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STATISTICAL MECHANICS OF MONOMER-DIMER MODELS ON COMPLETE AND ERDŐS-RÉNYI GRAPHS

Tesi di Laurea Magistrale in Meccanica Statistica

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Ai miei nonni con tanto tanto affetto

Contents

Introduzione in italiano

La Meccanica Statistica applica la Teoria delle Probabilità allo studio di sistemi composti da un grande numero di particelle, occupandosi di calcolare le grandezze macroscopiche come medie di grandezze microscopiche rispetto ad un'opportuna misura.
L'analisi delle grandezze medie nel limite termodinamico, cioè quando la taglia del sistema N tende all'infinito, è quindi di fondamentale importanza. I risultati più interessanti in questi studi, nonché le maggiori difficoltà, si devono solitamente all'interazione reciproca tra le particelle. In sistemi privi di interazione, infatti, ogni particella è indipendente dalle altre (la misura di probabilità si fattorizza) e dunque può essere analizzata singolarmente.
In generale si considera un sistema che può trovarsi in un numero finito di microstati s distinti. Ciascuno di questi si realizza con probabilità $\mu(s)$ ed è caratterizzato da un'energia $H(s)$. Nella Meccanica Statistica dell'equilibrio la distribuzione di probabilità dei microstati è strettamente legata ai loro livelli di energia. Precisamente, considerando l'Hamiltoniana H del sistema e fissando una temperatura inversa $\beta \geq 0$, la *misura di Boltzmann* sull'insieme dei microstati è definita come

$$\mu_{Boltz}(s) \;=\; \frac{1}{Z(\beta)}\,\exp\big(-\,\beta\,H(s)\big) \quad \forall\; s\; \textit{microstato}$$

dove $Z(\beta)$ è determinata dalla condizione $\sum_s \mu(s) = 1$.
La scelta della misura di Boltzmann per descrivere il comportamento microscopico del sistema all'equilibrio è legata al secondo principio della termodinamica. L'entropia di una misura μ è $S(\mu) = -\sum_s \mu(s)\,\log\mu(s)$, mentre l'energia interna del sistema è $U(\mu, H) \;=\; \sum_s H(s)\,\mu(s)$. Quando la tem-

peratura inversa β è fissata, queste grandezze possono essere combinate per definire l'energia libera:

$$F(\mu, H, \beta) = U - \frac{1}{\beta} S .$$

Il secondo principio della termodinamica afferma che all'equilibrio l'energia libera del sistema è minima. D'altro canto si può provare che la misura di Boltzmann μ_{Boltz} è la sola che minimizza $F(\mu, H, \beta)$ tra tutte le misure di probabilità μ.
E' anche possibile considerare una Hamiltoniana H aleatoria, ad esempio quando le particelle interagiscono su un grafo aleatorio. In questo caso la misura di Boltzmann è una misura aleatoria e dal punto di vista fisico è interessante studiare le grandezze "quenched" del sistema, ossia ottenute prima mediando su tutti i possibili microstati rispetto alla misura di Boltzmann (aleatoria) e successivamente mediando sulle possibili realizzazioni dell'Hamiltoniana H .

In questa tesi sono trattate due famiglie di modelli meccanico statistici su vari grafi:

- i modelli di spin ferromagnetici, anche detti modelli di Ising;

- i modelli di monomero-dimero.

I *modelli di spin ferromagnetici* descrivono il comportamento di un grande numero di particelle che ammettono due possibili orientazioni (± 1), sotto l'influenza di un campo esterno e di un'interazione imitativa reciproca.
La formulazione di questi modelli in termini matematici risale agli anni '20 e tutt'oggi essi costituiscono un ricco campo di ricerca a cui molti matematici e fisici si dedicano.
Di recente si è iniziato a studiare il modello di Ising su grafi aleatori. In particolare nel 2010 Dembo e Montanari sono riusciti a calcolare il limite termodinamico per l'energia libera sul grafo diluito alla Erdös-Rényi.

Il primo capitolo della tesi è dedicato principalmente allo studio del loro lavoro, con alcune generalizzazioni dovute a Dommers, Giardinà e Van Der Hofstad. Nonostante certi passaggi siano piuttosto tecnici, l'idea centrale è di sfruttare il fatto che i grafi di Erdös-Rényi diluiti tendono localmente ad essere privi di cicli. In quest'ottica è opportuno studiare l'energia interna del sistema che, grazie alla disuguaglianza di Griffiths-Kelly-Sherman, è ben approssimata da grandezze di tipo locale.

Nel secondo capitolo sono trattati i *modelli di monomero-dimero*, che descrivono la presenza di legami monogami in un ampio gruppo di particelle sotto l'influenza di una spinta a rimanere da soli e di varie tendenze opposte a formare una coppia.
L'origine di questi modelli si può far risalire agli anni '30, mentre negli anni '70 fu pubblicato un importante articolo di Heilmann e Lieb. Più di recente l'attenzione si è concentrata soprattutto sui reticoli 2-dimensionali.
Questa tesi ha l'obiettivo di dare un contributo nuovo alla teoria dei modelli di monomero-dimero, partendo dallo studio del lavoro di Heilmann e Lieb e dalle conoscenze sui più noti modelli di spin.
Dopo la definizione del modello, i principali argomenti trattati sono

- alcune disuguaglianze di correlazione che consentono di dimostrare in modo elegante la (già nota) esistenza del limite termodinamico per l'energia libera sui reticoli finito-dimensionali;

- l'espressione esplicita dell'energia libera su grafi ad albero con un numero uniforme di figli e sul grafo completo;

- la concentrazione dell'energia libera (aleatoria) intorno al proprio valor medio nel limite termodinamico sul grafo diluito di Erdös-Rényi.

Sugli alberi e sul grafo completo sono studiate le soluzioni esatte di Heilmann e Lieb per sistemi di taglia finita N: attraverso relazioni di ricorrenza esse coinvolgono rispettivamente i polinomi di Chebyshev e di Hermite. In seguito viene calcolato il limite per $N \to \infty$.

Sui grafi diluiti di Erdös-Rényi si utilizzano le martingale per dimostrare il risultato di concentrazione.
La tesi contiene anche un tentativo di applicare la tecnica dell'interpolazione di Guerra al modello di monomero-dimero sul grafo completo. Sarebbe interessante proseguire questi tentativi ed estenderli al grafo diluito con lo scopo di dimostrare l'esistenza del limite termodinamico per l'energia libera (quenched).
Un ulteriore obiettivo potrebbe essere il calcolo di tale limite, magari adattando al modello di monomero-dimero l'idea che Dembo e Montanari hanno avuto per il modello di Ising.

Introduction

Statistical Mechanics applies Probability Theory to study the behaviour of systems composed by a large number of particles, computing macroscopic quantities as averages of microscopic quantities with respect to an opportune measure.

Therefore it is important to investigate the average quantities in the thermodynamic limit, that is as the size of the system $N \to \infty$.

The most interesting results in this field (and the main difficulties) are usually due to the mutual interaction between particles. Indeed in non-interacting systems each particle is independent from the others (the probability measure factorizes) and so it can be studied individually.

In general one considers a system that may assume a finite number of different microstates s. Each of these is fulfilled with probability $\mu(s)$ and is characterized by an energy $H(s)$. In Equilibrium Statistical Mechanics the probability distribution of the microstates is strictly related to their energy levels. Precisely considering the Hamiltonian H of the system and fixing an inverse temperature $\beta \geq 0$, the *Boltzmann measure* on the microstates space is defined by

$$\mu_{Boltz}(s) \;=\; \frac{1}{Z(\beta)}\,\exp\big(-\,\beta\,H(s)\big) \quad \forall\, s \textit{ microstate}$$

where $Z(\beta)$ is determined by the condition $\sum_s \mu(s) = 1$.

The choice of the Boltzmann measure to describe the microscopic behaviour of the system at equilibrium is related to the second law of thermodynamics. The entropy of a measure μ is $S(\mu) = -\sum_s \mu(s)\,\log\mu(s)$, while the internal energy of the system is $U(\mu, H) = \sum_s H(s)\,\mu(s)$. When the inverse temperature β is fixed, these quantities can be combined to define the free energy:

$$F(\mu, H, \beta) = U - \frac{1}{\beta} S .$$

The second law of thermodynamic states that at the equilibrium the free energy of the system attains its minimum. And on the other hand the Boltzmann measure μ_{Boltz} is proven to be the only one that minimizes $F(\mu, H, \beta)$ over all probability measures μ.
It is also possible to consider a random Hamiltonian H, for example when the particles interact on a random graph. In this case the Boltzmann measure is a random measure and physically it is interesting to study the quenched quantities of the system, namely first take the average over all possible microstates w.r.t. the (random) Boltzmann measure and later take the average over all possible realisations of the hamiltonian H .

In this thesis two different families of statistical mechanical models are studied on several graphs:

- ferromagnetic spin models, also called Ising models;
- monomer-dimer models.

Ferromagnetic spin models describe the behaviour of a large number of particles which admit two different orientations (± 1), under the influence of an external field and an imitative interaction with one another.
A mathematical formulation of these models dates back to the 20's and they still constitute a reach area of research to which many physicists and mathematicians dedicate themselves.
Recently the Ising model has been studied on random graphs. In particular in 2010 Dembo and Montanari managed to compute the thermodynamic limit for the free energy on a diluted Erdös-Rényi graph.
The first chapter of this thesis is principally dedicated to the study of their work, with some generalisations due to Dommers, Giardinà and Van Der Hofstad. Although some steps are quite technical, the main idea is to use the

fact that diluted Erdös-Rényi graphs are locally tree-like. With this aim one investigates the internal energy that, thank to the Griffiths-Kelly-Sherman inequality, is well approximated by local quantities.

The second chapter treats the *monomer-dimer models*, which describe the presence of monogamous ties in a large group of particles under the influence of a boost to stay alone and several opposite boots to form a couple.
The origin of these model dates back to the 30's and an important paper was published in the 70's by Heilmann and Lieb. More recently the most attention has been focused on the two dimensional lattices.
This thesis has the purpose to make an original contribution to the theory of monomer-dimer models, starting from the study of the work by Heilmann and Lieb and from the proprieties of the better known spin models.
After the definition of the model, the main arguments treated are

- some correlation inequalities that allow to prove in an elegant way the (already known) existence of the thermodynamic limit for the free energy on finite dimensional lattices;
- the explicit expression of the free energy on the trees with a uniform offspring size and on the complete graph;
- the concentration of the (random) free energy around its expected value in the thermodynamic limit on a diluted Erdös-Rényi graph.

On the trees and on the complete graph the exact solutions by Heilmann and Lieb for systems of finite size N are studied: through a recurrence relation they involves respectively the Chebyshev and the Hermite polynomials. Further the limit as $N \to \infty$ is computed.
On the diluted Erdös-Rényi graph a martingale technique, described in the Appendix, is used to prove the concentration result.
The thesis also contains an attempt to apply the Guerra's interpolation technique to the monomer-dimer model on the complete graph. It would be interesting to continue these studies and extend them to the diluted Erdös-Rényi

graph with the aim of proving the existence of the thermodynamic limit for the (quenched) free energy.
A further purpose is to compute this quantity, maybe adapting to the monomer-dimer model the idea used by Dembo and Montanari for the Ising model.

Chapter 1

Ferromagnetic spin models

Let $G = (V, E)$ be a finite simple graph. Denote $N = |V|$.
Fix two kind of parameters: the *inverse temperature* $\beta \geq 0$ and the *external magnetic field* $\underline{B} = (B_i)_{i \in V} \in \mathbb{R}^V$ acting on each vertex.

Definition 1. A *spin configuration* on the graph G is a vector $\sigma = (\sigma_i)_{i \in V}$ such that

$$\sigma_i \in \{+1, -1\} \quad \forall\, i \in V\,.$$

We'll say that each vertex i of the graph is occupied by the *spin variable* σ_i, which may assume positive orientation ($\sigma_i = +1$) or negative orientation ($\sigma_i = -1$).

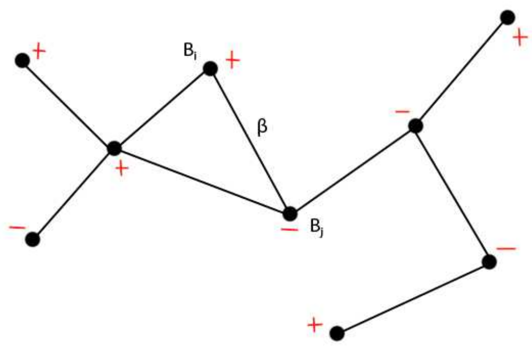


Figure 1.1: *Representation of a spin configuration on a graph G.*

We define the following probability measure on the set of all possible spin configuration on the graph G:

$$\begin{aligned}\mu(\sigma) \; :&= \frac{1}{Z(\beta,\underline{B})} \prod_{ij\in E} e^{\beta\,\sigma_i\sigma_j} \prod_{i\in V} e^{B_i\,\sigma_i} \\ &= \frac{1}{Z(\beta,\underline{B})} \; \exp\big(\beta \sum_{ij\in E}\sigma_i\sigma_j + \sum_{i\in V} B_i\,\sigma_i\big) \quad \forall\,\sigma\in\{\pm1\}^V\end{aligned} \tag{1.1}$$

where the normalizing factor is

$$Z(\beta,\underline{B}) \;:= \sum_{\sigma\in\{\pm1\}^V} \exp\big(\beta \sum_{ij\in E}\sigma_i\sigma_j + \sum_{i\in V} B_i\,\sigma_i\big)\,.$$

This is called a *ferromagnetic spin model* or an *Ising model* on the graph G. Intuitively in this model a spin configuration σ has an high probability to verify if:

i. neighbour spins have the same orientation (i.e. $\sigma_i=\sigma_j$ for $ij\in E$),

ii. each spin is oriented as the external field acting on it (i.e. $\sigma_i=\operatorname{sign} B_i$),

where the first condition assumes more importance if β is large, the second one if $|B_i|$ is large.
The expected value with respect to the measure μ will be denoted $\langle\,\cdot\,\rangle$, that is for any function f of the spin configuration we set

$$\langle f\rangle := \sum_{\sigma\in\{\pm1\}^V} f(\sigma)\,\mu(\sigma)\,.$$

The function $Z(\beta,\underline{B})$ defined above is called the *partition function* of the model. Its natural logarithm $P(\beta,\underline{B}) := \log Z(\beta,\underline{B})$ is called *pressure* or *free energy*.

1.1 Correlation inequalities

Interesting quantities of the Ising model are the *magnetisation* of each spin $\langle\sigma_i\rangle$ and the internal energy of the system $\sum_{ij\in E}\langle\sigma_i\sigma_j\rangle$. We'll see that these quantities can be computed as derivatives of the pressure, namely

$$\frac{\partial P}{\partial \beta} = \sum_{ij \in E} \langle \sigma_i \sigma_j \rangle \ , \qquad \frac{\partial P}{\partial B_i} = \langle \sigma_i \rangle \ .$$

But it's useful to work with a slightly more general model. In this section we'll consider a spin model with all possible interactions:

$$\mu(\sigma) := \frac{1}{Z(\underline{J})} \ \exp\left(\sum_{X \subseteq V} J_X \prod_{i \in X} \sigma_i \right) \ \ \forall\, s \in \{\pm 1\}^V ,$$

where $\underline{J} = (J_X)_{X \subseteq V}$ is a family of real parameters.
Notice that the Ising model defined by (1.1) is obtained taking for each vertex $i \in V$ $J_i = B_i$, for each couple of vertices $ij \in \mathcal{P}(V, 2)$

$$J_{ij} = \begin{cases} \beta & \text{if } ij \in E \\ 0 & \text{if } ij \notin E \end{cases} ,$$

and all the other coefficients J_X equal to zero.

Proposition 1. *Let $X, Y \subseteq V$ be two sets of vertices. The correlation between the spin variables of X and the centred correlation between the spin variables of X and those of Y are respectively:*

$$\left\langle \prod_{i \in X} \sigma_i \right\rangle = \frac{\partial P}{\partial J_X} ,$$

$$\left\langle \prod_{i \in X} \sigma_i \prod_{j \in Y} \sigma_j \right\rangle - \left\langle \prod_{i \in X} \sigma_i \right\rangle \left\langle \prod_{j \in X} \sigma_j \right\rangle = \frac{\partial^2 P}{\partial J_X \, \partial J_Y} .$$

Proof. Directly compute the derivatives:

$$\frac{\partial P}{\partial J_X} = \frac{1}{Z} \frac{\partial Z}{\partial J_X} = \frac{1}{Z} \sum_\sigma \left(\prod_{i \in X} \sigma_i \right) \exp\left(\sum_{A \subset V} J_A \prod_{k \in A} \sigma_k \right) = \left\langle \prod_{i \in X} \sigma_i \right\rangle ,$$

$$\begin{aligned} \frac{\partial^2 P}{\partial J_Y \, \partial J_X} &= \frac{\partial}{\partial J_Y} \frac{\sum_\sigma \left(\prod_{i \in X} \sigma_i \right) \exp\left(\sum_{A \subset V} J_A \prod_{k \in A} \sigma_k \right)}{Z} \\ &= \left\langle \prod_{i \in X} \sigma_i \prod_{j \in Y} \sigma_j \right\rangle - \left\langle \prod_{i \in X} \sigma_i \right\rangle \left\langle \prod_{j \in Y} \sigma_j \right\rangle . \end{aligned}$$

□

In case that all the coefficients J_X are non-negative, spin models are characterized by two fundamental inequalities.

Theorem 2 (Griffiths-Kelly-Sherman inequalities)**.**
Suppose $J_A \geq 0$ *for all* $A \subseteq V$*.*
Let X, $Y \subseteq V$ *be two sets of vertices. Then:*

- $\left\langle \prod_{i\in X} \sigma_i \right\rangle \geq 0$
- $\left\langle \prod_{i\in X} \sigma_i \prod_{j\in Y} \sigma_j \right\rangle - \left\langle \prod_{i\in X} \sigma_i \right\rangle \left\langle \prod_{j\in Y} \sigma_j \right\rangle \geq 0$

Proof. **1)** Observe that $Z > 0$ and

$$Z \left\langle \prod_{i\in X} \sigma_i \right\rangle = \sum_{\sigma\in\{\pm1\}^V} \left(\prod_{i\in X} \sigma_i \right) \exp\left(\sum_{A\subseteq V} J_A \prod_{j\in A} \sigma_j \right),$$

so that it suffices to investigate the right-hand term to determine the sign of $\left\langle \prod_{i\in X} \sigma_i \right\rangle$. Start expanding the exponential with its Taylor series and use the fact that $\sigma_i^2 = 1$:

$$\exp\left(\sum_{A\subseteq V} J_A \prod_{j\in A} \sigma_j \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{A\subseteq V} J_A \prod_{j\in A} \sigma_j \right)^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{A_1,\dots,A_k\subseteq V} J_{A_1} \cdots J_{A_k} \left(\prod_{j_1\in A_1} \sigma_{j_1} \right) \cdots \left(\prod_{j_k\in A_k} \sigma_{j_k} \right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{A_1,\dots,A_k\subseteq V} J_{A_1} \cdots J_{A_k} \left(\prod_{j\in A_1\Delta\dots\Delta A_k} \sigma_j \right)$$

where $A_1\Delta\dots\Delta A_k = \{i \in A_1\cup\cdots\cup A_k \,|\, i$ belongs to an odd number of A_s's$\}$ is the symmetric difference of the indicated sets.
Now observe that for any $Y \subseteq V$

$$\sum_{\sigma\in\{\pm1\}^V} \left(\prod_{i\in Y} \sigma_i \right) = \begin{cases} 2^{|V|} & \text{if } Y = \varnothing \\ 0 & \text{if } Y \neq \varnothing \end{cases}$$

indeed $\sigma \mapsto -\sigma$ is a bijection of $\{\pm1\}^V$, hence if $Y \neq \varnothing$ $\sum_\sigma \left(\prod_{i\in Y} \sigma_i \right) = \sum_\sigma \left(\prod_{i\in Y} -\sigma_i \right) = -\sum_\sigma \left(\prod_{i\in Y} \sigma_i \right)$.

Therefore one obtains

$$
\begin{aligned}
Z\,\big\langle \textstyle\prod_{i\in X}\sigma_i\big\rangle &= \sum_{\sigma\in\{\pm1\}^V} \big(\textstyle\prod_{i\in X}\sigma_i\big)\,\exp\big(\sum_{A\subseteq V} J_A \prod_{j\in A}\sigma_j\big)\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{A_1,\dots,A_k\subseteq V} J_{A_1}\cdots J_{A_k}\sum_{\sigma\in\{\pm1\}^V}\big(\textstyle\prod_{i\in X}\sigma_i\big)\big(\prod_{j\in A_1\Delta\dots\Delta A_k}\sigma_j\big)\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{A_1,\dots,A_k\subseteq V} J_{A_1}\cdots J_{A_k}\,2^{|V|}\,\mathbb{1}(X=A_1\Delta\dots\Delta A_k)\ \geq\ 0\,.
\end{aligned}
$$

2) For shortness denote $\sigma^X := \prod_{i\in X}\sigma_i$. To prove the second inequality observe that

$$
\begin{aligned}
&\langle\sigma^X\,\sigma^Y\rangle - \langle\sigma^X\rangle\,\langle\sigma^Y\rangle \ =\ \langle\sigma^{X\Delta Y}\rangle - \langle s^X\rangle\,\langle\sigma^Y\rangle\\
&= \frac{\sum_\sigma \sigma^{X\Delta Y}\exp(\sum_A J_A\,\sigma_A)}{Z}\ \frac{\sum_\tau \exp(\sum_A J_A\,\tau_A)}{Z}\ +\\
&\qquad\qquad - \frac{\sum_\sigma \sigma^{X}\exp(\sum_A J_A\,\sigma_A)}{Z}\ \frac{\sum_\tau \tau^Y\exp(\sum_A J_A\,\tau_A)}{Z}\\
&= \frac{1}{Z^2}\sum_{\sigma,\tau\in\{\pm1\}^V}(\sigma^{X\Delta Y}-\sigma^X\,\tau^Y)\,\exp\big(\sum_{A\subseteq V}J_A\,(\sigma^A+\tau^A)\big)
\end{aligned}
$$

Now, using the fact that $\sigma_i^2 = 1$, rewrite the quantities

$$
\begin{aligned}
\sigma^{X\Delta Y} - \sigma^X\,\tau^Y &= \sigma^{X\Delta Y}(1-\sigma^{X\Delta Y}\sigma^X\tau^Y) = \sigma^{X\Delta Y}(1-\sigma^Y\tau^Y)\,,\\
\sigma^A+\tau^A &= \sigma^A(1+\sigma^A\tau^A)\,,
\end{aligned}
$$

and observe that $(\sigma,\tau)\mapsto(\sigma,\,\sigma\,\tau) =: (\sigma,\,\zeta)$ is a bijection of $\{\pm1\}^V\times\{\pm1\}^V$, indeed $\sigma\,\zeta = \sigma^2\,\tau = \tau$.

Therefore, setting $\widetilde{J}_A(\zeta) := J_A(1+\zeta^A)\geq 0$, one finds

$$
\begin{aligned}
&\sum_{\sigma,\tau\in\{\pm1\}^V}(\sigma^{X\Delta Y}-\sigma^X\,\tau^Y)\,\exp\big(\sum_{A\subseteq V}J_A\,(\sigma^A+\tau^A)\big)\ =\\
&\sum_{\zeta\in\{\pm1\}^V}\underbrace{(1-\zeta^Y)}_{\geq 0}\ \underbrace{\sum_{\sigma\in\{\pm1\}^V}\sigma^{X\Delta Y}\exp\big(\sum_{A\subseteq V}\widetilde{J}_A(\zeta)\,\sigma^A\big)}_{\geq 0}\ \geq\ 0
\end{aligned}
$$

where the last step is due the first inequality of Griffiths-Kelly-Sherman we proved in **1)**. This concludes the proof. □

Remark 1. The proposition 1 allows to give an intuitive and very useful interpretation of the second G.K.S. inequality. Indeed observe

$$\frac{\partial}{\partial J_Y}\left\langle \prod_{i\in X}\sigma_i \right\rangle = \frac{\partial}{\partial J_Y}\frac{\partial P}{\partial J_X} = \left\langle \prod_{i\in X}\sigma_i \prod_{j\in Y}\sigma_j \right\rangle - \left\langle \prod_{i\in X}\sigma_i \right\rangle \left\langle \prod_{j\in Y}\sigma_j \right\rangle$$

Hence the second G.K.S. exactly states that if $\underline{J} \geq 0$, then for any X, $Y \subseteq V$

$$J_Y \mapsto \left\langle \prod_{i\in X}\sigma_i \right\rangle \text{ is an increasing function.}$$

That is if all the interaction coefficients are non-negative and one of them increases, then all the correlations between the spins increase.

Coming back to our Ising model defined by (1.1), the Griffiths-Kelly-Sherman inequalities can be restated as follows.

Corollary 3. *Let μ be the Ising measure on the graph $G = (V, E)$ with inverse temperature $\beta \geq 0$ and external field $B_i \geq 0\;\forall\, i \in V$.*
Let μ' be the Ising measure on the subgraph $G' = (V, E')$, $E' \subseteq E$, with inverse temperature $0 \leq \beta' \leq \beta$ and external field $0 \leq B'_i \leq B_i\;\forall\, i \in V$.
Then for all $X \subseteq V$

$$0 \leq \left\langle \prod_{i\in X}\sigma_i \right\rangle_{\mu'} \leq \left\langle \prod_{i\in X}\sigma_i \right\rangle_{\mu} .$$

So in particular in the Ising model the magnetisation $\langle\sigma_i\rangle$ and the internal energy $\langle\sigma_i\sigma_j\rangle$ increase with the inverse temperature, the magnetic field and the connection of the graph.
Another useful fact is that the magnetisation is a convex function of the magnetic field. We state this result without proving it.

Theorem 4 (Griffiths-Hurst-Sherman inequality)**.**
Consider the Ising model on the graph G with inverse temperature $\beta \geq 0$ and external field $B_i \geq 0\;\forall\, i \in V$. For any i, j, $k \in V$

$$\frac{\partial^2}{\partial B_j\,\partial B_k}\langle\sigma_i\rangle \leq 0 .$$

1.2 Ising model and number of cycles

This sections presents a nice expression for the partition function of an Ising model at zero magnetic field in term of the number of cycles of the graph (also called high temperature expansion).

Proposition 5. *Consider the Ising model on the graph $G = (V, E)$ with inverse temperature β and external field $B \equiv 0$. The partition function is*

$$Z(\beta, 0) \;=\; 2^{|V|} (\cosh \beta)^{|E|} \sum_{k=0}^{|E|} |\mathscr{C}_k| \, (\tanh \beta)^k$$

where

$$\mathscr{C}_k = \big\{\{i_1 j_1, \dots, i_k j_k\} \in \mathcal{P}(E, k) \,\big|\, \{i_1, j_1\} \Delta \dots \Delta \{i_k, j_k\} = \varnothing \big\}\,.$$

Notice that $|\mathscr{C}_k|$ is the number of cycles and unions of edge-disjoint cycles of total length k in the graph G.

Proof. Note that since $\sigma_i \sigma_j = \pm 1$, $\exp(\beta\, \sigma_i \sigma_j) = \cosh \beta + \sigma_i \sigma_j \sinh \beta$. Therefore

$$\begin{aligned}
Z(\beta, 0) \;&=\; \sum_{\sigma} \exp\big(\beta \sum_{ij \in E} \sigma_i \sigma_j\big) \;=\; \sum_{\sigma} \prod_{ij \in E} (\cosh \beta + \sigma_i \sigma_j \sinh \beta) \\
&=\; (\cosh \beta)^{|E|} \sum_{\sigma} \prod_{ij \in E} (1 + \sigma_i \sigma_j \tanh \beta) \\
&=\; (\cosh \beta)^{|E|} \sum_{\sigma} \sum_{A \subset E} \prod_{ij \in A} (\sigma_i \sigma_j \tanh \beta) \\
&=\; (\cosh \beta)^{|E|} \sum_{k=0}^{|E|} (\tanh \beta)^k \sum_{\substack{\{i_1 j_1, \dots, i_k j_k\} \\ \in \mathcal{P}(E,k)}} \sum_{\sigma} \sigma_{i_1} \sigma_{j_1} \cdots \sigma_{i_k} \sigma_{j_k}
\end{aligned}$$

Now observe that, as in the proof of theorem 2,

$$\sum_{\sigma \in \{\pm 1\}^V} \sigma_{i_1} \sigma_{j_1} \cdots \sigma_{i_k} \sigma_{j_k} = \begin{cases} 2^{|V|} & \text{if } \{i_1, j_1\} \, \Delta \dots \Delta \, \{i_k, j_k\} = \varnothing \\ 0 & \text{if } \{i_1, j_1\} \, \Delta \dots \Delta \, \{i_k, j_k\} \neq \varnothing \end{cases}$$

Hence one obtains

$$Z(\beta, 0) \;=\; 2^{|V|} (\cosh \beta)^{|E|} \sum_{k=0}^{|E|} (\tanh \beta)^k \, |\mathscr{C}_k|\,.$$

Finally remember that given a family of k distinct edges $i_1j_1,\dots,i_kj_k$, their symmetric difference is empty if and only if *each vertex is touched by an even number of those edges.* This means that a reordering of edges $i_1j_1,\dots,i_kj_k$ forms a cycle or a union of edge-disjoint cycles. □

Corollary 6. *Consider the Ising model on the graph $G=(V,E)$ with inverse temperature $\beta\geq 0$ and uniform external field $B\in\mathbb{R}$. The pressure per particle is bounded by*

$$\frac{1}{|V|}\,P(\beta,B)\;\geq\;-|B|+\log 2+\frac{|E|}{|V|}\,\log\cosh\beta\,,$$

$$\frac{1}{|V|}\,P(\beta,B)\;\leq\;|B|+\log 2+\frac{|E|}{|V|}\,\log\cosh\beta+\frac{|E|}{|V|}\,\log(1+\tanh\beta)\,.$$

Proof. First assume $B=0$. Since each $\mathscr{C}_k$ is a subset of $\mathcal{P}(E,k)$ clearly $0\leq|\mathscr{C}_k|\leq\binom{|E|}{k}$, in addition $|\mathscr{C}_0|=1$.
Therefore by the previous proposition,

$$2^{|V|}\,(\cosh\beta)^{|E|}\;\leq\;Z(\beta,0)\;\leq\;2^{|V|}\,(\cosh\beta)^{|E|}\sum_{k=0}^{|E|}\binom{|E|}{k}(\tanh\beta)^k$$

Using the Newton's binomial formula on the right-hand side and taking the logarithms, one obtains the desired bounds for $P(\beta,0)$.
Now for a general B , it suffices to observe that

$$Z(\beta,B)\;=\;\sum_{\sigma}\exp\big(\beta\sum_{ij\in E}\sigma_i\sigma_j+B\sum_{i\in V}\sigma_i\big)\;\begin{cases}\leq\;Z(\beta,0)\,e^{|B|\,|V|}\\ \geq\;Z(\beta,0)\,e^{-|B|\,|V|}\end{cases}.$$

□

1.3 The thermodynamic limit of Ising models: an overview

Consider a sequence of graphs $(G_N)_{N\in\mathbb{N}}$ such that $G_N=(V_N,E_N)$ with $|V_N|=N$. Fix a uniform magnetic field B and an inverse temperature β, renormalized to β/N if needed.

At size N we consider the Ising model on the graph G_N with the introduced parameters, and we denote Z_N its partition function, P_N its pressure and $p_N = \frac{1}{N}\,P_N$ the pressure per particle.
Statistical Mechanics is naturally interested in the behaviour of a system with a huge number of particles, approximated by the thermodynamic limit $N \to \infty$. As we have seen the pressure is a fundamental quantity, from which it's possible to deduce much information about the system, so it's important to know its behaviour in the thermodynamic limit.
Physically the free energy is an extensive quantity, namely it is of the order of the number of particles. Therefore a natural question is: there exists $\lim_{N\to\infty} \frac{1}{N}\,P_N$? And if so what is its value?

The first trivial case to study is a *non-interactive system.*
Suppose that all the vertices of the graph are isolated (i.e. $E_N = \varnothing$), or equivalently that the inverse temperature is $\beta = 0$. Then it' easy to compute

$$\begin{aligned} Z_N \;&=\; \sum_{\sigma\in\{\pm1\}^{V_N}} \exp\big(B \sum_{i\in V_N} \sigma_i\big) \;=\; \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_N=\pm1} e^{B\,\sigma_1}\cdots e^{B\,\sigma_N} \\ &= \big(\sum_{\sigma_1=\pm1} e^{B\,\sigma_1}\big)^N = (e^B + e^{-B})^N = 2^N\,(\cosh B)^N\,. \end{aligned}$$

Therefore for any $N\in\mathbb{N}$ (and for $N\to\infty$ too)

$$p_N \;=\; \frac{1}{N}\,\log Z_N \;=\; \log 2 + \log\cosh B\,.$$

The simplest systems with interactions are certainly the *trees.*
Suppose that each G_N is a tree, namely a connected graph with no cycles. For simplicity assume that the external field is $B = 0$. Therefore, as a tree has no cycles and $|E_N| = N-1$, by proposition 5 we find immediately

$$Z_N \;=\; 2^N\,(\cosh\beta)^{N-1}$$

hence

$$p_N \;=\; \frac{1}{N}\,\log Z_N \;\xrightarrow[N\to\infty]{}\; \log 2 + \log\cosh\beta\,.$$

On the *finite dimensional lattices* $\mathbb{Z}^d$, with $d \geq 2$ the computation of the thermodynamic limit is really hard and the only known exact solution is due to Onsager for $d = 2$.
Anyway using the G.K.S. inequality and the bounds for the pressure given by corollary 6, it's not difficult to prove that $\lim_{N\to\infty} p_N$ exists when each G_N is a hyper-cubic lattice of side $\sqrt[d]{N}$. Instead of proving it here we refer the reader to the second chapter of this thesis, where an analogous result is proven for the monomer-dimer model.

