-h| a_i^\dagger a_j |e-h\rangle = \langle GS| c^\dagger_{(n_F-1)/2} c_{(n_F+1)/2} c_i^\dagger c_j c^\dagger_{(n_F+1)/2} c_{(n_F-1)/2} |GS\rangle. \tag{6.77}$$

We can rewrite this correlation matrix as an expectation value of a string of six fermionic operators computed on the ground state that can be computed in the Fourier space using the relation 6.59 and Wick's theorem:

$$\tilde{A}_{ij} = \langle e-h| a_i^\dagger a_j |e-h\rangle = \frac{1}{L}\left[\sum_{k\in\Omega_\nu} e^{\frac{2\pi\imath}{L}(i-j)k} - e^{\frac{\pi\imath}{L}(n_F-1)} + e^{\frac{\pi\imath}{L}(n_F+1)}\right]. \tag{6.78}$$

In the case of a set Ω_ν of the type:

$$\Omega_\nu = \{-\frac{\nu L-1}{2};\ldots;-\frac{1}{2};+\frac{1}{2};\ldots;+\frac{\nu L-1}{2}\}. \tag{6.79}$$

We can simplify it using the symmetry of the set Ω_ν and we have:

$$\tilde{A}_{ij} = \frac{2}{L}\sum_{k=1/2}^{(n_F-1)/2} \cos\left(\frac{2\pi}{L}(i-j)k\right) - \frac{1}{L}e^{\frac{\pi\imath}{L}(n_F-1)} + \frac{1}{L}e^{\frac{\pi\imath}{L}(n_F+1)}, \tag{6.80}$$

where the sum starts from $k = 1/2$. The correlation matrix of the electron hole excitation has two extra pieces compared to the one of the ground state case. These extra pieces depend on the momenta of the annihilated and created particles, namely:

$$e^{\frac{\pi\imath}{L}(n_F+1)} - e^{\frac{\pi\imath}{L}(n_F-1)} = e^{\frac{2\pi\imath}{L}\left(\frac{n_F-1}{2}\right)} - e^{\frac{2\pi\imath}{L}\left(\frac{n_F+1}{2}\right)}. \tag{6.81}$$

This can be generalized to the excited state:

$$c^{\dagger}_{(n_F-1)/2-m} c_{(n_F-1)/2+n} \left|GS\right\rangle. \tag{6.82}$$

This will lead to two additional terms in the correlation matrix like:

$$e^{\frac{2\pi\imath}{L}\left(\frac{n_F-1}{2}-m\right)} - e^{\frac{2\pi\imath}{L}\left(\frac{n_F-1}{2}+n\right)}, \tag{6.83}$$

the electron hole is defined as $(m = 0; n = 1)$ and the Umklapp excitation, that is compact and it has the same entropy of the ground state as we saw in last section, is defined with $(m = 2\frac{n_F-1}{2}; n = 1)$.
We can do some numerical examples of chains and filling in order to fix the ideas about the set Ω_ν. In the case of $\nu = 1/3$ and a chain of length $L = 60$ we have $\nu L = 20$. We have to use a set of numbers symmetric with respect to the momentum 0 in order to find the ground state:

$$\Omega_{\nu=1/3;L=60} = \{-\frac{19}{2}; \ldots; -\frac{1}{2}; \frac{1}{2}; \ldots; \frac{19}{2}\}. \tag{6.84}$$

In the case of a a chain of length $L = 63$ and same filling the situation is a bit different, since in this case we have $\nu L = 31$ and the set of number symmetric respect to the zero is:

$$\Omega_{\nu=1/3;L=62} = \{-10; -9; \ldots; -1; 0; +1; \ldots; 9; 10\}. \tag{6.85}$$

The change is very small but when we work with short chains it could be very important since the correlation matrix has an extra term depending on L:

$$\tilde{A}_{ij} = \frac{2}{L} \sum_{k=0}^{(n_F-1)/2} \cos\left(\frac{2\pi}{L}(i-j)k\right) - \frac{1}{L} e^{\frac{\pi\imath}{L}(n_F-1)} + \frac{1}{L} e^{\frac{\pi\imath}{L}(n_F+1)} - \frac{1}{L}. \tag{6.86}$$

6.86 and 6.80 are expressions for the correlation matrices of the excited state, the entanglement entropy and Renyi entropy are obtained simply diagonalizing them and using the following expressions:

$$\tilde{S}(l) = -\sum_{i}^{l} \left[\tilde{\lambda}_i \ln \tilde{\lambda}_i + \left(1 - \tilde{\lambda}_i\right) \ln\left(1 - \tilde{\lambda}_i\right)\right], \tag{6.87}$$

where the $\tilde{\lambda}_i$ are the eigenvalues of the correlation matrix of the excited state.
The Renyi entropy is:

$$\tilde{S}^{(n)}(l) = \frac{1}{1-n} \sum_{i}^{l} \ln\left(\tilde{\lambda}_i^n + (1 - \tilde{\lambda}_i)^n\right). \tag{6.88}$$

6.4 Numerical Results

In our theoretical analysis of Chapt.5 we have shown that from conformal field theory arguments it is possible to predict the existence of unusual corrections to scaling like:

$$\left(\frac{L\sin(\pi x)}{\pi}\right)^{-\frac{2\Delta}{n}}. \tag{6.89}$$

In this chapter we present our numerical results that will be in perfect agreement with our theoretical predictions and we will also explore the form of the corrections to scaling comparing them with the ground state case that is well known in literature.
We have studied the case of the electron hole excitation, that is represented in the continuum limit by the operator $\imath\partial\phi$, since it has a very different behaviour from the Renyi entropy of the ground state.
We have shown how to compute the entanglement entropy and Renyi entropy of the excited states starting from the correlation matrix and in particular from its eigenvalues.
The method previously exposed allows us to compute the Renyi entropy of the excited state $\tilde{S}^{(n)}$ and the one of the ground state $S^{(n)}$. From these two quantities we can find the function $F^{(n)}_{\imath\partial\phi}(x)$ by its definition:

$$F^{(n)}_{\imath\partial\phi} = e^{(1-n)\left(\tilde{S}^{(n)}-S^{(n)}\right)}. \tag{6.90}$$

