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ASIAN OPTIONS ON ARITHMETIC AND HARMONIC AVERAGES

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Relatore:
Chiar.mo Prof.
ANDREA PASCUCCI

Presentata da:
ANDREA FONTANA

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INTRODUCTION

The first aim of this thesis is to develop approximation formulae expressed in terms of elementary functions for the density and the price of path dependent options of Asian style.
In the second part of this work, whereas, the purpose is to provide a theoretical bound estimation for the error committed by our approximations developed formerly.
Asian options are path dependent derivatives whose payoff depends on some form of averaging prices of the underlying asset.
In particular in this thesis they will be considered payoff functions depending on the Arithmetic average and on the Harmonic average.

In the first chapter, after having given some preliminary notions of stochastic analysis, it is described the strategy utilized to approximate the density and consequently the price of Asian options.
From the mathematician point of view, pricing an Asian option is equivalent to find the fundamental solution for a degenerate and not uniformly parabolic two-dimensional partial differential equation.
Since the degenerate nature of this differential operator it is not possible to find an analytic expression for its fundamental solution, therefore in order to price Asian options there is a need to use numeric methods or anyway others approximation methods.
The basic idea of the approximation method developed in this work is to consider the Taylor series of the differential operator and then use it to approximate the solution searched.

In the second chapter, Arithmetic averaged Asian options are considered and it is provided explicitly a first order approximation for the density and the price of these.
To simplify the computations it has been natural working on the adjoint op-

erators and in the Fourier space.
Finally some numeric results obtained by our approximations are shown and confronted with the ones of others pricing methods present in literature.

The third chapter is whereas focused on the pricing of Harmonic averaged Asian options.
In contrast to the Arithmetic case, in the Harmonic case some unintended difficulties and problems have arisen. For this reason more approaches have been tried to get reasonable approximation of the prices.
As formerly, in the last part of this chapter numeric results are shown and tested against the Monte Carlo method. In this case our results appear less accurate than in the Arithmetic case.

Eventually in the fourth and last chapter it is proved a theoretic estimation for the error committed by our approximations of order zero and order one in the Arithmetic average case.
The idea is to modify and adapt the original Levi parametrix method to get an error bound for our approximations. The parametrix method allows to construct a fundamental solution for the differential operator considered starting from our approximating function. In this way in order to compute the error it is sufficient to estimate the difference between the fundamental solution constructed and our approximating function.

Chapter 1

PRICES OF ASIAN OPTIONS

1.1 Preliminaries

We begin this chapter giving some preliminary definitions and results: We introduce the Ito formula and the stochastic differential equations; then we show quickly the relationship between stochastic differential equations and partial differential equations. In particular for the method of approximation we will develop further in this chapter we are interested in the relationship between Kolmogorov operators and linear stochastic differential equations.

Definition 1.1 (Ito process). Let W be a d-dimensional Brownian motion, μ, σ respectively N dimensional and $N \times d$ dimensional stochastic process in $\mathbb{L}^2_{loc}$. A stochastic process X is a N-dimensional *Ito process* if it holds:

$$X_t \;=\; X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s$$

We denote the Ito process X_t also with the following notation:

$$dX_t \;=\; \mu_t \, dt + \sigma_t \, dW_t \tag{1.1}$$

We now introduce in the following theorem the fundamental Ito formula; for its proof see for example [1]

Theorem 1.1.1 (Ito formula for an Ito process).
$F(t,x) \in C^2(\mathbb{R} \times \mathbb{R}^N)$, X_t an Ito process in $\mathbb{R}^N$ (as in (1.1)); then $F(t, X_t)$ is an Ito process and it holds:

$$dF(t,X_t) \;=\; \partial_t F(t,X_t) \, dt + < \nabla_x F(t,X_t), dX_t > + \frac{1}{2} \sum_{i,j=1}^{N} \partial^2_{x_i,x_j} F(t,X_t) \, (\sigma_t \, \sigma_t^*)_{ij} \, dt$$

Definition 1.2 (**Stochastic differential equation (SDE)**).
Let be given a vector $z \in \mathbb{R}^N$,
a function $b\,:\,[0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ called *drift coefficient*,
and another function $\sigma\,:\,[0,T] \times \mathbb{R}^N \to \mathbb{R}^{N \times d}$ called *diffusion coefficient.*
Then let W be a d-dimensional Brownian motion on a filtered probability space $\Big(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)\Big)$.
If X_t is an adapted stochastic process on $\Big(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)\Big)$, we say it is a solution to the *stochastic differential equation (SDE)* with coefficients (z, b, σ) with respect to the Brownian motion W if:

1. $b(t, X_t), \sigma(t, X_t) \;\in \mathbb{L}^2_{loc}$

2. $X_t \;=\; z + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s$

Or equivalently we write:

$$dX_t \;=\; b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t\;, \quad X_0 = z$$

We say the SDE with coefficients (z, b, σ) has a solution in the strong sense if for any Brownian motion there exists a solution.
We say instead that the SDE with coefficients (z, b, σ) has a unique solution in the strong way if any two solutions X_t, Y_t with respect to the same Brownian motion are indistinguishable, that is: $\mathbb{P}\Big(X_t \;=\; Y_t\,, \forall\, t\Big) \;=\; 1$

Theorem 1.1.2. *We consider a SDE with coefficients (z, b, σ); if the following two conditions hold:*

1. $\forall n \in \mathbb{N}\, \exists\, k_n > 0$ *such that:*

   $|b(t,x) - b(t,y)|^2 + |\sigma(t,x) - \sigma(t,y)|^2 \;\le\; k_n\,|x-y|^2 \qquad \forall t \in [0,T],$

   $\forall\, x, y\,:\,|x|, |y| \le n$

2. $\exists\, K > 0\,:\,|b(t,x)|^2 + |\sigma(t,x)|^2 \;\le\; K\,(1 + |x|^2) \quad \forall\, x, t$

Then the SDE has a unique solution in the strong way and it is unique in the strong way

For the proof see for example [1]

Definition 1.3 (**Linear stochastic differential equations (LSDE)**).
A stochastic differential equation of the following form:

$$dX_t \;=\; \Big(b(t) + B(t)\,X_t\Big)\,dt + \sigma(t)\,dW_t \tag{1.2}$$

with $b(t)$ deterministic function in $\mathbb{R}^N$, $B(t)$ in $\mathbb{R}^{N\times N}$ and $\sigma(t)$ in $\mathbb{R}^{N\times d}$ is called *linear stochastic differential equation (LSDE)*

Remark 1. A Linear stochastic differential equation always satisfies the conditions 1 and 2 of Theorem 1.1.2 and then has a solution in the strong way and it is unique in the strong way.

Theorem 1.1.3. *The solution $X = X_T^{t,x}$ to the LSDE* (1.2) *with initial condition $x \in \mathbb{R}^N$ at time t is given explicitly by:*

$$X_T \;=\; \Phi(t,T)\left(x + \int_t^T \Phi^{-1}(t,\tau)\,b(\tau)d\tau + \int_t^T \Phi^{-1}(t,\tau)\,\sigma(\tau)\,dW_\tau\right) \tag{1.3}$$

Where $T \to \Phi(t,T)$ is the matrix-valued solution to the deterministic Cauchy problem:

$$\begin{cases} \frac{d}{dT}\Phi(t,T) \;=\; B(T)\,\Phi(t,T) \\ \Phi(t,t) \;=\; I_N \end{cases} \tag{1.4}$$

Moreover $X_T^{t,x}$ is a Gaussian process (has multi-normal distribution) with expectation:

$$m_{t,x}(T) = M(t,T,x) \;:=\; \Phi(t,T)\left(x + \int_t^T \Phi^{-1}(t,\tau)\,b(\tau)\,d\tau\right) \tag{1.5}$$

and covariance matrix:

$$C(t,T) \;:=\; \Phi(t,T)\left(\int_t^T \Phi^{-1}(t,\tau)\,\sigma(\tau)\left(\Phi(t,\tau)^{-1}\,\sigma(\tau)\right)^{*} d\tau\right)\Phi(t,T)^{*} \tag{1.6}$$

For proof see for example [1]

Remark 2. The covariance matrix in (1.6) of the solution $X_T^{t,x}$ to the LSDE (1.2) is symmetric and positive semi-defined.

Remark 3. Under the hypothesis:

$(D1)$ the covariance matrix $C(t,T)$ is positive definite for any $t < T$

The solution $X_T^{t,x}$ to the LSDE (1.2) with initial condition $x \in \mathbb{R}^N$ at time t has a transition density given by

$$\Gamma(t,x,T,y) \;=\; \frac{1}{\sqrt{(2\,\pi)^N\;\det C(t,T)}}\, e^{-\frac{1}{2}<C^{-1}(t,T)\,(y-m_{t,x}(T)),y-m_{t,x}(T)>} \tag{1.7}$$

This means that for fixed $x \in \mathbb{R}^N$ and $t < T$, the density of the random variable $X_T^{t,x}$ is: $\quad y \mapsto \Gamma(t,x,T,y)$

We examine now the deep connection between stochastic differential equations and partial differential equations.
We consider the following SDE in $\mathbb{R}^N$

$$dX_t \;=\; b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t \tag{1.8}$$

and we assume that:

1. the coefficients b, σ are measurable and have at most linear growth in x

2. for every $(t,x) \in (0,T) \times \mathbb{R}^N$ there exists a solution $X_T^{t,x}$ of the SDE (1.8) relative to a d-dimensional Brownian motion W on the space $\Big(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)\Big)$

We define then the *characteristic operator* of the SDE (1.8) as:

$$\mathcal{A} \;=\; \frac{1}{2}\sum_{i,j=1}^{N} c_{ij}\,\partial_{x_i x_j} + \sum_{i=1}^{N} b_i\,\partial_{x_i} \tag{1.9}$$

where $c_{ij} = (\sigma\,\sigma^*)_{ij}$
The **Feynman-Kac formula** provides us a representation formula for the classical solution of the following Cauchy problem

$$\begin{cases} \mathcal{A}u + \partial_t u \;=\; f & \text{in} S_T := (0,T) \times \mathbb{R}^N \\ u(T,x) \;=\; \phi(x) \end{cases} \tag{1.10}$$

where f, ϕ are given real functions.
More precisely is valid the following theorem

Theorem 1.1.4 (Feynman-Kac formula).
Let $u \in C^2(S_T) \cap C(\bar{S_T})$ be a solution of the Cauchy problem (1.10).
We assume that the hypothesis 1, 2 above hold and at least one of the following conditions are in force:

1. *there exist two positive constant M, p such that:*

$$|u(t,x)| + |f(t,x)| \leq M\,(1+|x|)^p \quad (t,x) \in S_T$$

2. *the matrix σ is bounded and there exist two positive constant M, α such that:*

$$|u(t,x)| + |f(t,x)| \leq M\, e^{\alpha\,|x|^2} \quad (t,x) \in S_T$$

Then for every (t,x) in S_T we have the representation formula:

$$u(t,x) = \mathbb{E}\left[\phi(X_T^{t,x}) - \int_t^T f(s, X_s^{t,x})\, ds\right]$$

For proof see [1]

Remark 4. We now observe that if the operator $\mathcal{A} + \partial_t$ has a fundamental solution $\Gamma(t,x,T,y)$ then, for every $\phi \in C_b(\mathbb{R}^N)$, the function

$$u(t,x) = \int_{\mathbb{R}^N} \phi(y)\, \Gamma(t,x,T,y)\, dy \tag{1.11}$$

is the classical solution of the Cauchy problem (1.10) with $f = 0$, and so, by Feynman-Kac formula:

$$\mathbb{E}\left[\phi(X_T^{t,x})\right] = \int_{\mathbb{R}^N} \phi(y)\, \Gamma(t,x,T,y)\, dy \tag{1.12}$$

By the arbitrariness of ϕ this means that, for fixed $x \in \mathbb{R}^N$ and $t < T$, the function $y \mapsto \Gamma(t,x,T,y)$ is the density of the random variable $X_T^{t,x}$; this means $\Gamma(t,x,T,y)$ is the transition density of $X_T^{t,x}$
Vice versa if the stochastic process $X_T^{t,x}$ solution of (1.8) has a transition density $\Gamma(t,x,T,y)$ then it is the fundamental solution for the operator $\mathcal{A}+\partial_t$

Definition 1.4 (**Kolmogorov operator**). Let us consider the linear SDE in $\mathbb{R}^N$

$$dX_t \;=\; (B(t)\,X_t + b(t))\;dt + \sigma(t)\,dW_t \tag{1.13}$$

with $b(t)$ deterministic function in $\mathbb{R}^N$, $B(t)$ in $\mathbb{R}^{N\times N}$ and $\sigma(t)$ in $\mathbb{R}^{N\times d}$ Then we say that the differential operator in $\mathbb{R}^{n+1}$

$$K \;=\; \frac{1}{2}\sum_{i,j=1}^{N} c_{ij}(t)\partial_{x_i x_j} + < b(t) + B(t)\,x\;,\;\nabla > + \partial_t \tag{1.14}$$

where $c_{ij} = (\sigma\,\sigma^*)_{ij}$, is the Kolmogorov operator associated with the linear SDE (1.13)

Remark 5. Under hypothesis $(D1)$, the solution $X_T^{t,x}$ of the linear SDE (1.13) has a transition density $\Gamma(t,x,T,y)$ (1.7); then it is the fundamental solution of the Kolmogorov operator (1.14) associated with the linear SDE

1.2 Approximation methodology

We consider a standard market model where there is a risky asset S following the stochastic equation:

$$dS_t = (r(t) - q(t))\,S_t\,dt + \sigma(t,S_t)\,S_t\,dW_t \tag{1.15}$$

Where $r(t)$ and $q(t)$ denote the risk-free rate and the dividend yield at time t respectively, σ is the local volatility function and W is a standard real Brownian motion.

The averaging prices for an Asian option are usually described by the additional state process:

$$A_t \;=\; f(t,S_t) \tag{1.16}$$

Or equivalently:

$$dA_t \;=\; df(t,S_t) \tag{1.17}$$

Where

$$f(t,S_t) = g\Big(\frac{1}{t-t_0}\int_{t_0}^{t} g^{-1}(S_u)\,du\Big) \tag{1.18}$$

with g suitably regular real function .
Varying the function g we can obtain the various averages:

1. If $g(x) := x$ then:

$$f(t, S_t) = \frac{1}{t - t_0} \int_{t_0}^{t} S_u \, d_u \qquad \textit{Arithmetic average of } S_t$$

2. If $g(x) := e^x$ then:

$$f(t, S_t) = e^{\frac{1}{t-t_0} \int_{t_0}^{t} \log S_u \, d_u} \qquad \textit{Geometric average of } S_t$$

3. If $g(x) := \frac{1}{x}$ then:

$$f(t, S_t) = \Big(\frac{1}{t - t_0} \int_{t_0}^{t} \frac{1}{S_u} \, d_u\Big)^{-1} \qquad \textit{Harmonic average of } S_t$$

In this work we will concentrate on Arithmetic average and Harmonic average

By usual no-arbitrage arguments, the price of an European Asian option with payoff function ϕ is given by:

$$V(t, S_t, A_t) = e^{-\int_t^T r(\tau)\, d\tau} \, u(t, S_t, A_t)$$

where

$$u(t, S_t, A_t) = \mathbb{E}\left[\phi(S_t, A_t) \,|\, S_t, A_t\right] \tag{1.19}$$

In this work we will consider the stationary case in which the coefficients r, q, σ are constants, even if this methodology can include a generic case. We will consider the following payoff functions:

$$\phi(S, A) = \Big(A_T - K\Big)^+ \qquad \textit{fixed strike arithmetic Call} \tag{1.20}$$

$$\phi(S, H) = \Big(H_T - K\Big)^+ \qquad \textit{fixed strike harmonic Call} \tag{1.21}$$

By the Feynman-Kac representation (Theorem 1.1.4), the price function u in (1.19) is the solution to the Cauchy problem

$$\begin{cases} L\,u(t,s,a) \;=\; 0 & t < T \;\; s,a \,\in\, \mathbb{R}^+ \\ u(T,s,a) \;=\; \phi(s,a) & s,a \,\in\, \mathbb{R}^+ \end{cases} \tag{1.22}$$

Where L is the characteristic operator of the stochastic differential equation

$$\begin{cases} dS_t = (r(t) - q(t))\, S_t\, dt + \sigma(t, S_t)\, S_t\, dW_t \\ dA_t \;=\; df(t, S_t) \end{cases} \tag{1.23}$$

Now reminding the expression of f in (1.18), by the Ito formula (Theorem 1.1.1) we have:

$$df(t, S_t) \;=\; h(t, S_t)\, dt \tag{1.24}$$

where

$$h(t, S_t) = g'\Big(\frac{1}{t-t_0}\int_{t_0}^{t} g^{-1}(S_u)\, du\Big)\Big(-\frac{1}{(t-t_0)^2}\int_{t_0}^{t} g^{-1}(S_u)\, du + \frac{1}{t-t_0}\, g^{-1}(S_t)\Big) \tag{1.25}$$

Then the operator L related to the SDE in (1.23) is

$$L \;=\; \frac{\sigma^2(t,s)\, s^2}{2}\,\partial_{ss} + \mu(t)\, s\, \partial_s + h(t,s)\,\partial_a + \,\partial_t \tag{1.26}$$

where $\mu \;=\; r - q$
L is a degenerate parabolic operator and under suitable regularity and growth conditions, there exists a unique solution to the Cauchy problem (1.22)

We are ready to describe the method that we will use in this work both in the cases of arithmetic average and harmonic average to approximate the price of an European Asian option. We now show the general method for the arithmetic average, but the same method can be applied in the case of harmonic average.
We have seen that to compute the price of the option we have to find the solution to the Cauchy problem (1.22).
In the arithmetic average case, we can then shift the normalization constant $\frac{1}{t-t_0}$ from the averaging function to the payoff function. Thus we have:

$$f(t, S_t) \;=\; \int_{t_0}^{t} S_u\, d_u \tag{1.27}$$

While the payoff becomes:

$$\phi(S, A) = \left(\frac{A}{T} - K\right)^{+} \qquad \textit{fixed strike arithmetic Call} \tag{1.28}$$

Then since $h(t, S_t) = S_t$ we have:

$$L = \frac{\alpha(t, s)}{2}\,\partial_{ss} + \mu(t)\,s\,\partial_s + s\,\partial_a + \partial_t \tag{1.29}$$

where $\alpha(t, s) = \sigma^2(t, s)\,s^2$

We assume that α is a suitable smooth, positive function and we take the Taylor expansion of $\alpha(t, \cdot)$ about $s_0 \in \mathbb{R}^+$; then formally we get

$$L = L_0 + \sum_{k=1}^{\infty} (s - s_0)^k\,\alpha_k(t)\,\partial_{ss} \tag{1.30}$$

where, setting $\alpha_0(t) = \alpha(t, s_0)$

$$L_0 = \frac{\alpha_0(t)}{2}\,\partial_{ss} + \mu(t)\,s\,\partial_s + s\,\partial_a + \partial_t \tag{1.31}$$

is the leading term in the approximation of L and

$\alpha_k(t) = \frac{1}{2\,k!}\,\partial_s^k \alpha(t, s_0)$

Remark 6. We remark now that L_0 is the Kolmogorov operator associated to the system

$$\begin{cases} dS_t = \mu(t)\,S_t\,dt + \sqrt{\alpha_0(t)}\,dW_t \\ dA_t = S_t\,dt \end{cases} \tag{1.32}$$

The SDE in (1.32) is a linear stochastic differential equation in $X_t = (S_t, A_t)$, with $b(t) = 0$,

$$B(t) = \begin{pmatrix} \mu(t) & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma(t) = \begin{pmatrix} \sqrt{\alpha_0(t)} \\ 0 \end{pmatrix}$$

Then under hypothesis $(D1)$ by remark 3 and 5 we can explicitly compute the transition density of the solution X_t:

$$\Gamma_0(t, s, a, T, S, A) = \frac{1}{\sqrt{(2\,\pi)^N\,\det C(t, T)}}\,e^{-\frac{1}{2} < C^{-1}(t,T)\,((S,A) - m_{t,s,a}(T)),(S,A) - m_{t,s,a}(T) >}$$

where $m_{t,s,a}(T)$ and $C(t,T)$ are respectively as in (1.5), (1.6)
Furthermore $\Gamma_0(t,s,a,T,S,A)$ is the fundamental solution of the operator L_0 in (1.31)

In conclusion we know explicitly the fundamental solution of the operator L_0 that is the approximation of order zero of the operator L.

We define

$$G_0(t,s,a,T,S,A) := \Gamma_0(t,s,a,T,S,A) \qquad t<T,\ \ s,a,S,A \in \mathbb{R} \tag{1.33}$$

and for $n \geq 1$, $G_n(t,s,a,T,S,A)$ is defined recursively in terms of the following sequence of Cauchy problems:

$$\begin{cases} L_0 G_n(t,s,a,T,S,A) \;=\; -\sum_{k=1}^{n} (s-s_0)^k\, \alpha_k(t)\, \partial_{ss}\, G_{n-k}(t,s,a,T,S,A) \\ \\ G_n(T,s,a,T,S,A) \;=\; 0 \end{cases} \tag{1.34}$$

Now we can construct the fundamental solution of the operator L, $\Gamma(t,s,a,T,S,A)$, summing all the functions $G_n(t,s,a,T,S,A)$:

$$\Gamma(t,s,a,T,S,A) \;=\; \sum_{n=1}^{\infty} G_n(t,s,a,T,S,A) \tag{1.35}$$

Indeed:

$$L\Gamma(t,s,a,T,S,A) \;=\; L_0\Gamma(t,s,a,T,S,A) + \sum_{k=1}^{\infty}(s-s_0)^k\, \alpha_k\, \partial_{ss}\Gamma(t,s,a,T,S,A)$$

$$= \; L_0\left(\sum_{n=0}^{\infty} G_n(t,s,a,T,S,A)\right) + \sum_{k=1}^{\infty}(s-s_0)^k\, \alpha_k\, \partial_{ss}\left(\sum_{n=0}^{\infty} G_n(t,s,a,T,S,A)\right)$$

$$= \; L_0G_0(t,s,a,T,S,A) + L_0G_1(t,s,a,T,S,A) + L_0G_2(t,s,a,T,S,A) + \ \dots$$
$$+\sum_{k=1}^{\infty}(s-s_0)^k\, \alpha_k\, \partial_{ss}\left(\sum_{n=0}^{\infty} G_n(t,s,a,T,S,A)\right) \;=$$
$$L_0G_0(t,s,a,T,S,A) + L_0G_1(t,s,a,T,S,A) + L_0G_2(t,s,a,T,S,A) + \ \dots$$
$$+(s-s_0)\,\alpha_1\,\partial_{ss}G_0(t,s,a,T,S,A) + (s-s_0)\,\alpha_1\,\partial_{ss}G_1(t,s,a,T,S,A) + \dots$$

$$+(s-s_0)^2\,\alpha_2\,\partial_{ss}G_0(t,s,a,T,S,A)+(s-s_0)^2\,\alpha_2\,\partial_{ss}G_1(t,s,a,T,S,A)+\dots$$
$$+(s-s_0)^3\,\alpha_3\,\partial_{ss}G_0(t,s,a,T,S,A)+(s-s_0)^3\,\alpha_3\,\partial_{ss}G_1(t,s,a,T,S,A)+\dots$$
$$\vdots$$

Summing along the diagonal and by definitions of G_n in (1.34) we get zero. Moreover

$$\Gamma(T,s,a,T,S,A)\;=\;\sum_{n=1}^{\infty}G_n(T,s,a,T,S,A)\;=\;G_0(T,s,a,T,S,A)\;=\;\delta_{s,a}(S,A)$$

In conclusion $\Gamma(t,s,a,T,S,A)$ is the fundamental solution of the operator L

Thus, by (1.35) the N-th order approximation of Γ is given by

$$\Gamma(t,s,a,T,S,A)\;\approx\;\sum_{n=0}^{N}G_n(t,s,a,T,S,A)\;=:\;\Gamma_N(t,s,a,T,S,A) \tag{1.36}$$

Moreover we have the following N-th order approximation formula for the price of an arithmetic Asian option with payoff function ϕ

$$u(t,S_t,A_t)\;=\;\int_{\mathbb{R}2}\Gamma(t,s,a,T,S,A)\,\phi(S,A)\,dS\,dA$$

$$\approx\int_{\mathbb{R}^2}\Gamma_N(t,s,a,T,S,A)\,\phi(S,A)dSdA=\int_{\mathbb{R}^2}\sum_{n=0}^{N}G_n(t,s,a,T,S,A)\phi(S,A)dSdA \tag{1.37}$$

Furthermore we will see in the next chapter that the various G_n, solutions to the Cauchy problem (1.34), will be written as a differential operator $J^n_{t,T,s,a}$ applied to $G_0=\Gamma_0$.
Thus the integral in (1.37) is equal to

$$\int_{\mathbb{R}^2}\sum_{n=0}^{N}J^n_{t,T,s,a}\big(\Gamma_0(t,s,a,T,S,A)\big)\phi(S,A)dSdA\;=\;\sum_{n=0}^{N}J^n_{t,T,s,a}\,C_0(t,s,a) \tag{1.38}$$

Where:

$$C_0(t,s,a,T)\;=\;\int_{\mathbb{R}^2}\Gamma_0(t,s,a,T,S,A)\,\phi(S,A)\,dS\,dA \tag{1.39}$$

Chapter 2

ARITHMETIC AVERAGE

2.1 Explicit first order computation

In this chapter we explicitly apply the method seen in the previous chapter to the arithmetic average case and we get some numeric results for the approximation of order $0, 1$.
For simplicity it is considered only the case $\mu(t)$ and $\sigma(t,s)$ constants even if our method can be applied also in the general case; we define furthermore the following notations: $x := (s,a)$, $y = (S,A)$.
We proceed now to find the functions G_n defined in (1.33) and (1.34).
The function G_0 is already known:

$$G_0(t,s,a,T,S,A) \,=\, \Gamma_0(t,s,a,T,S,A) \,=$$

$$\frac{1}{\sqrt{(2\,\pi)^N\ \det C(t,T)}}\, e^{-\frac{1}{2} < C^{-1}(t,T)\,((S,A)-m_{t,s,a}(T)),(S,A)-m_{t,s,a}(T)>}$$

where $m_{t,s,a}(T)$ and $C(t,T)$ respectively as in (1.5), (1.6)

We recall then that $G_0 \,=\, \Gamma_0$ is the fundamental solution of the operator L_0:

$$L_0 \,=\, \frac{\alpha_0}{2}\,\partial_{ss} + \mu\, s\, \partial_s + s\,\partial_a +\, \partial_t \tag{2.1}$$

where $\alpha_0 = \sigma^2\, s_0^2$
We define then

$$L_k \,:=\, \alpha_k\,(s - s_0)^k\,\partial_{ss} \tag{2.2}$$

where $\alpha_k \,=\, \frac{1}{2\,k!}\,\partial_s^k(\sigma^2\, s^2)|_{s_0}$

Thus for $n \geq 1$, G_n is the solution of the following Cauchy problem:

$$\begin{cases} L_0 G_n(t,s,a,T,S,A) \,=\, -\sum_{k=1}^{n} L_k\, G_{n-k}(t,s,a,T,S,A) \\ \\ G_n(T,s,a,T,S,A) \,=\, 0 \end{cases} \tag{2.3}$$

Applying the Fourier transform to the operator L_0 with respect to the variable (s,a) (in (ξ,φ)) we get:

$$\hat{L_0}\hat{u} \,=\, -\frac{\alpha_0}{2}\,\xi^2\,\hat{u} - \mu\,\xi\,\partial_\xi\hat{u} - \varphi\,\partial_\xi\hat{u} - \mu\,\hat{u} + \partial_t\hat{u} \tag{2.4}$$

In this way we have transformed a second order operator in an one order operator solvable using the method of characteristics.
Hence the idea to solve the Cauchy problems in (2.3) is to apply the Fourier transform to the problems and then use the method of characteristics to solve them:

$$\begin{cases} \hat{L_0}\hat{G_n}(t,\xi,\varphi,T,S,A) \,=\, -\sum_{k=1}^{n} \hat{L_k}\, \widehat{G_{n-k}}(t,\xi,\varphi,T,S,A) \\ \\ \hat{G_n}(T,\xi,\varphi,T,S,A) \,=\, 0 \end{cases} \tag{2.5}$$

Finally applying the inverse Fourier transforms to $\hat{G_n}$ we get the solutions to the original Cauchy problems (2.3), G_n

We note now that the function Γ_0 is a Gaussian function in the variables (S,A) while it isn't properly a Gaussian function in the variables (s,a) since:

$$m_{t,s,a}(T) \,=\, M(t,T,x) \,=\, \Phi(t,T)\,x$$

with $\Phi(t,T)$ defined in (1.4).
For this reason it would be much more easy to work and consequently transform with respect to the variables (S,A) instead of (s,a).