An important case is when each graph G_N is *complete*, namely $E_N = \mathcal{P}(V_N, 2)$. Here the Ising model is also called *Curie-Weiss model.*
To keep the pressure of order N we need to normalize the inverse temperature, taking $\beta/(2N)$. The thermodynamic limit is proved to be

$$p_N \xrightarrow[N\to\infty]{} \log 2 - \frac{\beta}{2}\,(m^*)^2 + \log\cosh(\beta\, m^* + B)$$

where $m^* = m^*(\beta, B)$ is the solution of the following fixed point equation

$$m^* = \tanh(\beta\, m^* + B) \tag{1.2}$$

with the same sign of B. This m^* represents the magnetisation per particle.
If $B = 0$, the fixed point equation (1.2) admits a unique solution $m^* = 0$ for $0 \leq \beta \leq 1$, while it has two distinct symmetric solutions m^*, $-m^*$ for $\beta > 1$.
This fact entails that the system in the thermodynamic limit has a phase transition at $\beta = \beta_c = 1$, namely the magnetisation per particle is not differentiable w.r.t. β at β_c.
On the complete graph Guerra developed an important technique, called "interpolation", which allows to prove the monotone existence of the thermodynamic without computing it.
His idea is to break the complete graph G_N into two disjoint complete subgraphs G_{N_1}, G_{N_2} with $N_1 + N_2$ and to interpolate between the two situations (taking into account the normalisation of the parameters) with the aim of proving that the pressure P_N is super-additive (or sub-additive).

Finally the Ising model has been recently studied on *random graphs*.
For example let G_N be a *diluted Erdös-Rényi graph*, namely each pair of vertices $i,\, j \in V_N$ has probability $p = (2\,c)/(N-1)$ to be linked by an edge and probability $1-p$ not to be, independently of the others. Notice the background randomness of the graph structure is added to the specific randomness of the Ising model.
In 2010 Dembo and Montanari computed the thermodynamic limit for the Ising model on such a random graph. Their solution is quite technical but it is based on the fact that the graph G_N asymptotically has a locally tree-like structure. The rest of this chapter is dedicated to the study of their work.
In 2011 Contucci, Dommers, Giardinà and Starr applying an interpolation technique managed to prove that the anti-ferromagnetic spin model (i.e. $\beta < 0$) on the diluted Erdös-Rényi graph admits a monotone thermodynamic limit.

1.4 Ferromagnetic spin model on locally tree-like graphs

Now we'll study the Ising model on a random graph which locally tends to have no cycles. The whole section is based on the work of Dembo and Montanari and on its generalisations due to Dommers, Giardinà and Van Der Hofstad.

1.4.1 Definitions concerning the graph structure

Let $G_N = (V_N, E_N)$, $N \in \mathbb{N}$ be a sequence of finite random graphs. We suppose that the vertex set is $V_N = \{1, \ldots, N\}$, while the edge set is random. That is in general

$$E_N = \{ij \in \mathcal{P}(V_N, 2) \,|\, \varepsilon_{ij}^N = 1\}$$

where $\underline{\varepsilon} := (\varepsilon_{ij}^N)_{ij,\,N}$ is a family of random variables taking values 0 or 1 .
We'll denote $\mathbb{P}$, $\mathbb{E}[\,\cdot\,]$ respectively the probability measure and the expected

value with respect to the randomness of the graph sequence.
As usual on the graph G_N the distance between $i, j \in V_N$ is defined

$$d_N(i,j) = \inf\{\, l \in \mathbb{N} \,|\, \exists\, v_0, \dots, v_l \in V_N \text{ s.t. } v_0 = i,\ v_l = j,\ v_s v_{s+1} \in E_N \}\,.$$

For any $t \in \mathbb{N}$, denote $B_N(i,t)$ the sub-graph of G_N induced by the vertices

$$\{j \in V_N \,|\, d_N(j,i) \leq t\}\,.$$

For any vertex $i \in V_N$ denote $\partial_N i$ the sets of its neighbours in the graph G_N and denote its degree by

$$\deg_N(i) = |\partial_N i| = \mathit{Card}\{j \in V_N \,|\, ij \in E_N\}\,.$$

As we are interested in the asymptotic behaviour of G_N, we give the following

Definition 2. Let $P = (P_k)_{k \geq 0}$ be a probability distribution over the non-negative integers. We say that the graph sequence $(G_N)_{N \in \mathbb{N}}$ has *asymptotic degree distribution* P if

$$\frac{1}{N} \sum_{i \in V_N} \mathbb{1}(\deg_N(i) = k) \xrightarrow[N \to \infty]{} P_k \quad \text{a.s.} \quad \forall\, k \in \mathbb{N}\,.$$

Now we define an important infinite random graph, the *rooted random tree with independent offspring.*

Definition 3. Let P, ρ be two probability distributions over $\mathbb{N}$.
The random independent tree $\mathcal{T}(P, \rho, \infty)$ rooted at $\varnothing$ is the random tree graph defined recursively as follows.
Let L be a random variable with distribution P and let $(K_{t,i})_{t \geq 1,\, i \geq 1}$ be i.i.d. random variables with distribution ρ. Let L and $(K_{t,i})_{t \geq 1,\, i \geq 1}$ be independent.

1) Connect the root $\varnothing$ to L offspring, which form the 1^{st} generation

2) Connect each node (t,i) in the t^{th} generation to $K_{t,i}$ offspring; all together these nodes form the $(t+1)^{th}$ generation

Repeat recursively the step 2) for all $t \geq 1$.
We denote $T(P,\rho,t)$ the sub-tree of $\mathcal{T}(P,\rho,\infty)$ induced by the first t generations. Notice $\mathcal{T}(P,\rho,\infty)$ is locally finite, that is each $\mathcal{T}(P,\rho,t)$ is finite.
If the distribution P equals ρ, then we'll denote $\mathcal{T}(\rho,\infty)$.

And now come to the definition which characterizes the graphs we will study.

Definition 4. We say the random graph sequence $(G_N)_{N\in\mathbb{N}}$ *converges locally to the random tree* $\mathcal{T}(P,\rho,\infty)$ if for any $t \in \mathbb{N}$ and for any T rooted tree with t generations

$$\frac{1}{N}\sum_{i\in V_N} \mathbb{1}\big(B_N(i,t) \cong T\big) \xrightarrow[N\to\infty]{} \mathbb{P}\big(\mathcal{T}(P,\rho,t) \cong T\big) \text{ a.s.}$$

This definition is slightly different from that given by Dembo and Montanari, we adopt it because some proofs become simpler.

Remark 2. The following statements are equivalent:

i. for any T rooted tree with t generations

$$\frac{1}{N}\sum_{i\in V_N} \mathbb{1}\big(B_N(i,t) \cong T\big) \xrightarrow[N\to\infty]{} \mathbb{P}\big(\mathcal{T}(P,\rho,t) \cong T\big) \text{ a.s.}$$

ii. for any B realisation of the random graph $B_N(i,t)$

$$\frac{1}{N}\sum_{i\in V_N} \mathbb{1}\big(B_N(i,t) \cong B\big) \xrightarrow[N\to\infty]{} \mathbb{P}\big(\mathcal{T}(P,\rho,t) \cong B\big) \text{ a.s.}$$

iii. for any F invariant by isomorphisms bounded function of a graph

$$\frac{1}{N}\sum_{i\in V_N} F\big(B_N(i,t)\big) \xrightarrow[N\to\infty]{} \mathbb{E}\big[F\big(\mathcal{T}(P,\rho,t)\big)\big] \text{ a.s.}$$

Proof. $iii \Rightarrow i$ is obvious.
To prove that $ii \Rightarrow iii$, observe that the possible realisations of the random graph G_N are only a finite number (since $|V_N| = N$). Therefore we can write

$$F\big(B_N(i,t)\big) = \sum_B F(B)\, \mathbb{1}\big(B_N(i,t) \cong B\big)$$

where the sum is over all the possible realisations B of $B_N(i,t)$ identified by isomorphisms. Hence by hypothesis ii

$$\frac{1}{N}\sum_{i\in V_N} F\big(B_N(i,t)\big) = \sum_B F(B)\,\frac{1}{N}\sum_{i\in V_N}\mathbb{1}\big(B_N(i,t)\cong B\big) \xrightarrow[N\to\infty]{a.s.} \sum_B F(B)\,\mathbb{P}\big(\mathcal{T}(P,\rho,t)\cong B\big) = \mathbb{E}\big[F\big(\mathcal{T}(P,\rho,t)\big)\big]\,.$$

In the end to prove that $i\Rightarrow ii$, notice that

$$\sum_T \mathbb{P}\big(\mathcal{T}(P,\rho,t)\cong T\big) = 1$$

where the sum is over all the T rooted tree with t generations, up to isomorphisms. Hence for any B realisation of $B_N(i,t)$ which is not a tree (rooted with t generations) we have $\mathbb{P}\big(\mathcal{T}(P,\rho,t)\cong B\big)=0$, and on the other side using hypothesis i

$$\frac{1}{N}\sum_{i\in V_N}\mathbb{1}\big(B_N(i,t)\cong B\big) \leq 1-\sum_T\frac{1}{N}\sum_{i\in V_N}\mathbb{1}\big(B_N(i,t)\cong T\big) \xrightarrow[N\to\infty]{a.s.} 1-\sum_T\mathbb{P}\big(\mathcal{T}(P,\rho,t)\cong T\big) \;=\; 0\,.$$

Remark 3. If $(G_N)_{N\in\mathbb{N}}$ converges locally to $\mathcal{T}(P,\rho,\infty)$, then its asymptotic degree distribution is P. Indeed for all $k\in\mathbb{N}$

$$\frac{1}{N}\sum_{i\in V_N}\mathbb{1}\big(\deg_N(i)=k\big) = \frac{1}{N}\sum_{i\in V_N}\mathbb{1}\big(B_N(i,1)\ni k\textit{ vertices}\big) \xrightarrow[N\to\infty]{} \mathbb{P}\big(T(P,\rho,1)\ni k\textit{ vertices}\big) = P_k\,.$$

Consequently the number of edges $|E_N|$ is asymptotically equivalent to $N\overline{P}/2$:

$$\frac{|E_N|}{N} \;=\; \frac{1}{2N}\sum_{i\in V_N}\deg_N(i) \;=\; \frac{1}{2N}\sum_{i\in V_N}\sum_{k=0}^{\infty}k\,\mathbb{1}\big(\deg_N(i)=k\big) \xrightarrow[N\to\infty]{} \frac{1}{2}\sum_{k=0}^{\infty}k\,P_k \;=\; \frac{\overline{P}}{2}\,.$$

To end with definitions we specify the proprieties of the distributions $P\,,\rho$ which will characterize the random tree $\mathcal{T}(P,\rho,\infty)$ in the following results.

Let $P = (P_k)_{k\geq 0}$ be a probability distribution over the non-negative integers such that $\overline{P} := \sum_{k=0}^{\infty} k\, P_k < \infty$.

Definition 5. We define the *size-biased law* of P as the probability distribution over non-negative integers $\rho = (\rho_k)_{k\geq 0}$ with

$$\rho_k = \frac{(k+1)\, P_{k+1}}{\overline{P}} \,.$$

Notice that $\sum_{k=0}^{\infty} \rho_k = 1/\overline{P} \, \sum_{k=1}^{\infty} k\, P_k = 1/\overline{P} \, \sum_{k=0}^{\infty} k\, P_k = 1\,.$

Definition 6. Let $\epsilon > 0$. We say that P has *ϵ-strongly finite mean* if:

$$\sum_{k=n}^{\infty} P_k \; = O(n^{-(1+\epsilon)}) \quad \text{as } n \to \infty\,.$$

Notice this condition is satisfied if $P_k = O(k^{-(2+\epsilon)})$ as $k \to \infty$.

1.4.2 Definitions concerning the Ising model

From now on fix an inverse temperature $\beta \geq 0$ and a magnetic field $\underline{B}$.
On a graph $G = (V, E)$ we remind that the Ising model is defined by the following probability measure over all spin configurations $\sigma = (\sigma_i)_{i\in V} \in \{\pm 1\}^V$

$$\mu(\sigma) = \frac{\exp\left(\beta \sum_{ij\in E} \sigma_i\, \sigma_j \; + \sum_{i\in V} B_i\, \sigma_i\right)}{Z(\beta, \underline{B})} \,.$$

When more clarity is needed, we denote μ_G this probability measure and $\langle \cdot \rangle_G$ the associated expectation.

Notation. When the sequence of graphs $(G_N)_{N\in\mathbb{N}}$ is considered, we'll denote $Z_N(\beta, B)$, $P_N(\beta, B)$ respectively the partition function and the pressure of the Ising model on G_N. Further we set $p_N := \frac{1}{N}\, P_N$.

It is useful to consider a subgraph U of G. Given a spin configuration $\sigma \in \{\pm 1\}^V$ on the graph G, we denote its restriction to U by $\sigma_U = (\sigma_i \,|\, i \in U)$. Furthermore we denote $\mu_{G\to U}$ the marginal on U of the measure μ_G, that is

$$\mu_{G\to U}(\sigma_U) = \sum_{\sigma_{G-U}} \mu_G(\sigma_U, \sigma_{G-U})\,.$$

On the other hand μ_U will simply indicate the measure associated to the Ising model on the graph U.
We introduce the Ising model on U with positive boundary conditions. That is we define the measure

$$\mu_U^+(\sigma) = \frac{1}{Z_U^+(\beta,\underline{B})} \exp\big(\beta\sum_{ij\in U}\sigma_i\sigma_j + \sum_{i\in U-\partial U} B_i\sigma_i\big)\,\mathbb{1}(\sigma_i\!=\!1\ \forall\, i\!\in\!\partial U)$$

for all σ spin configuration on U.
As we are going to show, this model is equivalent to the Ising model on U without boundary conditions in the limit of positive infinite magnetic field on ∂U.

Proposition 7. *Let U be a subgraph of the graph G. Then in the limit $B_i \equiv B \to \infty$ for all $i \in \partial U$ we have*

i. $\mu_U(\sigma) \longrightarrow \mu_U^+(\sigma)$ *for all σ spin configuration on U;*

ii. $\mu_G(\sigma) \longrightarrow \mu_U^+(\sigma_U)\cdot\widetilde{\mu}_{G-U}(\sigma_{G-U})$ *for all σ spin configuration on G,*

where $\widetilde{\mu}_{G-U}$ is the measure associated to the Ising model on $G-U$ with magnetic field increased on $\partial(G-U)$, precisely

$$\widetilde{B}_i = \begin{cases} B_i + \beta & \text{if } i \in \partial(G-U) \\ B_i & \text{if } i \in (G-U) - \partial(G-U) \end{cases}$$

Proof. **i.** Fix a spin configuration σ on the graph U.
Suppose that $\partial U)$ contains n vertices, on which there are $p \geq 0$ spin variables σ_i with value -1 and $n-p$ with value 1.
Write the probability of σ in the Ising model on U with no boundary conditions, isolating the effect of the external field on the boundary:

$$\mu_U(\sigma) = \frac{\exp\big(\beta\sum_{ij\in U}\sigma_i\sigma_j + \sum_{i\in U}B_i\sigma_i\big)}{\sum_\tau \exp\big(\beta\sum_{ij\in U}\tau_i\tau_j + \sum_{i\in U}B_i\tau_i\big)} =$$

$$= \frac{\exp\big(\beta\sum_{ij\in U}\sigma_i\sigma_j + \sum_{i\in U-\partial U}B_i\sigma_i\big)\ e^{-pB}\ e^{(n-p)B}}{\sum_{q=0}^n\sum_{\tau\in\mathscr{C}_q}\exp\big(\beta\sum_{ij\in U}\sigma_i\sigma_j + \sum_{i\in U-\partial U}B_i\,\sigma_i\big)\ e^{-qB}\ e^{(n-q)B}}$$

where $\mathscr{C}_q$ is the set of spin configurations τ on U such that on ∂U there are q spin variables τ_i taking value -1 and $n-q$ taking value 1.
Observe that $\frac{e^{-pB}\, e^{(n-p)B}}{e^{-qB}\, e^{(n-q)B}} = \frac{1}{e^{2(p-q)B}}$ and

$$e^{2(p-q)B} \xrightarrow[B\to\infty]{} \begin{cases} 0 & \text{if } q > p \\ 1 & \text{if } q = p \\ \infty & \text{if } q < p \end{cases} .$$

Therefore if $p = 0$

$$\mu_U(\sigma) \xrightarrow[B\to\infty]{} \frac{\exp\left(\beta \sum_{ij\in U} \sigma_i\sigma_j + \sum_{i\in U-\partial U} B_i\sigma_i\right)}{\sum_{\tau\in\mathscr{C}_0} \exp\left(\beta \sum_{ij\in U} \sigma_i\sigma_j + \sum_{i\in U-\partial U} B_i\sigma_i\right)} ,$$

whereas if $p \geq 1$

$$\mu_U(\sigma) \xrightarrow[B\to\infty]{} 0 .$$

Hence in both cases it is $\lim_{B\to\infty} \mu_U(\sigma) = \mu_U^+(\sigma)$.

ii. Now let σ be a spin configuration on the graph G. Observe that by definition of ∂U, one can divide the following disjoint cases:

$$ij \in G \Leftrightarrow (ij \in U) \text{ xor } (ij \in G-U) \text{ xor } (ij \in G,\, i \in \partial U,\, j \in \partial(G-U)) ;$$

$$i \in G \Leftrightarrow (i \in U) \text{ xor } (i \in G-U) .$$

Therefore the probability of σ can be split in

$$\begin{aligned} \mu_G(\sigma) &= C \exp\Big(\beta \sum_{ij\in U} \sigma_i\sigma_j + \sum_{i\in U} B_i\,\sigma_i\Big)\cdot \\ &\quad \cdot \exp\Big(\beta \sum_{ij\in G-U} \sigma_i\sigma_j + \sum_{i\in G-U} B_i\,\sigma_i\Big) \cdot \exp\Big(\beta \sum_{\substack{ij\in G \\ i\in\partial U,\, j\in\partial(G-U)}} \sigma_i\sigma_j\Big) \end{aligned}$$

The first term up to a constant equals $\mu_U(\sigma_U)$, therefore as proven in **i.** it converges to $\mu_U^+(\sigma_U)$ as $B \to \infty$.
Since $\mu_U^+(\sigma_U)$ contains $\mathbb{1}(\sigma_{\partial U} \equiv 1)$, as $B \to \infty$ the third term can be substituted by $\exp\left(\sum_{j\in\partial(G-U)} \beta\,\sigma_j\right)$.
The second term multiplied by this new term equals $\widetilde{\mu}_{G-U}(\sigma_{G-U})$ with magnetic field increased of β on $\partial(G-U)$.
Thus it's proved that $\mu_G(\sigma) \longrightarrow \mu_U(\sigma_U) \cdot \widetilde{\mu}_{G-U}(\sigma_{G-U})$ as $B \to \infty$. □

1.4.3 Results on the trees: the root magnetisation

In this subsection we'll see some preliminary results about the Ising model on a tree (deterministic or random). These results concern in particular the magnetisation of the root. Indeed this will turn out to be the fundamental quantity to study in order to compute the thermodynamic limit on a sequence of graphs locally convergent to a tree.
Let start with a simple but important lemma which permits to restrict the Ising model on a tree to any sub-tree without difficulties.

Lemma 8. *Let T be a finite tree. Let U be a sub-tree of T.*
For every $i \in \partial U$, let T_i be the maximal sub-tree of $T - U + i$ rooted at i.
The marginal on U of the measure associated to the Ising model on T is the measure associated to an Ising model on U, with magnetic field increased on the boundary. Precisely:

$$\mu_{T\to U} = \widetilde{\mu}_U \qquad \text{with} \quad \widetilde{B}_i = \begin{cases} \operatorname{atanh}\langle\sigma_i\rangle_{T_i} \geq B_i & \text{if } i \in \partial U \\ B_i & \text{if } i \in U - \partial U \end{cases} .$$

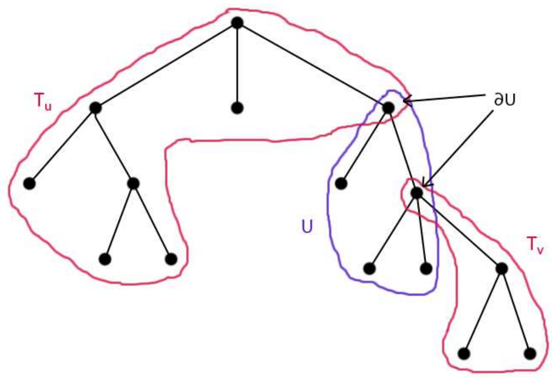


Figure 1.2: *The tree T and its sub-tree U. Each vertex $i \in \partial U$ is the root of a maximal sub-tree T_i of $T - U + i$.*

Proof. Denote $W := T - U$, the complementary forest of U in T.
Notice that the union of the trees T_i, $i \in \partial U$ is equal to $W + \partial U$, since the T_i's are maximal. Furthermore observe that for any $i, j \in \partial U$, $i \neq j$ the trees T_i and T_j are disjoints, because U is connected and T admits no cycles.

Now for any σ_U spin configuration on U

$$\mu_{T\to U}(\sigma_U) \stackrel{\text{def}}{=} \sum_{\sigma_W} \mu_T(\sigma_U,\, \sigma_W) \;=\; \frac{1}{Z_T} \sum_{\sigma_W} \exp\big(\beta \sum_{ij\in T} \sigma_i\sigma_j + \sum_{i\in T} B_i\,\sigma_i\big)$$

$$= \frac{1}{Z_T}\,\exp\big(\beta \sum_{ij\in U} \sigma_i\sigma_j + \sum_{i\in U-\partial U} B_i\,\sigma_i\big)\; \sum_{\sigma_W} \exp\big(\beta \sum_{ij\in W+\partial U} \sigma_i\sigma_j + \sum_{i\in W+\partial U} B_i\,\sigma_i\big)\,,$$

but since W is the disjoint union of the $T_i - i$'s with $i \in \partial U$

$$\sum_{\sigma_W} \exp\big(\beta \sum_{ij\in W+\partial U} \sigma_i\sigma_j \,+ \sum_{i\in W+\partial U} B_i\,\sigma_i\big) = \prod_{i\in\partial U} \sum_{\sigma_{T_i-i}} \exp\big(\beta \sum_{hk\in T_i} \sigma_h\sigma_k + \sum_{h\in T_i} B_h\sigma_h\big)$$

$$= \prod_{i\in\partial U} \sum_{\sigma_{T_i-i}} Z_{T_i}\; \mu_{T_i}(\sigma_{T_i}) = \prod_{i\in\partial U} Z_{T_i}\; \mu_{T_i\to i}(\sigma_i)\,,$$

hence it follows

$$\mu_{T\to U}(\sigma_U) = C\, \exp\big(\beta \sum_{ij\in U} \sigma_i\sigma_j + \sum_{i\in U-\partial U} B_i\,\sigma_i\big) \prod_{i\in\partial U} \mu_{T_i\to i}(\sigma_i)\,.$$

Compute $\mu_{T_i\to i}(\sigma_i)$. For any random variable X taking values ± 1 it's easy to check that

$$\mathbb{P}[X = \pm 1] = \frac{1 \pm \mathbb{E}[X]}{2}\,,$$

further observe that for any $-1 \leq \alpha \leq 1$

$$\frac{1 \pm \alpha}{(1-\alpha^2)^{1/2}} = \big(\,\frac{1+\alpha}{1-\alpha}\,\big)^{\pm 1/2} = \exp(\pm\,\mathrm{atanh}\,\alpha)\,,$$

therefore in our case:

$$\mu_{T_i\to i}(\sigma_i) = \frac{1 + \sigma_i\,\langle\sigma_i\rangle_{T_i}}{2} \;=\; \frac{(1-\langle\sigma_i\rangle_{T_i}^2)^{1/2}}{2}\; \exp\big(\sigma_i\;\mathrm{atanh}\langle\sigma_i\rangle_{T_i}\big)\,.$$

Substitute in the previous expression of $\mu_{T\to U}(\sigma_U)$ and find

$$\mu_{T\to U}(\sigma_U) = C\, \exp\big(\beta \sum_{ij\in U} \sigma_i\sigma_j + \sum_{i\in U-\partial U} B_i\,\sigma_i + \sum_{i\in\partial U} \widetilde{B}_i\,\sigma_i\big)\,,$$

with $\widetilde{B}_i = \mathrm{atanh}\langle\sigma_i\rangle_{T_i}$ for any $i \in \partial U$.
To conclude the proof it remains only to check that $\mathrm{atanh}\langle\sigma_i\rangle_{T_i} \geq B_i$ for any $i \in \partial U$. To do it use the G.K.S. inequality:

$$\langle\sigma_i\rangle_{T_i} \geq \langle\sigma_i\rangle_{\{i\}} = \frac{\sum_{\sigma_i=\pm 1} \sigma_i\, e^{B_i\sigma_i}}{\sum_{\sigma_i=\pm 1} e^{B_i\sigma_i}} = \frac{e^{B_i} - e^{-B_i}}{e^{B_i} + e^{-B_i}} = \tanh B_i\,. \quad \square$$

Notation. Given a rooted tree $T(t)$ of t generations we denote $\mathrm{Bd}\,T(t)$ the set of vertices composing its t^{th} generation.

Lemma 9. *Let $T(1)$ be a finite tree rooted at $\varnothing$ and with only one generation. The magnetisation of the root in the Ising model on $T(1)$ is:*

$$\langle \sigma_\varnothing \rangle_{T(1)} = \tanh\left[B_\varnothing + \sum_{i\ son\ of\ \varnothing} \operatorname{atanh}(\tanh\beta\,\tanh B_i)\right].$$

Proof. For brevity denote $T = T(1)$.
As T is a rooted tree with only one generation, $T = \varnothing + \mathrm{Bd}\,T$ and the only edges are those which link $\varnothing$ to its offspring.
With these remarks write $\langle \sigma_\varnothing \rangle_T$ developing the sums over $\sigma_\varnothing = \pm 1$:

$$\langle \sigma_\varnothing \rangle_T = \sum_{\sigma\in\{\pm1\}^T} \sigma_\varnothing\, \mu_T(\sigma_\varnothing) =$$

$$\frac{\sum_{\sigma\in\{\pm1\}^{\mathrm{Bd}\,T}}\left[\exp\big(\sum_{i\in\mathrm{Bd}\,T}\beta\,\sigma_i + \sum_{i\in\mathrm{Bd}\,T} B_i\sigma_i + B_\varnothing\big) - \exp\big(-\sum_{i\in\mathrm{Bd}\,T}\beta\,\sigma_i + \sum_{i\in\mathrm{Bd}\,T} B_i\sigma_i - B_\varnothing\big)\right]}{\sum_{\sigma\in\{\pm1\}^{\mathrm{Bd}\,T}}\left[\exp\big(\sum_{i\in\mathrm{Bd}\,T}\beta\,\sigma_i + \sum_{i\in\mathrm{Bd}\,T} B_i\sigma_i + B_\varnothing\big) + \exp\big(-\sum_{i\in\mathrm{Bd}\,T}\beta\,\sigma_i + \sum_{i\in\mathrm{Bd}\,T} B_i\sigma_i - B_\varnothing\big)\right]}$$

Now that all interactions are made explicit, it's simple to rewrite each term:

$$\sum_{\sigma\in\{\pm1\}^{\mathrm{Bd}\,T}} \exp\big(\pm\sum_{i\in\mathrm{Bd}\,T}\beta\,\sigma_i + \sum_{i\in\mathrm{Bd}\,T} B_i\sigma_i \pm B_\varnothing\big) = e^{\pm B_\varnothing} \sum_{\sigma\in\{\pm1\}^{\mathrm{Bd}\,T}} \prod_{i\in\mathrm{Bd}\,T} e^{(\pm\beta+B_i)\,\sigma_i}$$
$$= e^{\pm B_\varnothing} \prod_{i\in\mathrm{Bd}\,T} \sum_{\sigma_i=\pm1} e^{(\pm\beta+B_i)\,\sigma_i} = e^{\pm B_\varnothing} \prod_{i\in\mathrm{Bd}\,T} \left(e^{\pm\beta+B_i} + e^{-(\pm\beta+B_i)}\right)$$

hence, substituting in the previous equality,

$$\langle \sigma_\varnothing \rangle_T = \frac{e^{B_\varnothing}\prod_{i\in\mathrm{Bd}\,T}\left(e^{\beta+B_i}+e^{-\beta-B_i}\right) - e^{-B_\varnothing}\prod_{i\in\mathrm{Bd}\,T}\left(e^{-\beta+B_i}+e^{\beta-B_i}\right)}{e^{B_\varnothing}\prod_{i\in\mathrm{Bd}\,T}\left(e^{\beta+B_i}+e^{-\beta-B_i}\right) + e^{-B_\varnothing}\prod_{i\in\mathrm{Bd}\,T}\left(e^{-\beta+B_i}+e^{\beta-B_i}\right)}.$$

To conclude it's only matter of rewriting more compactly this equation. Start from here and use two times the fact that $\alpha = \frac{x-y}{x+y} \Leftrightarrow \frac{1+\alpha}{1-\alpha} = \frac{x}{y}$ to compute:

$$\frac{1+\langle \sigma_\varnothing \rangle_T}{1-\langle \sigma_\varnothing \rangle_T} = e^{2B_\varnothing} \prod_{i\in\mathrm{Bd}\,T} \frac{e^{\beta+B_i}+e^{-\beta-B_i}}{e^{-\beta+B_i}+e^{\beta-B_i}} = e^{2B_\varnothing} \prod_{i\in\mathrm{Bd}\,T} \frac{1+\tanh\beta\,\tanh B_i}{1-\tanh\beta\,\tanh B_i}$$

Therefore:

$$\begin{aligned}\operatorname{atanh}\langle\sigma_\varnothing\rangle_T &= \frac{1}{2}\log\frac{1+\langle\sigma_\varnothing\rangle_T}{1-\langle\sigma_\varnothing\rangle_T} = B_\varnothing + \sum_{i\in\operatorname{Bd}T}\frac{1}{2}\log\frac{1+\tanh\beta\,\tanh B_i}{1-\tanh\beta\,\tanh B_i} = \\ &= B_\varnothing + \sum_{i\in\operatorname{Bd}T}\operatorname{atanh}(\tanh\beta\,\tanh B_i)\;.\end{aligned}$$

□

Notation. Consider the Ising model on a finite rooted tree $T(t)$ composed of t generations, with magnetic field on its nodes $\underline{B} = (B_i)_{i\in T(t)}$ and the inverse temperature β. As usual the expected value w.r.t. the spin variables in this model is denoted

$$\langle\,\cdot\,\rangle_{T(t)}$$

Now consider the Ising model with magnetic filed increased only on $\operatorname{Bd}T(t)$ by a vector $\underline{H} = (H_i)_{i\in\operatorname{Bd}T(t)}$. We denote the expected value w.r.t. the spin variables in this new model by

$$\langle\,\cdot\,\rangle_{T(t),\,+\underline{H}}$$

The following lemma is a direct consequence of lemmas 8 and 9. This statement will be used often, so it's opportune to write it explicitly.

Lemma 10. *Let $T(t+1)$ be a finite tree rooted in $\varnothing$ and composed of $t+1$ generations. Let $T(t)$ be its sub-tree induced by the first t generations.*
The magnetisation of the root in the Ising model on $T(t+1)$ equals the magnetisation of the root in the Ising model on $T(t)$ with increased magnetic field on $\operatorname{Bd}T(t)$. Precisely:

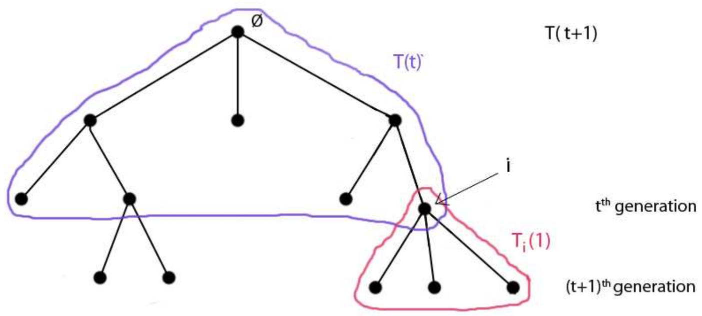
$$\langle\sigma_\varnothing\rangle_{T(t+1)} = \langle\sigma_\varnothing\rangle_{T(t),\,+\underline{H}} \quad \textit{with}\;\; H_i = \sum_{j\text{ son of }i}\xi_\beta(B_j)\;\;\forall\, i\in\operatorname{Bd}T(t)$$

where we set $\xi_\beta(x) := \operatorname{atanh}(\tanh\beta\,\tanh x)$.

Proof. Apply lemma 8 choosing $T = T(t+1)$ and $U = T(t) \ni \varnothing$:

$$\langle\sigma_\varnothing\rangle_{T(t+1)} \;=\; \langle\sigma_\varnothing\rangle_{T(t)}\,|\,B_i\to\widetilde{B}_i\;\;\forall\, i\in\partial T(t)$$

where for any $i \in \partial T(t)$ $\widetilde{B}_i = \operatorname{atanh}\langle\sigma_i\rangle_{T_i(1)}$ and $T_i(1)$ is the maximal sub-tree rooted in i and contained in $T(t+1) - T(t) + i$.
Notice $T_i(1)$ is the sub-tree induced by i and its offspring.



Therefore by lemma 9, compute for any $i \in \partial T(t)$

$$\widetilde{B}_i \;=\; \operatorname{atanh}\langle\sigma_i\rangle_{T_i} \;=\; B_i \;+\sum_{j\text{ son of }i}\xi_\beta(B_j)\,.$$

To conclude observe that

$$\operatorname{Bd}T(t) = \partial T(t) \,\sqcup\, \{i \in t^{th}\text{ generation of }T(t)\mid i\text{ has no sons}\}$$

hence there is no problem to write

$$\langle\sigma_\varnothing\rangle_{T(t+1)} \;=\; \langle\sigma_\varnothing\rangle_{T(t)}\,|\,B_i \to B_i \;+\sum_{j\text{ son of }i}\xi_\beta(B_j)\;\;\forall\, i\in\operatorname{Bd}T(t)\,. \quad\square$$

Now we are ready to write a recursive formula for the root magnetisation, a central result for the next investigations on random graphs.

