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Quantum groups and vertex models

Tesi di Laurea in Algebra

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Introduzione

Negli ultimi decenni la teoria delle algebre di Lie ha trovato ampie applicazioni in numerosi campi della fisica. In questa tesi vengono presentate alcune relazioni fra gruppi quantici e modelli reticolari. In particolare, si mostra come associare un modello vertex ad una rappresentazione di un'algebra inviluppante quantizzata affine e si osserva che, specializzando il parametro quantistico ad una radice dell'unità, si vengono a creare speciali simmetrie.

Il principale oggetto di studio è $\mathrm{U_q}(\mathfrak{g})$, l'algebra inviluppante universale quantizzata associata all'algebra di Kac-Moody $\mathfrak{g}$. Specializzando il parametro quantistico q ad una radice dispari dell'unità si ottiene una copia dell'algebra inviluppante non deformata standard $U(\mathfrak{g})$.

Si considera poi una rappresentazione di $\mathrm{U_q}(\mathfrak{g})$ su uno spazio vettoriale V. Lo spazio V può essere interpretato da un punto di vista fisico come uno spazio di stati ma, come sarà più chiaro in seguito, per definire l'azione di un modello vertex è necessario agire su una catena di spazi, vale a dire, a livello matematico, sul prodotto tensoriale di L copie di V e pertanto è necessario estendere la rappresentazione data su V ad una rappresentazione su $V^{\otimes L}$. A tal fine si munisce $\mathrm{U_q}(\mathfrak{g})$ della struttura di algebra di Hopf e si utilizza la comoltiplicazione.

Si osserva così una prima simmetria tra l'azione di $U(\mathfrak{g})$ su $V^{\otimes L}$ e l'operatore di shift sullo stesso spazio; questo risultato, descritto in primo luogo da un punto di vista puramente matematico, viene in un secondo momento interpretato come l'invarianza traslazionale del modello vertex associato a $U_q(\hat{\mathfrak{g}})$.

Il passo successivo consiste nell'associare un modello vertex ad una rappresentazione di valutazione dell'algebra quantizzata affine $U_q(\hat{\mathfrak{g}})$. Questo processo si compone a sua volta di due parti. La prima parte è la costruzione di una R-matrice $R(u) \in End(V \otimes V)$ a partire dalla R-matrice universale R di $U_q(\hat{\mathfrak{g}})$ e per una data famiglia di rappresentazioni di valutazione parametrizzate da u. $U_q(\hat{\mathfrak{g}})$ munito di R è un'algebra quasi-triangolare. Utilizzando questa particolare struttura si dimostra che $R(u)$ risolve l'equazione di Yang-Baxter quantistica e che la R-matrice permutata $\mathscr{R}(u)$ commuta con l'azione di $U_q(\hat{\mathfrak{g}})$. Nella seconda parte, di natura essenzialmente fisica, si esplicita la relazione tra funzione di partizione e matrice di trasferimento di un modello vertex rettangolare, suggerendo così una caratterizzazione del modello basata sulla R-matrice $R(u)$. Essendo $R(u)$ una soluzione dell'equazione di Yang-Baxter quantistica si può inoltre dimostrare che il modello così costruito è integrabile. Si mostra infine che la sua matrice di trasferimento commuta con l'azione dell'algebra inviluppante non deformata $U(\hat{\mathfrak{g}})$ costruita all'inizio. Questo risultato si basa sull'invarianza traslazionale e sull'invarianza rispetto all'ope-ratore di boost; quest'ultima viene ottenuta usando l'invarianza della R-matrice permutata rispetto a $U_q(\hat{\mathfrak{g}})$.

Il campo base utilizzato durante tutta la tesi sarà $\mathbb{C}$.

Introduction

In the last decades the theory of Lie algebras has found large application to various fields of physics. In this thesis we present some relations between quantum universal enveloping algebras and lattice models. In particular we want to associate a vertex model to a representation of an affine quantum enveloping algebra and show that special symmetries arise when specializing the quantum parameter at a root of unity.

We start studying $\mathrm{U_q}(\mathfrak{g})$, the quantum universal enveloping algebra associated to the Kac-Moody algebra $\mathfrak{g}$. Specializing the quantum parameter q at an odd root of unity one can then construct a copy of the classical non deformed enveloping algebra $U(\mathfrak{g})$.

We then consider a representation of $U_q(\mathfrak{g})$ over a vector space V. The space V can be physically interpreted as a space of states but, as it will be clear later, in order to define an action of a vertex model it is necessary to act over a chain of spaces, which is, mathematically speaking, the tensor product of L copies of V, i.e., it is necessary to extend the given representation on V to a representation on $V^{\otimes L}$. To this purpose we endow $U_q(\mathfrak{g})$ of a Hopf algebra structure and make use of its comultiplication.

This allows us to discover a first symmetry between the action of $U(\mathfrak{g})$ over $V^{\otimes L}$ and the shift operator over the same space; this result, which is first described from a completely mathematical point of view, will be later physically interpreted as the translational invariance of the vertex model associated to $\mathrm{U_q}(\hat{\mathfrak{g}})$.

The next step is to associate a vertex model to an evaluation representation of the affine quantum algebra $U_q(\hat{\mathfrak{g}})$. This process is divided into two parts. The first part is the mathematical construction of an R-matrix $R(u) \in End(V \otimes V)$ starting from the universal R-matrix R of $U_q(\hat{\mathfrak{g}})$ and for a given family of evaluation representations parametrized by u. $U_q(\hat{\mathfrak{g}})$ endowed with R is in particular a braided bialgebra. Using this particular structure one can prove that $R(u)$ solves the quantum Yang-Baxter equation and that the permuted R-matrix $\mathscr{R}(u)$ commutes with the action of the affine quantum group. The second part, more physically flavoured, shows the relation between the partition function and the transfer matrix of a rectangular vertex model, suggesting a characterization of the model based on the R-matrix $R(u)$. Being $R(u)$ a solution of the quantum Yang-Baxter equation we can also prove that the so obtained model is integrable.

