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INTRINSIC LIPSCHITZ GRAPHS IN HEISENBERG GROUPS AND SUBELLIPTIC PDE'S

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Introduzione

Un gruppo di Carnot è un gruppo di Lie connesso, semplicemente connesso e con algebra di Lie stratificata.

Un esempio non banale di gruppi di Carnot è dato dal gruppo di Heisenberg $\mathbb{H}^n$, l'ambiente in cui si sviluppano i risultati di questa tesi.

Siamo interessati a studiare una nozione di continuità Lipschitz, che dipenda solo dalle proprietà algebriche del gruppo di Heisenberg, per funzioni che hanno il grafico contenuto in $\mathbb{H}^n$.

Analogamente al caso euclideo, dobbiamo, prima di tutto, definire una "buona" decomposizione di $\mathbb{H}^n$ in sottogruppi complementari. Poichè vogliamo apprezzare la struttura algebrica di $\mathbb{H}^n$, consideriamo solo sottogruppi omogenei, ovvero sottogruppi che sono invarianti per dilatazioni del gruppo. Diciamo che $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ è una decomposizione in sottogruppi complementari se ogni punto $q \in \mathbb{H}^n$ si scrive in modo unico come il prodotto di un elemento di $\mathbb{G}_1$ e uno di $\mathbb{G}_2$.

Seguendo [17, 16, 15], studiamo i grafici di funzioni che agiscono tra sottogruppi complementari di $\mathbb{H}^n$. Intuitivamente, $S \subset \mathbb{H}^n$ è un *grafico intrinseco (sinistro)*, in direzione di un sottogruppo omogeneo $\mathbb{G}$, se S interseca ogni laterale sinistro di $\mathbb{G}$ in al più un punto.

Definiamo grafici *intrinsecamente Lipschitz* come grafici intrinseci che intersecano in al più un punto un oggetto costruito *ad hoc*: un *cono intrinseco*. Questo oggetto, definito a partire da una decomposizione adeguata di $\mathbb{H}^n$ in sottogruppi complementari, è invariante per dilatazioni del gruppo.

Dopo un primo capitolo in cui ricordiamo al lettore alcuni risultati riguardanti i gruppi di Carnot e le loro principali proprietà, nel secondo capitolo dell'elaborato presentiamo alcune caratterizzazioni e proprietà dei grafici intrinsecamente Lipschitz. Dopo aver studiato in generale questi grafici, ci restringiamo al caso di grafici di codimensione 1. Viene studiato ([17]) un teorema di estensione del tipo McShane-Whitney e si dimostra che i grafici intrinsecamente Lipschitz sono bordi di insiemi che hanno localmente $\mathbb{H}$-perimetro finito. Successivamente, introducendo una nozione di differenziabilità intrinseca, ancora una volta tratta da [17], viene presentata la prova di un teorema di tipo Rademacher per funzioni intrinsecamente Lipschitz.

L'ultima parte della tesi è dedicata alle applicazioni della teoria studiata nei primi due capitoli. Definiamo i domini con bordo intrinsecamente Lipschitz come insiemi aperti, connessi e limitati il cui bordo è localmente il grafico di una funzione intrinsecamente Lipschitz. Viene dimostrato che questi sono domini (ε, δ). Da tale risultato segue che i domini intrinsecamente Lipschitz sono anche domini di Boman. Siamo così autorizzati ad estendere alcuni risultati dell'Analisi Funzionale su questo tipo di insiemi, come ad esempio la Disuguaglianza di Poincaré e un Teorema di estensione per funzioni di Sobolev che prendono valori in un dominio intrinsecamente Lipschitz.

Un'ultima applicazione riguarda la teoria degli operatori subellittici. Se $\mathcal{L}$ è un sub-Laplaciano in $\mathbb{H}^n$ e $\Omega \subset \mathbb{H}^n$ è un dominio intrinsecamente Lipschitz, proviamo, utilizzando il *test di regolarità di Wiener*, che il problema al contorno

$$\begin{cases} \mathcal{L}u = 0, & \text{in } \Omega \\ u_{|\partial\Omega} = \varphi, \end{cases}$$

ammete una unica soluzione.

Introduction

A Carnot group $\mathbb{G}$ is a connected, simply connected Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = V_1 \oplus ... \oplus V_r.$$

In other words $\mathfrak{g}$ decomposes in a direct sum of r vector spaces. The first layer V_1 (called *horizontal layer*) generates the entire algebra $\mathfrak{g}$ and it can be identified with a linear subspace of the tangent space at the origin e of $\mathbb{G}$. Moreover, by left translation, V_1 determines a subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$. $H\mathbb{G}$ is called the *horizontal bundle*.

Through the exponential map, each Carnot group can be identified with $\mathbb{R}^n$ endowed with a (non commutative) group law and with a family of (non isotropic) dilations δ_t, $t > 0$.

A non trivial example of Carnot group is the Heisenberg group $\mathbb{H}^n$, the setting of this thesis. We denote (z,t) a point in $\mathbb{C}^n \times \mathbb{R}$. The *n-th Heisenberg group* $\mathbb{H}^n$ is given by $\mathbb{C}^n \times \mathbb{R}$ endowed with the following group law:

$$(z,t) \cdot (z',t') = \left(z + z', t + t' + \frac{1}{2}\Im(z \cdot \bar{z}') \right),$$

where $z \cdot \bar{z}'$ denotes the usual Hermitian product of $\mathbb{C}^n$. Dilations in $\mathbb{H}^n$ are defined as follows, for $\lambda \in \mathbb{R}^+$,

$$\begin{aligned} \delta_\lambda : \mathbb{C}^n \times \mathbb{R} &\longrightarrow \mathbb{C}^n \times \mathbb{R} \\ (z,t) &\longmapsto \left(\lambda z, \lambda^2 t\right). \end{aligned}$$

For an exhaustive introduction to Carnot groups, we refer to [3].

What about metric structures over a Carnot group $\mathbb{G}$? We can introduce over $\mathbb{G}$ a left invariant metric, the Carnot-Carathéodory metric: nothing else than the sub-Riemannian metric associated with $H\mathbb{G}$. Throughout this thesis, we will use this metric or other metrics which are topologically equivalent to the Carnot-Carathéodory one. The reader interested in general spaces endowed with a Carnot-Carathéodory metric is referred to [21] for further informations.

Recently, efforts were made at studying geometric measure theory on metric spaces and, in particular, Carnot groups. For instance, in a large amount of works, the main focus is on rectifiable sets, finite perimeters sets and their properties.

Let us think for a moment to the Euclidean case. Rectifiable sets are generalizations of regular and Lipschitz submanifolds. Going one step further, submanifolds are locally graphs.

When we translate these points into our setting, the Heisenberg group (Carnot groups, more generally), we need to pay attention to the algebraic structure of the group and to the fact that Carnot groups, in general, can not be viewed as Cartesian products of subgroups.

Hence, we define a good decomposition of $\mathbb{H}^n$ in complementary subgroups. Since we would like to stand out the algebraic structure of $\mathbb{H}^n$, we consider only homogeneous subgroups, i.e. subgroups which are invariant under group dilations. We say that $\mathbb{G}_1$ and $\mathbb{G}_2$ are *complementary subgroups* of $\mathbb{H}^n$ if they are homogeneous and if any element $p \in \mathbb{H}^n$ can be uniquely written as product of $p_{\mathbb{G}_1} \in \mathbb{G}_1$ and $p_{\mathbb{G}_2} \in \mathbb{G}_2$.

Thank to this decomposition, following [16, 15, 17], we can define graphs of functions acting between complementary subgroups of $\mathbb{H}^n$. This notion will be an *intrinsic* notion, i.e. a notion depending only on the group structure.

Definition. *Let* $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$. *We say that* $S \subset \mathbb{H}^n$ *is a* (left) intrinsic graph *over* $\mathbb{G}_1$ *along* $\mathbb{G}_2$ *if there exists a function* $f : \mathcal{E} \subset \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ *such that*

$$S = \{\xi \cdot f(\xi) \,|\, \xi \in \mathcal{E}\} .$$

In this case we write $S = \mathrm{graph}(f)$.

This definition is intrinsic in the following sense: S keeps being a graph after left translations and dilations of the group.

The purpose of this thesis is to study a special type of graphs: graphs of intrinsic Lipschitz continuous functions. This new notion, originally suggested in [15] and developed in [17], extends the notion of Lipschitz continuity but it depends only on the structure of $\mathfrak{h}^n$, the Lie algebra of $\mathbb{H}^n$.

Let us recall here the geometric approach to this new concept. First we need to define *intrinsic cones.*

Definition. *Let* $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ *and let* $q \in \mathbb{H}^n$ *and* $\alpha \in \mathbb{R}^+$ *be fixed. We call* intrinsic cone *of base* $\mathbb{G}_1$*, axis* $\mathbb{G}_2$*, vertex* q *and opening* α

$$C_{\mathbb{G}_1,\mathbb{G}_2}(q,\alpha) \,=\, q \cdot C_{\mathbb{G}_1,\mathbb{G}_2}(e,\alpha),$$

where

$$C_{\mathbb{G}_1,\mathbb{G}_2}(e,\alpha) \,:=\, \{p \in \mathbb{H}^n \,|\, \|p_{\mathbb{G}_1}\| \leq \alpha \|p_{\mathbb{G}_2}\|\}\,.$$

Once again, these geometric objects are invariant under group dilations and, analogously to the Euclidean case, they let us give the following

Definition. *We say that* $f : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ *is* intrinsic Lipschitz continuous *if, at each point* $q \in \text{graph}(f)$*, there exists an intrinsic cone with vertex* q *and fixed opening, intersecting* $\text{graph}(f)$ *only in* q.

Staying close to [17], we present some analytic and geometric characterizations and some general properties. In a second step, we restrict ourselves to 1-codimensional graphs: we decompose $\mathbb{H}^n$ in a vertical subgroup $\mathbb{W}$, that is a subgroup which contains the center $\mathbb{T}$ of $\mathbb{H}^n$, and a horizontal 1-dimensional subgroup $\mathbb{V}$, that is contained in $\{(z,t) \in \mathbb{H}^n \,|\, t = 0\}$.

We propose the proof of a McShane-Whitney type extension Theorem and we show that intrinsic Lipschitz graphs are boundaries of sets of locally finite $\mathbb{H}$-perimeter. In our presentation, we give a look also to an intrinsic notion of differentiability:

Definition. *We say that* $f : \mathbb{W} \longrightarrow \mathbb{V}$ *is* intrinsic differentiable *at* $g \in \mathbb{W}$ *if there is a homogeneous subgroup* $\mathbb{T}_g$ *of* $\mathbb{G}$ *such that, if* $p = g \cdot f(g) \in \text{graph}(f)$*,* $p \cdot \mathbb{T}_g$ *is the* tangent plane *to* $\text{graph}(f)$ *in* p.

With this notion, the authors in [17] proves a Rademacher type theorem for intrinsic Lipschitz functions.

Theorem. *Let* $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$, *with* $\dim\mathbb{V} = 1$, $\mathcal{E} \subset \mathbb{W}$ *be an open set and* $f : \mathcal{E} \longrightarrow \mathbb{V}$ *be an intrinsic Lipschitz continuous function. Then* f *is intrinsic differentiable* $(\mathcal{L}^{2n} \llcorner \mathbb{W})$*-a.e. in* $\mathcal{E}$.

The second part of the thesis is centered on domains in $\mathbb{H}^n$ whose boundaries are locally graphs of intrinsic Lipschitz functions. We call such domains *intrinsic Lipschitz domains.*

A more analytic characterization for these domains is given in [40]: we write, locally, the boundary of an intrinsic Lipschitz domain Ω as the zero set of a metric Lipschitz function acting from $\mathbb{H}^n$ into $\mathbb{R}$, endowed with the Euclidean metric. With this characterization, we show here that intrinsic Lipschitz domains are (ε, δ)-domains. This fact opens the way for other properties.

For example, since an (ε, δ)-domain is a Boman domain, we have automatically a Poincaré inequality. Moreover, once we defined Sobolev spaces $W^{k,p}_{\mathbb{H}}(\mathbb{G})$ over a Carnot group $\mathbb{G}$, we can adapt a Theorem, first proved in [35], which provides the existence of an extension operator for Sobolev functions over an intrinsic Lipschitz domain $\Omega \subset \mathbb{H}^n$:

Theorem. *Let* $\Omega \subset \mathbb{H}^n$ *be an intrinsic Lipschitz domain. If* $1 \leq p < \infty$, *then there exists an extension operator* Λ *on* Ω *such that*

$$\|\Lambda f\|_{W^{k,p}_{\mathbb{H}}(\mathbb{H}^n)} \leq C\|f\|_{W^{k,p}_{\mathbb{H}}(\Omega)},$$

for all $f \in W^{k,p}_{\mathbb{H}}(\Omega)$ *and where* C *is a positive constant not depending on* f.

A final application concerns Subelliptic PDE's. Let $\mathcal{L}$ be a sub-Laplacian of $\mathbb{H}^n$ and let $\Omega \subset \mathbb{H}^n$ be an intrinsic Lipschitz domain. Consider the boundary value problem

$$\begin{cases} \mathcal{L}u = 0, & \text{in } \Omega \\ u_{|\partial\Omega} = \varphi, \end{cases} \tag{1}$$

for $\varphi : \partial\Omega \longrightarrow \mathbb{R}$ a continuous function. The question is: has problem (1) a (unique) solution? Using the *Wiener's regularity test* we can give an affirmative answer. In concrete terms, each point of the boundary is a *regular point*, this means that the solution of (1) takes value on $\partial\Omega$.

Let us now briefly summarize the contents of the whole thesis.

Chapter 1 contains a brief introduction about Carnot groups and, more precisely, on some general aspects regarding their nature of stratified Lie groups, on the Carnot-Carathéodory metric and on Heisenberg groups and their homogeneous subgroups.

Chapter 2 is entirely dedicated to intrinsic Lipschitz graphs. In the first part of the Chapter we study general intrinsic graphs, then we concentrate to 1-codimensional graphs.

Chapter 3 presents some applications to the theory introduced in Chapter 2. After a self contained introduction to some particular domains and their geometric aspects, we define intrinsic Lipschitz domains and we study a pair of applications from Functional Analysis and from Subelliptic PDE's theory.

Three Appendices close the thesis with the aim of helping the reading.

Chapter 1

Main Notions

This first chapter is a brief introduction to some fundamental notions and results that we need through our thesis. We start with basic properties of Lie groups and Lie algebras, with a focus on Carnot groups, with the purpose of studying Carnot-Carathéodory metric and the notion of P-differentiability. In the second part we concentrate on Heisenberg groups, on the structure of their subgroups and, finally, on the notions of $BV_{\mathbb{H}}$-functions and $\mathbb{H}$-Caccioppoli sets.

1.1 Lie Groups and Lie Algebras

We recall some notations and results about Lie groups and their Lie algebras. Let us start with the definition of Lie group (for a comprehensive treatment and for references to the extensive literature on the subject one may refer to the book [39]).

Definition 1.1.1. *A* Lie group *is a differentiable manifold* $\mathbb{G}$ *endowed with a group structure, which is differentiable in the sense that the product* $(x, y) \mapsto x \cdot y$ *and the inverse* $x \mapsto x^{-1}$ *are smooth maps.*

Definition 1.1.2. *A* Lie subgroup *of* $\mathbb{G}$ *is an embedded submanifold of* $\mathbb{G}$ *which is also a subgroup of* $\mathbb{G}$.

Definition 1.1.3. *Let* $\mathbb{G}$ *and* $\mathbb{H}$ *be Lie groups. A* Lie homomorphism *from* $\mathbb{G}$ *to* $\mathbb{H}$ *is a* C^k*-map*

$$\varphi : \mathbb{G} \longrightarrow \mathbb{H}$$

that is also a group homomorphism.

Remark 1.1.4. A map $\varphi : \mathbb{G} \to \mathbb{H}$ is called a *Lie isomorphism* if it is a Lie homomorphism and also its inverse is a Lie homomorphism.

To understand the objects we are working with, let us treat two simple examples of Lie groups on Euclidean spaces: we will consider $\mathbb{R}^n$ endowed with an algebraic group structure such that $(\mathbb{R}^n, *)$ is a Lie group.

Example 1.1.1. Consider $\mathbb{R}^n$ with the usual operation of addiction. $(\mathbb{R}^n, +)$ is a Lie group, indeed the maps

$$\begin{aligned}(x\,,\,y) &\longmapsto x + y\\ x &\longmapsto x^{-1}\end{aligned}$$

are polynomials.

In the next example we introduce Heisenberg groups, which will be the setting of our studies. We postpone the general case and main properties until Section 1.4.

Example 1.1.2. We consider $\mathbb{R}^3$ identified to $\mathbb{C} \times \mathbb{R}$ and use the notation

$$x = (x_1, x_2, t) = (z, t) \in \mathbb{C} \times \mathbb{R},$$

with $z = x_1 + ix_2$. We give to $\mathbb{C} \times \mathbb{R}$ a Lie group structure with group law:

$$(z\,,\,t) \cdot (w\,,\,s) = \left(z + w\,,\,t + s + \frac{1}{2}\Im(z \cdot \bar{w})\right).$$

It is not difficult to check that the identity is 0 and that the inverse is given by $(z, t)^{-1} = (-z, -t)$. We call the Lie Group $\mathbb{H}^1 = (\mathbb{R}^3, \cdot)$ the *first Heisenberg Group.*

Definition 1.1.5. *Fixed $g \in \mathbb{G}$, we denote by*

$$\begin{aligned}\tau_g : \mathbb{G} &\longrightarrow \mathbb{G}\\ x &\longmapsto g \cdot x\end{aligned}$$

the left translation *by g on $\mathbb{G}$.*

By definition of Lie group, for each $g \in \mathbb{G}$, τ_g is a diffeomorphism of the group onto itself, that maps the identity e to g. We can interprete this fact to mean that a Lie group is a homogeneous space. This says (roughly speaking) that to study the local structure of a Lie group, it is sufficient to examine a neighbourhood of the identity. One must, therefore, notice that the tangent space of $\mathbb{G}$ at e, $T_e\mathbb{G}$, plays a key role: we will discover that, with a suitable operation, $T_e\mathbb{G}$ has a richer structure.

We start by giving the definition of smooth left invariant vector fields. For a complete understanding, we recommend to compare next results and definitions to Appendix A, in which the reader will be reminded of some basic theory about differential geometry.

Definition 1.1.6. *A smooth vector field $X \in \Gamma(T\mathbb{G})$ is said to be* left invariant *if, for every $g \in \mathbb{G}$,*

$$d\tau_g X = X \circ \tau_g, \tag{1.1}$$

where $d\tau_g : T\mathbb{G} \to T\mathbb{G}$ is the derivative map of the left translation τ_g.

Since τ_g is a Lie isomorphism (a diffeomorphism more generally), notice that $d\tau_g X$ is well defined as vector field. The condition (1.1) is equivalent to the following one:

$$(d_x\tau_g\,(X(x)) = X(g \cdot x),$$

for every $g, x \in \mathbb{G}$. If we apply the previous formula at the identity of $\mathbb{G}$, we obtain

$$d_e\tau_g\,(X(e)) = X(g),$$

for every $g \in \mathbb{G}$. Moreover, the condition of left invariance can be rewritten in this way

$$X\,(f \circ \tau_g)\,(x) = Xf \circ \tau_g(x),$$

for all $x, g \in \mathbb{G}$ and for all smooth function f on $\mathbb{G}$.

Definition 1.1.7. *Let $\mathbb{G}$ be a Lie group. We call the* Lie algebra *of $\mathbb{G}$, and write $\mathfrak{g}$, the set of all smooth left invariant vector fields on $\mathbb{G}$.*

Proposition 1.1.1. *$\mathfrak{g}$ is a* Lie Algebra[1] *under the Lie Bracket product define as*

$$[X, Y] = XYf - YXf,$$

for all $X, Y \in \mathfrak{g}$, and for all $f \in C^k(\mathbb{G})$.

Remark 1.1.8. The dimension of $\mathfrak{g}$ as vector space equals that of $\mathbb{G}$. Indeed, $\mathfrak{g}$ is canonically isomorphic to $T_e\mathbb{G}$ via the identification of X and $X(e)$.

Example 1.1.3. Return to the first Heisenberg group $\mathbb{H}^1$. It is not difficult to show that the vector fields

$$\begin{aligned} X_1 &= \partial_{x_1} + \frac{1}{2} x_2 \, \partial_t \\ X_2 &= \partial_{x_2} - \frac{1}{2} x_1 \, \partial_t \end{aligned}$$

are left invariant with respect to the group law.

1.1.1 The Exponential Map

Given a Lie group $\mathbb{G}$, we defined its Lie algebra. The question now is: is there a canonical way to associate each element of $\mathfrak{g}$ to a point of $\mathbb{G}$? The answer is given in the following paragraph. We start with a proposition (for more details we refer the reader to [3]):

Proposition 1.1.2. *The left invariant vector fields on a Lie group $\mathbb{G}$ are complete.*[2]

Given $g \in \mathbb{G}$ and $X \in \mathfrak{g}$, let us consider the solution of the following Cauchy problem:

$$\begin{cases} \dot{\gamma}_g(t) = X(\gamma_g(t)) \\ \gamma_g(0) = g. \end{cases}$$

[1] We recall that a vector space $\mathfrak{g}$ is a *Lie algebra* if there is a bilinear and antisymmetric map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the Jacobi's identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$$

for all $X, Y, Z \in \mathfrak{g}$.

[2] See Definition A.0.27 on page 109.

Remark 1.1.9. Notice that, by Proposition 1.1.2, the integral curve γ_g is defined for each $t \in \mathbb{R}$.

In the following, we set

$$\exp_X(t) \;:=\; \gamma_e(t).$$

Thanks to this notation, we can construct a canonical map which, with each vector field in $\mathfrak{g}$, associates a point of $\mathbb{G}$. We consider once again the integral curve of a fixed left invariant vector field X, we stop at time $t = 1$, that point will be the element of $\mathbb{G}$ associated with X:

Definition 1.1.10. *Let $\mathbb{G}$ be a Lie group with Lie algebra $\mathfrak{g}$, we set*

$$\begin{aligned} \exp \,:\, \mathfrak{g} &\longrightarrow \mathbb{G} \\ X &\longmapsto \exp(X) := \exp_X(1). \end{aligned}$$

This map is called exponential map related to the Lie group $\mathbb{G}$.

In the following proposition, we summarize some of the main properties of the exponential map and of integral curves more generally.

Proposition 1.1.3. *Let $\mathbb{G}$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Then*

(i) *The exponential map is a smooth map;*

(ii) *For every $X \in \mathfrak{g}$ and for any $t, s \in \mathbb{R}$, $\exp_X(t+s) \;=\; \exp_X(t)\exp_X(s)$;*

(iii) *The derivative map of the exponential map $d\exp : T_0\mathfrak{g} \to T_e\mathbb{G}$ is the identity map, under the canonical identification of both $T_0\mathfrak{g}$ and $T_e\mathbb{G}$ with $\mathfrak{g}$;*

(iv) *The exponential map is a local diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in $\mathbb{G}$.*

Proposition 1.1.4. *If $\mathbb{G}$ is a nilpotent,[3] simply connected Lie group, then the exponential map is a global diffeomorphism of $\mathfrak{g}$ onto $\mathbb{G}$. Moreover, if $\mathbb{H}$ is a Lie subgroup of $\mathbb{G}$, and $\mathfrak{h}$ is its Lie algebra, then $\mathbb{H} = \exp\mathfrak{h}$.*

[3] A Lie group is *nilpotent of step k* if its Lie algebra is nilpotent of step k, that is, defined the *descending central serie* of $\mathfrak{g}$, $\mathfrak{g}^{(1)} = \mathfrak{g}$ and $\mathfrak{g}^{(k+1)} = \big[\mathfrak{g}^{(k)}, \mathfrak{g}\big]$, for $k > 1$, there exists $r \in \mathbb{N}$ such that $\mathfrak{g}^{(r+1)} = 0$.

Consider now two vector fields $X, Y \in \mathfrak{g}$, we aim to reconstruct the group law of the Lie group associated with $\mathfrak{g}$; we define $C(X,Y) \in \mathfrak{g}$ setting

$$\exp(C(X,Y)) \;=\; \exp(X) \cdot \exp(Y).$$

It is possible to compute explicitly $C(X,Y)$. We start with some notations: let $\alpha = (\alpha_1, ..., \alpha_l)$ be a multiindex of non negative integers, we define

$$|\alpha| := \alpha_1 + ... + \alpha_l$$
$$\alpha! := \alpha_1! \cdot ... \cdot \alpha_l!,$$

we say that l is the *length of the multiindex* α. Let $\beta = (\beta_1, ..., \beta_l)$ another multiindex, with the same legth as α, such that $\alpha_l + \beta_l \geq 1$. We set

$$C_{\alpha,\beta}(X,Y) \;:=\; \begin{cases} (adX)^{\alpha_1}(adY)^{\beta_1} \cdot ... \cdot (adX)^{\alpha_l}(adY)^{\beta_l - 1}Y, & \text{if } \beta_l > 0 \\ (adX)^{\alpha_1}(adY)^{\beta_1} \cdot ... \cdot (adX)^{\alpha_l - 1}X, & \text{if } \beta_l = 0, \end{cases}$$

where $(adX)(Y) := [X,Y]$. Then the *Baker-Campbell-Hausdorff formula* states that

$$C(X,Y) \;:=\; \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} \sum_{\substack{\alpha,\beta \\ \alpha_l + \beta_l \geq 1}} \frac{1}{\alpha!\beta!(\alpha+\beta)} C_{\alpha,\beta}(X,Y), \tag{1.2}$$

whenever the serie at the right hand side makes sense. Moreover, it is clear that (1.2) holds in nilpotent Lie groups.

1.1.2 Carnot Groups

In this subsection we will approach to the setting of our studies. As already mentioned, we are interested in Heisenberg groups, which forms a particular family of Carnot groups. Therefore, we need a little background about them (for more details we refer, once more, the reader to [3]). We start with a definition:

Definition 1.1.11. *A Lie algebra* $\mathfrak{g}$ *is called* stratified *if it admits a* stratification*, i.e. there exists* $V_1, ..., V_r \subset \mathfrak{g}$ *subspaces such that*

$$\mathfrak{g} \;=\; V_1 \oplus ... \oplus V_r,$$

where

$$V_j = [V_1, V_{j-1}], \text{ for } j = 2, ..., r$$
$$[V_1, V_r] = \{0\}.$$

Remark 1.1.12. It is clear that V_r is contained in the center of $\mathfrak{g}$. We point out also that V_1 generates the whole Lie algebra. Because of its major role, we will call it *horizontal layer.*

Definition 1.1.13. *A group* $\mathbb{G}$ *is called* stratified *if its Lie algebra* $\mathfrak{g}$ *admits a stratification. Moreover, if the dimension of* $\mathbb{G}$ *is finite, then it is nilpotent of step* r*, exactly the number of subspaces in the stratification of* $\mathfrak{g}$.

From the definition of stratified Lie algebra, we can construct on $\mathfrak{g}$ a one parameter group of Lie homomorphisms, called *dilations* and denoted by $\{\delta_\lambda\}$. We fix $\lambda \geq 0$ and define, for $X \in V_j$:

$$\delta_\lambda X = \lambda^j X,$$

and then we extend this map over the entire $\mathfrak{g}$. Moreover, if $\lambda < 0$, we set

$$\delta_\lambda X = -\delta_{|\lambda|} X.$$

Proposition 1.1.5. *The following properties hold*

(i) $\delta_{\lambda\mu} = \delta_\lambda \circ \delta_\mu$;

(ii) $\delta_\lambda([X, Y]) = [\delta_\lambda X, \delta_\lambda Y]$;

(iii) $\delta_\lambda(C[X, Y]) = C(\delta_\lambda X, \delta_\lambda Y)$,

for any λ, μ *and for any* $X, Y \in \mathfrak{g}$.

Definition 1.1.14. *A* Carnot group $\mathbb{G}$ *is a finite dimensional, connected, simply connected Lie group, whose Lie algebra admits a stratification. If* r *is the step of the stratification, we say that* $\mathbb{G}$ *is of* step r.

Remark 1.1.15. We should stress that a Carnot group can admit more than one stratification. For example, consider again the first Heisenberg group $\mathbb{H}^1$. Its Lie algebra $\mathfrak{h}$ admits the following stratifications:

$$\begin{aligned}&\mathrm{span}\{X_1, X_2\} \oplus \mathrm{span}\{[X_1, X_2]\}\\&\mathrm{span}\{X_1 - 3[X_1, X_2], X_2\} \oplus \mathrm{span}\{[X_1, X_2]\}\\&\mathrm{span}\{X_1 + X_2, 3X_1 + [X_1, X_2]\} \oplus \mathrm{span}\{[X_1, X_2]\}.\end{aligned}$$

Definition 1.1.16. *Let $\mathbb{G}$ be a Carnot group with Lie algebra $\mathfrak{g}$. Let $\mathcal{V} = (V_1, ..., V_r)$ be a fixed stratification of $\mathfrak{g}$. We say that a basis $\mathcal{B}$ of $\mathfrak{g}$ is* adapted to $\mathcal{V}$ *if*

$$\mathcal{B} = \left(E_1^{(1)}, ..., E_{m_1}^{(1)}; ....; E_1^{(r)}, ..., E_{m_r}^{(r)}\right),$$

where, for $i = 1, ..., r$, we have $m_i := \dim(V_i)$ and $\left(E_1^{(i)}, ..., E_{m_i}^{(i)}\right)$ is a basis for V_i.

Notation 1.1.1. We say that $\mathbb{G}$ has m generators, where $m := \dim(V_1)$.

In Remark 1.1.15, we saw that a Lie algebra of a Carnot group could admit more than one stratification. In the following proposition we point out that the main algebraic aspects of a Carnot group do not depend on the choice of the stratification:

Proposition 1.1.6. *Let $\mathbb{G}$ a Carnot group and $\mathfrak{g}$ its Lie algebra. Let $(V_1, ..., V_r)$ and $(W_1, ..., W_r)$ be two stratifications of $\mathfrak{g}$. Then $r = s$ and $\dim V_i = \dim W_i$ for every $i = 1, ..., r$. Moreover, the real number*

$$Q := \sum_{i=1}^{r} i \cdot \dim V_i$$

depends only on the stratified nature of $\mathbb{G}$ and not on the particular stratification. Q is called homogeneous dimension of $\mathbb{G}$.

We conclude the section introducing on Carnot groups the so-called *exponential coordinates*. Let $(X_1, ..., X_n)$ a basis for the Lie algebra of $\mathbb{G}$, $\mathfrak{g}$. As usual, for general manifolds, in particular for Lie groups, we can write uniquely two vector fields in coordinates, setting $X = \sum_{i=1}^{n} x_i X_i$ and $Y = \sum_{i=1}^{n} y_i X_i$.[4] This fact permits us to give the following

[4]The reader should keep in mind that Carnot Groups are connected, simply connected and nilpotent. Then the exponential map, being a global diffeomorphism (Proposition 1.1.4), provides a global chart for the manifold.

Definition 1.1.17. *A system of* exponential coordinates *associated with* $X_1, ..., X_n$ *is the map*

$$\begin{aligned} \Psi \,:\, \mathbb{R}^n &\longrightarrow \mathbb{G} \\ (x_1, ..., x_n) &\mapsto \exp\left(\sum_{i=1}^n x_i X_i\right). \end{aligned} \tag{1.3}$$

We endow $\mathbb{R}^n$ with a group law, so that Ψ is a group isomorphism, that means $x \cdot y = z$ if and only if, using (1.1.1),

$$\sum_{i=1}^n z_i X_i = C\left(\sum_{i=1}^n x_i X_i, \sum_{i=i}^n y_i X_i\right).$$

With this group law, $\mathbb{R}^n$ is a Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$. Now, $\mathbb{G}$ and $\mathbb{R}^n$ are both nilpotent, connected and simply connected, so, by Proposition 1.1.4, Ψ is also a diffeomorphism. From now on, we identify abstract Carnot groups with Carnot groups on $\mathbb{R}^n$. We will refer to coordinates (1.3) as *graded exponential coordinates*.

As reminded before, the exponential map is a global diffeomorphism, then also its inverse in well defined. This allows us to introduce a one parameter group of automorphisms on $\mathbb{G}$. Using for simplicity the same notation of algebras case, we define *dilations* on $\mathbb{G}$, and write $\{\delta_\lambda\}$, as follows

$$\delta_\lambda(x) := \exp\left(\delta_\lambda(\exp^{-1}(x))\right),$$

for every $x \in \mathbb{G}$. With the same notation as in 1.1.16, if i is an index such that

$$m_1 + ... + m_{d_{i-1}} < i \leq m_1 + ... + m_{d_i},$$

for some $1 \leq d_i < k$, the coordinate x_i of $x = (x_1, ..., x_n) \in \mathbb{G}$ is said to have degree d_i. With this definition, group dilations $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ can be written as

$$\delta_\lambda(x) = \left(\lambda^{d_1} x_1, \lambda^{d_2} x_2, ..., \lambda^{d_n} x_n\right).$$

Using Proposition 1.1.5, one can prove the following properties:

(i) $\delta_{\lambda\mu} = \delta_\lambda \cdot \delta_\mu$;

(ii) $\delta_\lambda(x\,y) = \delta_\lambda(x) \cdot \delta_\lambda(y)$.

Using the notions introduced in Section 1.1, since the exponential map is a diffeomorphism from $\mathfrak{g}$ and $\mathbb{G}$, it follows, for each $x, y \in \mathbb{G}$,

$$x \cdot y \;=\; \exp\left(C(X,Y)\right) := P(x,y),$$

where X and $Y \in \mathfrak{g}$ are such that $\exp(X) = x$ and $\exp(Y) = y$. From this formula, one can prove some facts about group law:

Proposition 1.1.7. *There exists a polynomial vector function*

$$Q \,:\, \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}^n = \mathbb{R}^{m_1} \oplus ... \oplus \mathbb{R}^{m_r},$$

where $Q(x,y) = (Q_1(x,y), ..., Q_r(x,y))$, *and* $Q_i(x,y) = \left(Q_1^{(i)}(x,y), ..., Q_{m_i}^{(i)}(x,y)\right)$, *for all* $i = 1, ..., r$, *such that*

$$x \cdot y \;=\; x + y + Q(x,y).$$

Lemma 1.1.8. *The following properties hold:*

(i) *for all* $x, y \in \mathbb{R}^n$ $\lambda > 0$, $P(\delta_\lambda(x), \delta_\lambda(y)) = \delta(P(x,y))$*;*

(ii) *for all* $x \in \mathbb{R}^n$, $P(x,0) = P(0,x)$*;*

(iii) *for all* $x, y \in \mathbb{G}$, $Q^{(j)}(x,y) = 0$, *for* $j = 1, ..., m_1$, *and* $Q^{(j)}(x,0) = Q^{(j)}(0,x) = Q^{(j)}(x,x) = Q^{(j)}(x,x^{-1}) = 0$, *for* $j \geq m_1 + 1$.

Let us consider $\{X_1, ..., X_n\}$, a basis of $\mathfrak{g}$ as a vector space. We can write, for $j = 1, ..., n$,

$$X_j(x) \;=\; \sum_{i=1}^{n} a_{ij}(x) \partial_i,$$

with $a_{ij} \in C^\infty(\mathbb{R}^n)$. We assume also that $X_j(0) = \partial_j$, for each $j = 1, ..., n$. Then

$$a_{ij}(\delta_\lambda(x)) \;=\; \lambda^{d_i - d_j} a_{ij}(x), \tag{1.4}$$

for $\lambda > 0$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Indeed, let $\gamma :]-\delta, \delta[\to \mathbb{R}^n$ be a integral curve of ∂_j (γ is a regular curve such that $\gamma(0) = 0$ and $\dot{\gamma}(0) = \partial_j$), and let $f \in C^1(\mathbb{R}^n)$. We compute the derivative of f along X_j:

$$X_j f(x) \;=\; X_j(f \circ \tau_x)(e) \;=\; \lim_{t \to 0} \frac{f\left(P(x, \gamma(t))\right) - f\left(P(x, e)\right)}{t},$$

the first equality holds, since X_j is a left invariant vector field, the second comes from the fact that γ is an integral curve of ∂_j. Now,

$$\lim_{t\to 0} \frac{f\left(P(x,\gamma(t))\right) - f\left(P(x,e)\right)}{t} = \frac{\partial f}{\partial x}(x)\frac{\partial P}{\partial y}(x,0)\dot{\gamma}(0) = \frac{\partial f}{\partial x}(x)\frac{\partial P}{\partial y_j}(x,0).$$

Hence, $X_j f(x) = \frac{\partial f}{\partial x}(x)\frac{\partial P}{\partial y_j}(x,0)$, which is the same as

$$X_j(x) = \sum_{\partial y_j}^{n} \frac{\partial P_i}{\partial y_j}(x,0)\partial_i. \tag{1.5}$$

To conclude, if $\lambda > 0$, applying (i) of Lemma 1.1.8,

$$\begin{aligned} a_{ij}(\delta_\lambda(x)) &= \frac{\partial P_i}{\partial y_j}(\delta_\lambda(x),0) = \frac{\partial}{\partial y_j}(\delta_\lambda(P_i(x,0))) \\ &= \lambda^{d_i-d_j}\frac{\partial P_i}{\partial y_j}(x,0) = \lambda^{d_i-d_j}a_{ij}(x). \end{aligned}$$

1.2 Carnot-Carathéodory Metric

The purpose of this section is to introduce a metric on Carnot groups. We start by giving to $\mathbb{R}^n$ (supposed to not be provided with any particular structure) the so-called *Carnot-Carathéodory metric*, induced by a family of vector fields which satisfies certain conditions. After doing that, in Subsection 1.2.1, we will discover that, because of their peculiarities, Carnot groups can be naturally equipped with a Carnot-Carathéodory metric.

Let us consider a family of locally Lipschitz continuous vector fields on an open set $\Omega \subseteq \mathbb{R}^n$

$$X_j(x) = \sum_{i=1}^{n} a_{ij}(x)\partial_i, \quad j = 1,...,m.$$

As usual, we call *horizontal fiber at the point* x, and write $H_x\mathbb{R}^n$, the subspace of $T_x\mathbb{R}^n$ generated by $X_1(x),...,X_m(x)$. $H\mathbb{R}^N$ will be the *horizontal subbundle* of $T\mathbb{R}^n$.

Notation 1.2.1. We denote by

$$\mathcal{A} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

the matrix whose columns are the coefficients of the vector fields $\mathbb{X} := (X_1, ..., X_m)$.

Definition 1.2.1. *We say that a Lipschitz continuous curve* $\gamma : [0,T] \to \Omega$ *is* $\mathbb{X}$-admissible *if there exists a measurable vector function* $h = (h_1, ..., h_m) : [0,T] \to \Omega$ *such that*

(i) $\dot{\gamma}(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t))$, *for a.e.* $t \in [0,T]$;

(ii) $|h| \in L^\infty([0,T])$.

The curve γ *is called* $\mathbb{X}$-subunit *if it is* $\mathbb{X}$*-admissible and* $\|h\|_\infty \leq 1$, *for a.e.* $t \in [0,T]$.

Proposition 1.2.1. *A Lipschitz continuous curve* $\gamma : [0,T] \to \Omega$ *is* $\mathbb{X}$*-subunit if and only if*

$$\langle \dot{\gamma}(t), \xi \rangle^2 \leq \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2, \tag{1.6}$$

for all $\xi \in \mathbb{R}^n$ *and for a.e.* $t \in [0,T]$.

Proof. Let us consider $\gamma : [0,T] \to \Omega$ a subunit curve and fix $\xi \in \mathbb{R}^n$. Applying Schwarz' inequality,

$$\begin{aligned} \langle \dot{\gamma}(t), \xi \rangle^2 &= \left(\sum_{j=1}^{m} h_j(t) \langle X_j(\gamma(t)), \xi \rangle \right)^2 \\ &\leq \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2, \end{aligned}$$

for a.e. $t \in [0,T]$.