Proposition 11. *Let $T(t)$ be a finite tree with root in $\varnothing$ and composed of t generations.*
The atanh *of the magnetisation of the root in the Ising model on $T(t)$ is:*

$$\operatorname{atanh}\langle\sigma_\varnothing\rangle_{T(t)} = B_\varnothing + \sum_{i_1\text{ son of }\varnothing}\xi_\beta\Bigg(B_{i_1} + \sum_{i_2\text{ son of }i_1}\xi_\beta\big(\dots B_{i_{t-1}} + \sum_{i_t\text{ son of }i_{t-1}}\xi_\beta(B_{i_t})\big)\Bigg)$$

where the function ξ_β is defined by

$$\xi_\beta(x) := \operatorname{atanh}(\tanh\beta\,\tanh x)\,.$$

Proof. Proceed by induction on the number $t \in \mathbb{N}$ of generations of the tree. For $t = 0$ it's simply $T(0) = \{\varnothing\}$, then

$$\langle \sigma_\varnothing \rangle_{T(0)} = \frac{\sum_{\sigma_\varnothing = \pm 1} \sigma_\varnothing \, e^{B_\varnothing \sigma_\varnothing}}{\sum_{\sigma_\varnothing = \pm 1} e^{B_\varnothing \sigma_\varnothing}} = \frac{e^{B_\varnothing} - e^{-B_\varnothing}}{e^{B_\varnothing} + e^{-B_\varnothing}} = \tanh B_\varnothing \,.$$

Now assume the result is true for a $t \geq 0$ and prove it for $t+1$.
Consider $T(t)$, the sub-tree of $T(t+1)$ composed by the first t generations.
By lemma 10

$$\langle \sigma_\varnothing \rangle_{T(t+1)} = \langle \sigma_\varnothing \rangle_{T(t),\,+\underline{H}}$$

where for all $i_t \in \operatorname{Bd} T(t)$ $\;H_{i_t} = \sum_{i_{t+1} \text{ son of } i_t} \xi_\beta(B_{i_{t+1}})\,.$
On the other hand by inductive hypothesis

$$\begin{aligned} &\operatorname{atanh} \langle \sigma_\varnothing \rangle_{T(t),\,+\underline{H}} = \\ &B_\varnothing + \sum_{i_1 \text{ son of } \varnothing} \xi_\beta\Bigg(B_{i_1} + \sum_{i_2 \text{ son of } i_1} \xi_\beta\big(\ldots\, B_{i_{t-1}} + \sum_{i_t \text{ son of } i_{t-1}} \xi_\beta(B_{i_t} + H_{i_t})\big)\Bigg) \end{aligned}$$

Substitute in this expression the expression of H_{i_t} and the thesis is proved for $t+1$. □

The form of proposition 11 simplifies if we consider uniform magnetic field B and a random tree $\mathcal{T}(\rho, t)$ defined as in subsection 1.4.1.

Corollary 12. *Let $(K_{t,i})_{t \geq 0,\, i \geq 1}$ be i.i.d. integer r.v. with distribution ρ such that a.s. $0 \leq K_{t,i} < \infty$.*
Let $T(\rho, \infty)$ be a random tree rooted in $\varnothing$ and such that the offspring size of the i^{th} vertex of the t^{th} generation (root included) is $K_{t,i}$.
Suppose the inverse temperature is $0 \leq \beta < \infty$ and the external field is uniformly $B_i \equiv B \in \mathbb{R}$. And set for any $t \in \mathbb{N}$

$$\boxed{h^{(t)} := \operatorname{atanh} \langle \sigma_\varnothing \rangle_{T(\rho,\,t)}}$$

The probability distribution of the magnetisation of the root $\varnothing$ in the Ising model on the tree $\mathcal{T}(\rho, t)$ is such that

$$\boxed{\begin{cases} h^{(t+1)} \overset{d}{=} B + \sum_{i=1}^{K} \xi_\beta(h_i^{(t)}) \quad \forall\, t \in \mathbb{N} \\ h^{(0)} = B \end{cases}} \tag{1.3}$$

where:

- $\xi_\beta(x) = \operatorname{atanh}(\tanh\beta\,\tanh x)$
- $(h_i^{(t)})_{i\geq 1}$ *are i.i.d. r.v. with the distribution of* $h^{(t)}$
- K *is a r.v. of distribution* ρ*, independent of* $(h_i^{(t)})_{i\geq 1,\, t\geq 0}$ *.*

Proof. Remind the definition $h^{(t)} := \operatorname{atanh}\langle\sigma_\varnothing\rangle_{T(\rho,\,t)}$, apply the proposition 11 and use the independence of the numbers of offspring:

$$h^{(0)} = B\,,$$

$$h^{(1)} = B + \sum_{i_1=1}^{K_0} \xi_\beta(B) \overset{\mathrm{d}}{=} B + \sum_{i=1}^{K} \xi_\beta(h^{(0)})\,,$$

$$h^{(2)} = B + \sum_{i_1=1}^{K_0} \xi_\beta\big(\underbrace{B + \sum_{i_2=1}^{K_{1,\,i_1}} \xi_\beta(B)}_{\overset{\mathrm{d}}{=}\,h^{(1)}}\big) \overset{\mathrm{d}}{=} B + \sum_{i=1}^{K} \xi_\beta(h^{(1)})\,,$$

$$h^{(3)} = B + \sum_{i_1=1}^{K_0} \xi_\beta\Bigg(\underbrace{B + \sum_{i_2=1}^{K_{1,\,i_1}} \xi_\beta\big(B + \sum_{i_3=1}^{K_{2,\,i_2}} \xi_\beta(B)\big)}_{\overset{\mathrm{d}}{=}\,h^{(2)}}\Bigg) \overset{\mathrm{d}}{=} B + \sum_{i=1}^{K} \xi_\beta(h^{(2)})\,,$$

...etcetera. □

The previous corollary gives a distributional recurrence for the the root magnetisation of a random tree $\mathcal{T}(\rho,t)$. We are interested its behaviour when $t\to\infty$ and we expect to obtain a fixed point of the recurrence.
Using the Ising model with positive boundary conditions, we will be able to prove that there exists a unique positive fixed point.

Notation. Given a rooted tree $T(t)$ of t generations denote $\langle\,\cdot\,\rangle^{+}_{T(t)}$ the expected value w.r.t. the Ising model on $T(t)$ with positive conditions on $\operatorname{Bd} T(t)$.

Proposition 13. *Let $T(t)$ be a finite tree rooted in $\varnothing$ and composed of t generations. Suppose the external field is $\underline{B} = (B_i)_i,\ B_i \geq B_{\min} > 0$ and the inverse temperature is $0 \leq \beta \leq \beta_{\max} < \infty$.*
The effect of positive boundary conditions on the magnetisation of the root in the Ising model on $T(t)$ vanishes when t grows. Precisely:

$$0 \leq \langle \sigma_\varnothing \rangle^+_{T(t)} - \langle \sigma_\varnothing \rangle_{T(t)} \leq \frac{M}{t} \,,$$

with $M = M(B_{\min}, \beta_{\max}) = \sup_{0<x\leq\beta_{\max}} x/\big(\operatorname{atanh}(\tanh x \, \tanh B_{\min})\big) < \infty$.

Proof. For any $s \leq t$ denote $T(s)$ the sub-tree of $T(t)$ induced by the first s generations. Then, given an additional magnetic field $\underline{H}$ on $\operatorname{Bd} T(s)$, denote $\langle \cdot \rangle_{T(s),+\underline{H}}$ the expectation w.r.t. the Ising model on the tree $T(s)$ with the magnetic field $\underline{B}$ increased by $\underline{H}$ only on $\operatorname{Bd} T(s)$.
The first inequality is a direct consequence of G.K.S.:

$$\langle \sigma_\varnothing \rangle_{T(t)}+ = \langle \sigma_\varnothing \rangle_{T(t),+\infty} \geq \langle \sigma_\varnothing \rangle_{T(t)} \,.$$

To deal with the second inequality begin observing that the case $\beta = 0$ is trivial, since the root does not interact with the boundary. Formally:

$$\langle \sigma_\varnothing \rangle_{T(t)} = \frac{\sum_\sigma \sigma_\varnothing \exp\big(\sum_{i\in T(t)} B_i\,\sigma_i\big)}{\sum_\sigma \exp\big(\sum_{i\in T(t)} B_i\,\sigma_i\big)} = \frac{\sum_{\sigma_\varnothing} \sigma_\varnothing\, e^{B_\varnothing\,\sigma_\varnothing} \sum_{\sigma'} \exp\big(\sum_{i\neq\varnothing} B_i\,\sigma_i\big)}{\sum_{\sigma_\varnothing} e^{B_\varnothing\,\sigma_\varnothing} \sum_{\sigma'} \exp\big(\sum_{i\neq\varnothing} B_i\,\sigma_i\big)} =$$
$$= \frac{e^{B_\varnothing} - e^{-B_\varnothing}}{e^{B_\varnothing} + e^{-B_\varnothing}} = \tanh B_\varnothing \,,$$

where here σ' denotes the vector σ minus its component $\sigma_\varnothing$; and similarly one finds that also $\langle \sigma_\varnothing \rangle^+_{T(t)} = \tanh B_\varnothing$.
Now assume $\beta > 0$ and fix $s = 1, \dots, t$.
I) Start studying the positive boundary conditions case. Use the lemma 10 :

$$\langle \sigma_\varnothing \rangle^+_{T(s)} = \langle \sigma_\varnothing \rangle_{T(s),+\infty} = \langle \sigma_\varnothing \rangle_{T(s-1),+\underline{H}}$$

where for every $i \in \operatorname{Bd} T(s-1)$

$$H_i = \sum_{j \text{ son of } i} \operatorname{atanh}(\tanh\beta\, \tanh\infty) = \beta\,\Delta_i$$

and Δ_i denotes the number of sons of the node i. Hence, setting $\underline{\Delta} = (\Delta_i)_i$,

$$\langle \sigma_\varnothing \rangle^+_{T(s)} = \langle \sigma_\varnothing \rangle_{T(s-1),\, +\beta\,\underline{\Delta}} \tag{1.4}$$

And by G.K.S. it follows that $s \mapsto \langle \sigma_\varnothing \rangle^+_{T(s)}$ is *monotonically decreasing*:

$$\langle \sigma_\varnothing \rangle^+_{T(s)} = \langle \sigma_\varnothing \rangle_{T(s-1),\, +\beta\,\underline{\Delta}} \leq \langle \sigma_\varnothing \rangle_{T(s-1),\, +\infty} = \langle \sigma_\varnothing \rangle^+_{T(s-1)} \,.$$

II) Now study the free boundary conditions case. Use the G.K.S. inequality and the lemma 10 :

$$\langle \sigma_\varnothing \rangle_{T(s)} \geq \langle \sigma_\varnothing \rangle_{T(s),\, +B_{\min} - \underline{B}} = \langle \sigma_\varnothing \rangle_{T(s-1),\, +\underline{H}'}$$

where for all $i \in \mathrm{Bd}\, T(s-1)$

$$H'_i \;=\; \sum_{j \text{ son of } i} \operatorname{atanh}(\tanh\beta\, \tanh B_{\min}) \;=\; \xi_\beta(B_{\min})\, \Delta_i$$

Hence, setting $\xi^0_\beta := \xi_\beta(B_{\min}) > 0$,

$$\langle \sigma_\varnothing \rangle_{T(s)} \geq \langle \sigma_\varnothing \rangle_{T(s-1),\, +\xi^0_\beta\,\underline{\Delta}} \tag{1.5}$$

And by G.K.S. it follows that $s \mapsto \langle \sigma_\varnothing \rangle_{T(s)}$ is *monotonically increasing*:

$$\langle \sigma_\varnothing \rangle_{T(s)} \geq \langle \sigma_\varnothing \rangle_{T(s-1),\, +\xi^0_\beta\,\underline{\Delta}} \geq \langle \sigma_\varnothing \rangle_{T(s-1)} \,.$$

III) By the G.H.S. inequality, the function $h \mapsto \langle \sigma_\varnothing \rangle_{T(s-1),\, +h\,\underline{\Delta}} =: f(h)$ is concave. Hence, since $\xi^0_\beta = \operatorname{atanh}(\tanh\beta\, \tanh B_{\min}) < \beta$, it follows that

$$\frac{f(\beta) - f(0)}{\beta - 0} \;\leq\; \frac{f(\xi^0_\beta) - f(0)}{\xi^0_\beta - 0}$$

Rewriting this condition one obtains

$$\langle \sigma_\varnothing \rangle_{T(s-1),\, +\beta\,\underline{\Delta}} - \langle \sigma_\varnothing \rangle_{T(s-1)} \;\leq\; M\, \left(\langle \sigma_\varnothing \rangle_{T(s-1),\, +\xi^0_\beta\,\underline{\Delta}} - \langle \sigma_\varnothing \rangle_{T(s-1)} \right) \tag{1.6}$$

with $M := \sup\{x / \operatorname{atanh}(\tanh x\, \tanh B_{\min}) \,|\, 0 < x \leq \beta_{\max}\}$, which is finite because the function to maximize is increasing.

Now bound the effect of positive boundary conditions on the root magnetisation in the model on $T(s)$, using equations (1.4), (1.5), (1.6)

$$\begin{aligned}\langle\sigma_\varnothing\rangle^+_{T(s)} - \langle\sigma_\varnothing\rangle_{T(s)} &\le \langle\sigma_\varnothing\rangle_{T(s-1),\,+\beta\,\Delta} - \langle\sigma_\varnothing\rangle_{T(s-1)} \\ &\le M\left(\langle\sigma_\varnothing\rangle_{T(s-1),\,+\xi^0_\beta\,\Delta} - \langle\sigma_\varnothing\rangle_{T(s-1)}\right) \\ &\le M\left(\langle\sigma_\varnothing\rangle_{T(s)} - \langle\sigma_\varnothing\rangle_{T(s-1)}\right) .\end{aligned}$$

Then use the different monotonicity of $s \mapsto \langle\sigma_\varnothing\rangle^+_{T(s)}$ and $s \mapsto \langle\sigma_\varnothing\rangle_{T(s)}$ to conclude:

$$\begin{aligned}t\left(\langle\sigma_\varnothing\rangle^+_{T(t)} - \langle\sigma_\varnothing\rangle_{T(t)}\right) &\le \sum_{s=1}^{t}\left(\langle\sigma_\varnothing\rangle^+_{T(s)} - \langle\sigma_\varnothing\rangle_{T(s)}\right) \le M\sum_{s=1}^{t}\left(\langle\sigma_\varnothing\rangle_{T(s)} - \langle\sigma_\varnothing\rangle_{T(s-1)}\right) \\ &= M\left(\langle\sigma_\varnothing\rangle_{T(t)} - \langle\sigma_\varnothing\rangle_{T(0)}\right) \le M .\end{aligned}$$

□

Using corollary 12 and proposition 13 we can prove the following probability result, which characterizes the root magnetisation on the random tree $\mathcal{T}(\rho, t)$ as $t \to \infty$.

Proposition 14. *Let $B > 0$, $0 \le \beta < \infty$ and let ρ be a probability distribution over $\mathbb{N}$.*
Consider a sequence $(h^{(t)})_{t\in\mathbb{N}}$ of r.v.'s whose distributions are defined by the recursive relation (1.3), that is

$$\begin{cases} h^{(t+1)} \overset{d}{=} B + \sum_{i=1}^{K} \xi_\beta(h_i^{(t)}) & \forall\, t \in \mathbb{N} \\ h^{(0)} = B \end{cases}$$

where

- $\xi_\beta(x) = \operatorname{atanh}(\tanh\beta\,\tanh x)$
- $(h_i^{(t)})_{i\ge1}$ *are i.i.d. r.v. with the same distribution of* $h^{(t)}$
- K *is a r.v. of distribution ρ, independent of* $(h_i^{(t)})_{i\ge1,\,t\ge0}$.

Then:

i. $(h^{(t)})_{t\in\mathbb{N}}$ *is stochastically monotone (that is it admits a coupling which is monotone a.s.)*

ii. *there exists a r.v.* h^* *such that* $h^{(t)} \xrightarrow[t\to\infty]{d} h^*$

iii. h^* *is the only (in distribution) r.v. supported on* $[0,\infty[$ *such that*

$$\boxed{h^* \overset{d}{=} B + \sum_{i=1}^{K} \xi_\beta(h_i^*)} \tag{1.7}$$

with $(h_i^*)_{i\geq 1}$ *i.i.d., distributed as* h^**, independent of* K*.*

Proof. **I)** Consider the random rooted tree $\mathcal{T}(\rho,\infty)$ and the Ising model on it with inverse temperature β and magnetic field B.
By corollary 12 a coupling of $(h^{(t)})_{t\in\mathbb{N}}$ is given by

$$h^{(t)} := \operatorname{atanh}\langle\sigma_\varnothing\rangle_{\mathcal{T}(\rho,\,t)} \quad \forall\, t \in \mathbb{N}\,.$$

As seen in the proof of proposition 13 (or simply by G.K.S. inequality), the sequence $t \mapsto \langle\sigma_\varnothing\rangle_{\mathcal{T}(\rho,\,t)}$ is monotonically increasing. Hence also $t \mapsto h^{(t)}$ is monotonically increasing. Therefore there exists $h^* \geq B$ such that

$$h^{(t)} \xrightarrow[t\to\infty]{} h^* \ a.s.$$

and so also in distribution. Then, by definition, for any $\phi : \mathbb{R}_+ \to \mathbb{R}$ continuous and bounded

$$\mathbb{E}[\phi(h^*)] = \lim_{t\to\infty} \mathbb{E}[\phi(h^{(t)})] = \lim_{t\to\infty} \mathbb{E}\big[\phi\big(B + \sum_{i=1}^{K} \xi_\beta(h_i^{(t-1)})\big)\big]$$

Now since $h_i^{(t-1)} \xrightarrow{\mathrm{d}} h_i^*$ as $t\to\infty$ and $\big((h_i^{(t-1)})_i,\, K\big)$ are independent as well as $\big((h_i^*)_i,\, K\big)$, it is also true that $\big((h_i^{(t-1)})_i,\, K\big) \xrightarrow{\mathrm{d}} \big((h_i^*)_i, K\big)$ as $t\to\infty$ (this can be easily proven via the characteristic functions). Therefore by dominated convergence

$$\lim_{t\to\infty} \mathbb{E}\big[\phi\big(B + \sum_{i=1}^{K} \xi_\beta(h_i^{(t-1)})\big)\big] = \mathbb{E}\big[\phi\big(B + \sum_{i=1}^{K} \xi_\beta(h_i^*)\big)\big]\,.$$

Hence $\mathbb{E}[\phi(h^*)] = \mathbb{E}\big[\phi\big(B + \sum_{i=1}^K \xi_\beta(h_i^*)\big)\big]$, and by arbitrariness of ϕ continuous and bounded conclude that

$$h^* \overset{\mathrm{d}}{=} B + \sum_{i=1}^{K} \xi_\beta(h_i^*)\,.$$

II) It remains to prove that $h^* < \infty$ and that any other r.v. supported on $[0,\infty[$ and satisfying the same fixed point distributional equation is necessarily equal in distribution to h^*. Define

$$h^{(t),+} := \operatorname{atanh}\langle\sigma_\varnothing\rangle^+_{\mathcal{T}(\rho,\,t)} \quad \forall\, t \in \mathbb{N}\,.$$

Since the model with positive boundary conditions is equivalent to that one with no boundary conditions when the magnetic field on $\operatorname{Bd}\mathcal{T}(\rho,\,t)$ goes to infinity,

$$\begin{cases} h^{(t+1),+} \overset{\mathrm{d}}{=} B + \sum_{i=1}^K \xi_\beta(h_i^{(t),+}) \quad \forall\, t \in \mathbb{N} \\ h^{(0),+} = \infty \end{cases} \tag{1.8}$$

where $(h_i^{(t),+})_i$ are i.i.d. copies of $h^{(t),+}$ and they're independent of K.
As seen in the proof of proposition 13, the sequence $t \mapsto \langle\sigma_\varnothing\rangle^+_{\mathcal{T}(\rho,\,t)}$ is monotonically decreasing. Hence also $t \mapsto h^{(t),+}$ is monotonically decreasing and so there exists $h^{*,+} < \infty$ (notice $h^{(1),+} < \infty$) such that

$$h^{(t),+} \xrightarrow[t\to\infty]{} h^{*,+} \; a.s.$$

Proceeding as before one proves that

$$h^{*,+} \overset{\mathrm{d}}{=} B + \sum_{i=1}^{K} \xi_\beta(h_i^{*,+})$$

with $(h_i^{*,+})_i$ i.i.d. copies of $h^{*,+}$, independent of K.
Now let h^{**} be a r.v. supported on $[0,\infty]$ and verifying the same fixed point distributional equation 1.7. Without loss of generality assume it is defined on the same probability space as $\mathcal{T}(\rho,\infty)$.
Notice that, since $\xi_\beta \geq 0$ on $[0,\infty]$, necessarily $h^{**} \geq B$. Therefore

$$h^{(0)} = B \,\leq\, h^{**} \leq \infty = h^{(0),+}\,.$$

Now take i.i.d. copies of each of these three r.v.'s, independent of K and coupled so they still verify the previous order relation. Then apply the function $B+\sum_{i=1}^{K}\xi_\beta(\cdot)$ and, since it is a monotonically increasing function w.r.t. each variable, one gets $h^{(1)} \le h^{**} \le h^{(1),+}$.
Repeat this procedure t times and then let $t\to\infty$:

$$h^{(t)} \le h^{**} \le h^{(t),+}\ \forall\, t\in\mathbb{N} \quad\Rightarrow\quad h^* \le h^{**} \le h^{*,+}\ a.s.$$

But on the other hand by proposition 13

$$|\tanh(h^{(t),+}) - \tanh(h^{(t)})| = |\langle\sigma_\varnothing\rangle^+_{T(\rho,t)} - \langle\sigma_\varnothing\rangle_{T(\rho,t)}| \le \frac{M}{t} \xrightarrow[t\to\infty]{} 0$$

therefore $\tanh h^{*,+} = \tanh h^* \;\Rightarrow\; h^{*,+} = h^*$. Thus conclude

$$h^* = h^{**} = h^{*,+} \in [B,\infty[\ \ a.s.$$

and so also in distribution. □

We conclude this subsection about trees proving that the root magnetisation is Lipschitz continuous w.r.t. β uniformly in number of generations t, so that the Lipschitz continuity is preserved in the limit $t\to\infty$.

Notation. Consider the Ising model on a finite rooted tree $T(t)$ composed of t generations, with magnetic field $\underline{B}$ and inverse temperature β.
To make explicit the parameters, we'll use the following notation for the magnetisation of the root $\varnothing$:

$$m_t(\beta,\underline{B}) := \langle\sigma_\varnothing\rangle_{T(t)}\,.$$

Proposition 15. *Let $T(t)$ be a finite rooted tree rooted with t generations. Suppose the external field is fixed $\underline{B}=(B_i)_i$, $B_i \ge B_{\min} > 0$, while the inverse temperature can be $0<\beta_{min}\le\beta_1\le\beta_2<\infty$.*
The magnetisation of the root in the Ising model is Lipschitz continuous w.r.t. β, uniformly in t. Precisely:

$$0 \le m_t(\beta_2,B) - m_t(\beta_1,B) \le C\,(\beta_2-\beta_1)\,,$$

with $C = C(\beta_{\min},B_{\min}) = \sup_{\beta_{\min}\le x<\infty} 1/\big(\operatorname{atanh}(\tanh x\,\tanh(B_{\min})\big) < \infty$.

Before the proof we need the following lemma.

Lemma 16. *Let T be a finite tree rooted in $\varnothing$.*
Let ij be and edge of T and assume that node j is a son of node i.
Then for any Ising model on T there exists $\gamma \in [0,1]$ s.t.

$$\langle \sigma_\varnothing\, \sigma_i \sigma_j \rangle_T - \langle \sigma_\varnothing \rangle_T \, \langle \sigma_i \sigma_j \rangle_T \;=\; \gamma \left(\langle \sigma_\varnothing\, \sigma_i \rangle_T - \langle \sigma_\varnothing \rangle_T \, \langle \sigma_i \rangle_T \right) .$$

Proof. **I)** Denote T_j the sub-tree of $T(t)$ induced by j and all its descendants. Thanks to the property of absence of cycles which characterizes trees, ij is the unique path connecting T_j with $T(t) - T_j$. It follows that $\sigma_{T_j},\ \sigma_{T(t)-T_j}$ are conditionally independent given σ_i. Indeed

$$\sum_{hk \in T} \sigma_h \sigma_k = \sum_{hk \in T} \sigma_h \sigma_k \,+ \sum_{hk \in T - T_j} \sigma_h \sigma_k \,+\, \sigma_i \sigma_j$$

so that for any σ spin configuration on T and for a fixed $\varepsilon = \pm 1$

$$\mu(\sigma |\, \sigma_i = \varepsilon) = C\, \exp\big(\beta \sum_{hk \in T} \sigma_h \sigma_k + \sum_{h \in T} B_h\, \sigma_h \big)\, \mathbb{1}(\sigma_i = \varepsilon)$$
$$= C \exp\big(\beta \sum_{hk \in T_j} \sigma_h \sigma_k + \sum_{h \in T_j} B_h \sigma_h \big)\; e^{\beta\, \varepsilon\, \sigma_j} \exp\big(\beta \sum_{hk \in T - T_j} \sigma_h \sigma_k + \sum_{h \in T - T_j} B_h \sigma_h \big)\, \mathbb{1}(\sigma_i \!=\! \varepsilon)$$
$$= \mu(\sigma_{T_j} |\, \sigma_i = \varepsilon)\; e^{\beta\, \varepsilon\, \sigma_j}\; \mu(\sigma_{T - T_j} |\, \sigma_i = \varepsilon)\,,$$

where for brevity the marginal measures are still denoted μ.
In particular $\sigma_\varnothing,\ \sigma_j$ are conditionally independent given σ_i.

II) Now apply the total probability formula to $\langle \sigma_\varnothing\, \sigma_i \sigma_j \rangle$ to condition on σ_i and then use the conditional independence just proven:

$$\langle \sigma_\varnothing\, \sigma_i \sigma_j \rangle =$$
$$\langle \sigma_\varnothing\, \sigma_i |\, \sigma_i \!=\! 1 \rangle\, \langle \sigma_j |\, \sigma_i \!=\! 1 \rangle\, \mu_i(1) \;+\; \langle \sigma_\varnothing\, \sigma_i \sigma_j |\, \sigma_i \!=\! -1 \rangle\, \langle \sigma_j |\, \sigma_i \!=\! -1 \rangle\, \mu_i(-1)\,,$$

where μ_i denotes the marginal on i^{th} spin variable of measure μ.
Then notice the last quantity can be rewritten in two different ways;
firstly grouping together $\langle \sigma_j |\, \sigma_i = 1 \rangle$ artificially to obtain

$$\langle \sigma_\varnothing\, \sigma_i \rangle\, \langle \sigma_j |\, \sigma_i \!=\! 1 \rangle \;+\; \langle \sigma_\varnothing |\, \sigma_i \!=\! -1 \rangle\, \mu_i(-1) \left(\langle \sigma_j |\, \sigma_i \!=\! 1 \rangle - \langle \sigma_j |\, \sigma_i \!=\! -1 \rangle \right) \;=: A$$

secondly grouping together $\langle \sigma_j | \, \sigma_i - = 1 \rangle$ artificially to obtain

$$\langle \sigma_\varnothing \, \sigma_i \rangle \, \langle \sigma_j | \, \sigma_i = -1 \rangle \; + \; \langle \sigma_\varnothing | \, \sigma_i = 1 \rangle \, \mu_i(1) \left(\langle \sigma_j | \, \sigma_i = 1 \rangle - \langle \sigma_j | \, \sigma_i = -1 \rangle \right) \; =: B \, .$$

Now since $\langle \sigma_\varnothing \, \sigma_i \sigma_j \rangle = A$ as well as $\langle \sigma_\varnothing \, \sigma_i \sigma_j \rangle = B$, one can write

$$\langle \sigma_\varnothing \, \sigma_i \sigma_j \rangle = \frac{A+B}{2} = \langle \sigma_\varnothing \, \sigma_i \rangle \, \gamma + \langle \sigma_\varnothing \rangle \, \delta \qquad \star$$

where $\gamma = \frac{\langle \sigma_j | \, \sigma_i = 1 \rangle + \langle \sigma_j | \, \sigma_i = -1 \rangle}{2}$ and $\delta = \frac{\langle \sigma_j | \, \sigma_i = 1 \rangle - \langle \sigma_j | \, \sigma_i = -1 \rangle}{2}$.
On the other hand notice that $\langle \sigma_i \rangle \, = 2 \, \mu_i(1) - 1 \, = \, 1 - 2 \, \mu_i(-1)$, since σ_i takes values ± 1 . Use this fact and the definitions of γ, δ to compute

$$\begin{aligned} \gamma \, \langle \sigma_i \rangle &= \langle \sigma_j | \, \sigma_i = 1 \rangle \, \frac{2 \, \mu_i(1) - 1}{2} + \langle \sigma_j | \, \sigma_i = -1 \rangle \, \frac{1 - 2 \, \mu_i(1)}{2} \\ &= \langle \sigma_i \sigma_j | \, \sigma_i = 1 \rangle \, \frac{2 \, \mu_i(1) - 1}{2} - \langle \sigma_i \sigma_j | \, \sigma_i = -1 \rangle \, \frac{1 - 2 \, \mu_i(1)}{2} \; = \; \langle \sigma_i \sigma_j \rangle - \delta \, . \end{aligned}$$

Conclude using equality $\star$ together with this last equality:

$$\begin{aligned} \langle \sigma_\varnothing \, \sigma_i \sigma_j \rangle - \langle \sigma_\varnothing \rangle \, \langle \sigma_i \sigma_j \rangle \; &= \; \gamma \, \langle \sigma_\varnothing \, \sigma_i \rangle + \delta \, \langle \sigma_\varnothing \rangle - \langle \sigma_\varnothing \rangle \left(\gamma \, \langle \sigma_i \rangle + \delta \right) \\ &= \gamma \left(\langle \sigma_\varnothing \, \sigma_i \rangle - \langle \sigma_\varnothing \rangle \, \langle \sigma_i \rangle \right) \end{aligned}$$

and observing that $|\gamma| \leq 1$ but by G.K.S. both the left-hand difference and the right-hand difference are ≥ 0 . □

Proof of proposition 15. For any $s \leq t$ $T(s)$ will denote the sub-tree of $T(t)$ induced by the first s generations.
The first inequality is simply due to the G.K.S. inequality. Prove the second one. By Lagrange mean value theorem

$$m_t(\beta_2, B) - m_t(\beta_1, B) = \frac{\partial \, m_t}{\partial \beta}(\beta, B) \; (\beta_2 - \beta_1)$$

for some $\beta \in [\beta_1, \beta_2] \subseteq [\beta_{\min}, \infty[$.
From now on reason in the Ising model on $T(t)$ with magnetic field B and inverse temperature β. To begin it's easy to compute

$$\frac{\partial \, m_t}{\partial \beta}(\beta, B) \equiv \frac{\partial}{\partial \beta} \, \langle \sigma_\varnothing \rangle_{T(t)} = \sum_{ij \in T(t)} \left(\, \langle \sigma_\varnothing \, \sigma_i \sigma_j \rangle_{T(t)} - \langle \sigma_\varnothing \rangle_{T(t)} \, \langle \sigma_i \sigma_j \rangle_{T(t)} \right) \, .$$

For any edge $ij \in T(t)$ one may assume that j is a son of i and apply the previous lemma:

$$\langle \sigma_\varnothing \, \sigma_i \sigma_j \rangle_{T(t)} - \langle \sigma_\varnothing \rangle_{T(t)} \, \langle \sigma_i \sigma_j \rangle_{T(t)} \;\le\; \langle \sigma_\varnothing \, \sigma_i \rangle_{T(t)} - \langle \sigma_\varnothing \rangle_{T(t)} \, \langle \sigma_i \rangle_{T(t)} \,.$$

It's an easy computation to check that

$$\langle \sigma_\varnothing \, \sigma_i \rangle_{T(t)} - \langle \sigma_\varnothing \rangle_{T(t)} \, \langle \sigma_i \rangle_{T(t)} \;=\; \frac{\partial}{\partial B_i} \, \langle \sigma_\varnothing \rangle_{T(t)}$$

Therefore putting the things together

$$\frac{\partial}{\partial \beta} \, \langle \sigma_\varnothing \rangle_{T(t)} \;\le\; \sum_{s=0}^{t-1} \, \sum_{i \in \mathrm{Bd}\, T(s)} \Delta_i \, \frac{\partial}{\partial B_i} \, \langle \sigma_\varnothing \rangle_{T(t)} \tag{1.9}$$

where Δ_i denotes the number of sons of node i.
Now observe that $s \mapsto \frac{\partial}{\partial B_i} \langle \sigma_\varnothing \rangle_{T(s)}$ is a decreasing sequence. Indeed using lemma 10 and the G.H.S. inequality:

$$\frac{\partial}{\partial B_i} \, \langle \sigma_\varnothing \rangle_{T(s)} \;=\; \frac{\partial}{\partial B_i} \, \langle \sigma_\varnothing \rangle_{T(s-1), +\underline{H}} \;\le\; \frac{\partial}{\partial B_i} \, \langle \sigma_\varnothing \rangle_{T(s-1)}$$

where $\underline{H}^{s-1} = (H_i)_{i \in \mathrm{Bd}\, T(s-1)}$, $H_i := \sum_{j \text{ son of } i} \xi_\beta(B_j) \ge 0$.
Hence one is allowed to substitute $T(t)$ with $T(s+1)$ into inequality (1.9) :

$$\frac{\partial}{\partial \beta} \, \langle \sigma_\varnothing \rangle_{T(t)} \;\le\; \sum_{s=0}^{t-1} \, \sum_{i \in \mathrm{Bd}\, T(s)} \Delta_i \, \frac{\partial}{\partial B_i} \, \langle \sigma_\varnothing \rangle_{T(s+1)} \tag{1.10}$$

Now using again lemma 10 and the G.H.S. inequality observe

$$\frac{\partial}{\partial B_i} \, \langle \sigma_\varnothing \rangle_{T(s+1)} \;=\; \frac{\partial}{\partial B_i} \, \langle \sigma_\varnothing \rangle_{T(s), +\underline{H}^s} \;\le\; \frac{\partial}{\partial B_i} \, \langle \sigma_\varnothing \rangle_{T(s), +\xi^0_\beta \, \underline{\Delta}^s}$$

where $\underline{\Delta}^s = (\Delta_i)_{i \in \mathrm{Bd}\, T(s)}$ and $\xi^0_\beta := \xi_\beta(B_{\min}) \le \xi_\beta(B_i)$.
Hence inequality (1.10) becomes

$$\begin{aligned} \frac{\partial}{\partial \beta} \, \langle \sigma_\varnothing \rangle_{T(t)} \;&\le\; \sum_{s=0}^{t-1} \, \sum_{i \in \mathrm{Bd}\, T(s)} \Delta_i \, \frac{\partial}{\partial B_i} \, \langle \sigma_\varnothing \rangle_{T(s), +\xi^0_\beta \, \underline{\Delta}^s} \\ &=\; \sum_{s=0}^{t-1} \frac{\partial}{\partial h} \left(\langle \sigma_\varnothing \rangle_{T(s), +h \, \underline{\Delta}^s} \right)_{|h = \xi^0_\beta} \end{aligned} \tag{1.11}$$

Now by G.H.S. inequality $h \mapsto \langle\sigma_\varnothing\rangle_{T(s),\,+h\,\underline{\Delta}^s} =: f(h)$ is concave, hence

$$f'(\xi^0_\beta) \;\leq\; \frac{f(\xi^0_\beta) - f(0)}{\xi^0_\beta - 0}\;,$$

rewriting this relation, using the G.K.S. inequality and lemma 10, one finds

$$\frac{\partial}{\partial h}\left(\langle\sigma_\varnothing\rangle_{T(s),\,+h\,\underline{\Delta}^s}\right)_{|h=\xi^0_\beta} \;\leq\; \frac{\langle\sigma_\varnothing\rangle_{T(s),\,+\xi^0_\beta\,\underline{\Delta}^s} - \langle\sigma_\varnothing\rangle_{T(s)}}{\xi^0_\beta} \;\leq\; \frac{\langle\sigma_\varnothing\rangle_{T(s),\,+\underline{H}^s} - \langle\sigma_\varnothing\rangle_{T(s)}}{\xi^0_\beta}$$

$$= \frac{\langle\sigma_\varnothing\rangle_{T(s+1)} - \langle\sigma_\varnothing\rangle_{T(s)}}{\xi^0_\beta}\;.$$

Substitute into inequality (1.11) and obtain

$$\frac{\partial}{\partial\beta}\,\langle\sigma_\varnothing\rangle_{T(t)} \;\leq\; \sum_{s=0}^{t-1}\frac{\langle\sigma_\varnothing\rangle_{T(s+1)} - \langle\sigma_\varnothing\rangle_{T(s)}}{\xi^0_\beta} \;=\; \frac{\langle\sigma_\varnothing\rangle_{T(t)} - \langle\sigma_\varnothing\rangle_{T(0)}}{\xi^0_\beta} \;\leq\; \frac{1}{\xi^0_\beta}\;,$$

to conclude notice that $\frac{1}{\xi^0_\beta} \leq \sup_{\beta_{\min}\leq x<\infty}[\operatorname{atanh}(\tanh x\,\tanh(B_{\min})]^{-1}$, which is finite as the function to maximize is decreasing. □

Corollary 17. *Let $B \geq B_{min} > 0$ and $\beta_1,\, \beta_2 \geq \beta_{min} > 0$.*
Let $h^*(\beta_1, B)$ and $h^*(\beta_2, B)$ be the non-negative solutions of fixed point distributional equation (1.7) for (β_1, B) and (β_2, B) respectively.
Then there exists a coupling of $h^*(\beta_1, B)$, $h^*(\beta_2, B)$ s.t.

$$0 \;\leq\; \tanh h^*(\beta_2, B) - \tanh h^*(\beta_1, B) \;\leq\; C\,(\beta_2 - \beta_1)$$

where $C = C(\beta_{\min}, B_{\min}) < \infty$.