We have computed in the section 3.3 using conformal field theory the continuum limit of the function $F^{(n)}_{\mathrm{CFT}}$.
We will study the scaling behaviour of the corrections defined as:

$$\Delta^{(n)} \equiv F^{(n)} - F^{(n)}_{\mathrm{CFT}}. \tag{6.91}$$

The dependence on the index n is not restricted to the set of the integers $n > 1$ because we can use the expression 3.57 that is valid for a generic real index n and it will be very useful in order to find the corrections to the von Neumann entropy, that are defined as the limit $n \to 1^+$.
Our prediction is that we can write the corrections as:

$$\Delta^{(n)} = \left(\frac{L\sin(\pi x)}{\pi}\right)^{-\frac{2}{n}} \mathcal{F}_n(x;\nu), \tag{6.92}$$

where $\mathcal{F}(x;\nu)$ is an function of the variable $x = l/L$, l is the size of the subsystem and it does not depend on the length of the chain L. It is a known result that for the XX chain we have $\Delta = 1$, see Ref.[61]. This means that if we compute the values of $\Delta^{(n)}$ for chains with different lengths we will obtain different corrections. On the other hand when we study the quantity:

$$\Delta^{(n)}\left(\frac{L\sin(\pi x)}{\pi}\right)^{2/n}, \tag{6.93}$$

all the different corrections will collapse on one.
Let start from the simplest case of the half-filled chain. We have studied chains of length 40, 80, 120, 160 and 200 and here in the figures below we have plotted Δ_n as functions of $x = l/L$ for different values of n:

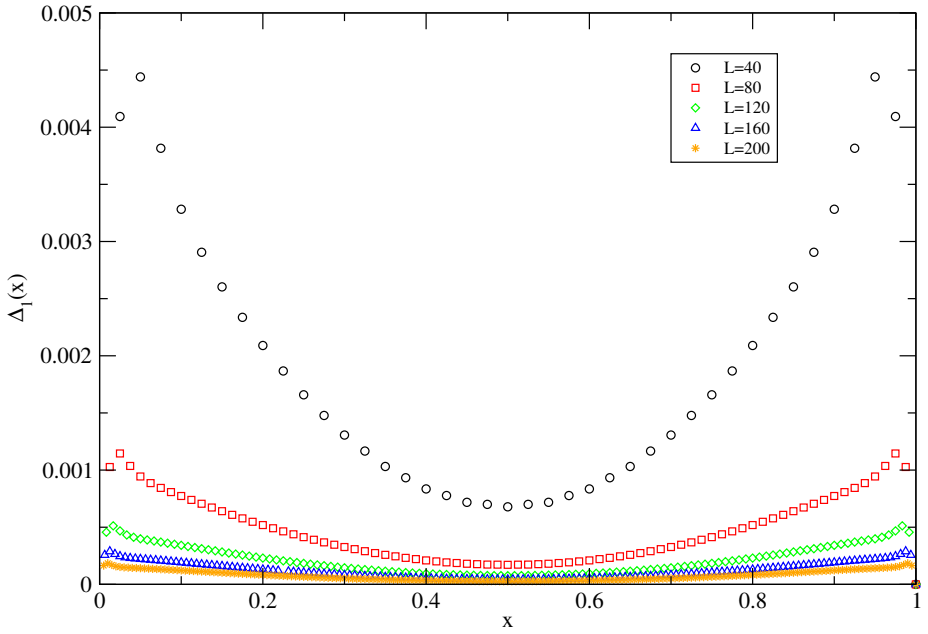

Figure 6.5: *Corrections to $F^{(1)}$ in the case of an electron-hole excitation in a half-filled chain.*

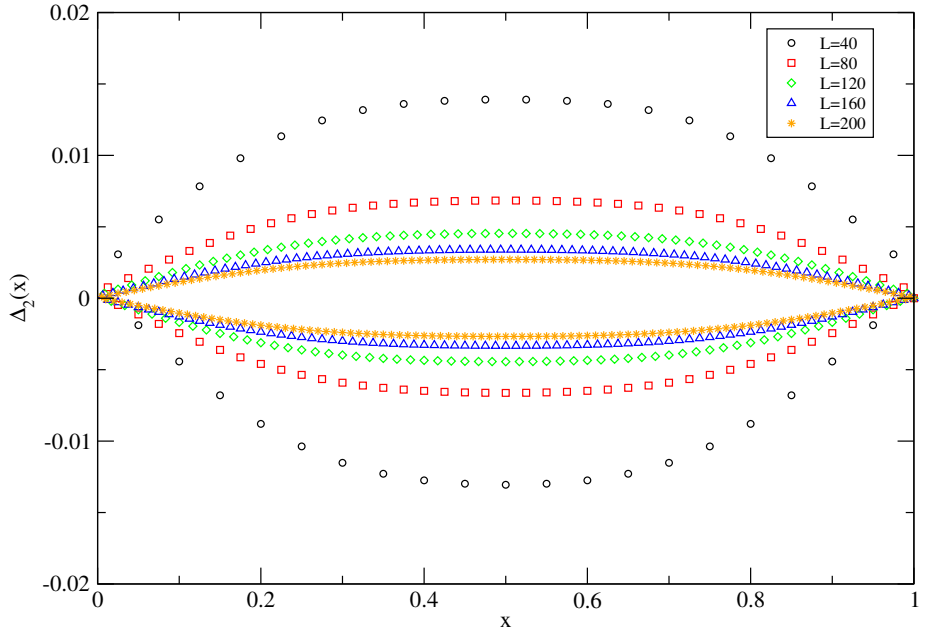

Figure 6.6: *Corrections to $F^{(2)}$ in the case of an electron-hole excitation in a half-filled chain.*

