Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
Corso di Laurea Magistrale in Fisica

$f(R)$ approximation in asymptotically safe quantum gravity

Relatore:
Dott. Alexandr Kamenchtchik

Correlatore:
Dott. Gian Paolo Vacca

Presentata da:
Demetrio Vilardi

Sessione I
Anno Accademico 2012/2013

Abstract

In questo lavoro viene seguito lo schema di sicurazze asintotica per la gravità quantistica. In tale approccio, è stata avanzata la possibilità che l'esistenza di un punto fisso non-Gaussiano nell'ultravioletto, con un numero finito di direzioni attrattive, ci permetta di considerare la teoria dei campi della relatività generale come un approccio consistente per la gravità quantistica. In questo lavoro, viene portato avanti un ansatz per l'azione effettiva mediata come funzione del solo scalare di curvatura $\Gamma_k \sim \int dx \sqrt{g} f_k(R)$. Attraverso tale scelta vengono utilizzate le tecniche del gruppo di rinormalizzazione funzionale per studiare il flusso della funzione $f_k(R)$; in particolare vengono utilizzati tre approcci differenti per il calcolo delle tracce funzionali nell'equazione di Wetterich's: la tecnica Heat Kernel e la somma spettrale sia attraverso una approssimazione asintotica sia utilizzando la formula di Eulero-Maclaurin per le somme finite. Nei primi due casi viene utilizzato uno schema di cutoff non-diagonale e vengono confermati risultati ottenuti già in precedenza. Invece, l'approssimazione di Eulero-Maclaurin permette di studiare il flusso della $f_k(R)$ con un cutoff diagonale attraverso una equazione differenziale del secondo ordine.

Abstract

In this thesis we follow the asymptotic safety program for quantum gravity. In this program, it has been proposed that the existence of a non-Gaussian UV fixed point with finite number of attractive directions for quantum Einstein's theory allows us to study the quantum field theory of general relativity as a self-consistent condidate for quantum gravity. In this work, we make an ansatz for average effective action as a function of scalar curvature only $\Gamma_k \sim \int dx \sqrt{g} f_k(R)$. With this choice we use the functional renormalization group formalism to study the flow of function $f_k(R)$ and use three different techniques to evaluate the functional traces in Wetterich's equation: the Heat Kernel technique, the spectral sums with the asymptotic behaviour approximation and with Euler-Maclaurin formula for finite sums. The first two techniques, which exploit a non-diagonal cutoff scheme, confirms the results given in previous works. Instead, Euler-Maclaurin approximation allows us to study the flow equation with a diagonal cutoff which gives a second order differential equation on $f_k(R)$ instead of a third order one, which can be used for a future numerical study.

Introduction

In the last century, two important theories changed our concepts on the Universe. First, quantum mechanics changes our point of view about microscopic world; on the other hand, the General theory of Relativity modifies our concept of spacetime.

Starting from the principles of quantum mechanics and the special theory of relativity, in the last 80 years, the quantum theory of fields has been developed and describes three of the fondamental interactions with great agreement with experiments. Although this success, quantum field theory, in the perturbative domain, can not be used to treat the quantum theory of gravity, since the counterterms in the action cannot be absorbed into a redefinition of fields or coupling constants; such a theory is said perturbatively non-renormalizable.

However, general relativity can be treated as an effective field theory, in the sense that one can compute the quantum effects due to graviton loops as long as the momenta of the particles in the loops are cut off at some scale. In this way it has been possible to calculate the quantum corrections to the non-relativistic Newton's potential [1]; but this is unrelated to the UV behaviour of the theory.

Hence, for quantum field theory of general relativity the concept of perturbative renormalization is not a powerfull method to predict the UV regime; so a non-perturbative approach is necessary to understand whether this theory is a consistent canditate for the quantum gravity problem.

Over the years, a series of different approaches propose a fondamental theory of quantum gravity. We can devide them into two categories: the *bottom-up* and *top-down* approaches. With the top-down approach, physicists try to replace the old theories with a new fondamental theory and verify that the low energy effective theory coincide with previous ones; as examples, for quantum gravity, physicists propose string theory, extradimensions and so on.

Contrary, the bottom-up approach has a different starting point. We know that quantum field theory and general relativity work so well in their domain; hence, the basic idea is to unify them starting from the principles of both theories. The asymptotic safety approach belongs to the latter category. This theory starts from the quantum field theory version of general relativity, considering the metric tensor $g_{\mu\nu}$ as fondamental degrees of freedom.

The main question about QFT of GR is whether this theory gives predictible quantities at all energies.

In fact, motivated by the analogy to the asymptotic freedom properties of non-Abelian gauge theories, the term "asymptotic safety" was suggested in [2] indicating that physical quantities are "safe" from divergencies as the cutoff is removed.

To quote from [2]: "A theory is said to be asymptotically safe if the essential coupling parameters approach a fixed point as the momentum scale of their renormalization point goes to infinity". Here, the "essential" couplings are those which are useful for the absorption of cutoff dependencies. If this criterion is valid for QFT of GR than the theory becomes predictive at all energies. Hence, to answer to the main question about QFT of GR, a non-perturbative approach is needed.

In the last 40 years, it has been shown that quantum field theory possesses an incredible power not only in high energy sector but also in statistical mechanics, for example, it has been used to understand critical phenomena and non-equilibrium conditions. This is due to Wilson's idea on Renormalization Group (RG), whose aim is to understand how physics changes when the typical lenght scale varies.

Within the asymptotic safety program, the basic idea is to derive the behaviour of quantum Einstein's theory with renormalization group approach, which can give us the possibility to study the running of the action functional when the momentum scale goes to infinity.

In fact, with renormalization group techniques we observe how the laws of physics change at different lenght scale. Hence, with this formalism we can relate micro- and macro-physics of the gravitational field. In particular, with the functional renomalization group approach, we can introduce an effective action Γ_k, which depends on typical momentum scale k and which give us all information about the system at lenght scale $l \sim \frac{1}{k}$ (in flat spacetime). This idea can be implemented considering the high and low momentum field modes of quantum fluctuations in a different way. In a path integral approach this corresponds to integrate only those fluctuation modes with momentum p less than k. In particular, one can construct an effective action, and study the flow towards of the so called *average effective action.*

As we shall see in the chapter 1, we can interpolate between the microscopic action S_B (at high energies) and the full quantum effective action Γ (at low energy) with all quantum flactuations taking into account and construct a functional Γ_k, which contains all informations about the physics at scale k. An important feature of this formalism is that we can determine the flow from the *bare action* S_B (for $k \to +\infty$) down to quantum action $\Gamma_{k\to 0}$ and observe, directly, how physics can change when the scale k varies. This can be implemented with Wetterich's equation [3].

In chapter 2, we generalize the functional RG technique for the Einstein's theory with the background field method, first used in non-Abelian Yang-Mills theory [4]. We employ an ansatz on average effective action $\Gamma_k \sim \int dx\sqrt{g} f_k(R)$ as a function only on scalar curvature R, neglecting more involved couplings such as $R^{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma}$ or $R^{\mu\nu}R_{\mu\nu}$. Finally, following [5], we rederive a third order differential equation on $f_k(R)$, which governs the flow of average effective action, and we extend the flow equation including the anomalous dimension of the fields. Contrary to [5], we use a different metric decomposition involving the Newton's constant and introduce the anomalous dimensions contribution to the flow equation. We study the polynomial truncation and verify that our results are compatible with that obtained in literature.

In chapter 3, we use a different mathematical technique, introduced in [6], and we obtain a different differential equation for $f_k(R)$. Authors in [6] evaluate the functional traces, present in Wetterich's equation, with spectral sums technique. In this thesis, we extend Benedetti and Caravelli's equation in general spacetime dimensions, verifing that the approximation used is still valid in d dimensions and including the anomalous dimensions contribution. We also study the polynomial truncation up to order $n = 5$.

In chapter 4 we introduce an alternative method for the evaluation of functional traces, given in a different, perhaps more physical, cutoff scheme. This cutoff choice is indipendent of $f_k''(R)$, so that the resulting flow equation is of second order, instead of third order, as in the previous works. The usual techniques for trace evaluation cannot be used in this context and we propose to employ the Euler-Maclaurin approximation for the spectral sums.

We studied a polynomial truncation and found a non-Gaussian UV fixed point with the same qualitative properties of that obtained with the previous flow equations, *i.e.* with a "third order" cutoff scheme and different trace approximation methods.

Contents

Introduction **vii**

1 Functional Renormalization Group **1**
1.1 Functionals Approach to Quantum Field Theory 1
1.2 Renormalization Group Flow 3
1.3 Need for truncation and projected theory-space 7
1.4 Asymptotic safety 8
1.5 Quantum field theory of General Relativity 10

2 Renormalization Group Flow for Quatum Gravity **13**
2.1 Construction of Functional RG for gravity 13
2.2 Tranverse-traceless decomposition 16
2.3 $f_k(R)$ trunctation ansatz 19
2.4 Cutoff scheme and gauge choice 23
2.5 Trace evaluation using the Heat Kernel technique 25
2.6 Possible closures for $f_k(R)$ RG equation 32
2.7 Polynomial truncation 33

3 Trace evaluation with spectral sums in d dimension **39**
3.1 Traces as spectral sums 45
3.2 Beta function for $f_k(R)$ 55
3.3 Possible closures for $f_k(R)$ RG equation 57
3.4 Polynomial truncation 57

4 Alternative flow equation for $f_k(R)$ with Euler-Maclaurin approximation **61**
4.1 Cutoff scheme 62
4.2 Spectral sums and Euler-Maclaurin approximation 64
4.3 Flow equation for f_k 73
4.4 Possible closures for $f_k(R)$ RG equation 75
4.5 Polynomial truncation 76
4.6 Spectral sums with Digamma function 78

4.7 Flow equation for f_k with Digamma function 82

Conclusion **85**

Appendices

Appendix A Riemann tensor variation **89**

Appendix B Calculation of $\Gamma_k^{(2)}$ in $f(R)$ approximation **93**

Appendix C Heat Kernel Technique **99**

Bibliography **105**

Acknowledgment **107**

Chapter 1

Functional Renormalization Group

1.1 Functionals Approach to Quantum Field Theory

In Quantum Field Theory all physical information, such as scattering amplitudes, is stored in Green functions or correlation functions. In Euclidean quantum field theory for a scalar field $\phi(x)$, described by the action $S[\phi]$, the n-point Green functions are defined by

$$\langle \phi(x_1)\phi(x_2)...\phi(x_n)\rangle := \mathcal{N}\int \mathcal{D}\phi\phi(x_1)\phi(x_2)...\phi(x_n)e^{-S[\phi]} \tag{1.1}$$

where $\mathcal{N}$ is such that $\langle 1\rangle = 1$. We suppose that there exists a regularized definition of the measure, in this case an ultraviolet cutoff Λ is imposed as a consequence of a spacetime lattice discretization; so $\int \mathcal{D}\phi$ is replaced by $\int_\Lambda \mathcal{D}\phi$. One can define the functional

$$Z[J] = \int_\Lambda \mathcal{D}\phi e^{-S[\phi]+\int J\phi} \tag{1.2}$$

where J is an external source coupled with ϕ through $\int J\phi$, which summarizes $\int d^dxJ(x)\phi(x)$. In terms of (1.2) the Green functions are obtained as

$$\langle \phi(x_1)\phi(x_2)...\phi(x_n)\rangle = \frac{1}{Z[0]}\left(\frac{\delta^{(n)}Z[J]}{\delta J(x_1)...\delta J(x_n)}\right)_{J=0} \tag{1.3}$$

for this reason $Z[J]$ is called full Green functions generating functional. Equation (1.3) tells us that $Z[J]$ contains all physical information about our scalar field theory.

One can also introduce another functional

$$W[J] := \ln Z[J] \tag{1.4}$$

which generates the *connected* Green functions or *connected correlators*, in analogous with $Z[J]$

$$\langle \phi(x_1)\phi(x_2)...\phi(x_n)\rangle_c = \left(\frac{\delta^{(n)}W[J]}{\delta J(x_1)...\delta J(x_n)}\right)_{J=0} \tag{1.5}$$

In most cases it is more convenient to do calculations with connected Green functions than with reducible Green functions (1.1). A simple, but important, example is the 2-point connected Green's function, called *non-perturbative propagator*

$$G^c(x_1, x_2) \equiv \langle \phi(x_1)\phi(x_2)\rangle_c = \langle \phi(x_1)\phi(x_2)\rangle - \langle \phi(x_1)\rangle\langle \phi(x_2)\rangle \tag{1.6}$$

where the last relation can be obtained inserting (1.4) in (1.5) and consider $n = 2$.

Whitin the functional approach in quantum field theory, there is a more efficient way to store the physical information, introducing the Legendre transform of $W[J]$, one starts defining

$$\varphi = \frac{\delta W}{\delta J} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J} = \langle \phi \rangle_J \tag{1.7}$$

which means that new variable φ corresponds to the expectation value of the scalar field ϕ in the presence of the source. Through the definition (1.7) one can finds explicitly the relation $J[\varphi]$. So the Legendre transform reads

$$\Gamma[\varphi] := \left(\int J\varphi - W[J] \right)_{J[\varphi]} \tag{1.8}$$

This is the *quantum effective action* for the scalar theory. Our definition of Γ guarantees that Γ itself is a convex functional (every Legendre transform does share this properties).

Taking the functional derivative of (1.8)

$$J = \frac{\delta \Gamma[\varphi]}{\delta \varphi} \tag{1.9}$$

and then setting to zero the source, one obtains

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi} = 0 \tag{1.10}$$

which is the *quantum equation of motion.* This equation governs the dynamics of the expectation value of the field taking into account its quantum fluctuations.

We can expand the effective action

$$\Gamma[\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 ... dx_n \Gamma^{(n)}(x_1, ..., x_n)\varphi(x_1)...\varphi(x_n) \tag{1.11}$$

where the coefficients $\Gamma^{(n)}$ are the n-point *one particle irreducible (1PI) Green functions* or *proper verteces.*

A simple, but foundamental, example is the 2-point 1PI Green function $\Gamma^{(2)}(x_1, x_2)$ which is also the inverse of nonperturbative propagator defined above in eq. (1.6), as the following relations show

$$\frac{\delta \varphi(x_1)}{J(x_2)} = \frac{\delta^{(2)} W[J]}{\delta J(x_1)\delta J(x_2)} = G^{(2)}(x_1, x_2) \tag{1.12}$$

$$\frac{\delta J(x_2)}{\delta \varphi(x_1)} = \Gamma^{(2)}(x_1, x_2) \tag{1.13}$$

so we conclude that

$$\int dz G^{(2)}(x,z)\Gamma^{(2)}(z,y) = \delta(x-y) \tag{1.14}$$

which tells us that $\Gamma^{(2)}$ is the inverse of the nonperturbative propagator.

Another way to define effective action, without introducing the Green's functions generator $W[J]$ and the Legendre transform, is the following: we perform a substitution variable $\phi \to \phi + \varphi$ in the functional integral in (1.2) and then impose $J = J[\varphi]$

$$e^{-\Gamma[\varphi]} = \int_\Lambda \mathcal{D}\phi \exp\left(-S[\phi] + \int \frac{\delta\Gamma[\varphi]}{\delta\varphi}(\phi - \varphi)\right) \tag{1.15}$$

If one use the expansion (1.11) in the last equation, one could find a infinitly system of coupled differential equations for the proper vertex known as Dyson-Schwinger equations.

The functional approach provides a perfectly defined non-perturbative method in quantum field theory, although an exact determination of $\Gamma[\varphi]$ is found only for special and rare case.

1.2 Renormalization Group Flow

Consider a large momentum scale Λ and our scalar field theory described by the *bare action* $S_B[\phi]$. The Functional Renormalization Group (FRG) approach is based on Wilson's idea to start with such a classical action S_B (at momentum scale Λ) and then to integrate out all flactuations successively from high to low momuntum scales. Once all flactuations are included one may cover the full quantum theory described above. For a review see [4].

Referring to the effective action Γ, we fix a momentum scale parameter k and construct an interpolating action Γ_k, the *average effective action* depending on the scale k, by imposing that $\Gamma_\Lambda \simeq S$ as initial condition and $\Gamma_{k\to 0} = \Gamma$. To construct the flow of Γ_k, from S to Γ, we modify the definition of generating funcionals (1.2) (1.4) by introducing an IR regulator as follows

$$e^{W_k[J]} = Z_k[J] := \int_\Lambda \mathcal{D}\phi e^{-S[\phi] - \Delta S_k[\phi] + \int J\phi} \tag{1.16}$$

It is convenient to choose

$$\Delta S_k[\phi] = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\phi(-p)R_k(p^2)\phi(p) \tag{1.17}$$

quadradic in the field ϕ so as to modify the mass term in the action. The regulator function $R_k(p^2)$ must satisfy

$$\lim_{p^2 \to 0} R_k(p^2) > 0 \tag{1.18}$$

which implies that for $p^2 \ll k^2$ the regulator behaves $R_k(p^2) \sim k^2$, so all modes with momentum lower then k acquire an effective mass $m \sim k$. This additional term acts as a screen for the IR modes and, as we shall see, modifies the full propagator of the theory. In the other regime

$$\lim_{p^2 \to \infty} R_k(p^2) = 0 \tag{1.19}$$

all the UV above the scale k are unaffected by the cutoff. Moreover we see that

$$\lim_{k \to 0} R_k(p^2) = 0 \tag{1.20}$$

which tells us that definition (1.16) for $k \to 0$ gives the standard generating functional (1.2) $Z_{k\to 0}[J] = Z[J]$, which implies (see below) $\Gamma_{k\to 0}[\varphi] = \Gamma[\varphi]$.

The last conditions we impose on the cutoff function R_k reads

$$\lim_{k^2 \to \Lambda \to \infty} R_k(p^2) = \infty \tag{1.21}$$

so that in the UV regime we have the condition $\Gamma_\Lambda[\phi] \simeq S[\phi]$ (see below). We now proceed to the definition of average effective action introducing

$$\varphi(x) = \frac{\delta W_k[J]}{\delta J(x)} = \langle \phi(x) \rangle_J \tag{1.22}$$

which allows to extract the functional $J = J_k[\varphi]$. The effective average action is then defined by

$$\Gamma_k[\varphi] = \left(\int J\varphi - W_k[J] \right)_{J_k[\varphi]} - \Delta_k S[\varphi] \tag{1.23}$$

which is a modified Legendre transform (so Γ_k is not a convex functional).

Following the same argument of the previous section, we can find a relation between the nonperturbative modified propagator $G_k^{(2)}(x_1, x_2)$, defined as the second functional derivative of the scale dependent functional $Z_k[\phi]$, and the average effective action.

Adapting eqs (1.12-1.13), we find

$$\int d^d z (\Gamma_k^{(2)} + \mathcal{R}_k)(x,z) G_k^{(2)}(z-y) = \delta^d(x-y)$$

or in matrix notation

$$(\Gamma_k^{(2)} + \mathcal{R}_k) G^{(2)} = \mathbf{1} \tag{1.24}$$

which tells us that, in the presence of the IR cutoff, the inverse propagator contains explicitly the cutoff function $\mathcal{R}_k$, as expected. Note that in the limit $k \to 0$ we have $\mathcal{R}_k \to 0$ and the inverse propagator turns out to be the full quantum one $\Gamma^{(2)}$.

From now on, the aim is to determine the interpolating itermadiate trajectory between the two limits of the average effective action, constructing a differential equation which captures the flow from the classical to the quantum action. Introducing the renormalization group time t

$$t = \ln \frac{k}{\Lambda} \qquad \partial_t = k\partial_k \tag{1.25}$$

taking the derivative of $W_k[J]$

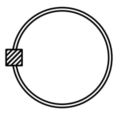
$$\begin{aligned}\partial_t W_k[J] &= -\frac{1}{2Z_k[J]} \int \mathcal{D}\phi \int \frac{d^d p}{(2\pi)^d} \phi(-p)\partial_t \mathcal{R}_k(p)\phi(p) e^{-S-\Delta S+\int J\phi} \\ &= -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \partial_t \mathcal{R}_k(p) G_k(p) + \partial_t \Delta S_k[\phi]\end{aligned} \tag{1.26}$$

where $G_k(p) = \langle\phi(-p)\phi(p)\rangle_k - \langle\phi(-p)\rangle\langle\phi(p)\rangle_k$ is the *modified connected propagator*. Taking the derivative with rispect to t in (1.23), using (1.26) and considering that the functional $J_k[\phi]$ depends explicitly on k, we obtain

$$\begin{aligned}\partial_t \Gamma_k[\varphi] &= \int \varphi \partial_t J_k[\varphi] - (\partial_t W_k)[J_k[\varphi]] - \int \frac{\delta W_k}{\delta J} \partial_k J_k[\varphi] - \partial_t \Delta S_k[\varphi] \\ &= \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \partial_t R_k(p) G_k(p)\end{aligned}$$

using (1.24) we find the Wetterich [3] equation

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} \operatorname{Tr} \left[\partial_t \mathcal{R}_k \left(\Gamma^{(2)}[\phi] + \mathcal{R}_k \right)^{-1} \right] \tag{1.27}$$

$$= \frac{1}{2}$$ 

This equation governs the flow starting from the bare action $S[\phi]$ down to $\Gamma[\varphi]$.

The importance of this equation can be summarized into the following properties

- No approximation are made in the derivation of Wetterich's equation, so one usually refers to (1.27) as Exact Renormalization Group Equation (ERGE).
- Contrary to Polchinsky equation, the microscopic action $S_\Lambda[\phi]$ appears only as initial condition at momuntum scale Λ.
- In this chapter we derived the Wetterich equation starting from the standard quantum field theory viewed through Wilson's eyes. Conversely, we can construct all properties of quantum field theory starting from that equation, remembering that all physical information are stored inside the effective average action, which can be obtained through the limit $k \to 0$ of solution of (1.27), at least formally. In the next section we show that an approximation scheme in this approach is required.

- In equation (1.27) the cutoff function $\mathcal{R}_k$ appears explicitly. So the consequence flow depends on the choice of the cutoff, which, fortunately (or not), in most cases can be choosen arbitrary, up to eqs (1.17-1.18-1.19-1.21), so to simplify the resulting equation. The function $\mathcal{R}_k$ introduces a scheme dependence, nevertheless the initial $\Gamma_\Lambda \sim S$ and final point $\Gamma_{k=0} = \Gamma$ of the flow are scheme independent thanks to relation (1.17-1.18-1.19).

- The ERGE has been derived via the path integral technique, defining the relative measure imposing the lattice regularization method, so the UV bare action has been considered. Since physics is stored into the renormalized, rather than bare, action, if we correctly know $\Gamma_{\hat{k}}$, given at a fixed momentum scale $\hat{k}$, typically found measuring the coupling costant at that energy scale, we can construct the relative flow, from $\hat{k}$ to a general scale k, using ERGE and find new physics at different energy scale. From this flow we can construct the limit $k \to \infty$ and search if the correct limit is found. Note that typically the bare action is not the fixed point action, it is just close to it. Starting from the fixed point one cannot move from it.

Starting from (1.27) one can construct "immediatly" the one-loop approximation for the full quantum effective action. Let us expand the effective average action into $\hbar$ expansion, yelding $\Gamma_k = S + \hbar\Gamma_k^{1-\text{loop}} + O(\hbar^2)$, so that to one-loop order $\Gamma_k^{(2)} = S^{(2)}$ and the *rhs* of (1.27) becomes a total derivative

$$\partial_t \Gamma_k^{1-loop} = \frac{1}{2}\partial_t \operatorname{Tr}\ln\left(S^{(2)} + \mathcal{R}_k\right)$$

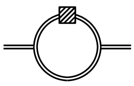
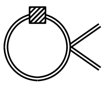
after integration between $k = 0$ and $k = \Lambda$ we finds

$$\Gamma^{1-\text{loop}} = S + \frac{1}{2}\operatorname{Tr}\ln S^{(2)} + \text{const.}$$

which is the standard formula given in many QFT books.

It is interesting to observe that, taking derivative of Wetterich's equation, one can obtain the flow for any proper vertex $\Gamma_k^{(n)}$. For example one has for $\Gamma^{(2)}$:

$$\partial_t \frac{\delta^{(2)}\Gamma_k}{\delta\varphi(x)\varphi(y)} = \operatorname{Tr}\left[\frac{1}{(\Gamma_k^{(2)} + R_k)}\frac{\delta\Gamma_k^{(2)}}{\delta\varphi(x)}\frac{1}{(\Gamma_k^{(2)} + R_k)}\frac{\delta\Gamma_k^{(2)}}{\delta\varphi(y)}\frac{\partial_t R_k}{(\Gamma_k^{(2)} + R_k)}\right] - \frac{1}{2}\operatorname{Tr}\left[\frac{\delta^{(2)}\Gamma_k^{(2)}}{\delta\varphi(x)\delta\varphi(y)}\frac{\partial_t R_k}{(\Gamma_k^{(2)} + R_k)}\right]$$

$= $  $- \frac{1}{2}$ 

where the three- and four-point verteces represent $\delta\Gamma_k^{(2)}/\delta\varphi$ and $\delta^{(2)}\Gamma_k/\delta\varphi\delta\varphi$, respectively. One can have an infinite system of coupled equations.

1.3 Need for truncation and projected theory-space

Wetterich's equation has been derived in an exact way, unfortunately there are no example in quantum field theory for which it can be resolved exatly. As we know, the average effective action, as the full quantum one, must contain *local* and *nonlocal* terms which depend on the mean field φ and only those which are allowed by the considered symmetry, for example gauge transformations for Yang-Mills theory and diffeomorphisms invariance for gravity. The difficulties arise since equation (1.27) is a highly complicated functional equation, so that the space to which the solution Γ_k belongs is an infinite dimensional space of all functionals of spacetime functions which is called *theory-space.* The usefullness of Wetterich's equation arise from the practicality when approximations are made. The appoximation which has been choosen in this work is the method of *operator truncation*; the starting point is to make an ansatz for the average effective action, for example

$$\Gamma_k[\varphi] = \sum_{n=1}^{N} g_n(k)\mathcal{O}_n[\varphi] \tag{1.28}$$

in which $\mathcal{O}_n[\varphi]$ are a finite set of local or non-local functional of its argument which may be choosen to not depend on scale k, whose dependence is stored only in the coefficients $g_{n,k}$. In other words, we project the full theory-space in a N-dimensional space coordinatised by the coefficients $g_{n,k}$; so the resulting flow equation can be obtained inserting (1.28) into equation (1.27). The resulting flow is governed by a system of coupled differential equation

$$\partial_t\Gamma_k[\varphi] = \sum_{n=1}^{N} \beta_n(k)\mathcal{O}_n[\varphi] \tag{1.29}$$

where $\beta_n(k) = \partial_t g_n(k)$ are the beta function associated with the coupling g_n. Expressing also the *l.h.s.* as a function of coupling g_n, and introducing the dimensionless couplings $\tilde{g}_n(k) = k^{d_n} g_n(k)$ in the spirit of renormalization group approach, one construct a new set of equations

$$k\partial_k \tilde{g}_i(k) = \mathcal{F}_i(\tilde{g}_n(k)) \tag{1.30}$$

whose solutions describe the flow for the coupling $\tilde{g}_i$, and so for the truncated average effective action (1.28). Equations (1.30) define a vector field $\boldsymbol{\mathcal{F}}$, with component $\mathcal{F}_i$, on the truncated N-dimensional theory space. Solution of equations (1.30) appears in the theory space as the integral curve of vector field $\boldsymbol{\mathcal{F}}$. As we shall see below, the aim, in this work, is not only to resolve completly equations (1.30), but to find a fixed point (see below for the definition) for the vector field $\boldsymbol{\mathcal{F}}$.

Other choices for the average effective action exist, for example we can truncate into a (still) infinite theory space making the ansatz

$$\Gamma_k[\varphi] = \sum_{n=1}^{N} \mathcal{O}_{n,k}[\varphi] \tag{1.31}$$

where $\mathcal{O}_{n,k}[\varphi]$ is a (choosen) set of functional of mean field φ. With this choice, contrary to ansatz (1.28), the operator $\mathcal{O}_k$ depends explicitly on momuntum scale k. Inserting (1.31) into Wetterich equation, one obtains other functional equations for the operators $\mathcal{O}_{i,k}$ which, we hope, are more simple than the starting flow equation.

1.4 Asymptotic safety

In standard perturbative quantum field theory, the Green's functions, defined in the first section, give infinities, as it is well known. The theory is said to be perturbative renormalizable if this kind of infinities can be eliminated in any physical obrservable with fields' or coupling constants' redefinition order by order in perturbation theory. The renormalizable theories give finite predictive physical quantities as expansion of coupling constants.

The problem of quantum field theory of gravity, as we shall see, is the non renormalizability in the perturbative domain. This conclusion may show us that standard methods in QFT are not consistent in the quantization of general relativity, leading to the expectation of new physics at small lenght scale near the Planck lenght. Before proposing this idea, the extension for renormalizability in the non perturbative regime may be considered; as we shall see, *asymptotically safe* theories replace the perturbatively renormalizable theories in QFT.

The concept of asymptotic safety was introduced for the first time by Nobel laureate Steven Weinberg [7] (for a review see [8]). For a practical introduction consider the scalar field theory analized in previous sections and collect the coupling running constant into the expression

$$\Gamma_k(\varphi, g_i) = \sum_i g_i(k)\mathcal{O}_i(\varphi) \tag{1.32}$$

The dependence of Γ_k on k is given by

$$\partial_t\Gamma_k(\varphi, g_i) = \sum_i \beta_i(k)\mathcal{O}_i(\varphi)$$

The two last relations seem identical to relations (1.28-1.29), but in this section we do not consider any truncation, in relation (1.32) the full average effective action is consider.

The beta functions $\beta_i(k)$ determine how the coupling running constants depend with the momentum scale. From dimensional analysis (see [8]) one can find that the beta functions for dimensionless couplings $\tilde{g}_i = g_i k^{-d_i}$ (d_i is the canonical dimension of g_i) are

$$\tilde{\beta}_i(\tilde{g}_j) \equiv \partial_t\tilde{g}_i = a_i(\tilde{g}_j) - d_i\tilde{g}_i$$

where

$$a_i(\tilde{g}_j) = k^{-d_i}\beta(k^{d_i}\tilde{g}_j; k)$$

Since $a_i(\tilde{g}_j)$ is dimensionless, it does not depend on k (remember that β functions are indipendent of Λ). So β_i depend on k only via the $\tilde{g}_i(k)$.

The difference between the two effective actions Γ_k and $\Gamma_{k-\delta k}$ is given by a functional integral over the field modes between k and $k-\delta k$; this functional integral do not give divergences and the beta functions are automatically finite at momentum scale $k-\delta k$. So if one knows the coupling constants at a scale k, the flow equation can be integrated and gives the finite running coupling at all energy scale. If the couplings g_i can be measured at the scale k_0, so one can construct the total RG trajectory in theory space and take the limit in either direction. The limit $k \to 0$ gives the full quantum effective action from which one obtains all quantum physical information, the other limit $k \to \infty$ give the UV properties of the theory.

If the trajectory can not be integrated beyond a momentum scale Λ and the limit $k \to \infty$ makes no sense. The QFT is said *non-perturbative non-rinormalizable* and the theory, valid only for scales $k < \Lambda$, is called *effective field theory.* Over the scale Λ some new physics is expected to appear.

When the limit $k \to \infty$ makes sense, the dimensionless coupling $\tilde{g}_i(k)$ tend to finite values $\tilde{g}^*$ and the physical dimensionless quantities remain finite for all momentum scale. In fact, cross-section and decay rates can by expressed as functions of only dimensionless quantities, for example the cross-section $\sigma = k^{-2}\tilde{\sigma}$, where the dimensionless cross-section $\tilde{\sigma}$ depends only on dimensionless kinematical variables and dimensionless couplings. So if the limit $k \to \infty$ gives finite couplings, the cross-section remains finite to all momentum scale. The correct limit can be reached if a fixed point (FP) for beta functions exists, *i.e.* by definition $\tilde{\beta}_i(\tilde{g}^*) = 0$.

Before giving a correct definition for "asymptotic safety", we must distinguish between relevant and irrelevant couplings. Let us define the *UV critical surface*, associated to our fixed point to be the set of points in theory space which is attracted towards the FP in the UV limit. We can compute the tangent space, at the FP, and obtain the flow in the vicinity of the fixed point through the linearization of flow equation

$$\partial_t \tilde{g}_i(k) = M_{ij}(\tilde{g}_j(k) - \tilde{g}_j^*) \tag{1.33}$$

with

$$M_{ij} \equiv \frac{\partial \tilde{\beta}_i}{\partial \tilde{j}}\bigg|_{\tilde{g}^*}$$

Making a linear transformation $z_i = U_{ij}(\tilde{g}_j(k) - \tilde{g}_j^*)$ we can diagonalize the system (1.33)

$$\frac{dz_i}{dt} = \lambda_i z_i$$

where the (complex) λ_i are the eigenvalues of M. The last equation can be integrated immediatly and the solutions are $z_i(t) = e^{\lambda_i t} z_i(0)$. The solutions with $\Re\lambda_i < 0$ converge towards the fixed point and the relative coupling z_i is said *relevant coupling.* The couplings with $\Re\lambda_i > 0$ is called *irrelevant* since the relative trajectory $z_i(t)$ does not converge into the fixed point. Last, for $\Re\lambda_i = 0$ no informations can be obtained with linearized analysis.

Since the dimension of the UV critical surface and of its tangent space is the same, then the dimensionality of UV critical surface is determined by the number of eigenvalues of M with negative real part. We expect that the measured dimensionless couplings lie in the UV critical surface, the physical sarface of theory space.

In fact, if the critical surface is (finite) n-dimensional, so the quantum theory is completly determined by the measurement of n couplings, which determine the other irrelevant couplings (if they exist). In the case of infinite dimensional critical surface, the theory can not be predictive.

From the last argument we can give the condition for a quantum field theory to be well behaved in the UV regime: *the Functional RG trajectory in theory space must possess an UV fixed point and the relative UV critical surface must have a finite number of relevant directions.* Such a theory is called *asymptotically safe*, so it is non-perturvative rinormalizable (free of divergences) and predictive. A simple example is represented by the non-Abelian gauge theory with $\mathfrak{su}(N)$ Lie algebra; this theories have an UV Gaussian FP (fixed point with $\tilde{g}^*_{YM} = 0$) and are asymptotically free, in the sense that the coupling g_{YM} tends to zero when the energy grow up, as it is known thanks to the standard perturbative methods of QFT.

In the next section we analize the "more complicated" case of quantum field theory of gravity.

1.5 Quantum field theory of General Relativity

With the success of perturbative renormalizability and relative application in the particle physics for gauge theory, many theoretical physicists try to apply the standard methods of quantum field theory using the dimensional regularization[9], successful for Yang-Mills theory.

In paper [10], the authors used the standard perturbative technique for pure gravity, in this case the counterterms at one loop level reads

$$\mathcal{L}^{1-\text{loop}}_{c.t.} = \frac{1}{\epsilon}\sqrt{g}\left(\frac{1}{120}R^2 + \frac{7}{20}R_{\mu\nu}R^{\mu\nu}\right) \tag{1.34}$$

but if we consider the equations of motion, the *On-shell* condition in pure gravity imposes

$$R_{\mu\nu} = 0 \qquad R = 0 \tag{1.35}$$

so that the counterterm (1.34) vanishes.

The two loop contribution to counterterms was calculated for the first time in [11], in which it was observed that not all the counterterms vanish for the On-shell condition (1.35). The non zero contribution gives

$$\mathcal{L}^{2-\text{loop}}_{c.t.} = \frac{209}{2880(4\pi)^2}\frac{1}{\epsilon}\sqrt{g}R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta} \tag{1.36}$$

No terms in Einstein-Hilbert Lagrangian can be redefined to inglobe this counterterm.

The case of gravity coupled to matter was analized also in [10], in which they consider only a scalar field. The one loop counterterms give (taking into account the relative On-shell conditions)

$$\mathcal{L}_{c.t.}^{1-\text{loop}} = \frac{203}{80\epsilon}\sqrt{g}R^2$$

We conclude that, in pure gravity non rinormalizable (in perturbative sense) counterterms appear at two loop level, while in gravity coupled to matter at one loop.

This conclusion allows us to exclude the perturbative treatment of the quantum field theory of General Relativity and lead us to take into account a full non-perturvative quantum theory of gravitation.

Althogh this discouraging results in perturbative gravity, in the original paper [7] was sugested that the quantum field theory of gravity can make sense in the non perturbative domain. In fact, for the first time in this paper Nobel laureate Steven Weinberg introduced the concept for an asymptotically safe field theory, which we discussed in the previous section.

Chapter 2

Renormalization Group Flow for Quatum Gravity

2.1 Construction of Functional RG for gravity

In order to extend the formalism of Functional Renormalization Group (FRG), applied in previous chapter for a scalar theory, two aspects have necessary to be point out. First, the full quantum metric $\gamma_{\mu\nu}$ must be decomposed into a general background metric $\bar{g}_{\mu\nu}$ plus the quantum fluctuations (not necessarily small) $h_{\mu\nu}$ because we want to use the background field method and keep the gauge invariance on it [1]

$$\gamma_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{2.1}$$

Second, in quantum field theory of gravity we must take into account a special *local* symmetry, the *general coordinates transformations*, or *diffeomorfisms invariance.* Consider an infinitesimal coordinates transformation $x'^{\mu}(x) = x^{\mu} - \epsilon(x)$, we know that the definition of local variation for the metric tensor is

$$\delta\gamma_{\mu\nu} = \gamma'_{\mu\nu}(x) - \gamma_{\mu\nu}(x) \tag{2.2}$$

and from the general coordinates transformation $\gamma'_{\mu\nu}(x') = \gamma_{\mu\nu}(x) + \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu$, we find

$$\delta\gamma_{\mu\nu} = \mathcal{L}_\epsilon\gamma_{\mu\nu} = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu \tag{2.3}$$

which links the local variation for the metric tensor and the Lie derivative associated to the vector $\epsilon^\mu\partial_\mu$. Following the trick by Faddeev and Popov, the (correct) definition of the functional generator of correlation functions reads

$$\begin{aligned} Z_k[\text{sources}] = \int \mathcal{D}h\mathcal{D}c\mathcal{D}\bar{c}\exp\Big\{ &- S_{EH}[\gamma] - S_{gf}[h;\bar{g}] - S_{gh}[\bar{c},c;\bar{g}] \\ &- \Delta_k S[h,\bar{c},c;\bar{g}] - S_{\text{sources}}\Big\} \end{aligned} \tag{2.4}$$

[1] For the construction of functional RG in gravity we follow Reuter [12]

where S_{EH} is the usual Einstein-Hilbert action we start with

$$S_{EH}[\gamma] = \frac{1}{16\pi G}\int d^dx\sqrt{\gamma}\,(2\Lambda - R) \tag{2.5}$$

Using the Faddeev and Popov method to quantize a dynamical system with a *local* symmetry, the gauge fixing S_{gf} and ghost S_{gh} actions appears in definition of (2.4). The gauge fixing action reads

$$S_{g.f.}[h;\bar{g}] = \frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}F_\alpha[h;\bar{g}]F_\beta[h;\bar{g}]\bar{g}^{\alpha\beta} \tag{2.6}$$

which implements the conditions $F^\alpha[h;\bar{g}] = 0$. Together with the gauge fixing term, one adds the *ghost action*, with Grassmann valued fields $\bar{c}^\mu$ and c^μ

$$S_{gh}[\bar{c},c;\bar{g}] = \int d^dx\sqrt{\bar{g}}\bar{c}_\mu\frac{\delta F^\mu}{\delta\epsilon^\rho}\bigg|_{\epsilon=0}c^\rho \tag{2.7}$$

the sources term

$$S_{\text{sources}}[t,\sigma,\bar{\sigma};\bar{g}] = \int d^dx\sqrt{\bar{g}}\,[t^{\mu\nu}h_{\mu\nu} + \bar{\sigma}^\mu c_\mu + \bar{c}^\mu\sigma_\mu] \tag{2.8}$$

and the cutoff dependent term

$$\Delta S_k[h,\bar{c},c;\bar{g}] = \frac{1}{2}\int d^dx\sqrt{\bar{g}}h_{\mu\nu}\mathcal{R}_k^{gr}[\bar{g}]^{\mu\nu\alpha\beta}h_{\alpha\beta} - \int d^dx\sqrt{\bar{g}}\bar{c}_\mu\mathcal{R}_k^{gh}[\bar{g}]^{\mu\nu}c_\nu \tag{2.9}$$

The most common choice for the gauge fixing condition is

$$F_\mu = \left(\bar{\nabla}^\rho h_{\rho\mu} - \frac{1+\rho}{d}\bar{\nabla}_\mu\right) \tag{2.10}$$

where the parameter ρ is gauge parameter as α. For $\rho = d/2-1$ in flat spacetime the condition $F^\alpha = 0$ reduces to the standard harmonic gauge condition $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$. The corresponding ghost action will be calculated later.

The next step is to define the functional generator for connected Green function as in the previous chapter

$$\begin{aligned}\exp\left(-W_k[t,\bar{\sigma},\sigma]\right) = \int \mathcal{D}h\mathcal{D}c\mathcal{D}\bar{c}\exp\Big\{ &- S_{EH}[\gamma] - S_{gf}[h;\bar{g}] - S_{gh}[\bar{c},c;\bar{g}] \\ &- \Delta_k S[h,\bar{c},c;\bar{g}] + S_{\text{sources}}[t,\sigma,\bar{\sigma};\bar{g}]\Big\}\end{aligned} \tag{2.11}$$

Given the functional W we introduce the classical fields

$$\bar{h}_{\mu\nu} = \frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta t^{\mu\nu}} \qquad \bar{C}_\mu = \frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta\sigma^\mu} \qquad C_\mu = \frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta\sigma^\mu}$$

where with Grassmann variables the left derivative is understood.