Vice versa, let $\gamma : [0,T] \to \Omega$ be a curve such that (1.6) holds. Let $t \in [0,T]$ a point at which γ is differentiable. We can write

$$\dot{\gamma}(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)) + \sum_{i=1}^{n} b_i(t) \partial_i,$$

for suitable coefficients $h(t) = (h_1(t), ..., h_m(t)) \in \mathbb{R}^m$ and $b(t) = (b_1(t), ..., b_n(t)) \in \mathbb{R}^n$. Since $m < n$, it is possible to select $\xi \in \mathbb{R}^n$ such that $\langle X_j(\gamma(t)), \xi \rangle = 0$, for every $j = 1, ..., m$. Then,

$$\begin{aligned} \langle \dot{\gamma}(t), \xi \rangle &= \sum_{j=1}^{m} h_j(t) \langle X_j(\gamma(t)), \xi \rangle + \sum_{i=1}^{n} b_i(t) \langle \partial_i, \xi \rangle \\ &= \langle b(t), \xi \rangle. \end{aligned}$$

By (1.6),

$$\langle b(t), \xi\rangle^2 \;=\; \langle \dot\gamma(t), \xi\rangle^2 \;\le \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi\rangle^2 = 0,$$

and thus $\langle b(t), \xi\rangle = 0$. This means that $\dot\gamma(t) \in \operatorname{span}\{X_1(\gamma(t)), ..., X_m(\gamma(t))\}$ and we can write $\dot\gamma(t) = \mathcal{A}(\gamma(t))h(t)$. Assuming that $h(t) = (\mathcal{A}(\gamma(t)))^T \cdot \xi$, for some $\xi \in \mathbb{R}^n$,

$$\begin{aligned} |h(t)|^4 \;&=\; \left\langle h(t), (\mathcal{A}(\gamma(t)))^T \cdot \xi \right\rangle^2 \\ &= \langle \mathcal{A}(\gamma(t)) \cdot h(t), \xi\rangle^2 \;=\; \langle \dot\gamma(t), \xi\rangle^2 \\ &\le \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi\rangle^2 \;=\; |\,(\mathcal{A}(\gamma(t)))^T \xi|^2 \;=\; |h(t)|^2, \end{aligned}$$

and this proves that $|h^2(t)| \le 1$, and $h(t)$ is defined for a.e. $t \in [0, T]$. □

Definition 1.2.2. *We define $d : \Omega \times \Omega \longrightarrow [0, \infty]$ as follows*

$$d(x, y) \;=\; \inf\{T \ge 0 \,|\, \exists\, \gamma : [0, T] \to \Omega \text{ subunit curve } :\; \gamma(0) = x,\ \gamma(T) = y\}.$$

If there exists no $\mathbb{X}$-subunit curve in Ω which joins x to y, then we write $d(x, y) = \infty$.

Definition 1.2.3. *We say that $\Omega \subseteq \mathbb{R}^n$ is* $\mathbb{X}$-connected *if for all $x, y \in \Omega$, there is a $\mathbb{X}$-subunit curve joining x to y.*

Theorem 1.2.2. *If $d(x, y) < \infty$ for all $x, y \in \Omega$, then (Ω, d) is a metric space. We call d the* Carnot-Carathéodory metric *on Ω (CC-metric for short).*

In order to prove this result we need a pair of lemmas.

Lemma 1.2.3. *Let $x_0 \in \Omega$ and $r > 0$ be such that $U := \{x \in \mathbb{R}^n \,|\, |x - x_0| < r\} \subset\subset \Omega$. Let $M := \sup_{x\in U} \|\mathcal{A}(x)\|$ and $\gamma : [0, T] \longrightarrow \Omega$ be a $\mathbb{X}$-subunit curve such that $\gamma(0) = x_0$. If $MT < r$, then $\gamma(t) \in U$ for all $t \in [0, T]$.*

Proof. We prove the Lemma by contradiction: we assume that, for some $t \in [0, T]$, $\gamma(t) \notin U$. We set

$$\bar t \;:=\; \inf\{\, t \in [0, T] \,|\, \gamma(t) \notin U\}\,.$$

Then

$$\begin{aligned}|\gamma(\bar{t}) - x_0| &= \left|\int_0^{\bar{t}} \dot{\gamma}(s)ds\right| = \left|\int_0^{\bar{t}} \mathcal{A}(\gamma(s))h(s)ds\right| \\ &= \int_0^{\bar{t}} |\mathcal{A}(\gamma(s)| \cdot |h(s)|ds \le \int_0^{\bar{t}} \|\mathcal{A}\| \cdot |h(s)|ds \\ &\le M \int_0^{\bar{t}} |h(s)|ds \le M\bar{t} \le M \cdot T < r.\end{aligned}$$

Therefore $\gamma(\bar{t}) \in U$, which is a contradiction. □

Lemma 1.2.4. *Let $K \subset\subset \Omega$ be a compact set. There exists a constant $C > 0$ such that*

$$d(x,y) \ge C|x-y| \tag{1.7}$$

for all $x, y \in K$.

Remark 1.2.4. Inequality (1.7) says that the Euclidean metric is continuous with respect to the Carnot-Carathéodory metric d.

Proof of Lemma 1.2.4. Let $\varepsilon \in \mathbb{R}^+$ be so that $K^\varepsilon \subset\subset \Omega$, where

$$K^\varepsilon := \left\{ x \in \Omega \;\middle|\; \min_{y \in K} |x-y| \le \varepsilon \right\}.$$

Let us take $x, y \in K$ and set $r = \min\{\varepsilon, |x-y|\}$ and $M := \sup_{z \in K_\varepsilon} \|\mathcal{A}(z)\|$. Let $\gamma : [0,T] \longrightarrow \Omega$ be a $\mathbb{X}$-subunit curve such that $\gamma(0) = x$ and $\gamma(T) = y$. Since $|\gamma(T) - \gamma(0)| = |y-x| \ge r$, by the previous Lemma, we have $T \cdot M \ge r$. If $r = \varepsilon$, then

$$T \ge \frac{\varepsilon}{M} \ge \frac{\varepsilon}{M} \frac{|x-y|}{\sup_{x,y \in K} |x-y|}.$$

If $r = |x-y|$, then

$$T \ge \frac{1}{M}|x-y|.$$

We chose an arbitrary $\mathbb{X}$-subunit curve γ, therefore, by definition of d, we get

$$d(x,y) \ge \min\left\{\frac{1}{M}, \frac{\varepsilon}{\sup_{x,y \in K} |x-y|}\right\}.$$

□

We are now ready to proceed to the proof of Proposition 1.2.2:

Proof of Proposition 1.2.2. Let us consider $x, y, x \in \Omega$. Firstly, we notice that $d(x, x) = 0$, indeed it is sufficient to consider the constant $\mathbb{X}$-subunit curve at x. Moreover, by (1.7), if $x \neq y$, then $d(x, y) > 0$.

The symmetry property follows from the fact that if $\gamma : [0, T] \to \Omega$ is a $\mathbb{X}$-subunit curve, then $\bar{\gamma}(t) := \gamma(T - t)$ is $\mathbb{X}$-subunit too.

Finally, if $\gamma_1 : [0, T_1] \longrightarrow \Omega$ and $\gamma_2 : [0, T_2] \longrightarrow \Omega$ are $\mathbb{X}$-subunit curves such that $\gamma_1(0) = x$, $\gamma_1(T_1) = y$ and $\gamma_2(0) = y$, $\gamma_2(T_2) = z$, then

$$\tilde{\gamma}(t) = \begin{cases} \gamma_1(t), & t \in [0, T_1] \\ \gamma_2(t), & t \in [T_1, T_1 + T_2] \end{cases}$$

is a $\mathbb{X}$-subunit curve such that $\tilde{\gamma}(0) = x$ and $\tilde{\gamma}(T_1 + T_2) = z$. Taking the infimum, we get $d(x, z) \leq d(x, y) + d(y, z)$. □

Notation 1.2.2. We can define, as usual, the metric balls with respect to the Carnot-Carathéodory metric setting, for $r > 0$,

$$U(x, r) := \{y \in \mathbb{R}^n \,|\, d(x, y) < r\} \quad \text{and} \quad B(x, r) := \{y \in \mathbb{R}^n \,|\, d(x, y) \leq r\}.$$

We point out that the metric d is finite on $\mathbb{R}^n$, and in general we can assume that the identity map between $(\mathbb{R}^n, d)$ and $(\mathbb{R}^n, |\cdot|)$ is a homeomorphism. This condition is satisfied when d is the CC-metric associated with a family of smooth vector fields $X_1, ..., X_m$ which satisfy *Hörmander condition*:

$$\operatorname{rank}\left(\mathcal{L}(X_1, ..., X_m)\right)(x) = n \tag{1.8}$$

for all $x \in \mathbb{R}^n$. With $\mathcal{L}(X_1, ..., X_m)$ we denote the Lie algebra generated by $X_1, ..., X_m$. Geometrically, condition (1.8) means that the vector fields $X_1, ..., X_m$ and their iterated brackets generate the whole tangent space at every point.

Theorem 1.2.5. *Let $X_1, ..., X_m$ be smooth vector fields on $\mathbb{R}^n$. Let $K \subset \mathbb{R}^n$ be a compact set and assume that, on K, Hörmander condition* (1.8) *is guaranteed by iterated commutators of length less or equal r. Then there exists a constant $C \in \mathbb{R}^+$ such that*

$$d(x, y) \leq C|x - y|^{\frac{1}{r}} \tag{1.9}$$

for all $x, y \in K$

Remark 1.2.5. Notice that the inequality (1.9), together with (1.7), ensures us that topology induced by CC-metric d is the same of topology induced by the Euclidean metric.

1.2.1 Carnot-Carathéodory Metric on Carnot Groups

Let us consider a Carnot group $\mathbb{G}$ with Lie algebra $\mathfrak{g}$. We know that we can represent $\mathbb{G}$ by $\mathbb{R}^n$, endowed with a Carnot structure, through a system of exponential coordinates, associated with a basis adapted to a stratification of $\mathfrak{g}$. Using the same notations as above, let $\mathfrak{g} = V_1 \oplus ... \oplus V_r$, $r \geq 2$, $m = \dim V_1$ and fix a basis $\mathbb{X} = (X_1, ..., X_m)$ of V_1. From definition of stratified algebra, V_1 generates the whole $\mathfrak{g}$ as an algebra. Hence, $X_1, ..., X_m$ satisfy Hörmander's condition, inducing a Carnot-Carathéodory metric on $\mathbb{G}$.

Proposition 1.2.6. *For all $x, y, z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^+$, the following properties hold:*

(i) $d(\tau_z(x), \tau_z(y)) = d(x, y)$;

(ii) $d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$.

Proof. Statements (i) follows from the fact that $\gamma : [0, T] \longrightarrow \mathbb{R}^n$ is a subunit path joining x to y, if and only if $\tilde{\gamma} := \tau_z(\gamma)$ is a subunit curve from $\tau_z(x)$ to $\tau_z(y)$.

Assume $\gamma(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(T))$, then

$$\begin{aligned}\dot{\tilde{\gamma}}(t) &= d\tau_z(\gamma(t)) \cdot \dot{\gamma(t)} = d\tau_z(\gamma(t)) \cdot \left(\sum_{j=1}^m h_j(t) X_j(\gamma(t)) \right) \\ &= \sum_{j=1}^m h_j(t) d\tau_z(\gamma(t)) X_j(\gamma(t)) = \sum_{j=1}^m h_j(t) X_j(\tau_z(\gamma(t))) \\ &= \sum_{j=1}^m h_j(t) X_j(\tilde{\gamma}(t)),\end{aligned}$$

which proves that $\tilde{\gamma}$ is a subunit curve.

We prove now (ii). Let us consider an $\mathbb{X}$-subunit curve $\gamma : [0, T] \longrightarrow \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma(T) = y$. We define $\gamma_\lambda : [0, \lambda T] \longrightarrow \mathbb{R}^n$ by $\gamma_\lambda(t) = \delta_\lambda(t)\gamma(\frac{t}{\lambda})$. Then, by

(1.4), since $d_j = 1$ for $j = 1, ..., m$ (using the notations introduced in Definition 1.2.1)

$$\begin{aligned}\dot\gamma(t) &= \sum_{i=1}^{n} \lambda^{d_i-1}\left(\sum_{j=1}^{m} h_j\left(\frac{t}{\lambda}\right) a_{ij}\left(\gamma\left(\frac{t}{\lambda}\right)\right)\right)\partial_i \\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{m} h_j\left(\frac{t}{\lambda}\right) a_{ij}(\gamma_\lambda(t))\right)\partial_i \\ &= \sum_{j=1}^{m} h_j\left(\frac{t}{\lambda}\right) X_j(\gamma_\lambda(t)).\end{aligned}$$

As $\gamma_\lambda(0) = \delta_\lambda(x)$, $\gamma_\lambda(\lambda T) = \delta_\lambda(y)$ and γ_λ is a subunit curve, one has that $d(\delta_\lambda(x), \delta_\lambda(y) \leq \lambda T$. Since γ is arbitrary, we can conclude that $d(\gamma_\lambda(x), \gamma_\lambda(y) \leq \lambda d(x, y)$, and the converse inequality can be obtained in the same way. □

We conclude this section with some remarks about measures and metrics.

If we denote $\mathcal{H}_d^k$ and $\mathcal{S}_d^k$ [5] the k-dimensional Hausdorff and Spherical Hausdorff measures associated with the Carnot-Caratéodory metric d, then one can check that

(i) $\mathcal{H}_d^k(x \cdot E) = \mathcal{H}_d^k(E)$,

(ii) $\mathcal{H}_d^k(\delta_\lambda E) = \lambda^k \mathcal{H}_d^k(E)$,

for every Lebesgue measurable set $E \subset \mathbb{R}^n$ and for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^+$. The same formulæhold for $\mathcal{S}_d^k$. The homogeneous dimension Q of $(\mathbb{R}^n, \cdot)$ is the Hausdorff dimension of $\mathbb{R}^n$ with respect to the CC-distance.

Moreover, we recall that the n-dimensional Lebesgue measure $\mathcal{L}^n$ is the Haar measure of the group. Therefore, the translation and dilation conditions read as follows:

Proposition 1.2.7. *Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. Then, for all $x \in \mathbb{R}^n$ and $\lambda \geq 0$.*

[5] We recall that, give $E \subset \mathbb{R}^n$ and $k \geq 0$, the *k-dimensional Hausdorff* and *Spherical Hausdorff measures* of E are defined, respectively, by

$$\begin{aligned}\mathcal{H}_d^k(E) &:= \lim_{\delta\searrow 0}\inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} E_i)^k \,\middle|\, E \subset \bigcup_{i=0}^{\infty} E_i,\ \operatorname{diam} E_i < \delta\right\} \\ \mathcal{S}_d^k(E) &:= \lim_{\delta\searrow 0}\inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} B_i)^k \,\middle|\, E \subset \bigcup_{i=0}^{\infty} B_i,\ \operatorname{diam} B_i < \delta,\ B_i \subset \mathbb{R}^n \text{ balls}\right\}.\end{aligned}$$

(i) $\mathcal{L}^n(x \cdot E) = \mathcal{L}^n(E \cdot x) = \mathcal{L}^n(E)$;

(ii) $\mathcal{L}^n(\delta_\lambda E) = \lambda^Q \mathcal{L}^n(E)$.

In particular $\mathcal{L}^n(B(x,r)) = r^Q \mathcal{L}^n(B(x,1))$.

Definition 1.2.6. *We say that a metric* ρ *on* $\mathbb{G}$ *is a* homogeneous distance *if, for all* $x, y, z \in \mathbb{G}$ *and* $\lambda \in \mathbb{R}^+$

(i) $\rho(x,y) = \rho(\tau_z(x), \tau_z(y))$;

(ii) $\rho(\delta_\lambda(x), \delta_\lambda(y)) = \lambda\rho(x,y)$.

In Proposition 1.2.6, we proved that the Carnot-Carathéodory metric is a homogeneous metric. We can construct other examples of homogeneous metrics. We start by defining the following quasi-metric

$$d_\infty(x,y) \;=\; \|y^{-1} \cdot x\|, \tag{1.10}$$

where $\|\cdot\|$ is a homogeneous norm. For example we can choose

$$\|x\|_\infty \;=\; \sum_{i=1}^{n} |x_i|^{\frac{1}{d_i}}$$

or

$$\|x\| \;=\; \max_i \left\{ \varepsilon_i |x_i|^{\frac{1}{d_i}} \right\},$$

where the ε_i's are suitable positive constants which depend on the group's structure and which let d_∞ be a distance on the group.

Remark 1.2.7. We notice that these homogeneous metrics induce the same topology as the Carnot-Carathéodory one.

Remark 1.2.8. In the case of Heisenberg group (see Section 1.4), we introduce the *Korányi norm*: if $p = (z,t) \in \mathbb{H}^n$,

$$\|p\| \;=\; \sqrt[4]{\|z\|^4_{\mathbb{R}^{2n}} + |t|^2}.$$

If it is not specified, through this thesis we will use this homogeneous norm. To verify that $d_\infty(x,y) = \|y^{-1} \cdot x\|$ is a metric, when $\|\cdot\|$ is the Korányi norm, one needs to prove the triangle inequality

$$d_\infty(x,y) \;\le\; d_\infty(x,z) + d_\infty(z,y). \tag{1.11}$$

This can be done by a direct computation.

First, we can replace $z^{-1} \cdot x$ with x and $y^{-1} \cdot z$ with y; then it is sufficient to prove (1.11) in the case when $z = e$ and to show that

$$\|x \cdot y\| \ \leq \ \|x\| + \|y\|.$$

Writing $x = (z,t)$ and $y = (w,s)$ and using the group law (1.12), we find

$$\begin{aligned}
\|x \cdot y\|^4 \ &= \ \|v + w\|^4_{\mathbb{R}^{2n}} + \left(t + w + \frac{1}{2}\Im(v \cdot \bar{w})\right)^2 \\
&= \ \left|\|v + w\|^2_{\mathbb{R}^{2n}} + 2i\left(t + w + \frac{1}{2}\Im(v \cdot \bar{w})\right)\right|^2 \\
&= \ \left|\|v\|^2_{\mathbb{R}^{2n}} + 2it + \bar{v} \cdot w + \|w\|^2_{\mathbb{R}^{2n}} + 2is\right|^2 \\
&\leq \ \left(\|x\|^2 + 2\|v\|^2_{\mathbb{R}^{2n}}\|w\|^2_{\mathbb{R}^{2n}} + \|y\|^2\right)^2 \\
&\leq \ (\|x\| + \|y\|)^4 \, .
\end{aligned}$$

1.3 Calculus on Carnot Groups

Definition 1.3.1. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be Carnot groups, with homogeneous norm $\|\cdot\|_1$, $\|\cdot\|_2$ and dilations δ^1_λ and δ^2_λ. We say that a function $L : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ is H-*linear *if L is a group homomorphism and if, for all $g \in \mathbb{G}_1$ and $\lambda \in \mathbb{R}^+$,*

$$L(\delta^1_\lambda g) \ = \ \delta^2_\lambda(g).$$

The set of all H-linear functions between $\mathbb{G}_1$ and $\mathbb{G}_2$ can be endowed with the norm

$$\|L\| \ := \ \sup_{\|g\|_1 \leq 1} \ \|L\,g\|_2.$$

Analogously to the classical case, we have the following

Proposition 1.3.1. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be Carnot groups with homogeneous norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If $L : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ is H-linear. Then L is continuous and*

$$\|L\,g\|_2 \ \leq \ \|L\| \cdot \|g\|_1,$$

for each $g \in \mathbb{G}_1$.

Proposition 1.3.2. *Let $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ be Carnot groups with homogeneous norm $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_3$, and let $L : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ and $M : \mathbb{G}_2 \longrightarrow \mathbb{G}_3$ be H-linear maps. Then $M \circ L : \mathbb{G}_1 \longrightarrow \mathbb{G}_3$ is H-linear and $\|L \circ M\| \leq \|L\| \, \|M\|$.*

We give the notion of *P-differentiability* for functions acting between Carnot groups. This key notion of differentiability was introduced by Pansu in [36].

Definition 1.3.2. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be Carnot groups with homogeneous norm $\|\cdot\|_1$ and $\|\cdot\|_2$. We say that $f : \mathcal{E} \subset \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ is* P-differentiable *in $g_0 \in \mathcal{E}$ if there exists a H-linear function, called* P-differential *of f at g_0,*

$$d_{g_0} f \, : \, \mathbb{G}_1 \, \longrightarrow \, \mathbb{G}_2$$

such that

$$\lim_{g \to g_0} \frac{\|(d_{g_0} f(g_0^{-1} \cdot g))^{-1} \cdot f(g_0)^{-1} \cdot f(g)\|_2}{\|g_0^{-1} \cdot g\|_1} \, = \, 0.$$

We say that $f : \mathcal{E} \longrightarrow \mathbb{G}_2$ is continuously P-differentiable *in $\mathcal{E}$ if f is P-differentiable at every point $g \in \mathcal{E}$ and $d_g f$ depends continuously on g. In this case we write $f \in C^1_H(\mathcal{E}, \mathbb{G}_2)$.*

1.4 Heisenberg Groups

In this section we study some peculiarities of the Heisenberg group, which is the most simple non Abelian Carnot group and the setting of this thesis. We start by recalling the definition and some general properties, then successively we will give a close look to the structure of subgroups.

Notation 1.4.1. We denote by (z, t) a point in $\mathbb{C}^n \times \mathbb{R}$, where $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $t \in \mathbb{R}$. If $z_j = x_j + iy_j$, we write $z = (x_1, ..., x_n, y_1, ..., y_n)$, with $x_j, y_j \in \mathbb{R}$ for $j = 1, ..., n$.

Let us consider in $\mathbb{C}^n \times \mathbb{R}$ the following composition law

$$(z, t) \cdot (z', t') \, = \, \left(z + z', t + t' + \frac{1}{2} \Im(z \cdot \bar{z'}) \right), \tag{1.12}$$

where $z \cdot \bar{z'}$ denotes the usual Hermitian product in $\mathbb{C}^n$:

$$z \cdot \bar{z'} = \sum_{j=1}^{n} (x_j + iy_j)(x'_j - iy'_j).$$

Remark 1.4.1. If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we can rewrite the operation law (1.12) in the following way:

$$(z,t)\cdot(z',t') \;=\; \left(z+z',t+t'+\frac{1}{2}\langle Jz,z'\rangle\right),$$

where J is the unit $(n\times n)$-symplectic matrix and $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$.

It is not difficult to verify that $(\mathbb{R}^{2n+1},\cdot)$ is a Lie group, whose identity is the origin of $\mathbb{R}^{2n+1}$ and the inverse is $(z,t)^{-1}=(-z,-t)$. We call this Lie group the *n-th Heisenberg group*, and we write $\mathbb{H}^n := (\mathbb{R}^{2n+1},\cdot)$.

The Heisenberg group $\mathbb{H}^n$ is the Lie group associated with the $2n+1$-dimensional real Lie algebra $\mathfrak{h}^n$ generated by $\{X_1,...,X_n,Y_1,...,Y_n,T\}$, that satisfies the relations

$$[X_i,X_j] \;=\; 0,\;\; [Y_i,Y_j] \;=\; 0,\;\; [X_j,Y_j] \;=\; T,$$

for every $i,j=1,...,n$. By the Jacobi's identity, we get that $[X_i,T]=[Y_i,T]=0$, for each $i=1,...,n$. This means that $\mathfrak{h}^n$ is a nilpotent Lie algebra. It is also clear that its center is $\mathrm{span}\{T\}$.

Let us denote

$$\mathfrak{h}_1 \;=\; \mathrm{span}\{X_1,...,X_n,Y_1,...,Y_n\} \;\text{ and }\; \mathfrak{h}_2 \;=\; \mathrm{span}\{T\}.$$

Then the Heisenberg algebra is stratified of step 2 with stratification

$$\mathfrak{h}^n \;=\; \mathfrak{h}_1\oplus\mathfrak{h}_2.$$ [6]

Remark 1.4.2. By the structure of $\mathfrak{h}^n$, we can say that the center of the group $\mathbb{H}^n$ is

$$\mathbb{T} \;=\; \{(z,t)\in\mathbb{R}^{2n+1}\,|\,z=0\},$$

and the homogeneous dilations are, for $\lambda\in\mathbb{R}^+$,

$$\begin{aligned}\delta_\lambda\;:\;\mathbb{R}^{2n+1} &\longrightarrow \mathbb{R}^{2n+1}\\ (z,t) &\longmapsto (\lambda z,\lambda^2 t).\end{aligned}$$

[6] Using exponential coordinates, one can prove that $\mathbb{H}^n$ is the unique simply connected, nilpotent Lie group associated with $\mathfrak{h}^n$.

We can realize the Heisenberg Lie algebra $\mathfrak{h}^n$ as an algebra of left invariant differential operators on $\mathbb{R}^{2n+1}$. For example, using formula (1.5), one can found $T = \partial_t$ and, consequently,

$$X_j = \partial_{x_j} + \frac{1}{2} y_j \partial_t, \quad Y_j = \partial_{y_j} - \frac{1}{2} x_j \partial_t,$$

for $j = 1, ..., n$. With this identification between vector fields and first order differetial operators, $X_1, ..., X_n, Y_1, ..., Y_n$ generate a vector bundle on $\mathbb{H}^n$, called *horizontal bundle* $H\mathbb{H}^n$. The horizontal bundle is a subbundle of the tangent bundle $T\mathbb{H}^n$. By definition of vector bundle, we know that we can identify canonically each fiber of $H\mathbb{H}^n$ with a vector subspace of $\mathbb{R}^{2n+1}$, so each section φ of $H\mathbb{H}^n$ can be identified with a map $\varphi : \mathbb{H}^n \longrightarrow \mathbb{R}^{2n+1}$. We denote by $H_x\mathbb{H}^n$ the fiber of $H\mathbb{H}^n$ at a point $x \in \mathbb{H}^n$. On $\mathbb{H}^n$ is defined a Sub-Riemannian structure: we can endow each fiber $H_x\mathbb{H}^n$ with a scalar product, denoted by $\langle \cdot, \cdot \rangle_x$, and the associated norm $| \cdot |_x$ that make the basis of $H_x\mathbb{H}^n$, $X_1(x), ..., X_n(x), Y_1(x), ..., Y_n(x)$, orthonormal; in other words, if we consider $v = \sum_{i=1}^n (v_i X_i(x) + v_{n+i} Y_i(x))$ and $w = \sum_{i=1}^n (w_i X_i(x) + w_{n+i} Y_i(x))$ vectors of $H_x\mathbb{H}^n$, then $\langle v, w \rangle_x := \sum_{i=1}^n v_i \cdot w_i + v_{n+i} \cdot w_{n+i}$ and $|v|_x^2 := \langle v, v \rangle_x$.

We end this subsection by giving a definition that will be useful throughout this thesis:

Definition 1.4.3. *Let (x, y, t) and x_0 be fixed points in $\mathbb{H}^n$. We set*

$$\pi_{x_0}((z, t)) = \sum_{j=1}^n x_j X_j(x_0) + \sum_{j=1}^n y_j Y_j(x_0).$$

The map $x_0 \longmapsto \pi_{x_0}((z, t))$. is a smooth section of $H\mathbb{H}^n$.

1.4.1 Homogeneous Subgroups of $\mathbb{H}^n$

In Chapter 2 we will study the notion of intrinsic Lipschitz graphs; roughly speaking we will study the graph of some special functions whose graphs lie in the Heisenberg group $\mathbb{H}^n$. We can compare these notions with the Euclidean case, in which we decompose $\mathbb{R}^n$ in the sum of subgroups. In Heisenberg setting, on the other side, we need more conditions for a “nice” decomposition in subgroups. We start with the definition of homogeneous subgroups, which are subgroups invariant under group dilations:

Definition 1.4.4. *We say that a subgroup $\mathbb{G}$ of $\mathbb{H}^n$ is a* homogeneous Lie subgroup *if, for all $g \in \mathbb{G}$ and $\lambda > 0$, $\delta_\lambda(g) \in \mathbb{G}$.*

We point out that Definition 1.4.4 can be stated also for a general Carnot group of step k. In this case, one can prove that each homogeneous subgroup is necessarily a graded subgroup with step at most k, but in general it is not a Carnot group.

Definition 1.4.5. *We say that $\mathbb{H}^n$ is a* semidirect product *of homogeneous subgroups $\mathbb{W}$ and $\mathbb{V}$, and we write $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, if $\mathbb{W} = \exp(\mathfrak{w})$ and $\mathbb{V} = \exp(\mathfrak{v})$, where $\mathfrak{w}$ and $\mathfrak{v}$ are homogeneous subalgebras of $\mathfrak{h}^n$ such that*

(i) $\mathfrak{h}^n = \mathfrak{w} \oplus \mathfrak{v}$;

(ii) $\mathfrak{w}$ *is an ideal in* $\mathfrak{h}^n$.

We will say that $\mathbb{W}$ and $\mathbb{V}$ are complementary subgroups *in $\mathbb{H}^n$.*

Remark 1.4.6. If $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, then $\mathbb{W} \cap \mathbb{V} = \{e\}$. From (ii), it follows also that $\mathbb{W}$ is a normal subgroup of $\mathbb{H}^n$.

Example 1.4.1. A simple example of a semidirect product is given by $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, where

$$\mathbb{V} = \{(x_1, 0, ..., 0) \,|\, x_1 \in \mathbb{R}\}$$

and

$$\mathbb{W} = \{(0, x_2, ..., x_n, y_1, ..., y_n, t) \,|\, x_i, y_j, t \in \mathbb{R},\ i = 2, ..., n, j = 1, ..., n\}.$$

Proposition 1.4.1. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Definition 1.4.5. Then each $q \in \mathbb{H}^n$ has unique components $q_\mathbb{W} \in \mathbb{W}$ and $q_\mathbb{V} \in \mathbb{V}$ such that $q = q_\mathbb{W} \cdot q_\mathbb{V}$.*

Proof. Let us assume that $q \in \mathbb{H}^n$ admits two decompositions: $q = q_\mathbb{W} \cdot q_\mathbb{V}$ and $q = q'_\mathbb{W} \cdot q'_\mathbb{V}$. Then, since e is the unique common element of $\mathbb{W}$ and $\mathbb{V}$,

$$(q'_\mathbb{W})^{-1} \cdot q_\mathbb{W} \;=\; q'_\mathbb{V} \cdot (q_\mathbb{V})^{-1} \;=\; e.$$

Thus, $q'_\mathbb{W} = q_\mathbb{W}$ and $q'_\mathbb{V} = q_\mathbb{V}$. □

Proposition 1.4.2. *Let $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ be a semidirect product as in Definition 1.4.5. Then the maps*

$$\begin{aligned} \Pi_{\mathbb{W}} : \mathbb{H}^n &\longrightarrow \mathbb{W} \\ q &\longmapsto q_{\mathbb{W}} \end{aligned}$$

and

$$\begin{aligned} \Pi_{\mathbb{V}} : \mathbb{H}^n &\longrightarrow \mathbb{V} \\ q &\longmapsto q_{\mathbb{V}} \end{aligned}$$

are continuous.

Proof. See [1], Proposition 3.4. □

Proposition 1.4.3. *All homogeneous subgroups of $\mathbb{H}^n$ are either horizontal, that are contained in $\{(z,t) \in \mathbb{H}^n \,|\, t = 0\}$, which can be identified with the horizontal fiber $H_e\mathbb{H}^n$, or vertical, that are containing the subgroup $\mathbb{T}$.*

A horizontal subgroup $\mathbb{V}$ has linear dimension k, with $1 \leq k \leq n$; moreover, $\mathbb{V}$ is algebraically isomorphic and isometric to $\mathbb{R}^k$.

A vertical subgroup $\mathbb{W}$ can have any dimension d, with $1 \leq d \leq n+1$, and its metric dimension is $d+1$.

Proof. Let $\mathbb{V} \subset \mathbb{H}^n$ be a homogeneous subgroup. Then there exists $\mathfrak{v}$, a homogeneous subalgebra of $\mathfrak{h}^n$, such that $\exp\mathfrak{v} = \mathbb{V}$. Then there exist linear independent vector fields $v_1, ..., v_k \in \mathfrak{h}^n$, with $\leq k \leq 2n+1$, such that $\mathfrak{v} = \mathrm{span}\{v_1, ..., v_n\}$ and $[v_i, v_j] \in \mathfrak{v}$, for each $i, j = 1, ..., k$. If $\mathbb{V}$ is horizontal, that means that $v_i \in \mathfrak{h}_1$, for all $i = 1, ..., k$, necessarily one has $[v_i, v_j] = 0$, for all $i, j = 1, ..., k$, and it must be $k \leq n$.

On the other hand, let T be a vector field in the center of $\mathfrak{h}^n$, and suppose there exists $v \in \mathfrak{h}_1$ such that $v + T \in \mathfrak{v}$. Then, for $\lambda \in \mathbb{R}$, $\lambda v + \lambda T \in \mathfrak{v}$ and, since $\mathbb{V}$ is a homogeneous subgroup, $\lambda v + \lambda^2 T \in \mathfrak{v}$ too. Hence, necessarily, it follows that $T \in \mathfrak{v}$, implying that $\mathbb{V}$ is a vertical subgroup.

Now, let us consider a horizontal subgroup $\mathbb{V}$ with $\dim\mathfrak{v} = k$, and let $x, y \in \mathbb{V}$. The points $x \cdot \delta_\lambda(x^{-1} \cdot y) \in \mathbb{V}$, for $\lambda \in [0,1]$, form a horizontal segment connecting x and y, then it follows that $\mathbb{V}$ is isomorphic and isometric to $\mathbb{R}^k$.

On the contrary, if $\mathbb{W}$ is a vertical subgroup with $\dim(\mathfrak{w}) = k$, then, in general, $\mathbb{W}$ is not isomorphic to $\mathbb{R}^k$ and it is never isometric to $\mathbb{R}^k$, indeed it has metric dimension equal to $k+1$. □

From the previous Proposition, we have directly

Proposition 1.4.4. *All possible couples $\mathbb{W}$ and $\mathbb{V}$ of complementary subgroups of $\mathbb{H}^n$ are of the type*

(i) $\mathbb{V}$ *horizontal of dimension k, $1 \leq k \leq n$,*

(ii) $\mathbb{W}$ *normal of dimension $2n+2-k$.*

Proof. See [1], Proposition 3.21. □

Remark 1.4.7. We aim to highlight that any horizontal subgroup $\mathbb{V}$ has a complementary normal subgroup $\mathbb{W}$; also the converse is true for normal subgroup $\mathbb{W}$ with linear dimension larger than $n+1$. On the contrary, normal subgroups of dimension less than or equal to n do not have complementary subgroups. For example the center $\mathbb{T}$ does not have a complementary subgroup.

Proposition 1.4.5. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Proposition 1.4.4. Then there exists a positive constant $C = C(\mathbb{W}, \mathbb{V})$ such that*

$$C\ (\|q_{\mathbb{V}}\| + \|q_{\mathbb{W}}\|) \leq \|q\| \leq (\|q_{\mathbb{V}}\| + \|q_{\mathbb{W}}\|)\,. \tag{1.13}$$

Moreover, $(q^{-1})_{\mathbb{V}} = (q_{\mathbb{V}})^{-1}$, $(q^{-1})_{\mathbb{W}} = q_{\mathbb{V}}^{-1} \cdot (q_{\mathbb{W}})^{-1} \cdot q_{\mathbb{V}}$, $(p \cdot q)_{\mathbb{V}} = p_{\mathbb{V}} \cdot q_{\mathbb{V}}$, *and* $(p \cdot q)_{\mathbb{W}} = p_{\mathbb{W}} \cdot p_{\mathbb{V}} \cdot q_{\mathbb{W}} \cdot p_{\mathbb{V}}^{-1}$.

Proof. We start by proving (1.13). By homogeneity, it is enough to show the left hand side of (1.13) when $\|q\| = 1$. In this case, the inequality holds. Indeed, by compactness, $\|q_{\mathbb{W}}\|$ and $\|q_{\mathbb{V}}\|$ have a maximum when $\|q\| = 1$. The right hand side of (1.13) is just the triangular inequality.

Coming to the second part of the Proposition, we notice that $q^{-1} = (q^{-1})_{\mathbb{W}} \cdot (q^{-1})_{\mathbb{V}}$, but also $q^{-1} = (q_{\mathbb{V}})^{-1} \cdot (q_{\mathbb{W}})^{-1} = (q_{\mathbb{V}})^{-1} \cdot (q_{\mathbb{W}})^{-1} \cdot q_{\mathbb{V}} \cdot (q_{\mathbb{V}})^{-1}$. The assertion follows by the uniqueness of the coordinates. The remaining equalities follow from similar arguments. □

Remark 1.4.8. By the previous Proposition, it follows that $\Pi_{\mathbb{V}}$ is a group homomorphism from $\mathbb{H}^n$ to $\mathbb{V}$; while, in general, $\Pi_{\mathbb{W}}$ is not a group homomorphism from $\mathbb{H}^n$ to $\mathbb{W}$. Moreover, one can notice that $\Pi_{\mathbb{V}} : \mathbb{H}^n \longrightarrow \mathbb{V}$ is a Lipschitz map. Indeed, let $p = (z, t)$ and $q = (z', t') \in \mathbb{H}^n$. Since $\mathbb{V}$ is isometric to $\mathbb{R}^k$,

$$\|\Pi_{\mathbb{V}}(p)^{-1} \cdot \Pi_{\mathbb{V}}(q)\| \;=\; \|\Pi_{\mathbb{V}}(q) - \Pi_{\mathbb{V}}(p)\|.$$

Then it follows

$$\|\Pi_{\mathbb{V}}(q)^{-1} - \Pi_{\mathbb{V}}(p)\| \;\leq\; \|z' - z\|_{\mathbb{R}^{2n}} \;\leq\; \|p^{-1} \cdot q\|.$$

On the contrary, $\Pi_{\mathbb{W}} : \mathbb{H}^n \longrightarrow \mathbb{W}$ is not, in general, a Lipschitz map. For example, consider the first Heisenberg group $\mathbb{H}^1$ decomposed in complementary subgroups as $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$, where $\mathbb{W} = \{(0, y, t)\}$ and $\mathbb{V} = \{(x, 0, 0)\}$. Now, for $\varepsilon \in \mathbb{R}^+$, let $p = (1, 0, 0)$ and $q = (0, \varepsilon, \frac{\varepsilon}{2})$. Then $p_{\mathbb{W}} = (0, 0, 0)$ and $q_{\mathbb{W}} = (0, \varepsilon, \frac{\varepsilon}{2})$. Thus,

$$\begin{aligned} \|\Pi_{\mathbb{W}}(q)^{-1} \cdot \Pi_{\mathbb{W}}(p)\| \;&=\; \|q_{\mathbb{W}}\| \;\approx\; \varepsilon^{\frac{1}{2}} \\ \|p^{-1} \cdot q\| \;&=\; \|(0, \varepsilon, 0)\| \;\approx\; \varepsilon \end{aligned}$$

Proposition 1.4.6. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Proposition 1.4.4 with $\dim(\mathfrak{v}) = k$, and let $p \in \mathbb{H}^n$ be fixed. Then there exists a positive constant $C = C(\mathbb{W}, \mathbb{V})$ such that, for $B(p, r) \subset \mathbb{H}^n$,*

$$\mathcal{L}^{2n+1-k}\left(\Pi_{\mathbb{W}}(B(p, r))\right) \;=\; C(\mathbb{W}, \mathbb{V})\, r^{2n+2-k}. \tag{1.14}$$

Proof. First of all we notice that, for every Lebesgue measurable set $E \subset \mathbb{H}^n$,

$$\mathcal{L}^{2n+1-k}\left(\Pi_{\mathbb{W}}(p \cdot E)\right) \;=\; \mathcal{L}^{2n+1-k}\left(\Pi_{\mathbb{W}}(E)\right). \tag{1.15}$$

Indeed, since

$$\Pi_{\mathbb{W}}(p \cdot E) \;=\; \left\{p_{\mathbb{W}} \cdot p_{\mathbb{V}} \cdot q_{\mathbb{W}} \cdot p_{\mathbb{V}}^{-1} \,|\, q \in E\right\} \;=\; p_{\mathbb{W}} \cdot p_{\mathbb{V}} \cdot \Pi_{\mathbb{W}}(E) \cdot p_{\mathbb{V}}^{-1},$$

the linear mapping

$$\begin{aligned} \mathbb{W} &\longrightarrow \mathbb{W} \\ w &\longmapsto p_{\mathbb{W}} \cdot p_{\mathbb{V}} \cdot w \cdot p_{\mathbb{W}}^{-1} \end{aligned}$$

has determinant equal to 1, therefore (1.15) holds. Now, by group dilations

$$\Pi_{\mathbb{W}}(B(e, r)) \;=\; \Pi_{\mathbb{W}}\left(\delta_r(B(e, 1))\right) \;=\; \delta_r\left(\Pi_{\mathbb{W}}(B(e, 1))\right),$$

thus

$$\mathcal{L}^{2n+1-k}\left(\Pi_{\mathbb{W}}(B(p,r))\right) = \mathcal{L}^{2n+1-k}\left(\Pi_{\mathbb{W}}(B(e,1))\right) r^{2n+2-k}.$$

Setting $C = C(\mathbb{W},\mathbb{V}) = \mathcal{L}^{2n+1-k}\left(\Pi_{\mathbb{W}}(B(e,1))\right)$ and using (1.15), one has the thesis. □

1.5 Rectifiability in Heisenberg Groups

In [12, 13, 14], the authors prove analogous statements of De Giorgi's results in the setting of Heisenberg group. In this section we aim to re-propose some of these results in order to use them in the next chapters. We start briefly by summarizing the classical results: If $E \subset \mathbb{R}^n$ is a Caccioppoli set,[7] then the associated perimeter measure $|\partial E|$ is concentrated on a portion of the boundary, the *reduced boundary* $\partial^* E \subset \partial E$, and this means that $\partial^* E$, up to a set of $\mathcal{H}^{n-1}$-measure zero, is a countable union of compact subset of C^1-manifolds.