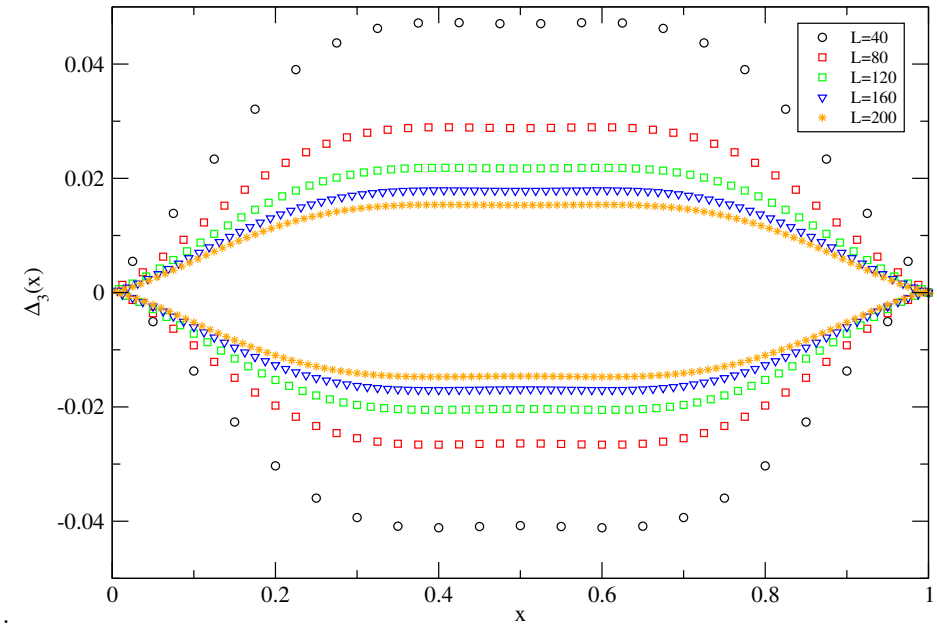

Figure 6.7: *Corrections to $F^{(3)}$ in the case of an electron-hole excitation in a half-filled chain.*

As we expect from our previous theoretical analysis we can see that the corrections to scaling approach to zero while length of the chain enhances.
All the corrections are pretty different as we expected, but if we plot the quantities:

$$\Delta_n(x)\left(\frac{L\sin(\pi x)}{\pi}\right)^{2/n}, \tag{6.94}$$

we observe that all the different corrections curves collapse on one curve:

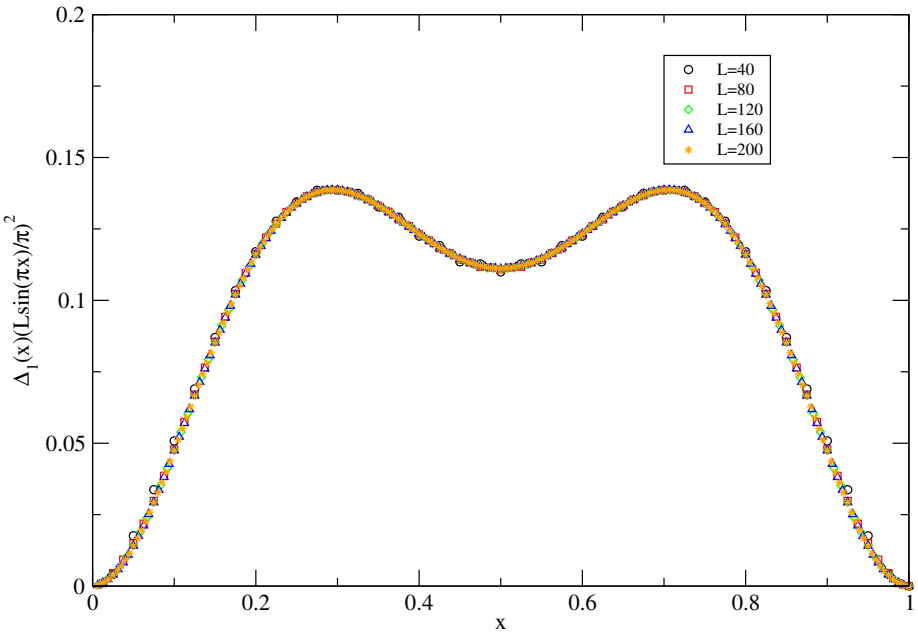

Figure 6.8: *Scaled corrections to $F^{(1)}$. All the different corrections in Fig.6.5 collapse on the same curve*

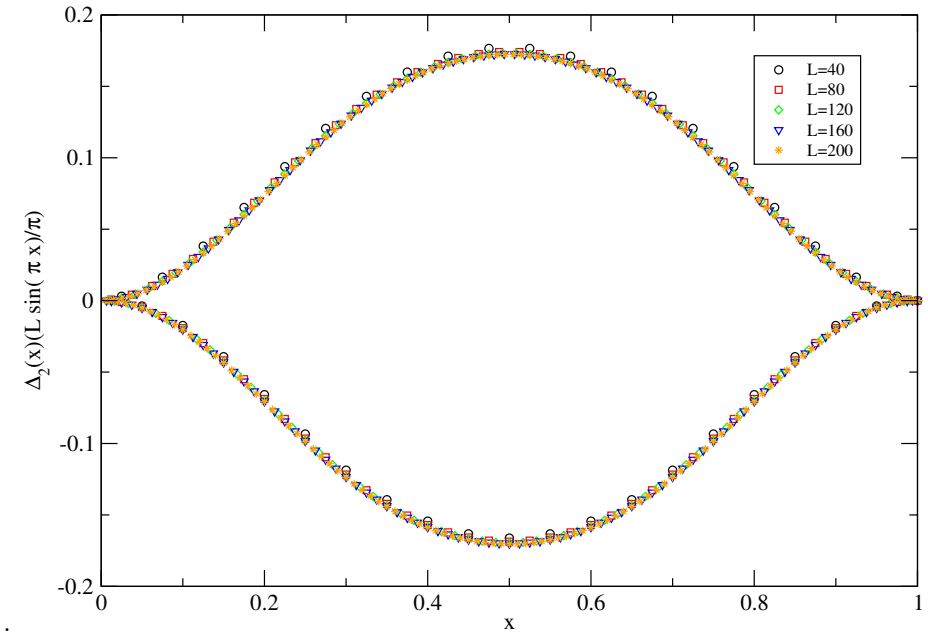

Figure 6.9: *Scaled corrections to $F^{(2)}$. All the different corrections in Fig.6.6 collapse on the same curve*

