ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

Scuola di Scienze
Corso di Laurea Magistrale in Fisica

Vector Fields in a Rindler Space

Relatore:
Prof. Roberto Soldati

Presentata da:
Caterina Specchia

Sessione I
Anno Accademico 2012/2013

Contents

Abstract

In questo lavoro ci si propone di studiare la quantizzazione del campo vettoriale, massivo e non massivo, in uno spazio-tempo di Rindler, considerando in particolare i gauge di Feynman e assiale. Le equazioni del moto vengono risolte esplicitamente in entrambi i casi; sotto opportune condizioni, é stato inoltre possibile trovare una base completa e ortonormale di soluzioni delle equazioni di campo in termini di modi normali di Fulling. Si é poi analizzata la quantizzazione dei campi vettoriali espressi in questa base.

Preface

The study of the quantization of Vector Fields as experienced by a Rindler observer, that is an accelerated noninertial observer, turns out to be really useful and important when considering the description of many physical phenomena.

First of all, it is known that accelerated observers are expected to experience the so-called Unruh Effect, according to which they measure a thermal bath of particles with respect to the inertial observers and vice-versa, with a characteristic temperature, called Unruh temperature, $T = \hbar a / 2\pi k_B$, where k_B is the Boltzmann's constant. Therefore, on the one hand, there is a great interest for the Unruh effect and its consequences regarding the modern Cosmology and Particle Physics. As a matter of fact, since the particle structure of the constituents of Dark Matter is so far unknown and is usually assumed to be a non-baryonic weakly interacting massive particle (WIMP), and since we live in an accelerated expanding universe, from the point of view of the accelerated observes co-moving with the galaxies, a cosmological thermal bath of WIMP particles is expected to be produced by the cosmic acceleration, with an Unruh temperature $T = \hbar H / 2\pi k_B$, where H is the Hubble's parameter related to the cosmic acceleration $a_{\text{cosmic}} = c\,H \approx 2.1 \times 10^{-9}\,\text{m s}^{-2}$. It is then important to consider the description of real scalar particles, Majorana spinors and real vector particles as experienced by some uniformly accelerated observers. The simplest class of such observers is given by the so-called *Rindler observers*. The quantization of the first two kind of particles in a Rindler space has been recently done [6], and thus the quantization of real vector particles acquires interest to complete the overview. On the other hand, many applications of the Unruh Effect concerning the Unruh-DeWitt detectors have been analyzed and have become of growing interest in the last years [8] [9], especially regarding the photon counting detectors.

Secondly, many aspects of the definition and properties of Black Holes can be easily understood and clarified considering physical phenomena in a flat

Minkowski spacetime, but from the point of view of a uniformly accelerated observer. In fact, according to the Principle of Equivalence, the physical laws in a local reference frame at rest in a gravitational field are equivalent to those in a uniformly accelerated frame in a flat spacetime. In particular, if we consider a small region near the event horizon of a Black Hole and use a frame at rest there, the local effects in that region can be described in an easier way from the point of view of a particular uniformly accelerated observer, that is the Rindler observer, since this rigid reference frame turns out to be equivalent to the former frame in the static gravitational field of the region under consideration.

This work is organized as follows. In the first chapter we review the quantization of both massive and massless vector fields in the usual Minkowski space, considering in particular the Feynman gauge. The second chapter is devoted to the analysis of Quantum Field Theory in a generical curved spacetime, focusing in particular on the Rindler space and the quantization of the scalar field in this space-like region. In the last chapter we present the quantization of the vector fields, both massive and massless, in a Rindler space considering several gauges; we also describe some recent features concerning photon counting detectors.

Chapter 1

Vector Fields in a Minkowski space

In this chapter we review the quantization of the Vector Fields in the usual Minkowski space. In the following we will denote the constant Minkowski metric tensor with $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$ and use the natural units with $c = \hbar = 1$.

It is known that the Action which leads to the Maxwell equations for the massless vector field is given by

$$\mathcal{S} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \right) \tag{1.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic antisymmetric tensor which is invariant under the gauge transformation of the first kind

$$A'_\mu(x) = A_\mu(x) + \partial_\mu f(x) \tag{1.2}$$

where $f(x)$ is any real function. The vacuum equations are recovered through the principle of least action and read

$$\partial_\mu F^{\mu\nu} = 0 \tag{1.3}$$

$$\partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0 \tag{1.4}$$

In order to find a unique solution, one must choose a particular gauge for the potential A_μ; a covariant choice is then given by the Lorenz condition $\partial_\mu A^\mu = 0$, which simplifies the field equations giving

$$\Box A_\mu = 0 \tag{1.5}$$

and therefore A_μ satisfies the usual D'Alembert wave equations. Of course, the gauge invariance is not a symmetry of the massive vector field, which is described by the following Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}(x)\, F^{\mu\nu}(x) + \frac{1}{2}\, m^2\, A_\mu(x)\, A^\mu(x) \tag{1.6}$$

and the consequent equations of motion

$$\partial_\mu F^{\mu\nu}(x) + m^2\, A^\nu(x) = 0 \tag{1.7}$$

$$\partial_\mu A^\mu = 0 \tag{1.8}$$

Now, in order to describe both the massive and massless vector fields, keeping the covariance manifest, we are left with the following Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}(x)\, F^{\mu\nu}(x) + \frac{1}{2}\, m^2\, A_\mu(x)\, A^\mu(x) + A^\mu(x)\, \partial_\mu B(x) + \frac{1}{2}\, \xi\, B^2(x) \tag{1.9}$$

where ξ is a real parameter and B is an auxiliary scalar field thanks to which one recovers the gauge fixing condition for the massless case. As a matter of fact, the field equations now read

$$\partial_\mu F^{\mu\nu}(x) + m^2\, A^\nu(x) + \partial^\nu B(x) = 0 \tag{1.10}$$

$$\partial_\nu A^\nu(x) = \xi\, B(x) \tag{1.11}$$

that can be recast as

$$\left[\eta^{\mu\nu}\left(\Box + m^2\right) - \left(1 - \frac{1}{\xi}\right)\partial^\mu \partial^\nu\right] A_\nu(x) = 0 \tag{1.12}$$

$$\partial_\nu A^\nu(x) = \xi\, B(x) \tag{1.13}$$

$$\left(\Box + m^2\, \xi\right) B(x) = 0 \tag{1.14}$$

It should be noticed that the auxiliary field B is a free real scalar field that satisfies the Klein-Gordon wave equation with a square mass $m^2\xi$, which is positive for $\xi > 0$, while for $\xi < 0$ it becomes tachyon-like, and therefore reveals that this field is not physical.

The field equations can be simplified choosing a particular value for the real parameter ξ. The choice $\xi = 0$ is called Lorenz-Landau gauge, while the Feynman gauge is given by $\xi = 1$ and is the one we will adopt. This way we get

$$\left(\Box + m^2\right) A_\nu(x) = 0 \tag{1.15}$$

$$\partial_\nu A^\nu(x) = B(x) \tag{1.16}$$

$$\left(\Box + m^2\right) B(x) = 0 \tag{1.17}$$

We will now analyze separately the Proca field and the massless field, then proceeding to the quantization.

1.1 Massive Vector Field

We first consider the quantization of the massive real vector field. In order to find the normal modes decomposition, it is useful to take the following gauge transformation of the vector potential, namely

$$A_\mu(x) = V_\mu(x) - \frac{1}{m^2} B(x) \tag{1.18}$$

so that the antisymmetric tensor becomes

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu &(1.19)\\ &= \partial_\mu V_\nu - \partial_\nu V_\mu &(1.20)\end{aligned}$$

and we recover the Proca field equations

$$\begin{aligned} &\left(\Box + m^2\right) V^\nu(x) = 0 &(1.21)\\ &\partial_\nu V^\nu(x) = 0 &(1.22)\\ &\left(\Box + m^2\right) B(x) = 0 &(1.23)\end{aligned}$$

The solutions can be found through the Fourier transform

$$V_\mu(x) = \int \frac{d^4k}{(2\pi)^{3/2}} \tilde{V}_\mu(k)\, e^{-ik_\mu x^\mu} \tag{1.24}$$

with the reality condition $\tilde{V}^*_\mu(k) = \tilde{V}_\mu(-k)$, and from the field equations (1.21, 1.22) we get

$$\tilde{V}_\mu(k) = \delta(k^2 - m^2)\, f_\mu(k) \tag{1.25}$$

where $f_\mu(k)$ are four arbitrary functions, regular on the hyperboloid $k^2 = m^2$ and which satisfy the reality condition $f^*_\mu(k) = f_\mu(-k)$ and the transversality condition $k^\mu f_\mu(k) = 0$, that means that there are only three indipendent functions. Therefore, we can introduce three indipendent real unit vectors, the linear polarization vectors $e^\mu_r(\mathbf{k})$ $(r = 1, 2, 3)$, which are dimensionless and determined by the following properties

- $k_\mu\, e^\mu_r(\mathbf{k}) = 0 \qquad r = 1, 2, 3 \qquad k_0 \equiv \omega_\mathbf{k} = (\mathbf{k}^2 + m^2)^{1/2}$
- $-\eta_{\mu\nu}\, e^\mu_r(\mathbf{k})\, e^\nu_s(\mathbf{k}) = \delta_{rs}$ (orthonormality relation)
- $\sum_{r=1}^3 e^\mu_r(\mathbf{k})\, e^\nu_r(\mathbf{k}) = -\eta^{\mu\nu} + k^\mu k^\nu / k^2 = -\eta^{\mu\nu} + k^\mu k^\nu / m^2$ (closure relation)

One suitable choice is then given by

$$e_r^\mu(\mathbf{k}) = (0,\, \mathbf{e}_r(\mathbf{k})), \quad \mathbf{e}_r(\mathbf{k})\cdot\mathbf{e}_s(\mathbf{k}) = \delta_{rs} \quad \mathbf{k}\cdot\mathbf{e}_r(\mathbf{k}) = 0 \quad r,s = 1,2$$
$$e_3^0(\mathbf{k}) = \frac{|\,\mathbf{k}\,|}{m} \qquad \mathbf{e}_3(\mathbf{k}) = \frac{\hat{\mathbf{k}}}{m}\,\omega_{\mathbf{k}} \tag{1.26}$$

so that we can expand the real vector field $V^\mu(x)$ in terms of the normal modes $u^\mu_{\mathbf{k},\,r}(x)$, namely

$$V^\mu(x) = \sum_{\mathbf{k},\,r}\left[f_{\mathbf{k},\,r}\,u^\mu_{\mathbf{k},\,r}(x) + f^*_{\mathbf{k},\,r}\,u^{\mu *}_{\mathbf{k},\,r}(x)\right] \tag{1.27}$$

$$u^\mu_{\mathbf{k},\,r}(x) = \frac{1}{\left[(2\pi)^3\,2\omega_{\mathbf{k}}\right]^{1/2}}\,e^\mu_r(\mathbf{k})\,e^{-i\omega_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}} \tag{1.28}$$

where we have used the shorthand notation

$$\sum_{\mathbf{k},\,r} \equiv \int d\mathbf{k}\sum_{r=1}^{3} \tag{1.29}$$

We notice also that the set of normal modes $u^\mu_{\mathbf{k},\,r}(x)$, which have canonical dimensions $\left[u^\mu_{\mathbf{k},\,r}\right] = \mathrm{cm}^{1/2}$ in natural units, is a complete orhonormal set which satisfies the following orthonormality and closure relations

$$\begin{aligned} -\eta_{\mu\nu}\,\left(u^\mu_{\mathbf{k},\,r},\,u^\nu_{\mathbf{p},\,s}\right) &= -\eta_{\mu\nu}\int d\mathbf{x}\,u^{\mu *}_{\mathbf{k},\,r}(x)\,i\,\overleftrightarrow{\partial_0}\,u^\nu_{\mathbf{p},\,s}(x) \\ &= \delta_{rs}\,\delta(\mathbf{k}-\mathbf{p}) \end{aligned} \tag{1.30}$$
$$\eta_{\mu\nu}\,\left(u^\mu_{\mathbf{k},\,r},\,u^{\nu *}_{\mathbf{p},\,s}\right) = \eta_{\mu\nu}\,\left(u^{\mu *}_{\mathbf{k},\,r},\,u^\nu_{\mathbf{p},\,s}\right) = 0 \tag{1.31}$$
$$\eta_{\mu\nu}\,\left(u^{\mu *}_{\mathbf{k},\,r},\,u^{\nu *}_{\mathbf{p},\,s}\right) = \delta_{rs}\,\delta(\mathbf{k}-\mathbf{p}) \tag{1.32}$$

$$\sum_{\mathbf{k},\,r}\left(u^\mu_{\mathbf{k},\,r}(x)\,u^{\nu *}_{\mathbf{k},\,r}(y)\right) = i\left(\eta^{\mu\nu} - \frac{\partial^\mu_x\partial^\nu_y}{m^2}\right)D^{(-)}(x-y) \tag{1.33}$$

where $D^{(-)}(x-y)$ is the positive frequency scalar distribution

$$D^{(-)}(x-y) = i\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\,\theta(k_0)\,e^{-ik\cdot(x-y)} \tag{1.34}$$

Moreover, for the auxiliary scalar field B, we find that the normal modes decomposition is given by

$$B(x) = m\sum_{\mathbf{k}}\left[b_{\mathbf{k}}\,u_{\mathbf{k}}(x) + b^*_{\mathbf{k}}\,u^*_{\mathbf{k}}(x)\right] \tag{1.35}$$

$$u_{\mathbf{k}}(x) = \frac{1}{\left[(2\pi)^3\,2\omega_{\mathbf{k}}\right]^{1/2}}\,e^{-i\omega_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}} \tag{1.36}$$

where we see that it has dimension $[B] = \text{cm}^{-2}$ in natural units which are not usual for a scalar field: this is another evidence of its unphysical nature.

1.1.1 Conserved Quantities of the Proca Field

In order to obtain a deeper insight of the Proca field, let us consider in more detail the field equations and the conserved quantities of the system. First of all, we notice that, using the gauge transformation (1.18), the Lagrangian density (1.9) splits into the sum of a Lagrangian referred only to the field $V^\mu(x)$ and one referred only to the auxiliary field $B(x)$, namely

$$\begin{aligned}
\mathcal{L}_{A,B} &= \mathcal{L}_V + \mathcal{L}_B \\
\mathcal{L}_V &= -\frac{1}{4}\, F_{\mu\nu}(x)\, F^{\mu\nu}(x) + \frac{1}{2}\, m^2\, V_\mu(x)\, V^\mu(x) && (1.37) \\
\mathcal{L}_B &= -\frac{1}{2m^2}\, \partial_\mu B(x)\, \partial^\mu B(x) + \frac{1}{2}\, \xi\, B^2(x) && (1.38)
\end{aligned}$$

which means that the two fields V^μ and B are decoupled. We can now write the canonical conjugate momenta of the fields:

$$\begin{aligned}
\Pi^\mu(x) &= \frac{\delta \mathcal{L}_V}{\delta \partial_0 V_\mu(x)} = \begin{cases} 0 & \text{for } \mu = 0 \\ -F^{0k} = E^k & \text{for } \mu = k = 1,2,3 \end{cases} && (1.39) \\
\Pi(x) &= \frac{\delta \mathcal{L}_B}{\delta \partial_0 B(x)} = -\frac{1}{m^2}\, \dot{B}(x) && (1.40)
\end{aligned}$$

and the consequent Poisson's brackets

$$\begin{aligned}
\left\{ V_k(t,\, \mathbf{x}), E^l(t,\, \mathbf{y}) \right\} &= \delta_k^l\, \delta(\mathbf{x} - \mathbf{y}) && (1.41) \\
\{ B(t,\, \mathbf{x}), \Pi(t,\, \mathbf{y}) \} &= \delta(\mathbf{x} - \mathbf{y}) && (1.42)
\end{aligned}$$

the other ones vanishing. The energy-momentum tensor is given by

$$\begin{aligned}
T^\mu_\nu &= \delta^\mu_\nu \left(\frac{1}{4}\, F^{\rho\sigma} F_{\rho\sigma} - \frac{1}{2}\, m^2\, V_\rho\, V^\rho \right) + m^2\, V_\nu\, V^\mu - F^{\mu\lambda} F_{\nu\lambda} - \partial_\lambda \left(V_\nu\, F^{\mu\lambda} \right) \\
&\quad -\frac{1}{m^2}\, \partial^\mu B\, \partial_\nu B + \delta^\mu_\nu \left(\frac{1}{2\, m^2} \partial^\lambda B\, \partial_\lambda B - \frac{1}{2}\, \xi\, B^2 \right) \\
&\equiv \Theta^\mu_\nu - \partial_\lambda \left(V_\nu\, F^{\mu\lambda} \right) && (1.43)
\end{aligned}$$

where Θ^μ_ν is the improved symmetric energy-momentum tensor, which satisfies $\partial_\mu T^{\mu\nu} = \partial_\mu \Theta^{\mu\nu} = 0$ and turns out to be the sum of the vector and the auxiliary scalar parts. Therefore, the total angular momentum density is given by

$$\begin{aligned} M^{\mu\rho\sigma} &= x^\rho\, T^{\mu\sigma} - x^\sigma\, T^{\mu\rho} - S^{\mu\rho\sigma} \\ &= x^\rho\, T^{\mu\sigma} - x^\sigma\, T^{\mu\rho} - F^{\mu\rho}\, V^\sigma + F^{\mu\sigma}\, V^\rho \\ &= x^\rho\, \Theta^{\mu\sigma} - x^\sigma\, \Theta^{\mu\rho} - \partial_\lambda \left(x^\rho\, V^\sigma\, F^{\mu\lambda} - x^\sigma\, V^\rho\, F^{\mu\lambda} \right) \end{aligned} \tag{1.44}$$

Since the last term does not contribute to the continuity equation $\partial_\mu M^{\mu\rho\sigma} = 0$, the total angular momentum tensor can always be written in the purely orbital form

$$M^{\rho\sigma} = \int d\mathbf{x}\, [x^\rho\, \Theta^{0\sigma}(t,\mathbf{x}) - x^\sigma\, \Theta^{0\rho}(t,\mathbf{x})] \tag{1.45}$$

which satisfies

$$\dot{M}^{\rho\sigma} = 0 \tag{1.46}$$

Therefore, we get the three spatial components

$$M^{ij} = \int d\mathbf{x}\, [x^i\, \Theta^{0j}(t,\mathbf{x}) - x^j\, \Theta^{0i}(t,\mathbf{x})] \tag{1.47}$$

which correspond to an orbital angular momentum, the sum of the Poynting vector and of the auxiliary scalar parts, and the spatial temporal components

$$\begin{aligned} M^{0k} &= \int d\mathbf{x}\, [x^0\, \Theta^{0k}(t,\mathbf{x}) - x^k\, \Theta^{00}(t,\mathbf{x})] \\ &= x^0 P^k - \int d\mathbf{x}\, x^k\, \Theta^{00}(t,\mathbf{x}) \end{aligned} \tag{1.48}$$

from which we can define the centre of energy for the total system of the massive vector and the auxiliary fields

$$X_t^k \equiv \int \frac{d\mathbf{x}}{P_0}\, x^k\, \Theta^{00}(t,\mathbf{x}) \tag{1.49}$$

that satisfies the particle velocity relationship

$$\dot{M}^{0k} = 0 \quad \Leftrightarrow \quad \dot{X}_t^k = \frac{P^k}{P_0} \tag{1.50}$$

1.1.2 Canonical Quantization of the Proca Field

Now we are ready to proceed to the canonical quantization of the system. One way is to keep the covariance manifest. If we follow this criterion, we can express the commutator between two vector fields in this covariant form:

$$
\begin{aligned}
[V_\mu(x),\, V_\nu(y)] &= i\left(\eta_{\mu\nu} - \frac{1}{m^2}\,\partial_{\mu,\,x}\partial_{\nu,\,y}\right) D(x-y;m) \\
&= \sum_{\mathbf{k},\,r}\left[u_{\mu;\,\mathbf{k},\,r}(x)\,u^*_{\nu;\,\mathbf{k},\,r}(y) - u^*_{\mu;\,\mathbf{k},\,r}(x)\,u_{\nu;\,\mathbf{k},\,r}(y)\right] \quad (1.51)
\end{aligned}
$$

where $D(x-y;m)$ is the Pauli-Jordan distribution for the scalar field, given by

$$
D(x-y;m) = i\int \frac{dk}{(2\pi)^3}\,\delta(k^2-m^2)\,\mathrm{sgn}(k_0)\,e^{-ik\cdot(x-y)} \quad (1.52)
$$

which is a Poincaré invariant solution of the Klein-Gordon wave equation, with the initial conditions

$$
\left(\Box_x + m^2\right) D(x-y) = 0 \quad (1.53)
$$

$$
\lim_{x_0\to y_0} D(x-y) = 0 \qquad \lim_{x_0\to y_0} \frac{\partial}{\partial x_0} D(x-y) = \delta(\mathbf{x}-\mathbf{y}) \quad (1.54)
$$

the second equivalence in (1.51) coming from the closure relation (1.33). Therefore, the above covariant commutator indeed fulfils all the fundamental requirement, i.e.