So, the average effective action for quantum gravity is defined by

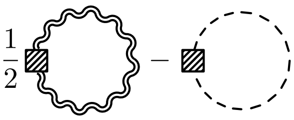
$$\Gamma_k[\bar{h},\bar{C},C;\bar{g}] = W_k[t,\sigma,\bar{\sigma};\bar{g}] - \int d^dx\sqrt{\bar{g}}\left[t^{\mu\nu}\bar{h}_{\mu\nu} + \bar{\sigma}^\mu C_\mu + \bar{C}^\mu\sigma_\mu\right] - \Delta S_k[\bar{h},\bar{C},C;\bar{g}] \tag{2.12}$$

Using the same algebra of chapter 1, one can construct the equation which governs the flow for the average effective action

$$\partial_t \Gamma_k[\bar{h}, \bar{C}, C] = \frac{1}{2} \operatorname{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}_{\bar{h}\bar{h}} (\partial_t \mathcal{R}_k)_{\bar{h}\bar{h}}\right] - \operatorname{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}_{\bar{C}C} (\partial_t \mathcal{R}_k)_{\bar{C}C}\right] \tag{2.13}$$

or

$$\partial_t \Gamma_k[\bar{h}, \bar{C}, C] = \frac{1}{2} \operatorname{STr}\left[\left(\boldsymbol{\Gamma}_k^{(2)} + \boldsymbol{\mathcal{R}}_k\right)^{-1} (\partial_t \boldsymbol{\mathcal{R}}_k)\right] = \frac{1}{2} \tag{2.14}$$



where we have introduced the short-hand notation

$$\left(\Gamma^{(2)}_{k,\bar{h}\bar{h}}\right)^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{\bar{g}}} \frac{\delta}{\delta h_{\mu\nu}} \frac{1}{\sqrt{\bar{g}}} \frac{\delta \Gamma_k}{\delta h_{\rho\sigma}}$$

$$\left(\Gamma^{(2)}_{k,\bar{C}C}\right)^{\mu\nu} = \frac{1}{\sqrt{\bar{g}}} \frac{\delta}{\delta C_{\mu}} \frac{1}{\sqrt{\bar{g}}} \frac{\delta \Gamma_k}{\delta \bar{C}_{\nu}}$$

and in diagrammatic rapresentation (2.14) the double wiggy line refers to graviton nonperturbative propagator and the dashed line to ghosts propagator. In equation (2.13), the ghosts trace part appears with a minus sign, which has a physical meaning. Ghost and anti-ghost fields appear in the definition of $Z[\text{sources}]$, together with gauge fixing term, with the aim of cancelling the redundant functional integration over the non physical gauge orbits. As we shall see below, thanks to the minus sign which appears in ghosts part of Wetterich's equation, the non physical ghosts degrees of freedom cancel almost exactly with the non physical degrees of freedom in graviton decomposition.

If we consider the average effective action as a functional of g and $\bar{g}$ instead of h, we can define

$$\hat{\Gamma}_k[g, \bar{g}, \bar{C}, C] = \Gamma_k[h = g - \bar{g}, \bar{C}, C; \bar{g}] \tag{2.15}$$

Since the cutoff $\mathcal{R}_k$ is constructed by giving us the full quantum effective action in the limit $k \to 0$

$$\Gamma[\bar{h}, \bar{g}] = \lim_{k \to 0} \Gamma_k[\bar{h}, \bar{C} = 0, C = 0; \bar{g}] \tag{2.16}$$

This quantum action is the generator of 1PI Off-shell Green functions, which depends on the background metric $\bar{g}_{\mu\nu}$. But the full quantum effective action which generates physical On-shell Green functions is not (2.16), instead of it it is obtained from taking the limit $k \to 0$ of (2.15) and imposing $h = 0$, and the result is diffeomorfisms invariant

$$\hat{\Gamma}[g] = \lim_{k \to 0} \hat{\Gamma}_k[g, \bar{g} = g, \bar{C} = 0, C = 0] = \lim_{k \to 0} \Gamma_k[\bar{h} = 0, \bar{C} = 0, C = 0, \bar{g} = g] \tag{2.17}$$

This definition for full quantum action is background indipendent and diffeomorfism invariant. The quantum equation for General Relativity reads [13]

$$\frac{\delta \hat{\Gamma}[g]}{\delta g_{\mu\nu}} = 0 \tag{2.18}$$

The average effective action satisfies an integro differential equation, as in the scalar case

$$\exp\left\{-\Gamma_k[\bar{h},\bar{C},C;\bar{g}]\right\} = \int \mathcal{D}h\mathcal{D}\bar{c}\mathcal{D}c \exp\Bigg\{ - S_{EH} - S_{gf} - S_{gh} - \int d^dx \Big[(h_{\mu\nu} - \bar{h}_{\mu\nu})\frac{\delta\Gamma_k}{\delta\bar{h}_{\mu\nu}} + (\bar{c}_\mu - \bar{C}_\mu)\frac{\delta\Gamma_k}{\delta\bar{C}_\mu} + \frac{\delta\Gamma_k}{\delta C_\mu}(c_\mu - C_\mu)\Big] - \Delta S_k[h-\bar{h},\bar{c}-\bar{C},c-C;\bar{g}]\Bigg\} \tag{2.19}$$

In the limit $k \to +\infty$ the leading term into the exponential is

$$\exp\{-\Delta S_k\} \sim \delta[h-\bar{h}]\delta[c-C]\delta[\bar{c}-\bar{C}] \tag{2.20}$$

so the functional integral became trivial and one finds

$$\Gamma_{k\to+\infty}[\bar{h},\bar{C},C;\bar{g}] = S_{EH}[\bar{g}+\bar{h}] + S_{gf}[\bar{h};\bar{g}] + S_{gh}[\bar{h},\bar{C},C;\bar{g}] \tag{2.21}$$

but the behaviour for On-shell 1PI Green functions generator (2.15) under the limit $k \to +\infty$ is

$$\hat{\Gamma}_{k\to\infty} = S_{EH} \tag{2.22}$$

2.2 Tranverse-traceless decomposition

In the contruction of a FRG equation for gravity, the inverse nonperturbative propagator $\mathbf{\Gamma}^{(2)} + \boldsymbol{\mathcal{R}}_k$ may depends on complicated composition of covariant derivatives, both in gravity and in ghost components. Transverse traceless decomposition, as we shall see, partially diagonalizes the argument of functional trace in FRGE; with this choice $\mathbf{\Gamma}^{(2)}$ depends only on Laplacian operator $\Delta = -g_{\mu\nu}\nabla^\mu\nabla^\nu$, hence the funcional trace can be approximated with Heat Kernel techniques. The (type-1) Transverse-Traceless (TT) decomposition for gravity fluctuations (used in [5, 14]) is defined by

$$h_{\mu\nu} = h^T_{\mu\nu} + \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu + \bar{\nabla}_\mu\bar{\nabla}_\nu\sigma - \frac{1}{d}\bar{g}_{\mu\nu}\bar{\nabla}^2\sigma + \frac{1}{d}\bar{g}_{\mu\nu}h \tag{2.23}$$

with the constraints

$$\bar{g}^{\mu\nu}h^T_{\mu\nu} = 0 \qquad \bar{\nabla}^\mu h^T_{\mu\nu} = 0 \qquad \bar{\nabla}^\mu\xi_\mu = 0 \tag{2.24}$$

For ghost fields

$$C_\mu = c^T_\mu + \bar{\nabla}_\mu c \qquad \bar{C}_\mu = \bar{c}^T_\mu + \bar{\nabla}_\mu\bar{c} \tag{2.25}$$

with conditions

$$\bar{\nabla}^\mu c^T_\mu = 0 \qquad \bar{\nabla}^\mu\bar{c}^T_\mu = 0 \tag{2.26}$$

The presence of covariant derivatives in this decomposition gives two consequences. First, since (2.23) and (2.25) is a substitution in the functional integral (2.11), non trivial Jacobians appears when we pass from $\mathcal{D}h$ to $\mathcal{D}h^T\mathcal{D}\xi\mathcal{D}\sigma\mathcal{D}h$ and the same for the ghost secto. Secondly, not all modes of the component fields contribute to the metric fluctuations and ghost fields. In (2.23), if ξ_μ is a Killing vector it does not contribute the modes of $h_{\mu\nu}$; the same for the constant mode of σ and the vector $C_\mu = \nabla_\mu \sigma$ which satisfies the conformal Killing equation

$$\bar{\nabla}_\mu C_\nu + \bar{\nabla}_\nu C_\mu - \frac{2}{d}\bar{g}_{\mu\nu}\bar{\nabla}_\alpha C^\alpha = 0 \tag{2.27}$$

These non-physical modes must be excluded in the computation of the functional trace in FRG equation. A deteiled analysis can be made if one chooses a d-dimensional sphere S^d as a background, which admits $d(d+1)/2$ Killing vectors, none of which do contribute to tensor $h_{\mu\nu}$ as explained. All and only Killing vectors are eigenvectors of $\Delta = -\bar{g}_{\mu\nu}\bar{\nabla}^\mu\bar{\nabla}^\nu$ corresponding to the degenerate eigenvalue $\bar{R}/4$, as the C.1 shows. For the scalar field σ, there exist $(d+2)$ modes which do not contribute to metric fluctuations, the first corresponds to the constant mode, the only eigenvector with null eigenvalue, as expected. The remaining $(d+1)$ modes correspond to the $(d+1)$-degenerate eigenvalue $\bar{R}/3$; this scalars are proportional to the Cartesian coordinates of R^{d+1}, the embedding for S^d.

The computation of the Jacobians for the decomposition (2.23) starts from the inner product for the metric tensor

$$\begin{aligned}
\langle h, h\rangle &\equiv \int d^dx\sqrt{\bar{g}}h_{\mu\nu}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}h_{\alpha\beta} \\
&= \int d^dx\sqrt{\bar{g}}\big[h^T_{\mu\nu}h^{T\mu\nu} - 2\xi_\mu\left(\bar{\nabla}^2 g^{\mu\nu} + \bar{R}^{\mu\nu}\right)\xi_\nu \\
&- 4\xi_\mu\bar{R}^{\mu\nu}\bar{\nabla}_\nu\sigma + \sigma\big[\frac{d-1}{d}(\bar{\nabla}^2)^2 + \bar{\nabla}_\mu\bar{R}^{\mu\nu}\bar{\nabla}_\nu\big] + \frac{1}{d}h^2\big]
\end{aligned} \tag{2.28}$$

which is orthogonal up to the mixing $\xi - \sigma$ terms. For the ghost fields the scalar product reads

$$\langle \bar{C}, C\rangle \equiv \int d^dx\sqrt{\bar{g}}\bar{C}_\mu C^\mu = \int d^dx\sqrt{\bar{g}}\left[\bar{C}^T_\mu C^{T\mu} - \bar{c}\bar{\nabla}^2 c\right] \tag{2.29}$$

Note that the mixing $\xi - \sigma$ terms vanishes when we fix a maximally symmetric spacetime ($\bar{R}_{\mu\nu} = \frac{\bar{R}}{d}\bar{g}_{\mu\nu}$) as a background.

With relations (2.28) and (2.29) we can compute the Jacobians considering

$$\begin{aligned}
&\int \mathcal{D}h_{\mu\nu}\exp\left[-\frac{1}{2}\langle h, h\rangle\right] = \\
&J_{gr}\int \mathcal{D}h^T\mathcal{D}\xi\mathcal{D}\sigma\mathcal{D}h\exp\left[-\frac{1}{2}\int d^dx\sqrt{\bar{g}}\left(h^T_{\mu\nu}h^{T\mu\nu} + \frac{1}{d}h^2 + [\xi_\mu,\sigma]M^{(\mu,\nu)}[\xi_\nu,\sigma]^T\right)\right]
\end{aligned} \tag{2.30}$$

where $M^{(\mu,\nu)}$ is a $(d+1)\times(d+1)$-matrix whose first d columns act on vector field ξ and the last columns acts on the σ field. The matrix $M^{(\mu,\nu)}$ reads

$$M^{(\mu,\nu)} = \begin{pmatrix} -2\left[\bar{g}^{\mu\nu}\bar{\nabla}^2 + \bar{R}^{\mu\nu}\right] & -2\bar{\nabla}^2\bar{\nabla}^\mu \\ 2\bar{\nabla}^\nu\bar{\nabla}^2 & \bar{\nabla}_\nu\bar{\nabla}^2\bar{\nabla}^\nu - \frac{1}{d}(\bar{\nabla}^2)^2 \end{pmatrix} \tag{2.31}$$

The h^T-dependent and h-dependent parts in (2.30) can be absorbed into the normalization of the measure, so that the Jacobians reads

$$J_{gr} = \left(\text{Det'}\left[M^{(\mu,\nu)}\right]\right)^{1/2} \equiv \left(\text{Det'}\,\hat{J}_{gr}\right)^{1/2} \tag{2.32}$$

where the prime in functional determinant remembers us to exclude the non-physical modes in the spectrum of eigenvalue. For the ghost fields we find

$$\begin{aligned} &\int \mathcal{D}C^\mu \mathcal{D}\bar{C}^\nu \exp\left[-\langle \bar{C}, C\rangle\right] = \\ &J_{gh} \int \mathcal{D}C^T \mathcal{D}\bar{C}^T \mathcal{D}c \mathcal{D}\bar{c} \exp\left[-\int d^d x \sqrt{\bar{g}}\left(\bar{C}^T_\mu C^{T\mu} - \bar{c}\bar{\nabla}^2 c\right)\right] \end{aligned} \tag{2.33}$$

so we can extract the Jacobians for the ghost sector

$$J_{gh} = \left(\text{Det'}\left[-\bar{\nabla}^2\right]\right)^{-1} \equiv \left(\text{Det'}\,\hat{J}_{gh}\right)^{-1} \tag{2.34}$$

There exist other choices for the decomposition for the gravity fluctuations. We recall what is used in [6], which modifies the previous decomposition (2.23) into

$$h_{\mu\nu} = h^T_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \bar{\nabla}_\mu \bar{\nabla}_\nu \sigma + \frac{1}{d}\bar{g}_{\mu\nu}\bar{h} \tag{2.35}$$

with the constraints

$$\bar{g}^{\mu\nu} h^T_{\mu\nu} = 0 \qquad \bar{\nabla}^\mu h^T_{\mu\nu} = 0 \qquad \bar{\nabla}^\mu \xi_\mu = 0 \tag{2.36}$$

The new variable $\bar{h}$ is related to the previous h by relation $\bar{h} = h - \bar{\nabla}^2\sigma$. Following the same argument for previous variables one can construct the Jacobians of transformation (2.35). First the scalar product

$$\begin{aligned} \langle h, h\rangle = {}& h^T_{\mu\nu} h^{T\mu\nu} - 2\xi_\mu\left(\bar{\nabla}^2 \bar{g}^{\mu\nu} + \bar{\nabla}^\mu \bar{\nabla}^\nu\right)\xi_\nu + \sigma \bar{\nabla}_\mu \bar{\nabla}^2 \bar{\nabla}^\nu \sigma \\ & + \frac{1}{d}\bar{h}^2 - 4\xi_\mu \bar{\nabla}^2 \bar{\nabla}^\mu \sigma + \frac{2}{d}\sigma \bar{\nabla}^2 \bar{h} \end{aligned}$$

with a maximally symmetric background spacetime we have

$$\langle h, h\rangle = h^T_{\mu\nu} h^{T\mu\nu} - 2\xi_\mu \left(\bar{\nabla}^2 + \frac{R}{d}\right)\xi^\mu + (\sigma, \bar{h})\left[T(\nabla^2)\right](\sigma, \bar{h})^T$$

with the scalar-scalar matrix

$$T(\nabla^2) = \begin{pmatrix} \left(\bar{\nabla}^2 + \frac{R}{d}\right)\bar{\nabla}^2 & \frac{1}{d}\bar{\nabla}^2 \\ \frac{1}{d}\bar{\nabla}^2 & \frac{1}{d} \end{pmatrix}$$

So we can calculate the Jacobians for gravity decomposition (2.35)

$$J_{gr,B1} = \left(\text{Det'}\left[-2\left(\bar{\nabla}^2 + \frac{R}{d}\right)\right]\right)^{-1} \equiv \left(\text{Det'}\,\hat{J}_{gh,B1}\right)^{-1} \tag{2.37}$$

for spin-one field ξ and

$$J_{gr,B0} = \left(\text{Det'}\left[\frac{d-1}{d^2}\left(\bar{\nabla}^2 + \frac{R}{d-1}\right)\bar{\nabla}^2\right]\right)^{-1} \equiv \left(\text{Det'}\,\hat{J}_{gh,B0}\right)^{-1} \tag{2.38}$$

for scalar part. Let us discuss how to treat this Jacobians for the FRG equation; there exist two methods. By the first method we can add addictional trace to FRG equation, as it is done in equation (2.65), with the same excluded modes of the original field. To explain the second method consider for example the Jacobians

$$J = \left(\text{Det}\left[-\bar{\nabla}^2\right]\right)^{1/2}$$

we can use the trick of Faddeev and Popov and exponentiate into the definition of Z_k[sources] introducing a scalar η and a complex Grassmann-valued $\zeta \quad \bar{\zeta}$ field and using the formula

$$\left(\text{Det}\left[-\bar{\nabla}^2\right]\right)^{1/2} = \left(\text{Det}\left[-\bar{\nabla}^2\right]\right)^{-1/2}\left(\text{Det}\left[-\bar{\nabla}^2\right]\right) = \int \mathcal{D}\eta\mathcal{D}\zeta\mathcal{D}\bar{\zeta}\exp\left\{-\int d^dx\sqrt{\bar{g}}\left(\frac{1}{2}\eta(-\bar{\nabla}^2)\eta + \bar{\zeta}(-\bar{\nabla}^2)\zeta\right)\right\}$$

So we will introduce the field $\eta, \zeta, \bar{\zeta}$ into the FRG equation as the standard fields.

2.3 $f_k(R)$ truncation ansatz

The aim of the equation (2.13) is to describe the flow from the action S_{EH} at a large momuntum scale down to the full quantum effective action $\Gamma_{k=0}$. First, from now on, we use an alternative metric decomposition, different from previous sections

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa_k h_{\mu\nu} \tag{2.39}$$

with we add κ_k which governs the vertex expansion. In fact, with this choice, the n-th proper vertex is proportional to $(\kappa_k)^{n-2}$. One impose $\kappa_k = 1$, which imply that (2.39) becomes the standard metric decomposition followed in most papers on asymptotically safety quantum gravity. Another choice may be the following, $\kappa_k = \sqrt{16\pi G_k}$, first used in [15]. From now, we consider the latter choice. For reasons explained above, we are forced to make an ansatz for the average effective action; in this thesis we choose the following projection for Γ_k

$$\Gamma_k[h, \bar{C}, C, b; \bar{g}] = \frac{1}{\kappa_k^2}\int d^dx\sqrt{g}f_k(R) + \Gamma_{k,g.f.} + \Gamma_{k,gh-c} + \Gamma_{k,gh-b} \tag{2.40}$$

As explained below, we have four fields in the argument of our effective action; first the quantum flactuation $h_{\mu\nu}$ for metric degrees of freedom, the standard ghost and antighost Grassmann-valued fields and an addictional (commuting) "third ghost" field b_μ. Since we want to study the anomalous dimensions that these fields can acquire in the vicinity of fixed point, in the spirit of renormalization group we redefine fields flactuations according to

$$h_{\mu\nu} \to Z_{k,h}^{1/2}h_{\mu\nu} \qquad C_\mu \to Z_{k,c}^{1/2}C_\mu \qquad \bar{C}_\mu \to Z_{k,c}^{1/2}\bar{C}_\mu \qquad b_\mu \to Z_{k,b}^{1/2}b_\mu \tag{2.41}$$

and define anomalous dimensions through the usual formulae

$$\eta_a = -\frac{\partial_t Z_{k,a}}{Z_{k,a}} \qquad a = h, c, b \tag{2.42}$$

Now, we explicite each contribution in ansatz (3.1). First, the gauge fixing condition reads (after fields redefinition)

$$\Gamma_{k,g.f.}[h;\bar{g}] = \frac{Z_{k,h}}{2} \int d^d x \sqrt{\bar{g}} F_\mu[h;\bar{g}] G^{\mu\nu} F_\nu[h;\bar{g}] \tag{2.43}$$

with

$$F_\mu[h;\bar{g}] = \bar{\nabla}^\rho h_{\rho\mu} - \frac{1+\rho}{d}\bar{\nabla}_\mu h \tag{2.44}$$
$$G^{\mu\nu} = (\alpha + \beta\bar{\nabla}^2)\bar{g}^{\mu\nu}$$

An addition term, proportional to $\beta\bar{\nabla}^2$, is added if one would consider higher derivative gravity[16]. If one consider a classical action proportional to high power of scalar curvature, the resulting equations of motion usually contains high derivative term. To be more specific, if we choose the bare action $S_{Bare} = \int d^d \sqrt{g} f(R)$, the corresponding equations of motion depends on four derivative of metric tensor. So it is natural to assume that also the gauge fixing term contains four covariant derivatives. We continue with a general exposition taking $\alpha, \beta \neq 0$ before to make a gauge choice in the trace evaluation.

If the bare action contains R^2 (through derivative) terms and not only Γ_k, one may be worried of ghost instabilities in the graviton propagator. It has been shown that this bad behaviour can be eliminated in maximally symmetric backgrounds by a suitable field redefinition.

The ghosts action follows from the gauge condition through the exponentiation of Faddeev-Popov functional determinant. With the choice (2.44), an addition ghost action is required [16], generally called *third ghost* term; so the total ghosts action becomes

$$\Gamma_{k,gh} = \Gamma_{k,gh-c} + \Gamma_{k,gh-b}$$

where $\Gamma_{k,gh-c}$ is the standard ghosts action obtained from the exponentiation of Faddeev-Popov determinant

$$\Gamma_{k,gh-c}[h,\bar{C},C;\bar{g}] = Z_{k,c} \int d^d x \sqrt{\bar{g}} \bar{C}_\mu G^{\mu\nu} \frac{\delta F_\nu}{\delta \epsilon^\rho}\bigg|_{\epsilon=0} C^\rho \tag{2.45}$$

where ϵ^μ corresponds to infinitesimal general coordinates transformation $x'^\mu = x^\mu - \epsilon^\mu$. While the additional third ghost reads

$$\Gamma_{k,gh-b}[b;\bar{g}] = \frac{Z_{k,b}}{2} \int d^d x \sqrt{\bar{g}} b_\mu (\alpha + \beta\bar{\nabla}^2) b^\mu \tag{2.46}$$

where the field b_μ, contrary to standard ghosts, is a commuting Lorentz vector field. Note that if we consider the gauge choice $\beta = 0$ the third ghost contribution (2.46) can be absorbed by the functional measure, as expected. Let us calculate explicitly the contribution (2.45) for the standard ghosts starting from the infinitesimal gauge transformation

$$\delta F_\mu = \bar{\nabla}^\rho \delta h_{\rho\mu} - \frac{1+\rho}{d}\bar{\nabla}_\nu \delta h$$

recalling that $\kappa_k \delta h_{\mu\nu} = \mathcal{L}_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$ and $\kappa_k \delta h = 2\bar{g}^{\mu\nu}\nabla_\mu \epsilon_\nu$ we get

$$\frac{\delta F^\mu[h;\bar{g}]}{\delta \epsilon^\nu} = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{\nabla}_\lambda(g_{\rho\nu}\nabla_\sigma + g_{\sigma\nu}\nabla_\rho) - \frac{2(1+\rho)}{d}\bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{\nabla}_\lambda g_{\sigma\nu}\nabla_\rho$$

which gives the ghosts action

$$\begin{aligned}\Gamma_{k,gh-c}[\bar{C},C;\bar{g}] = Z_{k,c}\int d^dx\sqrt{\bar{g}}\bar{C}_\mu(\alpha+\beta\bar{\nabla}^2)\Big[&\bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{\nabla}_\lambda(g_{\rho\nu}\nabla_\sigma + g_{\sigma\nu}\nabla_\rho) \\ &- \frac{2(1+\rho)}{d}\bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{\nabla}_\lambda g_{\sigma\nu}\nabla_\rho\Big]C^\nu\end{aligned} \tag{2.47}$$

According to FRG equation, only after the second variation we can put $g = \bar{g}$ inside the bracket and consequently calculations simplify. Since the second variation for (2.47) is trivial we can put from now $g = \bar{g}$ and obtain

$$\Gamma_{k,gh-c}[\bar{C},C;\bar{g}] = Z_{k,c}\int d^d\sqrt{\bar{g}}\bar{C}_\mu(\alpha+\beta\bar{\nabla}^2)\left[\bar{g}^{\mu\nu}\bar{\nabla}^2 + R^{\mu\nu} + \frac{d-2-2\rho}{d}\bar{\nabla}^\mu\bar{\nabla}^\nu\right]C_\nu \tag{2.48}$$

Only in higher order variations the dependence on the metric fluctuations would play a role. Remember that action (2.47) contains at least three and four point interactions between ghost and graviton field.

After imposing the ansatz (2.40), we proceed in construction of Wetterich's equation for gravity (2.13). The steps for calculation can be summarized in 3 points: 1. calculation of second variation for ansatz (3.1) and corresponding $\mathbf{\Gamma}_k^{(2)}$ 2. cutoff scheme and gauge choice in order to make important simplification for next step 3. calculation of trace with Heat Kernel technique, used in [5, 14] or with "sum of eigenvalue" approximation method [6]. Clearly, in both cases, the FRG equation reduces to a nonlinear partial differential equation for function $f_k(R)$.

Let us start with the first step, providing the calculation of second variation for our ansatz. Taking the second variation of $\int d^dx\sqrt{g}f_k(R)$

$$\begin{aligned}\delta^{(2)}\frac{1}{\kappa_k^2}\int d^dx\sqrt{g}f_k(R) = \int d^dx\Big[&\delta^{(2)}(\sqrt{g})\,f_k(R) + f_k'(R)2\delta(\sqrt{g})\,\delta R \\ &+f_k'(R)\sqrt{g}\delta^{(2)}R + \sqrt{g}f_k''(R)\,(\delta R)^2\,\Big]\end{aligned} \tag{2.49}$$

with the tensor variations, given in Appendix A, and taking into account (2.39) and redefinition

(4.2) we obtain

$$\begin{aligned}\delta^{(2)}\int d^dx\sqrt{g}f_k(R) = Z_{k,h}\int d^dx\sqrt{g}\Big[f_k''(R)\Big(R^{\alpha\beta}h_{\alpha\beta}R^{\mu\nu}h_{\mu\nu} - 2R^{\mu\nu}h_{\mu\nu}\nabla_\alpha\nabla_\beta h^{\alpha\beta}\\ + 2R^{\mu\nu}h_{\mu\nu}\nabla^2 h + h_{\mu\nu}\nabla^\mu\nabla^\nu\nabla^\alpha\nabla^\beta h_{\alpha\beta} - 2h\nabla^2\nabla_\beta\nabla_\alpha h^{\alpha\beta} + h(\nabla^2)^2 h\Big)\\ + f_k'(R)\Big(- R^{\mu\nu}h_{\mu\nu}h - \frac{1}{2}h\nabla^2 h + \frac{1}{2}h_{\mu\nu}\nabla^2 h^{\mu\nu} + h^{\mu\alpha}h_{\alpha\beta}R^\beta_\mu\\ + h_{\mu\nu}R^{\mu\rho\nu\sigma}h_{\rho\sigma} - h^\nu_\mu\nabla^\mu\nabla^\rho h_{\rho\nu} + h\nabla^\mu\nabla^\nu h_{\mu\nu}\Big) + f_k(R)\Big(\frac{1}{4}h^2 - \frac{1}{2}h_{\mu\nu}h^{\mu\nu}\Big)\Big]\end{aligned} \tag{2.50}$$

Only after the second variation (2.50) we impose $g = \bar{g}$ with a maximally symmetric background metric. From now on we eliminate the "bar" in metric and curvature tensors which refers to background; all geometric quantity refers to the d-dimensional sphere.

We use the transverse traceless decomposition (2.23) to diagonalize the second variation of $\mathbf{\Gamma}_k$. Also, this decomposition allows us to distinguish between the physical and non physical components in the quantum fluctuations $h_{\mu\nu}$. The explicitly calculation can be found in Appendix B

$$\Gamma_{h^T_{\mu\nu}h^T_{\alpha\beta}} = \frac{Z_{k,h}}{2}\left[f_k'(R)\left(\nabla^2 + \frac{2(d-2)}{d(d-1)}R\right) - f_k(R)\right]\delta^{\mu\nu,\alpha\beta} \tag{2.51}$$

where $\delta^{\mu\nu,\alpha\beta} = \frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha})$.

$$\Gamma^{(2)}_{\xi_\mu\xi_\nu} = Z_{k,h}\left(\nabla^2 + \frac{R}{d}\right)\left[(\alpha + \beta\nabla^2)\left(\nabla^2 + \frac{R}{d}\right) - \frac{2R}{d}f_k'(R) + f_k(R)\right]g^{\mu\nu} \tag{2.52}$$

The scalar part gives

$$\begin{aligned}\Gamma^{(2)}_{hh} =& Z_{k,h}\frac{d-2}{4d}\left[\frac{4(d-1)^2}{d(d-2)}f_k''(R)\left(\nabla^2 + \frac{R}{d-1}\right)^2 + \frac{2(d-1)}{d}f_k'(R)\left(-\nabla^2 - \frac{R}{d-1}\right)\right.\\ &\left. - \frac{2R}{d}f_k'(R) + f_k(R)\right] - \frac{\rho^2}{d^2}\left[\alpha - \beta\left(-\nabla^2 - \frac{R}{d}\right)\right]\nabla^2\end{aligned} \tag{2.53}$$

$$\begin{aligned}\Gamma^{(2)}_{h\sigma} =& Z_{k,h}\frac{d-1}{d^2}\left[(d-1)f_k''(R)\left(-\nabla^2 - \frac{R}{d-1}\right) + \frac{d-2}{2}f_k'(R)\right.\\ &\left. + \rho\left(\alpha - \beta\left(-\nabla^2 - \frac{R}{d}\right)\right)\right]\nabla^2\left(\nabla^2 + \frac{R}{d-1}\right)\end{aligned} \tag{2.54}$$

$$\begin{aligned}\Gamma^{(2)}_{\sigma\sigma} = Z_{k,h}\frac{d-1}{2d}\left[\frac{2(d-1)}{d}f_k''(R)\nabla^2\left(\nabla^2 + \frac{R}{d-1}\right) - \frac{d-2}{d}f_k'(R)\nabla^2 + \frac{2R}{d}f_k'(R) - f_k(R)\right.\\ \left. + \frac{2(d-1)}{d}\left(-\nabla^2 - \frac{R}{d-1}\right)\left(\alpha - \beta\left(-\nabla^2 - \frac{R}{d}\right)\right)\right]\nabla^2\left(\nabla^2 + \frac{R}{d-1}\right)\end{aligned} \tag{2.55}$$

At the begining of section we computed the total ghosts action, which contains two terms, first the standard (anti-commuting) vector-ghost term

$$\begin{aligned}\Gamma_{k,gh-c} &= Z_{k,c}\int d^dx\sqrt{\bar{g}}\bar{C}_\mu G^{\mu\nu}\frac{\delta F_\nu}{\delta\epsilon^\rho}C^\nu \\ &= \int d^dx\sqrt{\bar{g}}\bar{C}_\mu(\alpha+\beta\nabla^2)\left(g^{\mu\nu}\nabla^2+R^{\mu\nu}+\frac{d-2-2\rho}{d}\nabla^\mu\nabla^\nu\right)C_\nu\end{aligned} \tag{2.56}$$

and the (commuting) vector third ghost

$$\Gamma_{k,gh-b} = Z_{k,b}\frac{1}{2}\int d^dx\sqrt{\bar{g}}b_\mu(\alpha+\beta\nabla^2)b^\mu$$

As for metric fluctuations, we introduce a decomposition into transverse (c^T, b^T) and longitudinal (c, b) ghosts fields

$$\bar{C}_\mu = \bar{c}^T_\mu + \nabla_\mu\bar{c} \qquad C_\mu = c^T_\mu + \nabla_\mu c \qquad b_\mu = b^T_\mu + \nabla_\mu b$$

with the constraints

$$\nabla^\mu c^T_\mu = 0 \qquad \nabla^\mu c^T_\mu = 0 \qquad \nabla^\mu b^T_\mu = 0$$

According to this choice, the resulting variations give

$$\Gamma^{(2)}_{\bar{c}^T_\mu c^T_\nu} = Z_{k,c}(\alpha+\beta\nabla^2)\left(\nabla^2+\frac{R}{d}\right)g^{\mu\nu}$$

$$\Gamma^{(2)}_{\bar{c}c} = -Z_{k,c}\frac{2(d-1-\rho)}{d}\left(\alpha+\beta\left(\nabla^2+\frac{R}{d}\right)\right)\left(\nabla^2+\frac{1}{d-1-\rho}R\right)\nabla^2$$

$$\Gamma^{(2)}_{b^T_\mu b^T_\nu} = Z_{k,b}(\alpha+\beta\nabla^2)g^{\mu\nu}$$

$$\Gamma^{(2)}_{bb} = -Z_{k,b}\left(\alpha+\beta\left(\nabla^2+\frac{R}{d}\right)\right)\nabla^2$$

2.4 Cutoff scheme and gauge choice

The functional RG equation (2.13) requires the choice of a Cutoff function $\boldsymbol{\mathcal{R}}_k$ which, according to Wilson's idea analized in the first chapter, must be fixed in such a manner that relations (1.17-1.18-1.19) are safisfied. In the case of gravity, the cutoff $\boldsymbol{\mathcal{R}}_k$, is a matrix-valued function as $\boldsymbol{\Gamma}^{(2)}$. We choose the cutoff function in such a way that the calculation of the nonperturbative propagator $\left(\boldsymbol{\Gamma}^{(2)}_k+\boldsymbol{\mathcal{R}}_k\right)^{-1}$ becomes simple. We make the following choice

$$\boldsymbol{\Gamma}^{(2)}_k(-\nabla^2)+\boldsymbol{\mathcal{R}}_k(-\nabla^2) = \boldsymbol{\Gamma}^{(2)}_k\left(P_k(-\nabla^2)\right) \tag{2.57}$$

where $P_k(-\nabla^2) = -\nabla^2 + r_k(-\nabla^2)$ and $r_k(z)$ is the single-valued function which must obeys to relations (1.17), (1.18) and (1.19).

Following [5] we choose the gauge $\rho = 0$ and proceed in parallel with two different choices for parameters α and β. In one case we have $\alpha \to +\infty$ and $\beta = 0$, called $\alpha-$gauge, in the other case we impose $\beta \to \infty$ and $\alpha = 0$, called $\beta-$gauge.

Given the matrix elements of $\mathbf{\Gamma}_k^{(2)}$, obtained in the previous section, and using (2.57), we find for the tensorial part of cutoff function (from now on $d = 4$ and $\Delta = -\nabla^2$)

$$\mathcal{R}_k(\Delta)_{h^T_{\mu\nu},h^T_{\alpha\beta}} = -Z_{k,h}\frac{1}{2}f_k'(R)\left(P_k(\Delta) - \Delta\right)\delta^{\mu\nu,\alpha\beta} = -Z_{k,h}\frac{1}{2}f_k'(R)r_k(\Delta)\delta^{\mu\nu,\alpha\beta} \tag{2.58}$$

for the vector part in the $\alpha-$gauge and $\beta-$gauge rispectively

$$\mathcal{R}_k(\Delta)_{\xi_\mu,\xi_\nu} \overset{\alpha\to\infty}{=} Z_{k,h}\alpha\left[\left(P_k(\Delta) - \frac{R}{4}\right)^2 - \left(\Delta - \frac{R}{4}\right)^2\right]g^{\mu\nu} \tag{2.59}$$

$$\mathcal{R}_k(\Delta)_{\xi_\mu,\xi_\nu} \overset{\beta\to\infty}{=} -Z_{k,h}\beta\left[\left(P_k(\Delta) - \frac{R}{4}\right)^2 P_k(\Delta) - \left(\Delta - \frac{R}{4}\right)^2(\Delta)\right]g^{\mu\nu} \tag{2.60}$$

for the scalar part

$$\mathcal{R}_k(\Delta)_{\sigma,\sigma} \overset{\alpha\to\infty}{=} \frac{9Z_{k,h}}{16}\alpha\left[\left(P_k(\Delta) - \frac{R}{3}\right)^2 P_k(\Delta) - \left(\Delta - \frac{R}{3}\right)^2(\Delta)\right] \tag{2.61}$$

$$\begin{aligned}\mathcal{R}_k(\Delta)_{\sigma,\sigma} \overset{\beta\to\infty}{=} & -\frac{9Z_{k,h}}{16}\beta\left(P_k(\Delta) - \frac{R}{4}\right)\left(P_k(\Delta) - \frac{R}{3}\right)^2 P_k(\Delta) \\ & + \frac{9Z_{k,h}}{16}\beta\left(\Delta - \frac{R}{4}\right)\left(\Delta - \frac{R}{3}\right)^2\Delta\end{aligned} \tag{2.62}$$

$$\begin{aligned}\mathcal{R}_k(\Delta)_{h,h} =& \frac{9Z_{k,h}}{16}f_k''(R)\left[\left(P_k(\Delta) - \frac{R}{3}\right)^2 - \left(\Delta - \frac{R}{3}\right)^2\right] \\ & + \frac{3Z_{k,h}}{16}f_k'(R)\left(P_k(\Delta) - \Delta\right)\end{aligned} \tag{2.63}$$

$$\begin{aligned}\mathcal{R}_k(\Delta)_{h,\sigma} =& \frac{9Z_{k,h}}{16}f_k''(R)\left[\left(P_k(\Delta) - \frac{R}{3}\right)^2 P_k(\Delta) - \left(\Delta - \frac{R}{3}\right)^2\Delta\right] \\ & + \frac{3Z_{k,h}}{16}f_k'(R)\left[\left(P_k(\Delta) - \frac{R}{3}\right)P_k(\Delta) - \left(\Delta - \frac{R}{3}\right)\Delta\right]\end{aligned} \tag{2.64}$$

and similarly for ghosts contributions.

Taking into account the transverse traceless decomposition and relative Jacobians, the resulting Wetterich's equation (2.13) becomes

$$
\begin{aligned}
\partial_t\Gamma_k =& \frac{1}{2}\,\mathrm{Tr}_{(2)}\,\frac{\partial_t \boldsymbol{\mathcal{R}}_{h^T,h^T}}{\boldsymbol{\Gamma}^{(2)}_{h^T,h^T}+\boldsymbol{\mathcal{R}}_{h^T,h^T}} + \frac{1}{2}\,\mathrm{Tr}'_{(1)}\,\frac{\partial_t\mathcal{R}_{\xi,\xi}}{\Gamma^{(2)}_{\xi,\xi}+\mathcal{R}_{\xi,\xi}} \\
&+ \frac{1}{2}\,\mathrm{Tr}''_{(0)}\begin{pmatrix}\Gamma^{(2)}_{hh}(P_k) & \Gamma^{(2)}_{h\sigma}(P_k)\\ \Gamma^{(2)}_{\sigma h}(P_k) & \Gamma^{(2)}_{\sigma\sigma}(P_k)\end{pmatrix}^{-1}\begin{pmatrix}\partial_t\mathcal{R}_{hh} & \partial_t\mathcal{R}_{h\sigma}\\ \partial_t\mathcal{R}_{\sigma h} & \partial_t\mathcal{R}_{\sigma\sigma}\end{pmatrix} \\
&- \mathrm{Tr}_{(1)}\,\frac{\partial_t\mathcal{R}_{\bar{c}^T,c^T}}{\Gamma^{(2)}_{\bar{c}^T,c^T}+\mathcal{R}_{\bar{c}^T,c^T}} - \mathrm{Tr}'_{(0)}\,\frac{\partial_t\mathcal{R}_{\bar{c},c}}{\Gamma^{(2)}_{\bar{c},c}+\mathcal{R}_{\bar{c},c}} \\
&+ \frac{1}{2}\,\mathrm{Tr}_{(1)}\,\frac{\partial_t\mathcal{R}_{b^T,b^T}}{\Gamma^{(2)}_{b^T,b^T}+\mathcal{R}_{b^T,b^T}} + \frac{1}{2}\,\mathrm{Tr}'_{(0)}\,\frac{\partial_t\mathcal{R}_{b,b}}{\Gamma^{(2)}_{b,b}+\mathcal{R}_{b,b}} \\
&- \frac{1}{2}\,\mathrm{Tr}'_{(1)}\,\frac{\partial_t\mathcal{R}_{\hat{J}_V}}{\hat{J}_V+\mathcal{R}_{\hat{J}_V}} - \frac{1}{2}\,\mathrm{Tr}''_{(0)}\,\frac{\partial_t\mathcal{R}_{\hat{J}_s}}{\hat{J}_s+\mathcal{R}_{\hat{J}_s}} \\
&+ \mathrm{Tr}'_{(0)}\,\frac{\partial_t\mathcal{R}_{\hat{J}_c}}{\hat{J}_c+\mathcal{R}_{\hat{J}_c}} - \frac{1}{2}\,\mathrm{Tr}'_{(0)}\,\frac{\partial_t\mathcal{R}_{\hat{J}_b}}{\hat{J}_b+\mathcal{R}_{\hat{J}_b}} + \frac{1}{2}\sum_{l=0,1}\frac{\partial_t\mathcal{R}_{hh}(\lambda_l)}{\Gamma^{(2)}_{hh}(\lambda_l)+\mathcal{R}_{hh}(\lambda_l)}
\end{aligned}
\tag{2.65}
$$

We use the same notation appearing in [5], in which the n-prime in the trace operator means that in the calculation we must exclude first n mode, for example with $n = 2$

$$\mathrm{Tr}''\,W(\Delta) = \mathrm{Tr}\,W(\Delta) - W(\lambda_{i=0}) - W(\lambda_{i=1}) \tag{2.66}$$

As explained in previous section, in the trace for vector field ξ we explude the first mode and for the same reason the first two modes for σ field. Since σ scalar field appears in a non trivial mixing with trace field h, we exclude the first two modes in total mixing and then add the relative $l = 0, 1$ modes for h in the last line of (2.65). For ghosts fields we eclude the first mode in longitudinal component, as expected. The fifth and sixth lines give the contribution of Jacobians of transformations (2.23-2.25). The relative traces carry the same excluded modes as the fields in (2.23-2.25).