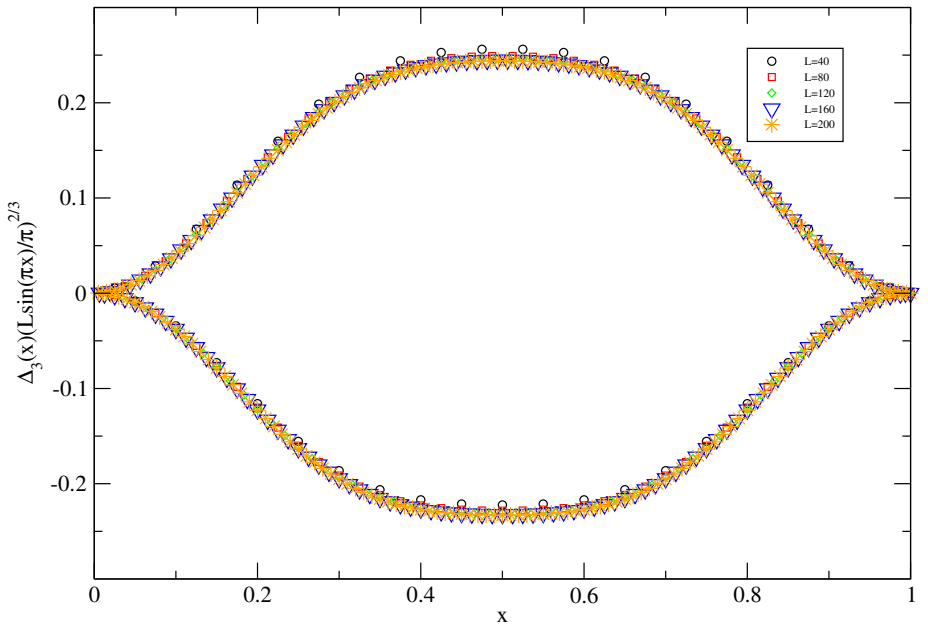

Figure 6.10: *Scaled corrections to $F^{(3)}$. All the different corrections in Fig.6.7 collapse on the same curve*

This very important since it confirms our predictions about the scaling of the corrections to scaling.
We have also studied many different fillings case but all the corrections obtained have a scaling behaviour that is in perfect agreement with our predictions.

In Fig.6.11, Fig.6.4, Fig.6.13 and Fig.6.14 are presented the correction in the case of filling $\nu = 1/3$. Here we have $\Delta_n(x)$ in $\nu = 1/3$ case:

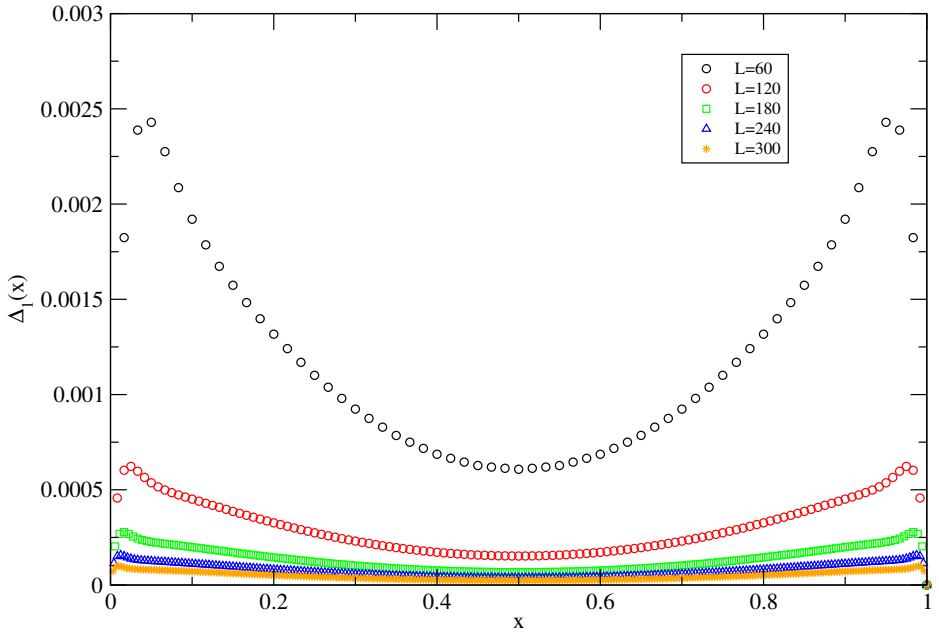

Figure 6.11: *Corrections to $F^{(1)}$ in the case of different chains with filling $\nu = 1/3$.*

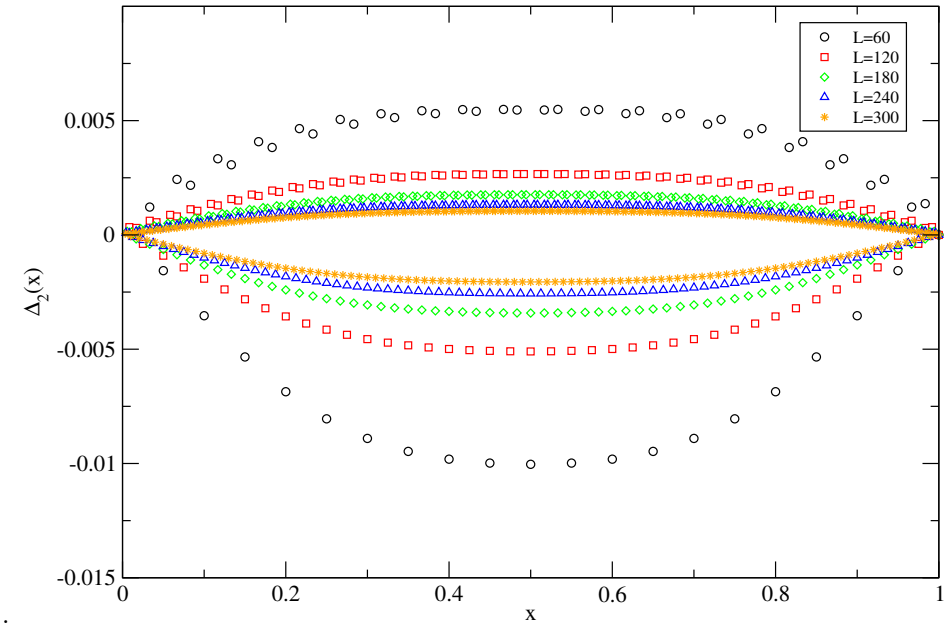

Figure 6.12: *Corrections to $F^{(2)}$ in the case of different chains with filling $\nu = 1/3$.*