- it is a solution of the field equations (1.21)

$$
\left(\Box_x + m^2\right)[V_\mu(x),\, V_\nu(y)] = \left(\Box_y + m^2\right)[V_\mu(x),\, V_\nu(y)] = 0 \quad (1.55)
$$

- it satisfies the transversality condition

$$
\partial_x^\mu\,[V_\mu(x),\, V_\nu(y)] = \partial_y^\nu\,[V_\mu(x),\, V_\nu(y)] = 0 \quad (1.56)
$$

- it fulfils the symmetry, hermiticity and microcausality properties

$$
\begin{aligned}
&[V_\mu(x),\, V_\nu(y)] = -\,[V_\mu(x),\, V_\nu(y)]^\dagger = -\,[V_\nu(y),\, V_\mu(x)] \\
&[V_i(x),\, V_j(y)] = 0 \qquad \forall (x-y)^2 < 0, \quad i,j = 1,2,3 \\
&\lim_{x_0\to y_0}[V_0(x),\, V_j(y)] = -\frac{i}{m^2}\,\nabla_j\,\delta(\mathbf{x}-\mathbf{y}) \quad (1.57)
\end{aligned}
$$

- the equal time canonical commutation relations that arise from the Dirac correspondence principle applied to the classical Poisson's brackets (1.41) are recovered: as a matter of fact, we have

$$\begin{aligned}\left[V_j(x),\, E^l(y)\right] &= [V_j(x),\, \partial_0 V_l(y) - \partial_l V_0(y)] \\ &= \frac{\partial}{\partial y_0}\,[V_j(x),\, V_l(y)] - \frac{\partial}{\partial y^l}\,[V_j(x),\, V_0(y)] \\ &= i\,\delta^l_j\,\frac{\partial}{\partial x_0}D(x-y;m) \qquad (1.58)\end{aligned}$$

that in the equal-time limit reduces to

$$\lim_{x_0\to y_0}\left[V_j(x),\, E^l(y)\right] = i\,\delta^l_j\,\delta(\mathbf{x}-\mathbf{y}) \qquad (1.59)$$

We can verify that also the other equal-time commutators are regained, namely

$$\lim_{x_0\to y_0}\,[V_j(x),\, V_i(y)] = 0 \qquad (1.60)$$

$$\lim_{x_0\to y_0}\left[E^j(x),\, E^l(y)\right] = 0 \qquad (1.61)$$

Therefore these covariant canonical commutation relations are the unique operator solution of the field equations which fulfil all the fundamental requirements. We can then proceed to the quantization of the Proca vector field, that now becomes an operator valued tempered distribution:

$$V^\mu(x) = \sum_{\mathbf{k},\,r}\left[f_{\mathbf{k},\,r}\,u^\mu_{\mathbf{k},\,r}(x) + f^\dagger_{\mathbf{k},\,r}\,u^{\mu *}_{\mathbf{k},\,r}(x)\right] \qquad (1.62)$$

The algebra of the creation and destruction operators can be immediatly derived from (1.51) and reads

$$[f_{\mathbf{k},\,r},\, f_{\mathbf{p},\,s}] = 0 \qquad \left[f^\dagger_{\mathbf{k},\,r},\, f^\dagger_{\mathbf{p},\,s}\right] = 0$$

$$\left[f_{\mathbf{k},\,r},\, f^\dagger_{\mathbf{p},\,s}\right] = \delta_{rs}\,\delta(\mathbf{k}-\mathbf{p}) \qquad (1.63)$$

It can be now verified that the energy momentum operator becomes diagonal when expressed in terms of the creation and destruction operators: it indeed corresponds to the sum over an infinite set of indipendent linear harmonic oscillators, one for each indipendent polarization and for each component of the wave vector $\mathbf{k}$,

$$P_0 = \sum_{\mathbf{k},\,r}\,\omega_{\mathbf{k}}\,f^\dagger_{\mathbf{k},\,r}\,f_{\mathbf{k},\,r}$$

$$\mathbf{P} = \sum_{\mathbf{k},\,r}\,\mathbf{k}\,f^\dagger_{\mathbf{k},\,r}\,f_{\mathbf{k},\,r} \qquad (1.64)$$

1.1.3 The Ghost Field

For what concerns the auxiliary scalar field, some features are to be considered in more detail. First of all, from (1.40) we see that the conjugate momentum of the field B has the wrong sign with respect to a usual scalar field:

$$\Pi(x) = \frac{\delta \mathcal{L}_B}{\delta \partial_0 B(x)} = -\frac{1}{m^2}\,\dot{B}(x) \tag{1.65}$$

This peculiarity is of great importance, as we shall see in a while. As a matter of fact, from the normal modes decomposition of both B and Π, which reads

$$B(x) = m \sum_{\mathbf{k}} \left[b_{\mathbf{k}}\, u_{\mathbf{k}}(x) + b_{\mathbf{k}}^{\dagger}\, u_{\mathbf{k}}^{*}(x) \right] \tag{1.66}$$

$$\Pi(x) = \frac{1}{m} \sum_{\mathbf{k}} i\,\omega'_{\mathbf{k}} \left[b_{\mathbf{k}}\, u_{\mathbf{k}}(x) - b_{\mathbf{k}}^{\dagger}\, u_{\mathbf{k}}^{*}(x) \right] \tag{1.67}$$

$$u_{\mathbf{k}}(x) = \frac{1}{\left[(2\pi)^3\, 2\omega'_{\mathbf{k}}\right]^{1/2}}\, e^{-i\omega'_{\mathbf{k}} t + i\mathbf{k}\cdot\mathbf{x}} \qquad \omega'_{\mathbf{k}} \equiv \left(\mathbf{k}^2 + m^2\,\xi\right)^{1/2} \tag{1.68}$$

in order to recover the ordinary canonical commutation relations between the auxiliary field and its conjugate momentum corresponding to the classical Poisson's brackets (1.42), i.e.

$$[B(t,\,\mathbf{x}),\,\Pi(t,\,\mathbf{y})] = i\,\delta(\mathbf{x}-\mathbf{y}) \tag{1.69}$$

we must require

$$\left[\, b_{\mathbf{k}},\, b_{\mathbf{p}}^{\dagger} \,\right] = -\,\delta(\mathbf{k}-\mathbf{p}) \tag{1.70}$$

all the other commutators vanishing. Moreover, the energy momentum operator takes the form

$$P_0 = -\sum_{\mathbf{k}} \omega'_{\mathbf{k}}\, b_{\mathbf{k}}^{\dagger}\, b_{\mathbf{k}} \equiv \mathcal{H}_B$$
$$\mathbf{P} = \sum_{\mathbf{k}} \mathbf{k}\, b_{\mathbf{k}}^{\dagger}\, b_{\mathbf{k}} \tag{1.71}$$

from which we see that for $\xi \geq 0$, $\mathcal{H}_B$ becomes negative definite and unbounded by below, while for $\xi < 0$ for low momenta $\mathbf{k}^2 < |\,\xi\,|\, m^2$ the energy becomes imaginary. Therefore, no physical meaning can be assigned to the hamiltonian operator of the auxiliary scalar field. Furthermore, defining the Fock space as usual, i.e. defining the vacuum state as

$$b_{\mathbf{k}}\,|0\rangle = 0 \qquad \forall\mathbf{k}\in\mathbb{R}^3 \tag{1.72}$$

and considering the proper 1-particle states given by

$$|b\rangle \equiv \int d\mathbf{k}\,\tilde{b}(\mathbf{k})\,b^{\dagger}_{\mathbf{k}}|0\rangle, \qquad \int d\mathbf{k}\,|\tilde{b}(\mathbf{k})|^2 = 1 \tag{1.73}$$

we see that they have negative norm:

$$\langle b|b\rangle = -\,1 \tag{1.74}$$

Hence the auxiliary scalar field B has no physical interpretation: this is why it is called *ghost* field. However, although its presence endows the whole Fock space of both the Proca vector field and the auxiliary scalar field with an indefinite metric, so that it contains states with positive, negative and null norm, the ghost field is necessary to build up a renormalizable theory. Neverthless, we have to select a physical subspace $\mathcal{H}_{phys}$ of the whole Fock space in which no quanta of the auxiliary field are allowed. We can then impose the following subsidiary condition

$$B^{(-)}(x)\,|\mathrm{phys}\rangle = 0 \qquad \forall\,|\mathrm{phys}\rangle\in\mathcal{H}_{phys} \tag{1.75}$$

where $B^{(-)}(x)$ is the positive frequency destruction part of the auxiliary scalar field,

$$B^{(-)}(x) = m\sum_{\mathbf{k}} b_{\mathbf{k}}\,u_{\mathbf{k}}(x) \tag{1.76}$$

the vacuum state becoming, this way, physical and cyclic, all the excited states being generated by the Proca creation operators $f^{\dagger}_{\mathbf{k},\,r}$.

1.1.4 The Feynman Propagator

We can then turn the attention to the propagators of the theory. Now, since the equations of motion of the Proca vector field and the auxiliary scalar field are decoupled, it can be easily verified that the following commutation relations hold, namely

$$[V_\mu(x),\,V_\nu(y)] = i\,\left(\eta_{\mu\nu} - m^{-2}\,\partial_{\mu,\,x}\partial_{\nu,\,y}\right)D(x-y;m) \tag{1.77}$$

$$[V_\mu(x),\,B(y)] = 0 \tag{1.78}$$

$$[B(x),\,B(y)] = im^2\,D(x-y;\xi\,m) \tag{1.79}$$

Then, considering the propagators, we find

$$\begin{aligned}
\langle 0|T(V_\mu(x)\,V_\nu(y))|0\rangle &= D^F_{\mu\nu}(x-y;m) = -\,\left(\eta_{\mu\nu} + m^{-2}\,\partial_\mu\partial_\nu\right)\,D^F(x-y;m)\\
\langle 0|T(B(x)\,B(y))|0\rangle &= -\,m^2\,D^F(x-y;\xi\,m)
\end{aligned} \tag{1.80}$$

where $D^F(x-y;m)$ is the Feynman propagator for the scalar field with the causal prescription

$$D^F(x-y;m) = i\int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k\cdot(x-y)}}{k^2-m^2+i\varepsilon} \tag{1.81}$$

so that the total propagator of the vector field $A_\mu(x)$ is given by

$$\begin{aligned}
D^F_{\mu\nu}(x-y;m,\xi) &= \langle 0|T(A_\mu(x)\,A_\nu(y))|0\rangle\\
&= i\int \frac{d^4k}{(2\pi)^4}\left[\frac{-\eta_{\mu\nu}+k_\mu\,k_\nu/m^2}{k^2-m^2+i\varepsilon} - \frac{k_\mu\,k_\nu/m^2}{k^2-\xi\,m^2+i\varepsilon'}\right]e^{-i\,k\cdot(x-y)}\\
&= i\int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k\cdot(x-y)}}{k^2-m^2+i\varepsilon}\left[-\eta_{\mu\nu} + \frac{(1-\xi)\,k_\mu k_\nu}{k^2-\xi\,m^2+i\varepsilon'}\right]
\end{aligned} \tag{1.82}$$

It is now apparent why the introduction of the auxiliary field is necessary: if we consider the leading asymptotic behaviour for large momenta of the momentum space Feynman propagator

$$\tilde{D}^F_{\mu\nu}(k;m,\xi) = \frac{i}{k^2-m^2+i\varepsilon}\left[-\eta_{\mu\nu} + \frac{(1-\xi)\,k_\mu k_\nu}{k^2-\xi\,m^2+i\varepsilon'}\right]$$

we see that it is like

$$\tilde{D}^F_{\mu\nu}(k;m,\xi) \sim k^{-2}d_{\mu\nu} \qquad (|k_\mu|\to\infty) \tag{1.83}$$

where $d_{\mu\nu}$ is a constant 4×4 matrix, and hence it decreases in a scale homogeneous quadratically way and with a momentum space isotropic law. Instead, if we considered only the Proca vector propagator, the leading behaviour would have presented a lack of scale homogeneity and naive power counting property: regarding the interacting theory, it would not have been renormalizable order by order in the perturbative expansion. Therefore, since the power counting property is one of the crucial necessary hypothesis for the perturbative order by order renormalizability of any interacting quantum field theory, the introduction of the ghost field appears to be unavoidable, even though in the interacting case the subsidiary condition to be imposed in order to decouple the auxiliary field from the physical sector will be non-trivial.

1.2 Massless Vector Field

We can now turn the attention to the real massless vector field. To begin with, let us rewrite the Lagrangian density in the Feynman gauge $\xi = 1$

$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}(x)F^{\mu\nu}(x) + A^{\mu}(x)\,\partial_{\mu}B(x) + \frac{1}{2}\,B^{2}(x) \tag{1.84}$$

The field equations read

$$\partial_{\mu}F^{\mu\nu}(x) + \partial^{\nu}B(x) = 0 \tag{1.85}$$

$$\partial_{\mu}A^{\mu}(x) = B(x) \tag{1.86}$$

which can be recast as

$$\Box A^{\nu}(x) = 0 \tag{1.87}$$

$$\Box B(x) = 0 \tag{1.88}$$

that means that both the vector and the auxiliary fields obey the D'Alembert wave equation. In particular, by means of the Fourier transform for $A^{\mu}(x)$

$$A^{\mu}(x) = \int \frac{d^{4}k}{(2\pi)^{3/2}}\,\tilde{A}^{\mu}(k)\,e^{-i\,k\cdot x} \tag{1.89}$$

we find that $\tilde{A}^{\mu}(k)$, which satisfies the reality conditions $\tilde{A}^{\mu *}(k) = \tilde{A}^{\mu}(-k)$, is of the form

$$\tilde{A}^{\mu}(k) = \delta(k^{2})\,f^{\mu}(k) \tag{1.90}$$

where $f^{\mu}(k)$ are four arbitrary functions regular on the light-cone $k^{2} = 0$, with the property $f^{\mu *}(k) = f^{\mu}(-k)$. Therefore, we look for four linearly indipendent real and dimensionless polarization vectors $\varepsilon^{\mu}_{A}(\mathbf{k})$ defined on the light-cone $k_{0} = \pm\,|\mathbf{k}|$. We can choose two of them orthogonal to the wave vector $\mathbf{k}$ and to each other, namely

$$k_{\mu}\,\varepsilon^{\mu}_{A}(\mathbf{k}) = 0, \qquad \varepsilon^{\mu}_{A}(\mathbf{k}) = (0,\,\varepsilon_{A}(\mathbf{k})), \qquad k_{0} \equiv \omega_{\mathbf{k}} = |\mathbf{k}|$$
$$\varepsilon_{A}(\mathbf{k})\cdot\varepsilon_{B}(\mathbf{k}) = \delta_{AB} \qquad A, B = 1, 2 \tag{1.91}$$

and one on le light-cone $k_{0} = |\mathbf{k}|$,

$$\varepsilon^{\mu}_{L}(\mathbf{k}) \equiv k^{\mu}/|\mathbf{k}|, \qquad \varepsilon^{\mu}_{L}(\mathbf{k})\,\varepsilon_{\mu,\,L}(\mathbf{k}) = 0 \tag{1.92}$$

so that $\eta_{\mu\nu}\,\varepsilon^{\mu}_{A}(\mathbf{k})\,\varepsilon^{\nu}_{L}(\mathbf{k}) = 0$. We can take the fourth one as another light-like vector, given by

$$\varepsilon^{\mu}_{S}(\mathbf{k}) \equiv \frac{k^{\mu}_{*}}{2|\mathbf{k}|}, \qquad k^{\mu}_{*} = (|\mathbf{k}|, -\mathbf{k}) \tag{1.93}$$

which satisfies $\varepsilon^{\mu}_{L}(\mathbf{k})\,\varepsilon_{\mu,\,S}(\mathbf{k}) = 1$, the labels L, S standing for *longitudinal* and *scalar* polarizations respectively. The orthonormality and closure relations are then

$$- \eta_{\mu\nu}\,\varepsilon^{\mu}_{A}(\mathbf{k})\,\varepsilon^{\nu}_{B}(\mathbf{k}) = \eta'_{AB} \tag{1.94}$$

$$\sum_{A,B=1,2,L,S} \eta'_{AB}\,\varepsilon^{\mu}_{A}(\mathbf{k})\,\varepsilon^{\mu}_{B}(\mathbf{k}) = -\eta^{\mu\nu} \tag{1.95}$$

$$k_{\mu}\,\varepsilon^{\mu}_{S}(\mathbf{k}) = |\mathbf{k}|, \qquad k_{\mu}\,\varepsilon^{\mu}_{A}(\mathbf{k}) = 0 \quad A = 1, 2, L \tag{1.96}$$

where

$$\eta'_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad (A, B = 1, 2, L, S) \tag{1.97}$$

The real massless vector field can now be written in terms of the normal modes decomposition,

$$A^{\mu}(x) = \sum_{\mathbf{k},\,A} \left[g_{\mathbf{k},\,A}\, u^{\mu}_{\mathbf{k},\,A}(x) + g^{*}_{\mathbf{k},\,A}\, u^{\mu *}_{\mathbf{k},\,A}(x) \right] \tag{1.98}$$

$$u^{\mu}_{\mathbf{k},\,A}(x) = \frac{1}{\left[(2\pi)^3\, 2|\mathbf{k}|\right]^{1/2}}\,\varepsilon^{\mu}_{A}(\mathbf{k})\, e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}} \tag{1.99}$$

while the auxiliary scalar field is given by

$$\begin{aligned} B(x) &= \partial_{\mu} A^{\mu}(x) = \\ &= -i \sum_{\mathbf{k}} \left[g_{\mathbf{k},\,S}\ k_{\mu}\, u^{\mu}_{\mathbf{k},\,S}(x) - g^{*}_{\mathbf{k},\,S}\ k_{\mu}\, u^{\mu *}_{\mathbf{k},\,S}(x) \right] \end{aligned} \tag{1.100}$$

We notice that the normal modes $u^{\mu}_{\mathbf{k},\,A}(x)$ form a complete orthonormal set of positive frequency solutions that satisfy the following orthonormality and closure relations

$$\begin{aligned} -\eta_{\mu\nu}\,\left(u^{\mu}_{\mathbf{k},\,A},\, u^{\nu}_{\mathbf{p},\,B}\right) &= -\eta_{\mu\nu} \int d\mathbf{x}\, u^{\mu *}_{\mathbf{k},\,A}(x)\, i \overleftrightarrow{\partial_0}\, u^{\nu}_{\mathbf{p},\,B}(x) \\ &= \eta'_{AB}\, \delta(\mathbf{k} - \mathbf{p}) \end{aligned} \tag{1.101}$$

$$\sum_{\mathbf{k},\,A} \left(u^{\mu}_{\mathbf{k},\,A}(x)\, u^{\nu *}_{\mathbf{k},\,A}(y)\right) = i\,\eta^{\mu\nu}\,\eta'_{AB}\,\Delta^{(-)}(x - y) \tag{1.102}$$

where $\Delta^{(-)}(x-y)$ is the scalar massless positive fequency distribution

$$\Delta^{(-)}(x-y) = i \int \frac{d^4k}{(2\pi)^3}\, \delta(k^2)\, \theta(k_0)\, e^{-i\,k\cdot(x-y)} \tag{1.103}$$

1.2.1 Canonical Quantization of the Massless Vector Field

We are now ready to consider the quantization of the massless vector field. First we notice that the canonical conjugate momenta are now given by

$$\Pi^\mu(x) = \frac{\delta\mathcal{L}}{\delta\partial_0 A_\mu(x)} = \begin{cases} 0 & \text{for } \mu = 0 \\ -F^{0k} = E^k & \text{for } \mu = k = 1,2,3 \end{cases} \tag{1.104}$$

$$\Pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0 B(x)} = A_0 \tag{1.105}$$

so that A_0 becomes the conjugate momentum of the auxiliary field. The consequent Poisson's brackets are

$$\left\{A_k(t,\,\mathbf{x}), E^l(t,\,\mathbf{y})\right\} = \delta_k^l\, \delta(\mathbf{x}-\mathbf{y}) \tag{1.106}$$

$$\{B(t,\,\mathbf{x}), \Pi(t,\,\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y}) \tag{1.107}$$

In order to quantize the massless vector field, as we did for the massive vector field, we follow the manifestly covariant formulation. This way, it turns out that the covariant commutation relations are given by

$$[A^\mu(x),\, A^\nu(y)] = i\,\eta^{\mu\nu}\,\Delta(x-y) \tag{1.108}$$

$$[B(x),\, A^\nu(y)] = i\,\partial_x^\nu \Delta(x-y) \tag{1.109}$$

$$[B(x),\, B(y)] = 0 \tag{1.110}$$

and in particular

$$[F^{\mu\rho}(x),\, A^\nu(y)] = i\,(\eta^{\rho\nu}\,\partial_x^\mu - \eta^{\mu\nu}\,\partial_x^\rho)\,\Delta(x-y) \tag{1.111}$$

$$[B(x),\, F^{\nu\rho}(y)] = 0 \tag{1.112}$$

where $\Delta(x-y)$ is the massless Pauli-Jordan real distribution,

$$\Delta(x-y) = \lim_{m\to 0} D(x-y;m) \tag{1.113}$$

which satisfies the Klein-Gordon equation together with the initial conditions

$$\Box\Delta(x-y) = 0 \tag{1.114}$$

$$\lim_{x_0\to y_0} \Delta(x-y) = 0, \qquad \lim_{x_0\to y_0} \partial_0\Delta(x-y) = \delta(\mathbf{x}-\mathbf{y}) \tag{1.115}$$

Thus, it can be verified that the above commutation relations fulfil all the fundamental requirements, i.e. they are solutions of the field equations, encode the symmetry, hemiticity and microcausality properties and also the equal-time commutation relations corresponding to the classical Poisson's brackets (1.106, 1.107) are recovered, namely

$$\left[A^k(t,\,\mathbf{x}),\,E^l(t,\,\mathbf{y})\right] = i\,\eta^{kl}\,\delta(\mathbf{x}-\mathbf{y}) \tag{1.116}$$

$$[B(t,\,\mathbf{x}),\,\Pi(t,\,\mathbf{y})] = i\,\delta(\mathbf{x}-\mathbf{y}) \tag{1.117}$$

In particular, the electric and magnetic components of the massless vector field do not commute at spacelike separations, i.e.

$$\left[B^i(t,\,\mathbf{x}),\,E^l(t,\mathbf{y})\right] = i\,\varepsilon^{ilk}\,\nabla_k\delta(\mathbf{x}-\mathbf{y}), \qquad \varepsilon_{123} = +1 \tag{1.118}$$

We notice also that for a usual massless scalar field $\phi(x)$ the commutators are given by

$$[\phi(x),\,\phi(y)] = -i\,\Delta(x-y) \tag{1.119}$$

so that for A_0 the relations (1.108) have the wrong sign: A_0 does not behave as a standard scalar field.

From the normal modes decomposition and the above commutation relations, it is easy to derive the algebra of the creation and destruction operators, that reads

$$\left[g_{\mathbf{k},\,A},\,g^\dagger_{\mathbf{p},\,B}\right] = \eta'_{AB}\,\delta(\mathbf{k}-\mathbf{p}) \tag{1.120}$$

all the other commutators vanishing. We can now proceed to the quantization, the massless vector field and the auxiliary field becoming operator valued tempered distributions

$$A^\mu(x) = \sum_{\mathbf{k},\,A}\left[g_{\mathbf{k},\,A}\,u^\mu_{\mathbf{k},\,A}(x) + g^\dagger_{\mathbf{k},\,A}\,u^{\mu*}_{\mathbf{k},\,A}(x)\right] \tag{1.121}$$

$$B(x) = -i\sum_{\mathbf{k}}\left[g_{\mathbf{k},\,S}\;k\cdot u_{\mathbf{k},\,S}(x) - g^\dagger_{\mathbf{k},\,S}\;k\cdot u^*_{\mathbf{k},\,S}(x)\right] \tag{1.122}$$

In order to define a physical Fock space, we first notice that the algebra of the creation and destruction operators (1.120) entails some not ordinary properties. As a matter of fact, while the tranverse polarizations $A = 1,2$ satisfy the usual commutation relations, owing to the indefinite metric η'_{AB}, the longitudinal and scalar polarizations encode negative and null norm states. In fact, consider for instance the following states

$$\frac{1}{\sqrt{2}}\left(g^{\dagger}_{\mathbf{k},\,L}+g^{\dagger}_{\mathbf{k},\,S}\right)|0\rangle, \qquad g^{\dagger}_{\mathbf{k},\,L}|0\rangle, \qquad g^{\dagger}_{\mathbf{k},\,S}|0\rangle \tag{1.123}$$

we see that the first one has negative norm,

$$\frac{1}{2}\langle 0|\,(g_{\mathbf{p},\,L}+g_{\mathbf{p},\,S})\,(g^{\dagger}_{\mathbf{k},\,L}+g^{\dagger}_{\mathbf{k},\,S})|0\rangle = -\,\delta(\mathbf{p}-\mathbf{k}) \tag{1.124}$$

while the other two have null norm. Clearly we must impose some conditions in order to define a physical Fock space. We notice also that the massless vector field has only two indipendent polarizations, which means that we have considered too many components so far. Therefore, we should reduce the whole Fock space to one containing only states with positive and null norm. To this aim, we can define a physical state by the following auxiliary condition

$$|\mathrm{phys}\rangle \in \mathcal{H}_{\mathrm{phys}} \qquad \Leftrightarrow \qquad B^{(-)}(x)\,|\mathrm{phys}\rangle = 0 \tag{1.125}$$

where $B^{(-)}(x)$ is the positive frequency part of the auxiliary field,

$$B^{(-)}(x) = -\,i\sum_{\mathbf{k}} k\cdot u_{\mathbf{k},\,S}(x)\,g_{\mathbf{k},\,S} \tag{1.126}$$

As a matter of fact, this way all states with negative norm are are excluded, since only the states with positive and null norm satisfy the conditions

$$B^{(-)}(x)\,g^{\dagger}_{\mathbf{k},\,A}\,|0\rangle = 0, \qquad A = 1,2,S$$
$$\langle 0|\,g_{\mathbf{p},\,B}\,g^{\dagger}_{\mathbf{k},\,A}\,|0\rangle = \begin{cases} \delta_{AB}\,\delta(\mathbf{p}-\mathbf{k}) & A,B=1,2 \\ 0 & A,B=S \text{ or } A,B=S,1,2 \end{cases}$$

Hence, $\mathcal{H}_{\mathrm{phys}}$ contains the transverse polarization states with positive norm plus an arbitrary number of states with null norm which do not change the probability densities: the Fock space is then partitioned in equivalence classes with respect to the states with null norm, and therefore has a semidefinite metric. We just point out that this feature represents the quantum mechanical counterpart of the classical gauge invariance of the first kind (1.2), which entails an equivalence class of gauge potentials obeying the invariant Lorenz condition.