We finally show that the model, or better its transfer matrix, commutes with the action of the non deformed enveloping algebra $U(\hat{\mathfrak{g}})$ constructed at the beginning. This result is based on the translational invariance and on the invariance with respect to the boost operator which is shown using the quantum group invariance of the permuted R-matrix.

Throughout the text the base field will be $\mathbb{C}$.

Chapter 1

Quantum universal enveloping algebras

1.1 Preliminary results and notation

In this first chapter we briefly recall the main definitions concerning Kac-Moody algebras, with particular attention to affine algebras. Definitions and notation introduced here will be used throughout the thesis. All the material of this section can be found in [10].

Definition 1.1. Let $n \in \mathbb{N}$ and $A = (a_{ij})_{1 \le i,j \le n}$ be a square matrix with integral entries. The matrix A is said to be a *generalized Cartan matrix* if:

1. $a_{ii} = 2$ for $1 \le i \le n$;
2. $a_{ij} \le 0$ if $i \ne j$;
3. $a_{ij} = 0$ if and only if $a_{ji} = 0$.

If there exists an invertible diagonal matrix D, such that DA is a symmetric matrix, A is called *symmetrizable.*

Definition 1.2. Let $A = (a_{ij})$ be a generalized Cartan matrix of order n. A *realization* of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ such that:

- $\mathfrak{h}$ is a $\mathbb{C}$-vector space of dimension $2n - rk(A)$;
- $\Pi = \{\alpha_1, ..., \alpha_n\} \subset \mathfrak{h}^*$ is a set of linearly independent elements;
- $\Pi^\vee = \{\alpha_1^\vee, ..., \alpha_n^\vee\} \subset \mathfrak{h}$ is a set of linearly independent elements;
- $\alpha_j(\alpha_i^\vee) = a_{ij}$ for $1 \leq i, j \leq n$.

Remark 1. There exists at least one realization of every generalized Cartan matrix. This realization is unique up to isomorphism.

Definition 1.3. Let A be a generalized Cartan matrix of order n and $(\mathfrak{h}, \Pi, \Pi^\vee)$ a realization of A. The Kac-Moody algebra $\mathfrak{g}(A)$ associated to A, is the Lie algebra over $\mathbb{C}$ with generators e_i, f_i, for $i = 1, ..., n$, and $\mathfrak{h}$ satisfying the following relations:

1. $[h, h'] = 0$ for $h, h' \in \mathfrak{h}$;
2. $[h, e_i] = \alpha_i(h) e_i$ for $h \in \mathfrak{h}, i = 1, ..., n$;
3. $[h, f_i] = -\alpha_i(h) f_i$ for $h \in \mathfrak{h}, i = 1, ..., n$;
4. $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$ for $1 \leq i, j \leq n$;
5. $ad_{e_i}^{1-a_{ij}}(e_j) = 0$ for $1 \leq i, j \leq n$ and $i \neq j$;
6. $ad_{f_i}^{1-a_{ij}}(f_j) = 0$ for $1 \leq i, j \leq n$ and $i \neq j$.

$\mathfrak{h}$ is called the Cartan subalgebra of $\mathfrak{g}(A)$.
When there is no matter of confusion we will denote the Kac-Moody algebra $\mathfrak{g}(A)$ simply by $\mathfrak{g}$.

Remark 2. The Kac-Moody algebra associated to a Cartan matrix A does not depend on the realization chosen but only on the matrix A; algebras obtained using two different realizations can in fact be proven to be isomorphic.

Remark 3. Let A be a symmetrizable generalized Cartan matrix. It is possible to define an invariant nondegenerate symmetric bilinear form on $\mathfrak{g}(A)$ whose restriction to its Cartan subalgebra $\mathfrak{h}$ is nondegenerate too. We can so define a nondegenerate bilinear form on $\mathfrak{h}^*$.

Kac-Moody algebras $\mathfrak{g}(A)$ can be classified on the basis of the properties of the Cartan matrix A; in particular three possible cases may occur:

- if A is a positive definite matrix then the associated Kac-Moody algebra is of *finite type* (i.e., it is a finite dimensional Lie algebra);
- if A is a positive semidefinite matrix then the associated Kac-Moody algebra is of *affine type*;
- if A is an indefinite matrix then the associated Kac-Moody algebra is of *indefinite type.*

Definition 1.4. A generalized Cartan matrix A is said to be *decomposable* if there exists a proper non empty subset $I \subset \{1, ..., n\}$ such that $a_{ij} = 0$ if $i \in I$ and $j \notin I$ or $i \notin I$ and $j \in I$. It is *indecomposable* otherwise.

Remark 4. If $\mathfrak{g}(A)$ is of finite type then A is always symmetrizable. If furthermore A is indecomposable then $\mathfrak{g}$(A) is simple.