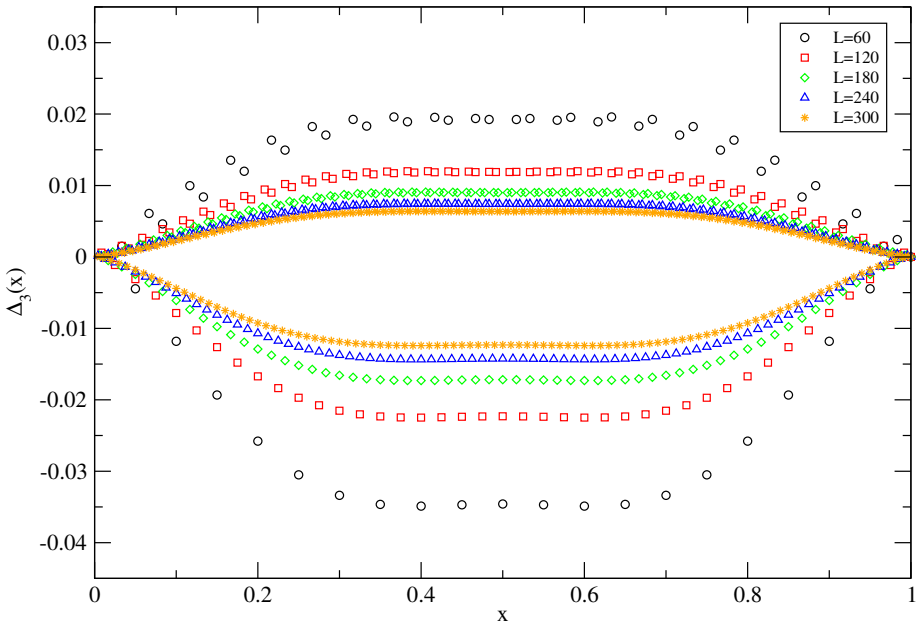

Figure 6.13: *Corrections to $F^{(3)}$ in the case of different chains with filling $\nu = 1/3$.*

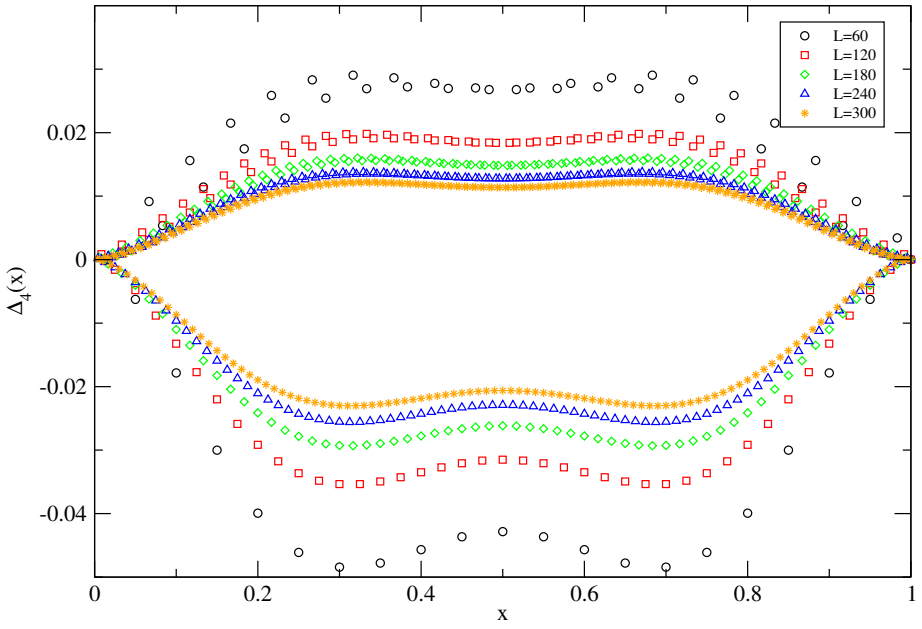

Figure 6.14: *Corrections to $F^{(4)}$ in the case of different chains with filling $\nu = 1/3$.*

If now we multiply all the previous corrections by their own proper scaling factor:

$$\left(\frac{L\sin(\pi x)}{\pi}\right)^{2/n} \tag{6.95}$$

We see that all the different corrections collapse on the same curve as we predicted:

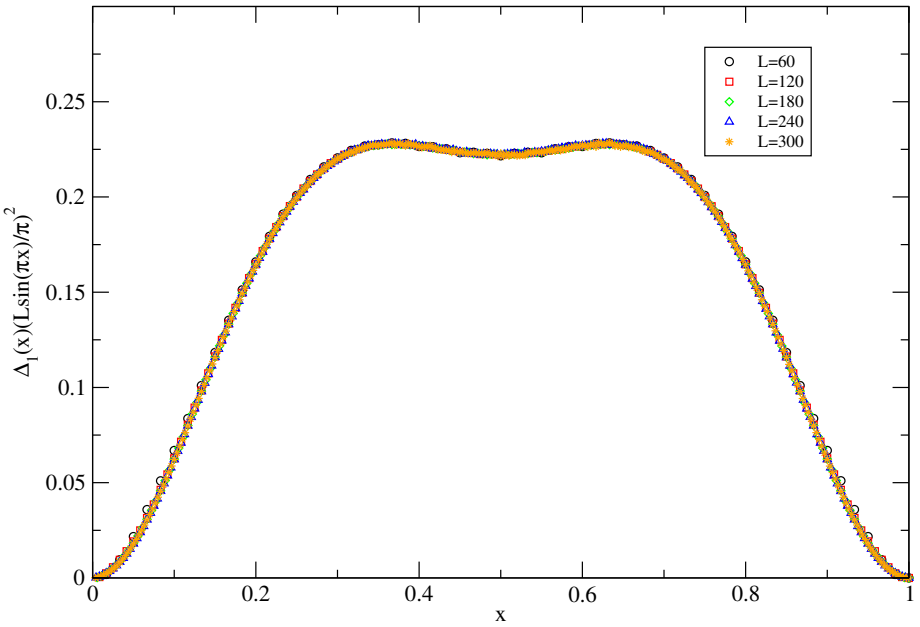

Figure 6.15: *Scaled corrections to $F^{(1)}$. All the different corrections in Fig.6.11 collapse on the same curve*