For the last step, the calculation of traces in FRG equation (2.65), two different methods can be followed. First, traces can be approximated with the Heat Kernel technique [5, 14] given in Appendix C and used in the next section. Second, an approximation on "sum over eingenvalues" [6] will be followed in the next chapter.

2.5 Trace evaluation using the Heat Kernel technique

We have derived the Wetterich's equation for the average effective action taking into account TT decomposition (2.65). The last step is the explicit evaluation of the functional trace using the methods of heat kernel introduced in Appendix C. From this section, we introduce dimensionless variables, defined as

$$R = k^2\tilde{R} \qquad f_k(R) = k^2\tilde{f}_k(R/k^2) \tag{2.67}$$

which imply

$$f'_k(R) = \tilde{f}'_k(R/k^2) \qquad f''_k(R) = k^{-2}\tilde{f}''_k(R/k^2) \tag{2.68}$$

$$\begin{aligned}
\partial_t f_k(R) &= k^2\left[\partial_t \tilde{f}_k(R/k^2) - 2\tilde{R}\tilde{f}'_k(R/k^2) + 2\tilde{f}_k(R/k^2)\right]\\
\partial_t f'_k(R) &= \left[\partial_t \tilde{f}'_k(R/k^2) - 2\tilde{R}\tilde{f}''_k(R/k^2)\right]\\
\partial_t f''_k(R) &= k^{-2}\left[\partial_t \tilde{f}''_k(R/k^2) - 2\tilde{R}\tilde{f}'''_k(R/k^2) - 2\tilde{f}''_k(R/k^2)\right]
\end{aligned} \tag{2.69}$$

Let us start with the trace evaluation for the tensor component h^T

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{Tr}_{(2)}\,\frac{\partial_t \boldsymbol{\mathcal{R}}_{h^T,h^T}}{\boldsymbol{\Gamma}^{(2)}_{h^T,h^T} + \boldsymbol{\mathcal{R}}_{h^T,h^T}} = \frac{1}{2}\,\mathrm{Tr}_{(2)}\,\frac{\partial_t\left(\boldsymbol{\Gamma}^{(2)}_{h^T,h^T}(P_k) - \boldsymbol{\Gamma}^{(2)}_{h^T,h^T}(\Delta)\right)}{\boldsymbol{\Gamma}^{(2)}_{h^T,h^T}(P_k)} = \\
&\frac{1}{2}\,\mathrm{Tr}_{(2)}\left[\frac{f'_k\partial_t P_k + (P_k-\Delta)(\partial_t f'_k - \eta_{k,h} f'_k)}{\left(P_k - \frac{R}{3}\right)f'_k + f_k}\right]
\end{aligned} \tag{2.70}$$

Imposing the β-gauge, the traces' vector part receives contribution from $\xi, \bar{c}^T, c^T, b^T, \hat{J}_V$. From now, we use the convetion

$$P_k^{(n)} = P_k - \frac{R}{n} \qquad \Delta^{(n)} = \Delta - \frac{R}{n} \tag{2.71}$$

- ξ trace part

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{Tr}'_{(1)}\,\frac{\partial_t\left(\Gamma^{(2)}_{\xi,\xi}(P_k) - \Gamma^{(2)}_{\xi,\xi}\right)}{\Gamma^{(2)}_{\xi,\xi}(P_k)} = \frac{1}{2}\,\mathrm{Tr}'_{(1)}\,\frac{\partial_t\left[Z_{k,h}\left(P_k^{(4)}\right)^2 P_k\right]}{Z_{k,h}\left(P_k^{(4)}\right)^2 P_k}\\
&= \frac{1}{2}\,\mathrm{Tr}'_{(1)}\,\frac{\partial_t r_k(\Delta)}{P_k(\Delta)} + \mathrm{Tr}'_{(1)}\,\frac{\partial_t r_k(\Delta)}{P_k^{(4)}(\Delta)} - \frac{1}{2}\,\mathrm{Tr}'_{(1)}\,\eta_{k,h}\theta(k^2-\Delta)
\end{aligned} \tag{2.72}$$

- $\bar{c}^T c^T$ trace part

$$\begin{aligned}
&\mathrm{Tr}_{(1)}\,\frac{\partial_t\left(\Gamma^{(2)}_{\bar{c}^T,c^T}(P_k) - \Gamma^{(2)}_{\bar{c}^T,c^T}\right)}{\Gamma^{(2)}_{\bar{c}^T,c^T}(P_k)} = -\,\mathrm{Tr}_{(1)}\,\frac{\partial_t\left[Z_{k,c}P_k^{(4)}P_k\right]}{Z_{k,c}P_k^{(4)}P_k}\\
&= -\,\mathrm{Tr}_{(1)}\,\frac{\partial_t r_k(\Delta)}{P_k(\Delta)} - \mathrm{Tr}_{(1)}\,\frac{\partial_t r_k(\Delta)}{P_k^{(4)}(\Delta)} + \mathrm{Tr}_{(1)}\,\eta_{k,c}\theta(k^2-\Delta)
\end{aligned} \tag{2.73}$$

- b^T trace part

$$\frac{1}{2}\,\mathrm{Tr}_{(1)}\,\frac{\partial_t\left(\Gamma^{(2)}_{b^T,b^T}(P_k) - \Gamma^{(2)}_{b^T,b^T}\right)}{\Gamma^{(2)}_{b^T,b^T}(P_k)} = \frac{1}{2}\,\mathrm{Tr}_{(1)}\,\frac{\partial_t r_k(\Delta)}{P_k(\Delta)} - \frac{1}{2}\,\mathrm{Tr}_{(1)}\,\eta_{k,b}\theta(k^2-\Delta) \tag{2.74}$$

- $\hat{J}_V$ trace part

$$-\frac{1}{2}\operatorname{Tr}'_{(1)}\frac{\partial_t r_k(\Delta)}{P_k^{(4)}(\Delta)} \tag{2.75}$$

So the total vector trace part gives

$$\begin{aligned}
&-\frac{1}{2}\operatorname{Tr}'_{(1)}\frac{\partial_t r_k(\Delta)}{P_k^{(4)}(\Delta)}-\frac{1}{2}\frac{\partial_t r_k(\lambda_{l=1})}{P_k(\lambda_{l=1})}D_{l=1,s=1}\\
&-\frac{\partial_t r_k(\lambda_{l=1})}{P_k^{(4)}(\lambda_{l=1})}D_{l=1,s=1}+\frac{\eta_{k,h}}{2}D_{l=1,s=1}+\frac{1}{2}\operatorname{Tr}_{(1)}\left[\theta(k^2-\Delta)(2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})\right]\\
&=-\frac{1}{2}\operatorname{Tr}'_{(1)}\frac{\partial_t r_k(\Delta)}{P_k^{(4)}(\Delta)}-5\frac{\partial_t r_k(\frac{R}{4})}{P_k(\frac{R}{4})}\\
&-10\frac{\partial_t r_k(\frac{R}{4})}{P_k^{(4)}(\frac{R}{4})}+5\eta_{k,h}+\frac{1}{2}\operatorname{Tr}_{(1)}\left[\theta(k^2-\Delta)(2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})\right]
\end{aligned} \tag{2.76}$$

where we used the eigenvalues and relative multiplicity of the Laplacian in S^d, as given in C.1.

For the trace of $h-\sigma$ scalar part, note that in the gauge $\rho=0$ only $\Gamma^{(2)}_{\sigma\sigma}$, and so $\mathcal{R}_{\sigma\sigma}$, dipends on β

$$\begin{aligned}
&\frac{1}{2}\operatorname{Tr}''_{(0)}\frac{\Gamma^{(2)}_{\sigma\sigma}\partial_t\mathcal{R}_{hh}-2\Gamma^{(2)}_{h\sigma}\partial_t\mathcal{R}_{h\sigma}+\Gamma^{(2)}_{hh}\partial_t\mathcal{R}_{\sigma\sigma}}{\Gamma^{(2)}_{\sigma\sigma}\Gamma^{(2)}_{hh}-\left(\Gamma^{(2)}_{h\sigma}\right)^2}+\frac{1}{2}\sum_{l=0,1}\frac{\partial_t\mathcal{R}_{hh}(\lambda_l)}{\Gamma^{(2)}_{hh}(\lambda_l)+\mathcal{R}_{hh}(\lambda_l)}\\
&=\frac{1}{2}\operatorname{Tr}_{(0)}\frac{\partial_t\mathcal{R}_{hh}(\Delta)}{\Gamma^{(2)}_{hh}(P_k(\Delta))}+\frac{1}{2}\operatorname{Tr}''_{(0)}\frac{\partial_t\mathcal{R}_{\sigma\sigma}}{\Gamma^{(2)}_{\sigma\sigma}(P_k(\Delta))}\\
&=\frac{1}{2}\operatorname{Tr}_{(0)}\frac{\partial_t\mathcal{R}_{hh}(\Delta)}{\Gamma^{(2)}_{hh}(P_k(\Delta))}+\frac{1}{2}\operatorname{Tr}''_{(0)}\left(\frac{\partial_t r_k(\Delta)}{P_k(\Delta)}+2\frac{\partial_t r_k(\Delta)}{P_k^{(3)}(\Delta)}+\frac{\partial_t r_k(\Delta)}{P_k^{(4)}(\Delta)}-\eta_{k,h}\theta(k^2-\Delta)\right)
\end{aligned} \tag{2.77}$$

The remaining scalar contribution $\bar{c},c,b,\hat{J}_s,\hat{J}_c,\hat{J}_b$ gives

- $\bar{c}c$

$$-\operatorname{Tr}'_{(0)}\frac{\partial_t\mathcal{R}_{\bar{c}c}(\Delta)}{\Gamma^{(2)}_{\bar{c}c}(P_k(\Delta))}=-\operatorname{Tr}'_{(0)}\left(\frac{\partial_t r_k(\Delta)}{P_k^{(4)}(\Delta)}+\frac{\partial_t r_k(\Delta)}{P_k^{(3)}(\Delta)}+\frac{\partial_t r_k(\Delta)}{P_k(\Delta)}-\eta_{k,c}\theta(k^2-\Delta)\right) \tag{2.78}$$

- b

$$\frac{1}{2}\operatorname{Tr}'_{(0)}\frac{\partial_t\mathcal{R}_{bb}(\Delta)}{\Gamma^{(2)}_{bb}(P_k(\Delta))}=\frac{1}{2}\operatorname{Tr}'_{(0)}\left(\frac{\partial_t r_k(\Delta)}{P_k^{(4)}(\Delta)}+\frac{\partial_t r_k(\Delta)}{P_k(\Delta)}-\eta_{k,b}\theta(k^2-\Delta)\right) \tag{2.79}$$

- $\hat{J}_c$, $\hat{J}_b$ and $\hat{J}_s$ gives

$$\frac{1}{2}\operatorname{Tr}'_{(0)}\frac{\partial_t r_k(\Delta)}{P_k(\Delta)}-\frac{1}{2}\operatorname{Tr}''_{(0)}\left(\frac{\partial_t r_k(\Delta)}{P_k^{(3)}(\Delta)}+\frac{\partial_t r_k(\Delta)}{P_k(\Delta)}\right) \tag{2.80}$$

The total scalar trace part gives

$$\begin{aligned}&\frac{1}{2}\,\mathrm{Tr}_{(0)}\,\frac{\partial_t\mathcal{R}_{hh}(\Delta)}{\Gamma^{(2)}_{hh}(P_k(\Delta))}-\frac{1}{2}\,\mathrm{Tr"}_{(0)}\,\frac{\partial_t r_k(\Delta)}{P_k^{(3)}}-5\frac{\partial_t r_k(\frac{R}{3})}{P_k^{(2)}(\frac{R}{3})}-\frac{5}{2}\frac{\partial_t r_k(\frac{R}{3})}{P_k^{(4)}(\frac{R}{3})}\\&+\frac{1}{2}\,\mathrm{Tr'}_{(0)}\left[\theta(k^2-\Delta)(2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})\right]+\frac{5}{2}\eta_{h,k}\end{aligned}\tag{2.81}$$

Last, we explicite the h trace part

$$\begin{aligned}&\frac{1}{2}\,\mathrm{Tr}_{(0)}\,\frac{\partial_t\mathcal{R}_{hh}}{\Gamma^{(2)}_{hh}(P_k)}=\\&\frac{1}{2}\,\mathrm{Tr}_{(0)}\left[\frac{\partial_t P_k\left(f'_k+6\left(P_k-\frac{R}{3}\right)f''_k\right)+(P_k-\Delta)(\partial_t f'_k-\eta_{k,h}f'_k+3(P_k+\Delta-\frac{2}{3}R)(\partial_t f''_k-\eta_{k,h}f''_k))}{\frac{2}{3}+\left(P_k-\frac{2}{3}R\right)f'_k-3f''_k\left(P_k-\frac{R}{3}\right)^2}\right]\end{aligned}\tag{2.82}$$

Adding all together, the ERGE becamos

$$\begin{aligned}\partial_t\Gamma_k=&\frac{1}{2}\,\mathrm{Tr}_{(2)}\left[\frac{f'_k\partial_t P_k+(P_k-\Delta)(\partial_t f'_k-\eta_{k,h}f'_k)}{\left(P_k-\frac{R}{3}\right)f'_k+f_k}\right]-\frac{1}{2}\,\mathrm{Tr'}_{(1)}\,\frac{\partial_t r_k(\Delta)}{P_k(\Delta)-\frac{R}{4}}-\frac{1}{2}\,\mathrm{Tr"}_{(0)}\,\frac{\partial_t r_k(\Delta)}{P_k(\Delta)-\frac{R}{3}}\\&+\frac{1}{2}\,\mathrm{Tr}_{(0)}\left[\frac{\partial_t P_k\left(f'_k+6\left(P_k-\frac{R}{3}\right)f''_k\right)+(P_k-\Delta)\left(\partial_t f'_k-\eta_{k,h}f'_k+3(P_k+\Delta-\frac{2}{3}R)(\partial_t f''_k-\eta_{k,h}f''_k)\right)}{\frac{2}{3}+\left(P_k-\frac{2}{3}R\right)f'_k-3f''_k\left(P_k-\frac{R}{3}\right)^2}\right]\\&+\frac{1}{2}\,\mathrm{Tr}_{(1)}\left[\theta(k^2-\Delta)(2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})\right]+\frac{1}{2}\,\mathrm{Tr'}_{(0)}\left[\theta(k^2-\Delta)(2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})\right]+\Sigma\end{aligned}\tag{2.83}$$

where

$$\Sigma=-5\frac{\partial_t r_k(\frac{R}{4})}{P_k(\frac{R}{4})}-10\frac{\partial_t r_k(\frac{R}{4})}{P_k^{(4)}(\frac{R}{4})}-5\frac{\partial_t r_k(\frac{R}{3})}{P_k^{(3)}(\frac{R}{3})}-\frac{5}{2}\frac{\partial_t r_k(\frac{R}{3})}{P_k^{(4)}(\frac{R}{3})}+\frac{15}{2}\eta_{k,h}\tag{2.84}$$

collects the residue modes which do not cancel when we sum non physical flactuation trace part (ξ,σ) and ghosts trace part $(\bar{c},c,b)$. Term by term, we must explicit the functional trace according to the technique described in the Appendix C. For the cutoff profile function $r_k(z)$ we choose the Litim's optimized cutoff [17] defined by

$$r_k(z)=(k^2-z)\theta(k^2-z)\tag{2.85}$$

With this choice the trace calculation drastically simplifies, but it introduces some non-smooth term in the final results.

Let us start with the spin one trace part

$$-\frac{1}{2}\,\mathrm{Tr'}_{(1)}\,\frac{\partial_t r_k(\Delta)}{P_k(\Delta)-\frac{R}{4}}=-\frac{1}{2}\,\mathrm{Tr}_{(1)}\,\frac{\partial_t r_k(\Delta)}{P_k(\Delta)-\frac{R}{4}}+\frac{1}{2}\frac{\partial_t r_k(\Delta)}{P_k-\frac{R}{4}}\bigg|_{\lambda_{l=1}}\tag{2.86}$$

Recalling the asymptotic Heat Kernel expansion for this case

$$\begin{aligned} -\frac{1}{2}\,\mathrm{Tr}_{(1)}\,\frac{\partial_t r_k(\Delta)}{P_k(\Delta)-\frac{R}{4}} &= -\frac{1}{2}\frac{V_{4S}}{(4\pi)^2}\left[Q_2\left(\frac{\partial_t r_k}{P_k-\frac{R}{4}}\right)\mathrm{tr}\,b_0+Q_1\left(\frac{\partial_t r_k}{P_k-\frac{R}{4}}\right)\mathrm{tr}\,b_2\right. \\ &\quad\left.+Q_0\left(\frac{\partial_t r_k}{P_k-\frac{R}{4}}\right)\mathrm{tr}\,b_4\right] \\ &= -\frac{1}{2}\frac{V_{4S}}{(4\pi)^2}\left[\frac{k^4}{1-\frac{\tilde R}{4}}\mathrm{tr}\,b_0+\frac{2k^2}{1-\frac{\tilde R}{4}}\mathrm{tr}\,b_2+\frac{2}{1-\frac{\tilde R}{4}}\mathrm{tr}\,b_4\right] \end{aligned} \tag{2.87}$$

where $Q_k(W)$ is introduced in Appendix C. We used formulae (C.15), dimensionless scalar curvature $R=k^2\tilde R$ and four sphere volume $V_{4S}=384\pi^2/R^2$. So the vector trace part becomes

$$-\frac{1}{2}\,\mathrm{Tr}'_{(1)}\,\frac{\partial_t r_k(\Delta)}{P_k(\Delta)-\frac{R}{4}}=\frac{48}{(\tilde R-4)\tilde R^2}\left[3+\frac{\tilde R}{2}-\frac{7}{720}\tilde R^2\right]-\frac{40}{\tilde R-4}\theta\left(1-\frac{\tilde R}{4}\right) \tag{2.88}$$

where we used the table C.2 in appendice for heat kernel coefficients traces.

The scalar trace

$$-\frac{1}{2}\,\mathrm{Tr}''_{(0)}\,\frac{\partial_t r_k(\Delta)}{P_k(\Delta)-\frac{R}{3}} \tag{2.89}$$

is analized as the vector trace. The heat kernel expansion reads

$$\begin{aligned} -\frac{1}{2}\,\mathrm{Tr}''_{(0)}\,\frac{\partial_t r_k(\Delta)}{P_k(\Delta)-\frac{R}{3}} &= -\frac{1}{2}\frac{V_{4S}}{(4\pi)^2}\left[Q_2\left(\frac{\partial_t r_k}{P_k-\frac{R}{3}}\right)\mathrm{tr}\,b_0+Q_1\left(\frac{\partial_t r_k}{P_k-\frac{R}{3}}\right)\mathrm{tr}\,b_2\right. \\ &\quad\left.+Q_0\left(\frac{\partial_t r_k}{P_k-\frac{R}{3}}\right)\mathrm{tr}\,b_4\right]+\frac{1}{2}\frac{\partial_t r_k}{P_k-\frac{R}{3}}\bigg|_{\frac{R}{4}}+\frac{5}{2}\frac{\partial_t r_k}{P_k-\frac{R}{3}}\bigg|_{\frac{R}{3}} \\ &= \frac{36}{(\tilde R-3)\tilde R^2}\left(1+\frac{\tilde R}{3}+\frac{29\tilde R^2}{1080}\right)-\frac{18}{\tilde R-3}\theta\left(1-\frac{\tilde R}{3}\right) \end{aligned} \tag{2.90}$$

Next, we calculate the tensor trace part

$$\frac{1}{2}\,\mathrm{Tr}_{(2)}\left[\frac{f'_k\partial_t P_k+(P_k-\Delta)(\partial_t f'_k-\eta_{k,h}f'_k)}{\left(P_k-\frac{R}{3}\right)f'_k+f_k}\right]\equiv\frac{1}{2}\,\mathrm{Tr}_{(2)}\,W_{(2)}(\Delta) \tag{2.91}$$

where we introduce the function $W_{(2)}$ defined by the last relation.

In this case the heat kernel expansion reads

$$\begin{aligned} \frac{1}{2}\,\mathrm{Tr}_{(2)}\,W_{(2)}(\Delta) =&\frac{1}{2}\frac{V_{4S}}{(4\pi)^2}\left[Q_2(W_{(2)})\,\mathrm{tr}\,b_0+Q_1(W_{(2)})\,\mathrm{tr}\,b_2\right. \\ &\left.+Q_0(W_{(2)})\,\mathrm{tr}\,b_4+Q_{-1}(W_{(2)})\,\mathrm{tr}\,b_6\right] \end{aligned} \tag{2.92}$$

Using (C.15)

$$Q_2(W_{(2)})=k^6\frac{f'_k+\frac{1}{6}(\partial_t f'_k-\eta_{k,h}f'_k)}{\left(k^2-\frac{R}{3}\right)f'_k+f_k} \tag{2.93}$$

$$Q_1(W_{(2)}) = k^4 \frac{2f'_k + \frac{1}{2}(\partial_t f'_k - \eta_{k,h} f'_k)}{\left(k^2 - \frac{R}{3}\right) f'_k + f_k} \tag{2.94}$$

$$Q_0(W_{(2)}) = k^2 \frac{2f'_k + (\partial_t f'_k - \eta_{k,h} f'_k)}{\left(k^2 - \frac{R}{3}\right) f'_k + f_k} \tag{2.95}$$

$$Q_{-1}(W_{(2)}) = \frac{(\partial_t f'_k - \eta_{k,h} f'_k)}{\left(k^2 - \frac{R}{3}\right) f'_k + f_k} \tag{2.96}$$

So that the total tensor trace part is

$$\frac{1}{2}\operatorname{Tr}_{(2)}\left[\frac{f'_k \partial_t P_k + (P_k - \Delta)(\partial_t f'_k - \eta_{k,h} f'_k)}{\left(P_k - \frac{R}{3}\right) f'_k + f_k}\right] = \frac{36}{\tilde{R}^2} \frac{1}{3f_k - (\tilde{R} - 3) f'_k} \left[f'_k \left(5 - \frac{5}{3}\tilde{R} - \frac{1}{216}\tilde{R}^2\right) \right.$$
$$\left. + (\partial_t f'_k - \eta_{k,h} f'_k)\left(\frac{5}{6} - \frac{5}{12}\tilde{R} - \frac{1}{432}\tilde{R}^2 + \frac{311}{54432}\tilde{R}^3\right)\right] \tag{2.97}$$

The remaining trace is the h-scalar part

$$\frac{1}{2}\operatorname{Tr}_{(0)}\left[\frac{\partial_t P_k\left(f'_k + 6\left(P_k - \frac{R}{3}\right) f''_k\right) + (P_k - \Delta)(\partial_t f'_k - \eta_{k,h} f'_k + 3(P_k + \Delta - \frac{2}{3}R)(\partial_t f''_k - \eta_{k,h} f''_k))}{\frac{2}{3} + \left(P_k - \frac{2}{3}R\right) f'_k - 3f''_k\left(P_k - \frac{R}{3}\right)^2}\right]$$
$$\equiv \frac{1}{2}\operatorname{Tr}_{(0)} W_s(\Delta) \tag{2.98}$$

This relation leads us to define the function $W_s(\Delta)$. The heat kernel expansion leads to

$$\frac{1}{2}\operatorname{Tr}_{(0)} W_s = \frac{1}{2}\frac{V_{4S}}{(4\pi)^2}\left[Q_2(W_s)\operatorname{tr} b_0 + Q_1(W_s)\operatorname{tr} b_2 + Q_0 \operatorname{tr} b_4 \right.$$
$$\left. + Q_{-1}(W_s)\operatorname{tr} b_6 + Q_{-2}(W_s)\operatorname{tr} b_8\right] \tag{2.99}$$

since third derivative of $W_s(z)$ in $z = 0$ vanishes.

Using (C.15) we find

$$Q_2(W_s) = k^4 \frac{f_k + 6\left(1 - \frac{R}{3k^2}\right) f''_k + \frac{1}{6}\left(\partial_t f'_k - \eta_{k,h} f'_k\right) + \left(\frac{3}{4} - \frac{R}{3k^2}\right)\left(\partial_t f''_k - \eta_{k,h} f''_k\right)}{\frac{2}{3} f_k + \left(1 - \frac{2}{3k^2}R\right) f'_k - 3f''_k\left(1 - \frac{R}{3}\right)^2}$$

$$Q_1(W_s) = k^4 \frac{2f'_k + 12\left(1 - \frac{R}{3}\right) f''_k + \frac{1}{2}\left(\partial_t f'_k - \eta_{k,h} f'_k\right) + \left(2 - \frac{R}{k^2}\right)(\partial_t f''_k - \eta_{k,h} f''_k)}{\frac{2}{3} f_k + \left(1 - \frac{2}{3k^2}R\right) f'_k - 3f''_k\left(1 - \frac{R}{3k^2}\right)^2}$$

$$Q_0(W_s) = k^2 \frac{2f'_k + 12\left(1 - \frac{R}{3k^2}\right) f''_k + \partial_t f'_k - \eta_{k,h} f'_k + 3\left(1 - \frac{2}{3k^2}R\right)\left(\partial_t f''_k - \eta_{k,h} f''_k\right.}{\frac{2}{3} f_k + \left(1 - \frac{2}{3k^2}R\right) f'_k - 3f''_k\left(1 - \frac{R}{3k^2}\right)^2}$$

$$Q_{-1}(W_s) = \frac{\partial_t f'_k - \eta_{k,h} f'_k - 2\frac{R}{k^2}\left(\partial_t f''_k - \eta_{k,h} f''_k\right)}{\frac{2}{3} f_k + \left(1 - \frac{2}{3k^2}R\right) f'_k - 3f''_k\left(1 - \frac{R}{3k^2}\right)^2}$$

$$Q_{-2}(W_s) = k^{-2} \frac{-6(\partial_t f'_k - \eta_{k,h} f'_k)}{\frac{2}{3} f_k + \left(1 - \frac{2}{3k^2} R\right) f'_k - 3 f''_k \left(1 - \frac{R}{3k^2}\right)^2}$$

So that the h scalar trace part becomes

$$\begin{aligned} \frac{1}{2} \operatorname{Tr}_{(0)} W_s = & \frac{1}{\tilde{R}^2 \left[\tilde{f}''_k (\tilde{R} - 3)^2 + 2\tilde{f} + (3 - 2\tilde{R}) \tilde{f}'_k \right]} \\ & \left[\tilde{f}'_k \left(36 + 12\tilde{R} + \frac{29}{30} \tilde{R}^2 \right) - \tilde{f}''_k \left(\frac{29}{15} \tilde{R}^3 + \frac{91}{5} \tilde{R}^2 - 216 \right) \right. \\ & - \left(\partial_t \tilde{f}''_k - 2\tilde{R} \tilde{f}'''_k - (\eta_{k,h} + 2) \tilde{f}''_k \right) \left(\frac{3801}{7056} \tilde{R}^4 + \frac{29}{30} \tilde{R}^3 + \frac{273}{60} \tilde{R}^2 - 27 \right) \\ & \left. + \left(\partial_t \tilde{f}'_k - 2\tilde{R} \tilde{f}''_k - \eta_{k,h} \tilde{f}'_k \right) \left(6 + 3\tilde{R} + \frac{29}{60} \tilde{R}^2 + \frac{37}{1512} \tilde{R}^3 \right) \right] \end{aligned} \tag{2.100}$$

The last trace contribution regards the anomalous dimensions for fields fluctuations

$$+ \frac{1}{2} \operatorname{Tr}_{(1)} \left[\theta(k^2 - \Delta)(2\eta_{k,c} - \eta_{k,h} - \eta_{k,b}) \right] + \frac{1}{2} \operatorname{Tr'}_{(0)} \left[\theta(k^2 - \Delta)(2\eta_{k,c} - \eta_{k,h} - \eta_{k,b}) \right] \tag{2.101}$$

Both traces reduce to the simple functional trace $\operatorname{Tr} \theta(k^2 - \Delta)$, which has the Heat Kernel expansion

$$\operatorname{Tr} \theta(k^2 - \Delta) = \frac{V_{4s}}{(4\pi^2)} \left[\frac{1}{2} k^4 \operatorname{tr} \boldsymbol{b}_0 + k^2 \operatorname{tr} \boldsymbol{b}_2 + \operatorname{tr} \boldsymbol{b}_4 \right] \tag{2.102}$$

where, clearly, the coefficients $\operatorname{tr} \boldsymbol{b}_n$ depend on the spin of the fields, as reported in Appendix C. The contribution (2.101) gives

$$\frac{1}{\tilde{R}^2} \left[24 + 5\tilde{R} + \frac{37}{360} \tilde{R}^2 \right] \tag{2.103}$$

Last, single modes Σ give

$$\Sigma = \frac{80}{\tilde{R} - 4} + \frac{30}{\tilde{R} - 3} - 10\theta \left(1 - \frac{\tilde{R}}{4} \right) + \frac{20}{\tilde{R} - 4} \theta \left(1 - \frac{\tilde{R}}{3} \right) \tag{2.104}$$

Collecting all contributions we find that the FRG equation in the $f_k(R)$ approximation for gravity becomes

$$
\begin{aligned}
\partial_t\Gamma_k = &\frac{36}{\tilde{R}^2}\frac{1}{\left[3\tilde{f}_k-(\tilde{R}-3)\tilde{f}'_k\right]}\left[\tilde{f}'_k\left(5-\frac{5}{3}\tilde{R}-\frac{1}{216}\tilde{R}^2\right)\right.\\
&\left.+(\partial_t\tilde{f}'_k-2\tilde{R}\tilde{f}''_k-\eta_{k,h}\tilde{f}'_k)\left(\frac{5}{6}-\frac{5}{12}\tilde{R}-\frac{1}{432}\tilde{R}^2+\frac{311}{54432}\tilde{R}^2\right)\right]-\frac{18}{\tilde{R}-3}\theta\Big(1-\frac{\tilde{R}}{3}\Big)\\
&+\frac{36}{(\tilde{R}-3)\tilde{R}^2}\Big[1+\frac{\tilde{R}}{3}+\frac{29\tilde{R}^2}{1080}\Big]+\frac{48}{(\tilde{R}-4)\tilde{R}^2}\Big[3+\frac{\tilde{R}}{2}-\frac{7}{720}\tilde{R}^2\Big]-\frac{40}{\tilde{R}-4}\theta\Big(1-\frac{\tilde{R}}{4}\Big)\\
&+\frac{1}{\tilde{R}^2\left[\tilde{f}''_k(\tilde{R}-3)^2+2\tilde{f}+(3-2\tilde{R})\tilde{f}'_k\right]}\left[\tilde{f}'_k\left(36+12\tilde{R}+\frac{29}{30}\tilde{R}^2\right)\right.\\
&-\tilde{f}''_k\left(\frac{29}{15}\tilde{R}^3+\frac{91}{5}\tilde{R}^2-216\right)-(\partial_t\tilde{f}''_k-2\tilde{R}\tilde{f}'''_k-(\eta_{k,h}+2)\tilde{f}''_k)\left(\frac{3801}{7056}\tilde{R}^4+\frac{29}{30}\tilde{R}^3+\frac{273}{60}\tilde{R}^2-27\right)\\
&\left.+(\partial_t\tilde{f}'_k-2\tilde{R}\tilde{f}''_k-\eta_{k,h}\tilde{f}'_k)\left(6+3\tilde{R}+\frac{29}{60}\tilde{R}^2+\frac{37}{1512}\tilde{R}^3\right)\right]+\Sigma\\
&+\frac{(2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})}{\tilde{R}^2}\left[24+5\tilde{R}+\frac{37}{360}\tilde{R}^2\right]
\end{aligned}
\tag{2.105}
$$

This equation is obtained with the gauge choice $\alpha=0$ and $\beta\to\infty$. But the same calculation can be done for the opposite gauge $\beta=0$ $\alpha\to\infty$; in this gauge we obtain the same equation up to the contribution Σ, which changes in

$$
\begin{aligned}
\Sigma_{\beta-\text{gauge}} &= -10\frac{\partial_t R_k(\frac{R}{4})}{P_k^{(4)}(\frac{R}{4})}-10\frac{\partial_t R_k(\frac{R}{3})P_k^{(6)}(\frac{R}{3})}{P_k(\frac{R}{3})P_k^{(3)}(\frac{R}{3})}+5\frac{\partial_t R_k(\frac{R}{3})}{P_k(\frac{R}{3})}\\
&=80\frac{\theta(4-\tilde{R})}{(\tilde{R}-4)}-10\theta(3-\tilde{R})\frac{\tilde{R}-6}{\tilde{R}-3}+10\theta(3-\tilde{R})
\end{aligned}
$$

2.6 Possible closures for $f_k(R)$ RG equation

In last sections we expand Wetterich's equation in the case of $f_k(R)$ approximation, and obtain the flow evolution equation (2.105); its solution, in principle, tells us the dependence of $f_k(R)$ both on k and on the scalar curvature R. After having a solution to equation (2.105), one can use relation (2.67) and get function $f_k(R)$ explicitly.

Equation (2.105), which is the starting point of any detailed analysis, contains anomalous dimensions of gravitational quantum flactuations and ghosts fields. Hence, one can choose different methods to close relation (2.105).

The same situations appears in scalar field theory where Wetterich's equation contains both the evolution of scalar potential $V_k(\varphi)$ and anomalous dimension η_φ. In this case, the equation can be closed considering the flow equation of $\Gamma_k^{(2)}$, which gives η_φ as a function of $V_k(\varphi=\text{const})$ and its derivatives.

The authors in [15] extend the previous method for scalar theory in gravity. They determine the flow of $\Gamma_k^{(2)}$ in the case of Einstein-Hilbert truncation and extract $\eta_{k,h}$ and $\eta_{k,c}$ (in this case we have only ghost and anti-ghost fields) as a function of $\tilde{G}$ and $\tilde{\Lambda}$, the dimensionless Newton's and cosmological constant.

A proposal for a future work is to extend the method in [15] for $f_k(R)$ approximation. Note that, on the contrary, the flow equation for $\Gamma_k^{(2)}$ depends on gravity-ghost-antighost proper vertex $\Gamma_k^{(1,1,1)}$ [2] and on gravity-gravity-ghost-antighost proper vertex $\Gamma_k^{(2,1,1)}$, which contain third and fourth derivative of $f_k(R)$.

To follow a consistent closure of (2.105), in this work we make two different ansatz for the values of $Z_{k,h}$, $Z_{k,c}$ and $Z_{k,b}$. The most simple ansatz, which we call *type I ansatz*, is the following

$$Z_{k,h} = \kappa_k^{-2} \qquad Z_{k,c} = Z_{k,b} = 1$$

which imply

$$\eta_{k,h} = -\frac{\beta_{\tilde{G}}}{\tilde{G}} - 2 \qquad \eta_{k,c} = 0 \qquad \eta_{k,b} = 0 \tag{2.106}$$

where$\kappa_k = \sqrt{16\pi G_k}$ and $\tilde{G} = k^2 G_k$ is the dimensionless Newton's constant. Note that type I ansatz (2.106) imply the following metric decomposition

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

which is the most used definition for quantum flactuations.

Another ansatz for the anomalous dimensions values, which we call type II ansatz, is the following

$$Z_{k,h} = Z_{k,b} = Z_{k,c} = 1$$

which implies

$$\eta_{k,h} = 0 \qquad \eta_{k,c} = 0 \qquad \eta_{k,b} = 0$$

Hence, with type I ansatz, the anomalous dimensions contribution in (2.105) is completly neglected. Note that, within this choice, we do not recover flow equation in [5], since metric decomposition $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa_k h_{\mu\nu}$ and truncation ansatz for Γ_k are different from this work.

2.7 Polynomial truncation

The most simple truncation for the average effective action is the standard Einstein-Hilbert action

$$\Gamma_k[h, \bar{C}, C, b; \bar{g}] = \frac{1}{16\pi G_k} \int d^d x \sqrt{g}(2\Lambda_k - R) + \Gamma_{k,gh} + \Gamma_{k,g.f.} \tag{2.107}$$

[2]Notation $\Gamma_k^{(1,0,0)}$ labels functional variation with respect to $h_{\mu\nu}$ while $\Gamma_k^{(0,0,1)}$ and $\Gamma_k^{(0,0,1)}$ with respect to ghost and antighost field, respectively.

where Λ_k is the running cosmological constant. This kind of truncation has been studied with different point of view [12, 18, 19, 20], in diverse cutoff choices [5] and in d spacetime dimension [21]. The choice (2.107) is consistent with previous truncation (3.1) if

$$f_k(R) = 2\Lambda_k - R \qquad \rightarrow \qquad f_k(0) = 2\Lambda_k \qquad f_k'(0) = -1$$

so one can construct the flow for gravitational and cosmological constants using (2.105).

For a first study on the dimensionality of the critical surface, the average effective action have to be modified introducing more interactions such as R^2, R^3..., or direct composition of Riemann tensor $R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}$.

Here, we consider the polynomial truncation, so that the effective action reads

$$\Gamma_k = \frac{1}{16\pi G_k}\int d^dx\sqrt{g}\sum_{i=0}^{n} g_i R^i + \Gamma_{k,gh} + \Gamma_{k,g.f.} \tag{2.108}$$

Clearly, for $n = 1$ we go back to Einstein-Hilbert form. This truncation ansatz is consistent with our initial assumption for the effective action as a function only on curvature scalar. To study the flow equation for the dimensionless couplings $\tilde{g}_i = k^{2-2i}g_i$, we use equation (2.105), which, as pointed out in the last section, necessitate of a closure to be solved for the presence of anomalous dimensions' contribution.

Type I closure: $Z_{k,h} = \kappa_k^{-2}$, $Z_{k,c} = Z_{k,b} = 1$

In table 2.1 the value of dimensionless couplings at the fixed point are reported for $n = 1$ to $n = 5$. First, we see that the value of $\tilde{G}^*$ and $\tilde{\Lambda}^*$ in the Einstein-Hilbert truncation ($n = 1$) agrees with the projection in the $\tilde{\Lambda} - \tilde{G}$ plane of other truncation ($n > 1$). For $n = 2$ we see a deviation of mean value of the fixed point position; the same deviation is observed also in [5]. As the reader has noticed yet, the only different between flow equation in [5] and equation (2.105) is a factor $Z_{k,h} = \kappa_k^{-1}$ in the regulator. Hence, this little expedient make the distribution of fixed point in different truncation more stable.

In table 2.2, the critical exponents are reported as a functions of the truncation. We first note that for $n > 1$ there are only three critical exponents with positive real part. Hence, we found an UV critical surface of dimension three for all $n > 1$; this is an important aspect which tells us that, within polynomial truncation up to $n = 5$, QFT of General Relativity is found to be asymptotically safe.

We report here some details of the Einstein-Hilbert approximation, which starts from ansatz (2.107). Inserting into equation (2.105), the *r.h.s.* becomes

$$\partial_t\Gamma_k = \frac{k^4}{16\pi\tilde{G}_k}\int d^dx\sqrt{g}\left[2\left(\beta_\Lambda - \frac{\tilde{\Lambda}_k}{\tilde{G}_k}\beta_G + 4\tilde{\Lambda}_k\right) + \left(\frac{\beta_G}{\tilde{G}_k} - 2\right)\frac{R}{k^2}\right]$$

where we introduced the beta function for the dimensionless Newton's constant $\beta_G = \partial_t\tilde{G}_k$ and for dimensionless cosmological constant $\beta_\Lambda = \partial_t\tilde{\Lambda}_k$. The calculation of *r.h.s.* of (2.105) is

Closure type I:$Z_{k,h} = \kappa_k^{-2}$, $Z_{k,c} = Z_{k,b} = 1$ $\eta_h^* = -2$, $\eta_{k,c} = \eta_{k,b} = 0$								
n	$\tilde{\Lambda}^*$	$\tilde{G}^*$	$\tilde{g}_0$	$\tilde{\kappa}^*$	$\tilde{g}_2$	$\tilde{g}_3$	$\tilde{g}_4$	$\tilde{g}_5$
1	0.194	0.7683	0.3879	6.2143				
2	0.2225	0.8894	0.445	6.6863	0.1087			
3	0.2057	0.8072	0.4113	6.3697	0.0829	-0.2929		
4	0.1986	0.7938	0.3972	6.3169	0.0753	-0.3028	-0.2084	
5	0.199	0.7946	0.3981	6.32	0.0758	-0.2824	-0.1943	-0.0597

Table 2.1: Couplings value at non-Gaussian fixed point as a functions of the order n of the truncation.