An important subclass of affine algebras, called *"nontwisted" affine algebras*, can be concretely constructed in terms of underlying simple finite dimensional Lie algebras. Let us briefly recall this construction to which we will refer as affinisation.
Let $\mathring{\mathfrak{g}}$ be a simple Lie algebra of dimension n and consider the so-called *loop algebra* $\mathscr{L}\mathring{\mathfrak{g}} := \mathbb{C}[t, t^{-1}] \otimes \mathring{\mathfrak{g}}$. This is not yet an affine algebra. If $[,]_{\mathring{\mathfrak{g}}}$ denotes the Lie bracket in $\mathring{\mathfrak{g}}$, this can be extended to a Lie bracket on $\mathscr{L}\mathring{\mathfrak{g}}$ by setting

$$[P \otimes x, Q \otimes y]_{\mathscr{L}\mathring{\mathfrak{g}}} = PQ \otimes [x, y]_{\mathring{\mathfrak{g}}} \quad \text{for } P, Q \in \mathbb{C}[t, t^{-1}] \text{ and } x, y \in \mathring{\mathfrak{g}}.$$

Now we extend the loop algebra adding a central element c, i.e. we define $\hat{\mathfrak{g}} := \mathscr{L}\mathring{\mathfrak{g}} \oplus \mathbb{C}c$. In order to extend the Lie bracket to $\hat{\mathfrak{g}}$ we recall that it is always possible to define an invariant symmetric non degenerate bilinear form over $\mathring{\mathfrak{g}}$, let us denote it by $(,)$. Then, for $P, Q \in \mathbb{C}[t, t^{-1}]$, $x, y \in \mathring{\mathfrak{g}}$ and $\lambda \in \mathbb{C}$, we set
$[P \otimes x, Q \otimes y]_{\hat{\mathfrak{g}}} = [P \otimes x, Q \otimes y]_{\mathscr{L}\mathring{\mathfrak{g}}} + Res(P'Q)(x, y)c$
$[P \otimes x + \lambda c, c]_{\hat{\mathfrak{g}}} = 0$

where $Res : \mathbb{C}[t, t^{-1}] \longrightarrow \mathbb{C}$ is the residue of the Laurent polynomial (in 0). Finally we extend $\hat{\mathfrak{g}}$ with a derivation element d so that we get $\tilde{\mathfrak{g}} := \hat{\mathfrak{g}} \oplus \mathbb{C}d$, and we define the bracket on $\tilde{\mathfrak{g}}$ in the following way:
$[a, b]_{\tilde{\mathfrak{g}}} = [a, b]_{\hat{\mathfrak{g}}}$ for $a, b \in \hat{\mathfrak{g}}$;
$[d, c] = 0$;
$[d, P \otimes x] = t\frac{dP}{dt} \otimes x$ for $P \in \mathbb{C}[t, t^{-1}]$ and $x \in \mathring{\mathfrak{g}}$.

In the literature the term Kac-Moody affine algebra is referred both to $\hat{\mathfrak{g}}$ and to $\tilde{\mathfrak{g}}$. In this thesis we will mainly deal with $\hat{\mathfrak{g}}$ so we will name it affine algebra; the difference between the two algebras consists in the dimension of their Cartan subalgebras, which in the case of $\tilde{\mathfrak{g}}$ is bigger and allows to define a non degenerate inner product. As we already pointed out, the Kac-Moody affine algebras constructed in this way are not all the possible affine algebras, but only a part of them. The Cartan matrix of $\hat{\mathfrak{g}}$ (or better of $\tilde{\mathfrak{g}}$) has order $n+1$ and contains the Cartan matrix of $\mathring{\mathfrak{g}}$ as a submatrix, so it is natural to call it *extended Cartan matrix.*

Given a Kac-Moody algebra $\mathfrak{g}$, we will denote by $U(\mathfrak{g})$ its universal enveloping algebra. Let us recall the definition of the evaluation representation for $U(\hat{\mathfrak{g}})$. Every representation π_V of $U(\mathring{\mathfrak{g}})$ over a $\mathbb{C}$-vector space V can be extended to a representation of $U(\hat{\mathfrak{g}})$ using the evaluation homomorphism $p_a, a \in \mathbb{C}^*$, defined by

$$\begin{aligned} p_a : \mathrm{U}(\hat{\mathfrak{g}}) &\longrightarrow U(\mathring{\mathfrak{g}}) \\ t^m \otimes x &\mapsto a^m x \quad \text{for } m \in \mathbb{Z}, x \in \mathring{\mathfrak{g}} \\ c &\mapsto 0 \end{aligned}$$

The map $\pi_V \circ p_a$ is a representation of $\mathrm{U}(\hat{\mathfrak{g}})$ called evaluation representation. In particular we point out that the evaluation morphism can not be naturally extended to $\mathrm{U}(\tilde{\mathfrak{g}})$ so it is not possible to associate in this way a representation of $U(\mathring{\mathfrak{g}})$ to a representation of $\mathrm{U}(\tilde{\mathfrak{g}})$. In fact, if we suppose there is an extension, for $P \in \mathbb{C}[t, t^{-1}]$ and $x \in \mathring{\mathfrak{g}}$, we get:

$[p_a(d), p_a(Px)] = p_a([d, Px]) = p_a(t\frac{dP}{dt}x) = aP'(a)x$
but
$[p_a(d), p_a(Px)] = [p_a(d), P(a)x] = P(a)[p_a(d), x]$
so $P(a)[p_a(d), x] = aP'(a)x$.
Assuming $x \neq 0$ and $P(a) \neq 0$ we then have that $\frac{P'(a)}{P(a)}$ does not depend on P, which is absurd.

1.2 Definition and general properties of $\mathrm{U_q}(\mathfrak{g})$

The main mathematical object studied in this thesis is the quantum universal enveloping algebra associated with an arbitrary symmetrizable Kac-Moody algebra. This is a deformation of the classical universal enveloping algebra, obtained by introducing in its defining relations the quantum parameter $q \in \mathbb{C}$. This algebra is sometimes named also quantum group, even if the term is a little more general and does not denote this definition uniquely. The notion of quantum group was introduced by V. G. Drinfeld during the International Congress of Mathematicians in 1986 in Berkeley and it refers also to additional structures of this algebra that will be presented later (the quasi-triangular structure).
We begin defining this algebra through a presentation.