Proof. Consider the random rooted tree $\mathcal{T}(\rho,\infty)$ and the two Ising models on it with magnetic field B and inverse temperature respectively β_1, β_2.
As seen in the proof of proposition 14 a coupling of $h^*(\beta_1, B)$, $h^*(\beta_2, B)$ is given by

$$h^*(\beta_i, B) = \lim_{t\to\infty} \operatorname{atanh} m_t(\beta_i, B) \text{ a.s.} \quad \forall\, i = 1, 2\;.$$

Now by proposition 15

$$0 \;\leq\; m_t(\beta_2, B) - m_t(\beta_1, B) \;\leq\; C\,(\beta_2 - \beta_1) \;\;\forall\, t \in \mathbb{N}\,.$$

Let $t \to \infty$ and since C does not depend on t obtain

$$0 \;\leq\; \tanh h^*(\beta_2, B) - \tanh h^*(\beta_1, B) \;\leq\; C\,(\beta_2 - \beta_1) \;\;\forall\, t \in \mathbb{N}\,. \quad \square$$

1.4.4 From trees to random graphs

In this subsection we'll manage to compute the thermodynamic limit for the pressure per particle p_N on a sequence of random graphs which locally converges to $\mathcal{T}(P, \rho, \infty)$.
Actually we would be already equipped to prove that the internal energy per particle $\partial p_N / \partial \beta$ converges. But in order to come back to p_N we need a technical result.

Definition 7. Fix P a probability distribution over $\mathbb{N}$ with finite mean $\overline{P}$. Let h be a random variable supported on $[0, \infty[$. Given $\beta \geq 0$, $B \in \mathbb{R}$ we define the following operator

$$\varphi_h(\beta, B) := \frac{\overline{P}}{2} \log\cosh\beta - \frac{\overline{P}}{2} \mathbb{E}\big[\log\big(1 + \tanh\beta\, \tanh h_1\, \tanh h_2\big)\big] + \\ + \mathbb{E}\big[\log\big(e^B \prod_{i=1}^{L} (1 + \tanh\beta\, \tanh h_i) + e^{-B} \prod_{i=1}^{L} (1 - \tanh\beta\, \tanh h_i)\big)\big] \tag{1.12}$$

where $(h_i)_{i \geq 1}$ are i.i.d. r.v.'s with the same distribution of h , while
L is an integer r.v. with distribution P and it is independent of $(h_i)_{i\geq 1}$.

Remark 4. Notice $\varphi_h(\beta, B)$ is well-defined and finite. Indeed the quantities under expectation are non-negative and furthermore, using Jensen inequality they are bounded respectively by

$$\log \mathbb{E}\big[1 + \tanh\beta\, \tanh h_1\, \tanh h_2\big] \leq \log(1 + \tanh\beta) \ ,$$

$$\log \mathbb{E}\big[e^B \textstyle\prod_{i=1}^{L} (1 + \tanh\beta\, \tanh h_i) + e^{-B} \prod_{i=1}^{L} (1 - \tanh\beta\, \tanh h_i)\big] \\ \leq \log\big(e^B\, \overline{P}\, (1 + \tanh\beta) + e^{-B}\, \overline{P}\, (1 - \tanh\beta)\big) \ .$$

Proposition 18. *Let B, $B' > 0$ and $0 \leq \beta$, $\beta' \leq \beta_{\max}$.*
Suppose P has ϵ-strongly finite mean for some $\epsilon > 0$.
Let $h^* := h^*(\beta, B)$ and $(h^*)' := h^*(\beta', B')$ be the non-negative solutions of fixed point distributional equations (1.7) for (β, B) and (β', B') respectively and choosing ρ the size-biased law of P (or $\rho = P$).

Then there exists a constant $\lambda = \lambda(\beta_{\max}) < \infty$ *s.t.*

$$\left|\varphi_{(h^*)'}(\beta, B) - \varphi_{h^*}(\beta, B)\right| \;\leq\; \lambda\, d_{MK}\big(\tanh(h^*)'\,,\, \tanh h^*\big)^{1+\eta}$$

where $\eta = \min\{1, \epsilon\}$ *and* d_{MK} *denotes the Monge-Kantorovich-Wasserstein distance between two r.v.'s* X, Y, *i.e. the infimum of* $\mathbb{E}[|\widetilde{X} - \widetilde{Y}|\,]$ *over all couplings* $(\widetilde{X}, \widetilde{Y})$ *of* X, Y.

Proof. Assume $0 < \epsilon < 1$ without loss of generality.
If $d_{\mathrm{MK}}\big(\tanh(h^*)'\,,\, \tanh h^*\big) = 0, +\infty$ then the inequality is obviously true.
Thus assume $0 < d_{\mathrm{MK}}\big(\tanh(h^*)'\,,\, \tanh h^*\big) < \infty$.
Let $\gamma > 1$. By definition of d_{MK} there exist $X \overset{\mathrm{d}}{=} \tanh h^*$, $Y \overset{\mathrm{d}}{=} \tanh(h^*)'$ defined on the same space s.t.

$$\mathbb{E}[|Y - X|\,] \;\leq\; \gamma\, d_{\mathrm{MK}}\big(\tanh(h^*)'\,,\, \tanh h^*\big)\;.$$

Then let $(X_i, Y_i)_{i\geq 1}$ be i.i.d. copies of (X, Y) and let $L \overset{\mathrm{d}}{\sim} P$, $K \overset{\mathrm{d}}{\sim} \rho$ independent of $(X_i, Y_i)_{i\geq 1}$.
Set $u := \tanh\beta$. For $l \geq 2$ and $0 \leq x_1, \dots, x_l \leq 1$ define

$$F_l(x_1, \dots, x_l) := \frac{l}{2}\,\log\cosh\beta - \frac{1}{l-1}\sum_{1\leq i<j\leq l}\log(1 + u\,x_i x_j) + \log\Big(e^B\prod_{i=1}^{l}(1 + u\,x_i) + e^{-B}\prod_{i=1}^{l}(1 - u\,x_i)\Big)$$

while let

$$F_1(x_1, x_2) := \frac{1}{2}\,\log\cosh\beta - \frac{1}{2}\,\log(1 + u\,x_1 x_2) + \frac{\log\big(e^B(1{+}\,u\,x_1) + e^{-B}(1{-}u\,x_1)\big)}{2} + \frac{\log\big(e^B(1{+}\,u\,x_2) + e^{-B}(1{-}u\,x_2)\big)}{2}\;,$$

and $F_0 := -\log 2 + \log\cosh B$.
Notice that

$$\varphi_{h^*}(\beta, B) = \mathbb{E}\big[F_L(X_1, \dots, X_L)\big]\,,\quad \varphi_{(h^*)'}(\beta, B) = \mathbb{E}\big[F_L(Y_1, \dots, Y_L)\big] \quad (1.13)$$

For shortness it's useful to set

$$G_l(x_2, \dots, x_l) := \tanh\Big(B + \sum_{j=2}^{l}\operatorname{atanh}(u\,x_j)\Big)\;.$$

Notice that by equation (1.7)

$$G_K(X_1,\dots,X_{i-1},X_{i+1},\dots,X_K) \stackrel{\mathrm{d}}{=} X \tag{1.14}$$

Now set $f(x,y) := u\,y/(1+u\,xy)$ and verify that for $l\geq 2,\ i=1,\dots,l$

$$\frac{\partial F_l}{\partial x_i}(x_1,\dots,x_l) = -\frac{1}{l-1}\sum_{\substack{j=1\\ j\neq i}}^{l} f(x_i,x_j) + f\big(x_i\,,\,G_l(x_1,\dots,\widehat{x_i},\dots,x_l)\big)\,;$$

to do this computation one need to observe that, using the logarithmic expression of atanh and the exponential expression of tanh,

$$\begin{aligned} G_l(x_2,\dots,x_l) &= \tanh\log\left(e^B+\prod_{j=2}^{l}\big(\frac{1+u\,x_j}{1-u\,x_j}\big)^{1/2}\right)\\ &= \frac{e^B\prod_{j=2}^{l}(1+u\,x_j)-e^{-B}\prod_{j=2}^{l}(1-u\,x_j)}{e^B\prod_{j=2}^{l}(1+u\,x_j)+e^{-B}\prod_{j=2}^{l}(1-u\,x_j)}\end{aligned}$$

and in consequence in a few steps one finds

$$f\big(x_1,\,G_l(x_2,\dots,x_l)\big) \;=\; u\;\frac{e^B\prod_{j=2}^{l}(1+u\,x_j)-e^{-B}\prod_{j=2}^{l}(1-u\,x_j)}{e^B\prod_{j=1}^{l}(1+u\,x_j)+e^{-B}\prod_{j=1}^{l}(1-u\,x_j)}\,.$$

Therefore by the found expression of the first derivatives $\frac{\partial F_l}{\partial x_i}(x_1,\dots,x_l)$, it'easy to bound them:

$$\sup_{[0,1]^l}\left|\frac{\partial F_l}{\partial x_i}\right| \;\leq\; 2\,\sup_{[0,1]^2}|f| \;=\; 2\,u\,. \tag{1.15}$$

Furthermore for $j=1,\dots,l,\ j\neq i$ compute

$$\begin{aligned}\frac{\partial^2 F_l}{\partial x_j\,\partial x_i}(x_1,\dots,x_l) &= -\,\frac{1}{l-1}\,\frac{\partial f}{\partial e_2}(x_i,x_j)\\ &\quad+\frac{\partial f}{\partial e_2}\,\big(x_i\,,G_l(x_1,\dots,\widehat{x_i},\dots,x_l)\big)\,\frac{\partial G_l}{\partial x_j}(x_1,\dots,\widehat{x_i},\dots,x_l)\end{aligned}$$

Therefore it's simple to bound it:

$$\sup_{[0,1]^l}\left|\frac{\partial^2 F_l}{\partial x_j\,\partial x_i}\right| \;\leq\; \sup_{[0,1]^2}\left|\frac{\partial f}{\partial y}\right|\left(\frac{1}{l-1}+\sup_{[0,1]^{l-1}}\left|\frac{\partial G_l}{\partial x_l}\right|\right) \;\leq\; u\,\big(1+\frac{1}{1-u^2}\big). \tag{1.16}$$

One can also check that for $l = 1$ and $i = 1, 2$

$$\frac{\partial F_1}{\partial x_i}(x_1, x_2) \;=\; -f(x_1, x_2) + f(x_1, G_1)\,,$$

and so the same bounds as before are valid.
Now let $\theta > 0$ (to be chosen later on) and split the variation of φ in two parts, depending on whether L is small or large:

$$I := \mathbb{E}\big[\big(F_L(Y_1,\dots,Y_L) - F_L(X_1,\dots,X_L)\big)\,\mathbb{1}_{\{L\geq\theta\}}\big]\;,$$

$$II := \mathbb{E}\big[\big(F_L(Y_1,\dots,Y_L) - F_L(X_1,\dots,X_L)\big)\,\mathbb{1}_{\{L<\theta\}}\big]\;.$$

Start to study I using the multivariate mean value theorem and the bound (1.15) for the first derivatives of F_l :

$$\begin{aligned} |\,I| &\leq\; \mathbb{E}\big[\,|F_L(Y_1,\dots,Y_L) - F_L(X_1,\dots,X_L)|\,\mathbb{1}_{\{L\geq\theta\}}\big] \\ &\leq\; \mathbb{E}\Bigg[\sum_{i=1}^{L}\,\sup_{[0,1]^L}\Big|\frac{\partial F_L}{\partial e_i}\Big|\;|Y_i - X_i|\;\mathbb{1}_{\{L\geq\theta\}}\Bigg] \\ &\leq\; 2\,u\;\mathbb{E}[\,L\,\mathbb{1}_{\{L\geq\theta\}}\,]\;\mathbb{E}[\,|Y-X|\,]\;, \end{aligned}$$

where the last step is possible also because each (X_i, Y_i) is independent of L and distributed as (X, Y).
Observe that for any integer r.v. $T \geq 1$ one can write $\sum_{k=1}^{\infty}\mathbb{1}_{\{T\geq k\}} = \sum_{k=1}^{T} 1 = T$, so that $\mathbb{E}[T] = \sum_{k=1}^{\infty}\mathbb{P}(T\geq k)$. Apply this fact to the r.v. $L\,\mathbb{1}_{\{L\geq\theta\}}$ and use the hypothesis that L has ϵ-strongly finite mean:

$$\begin{aligned} &\mathbb{E}[\,L\,\mathbb{1}_{\{L\geq\theta\}}\,] = \sum_{l=1}^{\infty}\mathbb{P}(L\,\mathbb{1}_{\{L\geq\theta\}}\geq l) = \sum_{l=1}^{\theta}\mathbb{P}(L\,\mathbb{1}_{\{L\geq\theta\}}\geq l) + \sum_{l=\theta+1}^{\infty}\mathbb{P}(L\,\mathbb{1}_{\{L\geq\theta\}}\geq l) \\ &= \theta\,\mathbb{P}(L\geq\theta) + \sum_{l=\theta+1}^{\infty}\mathbb{P}(L\geq l) \;\leq\; \theta\,\frac{C}{\theta^{\,1+\epsilon}} + \sum_{l=\theta+1}^{\infty}\frac{C}{l^{\,1+\epsilon}} \;\leq\; \frac{C}{\theta^{\,\epsilon}}\;, \end{aligned}$$

where C is a real constant depending only on the distribution P.
Therefore, substituting this bound in the previous expression,

$$|I| \leq C\,u\,\theta^{-\epsilon}\;\mathbb{E}[\,|Y-X|\,] \tag{1.17}$$

Now study II using a more refined estimate (proved in the remark after this proof):

$$|II| \leq \left|\mathbb{E}\left[\sum_{i=1}^{L}(Y_i - X_i)\int_0^1 \frac{\partial F_L}{\partial e_i}(X_1,\dots,tY_i+(1-t)X_i,\dots,X_L))\,dt\;\mathbb{1}_{\{L<\theta\}}\right]\right|$$
$$+\,\mathbb{E}\left[\sum_{\substack{i,j=1\\ i\neq j}}^{L}|Y_i - X_i|\,|Y_j - X_j|\,\sup_{[0,1]^L}\left|\frac{\partial^2 F_L}{\partial x_i\,\partial x_j}\right|\,\mathbb{1}_{\{L<\theta\}}\right],$$

denote II_1 the first addend and II_2 the second one.

The term II_2 can be studied using the bound (1.16) for the mixed second derivatives of F_L and the fact that for $i \neq j$ (X_i, Y_i), (X_j, Y_j) are independent of L, independent of one another and distributed as (X, Y) :

$$II_2 \leq u\left(1+\frac{1}{1-u^2}\right)\mathbb{E}[\,L(L-1)\,\mathbb{1}_{\{L<\theta\}}\,]\;\mathbb{E}[\,|Y-X|\,]^2$$

Observe that for any integer r.v. $T \geq 0$ one can write $\sum_{k=1}^{\infty}(2k-1)\,\mathbb{1}_{\{T\geq k\}} = \sum_{k=1}^{T}(2k-1) = T^2$ (by the formula for the sum of the first odd numbers), hence $\mathbb{E}[\,T^2] = \sum_{k=1}^{\infty}(2k-1)\,\mathbb{P}(T\geq k)$. Apply this fact to the r.v. $L^2\,\mathbb{1}_{\{L<\theta\}}$ and use the hypothesis that L has ϵ-strongly finite mean, $\epsilon<1$:

$$\mathbb{E}[L^2\,\mathbb{1}_{\{L<\theta\}}] = \sum_{l=1}^{\infty}(2\,l-1)\,\mathbb{P}(L\,\mathbb{1}_{\{L>\theta\}}\geq l) = \sum_{l=1}^{\theta}(2\,l-1)\,\mathbb{P}(L\,\mathbb{1}_{\{L>\theta\}}\geq l)$$
$$\leq \sum_{l=1}^{\theta}2\,l\,\mathbb{P}(L\geq l) \leq \sum_{l=1}^{\theta}2\,l\,\frac{C}{l^{1+\epsilon}} = C\,\theta^{1-\epsilon}\,.$$

Therefore, substituting this bound in the previous expression,

$$II_2 \leq C\,u\left(1+\frac{1}{1-u^2}\right)\theta^{1-\epsilon}\,\mathbb{E}[\,|Y-X|\,]^2 \tag{1.18}$$

Now to study II_1 split it again in two parts, the first one which involves L large and the other one which involves every value of L:

$$II_1 \leq \left|\mathbb{E}\left[\sum_{i=1}^{L}(Y_i - X_i)\int_0^1 \frac{\partial F_L}{\partial e_i}(X_1,\dots,tY_i+(1-t)X_i,\dots,X_L))\,dt\;\mathbb{1}_{\{L\geq\theta\}}\right]\right|$$
$$+\left|\mathbb{E}\left[\sum_{i=1}^{L}(Y_i - X_i)\int_0^1 \frac{\partial F_L}{\partial e_i}(X_1,\dots,tY_i+(1-t)X_i,\dots,X_L))\,dt\right]\right|,$$

call them respectively II_1' and II_1''.
Clearly II_1' can be bounded in the same way used for I, so

$$|II_1'| \le C\, u\, \theta^{-\epsilon}\, \mathbb{E}[\,|Y - X|\,] \tag{1.19}$$

On the other hand $II_1'' = 0$. To prove this claim observe that if ρ is the size-biased distribution of P, i.e. $l\, P_l = \overline{P}\, \rho_{l-1}$, then using also independence:

$$\begin{aligned}
&\mathbb{E}\big[\, L\; f\big(t\, Y_1 + (1-t)\, X_1\, , G_L(X_2, \dots, X_L)\big)\big] \\
&= \sum_{l=0}^{\infty} l\; \mathbb{E}\big[\, f\big(t\, Y_1 + (1-t)\, X_1\, , G_l(X_2, \dots, X_l)\big)\big]\; P_l \\
&= \sum_{l=1}^{\infty} \overline{P}\; \mathbb{E}\big[\, f\big(t\, Y_1 + (1-t)\, X_1\, , G_l(X_2, \dots, X_l)\big)\big]\; \rho_{l-1} \\
&= \overline{P}\; \mathbb{E}\big[\, f\big(t\, Y_1 + (1-t)\, X_1\, , G_{K+1}(X_2, \dots, X_{K+1})\big)\big] \\
&= \mathbb{E}\big[\, L\; f\big(t\, Y_1 + (1-t)\, X_1\, , G_{K+1}(X_2, \dots, X_{K+1})\big)\big]
\end{aligned}$$

on the other hand observe that if $\rho = P$ it is trivially true that

$$\begin{aligned}
&\mathbb{E}\big[\, L\; f\big(t\, Y_1 + (1-t)\, X_1\, , G_L(X_2, \dots, X_L)\big)\big] \\
&= \mathbb{E}\big[\, L\; f\big(t\, Y_1 + (1-t)\, X_1\, , G_K(X_2, \dots, X_K)\big)\big]
\end{aligned}$$

Now by the distributional equation (1.14) and by independence of $(X_i)_{i\ge 2}$ from X_1, both the previous equalities become

$$\mathbb{E}\big[L\; f\big(t\, Y_1 + (1-t)\, X_1\, , G_L(X_2, \dots, X_L)\big)\big] \;=\; \mathbb{E}\big[L\; f\big(t\, Y_1 + (1-t)\, X_1\, , X_2\big)\big] \tag{1.20}$$

Reminding the formulas for the first derivatives of F_l, it follows that

$$\mathbb{E}\big[L\, \frac{\partial F_L}{\partial e_1}(t\, Y_1 + (1-t)\, X_1,\; X_2, \dots, X_L)\big] = 0\;.$$

The same can be similarly proven for if one multiply by $(Y_1 - X_1)$ inside the expectation on both sides.
Therefore, using invariance of F_l under permutations, $(X_i, Y_i)_{i\ge 1}$ i.i.d. independent of L, and applying Fubini theorem, one finds

$$II_1'' = \mathbb{E}\Big[L\, (Y_1 - X_1) \int_0^1 \frac{\partial F_L}{\partial e_1}\big(t\, Y_1 + (1-t)\, X_1\, , X_2, \dots, X_L\big)\, dt\Big] = 0\;.$$

To conclude remember relations (1.13), use the bounds (1.17), (1.18), (1.19) and the fact that $II_1'' = 0$ and remember how X, Y were chosen:

$$
\begin{aligned}
&|\,\varphi_{h^*}(\beta,B) - \varphi_{(h^*)'}(\beta,B)| \leq |I| + |II_1'| + |II_1''| + |II_2| \\
&\leq C\,u\,\theta^{-\epsilon}\,\mathbb{E}[\,|Y-X|\,] \,+\, C\,u\,\big(1+\frac{1}{1-u^2}\big)\,\theta^{1-\epsilon}\,\mathbb{E}[\,|Y-X|\,]^2 \\
&\leq C\,u_{\max}\,\big[\theta^{-\epsilon}\,\gamma\,d_{\mathrm{MK}}(\tanh h^*,\tanh(h^*)') \,+\, \theta^{1-\epsilon}\,\gamma^2\,d_{\mathrm{MK}}(\tanh h^*,\tanh(h^*)')^2\,\big]\,,
\end{aligned}
$$

then choose $\theta = d_{\mathrm{MK}}(\tanh h^*,\tanh(h^*)')^{-1}$ and let $\gamma\to 1$:

$$|\,\varphi_{h^*}(\beta,B) - \varphi_{(h^*)'}(\beta,B)| \leq C\,u_{\max}\,d_{\mathrm{MK}}(\tanh h^*,\tanh(h^*)')^{1+\epsilon}\,. \quad \square$$

Remark 5. Here we'll prove the two different multivariate mean value theorems that we used in the previous proof to study respectively the large L term and the small L term.

1) If it suffices to stop at the first order, then just observe that by fundamental theorem of calculus

$$
\begin{aligned}
&F(y_1,\dots,y_l) - F(x_1,\dots,x_l) = \int_0^1 \frac{d}{dt}\,F\big(t\,y_1+(1-t)x_1,\dots,t\,y_l+(1-t)x_l\big)\,dt \\
&= \sum_{i=1}^{l}(y_i-x_i)\int_0^1 \frac{\partial F}{\partial x_i}\,\big(t\,y_1+(1-t)x_1,\dots,t\,y_l+(1-t)x_l\big)\,dt
\end{aligned}
$$

so that

$$|F(y_1,\dots,y_l) - F(x_1,\dots,x_l)| \leq \sum_{i=1}^{l}|y_i-x_i|\ \sup\left|\frac{\partial F}{\partial x_i}\right|\,.$$

2) For shortness here we denote $\sigma_s(x,y) = sy+(1-s)x$.
If it is convenient to reach the second order, one can use the following formula:

$$
\begin{aligned}
&|F(y_1,\dots,y_l) - F(x_1,\dots,x_l)| \leq \\
&\left|\sum_{i=1}^{l}(y_i-x_i)\int_0^1\frac{\partial F}{\partial e_i}\,\big(x_1,\dots,\sigma_t(x_i,y_i),\dots,x_l\big)\,dt\right| + \sum_{i\neq j}|y_i-x_i|\,|y_j-x_j|\ \sup\left|\frac{\partial^2 F}{\partial x_i\,\partial x_j}\right|.
\end{aligned}
$$

To prove it use twice the fundamental theorem of calculus computing

$$\begin{aligned}
&\sum_{i\neq j}(y_i - x_i)(y_j - x_j)\int_0^1\int_0^t \frac{\partial^2 F}{\partial e_i\,\partial e_j}\big(\sigma_s(x_1,y_1),\dots,\sigma_t(x_i,y_i),\dots,\sigma_s(x_l,y_l)\big)ds\,dt\\
&= \sum_{i=1}^{l}(y_i - x_i)\int_0^1\int_0^t \frac{d}{ds}\frac{\partial F}{\partial e_i}\big(\sigma_s(x_1,y_1),\dots,\sigma_t(x_i,y_i),\dots,\sigma_s(x_l,y_l)\big)\,ds\,dt\\
&= \sum_{i=1}^{l}(y_i - x_i)\int_0^1 \bigg[\frac{\partial F}{\partial e_i}\big(\sigma_t(x_1,y_1),\dots,\sigma_t(x_i,y_i),\dots,\sigma_t(x_l,y_l)\big)\\
&\qquad\qquad - \frac{\partial F}{\partial e_i}\big(x_1,\dots,\sigma_t(x_i,y_i),\dots,x_l\big)\bigg]\,dt\\
&= \int_0^1 \frac{d}{dt}\,F\big(\sigma_t(x_1,y_1),\dots,\sigma_t(x_i,y_i),\dots,\sigma_t(x_l,y_l)\big)\,dt\\
&\qquad\qquad - \sum_{i=1}^{l}(y_i - x_i)\int_0^1 \frac{\partial F}{\partial e_i}\big(x_1,\dots,\sigma_t(x_i,y_i),\dots,x_l\big)\,dt\\
&= F(y_1,\dots,y_l) - F(x_1,\dots,x_l) - \sum_{i=1}^{l}(y_i - x_i)\int_0^1 \frac{\partial F}{\partial e_i}\big(x_1,\dots,\sigma_t(x_i,y_i),\dots,x_l\big)dt\,.
\end{aligned}$$

Now let $G_N = (V_N, E_N)$ be a random graph with $|V_N| = N$ and for any $ij \in E_N$ denote $B_N(ij,t) = B_N(i,t) + B_N(j,t)$, the sub-graph of G_N induced by the vertices with distance $\leq t$ from i or j.
Let $\overline{\mathcal{T}}(\rho,\infty)$ be the random tree obtained linking by an extra edge the roots $\varnothing'$, $\varnothing''$ of two independent random trees $T'(\rho,\infty)$, $T''(\rho,\infty)$. Further denote $\overline{\mathcal{T}}(\rho,t)$ the subgraph of $\overline{\mathcal{T}}(\rho,\infty)$ induced by the first t generations of $T'(\rho,\infty)$ and $T''(\rho,\infty)$.

Lemma 19. *Suppose the sequence of graphs $(G_N)_{N\in\mathbb{N}}$ locally converges to the tree $\mathcal{T}(P,\rho,\infty)$.*
Suppose P has finite mean and ρ is the size-biased law of P.
Then for any $t \in \mathbb{N}$ and for any bounded function F of graphs, which is invariant by isomorphism,

$$\frac{1}{N}\sum_{ij\in E_N} F\big(B_N(ij,t)\big) \xrightarrow[N\to\infty]{} \frac{\overline{P}}{2}\,\mathbb{E}[F\big(\overline{\mathcal{T}}(\rho,t)\big)] \quad a.s.$$

Proof. Note that $E_N = \bigcup_{i\in V_N}\{ij \mid j \,\text{is a neighbour of}\, i \,\text{in}\, G_N\}$, where in the right-hand side each edge is counted twice.
Therefore one can use the hypothesis of local convergence:

$$\frac{1}{N}\sum_{ij\in E_N} F\big(B_N(ij,t)\big) = \frac{1}{2N}\sum_{i\in V_N}\sum_{j\in\partial_N i} F\big(B_N(ij,t)\big) =$$
$$\frac{1}{2N}\sum_{i\in V_N}\sum_{j\in\partial_N i} F\big(B_N(i,t)+B_N(j,t)\big) \xrightarrow[N\to\infty]{a.s.} \frac{1}{2}\,\mathbb{E}\big[\sum_{j\in\partial_{\mathcal{T}}\varnothing} F\big(\mathcal{T}(P,\rho,t)+\mathcal{T}_j(\rho,t)\big)\big]$$

where $\mathcal{T}_j(\rho,t)$ denotes the sub-tree of $\mathcal{T}(P,\rho,t{+}1)$ induced by the son j of $\varnothing$ and its descendants.
Now consider the tree $\mathcal{T}(P,\rho,\infty)$, let $L \overset{\mathrm{d}}{\sim} P$ be the offspring size of the root $\varnothing$ and let $K_j \overset{\mathrm{d}}{\sim} \rho$ be the offspring size of the son j of $\varnothing$.
Then consider separately the trees $\mathcal{T}'(\rho,\infty)$, $\mathcal{T}''(\rho,\infty)$ and let K', $K'' \overset{\mathrm{d}}{\sim} \rho$ be the offspring sizes of the roots $\varnothing'$, $\varnothing''$ respectively.
Note that, thank to the independence of all offspring sizes, *the conditional distribution of* $\mathcal{T}(P,\rho,t) + \mathcal{T}_j(\rho,t)$ *knowing* $(L = l\,)$ equals up to isomorphisms *the conditional distribution of* $\overline{\mathcal{T}}(\rho,t) = \mathcal{T}'(\rho,t) + \mathcal{T}''(\rho,t)$ *knowing* $(K' = l-1)$.

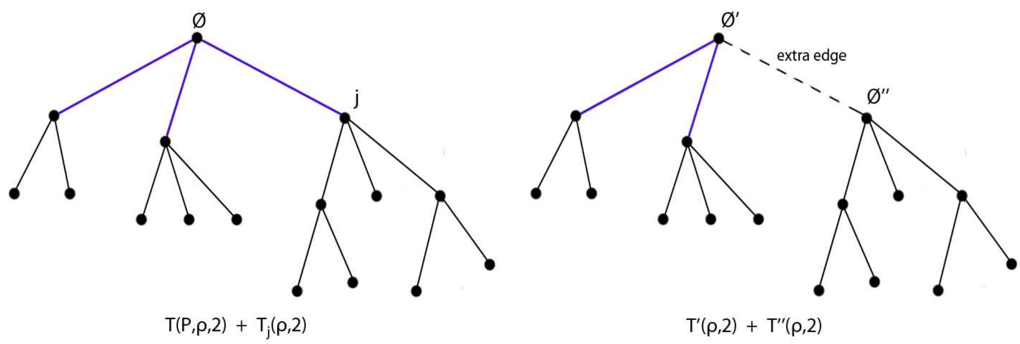


Figure 1.3: *Under the knowledge of* $(K' = L-1)$, $\mathcal{T}(P,\rho,t) + \mathcal{T}_j(\rho,t)$ *and* $\mathcal{T}'(\rho,t)+\mathcal{T}''(\rho,t)$ *have the same distribution, up to an isomorphism under which* $\varnothing$, j *correspond respectively to* $\varnothing'$, $\varnothing''$. *Indeed all the corresponding nodes, except* $\varnothing$ *and* $\varnothing'$, *share the same offspring distribution* ρ *and independence does the rest.*

Therefore, using the total probability formula and the fact that $l\,P_l = \overline{P}\,\rho_{l-1}$, one finds

$$\mathbb{E}\Big[\sum_{j\in\partial_{\mathcal{T}}\varnothing} F\big(\mathcal{T}(P,\rho,t)+\mathcal{T}_j(\rho,t)\big)\Big] = \sum_{l=0}^{\infty} l\,\mathbb{E}\big[F\big(\mathcal{T}(P,\rho,t)+\mathcal{T}_j(\rho,t)\big)\,\big|\,L=l\,\big]\,P_l$$

$$= \sum_{l=1}^{\infty} \overline{P}\;\mathbb{E}\big[F\big(\mathcal{T}(P,\rho,t)+\mathcal{T}_j(\rho,t)\big)\,\big|\,L=l\,\big]\,\rho_{l-1}$$

$$= \overline{P}\,\sum_{l=1}^{\infty}\mathbb{E}\big[F\big(\overline{\mathcal{T}}(\rho,t)\big)\,\big|\,K'=l-1\,\big]\,\rho_{l-1}$$

$$= \overline{P}\,\mathbb{E}\big[F\big(\overline{\mathcal{T}}(\rho,t)\big)\big]$$

□

We are now ready to prove the most important result of this chapter.

Theorem 20. *Consider the random graphs sequence $(G_N)_{N\in\mathbb{N}}$ and suppose*

i. $(G_N)_{N\in\mathbb{N}}$ *locally converges to the random tree* $\mathcal{T}(P,\rho,\infty)$,

ii. the asymptotic degree distribution P has strongly finite mean,

iii. ρ is the size-biased distribution of P.