Closure type I:$Z_{k,h} = \kappa_k^{-2}$, $Z_{k,c} = Z_{k,b} = 1$, $\eta_h^* = -2$, $\eta_{k,c} = \eta_{k,b} = 0$						
n	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5
1	3.2353+i0.4723	3.2353-i0.4723				
2	3.3791+i1.1891	3.3791-i1.1891	11.778			
3	4.8126	2.5069	1.9175	-6.5191		
4	3.6681+i0.8798	3.6681-i0.8798	89.1475	-6.3821	-3.4987	
5	4.6001	2.806	1.6701	-4.6191+i10.4134	-4.6191-i10.4134	-4.4418

Table 2.2: Critical exponents as a function of the order n of the truncation.

more involved; first, note that for a polynomial truncation all theta functions can be set to one, since the next step is to expand in Taylor series around $R = 0$ and collect only linear term. Now we have only contributions which are constant or linear in dimensionless scalar curvature $\tilde{R}$ in both side, so comparing the constant and scalar curvature coefficients we find the beta functions

$$\beta_G = -2\tilde{G}_k + \frac{12\tilde{G}_k^2\left(64\tilde{\Lambda}_k^3 - 210\tilde{\Lambda}_k^2 + 317\tilde{\Lambda}_k - 144\right)}{\tilde{G}_k\left(240\tilde{\Lambda}_k^3 - 336\tilde{\Lambda}_k^2 + 46\tilde{\Lambda}_k + 51\right) - 144\pi\left(1 - 2\tilde{\Lambda}_k\right)^2\left(4\tilde{\Lambda}_k - 3\right)} \tag{2.109}$$

$$\begin{aligned}\beta_\Lambda =& 2\tilde{\Lambda}_k - \frac{1}{4\pi(4\tilde{\Lambda}_k - 3)\left[144\pi(1 - 2\tilde{\Lambda}_k)^2\left(4\tilde{\Lambda}_k - 3\right) - \tilde{G}_k(240\tilde{\Lambda}_k^3 - 336\tilde{\Lambda}_k^2 + 46\tilde{\Lambda}_k + 51)\right]} \\ &\times \tilde{G}_k\Big[\tilde{G}_k(4992\tilde{\Lambda}_k^4 - 22024\tilde{\Lambda}_k^3 + 35502\tilde{\Lambda}_k^2 - 24465\tilde{\Lambda}_k + 6093) \\ &+ 48\pi(256\tilde{\Lambda}_k^5 - 1416\tilde{\Lambda}_k^4 + 3794\tilde{\Lambda}_k^3 - 4233\tilde{\Lambda}_k^2 + 1953\tilde{\Lambda}_k - 297)\Big]\end{aligned} \tag{2.110}$$

This coupled equations give immediatly a Gaussian fixed point

$$\tilde{\Lambda}^* = 0 \qquad \tilde{G}^* = 0$$

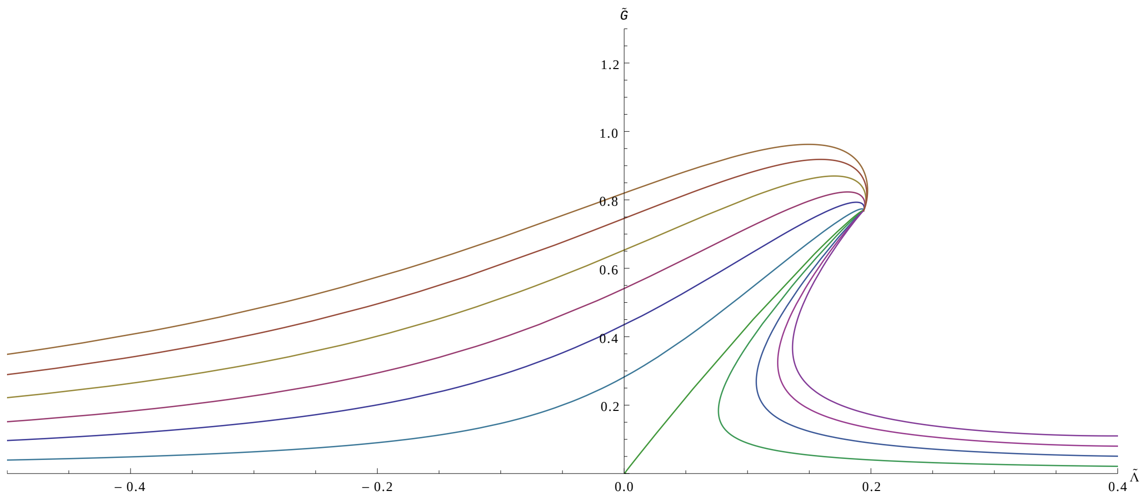


Figure 2.1: In this picture the flow given by beta functions (2.109-2.110) is represented in the $(\tilde{\Lambda}_k, \tilde{G}_k)$ plane.

Instead, a numerical study gives the following value for non Gaussian fixed point (NGFP)

$$\tilde{\Lambda}^* = 0.194 \qquad \tilde{G}^* = 0.7683 \tag{2.111}$$

with a stability coefficients, defined as the opposite of the eigenvalues of the stability matrix

$$\theta_{1,2} = 2.8764 \pm i1.8668 \tag{2.112}$$

which ensure us that NGFP is UV attractive in both direction in $(\tilde{\Lambda}_k, \tilde{G}_k)$ plane. This result is in agreement with [5] and is similar to that obtained with different cutoff scheme [12, 18, 5] and represents the first evidence for the asymptotic safety of quantum field theory of gravity. The main question is whether or not this UV fixed point persists in different (and more accurate) truncations. In papers [22, 23] an R^2 truncation is analized and non Gaussian fixed point is found with similar characteristics.

Our discussion within the $f_k(R)$ allows to consider a large polynomial truncation or $\ln R$ and R^{-n} contribution added in Einstein-Hilbert truncation[14]. In [5] a polynomial truncation up to order $n = 9$ is considered, a NGFP is found for $n = 9$ and the UV critical surface has dimension three, so the six irrelevant couplings can be expressed in terms of the remaining three relevant couplings. In paper [24] a polynomial truncation of orden $n = 35$ is considered and a NGFP is still found, with similar properties.

Type II closure: $Z_{k,h} = Z_{k,c} = Z_{k,b} = 1$, $\eta_{k,h} = \eta_{k,c} = \eta_{k,b} = 0$

In this section we present the result of polynomial truncation with a different choice for anomalous dimensions contribution. The type II closure for equation (2.105) reads

$$Z_{k,h} = Z_{k,c} = Z_{k,b} = 1 \quad \rightarrow \quad \eta_{k,h} = \eta_{k,c} = \eta_{k,b} = 0 \tag{2.113}$$

Closure type II: $Z_{k,h} = Z_{k,c} = Z_{k,b} = 1$ $\eta_{k,h} = \eta_{k,c} = \eta_{k,b} = 0$								
n	$\tilde{\Lambda}^*$	$\tilde{G}^*$	$\tilde{g}_0$	$\tilde{\kappa}^*$	$\tilde{g}_2$	$\tilde{g}_3$	$\tilde{g}_4$	$\tilde{g}_5$
1	0.0958	1.4451	0.1915	8.5229				
2	0.0645	1.8156	0.1289	9.5532	0.1099			
3	0.0954	1.5351	0.1908	8.7842	0.0527	-0.1211		
4	0.0962	1.5445	0.1923	8.8110	0.0539	-0.1408	0.0222	
5	0.0924	1.47136	0.1849	8.5999	0.0357	-0.1966	-0.0346	0.0493

Table 2.3: Couplings value at non-Gaussian fixed point as a functions of the order n of the truncation for closure (2.113).

Closure type II: $Z_{k,h} = Z_{k,c} = Z_{k,b} = 1$, $\eta_{k,h} = \eta_{k,c} = \eta_{k,b} = 0$						
n	$Re\theta_0 = Re\theta_1$	$Im\theta_0 = -Im\theta_1$	θ_2	$Re\theta_3 = Re\theta_4$	$Im\theta_3 = -Im\theta_4$	θ_5
1	2.3471	1.0908				
2	2.5343	0.3783	14.9101			
3	3.2492	0.3096	2.2444	-1.0945		
4	2.7630	1.1131	2.9292	-2.5628	-6.9273	
5	2.8721	1.3385	1.851	-3.0748	-5.9726	2.9515

Table 2.4: Critical exponents as a function of the order n of the truncation for closure (2.113).

The truncated effective average action is still (2.108) and the couplings values at fixed point is given in table 2.3.

First, we note that fixed point values for the couplings are different from type I closure. In fact, the two closures differ not only for regulator choice but also for metric decomposition; hence they give different values for the fixed point as expected, but the qualitative picture is the same.

Also in this case for $n = 2$ we have a deviation from the mean fixed point values; but the deviation is greater than in the previous case. Maybe, the previous choice on regulator term gives a more stable distribution of fixed point as a function of n.

In table 2.4 critical exponents are presented. For $n = 1$, the Einstein-Hilbert truncation, we have a pair of complex conjugate critical exponents; hence both directions in the $\tilde{\Lambda}_k - \tilde{G}_k$ plane are UV attractive. For $n > 1$ only three critical exponents have positive real part, so the UV critical surface has finite dimension, a result which is in common to the closure type I studied above.

Chapter 3

Trace evaluation with spectral sums in d dimension

The Heat Kernel approximation method is the most used in literature for trace evaluation of ERGE in gravity. We introduce now an approximation method, used in [6], which gives a different scaling equation for the $f(R)$.

The average effective action, as the starting point, is given by the ansatz[1]

$$\Gamma_k[h,\bar{C},C;\bar{g}] = \frac{1}{\kappa_k^2}\int d^dx\sqrt{g}f_k(R) + \Gamma_{k,gh} + \Gamma_{k,g.f.} \tag{3.1}$$

where, κ_k is a free parameter, which appears in the metric decomposition is $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa_k h_{\mu\nu}$, as in the previous chapter. At the end of calculation, two possible physical choice exist to fix its value, $\kappa_k = 1$ and $\kappa_k = \sqrt{16\pi G_k}$.

As in the previous chapter, we redefine field flactuations according to

$$h_{\mu\nu} \to Z_{k,h}^{1/2}h_{\mu\nu} \qquad C_\mu \to Z_{k,c}^{1/2}C_\mu \qquad \bar{C}_\mu \to Z_{k,c}^{1/2}\bar{C}_\mu \qquad b_\mu \to Z_{k,b}^{1/2}b_\mu \tag{3.2}$$

and define anomalous dimensions through the usual formulae

$$\eta_{k,a} = -\frac{\partial_t Z_{k,a}}{Z_{k,a}} \qquad a = h, c, b \tag{3.3}$$

Instead, contrary to the previous chapter, the gauge fixing condition reads (after redefinition) [2]

$$\Gamma_{k,g.f.}[h;\bar{g}] = \frac{Z_{k,h}}{2\alpha}\int d^dx\sqrt{\bar{g}}F_\mu[h;\bar{g}]F_\nu[h;\bar{g}]\bar{g}^{\mu\nu}$$

where

$$F_\mu[h;\bar{g}] = \bar{\nabla}^\rho h_{\rho\mu} - \frac{1}{d}\bar{\nabla}_\mu h \tag{3.4}$$

[1] Note that in [6] the ansatz is the same up to a running factor Z_k in the gravity and ghosts terms

[2] In the notation of the previous chapter we choose $\rho = 0$ and $\beta = 0$

Note that $F[h;\bar{g}]$ do not depend on field h, when the transverse-traceless decomposition is used; this observation is crucial in diagonalization of $\mathbf{\Gamma}_k^{(2)}$ within the gauge $\alpha \to 0$.

The corresponding standard ghost action reads

$$\Gamma_{k,gh-c}[h,\bar{C},C;\bar{g}] = Z_{k,c}\int d^dx\sqrt{\bar{g}}\bar{C}_\mu\left[\bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{\nabla}_\lambda(g_{\rho\nu}\nabla_\sigma + g_{\sigma\nu}\nabla_\rho) - \frac{2}{d}\bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{\nabla}_\lambda g_{\sigma\nu}\nabla_\rho\right]C^\nu \tag{3.5}$$

As it is pointed out in [25] the ghost action used is not (3.5). The aim is to provide a perfect cancellation between pure gauge degrees of freedom and ghosts sector in trace calculations. In order to achive this aim, it is crucial to note that the Faddeev-Popov determinant Det $\mathcal{M}$, which is implemented to a path integral over Grassmann-valued fields, is equivalent to $\sqrt{Det\mathcal{M}^2}$, which can be reprensented by a path integral over a vector Grassmann field and commuting real vector field and gives (after transversal and longitudinal decomposition)

$$\begin{aligned}\Gamma_{k,gh}[h=0,C,\bar{C},B;\bar{g}] =& \frac{Z_{k,c}}{\alpha}\int d^dx\sqrt{\bar{g}}\left[\bar{C}^T_\mu\left(\Delta - \frac{R}{d}\right)^2 C^{T\mu} + 4\left(\frac{d-1}{d}\right)^2\bar{c}\left(\Delta - \frac{R}{d-1}\right)^2\Delta c\right]\\ &+ \frac{Z_{k,b}}{\alpha}\int d^dx\sqrt{\bar{g}}\left[B^T_\mu\left(\Delta - \frac{R}{d}\right)^2 B^{T\mu} + 4\left(\frac{d-1}{d}\right)^2 b\left(\Delta - \frac{R}{d-1}\right)^2\Delta b\right]\end{aligned} \tag{3.6}$$

As we shall see, for this choice in ghost sector an exact cancellation of pure gauge and ghost degrees of freedom occurs. Here we do not consider higher derivative in the gauge fixind, so we do not need the third ghost term, contrary to the previous case.

We now follow the same steps of the previous chapter but adopting a different metric decomposition

$$h_{\mu\nu} = h^T_{\mu\nu} + \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu + \bar{\nabla}_\mu\bar{\nabla}_\nu\sigma + \frac{1}{d}\bar{g}_{\mu\nu}\bar{h} \tag{3.7}$$

with the constraints (2.36). The new variables $\bar{h}$ is related with the trace h by $\bar{h} = h - \bar{\nabla}^2\sigma$. The last difference with the previous treatment is the introduction of auxiliary fields. In previous sections, the role of Jacobians is taken into account directly in the calculation of functional trace. In paper [6], Jacobians are exponentiated with the trick introduced in 2.2. Hence Jacobian (2.37) for decomposition (3.7) gives the contribution to the total action ($\Delta = -\bar{\nabla}^2$)

$$\begin{aligned}S_{\text{aux-gr}} = \int d^dx\sqrt{\bar{g}}\Big[&2\bar{\chi}^T_\mu\left(\Delta - \frac{R}{d}\right)\chi^{T\mu} + \frac{d-1}{d^2}\bar{\chi}\left(\Delta - \frac{R}{d-1}\right)\Delta\chi\\ &+ 2\zeta^T_\mu\left(\Delta - \frac{R}{d}\right)\zeta^{T\mu} + \frac{d-1}{d^2}\zeta\left(\Delta - \frac{R}{d-1}\right)\Delta\zeta\end{aligned} \tag{3.8}$$

where χ^T and χ are Grassmann valued fields while ζ^T and ζ are real commuting fields. The Jacobian (2.38), belonging to the ghosts decomposition, gives

$$S_{\text{aux-gh}} = \frac{1}{2}\int d^dx\sqrt{\bar{g}}\phi\Delta\phi \tag{3.9}$$

We proceed in the construction of FRG equation with this decomposition. First, the calculation of a new second variation for the average effective action. Clearly, the transverse tensor and transverse vector components is the same of (2.51) and (2.52) for $\beta = 0$ (from now on all metric and Riemann tensor refer to the background)

$$\tilde{\Gamma}^{(2)}_{h^T_{\mu\nu}h^T_{\alpha\beta}} = -\frac{Z_{k,h}}{2}\left[-f_k'\left(\nabla^2 - \frac{2}{d(d-1)}R\right) + \left(f_k - \frac{2}{d}Rf_k'\right)\right]\delta^{\mu\nu,\alpha\beta} \tag{3.10}$$

$$\tilde{\Gamma}^{(2)}_{\xi_\mu\xi_\nu} = Z_{k,h}\left(\nabla^2 + \frac{R}{d}\right)\left[\frac{1}{\alpha}\left(\nabla^2 + \frac{R}{d}\right) - \left(\frac{2R}{d}f_k' - f_k\right)\right]g^{\mu\nu} \tag{3.11}$$

To distinguish from the previous $\mathbf{\Gamma}^{(2)}$ we add a tilde on new variations.

Since the decomposition (3.7) is different from (2.23) for the scalar part we rewrite the quadratic form $h^{\mu\nu}\Gamma^{(2)}_{\mu\nu\alpha\beta}h^{\alpha\beta}$, obtained in section 2.3, for the scalar contribution (2.53-2.54-2.55) and then substitute $h = \bar{h} + \nabla^2\sigma$

$$\begin{aligned} h\Gamma^{(2)}h + 2h\Gamma^{(2)}_{h\sigma}\sigma + \sigma\Gamma^{(2)}_{\sigma\sigma}\sigma = \sigma &\left[\Gamma^{(2)}_{\sigma\sigma} + \quad 2\Gamma^{(2)}_{h\sigma}\nabla^2 + \Gamma^{(2)}_{hh}(\nabla^2)^2\right]\sigma \\ &+ 2\bar{h}\left[\Gamma^{(2)}_{h\sigma} + \Gamma^{(2)}_{hh}\nabla^2\right]\sigma + \bar{h}\Gamma^{(2)}_{hh}\bar{h} \\ &\equiv \sigma\tilde{\Gamma}^{(2)}_{\sigma\sigma}\sigma + 2\bar{h}\tilde{\Gamma}^{(2)}_{\bar{h}\sigma}\sigma + \bar{h}\tilde{\Gamma}^{(2)}_{\bar{h}\bar{h}}\bar{h} \end{aligned}$$

Hence the resulting second variation for decomposition (3.7) in the scalar sector gives

$$\begin{aligned} \tilde{\Gamma}^{(2)}_{\bar{h}\bar{h}} =& Z_{k,h}\frac{(d-2)}{4d}\left[\frac{4(d-1)^2}{d(d-2)}f_k''(R)\left(\nabla^2 - \frac{R}{d-1}\right)^2 + \frac{2(d-1)}{d}f_k'(R)\left(-\nabla^2 - \frac{R}{d-1}\right)\right. \\ &\left. - \frac{2R}{d}f_k'(R) + f_k(R)\right] \end{aligned} \tag{3.12}$$

$$\tilde{\Gamma}^{(2)}_{\bar{h}\sigma} = -Z_{k,h}\frac{(d-2)}{2d^2}\left(Rf_k' - \frac{d}{2}f_k\right)\nabla^2 \tag{3.13}$$

$$\tilde{\Gamma}^{(2)}_{\sigma\sigma} = -Z_{k,h}\frac{(d-1)^2}{d^2\alpha}\left(\nabla^2 + \frac{R}{d-1}\right)^2\nabla^2 + \frac{1}{2d}\left(Rf_k' - \frac{d}{2}f_k\right)\left(\nabla^2 + \frac{2R}{d}\right)\nabla^2 \tag{3.14}$$

As in the previous case, only $\tilde{\Gamma}^{(2)}_{\sigma\sigma}$ and $\tilde{\Gamma}^{(2)}_{\xi\xi}$ depends on gauge parameter α. So in the limit $\alpha \to 0$ the 2×2 scalar part of $\tilde{\mathbf{\Gamma}}^{(2)}$ diagonalizes into $\sigma - \sigma$ and $\bar{h} - \bar{h}$ contributions as we have seen in 2.5.

In previous sections, the cutoff scheme was based on replecement rule

$$\Delta \to P_k(\Delta) = \Delta + r_k(\Delta) \tag{3.15}$$

which is encoded into the general cutoff scheme

$$\mathbf{\Gamma}^{(2)}_k(\Delta) + \boldsymbol{\mathcal{R}}_k(\Delta) = \mathbf{\Gamma}^{(2)}_k(P_k(\Delta)) \tag{3.16}$$

With this choice, in the previous chapter, we realized that unphysical singularities appear in the final equation (2.105). To understand why this unphysical infinities appear, consider for example the vector trace part in (2.5). Inside the $\Gamma^{(2)}_{\xi\xi}$ in (2.52), only the operator $\Delta - \frac{R}{d}$ appears after the gauge choice. With cutoff scheme (3.16) and with Litim's optimized cutoff the replecement rule (3.15) becomes ($P_k \sim k^2$)

$$\Delta - \frac{R}{d} \to k^2 - \frac{R}{d}$$

which is zero, and so not invertible, for $\tilde{R} = d$. The same conclusion is valid for the scalar part where operator $\Delta - \frac{R}{d-1}$ appears. To avoid this unphysical singularities, following [6], we introduce three new operators

$$\Delta_0 \equiv \Delta - \frac{R}{d-1} \qquad P_k^{(0)}(\Delta_0) \equiv \Delta_0 + r_k(\Delta_0) \tag{3.17}$$

$$\Delta_1 \equiv \Delta - \frac{R}{d} \qquad P_k^{(1)}(\Delta_1) \equiv \Delta_1 + r_k(\Delta_1) \tag{3.18}$$

$$\Delta_2 \equiv \Delta + \frac{2R}{d(d-1)} \qquad P_k^{(2)}(\Delta_2) \equiv \Delta_2 + r_k(\Delta_2) \tag{3.19}$$

To obtain a new cutoff scheme, we consider $\mathbf{\Gamma}_k^{(2)}$ as a functions only of operators (3.17-3.18-3.19) and then apply (3.16). This corresponds to a slightly different pattern of coarse-graining.

Hence the corresponding regulator functions in the limit $\alpha \to 0$ becomes

$$\mathcal{R}_{h^T_{\mu\nu}h^T_{\alpha\beta}}(\Delta_2) = -\frac{Z_{k,h}}{2} f_k' r_k(\Delta_2)\delta^{\mu\nu,\alpha\beta}$$

$$\mathcal{R}_{\xi_\mu\xi_\nu} = \frac{Z_{k,h}}{\alpha}\left[\left(P_k^{(1)}(\Delta_1)\right)^2 - \Delta_1^2\right]\bar{g}^{\mu\nu}$$

$$\mathcal{R}_{\bar{h}\bar{h}}(\Delta_0) = Z_{k,h}\frac{(d-1)}{d^2} f_k''\left[\left(P_k^{(0)}(\Delta_0)\right)^2 - \Delta_0^2\right] + Z_{k,h}\frac{(d-2)(d-1)}{2d^2} f_k' r_k(\Delta_0)$$

$$\mathcal{R}_{\sigma\sigma} = Z_{k,h}\frac{(d-1)^2}{\alpha d^2}\left[\left(P_k^{(0)}(\Delta_0)\right)^2\left(P_k^{(0)}(\Delta_0) + \frac{R}{d-1}\right) - \Delta_0^2\left(\Delta_0 + \frac{R}{d-1}\right)\right]$$

and for the ghosts and auxiliary fields part becomes

$$\mathcal{R}_{\bar{C}^T_\mu C^T_\nu} = \frac{Z_{k,c}}{\alpha}\left[\left(P_k^{(1)}(\Delta_1)\right)^2 - \Delta_1^2\right]\bar{g}^{\mu\nu}$$

$$\mathcal{R}_{B^T_\mu B^T_\nu} = \frac{Z_{k,b}}{\alpha}\left[\left(P_k^{(1)}(\Delta_1)\right)^2 - \Delta_1^2\right]\bar{g}^{\mu\nu}$$

$$\mathcal{R}_{\bar{\chi}^T_\mu \chi^T_\nu} = \mathcal{R}_{\zeta^T_\mu\zeta^T_\nu} = 2\left[\left(P_k^{(1)}(\Delta_1)\right)^2 - \Delta_1^2\right]\bar{g}^{\mu\nu}$$

$$\mathcal{R}_{\bar{c}c} = \frac{4Z_{k,c}}{\alpha}\frac{(d-1)^2}{d^2}\left[\left(P^{(0)}(\Delta_0)\right)^2\left(P_k^{(0)}(\Delta_0) + \frac{R}{d-1}\right) - \Delta_0^2\left(\Delta_0 + \frac{R}{d-1}\right)\right]$$

$$\mathcal{R}_{bb} = \frac{4Z_{k,b}}{\alpha}\frac{(d-1)^2}{d^2}\left[\left(P^{(0)}(\Delta_0)\right)^2\left(P_k^{(0)}(\Delta_0)+\frac{R}{d-1}\right)-\Delta_0^2\left(\Delta_0+\frac{R}{d-1}\right)\right]$$
$$\mathcal{R}_{\bar{\chi}\chi} = \mathcal{R}_{\zeta\zeta} = \frac{d-1}{d^2}\left[P^{(0)}(\Delta_0)\left(P_k^{(0)}(\Delta_0)+\frac{R}{d-1}\right)-\Delta_0\left(\Delta_0+\frac{R}{d-1}\right)\right]$$

and finally for the scalar auxiliary field ϕ, simply

$$\mathcal{R}_{\phi\phi} = r_k(\Delta_0)$$

We now proceed in the calculation of trace contributions starting from the tensor trace part which gives

$$\frac{1}{2}\,\mathrm{Tr}_{(2)}\left[\frac{\partial_t \mathcal{R}_{h^T h^T}}{\tilde{\Gamma}^{(2)}_{h^T h^T}+\mathcal{R}_{h^T h^T}}\right]$$

introducing dimensionless variables

$$R = k^2\tilde{R} \qquad f_k(R) = k^2\tilde{f}_k(R/k^2)$$

which imply

$$f_k'(R) = \tilde{f}_k'(R/k^2) \qquad f_k''(R) = k^{-2}\tilde{f}_k''(R/k^2) \tag{3.20}$$
$$\partial_t f_k(R) = k^2\left[\partial_t\tilde{f}_k(R/k^2)-2\tilde{R}\tilde{f}_k'(R/k^2)+2\tilde{f}_k(R/k^2)\right]$$
$$\partial_t f_k'(R) = \left[\partial_t\tilde{f}_k'(R/k^2)-2\tilde{R}\tilde{f}_k''(R/k^2)\right] \tag{3.21}$$
$$\partial_t f_k''(R) = k^{-2}\left[\partial_t\tilde{f}_k''(R/k^2)-2\tilde{R}\tilde{f}_k'''(R/k^2)-2\tilde{f}_k''(R/k^2)\right]$$

we obtain the total tensor trace contribution

$$\frac{1}{2}\,\mathrm{Tr}_{(2)}\left[\frac{\left(1-\frac{\Delta_2}{k^2}\right)\left(\partial_t\tilde{f}_k'-2\tilde{R}\tilde{f}_k''-\eta_{k,h}\tilde{f}_k'\right)+2\tilde{f}_k'}{\tilde{f}_k'+\tilde{f}_k-\frac{2}{d}\tilde{R}\tilde{f}_k'}\right]$$

where the optimized cutoff $r_k(\Delta_i) = (k^2-\Delta_i)\theta(k^2-\Delta_i)$ is used.

The vector trace contributions come from ξ, $\bar{C}^T$, C^T, B^T, $\bar{\chi}^T$, χ^T and ζ^T fields. First we note that pure gauge ξ cancel almost exactly with $\bar{C}^T C^T$ and B^T contributions

$$2\,\mathrm{Tr}_{(1)}^{'}\left[\frac{\partial_t r_k(\Delta_1)}{P_k^{(1)}(\Delta_1)}\right]\Big(\underbrace{\frac{1}{2}}_{\xi}\;\underbrace{-1}_{\bar{C}^T C^T}+\underbrace{\frac{1}{2}}_{B^T}\Big) = 0$$

Only the anomalous dimensions contributions remain

$$\frac{1}{2}\left(2\eta_{k,c}-\eta_{k,h}-\eta_{k,b}\right)\mathrm{Tr}_{(1)}^{'}\,\theta(k^2-\Delta_1) \tag{3.22}$$

This conclusion arises from the particular choice for ghosts sector, as analized at the beginning of this chapter.

The remaining spin 1 contribution, related to the auxiliary fields, gives

$$-\frac{1}{2}\,\mathrm{Tr}'_{(1)}\left[\frac{\partial_t r_k(\Delta_1)}{P_k^{(1)}(\Delta_1)}\right] = -\,\mathrm{Tr}'_{(1)}\left[\frac{k^2\theta(k^2-\Delta_1)}{\Delta_1+(k^2-\Delta_1)\theta(k^2-\Delta_1)}\right] = -\,\mathrm{Tr}'_{(1)}\,\theta(k^2-\Delta_1)$$

From now on, following notations in [6], we devide the total scalar contribution into np (non-physical) and $\bar{h}$ contributions; σ, $\bar{c}$, c, b, $\bar{\chi}$, χ, ζ and ϕ fields give the np trace part, the remaining $\bar{h}$ forms the "physical" contribution.

Let us start with non physical scalar contribution. First note that, once again, the component σ cancels almost exactly with the ghosts scalar degrees of freedom $\bar{c}$, c and b

$$\mathrm{Tr'}_{(0)}\left[\frac{\partial_t\left[\left(P_k^{(0)}(\Delta_0)\right)^2\left(P_k^{(0)}(\Delta_0)+\frac{R}{d-1}\right)\right]}{\left(P_k^{(0)}(\Delta_0)\right)^2\left(P_k^{(0)}(\Delta_0)+\frac{R}{d-1}\right)}\right]\Big(\underbrace{\frac{1}{2}}_{\sigma}-\underbrace{1}_{\bar{c}c}+\underbrace{\frac{1}{2}}_{b}\Big)=0$$

Only the anomalous dimensions contributions still remain

$$\frac{1}{2}\,(2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})\,\mathrm{Tr'}_{(0)}\,\theta(k^2-\Delta_0) \tag{3.23}$$

The scalar $\bar{\chi}$, χ and ζ fields give

$$-\frac{1}{2}\,\mathrm{Tr'}_{(0)}\left[\frac{\partial_t\left[P_k^{(0)}(\Delta_0)\left(P_k^{(0)}(\Delta_0)+\frac{R}{d-1}\right)\right]}{P_k^{(0)}(\Delta_0)\left(P_k^{(0)}(\Delta_0)+\frac{R}{d-1}\right)}\right]$$

and the remaining trace contribution for scalar ϕ field reads

$$\frac{1}{2}\,\mathrm{Tr'}_{(0)}\left[\frac{\partial_t r_k(\Delta_0)}{P_k^{(0)}(\Delta_0)+\frac{R}{d-1}}\right]$$

Adding all together, we obtain the total scalar non physical trace part

$$\begin{aligned}&-\frac{1}{2}\,\mathrm{Tr'}_{(0)}\left[\frac{\partial_t r_k(\Delta_0)}{P_k^{(0)}(\Delta_0)}\right]+\frac{1}{2}\,(2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})\,\mathrm{Tr''}_{(0)}\,\theta(k^2-\Delta_0)\\ &=-\,\mathrm{Tr'}_{(0)}\left[\theta(k^2-\Delta_0)\right]+\frac{1}{2}\,(2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})\,\mathrm{Tr''}_{(0)}\,\theta(k^2-\Delta_0)\end{aligned}$$

The scalar $\bar{h}$ contribution are more involved, using (3.20) and (3.21) we find

$$\mathrm{Tr}_{(0)}\left[\theta(k^2-\Delta_0)W_0^{\bar{h}}(\Delta_0/k^2)\right]$$

where

$$
\begin{aligned}
W_0^{\bar{h}}(z) = & \frac{1}{\left[2(d-1)^2\tilde{f}_k'' + (d-2)\tilde{f}_k'\big((d-1)-\tilde{R}\big) + \frac{d(d-2)}{2}\tilde{f}_k\right]} \\
& \times \left\{(d-1)^2(1-z^2)\left[\partial_t\tilde{f}_k'' - 2\tilde{R}\tilde{f}_k''' - 2\tilde{f}_k'' - \eta_{k,h}\tilde{f}_k''\right]\right. \\
& + (1-z)\frac{(d-1)(d-2)}{2}\left[\partial_t\tilde{f}_k' - 2\tilde{R}\tilde{f}_k'' - \eta_{k,h}\tilde{f}_k'\right] \\
& \qquad\qquad \left. + (d-1)\left[(d-2)\tilde{f}_k' + 4(d-1)\tilde{f}_k''\right]\right\}
\end{aligned}
\tag{3.24}
$$

At this level, the functional RG equation becomes

$$
\begin{aligned}
\partial_t\Gamma_k = & \mathrm{Tr}_{(2)}\left[\theta(k^2-\Delta_2)W_2(\Delta_2/k^2)\right] + \mathrm{Tr}'_{(1)}\left[\theta(k^2-\Delta_1)W_1(\Delta_1/k^2)\right] \\
& + \mathrm{Tr}''_{(0)}\left[\theta(k^2-\Delta_0)W_0^{np}(\Delta_0/k^2)\right] + \mathrm{Tr}_{(0)}\left[\theta(k^2-\Delta_0)W_0^{\bar{h}}(\Delta_0/k^2)\right] \\
& + \frac{1}{2}\left(2\eta_{k,c} - \eta_{k,h} - \eta_{k,b}\right)\left(\mathrm{Tr}'_{(1)}\,\theta(k^2-\Delta_1) + \mathrm{Tr}''_{(0)}\,\theta(k^2-\Delta_2)\right)
\end{aligned}
\tag{3.25}
$$

where

$$
W_2(z) = \frac{(z-1)\left(\partial_t\tilde{f}_k' - 2\tilde{R}\tilde{f}_k'' - \eta_{k,h}\tilde{f}_k'\right) - 2\tilde{f}_k'}{\left(\frac{4}{d}\tilde{R} - 2\right)\tilde{f}_k' - 2\tilde{f}_k}
$$

$$
W_0^{np}(z) = W_1(z) = -1
$$

and $W_0^{\bar{h}}(z)$ is given in (3.24).

3.1 Traces as spectral sums

In the last chapter we used the Heat Kernel technique for the evaluation of traces in functional RG equation, given in Appendix C. In this chapter we use an alternative calculation of traces which enables us to extend, in a natural manner, the exact RG equation in ansatz (3.1) to general d spacetime dimension. This new method, used in [6], is based on "sums over eigenvalues" in trace calculation and relies on the fact that we have choosen for the background manifold a sphere.

Consider a function $W(\Delta_s)$ with $i = 0, 1, 2$, the definition of functional trace reads

$$
\mathrm{Tr}\, W(\Delta_s) = \sum_{n=n_s}^{+\infty} W(\lambda_{n,s}) D_{n,s}
$$

where $\lambda_{n,s}$ and $D_{n,s}$ with $s = 0, 1, 2$ are the eigenvalues and relative multiplicities for operators (3.17-3.18-3.19), given in Table 3.1.

Operator	Eigenvalue $\lambda_{n,s}$	Multiplicity $D_{n,s}$
$\Delta_0 = \Delta - \frac{R}{d}$	$\frac{n(n+d-1)-d}{d(d-1)}R;\ n = 0, 1...$	$\frac{(2n+d-1)(n+d-2)!}{n!(d-1)!}$
$\Delta_1 = \Delta - \frac{R}{d-1}$	$\frac{n(n+d-1)-d}{d(d-1)}R;\ n = 1, 2...$	$\frac{n(n+d-1)(2n+d-1)(n+d-3)!}{(d-2)!(n+1)!}$
$\Delta_2 = \Delta + \frac{2R}{d(d-1)}$	$\frac{n(n+d-1)}{d(d-1)}R;\ n = 2, 3...$	$\frac{(d+1)(d-2)(n+d)(n-1)(2n+d-1)(n+d-3)!}{2(d-1)!(n+1)!}$

Table 3.1: Eigenvalue and their multiplicities of operators (3.17-3.18-3.19) on a d-sphere

However in equation (3.25) we have a theta function in the trace argument

$$\mathrm{Tr}\left[\theta(k^2 - \Delta_s)W(\Delta_s)\right] \tag{3.26}$$

The theta function enables us to truncate the sum to $\tilde{N}_s(\tilde{R})$, with

$$\tilde{N}_s(\tilde{R}) \equiv \max\{n \in \mathcal{N}; \lambda_{n,s} \leq k^2\} = \lfloor N_s(\tilde{R}) \rfloor \tag{3.27}$$

where, clearly, $N_s(\tilde{R})$ is such that $\lambda_{N_s,s} = k^2$ and $x \to \lfloor x \rfloor$ is the Floor function. So trace (3.26) becomes

$$\mathrm{Tr}\left[\theta(k^2 - \Delta_s)W(\Delta_s)\right] = \sum_{n=n_s}^{\tilde{N}_s(\tilde{R})} W(\lambda_{n,s})D_{n,s}$$

By definition (3.27) can be obtained from conditions $\lambda_{N_s,s} = k^2$, which give

$$N_2(\tilde{R}) = \frac{(1-d)\tilde{R} + \sqrt{(d-1)^2\tilde{R}^2 + 4\tilde{R}d(d-1)}}{2\tilde{R}} \tag{3.28}$$

$$N_0(\tilde{R}) = N_1(\tilde{R}) = \frac{(1-d)\tilde{R} + \sqrt{(d+1)^2\tilde{R}^2 + 4\tilde{R}d(d-1)}}{2\tilde{R}} \tag{3.29}$$

$$\tag{3.30}$$

Hence, the functional RG equation (3.25) becomes

$$\begin{aligned}
\partial_t \Gamma_k = & \sum_{n=2}^{\tilde{N}_2(\tilde{R})} W_2(\lambda_{n,2}/k^2)D_{n,2} + \sum_{n=2}^{\tilde{N}_1(\tilde{R})} W_1(\lambda_{n,1}/k^2)D_{n,1} \\
& + \sum_{n=1}^{\tilde{N}_0(\tilde{R})} W_0^{np}(\lambda_{n,0}/k^2)D_{n,0} + \sum_{n=0}^{\tilde{N}_0(\tilde{R})} W_0^{\bar{h}}(\lambda_{n,0}/k^2)D_{n,0} \\
& + \frac{1}{2}\left(2\eta_{k,c} - \eta_{k,h} - \eta_{k,b}\right)\left(\sum_{n=2}^{\tilde{N}_1(\tilde{R})} D_{n,1} + \sum_{n=1}^{\tilde{N}_0(\tilde{R})} D_{n,0}\right)
\end{aligned}$$

Note that tensor and "physical" scalar sums start from $n = 2$ and $n = 0$ as expected. Vector trace part starts from $n = 2$ instead of $n = 1$ since we must exclude modes relative to Killing vectors, which satisfy $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$. All $d(d-1)/2$ (in S^d) Killing vectors are collected into $n = 1$ eigenvectors of Δ_1. The scalar non-physical contribution starts from $n = 1$, the $n = 0$ eigenmode must be excluded since $\sigma = \text{const}$ and $\nabla_\mu \sigma$ do not contribute to metric fluctuations.

We analize trace calculation in details in the following sections.

Tensor trace contribution

For tensor trace contribution we have to evaluate the following sums

$$S_{t1}(N) = \sum_{n=2}^{N} D_{n,2} \tag{3.31}$$

$$S_{t2}(N) = \sum_{n=2}^{N} \left(\frac{\lambda_{n,2}}{k^2} - 1 \right) D_{n,2} \tag{3.32}$$

Let us start with the first sum

$$S_{t1}(N) = \sum_{n=2}^{N} \frac{(d+1)(d-2)(n+d)(n-1)(2n+d-1)(n+d-3)!}{2(d-1)!(n+1)!} \tag{3.33}$$

The exact value for this is

$$S_{t1}^{(exact)}(\tilde{R}) = S_{t1}(\lfloor N_2(\tilde{R}) \rfloor) \tag{3.34}$$

which is really difficult to be evaluated, so an approximation method is needed. It has been proposed in [6] to make the following approximation

$$S_{t1}^{(asymp)}(\tilde{R}) = S_{t1}^{(0)}(\tilde{R}) + S_{t1}^{(\infty)}(\tilde{R}) \tag{3.35}$$

where the two contributions $S_{t1}^{(0)}(\tilde{R})$ and $S_{t1}^{(\infty)}(\tilde{R})$ represents the asymptotic behaviour of $S_{t1}(N_2(\tilde{R}))$ for $\tilde{R} \to 0$ and $\tilde{R} \to \infty$ respectively.