Definition 1.5. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a symmetrizable generalized Cartan matrix and $\mathfrak{g}$ the associated Kac-Moody algebra. Let q be a complex variable and define for α_i simple root of $\mathfrak{g}$, $q_i := q^{\frac{(\alpha_i, \alpha_i)}{2}}$, $(,)$ being the invariant bilinear form on $\mathfrak{h}^*$ introduced in Remark 3. Then $U_q(\mathfrak{g})$ is the associative $\mathbb{Q}(q)$-algebra generated by the elements $e_i, f_i, K_i, K_i^{-1}, \quad i = 1, ..., n$ with the following relations considered for $i, j = 1, ..., n$:

1.
$$[K_i, K_j] = 0 \quad K_i K_i^{-1} = K_i^{-1} K_i = 1$$
$$K_i e_j K_i^{-1} = q_i^{a_{ij}} e_j \quad K_i f_j K_i^{-1} = q_i^{-a_{ij}} f_j$$

2.
$$[e_i, f_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

3.
$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix}_{q_i} e_i^n e_j e_i^{1-a_{ij}-n} = 0$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix}_{q_i} f_i^n f_j f_i^{1-a_{ij}-n} = 0$$

Remark 5. 1. Definition 1.5 makes use of the so-called q-integers, in particular of the q-binomial coefficient. The precise definition and some basic but useful properties can be found in the appendix.

2. Relations 1. and 2. are the quantum deformations of the classical Weyl (Chevalley) relations while relations 3. are the so-called quantum Serre relations.

3. Let α be a real root of a nontwisted affine algebra $\hat{\mathfrak{g}}$; we define the length of α by $l(\alpha) = (\alpha, \alpha)^{\frac{1}{2}}$. There are only two possible lengths for the roots of $\mathfrak{g}$, so, choosing by convention that for the short roots $(\alpha, \alpha) = 2$ and remembering that for two roots α and β we have that the ratio $\frac{(\alpha,\alpha)}{(\beta,\beta)}$ gets one of these values $1, 2, 3, \frac{1}{2}, \frac{1}{3}$, it follows immediately that, if α_i is a short root $q_i = q$ and if α_i is a long root $q_i = q^2, q^3$.

4. If A is indecomposable let $D = (d_1, ..., d_n)$ be the invertible diagonal matrix such that DA is a symmetric matrix. D is determined up to a scalar factor and we can choose D such that its diagonal elements d_i are positive and coprime integers. We can then define the q_i in an alternative but equivalent way, i.e. we can set $q_i = q^{d_i}$.

We now focus on the difference between the quantum affine algebra $\mathrm{U_q}(\hat{\mathfrak{g}})$ and the algebra $\mathrm{U_q}(\tilde{\mathfrak{g}})$.

Definition 1.6. Let $A = (a_{ij})_{1\leq i,j\leq n}$ be the Cartan matrix of a finite dimensional simple Lie algebra $\mathring{\mathfrak{g}}$ and $A' = (a_{ij})_{0\leq i,j\leq n}$ the extended Cartan matrix of rank n. Then $\mathrm{U_q}(\hat{\mathfrak{g}})$ is defined as in Definition 1.5 (i.e., with generators $e_i, f_i, K_i^{\pm 1}$ for $i = 0, ..., n$ and the usual quantum relations), while the algebra $\mathrm{U_q}(\tilde{\mathfrak{g}})$ is its extension obtained considering the generators $d, e_i, f_i, K_i^{\pm 1}$ for $i = 0, ..., n$ with the usual relations and

$$[d, K_i] = 0 \qquad [d, e_i] = \delta_{i,0} e_i \qquad [d, f_i] = \delta_{i,0} f_i$$

It is also possible to give a presentation of $\mathrm{U_q}(\hat{\mathfrak{g}})$ (and $\mathrm{U_q}(\tilde{\mathfrak{g}})$) as an extension of a loop algebra with a central element and a derivation element ([1]).

We shall now introduce some important relations among the elements of a quantum universal enveloping algebra.

Lemma 1.2.1. *Let $n \in \mathbb{N}$. We have*

$$[e_i, f_i^n] = [n]_{q_i} f_i^{n-1} \frac{K_i q_i^{1-n} - K_i^{-1} q_i^{n-1}}{q_i - q_i^{-1}}$$

$$[e_i^n, f_i] = [n]_{q_i} e_i^{n-1} \frac{K_i q_i^{n-1} - K_i^{-1} q_i^{1-n}}{q_i - q_i^{-1}}$$

Proof. We prove the first relation by induction on n.
If $n = 1$ it is true by definition.
Now let us suppose it is true for n and let us show it for $n + 1$.
Remembering that $K_i f_i = q_i^{-2} f_i K_i$ we have