We finally consider the physical observables present in this contest. We define a gauge invariant local observable $\mathcal{O}(x)$ as a self-adjoint operator that maps the physical Hilbert space $\mathcal{H}_{\rm phys}$ into itself, i.e.

$$\begin{aligned} &\mathcal{O}(x) : \mathcal{H}_{\rm phys} \to \mathcal{H}_{\rm phys}\,; \qquad \mathcal{O}(x)\,|{\rm phys}\rangle \in \mathcal{H}_{\rm phys} \quad \forall|{\rm phys}\rangle \in \mathcal{H}_{\rm phys} \\ &\mathcal{O}^{\dagger}(x) = \mathcal{O}(x) \end{aligned} \tag{1.127}$$

which implies that

$$B^{(-)}(x)\,\mathcal{O}(x)\,|{\rm phys}\rangle = \left[B^{(-)}(x),\,\mathcal{O}(x)\right]|{\rm phys}\rangle \propto B^{(-)}(x)\,|{\rm phys}\rangle = 0 \tag{1.128}$$

It follows that, for instance, the Maxwell field equations and the energy momentum tensor hold true only as matrix elements between physical states. In fact, $\forall|{\rm phys'}\rangle,\ |{\rm phys}\rangle \in \mathcal{H}_{\rm phys}$, we find

$$\begin{aligned} \langle {\rm phys'}|\partial_\mu F^{\mu\nu}(x) + \partial^\nu B(x)|{\rm phys}\rangle &= \langle {\rm phys'}|\partial_\mu F^{\mu\nu}(x)|{\rm phys}\rangle \\ \langle {\rm phys'}|\Theta_{\mu\nu}(x)|{\rm phys}\rangle &= \langle {\rm phys'}|\ \frac{1}{4}\,\eta_{\mu\nu}\,F^{\rho\sigma}F_{\rho\sigma} - F_\mu^\rho F_{\nu\rho}\,|{\rm phys}\rangle \end{aligned}$$

since

$$[B(x),\,F^{\nu\rho}(y)] = 0, \qquad [B(x),\,\Theta^{\nu\rho}(y)] = 0$$

while

$$[B(x),\,T^{\nu\rho}(y)] \neq 0 \tag{1.129}$$

Therefore, we can also verifiy that only the six components of the angular momentum operator, which is of purely orbital form, are the observable quantities, namely

$$M^{\lambda\mu\nu} = \int d\mathbf{x} : x^\mu\,\Theta^{\lambda\nu}(t,\mathbf{x}) - x^\nu\,\Theta^{\lambda\mu}(t,\mathbf{x}) : \tag{1.130}$$

Chapter 2

Field Theory in curved spacetime

In this chapter we will focus on the tools and problems regarding the quantization of arbitrary fields in a curved spacetime.

To start with, we recall the most important principles of General Relativity and their consequences, which stand as a basis for all the developments about Field Theory in any Riemannian space, and in particular in a Rindler space, the one about we will be concerned in this work. Of course, the first and most important principle from which all the other descend is the Principle of Equivalence, that rests on the equality between gravitational and inertial mass, and states that at every space-time point x^μ in an arbitrary gravitational field, there always exists a locally inertial coordinate system such that, within a sufficiently small region around x^μ where the field can be considered sensibly constant, the laws of Nature take the same form as in an unaccelerated Cartesian coordinate system in the absence of gravity: the laws of special relativity are then recovered. It should be noticed also that this principle actually reflects a geometrical property of the Riemannian spaces, i.e. every curved manifold is locally flat like a Minkowski space, where $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\mu_{\mu\lambda} = 0$. From this geometrical point of view, differential geometry gains a fundamental role in the description of all physical phenomena.

From the Principle of Equivalence, we gain a way to find the equations of motion for a system in a gravitational field: first write down the equations that hold in a locally inertial reference frame, and then perfom a coordinate transformation to the non-inertial reference frame of interest to obtain the corrisponding equations. Clearly, this method would be very tedious when the system under consideration is not simple; however, the Principle of General Covariance states that a physical equation holds in a general gravitational field iff

1. the equation is generally covariant, that is it preserves its form under a general coordinate transformation;
2. the equation holds in the absence of gravitation, that is it agrees to the laws of special relativity when the metric $g_{\mu\nu} = \eta_{\mu\nu}$ and the affine connection vanishes $\Gamma^{\mu}_{\mu\lambda} = 0$.

It should be noticed, however, that the Principle of General Covariance is not just a merely generalization of the Lorentz covariance for special relativity: this time we do not want to make any restriction on the equation we start with because the presence of $g_{\mu\nu}$ and $\Gamma^{\mu}_{\nu\lambda} \neq 0$ indeed represents the gravitational field and its effects on the system under study. Therefore, this is not an invariance principle, but it is a statement about the existence and consequences of the gravitational field. Moreover, it neither implies Lorentz invariance. Finally, we would like to remark that this principle actually applies only on scales small compared with the spacetime distances typical of the gravitational field: it is only on these scales that we are surely able to construct a locally inertial reference frame, thanks to the Principle of Equivalence.

One way to build up such physical equations is to use tensor equations. Neverthless, since differentiation of a tensor in general does not yeld another tensor, we have to define the so called *covariant derivative*, that for a controvariant vector V^{ν} is given by

$$\nabla_{\mu} V^{\nu} \equiv V^{\nu}_{\,;\,\mu} \equiv V^{\nu}_{\,,\,\mu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda} \tag{2.1}$$

and for a covariant vector

$$\nabla_{\mu} V_{\nu} \equiv V_{\nu\,;\,\mu} \equiv V_{\nu\,,\,\mu} - \Gamma^{\lambda}_{\nu\mu} V_{\lambda} \tag{2.2}$$

We just recall that the covariant derivative of the metric tensor vanishes, $g_{\mu\nu\,;\,\lambda} = 0$, since it vanishes in locally inertial coordinates, where $\Gamma^{\mu}_{\nu\lambda} = 0$ and $g_{\mu\nu\,,\,\lambda} = 0$. Then, as covariant differentiation converts tensors to other tensors and reduces to ordinary differentiation in the absence of gravity, we can safely say that, in order to find the equations of motion of a system in a general gravitational field, we can first write the corrisponding equations in the absence of gravity, as in special relativity, and then replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$ and all derivatives with covariant derivatives.

We can now turn the attention to the quantization of an arbitrary field in a curved spacetime. In general, it would be more useful to choose, if possible, a coordinate system in which the field equations can be solved by separation of variables, so as to quantize the resulting normal mode structure in analogy to the standard quantization of a free field in flat space and recover the positive and negative frequency parts of the solutions, in order to interpret them in terms of particles. However, as we shall see, even in this contest, the notions of particles and vacuum state would be completely different from the ones obtained in the usual Minkowki space. This ambiguity would also affect the definition of the enegy-momentum tensor, which is the most important observable of the system in a gravitational field.

As an example, let us consider the quantization of a scalar field in an arbitrary gravitational field. Consider the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}[-g(x)]^{-\frac{1}{2}} \left[g^{\mu\nu}\phi(x)_{,\mu}\,\phi(x)_{,\nu} - m^2\,\phi(x)^2\right] \tag{2.3}$$

whose equations of motion are

$$(g^{\mu\nu}\,\nabla_\mu\,\partial_\nu + m^2)\,\phi(x) = 0 \tag{2.4}$$

If we now introduce the following invariant inner product between two solutions of the above equation

$$(\phi_1,\,\phi_2) \equiv \oint_\Sigma \phi_1^*(x)\,i\,\overleftrightarrow{\partial_\lambda}\,\phi_2(x)\,d\Sigma^\lambda \tag{2.5}$$

where Σ^λ is a three-dimensional Cauchy hypersurface and

$$\begin{aligned} d\Sigma^\lambda &= \frac{1}{6}\,\varepsilon^{\lambda\mu\nu\rho}\,dx_\mu\,dx_\nu\,dx_\rho\,\sqrt{-g} \\ &= \frac{1}{6}\,\varepsilon^{\lambda\mu\nu\rho}\,g_{\mu\alpha}(x)\,g_{\nu\beta}(x)\,g_{\rho\gamma}(x)\,dx^\alpha\,dx^\beta\,dx^\gamma\,\sqrt{-g} \end{aligned} \tag{2.6}$$

is the invariant oriented hypersurface element with $\varepsilon^{0123} = 1$, it is possible to find a complete set of mode solutions $u_i(x)$ of (2.4) that are orthonormal in the above inner product:

$$(u_i, u_k) = \delta_{ik}, \qquad (u_i^*, u_k^*) = -\delta_{ik}, \qquad (u_i, u_k^*) = 0 \tag{2.7}$$

This way, the field $\phi(x)$ can be expanded as

$$\phi(x) = \sum_i [a_i\,u_i(x) + a_i^\dagger\,u_i^*(x)] \tag{2.8}$$

The covariant quantization is then implemented by adopting the canonical commutation relations

$$[\,a_i,\,a_k^\dagger\,] = \delta_{ik} \tag{2.9}$$

all the other commutators vanishing. It is therefore possible to construct a vacuum state, a Fock space and proceed in the same way as in the Minkowski case, but this time an ambiguity in the formalism arises: in the Minkowski space the metric is static, so the vector $\partial/\partial t$ is a Killing vector, orthogonal to the spacelike hypersurfaces $t = const$, and the modes

$$u_k(x) = \frac{1}{[2\omega(2\pi)^3]^{1/2}} e^{-i\omega t + i\mathbf{k}\ \mathbf{x}}$$

are its eingenfunctions with eingenvalues $-i\omega$ for $\omega > 0$; the vacuum is invariant under the Poincaré group. However, in curved spacetime the Poincaré group is no longer a symmetry group of the spacetime and in general there will not even be any Killing vectors with which define positive frequency modes. Actually, this is a consequence of the Principle of General Relativity: there does not exist a privileged coordinate system and a consequent natural mode decomposition based on separation of variables.

In order to show this ambiguity in more detail, consider, for instance, an other complete orthonormal set of modes $v_j(x)$ satisfying the equation (2.4). We can then expand ϕ as

$$\phi(x) = \sum_i [\, b_i\, v_i(x) + b_i^\dagger\, v_i^*(x)\,] \tag{2.10}$$

and we can define a new vacuum state $|0\rangle_b$

$$b_j|0\rangle_b = 0 \qquad \forall j \tag{2.11}$$

and the corresponding Fock space. Since both sets are complete, we can expand the new modes v_j in terms of the u_i: namely

$$v_j(x) = \sum_i (\alpha_{ji}\, u_i(x) - \beta_{ji}\, u_i^*(x)) \tag{2.12}$$

and conversely

$$u_i(x) = \sum_j (\alpha_{ij}^*\, v_j(x) + \beta_{ij}^*\, v_j^*(x)) \tag{2.13}$$

These relations are known as Bogolyubov transformations and α_{ij}, β_{ji} Bogolyubov coefficients, which can be evaluated as

$$\alpha_{ij} = (v_i, u_j) \qquad \beta_{ij} = -(u_i, v_j^*) \tag{2.14}$$

From (2.8, 2.10) and the above equations, we can also write

$$a_i = \sum_j (\alpha_{ji}\, b_j - \beta_{ji}^*\, b_j^\dagger), \tag{2.15}$$

$$b_j = \sum_i (\alpha_{ji}^*\, a_i + \beta_{ji}\, a_i^\dagger) \tag{2.16}$$

and find

$$[\, a_i, a_k^\dagger \,] = \sum_j (\alpha_{ij}\, \alpha_{kj}^* - \beta_{ij}^*\, \beta_{kj}) = \delta_{ik}, \tag{2.17}$$

$$[\, b_i, b_k^\dagger \,] = \sum_j (\alpha_{ji}\, \alpha_{jk}^* - \beta_{ji}\, \beta_{jk}^*) = \delta_{ik} \tag{2.18}$$

all the others vanishing. From (2.15) it is easy to see that the two Fock spaces belonging to the modes $u_i(x)$ and $v_j(x)$ respectively are different as long as $\beta_{ij} \neq 0$: as a matter of fact, a_i does not annichilate $|0\rangle_b$:

$$a_i|0\rangle_b = \sum_j \beta_{ij}^*\, |1\rangle_b \neq 0 \tag{2.19}$$

and the expectation value of the particle operator $N_i = a_i^\dagger a_i$ in the $|0\rangle_b$ states does not vanish:

$$_b\langle 0|\, N_i = a_i^\dagger a_i\, |0\rangle_b = \sum_j |\beta_{ij}|^2 \tag{2.20}$$

This means that the two vacuum states are actually different, that is, the vacuum of the v_j modes contains $\sum_j |\beta_{ij}|^2$ particles of the u_i modes. It should be noticed also that if any $\beta_{ij} \neq 0$, the v_j modes will contain a mixture of positive (say u_i) and negative (say u_i^*) frequency modes and particles of this kind will be present in the $|0\rangle_b$ vacuum state. If, instead, u_i are positive frequency modes with respect to some timelike Killing vector and v_j are a linear combination only of the u_i, then $\beta_{ij} = 0$ and the two vacuum state will be the same.

This phenomenon is the so called *Unruh Effect.* Of course, there is not a privileged set of modes which can give the closest description of a physical vacuum as our experience of no particles: in fact, the state of motion of the measuring device can affect the observation of particles. For instance, an accelerated detector will register some particles even in the vacuum state defined in the usual Minkowski space. What makes the Minkowski space so special is its global invariance under the Poincaré group, thanks to which all inertial measuring devices agree in defining the same vacuum state.

From these considerations, it is easy to realize that the particle concept becomes an observer-dependent quantity and looses its universal significance. In order to overcome this astonishing fact, in many problems the spacetime can be treated as asymptotically flat, recovering this way the Minkowski vacuum as usually defined and its well understood physical meaning, that is the absence of particles according to all inertial observers in the asymptotic regions. However, the two vacua, i.e. the asymptotic past and future ones, not necessarily coincide: this is what is commonly referred to as particle "creation" by the time dependent gravitational field.

2.1 Rindler spacetime

In our work we will focus on the behaviour of Vector Fields as experienced by a uniformly accelerated noninertial observer. The simplest, but very interesting coordinate system that describes such observers is the so called *Rindler space*. In the following we will set $c = \hbar = 1$. Let us introduce this spacetime.

Consider the four dimensional Minkowski spacetime with line element

$$ds^2 = \eta_{\alpha\beta}\, dX^\alpha\, dX^\beta \tag{2.21}$$

where $\eta_{\alpha\beta} = diag(+,-,-,-)$ is the constant metric tensor and $X^\alpha = (\tau, X, Y, Z)$ are the inertial coordinates. Let us consider now an observer in the right Rindler wedge,

$$\mathcal{M}_R = \{X^\mu \in \mathbb{R}^4;\, X \geq 0,\, \tau^2 \leq X^2\} \tag{2.22}$$

that is an uniformly accelerated noninertial observer, described by the coordinates $x^\mu = (t, x, y, z)$:

$$\begin{aligned}
\tau &= x \sinh(at) && (2.23)\\
X &= x \cosh(at) && (2.24)\\
Y &= y && (2.25)\\
Z &= z && (2.26)
\end{aligned}$$

where $a > 0$ is the constant acceleration. The above coordinate transformation can be readily inverted:

$$\begin{aligned}
t &= \frac{1}{a}\,\mathrm{arth}\left(\frac{\tau}{X}\right)\\
x &= \sqrt{X^2 - \tau^2} \geq 0\\
y &= Y\\
z &= Z
\end{aligned} \tag{2.27}$$

so that we can also write line element

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu = a^2\, x^2\, dt^2 - dx^2 - dY^2 - dZ^2 \tag{2.28}$$

Thus, in this region the metric takes the form

$$g_{\mu\nu}(x) = \begin{pmatrix} a^2x^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

with determinant $g = \det g_{\mu\nu} = -a^2x^2$. Let us denote $\xi = ax$, $\eta = at$; the change of coordinates is then simply given by

$$\frac{\partial X^\alpha}{\partial x^\mu} = \begin{pmatrix} \xi\cosh\eta & \sinh\eta & 0 & 0 \\ \xi\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \zeta^\alpha_\nu(\xi,\eta)$$

We can also find that, among the Christoffel symbols

$$\Gamma^\lambda_{\mu\nu}(x) = \frac{1}{2}\, g^{\lambda k}(x)\{\partial_\mu\, g_{\nu k}(x) + \partial_\nu\, g_{\mu k}(x) - \partial_k\, g_{\mu\nu}(x)\} \tag{2.29}$$

the only nonvanishing ones are

$$\Gamma^0_{10}(x) = 1/x = \Gamma^0_{01}(x) \tag{2.30}$$

$$\Gamma^1_{00}(x) = a^2 x \tag{2.31}$$

Note that the other spacelike region of the Minkowski spacetime is covered by changing both signs in (2.23, 2.24), i.e. performing a time reversal and a parity transformation:

$$\mathcal{M}_L = \{X^\mu \in \mathbb{R}^4;\ X \leq 0,\ \tau^2 \leq X^2\} \tag{2.32}$$

Some comments are now in order. First of all, we notice that a translation in the coordinate t, with x fixed, corresponds to a homogeneous Lorentz transformation in the (τ, X) space: this is the reason why the metric of flat space has an explicit static form with respect to the curvilinear coordinates (t, x). Then, it is easy to see that the classical Cauchy problem should be well posed for initial conditions on any hypersurface $t = const$. Secondly, tajectories with $x = const$ are hyperbolae and therefore represent the world lines of a uniformly accelerated observer (see Figure 2.1). In order to understand better the meaning of this coordinate system, let us write (x, t) in a slight different way (that is often used in many texts): namely

$$\tau = \frac{e^{a\xi}}{a}\sinh(a\eta) \tag{2.33}$$

$$X = \frac{e^{a\xi}}{a}\cosh(a\eta) \tag{2.34}$$

with $-\infty < \eta,\ \xi < \infty$; the line element becomes

$$ds^2 = e^{2a\xi}(d\eta^2 - d\xi^2) \tag{2.35}$$

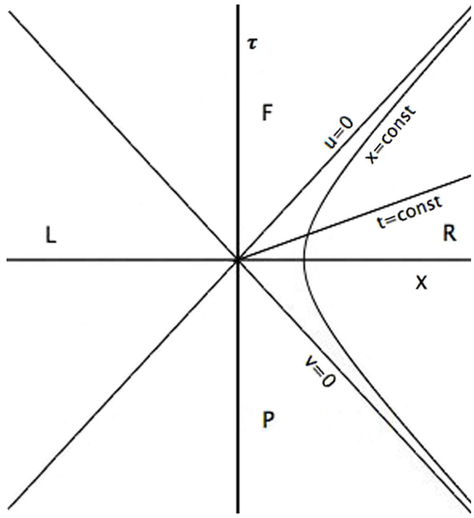

Figure 2.1: Rindler coordinates in Minkowski space: both in R and L lines with constant t are straight lines through the origin, while lines with $x = const$ are hyperbolae representing uniformly accelerated observers, with null asymptotes $u = 0$, $v = 0$. The Rindler coordinates are non-analytic across $u = 0$ and $v = 0$: the four regions R, L, F, P must be covered by separate coordinate patches.

from which we can see that it is a conformal transformation of the Minkowski metric. Of course, the coordinates (η, ξ) still cover the same region $\mathcal{M}_R$ of the Minkowski spacetime as (t, x). We can see as before that lines with constant η are straight, $X \propto \tau$, while lines of constant ξ are hyperbolae, $X^2 - \tau^2 = a^{-2}e^{2a\xi} = const$, where $ae^{-a\xi}$ is the proper acceleration. Thus, lines of large positive ξ, i.e. far from $X = \tau = 0$, represent weakly accelerated observers, while the hyperbolae that approach $X = \tau = 0$ carry a high proper acceleration. It should be noticed that all the hyperbolae are asymptotically null, that is, approach the null rays (let us call them as $u = 0, v = 0$) of the light cone of the Minkowski space: the accelerated observers get close to the speed of light as $\eta \to \pm\infty$. Therefore, it is easily understood that the causal structure of the Rindler wedge is not trivial. The Rindler observers approach, but do not cross the null rays $u = 0, v = 0$: these, then, act as event horizons and the two regions $\mathcal{M}_R$ and $\mathcal{M}_L$ are causally disjoint. No event from the region of the Minkowski spacetime beyond the null rays can be witnessed by the Rindler observers, neither can causally influence them. This can also be seen in the Penrose conformal diagram below (Figure 2.2), where timelike lines with $x = const$ do not intersect the vertices $i^{\pm}$ as they do in the Minkowski space, but the lower ones. Thus, for instance, the upper null ray acts as a future horizon and events in the portion marked F cannot causally influence the diamond shaped R region. Moreover, we notice that a three-dimensional hypersurface with $t = const$ describes events that are simultaneous from the point of view of the accelerated observers. This surface is a hyperplane with constant ratio τ/X. All such hyperplanes cross one another at $\tau = X = 0$, and therefore it is evident that the proper distance between world lines of Rindler observers is time indipendent, and thus the Rindler frame is rigid. Finally, we notice that, in order for the frame to be rigid, two trajectories of the family with different values of X must have different accelerations at a given moment of time τ; this means that the larger is the value of X, the smaller is the acceleration.

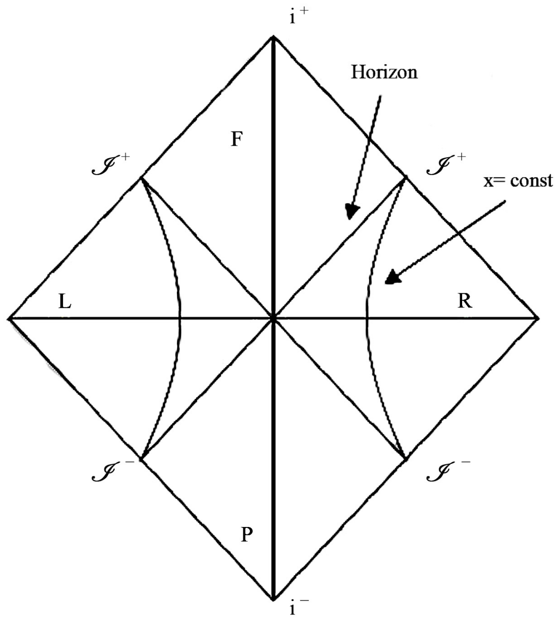

Figure 2.2: Penrose conformal diagram of Rindler system. Timelike lines with $x = const$ do not intersect $i^{\pm}$ as they do in Minkowski space: events in F cannot be witnessed in R, so that the null rays $u = 0, v = 0$ act as event horizons.