In the first subsection we recall the definition of functions with bounded $\mathbb{H}$-variation and introduce some properties, in the second part we present the restatement of De Giorgi's theory in the case of Heisenberg groups. For proofs and a more details we refer the reader to [12, 13, 14].

1.5.1 $BV_{\mathbb{H}}$-functions

We start with some notations. If Ω is an open set of $\mathbb{H}^n$ and k is a non negative integer, we denote by $C^k(\Omega)$ and $C^\infty(\Omega)$ the spaces of real valued continuously differentiable functions. We denote by $C^k(\Omega, H\mathbb{H}^n)$ the set of all C^k-sections of $H\mathbb{H}^n$, analogously we define $C^\infty(\Omega, H\mathbb{H}^n)$ and $C_0^\infty(\Omega, H\mathbb{H}^n)$.

Definition 1.5.1. *Let Ω be an open subset of $\mathbb{H}^n$, $\varphi = (\varphi_1, ..., \varphi_n) \in C^1(\Omega, H\mathbb{H}^n)$ and*

[7] We say that a set $E \subset \mathbb{R}^n$ is a set of *locally finite perimeter*, or a *Caccioppoli set*, if the total variation of its characteristic function is finite, i.e., if Ω i s any open set in $\mathbb{R}^n$,

$$|\partial E|(\Omega) := \sup \left\{ \int_\Omega \operatorname{div} \varphi \, dx \,\middle|\, \varphi \in C_0^1(\Omega)\,,\ |\varphi(x)| \le 1 \right\} < \infty.$$

We recall that the perimeter measure $|\partial E|$ defines a Radon measure on $\mathbb{R}^n$.

$f \in C^1(\Omega)$. *We define the* horizontal gradient of f

$$\nabla_{\mathbb{H}} f := (X_1 f, ..., X_n f, Y_1 f, ..., Y_n f)$$

and the horizontal divergence of φ

$$\mathrm{div}_{\mathbb{H}} \varphi := \sum_{j=1}^{n} (X_j \varphi_j + Y_j \varphi_{n+j}) .$$

Remark 1.5.2. We can define the horizontal gradient as a section of the horizontal bundle $H\mathbb{H}^n$:

$$\nabla_{\mathbb{H}} f = \sum_{j=1}^{n} ((X_j f) X_j + (Y_j f) Y_j) ,$$

whose coordinates are $(X_1 f, ..., X_n f, Y_1 f, ..., Y_n f)$. We point out that this definition depends on the choice of the basis of the first layer: if we choose a different basis, say $\left(\tilde{X}_1, ..., \tilde{X}_n, \tilde{Y}_1, ..., \tilde{Y}_n\right)$, then in general

$$\sum_{j=1}^{n} ((X_j f) X_j + (Y_j f)) \neq \sum_{j=1}^{n} \left(\left(\tilde{X}_j f\right) \tilde{X}_j + \left(\tilde{Y}_j f\right) \tilde{Y}_j\right) .$$

Only if the two basis are one orthonormal with respect to the scalar product induced by the other, we have that

$$\sum_{j=1}^{n} ((X_j f) X_j + (Y_j f)) = \sum_{j=1}^{n} \left(\left(\tilde{X}_j f\right) \tilde{X}_j + \left(\tilde{Y}_j f\right) \tilde{Y}_j\right) .$$

We are now ready for the definition of boundary $\mathbb{H}$-variation functions.

Definition 1.5.3. *Let $\Omega \subset \mathbb{H}^n$ be an open set. We say that $f : \Omega \to \mathbb{H}^n$ is of bounded $\mathbb{H}$-variation if $f \in L^1(\Omega)$ and*

$$\|\nabla_{\mathbb{H}} f\| (\Omega) := \sup \left\{ \int_{\Omega} f \, \mathrm{div}_{\mathbb{H}} \varphi \, dh \, \middle| \, \varphi \in C_0^1(\Omega, H\mathbb{H}^n) , \, |\varphi(x)|_x \leq 1 \right\} < \infty.$$ [8]

We denote by $BV_{\mathbb{H}}(\Omega)$ the space of all functions of bounded $\mathbb{H}$-variation. The space $BV_{\mathbb{H},loc}(\Omega)$ is the set of functions belonging to $BV_{\mathbb{H}}(U)$ for each open set $U \subset\subset \Omega$.

[8] We denote with dh the integration with respect the Haar measure of the group, that, as already point out, is the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

We present now some properties of functions with bounded $\mathbb{H}$-variation, $BV_{\mathbb{H}}$-functions for short. For details and proofs in the general context of subriemannian geometries we refer the reader to [19, 14].

Theorem 1.5.1. *If $f \in BV_{\mathbb{H}}(\Omega)$, then $\|\nabla_{\mathbb{H}} f\|$ is a Radon measure on Ω. Moreover, there exists $\|\nabla_{\mathbb{H}} f\|$-measurable horizontal section $\sigma_f : \Omega \to H\mathbb{H}^n$ such that $|\sigma_f(x)|_x = 1$ for $\|\nabla_{\mathbb{H}} f\|$-a.e. $x \in \Omega$ and*

$$\int_\Omega f \,\mathrm{div}_{\mathbb{H}} \varphi \, dh \;=\; -\int_\Omega \langle \varphi, \sigma_f \rangle \; d\|\nabla_{\mathbb{H}} f\|,$$

for $\varphi \in C_0^1(\Omega, H\mathbb{H}^n)$.

Finally, we can extend the notion of the horizontal gradient $\nabla_{\mathbb{H}}$ from regular functions to $BV_{\mathbb{H}}$-functions, defining $\nabla_{\mathbb{H}} f$ as vector valued function:

$$\nabla_{\mathbb{H}} f \;:=\; -\sigma_f \llcorner \|\nabla_{\mathbb{H}} f\| \;=\; \left(-(\sigma_f)_1 \llcorner \|\nabla_{\mathbb{H}} f\|, ..., -(\sigma_f)_{2n} \llcorner \|\nabla_{\mathbb{H}} f\|\right),$$

where $(\sigma_f)_1, ..., (\sigma_f)_{2n}$ are the components of σ_f with respect to the horizontal basis $X_1, ..., X_n, Y_1, ..., Y_n$.

Theorem 1.5.2. *$BV_{\mathbb{H},loc}(\Omega)$ is compactly embedded in $L^1_{loc}(\mathbb{H}^n)$ for $1 \le p < \frac{Q}{Q-1}$.*

1.5.2 $\mathbb{H}$-Caccioppoli Sets

Keeping in mind the classical definition of De Giorgi, we define the $\mathbb{H}$-Caccioppoli sets:

Definition 1.5.4. *A measurable set $E \subset \mathbb{H}^n$ is of* locally finite $\mathbb{H}$-perimeter*, or is a* $\mathbb{H}$-Caccioppoli set*, if the characteristic function $\mathbf{1}_E \in BV_{\mathbb{H},loc}(\mathbb{H}^n)$. In this case we call* $\mathbb{H}$-perimeter *of E the measure*

$$|\partial E|_{\mathbb{H}} \;:=\; \|\nabla_{\mathbb{H}} \mathbf{1}_E\|.$$

Remark 1.5.5. The value of $\mathbb{H}$-perimeter depends on the choice of the generating vector fields $X_1, ..., X_n, Y_1, ..., Y_n$. However the perimeters induced by different families of generating vector fields are equivalent as measures. As a consequence, the property of being a $\mathbb{H}$-Caccioppoli depends only on the group $\mathbb{H}^n$.

We point out that the $\mathbb{H}$-perimeter is invariant under group translation, that is

$$|\partial E|_{\mathbb{H}}(A) \;=\; |\partial(\tau_p(E)|_{\mathbb{H}}(\tau_p(A)),$$

for all $p \in \mathbb{H}^n$ and for every Borel set $A \subset \mathbb{H}^n$. Indeed $\mathrm{div}_{\mathbb{H}}$ is invariant under group translations and the Jacobian determinant of $\tau_p : \mathbb{H}^n \to \mathbb{H}^n$ is equal 1. It is important to remark also that the $\mathbb{H}$-perimeter is homogeneous of degree $Q-1$ with respect to the dilations of the group, that means

$$|\partial(\delta_\lambda E)|_{\mathbb{H}}(A) \;=\; \lambda^{1-Q}|\partial E|_{\mathbb{H}}\left(\delta_\lambda(A)\right),$$

for every Borel set $A \subset \mathbb{H}^n$.

Proposition 1.5.3. *The perimeter measure is lower semicontinuous with respect to the L^1 convergence of the characteristic functions of the sets.*

From the 1.5.1, in the special case when $f = \mathbf{1}_E$ and E is a $\mathbb{H}$-Caccioppoli set, the section $\sigma_{\mathbf{1}_E}$ can be interpreted as a generalized inward normal to the set E and shall be indicated as ν_E. Theorem 1.5.1 can be restated as follows:

Theorem 1.5.4. *There exists a $|\partial E|_{\mathbb{H}}$-measurable section ν_E of $H\mathbb{H}^n$ such that*

$$\int_E \mathrm{div}_{\mathbb{H}}\varphi\, dh \;=\; -\int_{\mathbb{H}^n} \langle \nu_E, \varphi\rangle\; d|\partial E|_{\mathbb{H}},$$

for all $\varphi \in C_0^\infty(\Omega, H\mathbb{H}^n)$, $|\nu_E(x)|_x = 1$ for $|\partial E|_{\mathbb{H}}$-a.e. $x \in \mathbb{H}^n$.

If $E \subset \mathbb{H}^n$, we introduce the *measure theoretic boundary* $\partial_{*,\mathbb{H}}E$, which is called also *essential boundary*, and the *reduced boundary* $\partial^*_{\mathbb{H}}E$.

Definition 1.5.6. *Ler $E \subset \mathbb{H}^n$ be a measurable set, we say that $x \in \partial_{*,\mathbb{H}}E$ if*

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^{2n+2}(E \cap U(x,r))}{\mathcal{L}^{2n+2}(U(x,r))} \;>\; 0$$

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^{2n+2}(E^c \cap U(x,r))}{\mathcal{L}^{2n+2}(U(x,r))} \;>\; 0.$$

Definition 1.5.7. *Let $E \subset \mathbb{H}^n$ be a $\mathbb{H}$-Caccioppoli set. We say that $x \in \partial^*_{\mathbb{H}}E$ if*

(i) $|\partial E|_{\mathbb{H}}(U(x,r)) > 0$ *for every $r > 0$;*

(ii) *there exists* $\lim_{r\to 0} \fint_{U(x,r)} \nu_E \, d|\partial E|_{\mathbb{H}}$;

(iii) $\left|\lim_{r\to 0} \fint_{U(x,r)} \nu_E \, d|\partial E|_{\mathbb{H}}\right| = 1$.

Remark 1.5.8. As after definition 1.5.4, we observe that the $\mathbb{H}$-reduced boundary of a set is invariant under group translation, that is $x \in \partial^*_{\mathbb{H}} E$ if and only of $\tau_p(x) \in \partial^*_{\mathbb{H}}(\tau_p E)$. Moreover $\nu_E(x) = \nu_{\tau_p E}(\tau_p(x))$, for all $p \in \mathbb{H}^n$.

Remark 1.5.9. Notice that two different but equivalent doubling distances on $\mathbb{H}^n$ yield the same measure theoretic boundary of E. On the contrary, the respectively induced reduced boundary can be different.

Next result states that $\mathbb{H}$-Caccioppoli sets have an approximate tangent plane at each point of their reduced boundary. First, let us fix some notations:

Notation 1.5.1. For any set $E \subset \mathbb{H}^n$, $x_0 \in \mathbb{H}^n$ and $r > 0$, we define

$$E_{r,x_0} := \{x \in \mathbb{H}^n \,|\, x_0 \cdot \delta_r(x) \in E\}.$$

Moreover, if $\nu \in H\mathbb{H}^n$. We define the hemispaces $S^+_{\mathbb{H}}(\nu)$ and $S^-_{\mathbb{H}}(\nu)$ as follows

$$\begin{aligned} S^+_{\mathbb{H}}(\nu) &:= \{x \in \mathbb{H}^n \mid \langle \pi_{x_0} x, \nu\rangle \geq 0\}, \\ S^-_{\mathbb{H}}(\nu) &:= \{x \in \mathbb{H}^n \mid \langle \pi_{x_0} x, \nu\rangle \leq 0\}. \end{aligned} \tag{1.16}$$

The common topological boundary $\mathbb{N}(\nu)$ of $S^+_{\mathbb{H}}(\nu)$ and $S^-_{\mathbb{H}}(\nu)$ is

$$\mathbb{N}(\nu) := \{x \in \mathbb{H}^n \mid \langle \pi_{x_0} x, \nu\rangle = 0\}.$$

Theorem 1.5.5. *Let $E \subset \mathbb{H}^n$ be a $\mathbb{H}$-Caccioppoli set and $x_0 \in \partial^*_{\mathbb{H}} E$. Let $\nu_E(x_o) \in H_{x_0}\mathbb{H}^n$ be the unit inward normal as defined above. Then*

$$\lim_{r\to 0} \mathbf{1}_{E_{r,x_0}} = \mathbf{1}_{S^+_{\mathbb{H}}(\nu_E(x_0))} \quad \textit{in } L^1_{loc}(\mathbb{H}^n).$$

Moreover, for every $R > 0$,

$$\lim_{r\to 0} |\partial E_{r,x_0}|_{\mathbb{H}}(U(0,R)) = |\partial S^+_{\mathbb{H}}(\nu_E(x_0))|_{\mathbb{H}}(U(0,R)).$$

Proposition 1.5.6. *Let E be a $\mathbb{H}$-Caccioppoli set. Then*

(i) $\partial^*_{\mathbb{H}} E \subseteq \partial_{*,\mathbb{H}} E \subseteq \partial E$;

(ii) $\mathcal{S}_d^{Q-1}\left(\partial_{*,\mathbb{H}}E \setminus \partial_{\mathbb{H}}^{*}E\right) = 0$.

Proposition 1.5.7. *Let E be a $\mathbb{H}$-Caccioppoli set. Then*

$$|\partial E|_{\mathbb{H}} = c\,\mathcal{S}_d^{2n+1} \llcorner \partial_{*,\mathbb{H}}E,$$

where c is a dimensional constant.

1.6 Convolution on Groups

In this last section, we would like to recall the classical technique of convolution in homogeneous groups (see [10]). Let $\mathbb{G}$ be a Carnot group with Lie algebra $\mathfrak{g}$. We fix over $\mathbb{G}$ a homogeneous metric. Let $\eta \in C_0^\infty(\mathbb{G})$ be such that

(i) $0 \leq \eta \leq 1$,

(ii) $\int_{\mathbb{G}} \eta(x)\, dh(x) = 1$,

(iii) $\mathrm{supp}(\eta) \subset U(0,1)$.

Let $\varepsilon \in \mathbb{R}^+$, we denote, for $x \in \mathbb{G}$,

$$\eta_\varepsilon(x) := \varepsilon^{-Q}\eta\left(\delta_{\frac{1}{\varepsilon}}(x)\right)$$

the *standard mollifier*. Thanks to this definition, we can construct a convolution as follows: let $f : \mathbb{G} \longrightarrow \mathbb{R}$,

$$\begin{aligned}(\eta_\varepsilon * f)(x) :&= \int_{\mathbb{G}} \eta_\varepsilon(y) f\left(y^{-1}\cdot x\right)\, dh(y) \\ &= \int_{\mathbb{G}} \eta_\varepsilon\left(x\cdot z^{-1}\right) f(z)\, dh(z).\end{aligned}$$

Remark 1.6.1. Analogously to the classical case, if $f \in L^p(\mathbb{G})$, $1 \leq p < \infty$, then $\eta_\varepsilon * f \in C^\infty(\mathbb{G})$.

Proposition 1.6.1. *The following properties hold:*

(i) *if $f \in C^0(\mathbb{G})$, for a suitable open set $\Omega \subset \mathbb{G}$, then $\eta_\varepsilon * f \longrightarrow f$ uniformly on compact subsets of Ω as $\varepsilon \to 0$;*

(ii) *if* $f \in L^p_{loc}(\mathbb{G})$, $1 \le p < \infty$, *then* $\eta_\varepsilon * f \longrightarrow f$ *in* $L^p_{loc}(\mathbb{G})$, *as* $\varepsilon \to 0$*;*

(iii) $X(\eta_\varepsilon * f) = \eta_\varepsilon * X(f)$, *for any* $f \in C^1(\mathbb{G})$ *and for any* $X \in \mathfrak{g}$.

Proof. We start with the proof of (i). Consider an open subset V of Ω and choose a compact set W, such that $V \subset W \subset \Omega$. Let $x \in V$ be fixed, we compute

$$\begin{aligned} \eta_\varepsilon * f(x) &= \int_{\mathbb{G}} \eta_\varepsilon(x \cdot y^{-1}) f(y)\, dh(y) \\ &= \varepsilon^{-Q} \int_{U(x,\varepsilon)} \eta\left(\delta_{\frac{1}{\varepsilon}}(x \cdot y^{-1})\right) f(y)\, dh(y) \\ &= \int_{U(0,1)} \eta(x) f\left((\delta_\varepsilon(z))^{-1} \cdot x\right)\, dh(z). \end{aligned} \tag{1.17}$$

Just applying the definition of mollifier, one has

$$|\eta_\varepsilon * f(x) - f(x)| \le \int_{U(0,1)} \eta(z) \cdot \left| f\left((\delta_\varepsilon(z))^{-1}\right) - f(x)\right|\, dh(z).$$

Now, f is continuous on W, therefore we can conclude, from the last estimate, that $\eta_\varepsilon * f \to f$ uniformly on V, and the proof of (i) follows.

Let us prove (ii). Consider $f \in L^p_{loc}(\Omega)$, $\Omega \subset \mathbb{G}$ open subset, and take $V \subset W \subset \Omega$, as above. We fix $x \in V$ and $\varepsilon \in \mathbb{R}^+$. For case $1 < p < \infty$:

$$\begin{aligned} |\eta_\varepsilon * f(x)| &\le \int_{U(0,1)} \left|\eta(z) f\left((\delta_\varepsilon(z))^{-1} \cdot x\right)\right|\, dh(z) \\ &= \int_{U(0,1)} |\eta(z)|^{\frac{1}{q}} \, |\eta(z)|^{\frac{1}{p}} \left| f\left((\delta_\varepsilon(z))^{-1} \cdot x\right)\right|\, dh(z), \end{aligned} \tag{1.18}$$

where $\frac{1}{q} + \frac{1}{p} = 1$. Using Hölder's inequality, we get:

$$\begin{aligned} |\eta_\varepsilon * f(x)| &\le \left(\int_{U(0,1)} \eta(z)\, dh(z)\right)^{\frac{1}{q}} \left(\int_{U(0,1)} \eta(z) \left| f\left((\delta_\varepsilon(z))^{-1} \cdot x\right)\right|^p dh(z)\right)^{\frac{1}{p}} \\ &= \left(\int_{U(0,1)} \eta(z) \left| f\left((\delta_\varepsilon(z))^{-1} \cdot x\right)\right|^p dh(z)\right)^{\frac{1}{p}}. \end{aligned} \tag{1.19}$$

Now, we need to extend also to the case $p = 1$:

$$\begin{aligned} \int_V |\eta_\varepsilon * f(x)|^p\, dh(x) &\le \int_{U(0,1)} \left(\int_V \left| f\left((\delta_\varepsilon(z))^{-1} \cdot x\right)\right|^p dh(x)\right) dh(z) \\ &\le \int_W |f(y)|^p\, dh(y), \end{aligned} \tag{1.20}$$

for $\varepsilon \in \mathbb{R}^+$ small enough. Now, fix $\delta \in \mathbb{R}^+$. Since $f \in L^p(W)$, there exists $g \in C^0\left(\overline{W}\right)$ such that

$$\|f - g\|_{L^p(W)} < \delta.$$

By inequality (1.20), we can also say that

$$\|\eta_\varepsilon * f - \eta_\varepsilon * g\|_{L^p(V)} < \delta.$$

Therefore,

$$\begin{aligned} \int_V |\eta_\varepsilon * f(x) - f(x)|^p \, dh(x) \;\le\; & \int_V |\eta_\varepsilon * f(x) - \eta_\varepsilon * g(x)|^p \, dh(x) + \\ & + \int_V |\eta_\varepsilon * g(x) - g(x)|^p \, dh(x) + \int_V |g(x) - f(x)|^p \, dh(x), \end{aligned} \tag{1.21}$$

and so $\|\eta_\varepsilon * f - f\|_{L^p(V)} \le 2\delta + \|\eta_\varepsilon * g - g\|_{L^p(V)}$. Using assertion (i), also (ii) is proved.

We conclude by proving (iii). Let $f \in C^1(\mathbb{G})$ and $X \in \mathfrak{g}$ be fixed. We compute

$$\begin{aligned} X(\eta_\varepsilon * f) &= X\left(\int_{\mathbb{G}} \eta_\varepsilon(y) f(y^{-1} \cdot x)\, dh(y)\right) \\ &= \int_{\mathbb{G}} X\left(\eta_\varepsilon(y) f(y^{-1} \cdot x)\right) dh(y). \end{aligned} \tag{1.22}$$

Now, since $X \in \mathfrak{g}$ is a left invariant vector field, we have

$$\begin{aligned} \int_{\mathbb{G}} X\left(\eta_\varepsilon(y) f(y^{-1} \cdot x)\right) dh(y) &= \int_{\mathbb{G}} \eta_\varepsilon(y)\,(Xt)\,(y^{-1} \cdot x)\, dh(y) \\ &= \eta_\varepsilon * X(f)(x), \end{aligned} \tag{1.23}$$

and the proof of the statement is complete. □

Lemma 1.6.2. *Any left invariant vector field $X \in \mathfrak{g}$ is self-adjoint, i.e.*

$$\int_{\mathbb{G}} v\, Xv = -\int_{\mathbb{G}} u\, Xv,$$

for any $u, v \in C_0^\infty(\mathbb{G})$.

Proposition 1.6.3. *Let $f : \mathbb{G} \longrightarrow \mathbb{R}$ be a continuous function and $X \in \mathfrak{g}$ be such that the distributional derivative Xf is represented by a continuous function on $\mathbb{G}$. Then*

$$X(\eta_\varepsilon * f) = \eta_\varepsilon * (Xf).$$

Proof. Since $\eta_\varepsilon * f \in C_0^\infty(\mathbb{G})$, it is sufficient to prove that

$$\langle X(\eta_\varepsilon * f)\,,\,g\rangle \;=\; \langle \eta_\varepsilon * (Xf)\,,\,g\rangle\,,$$ [9]

for any $g \in C_0^\infty(\mathbb{G})$.

Using Proposition 1.6.1 and Lemma 1.6.2, one has

$$\begin{aligned}\langle X(\eta_\varepsilon * f)\,,\,g\rangle \;&=\; -\,\langle \eta_\varepsilon * f\,,\,Xg\rangle \;=\; -\,\langle f\,,\,\eta_\varepsilon * (Xg)\rangle\\ &=\; -\,\langle f\,,\,X(\eta_\varepsilon * g)\rangle \;=\; \langle \eta_\varepsilon * (Xf)\,,\,g\rangle\,.\end{aligned}$$

□

[9] We use the classical notation

$$\langle u\,,\,v\rangle \;:=\; \int_{\mathbb{G}} u\,v,$$

for $u, v : \mathbb{G} \longrightarrow \mathbb{R}$.

Chapter 2

Intrinsic Lipschitz Graphs

In this Chapter, following [17, 16, 15], we study some characterizations and properties of graphs of functions acting between complementary subgroups of Heisenberg group $\mathbb{H}^n$. Let us think for a moment about Euclidean setting. We know that, locally, each submanifold of dimension k in $\mathbb{R}^n$ can be viewed as the graph of a function acting between $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ In the same way, a Lipschitz submanifold can be viewed, locally, as a graph of a Lipschitz function.

As already noticed in Section 1.4.1, the case of Heisenberg group (Carnot groups, more generally) needs more attention: we studied a semidirect product of homogeneous subgroups. With such a decomposition of the ambient space, we are allowed, analogously to the Euclidean case, to consider functions acting between subgroups and to study geometrical properties of their graphs.

In the first section we introduce the notion of *intrinsic graph*, where with "intrinsic" the authors in [16] mean properties defined only in terms of the group structure of $\mathbb{H}^n$ or its Lie algebra $\mathfrak{h}^n$.

In the second section, following more closely [18], we study *intrinsic Lipschitz graphs*: intrinsic graphs of intrinsic Lipschitz continuous functions.

2.1 Intrinsic Graphs

Definition 2.1.1. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be homogeneous subgroups of $\mathbb{H}^n$, with $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ a semidirect product. We say that $S \subset \mathbb{H}^n$ is a* (left) graph *over $\mathbb{G}_1$ along $\mathbb{G}_2$ (or from $\mathbb{G}_1$ to $\mathbb{G}_2$) if*

$$S \cap (\xi \cdot \mathbb{G}_2)$$

contains at most one point for all $\xi \in \mathbb{G}_1$.

Remark 2.1.2. An equivalent definition is the following: we say that $S \subset \mathbb{H}^n$ is a (left) graph from $\mathbb{G}_1$ to $\mathbb{G}_2$ if there exists a function $f : \mathcal{E} \subset \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ such that

$$S = \{\xi \cdot f(\xi) \mid \xi \in \mathcal{E}\}.$$

In this case we write $S = \text{graph}(f)$.

Proposition 2.1.1. *Let $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ be a semidirect product of $\mathbb{H}^n$ and let S be a graph from $\mathbb{G}_1$ to $\mathbb{G}_2$. Then, for all $\lambda \in \mathbb{R}^+$, $\delta_\lambda(S)$ is a graph.*

Proof. Since S is a graph, there exists a function $f : \mathcal{E} \subset \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ such that $S = \{\xi \cdot f(\xi) \mid \xi \in \mathcal{E}\}$. Let us define

$$f_\lambda := \delta_\lambda \circ f \circ \delta_{\frac{1}{\lambda}} : \delta_\lambda \mathcal{E} \subset \mathbb{G}_1 \longrightarrow \mathbb{G}_2.$$

Therefore, $\delta_\lambda S := \text{graph}(f_\lambda)$. Indeed

$$\begin{aligned} \delta_\lambda(S) &= \{\delta_\lambda(\xi \cdot f(\xi)) \mid \xi \in \mathcal{E}\} \\ &= \{\delta_\lambda(\xi) \cdot \delta_\lambda(f(\xi)) \mid \xi \in \mathcal{E}\}; \end{aligned}$$

setting $\eta := \delta_\lambda(\xi)$, $\eta \in \delta_\lambda(\mathcal{E})$, then

$$\left\{ \eta \cdot \delta_\lambda f\left(\delta_{\frac{1}{\lambda}}(\eta)\right) \,\middle|\, \eta \in \delta_\lambda(\mathcal{E}) \right\} = \{\eta \cdot f_\lambda(\eta) \mid \eta \in \delta_\lambda(\mathcal{E})\} = \text{graph}(f_\lambda).$$

□

In the following two propositions we prove that the translation of a graph is a graph as well.

Proposition 2.1.2. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Proposition 1.4.4. Let $S \subset \mathbb{H}^n$ be a left graph such that*

$$S = \{ \xi \cdot f(\xi) \,|\, \xi \in \mathcal{E} \subset \mathbb{W} \},$$

with $f : \mathcal{E} \subset \mathbb{W} \longrightarrow \mathbb{V}$. Then, for every $q \in \mathbb{H}^n$, there are

$$\begin{aligned} \mathcal{E}_q &= \{ q \cdot \xi \cdot (q_{\mathbb{V}})^{-1} \,|\, \xi \in \mathcal{E} \} \text{ and} \\ f_q &: \mathcal{E}_q \longrightarrow \mathbb{V},\ f_q(\eta) = q_{\mathbb{V}} \cdot f(q_{\mathbb{V}}^{-1} \cdot q_{\mathbb{W}}^{-1} \cdot \eta \cdot q_{\mathbb{V}}), \end{aligned}$$

such that

$$q \cdot S = \operatorname{graph}(f_q) = \{ \eta \cdot f_q(\eta) \,|\, \eta \in \mathcal{E}_q \}.$$

Proof. First of all we notice that $\mathcal{E}_q \subset \mathbb{W}$, because $\mathbb{W}$ is a normal subgroup. By definition of left graph and left translation, $q \cdot S = \{ q \cdot \xi \cdot f(\xi) \,|\, \xi \in \mathcal{E} \}$. We can write

$$\begin{aligned} q \cdot \xi \cdot f(\xi) &= q_{\mathbb{W}} \cdot q_{\mathbb{V}} \cdot \xi \cdot f(\xi) \\ &= q_{\mathbb{W}} \cdot q_{\mathbb{V}} \cdot \xi \cdot q_{\mathbb{V}}^{-1} \cdot q_{\mathbb{V}} \cdot f(\xi). \end{aligned}$$

Now, calling $\eta = q_{\mathbb{W}} \cdot q_{\mathbb{V}} \cdot \xi \cdot q_{\mathbb{V}}^{-1}$, we get $\xi = q_{\mathbb{V}}^{-1} \cdot q_{\mathbb{W}}^{-1} \cdot \eta \cdot q_{\mathbb{V}}$. Therefore $q \cdot \xi \cdot f(\xi) \in q \cdot S$ can be written as $\eta \cdot q_{\mathbb{V}} \cdot f(q_{\mathbb{V}}^{-1} \cdot q_{\mathbb{W}}^{-1} \cdot \eta \cdot q_{\mathbb{V}})$, and the proof is completed. □

Proposition 2.1.3. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Proposition 1.4.4. Let $S \subset \mathbb{H}^n$ be a left graph such that*

$$S = \{ \xi \cdot f(\xi) \,|\, \xi \in \mathcal{A} \subset \mathbb{V} \},$$

with $f : \mathcal{A} \subset \mathbb{V} \longrightarrow \mathbb{W}$. Then, for every $q \in \mathbb{H}^n$, there are

$$\begin{aligned} \mathcal{A}_q &= \{ q \cdot \xi \,|\, \xi \in \mathcal{A} \} \text{ and} \\ f_q &: \mathcal{A}_q \longrightarrow \mathbb{V},\ f_q(\eta) = \eta^{-1} \cdot q_{\mathbb{W}} \cdot \eta \cdot f(q_{\mathbb{V}}^{-1} \cdot \eta), \end{aligned}$$

such that

$$q \cdot S = \operatorname{graph}(f_q) = \{ \eta \cdot f_q(\eta) \,|\, \eta \in \mathcal{A}_q \}.$$

Proof. By definition of left graph

$$q \cdot S = \{ q \cdot \xi \cdot f(\xi) \,|\, \xi \in \mathcal{A} \},$$

Recalling Proposition 1.4.5, we rewrite

$$\begin{aligned} q\cdot\xi\cdot f(\xi) &= q_{\mathbb{W}}\cdot q_{\mathbb{V}}\cdot\xi\cdot f(\xi) \\ &= q_{\mathbb{V}}\cdot q_{\mathbb{V}}^{-1}\cdot q_{\mathbb{W}}\cdot q_{\mathbb{V}}\cdot\xi\cdot f(\xi) \\ &= q_{\mathbb{V}}\cdot\xi\cdot\xi^{-1}\cdot q_{\mathbb{V}}^{-1}\cdot q_{\mathbb{W}}\cdot q_{\mathbb{V}}\cdot\xi\cdot f(\xi). \end{aligned}$$

Notice that $q_{\mathbb{V}}\cdot\xi\in\mathbb{V}$ and $\xi^{-1}\cdot q_{\mathbb{V}}^{-1}\cdot q_{\mathbb{W}}\cdot q_{\mathbb{V}}\cdot\xi\in\mathbb{W}$. Then, setting $\eta := q_{\mathbb{V}}\cdot\xi$ and observing that

$$\xi^{-1}\cdot q_{\mathbb{V}}^{-1}\cdot q_{\mathbb{W}}\cdot q_{\mathbb{V}}\cdot\xi \;=\; \eta^{-1}\cdot q_{\mathbb{W}}\cdot q_{\mathbb{V}}\cdot q_{\mathbb{V}}^{-1}\cdot\xi \;=\; \eta^{-1}\cdot q_{\mathbb{W}}\cdot\eta,$$

$q\cdot\xi\cdot f(\xi)\in q\cdot S$ can be rewrite as

$$q\cdot\xi\cdot f(\xi) \;=\; \eta^{-1}\cdot q_{\mathbb{W}}\cdot\eta\cdot f(q_{\mathbb{W}}^{-1}\cdot\eta) \;=\; f_q(\eta).$$

□

Remark 2.1.3. Let $f:\mathcal{A}\subset\mathbb{V}\longrightarrow\mathbb{W}$ be such that $S=\{\,\xi\cdot f(\xi)\,|\,\xi\in\mathcal{A}\subset\mathbb{V}\,\}$ is a left graph in $\mathbb{H}^n$. Then S is also an Euclidean graph over $\mathbb{V}$. Indeed, recalling that $\mathbb{V}$ is isometric and isomorphic to $\mathbb{R}^k$, for some $0<k<2n+1$, we can identify $\mathbb{V}$ with a k-dimensional vector subspace of $\mathbb{R}^{2n+1}$. On the contrary, if $S=\mathrm{graph}(f)$, where $f:\mathcal{E}\subset\mathbb{W}\longrightarrow\mathbb{V}$, then, in general, S is not an Euclidean graph.

An example is given in [16]: consider the semidirect product $\mathbb{H}^1=\mathbb{W}\cdot\mathbb{V}$, where $\mathbb{W}=\{\,(0,y,t)\,|\,y,t\in\mathbb{R}\,\}$ and $\mathbb{V}=\{\,(x,0,0)\,|\,x\in\mathbb{R}\,\}$. Let us fix $\frac{1}{2}<\alpha<1$ and take $f:\mathbb{W}\longrightarrow\mathbb{V}$ defined as

$$f(0,y,t) \;=\; (|t|^\alpha,0,0).$$

It is clear that $\mathrm{graph}(f)=S$ is not an Euclidean graph near the origin:

$$S \;=\; \{\,\xi\cdot f(\xi)\,|\,\xi\in\mathbb{W}\,\} \;=\; \{\,(|t|^\alpha,y,t+2y|t|^\alpha)\,|\,t,y\in\mathbb{R}\,\}.$$

2.2 Intrinsic Lipschitz Graphs

In the previous Section we studied graphs of functions acting between complementary subgroups of $\mathbb{H}^n$. Now we aim to specialize to graphs of *intrinsic Lipschitz continuous functions*. This notion was originally suggested in [15], where the authors describe submanifolds in Heisenberg group. Let us see more details about, in order to justify the notion of intrinsic Lipschitz continuity.

Definition 2.2.1. *Let k be an integer, $1 \leq k \leq n$.*

(i) *We say that $S \subset \mathbb{H}^n$ is a k-dimensional* H-regular submanifold *if, for each $p \in S$, there exists an open neighbourhood U of p in $\mathbb{H}^n$, an open set $\mathcal{E} \subset \mathbb{R}^k$ and an injective continuously P-differentiable function $f : \mathcal{E} \longrightarrow U$, with injective P-differential, such that*

$$S \cap U \;=\; f(\mathcal{E}).$$

(ii) *We say that $S \subset \mathbb{H}^n$ is a* k-codimensional H-regular submanifold *if, for each $p \in S$, there exist an open neighbourhood $U \subset \mathbb{H}^n$ of p, $f \in C^1_H(U, \mathbb{R}^k)$, with surjective P-differential, such that*

$$S \cap U \;=\; \{\, x \in U \,|\, f(x) = 0 \,\}.$$

Remark 2.2.2. Although Definition 2.2.1 seems similar to the Euclidean case, we emphasize that, for example, k-codimensional H-regular submanifolds can be very irregular objects from the Euclidean point of view (for the details we refer the reader to [15]).

We state now the *Implicit function Theorem* for k-codimensional H-regular submanifolds, without giving the proof. It will be clear how this theorem suggested the notion of intrinsic Lipschitz continuity that we are going to introduce.

Theorem 2.2.1. *Let S be a k-codimensional H-regular submanifold, with $1 \leq k \leq n$. Then, for each $p_0 \in S$, there are an open set $U \subset \mathbb{H}^n$, with $p \in U$, and complementary subgroups $\mathbb{W}$ and $\mathbb{V}$ of $\mathbb{H}^n$ as in Proposition 1.4.4, such that*

$$(d_p f)\,|_{\mathbb{V}} \,:\, \mathbb{V} \longrightarrow \mathbb{R}^k$$

is injective for all $p \in U$. Moreover, there are an open set $\mathcal{E} \subset \mathbb{W}$[1] and continuous $\varphi : \mathcal{E} \longrightarrow \mathbb{V}$ such that

$$S \cap U \;=\; \{\, \xi \cdot \varphi(\xi) \,|\, \xi \in \mathcal{E} \,\}.$$

Finally, there is a positive constant L such that, for all ξ and $\bar{\xi} \in \mathcal{E}$,

$$\|\varphi(\bar{\xi})^{-1} \cdot \varphi(\xi)\| \;\leq\; L\, \|\varphi(\bar{\xi})^{-1} \cdot (\bar{\xi}^{-1} \cdot \xi) \cdot \varphi(\bar{\xi})\|. \tag{2.1}$$

[1]When we say that $\mathcal{E}$ is open in $\mathbb{W}$, we mean with respect to the relative topology of $\mathbb{W}$, induced by $\mathbb{H}^n$.

Exactly from inequality (2.1), we have the following

Definition 2.2.3. *Let* $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ *be a semidirect product. We say that*

$$f : \mathcal{E} \subset \mathbb{G}_1 \longrightarrow \mathbb{G}_2$$

is an intrinsic Lipschitz continuous function *if there exists a positive constant* L *such that, for all* $q \in \text{graph}(f)$,

$$\|f_{q^{-1}}(x)\| \leq L\,\|x\|, \tag{2.2}$$

for each $x \in \mathcal{E}_{q^{-1}}$. *As usual, we call the* intrinsic Lipschitz constant *of* f *the infimum of the numbers* L *such that* (2.2) *holds.*

Remark 2.2.4. Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Proposition 1.4.4. Using Propositions 2.1.2 and 2.1.3, we can specify the two cases:

(i) $f : \mathbb{W} \longrightarrow \mathbb{V}$ is said an intrinsic Lipschitz function, if there exists a positive constant L such that, for all $\xi, \bar{\xi} \in \mathbb{W}$,

$$\|f(\xi)^{-1} \cdot f(\bar{\xi})\| \leq L\,\|f(\xi)^{-1} \cdot \xi^{-1} \cdot \bar{\xi} \cdot f(\xi)\|;$$

(ii) $f : \mathbb{V} \longrightarrow \mathbb{W}$ is said an intrinsic Lipschitz function, if there exists a positive constant L such that, for all $\eta, \bar{\eta} \in \mathbb{V}$,

$$\|\bar{\eta}^{-1} \cdot \eta \cdot f(\eta)^{-1} \cdot \eta^{-1} \cdot \bar{\eta} \cdot f(\bar{\eta})\| \leq L\,\|\eta^{-1} \cdot \bar{\eta}\|.$$

Our aim now is to give a more geometrical definition of intrinsic Lipschitz continuity: if $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ is a semidirect product and $f : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$, we will say that f is intrinsic Lipschitz continuous if, at each $p \in \text{graph}(f)$, there is an intrinsic closed cone, with vertex p and axis $\mathbb{G}_2$, intersecting graph(f) only in p. Once again, "intrinsic cone" means that properties of the cone depend only on the structure of the Lie algebra $\mathfrak{h}^n$.