$$\begin{aligned}
[e_i, f_i^{n+1}] &= e_i f_i^n f_i - f_i^{n+1} e_i \\
&= f_i^n e_i f_i + [n]_{q_i} f_i^{n-1} \tfrac{K_i q_i^{1-n} - K_i^{-1} q_i^{n-1}}{q_i - q_i^{-1}} f_i - f_i^{n+1} e_i \\
&= f_i^{n+1} e_i + f_i^n \tfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}} + [n]_{q_i} f_i^{n-1} \tfrac{K_i q_i^{1-n} - K_i^{-1} q_i^{n-1}}{q_i - q_i^{-1}} f_i - f_i^{n+1} e_i \\
&= f_i^n (\tfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}} + \tfrac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} \tfrac{K_i q_i^{-1-n} - K_i^{-1} q_i^{n+1}}{q_i - q_i^{-1}}) \\
&= \tfrac{f_i^n}{q_i - q_i^{-1}} (\tfrac{K_i (q_i - q_i^{-1} + (q_i^n - q_i^{-n}) q_i^{-1-n}) - K_i^{-1} (q_i - q_i^{-1} + (q_i^n - q_i^{-n}) q_i^{n+1})}{q_i - q_i^{-1}}) \\
&= \tfrac{f_i^n}{q_i - q_i^{-1}} \tfrac{K_i q_i - K_i^{-1} q_i^{2n+1} - K_i q_i^{-2n-1} + K_i^{-1} q_i^{-1}}{q_i - q_i^{-1}} \\
&= \tfrac{f_i^n}{q_i - q_i^{-1}} \tfrac{(q_i^{n+1} - q_i^{-n-1})(K_i q_i^{1-(n+1)} - K_i^{-1} q_i^{(n+1)-1})}{q_i - q_i^{-1}} \\
&= [n+1]_{q_i} f_i^n \tfrac{K_i q_i^{1-(n+1)} - K_i^{-1} q_i^{(n+1)-1}}{q_i - q_i^{-1}}
\end{aligned}$$

The second relation can be proved in an analogous way. □

Definition 1.7. Let e_i, f_i and q_i be as in Definition 1.5 and let $r \in \mathbb{N}^+$. We will call *divided powers* the formal elements $e_i^{(r)}$ and $f_i^{(r)}$ defined as follows:

$$e_i^{(r)} = \frac{e_i^r}{[r]_{q_i}!} \tag{1.1}$$

$$f_i^{(r)} = \frac{f_i^r}{[r]_{q_i}!} \tag{1.2}$$

Lemma 1.2.2. *Let $n \in \mathbb{N}$. We have:*

$$[e_i^n, f_i^n] = \sum_{l=1}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{q_i}^2 f_i^{n-l} e_i^{n-l} [l]_{q_i}! \prod_{r=1}^{l} \frac{K_i q_i^{1-r} - K_i^{-1} q_i^{r-1}}{q_i - q_i^{-1}}$$

from which we immediately obtain the relation

$$[e_i^{(n)}, f_i^{(n)}] = \sum_{l=1}^{n} f_i^{(n-l)} e_i^{(n-l)} \prod_{r=1}^{l} \frac{K_i q_i^{1-r} - K_i^{-1} q_i^{r-1}}{q_i^r - q_i^{-r}}$$

Proof. The proof is by induction on n using Lemma 1.2.1. Computations are very long so we give only a sketch.
If $n = 1$ it is obviously true because we get the usual relation.
Now let us suppose it is true for n and let us show it for $n + 1$.
We have:

$$\begin{aligned}[e_i^{n+1}, f_i^{n+1}] &= e_i e_i^n f_i^n f_i - f_i^{n+1} e_i^{n+1} \\ &= e_i f_i^n e_i^n f_i + e_i [e_i^n, f_i^n] f_i - f_i^{n+1} e_i^{n+1} \\ &= f_i^n e_i^{n+1} f_i + [e_i, f_i^n] e_i^n f_i + e_i [e_i^n, f_i^n] f_i - f_i^{n+1} e_i^{n+1} \\ &= f_i^n [e_i^{n+1}, f_i] + [e_i, f_i^n] e_i^n f_i + e_i [e_i^n, f_i^n] f_i\end{aligned}$$

Now it is possible to complete the proof using the inductive hypothesis and Lemma 1.2.1. □

Lemma 1.2.3. *We recall the higher order Serre relations introduced by Lusztig (see [15]). Let $n \geq 1, m > -a_{ij}n$ and $s \in \mathbb{N}$.*
Then

$$e_i^{(m)} e_j^{(n)} = \sum_{k=0}^{-na_{ij}} C_{m-k}(q_i) e_i^{(k)} e_j^{(n)} e_i^{(m-k)} \tag{1.3}$$

where the coefficient function is

$$C_s(q_i) = \sum_{l=0}^{m+a_{ij}n-1} (-1)^{s+l+1} q_i^{-s(l+1-a_{ij}n-m)+l} \begin{bmatrix} s \\ l \end{bmatrix}_{q_i}$$

1.2.1 Evaluation representations of quantum affine algebras

In this paragraph we try to extend the notion of evaluation representation of $U(\hat{\mathfrak{g}})$, introduced in Section 1.1, to the quantum case.

We start introducing the automorphism D_u, $u \in \mathbb{C}$, given by

$$\begin{aligned} D_u : \mathrm{U_q}(\hat{\mathfrak{g}}) &\rightarrow \mathrm{U_q}(\hat{\mathfrak{g}}) \\ e_i &\mapsto e^{\delta_{i,0}u} e_i \\ f_i &\mapsto e^{-\delta_{i,0}u} f_i \\ K_i^{\pm 1} &\mapsto K_i^{\pm 1} \end{aligned}$$

We note that D_u is the identity on all the generators, except for e_0 and f_0. It is immediate to prove that we indeed defined an algebras automorphism, or in other words, that the defining relations of $U_q(\hat{\mathfrak{g}})$ are preserved.
Indeed we have, for example,
$D_u(K_i e_j K_i^{-1}) = D_u(q_i^{a_{ij}} e_j) = q_i^{a_{ij}} e^{\delta_{j,0}u} e_j = K_i e^{\delta_{j,0}u} e_j K_i^{-1} = D_u(K_i) D_u(e_j) D_u(K_i^{-1})$
and

$$\begin{aligned} D_u([e_i, f_j]) &= D_u(\delta_{ij} \tfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}) = \delta_{ij} \tfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}} = [e_i, f_j] = [e^{\delta_{i,0}u} e_i, e^{-\delta_{j,0}u} f_j] \\ &= [D_u(e_i), D_u(f_j)] \end{aligned}$$