2.2 Scalar Field in a Rindler Space

As a first example, let us consider the quantization of the Scalar Field in the right Rindler wedge $\mathcal{M}_R$, as the results achieved here will be useful for the study of the quantization of the Vector Field that we are going to explore in the next chapter.

We start with the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\,[-g(x)]^{-\frac{1}{2}}\,\left[g^{\mu\nu}(x)\,\phi(x)_{\,,\mu}\,\phi(x)_{\,,\nu} - m^2\,\phi(x)^2\right] \tag{2.36}$$

whose equations of motion are

$$(g^{\mu\nu}(x)\,\nabla_\mu\,\partial_\nu + m^2)\,\phi(x) = 0 \tag{2.37}$$

that can be explicitly rewritten as

$$\left[\frac{1}{a^2x^2}\,\frac{\partial}{\partial t^2} - \Delta - \frac{1}{x}\,\frac{\partial}{\partial x} + m^2\right]\phi(t,x,y,z) = 0 \tag{2.38}$$

where Δ is the Laplace operator. In order to find the solution, it is useful to introduce the partial Fourier Transform, given by

$$\phi(t,x,y,z) = \int_{-\infty}^{+\infty}\frac{dE}{\sqrt{2\pi}}\int\frac{d^2k_\perp}{2\pi}\,\tilde{\phi}(E,k_\perp,x)\,e^{-iEt+ik_\perp\cdot x_\perp} \tag{2.39}$$

where

$$x_\perp = (y,z), \qquad k_\perp = (k_y,k_z) \tag{2.40}$$

so that the equation (2.38) simplifies in

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\,\frac{d}{dx} + \frac{E^2}{a^2x^2} - (m^2 + k_\perp^2)\right]\tilde{\phi}(x,E,k_\perp) = 0 \tag{2.41}$$

the solutions of which are expressed in terms of Bessel functions of imaginary order, i.e.

$$\tilde{\phi}(x,E,k_\perp) = c_1(E,k_\perp)\,I_{i\alpha}(b\,x) + c_2(E,k_\perp)\,K_{i\alpha}(b\,x) \tag{2.42}$$

where

$$\alpha = \frac{E}{a}, \quad b = \sqrt{m^2 + k_\perp^2} \tag{2.43}$$

However, the solutions $I_{i\alpha}(b\,x)$ are usually rejected since they are exponentially increasing for large positive x, so that we must set $c_1(E,k_\perp) = 0$. We are thus left with the normal modes decomposition of the real Scalar Field in the right Rindler wedge: namely,

$$\phi(t,x,y,z) = \int_{-\infty}^{+\infty} \frac{dE}{\sqrt{2\pi}} \int \frac{d^2k_\perp}{2\pi} \left[f(E,k_\perp)\, K_{i\alpha}(b\,x)\, e^{-iEt+ik_\perp\cdot x_\perp} + \text{c.c.} \right] \tag{2.44}$$

where the reality conditions

$$f(-E,-k_\perp) = f^*(E,k_\perp) \tag{2.45}$$

and the property $K_{i\alpha}(b\,x) = K_{-i\alpha}(b\,x)$ are to be taken suitably into account.

We can now introduce an invariant inner product between any two solutions of the covariant Klein-Gordon equation (2.38) of the form

$$(\phi_1,\, \phi_2) \equiv \oint_\Sigma \phi_1^*(x)\, i\, \overleftrightarrow{\partial_\lambda}\, \phi_2(x)\, d\Sigma^\lambda \tag{2.46}$$

In particular, as a Cauchy hypersurface, let us consider the initial time three dimensional hypersurface with

$$d\Sigma^0 = -\frac{1}{a\,x}\,\theta(x)\,dx\,d^2x_\perp \qquad d\Sigma^i = 0, \quad i = 1,2,3 \tag{2.47}$$

A complete orthonormal set of positive frequency solutions of eq. (2.38) is then given by the generalization of the so called *Fulling modes*:

$$\tilde{\varphi}_{E,k_\perp}(x) = \frac{\theta(x)}{2\,\pi^2\sqrt{a}} \sqrt{\sinh\left(\frac{\pi E}{a}\right)}\, K_{i\alpha}(b\,x)\, e^{-iEt+ik_\perp\cdot x_\perp} \tag{2.48}$$

where the normalization constant can be fixed e.g. by the requirement

$$(\tilde{\varphi}_{E,k_\perp},\, \tilde{\varphi}_{E',k'_\perp}) = \delta(E-E')\,\delta^{(2)}(k_\perp - k'_\perp) \tag{2.49}$$

This way, our invariant scalar normal modes have standard canonical dimensions $[\tilde{\varphi}_{E,k_\perp}] = \text{cm}^{1/2}$ in natural units. We can therefore expand the scalar field $\phi(x)$ in terms of these modes and proceed to the quantization, obtaining

$$\phi(x) = \int_{-\infty}^{+\infty} dE \int d^2k_\perp \left[a_{E,k_\perp}\, \tilde{\varphi}_{E,k_\perp}(x) + a^\dagger_{E,k_\perp}\, \tilde{\varphi}^*_{E,k_\perp}(x) \right] \tag{2.50}$$

where the operators $a, a^\dagger$ satisfy the following commutation relations

$$\left[a_{E,k_\perp},\, a^\dagger_{E',k'_\perp} \right] = \delta(E-E')\,\delta^{(2)}(k_\perp - k'_\perp) \tag{2.51}$$

$$\left[a_{E,k_\perp},\, a_{E',k'_\perp} \right] = 0 \tag{2.52}$$

$$\left[a^\dagger_{E,k_\perp},\, a^\dagger_{E',k'_\perp} \right] = 0 \tag{2.53}$$

$$\forall k_\perp, k'_\perp \in \mathbb{R}^2 \text{ and } \forall E, E' \in \mathbb{R} \tag{2.54}$$

and have canonical dimensions $[a] = \mathrm{cm}^{3/2}$ in natural units. It should be noticed that the spinless and chargeless quanta

$$a^\dagger_{E,k_\perp}|0\rangle \qquad (E, k_\perp \in \mathbb{R}^3) \tag{2.55}$$

correspond to pseudoparticles, with indefinite energy $E \in \mathbb{R}$, but fixed transverse wave numbers $k_\perp \in \mathbb{R}^2$. Clearly, the same would be true for the multi-pseudoparticles completely symmetric states. [1]

[1] An explicit calculation of the Bogolyubov coefficients and the rigorous check of the completeness of the generalized Fulling normal modes (2.48) can be found in [6].

Chapter 3

Vector Fields in a Rindler Space

In this chapter we will focus on the quantization of the Vector Field, both massive and massless, in the right Rindler wedge $\mathcal{M}_R$. We will consider two gauges in particular, that is the Feynman gauge and the axial gauge, as they appear to be the most interesting ones in this contest.

Before starting, we would like to point out that in our metric the following properties hold true, namely

$$\begin{aligned} F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu \\ &= \partial_\mu A_\nu - \Gamma^\lambda_{\nu\mu} A_\lambda - \partial_\nu A_\mu + \Gamma^\lambda_{\mu\nu} A_\lambda \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{3.1}$$

while

$$\begin{aligned} F^{\mu\nu} &= g^{\rho\mu} g^{\sigma\nu} F_{\rho\sigma} = g^{\rho\mu} g^{\sigma\nu} (\partial_\rho A_\sigma - \partial_\sigma A_\rho) \\ &= g^{\rho\mu} \nabla_\rho g^{\sigma\nu} A_\sigma + g^{\rho\mu} g^{\sigma\nu} \Gamma^\lambda_{\sigma\rho} A_\lambda - g^{\sigma\nu} \nabla_\sigma g^{\rho\mu} A_\rho - g^{\rho\mu} g^{\sigma\nu} \Gamma^\lambda_{\rho\sigma} A_\lambda \\ &= \nabla^\mu A^\nu - \nabla^\nu A^\mu \\ &= g^{\mu\rho} \nabla_\rho A^\nu - g^{\nu\rho} \nabla_\rho A^\mu \\ &= g^{\mu\rho} (\partial_\rho A^\nu + \Gamma^\nu_{\rho\lambda} A^\lambda) - g^{\nu\rho} (\partial_\rho A^\mu + \Gamma^\mu_{\rho\lambda} A^\lambda) \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu + g^{\mu\rho} \Gamma^\nu_{\rho\lambda} A^\lambda - g^{\nu\rho} \Gamma^\mu_{\rho\lambda} A^\lambda \end{aligned} \tag{3.2}$$

and in particular, by taking two covariant derivatives of the field strenght tensor one gets

$$\nabla_\nu \nabla_\mu F^{\mu\nu} = \left(\partial_\nu \Gamma^\mu_{\mu\lambda}\right) F^{\lambda\nu} = 0 \tag{3.3}$$

Moreover, since the Rindler spacetime is flat, one obtains

$$[\nabla_\mu,\, \nabla_\nu]\, A^\rho = -R^\rho_{\lambda\mu\nu}\, A^\lambda = 0 \tag{3.4}$$

3.1 Quantization of the Vector Field in the Feynman gauge

Consider the Vector Field in the space-like region $\mathcal{M}_R$: it is described by the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left[-\frac{1}{4}\, F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{1}{2}\, m^2\, A_\mu(x)\, A^\mu(x) + A^\mu(x)\, \partial_\mu B + \frac{1}{2}\, \xi\, B^2(x)\right] \tag{3.5}$$

which leads to the equations of motion for A^ν and B, namely

$$\nabla_\mu F^{\mu\nu}(x) + m^2 A^\nu(x) + \partial^\nu B(x) = 0 \tag{3.6}$$
$$\nabla_\mu A^\mu(x) = \xi\, B(x) \tag{3.7}$$

These equations can be simplified choosing the Feynman gauge $\xi = 1$, so that they become

$$\nabla_\mu F^{\mu\nu}(x) + m^2\, A^\nu(x) + \partial^\nu B(x) = 0 \tag{3.8}$$
$$\nabla_\nu A^\nu(x) = \partial_\nu\, A^\nu(x) + \Gamma^\nu_{\nu\lambda}\, A^\lambda(x) = B(x) \tag{3.9}$$
$$(\nabla_\mu\, \partial^\mu + m^2) B(x) = 0 \tag{3.10}$$

which can be explicitly written in the form

$$\left[\frac{1}{a^2x^2}\, \partial_t^2 - \Delta + m^2 - \frac{3}{x}\, \partial_1\right] A^0(x) \;=\; -\frac{2}{a^2x^3}\, \partial_0\, A^1(x) \tag{3.11}$$
$$\left[\frac{1}{a^2x^2}\, \partial_t^2 - \Delta + m^2 - \frac{1}{x}\, \partial_1 + \frac{1}{x^2}\right] A^1(x) \;=\; -\frac{2}{x}\, \partial_0 A^0(x) \tag{3.12}$$
$$\left[\frac{1}{a^2x^2}\, \partial_t^2 - \Delta + m^2 - \frac{1}{x}\, \partial_1\right] A^\perp(x) \;=\; 0 \tag{3.13}$$

and

$$\left[\frac{1}{a^2x^2}\, \partial_t^2 - \Delta + m^2 - \frac{1}{x}\, \partial_1\right] B(x) = 0 \tag{3.14}$$

In particular, considering the second equation (3.12), if we sobstitute the transversality condition (3.9),

$$\partial_0\, A^0(x) = B(x) - \partial_1\, A^1(x) - \partial_\perp \cdot A^\perp(x) - \frac{1}{x}\, A^1(x) \tag{3.15}$$

the equation can be recast as

$$\left[\frac{1}{a^2x^2}\,\partial_t^2 - \Delta + m^2 - \frac{3}{x}\,\partial_1 - \frac{1}{x^2}\right] A^1(x) = -\frac{2}{x}\,\left(B(x) - \partial_\perp \cdot A^\perp(x)\right) \quad (3.16)$$

Now, in order to find the solutions, it is convenient to introduce the partial Fourier Transform for A^μ and B given by

$$A^\mu(t,x,y,z) = \int_{-\infty}^{+\infty} \frac{dk_0}{\sqrt{2\pi}} \int \frac{d^2k_\perp}{2\pi}\, \tilde{A}^\mu(k_0,k_\perp,x)\, e^{-ik_0t + ik_\perp\cdot x_\perp} \quad (3.17)$$

$$B(t,x,y,z) = \int_{-\infty}^{+\infty} \frac{dk_0}{\sqrt{2\pi}} \int \frac{d^2k_\perp}{2\pi}\, \tilde{B}(k_0,k_\perp,x)\, e^{-ik_0t + ik_\perp\cdot x_\perp} \quad (3.18)$$

In fact, this way the equations simplify, obtaining

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \frac{k_0^2}{a^2x^2} - (m^2 + k_\perp^2)\right] \tilde{B}(x,k_0,k_\perp) = 0 \quad (3.19)$$

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \frac{k_0^2}{a^2x^2} - (m^2 + k_\perp^2)\right] \tilde{A}^\perp(x,k_0,k_\perp) = 0 \quad (3.20)$$

$$\left[\frac{d^2}{dx^2} + \frac{3}{x}\frac{d}{dx} + \frac{k_0^2}{a^2x^2} - (m^2 + k_\perp^2)\right] \tilde{A}^0(x,k_0,k_\perp) = -\,\frac{2}{a^2x^3}\,i\,k_0\,\tilde{A}^1 \quad (3.21)$$

$$\left[\frac{d^2}{dx^2} + \frac{3}{x}\frac{d}{dx} + \frac{1}{x^2}\left(\frac{k_0^2}{a^2} + 1\right) - (m^2 + k_\perp^2)\right] \tilde{A}^1(x,k_0,k_\perp) = \frac{2}{x}\,(\tilde{B} - i\,k_\perp \cdot \tilde{A}^\perp) \quad (3.22)$$

Now we see that the equations for $\tilde{B}$ and $\tilde{A^\perp}$ are just the same as the ones obtained for the Scalar Field (see 2.41), so that we can immediatly write down the solutions for the real transverse vector field and the auxiliary field in terms of the normal modes, namely

$$B(t,x,y,z) = \int_{-\infty}^{+\infty} \frac{dk_0}{\sqrt{2\pi}} \int \frac{d^2k_\perp}{2\pi}\, \left[f(k_0,k_\perp)\, K_{i\alpha}(b\,x)\, e^{-ik_0t + ik_\perp\cdot x_\perp} + \text{c.c.}\right]$$

$$A^\perp(t,x,y,z) = \int_{-\infty}^{+\infty} \frac{dk_0}{\sqrt{2\pi}} \int \frac{d^2k_\perp}{2\pi}\, \left[f^\perp(k_0,k_\perp)\, K_{i\alpha}(b\,x)\, e^{-ik_0t + ik_\perp\cdot x_\perp} + \text{c.c.}\right] \quad (3.23)$$

where again the reality conditions

$$f^\perp(-k_0,-k_\perp) = f^{\perp *}(k_0,k_\perp) \quad \text{and} \quad f(-k_0,-k_\perp) = f^*(k_0,k_\perp) \quad (3.24)$$

and the property $K_{i\alpha}(bx) = K_{-i\alpha}(bx)$ are to be taken suitably into account. For what concerns $\tilde{A}^0$ and $\tilde{A}^1$, some more steps are necessary. Consider

$\tilde{A}^1$ first: the solution of the equation (3.22) is the sum of the homogeneous solution and of the inhomogeneous one. For what concerns the homogeneous equation, we see that it is a special case of the general differential equation

$$x^2\, y''(x) + (1-2s)\, x\, y'(x) + \left[\left(s^2 - r^2\, \nu^2\right) + a^2\, r^2\, x^{2r}\right] y(x) = 0 \tag{3.25}$$

whose most general solutions are of the form

$$y(x) = (\pm x)^s\, Z_\nu(\pm\, a\, x^r) \tag{3.26}$$

where Z_ν is a Bessel function of any kind: in our case we have to set

$$s = -1, \quad r = 1, \quad a = i\, b, \quad \nu = i\, \alpha = i\, k_0/a$$

so that, if we write

$$\tilde{A}^1(x, k_0, k_\perp) = \tilde{V}^1(x, k_0, k_\perp) + \tilde{\mathcal{I}}^1(x, k_0, k_\perp) \tag{3.27}$$

where $\tilde{V}^1$ is the solution of the homogeneous equation, we have

$$\tilde{V}^1(x, k_0, k_\perp) = f^1(k_0, k_\perp)\, \frac{K_{i\alpha}(b\, x)}{x} \tag{3.28}$$

while we can obtain the solution $\tilde{\mathcal{I}}^1$ through the Green function $G(x, x')$ which satisfies the differential equation

$$\left[x^2\, \frac{d^2}{dx^2} + 3\, x\, \frac{d}{dx} + \frac{k_0^2}{a^2} + 1 - x^2\,(m^2 + k_\perp^2)\right] G(x, x') = \delta(x - x') \tag{3.29}$$

By writing $G(x, x')$ in the form

$$G(x, x') = c_1(x')\, \theta(x - x')\, \frac{K_{i\alpha}(b\, x)}{x} + c_2(x')\, \theta(x' - x)\, \frac{I_{i\alpha}(b\, x)}{x} \tag{3.30}$$

one obtains

$$G(x, x') = \frac{1}{b\, x\, x'} \frac{\theta(x - x')\, I_{i\alpha}(b\, x') K_{i\alpha}(b\, x) + \theta(x' - x)\, K_{i\alpha}(b\, x')\, I_{i\alpha}(b\, x)}{K'_{i\alpha}(b\, x')\, I_{i\alpha}(b\, x') - I'_{i\alpha}(b\, x')\, K_{i\alpha}(b\, x')} \tag{3.31}$$

Thus, we have

$$\tilde{\mathcal{I}}^1(x, k_0, k_\perp) = \int_0^\infty dx'\, 2\, x'\, G(x, x')\, (\tilde{B}(x', k_0, k_\perp) - i\, k_\perp \cdot \tilde{A}^\perp(x', k_0, k_\perp)) \tag{3.32}$$

In order to evaluate this expression at least asymptotically, we recall the expansion of the Bessel functions given by

$$
\begin{aligned}
I_\nu(z) &\sim \frac{e^z}{\sqrt{2\pi z}}\left[1-\frac{1}{2\,z}\,\frac{\Gamma(\nu+\frac{3}{2})}{\Gamma(\nu-\frac{1}{2})}+\cdots\right] && (3.33)\\
K_\nu(z) &\sim \sqrt{\frac{\pi}{2z}}e^{-z}\left[1+\frac{1}{2\,z}\,\frac{\Gamma(\nu+\frac{3}{2})}{\Gamma(\nu-\frac{1}{2})}+\cdots\right] && (3.34)\\
I'_\nu(z) &= \frac{1}{2}\,[I_{\nu+1}(z)+I_{\nu-1}(z)]\\
&\sim \frac{e^z}{\sqrt{2\pi z}}\left\{1-\frac{1}{4\,z}\left[\frac{\Gamma(\nu+\frac{5}{2})}{\Gamma(\nu+\frac{1}{2})}+\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu-\frac{3}{2})}\right]+\cdots\right\} && (3.35)\\
K'_\nu(z) &= -\frac{1}{2}\,[K_{\nu+1}(z)+K_{\nu-1}(z)]\\
&\sim \sqrt{\frac{\pi}{2z}}e^{-z}\left\{1+\frac{1}{4\,z}\left[\frac{\Gamma(\nu+\frac{5}{2})}{\Gamma(\nu+\frac{1}{2})}+\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu-\frac{3}{2})}\right]+\cdots\right\} && (3.36)
\end{aligned}
$$

Now we can get for the leading behaviour

$$
\begin{aligned}
K_\nu(z)\,I'_\nu(z) &\sim \frac{1}{2\,z}\left\{1-\frac{1}{4\,z}\left[\frac{\Gamma(\nu+\frac{5}{2})}{\Gamma(\nu+\frac{1}{2})}+\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu-\frac{3}{2})}-2\,\frac{\Gamma(\nu+\frac{3}{2})}{\Gamma(\nu-\frac{1}{2})}\right]+\cdots\right\}\\
-I_\nu(z)\,K'_\nu(z) &\sim \frac{1}{2\,z}\left\{1+\frac{1}{4\,z}\left[\frac{\Gamma(\nu+\frac{5}{2})}{\Gamma(\nu+\frac{1}{2})}+\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu-\frac{3}{2})}-2\,\frac{\Gamma(\nu+\frac{3}{2})}{\Gamma(\nu-\frac{1}{2})}\right]+\cdots\right\}
\end{aligned}
$$

so that, for large x and x' we have

$$
\begin{aligned}
b^{-1}\left[K'_{i\alpha}(b\,x')\,I_{i\alpha}(b\,x')-I'_{i\alpha}(b\,x')\,K_{i\alpha}(b\,x')\right]^{-1} &\approx -x' && (3.37)\\
G(x,x') \approx \frac{1}{2\,b\,x}\,\frac{-1}{\sqrt{x\,x'}}\,&\left[\theta(x-x')\,e^{x'-x}+\theta(x'-x)\,e^{x-x'}\right] && (3.38)
\end{aligned}
$$

Then, the integral in (3.32) is well definite and convergent, although it cannot be expressed in closed form. However, this will not be a problem, as we shall see in the next section.