Definition 2.2.5. *Let* $\mathbb{H}^n$ *be the semidirect product of two subgroups* $\mathbb{G}_1$ *and* $\mathbb{G}_2$. *Let* $q \in \mathbb{G}$ *and* $\alpha \in \mathbb{R}^+$ *be fixed. We call* intrinsic closed cone *with base* $\mathbb{G}_1$, *axis* $\mathbb{G}_2$, *vertex* q *and opening* α

$$C_{\mathbb{G}_1,\mathbb{G}_2}(q, \alpha) := q \cdot C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha),$$

where

$$C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha) := \{p \in \mathbb{H}^n \mid \|p_{\mathbb{G}_1}\| \leq \alpha \|p_{\mathbb{G}_2}\|\}.$$

Proposition 2.2.2. *Let* $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ *be a semidirect product,* $t \in \mathbb{R}^+$ *and* $0 < \alpha < \beta$. *Then the following statements hold:*

(i) $C_{\mathbb{G}_1,\mathbb{G}_2}(e, 0) \; = \; \mathbb{G}_2$;

(ii) $C_{\mathbb{G}_1,\mathbb{G}_2}(q, \alpha) \; \subset \; C_{\mathbb{G}_1,\mathbb{G}_2}(q, \beta)$;

(iii) $\delta_t \left(C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha)\right) \; = \; C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha)$.

Proof. Since (i) and (ii) are trivial, we prove just (iii). Because of the uniqueness of the components $(\delta_t p)_{\mathbb{G}_i} = \delta_t(p_{\mathbb{G}_i})$, for $i = 1, 2$, then

$$\begin{aligned}
\delta_t(C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha)) \; &= \; \delta_t\{p \in \mathbb{H}^n \mid \|\delta_t(p)_{\mathbb{G}_1} \; \leq \; \alpha \, \|p_{\mathbb{G}_2}\|\} \\
&= \; \{\delta_t(p) \in \mathbb{H}^n \mid \|\delta_t(p)_{\mathbb{G}_1}\| \; \leq \; \alpha \, \|\delta_t(p)_{\mathbb{G}_2}\|\} \\
&= \; \{\delta_t(p) \in \mathbb{H}^n \mid t \, \|p_{\mathbb{G}_1}\| \; \leq \; \alpha \, t \, \|p_{\mathbb{G}_2}\|\} \\
&= \; \{\delta_t(p) \in \mathbb{H}^n \mid \|p_{\mathbb{G}_1}\| \; \leq \; \alpha \, \|p_{\mathbb{G}_2}\|\} \; = \; C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha).
\end{aligned}$$

□

Definition 2.2.6. *Let* $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ *be a semidirect product. We say that*

$$f \, : \, \mathcal{E} \, \subset \, \mathbb{G}_1 \, \longrightarrow \, \mathbb{G}_2$$

is intrinsic Lipschitz continuous *in* $\mathcal{E}$, *if there exists a positive constant* L *such that, for all* $q \in \mathrm{graph}(f)$,

$$C_{\mathbb{G}_1,\mathbb{G}_2}\left(q, \frac{1}{L}\right) \cap \mathrm{graph}(f) \; = \; \{q\}. \tag{2.3}$$

As usual we call the Lipschitz constant *of* f *in* $\mathcal{E}$ *the infimum of the numbers* L *such that* (2.3) *holds.*

We prove now the equivalence between the two definitions of intrinsic Lipschitz continuity:

Proposition 2.2.3. *Let* $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ *be as in Proposition 1.4.4. A function* $f : \mathbb{W} \longrightarrow \mathbb{V}$ *is intrinsic Lipschitz continuous according to Definition 2.2.3, with Lipschitz constant* L, *if and only if , for each* $q \in \mathrm{graph}(f)$ *and for all* α *such that* $0 \leq \alpha < \frac{1}{L}$,

$$C_{\mathbb{W},\mathbb{V}}(q, \alpha) \cap \mathrm{graph}(f) \; = \; \{q\}.$$

Proof. If $q \in \text{graph}(f)$,

$$C_{\mathbb{W},\mathbb{V}}(e,\alpha) \cap \text{graph}(f_{q^{-1}}) = \{e\},$$

hence, by Proposition 2.2.2,

$$\tau_q\left(C_{\mathbb{W},\mathbb{V}}(e,\alpha) \cap \text{graph}(f_{q^{-1}})\right) = \{q\}.$$

On the other hand,

$$\begin{aligned}\tau_q\left(C_{\mathbb{W},\mathbb{V}}(e,\alpha) \cap \text{graph}(f_{q^{-1}})\right) &= \tau_q\left(C_{\mathbb{W},\mathbb{V}}(e,\alpha)\right) \cap \tau_q\left(\tau_{q^{-1}}\text{graph}(f)\right)\\ &= C_{\mathbb{W},\mathbb{V}}(q,\alpha) \cap \text{graph}(f).\end{aligned}$$

□

Proposition 2.2.4. *Let $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ be as in Proposition 1.4.4. Then*

(i) *$f : \mathcal{E} \subset \mathbb{V} \longrightarrow \mathbb{W}$ is intrinsic Lipschitz continuous in $\mathcal{E}$, if and only if the parametrization map*

$$\Phi_f : \mathcal{E} \longrightarrow \mathbb{H}^n,$$

defined as $\Phi_f(v) = v\cdot f(v)$, is metric Lipschitz continuous

(ii) *$f : \mathcal{E} \subset \mathbb{V} \longrightarrow \mathbb{W}$ is intrinsic Lipschitz continuous in $\mathcal{E}$, if and only if there is a positive constant L such that, for all $\eta, \bar\eta \in \mathcal{E}$,*

$$\|f(\xi)^{-1}\cdot f(\bar\xi)\| \le L\,\|f(\xi)^{-1}\cdot \xi^{-1}\cdot \bar\xi^{-1}\cdot f(\xi)\|.$$

Proof. We start by proving statement (i). If $q = x\cdot f(x) \in \text{graph}(f)$, then, from Proposition 2.1.3, for each $\eta \in \mathcal{E}_{q^{-1}}$,

$$f_{q^{-1}}(\eta) = \eta^{-1}\cdot f(x)^{-1}\cdot \eta\cdot f(x\cdot\eta).$$

Assuming that Φ_f is metric Lipschitz continuous, denoting by $\tilde L$ its Lipschitz constant, for $\eta = x^{-1}\cdot v$, we have

$$\begin{aligned}\|f_{q^{-1}}(\eta)\| &= \|v^{-1}\cdot x\cdot f(x)^{-1}\cdot x^{-1}\cdot v\cdot f(v)\|\\ &\le \|v^{-1}\cdot x\| + \|f(x)^{-1}\cdot x^{-1}\cdot v\cdot f(v)\|\\ &= \|v^{-1}\cdot x\| + \|\Phi_f(x)^{-1}\cdot\Phi_f(v)\|\\ &\le \|v^{-1}\cdot x\| + \tilde L\,\|x^{-1}\cdot v\|\\ &= (1+\tilde L)\,\|x^{-1}\cdot v\| = (1+\tilde L)\,\|\eta\|.\end{aligned}$$

Vice versa, assume that f is intrinsic Lipschitz. We want to show that Φ_f is metric Lipschitz. We start writing, for $\bar{v} = x \cdot v$,

$$\begin{aligned} \Phi_f(v)^{-1} \cdot \Phi_f(\bar{v}) &= f(v)^{-1} \cdot v^{-1} \cdot \bar{v} \cdot f(\bar{v}) \\ &= f(v)^{-1} \cdot v^{-1} \cdot (x \cdot v) \cdot f(x \cdot v) \\ &= x \cdot x^{-1} \cdot f(v^{-1}) \cdot x \cdot f(x \cdot v) \\ &= x \cdot f_{q^{-1}}(x), \end{aligned}$$

where the last equality follows from Proposition 2.1.3. Now, taking the norm,

$$\begin{aligned} \|\Phi_f(v)^{-1} \cdot \Phi_f(\bar{v})\| &= \|x \cdot f_{q^{-1}}(x)\| \\ &\leq \|x\| + \|f_{q^{-1}}(x)\| \\ &\leq (2+L)\,\|x\| = (2+L)\,\|v^{-1} \cdot \bar{v}\|. \end{aligned}$$

Let us prove statement (ii). By Proposition 2.2.3 and Proposition 2.1.3, for each $\bar{x} \in \mathcal{E}$ and $y \in \mathcal{E}_{q^{-1}}$,

$$\|f_{q^{-1}}(y)\| = \left\|f(\bar{x})^{-1} \cdot f(\bar{x} \cdot f\left(\bar{x}\right) \cdot y \cdot f(\bar{x})^{-1})\right\| \leq L\,\|y\|.$$

Setting $x := \bar{x} \cdot f(\bar{x}) \cdot y \cdot f(\bar{x})^{-1}$, $y = f(\bar{x})^{-1} \cdot \bar{x}^{-1} \cdot x \cdot f(\bar{x})$, then it follows that

$$\|f(\bar{x})^{-1} \cdot f(x)\| \leq L\,\|f(\bar{x})^{-1} \cdot (\bar{x}^{-1} \cdot x) \cdot f(\bar{x})\|.$$

□

Remark 2.2.7. Since $\mathbb{W}$ and $\mathbb{V}$ are subsets of $\mathbb{H}^n$, they are metric spaces, then also the usual definition of *metric Lipschitz continuity* is available. We say that $f : \mathbb{W} \longrightarrow \mathbb{V}$, or $f : \mathbb{V} \longrightarrow \mathbb{W}$, is a *metric continuous Lipschitz function* if there is a constant $L > 0$ such that, for all $\eta, \bar{\eta} \in \mathbb{V}$,

$$\|f(\eta)^{-1} \cdot f(\bar{\eta})\| = d(f(\eta), f(\bar{\eta})) \leq L\,d(\eta, \bar{\eta}) = L\,\|\eta^{-1} \cdot \bar{\eta}\|. \tag{2.4}$$

As pointed out in [17], the definition of metric Lipschitz continuous function could seem to be more natural. Actually, intrinsic Lipschitz continuity is more appropriate for functions acting between complementary subgroups and, first and last, more closely related to the group structure. Indeed, intrinsic Lipschitz continuity is invariant under left translations:

Proposition 2.2.5. *Let $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ be a semidirect product. Then, for all $q \in \mathbb{H}^n$, $f : \mathcal{E} \subset \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ is intrinsic Lipschitz in $\mathcal{E}$, with constant $L > 0$, if and only if $f_q : \mathcal{E}_q \subset \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ is intrinsic Lipschitz in $\mathcal{E}_q$ with the same constant L.*

Proof. By definition, we know that $\mathrm{graph}(f_q) = q \cdot \mathrm{graph}(f)$. Hence, $p \in \mathrm{graph}(f_q)$ if and only if $p = q \cdot \bar{p}$, for some $\bar{p} \in \mathrm{graph}(f)$. First, assume that f is intrinsic Lipschitz, then

$$\begin{aligned}\{p\} &= \{q \cdot \bar{p}\} = q \cdot (C_{\mathbb{W},\mathbb{V}}(\bar{p}, \alpha) \cap \mathrm{graph}(g)) \\ &= C_{\mathbb{W},\mathbb{V}}(p, \alpha) \cap \mathrm{graph}(f_q).\end{aligned}$$

Hence f_q is intrinsic Lipschitz. The other implication can be analogously deduced keeping in mind that, if $p, q \in \mathbb{H}^n$,

$$\begin{aligned}\mathrm{graph}\left((f_p)_q\right) &= q \cdot \mathrm{graph}(f_p) = q \cdot (p \cdot \mathrm{graph}(f)) \\ &= (q \cdot p) \cdot \mathrm{graph}(f).\end{aligned}$$

□

Let us now consider two examples (Example 3.3 in [17]). In the first, we consider a function which is metric Lipschitz but not intrinsic Lipschitz. At the same time, we will show that metric Lipschitz continuity is not a left-invariant property.

Example 2.2.1. Let us consider the semidirect product $\mathbb{H}^1$ with the homogeneous norm $\|\cdot\|$ defined as in Section 1.2.1. Let $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$, where

$$\mathbb{W} = \{(0, y, t) \mid y, t \in \mathbb{R}\} \text{ and } \mathbb{V} = \{(x, 0, 0) \mid x \in \mathbb{R}\}.$$

Notice that, for $w = (0, y, t) \in \mathbb{W}$, $\|w\| = \max\{|y|, |t|^{\frac{1}{2}}\}$, and, for $v = (x, 0, 0) \in \mathbb{V}$, $\|v\| = |x|$. Let $f : \mathbb{W} \longrightarrow \mathbb{V}$ be defined as

$$f(0, y, t) = (1 + |t|^{\frac{1}{2}}, 0, 0).$$

It is a metric Lipschitz continuous function, indeed

$$\|f(0, y, t)^{-1} \cdot f(0, \bar{y}, \bar{t})\| = \left| |\bar{t}|^{\frac{1}{2}} - |t|^{\frac{1}{2}} \right| \leq |\, \bar{t} - t \,|^{\frac{1}{2}} = \|\, (0, y, t)^{-1} \cdot (0, \bar{y}, \bar{t}) \,\|.$$

On the contrary, f is not intrinsic Lipschitz continuous. To see this we translate graph(f) moving $p = (1, 0, 0) \in$ graph(f) to the origin e. We know, by Proposition 2.1.2, that the translated set is the graph of $f_{p^{-1}} : \mathbb{W} \longrightarrow \mathbb{V}$, which is, setting $\eta = (0, y, t)$,

$$f_{p^{-1}}(\eta) \;=\; (p^{-1})_{\mathbb{V}} \cdot f(p_{\mathbb{V}} \cdot p_{\mathbb{W}} \cdot \eta \cdot p_{\mathbb{V}}^{-1}) \;=\; (|\, y + t \,|^{\frac{1}{2}}, 0, 0).$$

The definition of intrinsic Lipschitz continuity, applied to this function, should be equivalent to the inequality

$$|\, y + t \,|^{\frac{1}{2}} \;\leq\; L \cdot \max\{|y|, |t|^{\frac{1}{2}}\}, \tag{2.5}$$

that is, in general, not true. This shows us also that property (2.4) is not invariant under graph translations, indeed the following inequality, because of (2.5), is false:

$$\begin{aligned} \|f_{p^{-1}}(0, y, t)^{-1} \cdot f_{p^{-1}}(0, \bar{y}, \bar{t})\| \;&=\; |(|\, \bar{y} + \bar{t} \,|^{\frac{1}{2}} \;+\; |\, y + t \,|^{\frac{1}{2}}, 0, 0)| \\ &\leq\; \tilde{L}\, \|(0, y, t)^{-1} \cdot (0, \bar{y}, \bar{t})\| \\ &=\; \tilde{L}\, |\, \bar{t} - t \,|^{\frac{1}{2}}, \end{aligned}$$

for some constant $\tilde{L}$.

The previous example shows us that metric Lipschitz continuous functions are not necessarily intrinsic Lipschitz. Also the converse is true: not each intrinsic Lipschitz continuous function is metric Lipschitz.

Example 2.2.2. Consider again the semidirect product in Example 2.2.1. Let $g : \mathbb{W} \longrightarrow \mathbb{V}$ be defined as follows

$$g((0, y, t)) \;:=\; \Big(1 + |\, t - y \,|^{\frac{1}{2}}, 0, 0\Big)\,.$$

We aim to show that it is not metric Lipschitz continuous but it is intrinsic Lipschitz. Let us start proving that it is not metric Lipschitz:

$$\begin{aligned} \|g^{-1}(0, y, t) \cdot g(0, \bar{y}, \bar{t})\| \;&=\; \Big|\Big(|\, \bar{t} - \bar{y} \,|^{\frac{1}{2}} \;-\; |\, t - y \,|^{\frac{1}{2}}, 0, 0\Big)\Big| \\ &=\; |\, \bar{t} \;-\; t \;-\; (\, \bar{y} \;-\; y \,) \,|^{\frac{1}{2}}\,, \end{aligned}$$

and there are no constants $\tilde{L} > 0$ such that

$$|\, \bar{t} \;-\; t \;-\; \bar{y} \;-\; y \,|^{\frac{1}{2}} \;\leq\; \tilde{L}\; \big\|\, (0, y, t)^{-1} \cdot (0, \bar{y}, \bar{t}) \,\big\|\,.$$

On the contrary, g is intrinsic Lipschitz continuous. Indeed, if $p = (1, 0, 0)$ and we define

$$\begin{aligned} \varphi \,:\, \mathbb{W} &\longrightarrow \mathbb{V} \\ \varphi(w) &:= \left(|t|^{\frac{1}{2}}, 0, 0 \right), \end{aligned}$$

we have that $g(w) = \varphi_p(w)$, therefore, since φ is trivially intrinsic Lipschitz continuous, by Poposition 2.2.5, g is intrinsic Lipschitz too.

In Proposition 1.4.3 on page 24, we studied all possible semidirect products in which one can decompose the Heisenberg group. In particular, we proved that $\mathbb{H}^n$ always can be decomposed in a horizontal subgroup $\mathbb{V}$ with linear dimension, say k, and a vertical subgroup $\mathbb{W}$ with metric dimension $2n + 2 - k$. The following proposition tells us what is the relation between metric dimension of the graph of an intrinsic Lipschitz function and metric dimension of the subgroup $\mathbb{W}$.

Proposition 2.2.6. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Proposition 1.4.4, and let $1 \leq k \leq n$ be the dimension of $\mathbb{V}$. If $f : \mathcal{E} \subset \mathbb{W} \longrightarrow \mathbb{V}$ is an intrinsic L-Lipschitz function and $\mathcal{E}$ is an open set $\mathbb{W}$, then* graph(f) *has metric dimension $2n + 2 - k$. Moreover, for any $Rin\mathbb{R}^+$ there is a positive geometric constant $C = C(\mathbb{W}, \mathbb{V}, L, R)$ such that, for all $p \in \mathbb{H}^n$,*

$$\mathcal{S}_d^{2n+2-k} \left(\mathrm{graph}(f) \cap B(p, R) \right) \,\leq\, C. \tag{2.6}$$

Simmetrically, if $f : \mathcal{E} \subset \mathbb{V} \longrightarrow \mathbb{W}$ is intrinsic L-Lipschitz and $\mathcal{E}$ is an open subset of $\mathbb{V}$, then graph(f) *has metric dimension k and, for all $p \in \mathbb{H}^n$ and $R \in \mathbb{R}^+$,*

$$\mathcal{S}_d^{k} \left(\mathrm{graph}(f) \cap B(p, R) \right) \,\leq\, C, \tag{2.7}$$

for $C = C(\mathbb{W}, \mathbb{V}, L, R) > 0$.

Proof. We prove in details only the case of $f : \mathbb{W} \longrightarrow \mathbb{V}$. The second case is analogous to the Euclidean one. Indeed, by Porposition 2.2.4, the parametrization of graph(f), $\Phi_f : \mathbb{V} \longrightarrow \mathbb{H}^n$, is a metric Lipschitz map, then $\mathcal{S}_d^k(\mathrm{graph}(f)) < \infty$, and in particular (2.7) holds. On the other hand, by Remark 1.4.8, $\Pi_{\mathbb{V}} : \mathbb{H}^n \longrightarrow \mathbb{V}$ is a metric Lipschitz map, then $\mathcal{S}_d^k(\mathrm{graph}(f)) > 0$, implying that k is the metric dimension of graph(f).

Let $f : \mathcal{E} \subset \mathbb{W} \longrightarrow \mathbb{V}$ be an intrinsic Lipschitz function. The lower bound for $\mathcal{S}_d^{2n+2-k}((\mathrm{graph}(f))$ is consequence of Proposition 1.4.6. Indeed, assume that

$$\mathcal{S}_d^{2n+2-k}(\mathrm{graph}(f)) < \infty.$$

Let us fix $\varepsilon \in \mathbb{R}^+$ and choose $r = r(\varepsilon) > 0$. We cover graph(f) with balls $B_i = B(p_i, r_i)$ such that $r_i \le r$. By definition of Hausdorff spherical measure (see Footnote 5 on page 17),

$$\sum_i r_i^{2n+2-k} \le \mathcal{S}^{2n+2-k}(\text{graph}(f)) + \varepsilon.$$

Hence, by (1.14),

$$\begin{aligned}\mathcal{L}^{2n+2-k}(\mathcal{E}) &\le \sum_i \mathcal{L}^{2n+2-k}(\Pi_{\mathbb{W}}(B_i)) \\ &= C(\mathbb{W},\mathbb{V}) \sum_i r_i^{2n+2-k} \\ &\le C(\mathbb{W},\mathbb{V})\, \mathcal{S}_d^{2n+2-k}(\text{graph}(f)) + \varepsilon.\end{aligned}$$

Now, to get inequality (2.6), it is enough to prove that for every $p \in \text{graph}(f)$, $R > 0$ and $\varepsilon > 0$, graph$(f) \cap B(p, R)$ can be covered by at most $N := c \cdot \left(\frac{1}{\varepsilon}\right)^{2n+2-k}$ metric balls with radius less than ε.[2] Notice that, here, the constant c depends on R, $\mathbb{W}$, $\mathbb{V}$ and L.

Without loss of generality, we assume that $p = e$ (remember that intrinsic Lipschitz continuity is invariant under left translations). Using a Vitali covering argument, we choose a covering of graph$(f) \cap B(e, R)$ with metric balls $\{B(p_i, 5\varepsilon)\}_i$, where $p_i = \bar{w}_i \cdot f(\bar{w}_i) \in \text{graph}(f)$, such that $B_i = B(p_i, \varepsilon)$ are pairwise disjoint.

Let us extimate the number N. We fix a notation:

$$\mathcal{E}_i := \left\{ w \in \mathbb{W} \;\middle|\; \left\| f(\bar{w}_i)^{-1} \cdot \bar{w}_i^{-1} \cdot w \cdot f(\bar{w}_i) \right\| < \frac{\varepsilon}{(1+L)(1+2L)} \right\},$$

for $1 \le i \le N$.

Because the balls $B(p_i, \varepsilon)$ are pairwise disjointed and $f : \mathcal{E} \subset \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz,

$$\begin{aligned}2\varepsilon &\le \left\| f(\bar{w}_i)^{-1} \cdot \bar{w}_i^{-1} \cdot \bar{w}_j \cdot f(\bar{w}_j) \right\| \\ &\le \left\| f(\bar{w}_i)^{-1} \cdot \bar{w}_i^{-1} \cdot \bar{w}_j \cdot f(\bar{w}_i) \right\| + \left\| f(\bar{w}_i)^{-1} \cdot f(\bar{w}_j) \right\| \\ &\le (1+L) \left\| f(\bar{w}_i)^{-1} \cdot \bar{w}_i^{-1} \cdot \bar{w}_j \cdot f(\bar{w}_i) \right\|\end{aligned}$$

[2] It is sufficient since $\mathbb{R}^m$, with the Carnot-Carathéodory metric, is a doubling space. Doubling spaces have the following covering property: there exists a function $\varphi :]0, \frac{1}{2}] \longrightarrow]0, \infty[$ such that every set of diameter, say d, can be covered by at most $\varphi(\varepsilon)$ sets of diameter less than or equal to $\varepsilon\, d$. The function φ is called a *covering function*, and can be chosen to be of the form $\varphi(\varepsilon) = C \left(\frac{1}{\varepsilon}\right)^{\beta}$, for some constant $C \ge 1$ and $\beta > 0$. In case of $\mathbb{R}^m$, the infimum of the constants β is exactly the Hausdorff dimension.

that is

$$\frac{2\varepsilon}{1+L} \leq \left\| f(\bar{w}_i)^{-1} \cdot \bar{w}_i^{-1} \cdot \bar{w}_j \cdot f(\bar{w}_i) \right\|.$$

Now assume, by contradiction, that there exists $w \in \mathcal{E}_i \cap \mathcal{E}_j$, then

$$\begin{aligned}
\frac{2\varepsilon}{1+L} &\leq \left\| f(\bar{w}_i)^{-1} \cdot \bar{w}_i^{-1} \cdot \bar{w}_j \cdot f(\bar{w}_i) \right\| \\
&\leq \left\| f(\bar{w}_i)^{-1} \cdot \bar{w}_i^{-1} \cdot w \cdot f(\bar{w}_i) \right\| + \left\| f(\bar{w}_i)^{-1} \cdot f(w) \right\| + \left\| f(w)^{-1} \cdot f(\bar{w}_j) \right\| \\
&+ \left\| f(\bar{w}_j)^{-1} \cdot w^{-1} \cdot \bar{w}_j \cdot f(\bar{w}_j) \right\| + \left\| f(\bar{w}_j)^{-1} \cdot f(w) \right\| + \left\| f(w)^{-1} \cdot f(\bar{w}_i) \right\| \\
&< (2+4L)\frac{\varepsilon}{(1+L)(1+2L)} = \frac{2\varepsilon}{1+L},
\end{aligned}$$

a contradiction. In this way we proved that

$$\mathcal{E}_i \cap \mathcal{E}_j = \emptyset,$$

for $i \neq j$. Now observe that

$$\mathcal{E}_i = \bar{w}_i \cdot f(\bar{w}_i) \cdot \left\{ w \in \mathbb{W} \;\middle|\; \|w\| < \frac{\varepsilon}{(1+L)(1+2L)} \right\} \cdot f(\bar{w}_i)^{-1},$$

and consider the map $\chi : \mathbb{W} \to \mathbb{W}$

$$w \mapsto \chi(w) := \bar{w} \cdot \bar{v} \cdot w \cdot \bar{v}^{-1},$$

for fixed $\bar{w} \in \mathbb{W}$ and $\bar{v} \in \mathbb{V}$. It is clear that its Jacobian determinant is identically equal to 1 (here $\mathbb{W}$ is identified with $\mathbb{R}^{2n+1-k}$). Therefore, it preserves the Lebesgue measure $\mathcal{L}^{2n+1-k} \llcorner \mathbb{W}$. Hence,

$$\mathcal{L}^{2n+1-k}(\mathcal{E}_i) = \mathcal{L}^{2n+1-k}\left(\left\{ w \in \mathbb{W} \;\middle|\; \|w\| < \frac{\varepsilon}{(1+L)(1+2L)} \right\} \right) = C\varepsilon^{2n+2-k}$$

Finally, the sets $\mathcal{E}_i$ are not only pairwise disjoint but also contained in a fixed bounded set (dependent on R, L but independent on $\varepsilon \in]0,1[$) because $\bar{w}_i$ are all contained in the bounded set $\Pi_{\mathbb{W}}(B(e,R))$ and $\|f(\bar{w}_i)\|$ is bounded by $L\|\bar{w}_i\|$. Thus, there is $C = C(\mathbb{W}, \mathbb{V}, L, R) > 0$ such that

$$N \leq c \left(\frac{1}{\varepsilon}\right)^{2n+2-k}.$$

□

2.2.1 1-Codimensional Intrinsic Lipschitz Graphs

In this section we restrict ourselves to 1-codimensional graphs of intrinsic Lipschitz continuous functions. According to Proposition 1.4.4, we can decompose the Heisenberg group $\mathbb{H}^n$ in the semidirect product of a horizontal subgroup $\mathbb{V}$, of dimension $1 \leq k \leq n$, and a vertical subgroup $\mathbb{W}$, of metric dimension $2n + 2 - k$. From now on, we assume that $k = 1$. In other words, we make the following

Assumption 2.2.7. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be a semidirect product as in Proposition 1.4.4, we assume that there exists a fixed vector field $V \in \mathfrak{h}_1$ such that*

$$\mathbb{V} \;=\; \{\, \exp(tV) \,|\, t \in \mathbb{R}\, \}.$$

Without loss of generality, we assume also that $|V| = 1$.

From this assumption, it follows also that if $f : \mathcal{E} \subset \mathbb{W} \longrightarrow \mathbb{V}$, then there exists a real-valued function $\varphi : \mathcal{E} \longrightarrow \mathbb{R}$ such that

$$f(w) \;=\; \exp(\varphi(w) \cdot V),$$

for every $w \in \mathcal{E}$.

We aim now to expound the proof of an Extension Theorem given in [18]. The authors follow the idea of the McShane-Whitney Extension Theorem for Lipschitz maps (see Appendix B).

We start with a technical Lemma, that holds for general step 2 Carnot groups:

Lemma 2.2.8. *Let $\mathbb{G}$ be a step 2 Carnot group and $p, q \in \mathbb{G}$. Then there exists a constant $c = c(\mathbb{G})$ such that*

$$\|p^{-1} \cdot q^{-1} \cdot p \cdot q\| \;\leq\; C\, \|p\|^{\frac{1}{2}} \cdot \|q\|^{\frac{1}{2}}. \tag{2.8}$$

From (2.8)*, it follows*

$$\|q^{-1} \cdot p \cdot q\| \;\leq\; \|p\| \;+\; C\, \|p\|^{\frac{1}{2}} \cdot \|q\|^{\frac{1}{2}}. \tag{2.9}$$

Proof. As usual, we identify $\mathbb{G}$ with $\mathbb{R}^n$ through exponential coordinates, so that the group law is written as

$$p \cdot q = p + q + Q(p, q).$$

Then

$$\begin{aligned} p^{-1} \cdot q^{-1} \cdot p \cdot q \; &= \; (-p-q+Q(p,q)) \cdot (p+q+Q(p,q)) \\ &= \; Q(-p,-q) + Q(p,q) \; + \; Q(-p-q+Q(p,q), p+q+Q(p,q)) \end{aligned}$$

By Lemma 1.1.8, we have

$$Q(-p-q+Q(-p,-q), p+q+Q(p,q)) \; = \; Q(-p-q, p+q) \; = \; 0.$$

Hence, since $Q_{m_1+1}, ..., Q_n$ are homogeneous polynomials of degree 2 containing only terms as $p_h \, q_k$, for $1 \leq k, h \leq m_1$,

$$\begin{aligned} \|p^{-1} \cdot q^{-1} \cdot p \cdot q\| \; &\leq \; \|Q(-p,-q)\| \; + \; \|Q(p,q)\| \\ &\leq \; \|p\|^{\frac{1}{2}} \cdot \|q\|^{\frac{1}{2}}. \end{aligned}$$

To conclude, we notice that

$$\begin{aligned} \|q^{-1} \cdot p \cdot q\| \; &\leq \; \|p\| \; + \; \|p^{-1} \cdot q^{-1} \cdot p \cdot q\| \\ &\leq \; \|p\| \; + \; \|p\|^{\frac{1}{2}} \cdot \|q\|^{\frac{1}{2}}. \end{aligned}$$

□

Lemma 2.2.9. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Assumption 2.2.7. For $L > 0$, we define $\varphi_L : \mathbb{W} \longrightarrow \mathbb{V}$ as*

$$\varphi_L(w) \; := \; \exp(L \, \|w\| \, V).$$

Then there exists a constant $L_1 = L_1(L, \mathbb{W}, \mathbb{V}) \geq L$ such that the function φ_L is intrinsic L_1-Lipschitz. The constant L_1 will be made precise during the proof.

Moreover,

$$\operatorname{graph}(\varphi_L) \; = \partial C^{+}_{\mathbb{W},\mathbb{V}}\left(e, \frac{1}{L}\right),$$

where

$$C^{+}_{\mathbb{W},\mathbb{V}}\left(e, \frac{1}{L}\right) \; := \; \exp\left(\{Z \in \mathfrak{h}^n \mid \langle Z, V\rangle \geq 0\}\right) \cap C_{\mathbb{W},\mathbb{V}}\left(e, \frac{1}{L}\right).$$

Proof. According to Remark 2.2.4, we need to prove that, for all $w, \eta \in \mathbb{W}$,

$$\left\|\varphi_L(w)^{-1} \cdot \varphi_L(\eta)\right\| \; \leq \; L_1 \left\|\varphi_L(w)^{-1} \cdot w^{-1} \cdot \eta \cdot \varphi_L(w)\right\|. \tag{2.10}$$

Notice that, since $\mathbb{V}$ is isometric to $\mathbb{R}$ and the isometry is realized by the exponential map,

$$\begin{aligned}\left\|\varphi_L(w)^{-1}\cdot\varphi_L(\eta)\right\| &= \|\exp(-L\|w\|\cdot V)\cdot\exp(L\|\eta\|\cdot V)\|\\ &= L\;\|\exp((\|\eta\|-\|w\|)\cdot V)\|\\ &= L\;|\|\eta\|-\|w\||\,.\end{aligned}$$

Now, let $\mathbb{V}=\exp(\mathfrak{v})=\exp(tV)$ and $\mathbb{W}=\exp(\mathfrak{w})=\exp(\text{span}\{W_1,...,W_{2n},T\})$, where $V,W_1,...,W_{2n}\in\mathfrak{h}_1$. We call M the $2n\times 2n$-matrix that gives the vector fields $V,W_1,...,W_{2n}$ in terms of the standard basis $X_1,...,X_n,Y_1,...,Y_n$. With respect to the new basis of $\mathfrak{v}$ and $\mathfrak{w}$, a fixed point $p\in\mathbb{H}^n$ can be written as

$$p=\exp\left(\tilde{p}_1+\sum_{i=1}^{2n}\tilde{p}_iW_i+p_{2n+1}T\right).$$

In this case, we use the notation

$$p\simeq(\tilde{p}_1,...,\tilde{p}_{2n},p_{2n+1})=(\tilde{p}',p_{2n+1}).$$

It turns out also that $p'=M^T\tilde{p}'$. Moreover, the group law of the Heisenberg group can be reformulated in this form

$$p\cdot q\simeq\left(\tilde{p}'+\tilde{q}',p_{2n+1}+q_{2n+1}-\frac{1}{2}\left\langle JM^T\tilde{p}',M^T\tilde{q}'\right\rangle_{\mathbb{R}^{2n}}\right),$$

and the Korányi norm

$$\|p\|=\left(\left\|M^T\tilde{p}'\right\|^4_{\mathbb{R}^{2n}}+p^2_{2n+1}\right)^{\frac{1}{4}}.$$

In these new coordinates, φ_L is as follows

$$\begin{aligned}\varphi_L(w) &= \exp\left(L\|w\|\cdot V\right)\simeq(L\|w\|,0,...,0)\\ &:= L\,\|w\|\cdot e_1.\end{aligned}$$

Therefore, for all w and $\eta\in\mathbb{W}$,

$$\begin{aligned}&\varphi_L(w)^{-1}\cdot w\cdot\eta\cdot\varphi_L(w)\\ &\quad\simeq\Big(0,\tilde{\eta}_2-\tilde{w}_2,...,\tilde{\eta}_{2n}-\tilde{w}_{2n},\eta_{2n+1}-w_{2n+1}\\ &\qquad\qquad+\frac{1}{2}\left\langle JM^T\tilde{w}',M^T\tilde{\eta}'\right\rangle_{\mathbb{R}^{2n}}-L\|w\|\left\langle JM^T(\tilde{\eta}'-\tilde{w}'),M^Te_1\right\rangle_{\mathbb{R}^{2n}}\Big).\end{aligned}$$

To simplify the notations, we set

$$\begin{aligned}
\mathcal{V} &:= \left(\eta_{2n+1} - w_{2n+1} + \frac{1}{2}\left\langle JM^T\tilde{w}', M^T\tilde{\eta}'\right\rangle_{\mathbb{R}^{2n}} - L\|w\|\left\langle JM^T(\tilde{\eta}' - \tilde{w}'), M^T e_1\right\rangle_{\mathbb{R}^{2n}}\right),\\
\mathcal{H} &:= \left\|M^t\,(\tilde{\eta}' \,-\, \tilde{w}')\right\|,\\
\mathcal{I} &:= \|\varphi_L(w)^{-1}\cdot w^{-1}\cdot\eta\cdot\varphi_L(w)\| \;=\; \sqrt[4]{\mathcal{H}^4+\mathcal{V}^2}.
\end{aligned}$$

Then, inequality (2.10) can be written as

$$\|\varphi_L(w)^{-1}\cdot\varphi_L(\eta)\| \;=\; L\;|\|\eta\| \,-\, \|w\|| \;\le\; L_1\,\mathcal{I}. \tag{2.11}$$

Now, without loss of generality, we assume that $\|\eta\| \le \|w\|$ and $\|w\| > 0$. Notice that inequality (2.10) is invariant by Heisenberg dilations. Hence, with

$$\eta \longmapsto \delta_{\frac{1}{\|w\|}}\eta, \quad w \longmapsto \delta_{\frac{1}{\|w\|}}w,$$

(2.11) is equivalent to

$$L\,(1-\|\eta\|) \;\le\; L_1\,\mathcal{I}, \tag{2.12}$$

for $\|\eta\| \le 1$ and $\|w\| = 1$.