Let now $p : U_q(\hat{\mathfrak{g}}) \longrightarrow U_q(\mathring{\mathfrak{g}})$ be an algebra homomorphism such that for $i = 1, ..., n$

$$\begin{aligned} p(e_i) &= e_i; \\ p(f_i) &= f_i; \\ p(K_i^{\pm 1}) &= K_i^{\pm 1}. \end{aligned}$$

The morphism $p_u = p \circ D_u$ is called evaluation homomorphism for $U_q(\hat{\mathfrak{g}})$, and every representation π_V of $U_q(\mathring{\mathfrak{g}})$ over a $\mathbb{C}$-vector space V can be naturally extended to a representation of $U_q(\hat{\mathfrak{g}})$ by $\pi_V \circ p_u$. The problem of this method is the construction of the map p, in fact it exists only for $\mathring{\mathfrak{g}} = \mathfrak{sl}_n$ (see [1],[5]). For this reason we will introduce an analogue of the evaluation representations for general quantum affine algebras.
To this purpose we need to recall the definition of highest weight module.

Definition 1.8. A $U_q(\mathfrak{g})$-module is a highest weight module if there exists $v_\lambda \in V$ such that $U_q(\mathfrak{g}).v_\lambda = V$, $e_i.v_\lambda = 0$ and $K_i.v_\lambda \in \mathbb{C}v_\lambda$.

Definition 1.9. A finite dimensional representation ρ_V of $U_q(\hat{\mathfrak{g}})$ over V is said to be an *evaluation representation* if all its irreducible $U_q(\mathring{\mathfrak{g}})$-subrepresentations are highest weight representations.

Thanks to the automorphism D_u we can then naturally define a family of evaluation representations.

Definition 1.10. Let $\rho_V : U_q(\hat{\mathfrak{g}}) \longrightarrow End(V)$ be an evaluation representation. Then $\rho_{V(u)} := \rho_V \circ D_u$ is a family of evaluation representations of $U_q(\hat{\mathfrak{g}})$, depending on the complex parameter u, which will be denoted by $V(u)$. Different representations are connected by the action of the automorphism D_u, in fact for $x \in \mathrm{U_q}(\hat{\mathfrak{g}})$ we have $\rho_{V(u+v)}(x) = \rho_{V(u)}(D_v(x))$.

We finally recall a result about the level of a representation which will be used later.

Definition 1.11. Given a representation of an affine Kac-Moody algebra (or (quantum) affine enveloping algebra), if the central element c, considering a loop algebra presentation, acts as the multiplication by the scalar k, then the representation is said to be of level k.

Proposition 1.2.4. *Every finite dimensional representation of the quantum affine algebra* $\mathrm{U_q}(\hat{\mathfrak{g}})$ *has level 0 (for a proof see [1]).*

1.3 The restricted specialization of $U_q(\mathfrak{g})$

In this section the concept of specialization of $U_q(\mathfrak{g})$ will be introduced in order to describe a particular specialization which will play a fundamental role in the following section. The idea behind the notion of specialization of $U_q(\mathfrak{g})$ is to construct an algebra starting from $U_q(\mathfrak{g})$ and "specializing" the quantum parameter q at a particular non-zero complex value.
The first complication in this process is that $U_q(\mathfrak{g})$ is a $\mathbb{Q}(q)$-algebra and so there can be problems with the existence of the rational functions when q

specializes to a complex value (the denominator can be 0). To bypass this difficulty we introduce the concept of integral form of $U_q(\mathfrak{g})$.

Definition 1.12. An integral form of $U_q(\mathfrak{g})$ is a $\mathbb{Z}[q,q^{-1}]$-subalgebra $U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g})$ such that the natural map

$$U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \to U_q(\mathfrak{g})$$

is an isomorphism of $\mathbb{Q}(q)$-algebras.
In other words we can say that the $\mathbb{Q}(q)$-algebra obtained from $U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g})$ with an extension of scalars (to $\mathbb{Q}(q)$) is $U_q(\mathfrak{g})$.

Now we can define a specialization starting from the integral form.

Definition 1.13. Let $U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g})$ be an integral form of $U_q(\mathfrak{g})$ and let $\varepsilon \in \mathbb{C}^*$. A specialization of $U_q(\mathfrak{g})$ is

$$U_\varepsilon(\mathfrak{g}) := U_{\mathbb{Z}[q,q^{-1}]} \otimes_{\mathbb{Z}[q,q^{-1}]} A$$

with the homomorphism $\mathbb{Z}[q,q^{-1}] \to A, \quad q \mapsto \varepsilon$ and where A can be $\mathbb{C}, \mathbb{Z}[\varepsilon,\varepsilon^{-1}]$ or $\mathbb{Q}(\varepsilon)$.

Now we define the $\mathbb{Z}[q,q^{-1}]$-subalgebra we will use, but we will not prove that it is an integral form. Using this integral form we will construct a particular specialization, called the restricted specialization.

Proposition 1.3.1. *Let $U^{res}_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g})$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by the elements $e_i^{(r)}, f_i^{(r)}, K_i$ and K_i^{-1} for $1 \le i \le n$ and $r \ge 1$.*
Then $U^{res}_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g})$ is an integral form of $U_q(\mathfrak{g})$.