Then, for all $0\leq\beta<\infty$ and $B\in\mathbb{R}$, there exists

$$\lim_{N\to\infty} p_N(\beta,B) = p(\beta,B) \quad a.s.$$

Furthermore its value for $B>0$ is

$$\boxed{\begin{aligned} p(\beta,B) = \frac{\overline{P}}{2}\,\log\cosh\beta - \frac{\overline{P}}{2}\,\mathbb{E}\big[\log\big(1+\tanh\beta\,\tanh h_1^*\,\tanh h_2^*\big)\big] \;+ \\ +\,\mathbb{E}\Big[\log\Big(e^B\prod_{i=1}^{L}(1+\tanh\beta\,\tanh h_i^*) + e^{-B}\prod_{i=1}^{L}(1-\tanh\beta\,\tanh h_i^*)\Big)\Big]\end{aligned}}$$

where

- $(h_i^*)_{i\geq1}$ *are i.i.d. r.v.'s with the same distribution of the positive solution* $h^*=h^*(\beta,B)>0$ *of the fixed point distributional equation (1.7)*

- *L is a random variable with distribution P, independent of $(h_i^*)_{i\geq 1}$.*

Finally for $B \leq 0$ the value of the limit is

$$p(\beta, B) = p(\beta, -B)\,, \qquad p(\beta, 0) = \lim_{B\to 0} p(\beta, B)\,.$$

Proof. Assume the function p is defined by the formula (**??**) and prove that $p_N = \frac{1}{N}\log Z_N \longrightarrow p$ as $N \to \infty$.
If $\beta = 0$, the system is non-interacting hence it's easy to check that

$$p_N(0, B) \;=\; \log 2 + \log\cosh B \;=\; p(0, B) \quad \forall\, N\in\mathbb{N}\,.$$

Henceforth assume $\beta > 0$ and $B > 0$.
Note that $\partial p_N/\partial\beta$, i.e. the internal energy per particle, is a mean of microscopic quantities. Indeed

$$\frac{\partial p_N}{\partial\beta} = \frac{1}{N}\sum_{ij\in E_N}\langle\sigma_i\sigma_j\rangle_{G_N}\,.$$

By the fundamental theorem of calculus, for any $N \in \mathbb{N}$

$$p_N(\beta, B) = p_N(0, B) + \int_0^\beta \frac{\partial p_N}{\partial\beta'}\,(\beta', B)\,d\beta' = p(0, B) + \int_0^\beta \frac{1}{N}\sum_{ij\in V_N}\langle\sigma_i\sigma_j\rangle_{G_N}\,d\beta'$$

$$p(\beta, B) \;=\; p(0, B) \;+\; \int_0^\beta \frac{\partial p}{\partial\beta'}\,(\beta', B)\,d\beta'\,.$$

The integrand in the first expression is a.s. bounded uniformly in N :

$$\left|\frac{1}{N}\sum_{ij\in E_N}\langle\sigma_i\sigma_j\rangle_{G_N}\right| \;\leq\; \frac{|E_N|}{N} \;\leq\; C \;\; a.s.$$

since by remark 3 the sequence $|E_N|/N$ converges a.s. Therefore it suffices to prove that

$$\frac{1}{N}\sum_{ij\in E_N}\langle\sigma_i\sigma_j\rangle_{G_N} \xrightarrow[N\to\infty]{} \frac{\partial p}{\partial\beta} \quad a.s. \tag{1.21}$$

to conclude that $p_N(\beta, B) \xrightarrow{a.s.} p(\beta, B)$ as $N \to \infty$ by dominated convergence.
We'll break the proof of 1.21 in two parts. Consider the Ising model on a

single edge with inverse temperature β and random magnetic fields h_1^*, h_2^* i.i.d. copies of $h^*(\beta, B)$

$$\nu(\sigma_1,\sigma_2) = \frac{1}{Z(\beta,\, h_1^*,\, h_2^*)}\ \exp(\beta\,\sigma_1\sigma_2 + h_1^*\,\sigma_1 + h_2^*\,\sigma_2) \qquad \forall\,\sigma_1,\sigma_2 = \pm 1\,.$$

First we'll prove that $1/N\ \sum_{ij\in E_N}\langle\sigma_i\sigma_j\rangle_{G_N} \xrightarrow{a.s.} \overline{P}/2\ \mathbb{E}[\langle\sigma_1\sigma_2\rangle_\nu]$ as $N\to\infty$. Second we'll show that $\partial p/\partial\beta = \overline{P}/2\ \mathbb{E}[\langle\sigma_1\sigma_2\rangle_\nu]\,.$

I) Let $ij\in E_N$ and $t\geq 1$. The first step is to localize the quantities to work with. On one side by the G.K.S. inequality

$$\langle\sigma_i\sigma_j\rangle_{G_N}\ \geq\ \langle\sigma_i\sigma_j\rangle_{B_N(ij,\,t)} \tag{1.22}$$

on the other side by the G.K.S. inequality and the proposition 7

$$\langle\sigma_i\sigma_j\rangle_{G_N}\ \leq\ \langle\sigma_i\sigma_j\rangle^+_{B_N(ij,\,t)}\ \underbrace{\langle\sigma_i\sigma_j\rangle^\sim_{G_N-B_N(ij,\,t)}}_{\leq 1}\ \leq\ \langle\sigma_i\sigma_j\rangle^+_{B_N(ij,\,t)} \tag{1.23}$$

Now as $(G_N)_{N\in\mathbb{N}}$ locally converges to $\mathcal{T}(P,\rho,\infty)$, by lemma 19

$$\frac{1}{N}\sum_{ij\in E_N}\langle\sigma_i\sigma_j\rangle_{B_N(ij,\,t)}\ \xrightarrow[N\to\infty]{a.s.}\ \frac{\overline{P}}{2}\ \mathbb{E}\big[\langle\sigma_{\varnothing'}\,\sigma_{\varnothing''}\rangle_{\overline{\mathcal{T}}(\rho,t)}\big] \tag{1.24}$$

By lemma 8 applied to $T=\overline{\mathcal{T}}(\rho,t)$ and U the subgraph induced by the single edge $\varnothing'\varnothing''$, one obtains

$$\langle\sigma_{\varnothing'}\,\sigma_{\varnothing''}\rangle_{\overline{\mathcal{T}}(\rho,t)}\ =\ \langle\sigma_{\varnothing'}\,\sigma_{\varnothing''}\rangle_{\varnothing'\varnothing''}\ \big|\, B_{\varnothing'}\to {h^{(t)}}',\ B_{\varnothing''}\to {h^{(t)}}''$$

where ${h^{(t)}}' := \operatorname{atanh}\langle\sigma_{\varnothing'}\rangle_{\mathcal{T}'(\rho,t)}$ and ${h^{(t)}}'' := \operatorname{atanh}\langle\sigma_{\varnothing''}\rangle_{\mathcal{T}''(\rho,t)}$.
By the proof of proposition 14 and thank to the independence of ${h^{(t)}}'$, ${h^{(t)}}''$

$$({h^{(t)}}',{h^{(t)}}'')\ \xrightarrow[t\to\infty]{d}\ ({h^*}',{h^*}'')$$

where ${h^*}'$, ${h^*}''$ are independent copies of the positive solution of fixed point distributional equation (1.7). Therefore

$$\langle\sigma_{\varnothing'}\,\sigma_{\varnothing''}\rangle_{\overline{\mathcal{T}}(\rho,t)}\ \xrightarrow[t\to\infty]{d}\ \langle\sigma_1\sigma_2\rangle_\nu \tag{1.25}$$

Starting from the lower bound (1.22) and using (1.24) and (1.25), one obtains

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{ij\in E_N} \langle\sigma_i\sigma_j\rangle_{G_N} \geq \frac{\overline{P}}{2}\, \mathbb{E}[\langle\sigma_1\sigma_2\rangle_\nu] \quad a.s.$$

Starting from the upper bound (1.23) and reasoning in a similar way, one finds

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{ij\in E_N} \langle\sigma_i\sigma_j\rangle_{G_N} \leq \frac{\overline{P}}{2}\, \mathbb{E}[\langle\sigma_1\sigma_2\rangle_\nu] \quad a.s.$$

So a.s. there exists $\lim_{N\to\infty} \frac{1}{N} \sum_{ij\in E_N} \langle\sigma_i\sigma_j\rangle_{G_N} = \frac{\overline{P}}{2}\, \mathbb{E}[\langle\sigma_1\sigma_2\rangle_\nu]$.

II) Now prove that $\frac{\partial p}{\partial \beta}$ exists and compute it.
Remind the definition 1.12 of the operator $h \mapsto \varphi_h(\beta, B)$, observing that

$$p(\beta, B) = \varphi_{h^*(\beta,B)}(\beta, B) \,,$$

where as usual $h^*(\beta, B)$ is the positive solution of the fixed point distributional equation (1.7).
Since the distribution P has ϵ-strongly finite mean (w.l.o.g. assume $0<\epsilon<1$), by corollary 17 and proposition 18

$$\begin{aligned} |\varphi_{h^*(\beta',B)}(\beta', B) - \varphi_{h^*(\beta,B)}(\beta', B)| &\leq \lambda\, d_{\mathrm{MK}}\big(\tanh h^*(\beta', B)\, , \tanh h^*(\beta, B)\big)^{1+\epsilon} \\ &\leq \lambda\, |\beta' - \beta|^{1+\epsilon} \end{aligned}$$

where $\lambda = \lambda(\beta_{\min}, \beta_{\max}, B_{\min}) < \infty$ is a constant valid for any $0 < \beta_{\min} \leq \beta, \beta' \leq \beta_{\max} < \infty$ and $B \geq B_{\min} > 0$.
This fact allows to derive p with no care about the dependence of h^* on β. Precisely

$$\begin{aligned} &\Big|\frac{p(\beta', B) - p(\beta, B)}{\beta' - \beta} - \frac{\varphi_{h^*(\beta,B)}(\beta', B) - \varphi_{h^*(\beta,B)}(\beta, B)}{\beta' - \beta}\Big| \\ &\leq \underbrace{\frac{|p(\beta', B) - \varphi_{h^*(\beta,B)}(\beta', B)|}{|\beta' - \beta|}}_{\leq\,\lambda\,|\beta'-\beta|^\epsilon} + \underbrace{\frac{|p(\beta, B) - \varphi_{h^*(\beta,B)}(\beta, B)|}{|\beta' - \beta|}}_{=\,0} \\ &\leq |\beta' - \beta|^\epsilon \xrightarrow[\beta'\to\beta]{} 0 \end{aligned}$$

so that there exists

$$\frac{\partial p}{\partial \beta}(\beta, B) = \frac{\partial \varphi_h}{\partial \beta}(\beta, B)\Big|_{h=h^*(\beta,B)} \tag{1.26}$$

Thus it remains to compute $\frac{\partial \varphi_h}{\partial \beta}$. Let $(X_i)_{i\geq 1}$ i.i.d. copies of $\tanh h^*(\beta, B)$, let $L \overset{\mathrm{d}}{\sim} P$ and $K \overset{\mathrm{d}}{\sim} \rho$ independent of $(X_i)_{i\geq 1}$.
Similarly to the proof of proposition 18, set $u := \tanh \beta$, $\bar{f}(x, y) := \frac{y}{1+u\,xy}$,

$$G_l(x_2, \dots, x_l) := \tanh(B + \sum_{j=2}^{l} \operatorname{atanh}(u\, x_j)).$$

From equation (1.12)

$$\frac{\partial \varphi_h}{\partial \beta}\Big|_{h=h^*} = \frac{\overline{P}}{2}\, u - \frac{\overline{P}}{2}(1-u^2)\, \mathbb{E}\Big[\frac{X_1\, X_2}{1 + u\, X_1\, X_2}\Big] +$$
$$+ (1-u^2)\, \mathbb{E}\Big[\sum_{i=1}^{L} X_i\, \frac{e^B \prod_{j\neq i}(1+u\, X_j) - e^{-B}\prod_{j\neq i}(1-u\,X_j)}{e^B\prod_{j=1}^{L}(1+u\,X_j) + e^{-B}\prod_{j=1}^{L}(1-u\,X_j)}\Big]$$

One can rewrite the last fraction as in the proof of proposition 18.
After that, together with independence, use $l\,P_l = \overline{P}\,\rho_{l-1}$ and the fact that $X_2 \overset{\mathrm{d}}{=} G_{K+1}(X_2, \dots, X_{K+1})$ as $h^*(\beta, B)$ solves equation (1.7) :

$$\mathbb{E}\Big[\sum_{i=1}^{L} X_i\, \frac{e^B \prod_{j\neq i}(1+u\, X_j) - e^{-B}\prod_{j\neq i}(1-u\,X_j)}{e^B\prod_{j=1}^{L}(1+u\,X_j) + e^{-B}\prod_{j=1}^{L}(1-u\,X_j)}\Big]$$
$$= \mathbb{E}\big[L\, X_1\, \bar{f}\big(X_1, G_L(X_2, \dots, X_L)\big)\big] = \overline{P}\,\mathbb{E}\big[X_1\, \bar{f}\big(X_1, G_{K+1}(X_2, \dots, X_{K+1})\big)\big]$$
$$= \overline{P}\,\mathbb{E}\big[X_1\, \bar{f}(X_1, X_2)\big]$$

Substituting in the previous expression after one finds

$$\frac{\partial \varphi_h}{\partial \beta}\Big|_{h=h^*} = \frac{\overline{P}}{2}\, \mathbb{E}\big[\frac{u + X_1\, X_2}{1 + u\, X_1\, X_2}\big] \tag{1.27}$$

And now it's easy to verify

$$\frac{u + X_1\, X_2}{1 + u\, X_1\, X_2} \overset{\mathrm{d}}{=} \frac{\tanh\beta + \tanh h_1^*\, \tanh h_2^*}{1 + \tanh\beta\, \tanh h_1^*\, \tanh h_2^*}$$
$$= \frac{e^{\beta+h_1^*+h_2^*} - e^{-\beta+h_1^*-h_2^*} - e^{-\beta-h_1^*+h_2^*} + e^{\beta-h_1^*-h_2^*}}{e^{\beta+h_1^*+h_2^*} + e^{-\beta+h_1^*-h_2^*} + e^{-\beta-h_1^*+h_2^*} + e^{\beta-h_1^*-h_2^*}} = \langle \sigma_1 \sigma_2 \rangle_\nu$$

Substituting into equality 1.27 and reminding relation 1.25, one finally finds

$$\frac{\partial p}{\partial \beta} = \frac{\overline{P}}{2}\,\mathbb{E}[\langle \sigma_1\sigma_2\rangle_\nu]\,.$$

Steps **I)** and **II)** show that the relation 1.21 is true, so that the proof is concluded in case $B > 0$.

Note that the partition function Z_N is invariant under the transformation $B \mapsto -B$, so p_N is too. Therefore for $B < 0$

$$p_N(\beta, B) = p_N(\beta, -B) \xrightarrow[N\to\infty]{a.s.} p(\beta, -B)\,.$$

To study the case $B = 0$, observe that p_N is Lipschitz continuous w.r.t. B uniformly in N:

$$|p_N(\beta, B_2) - p_N(\beta, B_1)| \;\leq\; |B_2 - B_1| \sup_{\beta\times[B_1,B_2]} \Big|\frac{\partial p_N}{\partial B}\Big| \;=$$
$$= |B_2 - B_1| \sup_{\beta\times[B_1,B_2]} \Big|\frac{1}{N}\sum_{i\in V_N}\langle\sigma_i\rangle_{G_N}\Big| \;\leq\; |B_2 - B_1|\,.$$

Therefore p is Lipschitz continuous too, hence there exists $\lim_{B\to 0} p(\beta, B)$. Furthermore [1]

$$|p_N(\beta,0) - p(\beta,B)| \leq |p_N(\beta,0) - p_N(\beta,B)| + |p_N(\beta,B) - p(\beta,B)|$$
$$\Rightarrow\; \limsup_{N\to\infty} |p_N(\beta,0) - p_(\beta,B)| \leq |B|$$
$$\Rightarrow\; \limsup_{N\to\infty} p_N(\beta,0) - p(\beta,B) \leq |B|\,,\;\; \liminf_{N\to\infty} p_N(\beta,0) - p(\beta,B) \geq -|B|$$

so that there exists $\lim_{N\to\infty} p_N(\beta,0) = \lim_{B\to 0} p(\beta,B)$. □

Remark 6. The core of this proof is certainly the part **I)**, in which the internal energy per particle $\partial p_N/\partial\beta = \frac{1}{N}\sum_{ij\in E_N}\langle\sigma_i\sigma_j\rangle_{G_N}$ is proved to converge a.s. in the thermodynamic limit to the quenched correlation of a two-vertex Ising model with random magnetic fields.

Differently from the rest of the proof, this part does not require any technicalities. The idea is to localize the correlations $\langle\sigma_i\sigma_j\rangle_{G_N}$ thank to the G.K.S.

[1] $\limsup|\cdot| \geq \{\limsup(\cdot)\,,\, -\limsup(\cdot) = \liminf(-\cdot)\}$

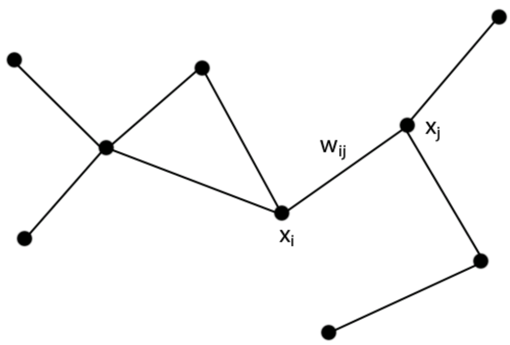
inequality: this enable to use the hypothesis of local convergence to a tree. Afterwards the study made about the root magnetisation of the random trees $\mathcal{T}(P, \rho, t)$ allows to write the limit in term of the fixed point h^* of the distributional equation (1.7).

On the contrary the part **II)** makes use of many technical results, which are basically needed to integrate the limit of $\partial p_N / \partial \beta$ and come back to the limit of p_N .

Chapter 2

Monomer-dimer models

Let $G = (V, E)$ be a finite simple graph. Denote $N = |V|$ the number vertices. Associate a positive weight to every vertex $\{x_i\}_{i\in V} = \underline{x}$ and to every edge $\{w_{ij}\}_{ij\in E} = \underline{w}$. These will be called *monomeric weights* and *dimeric weights* respectively.



Definition 8. A *dimeric configuration* on the graph G is a family of edges $D \subseteq E$ respecting the following bond of monogamy:

$$ij, hk \in D,\ ij \neq hk \ \Rightarrow\ i \neq h,\, i \neq k,\, j \neq h,\, j \neq k\,.$$

In words it means that two different edges belonging to D can't have a vertex in common.

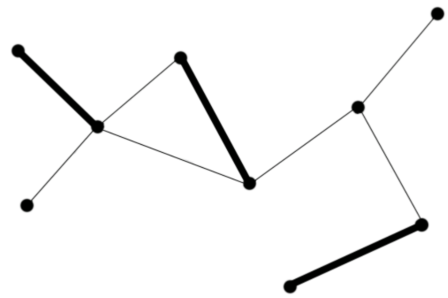


Figure 2.1: *The bold edges form a dimeric configuration on the graph*

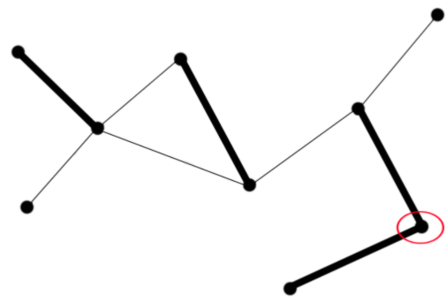


Figure 2.2: *The bold edges* ***do not*** *form a dimeric configuration.*

Given a dimeric configuration D, it's automatically determined the *monomeric configuration* composed by the free vertices:

$$M_D := \{i \in V | \forall j \in V \, ij \notin V\} .$$

We say that the edges in the dimeric configuration D are occupied by a *dimer*, while the vertices in the monomeric configuration M_D are occupied by a *monomer*.

Let $\mathscr{D}$ denote de set of all possible dimeric configurations on the graph G. We'll define a probability measure on $\mathscr{D}$, according to the dimeric and monomeric weights of the graph:

$$\mu(D) := \frac{1}{Z(\underline{w}, \underline{x})} \prod_{ij \in D} w_{ij} \prod_{k \in M_D} x_k \quad \forall D \in \mathscr{D} ,$$

where the normalising factor is

$$Z(\underline{w}, \underline{x}) := \sum_{D \in \mathscr{D}} \prod_{ij \in D} w_{ij} \prod_{k \in M_D} x_k .$$

This is called a *monomer-dimer model* on the graph G. From an intuitive point of view in this model a dimeric configuration D has an high probability to appear on the graph if it assigns a dimer to the edges with an high dimeric weight w_{ij} and a monomer to the vertices with an high monomeric weight x_i .
The expected value with respect to the measure μ will be denoted by $\langle \ \rangle$, that is for any function f of the dimeric configuration put

$$\langle f \rangle := \sum_{D \in \mathscr{D}} f(D)\, \mu(D)\,.$$

The polynomial function $Z(\underline{w}, \underline{x})$ defined above is called the *partition function* of the model. Its natural logarithm $P(\underline{w}, \underline{x}) := \log Z(\underline{w}, \underline{x})$ is called *pressure* or *free energy.*

Remark 7. If we assume that the monomeric and dimeric weights are uniform, i.e. $w_{ij} = w \ \forall\, ij \in E$ and $x_i = x \ \forall\, i \in V$, then the expression of the partition function becomes simpler.
Thank to the bond of monogamy that characterizes any dimeric configuration D, notice that $|M_D| = N - 2|D|$. Then it's easy to check that in case of uniform weights

$$Z(w, x) = \sum_{k=0}^{\lfloor N/2 \rfloor} \Lambda(k)\, w^k\, x^{N-2k}\,,$$

where $\Lambda(k) := \mathit{Card}\{D \in \mathscr{D}\ s.t.\ |D| = k\}$ is the number of possible dimeric configurations containing k dimers.

Remark 8. Continue assuming uniform monomeric and dimeric weights. In this case it's possible to study the partition function $Z(w, x)$ just fixing $w = 1$ and studying the dependence on x. Indeed using the formula of the previous remark, it's easy to verify that

$$Z(w, x) = w^{N/2}\, Z\big(1\,, \frac{x}{\sqrt{w}}\big)\,.$$

The consequent relation for the pressure is

$$P(w,x) = \frac{N}{2}\log w + P\big(1\,,\frac{x}{\sqrt{w}}\big)\,.$$

2.1 Correlation inequalities

An interesting quantity of the model is certainly the probability of having a dimer on a given edge ij. This probability can be expressed as $\langle \mathbb{1}_{ij\in D}\rangle$, where $\mathbb{1}$ denotes the indicator function of the event in subscript.
Another interesting quantity related to the previous one is the total number of dimers one expects to see on the graph, that is $\langle |D|\rangle$.
We'll see these quantities can be studied knowing the pressure of the model and its derivatives.

Proposition 21. *Given an edge $ij\in E$, the probability of having a dimer on ij is*

$$\langle \mathbb{1}_{ij\in D}\rangle = w_{ij}\,\frac{\partial P}{\partial w_{ij}}\,.$$

Whereas if the dimer and monomer weights are uniform, then the expected number of dimers on the graph is

$$\langle |D|\rangle = w\,\frac{\partial P}{\partial w} = -\frac{x}{2}\,\frac{\partial P}{\partial x} + \frac{N}{2}\,.$$

Proof. Observing that $\mathbb{1}_{ij\in D}\prod_{hk\in D} w_{hk}$ can be written as $w_{ij}\,\frac{\partial}{\partial w_{ij}}\big(\prod_{hk\in D} w_{hk}\big)$ one obtains

$$\langle \mathbb{1}_{ij\in D}\rangle = \frac{1}{Z}\sum_{D\in\mathscr{D}} \mathbb{1}_{ij\in D}\prod_{hk\in D} w_{hk}\prod_{l\in M_D} x_l = \frac{1}{Z}\,\frac{\partial Z}{\partial w_{ij}}\,w_{ij} = \frac{\partial P}{\partial w_{ij}}\,w_{ij}\,.$$

Now assume uniform weights $w_{ij}\equiv w$, $x_i\equiv x$. Using the expression of Z found in remark 7, compute

$$\langle |D|\rangle = \frac{1}{Z}\sum_{D\in\mathscr{D}}|D|\,w^{|D|}x^{|M_D|} = \frac{1}{Z}\sum_{k=0}^{\lfloor N/2\rfloor}\Lambda(k)\,k\,w^k\,x^{N-2k} = \begin{cases} \dfrac{1}{Z}\,\dfrac{\partial Z}{\partial w}\,w \\[2ex] -\dfrac{1}{Z}\,\dfrac{\partial Z}{\partial x}\,\dfrac{x}{2} + \dfrac{N}{2} \end{cases}$$

and conclude in both cases since $\partial Z/Z = \partial P$. □

Notice that, since the indicator function is non-negative, the previous proposition implies $\frac{\partial P}{\partial w_{ij}} \geq 0$.

We would like to find some correlation inequalities analogous to the Griffiths-Kelly-Sherman for ferromagnetic spin models.
We have just seen that an inequality like the G.K.S. 1 is easily satisfied by monomer-dimer models. Now we ask about an inequality like the G.K.S. 2. From an intuitive point of view, if we increase the dimeric weight of a given edge ij, we expect that the probability of having a dimer on ij increases as well. On the other side if we increase the dimeric weight of a different edge hk, we are not sure of what will happen.

Proposition 22. *Given two edges ij, $hk \in E$,*

- *if $ij = hk$ then*
$$\frac{\partial}{\partial w_{ij}} \langle \mathbb{1}_{ij\in D} \rangle \geq 0\,;$$
- *if $ij \neq hk$ then*
$$\frac{\partial}{\partial w_{hk}} \langle \mathbb{1}_{ij\in D} \rangle \geq 0 \iff Z^{(1)}(\underline{v},\underline{x})\, Z^{(2)}(\underline{v},\underline{x}) \geq Z^{(3)}(\underline{v},\underline{x})\, Z^{(4)}(\underline{v},\underline{x})\,.$$

Here $\underline{v}$ is the family of dimeric weights defined by $v_{ij} = v_{hk} = 1$ and $v_{lm} = w_{lm}\ \forall\, lm \in E$, $lm \neq ij, hk$.
And $Z^{(1)}$, $Z^{(2)}$, $Z^{(3)}$, $Z^{(4)}$ are respectively the conditional partition function w.r.t. "there's a dimer on ij but not on hk", "there's a dimer on hk but not on ij", "there's a dimer on both ij and hk", "there are no dimers on ij and hk".

Proof. **1)** Assume $ij = hk$. Write the partition function dividing the cases "there's a dimer on ij" and "there's not a dimer on ij":

$$Z(\underline{w},\underline{x}) = \sum_{D\in\mathscr{D}} \prod_{lm\in D} w_{lm} \prod_{p\in M_D} x_p = w_{ij} \sum_{\substack{D\in\mathscr{D}\\ ij\in D}} \prod_{\substack{lm\in D\\ lm\neq ij}} w_{lm} \prod_{p\in M_D} x_p + \sum_{\substack{D\in\mathscr{D}\\ ij\notin D}} \prod_{lm\in D} w_{lm} \prod_{p\in M_D} x_p$$

$$= w_{ij} \sum_{\substack{D\in\mathscr{D}\\ ij\in D}} \prod_{lm\in D} v_{lm} \prod_{p\in M_D} x_p + \sum_{\substack{D\in\mathscr{D}\\ ij\notin D}} \prod_{lm\in D} v_{lm} \prod_{p\in M_D} x_p = w_{ij}\, Z^{(3)}(\underline{v},\underline{x}) + Z^{(4)}(\underline{v},\underline{x})\,.$$

Hence, noticing $\underline{v}$ does not depend on the value of w_{ij} ,

$$\frac{\partial P}{\partial w_{ij}} = \frac{1}{Z}\frac{\partial Z}{\partial w_{ij}} = \frac{Z^{(3)}(\underline{v},\underline{x})}{w_{ij}\,Z^{(3)}(\underline{v},\underline{x}) + Z^{(4)}(\underline{v},\underline{x})} \,.$$

It follows that the function $w_{ij}\,\frac{\partial P}{\partial w_{ij}}$ is increasing w.r.t. the variable w_{ij} . Therefore

$$\frac{\partial}{\partial w_{ij}}\big(w_{ij}\,\frac{\partial P}{\partial w_{ij}}\big) \geq 0 \,.$$

Conclude using the fact that $\langle \mathbb{1}_{ij\in D}\rangle = w_{ij}\,\frac{\partial P}{\partial w_{ij}}$, by the previous proposition.

2) Now assume $ij \neq hk$. Dividing the four cases "there's a dimer on ij but not on hk", "there's a dimer on hk but not on ij", "there's a dimer on both ij and hk", "there are no dimers on ij and hk" and proceeding as before one finds

$$Z(\underline{w},\underline{x}) \;=\; w_{ij}\,Z^{(1)}(\underline{v},\underline{x}) + w_{hk}\,Z^{(2)}(\underline{v},\underline{x}) + w_{ij}\,w_{hk}\,Z^{(3)}(\underline{v},\underline{x}) + Z^{(4)}(\underline{v},\underline{x}) \,.$$

Notice $\underline{v}$ does not depend on the values of w_{ij}, w_{hk} and derive the pressure two times w.r.t. the variables w_{ij}, w_{hk}:

$$\frac{\partial^2 P}{\partial w_{hk}\partial w_{ij}} = \frac{1}{Z}\,\frac{\partial^2 Z}{\partial w_{hk}\partial w_{ij}} - \frac{1}{Z^2}\,\frac{\partial Z}{\partial w_{hk}}\,\frac{\partial Z}{\partial w_{ij}} = \frac{Z^{(3)}(\underline{v},\underline{x})Z^{(4)}(\underline{v},\underline{x}) - Z^{(1)}(\underline{v},\underline{x})Z^{(2)}(\underline{v},\underline{x})}{Z(\underline{w},\underline{x})^2}$$

In order to conclude remember that $\langle \mathbb{1}_{ij\in D}\rangle = w_{ij}\,\frac{\partial P}{\partial w_{ij}}$ by the previous proposition, so that

$$\frac{\partial}{\partial w_{hk}}\,\langle \mathbb{1}_{ij\in D}\rangle = w_{ij}\,\frac{\partial^2 P}{\partial w_{hk}\partial w_{ij}} \,.$$

□

Remark 9. On the contrary of the ferromagnetic spin models, in a monomer-dimer model we can't expect that increasing the dimeric weight of an edge hk always brings an increase of the probability of having a dimer on another edge ij. This is because of the bond of monogamy, which entails a competition between two neighbour edges in order to gain the common vertex.
In fact the last proposition allows us to prove this intuitive result. Suppose $h = i$ and $k \neq j$, then

$$\frac{\partial}{\partial w_{ik}}\,\langle \mathbb{1}_{ij\in D}\rangle \leq 0 \,.$$

Indeed there's no dimeric configuration D containing both ij and ik, hence $Z^{(3)} \equiv 0$. Using the non-negativity of any partition function, the previous proposition gives the result.

Now we'll imagine to break the graph $G = (V, E)$ into two disjoint subgraphs G' and G'' and we'll interpolate between the monomer-dimer models in the two different situations. With a good choice of the interpolation method, this technique brings a useful propriety of super-additivity for the pressure. Let V', V'' be two disjoint subsets of the vertex set V.
Let $E' = \{ij \in E | i \in V', j \in V'\}$, $E'' = \{ij \in E | i \in V'', j \in V''\}$, $E^{int} = E \smallsetminus (E' \cup E'')$, the induced edge sets. Let $G' = (V', E')$, $G'' = (V'', E'')$, the two disjoint subgraphs of G. Denote $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{D}''$ the sets of dimeric configurations on the graphs G, G', G'' respectively. Denote Z_G, $Z_{G'}$, $Z_{G''}$ and P_G, $P_{G'}$, $P_{G''}$ the partition functions and the pressures of the monomer-dimer model on the weighted graphs G, G', G'' respectively.

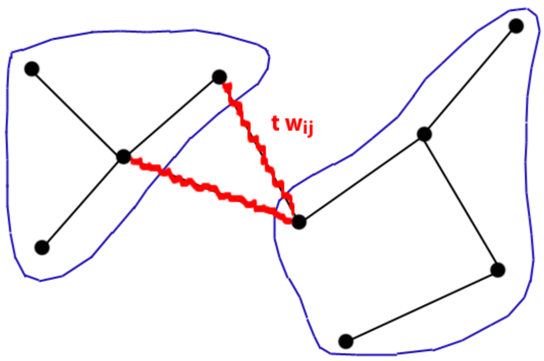


Figure 2.3: *Circled in blue the subgraphs G', G'', marked in red the edges of which we interpolate the dimeric weight*

Proposition 23. *Let G', G'' be two vertex-disjoint subgraphs of G. Then:*

$$P_G(\underline{w}, \underline{x}) \geq P_{G'}(\underline{w}, \underline{x}) + P_{G''}(\underline{w}, \underline{x}) \, .$$

Goes without saying that the dependence of the pressures in the right-hand side on $\underline{w}$, $\underline{x}$ is only by the appropriate restriction of these families.

Proof. Define the interpolating edge weights by

$$w_{ij}(t) := \begin{cases} w_{ij}\,, & \text{if } ij \in E' \text{ or } ij \in E'' \\ t\, w_{ij}\,, & \text{if } ij \in E^{int} \end{cases} \qquad \forall t \in [0,1]$$

The interpolating partition function is $Z_t := Z_G(\underline{w}(t), \underline{x})$. As usual the interpolating pressure will be $P_t := \log Z_t$.

Observe that given a dimeric configuration $D \in \mathscr{D}$ we can always think it as a disjoint union $D = D' \cup D'' \cup D^{int}$ with $D' = D \cap E \in \mathscr{D}'$, $D'' = D \cap E'' \in \mathscr{D}''$, $D^{int} = D \cap E^{int}$; whereas the associated monomeric configuration simply decomposes in $M_D = M'_{D'} \cup M''_{D''}$ with $M'_{D'} = \{i \in V' | \, \forall j \in V' \, ij \notin D'\}$ and $M''_{D''} = \{i \in V'' | \, \forall j \in V'' \, ij \notin D''\}$.

Moreover there's a bijection between the dimeric configurations $D \in \mathscr{D}$ such that $D^{int} = \varnothing$ and the disjoint unions of a dimeric configuration $D' \in \mathscr{D}'$ and a dimeric configuration $D'' \in \mathscr{D}''$.