Let us prove (2.12). Notice that

$$\begin{aligned}
0 \;\le\; 1-\|\eta\| \;&\le\; 1 \,-\, \sqrt[4]{\|M^T\tilde{\eta}'\|^4_{\mathbb{R}^{2n}} + |\eta_{2n+1}|^2}\\
&= 1 \,-\, \sqrt[4]{1-1+\|M^T\tilde{\eta}'\|^4_{\mathbb{R}^{2n}} + |\eta_{2n+1}|^2}\\
&\le 1 \,-\, \left\|M^T\tilde{\eta}'\right\|^4_{\mathbb{R}^{2n}} \,-\, \eta^2_{2n+1}\\
&= \|w\| \,-\, \left\|M^T\tilde{\eta}'\right\|^4_{\mathbb{R}^{2n}} \,-\, \eta^2_{2n+1}\\
&= \left\|M^T\tilde{w}'\right\| \,-\, \left\|M^T\tilde{\eta}'\right\|^4_{\mathbb{R}^{2n}} \,+\, w^2_{2n+1} \,-\, \eta^2_{2n+1}
\end{aligned}$$

Observe that, for $\|\eta\| \le 1$ and $\|w\| = 1$,

$$\begin{aligned}
\left\|M^T\tilde{w}'\right\|^4_{\mathbb{R}^{2n}} \,-\, \left\|M^T\tilde{\eta}'\right\|^4_{\mathbb{R}^{2n}} \;=\\
&= \left(\left\|M^T\tilde{w}'\right\|^2_{\mathbb{R}^{2n}} \,-\, \left\|M^T\tilde{\eta}'\right\|^2_{\mathbb{R}^{2n}}\right)\cdot\left(\left\|M^T\tilde{w}'\right\|^2_{\mathbb{R}^{2n}} \,+\, \left\|M^T\tilde{\eta}'\right\|^2_{\mathbb{R}^{2n}}\right)\\
&\le 2\,\left|\left\|M^T\tilde{w}'\right\|^2_{\mathbb{R}^{2n}} \,-\, \left\|M^T\tilde{\eta}'\right\|^2_{\mathbb{R}^{2n}}\right|\\
&= 2\,\left|\left\langle M^T(\tilde{w}'-\tilde{\eta}')\,,\, M^T(\tilde{w}'+\tilde{\eta}')\right\rangle_{\mathbb{R}^{2n}}\right|\\
&\le 4\,\left\|M^T(\tilde{w}'-\tilde{\eta}')\right\|_{\mathbb{R}^{2n}}\\
&= 4\,\mathcal{H} \;\le\; 4\,\mathcal{I}.
\end{aligned} \tag{2.13}$$

To conclude the proof, we need to extimate $\left|w_{2n+1}^2 - \eta_{2n+1}^2\right|$ in terms of $\mathcal{I}$. We divide into two cases:

$$\frac{1}{2}|\,\eta_{2n+1} - w_{2n+1}\,| \;<\; |\mathcal{V}| \quad \text{and} \quad \frac{1}{2}|\,\eta_{2n+1} - w_{2n+1}\,| \;\geq\; |\mathcal{V}|\,.$$

We start with the case

$$\frac{1}{2}|\,\eta_{2n+1} - w_{2n+1}\,| \;<\; |\mathcal{V}|\,.$$

We have, from definition of $\mathcal{V}$,

$$\begin{aligned}
\frac{1}{2}|\,\eta_{2n+1} - w_{2n+1}\,| \;&<\; \frac{1}{2}\left|\left\langle JM^T\tilde{w}'\,,\, M^T\tilde{\eta}'\right\rangle_{\mathbb{R}^{2n}}\right| \;+\; L\,\|w\|\,\left|\left\langle JM^T(\tilde{\eta}' - \tilde{w}')\,,\, M^Te_1\right\rangle_{\mathbb{R}^{2n}}\right| \\
&=\; \frac{1}{2}\left|\left\langle JM^T(\tilde{w}' - \tilde{\eta}')\,,\, M^Te_1\right\rangle_{\mathbb{R}^{2n}}\right| \\
&\leq\; \frac{1}{2}\,\|M^T\|\,(1+2L)\,\left\|M^T(\tilde{w}' - \tilde{\eta}'\right\|\,,
\end{aligned}$$

where the last inequality follows trivially from the Schwartz inequality. Therefore,

$$\begin{aligned}
|\,w_{2n+1}^2 - \eta_{2n+1}^2\,| \;&\leq\; 2\,|w_{2n+1} - \eta_{2n+1}\,| \\
&\leq\; 2\,\left\|M^T\right\|\,(1+2L)\,\mathcal{I}.
\end{aligned} \tag{2.14}$$

Consider now the second case

$$|\mathcal{V}| \;\geq\; \frac{1}{2}\,|\eta_{2n+1} - w_{2n+1}|.$$

We have

$$\begin{aligned}
|\,w_{2n+1}^2 - \eta_{2n+1}^2\,| \;&\leq\; 2\,|\,w_{2n+1} - \eta_{2n+1}\,| \\
&\leq\; 2\sqrt{2}\,|\,w_{2n+1} - \eta_{2n+1}\,|^{\frac{1}{2}} \\
&\leq\; 4\sqrt[4]{\mathcal{V}} \;\leq\; 4\,\mathcal{I}.
\end{aligned} \tag{2.15}$$

Then, from inequalities (2.13), (2.14) and (2.15), we get

$$\begin{aligned}
1 \,-\, \|\eta\| \;&\leq\; \left(\left\|M^T\tilde{w}'\right\|^4 \,-\, \left\|M^T\tilde{\eta}'\right\|^4\right) \,+\, |\,w_{2n+1}^2 - \eta_{2n+1}^2\,| \\
&\leq\; 2\,\max\left\{2, \left\|M^T\right\|\cdot(1+2L)\right\}\,\mathcal{I},
\end{aligned}$$

which, setting $L_1 = 2\,\max\left\{2, \left\|M^T\right\|\cdot(1+2L)\right\}$, completes the proof. □

Lemma 2.2.10. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Assumption 2.2.7. For each $\alpha > 0$, there exists $\alpha_1 = \alpha_1(\alpha, \mathbb{W}, \mathbb{V}) \leq \alpha$ such that, for all $v \in \exp(tV) \in \mathbb{V}$ with $t > 0$,*

$$C^+_{\mathbb{W},\mathbb{V}}(v, \alpha_1) \,:=\, v \cdot C^+_{\mathbb{W},\mathbb{V}}(e, \alpha_1) \,\subset\, C^+_{\mathbb{W},\mathbb{V}}(e, \alpha).$$

Proof. Let $p \in C^+_{\mathbb{W},\mathbb{V}}(e, \alpha_1)$ be fixed. By Proposition 1.4.5, $(v \cdot p)_{\mathbb{W}} = v \cdot p_{\mathbb{W}} \cdot v^{-1}$ and $(v \cdot p)_{\mathbb{V}} = v \cdot p_{\mathbb{V}}$. Moreover, since $\mathbb{V}$ is isometric to $\mathbb{R}$,

$$\|(v \cdot p)_{\mathbb{V}}\| \,=\, \|p_{\mathbb{V}}\| \,+\, \|v\|.$$

On the other hand, by (2.9)

$$\begin{aligned} \|(v \cdot p)_{\mathbb{W}}\| \,&=\, \|v \cdot p \cdot v^{-1}\| \\ &\leq\, \|p_{\mathbb{W}}\| \,+\, c\,\|v\|^{\frac{1}{2}}\,\|p_{\mathbb{W}}\|^{\frac{1}{2}}. \end{aligned}$$

Now, using the fact that $p \in C^+_{\mathbb{W},\mathbb{V}}(e, \alpha_1)$, one has

$$\begin{aligned} \|(v \cdot p)_{\mathbb{W}}\| \,&\leq\, \left(1 + \frac{c}{2}\right)\,\|p_{\mathbb{W}}\| \,+\, \frac{c}{2}\,\|v\| \\ &\leq\, \alpha_1\,\left(1 + \frac{c}{2}\right)\,\|p_{\mathbb{V}}\| \,+\, \frac{c}{2}\,\|v\| \\ &\leq\, \max\left\{\alpha_1\,\left(1 + \frac{c}{2}\right), \frac{c}{2}\right\}\,(\|p_{\mathbb{V}}\| \,+\, \|v\|) \\ &=\, \max\left\{\alpha_1\,\left(1 + \frac{c}{2}\right), \frac{c}{2}\right\}\,\|(v \cdot p)_{\mathbb{V}}\|. \end{aligned}$$

Hence, if we choose $\alpha > 0$ such that $\alpha = \max\left\{\alpha_1\left(1 + \frac{c}{2}\right)\,,\,\frac{c}{2}\right\}$, we have that $v \cdot p \in C^+_{\mathbb{W},\mathbb{V}}(e, \alpha)$. ☐

In classical proof of McShane and Whitney, we define a family $\{f_i\}_{i \in I}$ of Lipschitz functions $f_i : A \longrightarrow \mathbb{R}$. Lemma B.0.16 ensures us that $\inf_{i \in I} f_i(x)$ and $\sup_{i \in I} f_i(x)$, for each $x \in A$, are Lipschitz functions as well. It is possible to do the same for a family of intrinsic Lipschitz functions on $\mathbb{H}^n$. Since $\mathbb{V} = \{\exp(tV) \,|\, t \in \mathbb{R}\}$, it can be identified with $\mathbb{R}$ so that the exponential map induced an order on it. Then it makes sense to speak about the supremum oder the infimum of a collection of $\mathbb{V}$-valued functions.

In the next lemmas, our goal is to prove the analogous result of Lemma B.0.16.

Definition 2.2.8. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Assumption 2.2.7 and let $\{f_\alpha\}_{\alpha \in A}$ be a collection of $\mathbb{V}$-valued functions*

$$\begin{aligned} f_\alpha(w) \,&:\, \mathbb{W} \,\longrightarrow\, \mathbb{V} \\ f_\alpha(w) \,&=\, \exp(\varphi_\alpha(w)V), \end{aligned}$$

where $\varphi_\alpha : \mathbb{W} \longrightarrow \mathbb{R}$*. We define* $\inf_{\alpha\in A} f_\alpha : \mathbb{W} \longrightarrow \mathbb{V}$ *as*

$$\inf_{\alpha\in A} f_\alpha(w) := \exp\left(\inf_{\alpha\in A} \varphi_\alpha(w)\, V\right),$$

for all $w \in \mathbb{W}$ *such that* $\inf_{\alpha\in A} \varphi_\alpha(w)$ *is finite.*

Analogously, we define $\sup_{\alpha\in A} f_\alpha : \mathbb{W} \longrightarrow \mathbb{V}$ *as*

$$\sup_{\alpha\in A} f_\alpha(w) := \exp\left(\sup_{\alpha\in A} \varphi_\alpha(w)\, V\right).$$

In the same way, we define $\max\{f_\alpha, f_\beta\}$ *and* $\min\{f_\alpha, f_\beta\}$*.*

Notation 2.2.1. Let $f : \mathbb{W} \longrightarrow \mathbb{V}$ be a continuous function such that $f(w) = \exp(\varphi(w)\, V)$, for $\varphi : \mathbb{W} \longrightarrow \mathbb{R}$. We define the *subgraph* of f and the *supergraph* of f, respectively,

$$\begin{aligned} E^-(f) &= E^-_f := \{w \cdot \exp(tV) \,|\, w \in \mathbb{W},\ t < \varphi(w)\} \\ E^+(f) &= E^+_f := \{w \cdot \exp(tV) \,|\, w \in \mathbb{W},\ t > \varphi(w)\}. \end{aligned}$$

Notice that we can consider $\overline{E^-_f}$ and $\overline{E^+_f}$ just taking also equalities in the two sets.

Remark 2.2.9. Let $\{f_\alpha\}_{\alpha\in A}$ be a family of $\mathbb{V}$-valued functions. it is clear that

$$E^-\left(\inf_{\alpha\in A} f_\alpha\right) = \bigcap_{\alpha\in A} E^-_{f_\alpha} \quad \text{and} \quad E^+\left(\inf_{\alpha\in A} f_\alpha\right) = \bigcup_{\alpha\in A} E^+_{f_\alpha}.$$

In the next lemma we prove a characterization for 1-codimensional graphs of intrinsic Lipschitz functions:

Lemma 2.2.11. *Let* $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ *be as in Assumption 2.2.7. Then* $f : \mathbb{W} \longrightarrow \mathbb{V}$*,* $f(w) := \exp(\varphi(w)\, V)$*, where* $\varphi : \mathbb{W} \longrightarrow \mathbb{R}$*, is intrinsic* L*-Lipschitz, if and only if, for all* $w \in \mathbb{W}$*,*

$$C^+_{\mathbb{W},\mathbb{V}}\left(w \cdot f(w), \frac{1}{L}\right) \subset \overline{E^+_f} \quad \textit{and} \quad C^-_{\mathbb{W},\mathbb{V}}\left(w \cdot f(w), \frac{1}{L}\right) \subset \overline{E^-_f}.$$

Proof. The first implication is trivial. Indeed, if we assume that f is intrinsic L-Lipschitz, then, by definition, for all $w \in \mathbb{W}$ and $0 < \alpha < \frac{1}{L}$,

$$C_{\mathbb{W},\mathbb{V}}\left(w \cdot f(w), \alpha\right) \cap \operatorname{graph}(f) = \{w \cdot f(w)\}.$$

We prove the converse implication by contradiction: let $\bar{w} \in \mathbb{W}$ and $\bar{t} > \varphi(\bar{w}) \in \mathbb{R}$ be such that

$$\bar{w} \cdot \exp(\bar{t}\, V) \in C^{+}_{\mathbb{W},\mathbb{V}}(\bar{w} \cdot f(w), \alpha) \cap E^{-}_{f}. \tag{2.16}$$

Then again, for all $\bar{t} \geq t$

$$\exp(t\, V) \in C^{+}_{\mathbb{W},\mathbb{V}}(\bar{w} \cdot f(\bar{w}), \alpha).$$

In particular,

$$\bar{w} \cdot \exp(t\, V) \in C^{+}_{\mathbb{W},\mathbb{V}}(\bar{w} \cdot f(\bar{w}), \alpha)$$

and this contradicts (2.16). □

Lemma 2.2.12. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Assumption 2.2.7. Let $f_\alpha,\, f_\beta : \mathbb{W} \longrightarrow \mathbb{V}$ be intrinsic L-Lipschitz functions. Then there exists a constant $L_2 = L_2(L, \mathbb{W}, \mathbb{V}) \geq L$ such that $g := \max\{f_\alpha, f_\beta\}$ is intrinsic L_2-Lipschitz.*

Proof. Let $w \in \mathbb{W}$ be fixed. We apply Lemma 2.2.10, with $\alpha_1 = \frac{1}{L}$ and $L_2 := \frac{1}{\alpha_2}$ to have

$$C^{+}_{\mathbb{W},\mathbb{V}}\left(w \cdot g(w), \frac{1}{L_2}\right) \subset \overline{E^{-}_{f_\alpha}} \quad \text{and} \quad C^{+}_{\mathbb{W},\mathbb{V}}\left(w \cdot g(w), \frac{1}{L_2}\right) \subset \overline{E^{+}_{f_\beta}}.$$

Hence, by Remark 2.2.9,

$$C^{+}_{\mathbb{W},\mathbb{V}}\left(w \cdot g(w), \frac{1}{L_2}\right) \subset \overline{E^{+}_{f_\alpha}} \cap \overline{E^{+}_{f_\beta}} = \overline{E^{+}_{g}},$$

which, by Lemma 2.2.11, implies that g is intrinsic L_2-Lipschitz. □

Proposition 2.2.13. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Assumption 2.2.7. Let $\{f_\alpha\}_{\alpha \in A}$ be a collection of intrinsic L-Lipschitz functions. Then, there exists a constant $L_2 = L_2(L, \mathbb{W}, \mathbb{V}) \geq L$ such that*

$$f := \inf_{\alpha \in A} f_\alpha$$

is $f \equiv -\infty$ or f is defined on all $\mathbb{W}$ and it is intrinsic L_2-Lipschitz.

Proof. Using Remark 2.2.4, we can prove the assertion showing that, for $w, \eta \in \mathbb{W}$,

$$\left\| f(w)^{-1} \cdot f(\eta) \right\| \leq L_2 \left\| f(w)^{-1} \cdot w^{-1} \cdot \eta \cdot f(w) \right\|.$$

Let w and $\eta \in \mathbb{W}$ be fixed, and let $\varepsilon \in \mathbb{R}^+$. We choose $\alpha = \alpha(\varepsilon)$ and $\beta = \beta(\varepsilon) \in A$ such that

$$\left\| f_\alpha(w)^{-1} \cdot f(w) \right\| < \varepsilon \quad \text{and} \quad \left\| f_\beta(\eta)^{-1} \cdot f(\eta) \right\| < \varepsilon.$$

Notice also that it is always possible to choose such an α and β, because of the definition of infimum.

Now, define $g := \max\{f_\alpha, f_\beta\}$, and observe that, from Lemma 2.2.12, g is intrinsic L_2-Lipschitz. Moreover, g is such that

$$\|g^{-1}(w) \cdot f(w)\| < \varepsilon \quad \text{and} \quad \|g(\eta)^{-1} \cdot f(\eta)\| < \varepsilon.$$

Hence,

$$\begin{aligned}
\|f(w)^{-1} \cdot f(\eta)\| &= \|f(w)^{-1} \cdot g(w) \cdot g(w)^{-1} \cdot g(\eta) \cdot g(\eta)^{-1} \cdot f(\eta)\| \\
&\leq \|f(w)^{-1} \cdot g(w)\| + \|g(\eta)^{-1} \cdot f(\eta)\| + \|g(w)^{-1} \cdot g(\eta)\| \\
&\leq 2\,\varepsilon + L_2 \,\|g(w)^{-1} \cdot w^{-1} \cdot \eta \cdot g(w)\| \\
&= 2\,\varepsilon + L_2 \,\|g(w)^{-1} \cdot f(w) \cdot f(w)^{-1} \cdot w^{-1} \cdot \eta \cdot f(w) \cdot f(w)^{-1} \cdot g(w)\| \\
&\leq 2\,\varepsilon + 2\,\varepsilon\, L_2 + L_2 \,\|f(w)^{-1} \cdot w^{-1} \cdot \eta \cdot f(w)\|.
\end{aligned}$$

Choosing ε small enough, one has the assertion. □

We are now ready to give the proof of the Extension Theorem:

Theorem 2.2.14. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Assumption 2.2.7. Let $\mathcal{B}$ be a Borel subset of $\mathbb{W}$ and $f : \mathcal{B} \longrightarrow \mathbb{V}$ be an intrinsic L-Lipschitz function. Then there are $\tilde{f} : \mathbb{W} \longrightarrow \mathbb{V}$ and a constant $L_3 = L_3(L, \mathbb{W}, \mathbb{V}) \geq L$ such that*

(i) *$\tilde{f}$ is intrinsic L_3-Lipschitz;*

(ii) *$\tilde{f}(w) = f(w)$, for all $w \in \mathcal{B}$.*

Proof. Let $\bar{w} \in \mathcal{B}$. We define

$$\begin{aligned}
\varphi_{L,\bar{w}} \, : \, \mathbb{W} &\longrightarrow \mathbb{V} \\
w &\longmapsto f(\bar{w}) \cdot \varphi_L \left(f(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot f(\bar{w}) \right),
\end{aligned}$$

where φ_L is the function defined in Lemma 2.2.9. We notice that

$$\mathrm{graph}(\varphi_{L,\bar{w}}) \;=\; \partial C^{+}_{\mathbb{W},\mathbb{V}}\left(\bar{w}\cdot f(\bar{w}), \frac{1}{L}\right). \tag{2.17}$$

We can write, for real valued functions

$$\vartheta_{\bar{w}} \,:\, \mathbb{W} \longrightarrow \mathbb{R} \quad\text{and}\quad \varphi \,:\, \mathcal{B} \longrightarrow \mathbb{V},$$

that

$$\varphi_{L,\bar{w}}(w) \;=\; \exp(\vartheta_{\bar{w}}(w)\cdot V) \quad\text{and}\quad f(w) \;=\; \exp(\varphi(w)\cdot V).$$

Since, on $\mathcal{B}$, $\varphi_{L,\bar{w}} = f$, one has that

$$\vartheta_{\bar{w}}(\bar{w}) \;=\; \varphi(\bar{w}), \quad\text{for all } \bar{w}\in\mathcal{B}. \tag{2.18}$$

Let us define $\tilde{f} : \mathbb{W} \longrightarrow \mathbb{V}$ as

$$\tilde{f}(w) \;=\; \inf_{\bar{w}\in\mathcal{B}} \varphi_{L,\bar{w}}(W),$$

for all $w\in\mathbb{W}$.

From Lemma 2.2.9, we know that $\varphi_{L,\bar{w}}$ is intrinsic L_2-Lipschitz for all $\bar{w}\in\mathcal{B}$, where L_1 is the constant defined in Lemma 2.2.9. Therefore, by Proposition 2.2.13, one has that $\tilde{f}$ is intrinsic L_2-Lipschitz.

Our aim, now, is to show that $f(w) = \tilde{f}(w)$ for all $w\in\mathcal{B}$. Given (2.18), it is sufficient to show, for a fixes $w\in\mathcal{B}$, that

$$\vartheta_{\bar{w}}(w) \;\geq\; \varphi(w),$$

for all $w\in\mathcal{C}$. Now, f is intrinsic L-Lipschitz, then, since $t\longmapsto \exp(tV)$ is an isometry,

$$\begin{aligned} |\,\varphi(w) - \vartheta_{\bar{w}}(\bar{w})\,| \;&=\; |\,\varphi(w) - \varphi(\bar{w})\,| \\ &=\; \|f(\bar{w})^{-1}\cdot f(w)\| \\ &\leq\; L\,\|f(\bar{w})^{-1}\cdot\bar{w}^{-1}\cdot w\cdot f(\bar{w})\|. \end{aligned}$$

On the other hand, from (2.17),

$$\begin{aligned} |\,\vartheta_{\bar{w}}(w) - \vartheta_{\bar{w}}(\bar{w})| \;&=\; \|\varphi_{L,\bar{w}}(\bar{w})^{-1}\cdot\varphi_{L,\bar{w}}(w)\| \\ &=\; \|\varphi_{L,\bar{w}}(\bar{w})^{-1}\cdot w^{-1}\cdot w\cdot\varphi_{L,\bar{w}}(w)\| \\ &\leq\; L\,\|f(\bar{w})^{-1}\cdot\bar{w}^{-1}\cdot w\cdot f(\bar{w})\|. \end{aligned}$$

Hence,

$$|\varphi(w) - \vartheta_{\bar{w}}(\bar{w})| \le |\vartheta_{\bar{w}}(w) - \vartheta_{\bar{w}}(\bar{w})|,$$

which is exactly what we wanted. □

We aim to conclude this Section by proving that subgraphs of intrinsic Lipschitz functions are sets of locally finite $\mathbb{H}$-perimeter. We start with a useful characterization:

Theorem 2.2.15. *Let Ω be an open subset of $\mathbb{H}^n$.*

(i) *If $\mathcal{S}_d^{Q-1}(\partial\Omega)$ is locally finite in $\mathbb{H}$, then Ω has locally finite $\mathbb{H}$-perimeter;*

(ii) *if $\mathcal{S}_d^{Q-1}(\partial\Omega) < \infty$, then there exists a positive geometric constant $c > 0$ such that*

$$|\partial\Omega|_{\mathbb{H}} \le c\,\mathcal{S}_d^{Q-1} \llcorner \partial\Omega.$$

Proof. We divide the proof in three steps.

Step 1. First, suppose that Ω is bounded and $\mathcal{S}_d^{Q-1}(\partial\Omega) < \infty$. We have to show that $|\partial\Omega|_{\mathbb{H}}(\mathbb{H}^n) < \infty$. Let $\varepsilon \in \mathbb{R}^+$, we can cover $\partial\Omega$ with open metric balls $\{U_{j,\varepsilon}\}_{j\in\mathbb{N}}$, with radius $r_{j,\varepsilon} < \varepsilon$, such that

$$\sum_{j\in\mathbb{N}} r_{j,\varepsilon}^{Q-1} < \tilde{c}\,\mathcal{S}_d^{Q-1}(\partial\Omega) =: c < \infty.$$

Setting $S_\varepsilon := \bigcup_{j\in\mathbb{N}} U_{j,\varepsilon}$ and $\Omega_\varepsilon := \Omega \cap S_\varepsilon$, we have

$$\Omega_\varepsilon \longrightarrow \Omega \quad \text{in} \ \ L^1, \ \ \text{as} \ \varepsilon \longrightarrow 0. \tag{2.19}$$

Indeed,

$$\begin{aligned} \mathcal{L}^{2n+1}(\Omega_\varepsilon \triangle \Omega) &= \mathcal{L}^{2n+1}\left(\Omega_\varepsilon \setminus \Omega\right) \\ &\le \mathcal{L}^{2n+1}(S_\varepsilon) \\ &\le c \sum_{j\in\mathbb{N}} r_{j,\varepsilon}^{2n+2} \\ &\le c\,\varepsilon. \end{aligned}$$

Claim. The following inequality holds

$$|\partial\Omega_\varepsilon|_{\mathbb{H}}(\mathbb{H}^n) \le c\,|\partial S_\varepsilon|_{\mathbb{H}}(\mathbb{H}^n). \tag{2.20}$$

Assuming the Claim true, the proof of this first step follows from (2.19) and by the L^1-lower semicontinuity of the perimeter, since

$$\begin{aligned} |\partial S_\varepsilon|_{\mathbb{H}}(\mathbb{H}^n) &\leq \sum_{j\in\mathbb{N}} |\partial U_{j,\varepsilon}|_{\mathbb{H}}(\mathbb{H}^n) \\ &\leq c \sum_{j\in\mathbb{N}} r_{j,\varepsilon}^{2n+1} \\ &\leq c < \infty. \end{aligned}$$

Step 2. Let us prove the Claim. First observe that $\partial\Omega_\varepsilon \subset \bar{\Omega}^c$, $\mathrm{dist}(\partial\Omega_\varepsilon, \bar{\Omega}) > 0$ and $\Omega_\varepsilon \cap \bar{\Omega}^c$. Therefore,

$$\begin{aligned} |\partial\Omega_\varepsilon|_{\mathbb{H}}(\mathbb{H}^n) &= |\partial\Omega_\varepsilon|_{\mathbb{H}}(\bar{\Omega}^c) = |\partial(\Omega_\varepsilon \cap \bar{\Omega}^c)|_{\mathbb{H}}(\bar{\Omega}^c) \\ &= |\partial(S_\varepsilon \cap \bar{\Omega}^c)|_{\mathbb{H}}(\bar{\Omega}^c) = |\partial S_\varepsilon|_{\mathbb{H}}(\bar{\Omega}^c) \\ &\leq |\partial S_\varepsilon|_{\mathbb{H}}(\mathbb{H}^n). \end{aligned}$$

Step 3. We drop now the assumption of the boundedness of Ω. Let U be a fixed open ball, such that $\partial\Omega \cap U \neq \emptyset$ and $\mathcal{S}_d^{2n+1}(\partial\Omega \cap U) < \infty$. Now, since $\mathcal{S}_d^{2n+1}(U) < \infty$, we have

$$\begin{aligned} \mathcal{S}_d^{2n+1}(\partial(U \cap \Omega)) &\leq \mathcal{S}_d^{2n+1}(\partial U \cup (\partial\Omega \cap U)) \\ &\leq \mathcal{S}_d^{2n+1}(\partial U) + \mathcal{S}_d^{2n+1}(\partial\Omega \cap U) < \infty. \end{aligned}$$

Thus, from Step 1, it follows that

$$|\partial(U \cap \Omega)|_{\mathbb{H}}(\mathbb{H}^n) < \infty,$$

and consequently

$$\begin{aligned} |\partial\Omega|_{\mathbb{H}}(U) &= |\partial(\Omega \cap U)|_{\mathbb{H}}(U) \\ &= |\partial(\Omega \cap U)|_{\mathbb{H}}(\mathbb{H}^n) < \infty. \end{aligned}$$

□

Theorem 2.2.16. *Let $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ be as in Assumption 2.2.7. If $f : \mathbb{W} \longrightarrow \mathbb{V}$ is intrinsic Lipschitz, then the subgraph E_f^- is a set with locally finite $\mathbb{H}$-perimeter.*

Proof. By Proposition 2.2.6, it follows that $\mathcal{S}_d^{2n+1}(\partial E_f^-)$ is locally finite in $\mathbb{H}^n$. Therefore, statement (i) in Theorem 2.2.15 implies the assertion. □

Proposition 2.2.17. *Let Ω be an open set in $\mathbb{H}^n$. If U is an open ball such that $\partial\Omega \cap U$ is an intrinsic Lipschitz graph, then*

$$|\partial\Omega|_{\mathbb{H}}(U) \;=\; c\,\mathcal{S}^{2n+1}(\partial\Omega \cap U). \tag{2.21}$$

Proof. If $\partial\Omega \cap U$ is an intrinsic Lipschitz graph, then its measure theoretic boundary in U coincides with $\partial\Omega\cap U$ (this fact follows simply from the definition of intrinsic Lipschitz graph and from Definition 1.5.6). Proposition 1.5.7 implies directly (2.21). □

2.2.2 Intrinsic Differentiable Graphs and a Rademacher type Theorem

Following again [18], in this subsection we aim to apply ourselves to the proof of a Rademacher type results for intrinsic Lipschitz maps. First, we need a notion of differentiability which is dependent only of the group structure: the *intrinsic differentiability.*

As usual, differentiability is connected to the existence of approximating linear functions. Hence, we start with the definition of *intrinsic linear functions.*

Definition 2.2.10. *Let $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ be a semidirect product of homogeneous subgroups. We say that $L : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ is an* intrinsic linear function *if*

$$\mathrm{graph}(L) \;:=\; \{g \cdot L(g) \,|\, g \in \mathbb{G}_1\}$$

is a homogeneous subgroup of $\mathbb{H}^n$.

Remark 2.2.11. We point out that graph(L) is a closed set and that intrinsic linear functions are continuous functions from $\mathbb{G}_1$ to $\mathbb{G}_2$.

Once more, if we assume that $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ is as in Proposition 1.4.4, we have different characterizations depending if the intinsic linear map L is defined on $\mathbb{V}$ or on $\mathbb{W}$.

Proposition 2.2.18. *Let $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ be as in Proposition 1.4.4. Then*

(i) *$L : \mathbb{V} \longrightarrow \mathbb{W}$ is intrinsic linear if and only if the parametric map $\Phi_f : \mathbb{V} \longrightarrow \mathbb{H}^n$, $\Phi_f(g) := g \cdot f(g)$, is H-linear.*

(ii) $L : \mathbb{W} \longrightarrow \mathbb{V}$ *is intrinsic linear if and only if* L *is* H*-linear.*

Proof. See [1], Proposition 3.26. □

In the following example, again taken from [1], we highlight that it is really necessary, in Proposition 2.2.18, to distinguish the two cases:

Example 2.2.3. Let $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$ be as in Example 2.2.1, and let $a \in \mathbb{R}$ be fixed. The function $L : \mathbb{V} \longrightarrow \mathbb{W}$, define as

$$L((x,0,0)) \;=\; \left(0, ax, -\frac{ax^2}{2}\right),$$

is intrinsic linear. To prove this it is enough to show that graph(L) is a homogeneous subgroup of $\mathbb{H}^1$.

Let $(v_1, 0, 0) \in \mathbb{V}$. Then

$$\begin{aligned}(v_1,0,0)\cdot L((v_1,0,0)) \;&=\; (v_1,0,0)\cdot\left(0, av_1, -\frac{av_1^2}{2}\right)\\ &=\; \left(v_1, av_1, \frac{a}{2}v_1^2 - \frac{1}{2}\left(av_1^2\right)\right)\\ &=\; (v_1, av_1, 0).\end{aligned}$$

Hence, graph$(L) = \{(t, at, 0) \,|\, t \in \mathbb{R}\}$, which is clearly a horizontal 1-dimensional subgroup of $\mathbb{H}^1$. On the contrary, L is not a group homomorphism from $\mathbb{V}$ to $\mathbb{W}$.

Analogously, the function $L : \mathbb{W} \longrightarrow \mathbb{V}$, defined as

$$L((0,y,t)) \;=\; (ay, 0, 0),$$

is intrinsic linear. Indeed, graph$(L) = \{(at, t, s) \,|\, t, s \in \mathbb{R}\}$, which is a vertical subgroup of $\mathbb{H}^1$ of dimension 2. But the parametric function $\Phi_L : \mathbb{W} \longrightarrow \mathbb{V}$ acts as

$$\Phi_L((0,y,t)) \;=\; \left(ay, y, t - \frac{a}{2}y^2\right),$$

and, consequently, it is not a group homomorphism from $\mathbb{W}$ to $\mathbb{H}^1$.

Definition 2.2.12. *Let* $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ *be a semidirect product and* $f : \mathcal{E} \subset \mathbb{G}_1 \longrightarrow \mathbb{G}_2$*, with* $\mathcal{E}$ *an open subset of* $\mathbb{G}_1$*. For* $\bar{p} := \bar{g} \cdot f(\bar{g})$*, we consider the translated function* $f_{\bar{p}^{-1}}$

defined on $\mathcal{E}_{\bar{p}^{-1}}$. *We say that* f *is* intrinsic differentiable *in* $\bar{g} \in \mathcal{E}$ *if there exists an intrinsic linear map*

$$d_{\bar{g}}f \,:\, \mathbb{G}_1 \longrightarrow \mathbb{G}_2$$

such that, for all $g \in \mathcal{E}_{\bar{p}^{-1}}$,

$$\lim_{\|g\|\to 0} \frac{\|d_{\bar{g}}f(g)^{-1}\cdot f_{\bar{p}^{-1}}(g)\|}{\|g\|} \;=\; 0. \tag{2.22}$$

The map $d_{\bar{g}}f$ *is called the* intrinsic differential *of* f *in* $\bar{g}$.

Remark 2.2.13. Intrinsic differentiablity is invariant by left translations of the graph. Indeed, let $q_1 = g_1 \cdot f(g_1)$ and $q_2 = g_2 \cdot f(g_2) \in \operatorname{graph}(f)$. Now, f is intrinsic differentiable in $g_1 \in \mathbb{G}_1$ if and only if $f_{q_1^{-1}}$ is intrinsic differentiable in e. Consequently, f is intrinsic differentiable in g_1 if and only if $f_{q_2\cdot q_1^{-1}} \equiv \left(f_{q_1^{-1}}\right)_{q_2}$ is intrinsic differentiable in g_2.

Intrinsic differentiable functions, within $\mathbb{H}^n$, can be characterized in the following algebraic way.

Proposition 2.2.19. *Let* $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ *be as in Proposition 1.4.4. Then,*

(i) $f : \mathcal{E} \subset \mathbb{V} \longrightarrow \mathbb{W}$ *in intrinsic differentiable in* $\bar{g} \in \mathcal{E}$ *if and only if the parametric function* $\Phi_f : \mathcal{E} \longrightarrow \mathbb{H}^n$ *is* P*-differentiable in* $\bar{g}$*;*

(ii) $f : \mathcal{E} \subset \mathbb{W} \longrightarrow \mathbb{V}$ *is intrinsic differentiable in* $\bar{g} \in \mathcal{E}$ *if and only if there is an intrinsic linear map* $d_{\bar{g}}f : \mathbb{W} \longrightarrow \mathbb{V}$ *such that*

$$\left\|d_{\bar{g}}f(\bar{g}\cdot g)^{-1}\cdot f(\bar{g})^{-1}\cdot f(g)\right\| \;=\; o\left(\left\|f(\bar{g})^{-1}\cdot \bar{g}^{-1}\cdot g\cdot f(\bar{g})\right\|\right),$$

for $g \in \mathcal{E}$ *and* $\|f(\bar{g})^{-1}\cdot\bar{g}^{-1}\cdot g\cdot f(\bar{g})\| \longrightarrow 0$.

Remark 2.2.14. We should stress that, by assumption, $\mathbb{V}$ is a horizontal subgroup of $\mathbb{H}^n$. This implies, in particular, that $\mathbb{V}$ is isomorphic and isometric to $\mathbb{R}^k$ (Proposition 1.4.3 on page 24). Therefore, $\mathbb{V}$ is a Carnot group and it makes sense to speak about P-differentiability in (i).

Proof of Proposition 2.2.19. We start with case (i) and assume that f is intrinsic differentiable in $\bar{g}$ with intrinsic differential $d_{\bar{g}}f$. By Proposition 2.2.18, the map $g \longmapsto g\cdot d_{\bar{g}}f(g)$ is a homogeneous homomorphism from $\mathbb{V}$ to $\mathbb{H}^n$. We define $d_{\bar{g}}\Phi_f : \mathbb{V} \longrightarrow \mathbb{H}^n$ as follows

$$d_{\bar{g}}\Phi_f(g) \;:=\; g\cdot d_{\bar{g}}f(g).$$

Notice that, by Proposition 2.1.3 and using the same notations introduced in Definition 2.2.12,

$$d_{\bar g}f(\eta)^{-1}\cdot f_{\bar p^{-1}}(\eta) \;=\; d_{\bar g}f(\eta)^{-1}\cdot \eta^{-1}\cdot f(\bar g)^{-1}\cdot \eta\cdot f(\bar g\cdot\eta).$$

Therefore, setting $g := \bar g\cdot\eta$, we get

$$\begin{aligned} d_{\bar g}f(\bar g^{-1}\cdot g)\cdot f_{\bar p^{-1}}(\bar g^{-1}\cdot g) \;&=\; d_{\bar g}f(\bar g^{-1}\cdot g)^{-1}\cdot(\bar g^{-1}\cdot g)^{-1}\cdot f(\bar g)^{-1}\cdot \bar g^{-1}\cdot g\cdot f(g)\\ &=\; d_{\bar g}\Phi_f(\bar g^{-1}\cdot g)^{-1}\cdot \Phi_f(\bar g)^{-1}\cdot\Phi_f(g). \end{aligned}$$

Hence, the intrinsic differentiability of f implies

$$\begin{aligned} \left\|d_{\bar g}\Phi_f(\bar g^{-1}\cdot g)^{-1}\cdot \Phi_f(\bar g)^{-1}\cdot\Phi_f(g)\right\| \;&=\\ &=\; \left\|d_{\bar g}f(\bar g^{-1}\cdot g)^{-1}\cdot f_{\bar p^{-1}}(\bar g^{-1}\cdot g)\right\|\\ &=\; o\left(\|\bar g^{-1}\cdot g\|\right), \end{aligned}$$

as $\|\bar g^{-1}\cdot g\|\longrightarrow 0$.

Vice versa, assume that Φ_f is P-differentiable in $\bar g$. By definition of P-differentiability, we have that $d_{\bar g}\Phi_f:\mathbb{V}\longrightarrow\mathbb{H}^n$ is a homogeneous homomorphism.

We claim that

$$d_{\bar g}\Phi_f(g) \;=\; g\cdot L_{f,\bar g}(g), \tag{2.23}$$

for some intrinsic linear map $L_{f,\bar g}:\mathbb{V}\longrightarrow\mathbb{W}$.

Once (2.23) is proved, defining $L_{f,\bar g}$ as the intrinsic differential of f at $\bar g$, we have that

$$\begin{aligned} o\left(\|g^{-1}\cdot g\|\right) \;&=\; \left\|d_{\bar g}\Phi_f(\bar g^{-1}\cdot g)^{-1}\cdot \Phi_f(\bar g)^{-1}\cdot\Phi_f(g)\right\|\\ &=\; \left\|d_{\bar g}f(\bar g^{-1}\cdot g)^{-1}\cdot f_{\bar p^{-1}}(\bar g^{-1}\cdot g)\right\|, \end{aligned}$$

the intrinsic differentiability of f in $\bar g$.

Let us prove (2.23). By definition of P-differentiability and by Proposition 1.4.5, components along $\mathbb{V}$ and $\mathbb{W}$ of $d_{\bar g}\Phi_f(\bar g^{-1}\cdot g)^{-1}\cdot\Phi_f(\bar g)^{-1}\cdot\Phi_f(g)$ have to be $o\left(\|\bar g^{-1}\cdot g\|\right)$. In particular,

$$\begin{aligned} o\left(\|\bar g^{-1}\cdot g\|\right) \;&=\; \left\|(d_{\bar g}\Phi_f(\bar g^{-1}\cdot g)^{-1})_{\mathbb{V}}\cdot(\Phi_f(\bar g))_{\mathbb{V}}^{-1}\cdot(\Phi_f(g))_{\mathbb{V}}\right\|\\ &=\; \left\|(d_{\bar g}\Phi_f(\bar g^{-1}\cdot g)^{-1})_{\mathbb{V}}\cdot \bar g_{\mathbb{V}}^{-1}\cdot g_{\mathbb{V}}\right\|. \end{aligned}$$

Hence, since $(d_{\bar g}\Phi_f)_{\mathbb{V}}$ is a linear map from $\mathbb{V}$, identified with $\mathbb{R}^k$, to itself, we get that

$$(d_{\bar g}\Phi_f)_{\mathbb{V}} \;=\; I_{\mathbb{V}},$$

which provides (2.23).

We prove, now, case (ii). by Proposition 2.1.2, for each $\eta \in \mathbb{W}$, we have

$$d_{\bar{g}}f(\eta)^{-1} f_{\bar{p}^{-1}}(\eta) \;=\; d_{\bar{g}}f(\eta)^{-1} \cdot f(\bar{g})^{-1} \cdot f\left(\bar{g} \cdot f(\bar{g}) \cdot \eta \cdot f(\bar{g})^{-1}\right).$$

Let $g = \bar{g} \cdot f(\bar{g}) \cdot \eta \cdot f(\bar{g})^{-1}$. Then

$$\begin{aligned} d_{\bar{g}}f(\eta)^{-1} \cdot f(\bar{g})^{-1} \cdot f(\bar{g})^{-1} \cdot f\left(\bar{g} \cdot f(\bar{g}) \cdot \eta \cdot f(\bar{g})^{-1}\right) \;=& \\ =&\; d_{\bar{g}}f\left(f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\right)^{-1} \cdot f(\bar{g})^{-1} \cdot f(g) \\ =&\; d_{\bar{g}}f(\bar{g}^{-1} \cdot g)^{-1} \cdot f(\bar{g})^{-1} \cdot f(g), \end{aligned}$$

where the last inequality follows because $d_{\bar{g}}f$ is a homogeneous homomorphism and $\mathbb{V}$ ia an Abelian subgroup.