Remark 7. The Fourier transform of the function Γ_0 with respect to the variables (S,A) is:

$$\hat{\Gamma_0}(t,s,a,T,\xi,\varphi) \,=\, e^{i<M_{t,s,a}(T)\,,\,(\xi,\varphi)>-\frac{1}{2}<C(t,T)\,(\xi,\varphi)\,,\,(\xi,\varphi)>} \tag{2.6}$$

Moreover $\hat{\Gamma_0}$ is the characteristic function of a stochastic process with transition density Γ_0

Actually it is possible to work on the variables (S, A) instead of (s, a) using the adjoint operators.
Furthermore even using the adjoint operator, applying the Fourier transform we pass from a second order parabolic operator to a first order operator solvable using the method of characteristics.

Definition 2.1 (**Formal adjoint operator**). Let L be a linear differential operator:

$$L = \sum_{|\alpha|=0}^{n} A_\alpha(x)\, D_x^\alpha\, u \tag{2.7}$$

where α is a multi-indices and A_α are suitable regular functions in $\mathbb{R}$; the adjoint operator of L is the linear differential operator:

$$L^* = \sum_{|\alpha|=0}^{n} (-1)^{|\alpha|}\, D_x^\alpha\, (A_\alpha(x)\, u) \tag{2.8}$$

Remark 8. Let L be a linear differential operator, and L^* its adjoint operator; then for every $u, v \in C_0^\infty(\mathbb{R}^N)$, integrating by parts we obtain the following relation

$$\int_{\mathbb{R}^N} u\, Lv\, dx = \int_{\mathbb{R}^N} v\, L^* u\, dx$$

Remark 9. Let L_0 be as in (2.1), then L_0^* is

$$L_0^* = \frac{\alpha_0}{2}\, \partial_{ss} - \mu\, s\, \partial_s - s\, \partial_a - \mu - \partial_t \tag{2.9}$$

By a classical result (for instance, [2]) the fundamental solution $\Gamma_0(t, s, a, T, S, A)$ of L_0 is also the fundamental solution of L_0^* in the duals variables, that is:

$$\tilde{L_0} := L_0^{*(T,S,A)} = \frac{\alpha_0}{2}\, \partial_{SS} - \mu\, S\, \partial_S - S\, \partial_A - \mu - \partial_T \tag{2.10}$$

Theorem 2.1.1. *For any $k \geq 1$ and $(t, x) \in \mathbb{R}\times\mathbb{R}^2$, the function $G_n(t, x, \cdot, \cdot)$ in* (2.3) *is the solution of the following dual Cauchy problem on $]t, \infty[\times\mathbb{R}^2$*

$$\begin{cases} \tilde{L_0} G_n(t, x, T, y) = -\sum_{k=1}^{n} \tilde{L_k} G_{n-k}(t, x, T, y) \\ G_n(t, x, t, y) = 0 \end{cases} \tag{2.11}$$

where $\tilde{L_0}$ as in (2.10) *and $\tilde{L_k}$:*

$$\tilde{L_k} := L_k^{*(T,y)} = \alpha_k\, (S - s_0)^{k-2} \left(k\,(k-1) + 2\,k\,(S - s_0)\, \partial_S + (S - s_0)^2\, \partial_{SS} \right) \tag{2.12}$$

Proof. Since G_0 is the fundamental solution of the operator L_0, by the standard representation formula for the solution of the backward parabolic Cauchy problem (2.3), for $n \geq 1$ we have

$$G_n(t,x,T,y) \;=\; \sum_{k=1}^{n} \int_t^T \int_{\mathbb{R}^2} G_0(t,x,u,\eta)\, L_k^{(u,\eta)} G_{n-k}(u,\eta,T,y)\, du\, d\eta \quad (2.13)$$

Since G_0 is the fundamental solution also of the operator $\tilde{L_0}$, the assertion is equivalent to

$$G_n(t,x,T,y) \;=\; \sum_{k=1}^{n} \int_t^T \int_{\mathbb{R}^2} G_0(u,\eta,T,y)\, L_k^{*(u,\eta)} G_{n-k}(t,x,u,\eta)\, du\, d\eta \quad (2.14)$$

where here we have used the representation formula for the forward Cauchy problem (2.11) with $n \geq 1$
We proceed by induction and first prove (2.14) for $n = 1$. By (2.13) we have:

$$\begin{aligned} G_1(t,x,T,y) \;&=\; \int_t^T \int_{\mathbb{R}^2} G_0(t,x,u,\eta)\, L_1^{(u,\eta)} G_0(u,\eta,T,y)\, du\, d\eta \\ &= \int_t^T \int_{\mathbb{R}^2} G_0(u,\eta,T,y)\, L_1^{*(u,\eta)} G_0(t,x,u,\eta)\, du\, d\eta \end{aligned}$$

and this prove (2.14) for $n = 1$.
Next we assume that (2.14) holds for a generic $n > 1$ and we prove the thesis for $n+1$. Again, by (2.13) we have:

$$\begin{aligned} G_{n+1}(t,x,T,y) \;&=\; \sum_{k=1}^{n+1} \int_t^T \int_{\mathbb{R}^2} G_0(t,x,u,\eta)\, L_k^{(u,\eta)} G_{n+1-k}(u,\eta,T,y)\, du\, d\eta \\ &= \int_t^T \int_{\mathbb{R}^2} G_0(t,x,u,\eta)\, L_{n+1}^{(u,\eta)} G_0(u,\eta,T,y)\, du\, d\eta \\ &+ \sum_{k=1}^{n} \int_t^T \int_{\mathbb{R}^2} G_0(t,x,u,\eta)\, L_k^{(u,\eta)} G_{n+1-k}(u,\eta,T,y)\, du\, d\eta \end{aligned}$$

(by the inductive hypothesis)

$$\begin{aligned} &= \int_t^T \int_{\mathbb{R}^2} G_0(t,x,u,\eta)\, L_{n+1}^{(u,\eta)} G_0(u,\eta,T,y)\, du\, d\eta \\ &\qquad + \sum_{k=1}^{n} \int_t^T \int_{\mathbb{R}^2} G_0(t,x,u,\eta)\, L_k^{(u,\eta)} \cdot \end{aligned}$$

$$\Big(\sum_{h=1}^{n+1-k} \int_u^T \int_{\mathbb{R}^2} G_0(\tau,\varepsilon,T,y)\, L_h^{*(\tau,\varepsilon)} G_{n+1-k-h}(u,\eta,\tau,\varepsilon)\, d\tau\, d\varepsilon \Big)\, du\, d\eta$$

$$= \int_t^T \int_{\mathbb{R}^2} G_0(t,x,u,\eta)\, L_{n+1}^{(u,\eta)} G_0(u,\eta,T,y)\, du\, d\eta$$

$$+ \sum_{h=1}^{n} \sum_{k=1}^{n+1-h} \int_t^T \int_t^\tau \int_{\mathbb{R}^2\times\mathbb{R}^2} G_0(t,x,u,\eta)\, G_0(\tau,\varepsilon,T,y)\,\cdot$$

$$L_k^{(u,\eta)}\, L_h^{*(\tau,\varepsilon)} G_{n+1-k-h}(u,\eta,\tau,\varepsilon)\, d\eta\, d\varepsilon\, du\, d\tau$$

$$= \int_t^T \int_{\mathbb{R}^2} G_0(u,\eta,T,y)\, L_{n+1}^{*(u,\eta)} G_0(t,x,u,\eta)\, du\, d\eta$$

$$+ \sum_{h=1}^{n} \int_t^T \int_{\mathbb{R}^2} G_0(\tau,\varepsilon,T,y)\, L_h^{*(\tau,\varepsilon)}\cdot$$

$$\Big(\sum_{k=1}^{n+1-h} \int_t^\tau \int_{\mathbb{R}^2} G_0(t,x,u,\eta)\, L_k^{(u,\eta)}\, G_{n+1-k-h}(u,\eta,\tau,\varepsilon)\, d\eta\, du \Big)\, d\varepsilon\, d\tau \;=$$

(Again by (2.13))

$$= \int_t^T \int_{\mathbb{R}^2} G_0(u,\eta,T,y)\, L_{n+1}^{*(u,\eta)} G_0(t,x,u,\eta)\, du\, d\eta$$

$$+ \sum_{h=1}^{n} \int_t^T \int_{\mathbb{R}^2} G_0(\tau,\varepsilon,T,y)\, L_h^{*(\tau,\varepsilon)} G_{n+1-h}(t,x,\tau,\varepsilon)\, d\varepsilon\, d\tau$$

$$= \sum_{h=1}^{n+1} \int_t^T \int_{\mathbb{R}^2} G_0(\tau,\varepsilon,T,y)\, L_h^{*(\tau,\varepsilon)} G_{n+1-h}(t,x,\tau,\varepsilon)\, d\varepsilon\, d\tau$$

□

Remark 9 and Theorem 2.1.1 allow us to consider the duals variables (S,A) and apply the Fourier transform on these variables to find the functions G_n in (2.3) which we need to construct the approximation Γ_n of the fundamental solution Γ.
First of all we rewrite the operator L_0 and $\tilde{L_0}$ in vectorial form; this will

simplify the explicit computations we will do later.
Let us recall and introduce the following notations:

$$x = (s, a) \quad y = (S, A) \quad w = (\xi, \varphi) \quad B = \begin{pmatrix} \mu & 0 \\ 1 & 0 \end{pmatrix} \quad \sigma = \begin{pmatrix} \sqrt{\alpha_0} \\ 0 \end{pmatrix}$$

Then we have:

$$L_0\, u \;=\; \partial_t u + < \,B\, x\,,\, \nabla_x u > + \frac{1}{2} < \sigma\, \sigma^* \, \nabla_x u\,,\, \nabla_x u > \tag{2.15}$$

$$\tilde{L_0}\, u \;=\; -\partial_T u - < B\, y\,,\, \nabla_y u > + \frac{1}{2} < \sigma\, \sigma^* \, \nabla_y u\,,\, \nabla_y u > - tr(B)\, u \tag{2.16}$$

where $tr(B)$ is the trace of the matrix B.
Applying the Fourier transform to the operator $\tilde{L_0}$ with respect to the variable y we get:

$$K_0^{(T,w)}\, \hat{u} \;=\; -\partial_T\, \hat{u} + < B^*\, w\,,\, \nabla_w \hat{u} > - \frac{1}{2} < \sigma\, \sigma^*\, w\,,\, w > \,\hat{u} \tag{2.17}$$

While for $k \geq 1$ the Fourier transforms of the operators $\tilde{L_n}$ in (2.12) with respect to the variable y are:

$$K_n^{(T,w)}\, \hat{u} := \; F(\tilde{L_n})\, \hat{u} \;=$$
$$\alpha_n \Big(n\,(n-1)\,(-i\,\partial_\xi - s_0)^{n-2}\, \hat{u} + 2\, n\, (-i\,\partial_\xi - s_0)^{n-1}\, (-i\, \xi\, \hat{u}) + (-i\,\partial_\xi - s_0)^{n}\, (-\xi^2\, \hat{u}) \Big) \tag{2.18}$$

We now proceed to solve the first Cauchy problem aimed at find $\hat{G}_1(t, x, T, w)$

$$\begin{cases} K_0^{(T,w)}\, \hat{G}_1(t, x, T, w) \;=\; -K_1^{(T,w)}\, \hat{G}_0(t, x, T, w) \\ \hat{G}_1(t, x, t, y) \;=\; 0 \end{cases} \tag{2.19}$$

we recall

$$\hat{G}_0(t, x, T, w) = \hat{\Gamma}_0(t, x, T, w) \;=\; e^{i < m_{t,x}(T)\,,\, w> - \frac{1}{2} < C(t,T)\, w\,,\, w>} \tag{2.20}$$

with $C(t, T)$ as in (1.6) and $m_{t,x}(T) = \Phi(t, T) \cdot x$ with $\Phi(t, T)$ defined in (1.4).

The equation of the Cauchy problem (2.19) is equivalent to the following:

$$\partial_T\, \hat{G}_1 - < B^*\, w\,,\, \nabla_w \hat{G}_1 > + \frac{1}{2} < \sigma\, \sigma^*\, w\,,\, w > \, \hat{G}_1 \;=\; K_1^{(T,w)}\, \hat{G}_0 \tag{2.21}$$

We now find the characteristic curve solving the Cauchy problem:

$$\begin{cases} T'(\tau) = 1 \\ w(\tau)' \;=\; -B^* \, w(\tau) \\ \\ T(t) \;=\; 0 \\ w(t) \;=\; z \end{cases} \tag{2.22}$$

where $z \in \mathbb{R}^2$.
Thus we have $T = \tau$ and $w(\tau) = w(T, z) = e^{(t-T)\, B^*} \, z$.
Then along the characteristic curve $w(T, z)$ we have:

$$\frac{d}{dT} \hat{G}_1(t, x, T, w(T, z)) \;=\; \partial_T \, \hat{G}_1(t, x, T, w(T, z)) + \nabla_w \, \hat{G}_1(t, x, T, w(T, z)) \, \partial_T \, w(T, z)$$

$$= \partial_T \, \hat{G}_1(t, x, T, w(T, z)) - < B^* \, w(T, z) \, , \, \nabla_w \hat{G}_1(t, x, T, w(T, z)) >$$

Hence to solve the equation (2.21) we have to solve:

$$\begin{aligned} \frac{d}{dT} \, \hat{G}_1(t, x, T, w(T, z)) \;=\; & -\frac{1}{2} < \sigma \, \sigma^* \, w(T, z) \, , \, w(T, z) > \, \hat{G}_1(t, x, T, w(T, z)) \\ & + \left(K_1^{(T,w)} \, \hat{G}_0 \right) (t, x, T, w(T, z)) \end{aligned} \tag{2.23}$$

Which is an ordinary equation of the first order in one variable.
The solution along the characteristic curve is the following:

$$\hat{G}_1(t, x, T, w(T, z)) \;=\; \int_t^T e^{\frac{1}{2} \int_T^\tau < \sigma \, \sigma^* \, w(\theta, z) \, , \, w(\theta, z) > \, d\theta} \left(K_1 \, \hat{G}_0 \right) (t, x, \tau, w(\tau, z)) \, d\tau \tag{2.24}$$

Now to get $\hat{G}_1(t, x, T, w)$ we have to invert $w(T, z)$ with respect to z:
$z = e^{(T-t)\, B^*} \, w$
and substitute in $w(T, z)$ the expression found for z:

$$w(\tau, z(T, w)) \;=\; e^{(T-\tau)\, B^*} \, w \;=: \gamma_{(T,w)}(\tau) \tag{2.25}$$

In conclusion:

$$\hat{G}_1(t, x, T, w) \;=\; \int_t^T e^{\frac{1}{2} \int_T^\tau < \sigma \, \sigma^* \, \gamma_{(T,w)}(\theta) \, , \, \gamma_{(T,w)}(\theta) > \, d\theta} \left(K_1 \, \hat{G}_0 \right) (t, x, \tau, \gamma_{(T,w)}(\tau)) \, d\tau \tag{2.26}$$

Remark 10. Since we have assumed μ constant, also the matrix B is constant and then the solution Φ to the Cauchy problem:

$$\begin{cases} \frac{d}{dT}\Phi(t,T) \;=\; B\,\Phi(t,T) \\ \\ \Phi(t,t) \;=\; I_N \end{cases} \tag{2.27}$$

is given by:

$$\Phi(t,T) \;=\; e^{(T-t)\,B} \tag{2.28}$$

Consequently we have:

$$M_{t,s,a}(T) \;=\; \Phi(t,T)\cdot x \;=\; e^{(T-t)\,B}\cdot x \tag{2.29}$$

Remark 11.

$$\hat{G}_0(t,x,\tau,\gamma_{(T,w)}(\tau)) \;=\; e^{i\,<m_{t,x}(\tau)\,,\,\gamma_{(T,w)}(\tau)>-\frac{1}{2}<C(t,\tau)\,\gamma_{(T,w)}(\tau)\,,\,\gamma_{(T,w)}(\tau)>}$$

$$=\; e^{i\,<e^{(\tau-t)\,B}\cdot x\,,\,e^{(T-\tau)\,B^*}\,w>-\frac{1}{2}<C(t,\tau)\,e^{(T-\tau)\,B^*}\,w\,,\,e^{(T-\tau)\,B^*}\,w>}$$

$$=\; e^{i\,<e^{(T-t)\,B}\cdot x\,,\,w>-\frac{1}{2}<e^{(T-t)\,B}\left(\int_t^\tau \Phi^{-1}(t,\theta)\,\sigma\left(\Phi^{-1}(t,\theta)\,\sigma\right)^*\,d\theta\right)e^{(T-t)\,B^*}\,w\,,\,w>}$$

$$=\; e^{i\,<m_{t,x}(T)\,,\,w>-\frac{1}{2}\int_t^\tau<e^{(T-\theta)\,B}\,\sigma\,\sigma^*\,e^{(T-\theta)\,B^*}\,w\,,\,w>\,d\theta}$$

$$=\; e^{i\,<m_{t,x}(T)\,,\,w>-\frac{1}{2}\int_t^\tau<\sigma\,\sigma^*\,\gamma_{(T,w)}(\theta)\,,\,\gamma_{(T,w)}(\theta)>\,d\theta}$$

We proceed in the following way:

1. We show that for $n \geq 1$, $K_n\,\hat{G}_0 \;=\; \hat{G}_0\,p(\xi,\varphi)$, where $p(\xi,\varphi)$ is a polynomial function.

2. We show that $\hat{G}_1(t,x,T,w) \;=\; \hat{G}_0(t,x,T,w)\,\tilde{p}(\xi,\varphi)$, where $\tilde{p}(\xi,\varphi)$ is still a polynomial function

3. We apply to $\hat{G}_0(t,x,T,w)\,\tilde{p}(\xi,\varphi)$ the inverse Fourier transform with respect to w and we get $G_1(t,x,T,y) \;=\; \tilde{J^1}_{t,T,S,A}\,G_0(t,x,T,y)$ where $\tilde{J^1}_{t,T,S,A}$ is a differential operator in the variables (S,A).

To proceed with step one we need the following proposition:

Proposition 2.1.2. *We consider the function $e^{f(x)}$ with $f \in C^\infty(\mathbb{R},\mathbb{R})$ and such that $\partial_x^{(k)} f(x) = 0$ for $k > 2$;*
then is valid the following formula:

$$\partial_x^{(k)} e^{f(x)} = e^{f(x)} \sum_{n=0}^{\frac{k}{2}} \binom{k}{k-2\,n} (2\,n-1)!!\, (\partial_x f(x))^{k-2\,n}\, (\partial_{xx} f(x))^n \quad (2.30)$$

Proof. The thesis is proved by induction:
($k = 1$)

$$\partial_x^{(1)} e^{f(x)} = e^{f(x)} f'(x) = e^{f(x)} \binom{1}{1} (-1)!!\, (\partial_x f(x))^1$$

($k > 1$)
We assume the thesis holds for k and we prove it for $k+1$.
We assume furthermore k even, the case k odd can be proved in the same way.

$$\partial_x^{(k+1)} e^{f(x)} = \partial_x \left(\partial_x^{(k)} e^{f(x)} \right) =$$

(by inductive hypothesis)

$$= \partial_x \left(e^{f(x)} \sum_{n=0}^{\frac{k}{2}} \binom{k}{k-2\,n} (2\,n-1)!!\, (\partial_x f(x))^{k-2\,n}\, (\partial_{xx} f(x))^n \right)$$

$$= e^{f(x)} \sum_{n=0}^{\frac{k}{2}} \binom{k}{k-2\,n} (2\,n-1)!!\, (\partial_x f(x))^{k+1-2\,n}\, (\partial_{xx} f(x))^n$$

$$+ e^{f(x)} \sum_{n=0}^{\frac{k}{2}-1} \binom{k}{k-2\,n} (2\,n-1)!!\, (k-2\,n)\, (\partial_x f(x))^{k-2\,n-1}\, (\partial_{xx} f(x))^{n+1}$$

$$+ e^{f(x)} \sum_{n=1}^{\frac{k}{2}} \binom{k}{k-2\,n} (2\,n-1)!!\, (\partial_x f(x))^{k-2\,n}\, n\, (\partial_{xx} f(x))^{n-1}\, (\partial_x^{(3)} f(x))$$

(since $(\partial_x^{(3)} f(x)) = 0$)

$$= e^{f(x)} \sum_{n=0}^{\frac{k}{2}} \binom{k}{k-2\,n} (2\,n-1)!!\, (\partial_x f(x))^{k+1-2\,n}\, (\partial_{xx} f(x))^n$$

$$+e^{f(x)}\sum_{n=0}^{\frac{k}{2}-1}\binom{k}{k-2\,n}(2\,n-1)!!\,(k-2\,n)\,(\partial_x f(x))^{k-2\,n-1}\,(\partial_{xx}f(x))^{n+1}$$

$$=\ e^{f(x)}\left[\sum_{n=0}^{\frac{k}{2}}\binom{k}{k-2\,n}(2\,n-1)!!\,(\partial_x f(x))^{k+1-2\,n}\,(\partial_{xx}f(x))^{n}\right.$$

$$\left.+\sum_{n=1}^{\frac{k}{2}}\binom{k}{k-2\,(n-1)}(2\,(n-1)-1)!!\,(k-2\,(n-1))\,(\partial_x f(x))^{k-2\,(n-1)-1}\,(\partial_{xx}f(x))^{n}\right]$$

$$=\ e^{f(x)}\left[(\partial_x f(x))^{k+1}+\sum_{n=1}^{\frac{k}{2}}\binom{k}{k-2\,n}(2\,n-1)!!\,(\partial_x f(x))^{k+1-2\,n}\,(\partial_{xx}f(x))^{n}\right.$$

$$\left.+\sum_{n=1}^{\frac{k}{2}}\binom{k}{k-2\,(n-1)}(2\,n-3)!!\,(k-2\,(n-1))\,(\partial_x f(x))^{k+1-2\,n}\,(\partial_{xx}f(x))^{n}\right]$$

$$=\ e^{f(x)}\left[\sum_{n=1}^{\frac{k}{2}}\left(\binom{k}{k-2\,n}(2\,n-1)!!+\binom{k}{k-2\,(n-1)}(2\,n-3)!!(k-2\,n+2)\right)\right.$$

$$\left.\cdot\,(\partial_x f(x))^{k+1-2\,n}\,(\partial_{xx}f(x))^{n}\right]+e^{f(x)}\,(\partial_x f(x))^{k+1} \qquad (\star)$$

Observing that

$$\binom{k}{k-2\,n}(2\,n-1)!!+\binom{k}{k-2\,(n-1)}(2\,n-3)!!(k-2\,n+2)\ =$$

$$(2\,n-3)!!\left((2\,n-1)\,\frac{k!}{(k-2\,n)!\,(2\,n)!}+\frac{k!\,(k+2-2\,n)}{(k+2-2\,n)!\,(2\,n-2)!}\right)$$

$$=\ \frac{(2\,n-1)!!}{(2\,n-1)}\left((2\,n-1)\,\frac{k!}{(k-2\,n)!\,(2\,n)!}+\frac{k!}{(k+1-2\,n)!\,(2\,n-2)!}\right)$$

$$=\ \frac{(2\,n-1)!!}{(2\,n-1)}\left(\frac{(2\,n-1)\,(k+1-2\,n)\,k!+(2\,n)\,(2\,n-1)\,k!}{(k+1-2\,n)!\,(2\,n)!}\right)$$

$$=\ (2\,n-1)!!\,\frac{k!\,(k+1-2\,n+2\,n)}{(k+1-2\,n)!\,(2\,n)!}\ =\ (2\,n-1)!!\binom{k+1}{k+1-2\,n}$$

we get:

$$(\star)\ =\ e^{f(x)}\left[\,(\partial_x f(x))^{k+1}+\sum_{n=1}^{\frac{k}{2}}\binom{k+1}{k+1-2\,n}(2\,n-1)!!\,(\partial_x f(x))^{k+1-2\,n}\,(\partial_{xx}f(x))^{n}\right]$$