Now the interpolating partition function can be written

$$Z_t = \sum_{D \in \mathscr{D}} \prod_{ij \in D} w_{ij}(t) \prod_{k \in M_D} x_k \;= \sum_{D \in \mathscr{D}} t^{|D^{int}|} \prod_{ij \in D} w_{ij} \prod_{k \in M_D} x_k\,,$$

therefore the two extreme cases are

$$Z_0 = \sum_{\substack{D \in \mathscr{D} \\ D^{int} = \varnothing}} \prod_{ij \in D} w_{ij} \prod_{k \in M_D} x_k = \sum_{D' \in \mathscr{D}'} \sum_{D'' \in \mathscr{D}''} \prod_{ij \in D'} w_{ij} \prod_{ij \in D''} w_{ij} \prod_{k \in M'_{D'}} x_k \prod_{k \in M''_{D''}} x_k$$

$$= Z_{G'}(\underline{w}, \underline{x})\, Z_{G''}(\underline{w}, \underline{x})\,;$$

$$Z_1 = \sum_{D \in \mathscr{D}} \prod_{ij \in D} w_{ij} \prod_{k \in M_D} x_k \;= Z_G(\underline{w}, \underline{x})\,.$$

Multiplicative relations for the partition functions become additive relations for the pressures. Therefore, using also the fundamental theorem of calculus,

$$P_G(\underline{w}, \underline{x}) - \big[P_{G'}(\underline{w}, \underline{x}) + P_{G''}(\underline{w}, \underline{x})\big] = P_1 - P_0 = \int_0^1 \frac{dP_t}{dt}\, dt\,.$$

On the other hand observe

$$\frac{dP_t}{dt} = \frac{1}{Z_t}\frac{dZ_t}{dt} = \frac{\sum_{D \in \mathscr{D}} |D^{int}|\, t^{|D^{int}|-1} \prod_{ij \in D} w_{ij} \prod_{k \in M_D} x_k}{Z_t} = t^{-1} \langle |D^{int}| \rangle_t \geq 0\,.$$

□

Remark 10. We can think this proposition as an improvement of the proposition 21. Indeed the parameter t increasing brings the rise of some dimeric weights and therefore, since we already knew that $\frac{\partial P}{\partial w_{ij}} \geq 0$, it's natural that the pressure increases.

Remark 11. If the dimeric and monomeric weights are taken uniform on the graph, there's another nice way to prove the statement of proposition 23. Assume $w_{ij} = w \,\forall\, ij \in E$, $x_i = x \,\forall\, i \in V$, so that by remark 7 the partition functions can be written

$$Z_G(w,x) = \sum_{k=0}^{\lfloor N/2 \rfloor} \Lambda(k)\, w^k\, x^{N-2k}\,,$$

$$Z_{G'}(w,x) = \sum_{k_1=0}^{\lfloor N_1/2 \rfloor} \Lambda'(k_1)\, w^{k_1}\, x^{N_1-2k_1}\,,$$

$$Z_{G''}(w,x) = \sum_{k_2=0}^{\lfloor N_2/2 \rfloor} \Lambda''(k_2)\, w^{k_2}\, x^{N_2-2k_2}\,,$$

where $N_1 = |V'|$, $N_2 = |V''|$, while $\Lambda(k)$, $\Lambda'(k_1)$, $\Lambda''(k_2)$ are the number of possible dimeric configuration with k, k_1, k_2 dimers on the graph G, G', G'' respectively.

Notice that the (disjoint) union of a dimeric configuration on G' and a dimeric configuration on G'' is always a dimeric configuration on G, therefore

$$\Lambda(k) \geq \sum_{k_1+k_2=k} \Lambda'(k_1)\, \Lambda''(k_2) \quad \forall\, k = 0, \dots, \lfloor N/2 \rfloor\,.$$

Now multiplying the polynomials $Z_{G'}$ and $Z_{G''}$, one finds

$$Z_{G'}(w,x)\, Z_{G''}(w,x) = \sum_{k=0}^{\lfloor N/2 \rfloor} \Bigg(\sum_{k_1+k_2=k} \Lambda'(k_1)\, \Lambda''(k_2) \Bigg)\, w^k\, x^{N-2k} \leq Z_G(w,x)\,.$$

Taking the logarithms it follows that $P_{G'}(w,x) + P_{G''}(w,x) \leq P_G(w,x)$.

Remark 12. The proposition 23 (or equivalently the previous remark) will permit to prove the existence of the thermodynamic limit for the pressure of the monomer-dimer model on some sequences of graphs $(G_N)_{N\in\mathbb{N}}$ that do

not need any normalisation of the weights when N grows.
Actually it's important to notice that when we break the graph G into the two subgraphs G', G'', we did not change the weights.

Proposition 24. *If H is a subgraph of G then $P_H(\underline{w}, \underline{x}) \leq P_G(\underline{w}, \underline{x})$.*

Proof. Denote $\mathscr{D}_H$, $\mathscr{D}_G$ the set of all possible dimeric configurations on H, G respectively. If H is a subgraph of G, it is clear that $\mathscr{D}_H \subseteq \mathscr{D}_G$. Hence, since the weights are positive,

$$Z_G(\underline{w}, \underline{x}) = \sum_{D \in \mathscr{D}_G} \prod_{ij \in D} w_{ij} \prod_{k \in M_D} x_k \geq \sum_{D \in \mathscr{D}_H} \prod_{ij \in D} w_{ij} \prod_{k \in M_D} x_k = Z_H(\underline{w}, \underline{x}) .$$

And the statement follows by the monotonicity of the logarithm. □

2.2 General bounds for the pressure

We are interested in the behaviour of the system when the number of vertices N goes to infinity, i.e. in the thermodynamic limit.
Therefore we consider a sequence of graphs $G_N = (V_N, E_N)$, $N \in \mathbb{N}$ with $|V_N| = N$ and we provide them with dimeric and monomeric weights w_N, x_N (a priori depending on N, but chosen uniform on each graph for the sake of simplicity). We'll denote Z_N, P_N the partition function and the pressure of the monomer-dimer model on G_N.
It is clear that in such a generality we can't hope to obtain many results. A more precise sequence of graphs need to be chosen, and that's what we'll do in next sections. But for the moment let see how the weights should be chosen.
From a physical point of view we expect that the free energy grows linearly with the number of particles N. To guarantee that this condition is satisfied, it is important to find some bounds for the free energy of the monomer-dimer models.

Proposition 25.

$$\log x_N \leq \frac{P_N}{N}(w_N, x_N) \leq \log x_N + \frac{|E_N|}{N}\,\log\big(1 + \frac{w_N}{x_N^2}\big)\,.$$

Proof. By remark 7, the pressure can be written

$$P_N(w_N, x_N) = \log\big(\sum_{k=0}^{\lfloor N/2 \rfloor} \Lambda_N(k)\, w_N^k\, x_N^{N-2k}\big)\,,$$

where $\Lambda_N(k)$ is the number of possible dimeric configurations composed by k dimers on the graph G_N.

Start assuming $w_N = 1$. A lower bound is given choosing $k = 0$, that is the configuration with a monomer on each vertex of the graph:

$$\frac{P_N}{N}(1, x_N) = \frac{1}{N}\,\log\big(\sum_{k=0}^{\lfloor N/2 \rfloor} \Lambda_N(k)\, x_N^{N-2k}\big) \geq \frac{1}{N}\,\log(1 \cdot x_N^N) = \log x_N\,.$$

On the other side an upper bound can be found observing $\Lambda_N(k) \leq \binom{|E_N|}{k}$, since any dimeric configuration composed by k dimers is a (particular) subset of k edges. In addition notice that if $|E_N| < \lfloor N/2 \rfloor$ then $\Lambda(k) = 0\, \forall\, k \geq |E_N|$. Therefore:

$$\begin{aligned}\frac{P_N}{N}(1, x_N) &= \frac{1}{N}\,\log\big(\sum_{k=0}^{\lfloor N/2 \rfloor} \Lambda_N(k)\, x_N^{N-2k}\big) \leq \frac{1}{N}\,\log\big(\sum_{k=0}^{|E_N|} \binom{|E_N|}{k}\, x_N^{N-2k}\big)\\ &\overset{\star}{=} \frac{1}{N}\,\log\big(x_N^N\,(1 + x_N^{-2})^{|E_N|}\big) = \log x_N + \frac{|E_N|}{N}\,\log(1 + x_N^{-2})\,,\end{aligned}$$

where the Newton's binomial formula is used at $\star$.

Now use the remark 8 to deal with the general case. Since

$$\frac{P_N}{N}(w_N, x_N) = \log\sqrt{w_N} + \frac{P_N}{N}\big(1, \frac{x_N}{\sqrt{w_N}}\big)\,,$$

the previous bounds can be transformed in the desired ones.

□

Remark 13. In finite dimensional lattices or in diluted graphs, the number of edges is of the same order of the number of vertices. In this situation we

may choose the dimeric and monomeric weights independently on N.
Indeed if $|E_N| \leq CN$ and we set $w_N = w$, $x_N = x$, by the previous proposition we get

$$\log x \leq \frac{P_N}{N}(w,x) \leq \log x + C\,\log\big(1+\frac{w}{x^2}\big)\,,$$

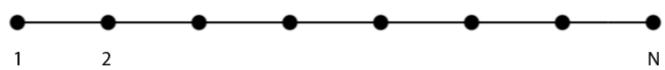
so that the free energy is of order N.
On the contrary in complete graphs the number of edges is of the order of the square of the number of vertices. In this situation we need to normalize the dimeric weight by N.
Indeed if $|E_N| \leq CN^2$ and we set $w_N = w/N$, $x_N = x$, by the previous proposition we get

$$\log x \leq \frac{P_N}{N}\big(\frac{w}{N}\,,x\big) \leq \log x + CN\,\log\big(1+\frac{w}{Nx^2}\big) \leq \log x + C\,\frac{w}{x^2}\,,$$

so that the free energy is again of order N.

2.3 Monomer-dimer model on the line

Consider a line of N points, each one interacting only with his two (or one at the extremes) neighbours. This can be represented by the graph $L_N = (V_N, E_N)$ with $V_N = \{1,\dots,N\}$ and $E_N = \{\{k,k+1\}|\,k = 1\dots N-1\}$.



For the sake of simplicity we want to consider uniform dimeric and monomeric weights. Therefore by remarks 8 and 13 we may take $w_{ij}^{(N)} = 1\ \forall\, ij \in E_N$ and $x_i^{(N)} = x\ \forall\, i \in V_N$ for every $N \in \mathbb{N}$. In this section we denote $Z_{L_N}(x)$, $P_{L_N}(x)$ the partition function and the pressure of the monomer-dimer model on the line L_N with the weights introduced.
We are interested in the behaviour of the pressure per particle P_{L_N}/N as N goes to infinity. With the aim of illustrating two different techniques, first we'll prove the existence of the limit and after we'll compute it.

2.3.1 Existence of the thermodynamic limit

Proposition 26. *There exists* $\lim_{N\to\infty} \frac{1}{N} P_{L_N}(x) = \sup_{N\in\mathbb{N}} \frac{1}{N} P_{L_N}(x) \in \mathbb{R}$.

Proof. Let N_1, N_2, $N \in \mathbb{N}$ such that $N_1 + N_2 = N$. Consider the line L_N and its two subgraphs L_{N_1} and $\widetilde{L}_{N_2}$ induced respectively by the first N_1 and the last N_2 vertices. Notice the graph $\widetilde{L}_{N_2}$ is isomorphic to L_{N_2}.

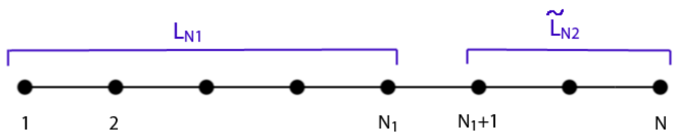


Figure 2.4: *The line graph L_N and its two subgraphs L_{N_1} and $\widetilde{L}_{N_2}$*

Proposition 23 applies, hence

$$P_{L_N}(x) \geq P_{L_{N_1}}(x) + P_{\widetilde{L}_{N_2}}(x) = P_{L_{N_1}}(x) + P_{L_{N_2}}(x)\,.$$

Therefore by Fekete's lemma

$$\exists \lim_{N\to\infty} \frac{P_{L_N}}{N}(x) = \sup_{N\in\mathbb{N}} \frac{P_{L_N}}{N}(x)\,.$$

Since $|E_N| = N - 1$, thank to proposition 25 this limit is finite. Precisely

$$\log x \leq \lim_{N\to\infty} \frac{P_{L_N}}{N}(x) \leq \log x + \log\left(1 + \frac{1}{x^2}\right)\,.$$

□

2.3.2 Exact solution

Now we'll compute the value of the polynomial $Z_{L_N}(x)$ for any $N \in \mathbb{N}$, using a recurrence technique due to Heilmann and Lieb. Then from this result we'll derive the value of the $\lim_{N\to\infty} \frac{1}{N} P_{L_N}(x)$.

It is convenient to introduce a new (real) polynomial related to the partition function:

$$Q_{L_N}(x) := i^{-N} Z_{L_N}(i\,x) = \sum_{k=0}^{\lfloor N/2 \rfloor} (-1)^k \,\Lambda_{L_N}(k)\; x^{N-2k}$$

where i is the imaginary unit and $\Lambda_{L_N}(k)$ denotes the number of possible dimeric configurations formed by k dimers on the line L_N.

Proposition 27. *The partition function of the monomer-dimer model on the line L_N is*

$$\boxed{Z_{L_N}(x) = i^N\, U_N(\frac{-i\,x}{2})}$$

where U_N is the N^{th} Chebyshev polynomial of the second kind.

Proof. Look for a recursive formula for Λ_{L_N}. For any dimeric configuration D on the line L_N two possibilities can be distinguished:

I. There is a monomer on the vertex N, i.e. $\{N-1, N\} \notin D$. This case is equivalent to say that $D = D'$ is a dimeric configuration on the sub-line L_{N-1}.

II. There is a dimer between the vertices N and $N-1$, i.e. $\{N-1, N\} \in D$. In order that this may happen it is necessary $\{N-2, N-1\} \notin D$. Hence this case is equivalent to say

$$D = \{N-1, N\} \cup D''\,,$$

where D'' is a dimeric configuration on the sub-line L_{N-2}.

Therefore for any $k = 0, \ldots, \lfloor N/2 \rfloor$

$$\Lambda_{L_N}(k) = \Lambda_{L_{N-1}}(k) + \Lambda_{L_{N-2}}(k-1)\,.$$

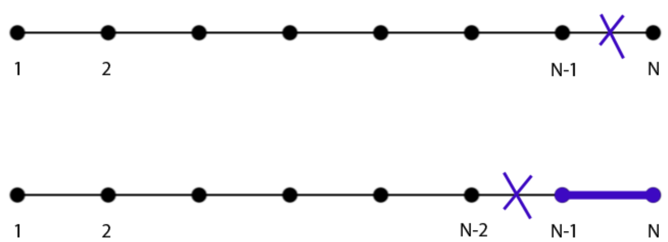


Figure 2.5: *The two kinds of dimeric configuration D on L_N: with a monomer on N or with a dimer between N and $N-1$.*

Now introduce the recurrence relation for the coefficients in the expression of the polynomial $Q_{L_N}(x)$ and find

$$Q_{L_N}(x) = x\, Q_{L_{N-1}}(x) - Q_{L_{N-2}}(x)\,.$$

Complete this recurrence relation with the initial conditions

$$Q_{L_1}(x) = x\,, \quad Q_{L_0}(x) = 1\,.$$

Now the Chebyshev polynomials of the second kind are defined as the solution of the following recurrence problem

$$\begin{cases} U_N(x) = 2x\, U_{N-1}(x) - U_{N-2}(x) \\ U_1(x) = 2x\,, \quad U_0(x) = 1 \end{cases}.$$

Therefore it's easy to check that $U_N(x/2)$ verifies the same recurrence problem as $Q_{L_N}(x)$. Hence

$$Q_{L_N}(x) = U_N\big(\frac{x}{2}\big)\,.$$

Conclude observing that by definition of Q_{L_N},

$$Z_{L_N}(x) = i^N\, Q_{L_N}(-i\,x)\,.$$

□

Corollary 28. *The partition function of the monomer-dimer model on the line L_N is*

$$Z_{L_N}(x) = \frac{\big(x + \sqrt{x^2+4}\,\big)^{N+1} - \big(x - \sqrt{x^2+4}\,\big)^{N+1}}{2^{N+1}\sqrt{x^2+4}}\,.$$

Proof. Remind the Chebyshev polynomials of the second kind admit the following explicit form:

$$U_N(x) = \frac{\big(x + \sqrt{x^2-1}\,\big)^{N+1} - \big(x - \sqrt{x^2-1}\,\big)^{N+1}}{2\sqrt{x^2-1}}\,.$$

Substitute this expression in the formula for $Z_{L_N}(x)$ given by the previous proposition and after few computations find the desired result. □

Set $p_L(x) := \lim_{N\to\infty} \frac{1}{N} P_{L_N}(x)$, the thermodynamic limit of the pressure per particle. By proposition 26 we already know that this limit exists, now we are going to compute it (it is not required to know the existence before).

Corollary 29. *The thermodynamic limit of the pressure on the line is*

$$\boxed{p_L(x) = -\log 2 + \log\left(x + \sqrt{x^2+4}\right)}$$

Proof. Use the explicit expression for the partition function $Z_{L_N}(x)$ found in the previous corollary to compute

$$\frac{1}{N} P_{L_N}(x) = \frac{1}{N} \log\left[\left(x+\sqrt{x^2+4}\right)^{N+1} - \left(x-\sqrt{x^2+4}\right)^{N+1}\right] + \\ - \frac{1}{N}\left((N+1)\log 2 + \log\sqrt{x^2+4}\right).$$

Denote S_1, S_2 respectively the first and the second addend in this expression. Clearly $S_2 \xrightarrow[N\to\infty]{} -\log 2$. To study S_1 isolate the leading part:

$$S_1 = \frac{1}{N}\log\left[1 - \left(\frac{x-\sqrt{x^2+4}}{x+\sqrt{x^2+4}}\right)^{N+1}\right] + \frac{1}{N}\log\left[\left(x+\sqrt{x^2+4}\right)^{N+1}\right]$$

and then observe $-1 < \frac{x-\sqrt{x^2+4}}{x+\sqrt{x^2+4}} < 0$ so that

$$\frac{1}{N}\log\left[1 - \left(\frac{x-\sqrt{x^2+4}}{x+\sqrt{x^2+4}}\right)^{N+1}\right] \xrightarrow[N\to\infty]{} 0\,,$$

while

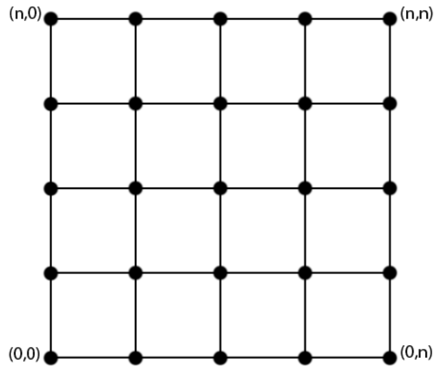
$$\frac{1}{N}\log\left[\left(x+\sqrt{x^2+4}\right)^{N+1}\right] = \frac{N+1}{N}\log\left(x+\sqrt{x^2+4}\right) \xrightarrow[N\to\infty]{} \log\left(x+\sqrt{x^2+4}\right)$$

This concludes the proof. □

2.4 Monomer-dimer model on the d-dimensional cube

Fix $d \in \mathbb{N}_{\geq 1}$. Consider the points of a d-dimensional lattice, each one interacting only with the nearest (in the standard euclidian metric) neighbours.

This situation can be represented by the graph $I_n = (V_n, E_n)$ with $V_n = \{0,1,\dots,n\}^d$ and $E_n = \Big\{\{(i_1,\dots,i_d),(j_1,\dots,j_d)\}\,\Big|\, 1 \le i_s, j_s \le n,\ \sum_{s=1}^d |i_s - j_s| = 1\Big\}$.



Observe that the numbers of vertices and edges of the d-dimensional cube I_n are respectively:

$$N_n := |V_n| = (n+1)^d\,, \quad |E_n| = d\,n\,(n+1)^{d-1}\,.$$

We will consider uniform dimeric and monomeric weights. Therefore by remarks 8 and 13 we may take $w_{ij}^{(N)} = 1\ \forall\, ij \in E_N$ and $x_i^{(N)} = x\ \forall\, i \in V_N$ for all $N \in \mathbb{N}$. We denote $Z_{I_n}(x)$ and $P_{I_n}(x)$ the partition function and the pressure of the monomer-dimer model on the d-dimensional cube I_n with the weights introduced.

Remark 14. Notice the bounds for the pressure found in proposition 25 here become

$$\log x \le \frac{1}{N_n}\,P_{I_n}(x) \le \log x + d\,\log(1+x^{-2})\,.$$

We will not approach the problem of computing the thermodynamic limit of the pressure per particle when $d \ge 2$. Anyway it is easy to show that it exists, thank to the super-additivity proved in proposition 23 .

Proposition 30. *There exists* $\lim_{n\to\infty} \frac{1}{N_n}\,P_{I_n}(x) = \sup_{n\in\mathbb{N}} \frac{1}{N_n}\,P_{I_n}(x) \in \mathbb{R}\,.$

Proof. Let $n, m \in \mathbb{N}$ with $n > m$. Do the euclidean division of n by m:

$$n = q\,m + r \quad \text{with } q \in \mathbb{N},\ q \geq 1,\ r \in \{0, 1, \dots, m-1\}\,.$$

First partition the cube I_n in the sub-cube $I_{q\,m}$ and in its complementary graph $I_n - I_{q\,m}$. By proposition 23

$$P_{I_n}(x) \geq P_{I_{qm}}(x) + P_{I_n - I_{qm}}(x)\,.$$

Now investigate the two terms separately.
1) Divide the cube I_{qm} in q^d cubes of side length m (hence isomorphic to I_m). Reducing by 1 vertex the sides of each of these cubes, we can suppose they are disjoint. Therefore by proposition 23

$$P_{I_{qm}}(x) \geq q^d\, P_{I_{m-1}}(x)\,.$$

2) On the other hand count the number of vertices contained in the graph $I_n - I_{q\,m}$, they are $N_n - N_{q\,m} = n^d - (q\,m)^d$. Then bound the pressure from below using proposition 25

$$P_{I_n - I_{qm}}(x) \geq \big(n^d - (q\,m^d)\big)\,\log x\,.$$

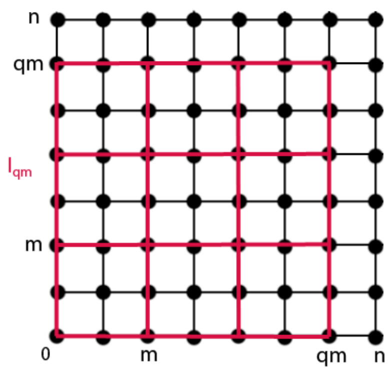


Figure 2.6: *The square lattice I_n ($d = 2$) and its sub-square I_{qm}. I_{qm} is again divided in 2^q squares isomorphic to I_m, which are "almost disjoint".*

In the end the inequality found, divided by $N_n = (n+1)^d$, is

$$\frac{1}{(n+1)^d}\,P_{I_n}(x)\ \geq\ \frac{q^d}{(n+1)^d}\,P_{I_{m-1}}(x)\ +\ \frac{n^d - (q\,m)^d}{(n+1)^d}\,\log x\,.$$

Keep m fixed and let $n \to \infty$; it's easy to check that the inequality becomes

$$\liminf_{n\to\infty} \frac{1}{(n+1)^d} P_{I_n}(x) \geq \frac{1}{m^d} P_{I_{m-1}}(x) + 0 ;$$

since m is arbitrary, it is true also for the supremum over m, hence

$$\liminf_{n\to\infty} \frac{1}{(n+1)^d} P_{I_n}(x) \geq \sup_{m\in\mathbb{N}} \frac{1}{m^d} P_{I_{m-1}}(x) \geq \limsup_{m\in\mathbb{N}} \frac{1}{m^d} P_{I_{m-1}}(x) .$$

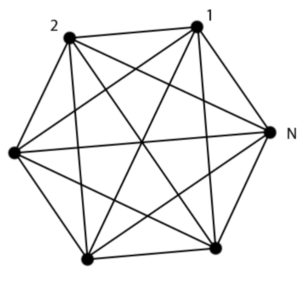
It is clear that $\limsup_{m\in\mathbb{N}} \frac{1}{m^d} P_{I_{m-1}}(x) = \limsup_{m\to\infty} \frac{1}{(m+1)^d} P_{I_m}(x)$. Therefore conclude that

$$\exists \lim_{n\to\infty} \frac{1}{(n+1)^d} P_{I_n}(x) = \sup_{n\in\mathbb{N}} \frac{1}{(n+1)^d} P_{I_n}(x)$$

and the previous remark guarantees that it is finite. □

2.5 Monomer-dimer model on the complete graph

Consider a set of N points each one interacting with all the others. That is the graph $K_N = (V_N, E_N)$ with $V_N = \{1, \dots, N\}$ and $E_N = \{\{i,j\} | \, i,j \in V_N\}$. For the sake of simplicity we want to consider uniform dimeric and monomeric weights. Therefore by remarks 8 and 13 we may take $w_{ij}^{(N)} = 1/(N-1) \; \forall \, ij \in E_N$ and $x_i^{(N)} = x \; \forall \, i \in V_N$ for all $N \in \mathbb{N}$.



In this section we denote $Z_{K_N}(x) \equiv Z_{K_N}\left(\frac{1}{N-1}, x\right)$ and $P_{K_N}(x) \equiv P_{K_N}\left(\frac{1}{N-1}, x\right)$ the partition function and the pressure of the monomer-dimer model on the complete graph K_N with the weights introduced.

As observed in remark 12, the proposition 23 cannot be used to prove the convergence of the pressure per particle P_{K_N}/N on the complete graph, since here the dimeric weights are normalised by N. We would need a more specific interpolation which takes account of the dependence of the weights on the size N... We'll treat this problem later. Let's first prove the existence of the thermodynamic limit directly computing it.

2.5.1 Exact solution

As usual we start computing the value of the polynomial $Z_{K_N}(x)$, using the recurrence technique due to Heilmann and Lieb. Define two supporting polynomials related to the partition function

$$Q_{K_N}\big(\frac{1}{N-1}\,,x\big) := i^{-N}\,Z_{K_N}(i\,x) = \sum_{k=0}^{\lfloor N/2\rfloor}(-1)^k\;\Lambda_{K_N}(k)\;\big(\frac{1}{N-1}\big)^k\;x^{N-2k}\,,$$

$$Q_{K_N}(1,x) := \sum_{k=0}^{\lfloor N/2\rfloor}(-1)^k\;\Lambda_{K_N}(k)\;x^{N-2k}\,,$$

where $\Lambda_{K_N}(k)$ denotes the number of possible dimeric configurations formed by k dimers on the complete graph K_N. Notice the coefficient $1/(N-1)^k$ in the first polynomial, due to the normalized dimeric weight.

Proposition 31. *The partition function of the monomer-dimer model on the complete graph K_N is*

$$\boxed{Z_{K_N}(x) = \frac{i^N}{(N-1)^{N/2}}\;H_N\big(-\,i\,\sqrt{N-1}\;x\big)}$$

where H_N is the N^{th} probabilistic Hermite polynomial.

Proof. For any D dimeric configuration on the graph K_N there are two distinct possibilities:

I. On the vertex N there is a monomer, i.e. $\{1,N\},\,\dots\,,\{N-1,N\}\notin D$. This case is verified if and only if $D=D'$ is a dimeric configuration on the complete sub-graph K_{N-1}.

II. There is a dimer between the vertex N and (exactly) one of the other vertices, i.e. $\exists!\, i = 1, \dots, N-1$ s.t. $\{i, N\} \in D$. In order that this may happen it's necessary for i not have any dimer with the other vertices. Thus this case is verified if and only if

$$D = \{i, N\} \cup D'' ,$$

where D'' is a dimeric configuration on $\widetilde{K}^{(i)}_{N-2}$, the complete subgraph of K_{N-1} induced removing the vertex i.

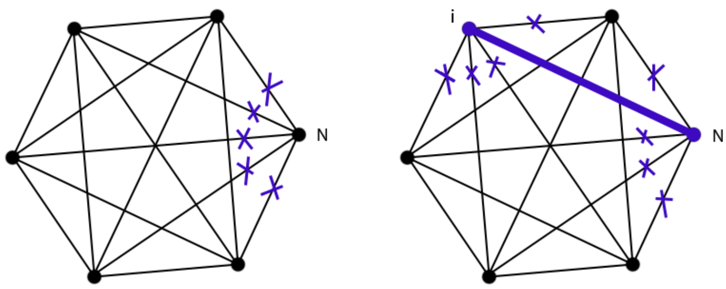


Figure 2.7: *The two kinds of dimeric configuration D on K_N: with a monomer on the vertex N or with a dimer between N and one of the previous vertices i.*

Notice any $\widetilde{K}^{(i)}_{N-2}$ is isomorphic to K_{N-2}. Therefore for any $k = 0, \dots, \lfloor N/2 \rfloor$

$$\Lambda_{K_N}(k) = \Lambda_{K_{N-1}}(k) + (N-1)\, \Lambda_{K_{N-2}}(k-1) .$$

Now introduce the recurrence relation for the coefficients in the expression of the polynomial $Q_{K_N}(1, x)$ and find

$$Q_{K_N}(1, x) = x\, Q_{K_{N-1}}(1, x) - (N-1)\, Q_{K_{N-2}}(1, x) .$$

Complete this recurrence relation with the initial conditions

$$Q_{K_1}(1, x) = x , \quad Q_{K_0}(1, x) = 1 .$$

Now the probabilistic Hermite polynomials are proven to be the solution of the following recurrence problem

$$\begin{cases} H_N(x) = x\, H_{N-1}(x) - (N-1)\, H_{N-2}(x) \\ H_1(x) = x , \quad H_0(x) = 1 \end{cases} .$$

Therefore $Q_{K_N}(1,x) = H_N(x)$. Notice, as previously done in remark 8, that $Q_{K_N}\big(\,1/(N-1)\,,\,x\,\big) = 1/(N-1)^{N/2}\; Q_{K_N}\big(\,1\,,\sqrt{N-1}\;x\,\big)$. Hence

$$Q_{K_N}\big(\frac{1}{N-1}\,,x\big) = \big(\frac{1}{N-1}\big)^{N/2}\; H_N\big(\sqrt{N-1}\;x\big)\,.$$

Conclude reminding that $Z_{K_N}(x) = i^N\; Q_{K_N}\big(\,1/(N-1)\,,-i\,x\,\big)$. □

Corollary 32. *The partition function of the monomer-dimer model on the complete graph K_N is*

$$Z_{K_N}(x) = \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{N!}{k!\,(N-2k)!}\; 2^{-k}\,(N-1)^{-k}\; x^{N-2k}\;.$$

Proof. Remind the probabilistic Hermite polynomials have the following expansion:

$$H_N(x) = \sum_{k=0}^{\lfloor N/2 \rfloor} (-1)^k\,\frac{N!}{k!\,(N-2k)!}\; 2^{-k}\; x^{N-2k}\;.$$

Substitute this expression in the formula for $Z_{K_N}(x)$ given by the previous proposition and after few computations find the desired result. □

Remark 15. This expansion of the partition function $Z_{K_N}(x)$ can be found also with a simple combinatorial argument. By definition

$$Z_{K_N}(x) = \sum_{k=0}^{\lfloor N/2 \rfloor} \Lambda_{K_N}(k)\;(N-1)^{-k}\; x^{N-2k}\,.$$

Notice any dimeric configuration D on K_N can be built following the iterative procedure here described:

- choose two different vertices i and j in V^s (notice it can be done in $\binom{|V^s|}{2}$ different ways) and link them by a dimer setting $D^s := D^{s-1} \cup ij$,
- set $V^{s+1} := V^s \smallsetminus \{i,j\}$;

repeat for $s = 1, \dots, k$, with initial sets $V^1 := V_N = \{1, \dots, N\}$, $D^0 := \varnothing$ and finally $D := D^k$.
Therefore the number $\Lambda_{K_N}(k)$ of possible dimeric configuration with k dimers on the complete graph K_N is

$$\Lambda_{K_N}(k) = \binom{N}{2}\binom{N-2}{2}\dots\binom{N-2(k-1)}{2} \Big/ k!\,,$$

where one divides by $k!$ as not interested in the order of the dimers.
This can be rewritten as

$$\Lambda_{K_N}(k) = \frac{N!}{k!\,(N-2k)!}\;2^{-k}\,,$$

giving the desired expression for Z_{K_N} .

Set $p_K(x) := \lim_{N\to\infty}\frac{1}{N}\,P_{K_N}(x)$, the thermodynamic limit of the pressure per particle. We are going to prove this limit exists directly computing it.

Corollary 33. *The thermodynamic limit of the pressure on the complete graph exists and it is*

$$p_K(x) = f(x)\,\big(1-\log f(x)-\log 2\big)+g(x)\,\big(1-\log g(x)+\log x\big)-1$$

where

$$f(x) = \frac{1}{4}\,\big(2+x^2-x\,\sqrt{x^2+4}\,\big)\quad,\quad g(x) = 1-2\,f(x)$$

Proof. It is convenient to set for $k = 0\dots\lfloor N/2\rfloor$

$$a_N(k,x) = \frac{N!}{k!\,(N-2k)!}\;2^{-k}\,(N-1)^{-k}\,x^{N-2k}\,,$$

$$M_N(x) = \max_{k=0\dots\lfloor N/2\rfloor} a_N(k,x)\,.$$

By the previous corollary (or remark) the explicit expansion of the partition function is

$$Z_{K_N}(x) = \sum_{k=0}^{\lfloor N/2\rfloor} a_N(k,x)\,,$$

hence

$$M_N(x) \;\leq\; Z_{K_N}(x) \;\leq\; \big(\frac{N}{2}+1\big)\, M_N(x)\,.$$

Taking the logarithms and dividing by N one gets

$$\frac{1}{N}\,\log M_N(x) \;\leq\; \frac{1}{N}\,P_{K_N}(x) \;\leq\; \underbrace{\frac{1}{N}\,\log\big(\frac{N}{2}+1\big)}_{\xrightarrow[N\to\infty]{}0} + \frac{1}{N}\,\log M_N(x)\,.$$

Therefore if one proves that $(\log M_N)/N \xrightarrow[N\to\infty]{} l$, it will follow that also $P_{K_N}/N \xrightarrow[N\to\infty]{} l$ thank to the squeeze theorem. So concentrate on the asymptotic behaviour of $\log M_N/N$.