Now, assume that f is intrinsic differentiable. Then

$$\left\| d_{\bar{g}}f(\eta)^{-1} \cdot f_{\bar{p}^{-1}}(\eta) \right\| \;=\; o\left(\|\eta\|\right),$$

for $\|\eta\| \longrightarrow 0$. This implies that

$$\begin{aligned} o\left(\|f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\|\right) \;=&\; \left\| d_{\bar{g}}f(\eta)^{-1} \cdot f_{\bar{p}^{-1}}(\eta) \right\| \\ =&\; \left\| d_{\bar{g}}f(\bar{g}^{-1} \cdot g)^{-1} \cdot f(\bar{g})^{-1} \cdot f(g) \right\|, \end{aligned}$$

for $\|f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\| \longrightarrow 0$. Analogously one can prove the converse. □

Once more following [18], our aim, now, is to study more geometric feature for intrinsic differentiability. Next theorem provides us two characterizations. The first is in terms of blow-ups: we approximate the graph of an intrinsic differentiable function f, from a vertical subgroup $\mathbb{W}$ to a horizontal one $\mathbb{V}$, with a vertical subgroup complementary to $\mathbb{V}$. The second characterization is in terms of intrinsic cones. First, we need the following

Lemma 2.2.20. *Let $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$ be a semidirect product. If $\mathbb{M}$ is another homogeneous subgroup such that $\mathbb{M} \cap \mathbb{G}_2 = \{e\}$, then there exists a continuous function*

$$\psi \;:\; \Pi_{\mathbb{G}_1}(\mathbb{M}) \;\longrightarrow\; \mathbb{G}_2$$

such that $\mathbb{M} = \mathrm{graph}(\psi)$.

In particular, if $\mathbb{W}$, $\mathbb{V}$ and $\mathbb{W}'$, $\mathbb{V}'$ are two couples of complementary subgroups as in Proposition 1.4.4, then there exists an intrinsic linear function $\psi : \mathbb{W} \longrightarrow \mathbb{V}$ such that $\mathbb{W}' = \mathrm{graph}(\psi)$.

Proof. We start by proving that $\Pi_{\mathbb{G}_1} : \mathbb{M} \longrightarrow \mathbb{G}_2$ is injective. Assume that $p, \bar{p} \in \mathbb{M}$ have the same projection on $\mathbb{G}_1$, that is $p = g_1 \cdot g_2$ and $\bar{p} = g_1 \cdot \bar{g}_2$. Since $\mathbb{M}$ is a group,

$$p^{-1} \cdot \bar{p} \;=\; g_2^{-1} \cdot g_2^{-1} \cdot g_1 \cdot \bar{g}_2 \;=\; g_2^{-1} \cdot \bar{g}_2 \,\in \mathbb{M} \cap \mathbb{G}_2.$$

Hence $p^{-1} \cdot \bar{p} = e$, which implies $p = \bar{p}$.

Now, any point $p \in \mathbb{M}$ can be written as $p = \Pi_{\mathbb{G}_1}(p) \cdot \Pi_{\mathbb{G}_2}(p)$. Hence, setting $g_1 := \Pi_{\mathbb{G}_1}(p) \in \Pi_{\mathbb{G}_1}(\mathbb{W})$, we can define $p = g_1 \cdot \psi(g_1)$, with $\psi = \Pi_{\mathbb{G}_2} \cdot (\Pi_{\mathbb{G}_1})^{-1}$. Clearly graph$(\psi) = \mathbb{M}$, then we have to show that ψ is a continuous mapping. By Proposition 1.4.2, $\Pi_{\mathbb{G}_1}$ is continuous, then the continuity of ψ follows from the continuity of $(\Pi_{\mathbb{G}_1})^{-1}$.

Let $\{q_n\}_{n\in\mathbb{N}} \subset \Pi_{\mathbb{G}_1}(\mathbb{M})$ be a sequence such that $q_n \to \bar{q} \in \Pi_{\mathbb{G}_1}(\mathbb{M})$, as $n \to +\infty$. Suppose for the moment that there exists a subsequence (not renamed) of $\left\{(\Pi_{\mathbb{G}_1})^{-1}(q_n)\right\}_{n\in\mathbb{N}}$ such that $(\Pi_{\mathbb{G}_1})^{-1}(q_n) \longrightarrow \bar{m}$, for some $\bar{m} \in \mathbb{H}^n$. Since $\mathbb{M}$ is closed, $\bar{m} \in \mathbb{M}$. Furthermore, by continuity of $\Pi_{\mathbb{G}_1}$,

$$q_n \;=\; \Pi_{\mathbb{G}_1} \circ (\Pi_{\mathbb{G}_1})^{-1}(q_n) \;\longrightarrow\; \Pi_{\mathbb{G}_1}(\bar{m}).$$

Hence, $\bar{q} = \Pi_{\mathbb{G}_1}(\bar{m})$ and $(\Pi_{\mathbb{G}_1})^{-1}(\bar{q}) = \bar{m}$, which gives the continuity of $(\Pi_{\mathbb{G}_1})^{-1}$.

The proof will be complete if we prove the existence of a convergent subsequence of $\left\{(\Pi_{\mathbb{G}_1}(q_n))^{-1}\right\}_{n\in\mathbb{N}}$. First, we notice that

$$(\Pi_{\mathbb{G}_1})^{-1} \,:\, \Pi_{\mathbb{G}_1}(\mathbb{M}) \;\longrightarrow\; \mathbb{M}$$

is homogeneous, that is $(\Pi_{\mathbb{G}_1})^{-1} \circ \delta_\lambda = \delta_\lambda \circ (\Pi_{\mathbb{G}_1})^{-1}$, for all $\lambda > 0$.

Now, let us consider $c_n := \left\|(\Pi_{\mathbb{G}_1})^{-1}(q_n)\right\|$. Striving for a contradiction, suppose that $\{c_n\}_{n\in\mathbb{N}}$ is unbounded. Then $p_n := \delta_{c_n^{-1}}(q_n) \longrightarrow e$ and

$$\left\|(\Pi_{\mathbb{G}_1})^{-1}\left(\delta_{c_n^{-1}}(q_n)\right)\right\| \;=\; c_n^{-1}\,\left\|(\Pi_{\mathbb{G}_1})^{-1}(q_n)\right\| \;=\; 1.$$

Hence, with the same argument as before, we get that

$$(\Pi_{\mathbb{G}_1})^{-1}\left(\delta_{c_n^{-1}}(q_n)\right) \;=\; \delta_{c_n^{-1}}\,(\Pi_{\mathbb{G}_1})^{-1}(q_n) \;\longrightarrow\; e,$$

as $n \longrightarrow +\infty$. Now we have the required contradiction since $\left\|(\Pi_{\mathbb{G}_1})^{-1}(p_n)\right\| = 1$, for each $n \in \mathbb{N}$. □

Theorem 2.2.21. *Let $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ be as in Proposition 1.4.4 and $f : \mathcal{E} \subset \mathbb{W} \longrightarrow \mathbb{V}$, with $\mathcal{E}$ an open subset of $\mathbb{W}$. Let $w_0 \in \mathcal{E}$ and $g_0 := w_0 \cdot f(w_0)$. Then the following statements are equivalent:*

(i) *f is intrinsic differentiable in $w_0 \in \mathcal{E}$.*

(ii) *There exists a vertical subgroup $\mathbb{T}_{f,g_0}$, complementary to $\mathbb{V}$, characterized by*

$$\lim_{\lambda\to+\infty} \delta_\lambda \left(g_0^{-1} \cdot \mathrm{graph}(f)\right) \;=\; \mathbb{T}_{f,g_0}, \tag{2.24}$$

in the sense of Hausdorff convergence[3] in compact subsets of $\mathbb{H}^n$.

(iii) *There exists a vertical subgroup $\mathbb{T}'_{f,g_0}$, complementary to $\mathbb{V}$, and, for any $\alpha > 0$, there exists $r_0 = r_0(f, w_0, \alpha) > 0$, such that*

$$C_{\mathbb{T}_{f,g_0},\mathbb{V}}(g_0, \alpha) \cap B(g_0, r_0) \cap \mathrm{graph}(f) \;=\; \{g_0\}.$$

Notice that $\mathbb{T}_{f,g_0} = \mathbb{T}'_{f,g_0} = \mathrm{graph}(d_{w_0} f)$.

Notation 2.2.2. We call $\mathbb{T}_{f,g_0}$ the *tangent subgroup* to graph(f) at g_0.

Proof. We can assume, without loss of generality, that $w_0 = e$ and $f(w_0) = g_0 = e$. Indeed, each condition (i), (ii) and (iii) depends on notions which are invariant by left translations.

First, we prove that (i) implies (iii). Denote by $\mathbb{T}'_{f,g_0} := \mathrm{graph}(d_e f)$. Let $p \in \mathbb{H}^n$. Then

$$\begin{aligned} p \;&=\; p_{\mathbb{W}} \cdot p_{\mathbb{V}} \\ &=\; \underbrace{p_{\mathbb{W}} \cdot (d_e f(p_{\mathbb{W}}))}_{\in\,\mathbb{T}'_{f,e}} \cdot \underbrace{(d_e f(p_{\mathbb{W}}))^{-1} \cdot p_{\mathbb{V}}}_{\in\,\mathbb{V}}. \end{aligned} \tag{2.25}$$

Moreover, if $x \cdot d_e f(x) \in \mathbb{V}$, for some $x \in \mathbb{W}$, then $x \in \mathbb{V}$, implying that $x = e$. Hence, $\mathbb{T}'_{f,e} \cap \mathbb{V} = \{e\}$.

[3]On the set $\mathcal{K}$ of all compact subsets of $\mathbb{H}^n$, we define the Hausdorff metric as follows:

$$d_H(E, F) \;:=\; \inf\left\{\varepsilon \in \mathbb{R}^+ \,|\, E^\varepsilon \subseteq F, F^\varepsilon \subseteq E\right\}.$$

With this metric, we say that a sequence $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ converges to $\tilde{K}$, if $d_H(K_n, \tilde{K}) \longrightarrow 0$, as $n \to 0$.

Our goal now is to prove

$$C_{\mathbb{T}_{f,e},\mathbb{V}}(e,\alpha)\cap B(e,r_0)\cap \operatorname{graph}(f) \;=\; \{e\}. \tag{2.26}$$

We have to show that, for all $\alpha > 0$, there exists $r_0 = r_0(f,e,\alpha) > 0$ such that if $0 < \|x\cdot f(x)\| < r_0$, then $x\cdot f(x)\notin C_{\mathbb{T}'_{f,e},\mathbb{V}}(e,\alpha)$. That is equivalent to prove that if $0 < \|x\cdot f(x)\| < r_0$, then $\left\|\Pi_{\mathbb{T}'_{f,e}}(x\cdot f(x))\right\| > \alpha\,\|\Pi_{\mathbb{V}}(x\cdot f(x)\|$. By a standard compactness argument, we know that the geometric constant

$$c := \inf_{\substack{v\in\mathbb{V}\\ x\in\mathbb{W},\,\|x\|=1}} \|x\cdot v\|$$

is positive (notice that it depends only on the group structure). Therefore, by homogeneity, we can conclude that

$$\|x\| \;\le\; \frac{1}{c}\,\|x\cdot v\|,$$

for all $x\in\mathbb{W}$ and for all $v\in\mathbb{V}$.

Let $\alpha\in\mathbb{R}^+$ be fixed and choose $\varepsilon = \frac{c}{\alpha}$. By equation (2.22), there exists $\delta\in\mathbb{R}^+$ such that, for each $\|x\| < \delta$,

$$\begin{aligned}\|\Pi_{\mathbb{V}}(x\cdot f(x))\| &= \|d_e f(x)^{-1}\cdot f(x)\|\\ &< \frac{c}{\alpha}\,\|x\| \;\le\; \frac{1}{\alpha}\,\|x\cdot d_e f(x)\|\\ &= \frac{1}{\alpha}\,\left\|\Pi_{\mathbb{T}_{f,e}}(x\cdot f(x)\right\|.\end{aligned}$$

Hence, choosing $r_0 := c\,\delta$, we have (2.26).

Let us prove that (iii) implies (ii). Fix $R\in\mathbb{R}^+$ and choose $\mathbb{T}_{f,e} = \mathbb{T}'_{f,e}$. Notice that, for each $\varepsilon > 0$, there exists a positive constant $\alpha = \alpha(\mathbb{T}_{f,e},\mathbb{V},R,\varepsilon)$ such that

$$B(e,R) \subset (\mathbb{T}_{f,e})^{\varepsilon} \cup C_{\mathbb{T}_{f,e},\mathbb{V}}(e,\alpha), \tag{2.27}$$

where $(\mathbb{T}_{f,e})^{\varepsilon}$ is an ε-neighbourhood of $\mathbb{T}_{f,e}$. Now, by hypothesis, we know that, for each $\alpha > 0$, there exists $r_0(\alpha) > 0$ such that, for all $0 < r < r_0$

$$C_{\mathbb{T}_{f,e},\mathbb{V}}(e,\alpha) \cap B(e,r) \cap \operatorname{graph}(f) \;=\; \{e\}.$$

Hence, for all $\lambda > \frac{R}{r_0}$, we have

$$C_{\mathbb{T}_{f,e},\mathbb{V}}(e,\alpha) \cap B(e,R) \cap \delta_\lambda\left(\operatorname{graph}(f)\right) \;=\; \{e\}, \tag{2.28}$$

which implies, together with (2.27), that, for all $\varepsilon \in \mathbb{R}^+$, there exists $\lambda_0 = \lambda_0(e, r_0, \varepsilon) > 0$ such that, for any $\lambda > \lambda_0$,

$$B(e,R) \cap \delta_\lambda(\text{graph}(f)) \subset (\mathbb{T}_{f,e})^\varepsilon .$$

For the converse inclusion, we need to prove that, for all $\varepsilon \in \mathbb{R}^+$ and $R > 0$, $\mathbb{T}_{f,e} \cap B(e,R)$ is contained in an ε-neighbourhood of $\delta_\lambda(\text{graph}(f))$, for $\lambda > \lambda_0(R, \varepsilon)$.

By Lemma 2.2.20, we know that there exists a continuous map $\psi : \Pi_{\mathbb{W}}(\mathbb{T}_{f,e}) \subset \mathbb{W} \longrightarrow \mathbb{V}$ such that $\mathbb{T}_{f,e} = \text{graph}(\psi)$. For simplicity of notation, we set $w = \delta_{\frac{1}{\lambda}}(\bar{w})$, where $\bar{p} = \bar{w} \cdot \psi(\bar{w})$. Since $\mathcal{E}$ is an open subset of $\mathbb{W}$, there exists $\bar{\lambda} > 0$, which depends on $\mathcal{E}$ and R but not on point $\bar{p}$, such that $w \in B(e, r_0) \cap \mathbb{W}$, for all $\lambda > \bar{\lambda}$. With this choice of $\bar{\lambda}$, we have

$$\delta_\lambda(w \cdot f(w)) = \bar{w} \cdot \delta_\lambda\left(f\left(\delta_{\frac{1}{\lambda}}(\bar{w})\right)\right).$$

The preceding observation leads to estimate $\text{dist}(\bar{p}, \delta_\lambda(\text{graph}(f)))$:

$$\begin{aligned}
\text{dist}(\bar{p}, \delta_\lambda(\text{graph}(f)) &\leq d(\bar{p}, \delta_\lambda(w \cdot f(w))) \\
&= \left\|\bar{p}^{-1} \cdot \delta_\lambda(w \cdot f(w))\right\| \\
&= \left\|\psi(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot \bar{w} \cdot \delta_\lambda\left(f\left(\delta_{\frac{1}{\lambda}}(\bar{w})\right)\right)\right\| \\
&= \left\|\psi(\bar{w})^{-1} \cdot \delta_\lambda(f(w))\right\| .
\end{aligned}$$

On the other side, from (2.28) and (iii), we know that, for all $\alpha > 0$ and $r_0 > 0$, there exists $\lambda_0 = \lambda_0(r_0, \alpha)$ such that, for all $\lambda > \lambda_0$,

$$C_{\mathbb{T}_{f,e},\mathbb{V}}(e, \alpha) \cap B(e, r_0) \cap \delta_\lambda(\text{graph}(f)) = \{e\},$$

which is equivalent, for all $\lambda > \lambda_0$ and for $w \cdot f(w) \in B(e, r_0)$, to

$$\alpha \cdot \|\Pi_{\mathbb{V}}(\delta_\lambda(w \cdot f(w)))\| < \left\|\Pi_{\mathbb{T}_{f,e}}(\delta_\lambda(w \cdot f(w)))\right\| . \tag{2.29}$$

We calculate explicitly the projections in order to rewrite (2.29):

$$\delta_\lambda(w \cdot f(w)) = \bar{w} \cdot \delta_\lambda f(w) = \underbrace{\bar{w} \cdot \psi(\bar{w})}_{\in \mathbb{T}_{f,e}} \cdot \underbrace{\psi(\bar{w})^{-1} \cdot \delta_\lambda(f(w))}_{\in \mathbb{V}} .$$

This completes the second part of the proof. Indeed, (2.29) can be rewritten in the form

$$\left\|\psi(\bar{w})^{-1} \cdot \delta_\lambda f(w)\right\| < \frac{1}{\alpha}\, \|\bar{w} \cdot \psi(\bar{w})\| \leq \frac{R}{\alpha}.$$

We come to the last part of the proof: (ii) implies (i). We have to show that there exists an intrinsic linear function $d_e f : \mathbb{W} \longrightarrow \mathbb{V}$, such that

$$\left\| d_e f(w)^{-1} \cdot f(w) \right\| \; = \; o\left(\|w\|\right), \tag{2.30}$$

as $\|w\| \longrightarrow 0$. Lemma 2.2.20 asserts that there exists a continuous intrinsic linear function $\psi : \mathbb{W} \longrightarrow \mathbb{V}$ such that $\mathbb{T}_{f,e} = \text{graph}(\psi)$. Then we can choose $d_e f(w) := \psi(w)$, for all $w \in \mathbb{W}$.

Let $\{w_n\}_{n\in\mathbb{N}} \subset \mathbb{W}$ be a sequence such that $w_n \neq e$, for all positive integer n, and $w_n \longrightarrow e$, as $n \longrightarrow \infty$. Let us define $\lambda_n := \frac{1}{\|w_n\|}$. Then from homogeneity of the distance

$$\begin{aligned}\lambda_n \, d\left(d_e f(w_n), f(w_n)\right) \; &= \; d\left(\delta_{\lambda_n}(w_n) \cdot d_e f\left(\delta_{\lambda_n} w_n\right), \delta_{\lambda_n}\left(w_n \cdot f(w_n)\right)\right) \\ &\leq \; d\left(\delta_{\lambda_n}(w_n) \cdot d_e f\left(\delta_{\lambda_n}(w_n) \cdot d_e f\left(\delta_{\lambda_n} w_n\right), p_n\right)\right) \\ &\qquad + \; d\left(p_n, \delta_{\lambda_n}\left(w_n \cdot f(w_n)\right)\right),\end{aligned} \tag{2.31}$$

where we have chosen $p_n = \eta_n \cdot d_e f(\eta_n) \in \mathbb{T}_{f,e} \cap B(e, 2)$ such that

$$\text{dist}\left(\delta_{\lambda_n}\left(w_n \cdot f(w_n)\right), \mathbb{T}_{f,e}\right) \; = \; d\left(\delta_{\lambda}(w_n \cdot f(w_n)), p_n\right).$$

Now, by assumption,

$$d\left(p_n, \delta_{\lambda_n}(w_n \cdot f(w_n))\right) \; := \; d\left(\eta_n \cdot d_e f(\eta_n), \delta_{\lambda_n}(w_n \cdot f(w_n))\right) \; \longrightarrow \; 0, \tag{2.32}$$

as $n \longrightarrow +\infty$. Hence, since projections are continuous maps (Proposition 1.4.2) and we remember that $\Pi_{\mathbb{W}}(p_n) = \eta_n$ and $\Pi_{\mathbb{W}}\left(\delta_{\lambda_n}\left(w_n \cdot f(w_n)\right)\right)$,

$$d\left(\delta_{\lambda_n} w_n, \eta_n\right) \; \longrightarrow \; 0 \tag{2.33}$$

as $n \longrightarrow +\infty$. It follows, because of continuity of $d_e f$, that

$$d\left(d_e f(\delta_\lambda w), d_e f(\eta_n)\right) \; \longrightarrow \; 0, \tag{2.34}$$

as $n \longrightarrow +\infty$. Thus, once more from (2.24), (2.33) and (2.34),

$$d\left(\delta_{\lambda_n}(w_n) \cdot d_e f(\delta_{\lambda_n} w_n), p_n\right) \; = \; d\left(\delta_{\lambda_n} w_n \cdot d_e f(\delta_{\lambda_n} w_n), \eta_n \, d_e f(\eta_n)\right) \; \longrightarrow \; 0, \tag{2.35}$$

as $n \longrightarrow +\infty$. The proof is complete; indeed, from (2.31), together with (2.35) and (2.31), we get

$$\lim_{n \to +\infty} \frac{d\left(d_e f(w_n), f(w_n)\right)}{\|w_n\|} \; = \; 0.$$

□

We are now ready to proceed to the main result of this section: the Rademacher-type Theorem. We confine ourselves to discussing the case of 1-codimensional graphs: from now on, we consider, once again, $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Assumption 2.2.7.

Let $f : \mathcal{E} \subset \mathbb{W} \longrightarrow \mathbb{V}$ be an intrinsic Lipschitz continuous function, where $\mathcal{E}$ is an open subset of $\mathbb{W}$. Analogously to the classical case, we aim to prove that f is intrinsic differentiable almost everywhere in $\mathcal{E}$.

Let us recall some notations introduced in Section 1.5.

Notation 2.2.3. Let $\nu \in \mathfrak{h}_1$ be a fixed horizontal vector field. We define the *vertical hemispaces* $S^+_{\mathbb{H}}(\nu)$ and $S^-_{\mathbb{H}}(\nu)$ and their common boundary $\mathbb{N}(\nu)$ as

$$\begin{aligned} S^+_{\mathbb{H}}(\nu) &:= \exp\left(\{ Z \in \mathfrak{h}^n \mid \langle Z, \nu \rangle \geq 0\}\right), \\ S^-_{\mathbb{H}}(\nu) &:= \exp\left(\{ Z \in \mathfrak{h}^n \mid \langle Z, \nu \rangle \leq 0\}\right), \\ \mathbb{N}(\nu) &:= \exp\left(\{ Z \in \mathfrak{h}^n \mid \langle Z, \nu \rangle = 0\}\right). \end{aligned}$$

Remark 2.2.15. Since $\{ Z \in \mathfrak{h}^n \mid \langle Z, \nu \rangle = 0\}$ is a 1-codimensional ideal of $\mathfrak{h}^n$, $\mathbb{N}(\nu)$ is a 1-codimensional normal subgroup of $\mathbb{H}^n$. Moreover, $\mathbb{L}(\nu) := \exp\{t\,\nu \mid t \in \mathbb{R}\}$ and $\mathbb{N}(\nu)$ are complementary subgroups in $\mathbb{H}^n$.

Proposition 2.2.22. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Assumption 2.2.7, and let $f : \mathbb{W} \longrightarrow \mathbb{V}$ be an intrinsic L-Lipschitz function. Consider also $p = w \cdot f(w) \in \mathrm{graph}(f)$.*

If there exists $\nu \in \mathfrak{h}_1$ such that

$$\lim_{r \to 0} \mathbf{1}_{\left(E_f^-\right)_{r,p}} = \mathbf{1}_{S^+_{\mathbb{H}}(\nu)} \tag{2.36}$$

in $L^1_{loc}(\mathbb{H}^n)$, where

$$\left(E_f^-\right)_{r,p} = \left\{ q \in \mathbb{H}^n \mid p \cdot \delta_r q \in E_f^- \right\},$$

then f is intrinsic differentiable in w.

The proof is a direct consequece of the following lemmas and of Theorem 2.2.21.

Lemma 2.2.23. *Under the same assumption of Proposition 2.2.22, there exists an intrinsic linear function $f_\infty : \mathbb{W} \longrightarrow \mathbb{V}$ such that*

(i) $\delta_{\frac{1}{r}} \circ f \circ \delta_r := f_{\frac{1}{r}} \longrightarrow f_\infty$*, as $r \longrightarrow \infty$, uniformly on compact sets;*

(ii) $\mathrm{graph}(f_\infty) = \mathbb{N}(\nu)$*;*

(iii) $\mathbb{N}(\nu)$ *and* $\mathbb{V}$ *are complementary subgroup in* $\mathbb{H}^n$.

Proof. For $M > 0$, we consider $\mathcal{F} := \left\{ f_{\frac{1}{r}} : B(e, M) \cap \mathbb{W} \longrightarrow \mathbb{V} \,\middle|\, r > 0 \right\}$. The collection $\mathcal{F}$ is equibounded. Indeed

$$\begin{aligned} \left\| f_{\frac{1}{r}}(w) \right\| &= \frac{1}{r} \, \|f(\delta_r(w))\| \le \frac{L}{r} \, \|\delta_r(w)\| \\ &= L\|w\| \le L\,M. \end{aligned}$$

Moreover, by Proposition 2.2.5, each element of $\mathcal{F}$ is intrinsic L-Lipschitz. This fact implies that $\mathcal{F}$ is equicontinuous. Indeed, for $r > 0$, by Remark 2.2.4,

$$\begin{aligned} \left\| f_{\frac{1}{r}}(w) \cdot f_{\frac{1}{r}}(\bar{w})^{-1} \right\| &\le L \left\| f_{\frac{1}{r}}(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot f_{\frac{1}{r}}(\bar{w}) \right\| \\ &\le L\,C \left(\left\| \bar{w}^{-1} \cdot w \right\| + \left\| f_{\frac{1}{r}}(\bar{w}) \right\|^{\frac{1}{2}} \cdot \|w' - \bar{w}'\|_{\mathbb{R}^{2n}}^{\frac{1}{2}} \right) \\ &\le C\,L\,M^{\frac{1}{2}} \left(1 + M^{\frac{1}{2}} \right), \end{aligned}$$

where in the second inequality we applied (2.9).

By Arzelà-Ascoli Theorem and a standard diagonal argument, we obtain that there exists a subsequence $\{r_k\}_{k\in\mathbb{N}}$, converging to zero as $k \longrightarrow +\infty$, such that $f_{\frac{1}{r_k}} \longrightarrow f_\infty$ uniformly on compact sets, as $k \longrightarrow +\infty$.

Statement (i) will follow from statement (ii), because of definition of intrinsic graph, which determines uniquely f_∞. Thus, we need to prove statement (ii). First we show that

$$\mathbb{N}(\nu) \subset \mathrm{graph}(f_\infty). \tag{2.37}$$

To do this, we take a point $\bar{p} \notin \mathrm{graph}(f_\infty)$; for instance, suppose $\bar{p} \in E^-_{f_\infty}$. We aim to show that $\mathbf{1}_{S^+_{\mathbb{H}}(\nu)}(\bar{p}) = 1$. Let us consider a ball B, centered at $\bar{p}$ with radius $\rho > 0$, such that

$$\mathrm{dist}\left(B, \mathrm{graph}(f_\infty)\right) > 0.$$

Since $\left\{ f_{\frac{1}{r_k}} \right\}_{k\in\mathbb{N}}$ is uniformly convergent on compact sets, we can assume that $B \subset E^-_{f_{1/r_k}}$, for k large enough. It follows that $\mathbf{1}_{E^-_{f_{1/r_k}}} \equiv 1$, for k large. By (2.36), since $E^-_{f_{1/r_k}} = \left(E^-_f\right)_{r_k,e}$, we have that $\mathbf{1}_{S^+_{\mathbb{H}}(\nu)} = 1$ almost everywhere in B. This implies that $\mathbf{1}_{S^+_{\mathbb{H}}(\nu)} = 1$ on B and hence $\mathbf{1}_{S^+_{\mathbb{H}}(\nu)} = 1$. The case $\bar{p} \in \left(E^-_{f_\infty}\right)^c \setminus \mathrm{graph}(f_\infty)$ can be handled in the same way, concluding the proof of (2.37).

To prove the converse inclusion, we notice that $\bar{p} \in \mathrm{graph}(f_\infty)$ is both the limit of a sequence $(p_n)_{n\in\mathbb{N}}$ in $E^-_{f_\infty}$ and of a sequence $(q_n)_{n\in\mathbb{N}}$ in $\left(E^-_{f_\infty}\right)^c \setminus \mathrm{graph}(f_\infty)$. Indeed, if $\bar{p} = \bar{w} \cdot f(\bar{w})$, recalling that $f(\bar{w}) = \exp(\varphi(\bar{w})\, V)$, it is enough to choose

$$p_n \;=\; \bar{w} \cdot \exp\left(\left(\varphi(\bar{w}) - \frac{1}{n}\right) \cdot V\right) \quad \text{and} \quad q_n \;=\; \bar{w} \cdot \exp\left(\left(\varphi(\bar{w}) + \frac{1}{n}\right) \cdot V\right).$$

On the other hand, we have just shown that $E^-_{f_\infty} \subset S^+_{\mathbb{H}}(\nu)$ and $\left(E^-_{f_\infty}\right)^c \setminus \mathrm{graph}(f_\infty) \subset S^-_{\mathbb{H}}(\nu)$, therefore $\bar{p} \in \overline{S^+_{\mathbb{H}}(\nu)} \cap \overline{S^-_{\mathbb{H}}(\nu)} = \mathbb{N}(\nu)$. This complete the proof of statements (i) and (ii).

Let us prove statement (iii). First we prove that if $v \in \mathbb{V}$, then $v \notin \mathrm{graph}(f_\infty)$. By assumption f is intrinsic L-Lipschitz, therefore, if we fix $0 < \alpha < \frac{1}{L}$, then

$$C_{\mathbb{W},\mathbb{V}}(e,\alpha) \cap \mathrm{graph}(f) \;=\; \{e\}.$$

Hence, since cones are invariant under group dilations ((iii) of Proposition 2.2.2),

$$C_{\mathbb{W},\mathbb{V}}(e,\alpha) \cap \mathrm{graph}\left(f_{\frac{1}{r}}\right) \;=\; \{e\},$$

for any $r > 0$. This implies that

$$C_{\mathbb{W},\mathbb{V}}\left(e, \frac{\alpha}{2}\right) \cap \mathrm{graph}(f_\infty) \;=\; \{e\}. \tag{2.38}$$

Indeed, take a point $p = w \cdot f_{\frac{1}{r}}(w)$, we have

$$p \;=\; \lim_{r\to 0} p_r.$$

Now, it is clear that $p_r \in \mathrm{graph}\left(f_{\frac{1}{r}}\right)$; we can also assume that $p_r \neq e$, for every $r > 0$. Then $p_r \notin C_{\mathbb{W},\mathbb{V}}(e,\alpha)$, for any $r > 0$. From (ii) of Proposition 2.2.2, we can coclude also that $p_r \notin C_{\mathbb{W},\mathbb{V}}\left(e, \frac{\alpha}{2}\right)$, hence (2.38).

Since $v \in C_{\mathbb{W},\mathbb{V}}\left(e, \frac{\alpha}{2}\right)$ and $v \neq e$, $v \notin \mathrm{graph}(f_\infty)$, implying that $\mathbb{N}(\nu) \cap \mathbb{V} = \{e\}$. This implies that, in $\mathfrak{h}^n$

$$\{Z \in \mathfrak{h}^n \mid \langle Z, \nu\rangle = 0\} \cap \mathrm{span}\{V\} \;=\; \{0\}.$$

Hence, we have necessarily that

$$\mathfrak{h}^n \;=\; \{Z \in \mathfrak{h}^n \mid \langle Z, \nu\rangle = 0\} \oplus \mathrm{span}\{V\},$$

that is exactly what we wanted. □

Lemma 2.2.24. *With the same assumption as in Proposition 2.2.22, for every $\alpha > 0$, there exists $\delta = \delta(\alpha) > 0$ such that*

$$\mathrm{graph}(f) \cap B(e,\delta) \cap C_{\mathbb{N}(\nu),\mathbb{V}}(e,\alpha) = \{e\}. \tag{2.39}$$

Proof. Let us assume, by contradiction, that (2.39) fails. Then we can find a sequence $p_n := w_n \cdot f(w_n) \in \mathrm{graph}(f) \cap C_{\mathbb{N}(\nu),\mathbb{V}}(e,\alpha)$ such that $p_n \neq e$, for every $n \in \mathbb{N}$, but

$$p_n \longrightarrow e, \qquad \text{as } n \longrightarrow +\infty.$$

For simplicity of notation, we set $\xi_n := \delta_{\frac{1}{\|w_n\|}}(w_n)$, for every $n \in \mathbb{N}$, and we assume, without loss of generality, that $\xi_n \longrightarrow \xi_0$, as $n \to +\infty$, with $\|\xi_0\| = 1$. We have

$$\delta_{\frac{1}{\|w_n\|}}(p_n) = \xi_n \cdot \delta_{\frac{1}{\|w_n\|}}\left(f\left(\delta_{\|w_n\|}(\xi_n)\right)\right) = \xi_n \cdot f_{\frac{1}{\|w_n\|}}(\xi_n);$$

hence $\delta_{\frac{1}{\|w_n\|}}(p_n) \in C_{\mathbb{N}(\nu),\mathbb{V}}(e,\alpha) \cap \mathrm{graph}\left(f_{\frac{1}{\|w_n\|}}\right)$. Applying Lemma 2.2.23, it follows

$$\xi_0 \cdot f_\infty(\xi_0) \in C_{\mathbb{N}(\nu),\mathbb{V}}(e,\alpha) \cap \mathrm{graph}(f_\infty) = C_{\mathbb{N}(\nu),\mathbb{V}}(e,\alpha) \cap \mathbb{N}(\nu) = \{e\}.$$

Thus, $\xi_0 \in \mathbb{W} \cap \mathbb{V} = \{e\}$, which contradicts the fact that $\|\xi_0\| = 1$. □

Theorem 2.2.25. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ be as in Assumption 2.2.7. Let $U \subset \mathbb{W}$ be an open subset and $f : U \longrightarrow \mathbb{V}$ be an intrinsic Lipschitz function. Then f is intrinsic differentiable $(\mathcal{L}^{2n} \llcorner \mathbb{W})$-a.e. in U.*

Proof. First, from Theorem 2.2.14, we can assume that f is intrinsic Lipschitz on all of $\mathbb{W}$. Now, by Theorem 1.5.5, we know that the reduced boundary

$$\partial^*_{\mathbb{H}} E_f^- \subset \partial E_f^- = \mathrm{graph}(f),$$

is such that

$$\left|\partial E_f^-\right| \left(\mathrm{graph}(f) \setminus \partial^*_{\mathbb{H}} E_f^-\right) = \left|\partial E_f^-\right|_{\mathbb{H}} \left(\partial E_f^- \setminus \partial^*_{\mathbb{H}} E_f^-\right) = 0, \tag{2.40}$$

and, for every $p \in \partial^*_{\mathbb{H}} E_f^-$, there exists $\nu = \nu(p) \in \mathfrak{h}_1$, $\|\nu(p)\| = 1$, the inward unit normal to E_f^- at p,

$$\lim_{r \to 0} \mathbf{1}_{\left(E_f^-\right)_{r,p}} = \mathbf{1}_{S^+_{\mathbb{H}}(\nu(p))} \quad \text{in } L^1_{loc}(\mathbb{H}^n).$$

From Proposition 2.2.22, f is intrinsic differentiable at every point $w \in \mathbb{W}$ such that $w \cdot f(w) \in \partial^*_{\mathbb{H}} E_f^-$.

The proof is concluded if we show that $\mathrm{graph}(f) \setminus \partial^*_{\mathbb{H}} E_f^-$ has $(\mathcal{L}^{2n} \llcorner \mathbb{W})$-measure zero. Since we know that (2.40) holds, it is enough to prove the following Lemma:

Lemma 2.2.26. *Under the same assumptions of Theorem 2.2.25, let $f : \mathbb{W} \longrightarrow \mathbb{V}$ be an intrinsic Lipschitz function and let $\Phi_f : \mathbb{W} \longrightarrow \mathbb{H}^n$ be the parametric map. Then there exists a positive constant $c = c(\mathbb{W}, \mathbb{V})$, such that*

$$(\Phi_f)_\sharp \left(\mathcal{L}^{2n} \llcorner \mathbb{W}\right) \;=\; c \cdot \langle \nu \,,\, V \rangle \; |\partial E_f^-|_{\mathbb{H}},^{4}$$

where ν denote the horizontal generalized inward normal to E_f^-.

Proof. We are assuming that there exists a vector field $V \in \mathfrak{h}_1$ such that

$$\mathbb{V} \;=\; \{\, \exp(tV) \,|\, t \in \mathbb{R} \,\} \,.$$

Therefore, we can find a non zero vector $a = (a_1, ..., a_{2n})$ such that

$$\mathbb{W} \;=\; \{\, p = (p', p_{2n+1}) \in \mathbb{H}^n \mid \langle a \,,\, p' \rangle_{\mathbb{R}^{2n}} = 0 \,\} \,.$$

Since a does not vanishes, we can assume, without loss of generality, that $a_1 = 1$.

By Theorem 2.2.16, we know that E_f^- has locally finite $\mathbb{H}$-perimeter, then

$$\int_{E_f^-} (V\, g)\; d\mathcal{L}^{2n+1} \;=\; \int_{\mathbb{H}^n} \langle \nu \,,\, V \rangle \; g\, d|\partial E_f^-|_{\mathbb{H}}, \tag{2.41}$$

for all $g \in C^1_0(\mathbb{H}^n)$.

Now, it is avantageous to define a change of variables. Let $\Phi : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2n+1} \equiv \mathbb{H}^n$ be defined as follows:

$$\Phi(\xi_1, 0, ..., 0) \;=\; \delta_{\xi_1} v \;\in\; \mathbb{V},$$
$$\Phi(0, \xi_2, ..., \xi_{2n+1}) \;=\; \left(-\sum_{i=2}^{2n} a_i \xi_i, \xi_2, ..., \xi_{2n+1} \right) \;\in\; \mathbb{W},$$

[4] We denote by $(\Phi_f)_\sharp \left(\mathcal{L}^{2n} \llcorner \mathbb{W}\right)$ the *push-forward* of the Lebesgue measure restricted to $\mathbb{W}$ under the map Φ_f.

and such that

$$\Phi(\xi_1,...,\xi_{2n+1}) \;=\; \Psi(0,\xi_2,...,\xi_{2n+1})\cdot\Psi(\xi_1,0,...,0).$$

Explicitly, the components of Ψ can be represented in the form

$$\begin{aligned}
\Psi_1(\xi) \;&=\; -\sum_{i=1}^{2n} a_i\xi_i \;+\; v_1\xi_1,\\
\Psi_j(\xi) \;&=\; \xi_j \;+\; \xi_1 v_j, \quad \text{for } j=2,...,2n,\\
\Psi_{2n+1}(\xi) \;&=\; \xi_{2n+1} \;-\; \frac{\xi}{2}\left(-\sum_{i=2}^{n}\xi_i v_{n+i} \;+\; \sum_{i=1}^{n}\xi_{n+i}v_i \;+\; \sum_{i=2}^{2n} a_i\xi_i\right).
\end{aligned}$$

Because of our requirements about the decomposition $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$, Ψ is injective. Furthermore,

$$\Psi^{-1}(\mathbb{W}) \;=\; \{\,\xi\in\mathbb{H}^n \,|\, \xi_1 = 0\,\},$$

which should be identified with a subset of $\mathbb{R}^{2n}$, and

$$\Psi(0,\cdot)\;:\;\mathbb{R}^{2n}\;\longrightarrow\;\mathbb{W}$$

is still an injective map. Now, the Jacobian matrix of Ψ is non singular, i.e. $c_1(\mathbb{W},\mathbb{V}) := \left|\det\frac{\partial\Psi}{\partial\xi}\right| \neq 0$. Indeed,

$$\det\frac{\partial\Psi}{\partial\xi} \;=\; \det\begin{pmatrix} v_1 & -a_2 & -a_3 & \cdots & -a_{2n} & 0\\ v_2 & 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0 & 0\\ v_{2n} & 0 & 0 & \cdots & 1 & 0\\ * & * & * & \cdots & * & 1 \end{pmatrix} \;=\; v_1 \;+\; \sum_{i=2}^{2n} a_i v_i \;\neq\; 0,$$

since $v\notin\mathbb{W}$.