$\Big($since k is even and consequently $\sum_{n=1}^{\frac{k}{2}} = \sum_{n=1}^{\frac{k+1}{2}}$ $\Big)$

$$= e^{f(x)} \sum_{n=0}^{\frac{k+1}{2}} \binom{k+1}{k+1-2\,n} (2\,n-1)!!\,(\partial_x f(x))^{k+1-2\,n}\,(\partial_{xx} f(x))^n$$

And the thesis results proved $\square$

Remark 12. We define

$$f(t,x,T,w) \;:=\; i\; < m_{t,x}(T)\,,\; w > -\frac{1}{2}\; < C(t,T)\,w\,,\; w > \tag{2.31}$$

Then we have:

$$\hat{G}_0(t,x,T,w) \;=\; e^{f(t,x,T,w)}$$

and:

$$\partial_\xi\, f(t,x,T,w) \;=\; i\;(m_{t,x}(T))_1 - (C(t,T)\cdot w)_1$$

$$\partial_\xi^{(2)}\, f(t,x,T,w) \;=\; -C_{11}(t,T)$$

$$\partial_\xi^{(k)}\, f(t,x,T,w) \;=\; 0 \qquad \text{for } k>2$$

Hence Proposition 2.0.2 yields:

$$\partial_\xi^{(k)}\,\hat{G}_0(t,x,T,w) \;=\; \hat{G}_0(t,x,T,w) \sum_{n=0}^{\frac{k}{2}} \binom{k}{k-2\,n} (2\,n-1)!!$$
$$\cdot\,(\partial_\xi f(t,x,T,w))^{k-2\,n}\,(\partial_\xi^{(2)} f(t,x,T,w))^n$$

Defining:

$$N^{(j)}(f) \;:=\; \sum_{n=0}^{\frac{j}{2}} \binom{j}{j-2\,n} (2\,n-1)!!\,(\partial_\xi f)^{j-2\,n}\,(\partial_{\xi\xi} f)^n \tag{2.32}$$

we can use the following notation:

$$\partial_\xi^{(j)}\,\hat{G}_0(t,x,T,w) \;=\; \hat{G}_0(t,x,T,w)\,N^{(j)}\Big(f(t,x,T,w)\Big) \tag{2.33}$$

Theorem 2.1.3. *For all $n \geq 1$ it holds:*

$$K_n^{(T,w)}\,\hat{G}_0(t,x,T,w) \;=\; \hat{G}_0(t,x,T,w)\,\alpha_n \Bigg(n\,(n-1)\,H_1^{n-2}\Big(f(t,x,T,w)\Big)$$

$$+2\,n\,H_2^{n-1}\Big(\xi, f(t,x,T,w)\Big) + H_3^{n}\Big(\xi, f(t,x,T,w)\Big)\Bigg)$$

with $f(t,x,T,w)$ *defined in* (2.31) *and*

$$H_1^k\Big(f(t,x,T,w)\Big) := \sum_{j=0}^{k}\binom{k}{j}(-i)^j\,(-s_0)^{k-j}\,N^{(j)}\Big(f(t,x,T,w)\Big)$$

$$H_2^k\Big(\xi, f(t,x,T,w)\Big) := \sum_{j=0}^{k}\binom{k}{j}(-i)^{j+1}\,(-s_0)^{k-j}\,\Bigg[\xi\,N^{(j)}\Big(f(t,x,T,w)\Big) + j\,N^{(j-1)}\Big(f(t,x,T,w)\Big)\Bigg]$$

$$H_3^k\Big(\xi, f(t,x,T,w)\Big) := -\sum_{j=0}^{k}\binom{k}{j}(-i)^{j}\,(-s_0)^{k-j}\,\Bigg[\xi^2\,N^{(j)}\Big(f(t,x,T,w)\Big) + 2\,j\,\xi\,N^{(j-1)}\Big(f(t,x,T,w)\Big)\Bigg]$$

where $N^{(j)}(f)$ *as in* (2.32)

Proof. It follow directly by definition of $K_n^{(T,w)}$, the Binomial theorem, Proposition 2.1.2 and Remark 12 □

Finally we define

$$\tilde{K}_n(t,x,T,w) := \alpha_n\left(n\,(n-1)\,H_1^{n-2}\Big(f(t,x,T,w)\Big) + 2\,n\,H_2^{n-1}\Big(\xi, f(t,x,T,w)\Big) + H_3^{n}\Big(\xi, f(t,x,T,w)\Big)\right) \tag{2.34}$$

in order to have:

$$K_n^{(T,w)}\,\hat{G}_0(t,x,T,w) \;=\; \hat{G}_0(t,x,T,w)\,\tilde{K}_n(t,x,T,w) \tag{2.35}$$

We are ready to proceed with step 2 and compute $\hat{G}_1(t, x, T, w)$.
In (2.26) we had:

$$\hat{G}_1(t,x,T,w) = \int_t^T e^{\frac{1}{2}\int_t^\tau <\sigma\,\sigma^*\,\gamma_{(T,w)}(\theta)\,,\,\gamma_{(T,w)}(\theta)>\,d\theta}\Big(K_1\,\hat{G}_0\Big)(t,x,\tau,\gamma_{(T,w)}(\tau))\,d\tau$$

(by Theorem 2.1.3)

$$= \int_t^T e^{\frac{1}{2}\int_t^\tau <\sigma\,\sigma^*\,\gamma_{(T,w)}(\theta)\,,\,\gamma_{(T,w)}(\theta)>\,d\theta}\,\hat{G}_0(t,x,\tau,\gamma_{(T,w)}(\tau))\,\tilde{K}_1(t,x,\tau,\gamma_{(T,w)}(\tau))\,d\tau$$

(by Remark 11)

$$= \int_t^T e^{i<m_{t,x}(T)\,,\,w>-\frac{1}{2}\int_t^T<\sigma\,\sigma^*\,\gamma_{(T,w)}(\theta)\,,\,\gamma_{(T,w)}(\theta)>\,d\theta}\,\tilde{K}_1(t,x,\tau,\gamma_{(T,w)}(\tau))\,d\tau$$

$$= \int_t^T e^{i<m_{t,x}(T)\,,\,w>-\frac{1}{2}<C(t,T)\,w\,,\,w>}\,\tilde{K}_1(t,x,\tau,\gamma_{(T,w)}(\tau))\,d\tau$$

$$= \hat{G}_0(t,x,T,w)\int_t^T \tilde{K}_1(t,x,\tau,\gamma_{(T,w)}(\tau))\,d\tau$$

$$= \hat{G}_0(t,x,T,w)\int_t^T \alpha_1\Big(-2\,i\,\gamma_1 + H_3^{(1)}\Big(\gamma_1, f(t,x,\tau,\gamma_{(T,w)}(\tau))\Big)\Big)d\tau$$

where γ_1 is a shorten notation for $\Big(\gamma_{(T,w)}(\tau)\Big)_1 = \Big(e^{(T-\tau)\,B^*}\cdot w\Big)_1$
We conclude step 2 remarking that by definition of $\gamma_1, f(t,x,T,w)$ and $H_3^{(1)}$,

$$\int_t^T \alpha_1\Big(-2\,i\,\gamma_1 + H_3^{(1)}\Big(\gamma_1, f(t,x,\tau,\gamma_{(T,w)}(\tau))\Big)\Big)d\tau$$

is a polynomial functions in the variables $(\xi,\varphi)=w$. In particular it holds:

$$\int_t^T \alpha_1\Big(-2\,i\,\gamma_1 + H_3^{(1)}\Big(\gamma_1, f(t,x,\tau,\gamma_{(T,w)}(\tau))\Big)\Big)d\tau =$$

$$\alpha_1\int_t^T -2\,i\,\gamma_1 + s_0\,\gamma_1^2 + 2\,i\,\gamma_1 + i\,\gamma_1^2\Big(i\,(m_{t,x}(\tau))_1 - (C(t,\tau)\cdot\gamma_{T,w}(\tau))_1\Big)d\tau$$

$$= \alpha_1\int_t^T \Big(s_0 - (m_{t,x}(\tau))_1\Big)\,\gamma_1^2 - i\,\Big((C(t,\tau)\cdot\gamma_{T,w}(\tau))_1\Big)\gamma_1^2\,d\tau$$

In conclusion:

$$\begin{aligned}\hat{G}_1(t,x,T,w) = &\alpha_1 \int_t^T \Big(s_0 - (m_{t,x}(\tau))_1\Big)\, \gamma_1^2\, \hat{G}_0(t,x,T,w) \\ &- i\Big(\big(C(t,\tau)\cdot \gamma_{T,w}(\tau)\big)_1\Big)\gamma_1^2\, \hat{G}_0(t,x,T,w)\, d\tau\end{aligned} \tag{2.36}$$

We have now to apply the inverse Fourier transform with respect to w to obtain $G_1(t,x,T,y)$

$$\gamma_1\, \hat{G}_0(t,x,T,w) \;=\; \Big(e^{(T-\tau)\,B^*}\cdot w\Big)_1\, \hat{G}_0(t,x,T,w)$$

$$= \Big(e^{(T-\tau)\,B^*}\Big)_{11}\, \xi\; \hat{G}_0(t,x,T,w) + \Big(e^{(T-\tau)\,B^*}\Big)_{12}\, \varphi\; \hat{G}_0(t,x,T,w)$$

Then applying the inverse Fourier transform:

$$\begin{aligned}\mathrm{F}^{-1}\Big(\gamma_1\, \hat{G}_0(t,x,T,w)\Big) \;=&\; i\Big(e^{(T-\tau)\,B^*}\Big)_{11}\, \partial_S\, G_0(t,x,T,y) \\ &+ i\Big(e^{(T-\tau)\,B^*}\Big)_{12}\, \partial_A\, G_0(t,x,T,y)\end{aligned}$$

Thus defining the following differential operator on the variables (S,A)

$$V_y(T,\tau) \;:=\; i\Big(e^{(T-\tau)\,B^*}\Big)_{11}\, \partial_S \;+\; i\Big(e^{(T-\tau)\,B^*}\Big)_{12}\, \partial_A \tag{2.37}$$

We can write:

$$\mathrm{F}^{-1}\Big(\gamma_1\, \hat{G}_0(t,x,T,w)\Big) \;=\; V_y(T,\tau)\, G_0(t,x,T,y)$$

Consequently:

$$\mathrm{F}^{-1}\Big(\gamma_1^2\, \hat{G}_0(t,x,T,w)\Big) \;=\; V_y(T,\tau)\, V_y(T,\tau)\, G_0(t,x,T,y) \;=$$

$$\begin{aligned}&-\Big(e^{(T-\tau)\,B^*}\Big)_{11}^2\, \partial_{SS}\, G_0(t,x,T,y) - \Big(e^{(T-\tau)\,B^*}\Big)_{12}^2\, \partial_{AA}\, G_0(t,x,T,y) \\ &-2\Big(e^{(T-\tau)\,B^*}\Big)_{11}\Big(e^{(T-\tau)\,B^*}\Big)_{12}\, \partial_{SA}\, G_0(t,x,T,y)\end{aligned}$$

On the other hand

$$\Big((C(t,\tau)\cdot\gamma_{T,w}(\tau))_1\Big)\gamma_1^2\,\hat{G}_0(t,x,T,w) \;=\; C_{11}(t,\tau)\,\gamma_1^3\,\hat{G}_0(t,x,T,w)$$

$$+C_{12}(t,\tau)\,\gamma_1^2\,\gamma_2\,\hat{G}_0(t,x,T,w)$$

where γ_2 is a is a shorten notation for $\Big(\gamma_{(T,w)}(\tau)\Big)_2$
Defining the following operator on the variables (S,A)

$$W_y(T,\tau) \;=\; i\left(e^{(T-\tau)\,B^*}\right)_{21}\partial_S \;+\; i\left(e^{(T-\tau)\,B^*}\right)_{22}\partial_A \tag{2.38}$$

it holds:

$$\mathrm{F}^{-1}\left((C(t,\tau)\cdot\gamma_{T,w}(\tau))_1\gamma_1^2\,\hat{G}_0(t,x,T,w)\right)=$$

$$C_{11}(t,\tau)\,V_y^3(T,\tau)\,G_0(t,x,T,y)+C_{12}(t,\tau)\,V_y^2(T,\tau)\,W_y(T,\tau)\,G_0(t,x,T,y)$$

Finally we are able to write explicitly the function $G_1(t,x,T,y)$

$$G_1(t,x,T,y) \;=\; \mathrm{F}^{-1}\Big(\hat{G}_1(t,x,T,w)\Big) \;=\; \alpha_1\int_t^T \Big(s_0-(m_{t,x}(\tau))_1\Big)$$

$$\cdot\,\mathrm{F}^{-1}\Big(\gamma_1^2\,\hat{G}_0(t,x,T,w)\Big) - i\,\mathrm{F}^{-1}\Big((C(t,\tau)\cdot\gamma_{T,w}(\tau))_1\gamma_1^2\,\hat{G}_0(t,x,T,w)\Big)\,d\tau$$

$$=\;\alpha_1\int_t^T \Big(s_0-(m_{t,x}(\tau))_1\Big)\,V_y^2(T,\tau)\,G_0(t,x,T,y)-i\,C_{11}(t,\tau)\,V_y^3(T,\tau)\,G_0(t,x,T,y)$$

$$-i\,C_{12}(t,\tau)\,V_y^2(T,\tau)\,W_y(T,\tau)\,G_0(t,x,T,y)$$

In conclusion we got the first order approximation of $\Gamma(t,s,a,T,S,A)$:

$$\Gamma_1(t,s,a,T,S,A) \;=\; G_0(t,s,a,T,S,A)+G_1(t,s,a,T,S,A)$$

Furthermore it holds:

$$\Gamma_1(t,s,a,T,S,A) \;=\; G_0(t,s,a,T,S,A)+\tilde{J}^1{}_{t,T,S,A}\,G_0(t,s,a,T,S,A) \tag{2.39}$$

Where $\tilde{J}^1{}_{t,T,S,A}$ is the differential operator:

$$\begin{aligned}\tilde{J}^1{}_{t,T,S,A} =& \alpha_1 \int_t^T \Big(s_0 - (m_{t,x}(\tau))_1\Big)\, V_y^2(T,\tau) \,-\, i\, C_{11}(t,\tau)\, V_y^3(T,\tau) \\ & -\, i\, C_{12}(t,\tau)\, V_y^2(T,\tau)\, W_y(T,\tau)\, d\tau\end{aligned} \tag{2.40}$$

Carrying on the computations we could go ahead in the same way with the higher orders approximation of $\Gamma(t,s,a,T,S,A)$

The first order approximation formula for the price of an arithmetic Asian option with payoff function ϕ is then given by:

$$u(t,S_t,A_t) \,\approx\, \int_{\mathbb{R}^2} \Big[\Big(1 + \tilde{J}^1{}_{t,T,S,A}\Big)\, \Gamma_0(t,s,a,T,S,A)\Big]\, \phi(S,A)\, dS\, dA \tag{2.41}$$

It is now convenient to transform the operator $\tilde{J}^1{}_{t,T,S,A}$ in an equivalent operator on the variables (s,a) in order to moving the differential operator outside the integral and in this way simplify significantly the computations. Since $\tilde{J}^1{}_{t,T,S,A}$ is applied on Γ_0, we can substitute it with an equivalent operator on the variables (s,a) by the following remark

Remark 13. For every $t < T$, $s,a,S,A \in \mathbb{R}$ it holds:

$$\nabla_{(S,A)}\, \Gamma_0(t,s,a,T,S,A) \,=\, \Big(-\, J^*_{m_{t,s,a}(T)}\Big)^{-1}\, \nabla_{(s,a)}\, \Gamma_0(t,s,a,T,S,A) \tag{2.42}$$

where $J_{m_{t,s,a}(T)}$ is the Jacobian matrix with respect to (s,a) of the function $m_{t,s,a}(T)$.

Indeed reminding

$$\Gamma_0(t,s,a,T,S,A) \,=\, \frac{1}{\sqrt{(2\,\pi)^N\, \det C(t,T)}}\, e^{-\frac{1}{2} < C^{-1}(t,T)\,((S,A) - m_{t,s,a}(T)),(S,A) - m_{t,s,a}(T) >}$$

we have:

$$\nabla_y\, \Gamma_0(t,x,T,y) \,=\, \Gamma_0(t,x,T,y)\, \Big(-\, C^{-1}(t,T)\, \Big(y - m_{t,x}(T)\Big)\Big)$$

$$\nabla_x\, \Gamma_0(t,x,T,y) \,=\, \Gamma_0(t,x,T,y)\, \Big(-\, C^{-1}(t,T)\, \Big(y - m_{t,x}(T)\Big)\Big) \cdot \Big(-\, J_{m_{t,x}(T)}\Big)$$

Therefore defining the following differential operators:

$$\begin{aligned} DS_{(s,a)}\, u \;&=< \Big(-\, J^{*}_{m_{t,s,a}(T)} \Big)^{-1} \nabla_{(s,a)}\, u\,,\, e_1 > \\ DA_{(s,a)}\, u \;&=< \Big(-\, J^{*}_{m_{t,s,a}(T)} \Big)^{-1} \nabla_{(s,a)}\, u\,,\, e_2 > \end{aligned} \tag{2.43}$$

where e_1, e_2 are respectively the vectors $(1,0)$ and $(0,1)$, we can substitute the derivatives of Γ_0 with respect to (S,A), with derivatives of Γ_0 with respect to (s,a) in the following way:

$$\begin{aligned} \partial_S\, \Gamma_0(t,s,a,T,S,A) \;&=\; DS_{(s,a)}\, \Gamma_0(t,s,a,T,S,A) \\ \partial_A\, \Gamma_0(t,s,a,T,S,A) \;&=\; DA_{(s,a)}\, \Gamma_0(t,s,a,T,S,A) \end{aligned} \tag{2.44}$$

Setting the relations above in the expression of V_y and W_y we get

$$\begin{aligned} \tilde{V}_x(T,\tau) \;&:=\; i \left(e^{(T-\tau)\,B^*} \right)_{11} DS_{(s,a)} \;+\; i \left(e^{(T-\tau)\,B^*} \right)_{12} DA_{(s,a)} \\ \tilde{W}_x(T,\tau) \;&:=\; i \left(e^{(T-\tau)\,B^*} \right)_{21} DS_{(s,a)} \;+\; i \left(e^{(T-\tau)\,B^*} \right)_{22} DA_{(s,a)} \end{aligned} \tag{2.45}$$

and

$$\begin{aligned} \tilde{V}_x(T,\tau)\, \Gamma_0(t,s,a,T,S,A) \;&=\; V_y(T,\tau)\, \Gamma_0(t,s,a,T,S,A) \\ \tilde{W}_x(T,\tau)\, \Gamma_0(t,s,a,T,S,A) \;&=\; W_y(T,\tau)\, \Gamma_0(t,s,a,T,S,A) \end{aligned} \tag{2.46}$$

Then substituting $V_y(T,\tau)$ and $W_y(T,\tau)$ in $\tilde{J}^1{}_{t,T,S,A}$ (2.40) respectively with $\tilde{V}_x(T,\tau)$ and $\tilde{W}_x(T,\tau)$ we get a differential operator $J^1_{t,T,s,a}$ on the variables (s,a) such that:

$$\tilde{J}^1{}_{t,T,S,A}\, \Gamma_0(t,s,a,T,S,A) \;=\; J^1_{t,T,s,a}\, \Gamma_0(t,s,a,T,S,A) \tag{2.47}$$

In conclusion the approximation of the price in (2.41) is equivalent to:

$$\begin{aligned} &\int_{\mathbb{R}^2} \Big[\Big(1 + J^1_{t,T,s,a} \Big)\, \Gamma_0(t,s,a,T,S,A) \Big]\, \phi(S,A)\, dS\, dA \\ &= \Big(1 + J^1_{t,T,s,a} \Big) \int_{\mathbb{R}^2} \Gamma_0(t,s,a,T,S,A)\, \phi(S,A)\, dS\, dA \\ &\qquad = C_0(t,s,a) + J^1_{t,T,s,a} \Big(C_0(t,s,a) \Big) \end{aligned}$$

with $C_0(t,s,a,T)$ as in (1.39):

$$C_0(t,s,a,T) \;=\; \int_{\mathbb{R}^2} \Gamma_0(t,s,a,T,S,A)\, \phi(S,A)\, dS\, dA$$

2.2 Numeric results

In this section we show some numeric results for the price of an Asian option obtained with the method described above.
The numeric results have been obtained using the computational software program *mathematica* to carry on the computations showed in the precedent section.
We simply provide results related to the approximations of order zero or one since for those of second order we haven't be able to nullify the numeric error. For this reason, with respect to the approximation of second order, we got numeric results which appeared less accurate than the ones obtained by the first order approximation.
The approximation of the prices function $u(t, S_t, A_t)$ we found in the precedent section is still a function on the variables t, s and a; hence we have to assign this values to get a price.
The case $t = 0$ can be considered without losing generality; furthermore it has been seen that $s = s_0$ and $a = a_0$ is a very convenient choice that allows to get very accurate results.
While s_0 is a free parameter, a_0 is constricted to be zero by the definition of the process A_t in (1.27)
As already said the payoff function used for the arithmetic Asian option is the following:

$$\phi(S, A) = \left(\frac{A}{T} - K\right)^+$$

where K is a free parameter.

In the following tables our approximation formulae are compared with other various methods:
Second order and third order approximation of Foschi, Pagliarani, Pascucci in [3] (FPP2 and FPP3), the method Linetsky in [4], the PDE method of Vecer in [5] and the matched asymptotic expansions of Dewynne and Shaw in [6] (MAE3 and MAE5).

Table 2.1 reports the interest rate r, the volatility σ, the time to maturity T, the strike K and the initial asset price s_0 for seven cases that will be tested.
In this first set of tests a null dividend rate is assumed: $q = 0$.

Table 2.1: Parameter values for seven test cases

Case	s_0	K	r	σ	T
1	2	2	0.02	0.1	1
2	2	2	0.18	0.3	1
3	2	2	0.0125	0.25	2
4	1.9	2	0.05	0.5	1
5	2	2	0.05	0.5	1
6	2.1	2	0.05	0.5	1
7	2	2	0.05	0.5	2

Table 2.2: Asian Call option prices when q=0 (parameters as in Table 2.1)

Case	Order zero	Order one	FPP3	Linetsky	Vecer	MAE 3
1	0.0560415	0.0559965	0.05598604	0.05598604	0.055986	0.05598596
2	0.219607	0.218589	0.21838706	0.21838755	0.218388	0.21836866
3	0.172939	0.172738	0.17226694	0.17226874	0.172269	0.17226265
4	0.188417	0.194468	0.19316359	0.19317379	0.193174	0.19318824
5	0.248277	0.247714	0.24640562	0.24641569	0.246416	0.24638175
6	0.314568	0.307579	0.30620974	0.30622036	0.306220	0.30613888
7	0.355167	0.35361	0.35003972	0.35009522	0.350095	0.34990862

In the following table the same seven tests are repeated with a dividend rate equal to the interest rate. In this case the results of Linetsky and Vecer are not reported: the former because these tests were not considered in his paper; the latter because Vecer's code cannot deal with that special case.

Table 2.3: Asian call option when q=r (parameters as in Table 2.1)

Case	Order zero	Order one	FPP2	FPP3	MAE3	MAE5
1	0.045153	0.045153	0.045143	0.045143	0.045143	0.045143
2	0.115432	0.115432	0.115188	0.115188	0.115188	0.115188
3	0.158846	0.158846	0.158378	0.158378	0.158378	0.158380
4	0.164030	0.170494	0.169238	0.169192	0.169238	0.169201
5	0.219096	0.219096	0.217805	0.217805	0.217805	0.217815
6	0.280735	0.274251	0.272868	0.272914	0.272869	0.272925
7	0.294737	0.294737	0.291263	0.291263	0.291264	0.291316

Next we tested our method with a low-volatility parameter $\sigma = 0.01$. Tables 2.4 shows the performances of the approximations against Monte Carlo 95% confidence intervals. These intervals are computed using 500000 Monte Carlo replication and an Euler discretization with 300 time-step for $T = 0.25$ and $T = 1$ and 1500 time-step for $T = 5$. In these experiments the initial asset level is $s_0 = 100$, the interest rate is $r = 0.05$ and the dividend yield is null $q = 0$. Table 2.5 shows the results of the methods FPP3, Vecer and MAE3 for the same parameters.

Table 2.4: Tests with low volatility: $\sigma = 0.01$, $s_0 = 100$, $r = 0.05$ and $q = 0$

T	K	Order zero	Order one	Euler-Monte Carlo method	
0.25	99	1.60739×10^0	1.60739×10^0	1.60849×10^0	1.61008×10^0
0.25	100	6.21366×10^{-1}	6.21359×10^{-1}	6.22333×10^{-1}	6.23908×10^{-1}
0.25	101	1.3492×10^{-2}	1.37603×10^{-2}	1.39301×10^{-2}	1.42436×10^{-2}
1.00	97	5.27190×10^0	5.27190×10^0	5.27670×10^0	5.27985×10^{-0}
1.00	100	2.41821×10^0	2.41821×10^0	2.42451×10^0	2.42767×10^0
1.00	103	6.99018×10^{-2}	7.2416×10^{-2}	7.44026×10^{-2}	7.54593×10^{-2}
5.00	80	2.61756×10^1	2.61756×10^1	2.61775×10^1	2.61840×10^1
5.00	100	1.05996×10^1	1.05996×10^1	1.06040×10^1	1.06105×10^0
5.00	120	7.80248×10^-7	2.55831×10^-6	1.41956×10^{-6}	1.38366×10^{-5}

Table 2.5: Tests with low volatility: $\sigma = 0.01$, $s_0 = 100$, $r = 0.05$ and $q = 0$

T	K	FPP3	Vecer	MAE3
0.25	99	1.60739×10^0	-4.18937×10^1	1.60739×10^0
0.25	100	6.21359×10^{-1}	5.40466×10^{-1}	6.21359×10^{-1}
0.25	101	1.37618×10^{-2}	-3.96014×10^{-2}	1.37615×10^{-2}
1.00	97	5.27190×10^0	-9.73504×10^0	5.27190×10^0
1.00	100	2.41821×10^0	2.37512×10^0	2.41821×10^0
1.00	103	7.26910×10^{-2}	7.25478×10^{-2}	7.24337×10^{-2}
5.00	80	2.61756×10^1	2.52779×10^1	2.61756×10^1
5.00	100	1.05996×10^1	1.05993×10^1	1.05996×10^1
5.00	120	2.06699×10^-5	1.07085×10^-5	5.73317×10^{-6}

Chapter 3

HARMONIC AVERAGE

3.1 Strategies tried

We consider a risky asset S following the equation in (1.15):

$$dS_t \;=\; \mu(t)\, S_t\, dt + \sigma(t, S_t)\, S_t\, dW_t \tag{3.1}$$

where $\mu(t) = r(t) - q(t)$, is the difference between the risk-free rate and the dividend yield at time t, σ is the local volatility function and W is a standard real Brownian motion.
Now we consider as averaging price for the Asian option the state process given by the harmonic average:

$$H_t \;=\; \Big(\frac{1}{t - t_0} \int_{t_0}^{t} \frac{1}{S_u}\, du\Big)^{-1} \tag{3.2}$$

For notational simplicity, we assume the starting time t_0 is equal to zero.
We also assume that μ and σ are constants and we try to compute the price for an Asian option using the method seen in Chapter 1.
We have tried in different ways to arrive to this result, facing many obstacles and problems; eventually we got the results searched, even if in this case we obtained less accurate numeric results with respect to the arithmetic average case.