I) The first step is to understand which is the maximum term of each sum, studying the trend of $a_N(k,x)$ as a function of $k \in \{0,\dots,\lfloor N/2\rfloor\}$. Simplifying the factorials and isolating k and k^2, on finds

$$\begin{aligned}
&a_N(k,x) \leq a_N(k+1,x) \quad\Longleftrightarrow\\
&4\,k^2 - 2\,\big(2N-1+(N-1)\,x^2\big)\,k + N^2 - N - 2\,(N-1)\,x^2 \geq 0 \quad (\star)
\end{aligned}$$

Solve this second degree equation in k, finding for all N sufficiently large

$$k \leq \frac{1}{4}\,\big(2N-1+(N-1)\,x^2\big) - \frac{1}{8}\,\sqrt{\Delta(N,x)} \;\;=: k_-(N,x) \quad \text{or}$$

$$k \geq \frac{1}{4}\,\big(2N-1+(N-1)\,x^2\big) + \frac{1}{8}\,\sqrt{\Delta(N,x)} \;\;=: k_+(N,x)\;,$$

with

$$\Delta(N,x) = 4\,\big[(x^4+4x^2)\,N^2 + (2x^2-2x^4)\,N + \,x^4 - 6x^2 + 1)\big] > 0\,.$$

As $N\to\infty$ one may estimate

$$k_\pm(N,x) \;=\; f_\pm(x)\,N + O(\sqrt{N}\,)\,,$$

with

$$f_\pm(x) = \frac{1}{4}\,\big(2 + x^2 \pm x\,\sqrt{x^2+4}\,\big)\,.$$

Observe that $f_+(x) > 1/2$ while $f_-(x) < 1/2$, hence for N sufficiently large $k_+(N,x) > N/2$ while $k_-(N,x) < N/2$.

Therefore the inequality ($\star$) with $k < N/2$ is equivalent to $k \leq k_-(N,x)$.
To resume for N sufficiently large

$$a_N(k,x) \leq a_N(k+1,x) \iff k \leq k_-(N,x) = f_-(x)\,N + O(\sqrt{N}\,)\;.$$

II) Now one knows the maximum term of the sum is that with index $k = k_{\max} = \lfloor k_-(N,x) \rfloor + 1$, one can compute

$$\begin{aligned} M_N(x) &= \max_{k=0\ldots\lfloor N/2\rfloor} a_N(k,x) = a_N(k_{\max},x) = a_N\big(f_-(x)\,N + O(\sqrt{N}\,)\,,x\big) \\ &= \frac{N!\,\big(2\,(N-1)\big)^{-f(x)\,N+O(\sqrt{N}\,)}}{\big(f(x)\,N + O(\sqrt{N}\,)\big)!\,\big(N - 2f(x)\,N + O(\sqrt{N}\,)\big)!}\;x^{N-2\,f(x)\,N+O(\sqrt{N}\,)} \end{aligned}$$

where $f(x) := f_-(x)$. Set also $g(x) := 1 - 2\,f(x)$.
Take the logarithm, divide by N and use the Stirling formula (in the form $\log(n!) = n\,\log n - n + O(\log n)$ as $n \to \infty$) to find

$$\begin{aligned} \frac{1}{N}\,\log M_N(x) \underset{N\to\infty}{\sim} &\frac{N\log N - N}{N} - \frac{f(x)N\,\log\big(f(x)N\big) - f(x)N}{N} + \\ &- \frac{g(x)N\,\log\big(g(x)N\big) - g(x)N}{N} - \frac{f(x)N\log(2N)}{N} + \frac{g(x)N\log x}{N} \end{aligned}$$

Simplifying N and isolating $\log N$ and then $f(x)$ and $g(x)$, one finds

$$\begin{aligned} \frac{1}{N}\,\log M_N(x) \underset{N\to\infty}{\sim} &\big(1 - f(x) - g(x) - f(x)\big)\log N + \\ &+ f(x)\,\big(-\log f(x) + 1 - \log 2\big) + g(x)\,\big(-\log g(x) + 1 + \log x\big) - 1 \end{aligned}$$

Notice here the coefficient of $\log N$ is zero, hence the limit of $(\log M_N(x))/N$ is found. As observed before $P_{K_N}(x)/N$ converges to the same limit and so the statement is proved. □

2.5.2 An open problem: monotonicity

We may ask if the thermodynamic limit $p_K(x)$ is reached in a monotone way, or more generally if $p_K(x) = \sup_{N\in\mathbb{N}} \frac{1}{N}\,P_{K_N}(x)$ or $p_K(x) = \inf_{N\in\mathbb{N}} \frac{1}{N}\,P_{K_N}(x)$. Super-additivity or sub-additivity of the sequence $(P_{K_N})_{N\in\mathbb{N}}$ would give this result, but since on the complete graph the dimeric weight depends on N the

proposition 23 can't be used to infer the super-additivity of P_{K_N}.
The first idea is to define a new interpolation, which takes account of the dependence of the weights on the size of the system.

Given $N,\ N_1,\ N_2 \in \mathbb{N}$ such that $N_1+N_2=N$, we defined the following linear interpolating dimeric weights for $t\in[0,1]$

$$w_{ij}(t) = w_{ij}^{(N,N_1,N_2)}(t) = \begin{cases} \frac{t}{N-1}+\frac{1-t}{N_1-1}\,, & \text{if } i,j\in\{1\dots N_1\} \\ \frac{t}{N-1}+\frac{1-t}{N_2-1}\,, & \text{if } i,j\in\{N_1+1\dots N\} \\ \frac{t}{N-1}\,, & \text{if } i\in\{1\dots N_1\}\,,\ j\in\{N_1+1\dots N\} \end{cases}$$

Notice that when $t=1$ we have the normalized dimeric weight on K_N, whereas when $t=0$ we have the normalised dimeric weights on the two complete subgraphs K_{N-1}, $\widetilde{K}_{N-2}$ and no connection (i.e. zero dimeric weight) between them.
Denote $Z_t := Z_{K_N}\big(\underline{w}(t),x\big)$ the interpolating partition function and $P_t := \log Z_t$. Along the lines of the proof of proposition 23, it's easy to check that

$$Z_0 = Z_{K_{N_1}}\big(\frac{1}{N_1-1}\,,x\big)\ Z_{K_{N_2}}\big(\frac{1}{N_2-1}\,,x\big) \quad , \quad Z_1 = Z_{K_N}\big(\frac{1}{N-1}\,,x\big)$$

and in consequence:

$$P_{K_N}\big(\frac{1}{N}\,,x\big) - \Big[P_{K_{N_1}}\big(\frac{1}{N_1-1}\,,x\big) + P_{K_{N_2}}\big(\frac{1}{N_2-1}\,,x\big)\Big] = P_1 - P_0 = \int_0^1 \frac{dP_t}{dt}\,dt\,.$$

But for the moment we did not manage to show if dP_t/dt has a constant sign.

Also if we didn't prove the result by interpolation, we conjecture the same that the thermodynamic limit is reached in a monotone way, with direction changing if the dimeric weight is normalised by $N-1$ or by N. Precisely, denoting with $p_K(x)$ the limit computed in corollary 33, we think that

Conjecture 34. *For every $x>0$ there exist $\overline{N}(x)\,,\overline{\overline{N}}(x)\in\mathbb{N}$ such that*

$$p_K(x) = \inf_{N>\overline{N}(x)} \frac{1}{N}\,P_{K_N}\big((N-1)^{-1},x\big) = \inf_{N\in 2\mathbb{N}} \frac{1}{N}\,P_{K_N}\big((N-1)^{-1},x\big)\,,$$

$$p_K(x) = \inf_{N>\overline{\overline{N}}(x)} \frac{1}{N}\,P_{K_N}(N^{-1},x) = \inf_{N\in 2\mathbb{N}+1} \frac{1}{N}\,P_{K_N}(N^{-1},x)\,.$$

It's easy to check that if we take dimeric weight $1/(N-1)$, as done in the previous subsection, or $1/N$, the limit of the pressure per particle as $N \to \infty$ is always $p_K(x)$ (in fact the proof of corollary 33 is the same).
Anyway from a graphical analysis it is clear that the direction from which this limit is reached does change.

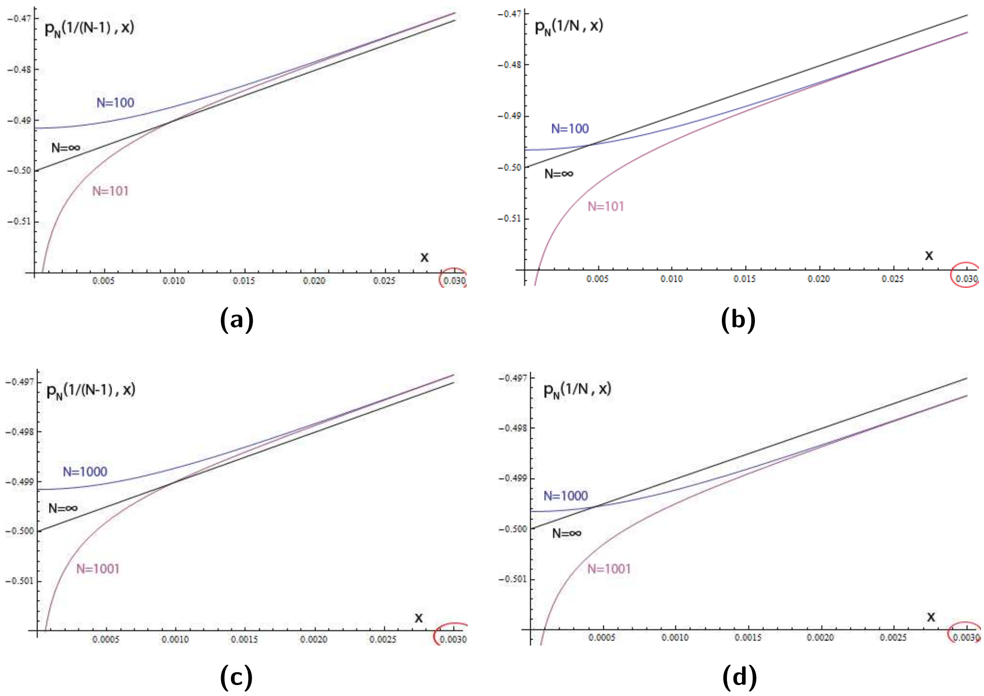


Figure 2.8: *The pressure per particle* $x \mapsto \frac{1}{N} P_{K_N}(w^{(N)}, x)$ *for some values of* N *and in the limit* $N \to \infty$*. We can compare the behaviour if the dimeric weight is* $w^{(N)} = 1/(N-1)$ *(graphs* **a**, **c***) or* $w^{(N)} = 1/N$ *(graphs* **b**, **d***). In the first case the limit is reached from above, in the second one it is reached from below.*
We can also see there is no uniformity w.r.t. x*, unless we restraint to* N *even in the first case or* N *odd in the second one.*

Another way to prove our conjecture could be to investigate directly the sub-additivity or super-additivity inequality, since we know the explicit expression of the partition function Z_{K_N}.
To fix the ideas let's take dimeric weight $1/(N-1)$. Let N, N_1, $N_2 \in \mathbb{N}$ such that $N_1 + N_2 = N$; obviously the sub-additivity of the pressure can be

rewritten as

$$P_{K_N}\big((N-1)^{-1},x\big) \leq P_{K_{N_1}}\big((N_1-1)^{-1},x\big) + P_{K_{N_2}}\big((N_2-1)^{-1},x\big) \iff$$
$$Z_{K_N}\big((N-1)^{-1},x\big) \leq Z_{K_{N_1}}\big((N_1-1)^{-1},x\big) \cdot Z_{K_{N_2}}\big((N_2-1)^{-1},x\big)$$

A sufficient (but not necessary) condition is given comparing coefficient by coefficient the left-hand polynomial and the right-hand one. That is, looking at the expression of the partition function given by corollary 32,

$$\frac{N!\,(N-1)^{-k}}{k!\,(N-2k)!} \leq \sum_{k_1+k_2=k} \frac{N_1!\,N_2!\,(N_1-1)^{-k_1}\,(N_2-1)^{-k_2}}{k_1!\,k_2!\,(N_1-2k_1)!\,(N_2-2k_2)!}$$

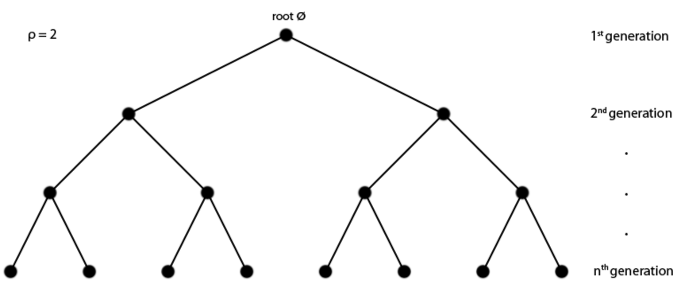
for every $k = 0,\, \dots,\, \lfloor N/2 \rfloor$.

The numerical investigation of this inequality gives good results: if we take N_1, N_2 even, it seems to be always verified.

2.6 Monomer-dimer model on a regular tree

Consider a tree of n generations (root included), in which each node has $\rho > 1$ sons except for those of the last generation. This is represented by the graph $T_n = (V_n, E_n)$ with $V_n = \{(k,i)\,|\,k = 1,\dots,n,\ i = 1,\dots,\rho^{k-1}\}$ and $E_n = \big\{\,\{(k,i)\,,\,(k{+}1,j)\}\,|\,k = 1,\dots,n{-}1,\ i = 1,\dots,\rho^{k-1},\ j = \rho\,(i{-}1){+}1,\dots,\rho\,i\big\}$.
Notice the number of vertices and the number of edges are respectively

$$N_n := |V_n| = 1 + \rho + \cdots + \rho^{n-1} = \frac{\rho^n - 1}{\rho - 1}\ , \quad |E_n| = N_n - 1\ .$$



We want to consider uniform dimeric and monomeric weights. Therefore by remarks 8 and 13 we may take $w_{ij}^{(n)} = 1\ \forall\, ij \in E_n$ and $x_i^{(n)} = x\ \forall\, i \in V_n$

for every $n \in \mathbb{N}$. In this section we denote $Z_{T_n}(x)$, $P_{T_n}(x)$ the partition function and the pressure of the monomer-dimer model on the tree T_n with the weights introduced.
The tree graphs are generalisations of the line, which is obtained for $\rho = 1$. So we'll proceed studying the monomer-dimer model on a tree along the lines of what we have done in section 2.3. Pay attention because the quantity to study here is the pressure per particle P_{T_n}/N_n as $n \to \infty$ (and not P_{T_n}/n).

2.6.1 Existence of the thermodynamic limit

Remark 16. Notice the bounds for the pressure found in proposition 25, here become the following

$$\log x \le \frac{1}{N_n} P_{T_n} \le \log x + \log(1 + x^{-2}) \,.$$

Proposition 35. *There exists*

$$\lim_{n\to\infty} \frac{1}{N_n} P_{T_n}(x) = \sup_{n\in\mathbb{N}} \Big(\frac{1}{\rho^n} \log x + \frac{1}{N_n} P_{T_n}(x) \Big) \in \mathbb{R} \,.$$

Notice that if $x \ge 1$ *this supremum equals* $\sup_{n\in\mathbb{N}} \frac{1}{N_n} P_{T_n}(x)$.

Proof. Let $n, m \in \mathbb{N}$ with $n > m$.
Divide the tree T_n in its sub-tree T_{n-m} (induced by the vertices from the root to $n-m^{th}$ generation) and in the complementary forest $T_n - T_{n-m}$. Observe this forest is composed by ρ^{n-m} disjoint trees, each one isomorphic to T_m. Therefore by proposition 23

$$P_{T_n}(x) \ge P_{T_{n-m}}(x) + \rho^{n-m} P_{T_m}(x) \,.$$

Now observe that by proposition 25

$$P_{T_{n-m}}(x) \ge \frac{\rho^{n-m} - 1}{\rho - 1} \log x \,.$$

Thus dividing by $N_n = (\rho^n - 1)/(\rho - 1)$ one obtains

$$\frac{\rho - 1}{\rho^n - 1} P_{T_n}(x) \ \ge\ \frac{\rho^{n-m} - 1}{\rho^n - 1} \log x \ +\ \frac{(\rho - 1)\, \rho^{n-m}}{\rho^{n-1}} P_{T_m}(x) \,.$$

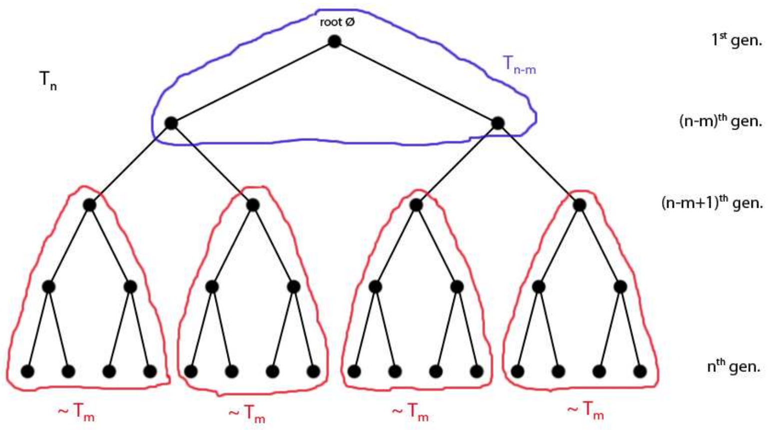


Figure 2.9: *Here $\rho = 2$. The tree T_n partitioned in the sub-tree T_{n-m} and in ρ^{n-m} isomorphic copies of the tree T_m.*

Keep m fixed and let n go to infinity:

$$\liminf_{n\to\infty} \frac{\rho-1}{\rho^n-1} P_{T_n}(x) \geq \rho^{-m} \log x + (\rho-1)\,\rho^{-m} P_{T_m}(x)\,;$$

then take the supremum over all m:

$$\liminf_{n\to\infty} \frac{\rho-1}{\rho^n-1} P_{T_n}(x) \geq \sup_{m\in\mathbb{N}} \big(\rho^{-m} \log x + (\rho-1)\,\rho^{-m} P_{T_m}(x)\big) \geq$$

$$\limsup_{m\in\mathbb{N}} \big(\rho^{-m} \log x + (\rho-1)\,\rho^{-m} P_{T_m}(x)\big) = \limsup_{m\in\mathbb{N}} \frac{\rho-1}{\rho^m-1} P_{T_m}(x)\,.$$

Therefore there exists

$$\lim_{n\to\infty} \frac{1}{N_n} P_{T_n}(x) = \sup_{n\in\mathbb{N}} \Big(\frac{1}{\rho^n} \log x + \frac{1}{N_n} P_{T_n}(x)\Big)$$

and by the previous remark it is finite. □

2.6.2 Exact solution

We'll compute the value of the polynomial $Z_{T_n}(x)$, using the recurrence technique due to Heilmann and Lieb. Define the supporting polynomial related to the partition function

$$Q_{T_n}(x) := i^{-N_n} Z_{T_n}(i\,x) = \sum_{k=0}^{\lfloor N_n/2 \rfloor} (-1)^k \Lambda_{T_n}(k)\; x^{N_n-2k}$$

where i is the imaginary unit and $\Lambda_{T_n}(k)$ denotes the number of possible dimeric configurations formed by k dimers on the tree T_n.

Proposition 36. *The partition function of the monomer-dimer model on the tree T_n is*

$$\boxed{Z_{T_n}(x) = i^{N_n}\,\rho^{n/2}\,U_n\big(\frac{-i\,x}{2\,\rho^{1/2}}\big)\,\prod_{j=1}^{n-1}\Big(\rho^{j/2}\,U_j\big(\frac{-i\,x}{2\,\rho^{1/2}}\big)\Big)^{(\rho-1)\,\rho^{n-1-j}}}$$

where U_n is the n^{th} Chebyshev polynomial of the second kind.

Proof. Denote $\varnothing$ the root of the tree T_n. Given a vertex v and an integer k small enough, denote by $T_k(v)$ the sub-tree of T_n rooted at v and composed by k generations. Furthermore denote $v^{(1)},\dots,v^{(\rho)}$ the sons of v.
Now look for a recursive formula for Λ_{T_n}. Let D be a dimeric configuration on the tree T_n. Distinguish two possibilities:

I. On the root $\varnothing$ there is a monomer, i.e. $\{\varnothing,\varnothing^{(1)}\}$, ..., $\{\varnothing,\varnothing^{(\rho)}\}\notin D$. This case is verified if and only if

$$D = D^1\,\cup\cdots\cup\,D^\rho\,,$$

where D^i is a dimeric configuration on the sub-tree $T_{n-1}(\varnothing^{(i)})$ for each $i=1,\dots,\rho$.

II. There is a dimer between the root $\varnothing$ and (exactly) one of its sons , i.e. $\exists\,!\,i=1,\dots,\rho$ s.t. $\{\varnothing,\varnothing^{(i)}\}\in D$. In order that this may happen it's necessary for $\varnothing^{(i)}$ not to have any dimer with its sons. Hence this case is verified if and only if

$$D = D^1\cup\cdots\cup D^{i-1}\cup\Big[\{\varnothing,\varnothing^{(i)}\}\cup D^{i,1}\cup\cdots\cup D^{i,\rho}\Big]\cup D^{i+1}\cup\cdots\cup D^\rho$$

where D^j is a dimeric configuration on $T_{n-1}(\varnothing^{(j)})$ for each $j=1,\dots,\rho$, $j\neq i$, whereas $D^{i,l}$ is a dimeric configuration on $T_{n-2}(\varnothing^{(i)(l)})$ for each $l=1,\dots,\rho$.

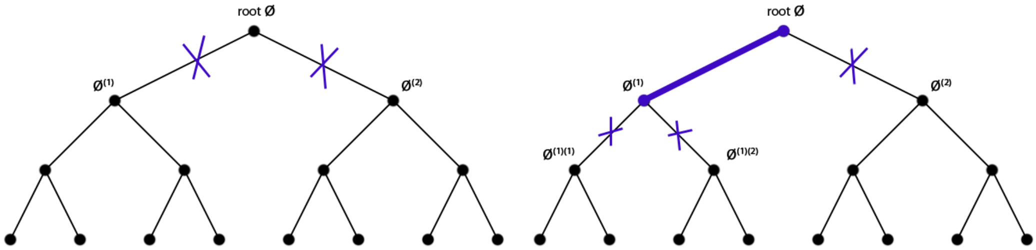


Figure 2.10: *The two kinds of dimeric configuration on the tree T_n: with monomer on the root $\varnothing$ or with a dimer between the root $\varnothing$ and one of its sons $\varnothing^{(i)}$.*

Notice the trees $T_{n-1}(\varnothing^{(i)})$ and $T_{n-2}(\varnothing^{(i,l)})$ are isomorphic to T_{n-1} and T_{n-2} respectively. Therefore for any $k = 0, \ldots, \lfloor N_n/2 \rfloor$

$$\Lambda_{T_n}(k) \;=\; \sum_{k_1+\cdots+k_\rho=k} \prod_{i=1}^{\rho} \Lambda_{T_{n-1}}(k_i) \;+\; \rho \sum_{\substack{k_1+\cdots+k_{\rho-1}+\\ +h_1+\cdots+h_\rho=k}} \prod_{j=1}^{\rho-1} \Lambda_{T_{n-1}}(k_j) \prod_{l=1}^{\rho} \Lambda_{T_{n-2}}(h_l)\,.$$

Substitute this identity in the expression of the polynomial $Q_{T_n}(x)$ and after some computations find

$$Q_{T_n}(x) = x\left(Q_{T_{n-1}}(x)\right)^{\rho} - \rho\left(Q_{T_{n-1}}(x)\right)^{\rho-1}\left(Q_{T_{n-2}}(x)\right)^{\rho}.$$

Now define a further supporting polynomial:

$$\widehat{Q}_{T_n}(x) := Q_{T_n}(x) \prod_{j=1}^{n-1} \left(Q_{T_j}(x)\right)^{-(\rho-1)} \qquad (\star)$$

From the recurrence relation for Q_{T_n} deduce the following one for $\widehat{Q}_{T_n}$:

$$\widehat{Q}_{T_n}(x) = x\,\widehat{Q}_{T_{n-1}}(x) - \rho\,\widehat{Q}_{T_{n-2}}(x)\,,$$

which is completed by the initial conditions

$$\widehat{Q}_{T_1}(x) = x\,, \quad \widehat{Q}_{T_0}(x) = 1\,.$$

Now remind the Chebyshev polynomials of the second kind are defined as the solution of the following recurrence problem

$$\begin{cases} U_n(x) = 2x\,U_{n-1}(x) - U_{n-2}(x) \\ U_1(x) = 2x\,, \quad U_0(x) = 1 \end{cases}.$$

Therefore it's easy to check that $\rho^{n/2}\,U_n\big(x/(2\rho^{1/2})\big)$ verifies the same recurrence problem as $\widehat{Q}_{T_n}(x)$. Hence

$$\widehat{Q}_{T_n}(x) = \rho^{n/2}\,U_n\big(\frac{x}{2\,\rho^{1/2}}\big) \qquad (\star'\,)$$

On the other side inverting the definition $(\star)$ one finds

$$Q_{T_n}(x) = \widehat{Q}_{T_n}(x) \prod_{j=1}^{n-1} \big(\widehat{Q}_{T_j}(x)\big)^{(\rho-1)\,\rho^{n-1-j}}$$

using the fact that $\rho^{n-1-j}-1 = (\rho-1)\,(\rho^{n-1-(j+1)}+\dots+\rho^{n-1-(n-1)})$ for every $j=1,\dots,n-2$.
To conclude substitute the expression $(\star'\,)$ in the last relation and remember that, by definition, $Z_{T_n}(x) = i^{N_n}\,Q_{T_n}(-i\,x)\,.$ □

Corollary 37. *The partition function of the monomer-dimer model on the tree* T_n *is*

$$Z_{T_n}(x) = \big(\frac{1}{2}\big)^{\frac{\rho^n-1}{\rho-1}}\;\psi_n(x,\rho)\prod_{j=1}^{n-1}\psi_j(x,\rho)^{(\rho-1)\,\rho^{n-1-j}}\,,$$

where

$$\psi_j(x,\rho) = \frac{(x+\sqrt{x^2+4\,\rho})^{j+1}-(x-\sqrt{x^2+4\,\rho})^{j+1}}{2\sqrt{x^2+4\,\rho}}\,.$$

Proof. Remind the Chebyshev polynomials of the second kind admit the following explicit form:

$$U_j(x) = \frac{\big(x+\sqrt{x^2-1}\,\big)^{j+1}-\big(x-\sqrt{x^2-1}\,\big)^{j+1}}{2\sqrt{x^2-1}}\,.$$

With few computations it follows that

$$U_j\big(\frac{-i\,x}{2\,\rho^{1/2}}\big) \;=\; \left(\frac{-i}{2\,\rho^{1/2}}\right)^{j}\,\psi_j(x,\rho)\,.$$

It is useful also to compute the following sum (related to the derivative of a geometric sum):

$$(\rho-1)\sum_{j=1}^{n-1} j\;\rho^{n-1-j} = -n+\frac{\rho^n-1}{\rho-1}\,.$$

Using these two results the formula for $Z_{T_n}(x)$ found in the previous proposition can be easily transformed in the desired one. □

Set $p_T(x) := \lim_{n\to\infty} \frac{1}{N_n} P_{T_n}(x)$, the thermodynamic limit of the pressure per particle on the trees sequence. Remind $N_n = (\rho^n - 1)/(\rho - 1)$.

Corollary 38. *The thermodynamic limit of the pressure on the tree exists and it is*

$$p_T(x) = -\log 2 - \frac{\rho-1}{\rho} \log\left(2\sqrt{x^2+4\rho}\right) + \frac{2\rho-1}{\rho} \log\left(x+\sqrt{x^2+4\rho}\right) + (\rho-1)^2 \sum_{j=1}^{\infty} \rho^{-(j+1)} \log\left[1 - \left(\frac{x-\sqrt{x^2+4\rho}}{x+\sqrt{x^2+4\rho}}\right)^{j+1}\right]$$

where the series is absolutely convergent.

Proof. Before starting compute the following two sums:

$$(\rho-1)\sum_{j=1}^{n-1} \rho^{n-1-j} = \rho^{n-1} - 1\,, \qquad (\rho-1)\sum_{j=1}^{n-1} j\,\rho^{n-1-j} = -n + \frac{\rho^n - 1}{\rho - 1}\,.$$

Use the explicit expression of the partition function $Z_{T_n}(x)$ found in the previous corollary to compute

$$P_{T_n}(x) = -N_n \log 2 + S_1 - S_2 + S_3 - S_4\,,$$

with

$$S_1 = \log\left[\left(x+\sqrt{x^2+4\rho}\right)^{n+1} - \left(x-\sqrt{x^2+4\rho}\right)^{n+1}\right], \quad S_2 = \log\left(2\sqrt{x^2+4\rho}\right),$$

$$S_3 = (\rho-1)\sum_{j=1}^{n-1} \rho^{n-1-j} \log\left[\left(x+\sqrt{x^2+4\rho}\right)^{j+1} - \left(x-\sqrt{x^2+4\rho}\right)^{j+1}\right],$$

$$S_4 = (\rho-1)\sum_{j=1}^{n-1} \rho^{n-1-j} \log\left(2\sqrt{x^2+4\rho}\right).$$

Divide by N_n and study the asymptotic behaviour addend by addend.
For S_1 notice $-1 < (x - \sqrt{x^2+4\rho})\,/\,(x+\sqrt{x^2+4\rho}) < 0$ and separate this vanishing part from the rest:

$$\frac{S_1}{N_n} = \frac{\rho-1}{\rho^n-1} \underbrace{\log\left[1-\left(\frac{x-\sqrt{\cdot}}{x+\sqrt{\cdot}}\right)^{n+1}\right]}_{\to 0} + \underbrace{\frac{\rho-1}{\rho^n-1}(n+1)}_{\to 0} \log\left[x+\sqrt{\cdot}\,\right] \xrightarrow[n\to\infty]{} 0\,.$$

S_2 and S_4 can be studied together, using the computation of the sum done at the beginning:

$$\frac{S_2+S_4}{N_n} = \frac{\rho-1}{\rho^n-1}\,\rho^{n-1}\log\left(2\sqrt{x^2+4\rho}\right) \xrightarrow[n\to\infty]{} \frac{\rho-1}{\rho}\log\left(2\sqrt{x^2+4\rho}\right).$$

For S_3 it's better to separate the vanishing part in the logarithm from the rest, that is write $S_3 = S_3' + S_3''$ with:

$$S_3' = (\rho-1)\sum_{j=1}^{n-1}\rho^{n-1-j}\log\left[1-\left(\frac{x-\sqrt{x^2+4\rho}}{x+\sqrt{x^2+4\rho}}\right)^{j+1}\right],$$

$$S_3'' = (\rho-1)\sum_{j=1}^{n-1}\rho^{n-1-j}(j+1)\log\left[x+\sqrt{x^2+4\rho}\right].$$

For S_3'' use the computation of the two sums done at the beginning:

$$\frac{S_3''}{N_n} = \frac{\rho-1}{\rho^n-1}\left(-n+\frac{\rho^n-1}{\rho-1}+\rho^{n-1}-1\right)\log\left(x+\sqrt{x^2+4\rho}\right) \xrightarrow[n\to\infty]{}$$

$$\left(0+1+(\rho-1)\rho^{-1}\right)\log\left(x+\sqrt{x^2+4\rho}\right) = \frac{2\rho-1}{\rho}\log\left(x+\sqrt{x^2+4\rho}\right).$$

In S_3' take ρ^n out of the sum:

$$\frac{S_3'}{N_n} = (\rho-1)^2\underbrace{\frac{\rho^n}{\rho^n-1}}_{\to 1}\sum_{j=1}^{n-1}\rho^{-j-1}\log\left[1-\left(\frac{x-\sqrt{x^2+4\rho}}{x+\sqrt{x^2+4\rho}}\right)^{j+1}\right]$$

and observe the remaining series is absolutely convergent as $n\to\infty$. Indeed $-1 < (x-\sqrt{x^2+4\rho})\,/\,(x+\sqrt{x^2+4\rho}) < 0$, so that

$$0 \le \log\left[1+\left(\frac{\sqrt{\cdot}-x}{\sqrt{\cdot}+x}\right)^{j+1}\right] \le \left(\frac{\sqrt{\cdot}-x}{\sqrt{\cdot}+x}\right)^{j+1},$$

$$\log\left[1-\frac{\sqrt{\cdot}-x}{\sqrt{\cdot}+x}\right] \le \log\left[1-\left(\frac{\sqrt{\cdot}-x}{\sqrt{\cdot}+x}\right)^{j+1}\right] \le -\left(\frac{\sqrt{\cdot}-x}{\sqrt{\cdot}+x}\right)^{j+1}.$$

This concludes the proof. □

2.7 Plots of the pressure per particle

Up to now we have computed explicitly the thermodynamic limits for the pressure per particle of the monomer-dimer model on the line, on the complete graph and on a regular tree with $\rho \in \mathbb{N}_{\geq 2}$ sons. The following plot compares them.

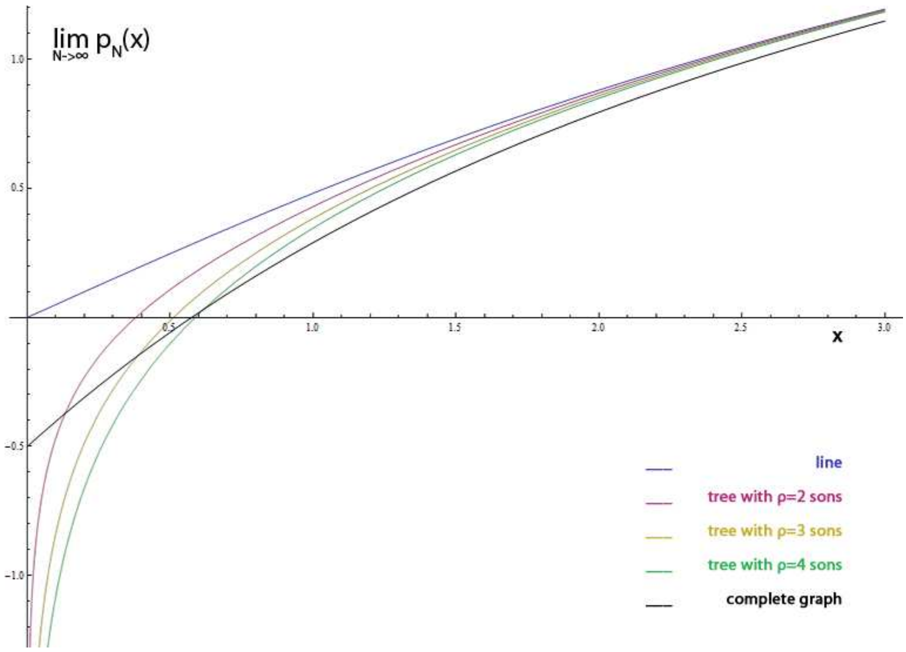


Figure 2.11: *Monomer-dimer models. The thermodynamic limit of the pressure per particle on the line, on the regular trees with* $\rho = 2, 3, 4$ *sons and on the complete graph is plotted as a function of the monomeric weight* x *(while the dimeric weight is fixed at* 1 *or* $1/N$ *for the complete graph).*

2.8 Monomer-dimer model on the Erdös-Rényi diluted graph

In this section we want consider a random graph in which each couple of vertices has a probability of being linked by an edge, independently from the other ones. Furthermore we want the linked couples to be "not too much".