3.1.1 Polarization Vectors

In order to introduce properly the polarization vectors, it is useful to consider a particular solution of the field equation with

$$B(x)-\partial_\perp\cdot A^\perp(x)=0 \tag{3.39}$$

Actually, this choice enables us to decompose the vector field A^μ only in terms of the homogeneous solutions of the equations (3.20 - 3.22): namely,

$$\tilde{A}^\perp(x, k_0, k_\perp) \equiv \tilde{V}^\perp(x, k_0, k_\perp) \tag{3.40}$$
$$\tilde{A}^1(x, k_0, k_\perp) = \tilde{V}^1(x, k_0, k_\perp) \tag{3.41}$$
$$\tilde{A}^0(x, k_0, k_\perp) = \tilde{V}^0(x, k_0, k_\perp) = -\frac{i}{k_0}\left(\frac{d}{dx} + \frac{1}{x}\right)\tilde{V}^1(x, k_0, k_\perp) \tag{3.42}$$

and equation (3.15) reduces to

$$\partial_t V^0 + \left(\partial_x + \frac{1}{x}\right) V^1 = 0 \tag{3.43}$$

which involves only the two components V^0 and V^1. We can then write the components of the vector field in terms of normal modes of the Fulling type, i.e.

$$\tilde{\varphi}^\perp_{E,k_\perp}(x) = f^\perp(E, k_\perp)\, K_{i\alpha}(b\,x)\, e^{-iEt+ik_\perp\cdot x_\perp} \tag{3.44}$$
$$\tilde{\varphi}^1_{E,k_\perp}(x) = f^1(E, k_\perp)\, \frac{K_{i\alpha}(b\,x)}{x}\, e^{-iEt+ik_\perp\cdot x_\perp} \tag{3.45}$$
$$\tilde{\varphi}^0_{E,k_\perp}(x) = -\frac{i}{E}\, f^1(E, k_\perp)\, \frac{b\,K'_{i\alpha}(b\,x)}{x}\, e^{-iEt+ik_\perp\cdot x_\perp} \tag{3.46}$$

We are now ready to define the inner product between any two solutions $\phi^\mu_r(x)$ of equations (3.8, 3.9). We proceed as follows: if we consider the covariant vector current

$$J_\lambda(x) = g_{\mu\nu}(x)\, \phi^{\mu*}_r(x)\, i\, \overleftrightarrow{\nabla}_\lambda\, \phi^\nu_s(x) \tag{3.47}$$

we can immediatly verify that, thanks to the equations of motion satisfied by $\phi^\mu_r(x)$, it is covariantly conserved, i.e. $\nabla_\lambda J^\lambda = 0$. Thus we can define the product

$$(\phi^\mu_r,\, \phi^\nu_s) = \oint_\Sigma d\Sigma^\lambda\, \phi^{\mu*}_r(x)\, i\, g_{\mu\nu}(x)\, \overleftrightarrow{\nabla}_\lambda\, \phi^\nu_s(x) \tag{3.48}$$

which is a straightforward generalization of the scalar inner product (2.5); in particular, we will consider the initial time three dimensional hypersurface with

$$d\Sigma^0 = -\frac{1}{a\,x}\,\theta(x)\, dx\, d^2x_\perp \qquad d\Sigma^i = 0, \quad i = 1, 2, 3 \tag{3.49}$$

so that (3.48) becomes

$$
\begin{aligned}
(\,\phi^{\nu}_{s}\,,\,\phi^{\mu}_{r}\,) &= \oint_{\Sigma} d\Sigma^{0}\, g_{\mu\nu}(x)\,\phi^{\nu *}_{s}(x)\, i\overleftrightarrow{\nabla}_{0}\,\phi^{\mu}_{r}(x)\\
&= \frac{1}{a}\int d^{2}x_{\perp}\int_{0}^{\infty}\frac{dx}{x}\, g_{\mu\nu}(x)\Big\{-\,\phi^{\nu *}_{s}(x)\, i\overleftrightarrow{\partial}_{t}\,\phi^{\mu}_{r}(x)\\
&+\; i\,\Gamma^{\nu}_{0\rho}(x)\,[\,\phi^{\mu *}_{s}(x)\,\phi^{\rho}_{r}(x)-\phi^{\rho *}_{s}(x)\,\phi^{\mu}_{r}(x)\,]\Big\}\\
&= \frac{1}{a}\int d^{2}x_{\perp}\int_{0}^{\infty} dx\,\Big\{\frac{1}{x}\,\phi^{\jmath *}_{s}(x)\, i\overleftrightarrow{\partial}_{t}\,\phi^{\jmath}_{r}(x)\\
&-\; a^{2}x\,\phi^{0 *}_{s}(x)\, i\overleftrightarrow{\partial}_{t}\,\phi^{0}_{r}(x)\\
&-\; 2i\, a^{2}\left[\,\phi^{1 *}_{s}(x)\,\phi^{0}_{r}(x)-\phi^{0 *}_{s}(x)\,\phi^{1}_{r}(x)\,\right]\Big\}
\end{aligned}
$$

In order to compute explicitly the above inner product, it is convenient to introduce the following orthogonal controvariant vectors

$$
e^{0}_{1}(x,E,k_{\perp})=\frac{-\,i}{E\,x}\,K'_{i\alpha}(b\,x)\quad e^{1}_{1}(x,E,k_{\perp})=\frac{1}{b\,x}\,K_{i\alpha}(b\,x),\quad e^{\perp}_{1}(x,E,k_{\perp})=0 \tag{3.50}
$$

$$
e^{\mu}_{2}(x,E,k_{\perp})=K_{i\alpha}(b\,x)\,\delta^{\mu}_{2},\qquad e^{\mu}_{3}(x,E,k_{\perp})=K_{i\alpha}(b\,x)\,\delta^{\mu}_{3} \tag{3.51}
$$

which clearly satisfy

$$
g_{\mu\nu}(x)\,e^{\mu}_{r}(x,E,k_{\perp})\,e^{\nu}_{s}(x,E,k_{\perp})=0\qquad \text{for } r\neq s,\quad r,s=1,2,3 \tag{3.52}
$$

the first one representing a longitudinal polarization along the acceleration axis, while the other two being transverse polarizations, orthogonal to each other and to the direction of the acceleration. We can thus build up the vector analogues of the Fulling scalar normal modes, i.e.

$$
u^{\mu}_{E,\,k_{\perp},\,r}(x)=\mathcal{N}_{r}\,e^{\mu}_{r}(x,E,k_{\perp})\,e^{-iEt+ik_{\perp}\cdot x_{\perp}} \tag{3.53}
$$

where $\mathcal{N}_r$ are real normalization constants to be suitable defined. Owing to (3.52), these vector normal modes are orthogonal to each other:

$$
g_{\mu\nu}(x)\,u^{\mu}_{E,\,k_{\perp},\,r}(x)\,u^{\nu}_{E,\,k_{\perp},\,s}(x)=0\qquad \text{for } r\neq s,\quad r,s=1,2,3 \tag{3.54}
$$

For the transverse polarizations, the normalization constants $\mathcal{N}_r$ can be set considering the inner product

$$
\begin{aligned}
&\Big(u^{j}_{\alpha',\,k'_{\perp},\,r}(x),\,u^{j}_{\alpha,\,k_{\perp},\,r}(x)\Big)=\int d^{2}x_{\perp}\int_{0}^{\infty}\frac{dx}{ax}\,u^{j}_{r,\,\alpha',\,k_{\perp}}(x)\,\overleftrightarrow{\partial_{0}}\,u^{j}_{s,\,\alpha',\,k'_{\perp}}(x)\\
&=(2\pi)^{2}\,\delta^{2}(k_{\perp}-k'_{\perp})\,(\alpha+\alpha')\,e^{-ia(\alpha-\alpha')t}\,\mathcal{N}^{2}_{r}\int_{0}^{\infty}\frac{dx}{ax}\,K_{i\alpha'}(bx)\,K_{i\alpha}(bx)\\
&=(2\pi)^{2}\,\delta^{(2)}(k_{\perp}-k'_{\perp})\,\mathcal{N}^{2}_{r}\,\pi^{2}\,\mathrm{csch}(-\pi\alpha)\,\delta(\alpha-\alpha')
\end{aligned}
\tag{3.55}
$$

so that we can define

$$\mathcal{N}_r = \frac{\sqrt{\sinh(\pi E/a)}}{2\sqrt{a}\,\pi^2} \tag{3.56}$$

and thus the normal modes become

$$u^{\mu}_{E,\,k_\perp,\,r}(x) = \frac{\sqrt{\sinh(\pi E/a)}}{2\sqrt{a}\,\pi^2}\,K_{i\alpha}(bx)\,\delta^{\mu}_{r}\,e^{-iEt+ik_\perp\cdot x_\perp} \qquad r=2,3 \tag{3.57}$$

which, as expected, have the same form as the normal modes obtained for the scalar field, since the equations satisfied by both are the same. It should be noticed also that these normal modes have canonical dimensions $[u^{\mu}_{E,\,k_\perp,\,r}] = \mathrm{cm}^{1/2}$ in natural units, as they have in the usual Minkowski space, while now satisfy

$$\begin{aligned}&\oint_{\Sigma} d\Sigma^{\lambda}\,u^{\mu\,*}_{E',\,k'_\perp,\,r}(x)\,i\,g_{\mu\nu}(x)\,\overleftrightarrow{\nabla}_{\lambda}\,u^{\nu}_{E,\,k_\perp,\,s}(x)\\ &= a^{-1}\,\delta^{(2)}(k_\perp - k'_\perp)\,\delta(\alpha-\alpha')\end{aligned} \tag{3.58}$$

Instead, for what concerns the longitudinal normal modes, the inner product turns out to be a little bit more complicated, since in our metric the covariant derivative involves directly these modes. We have

$$\begin{aligned}&\oint_{\Sigma} d\Sigma^{\lambda}\,g_{\mu\nu}(x)\,u^{\mu *}_{E',\,k'_\perp,\,1}(x)\,i\overleftrightarrow{\nabla}_{\lambda}\,u^{\nu}_{E,\,k_\perp,\,1}(x)\\ =\;& \frac{1}{a}\int d^2x_\perp \int_0^\infty dx\,\Big\{\frac{1}{x}\,u^{1*}_{E',\,k'_\perp,\,1}(x)\,i\overleftrightarrow{\partial}_t\,u^{1}_{E,\,k_\perp,\,1}(x)\\ -\;& a^2x\,u^{0\,*}_{E',\,k'_\perp,\,1}(x)\,i\overleftrightarrow{\partial}_t\,u^{0}_{E,\,k_\perp,\,1}(x)\\ -\;& 2i\,a^2\left[u^{1\,*}_{E',\,k'_\perp,\,1}(x)\,u^{0}_{E,\,k_\perp,\,1}(x) - u^{0\,*}_{E',\,k'_\perp,\,1}(x)\,u^{1}_{E,\,k_\perp,\,1}(x)\right]\Big\}\\ =\;& (2\pi)^2\,\delta^{(2)}(k_\perp-k'_\perp)\,\mathcal{N}_1^2\int_0^\infty \frac{dx}{x}\,\mathfrak{I}(\alpha,\alpha';bx)\,e^{-it(E-E')}\end{aligned} \tag{3.59}$$

where we have set

$$\begin{aligned}\mathfrak{I}(\alpha,\alpha';\zeta) \;\equiv\;& \Big\{\frac{\alpha+\alpha'}{\zeta^2}\,K_{i\alpha'}(\zeta)\,K_{i\alpha}(\zeta) - \frac{\alpha+\alpha'}{\alpha\alpha'}\,K'_{i\alpha'}(\zeta)\,K'_{i\alpha}(\zeta)\\ -\;& \frac{2}{\zeta}\left[K_{i\alpha'}(\zeta)\,K'_{i\alpha}(\zeta)\,\frac{1}{\alpha} + \frac{1}{\alpha'}\,K'_{i\alpha'}(\zeta)\,K_{i\alpha}(\zeta)\right]\Big\}\end{aligned} \tag{3.60}$$

with $\zeta = b\,x$. Notice that the integrand $\mathfrak{I}(\alpha,\alpha';\zeta)$ is even under the exchange of α and α', as it does. Anyway, the integral in (3.59) can be understood and evaluated thanks to the analytic continuation (see Appendix A). This way, the inner product between the longitudinal normal modes gives

$$\begin{aligned}
& \oint_\Sigma d\Sigma^\lambda\, g_{\mu\nu}(x)\, u^{\mu *}_{E',\,k'_\perp,\,1}(x)\, i\overleftrightarrow{\nabla}_\lambda\, u^{\nu}_{E,\,k_\perp,\,1}(x) \\
= & \ (2\pi)^2\, \delta(k_\perp - k'_\perp)\, \mathcal{N}_1^2 \int_0^\infty \frac{dx}{x}\, \mathfrak{I}(\alpha,\alpha';bx)\, e^{-\,it(E-E')} \\
= & \ (2\pi)^2\, \delta^{(2)}(k_\perp - k'_\perp)\, \mathcal{N}_1^2\, \frac{\pi^2}{\alpha^2}\, \mathrm{csch}(-\,\pi\alpha)\, \delta(\alpha-\alpha') \qquad (3.61)
\end{aligned}$$

and we can set

$$\mathcal{N}_1 = \frac{\alpha}{2\pi^2}\sqrt{a^{-1}\sinh(\pi\alpha)} \qquad (3.62)$$

We are thus left with a complete orthonormal set of normal modes of the Fulling-type, with three indipendent polarizations, i.e.

$$\begin{aligned}
u^{\mu}_{\alpha,\,k_\perp,\,r}(x) \ = & \ \frac{1}{2\pi^2}\sqrt{a^{-1}\sinh(\pi\alpha)}\, e^{\mu}_{r}(\alpha,k_\perp;x)\; e^{\,i\,\mathbf{k}\cdot\mathbf{x}-i\,a\,\alpha\,t} \qquad (x>0) \\
& \ r = 1,2,3 \qquad \alpha = E/\mathrm{a} \in \mathbb{R} \\
& \ k_\perp \in \mathbb{R}^2 \qquad b = \sqrt{k_\perp^2 + m^2} \qquad (3.63)
\end{aligned}$$

with

$$e^{\mu}_{1}(\alpha,k_\perp;x) \ = \ \frac{1}{x}\left\{\frac{\alpha}{b}\, K_{i\alpha}(bx)\,\delta^{\mu}_{1} + \frac{i}{2a}[\,K_{i\alpha-1}(bx)+K_{i\alpha+1}(bx)\,]\;\delta^{\mu}_{0}\right\} \qquad (3.64)$$

$$e^{\mu}_{r}(\alpha,k_\perp;x) \ = \ K_{i\alpha}(bx)\,\delta^{\mu}_{r} \qquad r = 2,3 \qquad (3.65)$$

which fulfil the orthonormality relations

$$\begin{aligned}
- & \ \oint_\Sigma d\Sigma^\lambda\, g_{\mu\nu}(x)\, u^{\mu *}_{E',\,k'_\perp,\,1}(x)\, i\overleftrightarrow{\nabla}_\lambda\, u^{\nu}_{E,\,k_\perp,\,1}(x) \\
= & \ a^{-1}\,\delta^{(2)}(k_\perp - k'_\perp)\,\delta(\alpha-\alpha')\,\delta_{rr'} \qquad (3.66)
\end{aligned}$$

together with the reduced Lorenz condition

$$\partial_0\, u^{0}_{\alpha,\,k_\perp,\,r}(x) + \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}\right) u^{1}_{\alpha,\,k_\alpha,\,r}(x) = 0 \qquad (3.67)$$

$$\alpha \in \mathbb{R} \qquad k_\perp \in \mathbb{R}^2 \qquad r = 1,2,3$$

Notice that the normal modes have canonical dimensions $[\,u^{\mu}_{\alpha,\,k_\perp,\,r}\,] = \sqrt{\mathrm{cm}}$, in natural units, just like the Fulling scalar functions (2.48) and their usual Minkowski counterparts (1.28).

3.1.2 Canonical Quantization

We can now write the Vector and the auxiliary fields in terms of the normal modes solutions just found, namely,

$$V^\mu(x) = \sum_{\alpha,\,k_\perp,\,r} \left[\, f_{\alpha,\,k_\perp,\,r}\, u^{\mu}_{\alpha,\,k_\perp,\,r}(x) \;+\; f^{*}_{\alpha,\,k_\perp,\,r}\, u^{\mu\,*}_{\alpha,\,k_\perp,\,r}(x) \,\right] \tag{3.68}$$

$$B(x) = a \sum_{\alpha,\,k_\perp} \left[\, b_{\alpha,\,k_\perp}\, u_{\alpha,\,k_\perp}(x) \;+\; b^{*}_{\alpha,\,k_\perp}\, u^{*}_{\alpha,\,k_\perp}(x) \,\right] \tag{3.69}$$

where we have introduced the shorthand notations

$$\sum_{\alpha,\,k_\perp,\,r} \equiv a \int_{-\infty}^{\infty} d\alpha \int d^2k_\perp \sum_{r=1}^{3} \qquad\qquad \sum_{\alpha,\,k_\perp} \equiv a \int_{-\infty}^{\infty} d\alpha \int d^2k_\perp \tag{3.70}$$

$f_{\alpha,\,k_\perp,\,r}$ and $b_{\alpha,\,k_\perp}$ being complex coefficients. The general solutions of the field equations (3.6, 3.7) are then given by

$$A^\mu(x) = V^\mu(x) + \Delta\, A^\mu(x) \tag{3.71}$$

$$\Delta\, A^2(x) = \Delta\, A^3(x) = 0 \tag{3.72}$$

$$\Delta\, A^1(x) = 2\,\delta^\mu_1 \int_{-\infty}^{\infty} dx'\, x'\; G(x,x')\, \mathfrak{B}(t,x',x_\perp) \tag{3.73}$$

$$\Delta\, A^0(x) = 2\delta^\mu_0 \int_{-\infty}^{\infty} \frac{d\alpha}{i\alpha}\; e^{-i\,a\,\alpha\,t} \;\times$$

$$\left\{ \left(\frac{d}{dx} + \frac{1}{x} \right) \int_{-\infty}^{\infty} dx'\, x'\; G(x,x')\; \widehat{\mathfrak{B}}(\alpha,x',x_\perp) - \tfrac{1}{2}\, \widehat{\mathfrak{B}}(\alpha,x,x_\perp) \right\} \tag{3.74}$$

$$\mathfrak{B}(x) = B(x) - \partial_\perp \cdot V^\perp(x) = a \int_{-\infty}^{\infty} d\alpha\; e^{-i\,a\,\alpha\,t}\; \widehat{\mathfrak{B}}(\alpha,x,x_\perp) \tag{3.75}$$

We can then proceed to the canonical quantization by replacing the classical field functions with the corrisponding operator valued distributions

$$V^\mu(x) = \sum_{\alpha,\,k_\perp,\,r} \left[\, f_{\alpha,k_\perp,r}\, u^{\mu}_{\alpha,k_\perp,r}(x) \;+\; f^{\dagger}_{\alpha,k_\perp,r}\, u^{\mu\,*}_{\alpha,k_\perp,r}(x) \,\right] \tag{3.76}$$

$$B(x) = a \sum_{\alpha,\,k_\perp} \left[\, b_{\alpha,\,k_\perp}\, u_{\alpha,k_\perp}(x) \;+\; b^{\dagger}_{\alpha,\,k_\perp}\, u^{*}_{\alpha,k_\perp}(x) \,\right] \tag{3.77}$$

which satisfy the canonical commutation relations

$$[\, f_{\alpha,\,k_\perp,\,r}\,,\; f^{\dagger}_{\alpha',\,k'_\perp,\,r'} \,] = a^{-1}\, \delta^{(2)}(k_\perp - k'_\perp)\, \delta(\alpha - \alpha')\, \delta_{rr'} \tag{3.78}$$

$$[\, f_{\alpha,\,k_\perp,\,r}\,,\; f_{\alpha',\,k'_\perp,\,r'} \,] = [\, f^{\dagger}_{\alpha,\,k_\perp,\,r}\,,\; f^{\dagger}_{\alpha',\,k'_\perp,\,r'} \,] = 0 \tag{3.79}$$

$$[\, b^{\dagger}_{\alpha,\, k_\perp} \,,\, b_{\alpha',\, k'_\perp} \,] = a^{-1}\, \delta^{(2)}(k_\perp - k'_\perp)\, \delta(\alpha - \alpha') \tag{3.80}$$

$$[\, b_{\alpha,\, k_\perp} \,,\, b_{\alpha',\, k'_\perp} \,] = [\, b^{\dagger}_{\alpha,\, k_\perp} \,,\, b^{\dagger}_{\alpha',\, k'_\perp} \,] = 0 \tag{3.81}$$

3.2 Quantization of the Vector Field in the axial gauge

Consider now the massless Vector Field in the space-like region $\mathcal{M}_R$: in the axial gauge, it is described by the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left[-\frac{1}{4}\, F_{\mu\nu}(x)\, F^{\mu\nu}(x) + A^\mu(x)\, \eta_\mu\, B(x)\right] \tag{3.82}$$

where $\eta_\mu = (0,1,0,0)$ is a constant four-vector, thanks to which we recover the condition $A^1 = 0$: in fact, the equations of motion for A^ν and B are

$$\nabla_\mu F^{\mu\nu}(x) + \eta^\nu\, B(x) = 0 \tag{3.83}$$

$$\eta_\mu\, A^\mu(x) = 0 \quad \Rightarrow A^1(x) = 0 \tag{3.84}$$

and taking one more covariant derivative of (3.83), we obtain

$$\eta^\nu\, \partial_\nu B(x) = 0 \tag{3.85}$$

from which, for suitable boundary conditions, can be set $B(x) = 0$. This way (3.83) becomes

$$\nabla_\mu F^{\mu\nu}(x) = 0 = \nabla_\mu(\nabla^\mu A^\nu(x) - \nabla^\nu A^\mu(x)\,) \tag{3.86}$$

and hence, making a contraction with η_ν yelds

$$\begin{aligned} &\eta_\nu\, (\nabla_\mu\, \nabla^\mu)\, A^\nu(x) - \eta_\nu\, g^{\nu\lambda}(x)\, \partial_\lambda\, (\nabla_\mu\, A^\mu(x)\,) = 0 \\ &g^{11}\, \partial_1\, (\nabla_\mu\, A^\mu(x)\,) = g^{11}\, \partial_1\, (\partial_\mu\, A^\mu(x)\,) = 0 \end{aligned} \tag{3.87}$$

that entails $\partial_\mu\, A^\mu = 0$, with opportune boundary conditions. The equations (3.83) can be explicitly written as

$$\left[\partial_\mu\, \partial^\mu - \frac{3}{x}\, \partial_1\right] A^0(x) - \frac{1}{a^2 x^3}\, \partial_0\, \partial_\mu\, A^\mu(x) = 0 \tag{3.88}$$

$$\left[\partial_\mu\, \partial^\mu - \frac{1}{x}\, \partial_1\right] A^\perp(x) + \partial_\perp\, \partial_\mu\, A^\mu(x) = 0 \tag{3.89}$$

which, using $\partial_\mu A^\mu = 0$, simplify, as they decouple:

$$\left[\partial_\mu \partial^\mu - \frac{1}{x}\partial_1\right] A^\perp(x) = 0 \tag{3.90}$$

$$\partial_0 A^0(x) = -\partial_\perp \cdot A^\perp(x) \tag{3.91}$$

the component A^0 being dependent on $A^\perp$ because of the condition $\partial_\mu A^\mu = 0$. Thus we are left with only two indipendent component of the massless vector field as we should. To proceed further, we first notice that the equation (3.90) satisfied by $A^\perp$ has just the same form as the one satisfied by the transverse vector field in the Feynman gauge, so that we can find the solutions in the same way as in the previous section using the partial Fourier transform. We have

$$A^\perp(t,x,y,z) = \int_{-\infty}^{+\infty} \frac{dE}{\sqrt{2\pi}} \int \frac{d^2k_\perp}{2\pi} \tilde{A}^\perp(E,k_\perp,x)\, e^{-iEt+ik_\perp\cdot x_\perp} \tag{3.92}$$

and this way the equation (3.90) becomes

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \frac{E^2}{a^2x^2} - k_\perp^2\right] \tilde{A}^\perp(x,E,k_\perp) = 0 \tag{3.93}$$

the solutions of which are again expressed in terms of the modified Bessel functions of imaginary order. We end up with

$$\tilde{A^\perp}(x) = c_1(E,k_\perp) I_{i\alpha}(\kappa x) + c_2(E,k_\perp) K_{i\alpha}(\kappa x) \tag{3.94}$$

where

$$\alpha = \frac{E}{a}, \qquad \kappa = |k_\perp|$$

However, as we have already pointed out, the solutions $I_{i\alpha}(\kappa x)$, are usually rejected since they are exponentially increasing for large positive x, so that we must set $c_1(E,k_\perp) = 0$. We are thus left with the normal modes decomposition of the real massless vector field in the axial gauge in the right Rindler wedge:

$$A^\perp(t,x,y,z) = \int_{-\infty}^{+\infty} \frac{dE}{\sqrt{2\pi}} \int \frac{d^2k_\perp}{2\pi} f^\perp(E,k_\perp) K_{i\alpha}(\kappa x) e^{-iEt+ik_\perp\cdot x_\perp} + \text{c.c.}$$

where the reality conditions

$$f^\perp(-E,-k_\perp) = f^{\perp *}(E,k_\perp) \tag{3.95}$$

and the property $K_{i\alpha}(\kappa x) = K_{-i\alpha}(\kappa x)$ are to be taken suitably into account.