In order to evaluate this two contributions we need the asymptotic behaviour of $N_2(\tilde{R})$, which can be obtained from (3.28)

$$N_2(\tilde{R}) \overset{\tilde{R}\to 0}{=} \left(\frac{d(d-1)}{\tilde{R}} \right)^{\frac{1}{2}} \tag{3.36}$$

$$N_2(\tilde{R}) \overset{\tilde{R}\to +\infty}{=} 0 \tag{3.37}$$

Now, using the limit $N_2(\tilde{R} \to 0) \to +\infty$, we can approximate sum (3.33) to evaluate $S_{t1}^{(0)}(\tilde{R})$

$$S_{t1}^{(0)}(\tilde{R}) \overset{\tilde{R}\to 0}{=} \frac{(d+1)(d-2)}{2(d-1)!} \sum_{n=2}^{N_2(\tilde{R})} n^{d-1} \overset{\tilde{R}\to 0}{=} \frac{(d+1)(d-2)}{2(d-1)!} (N_2(\tilde{R}))^d$$
$$= \frac{(d+1)(d-2)}{2(d-1)!} \left(\frac{d(d-1)}{\tilde{R}} \right)^{\frac{d}{2}}$$

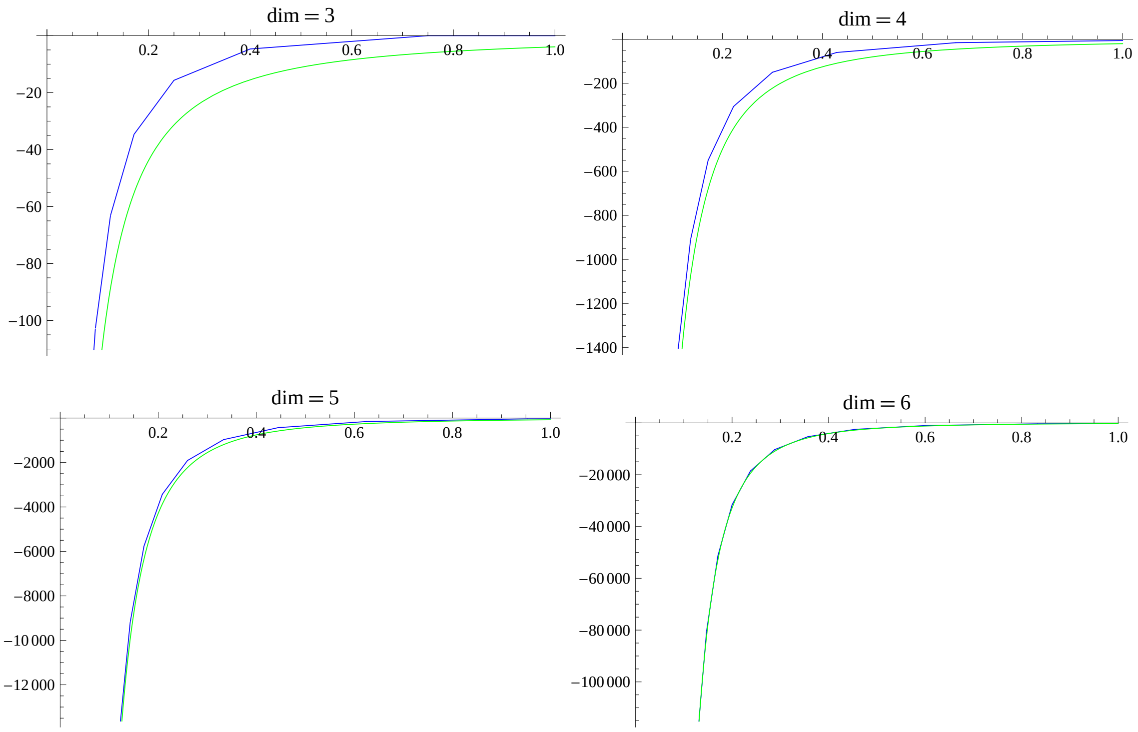


Figure 3.1: $S_{t2}^{(exact)}$ in blue, $S_{t2}^{(asymp)}$ in green for $d = 3$, $d = 4$, $d = 5$ and $d = 6$

Instead, $S_{t1}^{(\infty)}(\tilde{R})$ vanishes as a consequence of (3.37). Remember that for large N

$$\sum_{n=1}^{N} n^k \overset{N \to +\infty}{=} \frac{N^{k+1}}{k+1}$$

So, in the asymptotic approximation sum over multiplicities reads

$$\sum_{n=2}^{N_2(\tilde{R})} D_{n,2} = \frac{(d+1)(d-2)}{2(d-1)!} \left(\frac{d(d-1)}{\tilde{R}} \right)^{\frac{d}{2}}$$

Let us proceed to the $S_{t2}(N)$, which is quite different and can be devided in

$$S_{t2}(N) = \sum_{n=2}^{N} \frac{\lambda_{n,2}}{k^2} D_{n,2} - S_{t1}(N)$$

So we analize the first contribution on the right hand side, which reads

$$\sum_{n=2}^{N_2(\tilde{R})} \frac{(d+1)(d-2)n(n+d-1)(n+d)(n-1)(2n+d-1)(n+d-3)!}{2d!(d-1)(n+1)!} \tilde{R}$$

Following the same procedure of previous case, this sum can be approximated by

$$\frac{(d+1)(d-2)}{2(d-1)!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}}$$

Hence, the asymptotic approximation for sum (3.32) is given by

$$S_{t2}^{(asymp)}(\tilde{R}) = -2\frac{(d+1)(d-2)}{d(d+2)(d-1)!}\left(\frac{d(d-1)}{R}\right)^{\frac{d}{2}} \tag{3.38}$$

At this level, an analisys of the quality of the approximation made is needed. We have implemented a Mathematica program, in which the numerical calculation for the exact summation

$$S_{t2}^{(exact)}(\tilde{R}) = S_{t2}(\lfloor N_2(\tilde{R})\rfloor) \tag{3.39}$$

is done. Picture 3.1 plots $S_{t2}^{(exact)}(\tilde{R})$ in blue and $S_{t2}^{(asymp)}(\tilde{R})$ in green in different dimension, $d=3, d=4, d=5, d=6$. In all cases, the assumption approximates the exact result in a good way, giving an almost perfect smooth approximation.

Hence the total tensor trace gives the contribution

$$\begin{aligned}\sum_{n=2}^{N_2(\tilde{R})} W_2(\lambda_{n,2}/k^2, \tilde{R}) &\overset{(asymp)}{=} -2\frac{(d+1)(d-2)}{d(d+2)(d-1)!} \\ &\times \left[\frac{\partial_t \tilde{f}'_k - 2\tilde{R}\tilde{f}''_k - \eta_{k,h}\tilde{f}'_k + (d+2)\tilde{f}'_k}{\left(\frac{4}{d}\tilde{R}-2\right)\tilde{f}'_k - 2\tilde{f}_k}\right]\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}}\end{aligned}$$

Vector trace contribution

The vector, as scalar part, trace contribution is quite simply since the only sum needed is the following

$$S_v(N) = \sum_{n=2}^{N} D_{n,1} = \sum_{n=2}^{N}\frac{n(n+d-1)(2n+d-1)(n+d-3)!}{(d-2)!(n+1)!} \tag{3.40}$$

which give the exact value

$$S_v^{(exact)}(\tilde{R}) = S_v(\lfloor N_1(\tilde{R})\rfloor) \tag{3.41}$$

and using the asymptotic approximation we have

$$S_v^{(asymp)}(\tilde{R}) = \frac{2}{d(d-2)!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} \tag{3.42}$$

Note that the behaviour for $\tilde{R} \to +\infty$ implies $N_1(\tilde{R}) \to 1$ (see (3.29)), but sum (3.40) starts from $n=2$, so this limit does not give contribution.

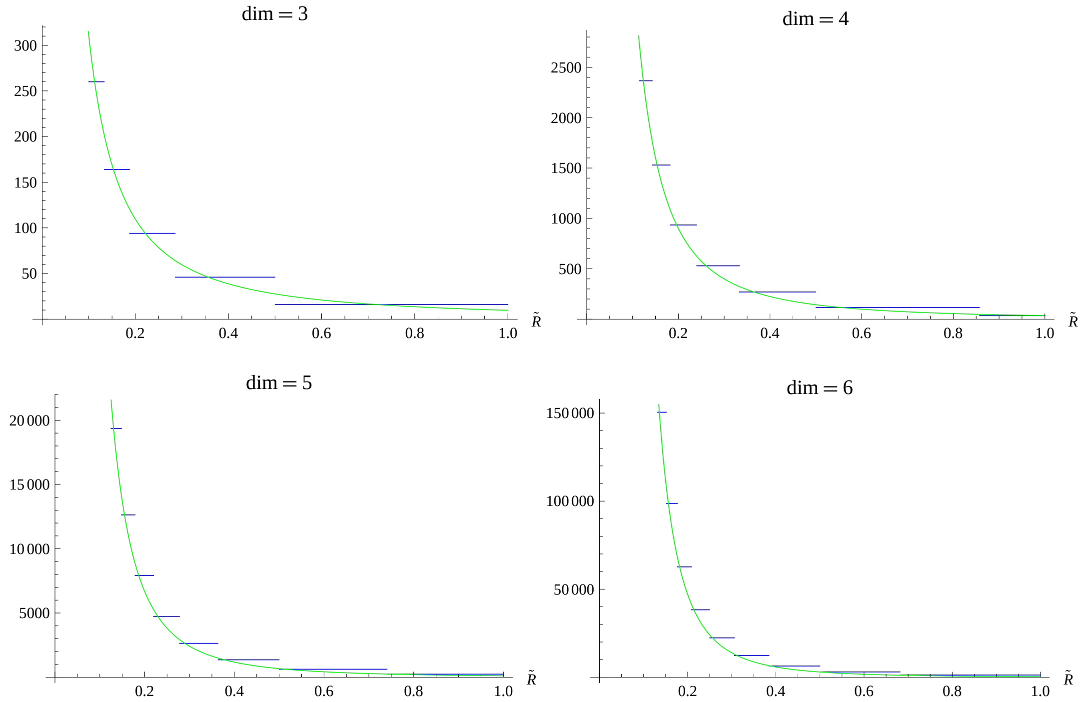


Figure 3.2: $S_v^{(exact)}$ in blue, $S_v^{(asymp)}$ in green for $d = 3$, $d = 4$, $d = 5$ and $d = 6$

A comparison between the exact value $S_v^{(exact)}(\tilde{R})$ and approximated one $S_v^{(asymp)}(\tilde{R})$ is given in figure 3.2. As in the previous case, the agreement between the exact and asymptotic approximation is excellent.

So the total vector trace is given by

$$\sum_{n=2}^{N_1(\tilde{R})} W_1(\lambda_{n,1}/k^2, \tilde{R}) = -\frac{2}{d(d-2)!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}}$$

"Non-physical" scalar contribution

As in the vector case, the "non-physical" scalar contribution is given by the simple sum over the multiplicities

$$S_{np}(N) = \sum_{n=1}^{N} D_{n,0} = \sum_{n=1}^{N} \frac{(n+d-2)!(2n+d-1)}{n!(d-1)!}$$

which can be written without approximation as

$$S_{np}^{(exact)}(\tilde{R}) = S_{np}(\lfloor N_0(\tilde{R})\rfloor)$$

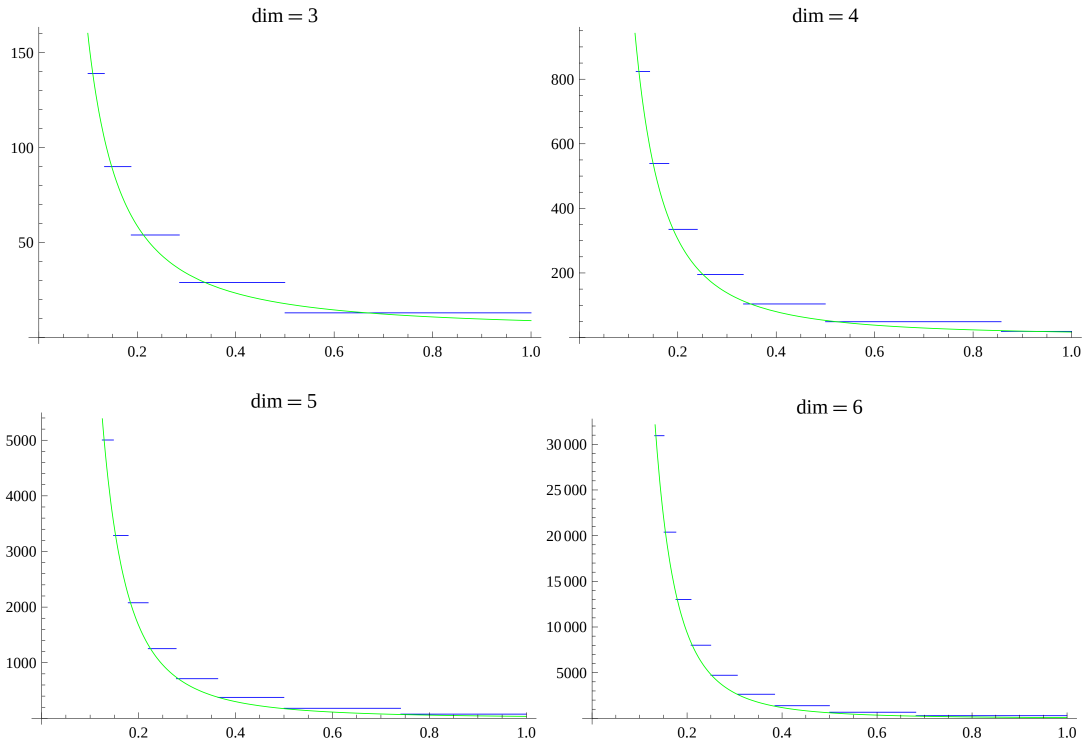


Figure 3.3: $S_{np}^{(exact)}$ in blue, $S_{np}^{(asymp)}$ in green for $d = 3$, $d = 4$, $d = 5$ and $d = 6$

The N_0 function has the following asymptotic behaviours

$$N_0(\tilde{R}) \overset{\tilde{R}\to 0}{=} \left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{1}{2}}$$
$$N_0(\tilde{R}) \overset{\tilde{R}\to+\infty}{=} 1$$

which give the following asymptotic approximation

$$\begin{aligned} S_{np}^{(asymp)}(\tilde{R}) &= \frac{2}{d!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} + D_{1,0} \\ &= \frac{2}{d!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} + d + 1 \end{aligned} \tag{3.43}$$

A comparison between the exact value $S_{np}^{(exact)}(\tilde{R})$ and approximated one $S_{np}^{(asymp)}(\tilde{R})$ is given in figure 3.3. As in the previous case, the agreement between the exact and asymptotic approximation is very good.

In asymptotic approximation the corresponding scalar trace becomes

$$\sum_{n=1}^{N_0} W_0^{np}(\lambda_{n,0}/k^2, \tilde{R}) D_{n,0} = -\frac{2}{d!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} - d - 1$$

$\bar{h}$-scalar trace contribution

The last contribution involves the scalar field $\bar{h}$. The relative functional trace is composed by the three following contribution

$$S_{\bar{h}1}(N) = \sum_{n=0}^{N} \left(1 - \frac{\lambda_{n,0}}{k^2}\right) D_{n,0} \tag{3.44}$$

$$S_{\bar{h}2}(N) = \sum_{n=0}^{N} \left(1 - \frac{\lambda_{n,0}^2}{k^4}\right) D_{n,0} \tag{3.45}$$

$$S_{\bar{h}3}(N) = \sum_{n=0}^{N} D_{n,0} \tag{3.46}$$

First, note that the last term is analized in previous section where the sum starts from $n = 1$

$$S_{\bar{h}3}(N) = S_{np}(N) + D_{0,0}$$

Hence, the $S_{\bar{h}3}^{(asymp)}$ can be calculated directly from (3.43)

$$S_{\bar{h}3}^{(asymp)}(\tilde{R}) = S_{np}^{(asymp)} + D_{0,0} = \frac{2}{d!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} + d + 2 \tag{3.47}$$

Let us consider the sum $S_{\bar{h}1}$, defined in (3.44). The exact value $S_{\bar{h}1}^{(exact)}$ reads

$$S_{\bar{h}1}^{(exact)}(\tilde{R}) = S_{\bar{h}1}(\lfloor N_0(\tilde{R})\rfloor)$$

For the asymptotic approximation we start deviding (3.44) into

$$S_{\bar{h}1}(N) = S_{\bar{h}3}(N) - \sum_{n=0}^{N} \frac{\lambda_{n,0}}{k^2} D_{n,0}$$

Since we know from (3.47) the asymptotic approximation for the first term on the right hand side, we move to the second contribution and take first the limit for large $N_0(\tilde{R})$

$$\begin{aligned}
\sum_{n=0}^{N_0} \frac{\lambda_{n,0}}{k^2} D_{n,0} &= \sum_{n=0}^{N_0} \frac{[n(n+d-1)-d]}{d(d-1)} \frac{(2n+d-1)(n+d-2)!}{n!(d-1)!}\tilde{R} \\
&\overset{N_0\to+\infty}{=} \frac{2}{d!(d-1)}\tilde{R}\sum_{n=0}^{N_0} n^{d+1} \overset{N_0\to+\infty}{=} \frac{2}{d!(d+2)(d-1)} N_0^{d+2} \\
&= \frac{2}{(d-1)!(d+2)}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}}
\end{aligned} \tag{3.48}$$

and then the limit $\tilde{R} \to +\infty$ (which implies $N_0(\tilde{R}) \simeq 1$)

$$\sum_{n=0}^{N_0\simeq 1} \frac{\lambda_{n,0}}{k^2} D_{n,0} = \frac{\lambda_{0,0}}{k^2} D_{0,0} + \frac{\lambda_{1,0}}{k^2} D_{1,0} = -\frac{\tilde{R}}{d-1}$$

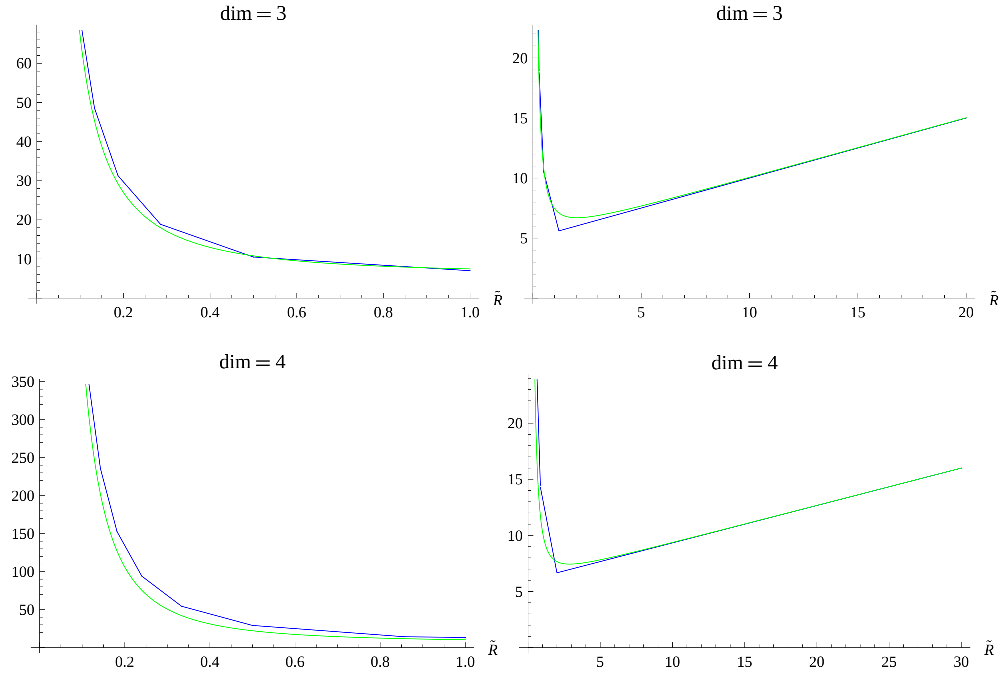


Figure 3.4: $S^{(exact)}_{\bar{h}1}$ in blue, $S^{(asymp)}_{\bar{h}1}$ in green for $d = 3$ and $d = 4$

So we find for the approximated sum $S^{(asymp)}_{\bar{h}1}(\tilde{R})$

$$S^{(asymp)}_{\bar{h}1}(\tilde{R}) = \frac{4}{d(d+2)(d-1)!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} + \frac{\tilde{R}}{d-1} + d + 2$$

A comparison between the exact value $S^{(exact)}_{\bar{h}1}(\tilde{R})$ and approximated one $S^{(asymp)}_{\bar{h}1}(\tilde{R})$ is given in figure 3.4. The pictures show both the behaviour for small range value of $\tilde{R}$ on the left and for a wide range of $\tilde{R}$ on the right. As in the previous case, the agreement between the exact and asymptotic approximation is optimal in both situations.

Now we consider the contribution

$$S_{\bar{h}2}(N) = S_{\bar{h}3} - \sum_{n=0}^{N} \frac{\lambda^2_{n,0}}{k^4} D_{n,0}$$

The second term on the right hand side have the following behaviour for $\tilde{R} \to 0$

$$\begin{aligned}\sum_{n=0}^{N_0} \frac{\lambda_{n,0}^2}{k^4} D_{n,0} &= \sum_{n=0}^{N_0} \frac{[n(n+d-1)-d]^2}{d^2(d-1)^2} \frac{(2n+d-1)(n+d-2)!}{n!(d-1)!} \tilde{R}^2 \\ &\overset{N_0 \to +\infty}{=} \frac{2}{d!d(d-1)^2} \tilde{R} \sum_{n=0}^{N_0} n^{d+3} \overset{N_0 \to +\infty}{=} \frac{2}{d!d(d+4)(d-1)^2} N_0^{d+4} \\ &= \frac{2}{(d-1)!(d+4)} \left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}}\end{aligned}$$

for the opposite limit $\tilde{R} \to +\infty$, so $N_0(\tilde{R}) \simeq 1$, we have

$$\sum_{n=0}^{N_0 \simeq 1} \frac{\lambda_{n,0}^2}{k^4} D_{n,0} = \frac{\lambda_{0,0}^2}{k^4} D_{0,0} + \frac{\lambda_{1,0}^2}{k^4} D_{1,0} = \frac{\tilde{R}^2}{(d-1)^2}$$

Hence, the asymptotic approximation for (3.45) $S_{\bar{h}2}^{(asymp)}$ becomes

$$S_{\bar{h}2}^{(asym)} = \frac{8}{d(d+4)(d-1)!} \left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} - \frac{\tilde{R}^2}{(d-1)^2} + d + 2$$

A comparison between the exact value $S_{\bar{h}2}^{(exact)}(\tilde{R})$ and approximated one $S_{\bar{h}2}^{(asymp)}(\tilde{R})$ is given in figure 3.5. The pictures show both the behaviour for small range value of $\tilde{R}$ on the left and for a wide range of $\tilde{R}$ on the right. As in the previous case, the agreement between the exact and asymptotic approximation is vary good in both situations.

Collecting all contribution we find for the scalar $\bar{h}$ trace part

$$\begin{aligned}T_0^{\bar{h}} \equiv \sum_{n=0}^{\tilde{N}_0(\tilde{R})} W_0^{\bar{h}}(\lambda_{n,0}/k^2) D_{n,0} &= \frac{1}{\left[2(d-1)^2 \tilde{f}_k'' + (d-2)\tilde{f}_k'(d-1-\tilde{R}) + \frac{d(d-2)}{2}\tilde{f}_k\right]} \\ \times \Bigg\{ &\left[\frac{8}{d(d+4)(d-1)!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} - \frac{\tilde{R}^2}{(d-1)^2} + d + 2\right](d-1)^2 \\ &\times \left[\partial_t \tilde{f}_k'' - 2\tilde{R}\tilde{f}_k''' - 2\tilde{f}_k'' - \eta_{k,h}\tilde{f}_k''\right] \\ + \frac{(d-2)(d-1)}{2} &\left[\frac{4}{d(d+2)(d-1)!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} + \frac{\tilde{R}}{d-1} + d + 2\right] \\ &\times \left(\partial_t \tilde{f}_k' - 2\tilde{R}\tilde{f}_k'' - \eta_{k,h}\tilde{f}_k'\right) \\ + (d-1) &\left[\frac{2}{d(d-1)!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} + d + 2\right]\left[(d-2)\tilde{f}_k' + 4(d-1)\tilde{f}_k''\right] \Bigg\}\end{aligned} \tag{3.49}$$

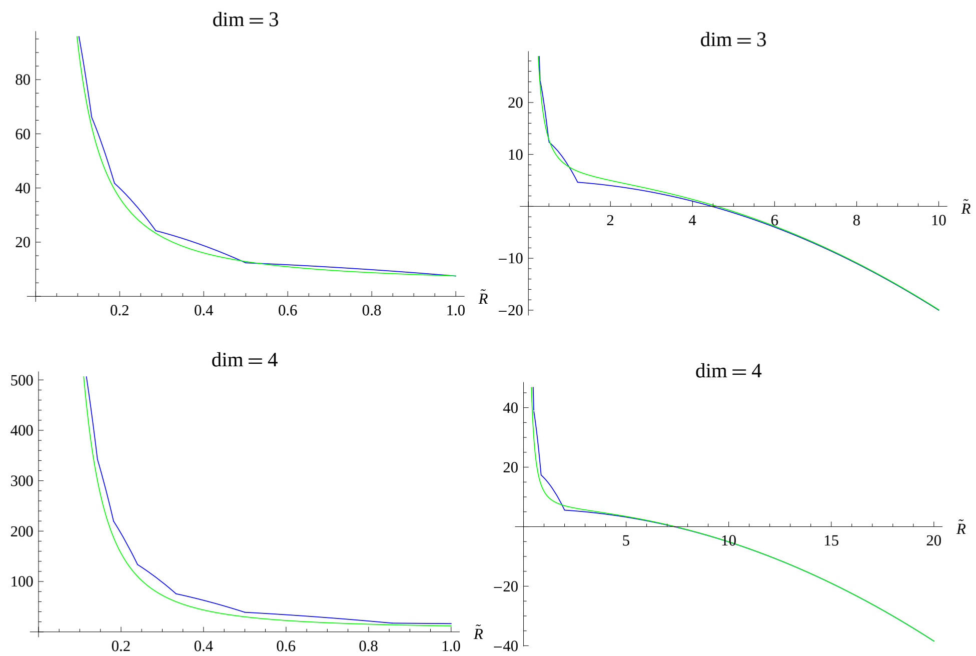


Figure 3.5: $S_{\bar{h}2}^{(exact)}$ in blue, $S_{\bar{h}2}^{(asymp)}$ in green for $d = 3$ andd $d = 4$

Anomalous dimensions trace contributions

Last, we have the anomalous dimensions contributions to the total trace calculation

$$T^{\eta} \equiv +\frac{1}{2}\left(2\eta_{k,c} - \eta_{k,h} - \eta_{k,b}\right)\left(\sum_{n=1}^{\tilde{N}_1(\tilde{R})} D_{n,1} + \sum_{n=1}^{\tilde{N}_0(\tilde{R})} D_{n,0}\right) \tag{3.50}$$

The sums have been evaluated within the asymptotic approximation in previous sections. Equations (3.43,3.42) give

$$T^{\eta} = \frac{1}{2}\left(2\eta_{k,c} - \eta_{k,h} - \eta_{k,b}\right)\left[\frac{2}{(d-1)!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} + d + 1\right] \tag{3.51}$$

3.2 Beta function for $f_k(R)$

Finally, collecting all trace contributions, we find the functional RG equation for $f_k(R)$ approximation with the spectral sums technique

$$\partial_t \Gamma_k = \frac{k^d}{\tilde{\kappa}_k^2} V_{dS}\left[\partial_t \tilde{f}_k(\tilde{R}) - 2\tilde{R}\tilde{f}_k'(\tilde{R}) + \left(d - 2\frac{\partial_t \tilde{\kappa}_k}{\tilde{\kappa}_k}\right)\tilde{f}_k(\tilde{R})\right] = T_2 + T_1 + T_0^{np} + T_0^{\bar{h}} + T^{\eta} \tag{3.52}$$

where we defined

$$T_2 = -2\frac{(d+1)(d-2)}{d(d+2)(d-1)!}\left[\frac{\partial_t \tilde{f}'_k - 2\tilde{R}\tilde{f}''_k + (d+2-\eta_{k,h})\tilde{f}'_k}{\left(\frac{4}{d}\tilde{R}-2\right)\tilde{f}'_k - 2\tilde{f}_k}\right]\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}}$$

$$T_1 = -\frac{2}{d(d-2)!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}}$$

$$T_0^{np} = -\frac{2}{d(d-1)!}\left(\frac{d(d-1)}{\tilde{R}}\right)^{\frac{d}{2}} - d - 1$$

while $T_0^{\bar{h}}$ and T^{η} is defined respectively in (3.49) and (3.51), and last

$$V_{dS} = (4\pi)^{\frac{d}{2}}\left(\frac{d(d-1)}{R}\right)^{\frac{d}{2}}\frac{\Gamma(\frac{d}{2})}{\Gamma(d)}$$

is the volume of a d dimensional sphere.

For completeness we report here the same equation for $d = 4$, recovering the well known spacetime dimensions

$$\begin{aligned}\partial_t\Gamma_k =& \frac{k^4}{\tilde{\kappa}_k^2}V_{4S}\left[\partial_t\tilde{f}_k(\tilde{R}) - 2\tilde{R}\tilde{f}'_k(\tilde{R}) + \left(4 - 2\frac{\partial_t\tilde{\kappa}_k}{\tilde{\kappa}_k}\right)\tilde{f}_k(\tilde{R})\right] \\ =& T_{2,d=4} + T_{1,d=4} + T^{np}_{0,d=4} + T^{\bar{h}}_{0,d=4}\end{aligned} \tag{3.53}$$

where we defined

$$T_{2,d=4} = -\frac{20\left(\partial_t\tilde{f}'_k - 2\tilde{R}\tilde{f}''_k + (6-\eta_{k,h})\tilde{f}'_k\right)}{\tilde{R}^2\left(\left(\tilde{R}-2\right)\tilde{f}'_k - 2\tilde{f}_k\right)}$$

$$T_{1,d=4} = -\frac{36}{\tilde{R}^2}$$

$$T^{np}_{0,d=4} = -\frac{12+5\tilde{R}^2}{\tilde{R}^2}$$

$$\begin{aligned}T^{\bar{h}}_{0,d=4} =& \frac{1}{2\tilde{R}^2\left[9\tilde{f}''_k + \tilde{f}'_k(3-\tilde{R}) + 2\tilde{f}_k\right]}\Big\{\left(-\tilde{R}^4 + 54\tilde{R}^2 + 54\right)\left[\partial_t\tilde{f}''_k - 2\tilde{R}\tilde{f}'''_k - (\eta_{k,h}+2)\tilde{f}''_k\right] \\ &+ \left(\tilde{R}^3 + 18\tilde{R}^2 + 12\right)\left(\partial_t\tilde{f}'_k - 2\tilde{R}\tilde{f}''_k - \eta_{k,h}\tilde{f}'_k\right) + 36\left(\tilde{R}^2+2\right)\left(\tilde{f}'_k + 6\tilde{f}''_k\right)\Big\}\end{aligned}$$

$$T^{\eta}_{d=4} = \frac{1}{2}\left(2\eta_{k,c} - \eta_{k,h} - \eta_{k,b}\right)\left(\frac{48}{\tilde{R}^2+5}\right) \tag{3.54}$$

and last $V_{4S} = 384\pi^2/R^2$ is the volume of a four sphere. Equation (3.53) recovers the previous results appeared in [6].

3.3 Possible closures for $f_k(R)$ RG equation

Equation (3.52) allows us to describe the flow of function $f_k(R)$ and also its dependence on the curvature scalar R. This equation contains fields anomalous dimensions which have to be determined in a different way. As explained in the previous chapter, a possible closure is the following: find anomalous dimensions contribution with flow of $\Gamma_k^{(2)}$. This possibility is a proposal for a future work.

To follow a consistent closure of (3.52), in this work we make two different natural ansatz for the values of $Z_{k,h}$, $Z_{k,c}$ and $Z_{k,b}$. The most simple ansatz, which we call *type I ansatz*, is the following

$$Z_{k,h} = \kappa_k^{-2} \qquad Z_{k,c} = Z_{k,b} = 1$$

which imply

$$\eta_{k,h} = -\frac{\beta_{\tilde{G}}}{\tilde{G}} - 2 \qquad \eta_{k,c} = 0 \qquad \eta_{k,b} = 0 \tag{3.55}$$

where $\kappa_k = \sqrt{16\pi G_k}$ and $\tilde{G} = k^2 G_k$ is the dimensionless Newton's constant. Note that type I ansatz (3.55) imply the following metric decomposition

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

which is the most used definition for quantum flactuations.

Another ansatz for the anomalous dimensions values, which we call type II ansatz, is the following

$$Z_{k,h} = Z_{k,b} = Z_{k,c} = 1$$

which implies

$$\eta_{k,h} = 0 \qquad \eta_{k,c} = 0 \qquad \eta_{k,b} = 0$$

Hence, with type I ansatz, the anomalous dimensions contribution in (3.52) is completly neglected. Note that, within this choice, we do not recover flow equation in [6], since metric decomposition $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa_k h_{\mu\nu}$ and truncation ansatz for Γ_k are different from this work.

3.4 Polynomial truncation

The most simple truncation for the average effective action is the standard Einstein-Hilbert action

$$\Gamma_k[h, \bar{C}, C, b; \bar{g}] = \frac{1}{16\pi G_k} \int d^d x \sqrt{g}(2\Lambda_k - R) + \Gamma_{k,gh} + \Gamma_{k,g.f.} \tag{3.56}$$

where Λ_k is the running cosmological constant. This kind of truncation has been studied with different point of view [12, 18, 19, 20], in diverse cutoff choices [5] and in d spacetime dimension [21]. The choice (3.56) is consistent with previous truncation (3.1) if

$$f_k(R) = 2\Lambda_k - R \qquad \rightarrow \qquad f_k(0) = 2\Lambda_k \qquad f_k'(0) = -1$$

Closure type I: $Z_{k,h} = \kappa_k^{-2}$, $Z_{k,c} = Z_{k,b} = 1$ $\eta_h^* = -2$, $\eta_{k,c} = \eta_{k,b} = 0$								
n	$\tilde{\Lambda}^*$	$\tilde{G}^*$	$\tilde{g}_0$	$\tilde{\kappa}^*$	$\tilde{g}_2$	$\tilde{g}_3$	$\tilde{g}_4$	$\tilde{g}_5$
1	0.2839	0.6616	0.5678	5.7667				
2	0.3038	0.7988	0.6077	6.3365	0.1053			
3	0.3388	0.6593	0.6777	5.7569	0.1377	-0.0705		
4	0.3428	0.6523	0.6857	5.7261	0.1426	-0.0796	-0.0549	
5	0.3414	0.6548	0.6828	5.7368	0.1411	-0.0791	-0.0642	-0.0365

Table 3.2: Couplings value at non-Gaussian fixed point as a functions of the order n of the truncation in $d = 4$.

so one can construct the flow for gravitational and cosmological constants using (3.52).

For a first study on the dimensionality of the critical surface, the average effective action have to be modified introducing more interactions such as R^2, R^3 and so on.

Here, we consider the polynomial truncation, so that the effective action reads

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^d x \sqrt{g} \sum_{i=0}^{n} g_i R^i + \Gamma_{k,gh} + \Gamma_{k,g.f.} \tag{3.57}$$

Clearly, for $n = 1$ we go back to Einstein-Hilbert form. This truncation ansatz is consistent with our initial assumption for the effective action as a function only on curvature scalar. To study the flow equation for the dimensionless couplings $\tilde{g}_i = k^{2-2i} g_i$, we use equation (3.52), which, as pointed out in the last section, necessitate of a closure to be solved for the presence of anomalous dimensions' contribution.

Type I closure: $Z_{k,h} = \kappa_k^{-2}$, $Z_{k,c} = Z_{k,b} = 1$

In table 3.2 the value of dimensionless couplings at the fixed point are reported for $n = 1$ to $n = 5$. First, we see that the value of $\tilde{G}^*$ and $\tilde{\Lambda}^*$ in the Einstein-Hilbert truncation ($n = 1$) agrees with the projection in the $\tilde{\Lambda} - \tilde{G}$ plane of other truncation ($n > 1$). For $n = 2$ we see a deviation of mean value of the fixed point position, already present in other approaches and in previous chapter.

In table 3.3, the critical exponents is reporter as a functions of the truncation. We first note that for $n > 1$ there are only three critical exponents with positive real part. Hence, we found an UV critical surface of dimension three for all $n > 1$; this is an important aspect which tells us that, within polynomial truncation up to $n = 5$, QFT of General Relativity is found to be asymptotically safe. The check that such a property is mantained for higher n is not done here for simplicity.

Closure type I:$Z_{k,h} = \kappa_k^{-2}$, $Z_{k,c} = Z_{k,b} = 1$, $\eta_h^* = -2$, $\eta_{k,c} = \eta_{k,b} = 0$						
n	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5
1	3.1443	5.749				
2	2.7194	9.0078	24.9525			
3	3.2284+i0.5134	3.2284-i0.5134	6.6439	-51.7959		
4	3.3275	2.0639	6.4078	-17.3671+i12.6919	-17.3671-i12.6919	
5	3.2416	2.2692	6.4239	-9.985+i17.7432	-9.985-i17.7432	12.5219

Table 3.3: Critical exponents as a function of the order n of the truncation.

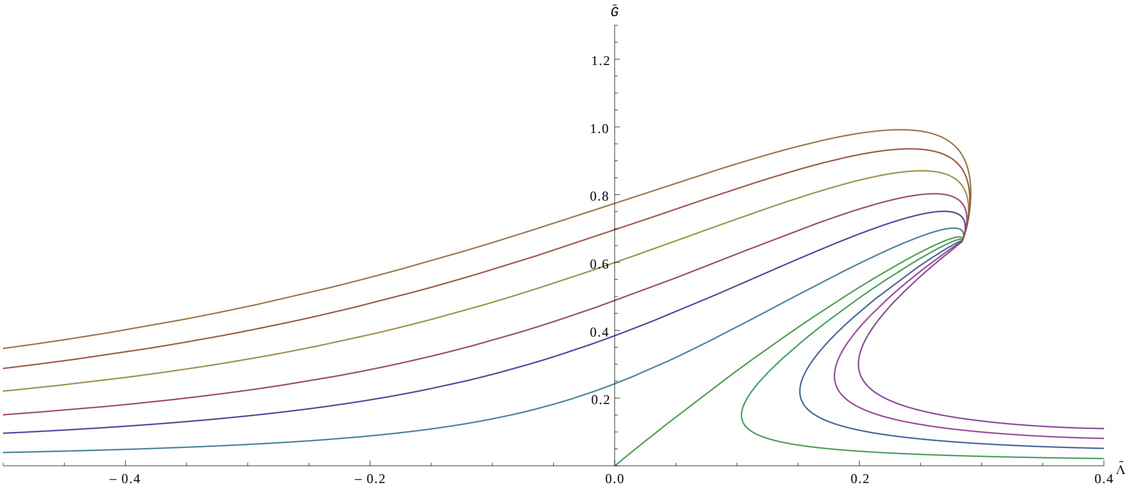


Figure 3.6: In this picture the flow given by equation (3.52), using the Einstein-Hilbert truncation, which projects in the $(\tilde{\Lambda}_k, \tilde{G}_k)$ plane.

Type II closure: $Z_{k,h} = Z_{k,c} = Z_{k,b} = 1$, $\eta_{k,h} = \eta_{k,c} = \eta_{k,b} = 0$

In this section we present the result for the polynomial truncation with a different choice of anomalous dimensions contribution. The type II closure for equation (3.52) reads

$$Z_{k,h} = Z_{k,c} = Z_{k,b} = 1 \quad \to \quad \eta_{k,h} = \eta_{k,c} = \eta_{k,b} = 0 \tag{3.58}$$

The truncated effective average action is still (3.57) and the couplings values at fixed point is given in table 3.4.

First, we note that fixed point values for the couplings are highly different from type I closure. In fact, the two closures differ not only for regulator choice but also for metric decomposition; hence they give different values for the fixed point as expected.

Also in this case for $n = 2$ we have a deviation from the mean fixed point values; but the deviation is greater than in the previous case. Maybe, the previous choice on regulator term gives a more stable distribution of fixed point as a function of n.

Closure type II: $Z_{k,h} = Z_{k,c} = Z_{k,b} = 1$ $\eta_{k,h} = \eta_{k,c} = \eta_{k,b} = 0$								
n	$\tilde{\Lambda}^*$	$\tilde{G}^*$	$\tilde{g}_0$	$\tilde{\kappa}^*$	$\tilde{g}_2$	$\tilde{g}_3$	$\tilde{g}_4$	$\tilde{g}_5$
1	0.1902	1.7674	0.3804	9.4253				
2	0.3199	1.4513	0.6397	8.5411	0.1732			
3	0.2441	1.8644	0.4883	9.6805	0.1096	-0.0705		
4	0.2358	1.8858	0.4717	9.7361	0.1073	-0.0726	-0.0299	
5	0.2363	1.8882	0.4725	9.7423	0.1075	-0.068	-0.0284	-0.005

Table 3.4: Couplings value at non-Gaussian fixed point as a functions of the order n of the truncation for closure (3.58).

Closure type II: $Z_{k,h} = Z_{k,c} = Z_{k,b} = 1$, $\eta_{k,h} = \eta_{k,c} = \eta_{k,b} = 0$						
n	$Re\theta_0 = Re\theta_1$	$Im\theta_0 = -Im\theta_1$	θ_2	$Re\theta_3 = Re\theta_4$	$Im\theta_3 = -Im\theta_4$	θ_5
1	2.1817	2.1665				
2	1.8697	3.2475	2.2156			
3	2.2667	2.1951	2.4279	-5.4926		
4	2.4394	2.2145	2.3254	-3.9547	-1.9701	
5	2.8721	1.3385	1.851	-3.0748	-5.9726	-2.9515

Table 3.5: Critical exponents as a function of the order n of the truncation for closure (3.58).

In table 3.5 critical exponents is presented. For $n = 1$, the Einstein-Hilbert truncation, we have a pair of complex conjugate critical exponents; hence both directions in the $\tilde{\Lambda}_k - \tilde{G}_k$ plane are UV attractive. For $n > 1$ only three critical exponents have positive real part, so the UV critical surface has finite dimension, a result which is in common to the closure type I studied above.

Chapter 4

Alternative flow equation for $f_k(R)$ with Euler-Maclaurin approximation

In previous chapters we derived two differential equation for the flow of $f_k(R)$ using two different methods in evaluating the functional trace present in the Wetterich's equation.

The first method is based on the Heat Kernel technique while the second one regards a particular approximation for the spectral sums. In this chapter we use a different approximation scheme to evaluate the trace based on Euler-Maclaurin formula for finite series. The aim of this work is to find a differential equation of second order for $f_k(R)$, instead of third order; this can be acheved with a "second order" cutoff scheme, as we have seen in previous chapters.

An equation with a second order derivatives in R would allow for a possibly simpler analysis of the fixed point and stability involving an infinite numer of coupling.

The starting point is the same as in the previous chapters. The metric decomposition $g = \bar{g} + \kappa_k h$ contains an extra factor $\kappa_k = \sqrt{16\pi G_k}$, so that the $n-th$ proper vertex depends on κ_k^{n-2}.