The first idea has been to describe the process (S_t, H_t) through a system of stochastic equations similar to the one that we got in the arithmetic average case.

We can do this shifting the constant of normalization $\frac{1}{t}$ from the average to the payoff function and defining the following stochastic processes:

$$X_t = \frac{1}{S_t} \qquad \text{and} \qquad Y_t = \frac{1}{H_t} \tag{3.3}$$

Then using the Ito formula (Theorem 1.1.1) we have:

$$\begin{aligned} dX_t &= d\left(\frac{1}{S_t}\right) = -\frac{1}{S_t^2}\,dS_t + \frac{1}{2}\,\frac{2}{S_t}\,\sigma^2\,S_t^2\,dt = \frac{1}{S_t}\,(\sigma^2-\mu)\,dt - \frac{1}{S_t}\,\sigma\,dW_t \\ &= (\sigma^2-\mu)\,X_t\,dt - \sigma\,X_t\,dW_t \end{aligned}$$

And

$$dY_t = d\left(\int_0^t X_u\,du\right) = X_t\,dt$$

Then the stochastic process is described by the following equations

$$\begin{aligned} dX_t &= (\sigma^2-\mu)\,X_t\,dt - \sigma\,X_t\,dW_t \\ dY_t &= X_t\,dt \end{aligned} \tag{3.4}$$

The characteristic operator of this SDE is:

$$L = \frac{1}{2}\,\sigma^2\,x^2\,\partial_{xx} + (\sigma^2-\mu)\,x\,\partial_x + x\,\partial_y + \partial_t \tag{3.5}$$

This partial differential equation is then analogous to that we had in the case of the arithmetic average.
So we could get an approximation of its fundamental solution simply repeating the same computations we have seen for the arithmetic average.
Despite that, we still couldn't get an estimation for the price of the Asian option. Indeed we have seen that if $\Gamma(t,s,a,T,S,A)$ is the fundamental solution for the characteristic operator of the SDE describing the process, and $\phi(S,A)$ is the payoff function, then the price is given by

$$v(t,T,s,a) = e^{-r\,(T-t)} \int_{\mathbb{R}^2} \Gamma(t,s,a,T,S,A)\,\phi(S,A)\,dS\,dA \tag{3.6}$$

Now we recall that for the harmonic average we consider the following payoff function:

$$\phi(S,A) = (H_T - K)^+ \tag{3.7}$$

In this case we have:

$$\phi(X,Y) = \left(\frac{T}{Y} - K\right)^+ \tag{3.8}$$

So since ϕ has a non-integrable singularity in $Y = 0$ and we have seen in Chapters 1 and 2 that our approximation $\Gamma_N(t, x, y, T, X, Y)$ of the fundamental solution for an operator like L is a Gaussian function in the variables (X, Y), we have that $\Gamma_N(t, x, y, T, X, Y)\, \phi(X, Y)$ diverge to infinity in $Y = 0$ and it is not integrable on $\mathbb{R}^2$.
In conclusion we can't use the processes X_t, Y_t in (3.3) to compute the price of an Harmonic averaged Asian option.

We have then tried to get an integrable payoff function without singularity using directly the dynamic of H_t instead of Y_t. In this case however, we haven't be able to shift the normalization factor $\frac{1}{t}$ from the average to the payoff function. Indeed if we consider only $H_t = \left(\int_0^t \frac{1}{S_u}\, du \right)^{-1}$, then the starting point H_0 isn't defined since $\int_0^0 \frac{1}{S_u}\, du = 0$.
On the contrary if we consider $H_t = \left(\frac{1}{t} \int_0^t \frac{1}{S_u}\, du \right)^{-1}$, then we can assume $H_0 = S_0$ since $\frac{1}{t} \int_0^t \frac{1}{S_u}\, du \to S_0^{-1}$ for $t \to 0$ under suitable regularity hypothesis.

Thus, starting again from the processes X_t, Y_t in (3.3), we have computed the dynamic of H_t:

$$dH_t = d\left(\frac{1}{Y_t}\right) = -\frac{1}{Y_t^2}\, dY_t$$

Where

$$dY_t = d\left(\frac{1}{t}\int_0^t X_u\, du\right) = -\frac{1}{t^2}\left(\int_0^t X_u\, du\right) dt + \frac{1}{t}\, X_t\, dt = \frac{1}{t}\, (X_t\, dt - Y_t\, dt)$$

Hence

$$dH_t = -\frac{1}{Y_t^2}\,\frac{1}{t}\, (X_t\, dt - Y_t\, dt) = \frac{1}{t}\, \left(H_t - H_t^2\, X_t\right)\, dt$$

Then the operator related to the processes (X_t, H_t) is:

$$L = \frac{1}{2}\,\sigma^2\, x^2\, \partial_{xx} + (\sigma^2 - \mu)\, x\, \partial_x + \frac{1}{t}\, (h - h^2\, x)\, \partial_h + \partial_t \tag{3.9}$$

Now we can proceed with our method computing as usual the Taylor series of $\sigma^2\, x^2$ with respect to the point x_0, and in addition, of h^2 with respect to the point h_0. In this way the Taylor expansion of L at order zero is given by

$$L_0 = \frac{1}{2}\,\sigma^2\, x_0^2\, \partial_{xx} + (\sigma^2 - \mu)\, x\, \partial_x + \frac{1}{t}\, h\, \partial_h - \frac{1}{t}\, h_0^2\, x\, \partial_h + \partial_t \tag{3.10}$$

Thus the approximation of order zero of the fundamental solution for L is given by the fundamental solution of L_0, which is the Kolmogorov operator associated to the linear system

$$d \begin{pmatrix} X_t^0 \\ H_t^0 \end{pmatrix} = \begin{pmatrix} \sigma^2 - \mu & 0 \\ -\frac{1}{t}\, h_0^2 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} X_t^0 \\ H_t^0 \end{pmatrix} dt + \begin{pmatrix} \sigma\, x_0 \\ 0 \end{pmatrix} dW_t \tag{3.11}$$

Then the fundamental solution for the operator L_0 is given by the transition density of the process (X_t^0, H_t^0).
In this case we haven't been able to compute explicitly the transition density, though.
Indeed the function $\phi(t, T)$ solution of the Cauchy problem (1.4) that is necessary to compute the expectation and the covariance matrix of the process (X_t^0, H_t^0) doesn't have an analytic expression.
In conclusion we haven't been able to get an analytic expression for the transition density of the process.

An attempt to bypass this problem has been to consider the dynamic of X_t as in (3.3) and $H_t = \left(\int_0^t X_t\, dt\right)^{-1}$ even if in this case H_0 isn't defined. By the Ito formula, the process is described by the stochastic equation

$$\begin{aligned} dX_t &= (\sigma^2 - \mu)\, X_t\, dt - \sigma\, X_t\, dW_t \\ dH_t &= -X_t\, H_t^2\, dt \end{aligned} \tag{3.12}$$

Then its characteristic operator is:

$$L = \frac{1}{2}\sigma^2\, x^2\, \partial_{xx} + (\sigma^2 - \mu)\, x\, \partial_x - h^2\, x\, \partial_h + \partial_t \tag{3.13}$$

And so computing again the Taylor series of $\sigma^2\, x^2$ with respect to the point x_0, and of h^2 with respect to the point h_0 we get:

$$L_0 = \frac{1}{2}\sigma^2\, x_0^2\, \partial_{xx} + (\sigma^2 - \mu)\, x\, \partial_x - h_0^2\, x\, \partial_h + \partial_t \tag{3.14}$$

In this case we can explicitly compute the transition density of the linear system associated to L_0 getting hence its fundamental solution Γ_0. Thus we can approximate the price computing explicitly the integral in (3.6) with Γ_0 instead of Γ.
To solve the problem of the non-definition of H_0 we have tried to assign to it large values and we have seen the trend of the numeric results as H_0 was larger. This results didn't converge to a sensible solution but remained extremely low, even increasingly consistently the value of H_0. In conclusion

they were totally disconnected from the ones we got using the Monte Carlo method.

Another attempt to compute the price of an Asian option with respect to the harmonic average has been to consider the following variable change

$$\begin{aligned} X_t &:= \log\left(\frac{S_t}{S_0}\right) \\ Z_t &:= \log Y_t = \log\frac{1}{H_t} = \log\frac{1}{t}\int_0^t \frac{1}{S_u}\,du \end{aligned} \tag{3.15}$$

Then by the Ito formula we have

$$dX_t = \frac{S_0}{S_t}\,dS_t + \frac{1}{2}\left(-\frac{S_0}{S_t^2}\,\sigma^2\,S_t^2\,dt\right) = (\mu - \frac{\sigma^2}{2})\,S_0\,dt + \sigma\,S_0\,dW_t$$

$$dY_t = d\left(\frac{1}{t}\int_0^t \frac{e^{-X_u}}{S_0}\,du\right) = -\frac{1}{t}\,Y_t\,dt + \frac{1}{t}\,\frac{e^{-X_t}}{S_0}\,dt$$

$$dZ_t = \frac{1}{Y_t}\,dt = -\frac{1}{t}\,dt + \frac{1}{t}\,e^{Z_0}\,e^{-X_t - Z_t}\,dt$$

Where $Z_0 = \log Y_0 = -\log S_0$; furthermore the payoff function is given by

$$\phi(X, Z) = \left(e^{-Z} - K\right)^+$$

Which together a Gaussian function is integrable on $\mathbb{R}^2$
The characteristic operator of the dynamic is then:

$$L = \frac{\sigma^2\,S_0^2}{2}\,\partial_{xx} + (\mu - \frac{\sigma^2}{2})\,S_0\,\partial_x - \frac{1}{t}\,\partial_z + \frac{e^{z_0 - x - z}}{t}\,\partial_z + \partial_t \tag{3.16}$$

Now we can consider as L_0 the following

$$L_0 = \frac{\sigma^2\,S_0^2}{2}\,\partial_{xx} + (\mu - \frac{\sigma^2}{2})\,S_0\,\partial_x - \frac{1}{t}\,\partial_z + \frac{e^{z_0 - x_0 - z_0}}{t}\,\partial_z + \partial_t \tag{3.17}$$

But in this case, since there isn't the term in $x\,\partial_z$ the covariance matrix of the stochastic process related isn't invertible and then it's transition density haven't got an analytic expression.
We can however approximate the term e^{-x-z} with $e^{-z0}\,(1-x)$ by the Taylor expansion of the exponential function; thus as L_0 we can take:

$$L_0 = \frac{\sigma^2\,S_0^2}{2}\,\partial_{xx} + (\mu - \frac{\sigma^2}{2})\,S_0\,\partial_x - \frac{1}{t}\,\partial_z + \frac{1}{t}\,(1 - x)\,\partial_z + \partial_t \tag{3.18}$$

Then we can compute explicitly its fundamental solution $\Gamma_0(t, x, z, T, X, Z)$ and thus approximate the price computing the integral in (3.6) with Γ_0 instead of Γ.
However the numeric results we have gotten haven't been sensible. Indeed first of all we note that the equation L_0 degenerate as t goes to zero, and so does Γ_0 and its integral with the payoff function.
Furthermore even for t not close to zero our results aren't comparable with the ones expected as shows the following table:

Table 3.1: Asian Call option prices when $t = 1, T = 2, q = 0, K = 2$

S_0	2	2	2	1.9	2	2.1
r	0.02	0.18	0.0125	0.05	0.05	0.05
σ	0.1	0.3	0.25	0.5	0.5	0.5
Price: Our Method	2.047	2.451	2.020	2.073	2.093	2.114
Price: Monte Carlo Method	0.025	0.084	0.049	0.075	0.082	0.089

Certainly to obtain the operator L_0 in (3.18) we have approximated an exponential coefficient with a linear coefficient in x and this surely make us lose some accuracy but not enough to justify the results obtained in Table 3.1.
Thus there must be also another reason that explain our incorrect results.
We have then understood that our attempt was wrong since we were using a Gaussian function Γ_0 to approximate a process H_t that had a Log-normal distribution.

Therefore we have tried to change our approximation strategy to get sensible results.
Thus we have started again from:

$$\begin{aligned} dS_t &= \mu\, S_t\, dt + \sigma\, S_t\, dW_t \\ H_t &= \left(\int_0^t \frac{1}{S_u}\, du \right)^{-1} \end{aligned} \tag{3.19}$$

We have assumed for simplicity $\mu = 0$ and considered:

$$X_t := \frac{1}{S_t} \quad , \quad Y_t := e^{\int_0^t X_u\, du} \tag{3.20}$$

Then by the Ito formula we have:

$$dX_t = \sigma^2 X_t\, dt - \sigma X_t\, dW_t$$

$$dY_t = Y_t X_t\, dt$$

$X_0 = \frac{1}{S_0}$ and $Y_0 = 1$, while the payoff function is:

$$\phi(X, Y) = \left(\frac{T}{\log Y} - K\right)^+ \tag{3.21}$$

The characteristic operator then is the following:

$$L = \frac{1}{2}\sigma^2 x^2 \partial_{xx} + \sigma^2 x\, \partial_x + x\, y\, \partial_y + \partial_t \tag{3.22}$$

Expanding as usual the term $\sigma^2 x^2$ by Taylor with respect to the point x_0, we have that the leading term of the Series is:

$$L_0 = \frac{1}{2}\sigma^2 x_0^2\, \partial_{xx} + \sigma^2 x\, \partial_x + x\, y\, \partial_y + \partial_t \tag{3.23}$$

We remark now that the operator L_0 is a different kind of operator with respect to the other operators considered so far. In particular it isn't a Kolmogorov operator related to a linear SDE.
Despite this we are able to compute explicitly its fundamental solution in the following way:
Let us define the following operator:

$$\tilde{L_0} = \frac{1}{2}\sigma^2 x_0^2\, \partial_{xx} + \sigma^2 x\, \partial_x + x\, \partial_y + \partial_t \tag{3.24}$$

This is a Kolmogorov operator and thus we can easily find its fundamental solution. Furthermore it is also easy to verify that if $\tilde{\Gamma_0}(t, x, y, T, X, Y)$ is its fundamental solution then

$$\Gamma_0(t, x, y, T, X, Y) := \tilde{\Gamma_0}(t, x, \log y, T, X, \log Y) \tag{3.25}$$

is the fundamental solution for the operator L_0 in (3.23). Moreover since $\tilde{\Gamma_0}$ is a Gaussian function in the variable X and Y, the function Γ_0 isn't a Gaussian function in Y but it is a Log-normal.
Anyway even in this case our approximation Γ_0 together the payoff function ϕ in (3.21) isn't integrable on $\mathbb{R}^2$; indeed the payoff function has a singularity in $Y = 1$ which isn't removed by the fundamental solution Γ_0.

As another attempt to apply our method to the harmonic average we have considered

$$X_t = \frac{1}{S_t} \qquad , \qquad A_t = \int_0^t X_u \, du$$

Then we have considered the process:

$$y_t := (A_t)^p \qquad p \in \mathbb{N} \tag{3.26}$$

Hence the price of an Asian option is described by the following process:

$$\begin{cases} dX_t = (\sigma^2 - \mu) \, X_t \, dt - \sigma \, X_t \, dW_t \\ dY_t = p \, (A_t)^{p-1} \, dA_t = p \, (Y_t)^{\frac{p-1}{p}} \, X_t \, dt \end{cases} \tag{3.27}$$

Then we have the following characteristic operator related:

$$L = \frac{1}{2} \sigma^2 \, x^2 \, \partial_{xx} + (\sigma^2 - \mu) \, x \, \partial_x + p \, y^{\frac{p-1}{p}} \, x \, \partial_y + \partial_t \tag{3.28}$$

And as usual L_0 is

$$L_0 = \frac{1}{2} \sigma^2 \, x_0^2 \, \partial_{xx} + (\sigma^2 - \mu) \, x \, \partial_x + p \, y^{\frac{p-1}{p}} \, x \, \partial_y + \partial_t \tag{3.29}$$

Now for p different from one we can't find its fundamental solution since it isn't a Kolmogorov operator. We have thought anyway that if p is large enough maybe we can approximate the term $p \, y^{1-\frac{1}{p}} \, x \, \partial_y$ with $p \, y \, x \, \partial_y$

So we have decided to study the following equation and see its behaviour as p goes to infinity.

$$L_p = \partial_{xx} + p \, x \, y \, \partial_y + \partial_t \tag{3.30}$$

We remark now that, as we have already seen, we are able to compute explicitly the fundamental solution of L_p.
Indeed if $\tilde{\Gamma}(t, x, y, T, X, Y)$ is the fundamental solution for the operator:

$$\tilde{L_p} := \partial_{xx} + p \, x \, \partial_y + \partial_t$$

Then

$$\Gamma_p(t, x, y, T, X, Y) := \tilde{\Gamma}(t, x, \log y, T, X, \log Y) \tag{3.31}$$

is the fundamental solution for the operator L_p.
The payoff function in this case is given by:

$$\phi(X, Y) = \left(\frac{T}{y^{\frac{1}{p}}} - K \right)^+ \tag{3.32}$$

ϕ has a singularity in $Y = 0$ but it is removed by Γ_p since it has a log-normal distribution in Y and it converges to zero more rapidly as Y tend to zero; in conclusion the integral on $\mathbb{R}^2$ of the payoff function with Γ_p is finite.
However we have a first problem given by the fact that for definition of the process Y_t, it must be $Y_0 = 0$ and the fundamental solution Γ_p is identically null for $y = 0$.
Thus we have decided to try with other initial dates in y and we have seen the behaviour of the integral

$$\int_{\mathbb{R}^2} \Gamma_p(t, x, y, T, X, Y) \left(\frac{T}{y^{\frac{1}{p}}} - K \right)^+ dX\, dY \tag{3.33}$$

as p growths to infinity and y approaches 0.

We fixed $t = 0$, $T = 1$, $x_0 = 2$, $K = 2$ and we obtained the following table:

Table 3.2: Numeric results of the integral (1.36)

	$p = 10$	$p = 20$	$p = 100$	$p = 1000$
$y = 1$	$1.898 \cdot 10^{-11}$	$8.49 \cdot 10^{-15}$	$5.38 \cdot 10^{-40}$	≈ 0
$y = 0.1$	$2.22 \cdot 10^{-10}$	$8.49 \cdot 2.95^{-14}$	$6.9 \cdot 10^{-40}$	≈ 0
$y = 0.01$	$1.9 \cdot 10^{-9}$	$9 \cdot 10^{-14}$	$8.83 \cdot 10^{-40}$	≈ 0
$y = 10^{-10}$	$7.5 \cdot 10^{-6}$	$8.6 \cdot 10^{-11}$	$5.67 \cdot 10^{-39}$	≈ 0

The table shows that both for p that growths to infinity and y that approaches 0, the fundamental solution Γ_p converge to the null function. This is confirmed even by the direct computation of the limit of $\Gamma_p(t, x, y, T, X, Y)$ for $p \to \infty$, that is zero independently by the other variables.
In conclusion even this last approach to the problem hasn't produced results.

As last attempt we have tried then to consider again the operator L in (3.9)

$$L = \frac{1}{2} \sigma^2 x^2 \partial_{xx} + (\sigma^2 - \mu) x \partial_x + \frac{1}{t} (h - h^2 x) \partial_h + \partial_t \tag{3.34}$$

We remind that in this case we have considered the following approximation of order zero:

$$L_0 = \frac{1}{2} \sigma^2 x_0^2 \partial_{xx} + (\sigma^2 - \mu) x \partial_x + \frac{1}{t} (h - h_0^2 x) \partial_h + \partial_t \tag{3.35}$$

but we haven't been able to compute its fundamental solution since the function $\phi(t,T)$ in (1.4) doesn't have an explicit expression.
We have tried therefore to approximate the function $\phi(t,T)$, and consequently the covariance matrix $C(t,T)$ using its Taylor expansion of the third order in the variable T with respect to t.
In this way it has been possible to get an explicit expression for its fundamental solution Γ_0 and consequently an approximation of order zero of the prices computing the following integral:

$$e^{-r\,(T-t)} \int_{\mathbb{R}^2} \Gamma_0(t,s,a,T,S,A)\,\phi(S,A)\,dS\,dA$$

Even in this case however the solution Γ_0 degenerate as t go to zero and consequently the prices approximation doesn't work for t close to zero.
Anyway repeating the argument seen in Chapter 2 we have arrived to an one order approximation for the prices:

$$e^{-r\,(T-t)} \int_{\mathbb{R}^2} \Gamma_1(t,x,h,T,X,H)\,\phi(X,H)\,dX\,dH$$

Where $\Gamma_1(t,x,h,T,X,H) = G_0(t,x,h,T,X,H) + G1(t,x,h,T,X,H)$, with $G0 = \Gamma_0$ and $G1$ the solution of the following Cauchy problem:

$$\begin{cases} L_0\,G_1(t,x,h,T,X,H) \;=\; -\Big(\sigma^2\,x_0\,(x-x_0)\,\partial_{xx} - 2\,\frac{h_0}{t}\,x\,(h-h_0)\,\partial_h\Big)\,\Gamma_0(t,x,h,T,X,H) \\ \\ G1(T,x,h,T,X,H) \;=\; 0 \end{cases}$$

Remark 14. In the special case $\sigma^2 = \mu$, the first raw of the matrix B in (1.4) is null, then in this case the function $\phi(t,T)$ in (1.4) can be computed explicitly and all the computations can be carried on without using the approximation with the Taylor series.

3.2 Numeric results

In this section we show some numeric results for the price of Asian options obtained with the last method described above.
The numeric results have been obtained using again the computational software program *mathematica* to carry on the computations showed in the precedent section.
Since the approximation of the prices function $u(t,X_t,H_t)$ we have found in the precedent section is still a function on the variables t, x and h we have

to assign this values to get a price.
The most reasonable choice is $x = x_0$ and $h = h_0$.
Where x_0 is a free parameter while h_0 is constricted to be again x_0 by the definition of the process H_t
Our results has been confronted with the ones obtained by the Monte Carlo method. For this one has been used an Euler discretization with 150 time-step and 100000 Monte Carlo replications.
Table 3.3 reports the interest rate r, the volatility σ, the time to maturity T, the strike K and the initial asset price x_0 for seven cases that will be tested. The time t has been considered fixed to 1 and a null dividend rate is assumed: $q = 0$.

Table 3.3: Parameter values for seven test cases

Case	x_0	K	r	σ	T
1	2	2	0.02	0.1	1.5
2	2	2	0.18	0.3	1.5
3	2	2	0.0125	0.25	2
4	1.9	2	0.05	0.5	1.5
5	2	2	0.05	0.5	1.5
6	2.1	2	0.05	0.5	1.5
7	2	2	0.05	0.5	2

Table 3.4: Asian Call option when $q = 0$ and $t = 1$ (Parameters as in Table 3.3)

Order one	Euler-Monte Carlo method	
1.72168×10^{-2}	1.17305×10^{-2}	1.19322×10^{-2}
5.39795×10^{-2}	3.82200×10^{-2}	3.87992×10^{-2}
1.28456×10^{-1}	4.94814×10^{-2}	5.04352×10^{-2}
3.35189×10^{-2}	1.03060×10^{-2}	1.06783×10^{-2}
7.46597×10^{-2}	4.21469×10^{-2}	4.29612×10^{-2}
1.34801×10^{-1}	1.00662×10^{-1}	1.01915×10^{-1}
2.60621×10^{-1}	8.32783×10^{-2}	8.49538×10^{-2}

This first table shows that our results aren't always accurate.

In particular we have noticed a greater accuracy when $x_0 > K$ and when $T - t \leq 0.5$. This can be shown in the following tables:

Table 3.5: Asian Call option when: $q = 0$, $r = 0.05$, $\sigma = 0.01$, $t = 1$ and $T = 1.5$

x_0	K	Order one	Euler-Monte Carlo method	
110	100	1.01053×10^1	1.01997×10^1	1.02015×10^1
150	100	4.92457×10^1	4.93737×10^1	4.93762×10^1
200	100	9.81713×10^1	9.83422×10^1	9.83455×10^1
103	100	3.25568×10^0	3.34405×10^0	3.34574×10^0
100	100	3.21290×10^{-1}	4.05598×10^{-1}	4.07237×10^{-1}
99.5	100	-1.75072×10^{-2}	2.09574×10^{-2}	2.15633×10^{-2}
4	2	1.96343×10^0	1.96687×10^0	1.96693×10^0
2	2	6.42580×10^{-3}	8.11072×10^{-3}	8.14351×10^{-3}
1.9	2	5.11929×10^{-120}	0.	0.

Table 3.6: Asian Call option when: $q = 0$, $r = 0.05$, $\sigma = 0.01$, $x_0 = 2$ and $K = 2$

t	T	Order one	Euler-Monte Carlo method	
1	1.2	1.61075×10^{-3}	1.65098×10^{-3}	1.66132×10^{-3}
1	1.4	4.84932×10^{-3}	5.59543×10^{-3}	5.62067×10^{-3}
1	1.5	6.42580×10^{-3}	8.12084×10^{-3}	8.15375×10^{-3}
1	1.6	7.64992×10^{-3}	1.09038×10^{-2}	1.0944×10^{-2}
1	1.8	8.13384×10^{-3}	1.70516×10^{-2}	1.71061×10^{-2}

Furthermore, as already said, this method doesn't work for t close to 0:

Table 3.7: Asian Call option as t goes to 0 ($q = 0$, $r = 0.05$, $\sigma = 0.01$, $x_0 = 4$ and $K = 2$)

t	T	Order one	Euler-Monte Carlo method	
1	1.3	1.97650×10^0	1.97705×10^0	1.97708×10^0
0.5	0.8	1.97804×10^0	1.98128×10^0	1.98134×10^0
0.1	0.3	1.95305×10^0	1.99332×10^0	1.99341×10^0
0.01	0.1	1.53698×10^0	1.99909×10^0	1.99917×10^0
0.001	0.1	-6.19201×10^4	1.99988×10^0	1.99996×10^0
0.00001	0.1	-3.38022×10^5	1.99995×10^0	2.00004×10^0

Finally we show some numeric results for the special case $\sigma^2 = \mu$; as already said, in this special case we haven't needed to use the Taylor series to get the approximation of the price; for this reason in this case we should have more accurate numeric results with respect to the general case; nevertheless, the limits came up formerly persist: the degeneracy of the method for t close to zero and the loss of accuracy for T distant from t or for x_0 not enough greater than K.