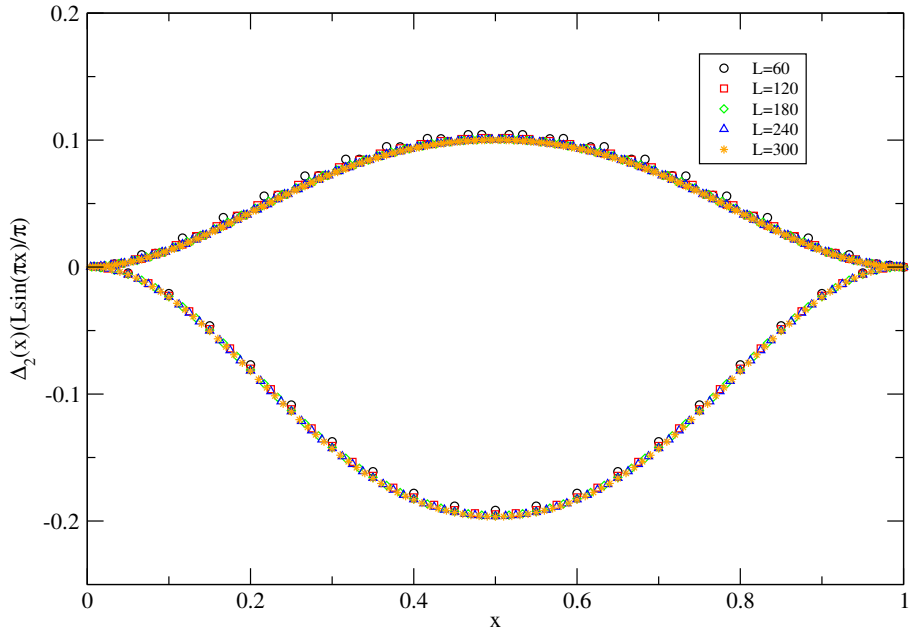

Figure 6.16: *Scaled corrections to $F^{(2)}$. All the different corrections in Fig.6.4 collapse on the same curve*

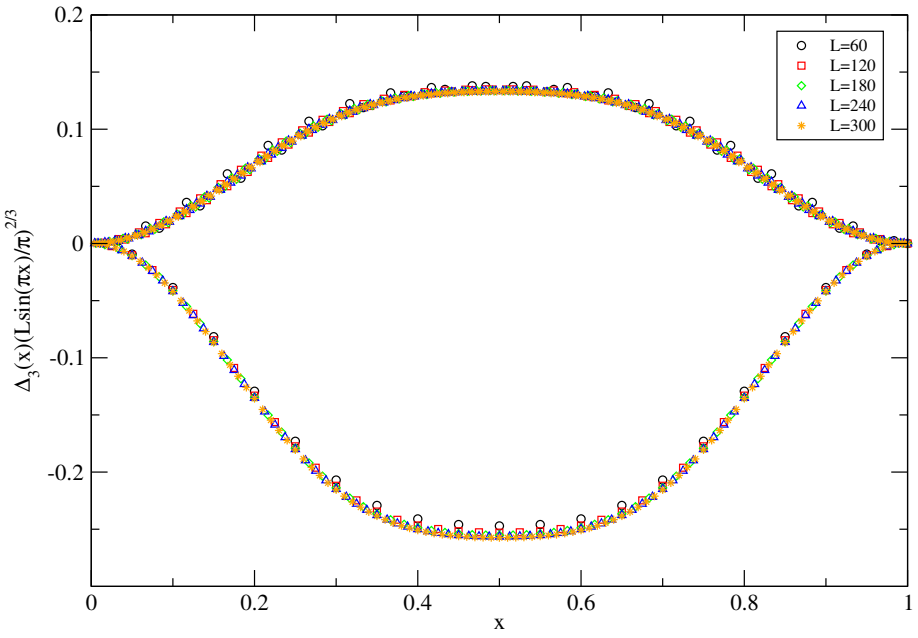

Figure 6.17: *Scaled corrections to $F^{(3)}$. All the different corrections in Fig.6.13 collapse on the same curve*

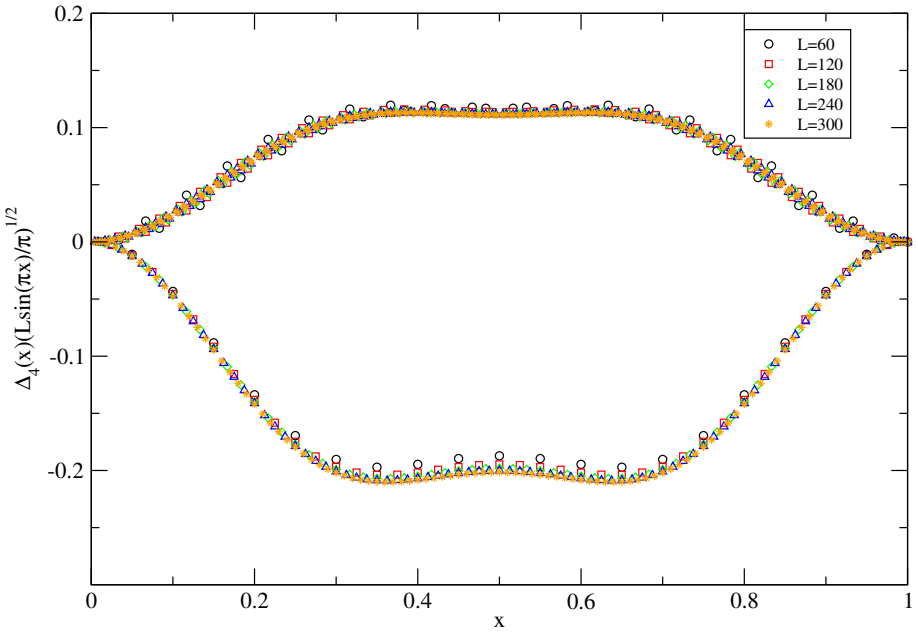

Figure 6.18: *Scaled corrections to $F^{(4)}$. All the different corrections in Fig.6.14 collapse on the same curve*

All these graphics have shown that the scaling of the correction is exactly the same as the ground state case as we predicted in Chapt.5.
It is important to notice that the scaling of the oscillations is in perfect agreement with numerical data for every filling and every Renyi index studied.
The corrections for different fillings are different and we can think that there is some relation between them. In particular we can try to compare the corrections of excited states with the ground state ones that are known.