Let us define the inner product between any two solutions $\phi_r^\mu(x)$ of equations (3.83, 3.84) as before (see section 3.1.1). It is now convenient to introduce the following orthogonal controvariant vectors

$$e_2^\mu(x,E,k_\perp) = K_{i\alpha}(\kappa x)\,\delta_2^\mu \tag{3.96}$$
$$e_3^\mu(x,E,k_\perp) = K_{i\alpha}(\kappa x)\,\delta_3^\mu \tag{3.97}$$

which clearly satisfy

$$g_{\mu\nu}(x)\,e_r^\mu(x,E,k_\perp)\,e_s^\nu(x,E,k_\perp) = 0 \qquad \text{for } r\neq s, \quad r,s=2,3 \tag{3.98}$$

and from which it is possible to build up the vector analogues of the Fulling scalar normal modes, i.e.

$$u^\mu_{E,\,k_\perp,\,r}(x) = \mathcal{N}_r\,e_r^\mu(x,E,k_\perp)\,e^{-iEt+ik_\perp\cdot x_\perp} \tag{3.99}$$

where again $\mathcal{N}_r$ are real normalization constants that can be defined in the same way as in section 3.1.1. We have

$$\mathcal{N}_r = \frac{\sqrt{\sinh(\pi E/a)}}{2\sqrt{a}\,\pi^2} \tag{3.100}$$

and thus

$$u^\mu_{E,\,k_\perp,\,r}(x) = \frac{\sqrt{\sinh(\pi E/a)}}{2\sqrt{a}\,\pi^2}\,K_{i\alpha}(\kappa\,x)\,\delta_r^\mu\,e^{-iEt+ik_\perp\cdot x_\perp} \qquad r=2,3 \tag{3.101}$$

We can therefore expand the real Vector field in terms of these Fulling modes

$$A^\mu(x) = \int_{-\infty}^{\infty} dE\int d^2k_\perp\sum_{r=1}^{2}[f_{r,E,k_\perp}u^\mu_{r,E,k_\perp}(x) + f^*_{r,E,k_\perp}u^{\mu\,*}_{r,E,k_\perp}(x)] \tag{3.102}$$

remembering that

$$\begin{cases} \partial_0 A^0 = -\partial_\perp\cdot A^\perp \\ A^1 = 0 \end{cases}$$

The canonical quantization is then obtained by replacing the classical field functions with operator valued tempered distributions

$$A^\mu(x) = \sum_{E,\,k_\perp,\,r}\left[f_{r,E,k_\perp}\,u^\mu_{r,E,k_\perp}(x) + f^\dagger_{r,E,k_\perp}\,u^{\mu\,*}_{r,E,k_\perp}(x)\right] \tag{3.103}$$

where the creation and destruction operators satisfy the canonical commutation relations

$$[\,f_{\alpha,\,k_\perp,\,r}\,,\,f^\dagger_{\alpha',\,k'_\perp,\,r'}\,] = a^{-1}\,\delta^{(2)}(k_\perp-k'_\perp)\,\delta(\alpha-\alpha')\,\delta_{rr'} \tag{3.104}$$

all the other commutators vanishing.

3.2.1 Massless Vector Field in the Lorenz-Landau gauge

In this section, we present the quantization of the massless vector field in the Lorenz-Landau gauge, following the method used in [7].

In order to simplify the calculations, we consider only the two dimensional Minkowki spacetime with metric

$$ds^2 = d\tau^2 - dX^2 = dx^- \, dx^+ \tag{3.105}$$

where $x^- = \tau - X$ and $x^+ = \tau + X$ are the standard light-cone coordinates. The two-dimensional right Rindler wedge $\mathcal{M}_R$ is then recovered by the following coordinate transformation

$$\begin{aligned} \tau &= \frac{1}{a} e^{a\,\xi} \sinh(a\eta) \qquad (|X| > \tau) \\ X &= \frac{1}{a} e^{a\,\xi} \cosh(a\eta) \end{aligned} \tag{3.106}$$

that in terms of the light-cone coordinates is given by

$$\begin{aligned} u &= \frac{1}{a} e^{-a\,(\eta-\xi)} = -\,x^- \\ v &= \frac{1}{a} e^{a\,(\eta+\xi)} = x^+ \end{aligned} \tag{3.107}$$

where $\eta,\, \xi \in \mathbb{R}$. The line element then becomes

$$ds^2 = e^{2\,a\,\xi} \left(d\eta^2 - d\xi^2\right) \tag{3.108}$$

The above coordinate transformations can be inverted, giving

$$\begin{aligned} \xi &= \frac{1}{a} \ln a \sqrt{u\,v} = \frac{1}{a} \ln a \sqrt{X^2 - \tau^2} \\ \eta &= \frac{1}{a} \ln \sqrt{\frac{v}{u}} = \frac{1}{a} \ln \sqrt{\frac{X+\tau}{X-\tau}} \end{aligned} \tag{3.109}$$

Moreover, we can define the following derivatives

$$\partial_- = -\frac{\partial \eta}{\partial u} \partial_\eta - \frac{\partial \xi}{\partial u} \partial_\xi = \frac{1}{2} e^{a\,(\eta-\xi)} \left(\partial_\eta - \partial_\xi\right) \tag{3.110}$$

$$\partial_+ = \frac{\partial \eta}{\partial v} \partial_\eta + \frac{\partial \xi}{\partial v} \partial_\xi = \frac{1}{2} e^{-a\,(\eta+\xi)} \left(\partial_\eta + \partial_\xi\right) \tag{3.111}$$

so that

$$\Box = \partial_- \partial_+ = \frac{1}{4} e^{-2a\xi} \left(\partial_\eta^2 - \partial_\xi^2\right) \tag{3.112}$$

We now first review the quantization of the massless vector field in the Lorenz gauge $\partial_\mu A^\mu = 0$ in the Minkoswki space. The equations of motion then become

$$\partial_\mu F^{\mu\nu} = 0 \qquad \Rightarrow \left(\partial_\tau^2 - \partial_X^2\right) A^\nu = 0 \tag{3.113}$$

$$\partial_\tau A^0 + \partial_X A^1 = 0 \tag{3.114}$$

that can be recast in the light-front form by defining

$$A^\pm = A^0 \pm A^1 = A_\mp \tag{3.115}$$

One obtains

$$\partial_u \partial_v A^\pm(u,\, v) = 0 \tag{3.116}$$

$$\partial_v A_+ - \partial_u A_- = 0 \qquad \Longleftrightarrow \qquad \partial_- A_+ + \partial_+ A_- = 0 \tag{3.117}$$

whose standard orthonormal set of positive frequency solutions is given by

$$\varphi_k^\nu(\tau,\, X) = e^\nu(k)\, \varphi_k(\tau,\, X) \qquad \partial_\nu \varphi_k^\nu(\tau,\, X) = 0 \tag{3.118}$$

$$\varphi_k(\tau,\, X) = \frac{1}{(4\pi\omega)^{1/2}}\, e^{-i\omega\tau + ikX} \qquad \omega = |k|, \quad k \in \mathbb{R} \tag{3.119}$$

$$e^\nu(k) = (\mathrm{sgn}\ k, 1) \tag{3.120}$$

For our purposes it is useful to introduce the light-front polarization vectors

$$e^\pm = e^0 \pm e^1 \qquad e^\pm = \pm\, 2\, \theta(\pm k) \tag{3.121}$$

This way, the normal modes with $k > 0$, which are right-moving waves with right polarization along the rays $x^- = const$, are given by

$$(8\pi\omega)^{-1/2}\, e^+\, e^{-i\omega x^-} \tag{3.122}$$

while, for $k < 0$, the left-moving waves with left polarization along the rays $x^+ = const$ can be written as

$$(8\pi\omega)^{-1/2}\, e^-\, e^{-i\omega x^+} \tag{3.123}$$

We can then expand the real vector field in terms of these normal modes and quantize, A^ν becoming an operator valued tempered distribution

$$A^\nu(\tau,\, X) = \int_{-\infty}^{+\infty} dk\ \left[f_k\, \varphi_k^\nu(\tau,\, X) + f_k^\dagger\, \varphi_k^{\nu\,*}(\tau,\, X)\right] \tag{3.124}$$

where the creation and destruction operators satisfy the canonical commutation relations

$$\left[f_k,\, f_p^\dagger\right] = \delta(k-p) \qquad \left[f_k,\, f_p\right] = \left[f_k^\dagger,\, f_p^\dagger\right] = 0 \tag{3.125}$$

Moreover, if we write the vector potential A^μ in terms of the light-cone fields, we find that they are a pair of tempered distributions with opposite light-front polarizations, namely

$$A^+(x^-) = \int_0^{+\infty} dk\, \frac{1}{(2\pi k)^{1/2}} \left[f_k\, e^{-ikx^-} + f_k^\dagger\, e^{ikx^-}\right] \tag{3.126}$$

$$A^-(x^+) = -\int_0^{+\infty} dk\, \frac{1}{(2\pi k)^{1/2}} \left[f_{-k}\, e^{-ikx^+} + f_{-k}^\dagger\, e^{ikx^+}\right] \tag{3.127}$$

which clearly satisfy both the field equations and the Lorenz condition

$$\partial_-\partial_+\, A^+(x^-) = 0 \qquad \partial_-\partial_+\, A^-(x^+) = 0 \tag{3.128}$$

$$\partial_+ A^+(x^-) = 0 \qquad \partial_- A^-(x^+) = 0 \tag{3.129}$$

A representation of the solutions in all the regions R, L, F, P (see Figure 2.1), that is the whole Minkowski space, can then be gained by writing

$$A^+(x^-) = \theta(-x^-)A^+(u) + \theta(x^-)A^+(-u) \tag{3.130}$$

$$A^-(x^+) = \theta(x^+)A^-(v) + \theta(-x^+)A^-(-v) \tag{3.131}$$

However, since the metric in $\mathcal{M}_R$ is conformal to the Minkowski one, one can follow an alternative procedure for the quatization of the vector potential based on the solutions obtained solving directly the field equations and the Lorenz condition in the right Rindler wedge, that, thanks to the conformal invariance, take now the form

$$\left(\partial_\eta^2 - \partial_\xi^2\right)\, A^\pm(\eta,\, \xi) = 0 \tag{3.132}$$

$$\left(\partial_\eta \pm \partial_\xi\right) A^\pm(\eta,\, \xi) = 0 \tag{3.133}$$

The solutions can be expressed again in terms of positive frequency normal modes

$$\phi_k(\eta,\, \xi) = \frac{1}{(4\pi\omega)^{1/2}}\, e^{\pm\, i\omega\eta + ik\xi} \qquad \omega = |k| > 0, \quad k \in \mathbb{R} \tag{3.134}$$

so that in the quantization, the two light-cone tempered distributions become

$$A^+(u) = \int_0^{+\infty} d\omega\, \frac{1}{(2\pi k)^{1/2}} \left[g_\omega\, (a\, u)^{i\omega/a} + g_\omega^\dagger\, (a\, u)^{-i\omega/a}\right] \tag{3.135}$$

$$A^-(v) = -\int_0^{+\infty} d\omega\, \frac{1}{(2\pi k)^{1/2}} \left[g_{-\omega}\, (a\, v)^{-i\omega/a} + g_{-\omega}^\dagger\, (a\, v)^{i\omega/a}\right] \tag{3.136}$$

where the creation and destruction operators satisfy the canonical commutation relations

$$\left[g_\omega,\, g^\dagger_{\omega'}\right] = \delta(\omega - \omega') \qquad [g_\omega,\, g_{\omega'}] = \left[g^\dagger_\omega,\, g^\dagger_{\omega'}\right] = 0 \tag{3.137}$$

The relation between the two set of modes, that is the Minkowski one and the Rindler one, can be obtained through the method presented in [7]. Consider then any real tempered distribution $f \in \mathcal{S}'(R)$. The Mellin transform of its restriction f_+ to the real positive half-line $v > 0$ is defined by

$$F_+(s) = \int_0^\infty dv\, f_+(v)\, v^{s-1} \qquad \Re\mathrm{e}\, s > 0 \tag{3.138}$$

for which exists the analytic continuation to a meromorphic function in the whole complex plane with simple poles at $s \in \mathbb{Z}_0^-$. The inversion formula is given by

$$\begin{aligned} f_+(v) &= \frac{v^{-\lambda}}{2\pi} \int_{-\infty}^\infty d\sigma\, F_+(\lambda + i\sigma)\, v^{-i\sigma} \\ &= \frac{v^{-\lambda}}{2\pi} \int_0^\infty d\sigma\, F_+(\lambda + i\sigma)\, v^{-i\sigma} + \text{c.c.} \end{aligned} \tag{3.139}$$

where $v, \lambda \in \mathbb{R}$, and therefore we can identify $\sigma = \omega/a$ for $\lambda \to 0$, obtaining

$$f_+(v) \equiv A_+(v) \qquad F_+(i\sigma) = -a^{1/2 - i\sigma} \sqrt{\frac{2\pi}{\sigma}}\, g_{-\sigma a} \qquad v > 0,\, \sigma > 0 \tag{3.140}$$

Moreover, from (3.138) and (3.127) we can write

$$-\int_0^\infty dv\, A_+(v)\, v^{s-1} = \int_0^\infty dv\, v^{s-1} \int_0^{+\infty} dk\, \frac{1}{(2\pi k)^{1/2}} \left[f_{-k}\, e^{-ikv} + f^\dagger_{-k}\, e^{ikv}\right] \tag{3.141}$$

Also, from the relation

$$\int_0^\infty dv\, e^{\pm ik}\, v^{s-1} = \Gamma(s)\, k^{-s}\, e^{\pm i\pi s/2} \tag{3.142}$$

we get

$$F_+(s) = -\frac{\Gamma(s)}{\sqrt{2\pi}} \int_0^\infty dk\, k^{-s-1/2} \left[f_{-k}\, e^{-i\pi s/2} + f^\dagger_{-k}\, e^{i\pi s/2}\right] \tag{3.143}$$

$$\begin{aligned} F_+(i\sigma) &= -\frac{\Gamma(i\sigma)}{\sqrt{2\pi}} \int_0^\infty dk\, k^{-i\sigma-1/2} \left[f_{-k}\, e^{\pi\sigma/2} + f^\dagger_{-k}\, e^{-\pi\sigma/2}\right] \\ &= -\, a^{1/2 - i\sigma} \sqrt{\frac{2\pi}{\sigma}}\, g_{-\sigma a} \end{aligned} \tag{3.144}$$

so that we have the Bogolyubov transformation

$$g_{-\omega} = \frac{\sqrt{\omega}}{2\pi a}\,\Gamma\left(\frac{i\omega}{a}\right)\int_0^\infty \frac{dk}{\sqrt{k}}\left(\frac{k}{a}\right)^{-i\omega/a}\left[f_{-k}\,e^{\pi\omega/2a} + f_{-k}^{\dagger}\,e^{-\pi\omega/2a}\right] \quad (3.145)$$

which leads to the Bogolyubov coefficients

$$\alpha_{-k,-\omega} = \frac{\sqrt{\omega}}{2\pi a}\,\Gamma\left(-\frac{i\omega}{a}\right)\left(\frac{k}{a}\right)^{i\omega/a}\frac{e^{\pi\omega/2a}}{\sqrt{k}} \quad (3.146)$$

$$\beta_{-k,-\omega} = \frac{\sqrt{\omega}}{2\pi a}\,\Gamma\left(\frac{i\omega}{a}\right)\left(\frac{k}{a}\right)^{-i\omega/a}\frac{e^{-\pi\omega/2a}}{\sqrt{k}} \quad (3.147)$$

The square modulus of the above coefficients corresponds the Bose-Einstein probability distributions at equilibrium temperature $T = a/2\pi k_B$, which is the Unruh temperature:

$$|\alpha_{-k,-\omega}|^2 \to \frac{e^{2\pi\omega/a}}{e^{2\pi\omega/a-1}} = 1 - N_{\omega,T} \quad (3.148)$$

$$|\beta_{-k,-\omega}|^2 \to \frac{1}{e^{2\pi\omega/a-1}} \equiv N_{\omega,T} \quad (3.149)$$

In a similar way, we can calculate the other Bogolyubov coefficients, obtaining first

$$\int_0^\infty du\,A^{+}(u)\,u^{s-1} = \int_0^\infty du\,u^{s-1}\int_0^{+\infty} dk\,\frac{1}{(2\pi k)^{1/2}}\left[f_k\,e^{-iku} + f_k^{\dagger}\,e^{iku}\right] \quad (3.150)$$

and then

$$F_{+}(s) = \frac{\Gamma(s)}{\sqrt{2\pi}}\int_0^\infty dk\,k^{-s-1/2}\left[f_k\,e^{-i\pi s/2} + f_k^{\dagger}\,e^{i\pi s/2}\right] \quad (3.151)$$

$$F_{+}(s) = \frac{\Gamma(i\sigma)}{\sqrt{2\pi}}\int_0^\infty dk\,k^{-i\sigma-1/2}\left[f_k\,e^{\pi\sigma/2} + f_k^{\dagger}\,e^{-\pi\sigma/2}\right]$$

$$= a^{1/2-i\sigma}\sqrt{\frac{2\pi}{\sigma}}\,g_{\sigma a}^{\dagger} \quad (3.152)$$

so that the Bogolyubov transformation now reads

$$g_{\omega}^{\dagger} = \frac{\sqrt{\omega}}{2\pi a}\,\Gamma\left(\frac{i\omega}{a}\right)\int_0^\infty \frac{dk}{\sqrt{k}}\left(\frac{k}{a}\right)^{-i\omega/a}\left[f_k\,e^{\pi\omega/2a} + f_k^{\dagger}\,e^{-\pi\omega/2a}\right] \quad (3.153)$$

and hence the Bogolyubov coefficient is given by

$$\alpha_{k,\omega} = \frac{\sqrt{\omega}}{2\pi a}\,\Gamma\left(\frac{i\omega}{a}\right)\left(\frac{k}{a}\right)^{-i\omega/a}\frac{e^{-\pi\omega/2a}}{\sqrt{k}} \quad (3.154)$$

$$|\alpha_{k,\omega}|^2 \to N_{\omega,T} \quad (3.155)$$

3.3 Photon counting detectors

In this section we present some applications of the Unruh Effect, focusing in particular on the detection of photons and the experimental devices used for this purpose.

Particle detectors have often been used as an evidence of the Unruh Effect and the consequent thermal bath experienced by some non-inertial observers, although, clearly, different detectors will give, in general, contrasting responses about the same feature of the bath. On the other hand, by the awareness of the existence of the Unruh Effect, many relations between the effects recorded by inertial and accelerated detectors can be investigated. In particular, recent researches [8] on photon counting devices have demonstrated that the coordinates of photons absorbed by a pair of counteraccelerating detectors in the two causally disconnected Rindler regions $\mathcal{M}_R$ and $\mathcal{M}_L$ are indeed correlated and that, when a photon is absorbed by a single accelerated detector, the Minkowski vacuum collapses into a state containing at least one photon that can eventually be absorbed by an inertial detector. Let us describe in more detail how these devices work.

The most common Unruh-DeWitt detector used to model an accelerated device is given by a two-level point monopole coupled to a real massless scalar field. In particular a semiconductor band structure that allows absorption of a wide band of frequencies can be used. Then, any photon crossing the surface of the device is eventually absorbed and, by using a bias of the semiconductor pn-junction, it is not reemitted, since the electric field separates the electron-hole pair created by the photon and emission can then be neglected in an ideal device. Moreover, in general, the accuracy in the measurement of the position of a photon is related to the size of the device. A pixel that would absorb the low-frequency photons characteristic of the Unruh Effect should then be large. A method for the description of the position measurement performed by a photon counting array detector is given by a positive operator valued measure (POVM), that are projectors onto a complete set of exactly localized states. An important feature of this basis is that it does not impose any limitations on the size of the detector, nor on the form of the field incident on it. Furthermore, these states turn out to be useful to calculate the probability densities for absorption and emission of photons as a function of spacetime location on a hypersurface. In fact, the probability density for the absorption is then given by the absolute square of the projection of the photon state vector onto the localized states. Moreover, since in Quantum Field Theory particles are counted on a spacelike

Cauchy surface, thanks to which it is possible to identify positive and negative frequencies and the creation and destruction operators can be defined, the elements of the POVM should be given by projectors onto the localized states on such spacelike Cauchy hypersurfaces. Although some problems arise regarding the mathematical definition of the exactly localized states, as the fact that the fields describing localized states are themselves nonlocal and spread throughout the space instantaneously and the fact that they lead to negative scalar products, for practical purposes, it can be verified that the spacetime probability amplitude turns out to be local, the nonlocal effects appearing only outside the spacelike hypersurface of interest, and that since positive frequencies are associated with absorption and negative frequencies with emission, the integral over the spacelike hypersurface gives the difference between the absorbed and emittend photons (a photon that is reemitted is not counted), which indeed is what is really important. Then, since the scalar product is invariant, inertial and accelerated detectors will agree on the number of atomic transitions and on the net absorption minus emission probability, even if they will not agree on the description of the phenomenon.