Using the same notation as above, we denote $f(w) = \exp\left(\varphi(w)\cdot V\right)$, with $\varphi:\mathbb{W}\longrightarrow\mathbb{R}$. We also write

$$\hat{E}_f^- \;:=\; \Psi^{-1}\left(E_f^-\right) \;=\; \left\{\,\xi\in\mathbb{R}^{2n+1}\,|\,\xi_1 < \varphi\left(\Psi(0,\xi_2,...,\xi_{2n+1})\right)\,\right\}.$$

We thus get

$$\begin{aligned}\int_{\hat{E}_f^-} \frac{\partial}{\partial \xi_1}(g \circ \Psi)\, d\mathcal{L}^{2n+1} &= \int_{\hat{E}_f^-} (V g) \circ \Psi \, d\mathcal{L}^{2n+1} \\ &= \frac{1}{c_1(\mathbb{W}, \mathbb{V})} \int_{E_f^-} V g \, d\mathcal{L}^{2n+1}.\end{aligned} \tag{2.42}$$

On the other hand, using Coarea Formula, we obtain

$$\int_{\hat{E}_f^-} \frac{\partial}{\partial \xi_1}(g \circ \Psi)\, d\mathcal{L}^{2n+1} = \int_{\{\xi_1 = 0\}} g \circ \Phi_f \circ \Psi \, d\mathcal{L}^{2n}. \tag{2.43}$$

If we combine this with Theorem 1.19, on page 16, of [30], we see that

$$\int_{\mathbb{H}^n} g \, d\left((\Phi_f \circ \Psi(0, \cdot))\left(\mathcal{L}^{2n} \llcorner \mathbb{W}\right)\right) = \int_{\{\xi_1 = 0\}} g \circ \Phi_f \circ \Psi(0, \cdot)\, d\mathcal{L}^{2n}.$$

Now, we apply classical Area Formulæfor linear maps: there exists a positive constant $c_2(\mathbb{W})$ such that, for all $g \in C_0^1(\mathbb{H}^n)$,

$$\int_{\{\xi_1 = 0\}} g \circ \Phi_f(0, \cdot)\, d\mathcal{L}^{2n} = c_2(\mathbb{W}) \int_{\mathbb{W}} g \circ \Phi_f \, d\left(\mathcal{L}^{2n} \llcorner \mathbb{W}\right).$$

Therefore, by (2.42) and (2.43), we obtain

$$\int_{\mathbb{H}^n} g \, d\left((\Phi_f)_\sharp \left(\mathcal{L}^{2n} \llcorner \mathbb{W}\right)\right) = c(\mathbb{W}, \mathbb{V}) \int_{\mathbb{H}^n} \langle \nu \, , \, V \rangle \; g \, d|\partial E_f^-|_{\mathbb{H}},$$

for all $g \in C_0^1(\mathbb{H}^n)$. This completes the proof. □

Corollary 2.2.27. *Under the same assumptions of Lemma 2.2.26, we have*

$$\left(\mathcal{L}^{2n} \llcorner \mathbb{W}\right)\left(\mathbb{W} \setminus \Pi_{\mathbb{W}}(\partial^* E_f^-)\right) = 0.$$

Proof. We apply the previous Lemma. We can write, using again Theorem 1.40 of [30],

$$\begin{aligned}\left(\mathcal{L}^{2n} \llcorner \mathbb{W}\right)\left(\mathbb{W} \setminus \Pi_{\mathbb{W}}(\partial^* E_f^-)\right) &= \int_{\mathbb{W} \setminus \Pi_{\mathbb{W}}(\partial^* E_f^-)} d\left(\mathcal{L}^{2n} \llcorner \mathbb{W}\right) \\ &= \int_{\mathbb{H}^n \setminus \partial^* E_f^-} d\,(\Phi_f)_\sharp \left(\mathcal{L}^{2n} \llcorner \mathbb{W}\right) \\ &= c(\mathbb{W}, \mathbb{V}) \int_{\mathbb{H}^n \setminus \partial^* E_f^-} \langle \nu \, , \, V \rangle \; d|\partial E_f^-|_{\mathbb{H}} = 0,\end{aligned}$$

because $|\partial E_f^-|_{\mathbb{H}}\left(\mathbb{H}^n \setminus \partial^* E_f^-\right) = 0$. □

Chapter 3

Intrinsic Lipschitz Domains and Applications

In this Chapter we study intrinsic Lipschitz domains, which are connected open subsets of $\mathbb{H}^n$ whose boundaries are locally graphs of intrinsic Lipschitz maps. The main point of the chapter is the proof of the fact that intrinsic Lipschitz domains are uniform domains. In our setting being a uniform domain means to be a Boman domain. This fact is of considerable importance because this condition plays a key role in proving theorems of classical Functional Analysis and, more precisely, in Potential Theory.

In the first Section we study some particular domains in general metric space. In the second Section we prove that intrinsic Lipschitz domains are uniform and we study some properties which follows from this fact. Then we conclude the Chapter with an application concerning the regularity of intrinsic Lipschitz domains for the Dirichlet problem.

3.1 Geometry of Domains

We start by fixing our setting: from now on (if it is not differently specified) we assume that (M, d) is a general metric space.

Definition 3.1.1. *We say that a metric space* (M, d) *is* with geodesics *if, for every couple of points* $x, y \in M$*, there exists a continuous rectifiable curve* $\gamma : [0, t] \longrightarrow M$ *such* *that* $\gamma(0) = x$ *and* $\gamma(t) = y$ *and* $\mathrm{length}(\gamma) = d(x, y)$.

In this Section we investigate some properties of domains in general metric spaces. Together with the definitions of different classes of domains, we state also some relations between them. To follow our presentation, it could be useful to keep in mind the diagram in Figure 3.1 on the facing page.

Let us introduce some basic properties used in definitions we will give.

Definition 3.1.2. *Let* $\Omega \subset M$ *be a bounded open set and let* $\alpha \geq 1$. *A sequence of balls* $B_0, B_1, ..., B_k \subset \Omega$ *is an* α-Harnack chain *of* Ω *if*

(i) $B_i \cap B_{i-1} \neq \emptyset$, *for all* $i = 1, ..., k$;

(ii) *every ball* B_i *is* α-non tangetial *in* Ω, *i.e.*

$$\frac{1}{\alpha}\,\mathrm{dist}\,(B_i, \partial\Omega) \;\leq\; \mathrm{diam}\,(B_i) \;\leq\; \alpha\,\mathrm{dist}\,(B_i, \partial\Omega)\,.$$

Definition 3.1.3. *Let* $\Omega \subset M$ *be a bounded open set. We say that* Ω *satisfies the* Harnack chain condition *if, for every* $\varepsilon \in \mathbb{R}^+$ *and* $x, y \in \Omega$ *with* $\mathrm{dist}\,(x, \partial\Omega) > \varepsilon$, $\mathrm{dist}\,(y, \partial\Omega) > \varepsilon$ *and* $d(x, y) < C\varepsilon$ *for some* $C > 0$, *there exists, for* $\alpha \geq 1$, *an* α*-Harnack chain joining* x *to* y *with length* $k = k(\varepsilon)$.

Definition 3.1.4. *Let* $E \subset M$ *be a general subset. We say that* E *satisfies the* interior corkscrew condition *(*exterior corkscrew condition*) if there exist* $r_0 > 0$ *and* $k \geq 1$ *such that, for every* $r \in]0, r_0]$ *and* $x \in \partial E$, *there is* $y \in E$ *(*$y \in M \setminus E$*) for which the following two inequalities hold*

$$\frac{r}{k} \;\leq\; \mathrm{dist}\,(y, \partial E) \;\; \text{and} \;\; d(x, y) \;\leq\; r.$$

As already announced in the introduction to the Chapter, these definitions play a significant role in Potential Theory. For example, the exterior corkscrew condition (see Definition 3.1.4) implies that the domain is regular for the Dirichlet problem (via the Wiener criterion). Another application in this kind of theory is the following: the Harnack chain condition (Definition 3.1.3) is used to compare values of positive harmonic functions at different points.

Definition 3.1.5. *Let* $\Omega \subset M$ *be a bounded open set. We say that* Ω *is a* non tangentially accessible domain*,* NTA domain *for short, if there exist* $r_0 > 0$ *and* $k \geq 1$ *such that*

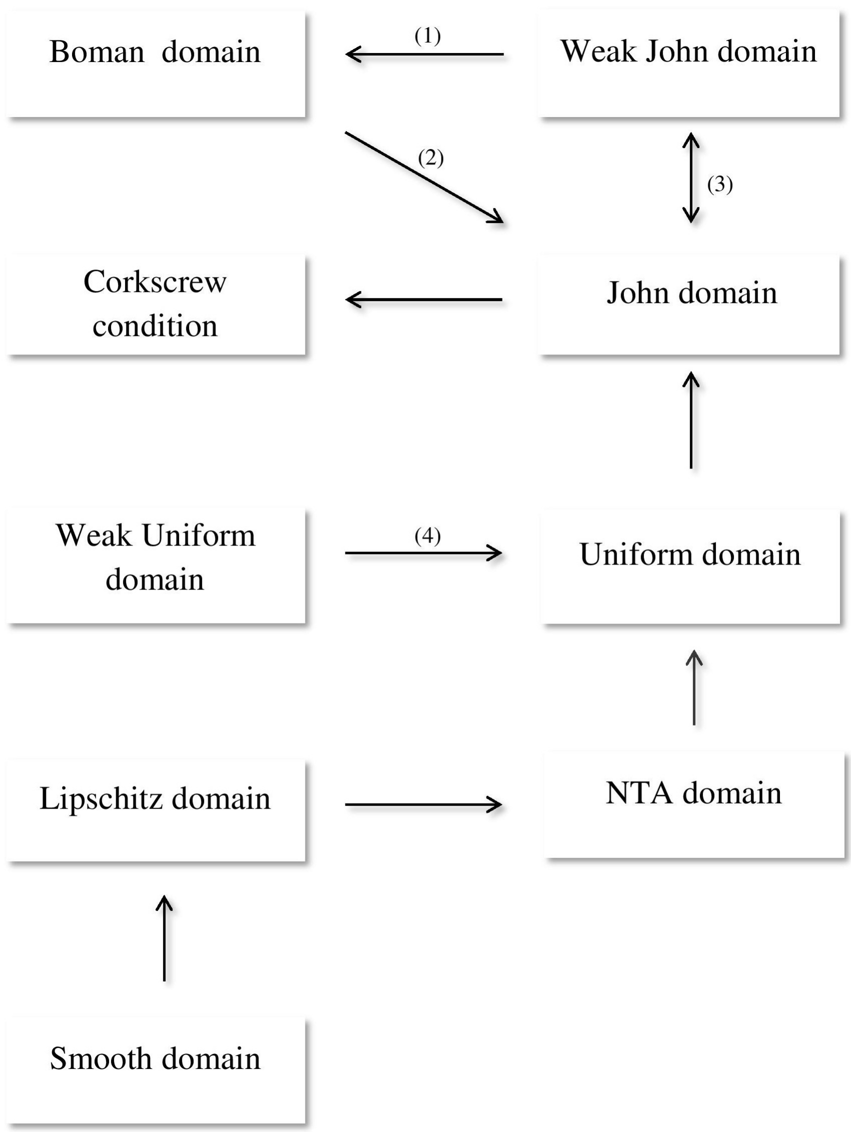

Figure 3.1: (1) If (Ω, d) is a doubling space; (2) If (Ω, d) is a doubling space with geodesics; (3) If (Ω, d) is a doubling space with geodesics; (4) If (Ω, d, μ) is an Ahlfors-regular metric space with geodesics.

(i) Ω *satisfies the corkscrew conditions;*

(ii) Ω *satisfies the Harnack chain condition.*

Let us consider a little example taken from [6], where the reader could find also more details.

Example 3.1.1. We aim to give an example of a domain which is not a NTA domain. In $\mathbb{H}^1$, endowed with the Carnot-Carathéodory metric, we consider an Euclidean cone

$$C = \left\{ (x, y, t) \in \mathbb{H}^1 \mid |x^2 + y^2|^{\frac{1}{2}} \leq \alpha t \right\},$$

for some fixed $\alpha > 0$.

The domain C does not satisfy the interior corkscrew condition near the vertex, hence it is not an NTA domain. Let us prove this fact. By contradiction, assume that there exist $r_0 > 0$ and $k \geq 1$ such that, for $r \in]0, r_0]$, there is $y_0 \in C$ which satisfies

(i) $\mathrm{dist}(y_0, 0) \leq r$;

(ii) $d(y_0, \partial C) \geq \frac{r}{k}$.

By (i), we have that $y_0 \in B(0, r)$, therefore

$$\Pi_{\mathbb{T}}(y_0) \leq \frac{2}{\pi} r^2.$$ [1]

Let us take an horizontal subunit segment γ from y_0 to $\mathbb{T}$ and continue until it meets ∂C. Let y_1 be the point of $\partial C \cap \gamma$. It is well known that horizontal segments are geodesics, then

$$d(y_0, \partial C) \leq d(y_0, y_1) \leq \mathrm{length}(\gamma) \leq \alpha \frac{2}{\pi} r^2,$$

and this contradicts (ii).

The class of NTA domains has been introduced by Jerison and Kenig in [25]. This class, which provides a generalization of several properties of Lipschitz domains, is a subset of the class of *uniform domain.* Uniform domains in Euclidean space $\mathbb{R}^n$, for $n \geq 2$, were introduced by Martio and Sarvas ([29]) and Jones ([27]).

[1] Compare with Appendix C.

Definition 3.1.6. *Let $\Omega \subset M$ be a (bounded) connected open set. We say that Ω is a* uniform domain *if there exists $\varepsilon \in \mathbb{R}^+$ such that, for every $x, y \in \Omega$, there exists a continuous curve $\gamma : [0,1] \longrightarrow \Omega$, joining x to y, with*

$$\text{length}(\gamma) \leq \frac{1}{\varepsilon} d(x,y), \tag{3.1}$$

and, for every $t \in [0,1]$,

$$\text{dist}\,(\gamma(t), \partial\Omega) \geq \varepsilon \min\Big\{\text{length}(\gamma_{|_{[0,t]}}), \text{length}(\gamma_{|_{[t,1]}})\Big\}. \tag{3.2}$$

Remark 3.1.7. Let us suppose that $M = \mathbb{R}^n$ and d is the Euclidean metric. Roughly speaking, a domain $\Omega \subset \mathbb{R}^n$ is uniform if each couple of points $x, y \in \Omega$ can be joined by a cigar which is not too thin (condition (3.2)) or too crooked (condition (3.1)). The reader might want to compare this approach with [38].

An example of a domain in $\mathbb{R}^2$ which is not a uniform domain is

$$\Omega = \mathbb{R}^2 \setminus \big\{ (x,0) \in \mathbb{R}^2 \,\big|\, x \leq 0 \big\}.$$

In this case the cigars are too crooked.

Remark 3.1.8. If conditions (3.1) and (3.2) hold for every $x, y \in \Omega$ such that $d(x,y) \leq \delta$, for some $\delta > 0$, then we say that Ω is an (ε, δ)*-domain*.

For completeness of exposition, we now introduce also the definition of *weak uniform domain*.

Definition 3.1.9. *Let $\Omega \subset M$ be a (bounded) connected open set. We say that Ω is a* weak uniform domain *if there exists $\varepsilon \in \mathbb{R}^+$ such that, for every $x, y \in \Omega$, there exists a continuous curve $\gamma : [0,1] \longrightarrow \Omega$, joining x to y, with*

$$\text{diam}(\gamma) \leq \frac{1}{\varepsilon} d(x,y), \tag{3.3}$$

and, for every $t \in [0,1]$,

$$\text{dist}\,(\gamma(t), \partial\Omega) \geq \varepsilon \min\Big\{\text{diam}(\gamma_{|_{[0,t]}}), \text{diam}(\gamma_{|_{[t,1]}})\Big\}. \tag{3.4}$$

Remark 3.1.10. It could be interesting to notice that, if we assume that (M, d) is an Ahlfors-regular[2] metric space with geodesics, then a weak uniform domain $\Omega \subset M$ is also a uniform domain.

Let us consider a pair of properties of uniform domains. For the detailed proofs we refer the reader to [34].

Lemma 3.1.1. *Let $\Omega \subset M$ be a bounded open set and let $0 < r < \operatorname{diam}(\Omega)$. If, for every $z \in \partial\Omega$ and for every $x, y \in \Omega \cap B(z, r)$, there exists a continuous rectifiable curve $\gamma : [0, 1] \longrightarrow \Omega$, joining x to y, such that (3.1) and (3.2) hold for some $\varepsilon \in \mathbb{R}^+$, which does not depend on z, then Ω is a uniform domain.*

Proposition 3.1.2. *Let us assume that there exists $0 < \delta \leq 2$ such that, for every $x \in M$ and $r \geq 0$, $\operatorname{diam}(B(x, r)) \geq \delta r$. If $\Omega \subset M$ is a uniform domain, then it is a Harnack domain.*

We now recall the basic definition and some general results concerning *John domains*, which have been introduced by John in [26].

Definition 3.1.11. *Let $\Omega \subset M$ be a bounded open set. We say that Ω is a* John domain *if there exists $x_0 \in \Omega$ and $C > 0$ such that, for every $x \in \Omega$, there exists a continuous and rectifiable curve $\gamma : [0, 1] \longrightarrow \Omega$, joining x to x_0, such that*

$$\operatorname{dist}(\gamma(t), \Omega) \geq C \operatorname{length}(\gamma_{|[0,t]}), \tag{3.5}$$

for all $t \in [0, 1]$.

Remark 3.1.12. Notice that we can assume that the curve γ is parametrized by arc length. In this case, (3.5) can be rewritten as

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq C\, t.$$

[2]A measure metric space (M, d, μ), where μ is a Borel measure, is *Ahlfors-regular* if there exist $Q > 0$ and $\alpha > 0$ such that, for every $x \in M$ and $r > 0$,

$$\frac{1}{\alpha} r^Q \leq \mu(B(x, r)) \leq \alpha\, r^Q.$$

Example 3.1.2. A simple example of John domains is provided by metric balls of a metric space with geodesics. Another example of a John domain is given by the interior of Von Koch's *snow flake*.

Analogously to Uniform domain, also for John domains we have a weaker definition:

Definition 3.1.13. *Let $\Omega \subset M$ be a bounded open set. We say that Ω is a* weak John domain *if there exists $x_0 \in \Omega$ and $0 < C \leq 1$ such that, for every $x \in \Omega$, there exists a continuous curve $\gamma : [0,1] \longrightarrow \Omega$, joining x to x_0, such that*

$$\operatorname{dist}(\gamma(t), \Omega) \geq C\, d(\gamma(t), x), \tag{3.6}$$

for all $t \in [0,1]$.

Remark 3.1.14. Notice that this definition is weaker than the John domain's one in the following sense: we do not require that γ is rectifiable, but we connect the points simply with continuous curves.

Example 3.1.3. Let (M, d) be a bounded arcwise connected metric space. If we take $\Omega = M$, then Ω is a weak John domain; indeed (3.6) is satisfied for any curve joining x to x_0.

Remark 3.1.15. Each John domain satisfies the interior corkscrew condition.

Indeed, let $\Omega \subset M$ be a John domain and choose $x_0 \in \partial\Omega$. From the definition of John domain, we know that for every $x \in \Omega$ there is a continuous and rectifiable curve $\gamma : [0,1] \longrightarrow \Omega$ such that $\gamma(0) = x_0$, $\gamma(1) = x$ and

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq C\, t.$$

This means that we can choose $t_0 \in [0,1]$, and denote $y := \gamma(t_0)$, so that

$$\operatorname{dist}(y, \partial\Omega) \geq C\, t_0.$$

Moreover, $d(x_0, y) \leq t_0$, because γ is rectifiable.

John domains and weak John domains are strictly connected, as we can see in the following

Theorem 3.1.3. *Let us assume that (M, d) is a doubling metric space with geodesics. Then $\Omega \subset M$ is a John domain if and only if it is a weak John domain.*

We conclude this Section by introducing the notion of *Boman domain*. We will discover that, under extra hypothesis, the definition of John domain is equivalent to that of Boman domain. The proof of this fact was originally given in [4]. The reader could find the detailed proof also in [19].

Definition 3.1.16. *Let $\Omega \subset M$ be an open set. We say that Ω is a* Boman domain *if there exist a covering with balls $\mathcal{F} = \{B_i\}_{i \in I}$ of Ω and constants $N \geq 1$, $\lambda > 1$ and $\nu \geq 1$ such that*

(i) *$\lambda B \subset \Omega$, for every $B \in \mathcal{F}$;*

(ii) *$\sum_{B \in \mathcal{F}} \chi_{\lambda B}(x) \leq N$, for every $x \in \Omega$;*

(iii) *there exists $B_0 \in \mathcal{F}$ such that, for any $B \in \mathcal{F}$, there exists $B_1, ..., B_k$ such that $B_k = B$, $\mu(B_1 \cap B_{i+1}) \geq \frac{1}{N} \max\{\mu(B_i), \mu(B_{i+1})\}$ and $B \subset \nu B_i$, for all $i = 1, ..., k$.*

Remark 3.1.17. Condition (ii) of the previous definition says that each point $x \in \Omega$ is covered by, at most, a fixed number N of dilated balls of $\mathcal{F}$. On the other hand, third condition means (roughly speaking) the following: if we start from a fixed region of Ω covered by a ball $B_0 \in \mathcal{F}$, then we can reach each point in Ω with a finite sequence of balls which are comparable one to each other.

Remark 3.1.18. In Definition 3.1.16, we can skip the requirement of boundedness for the domain Ω saying that $\Omega \subset M$ (unbounded) is a Boman domain if $\Omega = \cup_{i \in \mathbb{N}} \Omega_i$, with $\Omega_i \subset \Omega_{i+1}$ and Ω_i is a bounded Boman domain, for all $i \in \mathbb{N}$.

Theorem 3.1.4. *Let us assume that (M, d, μ) is a doubling metric space. Let $\Omega \subset M$ be a proper subset. If Ω is a weak John domain, then it is a Boman domain.*

Theorem 3.1.5. *Let (M, d, μ) be a doubling metric space with geodesics. If $\Omega \subset M$ is a Boman domain, then it is a John domain.*

Remark 3.1.19. Summarizing, in a doubling metric space with geodesics (M, d, μ), being a John domain is equivalent to being a Boman domain.

3.2 Intrinsic Lipschitz Domain

In this section we aim to apply results of Section 3.1 in order to prove that a bounded domain with intrinsic Lipschitz boundary is a uniform domain.

We immediately start by giving the definition of *intrinsic Lipschitz domain*. It could be interesting to compare this definition to the classic definition of Lipschitz domain; we refer the reader to the Definition on page 127 of [8].

Definition 3.2.1. *Let Ω be a bounded domain of $\mathbb{H}^n$. We say that Ω is an* intrinsic Lipschitz domain *if, for each $z \in \partial\Omega$, we can choose a decomposition $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Assumption 2.2.7, an intrinsic Lipschitz map $f : \mathcal{E} \subset \mathbb{W} \longrightarrow \mathbb{V}$, with $\mathcal{E}$ an open set in $\mathbb{W}$, and $r_0 \in \mathbb{R}^+$ such that*

$$\overline{\Omega} \cap U(z, r_0) \,=\, \overline{E_f^-} \cap U(z, r_0). \tag{3.7}$$

We recall that

$$\overline{E_f^-} \,:=\, \{w \cdot \exp(tV) \,|\, w \in \mathbb{W},\, t \leq \varphi(w)\,\},$$

where $f(w) = \exp(\varphi(w)V)$, for $\varphi : \mathbb{W} \longrightarrow \mathbb{R}$.

In different words, an intrinsic Lipschitz domain is an open, connected and bounded set in $\mathbb{H}^n$, whose boundary is locally the graph of an intrinsic Lipschitz map.

We can start with the proof that an intrinsic Lipschitz domain Ω is a uniform domain. To do that, we will use a characterization proved in [40]. We report here the results in which we are interested.

Let us denote, as in first Chapter, $\mathbb{X} := (X_1, ..., X_m)$ the vector fields which generate $\mathfrak{h}^n$, the Lie algebra of the Heisenberg group $\mathbb{H}^n$. Roughly speaking, we define an $\mathbb{X}$-Lipschitz domain as a domain whose boundary is locally the zero set of a metric Lipschitz function[3] which takes value in $\mathbb{R}$.

[3] Let $(\mathbb{R}^n, d)$ be a CC-space associated with a family of locally Lipschitz vector fields $\mathbb{X} = (X_1, ..., X_m)$. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is *metric L-Lipschitz* if , for all $x, y \in \mathbb{R}^n$,

$$|f(x) - f(y)| \,\leq\, L\, d(x, y).$$

Definition 3.2.2. *Let $\Omega \subset \mathbb{H}^n$ be a bounded open set. We say that Ω is an* $\mathbb{X}$-Lipschitz domain *if, for each $z \in \partial\Omega$, there exists a neighbourhood of z, say U, a metric L-Lipschitz function $F : U \longrightarrow \mathbb{R}$ and an index $j \in \{1, ..., m\}$ such that*

(i) $\Omega \cap U = \{x \in U \mid F(x) > 0\}$ *or* $\Omega \cap U = \{x \in U \mid F(x) < 0\}$*;*

(ii) *there exists a positive constant l such that $X_j F \geq l$, $\mathcal{L}^n$-a.e. on U.*

Theorem 3.2.1. *A set $\Omega \subset \mathbb{H}^n$ is an $\mathbb{X}$-Lipschitz domain if and only if it is an intrinsic Lipschitz domain according to Definition 3.2.1.*

The proof of this Theorem, given in [40], follows directly from the subsequent result, which we report here for completeness of exposition.

Proposition 3.2.2. *Let $S \subset \mathbb{G}$ be a hypersurface in a Carnot group $\mathbb{G}$. The following two statements are equivalent:*

(i) *for any $z \in S$ there exist an open neighbourhood $U \subset \mathbb{G}$, a Lipschitz function $F : U \longrightarrow \mathbb{R}$ and a positive constant l such that*

$$S \cap U = \{x \in U \mid F(x) = 0\}$$

and $X_1 F \geq l$, $\mathcal{L}^n$-a.e. on U.

(ii) *for any $z \in S$, there exist an open set $\mathcal{E} \subset \mathbb{W}$, a, $b \in \mathbb{R}$ and an intrinsic Lipschitz map $\varphi : \mathcal{E} \longrightarrow]a, b[\subset \mathbb{V}$ such that $z \in U := \mathcal{E} \cdot]a, b[$ and $S \cap U = \Phi_\varphi(\mathcal{E})$, where Φ_φ is the parametrization map $\Phi_\varphi(z) = z \cdot \varphi(z)$.*

Theorem 3.2.3. *Let $\Omega \subset \mathbb{H}^n$ be an intrinsic Lipschitz domain. Then Ω is a uniform domain.*[4]

Before giving the proof, we recall a result concerning metric Lipschitz continuous functions over CC-spaces. We give the general statement, for the proof we refer to [34]. The reader is invited to notice that our setting (the Heisenberg group with the CC-metric) is included in the one of the theorem.

[4] In order to define Uniform domain (Definition 3.1.6), we just need a metric space. So, for our setting, it does not matter if the metric is the CC-metric or another homogeneous one.

Theorem 3.2.4. *Let* $(\mathbb{R}^n, d)$ *be a CC-space associated with a family of locally Lipschitz continuous vector fields* $\mathbb{X} = (X_1, ..., X_m)$*. Let us assume that the metric* d *is continuous with respect to the Euclidean topology. If* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *is a metric* L*-Lipschitz function, then the derivatives* $X_j f$*,* $j = 1, ..., m$*, exist in distributional sense, are measurable functions and* $|X_j f(x)| \leq L$ *for a.e.* $x \in \mathbb{R}^n$*.*

Proof of Theorem 3.2.3. Let $z \in \partial\Omega$ be fixed. From Theorem 3.2.1, we know that there exist an open neighbourhood U of z and a metric L-Lipschitz function $F : U \subset \mathbb{H}^n \longrightarrow \mathbb{R}$ such that, using the same notations as in the Theorem, $XF \geq l$ $\mathcal{L}^n$-a.e. on U, for some $X \in \{X_1, ..., X_m\}$, and

$$\begin{aligned} \partial\Omega \cap U \ &= \ \{ x \in U \mid F(x) = 0 \}, \\ \Omega \cap U \ &= \ \{ x \in U \mid F(x) < 0 \}. \end{aligned}$$

Let $x, y \in \Omega \cap U$. We shall construct a rectifiable curve $\Gamma : [0,1] \longrightarrow \mathbb{H}^n$ such that properties (3.1) and (3.2) are satisfied. We divide the proof in a number of small steps.

Step 1. Let $R > 0$ be such that the ball $B(z, 2R)$ is entirely contained in the open neighbourhood U of z. Let $p \in B(z, R/2) \cap \Omega$. If $t \in]0, R/2[$, then

$$p \cdot \exp(tX) \ \in \ B(z, R).$$

Indeed, since the exponential map is an isometry along horizontal directions,

$$\begin{aligned} d\left(p \cdot \exp(tX), z\right) \ &\leq \ d\left(p \cdot \exp(tX), p\right) + d(p, z) \\ &< \ t + \frac{R}{2} < R. \end{aligned}$$

Step 2. Let again $p \in B(z, R/2) \cap \Omega$ and consider a point $q \in \partial\Omega$ which realizes the distance of p from $\partial\Omega$, i.e. $d(p, q) = \text{dist}(p, \partial\Omega)$. We take $\sigma : [0,1] \longrightarrow \mathbb{H}^n$ to be a geodesic joining p and q. If $\xi \in \sigma([0,1])$, then

$$\begin{aligned} d(\xi, z) \ &\leq \ d(\xi, p) + d(p, z) \\ &\leq \ d(p, q) + d(p, z) \\ &\leq 2d(p, z) \ < \ R. \end{aligned}$$

This chain of inequalities implies that the support of σ is entirely contained in $B(z, R)$.

Step 3. The idea now is to use the function F to measure how points inside Ω are far from the boundary. To do this, we remember that, along the integral curve of X, F is monotone. First, since F is metric L-Lipschitz continuous, we can write

$$\begin{aligned} |F(p)| &= |F(p) - F(q)| \leq L\, d(p,q) \\ &= L \operatorname{dist}(p, \partial\Omega). \end{aligned} \tag{3.8}$$

On the other hand, let us assume that $p \in B(z, \varepsilon R) \cap \Omega$, for some $\varepsilon \in]0, 1/2[$. We aim to prove that

$$|F(p)| \geq l \operatorname{dist}(p, \partial\Omega). \tag{3.9}$$

Being a metric L-Lipschitz function, F is continuous. On the contrary, we need to handle with regular functions, so we approximate F using mollifiers of Section 1.6. From Proposition 1.6.3, combined with Theorem 3.2.4, we can extimate, for $x \in B(z, R)$,

$$\begin{aligned} X(\eta_\varepsilon * F)(x) &= \int_{B(x,\varepsilon)} \eta_\varepsilon \left(x \cdot q^{-1}\right) (XF)(q)\, dh(q) \\ &\geq l \int_{B(x,\varepsilon)} \eta_\varepsilon \left(x \cdot q^{-1}\right) dh(q) \\ &\geq l. \end{aligned}$$

Notice that the first inequality holds because we are taking the integral over the ball $B(x, \varepsilon)$, which is entirely contained in $B(z, 2R) \subset U$. Now, using Proposition 1.6.1, we can write, for $t \in]0, 1/2[$,

$$\begin{aligned} F\left(p \cdot \exp(tX)\right) - F(p) &= \lim_{\varepsilon \to 0} \left[(\eta_\varepsilon * F)(p \cdot \exp(tX)) - (\eta_\varepsilon * F)(p)\right] \\ &= \lim_{\varepsilon \to 0} \int_0^t X(\eta_\varepsilon * F)(p \cdot \exp(sX))\, ds \\ &\geq l\, t. \end{aligned}$$

Therefore,

$$\begin{aligned} F(p \cdot \exp(tX)) &\geq lt + F(p) \\ &\geq lt + \min_{\bar\Omega \cap B(z,\varepsilon R)} F. \end{aligned}$$

By this inequality, since $F(z) = 0$, we can choose $\varepsilon \in \mathbb{R}^+$ such that

$$l\,\frac{R}{3} + \min_{\bar\Omega \cap B(z,\varepsilon R)} \; > \; 0.$$

So, from Intermediate Value Theorem, there exists $t_p \in]0, R/3[$ such that

$$p \cdot \exp(t_p X) \in \partial\Omega.$$

Now, $\mathrm{dist}(p, \partial\Omega) \leq d(p, p \cdot \exp(t_p X)) \leq t_p$; hence, if $p \in B(z, \varepsilon R) \cap \Omega$, using the same technique as above, together with the result in Step 2, we conclude

$$\begin{aligned} -F(p) &= F(p \cdot \exp(t_p X)) - F(p) \\ &\geq l\, t_p \geq l\, \mathrm{dist}(p, \partial\Omega). \end{aligned}$$

Step 4. We construct the rectifiable curve needed to prove the Theorem. We want to use Lemma 3.1.1. Let us assume that $x, y \in B(z, \delta)$, where $\delta < \varepsilon R$. We denote $d := d(x, y)$, $x' := x \cdot \exp(-dMX)$ and $y' := y \cdot \exp(-dMX)$, for some constant $0 < M < R/4$ to be determined. We construct the following curve

$$\Gamma(t) := \begin{cases} x \cdot \exp(-tMX)), & t \in [0, dM], \\ \gamma(t), & t \in [dM, dM + d(x', y')], \\ y \cdot \exp(-(\tilde{M} - t)X)), & t \in [dM + d(x', y'), \tilde{M}], \end{cases} \tag{3.10}$$

where γ is a geodesics joining x' to y' and $\tilde{M} := 2dM + d(x', y')$. Let us prove that such a Γ is the curve we are looking for.

Step 5. Using relation 2.9 on page 51, one has

$$\begin{aligned} \mathrm{length}(\Gamma) &\leq 2dM + d(x', y') \\ &\leq 2dM + C \left\| \exp(dMX) \cdot y^{-1} \cdot x \cdot \exp(-dMX) \right\| \\ &\leq 2dM + C_1 \left\| y^{-1} \cdot x \right\| + C_2 \left\| y^{-1} \cdot x \right\|^{\frac{1}{2}} \cdot |dM|^{\frac{1}{2}} \\ &\leq 2dM + C_3 d \left(1 + \sqrt{M}\right) \\ &= d(x, y) \left(2M + C_3 \left(1 + \sqrt{M}\right)\right), \end{aligned}$$

and this provides inequality (3.1).

Step 6. Let us prove inequality (3.2). We need to distinguish points in the three pieces of Γ. Let $t \in [0, dM]$ be fixed. First, we notice that, since $F(x) < 0$ for $x \in U \cap \Omega$,

$$\begin{aligned} -F(x \cdot \exp(-tX)) &= -F(x \cdot \exp(-tX)) + F(x) - F(x) \\ &\geq -F(x \cdot \exp(-tX)) + F(x) \\ &\geq l\, t. \end{aligned}$$

We now combine this inequality with (3.8). Since we have that

$$\begin{aligned} d(x \cdot \exp(-tX)), z) &\le d(x \cdot \exp(-tX), x) + d(x, z) \\ &\le dM + \delta \le \frac{R}{4} + \varepsilon R \\ &< \frac{1}{2} R, \end{aligned}$$

we can apply inequalities (3.8) and (3.9) of Step 3 in order to obtain

$$\operatorname{dist}(x \cdot \exp(-tX), \partial\Omega) \ge \frac{1}{L} lt \ge \frac{l}{L} \operatorname{length}\left(\Gamma_{|[0,t]}\right).$$

Exactly in the same way, one can prove the inequality also for $t \in [dM + d(x', y'), \tilde{M}]$.

Step 7. Let us consider $t \in [dM, dM + d(x', y')]$ and denote $\xi := \Gamma(t)$. If $\eta \in \partial\Omega$, we have, mimicking computations that we already did,

$$\begin{aligned} d(\xi, \eta) &\ge d(x', \eta) - d(\xi, x') \\ &\ge \operatorname{dist}(x', \partial\Omega) - d(x', y') \\ &\ge d(x, y) \left(\frac{l}{L} M - C_3 \left(1 + \sqrt{M}\right)\right). \end{aligned} \tag{3.11}$$

On the other hand,

$$\begin{aligned} \operatorname{length}\left(\Gamma_{|[0,t]}\right) &\le d(x', y') + dM \\ &\le d(x, y) \left(C_3 \left(1 + \sqrt{M}\right) + M\right) \\ &= C_4 d(x, y). \end{aligned} \tag{3.12}$$

Therefore, if we choose $M > 0$ such that $\frac{l}{L} M - C_3 \left(1 + \sqrt{M}\right) > 1$ (consequently one should choose $\delta \in \mathbb{R}^+$), we can combine (3.11) and (3.12) and the assertion follows, thank to Lemma 3.1.1. □

Remark 3.2.3. We point out that, in the proof, we showed relations (3.1) and (3.2) for each couple of points $x, y \in \Omega$ such that $d(x, y) \le \delta$. Therefore, more precisely, we proved that intrinsic Lipschitz domains are (ε, δ)-domains.

From Theorem 3.2.3, one can directly show the following

Proposition 3.2.5. *Let $\Omega \subset \mathbb{H}^n$ be an intrinsic Lipschitz domain. Then Ω is a Harnack domain.*

Proof. In Theorem 3.2.3 we proved that an intrinsic Lipschitz domain is a uniform domain. So, we can apply Proposition 3.1.2 and the proof follows, if we show that, in $\mathbb{H}^n$, there exists a constant $0 < \delta \leq 2$ such that $\text{diam}(B(x, 2r)) \geq 2r$, for each ball $B(x, 2r)$. This fact is the content of the following Lemma. □

Lemma 3.2.6. *Let d be a homogeneous metric in $\mathbb{H}^n$ (e.g. CC-metric, the homogeneous distance d_∞, etc...). Then, for each $x \in \mathbb{H}^n$ and $r \in \mathbb{R}^+$,*

$$\text{diam}(B(x, r)) \;=\; 2r.$$

Proof. Without loss of generality, we can assume that $x = 0$ and $r = 1$. It is clear that $\text{diam}(B(0, 1)) \leq 2$, so we need to prove the converse inequality.

Let $\xi = (t, 0, ..., 0)$ be such that $d(\xi, 0) = 1$. By translation invariance, we point out that $d(-\xi, 0) = 1$. This implies that $-\xi$ and $\xi \in B(0, 1)$. Now, because of the choice of ξ, we can conclude that

$$\begin{aligned} d(-\xi, \xi) \;&=\; d(0, \xi \cdot \xi) \;=\; d(0, \delta_2(\xi)) \\ &=\; 2\, d(0, \xi) \;=\; 2. \end{aligned}$$

□

3.2.1 Applications

Theorem 3.2.3 allows us to extend some classic results on intrinsic Lipschitz domains. A first application is given by the Poincaré inequality. As in Euclidean setting, we have a Poincaré inequality also in open sets different from balls. But it is well known that not any open set admits such a property. Restricting ourselves to the Heisenberg group, we state a theorem, proved in [11], which gives a special class of open sets for which the Poincaré inequality holds.

Notation 3.2.1. Let $f : \Omega \subset \mathbb{H}^n \longrightarrow \mathbb{R}$. We denote the *average* of f in Ω

$$f_\Omega \;:=\; \frac{1}{|\Omega|} \int_\Omega f(x)\, dh(x).$$

Theorem 3.2.7. *Let $\Omega \subset \mathbb{H}^n$ be a Boman domain. Let $1 \leq p < q < \infty$ be such that the following balance condition holds:*

$$\frac{r(\tilde{U})}{r(U)} \left(\frac{|\tilde{U}|}{|U|} \right)^{\frac{1}{q}} \leq C \left(\frac{|\tilde{U}|}{|U|} \right)^{\frac{1}{p}},$$

for any balls $\tilde{U} \subset U$, with radius $r(\tilde{U})$ and $r(U)$ respectively, centered in a neighbourhood of $\bar{\Omega}$, with $r(U) < r_0$, $r_0 < \infty$ fixed.

Then

$$\left(\int_\Omega |f - f_\Omega|^q \, dh \right)^{\frac{1}{q}} \leq C_\Omega \left(\int_\Omega |\nabla_{\mathbb{H}} f|^p \, dh \right)^{\frac{1}{p}}$$

with a constant $C_\Omega > 0$ independent on f.

If $1 \leq p < Q$, we can always choose $q = p^* := \frac{pQ}{Q-p}$.

Thanks to the following Lemma (we refer the reader to [20] for the proof), we can say that intrinsic Lipschitz domains have a Poincaré inequality.