Table 3.8: Asian Call option when $r = \sigma^2 = 1$ $(q = 0)$

x_0	K	t	T	Order one	Euler-Monte Carlo method	
110	100	1	1.5	8.63914×10^0	8.43662×10^0	8.51875×10^0
150	100	1	1.5	3.29424×10^1	3.19719×10^1	3.21151×10^1
103	100	1	1.5	5.19596×10^0	4.98402×10^0	5.04718×10^0
100	100	1	1.5	3.92878×10^0	3.72348×10^{-1}	3.77731×10^{-1}
99.5	100	1	1.5	3.73226×10^0	3.50772×10^0	3.56021×10^0
4	2	1	1.5	1.28589×10^0	1.25618×10^0	1.26005×10^0
2	2	1	1.5	7.85756×10^{-2}	7.39179×10^{-2}	749966×10^{-2}
2	2	1	1.2	2.95785×10^{-2}	2.90088×10^{-2}	2.94626×10^{-2}
2	2	1	1.4	6.36225×10^{-2}	6.13415×10^{-2}	6.22494×10^{-2}
2	2	1	2	1.29576×10^{-1}	1.08926×10^{-1}	1.10415×10^{-1}
4	2	0.1	0.3	1.76253×10^0	1.73339×10^0	1.74028×10^0
4	2	0.01	0.1	8.07567×10^{-1}	1.91364×10^0	1.92082×10^0
4	2	0.001	0.1	-1.67743×10^1	1.92653×10^0	1.93484×10^0
4	2	0.00001	0.1	-5.99884×10^2	1.93009×10^0	1.93857×10^0

Chapter 4

ERROR BOUNDS FOR ARITHMETIC AVERAGE

4.1 Preliminaries

We have seen that the problem of compute the price of an Asian Option respect to the arithmetic average is equivalent to find the fundamental solution of the following Kolmogorov operator:

$$L(u) = \frac{1}{2}\,\sigma^2\,s^2\,\partial_{ss}u + \mu\,s\,\partial_s u + s\,\partial_a u + \partial_t u \tag{4.1}$$

In this chapter it will be provided a theoretic estimation for the error committed by our method of approximation of this fundamental solution seen in Chapters 1 and 2.
Let us consider a generic Kolmogorov operator of the following type:

$$K \;=\; \frac{1}{2}\,\alpha(s)\,\partial_{ss} + \mu\,s\,\partial_s + s\,\partial_a + \partial_t \tag{4.2}$$

And we assume that K satisfies the following hypothesis:

$(H1)$ *There exist two positive constants* a , A *such that:*

$$a \le \alpha(s) \le A \qquad \forall s \in \mathbb{R}$$

Let then K_{s_0} be the operator K defined in (4.2) with $\alpha(s)$ fixed in a point s_0, that is:

$$K_{s_0} = \frac{1}{2}\,\alpha(s_0)\,\partial_{ss} + \mu\,s\,\partial_s + s\,\partial_a + \partial_t \tag{4.3}$$

At last we denote with $\Gamma(t,s,a,T,S,A)$ and $\Gamma_{s_0}(t,s,a,T,S,A)$ respectively the fundamental solutions for the operators K and K_{s_0}
where $t \geq 0$, $t < T$ and $s, a, S, A \in \mathbb{R}$

We remark that in the case of $\alpha(s) = \sigma^2 s^2$ the operator K is the operator L (in(4.1)) related to the arithmetic average; thus computing the Taylor series of L respect to the point s_0, as we did in our approximation method, we obtain that K_{s_0} and $\Gamma_{s_0}(t,s,a,T,S,A)$ are exactly our approximation of order zero of the operator L and of its fundamental solution.
Thus we have that the module of the difference between $\Gamma(t,s,a,T,S,A)$ and $\Gamma_{s_0}(t,s,a,T,S,A)$ is the global error committed by our method of approximation at order zero.

We remark also that $\sigma^2 s^2 = 0$ when $s = 0$ so the operator L doesn't properly satisfies the hypothesis $(H1)$ demanded for the operators K.

We will prove in **Theorem 4.4.2** and **Theorem 4.4.6** of this chapter that if we take $s_0 = S$ then for arbitrary $\epsilon > 0$ and $\bar{T} > 0$ there exists a constant positive C such that:

$$|\Gamma(t,s,a,T,S,A) - \Gamma_{s_0}(t,s,a,T,S,A)| \,\leq\, C\,(T-t)^{\frac{1}{2}}\,\Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

$$|\Gamma(t,s,a,T,S,A) - \Gamma^1_{s_0}(t,s,a,T,S,A)| \,\leq\, C\,(T-t)\,\Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

$\forall\, s, S, a, A \in \mathbb{R}$, and $t, T\; such\; that\, 0 < T - t \leq \bar{T}$,

where $\Gamma^1_{s_0}$ is the first order approximation with respect to s_0 of Γ and $\Gamma_{A+\epsilon}$ is the fundamental solution of the constant coefficients operator:
$\frac{1}{2}\,(A+\epsilon)\,\partial_{ss} + \mu\,s\,\partial_s + s\,\partial_a + \partial_t$

For notation simplicity we define $\tau := T - t$ with $T > t$ and the following vectors:
$x := (s,a)$, $y := (S,A)$.

We denote with $K_{\bar{s}}$, the operator K (in (4.2)) with α freezed in $\bar{s}$

$$K_{\bar{s}} \;=\; \frac{1}{2}\,\alpha(\bar{s})\,\partial_{ss} + \mu\, s\,\partial_s + s\,\partial_a + \partial_t \tag{4.4}$$

and we remind that we have seen in Chapter 1 that since $K_{\bar{s}}$ is a constant coefficients Kolmogorov operator we can compute explicitly its fundamental solution.

Indeed we have seen that $K_{\bar{s}}$ is related to the following **linear stochastic differential equation**:

$$dX_t \;=\; B\,X_t\,dt + \Sigma_{\bar{s}}\,dW_t \tag{4.5}$$

Where W is a mono-dimensional Brownian motion, $B = \begin{pmatrix} \mu & 0 \\ 1 & 0 \end{pmatrix}$ and $\Sigma_{\bar{s}} = \begin{pmatrix} \sqrt{\alpha(\bar{s})} \\ 0 \end{pmatrix}$

Now we observe that the expectation $M(t,T,x) = M(\tau,x)$ of the Gaussian process $X_T^{t,x}$ solution of (4.5) doesn't depend on $\bar{s}$, while the covariance matrix of $X_T^{t,x}$ depends on $\bar{s}$, so we denote it with:
$C_{\bar{s}}(t,T) = C_{\bar{s}}(T-t)$

We recall that the fundamental solution of $K_{\bar{s}}$ is:

$$\Gamma_{\bar{s}}(t,x,T,y) = \frac{1}{2\pi\sqrt{\det\, C_{\bar{s}}(t,T)}}\, e^{-\frac{1}{2}\,<C_{\bar{s}}^{-1}(t,T)\,(y-M(t,T,x))\,,\,y-M(t,T,x)>} \tag{4.6}$$

In particular the fundamental solution of K_{s_0} in (4.3) is given by

$$\Gamma_{s_0}(t,x,T,y) = \frac{1}{2\pi\sqrt{\det\, C_{s_0}(t,T)}}\, e^{-\frac{1}{2}\,<C_{s_0}^{-1}(t,T)\,(y-M(t,T,x))\,,\,y-M(t,T,x)>} \tag{4.7}$$

We define finally the following notation $z := y - M(t,T,x)$

4.2 Derivative estimations for $\mu = 0$

In this section we provide some estimations for $\Gamma_{\bar{s}}(t,x,T,y)$ and its derivatives.
We consider first the easiest case in which $\mu = 0$ in (4.2) and we provide those estimations in this case; successively, in the next section, we prove the same estimations for the general case $\mu \neq 0$.

In the case $\mu = 0$ we have:

$$K = \alpha(s)\,\partial_{ss} + s\,\partial_a + \partial_t \tag{4.8}$$

$$K_{s_0} = \alpha(s_0)\,\partial_{ss} + s\,\partial_a + \partial_t \tag{4.9}$$

Moreover the matrix B of the **linear stochastic differential equation** related to the operator (4.4) is:

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

In this case then it is very easy to explicitly compute $M(t,T,x)$ and $C_{s_0}(t,T)$ in (4.7):

$$M(\tau, s, a) = (s, a + s\,\tau) \tag{4.10}$$

$$C_{s_0}(\tau) = \alpha(s_0)\begin{pmatrix} \tau & \frac{1}{2}\tau^2 \\ \frac{1}{2}\tau^2 & \frac{1}{3}\tau^3 \end{pmatrix} \tag{4.11}$$

We remind finally that we have seen in Chapter 1 that the matrix $C_{\bar{s}}(\tau)$ is symmetric and positive defined $\forall\,\tau > 0, \bar{s} \in \mathbb{R}$; moreover in this case also $C_{s_0}^{-1}(\tau)$ assumes a simple expression:

$$C_{s_0}^{-1}(\tau) = \frac{1}{\alpha(s_0)}\begin{pmatrix} \frac{4}{\tau} & -\frac{6}{\tau^2} \\ -\frac{6}{\tau^2} & \frac{12}{\tau^3} \end{pmatrix} \tag{4.12}$$

We obtain Analogous formulae to (4.11) and (4.12) for $C_{\bar{s}}(\tau)$ and their inverse, while B and $M(t,T,x)$ are exactly the same in both the cases.

Remark 15. The formulae (4.11) and (4.12) show that for all $\bar{s} \in \mathbb{R}$, $0 \le t < T$ it holds that:

$$C_{\bar{s}}(t,T) \;=\; \alpha(\bar{s})\, C_1(t,T) \tag{4.13}$$

and

$$C_{\bar{s}}^{-1}(t,T) \;=\; \frac{1}{\alpha(\bar{s})}\, C_1^{-1}(t,T) \tag{4.14}$$

Corollary 4.2.1. *($\mu = 0$)*
For all $\bar{s} \in \mathbb{R}$, $\tau > 0$ we have the following inequalities for the quadratic forms associated to the covariance matrices:

$$a\, C_1(\tau) \le C_{\bar{s}}(\tau) \le A\, C_1(\tau) \tag{4.15}$$

$$\frac{1}{A}\, C_1^{-1}(\tau) \le C_{\bar{s}}^{-1}(\tau) \le \frac{1}{a}\, C_1^{-1}(\tau) \tag{4.16}$$

Proof. It follows directly by remark 15 and hypothesis $(H1)$ □

Proposition 4.2.2. *($\mu = 0$)*
For all $\bar{s} \in \mathbb{R}$, $x, y \in \mathbb{R}^2$ and $\tau > 0$ it holds that

$$\frac{a}{A}\, \Gamma_a(t,x,T,y) \;\le\; \Gamma_{\bar{s}}(t,x,T,y) \;\le\; \frac{A}{a}\, \Gamma_A(t,x,T,y) \tag{4.17}$$

Proof. We only prove the second inequality because the first is analogous.

We call $z := y - M(\tau,x) = (S,A) - M(t,T,s,a)$, then we have that

$$\Gamma_{\bar{s}}(\tau,x,y) = \frac{1}{2\pi\sqrt{\det\, C_{\bar{s}}(\tau)}}\, e^{-\frac{1}{2} < C_{\bar{s}}^{-1}(\tau)\, z\,,\, z >}$$

Corollary 4.2.1 yelds the following inequality:

$$-\frac{1}{2}\; < C_{\bar{s}}^{-1}(\tau)\, z, z > \;\le\; -\frac{1}{2A}\; < C_1^{-1}(\tau)\, z, z >$$

While by Remark 15 and hypothesis $(H1)$ we have

$$\det C_{\bar{s}}(\tau) \;=\; \alpha(\bar{s})^2\, \det C_1(\tau) \;\geq\; a^2\, \det C_1(\tau)$$

Thus we have

$$\Gamma_{\bar{s}}(\tau,x,y) = \frac{1}{2\pi\sqrt{\det\, C_{\bar{s}}(\tau)}}\, e^{-\frac{1}{2}<C_{\bar{s}}^{-1}(\tau)\, z\,,\, z>}$$

$$\leq\; \frac{1}{a}\,\frac{A}{2\pi\sqrt{\det\, C_A(\tau)}}\, e^{-\frac{1}{2}<C_A^{-1}(\tau)\, z, z>} = \frac{A}{a}\,\Gamma_A(\tau,x,y)$$

□

We define now the family $(D_0(\lambda))_{\lambda>0}$:

$$D_0(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^3 \end{pmatrix} \tag{4.18}$$

That is $D_0(\lambda)\,(s,a) \;=\; (\lambda\, s, \lambda^3\, a)$

Remark 16. In the case $\mu = 0$, $\forall\, \bar{s} \,\in\, \mathbb{R}\,,\, \tau > 0$ it holds that:

$$C_{\bar{s}}^{-1}(\tau) \;=\; D_0(\frac{1}{\sqrt{\tau}})\, C_{\bar{s}}^{-1}(1)\, D_0(\frac{1}{\sqrt{\tau}}) \tag{4.19}$$

Indeed we have

$$D_0(\frac{1}{\sqrt{\tau}})\, C_{\bar{s}}^{-1}(1)\, D_0(\frac{1}{\sqrt{\tau}}) = \frac{1}{\alpha(\bar{s})} \begin{pmatrix} \frac{1}{\sqrt{\tau}} & 0 \\ 0 & \frac{1}{\sqrt{\tau^3}} \end{pmatrix} \begin{pmatrix} \frac{4}{1} & -\frac{6}{1} \\ -\frac{6}{1} & \frac{12}{1} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\tau}} & 0 \\ 0 & \frac{1}{\sqrt{\tau^3}} \end{pmatrix}$$

$$= \frac{1}{\alpha(\bar{s})} \begin{pmatrix} \frac{4}{\sqrt{\tau}} & -\frac{6}{\sqrt{\tau}} \\ -\frac{6}{\sqrt{\tau^3}} & \frac{12}{\sqrt{\tau^3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\tau}} & 0 \\ 0 & \frac{1}{\sqrt{\tau^3}} \end{pmatrix} = \frac{1}{\alpha(\bar{s})} \begin{pmatrix} \frac{4}{\tau} & -\frac{6}{\tau^2} \\ -\frac{6}{\tau^2} & \frac{12}{\tau^3} \end{pmatrix} = C_{\bar{s}}^{-1}(\tau)$$

Corollary 4.2.3. *($\mu = 0$)*
$\forall\, \tau > 0\,,\; \bar{s} \,\in\, \mathbb{R}$ *and* $v \,\in\, \mathbb{R}^2$

1.

$$|(C_{\bar{s}}^{-1}(\tau)\,v)_1| \leq C\,\frac{|D_0(\frac{1}{\sqrt{\tau}})\,v|}{\sqrt{\tau}}$$

2.

$$|(C_{\bar{s}}^{-1}(\tau)\,v)_2| \leq C\,\frac{|D_0(\frac{1}{\sqrt{\tau}})\,v|}{\tau^{\frac{3}{2}}}$$

Where $(v)_1$, $(v)_2$ indicate respectively the first and the second component of the vector v and $C = \frac{|C_1^{-1}(1)|}{a}$

Proof. We prove only the first inequality, the second can be proved in the same way.
Equality (4.19) yelds:

$$|(C_{\bar{s}}^{-1}(\tau)\,v)_1| \;=\; \frac{1}{\sqrt{\tau}}\,|(C_{\bar{s}}^{-1}(1)\,D_0(\frac{1}{\sqrt{\tau}})\,v)_1|$$

Then by Corollary 4.2.1

$$\frac{1}{\sqrt{\tau}}\,|(C_{\bar{s}}^{-1}(1)\,D_0(\frac{1}{\sqrt{\tau}})\,v)_1| \leq \frac{1}{\sqrt{\tau}}\,\frac{1}{a}\,|C_1^{-1}(1)|\,|D_0(\frac{1}{\sqrt{\tau}})\,v|$$

□

Remark 17. It follows directly by the explicit expression of the covariance matrix in (4.12) that there exists a constant C such that:

$$\begin{aligned} |(C_{\bar{s}}^{-1}(\tau))_{11}| &\leq \frac{C}{\tau} \\ |(C_{\bar{s}}^{-1}(\tau))_{12}| &= |(C_{\bar{s}}^{-1}(\tau))_{21}| \leq \frac{C}{\tau^2} \\ |(C_{\bar{s}}^{-1}(\tau))_{22}| &\leq \frac{C}{\tau^3} \end{aligned}$$

Lemma 4.2.4. *Let p be a polynomial function and k a positive constant, then there exists a positive constant C such that:*

$$|p(x)|\,e^{-kx^2} \leq C \qquad \forall\, x \in \mathbb{R}$$

Proof. Since

$$\lim_{x\to\infty} |p(x)|\,e^{-kx^2} \;=\; \lim_{x\to-\infty} |p(x)|\,e^{-kx^2} \;=\; 0$$

and $|p(x)|\,e^{-kx^2} \in C(\mathbb{R})$, it is a bounded function □

Proposition 4.2.5. *($\mu = 0$)*
Let p be a polynomial function in $|\eta|$ where $\eta = D_0(\frac{1}{\sqrt{\tau}})\,(y - M(\tau, x))$, then for every $\epsilon > 0$ there exists a constant C dependent on ϵ and p such that:

$$|p\,(|\eta|)|\,\Gamma_{\bar{s}}(t, x, T, y) \;\le\; C\,\Gamma_{A+\epsilon}(t, x, T, y)$$

For every $\bar{s} \in \mathbb{R}$, $x, y \in \mathbb{R}^2$, $\tau > 0$

Proof.

$$|p\,(|\eta|)|\,\Gamma_{\bar{s}}(t, x, T, y) \;\le\; \frac{A}{a}\,|p\,(|\eta|)|\,\Gamma_A(t, x, T, y)$$

(by Proposition 4.2.2)

$$\frac{A}{a}\,|p\,(|\eta|)|\,\Gamma_A(t, x, T, y) \;=\; \frac{A}{a}\,|p\,(|\eta|)|\,\frac{1}{2\pi\sqrt{\det\,C_A(\tau)}}\,e^{-\frac{1}{2}<C_A^{-1}(\tau)\,z\,,\,z>}$$

$$=\;\frac{A}{a}\,|p\,(|\eta|)|\,\frac{1}{2\pi\,A\sqrt{\det\,C_1(\tau)}}\,e^{-\frac{1}{2A}<C_1^{-1}(1)\,D_0(\frac{1}{\sqrt{\tau}})\,z\,,\,D_0(\frac{1}{\sqrt{\tau}})\,z>}$$

$$=\;|p\,(|\eta|)|\,\frac{A}{a}\,\frac{A+\epsilon}{A}\,\frac{1}{2\pi\,(A+\epsilon)\sqrt{\det\,C_1(\tau)}}\,e^{-\frac{1}{2}<C_1^{-1}(1)\,\eta\,,\eta>\,(\frac{1}{A}-\frac{1}{A+\epsilon}+\frac{1}{A+\epsilon})} \qquad (\star)$$

Now as $C_1^{-1}(1)$ is positive definite there exists a positive constant k such that

$< C_1^{-1}(1)\,v\,,\,v > \ge\; k\,|v|^2\;\forall\,v \in \mathbb{R}^2$
Thus we have

$$(\star)\;\le\;\frac{A+\epsilon}{a}\,\Gamma_{A+\epsilon}(t, x, T, y)\,|p\,(|\eta|)|\,e^{-\frac{k\epsilon}{2A(A+\epsilon)}\,|\eta|^2}$$

And the thesis follow by Lemma 4.2.4 □

Theorem 4.2.6. *($\mu = 0$)*
For every $\epsilon > 0$, $k \in \mathbb{N}$ there exists a constant C depends on ϵ, k such that

1. $$|\partial_s^{(k)}\Gamma_{\bar{s}}(t, x, T, y)| \;\le\; \frac{C_{k,\epsilon}}{\tau^{\frac{k}{2}}}\,\Gamma_{A+\epsilon}(t, x, T, y)$$

2. $$|\partial_a^{(k)}\Gamma_{\bar{s}}(t, x, T, y)| \;\le\; \frac{C_{k,\epsilon}}{\tau^{\frac{3\,k}{2}}}\,\Gamma_{A+\epsilon}(t, x, T, y)$$

For every $\bar{s} \in \mathbb{R}$, $x, y \in \mathbb{R}^2$, $\tau > 0$

Proof. We only prove (1) since the proof of (2) is analogous.

We have to prove directly the case $k = 1$ and $k = 2$, then we will derive the general case from them.

$(k = 1)$

$$|\partial_s \Gamma_{\bar{s}}(t,x,T,y)| \;=\; \Gamma_{\bar{s}}(t,x,T,y)\, |(C_{\bar{s}}^{-1}(\tau)\, z\, \phi(\tau))_1|$$

By (4.6) and since $M(\tau, x) = \phi(\tau)\, x$ with $\phi(\tau)$ as in (1.4).

In the case $\mu = 0$ we have

$$\phi(\tau) \;=\; \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} \tag{4.20}$$

Thus we have:

$$|(C_{\bar{s}}^{-1}(\tau)\, z\, \phi(\tau))_1| \;\le\; |(C_{\bar{s}}^{-1}(\tau)\, z)_1| + |\tau\, (C_{\bar{s}}^{-1}(\tau)\, z)_2|$$

Now $|(C_{\bar{s}}^{-1}(\tau)\, z)_1| \le \frac{\bar{C}}{\sqrt{\tau}} \left| D_0\left(\frac{1}{\sqrt{\tau}}\right) z \right|$ by Corollary 4.2.3

and the same estimation can be obtained for $|\tau\, (C_{\bar{s}}^{-1}(\tau)\, z)_2|$ by Corollary 4.2.3 too. Then:

$$|\partial_s \Gamma_{\bar{s}}(t,x,T,y)| \;\le\; 2\, \Gamma_{\bar{s}}(t,x,T,y)\, \frac{\bar{C}}{\sqrt{\tau}} \left| D_0\left(\frac{1}{\sqrt{\tau}}\right) z \right| \;\le\; \frac{C}{\sqrt{\tau}}\, \Gamma_{A+\epsilon}(t,x,T,y)$$

By Proposition 4.2.5

$(k = 2)$

$$|\partial_{ss} \Gamma_{\bar{s}}(t,x,T,y)| \;=\; \Gamma_{\bar{s}}(t,x,T,y) \left(|(C_{\bar{s}}^{-1}(\tau)\, z\, \phi(\tau))_1|^2 + |(\phi^T(\tau)\, C_{\bar{s}}^{-1}(\tau)\, \phi(\tau)_1)_1| \right)$$

We have seen in the precedent part of this proof that:

$$|(C_{\bar{s}}^{-1}(\tau)\, z\, \phi(\tau))_1|^2 \;\le\; \frac{\bar{C}}{\tau} \left| D_0\left(\frac{1}{\sqrt{\tau}}\right) z \right|^2$$

While

$$|(\phi^T(\tau)\, C_{\bar{s}}^{-1}(\tau)\, \phi(\tau)_1)_1| \;\le\; |\phi_{11}(\tau)^2\, (C_{\bar{s}}^{-1}(\tau))_{11}| + |\phi_{11}(\tau)\, \phi_{21}(\tau)\, (C_{\bar{s}}^{-1}(\tau))_{21}|$$

$$+|\phi_{11}(\tau)\,\phi_{21}(\tau)\,(C_{\bar{s}}^{-1}(\tau))_{12}| + |\phi_{21}(\tau)^2\,(C_{\bar{s}}^{-1}(\tau))_{22}| \qquad (\star)$$

Directly by (4.12) and (4.20) we have

$$(\star) \;=\; \frac{1}{\alpha(\bar{s})}\left(\frac{4}{\tau} + 2\,\frac{6}{\tau} + \frac{12}{\tau}\right) \;\leq\; \frac{\hat{C}}{\tau}$$

In conclusion by Proposition 4.2.5

$$|\partial_{ss}\Gamma_{\bar{s}}(t,x,T,y)| \;\leq\; \frac{C}{\tau}\,\Gamma_{A+\epsilon}(t,x,T,y)$$

$(k > 2)$
We remark that

$$\Gamma_{\bar{s}}(t,x,T,y) \;=\; \frac{1}{2\,\pi\,\sqrt{\det C_{\bar{s}(\tau)}}}\,e^{\,f_{\bar{s}}(t,x,T,y)}$$

where

$$f_{\bar{s}}(t,x,T,y) \;=\; -\frac{1}{2} < C_{\bar{s}}^{-1}(\tau)\,(y - M(\tau,x))\,,\,(y - M(\tau,x)) \;>$$

and $\partial_s^{(k)} f_{\bar{s}}(t,x,T,y) = 0$ for $k > 2$; then by Proposition 2.1.2 we have that:

$$|\partial_s^{(k)}\Gamma_{\bar{s}}(t,x,T,y)| \;=$$

$$\Gamma_{\bar{s}}(t,x,T,y)\sum_{n=0}^{\frac{k}{2}}\binom{k}{k-2\,n}(2\,n-1)!!\;|\partial_s f_{\bar{s}}(t,x,T,y)|^{k-2\,n}\;|\partial_{ss} f_{\bar{s}}(t,x,T,y)|^{n}$$

$$\leq\; C_k'\,\Gamma_{\bar{s}}(t,x,T,y)\sum_{n=0}^{\frac{k}{2}}|\partial_s f_{\bar{s}}(t,x,T,y)|^{k-2\,n}\;|\partial_{ss} f_{\bar{s}}(t,x,T,y)|^{n} \qquad (\star)$$