Notation 1. For $c \in \mathbb{Z}$, $s \in \mathbb{N}$ and $i = 1, ..., n$ we introduce the notation:

$$\begin{bmatrix} K_i; c \\ s \end{bmatrix}_{q_i} = \prod_{r=1}^{s} \frac{K_i q_i^{c+1-r} - K_i^{-1} q_i^{r-1-c}}{q_i^r - q_i^{-r}} \tag{1.4}$$

Notation 2. For $N \in \mathbb{N}$ we set

$$N' := \begin{cases} N & \text{if } N \text{ odd} \\ \frac{N}{2} & \text{if } N \text{ even} \end{cases}$$

Definition 1.14. Let $\varepsilon \in \mathbb{C}^*$.
We define the so-called restricted specialization of $\mathrm{U_q}(\mathfrak{g})$ as follows

$$U_\varepsilon^{\mathrm{res}}(\mathfrak{g}) = U_{\mathbb{Z}[q,q^{-1}]}^{\mathrm{res}}(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(\varepsilon)$$

Remark 6. When specializing q to a root of unity of order N relations (1.1), (1.2) and (1.4) have no more meaning for $r \geq N'$, because the denominator goes to 0. We can anyway rewrite them bringing the denominator to the left side in order to avoid this problem, so, for example, 1.1 becomes $[r]_{q_i}!e_i^{(r)} = e_i^r$ and so on.
It follows immediately that, in $U_\varepsilon^{\mathrm{res}}(\mathfrak{g})$, $e_i^{N'} = 0$ and $f_i^{N'} = 0$.
Specializing the rewriting of 1.4 for $c = 0$ we get also $K_i^{2N'} = 1$, indeed we have $K_i q_i^{1-r} = K_i^{-1} q_i^{r-1}$, so $K_i^2 = q_i^{2(1-r)}$ and $K_i^{2N'} = q_i^{2N'(1-r)} = 1$.
We finally remark that K_i^N is central, because $K_i^N e_i = q_i^{Na_{ij}} e_i K_i^N = e_i K_i^N$ and $K_i^N f_i = q_i^{-Na_{ij}} f_i K_i^N = f_i K_i^N$

1.4 From $U_q(\mathfrak{g})$ to $U(\mathfrak{g})$

In this section we will show how to construct the non deformed enveloping algebra $U(\mathfrak{g})$ as a quotient (of a subalgebra) of the restricted specialisation of $U_q(\mathfrak{g})$ at a root of unity.

Before stating the main theorem, we present some technical results that will be used in its proof.
The following lemma can be proved by induction on r for fixed c.

Lemma 1.4.1. *In* $\mathrm{U_q}(\mathfrak{g})$ *the following relation holds:*

$$\begin{bmatrix} K_i; -c \\ r \end{bmatrix}_{q_i} = \sum_{0 \leq s \leq r} (-1)^s q_i^{c(r-s)} \begin{bmatrix} c+s-1 \\ s \end{bmatrix}_{q_i} K_i^s \begin{bmatrix} K_i; 0 \\ r-s \end{bmatrix}_{q_i} \tag{1.5}$$

for $c \in \mathbb{N}^+$ *and* $r \in \mathbb{N}$.

In the following remarks we will study the particular values of the q-binomial coefficients appearing in relation (1.5) when $q^N = 1$ and c is a multiple of N'.

Remark 7. Let $p \in \mathbb{N}, 1 \leq l \leq N' - 1$ and $q^N = 1$. Then we have:

$$\frac{[pN'+l]_{q_i}}{[l]_{q_i}} = \frac{q_i^{pN'+l} - q_i^{-pN'-l}}{q_i^l - q_i^{-l}} = q_i^{pN'} \frac{q_i^l - q_i^{-l}}{q_i^l - q_i^{-l}} = q_i^{pN'}$$

Furthermore, for a generic parameter q, we have:

$$\frac{[pN']_{q_i}}{[N']_{q_i}} = \frac{q_i^{pN'} - q_i^{-pN'}}{q_i^{N'} - q_i^{-N'}} = \frac{(q_i^{N'})^p - (q_i^{N'})^{-p}}{q_i^{N'} - q_i^{-N'}} = [p]_{q_i^{N'}}$$

Note that, if $q^N = 1$, then $[p]_{q_i^{N'}} = pq_i^{N'(p-1)}$

Remark 8. If $q^N = 1$, for $t \in \mathbb{N}^+$ and $s \in \mathbb{N}$, we have

$$\begin{bmatrix} N't+s-1 \\ s \end{bmatrix}_{q_i} = \begin{cases} 1 & \text{if } s = 0 \\ 0 & \text{if } 1 \leq s \leq N'-1 \\ tq_i^{N'(N't-1)} & \text{if } s = N' \end{cases}$$

For $s = 0$, $\left[\begin{smallmatrix} N't-1 \\ 0 \end{smallmatrix}\right]_{q_i} = 1$ by definition.
If $1 \leq s \leq N' - 1$ then

$$\begin{bmatrix} N't+s-1 \\ s \end{bmatrix}_{q_i} = \frac{[N't+s-1]_{q_i}!}{[s]_{q_i}![N't-1]_{q_i}!} = \frac{[N't]_{q_i}[N't+1]_{q_i}...[N't+s-1]_{q_i}}{[s]_{q_i}[1]_{q_i}...[s-1]_{q_i}} = 0$$

because $[N't]_{q_i} = 0$ and the other q-integer are all non zero.
In a similar way we proceed for the case $s = N'$ but the result is completely different. Thanks to Remark 7 we have

$$\begin{aligned} \left[\begin{smallmatrix} N't+N'-1 \\ N' \end{smallmatrix}\right]_{q_i} &= \tfrac{[N't+N'-1]_{q_i}!}{[N']_{q_i}![N't-1]_{q_i}!} \\ &= \tfrac{[N't]_{q_i}[N't+1]_{q_i}...[N't+N'-1]_{q_i}}{[N']_{q_i}[1]_{q_i}...[N'-1]_{q_i}} \\ &= [t]_{q_i^{N'}} \tfrac{[N't+1]_{q_i}...[N't+N'-1]_{q_i}}{[1]_{q_i}...[N'-1]_{q_i}} \\ &= tq_i^{N'(t-1)} q_i^{N'(N'-1)t} \\ &= tq_i^{N'(N't-1)} \end{aligned}$$

where the first three equalities hold for a generic parameter q while the last two hold under the hypothesis $q^N = 1$.