What we described is a diluted random graph à la Erdös-Rényi.
That is a simple graph $G_N = (V_N, E_N)$ such that $V_N = \{1, \dots, N\}$ and $E_N = \{\{i,j\} \mid i,j \in V_N,\ \varepsilon_{ij}^N = 1\}$, where

$$(\varepsilon_{ij}^N)_{ij} \text{ i.i.d. random variables } \overset{\mathrm{d}}{\sim} \mathcal{B}ernoulli\Big(\frac{2\,c}{N-1}\Big)$$

and $c > 0$ is a fixed constant.
For brevity set $\underline{\varepsilon}^N = (\varepsilon_{ij}^N)_{ij}$. Often we'll drop the index N.
We denote $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space on which the ε_{ij}^N are defined, and $\mathbb{E}[\cdot]$ the expected value w.r.t. to the measure $\mathbb{P}$. Namely for any function f of the graph structure,

$$\mathbb{E}[f(\underline{\varepsilon})] \ = \sum_{\underline{e} \in \{0,1\}^N} f(\underline{e})\ \mathbb{P}(\underline{\varepsilon} = \underline{e})\,.$$

Reminding remarks 8 and 13, we introduce the random uniform dimeric weight $w_{ij}^{(N)} = 1\ \forall\, ij \in E_N$ and the deterministic uniform monomeric weight $x_i^{(N)} = x > 0\ \forall\, i \in V_N$ for every $N \in \mathbb{N}$. We'll denote Z_{G_N}, P_{G_N} the partition function and the pressure of the monomer-dimer model on the diluted random graph G_N with the random weights introduced.

Remark 17. Notice the mean number of edges in the graph G_N is of the order of the number of vertices. Precisely:

$$\mathbb{E}[|E_N|\,] = \binom{N}{2}\,\mathbb{P}(\varepsilon_{ij} = 1) = \binom{N}{2}\,\frac{2\,c}{N-1} = c\,N\,.$$

This is the reason why this random graph is called diluted.
Therefore our choice of the dimeric and monomeric weights was well done, in the sense that the pressure per particle is bounded. Indeed, applying the expectation to the bounds found in remark 13, we obtain

$$\frac{1}{N}\,\mathbb{E}[|P_{G_N}|\,] \leq |\log x| + c\,\log\Big(1 + \frac{1}{x^2}\Big) < \infty\,.$$

Notice the partition function Z_{G_N} is a random variables w.r.t. the measure $\mathbb{P}$, since it depends on the structure assumed by the graph. Nevertheless, as we'll prove in the next proposition, when N grows the pressure per particle gradually looses any random behaviour, concentrating around its mean value.

Lemma 39. *Let $G = (V, E)$ be a graph, $|V| = N - 1$. Let $i \notin V$.*
For every $V' \subseteq V$, denote $G[\, i, V']$ the graph obtained from G by adding the new vertex i and linking it to the vertices in V'.
The maximum variation of the pressure of the monomer-dimer models on these graphs with fixed weights is

$$\sup_{V',V''\subseteq V} \left| P_{G[\,i,V']}(1,x) - P_{G[\,i,V'']}(1,x) \right| \;\leq\; \log(x + N - 1)\,.$$

Proof. Notice that if $V'' \subseteq V'$ then $G[\, i, V'']$ is a subgraph of $G[\, i, V']$ and so $P_{G[\,i,V'']}(1,x) \leq P_{G[\,i,V']}(1,x)$ (see proposition 24).
Therefore, being V and $\varnothing$ respectively the maximum and the minimum subsets of V, it follows that

$$\sup_{V',V''\subseteq V} \left| P_{G[\,i,V']}(1,x) - P_{G[\,i,V'']}(1,x) \right| \;=\; P_{G[\,i,V]}(1,x) - P_{G[\,i,\varnothing]}(1,x)\,.$$

Denote $\overline{G} := G[\, i, V]$ the graph obtained from G by adding the vertex i and linking it to all the others. Notice the monomer-dimer model on the graph $G[\, i, \varnothing]$, obtained from G by adding the vertex i and leaving it isolated, is equivalent to the monomer-dimer model on the graph $\overline{G} - i = G$.

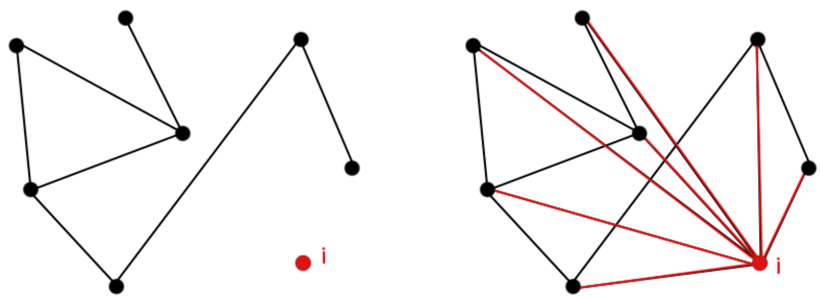


Figure 2.12: *In black the graph G; on the left the graph $G[\, i, \varnothing]$, on the right the graph $G[\, i, V]$. Our task is to compute the difference of the pressure of the monomer-dimer model on these two graphs.*

Now let D be a dimeric configuration on $\bar{G}$. Distinguish two possibilities:

I. There is monomer on the vertex i. This case is equivalent to say D is a dimeric configuration on $\overline{G} - i$.

II. There exists one vertex $v \in V$ such that there is a dimer on $\{i, k\}$. This case is equivalent to say

$$D = \{i, v\} \cup D'$$

with D' dimeric configuration on the graph $\overline{G} - i - v$, obtained from $\overline{G}$ by removing the vertices i, v and all their links.

Therefore for every $k = 0, \ldots, N$, the number $\Lambda_{\overline{G}}(k)$ of possible dimeric configurations composed by k dimers on the graph $\overline{G}$ is

$$\Lambda_{\overline{G}}(k) = \Lambda_{\overline{G}-i}(k) + \sum_{v \in V} \Lambda_{\overline{G}-i-v}(k-1)\,.$$

Introducing this relation in the expression of the partition function given by remark 7, one finds

$$Z_{\overline{G}}(1, x) = x\, Z_{\overline{G}-i}(1, x) + \sum_{v \in V} Z_{\overline{G}-i-v}(1, x)\,.$$

It follows that

$$\frac{Z_{\overline{G}}(1, x)}{Z_{\overline{G}-i}(1, x)} = x + \sum_{v \in V} \frac{Z_{\overline{G}-i-v}(1, x)}{Z_{\overline{G}-i}(1, x)} \leq x + |V| = x + N - 1\,,$$

using the fact that $Z_{\overline{G}-i-v} \leq Z_{\overline{G}-i}$ as $\overline{G} - i - v$ is a subgraph of $\overline{G} - i$ (see again proposition 24) . Finally take the logarithm

$$P_{\overline{G}}(1, x) - P_{\overline{G}-i}(1, x) = \log \frac{Z_{\overline{G}}(1, x)}{Z_{\overline{G}-i}(1, x)} \leq \log(x + N - 1)\,. \quad \square$$

Proposition 40. *For every $\epsilon > 0$*

$$\mathbb{P}\Big(\frac{1}{N}\,|P_{G_N} - \mathbb{E}[P_{G_N}]| \geq \epsilon\Big) \leq \exp\Big(-\frac{\epsilon^2}{2}\,\frac{N}{(\log(x+N))^2}\Big) \xrightarrow[N\to\infty]{} 0\,.$$

In consequence, assuming the whole graph sequence $(G_N)_{N\in\mathbb{N}}$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\boxed{\mathbb{P}\Big(\frac{1}{N}\,|P_{G_N} - \mathbb{E}[P_{G_N}]| \xrightarrow[N\to\infty]{} 0\Big) = 1}$$

Proof. **I)** Fix $N \in \mathbb{N}$.
Consider the filtration $(\mathcal{F}_i)_{i=1\dots N}$ such that at the i^{th} step one knows the subgraph of G_N induced by the vertices $1, \dots, i$. That is set

$$\mathcal{F}_1 = \{\Omega, \varnothing\}\,, \quad \mathcal{F}_2 = \sigma(\varepsilon_{2,1})\,, \quad \mathcal{F}_i = \mathcal{F}_{i-1} \cup \sigma(\varepsilon_{i,1}\,, \varepsilon_{i,2}\,, \dots, \varepsilon_{i,i-1})$$

for every $i = 2, \dots, N$. After that define the Doob martingale of P_{G_N}/N with respect to the filtration $(\mathcal{F}_i)_{i=1\dots N}$. That is

$$A_i := \frac{1}{N}\,\mathbb{E}[P_{G_N}|\mathcal{F}_i\,]\,, \quad \forall\, i = 1, \dots, N\,.$$

Observe that A_i is well defined since by the previous remark $P_{G_N} \in \mathbb{L}^1(\mathbb{P})$. Furthermore notice:

- $(A_i)_{i=1\dots N}$ is a martingale w.r.t. $(\mathcal{F}_i)_{i=1\dots N}$; indeed from the definition of conditional expectation A_i is $\mathcal{F}_i$-measurable and belongs to $\mathbb{L}^1(\mathbb{P})$, and by the tower propriety $\mathbb{E}[A_i|\mathcal{F}_{i-1}] = A_{i-1}$.

- $A_N - A_1 = (P_{G_N} - \mathbb{E}[P_{G_N}\,])/N$, since the entire graph G_N is $\mathcal{F}_N$-measurable while $\mathcal{F}_1$ is the trivial σ-algebra.

Let $i = 2, \dots, N$. Now imagine all the edges of the graph G_N are fixed, except for those depending on the σ-algebra $\mathcal{F}_i \smallsetminus \mathcal{F}_{i-i}$ (i.e. linking the vertex i with a previous one): each of these edges may be present or not in the graph. Bound the maximum variation of the pressure under these constraints:

$$\begin{aligned}
&\sup\Big\{\,|P_{G_N}(\underline{\varepsilon} = \underline{e}) - P_{G_N}(\underline{\varepsilon} = \underline{e}')|\;\Big|\;\underline{e},\,\underline{e}' \in \{0,1\}^{\binom{N}{2}} \text{ may differ on } i\,1, \dots, i(i-1)\Big\}\\
&\leq\; \sup\Big\{\,|P_{G_N}(\underline{\varepsilon} = \underline{e}) - P_{G_N}(\underline{\varepsilon} = \underline{e}')|\;\Big|\;\underline{e},\,\underline{e}' \in \{0,1\}^{\binom{N}{2}} \text{ may differ on } i\,1, \dots, iN\Big\}\\
&\leq\; \log(x + N - 1) \;\leq\; \log(x + N)\,,
\end{aligned}$$

where the second last inequality is due to the previous lemma.
Since the random variables $(\varepsilon_{lm})_{lm}$ are independent, it follows by proposition 44 in the appendix that

$$|A_i - A_{i-1}| = \frac{1}{N}\,\big|\,\mathbb{E}[P_{G_N}|\mathcal{F}_i\,] - \mathbb{E}[P_{G_N}|\mathcal{F}_{i-1}]\big| \;\leq \frac{1}{N}\,\log(x + N)\,.$$

Therefore by the Azuma-Hoeffding inequality applied to the martingale $(A_i)_{i=1\dots N}$

$$\mathbb{P}\big(\frac{1}{N}\,|P_{G_N}-\mathbb{E}[P_{G_N}]|\geq\epsilon\big)=\mathbb{P}\big(|A_N-A_1|\geq\epsilon\big)\leq\exp\Bigg(-\frac{\epsilon^2}{2\sum_{i=1}^N\big(\frac{\log(x+N)}{N}\big)^2}\Bigg)$$

$$=\exp\big(-\frac{\epsilon^2\,N}{2\,(\log(x+N))^2}\big)\,.$$

II) To conclude notice the founded bound is summable with respect to N, indeed one can easily prove that it is a $O(1/N^2)$. Hence

$$\sum_{N=1}^{\infty}\mathbb{P}\big(\frac{1}{N}\,|P_{G_N}-\mathbb{E}[P_{G_N}]|\geq\epsilon\big)\leq\sum_{N=1}^{\infty}\exp\big(-\frac{\epsilon^2\,N}{2\,(\log(x+N))^2}\big)<\infty\,.$$

Therefore by Borel-Cantelli lemma

$$\mathbb{P}\big(\exists\text{ infinitely many }N\text{ s.t. }\frac{1}{N}\,|P_{G_N}-\mathbb{E}[P_{G_N}]|\geq\epsilon\big)=0$$

and so by arbitrariness of $\epsilon>0$

$$\mathbb{P}\big(\forall\,\epsilon\in\mathbb{Q}_+\;\exists\,\bar{N}_\epsilon\in\mathbb{N}\text{ s.t. }\forall\,N>\bar{N}_\epsilon\;\frac{1}{N}\,|P_{G_N}-\mathbb{E}[P_{G_N}]\,|<\epsilon\big)=1\,,$$

that is $\mathbb{P}\big(\frac{1}{N}\,|P_{G_N}-\mathbb{E}[P_{G_N}]\,|\xrightarrow[N\to\infty]{}0\big)=1$. □

Remark 18. Thank to this proposition the problem of determine the behaviour of the pressure per particle P_{G_N}/N in the thermodynamic limit is reduced to the study of its expectation $\mathbb{E}[P_{G_N}]/N$.
An interesting challenge for the future will be to prove that this quantity converges and to compute its limit. Two possible ways could be:

1) to use an interpolation method which takes into account the dependence on N of the edges distribution, indeed remind $\varepsilon_{ij}^N\overset{\text{d}}{\sim}\mathcal{B}ernoulli\big(\frac{2\,c}{N-1}\big)$. Also for this purpose it would be important to find a good interpolation method on the complete graph, as tried in section 2.5.2.

2) to play on the fact that a diluted random graph à la Erdös-Rényi has an asymptotically tree-like structure, as done by Dembo and Montanari for the ferromagnetic spin models.

Appendix A

In this appendix we introduce briefly the Graph Theory, with particular attention to the definitions and the notations that are frequently used in the previous chapters.

Definition 9. A *simple graph* G is an ordered pair (V, E) such that

i. V is a non-empty set, whose elements are called *vertices*;

ii. E is a subset of $\mathcal{P}(V, 2)$ (the family of subset of V composed by two distinct elements), whose elements are called *edges*.

If $ij = \{i, j\}$ belongs to E, we say that in the graph G the vertices i and j are linked (by the edge ij).
We say that G is *finite* if both the sets V and E are finite.

In the thesis and in the following we usually write "graph" for "finite simple graph". For shortness often we write $i \in G$ meaning $i \in V$ and $ij \in G$ meaning $ij \in E$. This should not create confusion.
Clearly a graph can be represented by drawing a point for each vertex and a line that connects each pair of vertices linked by an edge.
We expect that any meaningful propriety of a graph does not depend on the name given to the vertices, hence we give the following definition.

Definition 10. Two graphs $G = (V, E)$, $G' = (V', E')$ are *isomorphic* ($G \cong G'$) if there exist two bijections $f : V \to V'$ and $g : E \to E'$ such that

$$g(ij) = f(i)f(j) \quad \forall\, i, j \in V\,.$$

Definition 11. A graph $U = (\widetilde{V}, \widetilde{E})$ is a *subgraph* of the graph $G = (V, E)$ if $\widetilde{V} \subseteq V$ and $\widetilde{E} \subseteq E$.

Definition 12. Two subgraphs $G' = (V', E')$, $G'' = (V'', E'')$ of the graph G are said:

i. *vertex-disjoint* or simply *disjoint* if $V' \cap V'' = \varnothing$;

ii. *edge-disjoint* if $E' \cap E'' = \varnothing$;

Note that to be vertex-disjoint implies to be also edge-disjoint, whereas the contrary is false in general.

Definition 13. Let $G = (V, E)$ be a graph. Consider a set of vertices $\widetilde{V} \subseteq V$. The *subgraph of* G *induced by the vertices in* $\widetilde{V}$ is the graph $U = (\widetilde{V}, \widetilde{E})$ with

$$\widetilde{E} = \{ij \in E \,|\, i \in \widetilde{V},\, j \in \widetilde{V}\}\,.$$

In particular given two subgraphs A, B of the graph G we denote:

i. $A - B$ the subgraph of G induced by the vertices of A which are not vertices of B;

ii. $A + B$ the subgraph of G induced by the vertices of A and those of B.

Note that the edge set of $A + B$ is in general bigger than the union of the edge sets of A and B.

Definition 14. Let G be a graph and consider a subgraph U. We call *boundary* of U (in G) the set

$$\partial U := \{i \in U \,|\, \exists j \in G - U \ \textit{s.t.}\ ij \in G\}\,.$$

Definition 15. The *neighbourhood* of a vertex i in the graph $G = (V, E)$ is

$$\partial i = \{j \in V \,|\, ij \in E\}\,.$$

The *degree* of i is

$$\deg(i) = |\partial i|\,.$$

Notice that $\sum_{i \in V} \deg(i) = 2\,|E|$.

An important fact is that it's possible to introduce a distance an a graph.

Definition 16. Let $G = (V, E)$ be a graph. A *walk* on G is a sequence of vertices

$$W = v_0,\, v_1,\, \dots,\, v_k$$

such that $v_s v_{s+1} \in E$ for all $s = 0 \dots k-1$.
We say that such a walk W connects the vertices v_0 and v_k. Further k is called the *length* of W.

The graph G is *connected* if for any pair of vertices i, j there exists a walk that connects them.

Definition 17. The *distance* between the vertices i, j in the graph G is

$$d(i,j) = \begin{cases} 0 & \text{if } i = j \\ \min\{lenght(W) \,|\, W \text{is a walk on G}\} & \text{if } i \neq j \text{ are connected} \\ +\infty & \text{if } i \neq j \text{ are not connected} \end{cases}$$

If the graph G is connected it's easy to verify that d satisfies the proprieties of a metric.

Definition 18. Let i be a vertex of the graph G. Let $r \in \mathbb{N}$.
The ball of center i and radius r, denoted $B(i,r)$, is the subgraph of G induced by the vertices which belong to

$$\{j \in G \,|\, d(j,i) \leq r\}.$$

It's useful to distinguish different kinds of walks: those which never pass twice through the same edge, those which don't touch twice the same vertex...

Definition 19. A walk $W = v_0,\, v_1 \dots v_k$ on the graph G is called

- a *trail*, if $v_s v_{s+1} \neq v_t v_{t+1}$ for any $s \neq t$;

- a *path*, if $v_s \neq v_t$ for any $s \neq t$;

- a *closed walk*, if $v_0 = v_k$;
- a *cycle*, if $v_0 = v_k$ and $v_s \neq v_t$ for any $s \neq t$ with $1 \leq t \leq k-1$.

Definition 20. A graph without cycles is called a *forest.*
A connected graph without cycles is called a *tree.*

On a tree the absence of cycles entails that there exists exactly one path connecting any given pair of vertices. This permits to introduce an order relation on the tree.
Precisely given a tree T and fixed a vertex $\varnothing \in V$, called the *root*, we say that

- the vertices linked by an edge to $\varnothing$ form the 1^{st} *generation*;
- the vertices linked by an edge to a vertex of the 1^{st} generation and which are different from $\varnothing$ compose all together the 2^{nd} *generation*;
  ⋮
- the vertices linked by an edge to a vertex of the k^{th} generation and which does not belong the $(k-1)^{th}$ generation compose all together the $(k+1)^{th}$ *generation*;
  ⋮

We stop at step t, when there are no more vertices remained.
A tree provided with such an order is called a *rooted tree* with t generations.
Note that each vertex in the k^{th} generation with $1 \leq k \leq t-1$ is linked by an edge to exactly one vertex in the $(k-1)^{th}$ generation, called its *father*, and to some (maybe zero) vertices in the $(k+1)^{th}$ generation, called its *offspring.*

Appendix B

In this appendix we'll prove the Azuma-Hoeffding inequality for martingales and show how it can be applied to show that a random quantity concentrates around its mean.

Lemma 41. *Let Z be a real random variable such that $\mathbb{E}[Z] = 0$ and $|Z| \le c$ for some constant $c > 0$. Then for every $\lambda > 0$*

$$\mathbb{E}[e^{\lambda Z}] \le e^{\lambda^2 c^2/2} .$$

Proof. The function $f : x \longmapsto e^{\lambda x}$ is convex. Therefore given the two points $\big(-c, f(-c)\big)$, $\big(c, f(c)\big)$, the straight line $r : x \longmapsto m\,x + q$ which links them stays always above f. That is for every $x \in [-c, c]$

$$f(x) \le m\,x + q$$

with

$$m = \frac{f(c) - f(-c)}{2\,c} \ , \qquad q = \frac{f(c) + f(-c)}{2} \ .$$

Since $-c \le Z \le c$, one may choose $x = Z$ and take the expectation:

$$\mathbb{E}[e^{\lambda Z}] \le m \underbrace{\mathbb{E}[Z]}_{=0} + q = \frac{e^{\lambda c} + e^{-\lambda c}}{2} \ .$$

To conclude set $\lambda\, c =: a$ and use the Taylor series expansions:

$$\frac{e^{a} + e^{-a}}{2} = \sum_{k=0}^{\infty} \frac{1}{k!}\,\frac{a^k + (-a)^k}{2} \le \sum_{k=0}^{\infty} \frac{1}{k!}\,\Big(\frac{a^2}{2}\Big)^k = e^{a^2/2} \ ,$$

where the middle inequality is easily verified term by term. □

Theorem 42 (Azuma-Hoeffding inequality).
Let $M = (M_i)_{i=1,\dots,n}$ be a real martingale on the filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_i)_{i=1,\dots,n}\big)$. Suppose that for every $i = 2, \dots, n$ there exists a real constant $c_i > 0$ such that

$$|M_i - M_{i-1}| \leq c_i \,.$$

Then for every $\epsilon > 0$

$$\mathbb{P}(|M_n - M_1| > \epsilon) \; \leq \; 2 \, \exp\Big(- \frac{\epsilon^2}{2 \sum_{i=1}^n c_i^2} \Big) \,.$$

Proof. It suffices to prove $\mathbb{P}(M_n - M_1 > \epsilon) \; \leq \; \exp\big(- \epsilon^2/(2 \sum_{i=1}^n c_i^2)\big)$, indeed reasoning with the martingale $-M$ one will obtain the same bound for $\mathbb{P}(M_n - M_1 < -\epsilon)$ and then conclude as $P(|M_n - M_1| > \epsilon) \leq \mathbb{P}(M_n - M_1 > \epsilon) + \mathbb{P}(M_n - M_1 < -\epsilon)$.
Let $\lambda > 0$. To estimate $\mathbb{P}(M_n - M_1 > \epsilon)$, first pass to the exponential and then use the Markov inequality:

$$\mathbb{P}(M_n - M_1 > \epsilon) \; = \; \mathbb{P}(e^{\lambda\,(M_n - M_1)} > e^{\lambda\,\epsilon}) \; \leq \; \frac{\mathbb{E}[e^{\lambda\,(M_n - M_1)}]}{e^{\lambda\,\epsilon}} \tag{2.1}$$

Now set $D_i := M_i - M_{i-1} \; \forall \, i = 2, \dots, n$ and write

$$\begin{aligned} \mathbb{E}[e^{\lambda\,(M_i - M_1)}] \; &= \; \mathbb{E}[e^{\lambda\,(M_{i-1} - M_1)}\, e^{\lambda\,D_i}] \; = \; \mathbb{E}\big[\mathbb{E}[e^{\lambda\,(M_{i-1} - M_1)}\, e^{\lambda\,D_i} |\, \mathcal{F}_{i-1}]\big] \\ &= \; \mathbb{E}\big[e^{\lambda\,(M_{i-1} - M_1)}\, \mathbb{E}[e^{\lambda\,D_i} |\, \mathcal{F}_{i-1}]\big] \end{aligned} \tag{2.2}$$

where the last equality is true as $M_{i-1} - M_1$ is $\mathcal{F}_{i-1}$-measurable.
A good upper bound for $\mathbb{E}[e^{\lambda\,D_i} |\, \mathcal{F}_{i-1}]$ is given by the previous lemma. Notice D_i satisfies the hypothesis w.r.t. the conditional expectation $\mathbb{E}[\,\cdot\,|\mathcal{F}_{i-1}]$, indeed:

$$\mathbb{E}[D_i|\mathcal{F}_{i-1}] = \mathbb{E}[M_i - M_{i-1}|\mathcal{F}_{i-1}] = 0 \quad \text{since } M \text{ is a martingale}\,,$$

$$|D_i| = |M_i - M_{i-1}| \leq c_i \quad \text{by the hypotesis}\,.$$

Therefore by the lemma:

$$\mathbb{E}[e^{\lambda\,D_i} |\, \mathcal{F}_{i-1}] \leq e^{\lambda^2\, c_i^2/2} \,.$$

Substitute into the equality (2.2) and proceed by induction:

$$\mathbb{E}[e^{\lambda\,(M_n - M_1)}] \le \mathbb{E}\big[e^{\lambda\,(M_{n-1}-M_1)}\big]\, e^{\lambda^2\, c_n^2/2} \le \cdots \le \mathbb{E}\big[e^{\lambda\,(M_1 - M_1)}\big] \prod_{i=1}^{n} e^{\lambda^2\, c_i^2/2}$$
$$= e^{\lambda^2\, c/2}\,,$$

where $c := \sum_{i=0}^{n} c_i^2$. Now substitute into the inequality (2.1) :

$$\mathbb{P}(M_n - M_1 > \epsilon) \le \frac{\mathbb{E}[e^{\lambda\,(M_n-M_1)}]}{e^{\lambda\,\epsilon}} \le \frac{e^{\lambda^2\, c/2}}{e^{\lambda\,\epsilon}}\,.$$

Remind $\lambda > 0$ is arbitrary. Deriving it's easy to check that the last ratio attains its minimum for $\lambda = \epsilon/c$. Hence choose this value of λ and obtain:

$$\mathbb{P}(M_n - M_1 > \epsilon) \le \frac{e^{(\epsilon/c)^2\, c/2}}{e^{(\epsilon/c)\,\epsilon}} = e^{-\epsilon^2/(2\,c)}\,. \quad \square$$

The Azuma-Hoeffding inequality is a powerful result. It is often used to show that a random variable concentrates around its expected value. We are going to state a corollary that makes this possibility more explicit.
Let $X_1, \ldots, X_n$ be random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values respectively in $\Xi_1, \ldots, \Xi_n$. For brevity denote $\underline{X} = (X_1, \ldots, X_n)$ and $\underline{X}_j = (X_1, \ldots, X_j)$, $\underline{X}^j = (X_j, \ldots, X_n)$.
Let $f : \Xi_1 \times \cdots \times \Xi_n \longrightarrow \mathbb{R}$ be a real function such that $f(\underline{X}) \in \mathbb{L}^1(\mathbb{P})$.

Corollary 43. *Suppose that for every $i = 2, \ldots, n$ there exists a real constant $c_i > 0$ such that*

$$\big|\,\mathbb{E}[f(\underline{X})\,|\,X_1, \ldots, X_{i-1}, X_i] - \mathbb{E}[f(\underline{X})\,|\,X_1, \ldots, X_{i-1}]\big| \le c_i\,.$$

Then for every $\epsilon > 0$

$$\mathbb{P}(|f(\underline{X}) - \mathbb{E}[f(\underline{X})]| > \epsilon) \le 2\,\exp\Big(-\frac{\epsilon^2}{2\sum_{i=1}^{n} c_i^2}\Big)\,.$$

Proof. Set $\mathcal{F}_1 := \{\Omega, \varnothing\}$, $\mathcal{F}_i := \mathcal{F}_{i-1} \cup \sigma(X_1, \ldots, X_i)$. Clearly $(\mathcal{F}_i)_{i=1,\ldots,n}$ is a filtration. Set $M_i := \mathbb{E}[f(\underline{X})|\mathcal{F}_i]$ and check that $(M_i)_{i=1,\ldots,n}$ is a martingale with respect to $(\mathcal{F}_i)_{i=1,\ldots,n}$:

M_i is $\mathcal{F}_i$-measurable and integrable by definition of conditional expectation; $\mathbb{E}[M_i|\mathcal{F}_{i-1}] = M_{i-1}$ by the tower propriety of conditional expectation.
This is called a Doob martingale. Now the result follows by the previous theorem, observing that $M_n = f(\underline{X})$ (as $f(\underline{X})$ is $\mathcal{F}_n$-measurable) and $M_1 = \mathbb{E}[f(\underline{X})]$ (as $\mathcal{F}_1$ is trivial). □

In general it can be difficult to bound the averaged differences, as required by the corollary. That is why we'll show some easier-to-verify conditions that leave the corollary true.
Assume that $X_1, \ldots, X_n$ take discrete values. Fix an index $i = 1, \ldots, n$ and let $c_i > 0$ be a real constant. Consider the following statements:

1. $\big|\,\mathbb{E}[f(\underline{X})|\,\underline{X}_i] - \mathbb{E}[f(\underline{X})|\,\underline{X}_{i-1}]\big| \leq c_i$

2. For every $a_i,\, a_i' \in \Xi_i$

$$\big|\,\mathbb{E}[f(\underline{X})|\,\underline{X}_{i-1},\, X_i = a_i] - \mathbb{E}[f(\underline{X})|\,\underline{X}_{i-1},\, X_i = a_i']\big| \leq c_i$$

3. For every $a_1 \in \Xi_1,\, \ldots, a_i, a_i' \in \Xi_i, \ldots, a_n \in \Xi_n$

$$\big|\, f(a_1, \ldots, a_{i-1},\, a_i,\, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_{i-1},\, a_i',\, a_{i+1}, \ldots, a_n)\big| \leq c_i$$

Proposition 44.

- *If f verifies 3. and the $(X_j)_{j=1,\ldots,n}$ are independent, then f verifies 2.*
- *If f verifies 2., then f verifies 1.*

Proof. Suppose that the $(X_j)_{j=1,\ldots,n}$ are independent and that f verifies the condition 3. Let $a \in \Xi_i$. By the formula of total probability write

$$\begin{aligned}
&\mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1},\, X_i = a\,] \; = \\
&= \sum_{a_{i+1}\ldots a_n} \mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1},\, X_i = a,\, \underline{X}^{i+1} = \underline{a}^{i+1}]\; \mathbb{P}(\underline{X}^{i+1} = \underline{a}^{i+1}|\,\underline{X}_{i-1}, X_i = a) \\
&= \sum_{a_{i+1}\ldots a_n} \mathbb{E}[f(\underline{X}_{i-1},\, a,\, \underline{a}^{i+1})\,|\,\underline{X}_{i-1},\, X_i = a,\, \underline{X}^{i+1} = \underline{a}^{i+1}]\; \mathbb{P}(\underline{X}^{i+1} = \underline{a}^{i+1}|\,\underline{X}_{i-1}, X_i = a) \\
&= \sum_{a_{i+1}\ldots a_n} \mathbb{E}[f(\underline{X}_{i-1},\, a,\, \underline{a}^{i+1})\,|\,\underline{X}_{i-1}]\; \mathbb{P}(\underline{X}^{i+1} = \underline{a}^{i+1})\;,
\end{aligned}$$

where the last equality is true since the $(X_j)_{j=1,\dots,n}$ are independent.
Choose $a = a_i,\, a'_i$ successively and take the difference. Then:

$$\Big|\,\mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1},\, X_i = a_i\,] - \mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1},\, X_i = a'_i\,]\Big| \;\le$$

$$\le \sum_{a_{i+1}\dots a_n} \Big|\,\mathbb{E}[f(\underline{X}_{i-1},\, a_i,\, \underline{a}^{i+1})|\,\underline{X}_{i-1}] - E[f(\underline{X}_{i-1},\, a'_i,\, \underline{a}^{i+1})|\,\underline{X}_{i-1}]\Big|\; \mathbb{P}(\underline{X}^{i+1}\!=\!\underline{a}^{i+1})$$

$$= \sum_{\substack{a_1\dots a_{i-1}\\ a_{i+1}\dots a_n}} \Big|\, f(\underline{a}_{i-1},\, a_i,\, \underline{a}^{i+1}) - f(\underline{a}_{i-1},\, a'_i,\, \underline{a}^{i+1})\Big|\; \mathbb{1}(\underline{X}_{i-1}\!=\!\underline{a}_{i-1})\; \mathbb{P}(\underline{X}^{i+1}\!=\!\underline{a}^{i+1})$$

$$\le \sum_{\substack{a_1\dots a_{i-1}\\ a_{i+1}\dots a_n}} c_i\; \mathbb{1}(\underline{X}_{i-1}\!=\!\underline{a}_{i-1})\; \mathbb{P}(\underline{X}^{i+1}\!=\!\underline{a}^{i+1}) \;=\; c_i\;,$$

where the second last passage is due to the hypothesis 3. Thus it is proven that f verifies the condition 2.

Now suppose that f satisfies the condition 2.
By the formula of total probability

$$\mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1}] = \sum_{a_i} \mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1},\, X_i = a_i]\; \mathbb{P}(X_i = a\,|\underline{X}_{i-1}) =$$

$$= \sum_{a_i,\, a'_i} \mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1},\, X_i = a_i]\; \mathbb{P}(X_i = a\,|\underline{X}_{i-1})\; \mathbb{1}(X_i = a'_i)\;;$$

while developing the conditional expectation w.r.t. X_i

$$\mathbb{E}[f(\underline{X})\,|\,\underline{X}_i] = \sum_{a'_i} \mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1},\, X_i = a'_i]\; \mathbb{1}(X_i = a'_i) =$$

$$= \sum_{a_i,\, a'_i} \mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1},\, X_i = a'_i]\; \mathbb{1}(X_i = a'_i)\; \mathbb{P}(X_i = a_i\,|\underline{X}_{i-1})\;.$$

Therefore taking the difference:

$$\Big|\mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1}] - \mathbb{E}[f(\underline{X})\,|\,\underline{X}_i]\Big| \;\le$$

$$\le \sum_{a_i,\, a'_i} \Big|\mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1},\, X_i = a_i\,] - \mathbb{E}[f(\underline{X})\,|\,\underline{X}_{i-1},\, X_i = a'_i\,]\Big|\; \mathbb{P}(X_i = a\,|\underline{X}_{i-1})\; \mathbb{1}(X_i = a'_i)$$

$$\le \sum_{a_i,\, a'_i} c_i\; \mathbb{P}(X_i = a\,|\underline{X}_{i-1})\; \mathbb{1}(X_i = a'_i) \;=\; c_i\;,$$

where the second last passage is due to the hypothesis 2. Thus it is proven that f satisfies the condition 1. □

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