We have used the XX model that is integrable and there are some very general and analytical results about it, in particular Calabrese, Mintchev and Vicari have studied in Ref.[61] its corrections to the scaling of entanglement entropy in the ground state case.
They found an analytical relation that describes the corrections to the scaling:

$$d_n(N) \equiv S_n(N) - S_n^{a\to 0}, \tag{6.96}$$

this quantity can be computed using the Fisher-Hartwing conjecture, that in this case is not a conjecture since it has been rigorously demonstrated. The expression for the corrections, valid up the order N^{-3}, is the following one:

$$d_n(N) = \frac{2}{1-n}\sum_{p,q=1}^{\infty}(-1)^p L_N^{-\frac{2p(2q-1)}{n}}(Q_q)^p\left[\frac{\cos(2p\pi Nl/L)}{p} + \frac{A_q\sin(2p\pi Nl/L)}{L_N}\right.$$
$$\left. + \frac{\left[B_{p,q}e^{2\imath\pi pNl/L} + h.c.\right]}{L_N^2}\right] + \frac{1}{L_N^2}\frac{n+1}{285n^3}\left(15(3n^2-7) + (49-n^2)\sin^2(\pi l/L)\right) + O(L_N^{-3}). \tag{6.97}$$

In this expression we have used the definitions:

$$N = L\sin(\pi\nu), \tag{6.98}$$
$$L_N = 2N\sin(\pi l/L), \tag{6.99}$$
$$A_q = \left[1 + 3\left(\frac{2q-1}{n}\right)^2\right]\cos(\pi l/L), \tag{6.100}$$
$$Q_q = \left[\frac{\Gamma(\frac{1}{2}+\frac{2q-1}{2n})}{\Gamma(\frac{1}{2}+-\frac{2q-1}{2n})}\right]^2, \tag{6.101}$$
$$B_{p;q} = \frac{2q-1}{6n}\left[\left(5 + 7\frac{(2q-1)^2}{n^2}\right)\sin(\pi l/L) - 15\left(\frac{(2q-1)^2}{n^2} + 1,\right)\right]$$
$$- \frac{p}{4}\left[\left(1 + 3\frac{(2q-1)^2}{n^2}\right)\cos(\pi l/L)\right]^2. \tag{6.102}$$

The method used to obtain these expressions is based on the fact that the correlation matrix of the ground state is in a particular form called Toeplitz matrix.
If we look at the correlation matrix in the excited state case we have that the correlation matrix is not a Toeplitz matrix and it is not possible to compute analytically the corrections. Anyway we could try to compare the two cases in order to have an idea o f t he f or m o f the corre ct ions in the excited state case.
This further study of the excitation is suggested by the fact that the functions $\mathcal{F}_n(x)$ for different fillings ν and for different values of n are similar even if not equal. In particular we can notice that the corrections to the entropy in the ground state with $\nu = 1/2$ oscillates:

$$d_n(N) \sim \cos(2k_F l). \tag{6.103}$$

In this specific case $2k_F = 2\pi\nu = \pi$ that simplify the previous expression:

$$d_n(N) \sim \cos(\pi l), \tag{6.104}$$

since l is an integer number, it labels the number of sites that composes the subsystem under study, we have:

$$d_n(N) \sim (-1)^l. \tag{6.105}$$

This means that the corrections are one negative and one positive in the ground state case. This happens also in excited states as we can see from all the figures of the half filling case: Fig. 6.8, Fig. 6.9 and Fig. 6.10.
This suggests that a comparison between the two cases should lead us to something interesting as the ν behaviour of the functions $\mathcal{F}_n(x;\nu)$.
We can truncate the expression 6.97 at the leading order in N^{-1}:

$$d_n(N) = \frac{2\cos(2k_F l)}{1-n}\left(2N\sin(\pi l/L)\right)^{-2/n}\left[\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2n}\right)}\right]^2. \tag{6.106}$$

We can try to look at the case of the excited state where we expect that the quantity:

$$R_n(x) = \frac{(1-n)\Delta_n(n;\nu)(L\sin(\pi x)/\pi)^{2/n}(2\pi)^{2/n}}{2\cos(2k_F l)\left[\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2n}\right)}\right]^2}, \tag{6.107}$$

has to be an universal quantity that does not depend on the fillings ν and on the lengths L of different chains.
In order to check our last statement we plot the scaled corrections of different chains with different filling factors, as we expect from the cases $\nu = 1/3$ and $\nu = 1/2$ all the scaled corrections have a different shape:

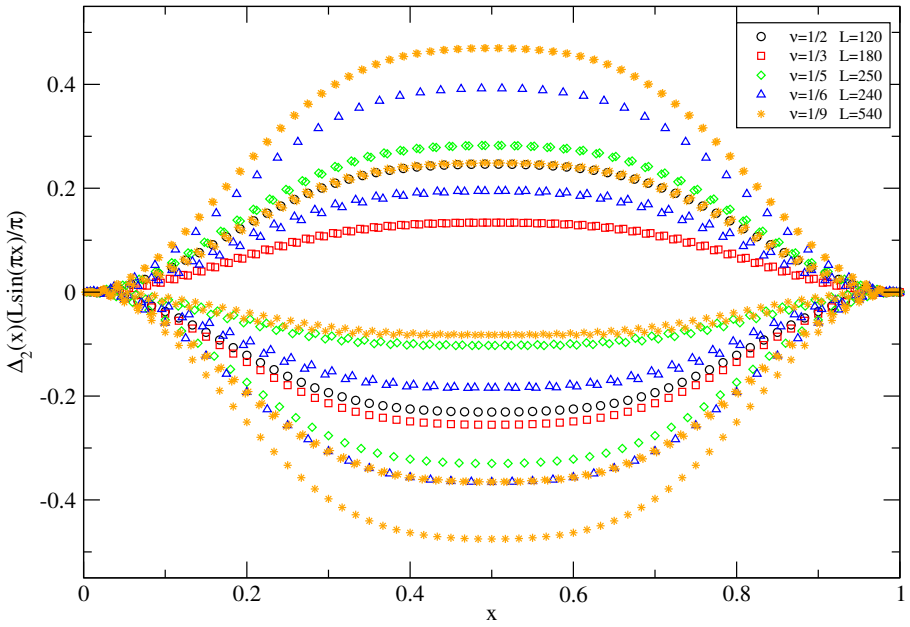

Figure 6.19: *Comparison between the scaled corrections to the $n = 2$ Renyi entropy of different fillings XX spin chains.*

If we plot now the expressions 6.107 for all chains we have that all the previous quantities collapse on one:

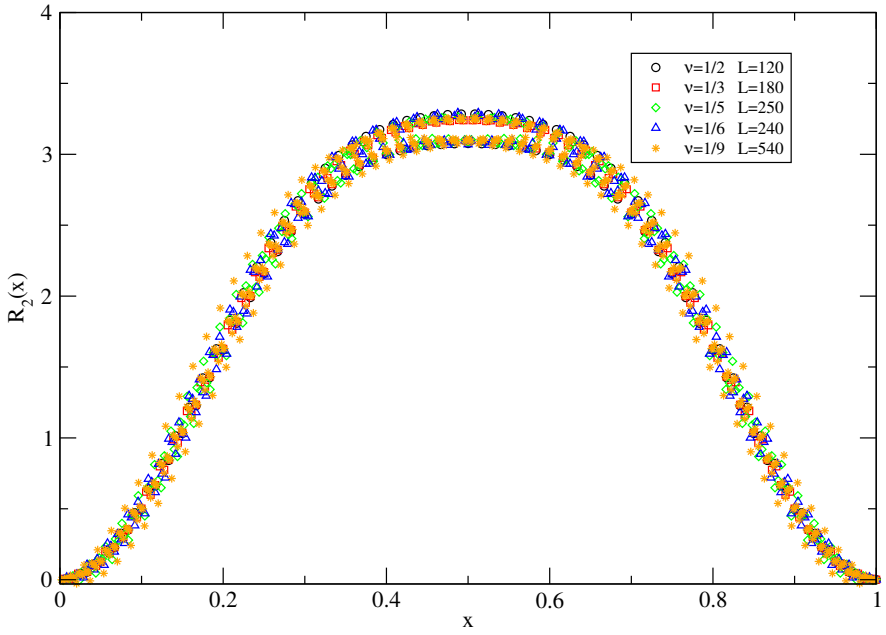

Figure 6.20: *Plot of the expression 6.107 for previous exposed different filling spin chains.*

The oscillations that affect these figures are due to the next to leading order in N^{-1} that is composed by other oscillating terms, the reader can find them in the expression 6.97. If we make computations with very long chains these contributions have to be suppressed and we will remain only with a smooth curve.
Here we have the asymptotic behaviour for the case of $n = 2$:

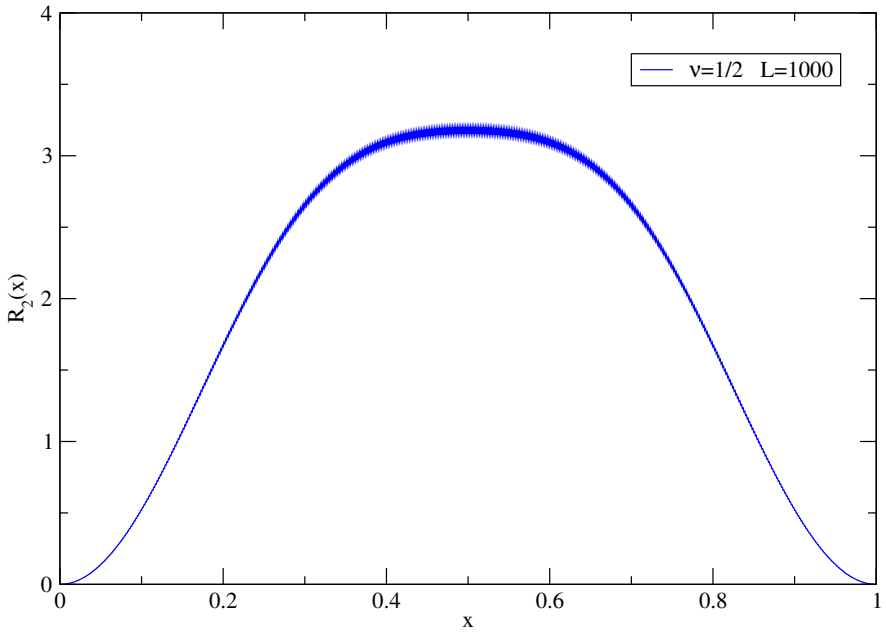

Figure 6.21: Shape of the function $R_2(x)$ obtained using a half-filled chain of length $L = 1000$

This figure confirm that the behaviour of the corrections to scaling in the excited state case is the same as in the ground state case. We can write the correction of the excited states in this way:

$$\Delta_n = \frac{2\cos(2k_F l)}{1-n}\left(2N\sin(\pi l/L)\right)^{-2/n}\left[\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2n}\right)}\right]^2 R_n(x), \tag{6.108}$$

where $R_n(x)$ is a function depending only on the fraction x.

Conclusions and outlooks

In this work we have studied the presence of unusual corrections to scaling in the entanglement entropy of excited states of conformal field theory.
Initially we have described the Calabrese and Cardy approach to the entanglement entropy of 1D critical quantum systems, see Ref.[10], that is reminiscent of the Replica Trick used by Holzey in Ref.[13]. The main result of the approach is the *universal* logarithmic scaling of the entanglement entropy of an 1D critical quantum system. This scaling is one of the most beautiful example of universality in the physics of many body systems and it has been observed in many works, see Ref.[62], Ref.[63] and Ref.[64] for example.
Besides the logarithmic scaling behaviour there also corrections that affect the entanglement entropy. These corrections have an unusual scaling behaviour that has been intensively studied in literature, see Ref.[16], Ref.[17] and Ref.[18] for example. In Ref.[15] Calabrese and Cardy demonstrated that the particular scaling of corrections is due to the local breaking of the conformal invariance of the system around the conical singularities used to define the Riemann surface. Furthermore the Calabrese and Cardy approach can be used to explore the entanglement entropy of excited states of conformal field theory, see Ref.[19] and Ref.[64], and it gives analytic results in perfect agreement with numerical computations both for fermionic and bosonic theories.
In our work we demonstrated that the entanglement entropy of the excited states is also affected by corrections that has the same scaling of the ground state ones. These corrections are due to the local breaking of conformal invariance around the conical singularities of the Riemann surface as well as the ones of the ground state. We used numerical computations in order to confirm our theoretical predictions and they are all in perfect agreement with analytic computations. We analysed the entanglement entropy of the excited states of the XX model and we compare the form of their corrections with the ones of the ground state finding the same dependence on the filling factor ν, see Ref.[61].
The research on the excited states of conformal field theory is a very active field of study and it would be possible to extract other interesting results from our analysis in the future.
As we said in Chapt.1 the entanglement spectrum, that is composed by the eigenvalues of the entanglement Hamiltonian, encodes useful information about the amount of entanglement in the quantum state of the system. A detailed description of the entanglement spectrum of excited states can give us other information about their physics and their entanglement amount.
Another study could be the time evolution of the excited states after a quantum quench. This is one of the protocols used to take a system out of equilibrium and it is particularly interesting because it can be realised in real experiments. This protocol consists in a system initially prepared in one of the eigenstates, generally the ground state, of an initial Hamiltonian H_0. Then a sudden change in the initial hamiltonian, namely $H_0 \to H_1$, takes the system out of equilibrium and the time evolution of the system is observed. It would be interesting to study the time evolution, after a quantum quench, of a system prepared in an excited state of the initial hamiltonian H_0 instead of the ground state.

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