Now we have seen in the previous cases that

$$|\partial_s f_{\bar{s}}(t,x,T,y)| \;=\; |(C_{\bar{s}}^{-1}(\tau)\,z\,\phi(\tau))_1| \;\leq\; \frac{C_1}{\sqrt{\tau}}\left|D_0\left(\frac{1}{\sqrt{\tau}}\right)\,z\right|$$

$$|\partial_{ss} f_{\bar{s}}(t,x,T,y)| \;=\; \Big(|(C_{\bar{s}}^{-1}(\tau)\,z\,\phi(\tau))_1|^2 \;+\; |(\phi^T(\tau)\,C_{\bar{s}}^{-1}(\tau)\,\phi(\tau)_1)_1|\Big)$$

$$\leq\; \frac{C_2}{\tau} + \frac{C_2}{\tau}\left|D_0\left(\frac{1}{\sqrt{\tau}}\right)\,z\right|^2$$

Thus

$$(\star) \;\le\; C'_k\,\Gamma_{\bar{s}}(t,x,T,y)\sum_{n=0}^{\frac{k}{2}}\left(\frac{C_1}{\sqrt{\tau}}\right)^{k-2n}|\eta|^{k-2n}\left(\frac{C_2}{\tau}\right)^{n}\left(1+|\eta|^2\right)^n$$

$$\le\; C''_k\,\Gamma_{\bar{s}}(t,x,T,y)\sum_{n=0}^{\frac{k}{2}}\left(\frac{1}{\tau}\right)^{\frac{k-2n}{2}+n}|\eta|^{k-2n}\left(1+|\eta|^2\right)^n$$

$$=\; C''_k\,\Gamma_{\bar{s}}(t,x,T,y)\sum_{n=0}^{\frac{k}{2}}\left(\frac{1}{\tau}\right)^{\frac{k}{2}}p(|\eta|)$$

where p is a polynomial function; finally by Proposition 4.2.5

$$C''_k\,\Gamma_{\bar{s}}(t,x,T,y)\sum_{n=0}^{\frac{k}{2}}\left(\frac{1}{\tau}\right)^{\frac{k}{2}}p(|\eta|)\;\le\;\frac{C_{k,\epsilon}}{\tau^{\frac{k}{2}}}\,\Gamma_{A+\epsilon}(t,x,T,y)$$

□

Corollary 4.2.7. *For every $\epsilon > 0$, $k, h \in \mathbb{N}$ there exists a constant $C > 0$ depends on ϵ, k, h such that:*

$$|\partial_a^{(h)}\,\partial_s^{(k)}\Gamma_{\bar{s}}(t,x,T,y)| \;\le\; \frac{C_{k,h,\epsilon}}{\tau^{\frac{k}{2}+\frac{3h}{2}}}\,\Gamma_{A+\epsilon}(t,x,T,y)$$

For every $\bar{s} \in \mathbb{R}$, $x, y \in \mathbb{R}^2$, $\tau > 0$

Proof. As in the previous theorem, by proposition 2.1.2 we have

$$|\partial_a^{(h)}\,\partial_s^{(k)}\Gamma_{\bar{s}}(t,x,T,y)| \;=\; \Bigg|\partial_a^{(h)}\Bigg(\Gamma_{\bar{s}}(t,x,T,y)\sum_{n=0}^{\frac{k}{2}}\binom{k}{k-2n}(2n-1)!!$$

$$\cdot\,(\partial_s f_{\bar{s}}(t,x,T,y))^{k-2n}\,(\partial_{ss} f_{\bar{s}}(t,x,T,y))^n\Bigg)\Bigg| \qquad (*)$$

where

$$\partial_s\, f_{\bar{s}}(t,x,T,y) \;=\; \Big(C_{\bar{s}}^{-1}(\tau)\,(y-M(\tau,x))\,\phi(\tau)\Big)_1$$

and

$$\partial_{ss}\, f_{\bar{s}}(t,x,T,y) \;=\; \Big(\phi^T(\tau)\,C_{\bar{s}}^{-1}(\tau)\,\phi(\tau)_1\Big)_1$$

Therefore:

$$\partial_a\Big(\partial_s\, f_{\bar{s}}(t,x,T,y)\Big) \;=\; \Big(\phi^T(\tau)\,C_{\bar{s}}^{-1}(\tau)\,\phi(\tau)_2\Big)_1 \quad\text{and}\quad \partial_a\Big(\partial_{ss}\, f_{\bar{s}}(t,x,T,y)\Big) \;=\; 0$$

while

$$\partial_a^{(j)}\Big(\partial_s\, f_{\bar s}(t,x,T,y)\Big) \;=\; \partial_a^{(j)}\Big(\partial_{ss}\, f_{\bar s}(t,x,T,y)\Big) \;=\; 0 \qquad \text{for } j>1$$

Then:

$$(*) \;=\; \Gamma_{\bar s}(t,x,T,y)\,\sum_{n=0}^{\frac{h}{2}} \binom{h}{h-2\,n}\,(2\,n-1)!!\; |\partial_a f_{\bar s}(t,x,T,y)|^{h-2\,n}$$

$$\cdot\; |\partial_{aa}\, f_{\bar s}(t,x,T,y)|^n \sum_{n=0}^{\frac{k}{2}} \binom{k}{k-2\,n}\,(2\,n{-}1)!!\; |\partial_s f_{\bar s}(t,x,T,y)|^{k-2\,n}\; |\partial_{ss}\, f_{\bar s}(t,x,T,y)|^n$$

$$+\,h\,\Gamma_{\bar s}(t,x,T,y)\sum_{n=0}^{\frac{h-1}{2}} \binom{h-1}{h-1-2\,n}\,(2\,n-1)!!\; |\partial_a f_{\bar s}(t,x,T,y)|^{h-1-2\,n}$$

$$\cdot\; |\partial_{aa}\, f_{\bar s}(t,x,T,y)|^n \sum_{n=0}^{\frac{k}{2}} \binom{k}{k-2\,n}\,(2\,n-1)!!\,(k-2\,n)\; |\partial_s f_{\bar s}(t,x,T,y)|^{k-1-2\,n}$$

$$\cdot\; |\partial_{as} f_{\bar s}(t,x,T,y)|\; |\partial_{ss}\, f_{\bar s}(t,x,T,y)|^n \qquad (\star)$$

Finally repeating the argument of the third part of Theorem 4.2.6 and since

$|\partial_{as} f(t,x,T,y)| \le \frac{\hat C}{\tau^2}$ we can conclude:

$$(\star) \;\le\; \frac{C_{k,h,\epsilon}}{\tau^{\frac{k}{2}+\frac{3\,h}{2}}}\,\Gamma_{A+\epsilon}(t,x,T,y)$$

□

Remark 18. Repeating the argument used in Theorem 4.2.6 and Corollary 4.2.7, the same estimations can be proved for the derivatives of $\Gamma_{\bar s}$ with respect also to the dual variables (S,A), that is:
For every $\epsilon>0$, $k,h,j,n \in \mathbb{N}$ there exists a positive constants C depends on ϵ,k,h,j,n such that:

$$|\partial_s^{(k)}\,\partial_S^{(j)}\,\partial_a^{(h)}\,\partial_A^{(n)}\Gamma_{\bar s}(t,x,T,y)| \;\le\; \frac{C}{\tau^{\frac{k+j}{2}+\frac{3\,(h+n)}{2}}}\,\Gamma_{A+\epsilon}(t,x,T,y)$$

For every $\bar s \in \mathbb{R}$, $x,y \in \mathbb{R}^2$, $\tau>0$

4.3 Derivative estimations for the general case

We consider now the general case $\mu \neq 0$ and for arbitrary $\bar{T} > 0$ we obtain again the estimations just found for $0 < \tau \leq \bar{T}$

First of all we remark that as $\Sigma_{\bar{s}} = \begin{pmatrix} \sqrt{\alpha(\bar{s})} \\ 0 \end{pmatrix}$, by the formula of the covariance matrix in (1.6) we still have $C_{\bar{s}}(\tau) = \alpha(\bar{s})\, C_1(\tau) \qquad \forall\, \bar{s} \in \mathbb{R}, \tau > 0$

This yields then that the identity $C_{\bar{s}}^{-1}(\tau) = \frac{1}{\alpha(\bar{s})}\, C_1^{-1}(\tau)$, **Corollary 4.2.1** and **Proposition 4.2.2** still hold. Moreover:

$$C_{\bar{s}}^{-1}(\tau) = \frac{1}{\alpha(\bar{s})} \begin{pmatrix} \frac{\mu\,(3-4e^{\tau\,\mu}+e^{2\,\tau\,\mu}+2\,\tau\,\mu)}{(-1+e^{\tau\,\mu})\,(2+\tau\,\mu+e^{\tau\,\mu}\,(-2+\tau\,\mu))} & \frac{\mu^2\,(-1+e^{\tau\,\mu})}{2+\tau\,\mu+e^{\tau\,\mu}\,(-2+\tau\,\mu)} \\ \\ \frac{\mu^2\,(-1+e^{\tau\,\mu})}{2+\tau\,\mu+e^{\tau\,\mu}\,(-2+\tau\,\mu)} & \frac{\mu^3\,(1+e^{\tau\,\mu})}{2+\tau\,\mu+e^{\tau\,\mu}\,(-2+\tau\,\mu)} \end{pmatrix} \tag{4.21}$$

Remark 19. For every $\bar{s} \in \mathbb{R}$ the covariance matrix $C_{\bar{s}}^{-1}(\tau)$ is C^∞ for $\tau > 0$. This is because the denominator is zero only when $\tau = 0$. Indeed:

$(-1 + e^{\tau\,\mu}) = 0$ if and only if $\tau = 0$ as $\mu \neq 0$. While it can be shown by a studying of the derivatives that:
$(2+\tau\,\mu+e^{\tau\,\mu}\,(-2+\tau\,\mu))$ is a strictly monotone function in the variable $\tau\,\mu$; then, since this function has a zero in $\tau\,\mu = 0$, it must be the only zero of the function; finally since $\mu \neq 0$, we have that the function is zero only if $\tau = 0$.

Now if $C_{\bar{s}}(\tau)$ is the covariance matrix related to the operator $K_{\bar{s}}$ in (4.4) we define $C_{\bar{s},0}(\tau)$ as the covariance matrix related to the same operator $K_{\bar{s}}$ but with $\mu = 0$
Then for $C_{\bar{s},0}(\tau)$ they hold all the properties seen in section two of this chapter and moreover $C_{\bar{s}}(\tau)$ and $C_{\bar{s},0}(\tau)$ are related in the following way:

Lemma 4.3.1. *There exist $C_\mu, t_0 > 0$ such that $\forall\, \tau\,,\ 0 < \tau \leq t_0$ it holds:*

$$(1 - C_\mu\,\tau)\, C_{\bar{s},0}(\tau) \;\leq\; C_{\bar{s}}(\tau) \;\leq\; (1 + C_\mu\,\tau)\, C_{\bar{s},0}(\tau)$$

In the sense of the quadratic form associated to the matrices

Proof. We define $G := \frac{<(\,C_{\bar{s}}(\tau) - C_{\bar{s},0}(\tau)\,)\,v, v>}{<C_{\bar{s},0}(\tau)\,v, v>}$ with $v \in \mathbb{R}^2 \setminus \{0\}$ and we set $v = D_0(\frac{1}{\sqrt{\tau}})\,v'$ $\qquad$ Then:

$$G = \frac{<D_0(\frac{1}{\sqrt{\tau}})\,(\,C_{\bar{s}}(\tau) - C_{\bar{s},0}(\tau)\,)\,D_0(\frac{1}{\sqrt{\tau}})\,v', v'>}{<C_{\bar{s},0}(1)\,v', v'>}$$

$$\leq \frac{|D_0(\frac{1}{\sqrt{\tau}})\,(\,C_{\bar{s}}(\tau) - C_{\bar{s},0}(\tau)\,)\,D_0(\frac{1}{\sqrt{\tau}})|\,|v'|^2}{k\,|v'|^2} = \frac{|h(\tau)|}{k} \qquad (\star)$$

Where

$$h(\tau) := D_0(\frac{1}{\sqrt{\tau}})\,(\,C_{\bar{s}}(\tau) - C_{\bar{s},0}(\tau)\,)\,D_0(\frac{1}{\sqrt{\tau}}) \tag{4.22}$$

We show now that there exist $c, t_0 > 0$ such that $|h(\tau)| \leq c\,\tau \quad \forall\, 0 < \tau \leq t_0$

First of all we remark that $\forall\, i, j = 1, 2$

$$(C_{\bar{s}}(\tau))_{ij} \;=\; (C_{\bar{s},0}(\tau))_{ij}\,(1 + \tau\, O(1)) \quad for\ \tau \to 0^+$$

Indeed:

$$\lim_{\tau \to 0^+} \frac{(C_{\bar{s}}(\tau))_{ij} - (C_{\bar{s},0}(\tau))_{ij}}{\tau\,(C_{\bar{s},0}(\tau))_{ij}} \;=\; \begin{cases} \mu & \text{if } (i,j) \neq (2,2) \\ \frac{3\mu}{4}, & \text{if } (i,j) = (2,2) \end{cases}$$

Then we have:

$$\begin{aligned} h(\tau)_{ij} \;&=\; \Big(D_0(\frac{1}{\sqrt{\tau}})\Big)_{ii}\,(C_{\bar{s},0}(\tau))_{ij}\;O(\tau)\,\Big(D_0(\frac{1}{\sqrt{\tau}})\Big)_{jj} \\ &=\; (C_{\bar{s},0}(1))_{ij}\;O(\tau) \quad for\ \tau \to 0^+ \end{aligned}$$

Thus $\exists\, t_0 > 0$ such that:

$$|h(\tau)| \;\leq\; c\,\tau \qquad for\ 0 < \tau < t_0 \tag{4.23}$$

And then:

$$G \;\leq\; \frac{|h(\tau)|}{k} \;\leq\; \frac{c\,\tau}{k}$$

Which for definition of G is equivalent to:

$$< C_{\bar{s}}(\tau)\,v, v > \leq\; C_\mu\,\tau\; < C_{\bar{s},0}(\tau)\,v, v > + < C_{\bar{s},0}(\tau)\,v, v > \qquad \forall\, v \in \mathbb{R}^2,\ \tau :\ 0 < \tau \leq t_0$$

That is:

$$C_{\bar{s}}(\tau) \;\leq\; (1 + C_\mu\,\tau)\,C_{\bar{s},0}(\tau)$$

The other inequality can be proved in the same way so its proof is omitted.

□

A relation symmetric to the previous holds for the matrices

$C^{-1}_{\bar{s}}(\tau)$, $C^{-1}_{\bar{s},0}(\tau)$

Lemma 4.3.2. *There exist $C_\mu, t_0 > 0$ such that $\forall\, \tau$, $0 < \tau \le t_0$ it holds:*

$$(1 - C_\mu\, \tau)\, C_{\bar{s}}^{-1}(\tau) \;\le\; C_{\bar{s},0}^{-1}(\tau) \;\le\; (1 + C_\mu\, \tau)\, C_{\bar{s}}^{-1}(\tau)$$

In the sense of the quadratic form associated to the matrices

Proof. This time we define $G := \frac{<(\, C_{\bar{s}}^{-1}(\tau) - C_{\bar{s},0}^{-1}(\tau)\,)\, v, v>}{<C_{\bar{s}}^{-1}(\tau)\, v, v>}$

with $v \in \mathbb{R}^2 \setminus \{0\}$ and we set $v = C_{\bar{s}}(\tau)\, v'$ Then:

$$G \;=\; \frac{<\,(\, C_{\bar{s}}(\tau)\, v', v'\,> - <\,(\, C_{\bar{s}}(\tau)\, C_{\bar{s},0}^{-1}(\tau)\, C_{\bar{s}}(\tau) v', v'\,>}{<\, C_{\bar{s}}(\tau)\, v', v'\,>}$$

$$= \frac{<\,(\, (C_{\bar{s}}(\tau) - C_{\bar{s},0}(\tau))\, (-C_{\bar{s},0}^{-1}(\tau))\, (C_{\bar{s}}(\tau) - C_{\bar{s},0}(\tau))\, v', v'\,>}{<\, C_{\bar{s}}(\tau)\, v', v'\,>}$$

$$+ \frac{<\,(\, (C_{\bar{s}}(\tau) - C_{\bar{s},0}(\tau))\, v', v'\,>}{<\,(C_{\bar{s}}(\tau)\, v', v'\,>} \;=:\; R(\tau, v') + S(\tau, v')$$

$$S(\tau, v') \;\le\; \frac{k\tau <\, C_{\bar{s},0}(\tau)\, v', v'\,>}{<\, C_{\bar{s}}(\tau)\, v', v'\,>} \;\le\; k\, \tau\, \frac{1}{1 - k\, \tau}$$

Both by Lemma 4.3.1. Now, since $k\, \tau\, \frac{1}{1 - k\, \tau} \to 1$ as $\tau \to 0^+$, for τ suitably small ($\le t_0$) we have $k\, \tau\, \frac{1}{1 - k\, \tau} \le \bar{k}\, \tau$

While setting $v' = D_0(\frac{1}{\sqrt{\tau}})\, v''$ we have:

$$R(\tau, v'') \;=\; \frac{<\, h(\tau)\, (-C_{\bar{s},0}^{-1}(1))\, h(\tau)\, v'', v''\,>}{<\, C_{\bar{s}}(1)\, v'', v''\,>}$$

with $h(\tau)$ defined in (4.22); hence:

$$R(\tau, v'') \le \frac{|h(\tau)|^2\, |C_{\bar{s},0}^{-1}(1)|}{\lambda} \;\le\; \hat{k}\, \tau \quad for\ 0 < \tau \le t_0 \qquad (by (4.25))$$

In conclusion for definition of G we have:

$$(1 - C_\mu\, \tau)\, C_{\bar{s}}^{-1}(\tau) \;\le\; C_{\bar{s},0}^{-1}(\tau)$$

The proof of the second inequality is analogous so it is omitted □

Remark 20. Actually in the inequalities seen in the two precedent lemmas we can also switch $C_{\bar{s}}$ with $C_{\bar{s},0}$ and $C_{\bar{s}}^{-1}$ with $C_{\bar{s},0}^{-1}$
Indeed if it holds

$$(1 - C_\mu\,\tau)\,C_{\bar{s},0}(\tau)\;\le\;C_{\bar{s}}(\tau)\;\le\;(1 + C_\mu\,\tau)\,C_{\bar{s},0}(\tau)$$

Then in particular $C_{\bar{s},0}(\tau)\;\ge\;C_{\bar{s}}(\tau)\,\frac{1}{1+C_\mu\,\tau}\;=\;C_{\bar{s}}(\tau)\,(1 - \frac{C_\mu\tau}{1+C_\mu\,\tau})$
Now, since $1 - \frac{C_\mu\,\tau}{1+C_\mu\,\tau}\;\ge\;1 - C_\mu\,\tau$, we have:

$$C_{\bar{s},0}(\tau) \ge C_{\bar{s}}(\tau)\,(1 - \frac{C_\mu\,\tau}{1 + C_\mu\,\tau})\;\ge\;(1 - C_\mu\,\tau)\,C_{\bar{s}}(\tau)$$

In the same way (for τ suitably small):

$$C_{\bar{s},0}\;\le\;C_{\bar{s}}(\tau)\,\frac{1}{1 - C_\mu\,\tau}\;=\;C_{\bar{s}}(\tau)\,(1 + \frac{C_\mu\,\tau}{1 - C_\mu\,\tau})\;\le\;C_{\bar{s}}(\tau)\,(1 + C_{\mu,2}\,\tau)$$

Since $\frac{C_\mu}{1-C_\mu\,\tau}$ is bounded for $0 < \tau \le t_0$

In the same way can be proved a symmetric relation to which of Lemma 4.3.2 for $C_{\bar{s},0}^{-1}$, $C_{\bar{s}}^{-1}$

In conclusion, for t_0 suitably small and taking the greatest of the constant found in the various relations, we have all the four following inequalities:

$$(1 - C_\mu\,\tau)\,C_{\bar{s},0}(\tau)\;\le\;C_{\bar{s}}(\tau)\;\le\;(1 + C_\mu\,\tau)\,C_{\bar{s},0}(\tau) \tag{4.24}$$

$$(1 - C_\mu\,\tau)\,C_{\bar{s}}(\tau)\;\le\;C_{\bar{s},0}(\tau)\;\le\;(1 + C_\mu\,\tau)\,C_{\bar{s}}(\tau) \tag{4.25}$$

$$(1 - C_\mu\,\tau)\,C_{\bar{s},0}^{-1}(\tau)\;\le\;C_{\bar{s}}^{-1}(\tau)\;\le\;(1 + C_\mu\,\tau)\,C_{\bar{s},0}^{-1}(\tau) \tag{4.26}$$

$$(1 - C_\mu\,\tau)\,C_{\bar{s}}^{-1}(\tau)\;\le\;C_{\bar{s},0}^{-1}(\tau)\;\le\;(1 + C_\mu\,\tau)\,C_{\bar{s}}^{-1}(\tau) \tag{4.27}$$

For all $\tau\,:\,0 < \tau \le t_0$

Corollary 4.3.3. *There exists $t_1 > 0$ such that for $\tau\,:\,0 < \tau \le t_1$ it hold:*

$$\frac{1}{2}\,C_{\bar{s},0}(\tau)\;\le\;C_{\bar{s}}(\tau)\;\le\;2\,C_{\bar{s},0}(\tau) \tag{4.28}$$

$$\frac{1}{2}\,C_{\bar{s},0}^{-1}(\tau)\;\le\;C_{\bar{s}}^{-1}(\tau)\;\le\;2\,C_{\bar{s},0}^{-1}(\tau) \tag{4.29}$$

Proof. By Remark 20 since $(1 - C_\mu\,\tau) \to 1$ and $(1 + C_\mu\,\tau) \to 1$ as $\tau \to 0$ $\square$

We can now repeat Corollary 4.2.3 and Remark 17 for the general case:

Lemma 4.3.4. *Given $\bar{T} > 0$, there exists $C > 0$ depends on $\mu, \bar{T}$ such that for all $\bar{s} \in \mathbb{R}$ it hold:*

1) $|(C_{\bar{s}}^{-1}(\tau)\, v)_1| \le \frac{C\,|D_0(\frac{1}{\sqrt{\tau}})\,v|}{\sqrt{\tau}} \qquad \forall\, \tau\,,\ 0 < \tau \le \bar{T}\,,\ v \in \mathbb{R}^2$

2) $|(C_{\bar{s}}^{-1}(\tau)\, v)_2| \le \frac{C\,|D_0(\frac{1}{\sqrt{\tau}})\,v|}{\tau^{\frac{3}{2}}} \qquad \forall\, \tau\,,\ 0 < \tau \le \bar{T}\,,\ v \in \mathbb{R}^2$

3) $|(C_{\bar{s}}^{-1}(\tau))_{11}| \le \frac{C}{\tau} \qquad \forall\, \tau\,,\ 0 < \tau \le \bar{T}$

4) $|(C_{\bar{s}}^{-1}(\tau))_{12}| = |(C_{\bar{s}}^{-1}(\tau))_{21}| \le \frac{C}{\tau^2} \qquad \forall\, \tau\,,\ 0 < \tau \le \bar{T}$

5) $|(C_{\bar{s}}^{-1}(\tau))_{22}| \le \frac{C}{\tau^3} \qquad \forall\, \tau\,,\ 0 < \tau \le \bar{T}$

Proof.
1) Let t_1 be as in Corollary 4.3.3; we consider first the case $\tau < t_1$, then:

$$|(C_{\bar{s}}^{-1}(\tau)\, v)_1| \le \left|\Big((C_{\bar{s}}^{-1}(\tau) - C_{\bar{s},0}^{-1}(\tau))\, v\Big)_1\right| + \left|\Big(C_{\bar{s},0}^{-1}(\tau)\, v\Big)_1\right| =$$

$$\frac{1}{\sqrt{\tau}}\left|\Big(D_0(\sqrt{\tau})\,(C_{\bar{s}}^{-1}(\tau) - C_{\bar{s},0}^{-1}(\tau))\, D_0(\sqrt{\tau})\, D_0(\frac{1}{\sqrt{\tau}})\, v\Big)_1\right| + \frac{1}{\sqrt{\tau}}\left|\Big(C_{\bar{s},0}^{-1}(1)\, D_0(\frac{1}{\sqrt{\tau}})\, v\Big)_1\right|$$

Now we have already seen that:

$$\frac{1}{\sqrt{\tau}}\left|\Big(C_{\bar{s},0}^{-1}(1)\, D_0(\frac{1}{\sqrt{\tau}})\, v\Big)_1\right| \le \frac{|C_{1,0}^{-1}(1)|}{a\,\sqrt{\tau}}\, |D_0(\frac{1}{\sqrt{\tau}})\, v|$$

While for the first term of the sum we have:

$$\frac{1}{\sqrt{\tau}}\left|D_0(\sqrt{\tau})\,\Big((C_{\bar{s}}^{-1}(\tau) - C_{\bar{s},0}^{-1}(\tau))\, D_0(\sqrt{\tau})\, D_0(\frac{1}{\sqrt{\tau}})\, v\Big)_1\right| \le$$

$$\frac{1}{\sqrt{\tau}}\left|D_0(\sqrt{\tau})\,(C_{\bar{s}}^{-1}(\tau) - C_{\bar{s},0}^{-1}(\tau))\, D_0(\sqrt{\tau})\right| \left|D_0(\frac{1}{\sqrt{\tau}})\, v\right|$$

In particular $\left|D_0(\sqrt{\tau})\,(C_{\bar{s}}^{-1}(\tau) - C_{\bar{s},0}^{-1}(\tau))\, D_0(\sqrt{\tau})\right|$

$$= \sup_{|v|=1}\left| < (C_{\bar{s}}^{-1}(\tau) - C_{\bar{s},0}^{-1}(\tau))\, D_0(\sqrt{\tau})\, v,\ D_0(\sqrt{\tau})\, v > \right| \le$$

$$\sup_{|v|=1}\left| < C_{\bar{s},0}^{-1}(\tau)\, D_0(\sqrt{\tau})\, v,\ D_0(\sqrt{\tau})\, v > \right| \qquad (\textit{By } (4.29) \textit{ since } \tau < t_1)$$

$$= \sup_{|v|=1}\left| < C_{\bar{s},0}^{-1}(1)\, v, v > \right| \le \frac{|C_{1,0}^{-1}(1)|}{a}$$

Thus for $\tau < t_1$ we have:

$$|(C_{\bar{s}}^{-1}(\tau)\, v)_1| \;\leq\; \frac{C\,|D_0(\frac{1}{\sqrt{\tau}})\, v|}{\sqrt{\tau}}$$