Lemma 3.2.8. *Let $\Omega \subset \mathbb{H}^n$ be an (ε, δ)-domain. Then Ω is a Boman domain.*

The second application concerns with an extension theorem for Sobolev functions defined over an (ε, δ)-domain. Let us recall the definition of Sobolev spaces over a Carnot group $\mathbb{G}$. Let $(X_1, ..., X_m)$ be generators for the first layer and let N be the dimension of $\mathfrak{g}$ as vector space.

Notation 3.2.2. Let $I = (i_1, ..., i_N)$ be a multiindex. We define the differential operator

$$X^I := X_1^{i_1} \cdot ... \cdot X_N^{i_N}. \tag{3.13}$$

We call $|I| = i_1 + ... + i_N$ the *order* of the differential operator and $d(I) = d_1 i_1 + ... + d_N i_N$ its *degree of homogeneity.*

We can construct also more general differential operators: if $J = (j_1, ..., j_k)$ be a multiindex, with $|J| < |I|$, we define

$$X^J := X_{\alpha_1}^{j_1} \cdot ... \cdot X_{\alpha_k}^{j_k},$$

for $\alpha_1, ..., \alpha_k \in \{1, ..., m\}$.

Remark 3.2.4. Differential operators of this kind can be always expressed as linear combination of operators of the special form (3.13).

Definition 3.2.5. *Let k be a positive integer, $1 < p < \infty$ and Ω be an open subset of $\mathbb{G}$. The* Sobolev space $W^{k,p}_{\mathbb{H}}(\Omega)$ *associated with the vector fields $X_1, ..., X_m$ is defined as the set of all functions $f \in L^p(\Omega)$ with distributional derivatives $X^I f \in L^p(\Omega)$, for every X^I defined above with $d(I) \leq m$.*

We say that the distributional derivative $X^I f$ exists and it is equal to a locally integrable function g_I if, for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega f \cdot X^I \varphi \, dh \; = \; (-1)^{d(I)} \int_\Omega g_I \cdot \varphi \, dh.$$

Definition 3.2.6. *Let $\Lambda : W^{k,p}_{\mathbb{H}}(\Omega) \longrightarrow W^{k,p}_{\mathbb{H}}(\mathbb{G})$ be a bounded linear operator. We say that Λ is a* bounded extension operator *on $W^{k,p}_{\mathbb{H}}(\Omega)$ if*

$$\Lambda f_{|\Omega} \; = \; f,$$

for each $f \in W^{k,p}_{\mathbb{H}}(\Omega)$. Moreover, we denote the operator norm of Λ as follows

$$\|\Lambda\| \; := \sup_{\|f\|_{W^{k,p}_{\mathbb{H}}(\Omega)}=1} \|\Lambda f\|_{W^{k,p}_{\mathbb{H}}(\mathbb{H}^n)} \, .$$

We are now ready to state the extension theorem whose proof was given firstly in [35] and, later, for a more general setting, in [20].

Theorem 3.2.9. *Let $\Omega \subset \mathbb{G}$ be an (ε, δ)-domain and let k be a positive integer. If $1 \leq p < \infty$, then there exists an extension operator Λ on Ω such that*

$$\|\Lambda f\|_{W^{k,p}_{\mathbb{H}}(\mathbb{G})} \; \leq \; C \, \|f\|_{W^{k,p}_{\mathbb{H}}(\Omega)}, \tag{3.14}$$

for all $f \in W^{k,p}_{\mathbb{H}}(\mathbb{H}^n)$, where C is a positive constant which does not depend on f.

Again, if $\Omega \subset \mathbb{H}^n$ is intrinsic Lipschitz, then it is an (ε, δ)-domain. Hence, if f is a Sobolev function defined over Ω, then it admits a unique extension over the entire $\mathbb{H}^n$ such that (3.14) holds.

3.3 Intrinsic Lipschitz Domains and Subelliptic PDE's

We conclude our work with an application concerning subelliptic PDE's. Our goal is to prove that intrinsic Lipschitz domains have a good behaviour with respect to the Dirichlet problem. Therefore, before coming to the central topic of this Section, we just recall some fundamental definitions and results about sub-Laplacian operators and their fundamental solutions. A detailed exposition, more suited to the purposes of the present thesis, is given in [3].

Let $\mathbb{G}$ be a Carnot group and let $(X_1, ..., X_m)$ be generators for the first layer V_1 of $\mathfrak{g}$, the Lie algebra of $\mathbb{G}$. The second order differential operator

$$\mathcal{L} = \sum_{i=1}^{m} X_i^2$$

is called a *sub-Laplacian*, or *subelliptic Laplacian*, on $\mathbb{G}$.

Thanks to the celebrated Hörmander's Theorem ([24]), $\mathcal{L}$ is hypoelliptic. This means that if $f \in C^\infty(\Omega)$, for $\Omega \subset \mathbb{G}$ an open set, then the distributional solution of $\mathcal{L}u = f$ is of class C^∞.

We now give the definition of fundamental solution of a sub-Laplacian $\mathcal{L}$ on a Carnot group.

Definition 3.3.1. *Let $\mathbb{G} = (\mathbb{R}^n, \cdot)$ be a Carnot group. Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. A function $\Gamma : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$ is a* fundamental solution *for $\mathcal{L}$ if*

(i) $\Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$*;*

(ii) $\Gamma \in L^1_{loc}(\mathbb{R}^n)$ *and* $\Gamma(x) \longrightarrow 0$*, as* $x \to \infty$*;*

(iii) $\mathcal{L}\Gamma = -Dir_0$*, where* Dir_0 *is the Dirac measure supported at* $\{0\}$*.*

Remark 3.3.2. One can prove, but it is hard work, that, on a Carnot group with $Q > 2$, the fundamental solution of $\mathcal{L}$ exists and it is unique.

Let us summarize here some main properties of the fundamental solution:

(i) $\Gamma(x^{-1}) = \Gamma(x)$, for any $x \in \mathbb{G} \setminus \{e\}$;

(ii) $\Gamma(\delta_\lambda(x)) = \lambda^{2-Q}\Gamma(x)$, for any $x \in \mathbb{G} \setminus \{e\}$ and $\lambda \in \mathbb{R}^+$;

(iii) $\Gamma(x) > 0$, for any $x \in \mathbb{G} \setminus \{e\}$.

On every Carnot group one can find some particular homogeneous norms which play a key role for sub-Laplacian. These norms, which are smooth and symmetric, are called *gauges*:

Definition 3.3.3. *Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. We call* $\mathcal{L}$-gauge *on $\mathbb{G}$ a homogeneous, symmetric norm d which is smooth on $\mathbb{G} \setminus \{e\}$ and satisfies*

$$\mathcal{L}(d^{2-Q}) = 0 \quad in\ \mathbb{G} \setminus \{e\}.$$

Fundamental solutions and $\mathcal{L}$-gauges are strictly related, as the following Proposition states:

Proposition 3.3.1. *Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$ and Γ be the fundamental solution of $\mathcal{L}$. Then*

$$d(x) := \begin{cases} \Gamma(x)^{\frac{1}{2-Q}}, & x \in \mathbb{G} \setminus \{e\}, \\ 0, & x = e \end{cases}$$

is an $\mathcal{L}$-gauge on $\mathbb{G}$.

Proof. See [3], Proposition 5.4.2. □

Remark 3.3.4. Also the converse of Proposition 3.3.1 is true. Indeed, if d is an $\mathcal{L}$-gauge on $\mathbb{G}$, then there exists a positive constant C_d such that $\Gamma = C_d\, d^{2-Q}$ is the fundamental solution of $\mathcal{L}$. This fact implies that the $\mathcal{L}$-gauge is unique up to a multiplicative constant.

Example 3.3.1. Let $\mathbb{H}^n$ be the n-th Heisenberg group. Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{H}^n$. Then, an $\mathcal{L}$-gauge is given by the Korányi norm

$$d((z,t)) = \sqrt[4]{\|z\|^4_{\mathbb{R}^{2n}} + |t|^2}.$$

As already said, in this section we aim to study the Dirichlet problem over an intrinsic Lipschitz domain $\Omega \subset \mathbb{H}^n$. In other words, we want to prove that Ω is regular with respect to a sub-Laplacian $\mathcal{L}$. What does it mean to be $\mathcal{L}$-regular? Let us define immediately this concept in a general Carnot group $\mathbb{G}$.

Definition 3.3.5. *Let $\Omega \subset \mathbb{G}$ be a bounded open set. We say that Ω is* $\mathcal{L}$-regular *if the boundary value problem*

$$\begin{cases} \mathcal{L}u = 0, & \text{in } \Omega \\ u_{|\partial\Omega} = \varphi \end{cases} \tag{3.15}$$

has a (unique) solution u for every continuous function $\varphi : \partial\Omega \longrightarrow \mathbb{R}$.

We say that u solves (3.15) *if it is $\mathcal{L}$-harmonic in Ω and*

$$\lim_{y \to x} u(y) = \varphi(x),$$

for every $x \in \partial\Omega$.

To show that an intrinsic Lipschitz domain $\Omega \subset \mathbb{H}^n$ is $\mathcal{L}$-regular, we will use the *Wiener's regularity test* for $\mathcal{L}$. For the sake of completeness, we study a little bit the background material concerning general notions and results which play a role in this kind of theory. Once more, the interested reader is referred to [3] for further informations.

Notation 3.3.1. Let $\Omega \subset \mathbb{G}$ be a bounded and connected open set. Following the notations used in [3], $\mathcal{M}(\Omega)$ denotes the set of Radon measures μ on Ω. We denote by $\mathcal{M}_0(\Omega)$ the subset of $\mathcal{M}(\Omega)$ of compactly supported Radon measures.

$\bar{\mathcal{S}}(\Omega)$ denotes the set of $\mathcal{L}$-superharmonic functions in Ω, and $\bar{\mathcal{S}}^+(\Omega)$ of non negative $\mathcal{L}$-superharmonic functions.

Definition 3.3.6. *Let $K \subset\subset \mathbb{G}$ be a compact set. For any $\mu \in \mathcal{M}(K)$, we say that*

$$I(\mu) := \int_{\mathbb{G}} (\Gamma * \mu)(x)\, d\mu(x) = \int_K \int_K \Gamma(y^{-1} \cdot x)\, d\mu(x) d\mu(y)$$

is the $\mathcal{L}$-energy *of μ. We define also the* $\mathcal{L}$-equilibrium value *of K as*

$$V(K) := \inf\{I(\mu) \,|\, \mu \in \mathcal{M}(K),\, \mu(K) = 1\}.$$

Theorem 3.3.2. *Let $K \subset\subset \mathbb{G}$ be a compact set. Then there exists a Radon measure $\bar{\mu} \in \mathcal{M}(K)$ such that $\bar{\mu}(K) = 1$ and*

$$I(\bar{\mu}) = V(K).$$

We call such a measure $\bar{\mu}$ an $\mathcal{L}$-equilibrium distribution *for K and the related potential $\Gamma * \bar{\mu}$ an* $\mathcal{L}$-equilibrium potential *for K.*

Definition 3.3.7. *Let $A \subseteq \mathbb{G}$ and $u \in \bar{\mathcal{S}}^+(\mathbb{G})$. The $\mathcal{L}$-*reduced function *of u relative to A is*

$$\mathbf{R}^u_A := \inf\{f \mid f \in \Phi^u_A\},$$

*where $\Phi^u_A := \{f \in \bar{\mathcal{S}}^+(\mathbb{G}) \mid f \geq u \text{ in } A\}$. We define also the $\mathcal{L}$-*balayage *of u relative to A as the lower semicontinuous regularization $\hat{\mathbf{R}}^u_A$ of $\mathbf{R}^u_A$, i.e. for any $x \in \mathbb{G}$,*

$$\hat{\mathbf{R}}^u_A := \liminf_{y \to x} \mathbf{R}^u_A(y).$$

We now list some properties of the $\mathcal{L}$-reduced function and the $\mathcal{L}$-balayage:

(i) $\hat{\mathbf{R}}^f_A$ is $\mathcal{L}$-subharmonic in $\mathbb{G}$ and $\mathcal{L}$-harmonic in $\mathbb{G} \setminus \bar{A}$;

(ii) if $0 \leq f \leq g$, then $\mathbf{R}^f_A \leq \mathbf{R}^g_A$, $\hat{\mathbf{R}}^f_A \leq \hat{\mathbf{R}}^g_A$;

(iii) if $A \subseteq B \subseteq \mathbb{G}$, then $\mathbf{R}^f_A \leq \mathbf{R}^f_B$, $\hat{\mathbf{R}}^f_A \leq \hat{\mathbf{R}}^f_B$;

(iv) $\hat{\mathbf{R}}^f_A = \mathbf{R}^f_A$ almost everywhere in $\mathbb{G}$.

It could be interesting to notice that, when $A \subseteq \mathbb{G}$ is a compact set, then the $\mathcal{L}$-balayage takes a particular form. This fact is the content of the following theorem:

Theorem 3.3.3. *Let $K \subset\subset \mathbb{G}$ be a compact set and let $u \in \bar{\mathcal{S}}^+(\mathbb{G})$. Then there exists a Radon measure μ in $\mathbb{G}$ such that*

$$\hat{\mathbf{R}}^u_f = \Gamma * \mu.$$

We are now ready to give the definition of $\mathcal{L}$-capacity. We start with the case of compact sets. Then, just taking an approximation from inside with compact sets, we can define the $\mathcal{L}$-capacity also for general sets.

Definition 3.3.8. *Let $K \subset\subset \mathbb{G}$ be a compact set. We define*

$$W_K := \mathbf{R}^1_K, \ \ V_K := \hat{\mathbf{R}}^1_K.$$

*By Theorem 3.3.3, there exists a measure μ_K, called $\mathcal{L}$-*capacitary distribution *for K, such that $V_K = \Gamma * \mu_K$. We call V_K the $\mathcal{L}$-*capacitary potential *of K.*

*We define the $\mathcal{L}$-*capacity *of K as*

$$\mathcal{C}(K) := \mu_K(K).$$

Let us give a look to some characterizations of the $\mathcal{L}$-capacity of compact sets.

Theorem 3.3.4. *Let $K \subset\subset \mathbb{G}$ be a compact set. Then*

$$\mathcal{K} = \max\{\mu(K) \,|\, \mu \in \mathcal{M}(K),\, \Gamma * \mu \leq 1 \text{ in } \mathbb{G}\}.$$

Theorem 3.3.5. *Let $K \subset\subset \mathbb{G}$ be a compact set and let $\mathcal{C}(K)$ and $V(K)$ be the $\mathcal{L}$-capacity and the $\mathcal{L}$-equilibrium value of K. Then*

$$\mathcal{C}(K) = (V(K))^{-1}.$$

Proposition 3.3.6. *Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{G}$. Then the following properties hold:*

(i) *if $K_1 \subseteq K_2$, then $\mathcal{C}(K_1) \leq \mathcal{C}(K_2)$;*

(ii) *if $\{K_j\}_j$ is a decreasing sequence, then $\mathcal{C}(K_n) \searrow \mathcal{C}(\cap_j K_j)$, as $n \to \infty$;*

(iii) $\mathcal{C}(K_1 \cup K_2) + \mathcal{C}(K_1 \cap K_2) \leq \mathcal{C}(K_1) + \mathcal{C}(K_2)$.

Proposition 3.3.7. *Let $K \subset\subset \mathbb{G}$ be a compact set. Then, for any $\lambda \in \mathbb{R}^+$ and $g \in \mathbb{G}$,*

$$\mathcal{C}(\delta_\lambda(K)) = \lambda^{Q-2}(K) \;\; \mathcal{C}(g \cdot K) = \mathcal{C}(K).$$

We now can extend the notion of $\mathcal{L}$-capacity to a larger class of sets.

Definition 3.3.9. *Let $E \subseteq \mathbb{G}$ be a non empty set. We define the* interior $\mathcal{L}$-capacity *of E as*

$$\mathcal{C}_*(E) := \sup\{\mathcal{C}(K) \,|\, K \text{ compact},\, K \subseteq E\},$$

and the exterior $\mathcal{L}$-capacity *as*

$$\mathcal{C}^*(E) := \inf\{\mathcal{C}_*(\Omega) \,|\, \Omega \text{ open},\, E \subseteq \Omega\}.$$

We say that $E \subseteq \mathbb{G}$ is $\mathcal{L}$-capacitable *if $\mathcal{C}_*(E) = \mathcal{C}^*(E)$. In this case, we denote this common value by $\mathcal{C}(E)$ and it is called the* $\mathcal{L}$-capacity *of E.*

Remark 3.3.10. Any Borel set is $\mathcal{L}$-capacitable.

We now introduce the following notation: let $y \in \mathbb{G}$ and let $\alpha > 1$ be a constant. For every $n \in \mathbb{N}$ we set

$$C_n := \{x \in \mathbb{G} \,|\, \alpha^n \le \Gamma(y^{-1} \cdot x) \le \alpha^{n+1}\}.$$

The Wiener's regularity test for the sub-Laplacian $\mathcal{L}$ reads as follows:

Theorem 3.3.8 (Wiener's regularity test for $\mathcal{L}$)**.** *Let Ω be an open and connected subset of $\mathbb{G}$ and let $y \in \partial\Omega$. Then the following statements are equivalent:*

(i) *y is a $\mathcal{L}$-regular point for Ω;*

(ii) *it holds $\sum_{n=1}^{\infty} \alpha^n \cdot \mathcal{C}^*(C_n \setminus \Omega) = \infty$;*

(iii) *it holds $\sum_{n=1}^{\infty} \hat{\mathbf{R}}^1_{C_n \setminus \Omega}(y) = \infty$;*

(iv) *it holds $\int_\alpha^\infty \mathcal{C}^* \left(\{x \in \Omega \,|\, \Gamma(y^{-1} \cdot x) \ge t\}\right) dt = \infty$.*

Proof. See [3], Theorem 12.4.3. □

As already announced, our final step is to use this test to show that an intrinsic Lipschitz domain in the Heisenberg group $\mathbb{H}^n$ is $\mathcal{L}$-regular.

Theorem 3.3.9. *Let $\Omega \subset \mathbb{H}^n$ be an intrinsic Lipschitz domain. The Ω is $\mathcal{L}$-regular.*

Proof. Let $y \in \partial\Omega$ be fixed. We need to show that y is an $\mathcal{L}$-regular point for Ω. Without loss of generality, we can assume that $y = e$. We can make this assumption because intrinsic Lipschitz graphs are invariant under left translations and because of Proposition 3.3.7.

Let us prove that property (i) in Theorem 3.3.8 holds. First, we notice that, since $\mathcal{L}$-capacity is monotone increasing, it is sufficient to prove (ii) for an intrinsic cone, say C, which realizes the definition of intrinsic Lipschitz graph near $e \in \partial\Omega$.

Let

$$\beta := \left(\frac{1}{\alpha}\right)^{\frac{1}{Q-2}}.$$

Then, keeping in mind Proposition 3.3.1, we can rewrite

$$C_n = \overline{B(e, \beta^{-n})} \setminus B(e, \beta^{-n-1}).$$

Now, intrinsic cones are invariant under group dilations, therefore, using Proposition 3.3.7,

$$\begin{aligned}\mathcal{C}^*(C_n \setminus \Omega) &= \mathcal{C}^*\left(\delta_{\beta^{-n}}(C_1 \setminus C)\right) \\ &= (\beta^{-n})^{Q-2}\mathcal{C}^*(C_1 \setminus C).\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \alpha^n (\beta^{-n})^{Q-2}\mathcal{C}^*(C_1 \setminus C) = \sum_{n=1}^{\infty} \mathcal{C}^*(C_1 \setminus C) = \infty,$$

and this proves the assertion. □

Appendix A

Basic Notions on Differential Geometry

In this appendix we recall some definitions and fundamental results from differential geometry.

Definition A.0.11. *We say that a topological space M is* locally Euclidean of dimension *n if every point $p \in M$ has a neighbourhood U homeomorphic to an open subset of $\mathbb{R}^n$. We call a* chart *of M the couple $(U, \varphi : U \to \mathbb{R}^n)$, where $\varphi : U \longrightarrow \mathbb{R}^n$ is a homeomorphism.*

Definition A.0.12. *We call* topological manifold *M a Hausdorff, second countable, locally Euclidean space of dimension n.*

Definition A.0.13. *We say that two charts $(U, \varphi : U \longrightarrow \mathbb{R}^n)$, $(V, \psi : V \longrightarrow \mathbb{R}^n)$ of a topological manifold M are* C^∞-compatible *if the two maps*

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \varphi(U \cap V)$$
$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

are C^∞.

Definition A.0.14. *Let M be a topological manifold. We call a* C^∞-atlas *a collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ of pairwice C^∞-compatible charts such that $M = \bigcup_\alpha U_\alpha$.*

Definition A.0.15. *We say that a* C^∞-manifold *is a topological manifold* M *together with a maximal atlas.*[1]

Remark A.0.16. One can prove that if M is a C^∞-manifold with a maximal atlas $\mathcal{A}$, and $(U, \varphi) \in \mathcal{A}$, then $\varphi : U \longrightarrow \varphi(U)$ is a diffeomorphism.

Definition A.0.17. *Let* N *and* M *be* C^∞*-manifolds of dimension* n *and* m*, respectively. A continuous map* $F : N \longrightarrow M$ *is* C^∞ *at a point* $p \in N$ *if there exists* (V, ψ)*, a chart of* M *in* $F(p)$*, and* (U, φ)*, a chart of* N *in* p*, such that*

$$\psi \circ F \circ \varphi^{-1} \,:\, \varphi\big(U \cap F^{-1}(V)\big) \longrightarrow \mathbb{R}^m$$

is a C^∞ *in Euclidean sense.*

Let M be a C^∞-manifold. A *germ* of a C^∞ function at p in M an equivalence class of C^∞ functions, defined on a neighbourhood of p in M, which agree on some open neighbourhood of p. Denoting by $C_p^\infty(M)$ the set of such equivalence classes, we call a *derivation* on M at p a linear map

$$D \,:\, C_p^\infty(M) \longrightarrow \mathbb{R}$$

such that

$$D(f \cdot g) \,=\, (Df) \cdot g(p) \,+\, f(p) \cdot Dg.$$

Definition A.0.18. *We call a* tangent vector *to* M *at a point* $p \in M$ *a derivation at* p*. We denote by* T_pM *the set of all tangent vector to* M *at* p*.*

Definition A.0.19. *Let* $F : N \longrightarrow M$ *be a* C^∞ *map between* C^∞*-manifolds. We define the* differential *of* F *at a point* $p \in N$ *the linear map*

$$d_pF \,:\, T_pN \longrightarrow T_{F(p)}M,$$

defined as follows

$$d_pF\,(X_p)\,(f) \,=\, X_p(f \circ F),$$

for all $X_p \in T_pN$ *and* $f \in C^\infty_{F(p)}(M)$*.*

[1] A *maximal atlas* is an atlas which is not contained in another atlas.

Theorem A.0.10. *If $F : N \longrightarrow M$ and $G : M \longrightarrow P$ are C^∞-maps between manifolds, and $p \in N$ is fixed, then the following* chain rule *holds*

$$d_p(g \circ F) \;=\; d_{F(p)}G \circ d_pF.$$

Let us consider a smooth manifold M of dimension n and fix a point $p \in M$. By definition of C^∞-manifold, there exists a chart $(U, \varphi) = (U, x^1, ..., x^n)$ of M at p. Since $\varphi : U \longrightarrow \varphi(U)$ is a diffeomorphism,

$$d_p\varphi \;:\; T_pM \;\longrightarrow\; T_{\varphi(p)}\mathbb{R}^n$$

is an isomorphism of vector spaces.

Proposition A.0.11. *If $(U, \varphi) = (U, x^1, ..., x^n)$ is a chart of a C^∞-manifold M at a point p, then*

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, ..., \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

is a basis for the vector space T_pM, where, if with $r^1, ..., r^n$ we denote the standard coordinates of $\mathbb{R}^n$, $\frac{\partial}{\partial x^i}\big|_p \in T_pM$ is defined by

$$d_p\varphi \left(\frac{\partial}{\partial x^i}\bigg|_p \right) \;=\; \frac{\partial}{\partial r^i}\bigg|_{\varphi(p)}.$$

Definition A.0.20. *Let M be a C^∞-manifold. A* smooth curve *in M is a smooth map $c :]a, b[\subset \mathbb{R} \longrightarrow M$. We call the* velocity vector *$c'(t_0)$ of the curve c at time $t_0 \in]a, b[$ the tangent vector*

$$c'(t_0) \;:=\; d_{t_0}c \left(\frac{d}{dt}\bigg|_{t_0} \right) \;\in\; T_{c(t_0)}M.$$

Proposition A.0.12. *For every point p on a C^∞-manifold M and for every $X_p \in T_pM$, there exists $\varepsilon \in \mathbb{R}^+$ and a smooth curve $c :]-\varepsilon, \varepsilon[\longrightarrow M$ such that $c(0) = p$ and $c'(0) = X_p$.*

Proposition A.0.13. *Let M be a C^∞-manifold and X_p be a tangent vector at a point $p \in M$. If $f \in C_p^\infty(M)$ and $c :]-\varepsilon, \varepsilon[\longrightarrow M$ is a smooth curve starting at p with $c'(0) = X_p$. Then*

$$X_pf \;=\; \frac{d}{dt}(f \circ c)\bigg|_{t=0}.$$

Proposition A.0.14. *Let $F : N \longrightarrow M$ be a smooth map between C^∞-manifolds, $p \in N$ and X_p be a tangent vector to N at p. If $c :]-\varepsilon, \varepsilon[\longrightarrow N$ is a smooth curve in N starting at p with velocity X_p, then*

$$d_pF\,(X_p) \;=\; \frac{d}{dt}(F \circ c)(t)\bigg|_{t=0}.$$

Let M and E be two C^∞-manifolds. Given a map $\pi : E \longrightarrow M$, we call the inverse image $\pi^{-1}(\{p\}) =: E_p$ of a point $p \in M$ the *fiber at p*.

For any two maps $\pi : E \longrightarrow M$ and $\tilde{\pi} : \tilde{E} \longrightarrow M$, a map $\varphi : E \longrightarrow \tilde{E}$ is said to be *fiber preserving* if $\varphi(E_p) \subset \tilde{E}_p$, for each $p \in M$.

A surjective smooth map $\pi : E \longrightarrow M$ of manifolds is said to be *locally trivial of rank r* if

(i) each fiber E_p has a structure of vector space of dimension r;

(ii) for each $p \in M$, there are an open neighbourhood U of p and a fiber preserving diffeomorphism $\varphi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^r$ such that, for every $q \in U$, the restriction

$$\varphi|_{\pi^{-1}(q)} \;:\; E_q \;\longrightarrow\; \{q\} \times \mathbb{R}^r$$

is a vector space isomorphism.

Definition A.0.21. *We call a C^∞-*vector bundle of rank r *a triplet (E, M, π) consisting of two C^∞-manifolds E and M and a surjective smooth map $\pi : E \longrightarrow M$ that is locally trivial of rank r. The manifold E is called the* total space *of the vector bundle and M the* base space. *By abuse of language, we say that E is a vector bundle over M.*

Definition A.0.22. *Let M be a C^∞-manifold, we call the* tangent bundle *of M the triplet (TM, M, π)[2] where $TM = \bigcup_{p \in M} \{p\} \times T_pM$ and $\pi : TM \longrightarrow M$ is defined as $\pi(v) = p$, if $v \in T_pM$.*

Definition A.0.23. *Let (E, M, π) be a vector bundle. We say that a map $s : M \longrightarrow E$ is a* section *of the vector bundle if $\pi \circ s = id_M$. We say that the section is smooth if it is smooth as a map.*

[2]One can endow TM with a structure of C^∞-manifold; the topology is just the topology induced by M through π.

Definition A.0.24. *Let* (E, M, π) *be a vector bundle, and let* U *be an open subset of* M*. We call a* frame *for the vector bundle* (E, M, π) *over* U *a collection of sections* $s_1, ..., s_r$ *such that, for each* $p \in U$*,* $\{s_1(p), ..., s_r(p)\}$ *is a basis for the fiber* E_p*.*

Definition A.0.25. *We call a* vector field X *on a* C^∞*-manifold* M *a section of the tangent bundle* (TM, M, π)*, and we write* $X \in \Gamma(TM)$*. We say that a vector field* $X : M \longrightarrow TM$ *is smooth if it is a smooth section.*

Let $p \in M$ be a fixed point and let $(U, \varphi) = (U, x^1, ..., x^n)$ be a chart of M around p. Then the value of the vector field at $p \in M$ is given by

$$X(p) \;=\; \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p .$$

We notice that as p varies in U, the coefficients a^i can be considered as functions on U.

Proposition A.0.15. *Let* M *be a* C^∞*-manifold and let* $(U, \varphi) = (U, x^1, ..., x^n)$ *be a chart of* M *at a fixed point* $p \in M$*. A vector field* $X = \sum_i a^i \frac{\partial}{\partial x^i}$ *on* U *is smooth if and only if the coefficients* a^i *are smooth functions on* U*.*

Definition A.0.26. *Let* X *be a smooth vector field over a* C^∞*-manifold* M*. We say that a smooth curve* $c :]a, b[\subset \mathbb{R} \longrightarrow M$ *is an* integral curve *of* X *if it is such that* $c'(t) = X(c(t))$ *for all* $t \in]a, b[$*. Usually, we assume that* $0 \in]a, b[$*. In this case, if* $c(0) = p$*, we say that* c *is an integral curve* starting at p*. We say that an integral curve is* maximal *if its domain can not be extended to a larger interval.*

We can also study the dependence of the integral curve of a vector field on its starting point:

Definition A.0.27. *A* local flow *about a point* p *in an open set* U *of a manifold* M *is a* C^∞ *function*

$$F \;:]-\varepsilon, \varepsilon\,[\times W \longrightarrow U,$$

where $\varepsilon > 0$ *and* W *is a neighborhood of* p *contained in* U*, such that writing* $F_t(q) = F(t, q)$*, we have*

(i) $F_0(q) = q$ *for all* $q \in W$*,*

(ii) $F_t(F_s(q)) = F_{t+s}(q)$ *(when both sides are defined).*

If a local flow is defined on $\mathbb{R} \times M$*, then it is called a* global flow*. A vector field having a global flow is called a* complete vector field.

Appendix B

McShane and Whitney Extension Theorem

We prove the classical McShane-Whitney extension Theorem. For further informations and for proofs, the reader should look in [23]. We start with a simple lemma

Lemma B.0.16. *Let $A \subset \mathbb{R}^n$ and $\{f_i\}_{i\in I}$ be a family of Lipschitz functions, $f_i : A \longrightarrow \mathbb{R}$, with the same Lipschitz constant $L > 0$. Then the functions*

$$\begin{aligned} A &\longrightarrow \mathbb{R} \\ x &\longmapsto \inf_{i\in I} f_i(x) \end{aligned}$$

and

$$\begin{aligned} A &\longrightarrow \mathbb{R} \\ x &\longmapsto \sup_{i\in I} f_i(x) \end{aligned}$$

are L-Lipschitz on A, if they are finite at one point.

Theorem B.0.17 (McShane-Whitney extension Theorem)**.** *Let $A \subset \mathbb{R}^n$ and $f : A \longrightarrow \mathbb{R}$ be a L-Lipschitz function. The there exists a L-Lipschitz function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $F|_A = f$.*

Proof. Let us define, for each $a \in A$, the following function

$$f_a(x) := f(a) + L\,|\,x - a\,|.$$

It is clear that it is a Lipschitz function on $\mathbb{R}^n$, indeed,

$$\begin{aligned} |f_a(x) - f_a(y)| &= L \, \big| |x-a| - |y-a| \big| \\ &\leq L\,|x - a - y + a| = L\,|x-y|. \end{aligned}$$

Hence, by Lemma B.0.16, the function

$$\begin{aligned} F &: \mathbb{R}^n \longrightarrow \mathbb{R} \\ F(x) &:= \inf_{a \in A} f_a(x), \end{aligned}$$

is Lipschitz with constant L. For such a F, it is also true that $F(a) = f(a)$, for each $a \in A$. $\square$

Appendix C

Geodesics and Balls in Heisenberg Group

In this Appendix we aim to present some useful ideas and notions about geodesics and metric balls in Heisenberg group. This part is entirely taken from [33].

We start by recalling the definition of geodesics, we will use the same notations as in Section 1.2.:

Definition C.0.28. *Let $\mathbb{R}^n$ be endowed with a Carnot-Carathéodory metric d induced by a Hörmander system of vector field. We say that a Lipschitz continuous $\mathbb{X}$-subunit curve $\gamma : [0,T] \longrightarrow \mathbb{R}^n$ is a* geodesic *if $d(\gamma(0),\gamma(T)) = T$.*

Theorem C.0.18. *If $(\mathbb{R}^n, d)$ is $\mathbb{X}$-connected then, for each couple $x, y \in \mathbb{R}^n$, there exists a geodesics γ which joins x to y.*

We state now the Pontryagin Maximum Principle, which can be used to derive equations of geodesics. First we need to fix some notations and introduce a *control problem*, whose solutions are exactly the geodesics.

If $h \in L^\infty([0,1];\mathbb{R}^m)$ is the *control*, we define the *pay-off functional* as follows

$$J(h) = \frac{1}{2}\int_0^1 |h(s)|^2\, ds. \tag{C.1}$$

The *state equation* is

$$\dot{x} = \sum_{j=1}^{m} h_j \cdot X_j(x), \tag{C.2}$$

and we are looking for a Lipschitz continuous function x over $[0,1]$ with values in $\mathbb{R}^m$. Moreover, we need also to fix initial and final data (*constraints*)

$$x(0) = x_0, \quad x(1) = x_1.$$

Remark C.0.29. We point out that we introduced a well posed problem. Indeed, Theorem C.0.18 ensures the existence of a curve which minimizes J.

Definition C.0.30. *Given* (C.1) *and* (C.2), *we say that the pair* (x, h) *is* optimal *if the control* h *minimizes* J *and* x *is almost everywhere a solution of* (C.2) *together with the constraints.*

Theorem C.0.19 (Pontryagin Maximum Principle)**.** *Assume that* (x, h) *is optimal for our problem. Then there exist a function* $\xi : [0,1] \longrightarrow \mathbb{R}^n$ *and a constant* $\lambda = 0$ *or* $\lambda = 1$ *such that*

(i) $|\xi(s)| + \lambda \neq 0$, *for every* $s \in [0,1]$*;*

(ii) $\dot{\xi} = -\frac{\partial}{\partial x} \langle \mathcal{A}(x)h, \xi\rangle$, *a.e. on* $[0,1]$*;*

(iii) $\langle \xi(s), \mathcal{A}(x)\xi(s)\rangle - \lambda\frac{1}{2}|h|^2 = \max_{u\in\mathbb{R}^m} \left(\langle \mathcal{A}(x)u, \xi(s)\rangle - \lambda\frac{1}{2}|u|^2\right)$, *for a.e.* $s \in [0,1]$.

Remark C.0.31. Equations (ii) in Theorem C.0.19, combined with (C.2), transform in the *Hamilton system*

$$\begin{cases} \dot{x} = \dfrac{1}{2}\dfrac{\partial H(x,\xi)}{\partial \xi} \\ \dot{\xi} = -\dfrac{1}{2}\dfrac{\partial H(x,\xi)}{\partial x}, \end{cases} \tag{C.3}$$

where the *Hamiltonian function* is

$$H(x,\xi) = \sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2 .$$

Using Pontryagin Maximum Principle, one can prove, in the setting of the Heisenberg group, the following

Theorem C.0.20. *Geodesics in* $\mathbb{H}^n$ *are curves of class* C^∞*.*

Remark C.0.31 and Theorem C.0.20 imply that geodesics in $\mathbb{H}^n$ are solutions of the Hamiltonian system (C.3) with

$$H\left((z,t),(\zeta,\tau)\right) \;=\; \sum_{j=1}^{n}\left((\xi_j+2y_j\tau)^2+(\eta_j-2x_j\tau)^2\right),$$

where we denote two generic points of $\mathbb{H}^n$ as $(z,t)=(x,y,t)$ and $(\zeta,\tau)=(\xi,\eta,\tau)$.

Therefore, we write explicitly the equations

$$\begin{cases} \dot{x} \;=\; \xi_j+2\tau y_j \\ \dot{y} \;=\; \eta_j-2\tau x_j \\ \dot{t} \;=\; \sum_{j=1}^{n} 2y_j\xi_j+4\tau y_j^2-2x_j\eta_j+4\tau x^2 \\ \dot{\xi}_j \;=\; 2\tau\eta_j-4\tau^2 x_j \\ \dot{\eta}_j \;=\; -2\tau\xi_j-4\tau^2 y_j \\ \dot{\tau} \;=\; 0, \end{cases}$$

for $j=1,...,n$. As initial data we take

$$(z(0),t(0))=(0,0) \quad \text{and} \quad (\zeta(0),\tau(0))=\left(B_1,...,B_n,A_1,...,A_n,\frac{\varphi}{4}\right),$$

which provide the solution

$$\begin{cases} x_j(s) \;=\; \dfrac{A_j(1-\cos(\varphi s))+B_j\sin(\varphi s)}{\varphi} \\ y_j(s) \;=\; \dfrac{-B_j(1-\cos(\varphi s))+A_j\sin(\varphi s)}{\varphi} \\ t(s) \;=\; 2\,\dfrac{\varphi s-\sin(\varphi s)}{\varphi^2}\displaystyle\sum_{j=1}^{n}\left(A_j^2+B_j^2\right). \end{cases} \tag{C.4}$$

Clearly, a posteriori, the correct normalization is $\sum_{j=1}^{n}\left(A_j^2+B_j^2\right)=1$.

Using equation (C.4), we can deduce a parametrization of the unitary metric ball centered at the origin. We restict the discussion to $\mathbb{H}^1$. Let

$$S=\{(x,y,t)\in\mathbb{H}^1 \,|\, d((x,y,t),0)=1\}.$$

The normalization $(A^2 + B^2) = 1$ allows us to choose $A = \cos\vartheta$ and $B = \sin\vartheta$. Therefore, if we choose $s = 1$, we obtain

$$\begin{cases} x(\vartheta,\varphi) &= \dfrac{\cos\vartheta(1-\cos\varphi)+\sin\vartheta\sin\varphi}{\varphi} \\ y(\vartheta,\varphi) &= \dfrac{-\sin\vartheta(1-\cos\varphi)+\cos\vartheta\sin\varphi}{\varphi} \\ t(\vartheta,\varphi) &= 2\,\dfrac{\varphi-\sin\varphi}{\varphi^2}, \end{cases} \tag{C.5}$$

with $0 \le \vartheta \le 2\pi$ and $-2\pi \le \varphi \le 2\pi$.

Remark C.0.32. Equations (C.5) imply that the surface S is of class C^1 where $(x, y) \neq (0, 0)$.

Remark C.0.33. Set

$$\mathcal{E} = \left\{ (\vartheta,\varphi,\rho) \in \mathbb{R}^3 \,\middle|\, -\frac{2\pi}{\rho} \le \varphi \le \frac{2\pi}{\rho},\ \rho \ge 0 \right\}$$

and define $\Phi : \mathcal{E} \longrightarrow \mathbb{H}^1$ by $\Phi(\vartheta,\varphi,\rho) = (x(\vartheta,\varphi,\rho), y(\vartheta,\varphi,\rho), t(\vartheta,\varphi,\rho))$, where

$$\begin{cases} x(\vartheta,\varphi,\rho) &= \dfrac{\cos\vartheta(1-\cos(\varphi\rho))+\sin\vartheta\sin(\varphi\rho)}{\varphi} \\ y(\vartheta,\varphi,\rho) &= \dfrac{-\sin\vartheta(1-\cos(\varphi\rho))+\cos\vartheta\sin(\varphi\rho)}{\varphi} \\ t(\vartheta,\varphi,\rho) &= 2\,\dfrac{\varphi\rho-\sin(\varphi\rho)}{\varphi^2}. \end{cases} \tag{C.6}$$

If $\rho > 0$ is fixed, the the equations (C.6), with $\vartheta \in [0, 2\pi[$ and $-\frac{2\pi}{\rho} \leqq \varphi \le \frac{2\pi}{\rho}$, parametrize $\partial B(0,\rho)$.

We point out that the t-components of the two poles are given by

$$t\left(2\pi, \pm\frac{2\pi}{\rho}, \rho\right) = \pm 2\,\frac{\rho^2}{2\pi}. \tag{C.7}$$

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