The truncation ansatz still reads

$$\Gamma_k[h, C, \bar{C}, b; \bar{g}] = \frac{1}{\kappa_k^2}\int d^dx\sqrt{g}f_k(R) + \Gamma_{k,g.f.}[h; \bar{g}] + \Gamma_{k,gh}[h, C, \bar{C}, b; \bar{g}] \tag{4.1}$$

In order to study the anomalous dimensions that fields can aquire, in the spirit of renormalization group, we make the following redefinition

$$h_{\mu\nu} \to Z_{k,h}^{1/2}h_{\mu\nu} \qquad C_\mu \to Z_{k,c}^{1/2}C_\mu \qquad \bar{C}_\mu \to Z_{k,c}^{1/2}\bar{C}_\mu \qquad b_\mu \to Z_{k,b}^{1/2}b_\mu \tag{4.2}$$

and define anomalous dimensions through the usual formulae

$$\eta_{k,a} = -\frac{\partial_t Z_{k,a}}{Z_{k,a}} \qquad a = h, c, b \tag{4.3}$$

The gauge fixing condition is the same as before

$$\Gamma_{k,g.f.}[h;\bar{g}] = \frac{Z_{k,h}}{2\alpha}\int d^dx\sqrt{\bar{g}}F_\mu[h;\bar{g}]F_\nu[h;\bar{g}]\bar{g}^{\mu\nu}$$

where

$$F_\mu[h;\bar{g}] = \bar{\nabla}^\rho h_{\rho\mu} - \frac{1}{d}\bar{\nabla}_\mu h \tag{4.4}$$

Note that $F[h;\bar{g}]$ does not depend on the scalar mode h when the transverse traceless decomposition is taking into account; this observation is crucial in diagonalization of $\mathbf{\Gamma}_k^{(2)}$ within the gauge $\alpha \to 0$.

As in chapter 3, we do not use the standard Faddeev-Popov ghosts contribution; instead, we used the following choice

$$\begin{aligned}\Gamma_{k,gh}[h=0,C,\bar{C},B;\bar{g}] =&\frac{Z_{k,c}}{\alpha}\int d^dx\sqrt{\bar{g}}\Big[\bar{C}^T_\mu\left(\Delta-\frac{R}{d}\right)^2 C^{T\mu} + 4\left(\frac{d-1}{d}\right)^2\bar{c}\left(\Delta-\frac{R}{d-1}\right)^2\Delta c\Big]\\ &+\frac{Z_{k,b}}{\alpha}\int d^dx\sqrt{\bar{g}}\Big[B^T_\mu\left(\Delta-\frac{R}{d}\right)^2 B^{T\mu} + 4\left(\frac{d-1}{d}\right)^2 b\left(\Delta-\frac{R}{d-1}\right)^2\Delta b\Big]\end{aligned} \tag{4.5}$$

As it was pointed out in the previous chapter, using this ghosts contribution, there exists an almost perfect cancellation between pure gauge degrees of freedom and ghosts sector in the trace calculation; only the anomalous dimensions' contribution still remains.

We rewrite for convenience the auxiliary field action resulting from the exponentiation of the Jacobians for the transverse-traceless decomposition, as discussed in chapter 2.

$$\begin{aligned}S_{\text{aux-gr}} = \int d^dx\sqrt{\bar{g}}\Big[&2\bar{\chi}^T_\mu\left(\Delta-\frac{R}{d}\right)\chi^{T\mu} + \frac{d-1}{d^2}\bar{\chi}\left(\Delta-\frac{R}{d-1}\right)\Delta\chi\\ &+2\zeta^T_\mu\left(\Delta-\frac{R}{d}\right)\zeta^{T\mu} + \frac{d-1}{d^2}\zeta\left(\Delta-\frac{R}{d-1}\right)\Delta\zeta\end{aligned} \tag{4.6}$$

where χ^T and χ are Grassmann valued fields while ζ^T and ζ are real commuting fields.

The calculation of $\mathbf{\Gamma}^(2)_k$ is given in the previous chapter, in particular for the gravity sector it is given by equations (3.10-3.11-3.12-3.13-3.14).

4.1 Cutoff scheme

In previous chapters we employed a particular cutoff scheme in order to simplify the corresponding functional traces, which had the effect of making the right hand side of Wetterich equation dependent on $\tilde{f}_k'''$. Instead, in this chapter we use a simplified version for cutoff scheme, called "second order", which reduce the dependence only on derivative up to second one. The "second order" cutoff is used, for the first time related to $f_k(R)$ approximation, in

[26]. In this paper a cutoff $\mathcal{R}(\Delta_s + \alpha_s R)$ with argument $\Delta_s + \alpha_s R$, where Δ_s $(s = 0, 1, 2)$ is given by (3.17-3.18-3.19) and α_s is fixed by the requirement that the argument $\Delta_s + \alpha_s R$ is positive or semi-positive definite. For $s = 1, 2$ we can impose $\alpha_s = 0$, while for $s = 0$ we choose $\alpha_0 = \frac{1}{d-1}$, so that $\Delta_0 + \alpha_0 R = \Delta_0 + \frac{R}{d-1} = \Delta$ is semi-positive definite.

To be more precise, we choose the following regulator

$$
\begin{aligned}
&\mathcal{R}_{h^T T}(\Delta_2) = -\frac{1}{2} f_k' Z_{k,h} r_k(\Delta_2) \\
&\mathcal{R}_{\xi}(\Delta_1) = \mathcal{R}_{\bar{C}^T C^T}(\Delta_1) = \mathcal{R}_{B^T} = \frac{Z_{k,h}}{\alpha}(k^4 - \Delta_1^2)\theta(k^4 - \theta_1^2) \\
&\mathcal{R}_{\bar{h}}(\Delta) = -\frac{d-1}{d^2} Z_{k,h} r_k(\Delta) \\
&\mathcal{R}_{\sigma}(\tilde{\Delta}) = \mathcal{R}_{\bar{c}c}(\tilde{\Delta}) = \mathcal{R}_b(\tilde{\Delta}) = \frac{(d-1)^2}{d^2} \frac{Z_{k,h}}{\alpha}(k^6 - \tilde{\Delta})\theta(k^6 - \tilde{\Delta}) \\
&\mathcal{R}_{\bar{\chi}^T \chi^T}(\tilde{\Delta}_{np}) = \mathcal{R}_{\zeta^T \zeta^T}(\tilde{\Delta}_{np}) = \frac{d-1}{d^2}(k^4 - \tilde{\Delta}_{np})\theta(k^4 - \tilde{\Delta}_{np}) \\
&\mathcal{R}_{\bar{\chi}\chi}(\Delta) = \mathcal{R}_{\zeta\zeta} = k^2 r_k(\Delta) \\
&\mathcal{R}_{\phi}(\Delta) = r_k(\Delta)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_1 &= \Delta - \frac{R}{d} \\
\Delta_2 &= \Delta + \frac{2R}{d(d-1)} \\
\tilde{\Delta} &= \left(\Delta - \frac{R}{d-1}\right)^2 \Delta \\
\tilde{\Delta}_{np} &= \left(\Delta - \frac{R}{d-1}\right) \Delta
\end{aligned}
$$

and last

$$
r_k(z) = (k^2 - z)\theta(k^2 - z)
$$

is the standard Litim's optimized cutoff.

Inserting previous regulators into Wetterich's equation, the *r.h.s.* gives the following contribution (divided by spin components)

$$
\partial_t \Gamma_k = T_2 + T_1 + T^{np} + T^{\bar{h}} + T^{\eta} \tag{4.7}
$$

where the spin 2 component reads

$$
T_2 = \frac{1}{2} \mathrm{Tr}_{(2)} \left[\frac{\left(1 - \frac{\Delta_2}{k^2}\right)\left(\partial_t \tilde{f}_k' - 2\tilde{R}\tilde{f}_k'' - \eta_{k,h}\tilde{f}_k'\right) + 2\tilde{f}_k'}{\tilde{f}_k' + \tilde{f}_k - \frac{2}{d}\tilde{R}\tilde{f}_k'} \right]
$$

Operator	Eigenvalue $\lambda_{n,s}$	Multiplicity $D_{n,s}$
Δ	$\frac{n(n+d-1)}{d(d-1)}R;\ n=0,1...$	$\frac{(2n+d-1)(n+d-2)!}{n!(d-1)!}$
$\Delta_1=\Delta-\frac{R}{d-1}$	$\frac{n(n+d-1)-d}{d(d-1)}R;\ n=1,2...$	$\frac{n(n+d-1)(2n+d-1)(n+d-3)!}{(d-2)!(n+1)!}$
$\Delta_2=\Delta+\frac{2R}{d(d-1)}$	$\frac{n(n+d-1)}{d(d-1)}R;\ n=2,3...$	$\frac{(d+1)(d-2)(n+d)(n-1)(2n+d-1)(n+d-3)!}{2(d-1)!(n+1)!}$

Table 4.1: Eigenvalue and their multiplicities of operators (3.17-3.18-3.19) on a d-sphere

the spin 1 is given by

$$T_1 = -\,\mathrm{Tr}'_{(1)}\,\theta(k^2-\Delta_1) \tag{4.8}$$

the non-physical scalar contribution is

$$T^{np} = -\,\mathrm{Tr'}_{(0)}\,\theta(k^4-\tilde{\Delta}_{np}) + \mathrm{Tr'}_{(0)}\,\theta(k^2-\Delta) \tag{4.9}$$

and the $\bar{h}$ contribution reads

$$T^{\bar{h}} = -\frac{1}{2}\,\mathrm{Tr}_{(0)}\left[\frac{\left[2-\eta_{k,h}+\eta_{k,h}\frac{\Delta}{k^2}\right]\theta(k^2-\Delta)}{(d-1)^2\tilde{f}''_k\left(\frac{\Delta}{k^2}-\frac{\tilde{R}}{d-1}\right)^2+(d-1)\tilde{f}'_k\left(\frac{\Delta}{k^2}-\frac{\tilde{R}}{d-1}\right)+\left(2\tilde{f}_k-\tilde{R}\tilde{f}'_k\right)+\frac{\Delta}{k^2}-1}\right] \tag{4.10}$$

Last, we give the anomalous dimensions' functional trace

$$T^{\eta} = (2\eta_c-\eta_h-\eta_b)\left\{\mathrm{Tr}'_{(1)}\,\theta(k^4-\Delta_1^2)+\mathrm{Tr'}_{(0)}\,\theta(k^6-\tilde{\Delta})\right\} \tag{4.11}$$

4.2 Spectral sums and Euler-Maclaurin approximation

As discussed in previous chapters, the *r.h.s.* of Wetterich's equation involves functional traces over functions of Laplacian $\Delta=-g_{\mu\nu}\nabla^\mu\nabla^\nu$ on a d-dimensional sphere.

In Chapter 2 we used the Heat Kernel technique implemented by a local expansion of the Heat Kernel operator. In Chapter 3 we adopted the *asymptotic approximation* used in [6] generalizing the method in d-spacetime dimensions.

Here, we introduce an alternative new approximation method based on the Euler-Maclaurin formula. The starting point is the standard definition of functional trace

$$\mathrm{Tr}\,\tilde{W}(\Delta_s) = \sum_{n=n_s}\tilde{W}(\lambda_{n,s})D_{n,s} \tag{4.12}$$

where, as pointed out above, function $\tilde{W}$ depends on Δ, Δ_1 or Δ_2.

First, note that in all cases we have $\tilde{W}(\Delta_s) = W(\Delta_s)\theta(k^2 - \Delta_s)$; hence we can truncate the sums to the values $\tilde{N}_s = \lfloor N_s \rfloor$, where

$$N_0(\tilde{R}) = N_2(\tilde{R}) = \frac{(1-d)\tilde{R} + \sqrt{(d-1)^2\tilde{R}^2 + 4\tilde{R}d(d-1)}}{2\tilde{R}} \tag{4.13}$$

$$N_1(\tilde{R}) = \frac{(1-d)\tilde{R} + \sqrt{(d+1)^2\tilde{R}^2 + 4\tilde{R}d(d-1)}}{2\tilde{R}} \tag{4.14}$$

$$\tag{4.15}$$

as it was discussed in the previous chapter. Formally the Wetterich's equation with "second order" cutoff and spectral sums technique becomes

$$\begin{aligned}\partial_k\Gamma_k =& \sum_{n=2}^{\tilde{N}_2} W_2(\lambda_{n,2})D_{n,2} + \sum_{n=2}^{\tilde{N}_1} W_1(\lambda_{n,1})D_{n,1} + \sum_{n=1}^{\tilde{N}_0} W^{np}(\lambda_{n,0})D_{n,0} \\ &+ \sum_{n=0}^{\tilde{N}_0} W^{\bar{h}}(\lambda_{n,0})D_{n,0} + \sum_{n=1}^{\tilde{N}_1} W^{\eta 1}(\lambda_{n,1})D_{n,1} + \sum_{n=1}^{\tilde{N}_0} W^{\eta 0}(\lambda_{n,0})D_{n,0}\end{aligned} \tag{4.16}$$

where $\lambda_{n,1}$ and $\lambda_{n,2}$ are the eigenvalues of operators Δ_1 and Δ_2, given in 3.1, while $\lambda_{n,0}$ are the eigenvalues of Δ with spin 0 and are reported in table C.1. Note that the spin 1 sum starts from $n = 2$ since all $d(d-1)/2$ Killing vectors do not contribute to the functional trace. Similarly, the np scalar contribution starts from $n = 1$ (contrary to [5]), since $\sigma = const$ does not contribution to quantum flactuations while $\nabla_\mu\sigma$, corresponding to $n = 1$ modes, does.

The functions $W(z)$ in the argument of the sums are given by the following expressions:

$$W_2(z) = \frac{1}{2}\frac{(1-z)\left(\partial_t\tilde{f}'_k - 2\tilde{R}\tilde{f}''_k - \eta_{k,h}\tilde{f}'_k\right) + 2\tilde{f}'_k}{\tilde{f}'_k + \tilde{f}_k - \frac{2}{d}\tilde{R}\tilde{f}'_k}$$

Note that when one consider $\tilde{R} = 0$ and impose $\tilde{f}'_k(0) = -1$ the denominator does not depend on z.

$$W_1(z) = -1$$

$$W^{np}(z) = -1$$

$$W^{\bar{h}} = -\frac{1}{2}\frac{2 - \eta_{k,h} + \eta_{k,h}z}{(d-1)^2\tilde{f}''_k\left(z - \frac{\tilde{R}}{d-1}\right)^2 + (d-1)\tilde{f}'_k\left(z - \frac{\tilde{R}}{d-1}\right) + 2\tilde{f}_k - \tilde{R}\tilde{f}'_k + z - 1}$$

$$W^{\eta 1}(z) = (2\eta_{k,c} - \eta_{k,h} - \eta_{k,b})$$

$$W^{\eta 2}(z) = (2\eta_{k,c} - \eta_{k,h} - \eta_{k,b})$$

To understand the Euler-Maclaurin approximation method, consider first the generic sum

$$S = \sum_{n=a}^{b} g(n)$$

The Euler-Maclaurin formula gives

$$\sum_{n=a}^{b} g(n) \simeq \int_a^b g(x)dx + \frac{g(a)+g(b)}{2} + \text{Remainder} \tag{4.17}$$

where the remainder term is given by

$$\text{Remainder} = \sum_{i=1}^{+\infty} \frac{B_{2i}}{(2i)!}\left(g^{(2i-1)}(b) - g^{(2i-1)}(a)\right)$$

and B_{2k} are Bernoulli coefficients: $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$ and so on. Using Euler-Maclaurin formula (4.17), one can approximate functional trace in (4.16) but the remainder term R gives a non-trivial contribution. Adding to (4.17) another suitable integral form would establish an equality.

The idea is to provide a truncation for the remainder term

$$\text{Remainder}\Big|_{\text{trunc.}} = R_l^{(t)} = \sum_{i=1}^{l} \frac{B_{2i}}{(2i)!}\left(g^{(2i-1)}(b) - g^{(2i-1)}(a)\right) \tag{4.18}$$

hence the truncated Euler-Maclaurin approximation becomes

$$S \simeq \int_a^b g(x)dx + \frac{g(a)+g(b)}{2} + R_l^{(t)} \tag{4.19}$$

where l can be chosen such that the truncated Euler-Maclaurin method gives a good approximation for functional trace.

We had verified that, for any value of l, the remainder contribution gives no important correction. As we shall show, the Euler-Maclaurin scheme gives a high quality approximation for functional traces considered in this work. Hence, from now on, we do not consider the remainder correction in the subsequent calculations.

Tensor trace contribution

First, we apply the truncated Euler-Maclaurin (tEM) approximation formula (4.19) to spin 2 trace contribution, which reads

$$S_2 = \sum_{n=2}^{N} W_2(\lambda_{n,2})D_{n,2} = \frac{1}{2}\sum_{n=2}^{N} \frac{\left(1-\frac{\lambda_{n,2}}{k^2}\right)\left(\partial_t \tilde{f}'_k - 2\tilde{R}\tilde{f}''_k - \eta_{k,h}\tilde{f}'_k\right) + 2\tilde{f}'_k}{\tilde{f}'_k + \tilde{f}_k - \frac{2}{d}\tilde{R}\tilde{f}'_k} D_{n,2} \tag{4.20}$$

where $\lambda_{n,2}$

In order to apply (4.19) first we devide the approximation formula as follows

$$S_2 \simeq I_2(N) + M_2(N)$$

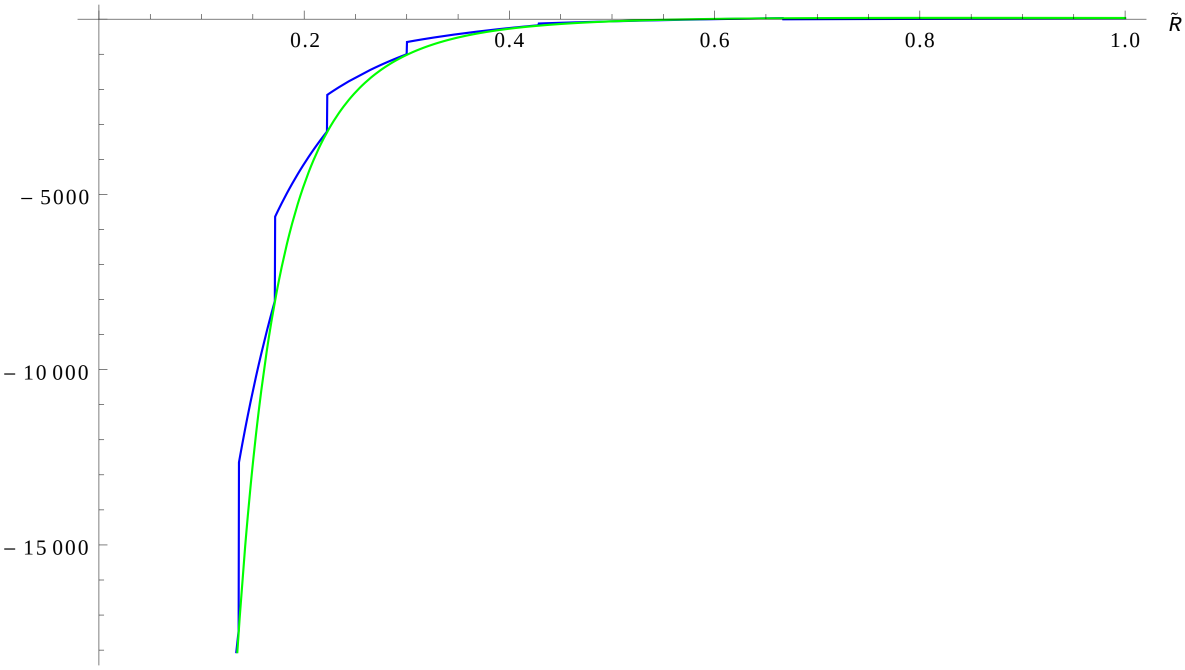


Figure 4.1: In this figure we plot the exact calculation of spin 2 trace contribution and the approximation (4.23) with a sample function function $\tilde{f}(\tilde{R}) = \tilde{R}^2 - \tilde{R} + 1$ and with $\eta_{k,h} = -2$.

where $I_2(N)$ represents the integral contribution and $M_2(N)$ is the second term of *r.h.s.* of (4.17). Regarding the spin 2 trace evaluation, the integral part gives, considering the replacement $N \to N_2(\tilde{R})$

$$I_2(\tilde{R}) \equiv I_2(N_2(\tilde{R})) = \frac{5(5\tilde{R}-6)^2(2\tilde{R}+3)}{27\tilde{R}^2} \frac{18\tilde{f}'_k \frac{(\tilde{R}+6)}{(5\tilde{R}-6)(2\tilde{R}+3)} + 2\tilde{f}''_k\tilde{R} - \partial_t \tilde{f}'_k + \eta_{k,h}\tilde{f}'_k}{\tilde{f}'_k\tilde{R} - 2(\tilde{f}_k + \tilde{f}')} \tag{4.21}$$

while for M_2 contribution we obtain

$$M_2(\tilde{R}) \equiv M_2(N_2(\tilde{R})) = \frac{5}{12\tilde{R}^2(\tilde{f}'_k\tilde{R} - 2(\tilde{f}_k + \tilde{f}'_k))} \Big[14\tilde{f}''_k(6-5\tilde{R})\tilde{R}^3 + 4\tilde{f}'_k\left(2(\tilde{R}-3)\sqrt{3}\sqrt{\tilde{R}(3\tilde{R}+16)} - 21\tilde{R}^2\right) + 7\tilde{R}^2(5\tilde{R}-6)\partial_t\tilde{f}'_k - \eta_{k,h}\tilde{f}'_k \Big] \tag{4.22}$$

Hence, the total spin 2 trace contribution that we consider reads

$$T^2 = I_2(N_2(\tilde{R})) + M_2(N_2(\tilde{R})) \tag{4.23}$$

We show the quality of the approximation in figure **??**fig:t2EM) by examing eq. (4.23) for a particular value of $f_k(R)$.

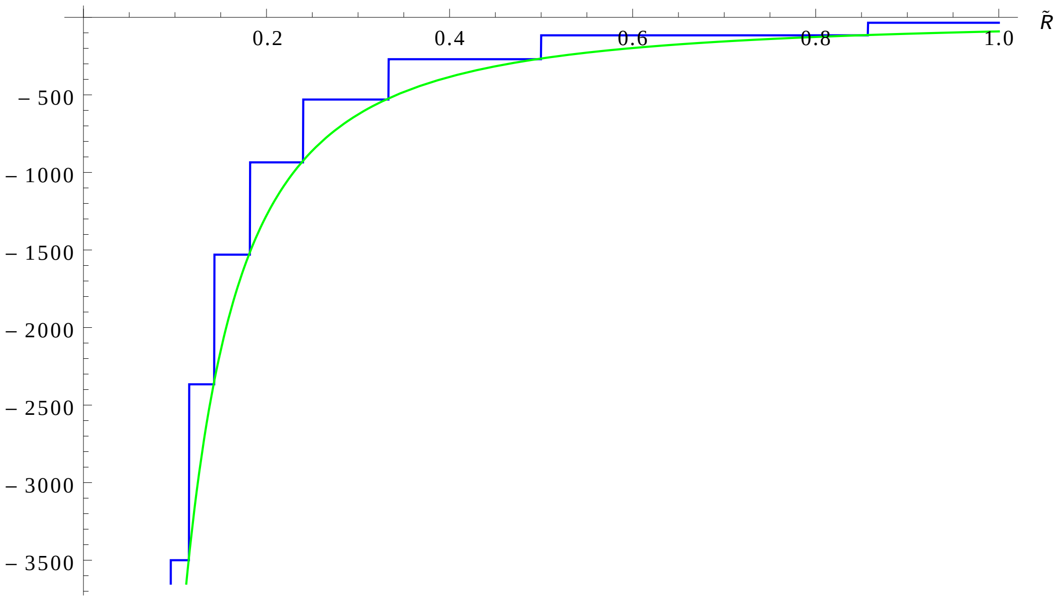


Figure 4.2: In this figure we plot the exact calculation of spin 1 trace contribution and the approximation (4.27).

Vector trace contribution

We now calculate the spin 1 trace part within the Euler-Maclaurin approximation scheme. The spin 1 spectral sum reads

$$S_1 = \sum_{n=2}^{N} W_1(\lambda_{n,1}) D_{n,1} = -\sum_{n=2}^{N} D_{n,1} \tag{4.24}$$

The approximation formula gives

$$S_1 \simeq I_1(N) + M_1 + R_l^{(t)}$$

where, as before, I_1 represents the integral contribution while M_1 the second term in (4.17) of Euler-Maclaurin formula.

The vector sum is trivial; in fact, the integral part gives

$$I_1(N) = -\int_2^N D_{x,1} = -\frac{N^4}{4} - \frac{3}{2}N^3 - \frac{9}{4}N^2 + 25 \tag{4.25}$$

while the M_1 contribution becomes

$$M_1(N) = -\frac{1}{2}\left(35 + \frac{1}{2}N(N+3)(2N+3)\right) \tag{4.26}$$

so that the total vector trace contribution gives

$$T^1 = I_1(N_1(\tilde{R})) + M_1(N_1(\tilde{R})) \tag{4.27}$$

Note that if one choose $l = 1$ then obtain the exact calculation of sum (4.24). In fact, for $l = 1$ we have the following contribution with Euler-Maclaurin approximation

$$S_1 = \frac{1}{4}\left(-N^4 - 8N^3 - 19N^2 - 12N + 40\right)$$

which is the correct evaluation of sum (4.24).

In figure 4.2 we plot the exact evaluation of vector trace (in blue) and the resulting approximation with Euler-Maclaurin formula (in green). This picture shows the great quality of approximation (4.27) already with $l = 0$.

Non-physical scalar contribution

The next term regards the non-physical contribution, which can be represented by the following sum

$$S_{np} = \sum_{n=1}^{N} W_{np}(\lambda_{n,0}/k^2) D_{n,0} = -\sum_{n=1}^{N} D_{n,0} \tag{4.28}$$

also this time, the functional trace reduces to simple sum over multiplicity of operator Δ.

The same consideration and procedure of previous section can be applied. The integral contribution to Euler-Maclaurin formula gives

$$I_{np} = -\int_1^N D_{x,0}dx = \frac{1}{12}(32 - 12N - 13N^2 - 6N^3 - N^4) \tag{4.29}$$

so that sum (4.28) can be approximated by

$$S_{np} \simeq -\frac{1}{12}(N^4 + 8N^3 + 22N^2 + 25N + 4) + R_l^{(t)} \tag{4.30}$$

The total *non-physical* scalar contribution gives

$$T^{np} = -\frac{1}{12}(N_0(\tilde{R})^4 + 8N_0(\tilde{R})^3 + 22N_0(\tilde{R})^2 + 25N_0(\tilde{R}) + 4) \tag{4.31}$$

In figure 4.3 the exact value of sum (4.28) (in Blue) and the approximation (4.31) are represented for $l = 0$.

$\bar{h}$ scalar contribution

The $\bar{h}$ trace part is the most complicated. It can be represented by the following sum

$$S_{\bar{h}} = \sum_{n=0}^{N} W_{\bar{h}}(\lambda_{n,0}/k^2) D_{n,0} = -\frac{1}{2}\sum_{n=0}^{N} \frac{\left(2 - \eta_{k,h} + \eta_{k,h}\frac{\lambda_{n,0}}{k^2}\right) D_{n,0}}{9\tilde{f}_k''\left(\frac{\lambda_{n,0}}{k^2} - \frac{\tilde{R}}{3}\right)^2 + 3\tilde{f}_k'\left(\frac{\lambda_{n,0}}{k^2} - \frac{\tilde{R}}{3}\right) + 2\tilde{f}_k - \tilde{R}\tilde{f}_k' + \frac{\lambda_{n,0}}{k^2} - 1} \tag{4.32}$$

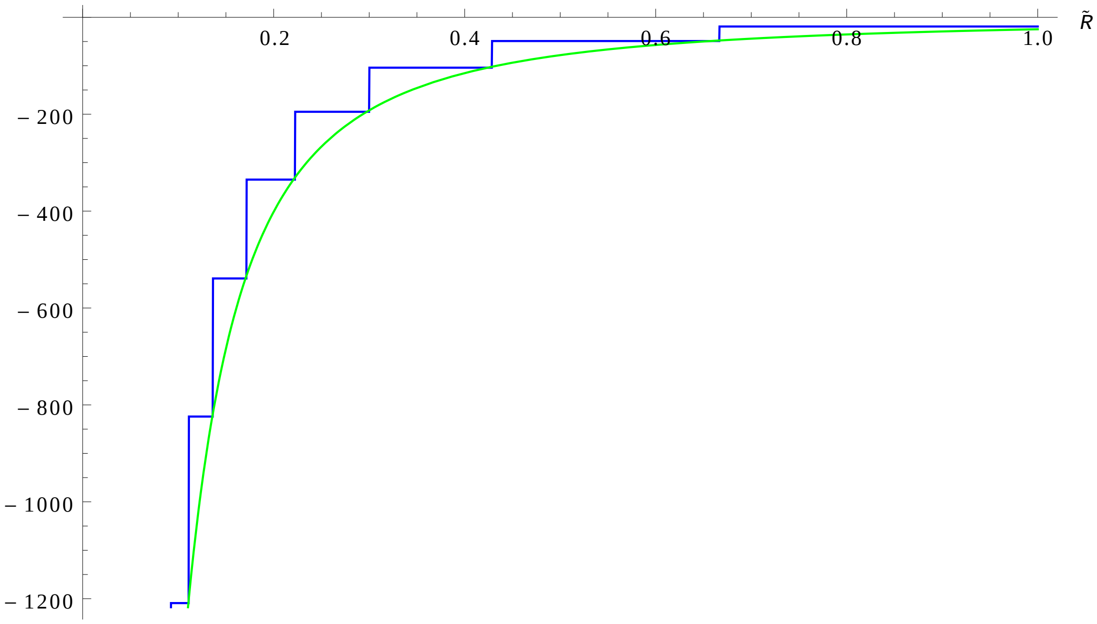


Figure 4.3: In this figure we plot the exact calculation of scalar non-physical trace contribution and the approximation (4.31).

We split the sum into two contribution

$$\sum_{n=0}^{N} W_{\bar{h}}(\lambda_{n,0}/k^2) D_{n,0} = W_{\bar{h}}(\lambda_{0,0}/k^2) D_{0,0} + \sum_{n=1}^{N} W_{\bar{h}}(\lambda_{n,0}/k^2) D_{n,0} \tag{4.33}$$

and treat only the second part of *r.h.s.* with the standard approximation scheme. Next, we apply the Euler-Maclaurin formula for the *r.h.s.* of (4.33) and calculate the integral contribution

$$I_{\bar{h}}(N) = \int_1^N W_{\bar{h}}(\lambda_{x,0}/k^2) D_{x,0} = A_{\bar{h}} + B_{\bar{h}}(N) \tag{4.34}$$

where

$$
\begin{aligned}
A_{\bar{h}} =& \frac{2}{3\tilde{f}_k''^2 \tilde{R}^2} \Bigg\{ \text{Arctanh} \left[\frac{\tilde{f}_k' + 1}{\sqrt{\tilde{f}_k'(\tilde{f}_k' + 4\tilde{f}_k'' + \tilde{R} + 2) - 4\tilde{f}_k''(2\tilde{f} + \tilde{R} - 3) + 1}} \right] \\
& \times \Big[12\tilde{f}_k''(1 + \tilde{f}_k' - 3\tilde{R}\tilde{f}_k'') \\
& - \eta_{k,h} \left(2 \left(\tilde{f}_k''(3\tilde{f}_k' - 4\tilde{f} + 9) + (\tilde{f}_k' + 1)^2 \right) - \tilde{f}_k'' \tilde{R}(\tilde{f}_k' + 18\tilde{f}_k'' + 9) + 6\tilde{f}_k''^2 \tilde{R}^2 \right) \Big] \Bigg\} \Bigg/ \\
& \left[\sqrt{\tilde{f}_k'(\tilde{f}_k' + 4\tilde{f}_k'' + \tilde{R} + 2) - 4\tilde{f}_k''(2\tilde{f} + \tilde{R} - 3) + 1} \right] \\
& + \frac{1}{3\tilde{f}_k''^2 \tilde{R}^2} \ln \left[256(\tilde{f}_k'\tilde{R} - \tilde{R} - 2\tilde{f}_k + 3) \right] \left(12\tilde{f}_k'' + \eta_{k,h}(\tilde{f}_k''(6 - 5\tilde{R}) + 2\tilde{f}_k' + 2) \right)
\end{aligned}
\tag{4.35}
$$

$$
\begin{aligned}
B_{\bar{h}}(N) =& \frac{\eta_{k,h}}{3\tilde{f}_k'' \tilde{R}^2}(N+4)(N-1) \\
& - \frac{2}{3\tilde{f}_k''^2 \tilde{R}^2} \Bigg\{ \text{Arctanh} \left[\frac{2\tilde{f}_k' + 2 + \tilde{f}_k''\tilde{R}(N+4)(N+1)}{\sqrt{2\tilde{f}_k'(\tilde{f}_k' + 4\tilde{f}_k'' + \tilde{R} + 2) - 4\tilde{f}_k''(2\tilde{f} + \tilde{R} - 3) + 1}} \right] \\
& \times \Big[12\tilde{f}_k''(1 + \tilde{f}_k' - 3\tilde{R}\tilde{f}_k'') \\
& - \eta_{k,h} \left(2 \left(\tilde{f}_k''(3\tilde{f}_k' - 4\tilde{f} + 9) + (\tilde{f}_k' + 1)^2 \right) - \tilde{f}_k'' \tilde{R}(\tilde{f}_k' + 18\tilde{f}_k'' + 9) + 6\tilde{f}_k''^2 \tilde{R}^2 \right) \Big] \Bigg\} \Bigg/ \\
& \left[\sqrt{\tilde{f}_k'(\tilde{f}_k' + 4\tilde{f}_k'' + \tilde{R} + 2) - 4\tilde{f}_k''(2\tilde{f} + \tilde{R} - 3) + 1} \right] \\
& - \frac{1}{3\tilde{f}_k''^2 \tilde{R}^2} \ln \Big[16 \Big(48 - 32\tilde{f}_k - \tilde{f}_k''\tilde{R}^2(N+4)^2(N-1)^2 \\
& \quad - 4\tilde{R} \left(\tilde{f}_k'(N(N+3) - 8) + N(N+3) \right) \Big) \Big] \\
& \times \left(12\tilde{f}_k'' + \eta_{k,h}(\tilde{f}_k''(6 - 5\tilde{R}) + 2\tilde{f}_k' + 2) \right)
\end{aligned}
\tag{4.36}
$$

Hence, the total $\bar{h}$ scalar trace is approximated by

$$
T^{\bar{h}} = M_{\bar{h}}(N_0(\tilde{R})) + A_{\bar{h}} + B_{\bar{h}}(N_0(\tilde{R})) \tag{4.37}
$$

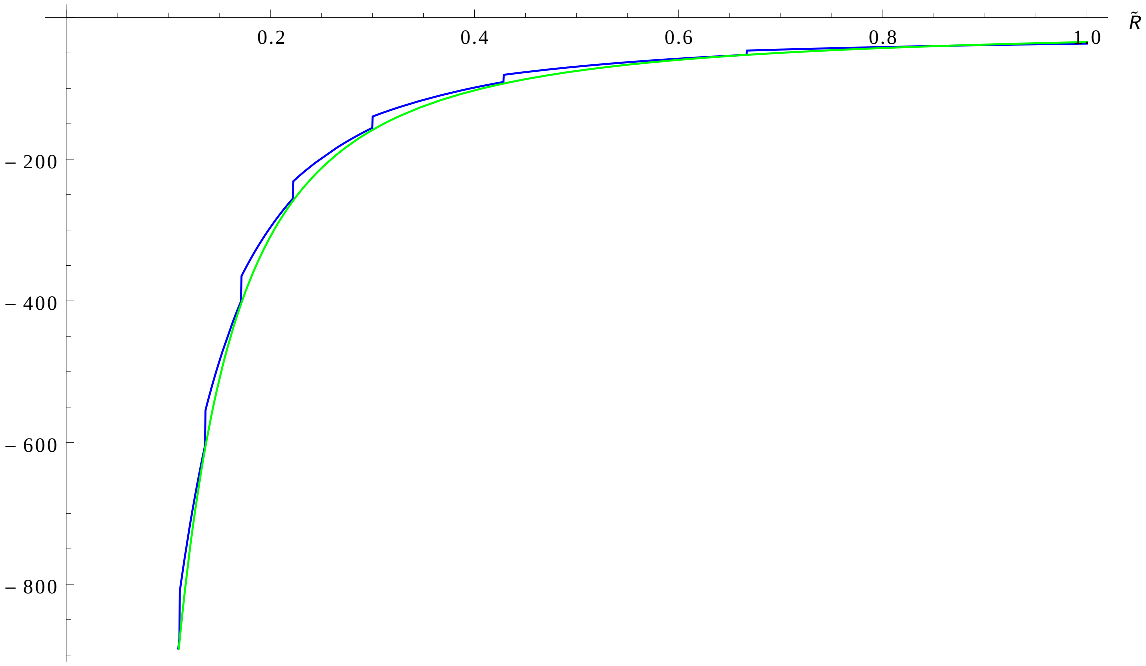


Figure 4.4: In this figure we plot the exact calculation of $\bar{h}$ trace contribution and the approximation (4.37) with a sample function function $\tilde{f}(\tilde{R}) = \tilde{R}^2 - \tilde{R} + 1$ and with $\eta_{k,h} = -2$.

where

$$M_{\bar{h}}(N) = \frac{3}{2}\frac{\eta_{k,h} - 2}{-3 + 2\tilde{f}_k - 2\tilde{f}'_k\tilde{R} + \tilde{f}''_k\tilde{R}^2} - \frac{5}{4}\frac{6 + (\tilde{R}-3)\eta_{k,h}}{-3 + 2\tilde{f}_k - \tilde{R}\tilde{f}'_k + \tilde{R}} + \\ + 2\frac{(N+1)(N+2)(2N+3)(2 - \eta_{k,h} + \frac{1}{12}N(3+N)\tilde{R}\eta_{k,h})}{(-48 + 32\tilde{f}_k - 16\tilde{f}'_k\tilde{R} - 4N(N+3)\tilde{R} + 4\tilde{f}'_k(N^2+3N-4)\tilde{R} + \tilde{f}''_k(N^2+3N-4)^2\tilde{R}^2)} \tag{4.38}$$

In figure 4.4 the value of sum (4.37) (with N replaced by $N_0(\tilde{R})$) and its approximation (4.37), where we choose (for sample) $\tilde{f}(\tilde{R}) = \tilde{R}^2 + \tilde{R} + 2$ and $l = 0$.

Note that the formula obtained for the $\bar{h}$ trace part with the Euler-Maclaurin approximation is valid with the condition $\tilde{f}''_k \neq 0$, since in the evaluation of integral (4.34) the previous the previous relation must hold.