Let τ now be in $[t_1, \bar{T}]$ then:

$$|(C_{\bar{s}}^{-1}(\tau)\, v)_1| \;=\; \frac{1}{\sqrt{\tau}}\left|\Big(D_0(\sqrt{\tau})\, C_{\bar{s}}^{-1}(\tau)\, D_0(\sqrt{\tau})\, D_0(\frac{1}{\sqrt{\tau}})\, v\Big)_1\right| \;\leq$$

$$\frac{1}{a\,\sqrt{\tau}}\max_{\tau\in[t_1,\bar{T}]}\Big(\left|D_0(\sqrt{\tau})\, C_1^{-1}(\tau)\, D_0(\sqrt{\tau})\right|\Big)\left|D_0(\frac{1}{\sqrt{\tau}})\, v\right| \;\leq\; \frac{C\,|D_0(\frac{1}{\sqrt{\tau}})\, v|}{\sqrt{\tau}}$$

2) The proof is analogous so it is omitted
3) Let t_1 be as in Corollary 4.3.3; then for τ , $t_1 \;\leq\; \tau \;\leq\; \bar{T}$:

$$|(C_{\bar{s}}^{-1}(\tau))_{11}| \;=\; \frac{1}{\tau}\,|\tau\,(C_{\bar{s}}^{-1}(\tau))_{11}| \;\leq\; \frac{C}{\tau}\max_{t_1\leq\tau\leq\bar{T}}|\tau\,(C_1^{-1}(\tau))_{11}|$$

While if $\tau < t_1$ we have:

$$|(C_{\bar{s}}^{-1}(\tau))_{11}| \;\leq\; \left|\Big(C_{\bar{s}}^{-1}(\tau) - C_{\bar{s},0}^{-1}(\tau)\Big)_{11}\right| + \left|\Big(C_{\bar{s},0}^{-1}(\tau)\Big)_{11}\right| \;=:\; I_1(\tau,\bar{s}) + I_2(\tau,\bar{s})$$

Now $I_2(\tau,\bar{s}) \leq \frac{C}{\tau}$ by Remark 17, while:

$$I_1(\tau,\bar{s}) \;=\; \frac{1}{\tau}\left|\Big(D_0(\sqrt{\tau})\,(C_{\bar{s}}^{-1}(\tau) - C_{\bar{s},0}^{-1}(\tau))\, D_0(\sqrt{\tau})\Big)_{11}\right|$$

$$\leq\; \frac{1}{\tau}\left|D_0(\sqrt{\tau})\,(C_{\bar{s}}^{-1}(\tau) - C_{\bar{s},0}^{-1}(\tau))\, D_0(\sqrt{\tau})\right| \;\leq\; \frac{C}{\tau}$$

Since in point 1) of this proof we have seen that:

$$\left|D_0(\sqrt{\tau})\,(C_{\bar{s}}^{-1}(\tau) - C_{\bar{s},0}^{-1}(\tau))\, D_0(\sqrt{\tau})\right| \;\leq\; C$$

4) , 5) can be proved in the same way of 3 $\square$

We can now extend Proposition 4.2.5 to the general case $\mu \neq 0$

Proposition 4.3.5. *Let p be a polynomial function in $|\eta|$ where $\eta = D_0(\frac{1}{\sqrt{\tau}})\;(y - M(\tau,x))$, then for every $\epsilon > 0$ there exists a constant C dependent on ϵ and p such that:*

$$|p\,(|\eta|)|\,\Gamma_{\bar{s}}(t,x,T,y) \;\leq\; C\,\Gamma_{A+\epsilon}(t,x,T,y)$$

For every $\bar{s} \in \mathbb{R}$, $x, y \in \mathbb{R}^2$, $\tau > 0$

Proof. We can choose t_0 suitably small such that (4.26), (4.27) hold and

$$(1 - C_\mu \, t_0)^2 \geq \frac{A + \frac{\epsilon}{2}}{A + \epsilon} \tag{4.30}$$

Then, for $0 < \tau \leq t_0$ we have:

$$\left| p\left(|\eta|\right) \right| \Gamma_{\bar{s}}(t, x, T, y) \leq \frac{k}{\sqrt{\det C_1(\tau)}} \left| p\left(|\eta|\right) \right| e^{-\frac{1}{2} < C_{\bar{s}}^{-1}(\tau)\, z, z>}$$

$$\leq \frac{k}{\sqrt{\det C_1(\tau)}} \left| p\left(|\eta|\right) \right| e^{-\frac{1}{2} <(1 - C_\mu \, t_0)\, C_{\bar{s},0}^{-1}(\tau)\, z, z>} \qquad (by \ (4.26))$$

$$= \frac{k}{\sqrt{\det C_1(\tau)}} \left| p\left(|\eta|\right) \right| e^{-\frac{1}{2} \frac{(1 - C_\mu \, t_0)}{\alpha(\bar{s})} < C_{1,0}^{-1}(1)\, \eta, \eta>} \leq$$

$$\frac{k}{\sqrt{\det C_1(\tau)}} \left| p\left(|\eta|\right) \right| e^{-\frac{1}{2} \frac{(1 - C_\mu \, t_0)}{A} < C_{1,0}^{-1}(1)\, \eta, \eta>} \qquad (\star)$$

Now repeating the idea of the proof of Proposition 4.2.5 we have

$$(\star) \leq \frac{k}{\sqrt{\det C_1(\tau)}} e^{-\frac{1}{2} \frac{(1 - C_\mu \, t_0)}{A + \frac{\epsilon}{2}} < C_{1,0}^{-1}(1)\, \eta, \eta>} = \frac{k}{\sqrt{\det C_1(\tau)}} e^{-\frac{1}{2} \frac{(1 - C_\mu \, t_0)}{A + \frac{\epsilon}{2}} < C_{1,0}^{-1}(\tau)\, z, z>}$$

$$\leq \frac{k}{\sqrt{\det C_1(\tau)}} e^{-\frac{1}{2} \frac{(1 - C_\mu \, t_0)^2}{A + \frac{\epsilon}{2}} < C_1^{-1}(\tau)\, z, z>} \qquad (by \ (4.27))$$

$$\leq \frac{\bar{k}}{\sqrt{\det C_{A+\epsilon}(\tau)}} e^{-\frac{1}{2} \frac{1}{A+\epsilon} < C_1^{-1}(\tau)\, z, z>} \qquad (by \ (4.30))$$

$$= \bar{k} \; \Gamma_{A+\epsilon}(t, x, T, y)$$

$\square$

Remark 21. In the case $\mu \neq 0$, the matrix $\phi(\tau)$ defined in 1.4 is

$$\phi(\tau) = \begin{pmatrix} e^{\tau \, \mu} & 0 \\ \frac{-1 + e^{\tau \, \mu}}{\mu} & 1 \end{pmatrix}$$

It is trivial then that $\phi(\tau)$ is a C^∞ function and it is bounded for τ : $0 \leq \tau \leq \bar{T}$, with $\bar{T} > 0$ arbitrary.

In particular, given $\bar{T} > 0$, by Lemma 4.3.4 and the regularity of $\phi(\tau)$ there exists a positive constant C such that for all $\bar{s} \in \mathbb{R}$, $0 < \tau \leq \bar{T}$ it holds:

$$\begin{aligned} |(C_{\bar{s}}^{-1}(\tau))_{11}\, \phi_{11}(\tau)\, \phi_{11}(\tau)| \, &\leq \, \frac{C}{\tau} \\ |(C_{\bar{s}}^{-1}(\tau))_{22}\, \phi_{22}(\tau)\, \phi_{22}(\tau)| \, &\leq \, \frac{C}{\tau^3} \end{aligned} \tag{4.31}$$

Moreover, computing the limit as $\tau \to 0^+$ of

$$\frac{|(C_{\bar{s}}^{-1}(\tau))_{12}\, \phi_{11}(\tau)\, \phi_{21}(\tau)|}{|(C_{\bar{s}}^{-1}(\tau))_{11}|} \quad \text{and} \quad \frac{|(C_{\bar{s}}^{-1}(\tau))_{22}\, \phi_{21}^2(\tau)|}{|(C_{\bar{s}}^{-1}(\tau))_{11}|}$$

It can be verified that $|(C_{\bar{s}}^{-1}(\tau))_{12}\, \phi_{11}(\tau)\, \phi_{21}(\tau)|$ and $|(C_{\bar{s}}^{-1}(\tau))_{22}\, \phi_{21}^2(\tau)|$ are a Big O of $|(C_{\bar{s}}^{-1}(\tau))_{11}|$ as $\tau \to 0^+$

hence there exist $M, t_* > 0$ such that $|(C_{\bar{s}}^{-1}(\tau))_{12}\, \phi_{11}(\tau)\, \phi_{21}(\tau)| \, \leq \, M\, |(C_{\bar{s}}^{-1}(\tau))_{11}|$

and $|(C_{\bar{s}}^{-1}(\tau))_{22}\, \phi_{21}^2(\tau)| \, \leq \, M\, |(C_{\bar{s}}^{-1}(\tau))_{11}|$ for $0 < \tau < t_*$

Then by Lemma 4.3.4 for $0 < \tau < t_*$ we have:

$$|(C_{\bar{s}}^{-1}(\tau))_{12}\, \phi_{11}(\tau)\, \phi_{21}(\tau)| \, \leq \, M\, |(C_{\bar{s}}^{-1}(\tau))_{11}| \leq \, \frac{C}{\tau}$$

$$|(C_{\bar{s}}^{-1}(\tau))_{22}\, \phi_{21}^2(\tau)| \, \leq \, M\, |(C_{\bar{s}}^{-1}(\tau))_{11}| \leq \, \frac{C}{\tau}$$

Furthermore both $\tau\, |(C_{\bar{s}}^{-1}(\tau))_{12}\, \phi_{11}(\tau)\, \phi_{21}(\tau)|$ and $\tau\, |(C_{\bar{s}}^{-1}(\tau))_{22}\, \phi_{21}^2(\tau)|$ are

C^∞ and bounded functions for τ in $[t_*, \bar{T}]$; thus also for τ, $t_* \, \leq \, \tau \, \leq \, \bar{T}$ it holds

$$\begin{aligned} |(C_{\bar{s}}^{-1}(\tau))_{12}\, \phi_{11}(\tau)\, \phi_{21}(\tau)| \, &\leq \, \frac{C}{\tau} \\ |(C_{\bar{s}}^{-1}(\tau))_{22}\, \phi_{21}^2(\tau)| \, &\leq \, \frac{C}{\tau} \end{aligned}$$

In conclusion for $0 < \tau \leq \bar{T}$ we have:

$$|(C_{\bar{s}}^{-1}(\tau))_{12}\, \phi_{11}(\tau)\, \phi_{21}(\tau)| \, \leq \, \frac{C}{\tau} \tag{4.32}$$

$$|(C_{\bar{s}}^{-1}(\tau))_{22}\, \phi_{21}^2(\tau)| \, \leq \, \frac{C}{\tau} \tag{4.33}$$

We are ready now to estimate the derivatives of $\Gamma_{\bar{s}}(t,x,T,y)$ and thus extend the Theorem 4.2.6 to the general case

Theorem 4.3.6. *For every $\epsilon, \bar{T} > 0$, $k \in \mathbb{N}$ there exists a constant C depends on $\epsilon, \bar{T}, \mu, k$ such that*

1.

$$|\partial_s^{(k)}\Gamma_{\bar{s}}(t,x,T,y)| \leq \frac{C}{\tau^{\frac{k}{2}}}\,\Gamma_{A+\epsilon}(t,x,T,y)$$

2.

$$|\partial_a^{(k)}\Gamma_{\bar{s}}(t,x,T,y)| \leq \frac{C}{\tau^{\frac{3k}{2}}}\,\Gamma_{A+\epsilon}(t,x,T,y)$$

For every $\bar{s} \in \mathbb{R}$, $x, y \in \mathbb{R}^2$, $0 < \tau \leq \bar{T}$

Proof. It is proved only (1) since the proof of (2) is analogous.
We prove directly the thesis for $k = 1, 2$. The general case follows then by Proposition 2.1.2, and Proposition 4.3.5 repeating the identical proof seen in Theorem 4.2.6 for $k > 2$

$$(k=1) \qquad |\partial_s\Gamma_{\bar{s}}(t,x,T,y)| = \Gamma_{\bar{s}}(t,x,T,y)\,|(C_{\bar{s}}^{-1}(\tau)\,z\,\phi(\tau))_1| \leq$$

$$\Gamma_{\bar{s}}(t,x,T,y)\,|\phi_{11}(\tau)|\,|(C_{\bar{s}}^{-1}(\tau)\,z)_1| + \Gamma_{\bar{s}}(t,x,T,y)\,|\phi_{21}(\tau)|\,|(C_{\bar{s}}^{-1}(\tau)\,z)_2|$$

Now by 1) of Lemma 4.3.4 and since $\phi_{11}(\tau)$ is bounded for $\tau,\ 0 < \tau \leq \bar{T}$ we have:

$$|\phi_{11}(\tau)|\,|(C_{\bar{s}}^{-1}(\tau)\,z)_1| \leq \frac{C_1\,||D_0(\frac{1}{\sqrt{\tau}})\,z||}{\sqrt{\tau}}$$

Furthermore, computing the limit as $\tau \to 0^+$ as in Remark 21 it is easy to verify that $|\phi_{21}(\tau)\,(C_{\bar{s}}^{-1}(\tau)\,z)_2|$ is asymptotic equivalent to $|(C_{\bar{s}}^{-1}(\tau)\,z)_1|$
Thus there exist $M, t_* > 0$ such that:

$$|\phi_{21}(\tau)\,(C_{\bar{s}}^{-1}(\tau)\,z)_2| \leq M\,|(C_{\bar{s}}^{-1}(\tau)\,z)_1| \leq \frac{C_2\,||D_0(\frac{1}{\sqrt{\tau}})\,z||}{\sqrt{\tau}} \quad \text{for } 0 < \tau < t_*$$

While the function $\frac{|\phi_{21}(\tau)\,(C_{\bar{s}}^{-1}(\tau)\,z)_2|\,\sqrt{\tau}}{||D_0(\frac{1}{\sqrt{\tau}})||}$ is bounded for $\tau \in [t_*, \bar{T}]$
Hence in conclusion it holds:

$$|\phi_{21}(\tau)\,(C_{\bar{s}}^{-1}(\tau)\,z)_2| \;\leq\; \frac{C_3\,||D_0(\frac{1}{\sqrt{\tau}})\,z||}{\sqrt{\tau}} \quad \text{for } 0 < \tau \leq \bar{T}$$

Then for $\tau, 0 < \tau \leq \bar{T}$ we have:

$$|\partial_s \Gamma_{\bar{s}}(t,x,T,y)| \;\leq\; 2\,\Gamma_{\bar{s}}(t,x,T,y)\,\frac{C_4\,||D_0(\frac{1}{\sqrt{\tau}})\,z||}{\sqrt{\tau}} \;\leq\; \frac{C}{\sqrt{\tau}}\,\Gamma_{A+\epsilon}(t,x,T,y)$$

By Proposition 4.3.5

$$(k=2) \quad |\partial_{ss}\Gamma_{\bar{s}}(t,x,T,y)| = \Gamma_{\bar{s}}(t,x,T,y)\left(|(C_{\bar{s}}^{-1}(\tau)\,z\,\phi(\tau))_1|^2 + |(\phi^T(\tau)\,C_{\bar{s}}^{-1}(\tau)\,\phi(\tau)_1)_1|\right)$$

We have seen in point 1) of this proof that:

$$|(C_{\bar{s}}^{-1}(\tau)\,z\,\phi(\tau))_1|^2 \;\leq\; \frac{\bar{C}}{\tau}\,||D_0(\frac{1}{\sqrt{\tau}})\,z||^2$$

While

$$|(\phi^T(\tau)\,C_{\bar{s}}^{-1}(\tau)\,\phi(\tau)_1)_1| \;\leq\; |\phi_{11}(\tau)^2\,(C_{\bar{s}}^{-1}(\tau))_{11}| + |\phi_{11}(\tau)\,\phi_{21}(\tau)\,(C_{\bar{s}}^{-1}(\tau))_{21}|$$

$$+|\phi_{11}(\tau)\,\phi_{21}(\tau)\,(C_{\bar{s}}^{-1}(\tau))_{12}| + |\phi_{21}(\tau)^2\,(C_{\bar{s}}^{-1}(\tau))_{22}| \;=$$

$$|\phi_{11}(\tau)^2\,(C_{\bar{s}}^{-1}(\tau))_{11}| + 2\,|\phi_{11}(\tau)\,\phi_{21}(\tau)\,(C_{\bar{s}}^{-1}(\tau))_{21}| + |\phi_{21}(\tau)^2\,(C_{\bar{s}}^{-1}(\tau))_{22}| \;\leq\; \frac{\hat{C}}{\tau}$$

By (4.33), (4.34), (4.35)
Then in conclusion by Proposition 4.3.5 :

$$|\partial_{ss}\Gamma_{\bar{s}}(t,x,T,y)| \;\leq\; \frac{C}{\tau}\,\Gamma_{A+\epsilon}(t,x,T,y)$$

□

Corollary 4.3.7. *For every $\epsilon, \bar{T} > 0$, k, $h \in \mathbb{N}$ there exists a constant C depends on $\epsilon, \bar{T}, \mu, k, h$ such that*

$$|\partial_a^{(h)}\partial_s^{(k)}\Gamma_{\bar{s}}(t,x,T,y)| \;\leq\; \frac{C}{\tau^{\frac{k}{2}+\frac{3\,h}{2}}}\,\Gamma_{A+\epsilon}(t,x,T,y)$$

For every $\bar{s} \in \mathbb{R}$, $x, y \in \mathbb{R}^2$, $0 < \tau \leq \bar{T}$

Proof. The proof is analogous to that of Corollary 4.2.7 so it is omitted □

Remark 22. As in the case $\mu = 0$ the same estimations can be obtained for the derivatives of $\Gamma_{\bar{s}}$ with also respect to the dual variables (S, A), that is:

$$|\partial_s^{(k)} \partial_S^{(j)} \partial_a^{(h)} \partial_A^{(n)} \Gamma_{\bar{s}}(t, x, T, y)| \leq \frac{C}{\tau^{\frac{k+j}{2} + \frac{3(h+n)}{2}}} \Gamma_{A+\epsilon}(t, x, T, y) \tag{4.34}$$

For every $\bar{s} \in \mathbb{R}$, $x, y \in \mathbb{R}^2$, $0 < \tau \leq \bar{T}$

4.4 The parametrix method and error bound estimations

In this section we modify and adapt the original parametrix method to get an estimation of the error committed by our approximation of order N of the fundamental solution for the operator K in (4.2).
Let $\Gamma_{s_0}^N(t, x, T, y)$ be our approximation of order N seen in Chapter 1.
Our modification of the parametrix method allows to construct a fundamental solution Γ for K starting from $\Gamma_{s_0}^N(t, x, T, y)$; then, in this way, we can estimate the difference between Γ and $\Gamma_{s_0}^N$ and thus compute the error committed by our approximation.

We remind that for $\bar{s} \in \mathbb{R}$, $\Gamma_{\bar{s}}$ denotes the fundamental solution for the "frozen" Kolmogorov operator:

$$K_{\bar{s}} = \frac{1}{2} \alpha(\bar{s}) \partial_{ss} + \mu s \partial_s + s \partial_a + \partial_t$$

The **parametrix** is defined as the function:

$$P(t, s, a, T, S, A) := \Gamma_S(t, s, a, T, S, A)$$

Our idea is to use the N^{th}-order approximation Γ^N as a parametrix; that is:

$$P^N(t, s, a, T, S, A) := \Gamma_S^N(t, s, a, T, S, A) \tag{4.35}$$

Or equivalently:

$$P^N(t, x, T, y) := \Gamma_S^N(t, x, T, y) \tag{4.36}$$

We now look for the fundamental solution Γ in the form:

$$\Gamma(t, x, T, y) = P^N(t, x, T, y) + J^N(t, x, T, y) \tag{4.37}$$

The function J^N is unknown and supposed to be of the form:

$$J^N(t,x,T,y) \;=\; \int_t^T \int_{\mathbb{R}^2} P^0(t,x,\sigma,\xi)\, \Phi^N(\sigma,\xi,T,y)\, d\sigma d\xi \tag{4.38}$$

Where Φ^N has to be determined by imposing that Γ is solution to K:

$$0 \;=\; K\,\Gamma(t,x,T,y) \;=\; K\,P^N(t,x,T,y) + K\,J^N(t,x,T,y) \;=$$

$$K\,P^N(t,x,T,y) + \int_t^T \int_{\mathbb{R}^2} K\left(P^0(t,x,\sigma,\xi)\right) \Phi^N(\sigma,\xi,T,y)\, d\sigma d\xi - \Phi^N(t,x,T,y)$$

since $\partial_t\Big(\int_t^T \Gamma_{\xi_1}(0,0,\sigma,\xi)d\sigma\Big)_{t=0} \;=\; -\delta_0(\xi)$

Thus we have:

$$\Phi^N(t,x,T,y) \;=\; K\,P^N(t,x,T,y) + \int_t^T \int_{\mathbb{R}^2} K\left(P^0(t,x,\sigma,\xi)\right) \Phi^N(\sigma,\xi,T,y)\, d\sigma d\xi$$

We can see then the function Φ^N as the fixed point of the operator G

$$G(u) \;:=\; K\,P^N(t,x,T,y) + \int_t^T \int_{\mathbb{R}^2} K\left(P^0(t,x,\sigma,\xi)\right) u(\sigma,\xi,T,y)\, d\sigma d\xi \tag{4.39}$$

So we can determinate Φ^N as the limit of the following iterative process:

$$\Phi_1^N(t,x,T,y) \;:=\; K\,P^N(t,x,T,y)$$

$$\Phi_{n+1}^N(t,x,T,y) \;:=\; K\,P^N(t,x,T,y) + \int_t^T \int_{\mathbb{R}^2} K\left(P^0(t,x,\sigma,\xi)\right) \Phi_n^N(\sigma,\xi,T,y)\, d\sigma d\xi$$

Now defining iteratively:

$$\begin{aligned} Z_1^N \;&:=\; K\,P^N(t,x,T,y) \\ Z_{n+1}^N \;&:=\; \int_t^T \int_{\mathbb{R}^2} K\left(P^0(t,x,\sigma,\xi)\right) Z_n^N(\sigma,\xi,T,y)\, d\sigma d\xi \end{aligned} \tag{4.40}$$

We have that for every $n \in \mathbb{N}$

$$\Phi_n^N(t,x,T,y) \;=\; \sum_{k=1}^{n} Z_k^N(t,x,T,y)$$

So we can determinate Φ^N as the limit of the following series:

$$\Phi^N(t,x,T,y) \;=\; \sum_{n=1}^{\infty} Z_n^N(t,x,T,y) \tag{4.41}$$

We consider now the family of dilatations $(D(\lambda))_{\lambda>0}$ defined by:

$$D(\lambda) := (\lambda^2, D_0(\lambda)) = diag(\lambda^2, \lambda, \lambda^3) \tag{4.42}$$

Remark 23. If $\mu = 0$, the Kolmogorov operator $K_{\bar{s}}$ is homogeneous of degree two with respect to the dilatations $(D(\lambda))_{\lambda>0}$; that means:

$$K_{\bar{s}}\Big(u\,(D(\lambda)(t,s,a))\Big) = \lambda^2\,(K_{\bar{s}}\,u)\,(D(\lambda)(t,s,a))$$

Indeed:

$$K_{\bar{s}}\Big(u\,(D(\lambda)(t,s,a))\Big) = K_{\bar{s}}\Big(u(\lambda^2\,t, \lambda\,s, \lambda^3\,a)\Big) =$$

$$\frac{1}{2}\,\alpha(\bar{s})\,\partial_{ss}u(\lambda^2 t, \lambda s, \lambda^3 a) + s\,\partial_a u(\lambda^2\,t, \lambda\,s, \lambda^3\,a) + \partial_t u(\lambda^2\,t, \lambda\,s, \lambda^3\,a) =$$

$$\frac{\lambda^2}{2}\,\alpha(\bar{s})\,\partial_{s's'}u(\lambda^2 t, \lambda s, \lambda^3 a) + s\,\lambda^3\,\partial_{a'}u(\lambda^2\,t, \lambda\,s, \lambda^3\,a) + \lambda^2\,\partial_{t'}u(\lambda^2\,t, \lambda\,s, \lambda^3\,a)$$

$$= \lambda^2\Big(\frac{1}{2}\,\alpha(\bar{s})\,\partial_{s's'}u(\lambda^2 t, \lambda s, \lambda^3 a) + s\,\lambda\,\partial_{a'}u(\lambda^2\,t, \lambda\,s, \lambda^3\,a) + \partial_{t'}u(\lambda^2\,t, \lambda\,s, \lambda^3\,a)\Big)$$

$$= \lambda^2\,(K_{\bar{s}}\,u)\,(D(\lambda)(t,s,a))$$

Then we define the following norm on $\mathbb{R}^3$ homogeneous of degree one with respect to the dilatations $(D(\lambda))$

$$||(t,s,a)||_K := |t|^{\frac{1}{2}} + |s| + |a|^{\frac{1}{3}} \tag{4.43}$$

Finally we define the following operation $\circ$ from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}^3$

$$\begin{aligned}(t,s,a)\circ(t',s',a') &= (t,x)\circ(t',x') := (t+t', s + (e^{tB}\,x')_1, a + (e^{tB}\,x')_2)\\ &= (t+t', x + e^{tB}\,x')\end{aligned} \tag{4.44}$$

Where B is the matrix in (4.5), $x = (s,a)$ and $x' = (s',a')$

Remark 24. Let K_w be a constant coefficients Kolmogorov operator

$$K_w = \frac{1}{2}\,w\,\partial_{ss} + \mu\,s\,\partial_s + s\,\partial_a + \partial_t$$

whit w constant; then it has invariant solution with respect to the translation defined by $\circ$.
This means that if $u(t,x)$ is a solution of K_w then $v(t,x) := u(t+t', x+e^{tB}\,x')$ is still a solution of K_w for every $t' > 0, x' \in \mathbb{R}^2$.