Firstly, let us recall and define some operators describing the massless vector field in the Lorenz gauge in the Minkowski space. As in the subsection (3.2.1), we will consider only the two-dimensional case for simplicity, so that the potential vector $A^\mu(x)$ reduces to just one indipendent component, say $\phi(\tau,\, X)$, which satisfies the massless Klein Gordon wave equation $\Box\phi = 0$. The electric field operator is then given by $E = -\,\partial_t\phi$. We can define the absorption and the emission density operators as

$$n^{(+)}(\tau,\, X) = i\,\phi^{(-)}(\tau,\, X)\,\overleftrightarrow{\partial_\tau}\,\phi^{(+)}(\tau,\, X) \tag{3.156}$$

$$n^{(-)}(\tau,\, X) = i\,\phi^{(+)}(\tau,\, X)\,\overleftrightarrow{\partial_\tau}\,\phi^{(-)}(\tau,\, X) \tag{3.157}$$

where $\phi^{(\pm)}$ are the positive and negative frequency parts of the scalar field ϕ, and for instance

$$\phi^{(+)}(\tau, X) = \int_{-\infty}^{\infty} dk\, u_{k,M}(\tau, X)\, a_{k,M} \tag{3.158}$$

$$u_{k,M}(\tau, X) = \frac{e^{i\,k(X-\varepsilon_k\,\tau)}}{\sqrt{4\pi\,|k|}} \qquad \varepsilon_k = k/|k| \tag{3.159}$$

the subscript M standing for “Minkowski”. Then, the probability density to count a photon at $(\tau',\, X')$ for an electromagnetic field initially in the state $|\psi\rangle$ is

$$w^{(+)}(\tau',\, X') = \langle\psi|\, n^{(+)}(\tau',\, X')\,|\psi\rangle \tag{3.160}$$

while, since in different pixels the field operators commute, the two photon correlation function is simply given by

$$w_2^{(+)} = \langle\psi|\, n^{(+)}(\tau',\, X')\, n^{(+)}(\tau'',\, X'')\, |\psi\rangle \tag{3.161}$$

The invariant scalar inner product evaluated on a hypersurface with $\tau = const$ is then

$$(\phi_1,\, \phi_2) = i \int_{-\infty}^{\infty} dX\, \phi_1^*(\tau, X)\, \overleftrightarrow{\partial_t}\, \phi_2(\tau, X) \tag{3.162}$$

and the orthonormality relations satisfied by the plane waves $u_{k,M}$ are

$$\begin{aligned}
&(u_{k,M},\, u_{k',M}) = \delta(k-k') \\
&\left(u^*_{k,M,},\, u^*_{k',M}\right) = -\,\delta(k-k') \\
&\left(u^*_{k,M},\, u_{k',M}\right) = \left(u_{k,M},\, u^*_{k',M}\right) = 0
\end{aligned} \tag{3.163}$$

For what concerns the positive frequency Rindler plane waves, we have

$$u_{K,\mathcal{M}_R}(\eta,\xi) \;=\; \begin{cases} \frac{e^{i\,K(\xi-\varepsilon_K\,\eta)}}{\sqrt{4\pi\,|K|}} & \text{in } \mathcal{M}_R \\ 0 & \text{in } \mathcal{M}_L \end{cases} \tag{3.164}$$

$$u_{K,\mathcal{M}_L}(\eta,\xi) \;=\; \begin{cases} 0 & \text{in } \mathcal{M}_R \\ \frac{e^{i\,K(\xi+\varepsilon_K\,\eta)}}{\sqrt{4\pi\,|K|}} & \text{in } \mathcal{M}_L \end{cases} \tag{3.165}$$

where $\varepsilon_K = K/|K|$, $\omega = |K|$. In both wedges, waves with $K > 0$ are seen by an inertial observer as outward propagating waves, while $K < 0$ as inward propagating waves. On any hupersurface with $\eta = const$, the Rindler plane waves satisfy analogous orthonormality relations as (3.163). The Bogolyubov coefficients that relate the Minkowski and Rindler plane waves, evaluated on the hypersurface $\tau = \eta = 0$ for convenience, since there the Rindler POVM is at rest, are then

$$\begin{aligned}
\alpha_{k,\omega;\,\mathcal{M}_R} &= (u_{k,M},\, u_{K,\mathcal{M}_R}) \\
&= \int_{X_0}^{\infty} dX\, \frac{\omega/(aX) + |k|}{4\pi\sqrt{|k|\omega}}\, e^{-i\,kX}\, (aX)^{i\,K/a} \\
\alpha_{k,\omega;\,\mathcal{M}_R} &= \frac{\sqrt{\omega}}{2\pi a}\,\Gamma\left(\frac{iK}{a}\right)\left(\frac{|k|}{a}\right)^{-i\,\omega/a} \frac{e^{\pi\omega/2a}}{\sqrt{|k|}}\,\delta_{\varepsilon_K,\varepsilon_k} + F(K)\,(aX_0)^{i\,K}
\end{aligned} \tag{3.166}$$

$$
\begin{aligned}
\beta_{k,\omega;\,\mathcal{M}_R} &= \left(u^*_{k,M},\, u_{K,\mathcal{M}_R}\right) \\
&= \int_{X_0}^{\infty} dX\, \frac{\omega/(aX) - |k|}{4\pi\sqrt{|k|\omega}}\, e^{i\,kX}\, (aX)^{i\,K/a} \\
&= \frac{\sqrt{\omega}}{2\pi a}\, \Gamma\left(\frac{iK}{a}\right) \left(\frac{|k|}{a}\right)^{-i\,\omega/a} \frac{e^{-\pi\omega/2a}}{\sqrt{|k|}}\, \delta_{\varepsilon_K,\varepsilon_k} + F(K)\,(aX_0)^{i\,K}
\end{aligned}
\tag{3.167}
$$

where the lower limit X_0 was introduced to have a well definite integral and the factor $\delta_{\varepsilon_K,\varepsilon_k}$ excludes antiparallel Minkowski and Rindler wave vectors. These relations are in accordance with (3.146), (3.147), and (3.154). In $\mathcal{M}_L$ the Bogolyubov coefficients are of the same form as the ones above, but with $X \to |X|$ and $\delta_{\varepsilon_K,\varepsilon_k} \to \delta_{\varepsilon_{-K},\varepsilon_k}$. The probability density for absorption is then given by

$$
|\beta_{k,\omega;\,\mathcal{M}_{R,L}}|^2 = \frac{1}{2\pi a|k|}\, \frac{1}{e^{2\pi\omega/a} - 1} \tag{3.168}
$$

from which we see, as already pointed out in subsection (3.2.1), that the Minkowski vacuum is indeed a thermal bath of particles with temperature $T = a/2\pi$. In particular, it can be verified [9] that for descrete wave vectors, the Minkowski vacuum can be written in terms of the Rindler plane waves as

$$
|0_M\rangle = \prod_{j=-\infty}^{\infty} \sqrt{1 - e^{-2\pi\omega_j/a}} \sum_{n_{K_j}=0}^{\infty} e^{-n_{K_j}\pi\omega_j/a}\, |n_{K_j}, \mathcal{M}_R\rangle \otimes |n_{-K_j}, \mathcal{M}_L\rangle \tag{3.169}
$$

We notice that in this description both Rindler and Minkowski plane waves are respectively functions of the variables ξ, X, with K, k fixed. In order to build up the localized states in momentum space, it is sufficient to replace the variables with K, k, while fixing a particular value of ξ, X, so that the Minkowski localized states on any hypersurface with $\tau = const$ become

$$
u_{X,M} = \frac{\sqrt{2|k|}\, e^{-i\,k\,(X - \varepsilon_k\,\tau)}}{\sqrt{2\pi}} \qquad k \in \mathbb{R} \tag{3.170}
$$

The localized state at (τ', X') will be given by

$$
u_{X',M}(\tau, X) = \int_{-\infty}^{\infty} dk\, \frac{e^{-i\,k\,[(X-X') - \varepsilon_k\,(\tau-\tau')]}}{2\pi\sqrt{2|k|}} \tag{3.171}
$$

which, at $\tau = \tau'$, can be integrated, giving $u_{X',M}(\tau, X) = \sqrt{2\pi}\,|X - X'|^{-1/2}$, that is nonlocal. It can be easily verified that these localized states are orthonormal, satisfying

$$
\begin{aligned}
(u_{X,M},\, u_{X',M}) &= \delta(X - X') \\
\big(u^*_{X,M,},\, u^*_{X',M}\big) &= -\,\delta(X - X') \\
\big(u^*_{X,M},\, u_{X',M}\big) &= \big(u_{X,M}, u^*_{X',M}\big) = 0
\end{aligned}
\tag{3.172}
$$

where now the inner product is done in the momentum space. In the same way, the orthonormal Rindler localized states on a hypersurface with $\eta = const$ will be given by

$$
u_{\xi,\mathcal{M}_R}(\eta,\xi) = \begin{cases} \frac{\sqrt{2\omega}\, e^{-i\,K\,(\xi - \varepsilon_K\,\eta)}}{\sqrt{2\pi}} & \text{in } \mathcal{M}_R \\ 0 & \text{in } \mathcal{M}_L \end{cases} \tag{3.173}
$$

$$
u_{\xi,\mathcal{M}_L}(\eta,\xi) = \begin{cases} 0 & \text{in } \mathcal{M}_R \\ \frac{\sqrt{2\omega}\, e^{-i\,K(\xi + \varepsilon_K\,\eta)}}{\sqrt{2\pi}} & \text{in } \mathcal{M}_L \end{cases} \tag{3.174}
$$

with $K \in \mathbb{R}$. In order to calculate the probability amplitudes for the Rindler localized states as seen by an inertial observer, it is first useful to expand the Rindler plane waves in terms of the Minkowski localized states. To this aim, we find the following probability amplitudes, namely

$$
\begin{aligned}
(u_{X,M},\, u_{K,\mathcal{M}_R}) &= \int_{-\infty}^{\infty} dk\ \alpha_{k,\omega;\,\mathcal{M}_R}\, \frac{e^{i\,kX}}{\sqrt{2\pi}} \\
\big(u^*_{X,M},\, u_{K,\mathcal{M}_R}\big) &= \int_{-\infty}^{\infty} dk\ \beta_{k,\omega;\,\mathcal{M}_R}\, \frac{e^{-i\,kX}}{\sqrt{2\pi}}
\end{aligned}
$$

where $\alpha_{k,\omega;\,\mathcal{M}_R}$, $\beta_{k,\omega;\,\mathcal{M}_R}$ are the Bogolyubov coefficients found before (3.166), (3.167). These integrals can be evaluated, giving for $X > 0$

$$
(u_{X,M},\, u_{K,\mathcal{M}_R}) = \frac{(aX)^{i\,K/a}}{\sqrt{2\pi a X}}\, e^{2\pi\omega/a - \epsilon\,\omega/a}\, g(K) + f(K) \tag{3.175}
$$

$$
\big(u^*_{X,M},\, u_{K,\mathcal{M}_R}\big) = \frac{(aX)^{i\,K/a}}{\sqrt{2\pi a X}}\, i\,\varepsilon_K\, g(K) - f(K) \tag{3.176}
$$

while for $X < 0$

$$
(u_{X,M},\, u_{K,\mathcal{M}_R}) = -\,i\,\frac{|aX|^{i\,K/a}}{\sqrt{2\pi a X}}\,\varepsilon_K\, e^{\pi\omega/a}\, g(K) + f(K) \tag{3.177}
$$

$$
\big(u^*_{X,M},\, u_{K,\mathcal{M}_R}\big) = \frac{|aX|^{i\,K/a}}{\sqrt{2\pi a X}}\, e^{\pi\omega/a} g(K) - f(K) \tag{3.178}
$$

where

$$g(K) \equiv \frac{(-1)^{1/4}}{2\pi}\, \varepsilon_K \sqrt{\frac{\omega}{a}}\, \Gamma\left(\frac{1}{2} - i\,\frac{K}{a}\right)\, \Gamma\left(i\,\frac{K}{a}\right)\, e^{-\pi\omega/a} \tag{3.179}$$

$$|g(K)|^2 = \left(e^{4\pi\omega/a} - 1\right)^{-1} \tag{3.180}$$

The small constant ϵ was introduced to give a convergent integral and a finite linewidth. $f(K)$ is the integral term of $F(K)$ in (3.166), that for $X_0 \ll X$ reduces to

$$f(K) = \frac{i\,(aX_0)^{i\,K/a}}{\sqrt{2\pi\omega/a}} \tag{3.181}$$

Here (3.180) suggests a temperature for the thermal bath of $T = a/4\pi$, which is half the Unruh temperature found before. However, since the creation and destruction operators are the same as before, the thermal bath seen by the Rindler observer is still characterized by the temperature $T = a/2\pi$. The difference in the result just found comes from the fact that $g(K)$ takes into account the nonlocality of the fields. Therefore, for $X > 0$ the emission and absorption probabilities seem to have a temperature of $T = a/4\pi$, while for $X < 0$, they are equal.

Now we can find the Bogolyubov coefficients between the Rindler and Minkowski localized states, namely

$$\begin{aligned} \alpha_{X,\xi;\,\mathcal{M}_R} &= (u_{X,M},\, u_{\xi,\mathcal{M}_R}) \\ &= \int_{-\infty}^{\infty} dK\, \left(u_{X,M},\, u_{K,\mathcal{M}_R}\right)\, \frac{e^{-i\,K\xi}}{\sqrt{2\pi}} \end{aligned} \tag{3.182}$$

$$\begin{aligned} \beta_{X,\xi;\,\mathcal{M}_R} &= \left(u^*_{X,M},\, u_{\xi,\mathcal{M}_R}\right) \\ &= \int_{-\infty}^{\infty} dK\, \left(u^*_{X,M},\, u_{K,\mathcal{M}_R}\right)\, \frac{e^{-i\,K\xi}}{\sqrt{2\pi}} \end{aligned} \tag{3.183}$$

while in $\mathcal{M}_L$ they are given by $\alpha_{X,\xi;\,\mathcal{M}_L} = -\alpha_{|X|,\xi;\,\mathcal{M}_R}$ and $\beta_{X,\xi;\,\mathcal{M}_L} = -\beta_{|X|,\xi;\,\mathcal{M}_R}$. It can be verified numerically that if the Minkowski observer sees the Rindler localized states as exactly localized, then the graph of $|\,(u_{X,M},\, u_{K,\mathcal{M}_R})\,|$ should be flat with respect to K, while the other K integrands should be zero. This feature is fulfilled for all K, but near $K = 0$, where the above integrals diverge as $\omega^{-1/2}$. Then, the flat regions lead to a delta function proportional to $(\ln|aX| - a\xi)/|aX|^{1/2}$, while the thermal peaks give additional delocalized components to all curves, which are largest in $\mathcal{M}_L$.

Let us consider now another feature of these photon counting devices. Photons are counted when they cross the detector surface and are absorbed within a penetration depth of a few wavelenghts; at normal incidence, it is possible to describe the probability per unit time to absorb a photon in terms of the number density $J^{(+)}$ in the localized states basis (on a spacelike hypersurface), namely

$$\langle\psi|\, n^{(+)}(\tau, X)\,|\psi\rangle = \pm\,\frac{1}{c}\,\langle\psi|\, J^{(+)}(\tau, X)\,|\psi\rangle \tag{3.184}$$

where the positive flux is for left to right propagation, while the converse is true for the negative flux. It is also assumed that only photons from one directions are counted, the others in the opposite direction being traced out. In particular, since the Minkowski vacuum state (3.169) in first approximation can be described by a pair of photon with opposite wave vectors in the two Rindler wedges, it is possible to study the entanglement transfer from the vacuum to the that pair of counteraccelerated detectors, which is due to the fact that, although the two causally disconnected Rindler regions cannot commucate, a Minkowski observer can receive signals from both wedges. What is found is that the coincidence rate for absorption of correlated photons in the two Rindler wegdes $\mathcal{M}_R$ and $\mathcal{M}_L$ is a Lorenzian function of the difference between the null Rindler coordinates (say $v' - v''$, $v = \eta + \xi$), with linewidth $2\pi/a$, where a is the proper acceleration on $\xi = 0$.

In fact, if we consider only the first term with $n_K = 1$ in (3.169), for photons propagating from right to left described by the Rindler null coordinate $v = \eta + \xi$, the probability amplitude for two photon absorption is given by

$$\begin{aligned}\langle u_{v,\mathcal{M}_R}, u_{v',\mathcal{M}_L}|0_M\rangle &= \int_{-\infty}^{0} dK\, e^{-\pi\omega/a}\,\frac{e^{i\,K\,(v-v')}}{2\pi} \\ &= \frac{1}{2\pi}\,\frac{1}{i\,(v-v') + \pi/a}\end{aligned} \tag{3.185}$$

that represents Lorentzian spacetime correlations with linewidth $2\pi/a$, which, for $a \to 0$ it becomes infinite, while for $a \to \infty$ we have

$$\langle u_{v,\mathcal{M}_R}, u_{v',\mathcal{M}_L}|0_M\rangle \to -\frac{1}{2}\,\delta(v-v') - \frac{i}{2\pi}\,PV\left(\frac{1}{v-v'}\right) \tag{3.186}$$

where PV is the Principal Value distribution. Then, including both positive and negative K, the above probability amplitude tends to the delta function $-\delta(v - v')$, that is, the spacetime correlations are exact in the infinite acceleration limit. In particular, if the photon in $\mathcal{M}_L$ is not detected, the probability density to count a photon in $\mathcal{M}_R$ is given by tracing out over v' the absolute square of (3.185), namely,

$$\int_{-\infty}^{\infty} dv'\,|\langle u_{v,\mathcal{M}_R}, u_{v',\mathcal{M}_L}|0_M\rangle|^2 = \frac{a}{4\pi^2} \tag{3.187}$$

We notice that the above quantity gives the probability per unit Rindler time. Then, the probability per unit proper time for a detector with an absorbing surface at ξ to count a photon is simply equal to $\bar{a}/4\pi$, where $\bar{a} = a\, e^{-a\xi}$ is the proper acceleration and $d\bar{t} = \pm(a/\bar{a})\, d\eta$ is the proper time interval. It can be also verified that the photon state prepared when an accelerated detector in $\mathcal{M}_R$ counts a photon can lead to absorption by either an inertial detector or an accelerated one in $\mathcal{M}_L$.

If we now include all n_j terms of (3.169), we will see that the probability that an accelerated detector will absorb a photon from the Minkowski vacuum in enhanced. To show this, let us introduce the Rindler number density operators

$$n_j^{(+)}(\xi) = i\, \phi_j^{(-)} \overleftrightarrow{\partial_\eta}\, \phi_j^{(+)} \qquad j = \mathcal{M}_R,\, \mathcal{M}_L \tag{3.188}$$

where now

$$\phi_j^{(+)}(\eta, \xi) = \int_{-\infty}^{\infty} dK\, u_{K,j}(\eta, \xi)\, b_{K,j} \qquad \phi_j^{(-)} = \phi_j^{(+)\dagger} \tag{3.189}$$

For right to left propagation, the probability density to count a photon for an electromagnetic field on the Minkowski vacuum is given by

$$\begin{aligned} w_{\mathcal{M}_R}^{(+)} &= \langle 0_M|\, n_{v,\mathcal{M}_R}^{(+)}\, |0_M\rangle \\ &= \int_{-\infty}^{0} \frac{dK}{2\pi}\, C_K^2 \sum_{n_K=0}^{\infty} e^{-2\pi\omega n_K/a}\, n_K \\ &= \int_{-\infty}^{0} \frac{dK}{2\pi} \left(e^{2\pi\omega/a} - 1\right)^{-1} \end{aligned} \tag{3.190}$$

which diverges at $\omega = 0$ and where $C_K = \sqrt{1 - e^{-2\pi\omega/a}}$. However, any real detector has a lower limit to the frequency response, so that we can introduce it as $\Omega_0 = |K_0|$, thus finding

$$w_{\mathcal{M}_R}^{(+)} = -\, 2a\, \ln\left(1 - e^{-2\pi\Omega_0/a}\right) \tag{3.191}$$

We see that in the limit $2\pi\Omega_0/a \ll 1$, $w_{\mathcal{M}_R}^{(+)} \simeq -\, 2a\, \ln(2\pi\Omega_0/a)$, which diverges as $\Omega_0 \to 0$. The two photon correlation function is given by

$$w_{\mathcal{M}_R,\mathcal{M}_L}^{(+)} = \langle 0_M|\, n_{v,\mathcal{M}_R}^{(+)}\, n_{v,\mathcal{M}_L}^{(+)}\, |0_M\rangle \tag{3.192}$$

which can be evaluated considering that

$$b_{-K,\mathcal{M}_L}\, b_{K,\mathcal{M}_R}\, |0_M\rangle = C_K \sum_{n_K=0}^{\infty} n_K\, e^{-n_K\, \pi\omega/a}\, |\, n_K - 1, \mathcal{M}_R\rangle \otimes |\, n_{-K} - 1, \mathcal{M}_L\rangle \tag{3.193}$$

Then we have

$$w^{(+)}_{\mathcal{M}_R,\mathcal{M}_L} = \left| \int_{-\infty}^{-\Omega_0} dK \, \frac{e^{\pi\omega/a - i\,K\,(v'-v)}}{2\pi(e^{2\pi\omega/a} - 1)} \right|^2 + w^{(+)}_{\mathcal{M}_R}\, w^{(+)}_{\mathcal{M}_L} \tag{3.194}$$

where the second term does not depend on the spacetime coordinates. Furthermore, in order to obtain the probability densities per unit proper time, we have to replace a with $\bar{a}$. However, now the minimum frequency is $\omega_0 = \Omega_0\, e^{-a\xi}$, so that $\Omega_0/a = \omega_0/\bar{a}$. In particular, using the SI units, we can replace a with the acceleration frequency a/c with dimesions of s^{-1}. An Unruh Temperature of $T = a\hbar/2\pi c k_B = 1K$ corresponds to an acceleration frequency of $a/c = 2 \times 10^{12}\, s^{-1}$. Then, for instance, for $\Omega_0\, c/a = 0.01$ a temperature of $1K$ requires a detector that can absorb a photon with angular frequencies greater than $2 \times 10^{10}\, rad/s$.

Conclusions

In this work we studied the quantization of both massive and massless vector fields in a Rindler space. Regarding the massive vector field, as in the usual Minkowski space, we focused on the Lagrangian density which led to a generalized linear and invariant transversality condition (eqn. 3.7). However, in this case, a different choice of the auxiliary field B turned out to be more convenient. As shown in the first chapter, in the usual Mikowski space the Proca vector field does not admit any gauge symmetry; therefore the field B can be chosen to be either $B = 0$, or, in order to build up a renormalizable theory, its insertion becomes necessary, even though one can always choose a particular gauge with $A^\mu = V^\mu - \partial^\mu B/m^2$, thanks to which one recovers the Proca field equations, and the two fields V^μ and B appear to be decoupled. In the Rindler case, instead, it is more useful to make another choice for B, namely $B - \partial_\perp \cdot A^\perp = 0$ (these two fields have formally the same form, so it is always possible to impose this condition), so that the vector field can be expressed only in terms of the known homogeneous solutions of the field equations. Moreover, this condition is suggested by the fact that in the 2-dimensional Rindler space, where $A^\perp = 0 = B$, the Proca vector field has only one indipendent component, i.e. the longitudinal one, in the direction of the acceleration, which satisfies the same equation as the homogeneous solution of the 4-dimensional vector field. With this choice, that in a way removes the auxiliary field from the physical relevant aspects, an explicit subsidiary condition to select a physical subspace is no more necessary. In the general case, an analogous condition as (1.75) can be used; then, the inhomogeneous solutions of the vector field will depend only upon the transverse field, which formally has the same form as the field B.

For what concerns the massless vector field, we preferred not to consider the Feynman gauge, since in this case the theory turned out to be more complicated. As a matter of fact, in the Rindler case the mass m plays just the role of a parameter, the presence or absence of which in the equations of motion is irrelevant. There is not any mass-shell condition. Thus, in the limit of $m \to 0$, the equations and the results found before for the massive vector field do not change their form, nor any complication or divergence appears. Neither the coupling to the auxiliary field changes. Then, trying to reduce the indipendent polarizations from three (massive case) to two (massless case) becomes a more complicated problem than it is in the Minkowski case. Therefore, it turned out to be more convenient to investigate the solutions for the massless vector field considering another gauge. We focused our attention on the axial gauge in particular, since this way one component, the longitudinal component, of the vector field cancels immediatly thanks to the equations of motion. Then, only two indipendent components are left, the other one being related to these two by the condition $\partial_\mu A^\mu = 0$. As one could intuitively expect, these two polarizations are transverse to the direction of the acceleration, as the ones found in the massive case, where the third one was longitudinal to the acceleration axis. We notice that the orthogonal polarization vectors could have been chosen in a different way. Our choice was due to convenience and simplicity, since this way the orthonormality and closure relations for the normal modes of the vector field followed directly from the generalization of the Fulling normal modes for the scalar field (see Appendix A). We considered also the Lorenz-Landau gauge in a 2-dimensional spacetime for the massless vector field, as in this case the computation of the Bogolyubov coefficients turned out to be quite simple and in full accordance with [7] for the scalar field (in a 2-dimensional spacetime, the vector field reduces to only one indipendent component, i.e. a scalar field).

Finally, the canonical quantization for both the massive and massless vector fields was obtained by replacing the classical field functions with operator valued tempered distributions, whose creation and destruction operators satisfy the commutation relations (3.78, 3.79, 3.80, 3.81, 3.104), analogous to the ones found for the scalar field.