For the case $\tilde{f}''_k = 0$, which we call "Einstein-Hilbert" case, a modified evaluation of the integral (4.34) is necessary. With $\tilde{f}_k''=0$, we define

$$
\begin{aligned}
I_{\bar{h}}^{EH}(N) =& \frac{2}{3(3\tilde{f}_k'-16)^3}\Bigg\{2(16-3\tilde{f}_k')^2\left(\left(N+\frac{3}{2}\right)^4-\frac{625}{16}\right)\eta_{k,h} \\
&+\frac{96}{\tilde{R}^2}(12\tilde{f}_k+(16-15\tilde{f}_k')\tilde{R}+96)\left(\eta_{k,h}(2\tilde{f}+\tilde{f}_k'(3-2\tilde{R}))-6\tilde{f}_k'+32\right) \\
&\times\ln\left[\frac{96\tilde{f}+4\tilde{R}\left(3\tilde{f}_k'\left(N^2+3N-8\right)-16N(N+3)\right)+768}{16(6\tilde{f}-(3\tilde{f}_k'+16)\tilde{R}+48)}\right] \\
&+\frac{1}{\tilde{R}^2}(3\tilde{f}_k'-16)\left(\left(N+\frac{3}{2}\right)^2-\frac{25}{4}\right)\left(\eta_{k,h}(93\tilde{f}_k'\tilde{R}-96\tilde{f}_k-144\tilde{f}_k'+16\tilde{R})+96(3\tilde{f}_k'-16)\right)\Bigg\}
\end{aligned}
\tag{4.39}
$$

Hence, for $\tilde{f}_k''=0$, the scalar $\bar{h}$ trace can be approximated by

$$
T_{EH}^{\bar{h}} = M_{\bar{h}}(N_0(\tilde{R}))+I_{\bar{h}}^{EH}(N_0(\tilde{R})) \tag{4.40}
$$

Anomalous dimensions' contribution

Finally, we consider the contribution to the traces given by the anomalous dimensions' part of Wetterich's equation, which can be represented by the following sums

$$
S_0^\eta = \sum_{n=1}^{N} W(\lambda_{n,0}/k^2)D_{n,0} = (2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})\sum_{n=1}^{N} D_{n,0} \tag{4.41}
$$

$$
S_1^\eta = \sum_{n=1}^{N} W(\lambda_{n,1}/k^2)D_{n,1} = (2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})\sum_{n=1}^{N} D_{n,1} \tag{4.42}
$$

The previous sums have been evaluated in the vector and non-physical scalar sector. Here, we report directly the results of Euler-Maclaurin approximation scheme for the total anomalous dimensions trace part

$$
T^\eta = \frac{1}{2}(2\eta_{k,c}-\eta_{k,h}-\eta_{k,b})(N_0(\tilde{R})^4+8N_0(\tilde{R})^3+20N_0(\tilde{R})^2+17N_0(\tilde{R})+4) \tag{4.43}
$$

4.3 Flow equation for f_k

Collecting all the expressions for the terms in the Euler-Maclaurin approximation scheme, the Wetterich's equation becomes

$$
\partial_t\Gamma_k = \frac{k^4}{\tilde{\kappa}_k^2}V_{4S}\left[\partial_t\tilde{f}_k(\tilde{R})-2\tilde{R}\tilde{f}_k'(\tilde{R})+\left(4-2\frac{\partial_t\tilde{\kappa}_k}{\tilde{\kappa}_k}\right)\tilde{f}_k(\tilde{R})\right] = T^2+T^1+T^{np}+T^{\bar{h}}+T^\eta \tag{4.44}
$$

with

$$T^2 = I_2(\tilde{R}) + M_2(\tilde{R}) \tag{4.45}$$

with

$$I_2(\tilde{R}) = \frac{5(5\tilde{R}-6)^2(2\tilde{R}+3)}{27\tilde{R}^2} \frac{18\tilde{f}'_k \frac{(\tilde{R}+6)}{(5\tilde{R}-6)(2\tilde{R}+3)} + 2\tilde{f}''_k\tilde{R} - \partial_t\tilde{f}'_k + \eta_{k,h}\tilde{f}'_k}{\tilde{f}'_k\tilde{R} - 2(\tilde{f}_k + \tilde{f}')} \tag{4.46}$$

and

$$\begin{aligned} M_2(\tilde{R})) =& \frac{5}{12\tilde{R}^2(\tilde{f}'_k\tilde{R} - 2(\tilde{f}_k + \tilde{f}'_k))} \Big[14\tilde{f}''_k(6-5\tilde{R})\tilde{R}^3 \\ &+ 4\tilde{f}'_k\left(2(\tilde{R}-3)\sqrt{3}\sqrt{\tilde{R}(3\tilde{R}+16)} - 21\tilde{R}^2\right) + 7\tilde{R}^2(5\tilde{R}-6)\partial_t\tilde{f}'_k - \eta_{k,h}\tilde{f}'_k\Big] \end{aligned} \tag{4.47}$$

$$T^1 = -\frac{1}{4}\left(N_1(\tilde{R}^4) + 8N_1(\tilde{R})^3 + 18N_1(\tilde{R})^2 + 9N_1(\tilde{R}) - 30\right) \tag{4.48}$$

$$T^{np} = -\frac{1}{12}\left(N_0(\tilde{R})^4 + 8N_0(\tilde{R})^3 + 22N_0(\tilde{R})^2 + 25N_0(\tilde{R}) + 4\right) \tag{4.49}$$

$$T^\eta = \frac{1}{2}(2\eta_{k,c} - \eta_{k,h} - \eta_{k,b})(N_0(\tilde{R})^4 + 8N_0(\tilde{R})^3 + 20N_0(\tilde{R})^2 + 17N_0(\tilde{R}) + 4) \tag{4.50}$$

and last

$$T^{\bar{h}} = M_{\bar{h}}(\tilde{R}) + A_{\bar{h}}(\tilde{R}) + B_{\bar{h}}(\tilde{R}) \tag{4.51}$$

where

$$\begin{aligned} M_{\bar{h}}(N) =& \frac{3}{2}\frac{\eta_{k,h}-2}{-3+2\tilde{f}_k - 2\tilde{f}'_k\tilde{R} + \tilde{f}''_k\tilde{R}^2} - \frac{5}{4}\frac{6+(\tilde{R}-3)\eta_{k,h}}{-3+2\tilde{f}_k - \tilde{R}\tilde{f}'_k + \tilde{R}} + \\ &- \frac{\sqrt{3}}{\tilde{R}^2}\frac{(\tilde{R}+6)\sqrt{\tilde{R}(3\tilde{R}+16)})}{(2\tilde{f}_k - \tilde{f}'_k(2\tilde{R}-3) + \tilde{f}''_k(\tilde{R}-3)^2)} \end{aligned} \tag{4.52}$$

$$\begin{aligned} A_{\bar{h}}(\tilde{R}) =& \frac{2}{3\tilde{f}''^2_k\tilde{R}^2}\Bigg\{ \text{Arctanh}\left[\frac{\tilde{f}'_k + 1}{\sqrt{\tilde{f}'_k(\tilde{f}'_k + 4\tilde{f}''_k + \tilde{R} + 2) - 4\tilde{f}''_k(2\tilde{f} + \tilde{R} - 3) + 1}}\right] \\ &\times \Big[12\tilde{f}''_k(1 + \tilde{f}'_k - 3\tilde{R}\tilde{f}''_k) \\ &- \eta_{k,h}\left(2\left(\tilde{f}''_k(3\tilde{f}'_k - 4\tilde{f} + 9) + (\tilde{f}'_k + 1)^2\right) - \tilde{f}''_k\tilde{R}(\tilde{f}'_k + 18\tilde{f}''_k + 9) + 6\tilde{f}''^2_k\tilde{R}^2\right)\Big]\Bigg\} \Big/ \\ &\left[\sqrt{\tilde{f}'_k(\tilde{f}'_k + 4\tilde{f}''_k + \tilde{R} + 2) - 4\tilde{f}''_k(2\tilde{f} + \tilde{R} - 3) + 1}\right] \\ &+ \frac{1}{3\tilde{f}''^2_k\tilde{R}^2}\ln\left[256(\tilde{f}'_k\tilde{R} - \tilde{R} - 2\tilde{f}_k + 3)\right]\left(12\tilde{f}''_k + \eta_{k,h}(\tilde{f}''_k(6-5\tilde{R}) + 2\tilde{f}'_k + 2)\right) \end{aligned} \tag{4.53}$$

$$
\begin{aligned}
B_{\bar{h}}(\tilde{R}) =&\frac{\eta_{k,h}}{3\tilde{f}_k''\tilde{R}^2}\left(\frac{12}{\tilde{R}}-4\right)\\
&-\frac{2}{3\tilde{f}_k''^2\tilde{R}^2}\Bigg\{\text{Arctanh}\left[\frac{2\tilde{f}_k'+2+\tilde{f}_k''(12\tilde{R}-4)}{\sqrt{2\tilde{f}_k'(\tilde{f}_k'+4\tilde{f}_k''+\tilde{R}+2)-4\tilde{f}_k''(2\tilde{f}+\tilde{R}-3)+1}}\right]\\
&\times\Big[12\tilde{f}_k''(1+\tilde{f}_k'-3\tilde{R}\tilde{f}_k'')\\
&-\eta_{k,h}\Big(2\left(\tilde{f}_k''(3\tilde{f}_k'-4\tilde{f}+9)+(\tilde{f}_k'+1)^2\right)-\tilde{f}_k''\tilde{R}(\tilde{f}_k'+18\tilde{f}_k''+9)+6\tilde{f}_k''^2\tilde{R}^2\Big)\Big]\Bigg\}\Bigg/\\
&\left[\sqrt{\tilde{f}_k'(\tilde{f}_k'+4\tilde{f}_k''+\tilde{R}+2)-4\tilde{f}_k''(2\tilde{f}+\tilde{R}-3)+1}\right]\\
&-\frac{1}{3\tilde{f}_k''^2\tilde{R}^2}\ln\Big[16\Big(-32\tilde{f}_k-\tilde{f}_k''(12\tilde{R}-4)^2\\
&\quad+16\tilde{f}_k'(2\tilde{R}-3)\Big)\Big]\\
&\times\Big(12\tilde{f}_k''+\eta_{k,h}(\tilde{f}_k''(6-5\tilde{R})+2\tilde{f}_k'+2)\Big)
\end{aligned}
\tag{4.54}
$$

4.4 Possible closures for $f_k(R)$ RG equation

Equation (4.65) allow us to describe the flow of function $f_k(R)$ and also its dependence of curvature scalar R. This equation contains fields anomalous dimensions which have to be determined in a different way. As explained in the previous chapter, a possible closure is the following: find anomalous dimensions contribution from the flow of $\Gamma_k^{(2)}$. This possibility is a proposal for a future work.

To follow a simple and consistent closure of (4.65), in this work we make two different ansatz for the values of $Z_{k,h}$, $Z_{k,c}$ and $Z_{k,b}$. The most simple ansatz, which we call *type I ansatz*, is the following

$$Z_{k,h}=\kappa_k^{-2}\qquad Z_{k,c}=Z_{k,b}=1$$

which implies

$$\eta_{k,h}=-\frac{\beta_{\tilde{G}}}{\tilde{G}}-2\qquad \eta_{k,c}=0\qquad \eta_{k,b}=0 \tag{4.55}$$

where $\kappa_k=\sqrt{16\pi G_k}$ and $\tilde{G}=k^2G_k$ is the dimensionless Newton's constant. Note that type I ansatz (4.55) is associated to the following metric decomposition

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}$$

which is the most used definition for quantum flactuations.

Another ansatz for the anomalous dimensions values, which we call type II ansatz, is the following

$$Z_{k,h} = Z_{k,b} = Z_{k,c} = 1$$

which implies

$$\eta_{k,h} = 0 \qquad \eta_{k,c} = 0 \qquad \eta_{k,b} = 0$$

Hence, with type I ansatz, the anomalous dimensions contribution in (4.65) is completly neglected.

4.5 Polynomial truncation

The most simple truncation for the average effective action is the standard Einstein-Hilbert action

$$\Gamma_k[h, \bar{C}, C, b; \bar{g}] = \frac{1}{16\pi G_k} \int d^d x \sqrt{g}(2\Lambda_k - R) + \Gamma_{k,gh} + \Gamma_{k,g.f.} \tag{4.56}$$

where Λ_k is the running cosmological constant. The choice (4.56) is consistent with previous truncation (3.1) if

$$f_k(R) = 2\Lambda_k - R \qquad \to \qquad f_k(0) = 2\Lambda_k \qquad f_k'(0) = -1$$

so one can construct the flow for gravitational and cosmological constants using (4.65).

For a first study on the dimensionality of the critical surface, the average effective action have to be modified introducing more interactions such as R^2, R^3 and so on.

Here, we consider the polynomial truncation, so that the effective action reads

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^d x \sqrt{g} \sum_{i=0}^{n} g_i R^i + \Gamma_{k,gh} + \Gamma_{k,g.f.} \tag{4.57}$$

Clearly, for $n = 1$ we go back to Einstein-Hilbert form. This truncation ansatz is consistent with our initial assumption for the effective action as a function only on curvature scalar. To study the flow equation for the dimensionless couplings $\tilde{g}_i = k^{2-2i} g_i$, we use equation (4.65), which, as pointed out in the last section, necessitate of a closure to be solved for the presence of anomalous dimensions' contribution.

Closure type: $Z_{k,h} = \kappa_k^{-2}$, $Z_{k,b} = Z_{k,c} = 1$

With equation (4.65) and in the polynomial truncation the numerical studies are really involved. From $n > 1$ the corresponding beta functions contain complicated expression with Arctanh and Log functions. Here, we present first the results for $n = 1$, where we find a non-Gaussian fixed point with values

$$Z_{k,h} = \kappa_k^{-2} \quad \to \qquad \tilde{G}^* = 0.61902 \quad \tilde{\Lambda}^* = 0.217946$$

Closure type I:$Z_{k,h}=\kappa_k^{-2}$, $Z_{k,c}=Z_{k,b}=1$, $\eta_h^*=-2$, $\eta_{k,c}=\eta_{k,b}=0$									
n	$\tilde{\Lambda}^*$	$\tilde{G}^*$	$\tilde{g}_0$	$\tilde{\kappa}^*$	$\tilde{g}_2$	$\tilde{g}_3$	$\tilde{g}_4$	$\tilde{g}_5$	$\tilde{g}_6$
1	0.2179	0.619	0.4358	5.578					
2	?	?	?	?	?				
3	0.2727	0.5144	0.5454	5.0852	0.1048	0.0114			
4	0.1563	0.5396	0.3126	5.2082	-0.3669	-1.0033	-3.2592		
5	0.1387	0.5021	0.2773	5.0236	-0.6453	-2.0609	-5.3371	4.5353	
6	0.1339	0.4907	0.2677	4.9663	-0.7478	-2.4707	-5.9841	6.1949	-12.8759

Table 4.2: Couplings value at non-Gaussian fixed point as a functions of the order n of the truncation.

Closure type I:$Z_{k,h}=\kappa_k^{-2}$, $Z_{k,c}=Z_{k,b}=1$, $\eta_h^*=-2$, $\eta_{k,c}=\eta_{k,b}=0$							
n	$Re\theta_0=Re\theta_1$	$Im\theta_0=-Im\theta_1$	θ_2	$Re\theta_3=Re\theta_4$	$Im\theta_3=-Im\theta_4$	θ_5	θ_6
1	1.7562	3.7246					
2	?	?	?				
3	1.0357	5.1787	1865	-21552			
4	2.7262	3.4013	8.9136	-119.933	-1.1675		
5	2.6887	3.0439	5.021	-91.9527	-2.5599	1.5174	
6	2.6975	3.0381	10.4644	-104.339	-5.0858	2.6116	-2.1647

Table 4.3: Critical exponents as a function of the order n of the truncation.

with critical exponents

$$Z_{k,h}=\kappa_k^{-2} \quad\to\quad \theta_\pm = 1.3725 \pm i3.4262$$

These values are compatible with previous results given in second and third chapters. Note that the two closures have very different values for the fixed point solution, feature in common with the two previous $f_k(R)$ equations.

In table 4.2 the values of coupling constants at the fixed point are given; instead, in table 4.3 the critical exponents are reported up to $n=6$. It is evident that the UV critical surface has dimension three, which confirms the results of previous equations.

For $n=2$, we find no NGFP with positive Newton's constant. This does not require that a fixed point for R^2 truncation does not exist since, simply, the numerical algorithm does not converge near the starting values we propose.

Closure type II: $Z_{k,h} = Z_{k,c} = Z_{k,b} = 1$ $\eta_{k,h} = \eta_{k,c} = \eta_{k,b} = 0$									
n	$\tilde{\Lambda}^*$	$\tilde{G}^*$	$\tilde{g}_0$	$\tilde{\kappa}^*$	$\tilde{g}_2$	$\tilde{g}_3$	$\tilde{g}_4$	$\tilde{g}_5$	$\tilde{g}_6$
1	0.0813	1.3219	0.1626	8.1514					
2	?	?	?	?	?				
3	0.0291	0.8838	0.0582	6.6651	-0.9152	-3.144			
4	0.0473	1.1084	0.0945	7.4643	-0.2684	-0.5262	-1.2757		
5	0.0372	1.0069	0.0744	7.1141	-0.5444	-1.2611	-2.3027	2.6717	
6	0.0351	0.9801	0.0702	7.019	-0.636	-1.5307	-2.4963	3.751	-5.0053

Table 4.4: Couplings value at non-Gaussian fixed point as a functions of the order n of the truncation.

Closure type: $Z_{k,h} = Z_{k,b} = Z_{k,c} = 1$

Here, we present first the results for $n = 1$, where we set $\eta_{k,i} = 0$. We find a non-Gaussian fixed point with values

$$Z_{k,h} = 1 \quad \rightarrow \quad \tilde{G}^* = 1.3219 \quad \tilde{\Lambda}^* = 0.0813$$

and critical exponents

$$Z_{k,h} = 1 \quad \rightarrow \quad \theta_{\pm} = 2.3839 \pm i0.8614$$

In table 4.4 we report the result of numerical studies also for $n > 1$. First note that we did not find any suitable fixed for $n = 2$; we stress that this does not require that a fixed point in R^2 truncation does not exists. The numerical algorithm does not converge nor give any values as the solution of the system. Maybe, the fixed point has values such that indesiderable poles appears in the integral of the Euler-Maclaurin approximation formula.

In table 4.5 we report the values of critical exponents. Here, we note that the UV critical surface has dimension three, which confirms the previous results obtained with different flow equation.

A general remark is in order: even we have choosen a pretty different cutoff scheme we notice that the qualitative picture for the UV critical behaviour does not change.

4.6 Spectral sums with Digamma function

There exists an alternative method to evaluate the spectral sums reported in previous sections. This method makes use of the special function called *Digamma* function. This special function is defined through the well known Gamma function

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \tag{4.58}$$

Closure type II: $Z_{k,h} = Z_{k,c} = Z_{k,b} = 1$, $\eta_{k,h} = \eta_{k,c} = \eta_{k,b} = 0$						
n	θ_0	θ_1	θ_2	θ_3	θ_4	$\theta_5 \pm i\theta_6$
1	2.3839+i 0.8614	2.3839-i 0.8614				
2	?	?	?			
3	2.9873	2.4264	1.1955	-43.9298		
4	2.8011+i 0.4986	2.8011-i 0.4986	9.898	-70.9492	-1.194	
5	2.6638+i 0.2646	2.6638-i 0.2646	5.1064	-58.499	-2.0632	-5.1065
6	2.7041+i 0.2533	2.7041-i 0.2533	9.3296	-65.6745	1.7447	-5.8279 ±0.9337

Table 4.5: Critical exponents as a function of the order n of the truncation.

where

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$$

Note that this definition is valid only for $\Re z > 0$, but can be extended also in the domain $\Re z < 0$ by analytic continuation, see [27].

One useful property of the Digamma function is the following

$$\psi(n+z) - \psi(z+1) = \sum_{k=1}^{n-1} \frac{1}{z+1} \tag{4.59}$$

for n integer value.

With formula (4.59) we can express the functional trace contribution into explicitly expression involving only the Digamma function.

We start from the spin 2 trace contribution. Consider the sums (4.20) which we report here

$$S_2 = \sum_{n=2}^{N} W_2(\lambda_{n,2}) D_{n,2} = \frac{1}{2} \sum_{n=2}^{N} \frac{\left(1 - \frac{\lambda_{n,2}}{k^2}\right)\left(\partial_t \tilde{f}'_k - 2\tilde{R}\tilde{f}''_k - \eta_{k,h}\tilde{f}'_k\right) + 2\tilde{f}'_k}{\tilde{f}'_k + \tilde{f}_k - \frac{2}{d}\tilde{R}\tilde{f}'_k} D_{n,2} \tag{4.60}$$

where the eigenvalues and relative multiplicities are given by

$$\lambda_{n,2} = \frac{n(n+3)}{12} R \qquad D_{n,2} = \frac{5}{6}(n-1)(n+4)(2n+3)$$

The sum given in (4.60) can be evaluated exactly and the result is

$$\begin{aligned} S_2(N) =& \frac{5N(N-1)(N+4)(N+5)\left(\tilde{R}(N+1)(N+3) - 18\right)}{216\left(2(\tilde{f}_k + \tilde{f}'_k) - \tilde{R}\tilde{f}'_k\right)} \\ & \times \left[\frac{36\tilde{f}''_k}{\left(\tilde{R}(N+1)(N+3) - 18\right)} - \partial_t \tilde{f}'_k + \eta_{k,h}\tilde{f}'_k\right] \end{aligned} \tag{4.61}$$

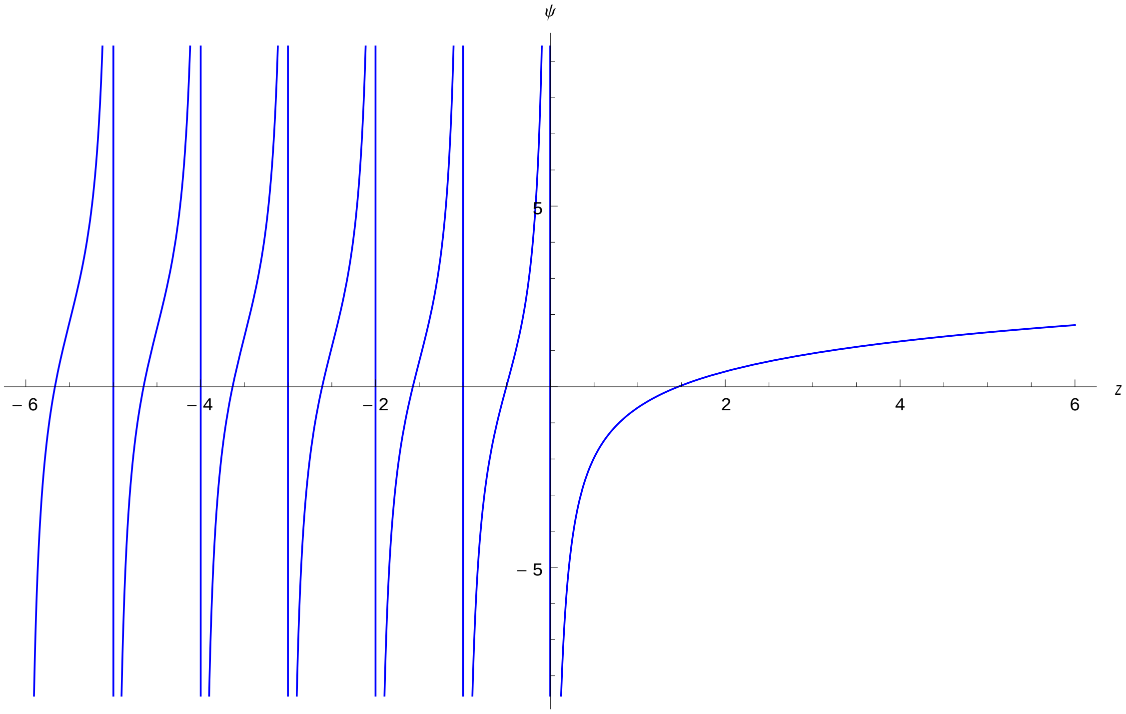


Figure 4.5: In this figure we plot Digamma function $\psi(z)$, defined in (4.58) , in the case of a real variable z.

One must stress that this result is exact and not obtained with an approximation scheme. But, there is a comment that allow us to prefer the Euler-Maclaurin approximation than the exact expression.

To recover the functional trace contribution in the spin 2 sector, we have to replace N with $N_2(\tilde{R})$

$$T^2 = S_2(N_2(\tilde{R})) \tag{4.62}$$

where

$$N_2(\tilde{R}) = \frac{-3\tilde{R} + \sqrt{9\tilde{R}^2 + 48\tilde{R}}}{2\tilde{R}}$$

First, when we consider the sum S_2 as function of $\tilde{R}$ (with the replecement rule $N \to N_2(\tilde{R})$), we do not use the Floor function in the argument of exact sum S_2, because this would imply a non analiticity in the equation and subsequent complication in numerical studies. This fact introduces an approximation into our scheme which gives us the same quality than the Euler-Maclaurin approximation.

In figure 4.6 we plot the exact value of spin 2 functional trace and the relative approximation with Digamma technique. The quality of the approximation scheme is the same as in Euler-Maclaurin technique.

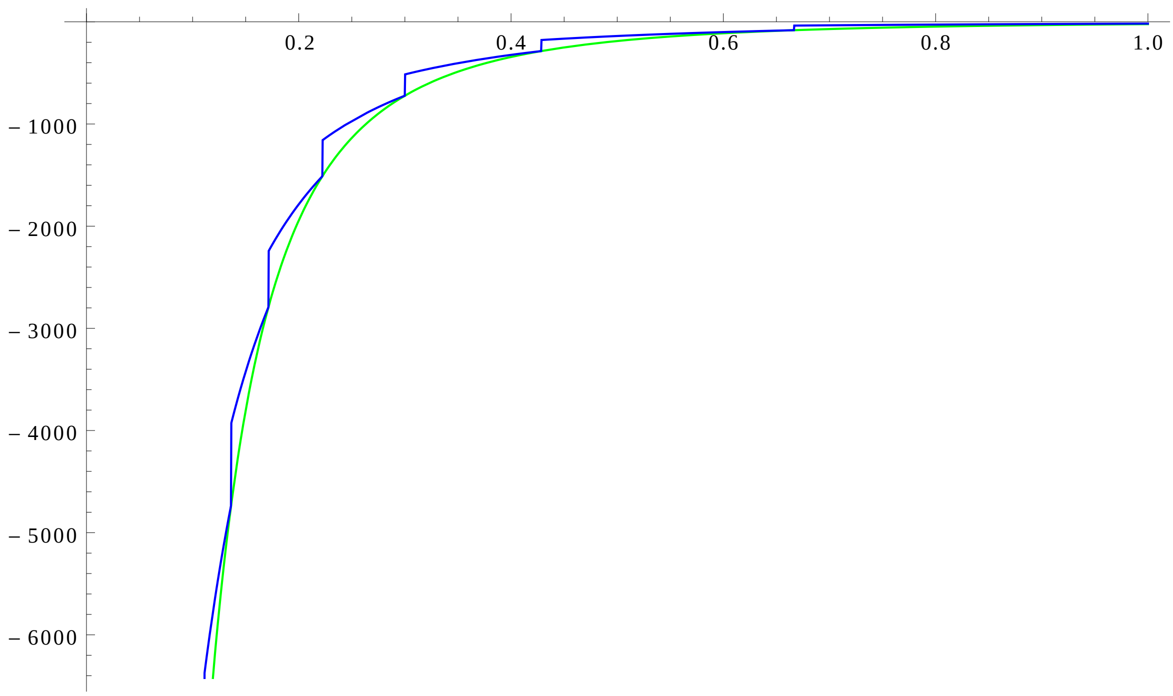


Figure 4.6: In this figure we plot the exact calculation of spin 2 trace contribution and the approximation (4.62) with a sample function function $\tilde{f}(\tilde{R}) = \tilde{R}^2 - \tilde{R} + 1$ and with $\eta_{k,h} = -2$.

Vector, scalar np and anomalous dimensions' contribuions

As it have been pointed out in previous sections, the vector, non-physical scalar and anomalous dimensions' contributions involves only a sum over multiplicities. This sums can be evaluated exactly again in simple term, since $D_{n,s}$ is a polynomial in n. Here we report the resulting trace contributions

$$T^1 = -\sum_{n=1}^{N_1(\tilde{R})} D_{n,1} = \frac{1}{4}(-N_1(\tilde{R})^4 - 8N_1(\tilde{R})^3 - 19N_1(\tilde{R})^2 - 12N_1(\tilde{R}) + 40)$$

$$T^{np} = -\sum_{n=1}^{N_0(\tilde{R})} D_{n,0} = -\frac{1}{12}(N_0(\tilde{R})^4 + 8N_0(\tilde{R})^3 + 23N_0(\tilde{R})^2 - 28N_0(\tilde{R}))$$

$$\begin{aligned} T^\eta =& (2\eta_{k,c} - \eta_{k,h} - \eta_{k,b}) \left[3\sum_{n=1}^{N_0(\tilde{R})} D_{n,0} + \sum_{n=1}^{N_1} D_{n,1} \right] \\ =& \frac{1}{2}(2\eta_{k,c} - \eta_{k,h} - \eta_{k,b}) N_0(\tilde{R}) \left(N_0(\tilde{R}) + 4 \right) \left(5 + N_0(\tilde{R})(N_0(\tilde{R}) + 4) \right) \end{aligned}$$

where

$$N_1(\tilde{R}) = \frac{-3\tilde{R} + \sqrt{16\tilde{R}^2 + 48\tilde{R}}}{2\tilde{R}}$$

$$N_0(\tilde{R}) = \frac{-3\tilde{R} + \sqrt{9\tilde{R}^2 + 48\tilde{R}}}{2\tilde{R}}$$

Scalar $\bar{h}$ contribution

Last, the $\bar{h}$ scalar contribution can be written in terms of the following sum

$$S_{\bar{h}} = \sum_{n=0}^{N} W_{\bar{h}}(\lambda_{n,0}/k^2) D_{n,0} = -\frac{1}{2}\sum_{n=0}^{N} \frac{\left(2 - \eta_{k,h} + \eta_{k,h}\frac{\lambda_{n,0}}{k^2}\right) D_{n,0}}{9\tilde{f}''_k\left(\frac{\lambda_{n,0}}{k^2} - \frac{\tilde{R}}{3}\right)^2 + 3\tilde{f}'_k\left(\frac{\lambda_{n,0}}{k^2} - \frac{\tilde{R}}{3}\right) + 2\tilde{f}_k - \tilde{R}\tilde{f}'_k + \frac{\lambda_{n,0}}{k^2} - 1} \tag{4.63}$$

Decomposing the denominator in terms of its roots, we find the exact evaluation of sum

$$\begin{aligned} S_{\bar{h}}(N) = &- \frac{(N+1)\eta_{k,h}}{3\tilde{f}''_k\tilde{R}}\left(N + 15 + 2x_1 + 2x_2 + 2x_3 + 2x_4\right) \\ &+ \sum_{i=1}^{4} \frac{\left(24 + \eta_{k,h}\tilde{R}x_i(x_i+3)\right)}{3\tilde{f}''_k\tilde{R}^2} \frac{(x_i+1)(x_i+2)(2x_i+3)}{\prod_{i\neq j}(x_i - x_j)} \left(\psi(1 + N - x_i) - \psi(-x_i)\right) \end{aligned} \tag{4.64}$$

in terms of Digamma function, where x_i $(i = 1, 2, 3, 4)$ are the roots of denominator of function $W_{\bar{h}}(z)$, which can be obtained from the four combinations of plus and minus sign of the following expression

$$x_i = -\frac{3}{2} \pm \sqrt{\frac{25\tilde{f}''_k\tilde{R} - 8\tilde{f}'_k + \pm 8\sqrt{\tilde{f}'_k(4\tilde{f}''_k\tilde{R} + \tilde{f}'_k + 2) - 4\tilde{f}''_k(2\tilde{f} + \tilde{R} - 3) + 1}}{12\tilde{f}''_k\tilde{R}}}$$

4.7 Flow equation for f_k with Digamma function

The Wetterich's equation for $\tilde{f}_k$ with Digamma special function becomes

$$\partial_t\Gamma_k = \frac{k^4}{\tilde{\kappa}_k^2}V_{4S}\left[\partial_t\tilde{f}_k(\tilde{R}) - 2\tilde{R}\tilde{f}'_k(\tilde{R}) + \left(4 - 2\frac{\partial_t\tilde{\kappa}_k}{\tilde{\kappa}_k}\right)\tilde{f}_k(\tilde{R})\right] = T^2 + T^1 + T^{np} + T^{\bar{h}} + T^{\eta} \tag{4.65}$$

where

$$\begin{aligned} T^2 =& \frac{5N(N-1)(N+4)(N+5)\left(\tilde{R}(N+1)(N+3) - 18\right)}{216\left(2(\tilde{f}_k + \tilde{f}'_k) - \tilde{R}\tilde{f}'_k\right)} \\ &\times \left[\frac{36\tilde{f}''_k}{\left(\tilde{R}(N+1)(N+3) - 18\right)} - \partial_t\tilde{f}'_k + \eta_{k,h}\tilde{f}'_k\right] \end{aligned} \tag{4.66}$$

$$T^1 = -\frac{1}{4}(N_1(\tilde{R})^4 + 8N_1(\tilde{R})^3 + 19N_1(\tilde{R})^2 + 12N_1(\tilde{R}) - 40)$$

$$T^{np} = -\frac{1}{12}(N_0(\tilde{R})^4 + 8N_0(\tilde{R})^3 + 23N_0(\tilde{R})^2 - 28N_0(\tilde{R}))$$

$$T^{\eta} = \frac{1}{2}(2\eta_{k,c} - \eta_{k,h} - \eta_{k,b})N_0(\tilde{R})\left(N_0(\tilde{R}) + 4\right)\left(5 + N_0(\tilde{R})(N_0(\tilde{R}) + 4)\right)$$

$$\begin{aligned} T^{\bar{h}}(\tilde{R}) = &- \frac{(N_0\tilde{R} + 1)\eta_{k,h}}{3\tilde{f}_k''\tilde{R}}\left(N_0\tilde{R} + 15 + 2x_1 + 2x_2 + 2x_3 + 2x_4\right) \\ &+ \sum_{i=1}^{4} \frac{\left(24 + \eta_{k,h}\tilde{R}x_i(x_i+3)\right)}{3\tilde{f}_k''\tilde{R}^2} \frac{(x_i+1)(x_i+2)(2x_i+3)}{\prod_{i\neq j}(x_i - x_j)}\left(\psi(1 + N_0(\tilde{R}) - x_i) - \psi(-x_i)\right) \end{aligned} \tag{4.67}$$

where x_i $(i = 1, 2, 3, 4)$ are the roots of denominator of function $W_{\bar{h}}(z)$, which can be obtained from the four combinations of plus and minus sign of the following expression

$$x_i = -\frac{3}{2} \pm \sqrt{\frac{25\tilde{f}_k''\tilde{R} - 8\tilde{f}_k' + \pm 8\sqrt{\tilde{f}_k'(4\tilde{f}_k''\tilde{R} + \tilde{f}_k' + 2) - 4\tilde{f}_k''(2\tilde{f} + \tilde{R} - 3) + 1}}{12\tilde{f}_k''\tilde{R}}}$$

and last the upper bound of the sums

$$N_1(\tilde{R}) = \frac{-3\tilde{R} + \sqrt{16\tilde{R}^2 + 48\tilde{R}}}{2\tilde{R}}$$

$$N_2(\tilde{R}) = N_0(\tilde{R}) = \frac{-3\tilde{R} + \sqrt{9\tilde{R}^2 + 48\tilde{R}}}{2\tilde{R}}$$

One may use this equation as a starting point for an alternative numerical analysis. Again with the choice done for the cutoff scheme the flow equation depends on derivative of $f_k(R)$ of order non higher than 2.

Conclusions

This thesis is devoted to the functional RG approach to the quantum field theory of general relativity. To be a consistent candidate to the quantum theory of gravitation the QFT of GR have to be an asymptotically safe theory, a paradigm which extend the well known notion of asymptotic freedom in the non-perturbative domain. According to Weinberg, a quantum field theory can be called asymptotically safe if the dimensionless coupling constants tend to fixed values as the momentum scale goes to infinity; then all measurable quantities remain finite at all energies.

The functional RG approach is a powerfull method to study the non-perturbative regime of any quantum field theory. To extend this formalism to curved space time in a diffeomorfisms invariant way we use the background field method, first introduced in non-Abelian gauge theories.

As it is pointed out in chapter 1, we choose a FRG technique for the average effective action and consider the flow equation for it. The exact equation is impossible to be solved so that one is forced to make an ansatz on the average effective action; in this work we choose a function of the scalar curvature R only, as in $\Gamma_k \sim \int dx\sqrt{g} f_k(R)$. The first work in this direction was carried on in [5, 14], wherein the authors use a "third order" cutoff and derive a flow equation for the function $f_k(R)$ which is a third order differential equation, where the Heat Kernel technique is used in the trace evaluations.

In chapter 2 and 3 we extend the equation given in [5, 6] considering a different metric decomposition and introducing the anomalous dimensions contributions to the flow equationi. We derive the corresponding differential equation, which is of third order due to the cutoff scheme choice, with the Heat Kernel technique in chapter 2 and with "asymptotic behaviour" approximation in chapter 3, as used for the first time in [5, 14] and [6], respectively.

In chapter 3, we also extend the flow equation to general d spacetime dimensions and verify, term by term, that the approximation used is still valid in general dimensions.

The $f_k(R)$ truncation ansatz allows also to study the polynomial truncation where the function f_k is expanded in power series of scalar curvature; in [5], for the first time, they considered a polynomial troncation up to order $n = 9$ and found a non-Gaussian UV fixed point solution with three attractive directions.

In chapter 4 we move to study a new cutoff scheme which is able to lead to a differential flow equation of second order in $f_k(R)$. Hence, in this thesis, we consider the case of "second order" cutoff in the sense that does not depend on $f_k''(R)$ and propose an alternative evaluation for the functional traces present in the flow equation. With Litim's optimized cutoff function the functional traces can be considered as finite sums where the upper bound depends on the dimensionless scalar curvature.

We consider two approximation schemes for the finite sums. First, we used the Euler-Maclaurin formula and observed that the quality of the approximation is very good. With this cutoff scheme, we derive a second order differential equation for $f_k(R)$ instead of third order one, which can be used as a proposal for a future work based on numerical investigation for searching a global solution for the scaling equation. We also propose a numerical study for the polynomial truncation and compare our results with those obtained in the previous chapters and in literature. We found, as expected, the presents of a non-Gaussian UV fixed point up to order $n = 6$ with three attractive directions, which confirms the previous results.

Secondly, we propose an alternative traces evaluations of spectral sums with expression containing the Digamma function. Althought the sums are computed analytically, this method introduces an approximation when we consider that the upper bound depends on dimensionless scalar curvature. Contrary to the Euler-Maclaurin approximation, this second technique involves a special function which complicate the flow equation. Hence, we consider the Euler-Maclaurin approximation method a very good tool for the trace evaluations.