Now we assume that the operator K in (4.2) satisfies also the following hypothesis of Lipschitz-continuity with respect to the operation $\circ$ and the norm $||\dot{}||_K$:

$(H2)$ *There exists a positive constants* L *such that:*
$|\alpha(s)-\alpha(s')| \leq L\,||(t,s,a)\circ(t',s',a')^{-1}||_K \qquad \forall (t,s,a),(t',s',a') \in \mathbb{R}^+\times\mathbb{R}^2$

Remark 25. The inverse of (t,s,a) with respect to the operation $\circ$, $(t,s,a)^{-1}$, is equal to $(-t, -e^{-tB}x)$. Thus, with respect to our notations we have:

$$(T,S,A)\,\circ\,(t,s,a)^{-1} \;=\; (T-t, y-e^{TB}\,(e^{-tB}x)) \;=\; (\tau, z) \tag{4.45}$$

since $y - M(t,T,x) = y - e^{\tau B}\,x$ by (2.29)
Then we have:

$$||(T,S,A)\circ(t,s,a)^{-1}||_K = ||(\tau,z)||_K = \tau^{\frac{1}{2}} + |S-(e^{\tau B}\,x)_1| + |A-(e^{\tau B}\,x)_2|^{\frac{1}{3}} =$$

$$\tau^{\frac{1}{2}}\left(1+\frac{1}{\tau^{\frac{1}{2}}}\,|S-(e^{\tau B}\,x)_1| + \frac{1}{\tau^{\frac{1}{2}}}\,|A-(e^{\tau B}\,x)_2|^{\frac{1}{3}}\right) = \tau^{\frac{1}{2}}\left|\left|\left(1, D_0\!\left(\frac{1}{\sqrt{\tau}}\right)\right) z\right|\right|_K$$

$$= \tau^{\frac{1}{2}}\,||(1,\eta)||_K$$

We now take $N = 0$ and we provide in Theorem 4.4.2 a theoretic error bound for the approximation of order zero

Proposition 4.4.1. *For every $\epsilon, \bar{T} > 0$, there exists a positive constant C depends on $\epsilon, \bar{T}, \mu$, such that:*

$$|Z_n^0(t,x,T,y)| \;\leq\; \frac{M_n\,\Gamma_{A+\epsilon}(t,x,T,y)}{\tau^{1-\frac{n}{2}}}$$

For every $n \in \mathbb{N}$, $x, y \in \mathbb{R}^2$, $0 < \tau \leq \bar{T}$. Where

$$M_n = \frac{C^n \, \Gamma_E^n(\frac{1}{2})}{\Gamma_E(\frac{n}{2})} \tag{4.46}$$

With Γ_E the Euler Gamma function

Proof. By induction on n

If $n = 1$, for $x, y \in \mathbb{R}^2$, $0 < \tau \leq \bar{T}$ we have:

$$|Z_1^0(t,x,T,y)| = |K \, P^0(t,x,T,y)| = |(K - K_S) \, \Gamma_S(t,x,T,y)|$$

(since $K_S \, \Gamma_S = 0$)

$$= \left| \frac{\alpha(s)}{2} - \frac{\alpha(S)}{2} \right| |\partial_{ss} \Gamma_S(t,x,T,y)| \leq L \, \tau^{\frac{1}{2}} \, ||(1,\eta)||_K \, \frac{\bar{C} \, \Gamma_{A+\epsilon}(t,x,T,y)}{\tau}$$

(by hypothesis ($H2$), Remark 25 and Theorem 4.3.6)

$$\leq C \, \frac{\Gamma_{A+\epsilon}(t,x,T,y)}{\tau^{\frac{1}{2}}} \qquad \text{(by Proposition 4.3.5)}$$

In particular: $M_1 = C$

We now assume that the thesis holds for n and prove it for $n+1$. By definition of Z_{n+1}^0 and inductive hypothesis we have:

$$|Z_{n+1}^0(t,x,T,y)| \leq \int_t^T \int_{\mathbb{R}^2} \frac{C \, \Gamma_{A+\epsilon}(t,x,\sigma,\xi)}{(\sigma - t)^{\frac{1}{2}}} \, M_n \, \frac{\Gamma_{A+\epsilon}(\sigma,\xi,T,y)}{(T-\sigma)^{1-\frac{n}{2}}} d\sigma d\xi$$

$$= C \, M_n \, \Gamma_{A+\epsilon}(t,x,T,y) \int_t^T \frac{1}{(\sigma - t)^{\frac{1}{2}}} \, \frac{1}{(T-\sigma)^{1-\frac{n}{2}}} d\sigma$$

(by the reproduction property of $\Gamma_{A+\epsilon}$)

$$= C \, M_n \, \Gamma_{A+\epsilon}(t,x,T,y) \frac{\Gamma_E(\frac{1}{2}), \Gamma_E(\frac{n}{2})}{\Gamma_E(\frac{n+1}{2})} \, \frac{1}{\tau^{1-\frac{n+1}{2}}}$$

Now:

$$C \, M_n \, \frac{\Gamma_E(\frac{1}{2}), \Gamma_E(\frac{n}{2})}{\Gamma_E(\frac{n+1}{2})} = C^{n+1} \, \frac{\Gamma_E^{n+1}(\frac{1}{2})}{\Gamma_E(\frac{n}{2})} \, \frac{\Gamma_E(\frac{n}{2})}{\Gamma_E(\frac{n+1}{2})} = M_{n+1}$$

Thus in conclusion:

$$|Z^0_{n+1}(t,x,T,y)| \le \frac{M_{n+1}}{\tau^{1-\frac{n+1}{2}}}\,\Gamma_{A+\epsilon}(t,x,T,y)$$

□

Theorem 4.4.2. *For every $\epsilon > 0$ and $\bar{T} > 0$ there exists a positive constant C such that:*

$$|\Gamma(t,s,a,T,S,A) - \Gamma_S(t,s,a,T,S,A)| \le C\,(T-t)^{\frac{1}{2}}\,\Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

$\forall\, s,S,a,A \in \mathbb{R},$ *and* t,T *such that* $0 < T-t \le \bar{T},$

Proof.

$$|\Gamma(t,x,T,y) - \Gamma_S(t,x,T,y)| = \Big|\int_t^T\int_{\mathbb{R}^2}\Gamma_S(t,x,\sigma,\xi)\,\Phi^0(\sigma,\xi,T,y)d\sigma d\xi\Big|$$

by (4.39), (4.40); then by (4.43):

$$\le \int_t^T\int_{\mathbb{R}^2}\Gamma_{A+\epsilon}(t,x,\sigma,\xi)\,\Big|\sum_{n=1}^{\infty}Z^0_n(\sigma,\xi,T,y)\Big|d\sigma d\xi$$

(by *Proposition* 4.4.1) $$\le \sum_{n=1}^{\infty}\int_t^T\int_{\mathbb{R}^2}\Gamma_{A+\epsilon}(t,x,\sigma,\xi)\;M_n\,\frac{\Gamma_{A+\epsilon}(\sigma,\xi,T,y)}{(T-\sigma)^{1-\frac{n}{2}}}d\sigma d\xi$$

(by the reproduction property of $\Gamma_{A+\epsilon}$)

$$= \Gamma_{A+\epsilon}(t,x,T,y)\sum_{n=1}^{\infty}\int_t^T M_n\,(T-\sigma)^{\frac{n}{2}-1}d\sigma$$

$$= \Gamma_{A+\epsilon}(t,x,T,y)\sum_{n=1}^{\infty}M_n\frac{2}{n}\,(T-t)^{\frac{n}{2}} =$$

$$\Gamma_{A+\epsilon}(t,x,T,y)\,(T-t)^{\frac{1}{2}}\sum_{n=0}^{\infty}M_{n+1}\frac{2}{n+1}\,(T-t)^{\frac{n}{2}} \le$$

$$\Gamma_{A+\epsilon}(t,x,T,y)\,(T-t)^{\frac{1}{2}}\sum_{n=0}^{\infty}M_{n+1}\frac{2}{n+1}\,\bar{T}^{\frac{n}{2}} \qquad (\star)$$

We compute now the radius of convergence of the power series using the ratio test

$$\Big|\frac{a_n}{a_{n-1}}\Big| = \bar{T}^{\frac{1}{2}}\,\frac{n}{n+1}\,C\,\Gamma_E\Big(\frac{1}{2}\Big)\,\frac{\Gamma_E(\frac{n}{2})}{\Gamma_E(\frac{n+1}{2})}$$

$\frac{n}{n+1} \to 1$ as $n \to \infty$, so we have to compute $\frac{\Gamma_E(\frac{n}{2})}{\Gamma_E(\frac{n+1}{2})}$

We utilize the following Stirling's approximation formula for the Euler Gamma function (see [7])

$$\Gamma_E(x) \sim \sqrt{2\,\pi\,(x-1)}\,\left(\frac{x-1}{e}\right)^{x-1} \quad \text{as } x \to \infty$$

Then

$$\frac{\Gamma_E(\frac{n}{2})}{\Gamma_E(\frac{n+1}{2})} \sim \frac{\sqrt{2\,\pi\,(\frac{n}{2}-1)}\,\left(\frac{\frac{n}{2}-1}{e}\right)^{\frac{n}{2}-1}}{\sqrt{2\,\pi\,(\frac{n+1}{2}-1)}\,\left(\frac{\frac{n+1}{2}-1}{e}\right)^{\frac{n+1}{2}-1}} = \sqrt{\frac{n-2}{n-1}}\,\left(\frac{n-2}{n-1}\right)^{\frac{n}{2}-1}\,\left(\frac{2\,e}{n-1}\right)^{\frac{1}{2}}$$

Now

$$\sqrt{\frac{n-2}{n-1}} \to 1\;,\;\left(\frac{2\,e}{n-1}\right)^{\frac{1}{2}} \to 0 \qquad \text{as } n \to \infty$$

While

$$\lim_{n\to\infty}\left(\frac{n-2}{n-1}\right)^{\frac{n}{2}-1} = \lim_{n\to\infty}\frac{1}{\left(\frac{n-2+1}{n-2}\right)^{\frac{n-2}{2}}} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n-2}\right)^{\frac{n-2}{2}}} = \frac{1}{\sqrt{e}}$$

Thus $\frac{\Gamma_E(\frac{n}{2})}{\Gamma_E(\frac{n+1}{2})} \to 0$ as $n \to \infty$ and so the series has radius of convergence equal to infinity.
In particular there exists a positive constant C such that

$$\sum_{n=0}^{\infty} M_{n+1}\frac{2}{n+1}\,\bar{T}^{\frac{n}{2}} \le C$$

And then finally

$$(\star) \le C\,(T-t)^{\frac{1}{2}}\;\Gamma_{A+\epsilon}\,(t,x,T,y)$$

□

In conclusion of this work we provide a theoretic error bound for the approximation of order one.
First of all we remind our approximation of order one is:

$$\Gamma^1_{s_0}(t,x,T,y) = G^0_{s_0}(t,x,T,y) + G^1_{s_0}(t,x,T,y)$$

where $G^0_{s_0}(t,x,T,y) = \Gamma^0_{s_0}(t,x,T,y)$, and $G^1_{s_0}$ is the solution of the Cauchy problem:

$$\begin{cases} K_{s_0}\,G^1_{s_0}(t,x,T,y) \;=\; -K^1_{s_0}\,G^0_{s_0}(t,x,T,y) \\ \\ G^1_{s_0}(T,x,T,y) \;=\; 0 \end{cases} \tag{4.47}$$

with

$$K^1_{s_0} = \frac{1}{2}\,\alpha'(s_0)\,(s-s_0)\,\partial_{ss} \tag{4.48}$$

We recall finally that in Chapter 2 we found:

$$G^1_{\bar s}(t,s,a,T,S,A) = \tilde J^1{}_{t,T,S,A}\,\Gamma_{\bar s}(t,s,a,T,S,A)$$

where

$$\begin{aligned}\tilde J^1{}_{t,T,S,A} =& \alpha_1 \int_t^T \Big(\bar s - (m_{t,x}(\sigma))_1\Big)\, V_y^2(T,\sigma) - i\,C_{11}(t,\sigma)\,V_y^3(T,\sigma) \\ &- i\,C_{12}(t,\sigma)\,V_y^2(T,\sigma)\,W_y(T,\sigma)\,d\sigma\end{aligned} \tag{4.49}$$

and $V_y(T,\sigma)$ and $W_y(T,\sigma)$ are the differential operators:

$$V_y(T,\sigma) := i\left(e^{(T-\sigma)\,B^*}\right)_{11}\partial_S + i\left(e^{(T-\sigma)\,B^*}\right)_{12}\partial_A \tag{4.50}$$

$$W_y(T,\sigma) = i\left(e^{(T-\sigma)\,B^*}\right)_{21}\partial_S + i\left(e^{(T-\sigma)\,B^*}\right)_{22}\partial_A \tag{4.51}$$

with B the matrix in (4.5)

Remark 26. Computing explicitly $e^{(T-\sigma)\,B^*}$ we have:

$$e^{(T-\sigma)\,B^*} = \begin{pmatrix} 1 & T-\sigma \\ 0 & 1 \end{pmatrix} \qquad \text{if} \quad \mu = 0$$

$$e^{(T-\sigma)\,B^*} = \begin{pmatrix} e^{(T-\sigma)\,\mu} & \frac{e^{(T-\sigma)\,\mu}-1}{\mu} \\ 0 & 1 \end{pmatrix} \qquad \text{if} \quad \mu \neq 0$$

It is trivial then to conclude that in both the cases there exists a positive constant C_μ depending on μ such that:

$$\left(e^{(T-\sigma)\,B^*}\right)_{11} \le C_\mu \quad , \quad \left(e^{(T-\sigma)\,B^*}\right)_{12} \le C_\mu\,(T-t) \tag{4.52}$$

For every $\sigma\,:\,t \le \sigma \le T$, $0 < \tau \le \bar T$. While

$$\left(e^{(T-\sigma)\,B^*}\right)_{21} = 0 \quad , \quad \left(e^{(T-\sigma)\,B^*}\right)_{21} = 1 \tag{4.53}$$

Proposition 4.4.3. *For every $\epsilon, \bar T > 0$, $k\,,\,h\,,\,j \in \mathbb{N}$ there exists a constant C depends on $\epsilon, \bar T, \mu, k, h, j$ such that*

$$1) \quad \left| V_y^{(k)}(T,\sigma)\,\partial_s^{(j)}\,\Gamma_{\bar s}(t,s,a,T,S,A)\right| \le \frac{C}{\tau^{\frac{k+j}{2}}}\,\Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

$$2) \quad \left| W_y^{(h)}(T,\sigma)\,\partial_s^{(j)}\,\Gamma_{\bar{s}}(t,s,a,T,S,A) \right| \le \frac{C}{\tau^{\frac{3h+j}{2}}}\,\Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

$$3) \quad \left| V_y^{(k)}(T,\sigma)\,W_y^{(h)}(T,\sigma)\,\partial_s^{(j)}\,\Gamma_{\bar{s}}(t,s,a,T,S,A) \right| \le \frac{C}{\tau^{\frac{k+j}{2}+\frac{3h}{2}}}\,\Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

For every $\bar{s} \in \mathbb{R},\ s,a,S,A \in \mathbb{R},\ 0 < \tau \le \bar{T},\ t \le \sigma \le T$

Proof. We only prove 1 since the proofs of 2 and 3 are analogous.

$$\left|V_y^{(k)}(T,\sigma)\,\partial_s^{(j)}\,\Gamma_{\bar{s}}(t,s,a,T,S,A)\right| = \Big|\sum_{n=0}^{k}\binom{k}{n}\Big(i\left(e^{(T-\sigma)B^*}\right)_{11}\Big)^{k-n}\Big(i\left(e^{(T-\sigma)B^*}\right)_{12}\Big)^{n}$$

$$\cdot\,\partial_S^{(k-n)}\,\partial_A^{(n)}\,\partial_s^{(j)}\,\Gamma_{\bar{s}}(t,s,a,T,S,A)\Big| \le \sum_{n=0}^{k}\binom{k}{n}\left|\left(e^{(T-\sigma)B^*}\right)_{11}\right|^{k-n}\left|\left(e^{(T-\sigma)B^*}\right)_{12}\right|^{n}$$

$$\cdot\left|\partial_S^{(k-n)}\,\partial_A^{(n)}\,\partial_s^{(j)}\,\Gamma_{\bar{s}}(t,s,a,T,S,A)\right|$$

(by Remark 26 and Remark 22)

$$\le \sum_{n=0}^{k}\binom{k}{n}\tilde{C}\,\tau^{n}\,\frac{\hat{C}}{\tau^{\frac{k-n}{2}+\frac{3n}{2}+\frac{j}{2}}}\,\Gamma_{A+\epsilon}(t,s,a,T,S,A) = \frac{C}{\tau^{\frac{k+j}{2}}}\,\Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

□

Proposition 4.4.4. *For every* $\epsilon, \bar{T} > 0$ *there exists a positive constant* C *depends on* $\epsilon, \bar{T}, \mu$ *such that*

$$|\partial_{ss} G_S^1(t,s,a,T,S,A)| \le \frac{C}{\sqrt{\tau}}\,\Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

For every $s,a,S,A \in \mathbb{R},\ 0 < T-t \le \bar{T}$

Proof.

$$\partial_{ss} G_S^1(t,s,a,T,S,A) = \partial_{ss}\Bigg(\alpha_1\int_t^T \Big(S-(m_{t,x}(\sigma))_1\Big)\,V_y^2(T,\sigma) - i\,C_{11}(t,\sigma)\,V_y^3(T,\sigma)$$

$$-i\,C_{12}(t,\sigma)\,V_y^2(T,\sigma)\,W_y(T,\sigma)\,d\sigma\ \ \Gamma_S(t,s,a,T,S,A)\Bigg) =$$

$$\partial_{ss}\Big(\alpha_1\int_t^T \Big(S-(m_{t,x}(\sigma))_1\Big)\,V_y^2(T,\sigma)\,d\sigma\,\Gamma_S(t,s,a,T,S,A)\Big) +$$

$$\alpha_1 \int_t^T -i\, C_{11}(t,\sigma)\, V_y^3(T,\sigma)\, d\sigma\, \partial_{ss}\, \Gamma_S(t,s,a,T,S,A) +$$

$$\alpha_1 \int_t^T -i\, C_{12}(t,\sigma)\, V_y^2(T,\sigma)\, W_y(T,\sigma)\, d\sigma\, \partial_{ss}\, \Gamma_S(t,s,a,T,S,A) \;=\; I+II+III$$

We now estimate the three terms separately.
Reminding $m_{t,x}(\sigma) = e^{(\sigma-t)\,B}\, x$ we have:

$$I \;=\; -2\,\alpha_1 \int_t^T \left(e^{(\sigma-t)\,B}\right)_{11} V_y^2(T,\sigma)\, d\sigma\, \partial_s\, \Gamma_S(t,s,a,T,S,A) +$$

$$\alpha_1 \int_t^T \Big(S - (m_{t,x}(\sigma))_1\Big)\, V_y^2(T,\sigma)\, d\sigma\;\, \partial_{ss}\, \Gamma_S(t,s,a,T,S,A)$$

Remark 26 and Proposition 4.4.3 yield:

$$|I| \;\leq\; C_1\,(T-t)\,\frac{C_2}{(T-t)^{\frac{2+1}{2}}}\,\Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

$$+\,\alpha_1 \int_t^T \Big|S - (m_{t,x}(\sigma))_1\Big|\, d\sigma\, \frac{C_3}{(T-t)^{\frac{2+2}{2}}}\, \Gamma_{A+\frac{\epsilon}{2}}(t,s,a,T,S,A)$$

We now observe that

$$\int_t^T \Big|S-(m_{t,x}(\sigma))_1\Big|\, d\sigma = \int_t^T \Big|S-\left(e^{(\sigma-t)\,B}\right)_{11} s\Big|\, d\sigma \leq |S(T-t)| + \Big|\left(e^{(T-t)\,B}\right)_{21} s\Big|$$

Then

$$\alpha_1 \int_t^T \Big|S - (m_{t,x}(\sigma))_1\Big|\, d\sigma\, \frac{C_3}{(T-t)^{\frac{2+2}{2}}}\, \Gamma_{A+\frac{\epsilon}{2}}(t,s,a,T,S,A)$$

$$\leq\; \alpha_1\, \frac{C_3}{(T-t)^{\frac{2+2}{2}}}\,(T-t)^{\frac{3}{2}} \left(\frac{|S|}{(T-t)^{\frac{1}{2}}}\, \Gamma_{A+\frac{\epsilon}{2}}(t,s,a,T,S,A) \right.$$

$$\left. + \frac{\Big|\left(e^{(T-t)\,B}\right)_{21} s\Big|}{(T-t)^{\frac{3}{2}}}\, \Gamma_{A+\frac{\epsilon}{2}}(t,s,a,T,S,A) \right)$$

(by Proposition 4.3.5)

$$\leq\; \alpha_1\, \frac{C_3}{(T-t)^{\frac{1}{2}}}\, 2\,\Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

In conclusion:

$$|I| \leq \frac{C_4}{(T-t)^{\frac{1}{2}}} \, \Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

While, always by Proposition 4.4.3

$$|II| \leq \alpha_1 \int_t^T C_{11}(t,\sigma)\, d\sigma \, \frac{C_5}{(T-t)^{\frac{5}{2}}} \, \Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

Where

$$C_{11}(t,\sigma) = \alpha(S)\,(\sigma - t) \qquad \text{if } \mu = 0$$

$$C_{11}(t,\sigma) = \alpha(S)\, \frac{e^{2\,\mu\,(\sigma-t)} - 1}{2\,\mu} \qquad \text{if } \mu \neq 0$$

Hence for $t, T \, : \, 0 < T - t < \bar{T}$, in both the cases it holds:

$$\int_t^T C_{11}(t,\sigma)\, d\sigma \leq C_6\,(T-t)^2$$

In conclusion:

$$|II| \leq \frac{C_7}{(T-t)^{\frac{1}{2}}} \, \Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

Finally, alway by Proposition 4.4.3

$$|III| \leq \alpha_1 \int_t^T C_{12}(t,\sigma)\, d\sigma \, \frac{C_8}{(T-t)^{\frac{7}{2}}} \, \Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

Where

$$C_{12}(t,\sigma) = \alpha(S)\, \frac{1}{2}\,(\sigma - t)^2 \qquad \text{if } \mu = 0$$

$$C_{12}(t,\sigma) = \alpha(S)\, \frac{(e^{\mu\,(\sigma-t)} - 1)^2}{2\,\mu^2} \qquad \text{if } \mu \neq 0$$

Hence again for $t, T \, : \, 0 < T - t < \bar{T}$, in both the cases it holds:

$$\int_t^T C_{12}(t,\sigma)\, d\sigma \leq C_9\,(T-t)^3$$

In conclusion:

$$|III| \leq \frac{\tilde{C}}{(T-t)^{\frac{1}{2}}} \, \Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

□

Proposition 4.4.5. *For every $\epsilon, \bar{T} > 0$, there exists a positive constant C depends on $\epsilon, \bar{T}, \mu$, such that:*

$$|Z_n^1(t,x,T,y)| \leq M_n \, \tau^{\frac{n-1}{2}} \, \Gamma_{A+\epsilon}(t,x,T,y)$$

For every $n \in \mathbb{N}$, $x, y \in \mathbb{R}^2$, $0 < \tau \leq \bar{T}$. Where

$$M_n = \frac{C^n \, \Gamma_E^n(\frac{1}{2})}{\Gamma_E(\frac{n}{2})} \tag{4.54}$$

With Γ_E the Euler Gamma function

Proof.
By induction on n

If $n = 1$, for $x, y \in \mathbb{R}^2$, $0 < \tau \leq \bar{T}$ we have:

$$|Z_n^1(t,x,T,y)| = |K \, P^1(t,x,T,y)| = |K \, \Gamma_S(t,x,T,y) + K \, G_S^1(t,x,T,y)|$$

$$= \left| \Big(K - (K_S + K_S^1)\Big) \, \Gamma_S(t,x,T,y) + (K - K_S) \, G_S^1(t,x,T,y) \right|$$

$$\leq \left| \Big(K - (K_S + K_S^1)\Big) \, \Gamma_S(t,x,T,y) \right| + |(K - K_S) \, G_S^1(t,x,T,y)| = I + II$$

$$I = \left| \Big(\frac{1}{2} \, \alpha(s) \, \partial_{ss} - \frac{1}{2} \, \alpha(S) \, \partial_{ss} - \frac{1}{2} \, \alpha'(S) \, (s - S) \, \partial_{ss}\Big) \, \Gamma_S(t,x,T,y) \right|$$

$$= \frac{1}{2} \Big| \sum_{k=2}^{\infty} \partial_s^{(k)} \alpha(S) \, (s - S)^k \, \partial_{ss} \, \Gamma_S(t,x,T,y) \Big| \leq \frac{\tilde{C}}{2} \left| (s - S)^2 \, \partial_{ss} \, \Gamma_S(t,x,T,y) \right|$$

(by Theorem 4.3.6)

$$\leq \frac{\tilde{C}}{2} \left| s - S \right|^2 \frac{C_1}{\tau} \, \Gamma_{A+\frac{\epsilon}{2}}(t,x,T,y) \leq \frac{\tilde{C}}{2} \left(\frac{|s|^2}{\tau} + \frac{|S|^2}{\tau} \right) C_1 \, \Gamma_{A+\frac{\epsilon}{2}}(t,x,T,y)$$

(by Proposition 4.3.5)

$$\leq C \, \Gamma_{A+\epsilon}(t,x,T,y)$$

$$II = \left| \Big(\frac{1}{2} \, \alpha(s) \, \partial_{ss} - \frac{1}{2} \, \alpha(S) \, \partial_{ss}\Big) \, G_S^1(t,x,T,y) \right| = \frac{1}{2} \, |\alpha(s) - \alpha(S)| \, |\partial_{ss} \, G_S^1(t,x,T,y)|$$

(by hypothesis $(H2)$, Remark 25 and Theorem 4.4.4)

$$\leq L\,\tau^{\frac{1}{2}}\,||(1,\eta)||_K\,\frac{C_2}{\tau^{\frac{1}{2}}}\,\Gamma_{A+\frac{\epsilon}{2}}(t,x,T,y)$$

(by Proposition 4.3.5)

$$\leq C\,\Gamma_{A+\epsilon}(t,x,T,y)$$

In conclusion:

$$|Z_n^1(t,x,T,y)| \leq C\,\Gamma_{A+\epsilon}(t,x,T,y)$$

and the thesis results proved for $n=1$.
If the thesis is true for n then it can be proved it is true for $n+1$ repeating the identical argument used in Proposition 4.4.1

□

Theorem 4.4.6. *For every $\epsilon > 0$ and $\bar{T} > 0$ there exists a positive constant C such that:*

$$|\Gamma(t,s,a,T,S,A) - \Gamma_S^1(t,s,a,T,S,A)| \leq C\,(T-t)\,\Gamma_{A+\epsilon}(t,s,a,T,S,A)$$

$\forall\, s,S,a,A \in \mathbb{R},$ *and* t,T *such that* $0 < T-t \leq \bar{T}$,

Proof. The proof is analogous to the one of theorem 4.4.2 □

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