Appendix A

The Inner Products of the Vector Normal Modes

In this section we present the computation of the inner products between the normal modes of the vector field on a Rindler space. To start with, we recall [3] the basic integral: namely,

$$\begin{aligned} \int_0^\infty \zeta^{-\lambda}\, K_\mu(\zeta)\, K_\nu(\zeta)\, \mathrm{d}\zeta &= \frac{2^{-2-\lambda}}{\Gamma(1-\lambda)} \\ &\times\ \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \\ &\times\ \Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right) \end{aligned}$$

with $\mathrm{Re}\,\lambda < 1 - |\mathrm{Re}\,\mu| - |\mathrm{Re}\,\nu|$. Next, let us first show how to get the normalization of the Fulling scalar normal modes (2.48), since the inner products in the vectorial case can be handled in very same way. Therefore, after setting $E/\mathrm{a} = \upsilon$, $E'/\mathrm{a} = \upsilon'$ we can write

$$\begin{aligned} &\frac{E+E'}{2\pi\mathrm{a}} \int_0^\infty \frac{x}{x}\, K_{iE'/\mathrm{a}}(\kappa x)\, K_{iE/\mathrm{a}}(\kappa x) \\ =&\ \lim_{\epsilon\to 0^+} \frac{\upsilon+\upsilon'}{16\pi\Gamma(\epsilon)}\, \Gamma\left(\frac{\epsilon+i\upsilon+i\upsilon'}{2}\right)\Gamma\left(\frac{\epsilon-i\upsilon+i\upsilon'}{2}\right) \\ \times&\ \Gamma\left(\frac{\epsilon-i\upsilon-i\upsilon'}{2}\right)\Gamma\left(\frac{\epsilon+i\upsilon-i\upsilon'}{2}\right) \end{aligned}$$

Now, it turns out that for $\upsilon \neq \upsilon'$ $(\upsilon, \upsilon' \in \mathbb{R})$ the above expression is an analytic function of $\epsilon > 0$, which vanish in the limit $\epsilon \to 0^+$. Then we can safely write

$$\frac{E+E'}{2\pi\mathrm{a}}\int_0^\infty \frac{x}{x}\,K_{iE'/\mathrm{a}}(\kappa x)\,K_{iE/\mathrm{a}}(\kappa x)=0 \qquad \text{for} \quad E\neq E^{'} \tag{A.1}$$

Moreover, recalling the following properties

$$\Gamma(\varepsilon+z)=\Gamma(z)[\,1+\varepsilon\,\psi(z)+\cdots\,] \qquad (0\leq\varepsilon\ll 1) \tag{A.2}$$

$$|\,\Gamma(iy)\,|^2=\frac{\pi}{y\sinh(\pi y)} \qquad (y\in\mathbb{R})$$

we can write

$$\begin{aligned} \frac{\upsilon+\upsilon'}{2\pi}\int_0^\infty \frac{x}{x}\,K_{i\upsilon'}(\kappa x)\,K_{i\upsilon}(\kappa x) \;&=\; \lim_{\epsilon\to 0^+}\frac{1}{16\pi\Gamma(\epsilon)} \\ \times\quad & \frac{2\pi}{\sinh[\,\pi(\upsilon+\upsilon')/2\,]}\cdot\frac{2\pi}{(\upsilon-\upsilon')\sinh[\,\pi(\upsilon-\upsilon')/2\,]} \\ \times\quad & \left\{1+\frac{\epsilon}{2}\,\psi\left(\frac{-i\upsilon-i\upsilon^{'}}{2}\right)+\frac{\epsilon}{2}\,\psi\left(\frac{i\upsilon+i\upsilon^{'}}{2}\right)\right. \\ +\quad & \left.\frac{\epsilon}{2}\,\psi\left(\frac{-i\upsilon+i\upsilon^{'}}{2}\right)+\frac{\epsilon}{2}\,\psi\left(\frac{i\upsilon-i\upsilon^{'}}{2}\right)\;+\;O(\epsilon^2)\right\} \end{aligned}$$

Next, if we replace $\upsilon^{'}-\upsilon\;\mapsto\;\eta+i\epsilon$ and contextually $\upsilon\simeq\upsilon^{'}$, then we can recast the above expression in the form

$$\begin{aligned} \frac{\upsilon+\upsilon'}{2\pi}\int_0^\infty \frac{x}{x}\,K_{i\upsilon'}(\kappa x)\,K_{i\upsilon}(\kappa x) \;&=\; \lim_{\epsilon\to 0^+}\frac{\pi}{4\,\Gamma(\epsilon)} \\ \times\quad & \Re\mathrm{e}\,\frac{1}{\sinh(\pi\upsilon)\,\sinh\left[\,\frac{1}{2}\pi(\eta+i\epsilon)\,\right]}\cdot\left[\mathrm{PV}\left(\frac{1}{\eta}\right)-i\pi\delta(\eta)\right] \\ \times\quad & \left\{1+\tfrac{1}{2}\,\epsilon\,[\,\psi(-i\upsilon)+\psi(i\upsilon)+\psi(1-i\eta)+\psi(1+i\eta)\,]+O(\epsilon^2)\right\} \end{aligned}$$

where PV denotes the Cauchy-Hadamard principal value prescription. It follows that the leading singular part for $\eta\to 0$ with $\epsilon\neq 0$ of the RHS of the above equality becomes

$$\begin{aligned} & \Re\mathrm{e}\,\frac{\mathrm{csch}(\pi\upsilon)}{4i\Gamma(\epsilon)}\left[\mathrm{PV}\left(\frac{1}{\eta}\right)-i\pi\delta(\eta)\right]\left(\frac{2}{\epsilon}+\frac{\pi^2}{12}\,\epsilon\;+\;\cdots\right) \\ \times\quad & \left\{1+\tfrac{1}{2}\,\epsilon\,[\,\psi(-i\upsilon)+\psi(i\upsilon)-2\mathbf{C}\,]+O(\epsilon^2)\right\} \\ \stackrel{\epsilon\downarrow 0}{\longrightarrow}\quad & -\tfrac{1}{2}\,\pi\,\mathrm{csch}(\pi\upsilon)\;\delta(\upsilon-\upsilon'\,) \end{aligned}$$

where $\mathbf{C}$ is the Euler-Mascheroni constant, while the limit $\epsilon\downarrow 0$ must be taken at the very end in the sense of the distributions.

Now we are ready to handle the inner product of the normal modes for the vector field. In particular, we consider the normalization of the longitudinal polarization, computing all the addenda in the expression (3.60) following the method just described. To start with, for $\lambda = 3-\omega$ and $\mu = iv\,,\ \nu = iv'$ we obtain

$$\begin{aligned}
\mathcal{I}_1(v,v'\,;\omega) &= (v+v')\int_0^\infty \zeta^{\,\omega-3}\,K_{iv}(\zeta)\,K_{iv'}(\zeta)\,\mathrm{d}\zeta \\
&= \frac{2^{\,\omega-5}}{\Gamma(\omega-2)}\,(v+v')\ \times \\
&\times\ \Gamma\left(\frac{\omega-2+iv+iv'}{2}\right)\Gamma\left(\frac{\omega-2-iv+iv'}{2}\right) \\
&\times\ \Gamma\left(\frac{\omega-2+iv-iv'}{2}\right)\Gamma\left(\frac{\omega-2-iv-iv'}{2}\right)
\end{aligned}$$

with $\mathrm{Re}\,\omega > 2$. It turns out that, for $v \neq v'$, the RHS of the above equality is an analytic function of the complex variable $\omega \in \mathbb{C}$, with simple zeros for $\omega \in \mathbb{Z}_- \cup \{0,1,2\}$. Hence, for $v \neq v'$, it does represent an analytic continuation, say $\overline{\mathcal{I}}_1(v,v'\,;\omega)$, of the integral (A.3) in the half-plane $\mathrm{Re}\,\omega < 2$, which includes the physical value $\omega = 0$. Then we can write

$$\lim_{\omega\to 0^+}\overline{\mathcal{I}}_1(v,v'\,;\omega) = 0 \qquad v \neq v' \tag{A.3}$$

Furthermore for $\omega \in \mathbb{R}$, $\omega < 2$, we have

$$\begin{aligned}
\overline{\mathcal{I}}_1(v,v'\,;\omega) &= \frac{2^{\,\omega-5}}{\Gamma(\omega-2)}\,(v+v')\ \times \\
&\times\ \left[\left(\frac{\omega-2+iv+iv'}{2}\right)\left(\frac{\omega-2-iv+iv'}{2}\right)\right. \\
&\times\ \left.\left(\frac{\omega-2+iv-iv'}{2}\right)\left(\frac{\omega-2-iv-iv'}{2}\right)\right]^{-1} \\
&\times\ \left|\Gamma\left(\frac{\omega+iv+iv'}{2}\right)\Gamma\left(\frac{\omega+iv-iv'}{2}\right)\right|^2
\end{aligned} \tag{A.4}$$

Now it is convenient to set

$$\omega = \epsilon \quad (0<\epsilon\ll 1) \qquad v'-v = \xi + i\epsilon \tag{A.5}$$

so that for $\epsilon \to 0^+$ and $\xi \sim 0$ we obtain the leading behaviour

$$\begin{aligned}
\overline{\mathcal{I}}_1(\upsilon,\xi;\epsilon) \;&=\; \Re\mathrm{e}\,\frac{1}{8\Gamma(\epsilon-2)}\left[\left(\frac{\epsilon-2+2i\upsilon}{2}\right)\right.\\
\times \quad &\left.\left(\frac{-2+i\xi}{2}\right)\left(\frac{2\epsilon-2-i\xi}{2}\right)\left(\frac{\epsilon-2-2i\upsilon}{2}\right)\right]^{-1}\\
\times \quad &\frac{\pi^2\,\mathrm{csch}(\pi\upsilon)}{\sinh[\,\pi(\xi+i\epsilon)/2\,]}\left[\mathrm{PV}\left(\frac{1}{\xi}\right)-i\pi\,\delta(\xi)\right]\Bigg\}\\
\overset{\xi\to 0}{\sim} \quad &\Re\mathrm{e}\,\frac{\epsilon-2}{\Gamma(1+\epsilon)}\cdot\frac{\pi i\,\mathrm{csch}(\pi\upsilon)}{(\epsilon-2)^2+4\upsilon^2}\left[\mathrm{PV}\left(\frac{1}{\xi}\right)-i\pi\,\delta(\xi)\right]\\
\xrightarrow{\epsilon\downarrow 0} \quad &-\,\frac{\pi^2\,\mathrm{csch}(\pi\upsilon)}{2\,(1+\upsilon^2)}\,\delta(\upsilon-\upsilon'\,)
\end{aligned}$$

Thus we eventually come to the physical limit value for $\omega\downarrow 0$

$$\lim_{\omega\to 0^+}\overline{\mathcal{I}}_1(\upsilon,\upsilon'\,;\omega)\equiv\overline{\mathcal{I}}_1(\upsilon,\upsilon'\,)=-\,\frac{\pi^2\mathrm{csch}(\pi\upsilon)}{2(1+\upsilon^2)}\,\delta(\upsilon-\upsilon'\,) \tag{A.6}$$

Moreover for $\lambda=1-\omega$ and $\mu=i\upsilon\pm 1\,,\ \nu=i\upsilon'\pm 1$ we get

$$\begin{aligned}
\mathcal{I}_2(\upsilon,\upsilon'\,;\omega) \;&=\; \frac{\upsilon+\upsilon'}{4\upsilon\upsilon'}\int_0^\infty \mathrm{d}\zeta\,\zeta^{\,\omega-1}\left[\,K_{i\upsilon'-1}(\zeta)+K_{i\upsilon'+1}(\zeta)\,\right]\\
\times \quad &\left[\,K_{i\upsilon-1}(\zeta)+K_{i\upsilon+1}(\zeta)\,\right] \;=\; 2^{\,\omega-5}\,\frac{\upsilon+\upsilon'}{\upsilon\upsilon'\,\Gamma(\omega)}\,\times\\
\times \quad &\left\{\Gamma\left(\frac{\omega-2+i\upsilon+i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-i\upsilon+i\upsilon'}{2}\right)\right.\\
\times \quad &\Gamma\left(\frac{\omega+i\upsilon-i\upsilon'}{2}\right)\Gamma\left(\frac{\omega+2-i\upsilon-i\upsilon'}{2}\right)\\
+ \quad &\left[\Gamma\left(\frac{\omega+i\upsilon+i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-2-i\upsilon+i\upsilon'}{2}\right)\right.\\
\times \quad &\left.\Gamma\left(\frac{\omega+2+i\upsilon-i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-i\upsilon-i\upsilon'}{2}\right)\;+\upsilon\;\longleftrightarrow\;\upsilon'\right]\\
+ \quad &\Gamma\left(\frac{\omega+2+i\upsilon+i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-i\upsilon+i\upsilon'}{2}\right)\\
\times \quad &\left.\Gamma\left(\frac{\omega+i\upsilon-i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-2-i\upsilon-i\upsilon'}{2}\right)\right\}\\
\equiv \quad &\mathcal{I}_2^{(-)}(\upsilon,\upsilon'\,;\omega)+\mathcal{I}_2^{(\circ)}(\upsilon,\upsilon'\,;\omega)+\mathcal{I}_2^{(+)}(\upsilon,\upsilon'\,;\omega)
\end{aligned}$$

always with $\mathrm{Re}\,\omega > 2$. The analytic continuation for $\mathrm{Re}\,\omega < 2$ yields

$$\begin{aligned}\overline{\mathcal{I}}_2^{(-)}(\upsilon,\upsilon';\omega) &= 2^{\omega-5}\,\frac{\upsilon+\upsilon'}{\upsilon\upsilon'\,\Gamma(\omega)}\cdot\frac{\omega-i\upsilon-i\upsilon'}{\omega-2+i\upsilon+i\upsilon'}\\ &\times\left|\Gamma\left(\frac{\omega+i\upsilon+i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-i\upsilon+i\upsilon'}{2}\right)\right|^2\end{aligned}$$

so that

$$\lim_{\omega\to 0^+}\overline{\mathcal{I}}_2^{(-)}(\upsilon,\upsilon';\omega)=0 \qquad \upsilon\neq\upsilon' \tag{A.7}$$

while

$$\begin{aligned}\overline{\mathcal{I}}_2^{(-)}(\upsilon,\xi;\epsilon) &= \Re\mathrm{e}\,2^{\epsilon-5}\,\frac{2}{\upsilon\,\Gamma(\epsilon)}\cdot\frac{\epsilon-2i\upsilon}{\epsilon-2+2i\upsilon}\\ \times\quad & \frac{\pi}{\upsilon\sinh(\pi\upsilon)}\cdot\frac{2\pi}{\sinh[\pi(\xi+i\epsilon)/2]}\left[\mathrm{PV}\left(\frac{1}{\xi}\right)-i\pi\,\delta(\xi)\right]\\ \overset{\xi\to 0}{\sim}\quad & \Re\mathrm{e}\,2^{\epsilon-2}\,\frac{-i}{\upsilon\,\Gamma(1+\epsilon)}\cdot\frac{\epsilon-2i\upsilon}{\epsilon-2+2i\upsilon}\cdot\frac{\pi}{\upsilon\sinh(\pi\upsilon)}\\ \times\quad & \left[\mathrm{PV}\left(\frac{1}{\xi}\right)-i\pi\,\delta(\xi)\right] \qquad\qquad (A.8)\\ \xrightarrow{\epsilon\downarrow 0}\quad & \frac{\pi^2}{4\,(1+\upsilon^2)\sinh(\pi\upsilon)}\;\delta(\upsilon-\upsilon') \qquad\qquad (A.9)\end{aligned}$$

Furthermore we find for $\mathrm{Re}\,\omega < 2$

$$\begin{aligned}\overline{\mathcal{I}}_2^{(\circ)}(\upsilon,\upsilon';\omega) &= 2^{\omega-5}\,\frac{\upsilon+\upsilon'}{\upsilon\upsilon'\,\Gamma(\omega)}\cdot\frac{\omega+i\upsilon-i\upsilon'}{\omega-2-i\upsilon+i\upsilon'}\\ &\times\left|\Gamma\left(\frac{\omega+i\upsilon+i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-i\upsilon+i\upsilon'}{2}\right)\right|^2\\ &+\;\upsilon\longleftrightarrow\upsilon'\end{aligned}$$

that yields

$$\lim_{\omega\to 0^+}\overline{\mathcal{I}}_2^{(\circ)}(\upsilon,\upsilon';\omega)=0 \qquad \upsilon\neq\upsilon' \tag{A.10}$$

together with

$$\begin{aligned}\overline{\mathcal{I}}_2^{(\circ)}(\upsilon,\xi;\epsilon) &= \Re\mathrm{e}\,2^{\epsilon-5}\,\frac{2}{\upsilon\,\Gamma(\epsilon)}\cdot\frac{2\epsilon-i\xi}{-2+i\xi}\\ &\times\frac{\pi}{\upsilon\sinh(\pi\upsilon)}\cdot\frac{2\pi}{\sinh[\,\pi(\xi+i\epsilon)/2\,]}\\ &\times\left[\mathrm{PV}\left(\frac{1}{\xi}\right)-i\pi\,\delta(\xi)\right]\;+\;\xi\longleftrightarrow-\xi\\ &\overset{\xi\to 0}{\sim}\frac{\epsilon\,2^{\epsilon}\,\pi^2\,\delta(\xi)}{2\Gamma(1+\epsilon)\,\upsilon^2\sinh(\pi\upsilon)}\;\xrightarrow{\epsilon\downarrow 0}\;0\end{aligned}$$

Moreover we get

$$\overline{\mathcal{I}}_2^{(+)}(\upsilon,\upsilon';\omega) = 2^{\omega-5}\,\frac{\upsilon+\upsilon'}{\upsilon\upsilon'\,\Gamma(\omega)}\cdot\frac{\omega+i\upsilon+i\upsilon'}{\omega-2-i\upsilon-i\upsilon'} \times \left|\Gamma\left(\frac{\omega+i\upsilon+i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-i\upsilon+i\upsilon'}{2}\right)\right|^2$$

whence we obtain

$$\lim_{\omega\to 0^+}\overline{\mathcal{I}}_2^{(+)}(\upsilon,\upsilon';\omega)=0 \qquad \upsilon\neq\upsilon' \tag{A.11}$$

while

$$\begin{aligned}
\overline{\mathcal{I}}_2^{(+)}(\upsilon,\xi;\epsilon) &= \Re\mathrm{e}\,2^{\epsilon-5}\,\frac{2}{\upsilon\,\Gamma(\epsilon)}\cdot\frac{\epsilon+2i\upsilon}{\epsilon-2-2i\upsilon}\\
&\times \frac{\pi}{\upsilon\sinh(\pi\upsilon)}\cdot\frac{2\pi}{\sinh[\pi(\xi+i\epsilon)/2]}\left[\mathrm{PV}\left(\frac{1}{\xi}\right)-i\pi\,\delta(\xi)\right]\\
&\overset{\xi\to 0}{\sim} \Re\mathrm{e}\,2^{\epsilon-2}\,\frac{-i}{\upsilon\,\Gamma(1+\epsilon)}\cdot\frac{\epsilon+2i\upsilon}{\epsilon-2-2i\upsilon}\cdot\frac{\pi}{\upsilon\sinh(\pi\upsilon)}\\
&\times \left[\mathrm{PV}\left(\frac{1}{\xi}\right)-i\pi\,\delta(\xi)\right] \qquad (A.12)\\
&\xrightarrow{\epsilon\downarrow 0} \frac{\pi^2}{4\,(1+\upsilon^2)\sinh(\pi\upsilon)}\,\delta(\upsilon-\upsilon') \qquad (A.13)
\end{aligned}$$

By summing up the three addenda we eventually get

$$\overline{\mathcal{I}}_2(\upsilon,\upsilon') = \frac{\pi^2\,\mathrm{csch}(\pi\upsilon)}{2(1+\upsilon^2)}\,\delta(\upsilon-\upsilon') \tag{A.14}$$

Finally, for $\lambda=2-\omega$ and $\mu=i\upsilon$, $\nu=i\upsilon'\pm 1$ we have

$$\begin{aligned}
\mathcal{I}_3(\upsilon,\upsilon';\omega) &= \frac{1}{\upsilon}\int_0^\infty \zeta^{\,\omega-2}\,K_{i\upsilon}(\zeta)\,[\,K_{i\upsilon'-1}(\zeta)+K_{i\upsilon'+1}(\zeta)\,]\,\mathrm{d}\zeta\\
+\quad \upsilon\longleftrightarrow\upsilon' &= \frac{2^{\,\omega-4}}{\upsilon\Gamma(\omega-1)}\times\\
&\times\left\{\Gamma\left(\frac{\omega-2+i\upsilon+i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-2-i\upsilon+i\upsilon'}{2}\right)\right.\\
&\times\Gamma\left(\frac{\omega+i\upsilon-i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-i\upsilon-i\upsilon'}{2}\right)\\
&+\Gamma\left(\frac{\omega+i\upsilon+i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-i\upsilon+i\upsilon'}{2}\right)\\
&\times\left.\Gamma\left(\frac{\omega-2+i\upsilon-i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-2-i\upsilon-i\upsilon'}{2}\right)\right\}+\upsilon\longleftrightarrow\upsilon'
\end{aligned}$$

always with $\mathrm{Re}\,\omega > 2$. Once again the analytic continuation drives to

$$\begin{aligned} \overline{\mathcal{I}}_3(\upsilon,\upsilon'\,;\omega) \;&=\; \frac{(\omega-1)\,2^{\omega}}{2\upsilon\Gamma(\omega)} \;\times \\ \times\;\; &\left|\Gamma\left(\frac{\omega+i\upsilon+i\upsilon'}{2}\right)\Gamma\left(\frac{\omega-i\upsilon+i\upsilon'}{2}\right)\right|^2 \\ \times\;\; &\frac{(\omega-2)^2+(\upsilon+\upsilon'\,)(\upsilon-\upsilon'\,)}{[\,(\omega-2)^2+(\upsilon+\upsilon'\,)^2\,][\,(\omega-2)^2+(\upsilon-\upsilon'\,)^2\,]} \;+\; \upsilon \longleftrightarrow \upsilon' \end{aligned}$$

and thereby

$$\overline{\mathcal{I}}_3(\upsilon,\upsilon'\,) = \frac{\pi^2\,\mathrm{csch}(\pi\upsilon)}{\upsilon^2(1+\upsilon^2)}\,\delta(\upsilon-\upsilon'\,) \tag{A.15}$$

Bibliography

[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, 1972.

[2] N. D. Birrel and P. C. W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.

[3] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Fourth Edition, Alan Jeffrey Editor, Academic Press, 1965.

[4] A. L. Rabenstein, *Introduction to Ordinary Differential Equations*, Second Edition, Academic Press, 1972.

[5] S. A. Fulling, *Nonuniqueness of Canonical Field Quantization in Riemannian Space-Time*, Physical Review D, 7, 2580 (1973).

[6] P. Longhi and R. Soldati, e-Print arXiv: 1210.7378

[7] I.Ya. Areféva and I.V. Volovich, e-Print arXiv:1302.6699v1 [hep-th] 27 Feb 2013.

[8] Margaret Hawton, e-Print arXiv:1304.5138v1 [quant-ph] 18 Apr 2013.

[9] L. C. B. Crispino, A. Higuchi, G. E. A. Matsas, e-Print arXiv:0710.5373v1 [gr-qc] 29 Oct 2007.

[10] V. P. Frolov and I. D. Novikov, *Black Hole Physics: Basic Concepts and New Developments*, Kluwer Academic Publishers, 1998.