Appendices

Appendix A

Riemann tensor variation

Riemann tensor and coefficients of Levi-Civita connection

$$\Gamma^\rho{}_{\alpha\beta} = \frac{1}{2} g^{\rho\sigma}(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta})$$

$$R^\rho{}_{\sigma\alpha\beta} = \partial_\alpha \Gamma^\rho{}_{\beta\sigma} - \partial_\beta \Gamma^\rho{}_{\alpha\sigma} + \Gamma^\rho{}_{\lambda\alpha}\Gamma^\lambda{}_{\beta\sigma} - \Gamma^\rho{}_{\lambda\beta}\Gamma^\lambda{}_{\alpha\sigma}$$

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} \qquad\qquad R = g^{\mu\nu} R_{\mu\nu}$$

Let us calculate metric tensor, coefficients of connection and Riemann tensor variation, starting from

$$\delta g_{\mu\nu} = h_{\mu\nu} \qquad \delta g^{\mu\nu} = -h^{\mu\nu}$$

$$\begin{aligned}
\delta\Gamma^\rho{}_{\alpha\beta} =& \frac{1}{2} g^{\rho\sigma}(\partial_\alpha h_{\sigma\beta} + \partial_\beta h_{\sigma\alpha} - \partial_\sigma h_{\alpha\beta}) - \frac{1}{2} h^{\rho\sigma}(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}) \\
=& \frac{1}{2}\left[g^{\rho\sigma}\partial_\alpha h_{\sigma\beta} + g^{\rho\sigma}\partial_\beta h_{\sigma\alpha} - g^{\rho\sigma}\partial_\sigma h_{\alpha\beta}\right] - g^{\rho\sigma}\Gamma^\lambda{}_{\alpha\beta} h_{\sigma\lambda} \\
=& \frac{1}{2} g^{\rho\sigma}\left(\nabla_\alpha h_{\sigma\beta} + \nabla_\beta h_{\alpha\sigma} - \nabla_\sigma h_{\alpha\beta}\right)
\end{aligned}$$

which, as expected, reveals that the variation of a connection is manifestly a tensor. In derivation we considered $[\delta, \partial_\alpha] = 0$, but $[\delta, \nabla_\alpha] \neq 0$, since

$$\begin{aligned}
&[\delta, \nabla_\alpha] v_\beta = -\delta\Gamma^\lambda{}_{\alpha\beta} v_\lambda \\
&[\delta, \nabla_\alpha] t_{\rho\sigma} = -\delta\Gamma^\lambda{}_{\alpha\rho} t_{\lambda\sigma} - \delta\Gamma^\lambda{}_{\alpha\sigma} t_{\rho\lambda}
\end{aligned}$$

Riemann tensor variation

$$\begin{aligned}
\delta R^\rho{}_{\sigma\alpha\beta} =& \partial_\alpha \delta\Gamma^\rho{}_{\beta\sigma} - \partial_\beta \delta\Gamma^\rho{}_{\alpha\sigma} + (\delta\Gamma^\rho{}_{\lambda\alpha})\Gamma^\lambda{}_{\beta\sigma} + \Gamma^\rho{}_{\lambda\alpha}\delta\Gamma^\lambda{}_{\beta\sigma} - (\delta\Gamma^\rho{}_{\lambda\beta})\Gamma^\lambda{}_{\alpha\sigma} - \Gamma^\rho{}_{\lambda\beta}\delta\Gamma^\lambda{}_{\alpha\sigma} \\
=& \nabla_\alpha(\delta\Gamma^\rho{}_{\beta\sigma}) - \nabla_\beta(\delta\Gamma^\rho{}_{\alpha\sigma}) \\
=& \frac{1}{2}\left[\nabla_\alpha\nabla_\sigma h^\rho{}_\beta + \nabla_\beta\nabla^\rho h_{\alpha\sigma} - \nabla_\alpha\nabla^\rho h_{\beta\sigma} - \nabla_\beta\nabla_\sigma h^\rho{}_\alpha + R^\rho{}_{\lambda\alpha\beta} h^\lambda{}_\sigma - R^\lambda{}_{\sigma\alpha\beta} h^\rho{}_\lambda\right]
\end{aligned}$$

Ricci and scalar curvature variation follow

$$\begin{aligned}\delta R_{\mu\nu} = \delta R^{\rho}{}_{\mu\rho\nu} =& \nabla_\rho(\delta\Gamma^{\rho}{}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^{\rho}{}_{\mu\rho})\\ =& \frac{1}{2}\left[\nabla_\rho\nabla_\mu h^{\rho}{}_{\nu} + \nabla_\nu\nabla_\rho h^{\rho}{}_{\mu} - \nabla^2 h_{\mu\nu} - \nabla_\nu\nabla_\mu h + R_{\rho\nu}h_{\mu}{}^{\rho} - R^{\lambda}{}_{\mu\rho\nu}h_{\lambda}{}^{\rho}\right]\end{aligned}$$

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = -h^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} = -h^{\mu\nu}R_{\mu\nu} + \nabla_\rho\nabla_\sigma h^{\rho\sigma} - \nabla^2 h$$

Let us start with second variation, taking into account

$$\delta^{(2)}g_{\mu\nu} = \delta h_{\mu\nu} = 0$$

which implies

$$\delta^{(2)}g^{\mu\nu} = -\delta h^{\mu\nu} = -\delta(g^{\mu\rho}g^{\nu\sigma}h_{\rho\sigma}) = 2h^{\mu\alpha}h^{\nu}{}_{\alpha}$$

and

$$\delta h = \delta(g_{\mu\nu}h^{\mu\nu}) = -h_{\mu\nu}h^{\mu\nu}$$

From the commutator (A) we can construct the second variation for coefficients of connection

$$\begin{aligned}\delta^{(2)}\Gamma^{\rho}{}_{\alpha\beta} &= -\frac{1}{2}h^{\rho\sigma}(\nabla_\alpha h_{\beta\sigma} + \nabla_\beta h_{\alpha\sigma} - \nabla_\sigma h_{\alpha\beta}) + \frac{1}{2}g^{\rho\sigma}\delta(\nabla_\alpha h_{\beta\sigma} + \nabla_\beta h_{\alpha\sigma} - \nabla_\sigma h_{\alpha\beta})\\ &= -\frac{1}{2}h^{\rho\sigma}(\nabla_\alpha h_{\beta\sigma} + \nabla_\beta h_{\alpha\sigma} - \nabla_\sigma h_{\alpha\beta}) - g^{\rho\sigma}\delta\Gamma^{\lambda}{}_{\alpha\beta}h_{\sigma\lambda}\\ &= -h^{\rho\sigma}(\nabla_\alpha h_{\beta\sigma} + \nabla_\beta h_{\alpha\sigma} - \nabla_\sigma h_{\alpha\beta})\end{aligned}$$

and for Ricci curvature scalar

$$\begin{aligned}\delta^{(2)}R =& -R_{\mu\nu}\delta h^{\mu\nu} - h^{\mu\nu}\delta R_{\mu\nu} + \delta\nabla_\rho\nabla_\sigma h^{\rho\sigma} - \delta\nabla^2 h\\ =& \, 2h^{\mu\nu}h_{\alpha\nu}R^{\alpha}{}_{\mu} - h^{\mu\nu}\delta R_{\mu\nu} + \nabla_\rho\delta\nabla_\sigma h^{\rho\sigma} - \delta\Gamma^{\lambda}{}_{\rho\sigma}\nabla_\lambda + \delta\Gamma^{\rho}{}_{\rho\lambda}\nabla_\sigma h^{\lambda\sigma} + \delta\Gamma^{\sigma}{}_{\rho\lambda}\nabla_\sigma h^{\rho\lambda}\\ & - \nabla_\rho\delta\nabla^{\rho}h - \delta\Gamma^{\rho}{}_{\rho\lambda}\nabla^{\lambda}h\\ =& \, 2h^{\mu\nu}h_{\alpha\nu}R^{\alpha}{}_{\mu} - h^{\mu\nu}\delta R_{\mu\nu} + \nabla_\rho\nabla_\sigma\delta h^{\rho\sigma} + \nabla_\rho\left[\delta\Gamma^{\rho}{}_{\lambda\sigma}h^{\lambda\sigma}\right] + \nabla_\rho\left[\delta\Gamma^{\sigma}{}_{\lambda\sigma}h^{\rho\lambda}\right]\\ & + \delta\Gamma^{\rho}{}_{\lambda\rho}\nabla_\sigma h^{\lambda\sigma} - \nabla^2\delta h - \delta\Gamma^{\rho}{}_{\lambda\rho}\nabla^{\lambda}h\\ =& \, 2h^{\mu\nu}h_{\alpha\nu}R^{\alpha}{}_{\mu} - \frac{1}{2}h^{\mu\nu}\left[\nabla_\lambda\nabla_\mu h^{\lambda}{}_{\nu} + \nabla_\nu\nabla^{\lambda}h_{\mu\lambda} - \nabla^2 h_{\mu\nu} - \nabla_\mu\nabla_\nu h\right.\\ & \left. + R_{\lambda\nu}h^{\lambda}{}_{\mu} - R^{\lambda}{}_{\mu\sigma\nu}h^{\sigma}{}_{\lambda}\right] - \frac{1}{2}h_{\mu\nu}\nabla^2 h^{\mu\nu} + \frac{1}{2}h^{\rho\sigma}\nabla_\rho\nabla_\sigma h + h^{\alpha\sigma}\nabla_\rho\nabla_\sigma h^{\rho}{}_{\alpha}\\ & + (\nabla_\rho h^{\rho\lambda})(\nabla_\lambda h) - \frac{1}{2}(\nabla_\sigma h)(\nabla^{\sigma}h) + (\nabla_\sigma h^{\rho}{}_{\lambda})(\nabla_\rho h^{\lambda\sigma}) - \frac{1}{2}(\nabla_\rho h^{\sigma\lambda})(\nabla^{\rho}h_{\sigma\lambda})\\ & - 2\nabla_\rho\nabla_\sigma[h^{\rho}{}_{\alpha}h^{\sigma\alpha}] + \nabla^2[h_{\mu\nu}h^{\mu\nu}]\\ =& \, h^{\mu\nu}\nabla_\mu\nabla_\nu h + \frac{3}{2}h^{\mu}h_{\alpha\nu}R^{\alpha}{}_{\mu} + \frac{1}{2}h_{\mu\nu}R^{\mu\rho\nu\sigma}h_{\rho\sigma} - \frac{3}{2}h^{\mu\nu}\nabla_\lambda\nabla_\mu h^{\lambda}{}_{\nu}\\ & - \frac{5}{2}h_{\mu\nu}\nabla^{\nu}\nabla^{\lambda}h_{\lambda}{}^{\mu} + (\nabla_\rho h^{\rho\lambda})(\nabla_\lambda h) - \frac{1}{2}(\nabla_\sigma h)(\nabla^{\sigma}h) + \frac{3}{2}(\nabla_\rho h^{\lambda\sigma})(\nabla^{\rho}h_{\lambda\sigma})\\ & + 2h_{\mu\nu}\nabla^2 h^{\mu\nu} - 2(\nabla_\rho h^{\rho\lambda})(\nabla_\sigma h_{\lambda}{}^{\sigma}) - (\nabla_\rho h^{\lambda\sigma})(\nabla_\sigma h^{\rho}{}_{\lambda})\end{aligned}$$

For relation (2.50) we need the quantities

$$\delta(\sqrt{g}) = \frac{1}{2\sqrt{g}}\delta g = \frac{1}{2}\sqrt{g}g^{\mu\nu}h_{\mu\nu}$$
$$\delta^{(2)}(\sqrt{g}) = \frac{1}{4}\sqrt{g}\left(g^{\mu\nu}h_{\mu\nu}g^{\rho\sigma}h_{\rho\sigma} - 2h^{\mu\nu}h_{\mu\nu}\right)$$

which allows us to construct

$$\begin{aligned}
2\frac{\delta(\sqrt{g})}{\sqrt{g}} + \delta^{(2)}R &= h\delta R + \delta^{(2)}R \\
&= -R^{\mu\nu}h_{\mu\nu}h + h\nabla_\mu\nabla_\nu h^{\mu\nu} - h\nabla^2 h + \frac{3}{2}h^{\mu\nu}h_{\alpha\nu}R^\alpha{}_\mu \\
&\quad + \frac{1}{2}h_{\mu\nu}R^{\mu\rho\nu\sigma}h_{\rho\sigma} - \frac{3}{2}h^{\mu\nu}\nabla_\lambda\nabla_\mu h^\lambda{}_\nu - \frac{5}{2}h_{\mu\nu}\nabla^\nu\nabla^\lambda h_\lambda{}^\mu \\
&\quad + (\nabla_\rho h^{\rho\lambda})(\nabla_\lambda h) - \frac{1}{2}(\nabla_\sigma h)(\nabla^\sigma h) + \frac{3}{2}(\nabla_\rho h^{\lambda\sigma})(\nabla^\rho h_{\lambda\sigma}) \\
&\quad + 2h_{\mu\nu}\nabla^2 h^{\mu\nu} - 2(\nabla_\rho h^{\rho\lambda})(\nabla_\sigma h_\lambda{}^\sigma) - (\nabla_\rho h^{\lambda\sigma})(\nabla_\sigma h^\rho{}_\lambda) \\
&= -R^{\mu\nu}h_{\mu\nu}h + h\nabla_\mu\nabla_\nu h^{\mu\nu} - h\nabla^2 h + \frac{3}{2}h^{\mu\nu}h_{\alpha\nu}R^\alpha{}_\mu \\
&\quad + \frac{1}{2}h_{\mu\nu}R^{\mu\rho\nu\sigma}h_{\rho\sigma} - \frac{1}{2}h^{\mu\nu}\nabla_\lambda\nabla_\mu h^\lambda{}_\nu - \frac{1}{2}h^{\mu\nu}\nabla_\mu\nabla_\lambda h^\lambda{}_\nu \\
&= -R^{\mu\nu}h_{\mu\nu}h - \frac{1}{2}h\nabla^2 h + \frac{1}{2}h_{\mu\nu}\nabla^2 h^{\mu\nu} + h^{\mu\alpha}h_{\alpha\beta}R^\beta_\mu \\
&\quad + h_{\mu\nu}R^{\mu\rho\nu\sigma}h_{\rho\sigma} - h^\nu_\mu\nabla^\mu\nabla^\rho h_{\rho\nu} + h\nabla^\mu\nabla^\nu h_{\mu\nu}
\end{aligned}$$

and

$$\begin{aligned}
(\delta R)^2 &= \left[-R^{\mu\nu}h_{\mu\nu} + \nabla^\mu\nabla^\nu h_{\mu\nu} - \nabla^2 h\right]\left[-R^{\alpha\beta}h_{\alpha\beta} + \nabla^\alpha\nabla^\beta h_{\alpha\beta} - \nabla^2 h\right] \\
&\doteq -R^{\mu\nu}R^{\alpha\beta}h_{\mu\nu}h_{\alpha\beta} - 2R^{\mu\nu}h_{\mu\nu}\nabla_\alpha\nabla_\beta h^{\alpha\beta} + 2R^{\mu\nu}h_{\mu\nu}\nabla^2 h \\
&\quad + (\nabla^\mu\nabla^\nu h_{\mu\nu})(\nabla^\alpha\nabla^\beta h_{\alpha\beta}) + h(\nabla^2)^2 h - 2h\nabla^2\nabla_\mu\nabla_\nu h^{\mu\nu}
\end{aligned}$$

where "$\doteq$" means "$=$" up to four divergences.

Appendix B

Calculation of $\Gamma_k^{(2)}$ in $f(R)$ approximation

In this appendix we explicite the calculation for second variation of average effective action within the ansatz (3.1). Starting from (2.50) we will derive the expressions (2.51- 2.55) using the transverse-traceless decomposition (2.23).

We start from the transverse-traceless component of $\mathbf{\Gamma}_k^{(2)}$, we get term by term

$$
\begin{aligned}
&f_k(R)\left(\frac{1}{2}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right) \to -\frac{1}{2}h^T_{\mu\nu}h^{T\mu\nu}f_k(R)\\
&\frac{1}{2}f'_k(R)h_{\mu\nu}h^{\mu\nu} \longrightarrow \frac{1}{2}f'_k(R)h^T_{\mu\nu}h^{T\mu\nu}\\
&f'_k(R)h^{\mu\alpha}h_{\alpha\beta}R^\beta_\mu \longrightarrow f'_k(R)h^{\mu\alpha}h_{\alpha\beta}R^\beta_\mu = \frac{R}{d}h^T_{\mu\nu}h^{T\mu\nu}\\
&h_{\mu\nu}R^{\mu\rho\nu\sigma}h_{\rho\sigma} \longrightarrow h^T_{\mu\nu}R^{\mu\rho\nu\sigma}h^T_{\rho\sigma} = -\frac{R}{d(d-1)}h^T_{\mu\nu}h^{T\mu\nu}
\end{aligned}
$$

the other terms are identically zero for the transverse or traceless properties (2.24) of $h^T_{\mu\nu}$. summing all together, the transverse traceless component gives

$$
h^T_{\mu\nu}f'_k(R)\left(\frac{1}{2}\nabla^2 + \frac{d-2}{d(d-1)}R\right) - \frac{1}{2}f_k(R)h^T_{\mu\nu}h^{T\mu\nu} \tag{B.1}
$$

which gives (2.51).

For the vector part we have

$$
\begin{aligned}
&f_k(R)\left(-\frac{1}{2}h_{\mu\nu}h^{\mu\nu}\right) \to f_k\xi_\mu\nabla^2\xi^\mu + f_k\xi_\nu\nabla_\mu\nabla^\nu\xi^\mu = f_k\xi_\mu\left(\nabla^2+\frac{R}{d}\right)\xi^\mu \\
&\frac{1}{2}f_k'(R)h_{\mu\nu}\nabla^2h^{\mu\nu} \to -f_k'\xi_\mu\nabla_\nu\nabla^2\nabla^\nu\xi^\mu - f_k'\xi_\mu\nabla_\nu\nabla^2\nabla^\mu\xi^\nu = \\
&\qquad = -f_k'\xi_\mu\left[(\nabla^2)^2+\frac{R}{d}\nabla^2+\frac{2R^2}{d^2(d-1)}\right]\xi^\mu = -f_k'\xi_\mu\left[\frac{R}{d}\nabla^2+\frac{R^2}{d^2}+\frac{2R}{d(d-1)}\nabla^2\right]\xi^\mu \\
&f_k'h^{\mu\alpha}h_{\alpha\beta}R^\beta_\mu \to -f_k'\frac{2R}{d}\xi_\mu\left(\nabla^2+\frac{R}{d}\right)\xi^\mu \\
&f_k'h_{\mu\nu}R^{\mu\rho\nu\sigma}h_{\rho\sigma} \to f_k'\frac{2R}{d(d-1)}\xi_\mu\left(\nabla^2+\frac{R}{d}\right)\xi^\mu \\
&-f_k'h^\nu_\mu\nabla^\mu\nabla^\rho h_{\rho\nu} \to f_k'\xi_\mu\left[(\nabla^2)^2+\frac{2R}{d}\nabla^2+\frac{R^2}{d^2}\right]\xi^\mu
\end{aligned}
$$

gauge fixing term

$$
\begin{aligned}
&F_\mu = \nabla^\rho h_{\rho\mu} - \frac{1+\rho}{d}\nabla_\mu h \to \nabla^\rho(\nabla_\rho\xi_\mu+\nabla_\mu\xi_\rho) = \left(\nabla^2+\frac{R}{d}\right)\xi_\mu \\
&F_\mu(\alpha+\beta\nabla^2)F^\mu \to \xi_\mu\left(\nabla^2+\frac{R}{d}\right)(\alpha+\beta\nabla^2)\left(\nabla^2+\frac{R}{d}\right)\xi^\mu
\end{aligned}
$$

all togheder gives

$$
-\frac{2R}{d}\xi_\mu\left(\nabla^2+\frac{R}{d}\right)\xi^\mu + \xi_\mu\left(\nabla^2+\frac{R}{d}\right)(\alpha+\beta\nabla^2)\left(\nabla^2+\frac{R}{d}\right)\xi^\mu \tag{B.2}
$$

which is equivalent to (2.52).

The scalar $h-h$ part gives

$$
\begin{aligned}
&f_k\left(\frac{1}{4}h^2-\frac{1}{2}h_{\mu\nu}h^{\mu\nu}\right)\to\frac{d-2}{4d}f_kh^2\\
&-f_k'R_{\mu\nu}h^{\mu\nu}h\to-\frac{R}{d}h^2\\
&-\frac{f_k'}{2}h\nabla^2h\\
&f_k'\frac{1}{2}h_{\mu\nu}\nabla^2h^{\mu\nu}\to f_k'\frac{1}{2d}h\nabla^2h\\
&f_k'h^{\mu\alpha}h_{\alpha\beta}R^\beta_\mu\to f_k'\frac{R}{d^2}h^2\\
&f_k'h_{\mu\nu}R^{\mu\rho\nu\sigma}h_{\rho\sigma}\to f_k'\frac{R}{d^2}h^2\\
&-f_k'h^\nu_\mu\nabla^\mu\nabla^\rho h_{\rho\nu}\to-\frac{f_k'}{d}h\nabla^2h\\
&f_k'h\nabla^\mu\nabla^\nu h_{\mu\nu}\to\frac{f_k'}{d}h\nabla^2h\\
&f_k''R^{\alpha\beta}h_{\alpha\beta}R^{\mu\nu}h_{\mu\nu}\to f_k''\frac{R^2}{d^2}h^2\\
&-2f_k''R^{\mu\nu}h_{\mu\nu}\nabla^\alpha\nabla^\beta h_{\alpha\beta}\to-2f_k''\frac{R}{d^2}h\nabla^2h\\
&2f_k''R^{\mu\nu}h_{\mu\nu}\nabla^2h\to2f_k''\frac{R}{d}h\nabla^2h\\
&f_k''h_{\mu\nu}\nabla^\mu\nabla^\nu\nabla^\alpha\nabla^\beta h_{\alpha\beta}\to\frac{f_k''}{d^2}h(\nabla^2)^2h\\
&-2f_k''h\nabla^2\nabla^\alpha\nabla^\beta h_{\alpha\beta}\to-\frac{2f_k''}{d}h(\nabla^2)^2h\\
&f_k''h(\nabla^2)^2h
\end{aligned}
$$

summing all we find

$$
\begin{aligned}
\frac{f_k''}{d^2}h\Big[(d-1)^2(\nabla^2)^2+2(d-1)R\nabla^2+R^2\Big]h+\frac{d-2}{4d}f_kh^2\\
-f_k'\frac{(d-1)(d-2)}{2d^2}h\left(\nabla^2+\frac{2R}{d-1}\right)h
\end{aligned}
$$

equivalent to (2.53).

For the scalar σ part we use relations

$$
\nabla^2\nabla_\mu\left[\sigma f^\mu\right]=f^\mu\nabla^2\nabla_\mu\sigma+\sigma\nabla_\mu\nabla^2f^\mu+\text{four divergences}
$$

$$
\nabla^\rho h_{\rho\nu}\to\nabla^2\nabla_\nu\sigma-\frac{1}{d}\nabla_\nu\nabla^2\sigma=\frac{d-1}{d}\nabla_\nu\left(\nabla^2+\frac{R}{d-1}\right)\sigma
$$

where f_μ is a four vector and s is a scalar. So that

$$
\begin{aligned}
&-\frac{f_k}{2}h_{\mu\nu}h^{\mu\nu} \to -f_k\frac{d-1}{2d}\sigma\left(\nabla^2+\frac{R}{d-1}\right)\nabla^2\sigma \\
&\frac{f_k'}{2}h_{\mu\nu}\nabla^2 h^{\mu\nu} \to \frac{f_k'}{2}\sigma\nabla_\mu\nabla_\nu\nabla^2\nabla^\mu\nabla^\nu - \frac{f_k'}{2d}\sigma(\nabla^2)^3\sigma \\
&\quad = \frac{f_k'}{2}\sigma\nabla_\mu(\nabla^2)^2\nabla^\mu\sigma + f_k'\frac{R}{2d}\sigma\nabla_\mu\nabla^2\nabla^\mu\sigma + f_k'\frac{R^2}{d^2(d-1)}\sigma\nabla^2\sigma - \frac{f_k'}{2d}\sigma(\nabla^2)^3\sigma \\
&\quad = f_k'\frac{(d-1)}{2d}\sigma(\nabla^2)^3\sigma + \frac{3f_k'}{2}\frac{R}{d}\sigma(\nabla^2)^2\sigma + f_k'\frac{R^2}{d(d-1)}\sigma\nabla^2\sigma \\
&f_k' h^{\mu\alpha}h_{\alpha\beta}R^\beta_\mu = f_k'\frac{R}{d}h_{\mu\nu}h^{\mu\nu} \to f_k'\frac{R}{d}\sigma\left[\frac{d-1}{d}\nabla^2+\frac{R}{d}\right]\nabla^2\sigma \\
&f_k' h_{\mu\nu}R^{\mu\rho\nu\sigma}h_{\rho\sigma} = -f_k'\frac{R}{d(d-1)}h_{\mu\nu}h^{\mu\nu} \to -f_k'\frac{R}{d^2}\sigma(\nabla^2)^2\sigma - f_k'\frac{R^2}{d^2(d-1)}\sigma\nabla^2\sigma \\
&-f_k' h^\nu_\mu\nabla^\mu\nabla^\rho h_{\rho\nu} \to -f_k'\frac{(d-1)^2}{d^2}\left(\nabla^2+\frac{R}{d-1}\right)^2\nabla^2\sigma \\
&f_k'' h_{\mu\nu}\nabla^\mu\nabla^\nu\nabla^\alpha\nabla^\beta h_{\alpha\beta} \doteq f_k''(\nabla^\mu\nabla^\nu h_{\mu\nu})^2 \to f_k''\frac{(d-1)^2}{d^2}\sigma\left(\nabla^2+\frac{R}{d-1}\right)^2(\nabla^2)^2\sigma
\end{aligned}
$$

for gauge fixing term

$$
\begin{aligned}
&(\nabla^\rho h_{\rho\sigma})(\alpha+\beta\nabla^2)(\nabla_\mu h^{\mu\sigma}) \to \frac{(d-1)^2}{d^2}\sigma\left(\nabla^2+\frac{R}{d-1}\right)\nabla_\sigma(\alpha+\beta\nabla^2)\nabla^\sigma\left(\nabla^2+\frac{R}{d-1}\right)\sigma \\
&= \frac{(d-1)^2}{d^2}\sigma\left(\nabla^2+\frac{R}{d-1}\right)^2\left(\alpha+\beta\left(\nabla^2+\frac{R}{d}\right)\right)\nabla^2\sigma
\end{aligned}
$$

where "$\doteq$" means "$=$" up to four divergences. The total σ contribution gives

$$
\begin{aligned}
&-f_k'\sigma\left[\frac{(d-1)(d-2)}{2d^2}(\nabla^2)^3 - \frac{R}{2d}(\nabla^2)^2 - \frac{R^2}{d^2}\nabla^2\right] \\
&\quad - f_k\frac{d-1}{2d}\sigma\left(\nabla^2+\frac{R}{d-1}\right)\nabla^2\sigma + f_k''\frac{(d-1)^2}{d^2}\sigma\left(\nabla^2+\frac{R}{d-1}\right)^2(\nabla^2)^2 \\
&\quad + \frac{(d-1)^2}{d^2}\sigma\left(\nabla^2+\frac{R}{d-1}\right)^2\left(\alpha+\beta\left(\nabla^2+\frac{R}{d}\right)\right)\nabla^2\sigma
\end{aligned}
$$

Last, the scalar mixing $h-\sigma$ contribution

$$
\begin{aligned}
&-f_k' h_\mu^\nu \nabla^\mu \nabla^\rho h_{\rho\nu} \to -2f_k' h \left[\frac{d-1}{d^2}\nabla^2 + \frac{R}{d^2}\right]\nabla^2\sigma \\
&f_k' h \nabla^\mu\nabla^\nu h_{\mu\nu} \to f_k' h \nabla^\mu\nabla^\nu l \left(\nabla_\mu\nabla_\nu - \frac{1}{d}g_{\mu\nu}\nabla^2\right)\sigma = f_k' \frac{d-1}{d} h \nabla^2\left(\nabla^2 + \frac{R}{d-1}\right)\sigma \\
&-2f_k'' R^{\mu\nu} h_{\mu\nu}\nabla^\alpha\nabla^\beta h_{\alpha\beta} \to -2f_k''\frac{R}{d} h\nabla_\mu\nabla^2\nabla^\mu\sigma + 2f_k''\frac{R}{d^2}h(\nabla^2)^2\sigma \\
&\qquad\qquad - 2f_k''\frac{R}{d}h(\nabla^2)^2\sigma + 2f_k''\frac{R}{d^2}h(\nabla^2)^2\sigma - 2f_k''\frac{R^2}{d^2}h\nabla^2\sigma \\
&-2f_k'' h\nabla^2\nabla_\beta\nabla_\alpha h^{\alpha\beta} \to -2f_k''\frac{d-1}{d}h(\nabla^2)^3\sigma - 2f_k''\frac{R}{d}h(\nabla^2)^2\sigma \\
&f_k''(\nabla^\mu\nabla^\nu h_{\mu\nu})^2 \to 2f_k''\frac{d-1}{d^2}h(\nabla^2)^3\sigma + 2f_k''\frac{R}{d^2}h(\nabla^2)^2\sigma
\end{aligned}
$$

for gauge fixing term

$$
\begin{aligned}
&F_\mu \to \frac{d-1}{d}\nabla_\mu\left[\left(\nabla^2 + \frac{R}{d-1}\right)\sigma - \frac{\rho}{d-1}h\right] \\
&F_\mu(\alpha+\beta\nabla^2)F^\mu \to -\frac{(d-1)^2}{d^2}\left[\sigma\left(\nabla^2+\frac{R}{d-1}\right) - \frac{\rho}{d-1}h\right]\nabla_\mu(\alpha+\beta\nabla^2)\nabla^\mu \\
&\qquad\qquad \times\left[\left(\nabla^2+\frac{R}{d-1}\right)\sigma - \frac{\rho}{d-1}h\right] \\
&\quad \to 2\rho\frac{d-1}{d^2}\sigma\left(\alpha+\beta\left(\nabla^2+\frac{R}{d}\right)\right)\nabla^2\left(\nabla^2+\frac{R}{d-1}\right)h
\end{aligned}
$$

The total $h-\sigma$ contribution reads

$$
\begin{aligned}
&-2f_k''\frac{d-1}{d^2}h\left(\nabla^2+\frac{R}{d-1}\right)^2\nabla^2\sigma + f_k' h\left[\frac{(d-1)(d-2)}{d^2}\nabla^2 + R\frac{d-2}{d^2}\right]\nabla^2\sigma \\
&2\rho\frac{d-1}{d^2}\sigma\left(\alpha+\beta\left(\nabla^2+\frac{R}{d}\right)\right)\nabla^2\left(\nabla^2+\frac{R}{d-1}\right)h
\end{aligned}
$$

which gives (2.54).

Next, the trasverse $\bar{C}^T C^T$ ghost part

$$
S_{gh,c} \to \bar{C}^{T\mu}(\alpha+\beta\nabla^2)\left(\nabla^2+\frac{R}{d}\right)C^T_\nu
$$

the longitudinal c part

$$
\begin{aligned}
S_{gh,c} \to &-\bar{c}\nabla_\mu(\alpha+\beta\nabla^2)\left[\left(\nabla^2+\frac{R}{d}\right)\nabla^\mu c + \frac{(d-2-2\rho)}{d}\nabla^\mu\nabla^2 c\right] \\
&-\bar{c}\left(\alpha+\beta\left(\nabla^2+\frac{R}{d}\right)\right)\nabla_\mu\left[\left(\nabla^2+\frac{R}{d}\right)\nabla^\mu c + \frac{(d-2-2\rho)}{d}\nabla^\mu\nabla^2 c\right] \\
&-\bar{c}\left(\alpha+\beta\left(\nabla^2+\frac{R}{d}\right)\right)\left[\left(\nabla^2+\frac{2R}{d}\right) + \frac{(d-2-2\rho)}{d}\nabla^2\right]\nabla^2 c
\end{aligned}
$$

the transverse b^T third ghost

$$S_{gh,b} \to b^T_\mu (\alpha + \beta \nabla^2) b^{T\mu}$$

and the longitudinal b third ghost

$$S_{gh,b} \to -b \nabla_\mu (\alpha + \beta \nabla^2) \nabla^\mu b = -b \left(\alpha + \beta \left(\nabla^2 + \frac{R}{d} \right) \right) \nabla^2 b$$

which concludes the matrix elements of $\mathbf{\Gamma}_k^{(2)}$.

Appendix C

Heat Kernel Technique

In the construction of Wetterich equation, the calculation of functional trace is needed; in particular we find the following expression

$$\mathrm{Tr}\left[W(\Delta)\right] \tag{C.1}$$

in which $W(\Delta)$ is a generic function of the Laplacian $\Delta = -g_{\mu\nu}\nabla^\mu\nabla^\nu$ in a four dimensional sphere S^4. From new on, we consider a d-dimensional sphere S^d e the Laplacian Δ on it. By definition the functional trace reads

$$\mathrm{Tr}\left[W(\Delta)\right] = \sum_i W(\lambda_i) \tag{C.2}$$

where λ_i are the eigenvalue of the Laplacian in S^d.

The first step for the evaluation of trace is to write W in terms of its Laplace anti-transform

$$\mathrm{Tr}\left[W(\Delta)\right] = \mathrm{Tr}\int_0^{+\infty} ds e^{-s\Delta}\tilde{W}(s) = \int_0^{+\infty} ds\tilde{W}(s)\,\mathrm{Tr}\left(e^{-s\Delta}\right) \tag{C.3}$$

so in order to calculate the functional trace of a generic function of Δ we must know only the trace $\mathrm{Tr}\left(e^{-s\Delta}\right)$. The exponential $H(s) = e^{-s\Delta}$ satisfies the heat equation

$$\left(\partial_s + \Delta\right) H(s) = 0 \tag{C.4}$$

Defining the function $H(x, y; s) = \langle x|H(s)|y\rangle$, it satisfies

$$\left(\partial_s + \Delta_x\right) H(x, y; s) = 0 \tag{C.5}$$

which posses a simple solution in the flat case

$$H(x, y; s)\Big|_{\sim flat} = \frac{1}{(4\pi s)^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{4s}} \tag{C.6}$$

Following De Witt, we make an ansatz for the general solution of (C.5)

$$H(x, y; s) = \frac{1}{(4\pi s)^{\frac{d}{2}}} e^{-\frac{\sigma(x,y)}{2s}} \Omega(x, y; s) \tag{C.7}$$

where $\sigma(x, y)$ is a generalization of the Minkowski square distance $(x - y)^2$ and satisfies

$$\frac{1}{2}\nabla_\mu \sigma \nabla^\mu \sigma = \sigma$$

for $x \neq y$ and $\sigma(x, y) = 0$ for $x = y$. The next step is to substitute the ansatz (C.7) into the equation (C.5) and suppose that the function $\Omega(x, y; s)$ can be expanded in Taylor series with respect to s (*local expansion*)

$$\Omega(x, y; s) = \sum_{n \geq 0} A_{2n}(x, y) s^n$$

After some algebra, one obtains a system of coupled equations for the coefficients $A_{2n}(x, y)$, which can be solved recursively in the limit $x \to y$. Fortunately only the coefficients $\boldsymbol{b}_{2n}(x) = A_{2n}(x, x)$ is needed for the evaluation of the trace for $e^{-s\Delta}$

$$\mathrm{Tr}\, H(s) = \int d^d x \sqrt{g} \langle x | H(s) | x \rangle = \frac{1}{(4\pi s)^{\frac{d}{2}}} \sum_n B_{2n} s^n \tag{C.8}$$

where

$$B_n = \int d^d x \sqrt{g}\, \mathrm{tr}\, \boldsymbol{b}_n$$

We report here the first three coefficients negletting the coupling with matter

$$\begin{aligned} \boldsymbol{b}_0 &= \mathbf{1} \\ \boldsymbol{b}_2 &= \frac{R}{6}\mathbf{1} \\ \boldsymbol{b}_4 &= \frac{1}{180}\left(R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} - R^{\mu\nu} R_{\mu\nu} + \frac{5}{2} R^2 + 6\nabla^2 R \right) \mathbf{1} \end{aligned} \tag{C.9}$$

where the **1** is the identity which acts on the space of fields. The relative coefficients with matter and the b_6 can be found in [5] while b_8 in [29].

Inserting (C.8) into (C.3) The functional trace for the generic function $W(\Delta)$ reads

$$\mathrm{Tr}\, W(\Delta) = \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{+\infty} B_{2n} Q_{\frac{d}{2}-n}(W) \tag{C.10}$$

with

$$Q_k(W) = \int_0^{+\infty} dz z^{-k} \tilde{W}(z) \tag{C.11}$$

Spin s	Eigenvalue $\lambda_{l,s}$	Multiplicity $D_{l,s}$
0	$\frac{l(l+d-1)}{d(d-1)}R;\ l=0,1...$	$\frac{(2l+d-1)(l+d-2)!}{l!(d-1)!}$
1	$\frac{l(l+d-1)-1}{d(d-1)}R;\ l=1,2...$	$\frac{l(l+d-1)(2l+d-1)(l+d-3)!}{(d-2)!(l+1)!}$
2	$\frac{l(l+d-1)-2}{d(d-1)}R;\ l=2,3...$	$\frac{(d+1)(d-2)(l+d)(l-1)(2l+d-1)(l+d-3)}{2(d-1)!(l+1)!}$

Table C.1: Eigenvalue and their multiplicities of the Laplacia $\Delta = -g_{\mu\nu}\nabla^\mu\nabla^\nu$ on a d-sphere

One can connect the quantity $Q_k(W)$ to the function $W(z)$ directly and not through its Laplace anti-transform. For integer $k > 0$ we introduce the gamma function in its Euler rapresentation

$$\Gamma(k) = \int_0^{+\infty} dss^{k-1}e^{-s} \tag{C.12}$$

inserting in (C.11)

$$Q_k(W) = \frac{1}{\Gamma(k)}\int_0^{+\infty} dzz^{k-1}W(z) \tag{C.13}$$

For integer $k \leq 0$ it's sufficient to construct the derivative of $W(z)$ in $z = 0$

$$W^{(i)}(0) = (-1)^i \int_0^{+\infty} dss^i\tilde{W}(z) \tag{C.14}$$

So for integer k and m we have

$$\begin{aligned} Q_k(W) &= \frac{1}{\Gamma(k)}\int_0^{+\infty} dzz^{k-1}W(z) \qquad k > 0 \\ Q_{-m}(W) &= (-1)^m W^{(k)}(0) \qquad m \geq 0 \end{aligned} \tag{C.15}$$

Finally the trace (for an even d dimensional-sphere) can be calculated with equation (C.10). For odd dimensional space see the review [5].

When we consider a transverse-traceless decomposition of type (2.23) or (2.25), a reconsideration of Heat Kernel coefficients b_n is needed. For example, consider the decomposition in transverse and longitudinal component as in ghosts sector

$$C_\mu = c_\mu^T + \nabla_\mu c \tag{C.16}$$

One can relates the spectrum of $-\nabla^2$ on longitudinal part to the spectrum of $-\nabla^2 - \frac{R}{d}$ considering the formula (we are on d-dimensional sphere)

$$-\nabla^2\nabla_\mu c = -\nabla_\mu\left(\nabla^2 + \frac{R}{d}\right)c$$

	S	V	VT	T	TT	TTT
tr $\boldsymbol{b}_0$	1	4	3	10	9	5
tr $\boldsymbol{b}_2$	$\frac{R}{6}$	$\frac{2}{3}R$	$\frac{R}{4}$	$\frac{5}{3}R$	$\frac{3}{2}R$	$-\frac{5}{6}R$
tr $\boldsymbol{b}_4$	$\frac{29R^2}{2160}$	$\frac{43R^2}{1080}$	$-\frac{7R^2}{1440}$	$\frac{11R^2}{216}$	$\frac{81R^2}{2160}$	$-\frac{R^2}{432}$
tr $\boldsymbol{b}_6$	$\frac{37R^3}{54432}$	$-\frac{R^3}{17010}$	$-\frac{541R^3}{362880}$	$-\frac{1343R^3}{136080}$	$-\frac{319R^3}{30240}$	$\frac{311R^3}{54432}$
tr $\boldsymbol{b}_8$	$\frac{149R^4}{6531840}$	$-\frac{2039R^4}{13063680}$	$-\frac{157R^4}{2488320}$	$-\frac{2999R^4}{3265920}$	$\frac{683R^4}{725760}$	$\frac{109R^4}{1306368}$

Table C.2: Heat kernel coefficients for S^4. The scalar (S), vector (V) and tensor (T) trace can be obtained from relations (C.9), while transverse vector (VT) and transverse-traceless tensor (TTT) expanding relations (C.17) and (C.19). (TT) means traceless tensor. Note that tr $\boldsymbol{b}_0$ is simply the dimension of field space where $\boldsymbol{b}_n$ acts.

So one can obtain the corresponding Heat Kernel coefficients by the relation

$$\mathrm{Tr}\, e^{-s\Delta}\Big|_{C_\mu} = \mathrm{Tr}\, e^{-s\Delta}\Big|_{c^T_\mu} + \mathrm{Tr}\, e^{-s\left(\Delta-\frac{R}{d}\right)}\Big|_{c} - e^{s\frac{R}{d}} \tag{C.17}$$

from which and from the spectrum of Δ on scalar and vector we find the relative Heat Kernel coefficients for decomposition (C.16) and are reported in Tabella bla bla.

The same argument ca be applied with decomposition on gravity fluctuations

$$h_{\mu\nu} = h^T_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu + \nabla_\mu\nabla_\nu\sigma - \frac{1}{d}g_{\mu\nu}\nabla^2\sigma + \frac{1}{d}g_{\mu\nu}h \tag{C.18}$$

We use the relations

$$-\nabla^2(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) = \nabla_\mu\left(-\nabla^2 - \frac{d+1}{d(d-1)}R\right)\xi_\nu + \nabla_\nu(-\nabla^2 - \frac{d+1}{d(d-1)}R)\xi_\mu$$

and

$$-\nabla^2\left(\nabla_\mu\nabla_\nu - \frac{1}{d}g_{\mu\nu}\nabla^2\right)\sigma = \left(\nabla_\mu\nabla_\nu - \frac{1}{d}g_{\mu\nu}\nabla^2\right)\left(-\nabla^2 - \frac{2}{d-1}R\right)\sigma$$

Noting that the $\frac{d(d+1)}{2}$ Killing vectors do not contribute to spectrum for $-\nabla^2$, although they are eigenvectors of $-\nabla^2 - \frac{d+1}{d(d-1)}R$ and the same for the constant and first modes of scalar σ we have

$$\begin{aligned}\mathrm{Tr}\, e^{-s\Delta}\Big|_{h_{\mu\nu}} &= \mathrm{Tr}\, e^{-s\Delta}\Big|_{h^T} + \mathrm{Tr}\, e^{-s\left(\Delta-\frac{d+1}{d(d-1)}R\right)}\Big|_{\xi} + \mathrm{Tr}\, e^{-s\Delta}\Big|_{h} \\ &+ \mathrm{Tr}\, e^{-s\left(\Delta-\frac{2R}{d-1}\right)}\Big|_{\sigma} - e^{\frac{2R}{d-1}s} - \frac{d(d+1)}{2}e^{\frac{2R}{d(d-1)s}} - (d+1)e^{\frac{R}{d-1}s}\end{aligned} \tag{C.19}$$

With this relation we can construct the coefficients $\mathrm{tr}\, b_n$ for different component of metric fluctuations which are reported in C.2.

Bibliography

[1] J. F. Donoghue, Phys.Rev.Lett. **72**, 2996 (1994), gr-qc/9310024.

[2] S. Weinberg, (1980).

[3] C. Wetterich, Phys.Lett. **B301**, 90 (1993).

[4] H. Gies, Lect.Notes Phys. **852**, 287 (2012), hep-ph/0611146.

[5] A. Codello, R. Percacci, and C. Rahmede, Annals Phys. **324**, 414 (2009), 0805.2909.

[6] D. Benedetti and F. Caravelli, JHEP **1206**, 017 (2012), 1204.3541.

[7] S. Weinberg, Ultraviolet Divergences in Quantum Theories of Gravitation, in *General Relativity: An Einstein centenary survey*, edited by W. I. S. W. Hawking, chap. 16, pp. 790–831, Cambridge University Press, 1979.

[8] R. Percacci, Asymptotic Safety, in *Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter*, edited by D. Oriti, Cambridge University Press, 2009.

[9] G. 't Hooft and M. Veltman, Nucl.Phys. **B44**, 189 (1972).

[10] G. 't Hooft and M. Veltman, Annales Poincare Phys.Theor. **A20**, 69 (1974).

[11] M. H. Goroff and A. Sagnotti, Nucl.Phys. **B266**, 709 (1986).

[12] M. Reuter, Phys.Rev. **D57**, 971 (1998), hep-th/9605030.

[13] M. Niedermaier and M. Reuter, Living Rev.Rel. **9**, 5 (2006).

[14] P. F. Machado and F. Saueressig, Phys.Rev. **D77**, 124045 (2008), 0712.0445.

[15] A. Codello, G. D'Odorico, and C. Pagani, (2013), 1304.4777.

[16] I. Buchbinder, S. Odintsov, and I. Shapiro, (1992).

[17] D. F. Litim, Phys.Rev. **D64**, 105007 (2001), hep-th/0103195.

[18] W. Souma, Prog.Theor.Phys. **102**, 181 (1999), hep-th/9907027.

[19] O. Lauscher and M. Reuter, Phys.Rev. **D65**, 025013 (2002), hep-th/0108040.

[20] M. Reuter and F. Saueressig, Phys.Rev. **D65**, 065016 (2002), hep-th/0110054.

[21] D. F. Litim, Phys.Rev.Lett. **92**, 201301 (2004), hep-th/0312114.

[22] O. Lauscher and M. Reuter, Phys.Rev. **D66**, 025026 (2002), hep-th/0205062.

[23] O. Lauscher and M. Reuter, Class.Quant.Grav. **19**, 483 (2002), hep-th/0110021.

[24] K. Falls, D. Litim, K. Nikolakopoulos, and C. Rahmede, (2013), 1301.4191.

[25] D. Benedetti, New J.Phys. **14**, 015005 (2012), 1107.3110.

[26] D. Benedetti, Europhys.Lett. **102**, 20007 (2013), 1301.4422.

[27] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Fifth ed. (Dover, New York, 1964).

[28] J. A. Dietz and T. R. Morris, JHEP **1301**, 108 (2013), 1211.0955.

[29] I. Avramidi, Lect.Notes Phys. **M64**, 1 (2000).

Acknowledgment

I would like to express my sincere gratitude to Dr. Gian Paolo Vacca for his continuous support and encouragement.