We now consider Lusztig's formula given in Lemma 1.2.3 with $m = N'(1-a_{ij})$ and $n = N'$. In this case the coefficient function $C_s(q_i)$ becomes:

$$C_s(q_i) = \sum_{l=0}^{N'-1} (-1)^{s+l+1} q_i^{-s(l+1-N')+l} \begin{bmatrix} s \\ l \end{bmatrix}_{q_i} \tag{1.6}$$

In particular we are interested in the values it gets adding the hypothesis $q^N = 1$.

Lemma 1.4.2. *If $q^N = 1$ the coefficient function becomes*

$$C_s(q_i) := \begin{cases} (-1)^{s+1} q_i^{s(N'-1)} & \textit{if } s \equiv 0 \pmod{N'} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We note that, if $s \equiv 0 \pmod{N'}$ and $0 < l < N'$, then the q-binomial coefficient appearing in formula (1.6) is 0. In fact, if we set $s = N't$ we have

$$\begin{bmatrix} N't \\ l \end{bmatrix}_{q_i} = \frac{[N't]_{q_i}!}{[N't-l]_{q_i}![l]_{q_i}!} = \frac{q_i^{N't} - q_i^{-N't}}{q_i - q_i^{-1}} \frac{[N't-1]_{q_i}!}{[N't-l]_{q_i}![l]_{q_i}!} = 0$$

so, all the summands in the definition of $C_s(q_i)$ are 0, except for $l = 0$ and we get the result.

If $s \neq 0 \pmod{N'}$ the situation is not so easy. We begin by noticing that, if $s = rN' + k, 1 \leq k < N'$, then the binomial coefficient $\left[\begin{smallmatrix} s \\ l \end{smallmatrix}\right]_{q_i}$ can be simplified as follows:

$$\begin{bmatrix} s \\ l \end{bmatrix}_{q_i} := \begin{cases} q_i^{rN'l} \left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_{q_i} & \text{if } l \leq k \\ 0 & \text{if } l > k \end{cases}$$

where we use that $[rN' + t]_{q_i} = q_i^{rN'}[t]_{q_i}$ for $1 \leq t \leq k$ and that $[rN']_{q_i} = 0$. We can then write the coefficient function as

$$\begin{aligned} C_s(q_i) &= \textstyle\sum_{l=0}^{k} (-1)^{rN'+k+l+1} q_i^{-(rN'+k)(l+1-N')+l} q_i^{rN'l} \left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_{q_i} \\ &= (-1)^{rN'+1} q_i^{-rN'+rN'^2} \textstyle\sum_{l=0}^{k} (-1)^{k+l} q_i^{-k(l+1-N')+l} \left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_{q_i} \end{aligned}$$

In order to complete the proof it is now enough to show that

$$\sum_{l=0}^{k} (-1)^{k+l} q_i^{-k(l+1-N')+l} \begin{bmatrix} k \\ l \end{bmatrix}_{q_i} = 0$$

If $k=1$ we have $(-1)q_i^{N'-1}+q_i^{N'-1}=0$
In general for $k>1$ the formula can be split using the q-Pascal identity (A.0.4-(4)) so we get

$$\sum_{l=0}^{k}(-1)^{k+l}q_i^{-k(l+1-N')+l}\begin{bmatrix}k\\l\end{bmatrix}_{q_i}=$$
$$\sum_{l=0}^{k-1}(-1)^{k+l}q_i^{-k(l+1-N')+l}q_i^{-l}\begin{bmatrix}k-1\\l\end{bmatrix}_{q_i}+\sum_{l=1}^{k}(-1)^{k+l}q_i^{-k(l+1-N')+l}q_i^{k-l}\begin{bmatrix}k-1\\l-1\end{bmatrix}_{q_i}=$$
$$\sum_{l=0}^{k-1}(-1)^{k+l}q_i^{-k(l+1-N')}\begin{bmatrix}k-1\\l\end{bmatrix}_{q_i}+\sum_{l=1}^{k}(-1)^{k+l}q_i^{-k(l-N')}\begin{bmatrix}k-1\\l-1\end{bmatrix}_{q_i}=$$
$$\sum_{l=0}^{k-1}(-1)^{k+l}q_i^{-k(l+1-N')}\begin{bmatrix}k-1\\l\end{bmatrix}_{q_i}-\sum_{l=0}^{k-1}(-1)^{k+l}q_i^{-k(l+1-N')}\begin{bmatrix}k-1\\l\end{bmatrix}_{q_i}=0$$

□

Finally we need a relation involving the divided powers.

Remark 9. Let $s\in\mathbb{N}, s>0$.
We have

$$e_i^{(N's)}=\frac{e_i^{N's}}{[N's]_{q_i}!}\frac{[N']_{q_i}!^s}{[N']_{q_i}!^s}=\frac{[N']_{q_i}!^s}{[N's]_{q_i}!}\frac{(e_i^{N'})^s}{[N']_{q_i}!^s}=\frac{[N']_{q_i}!^s}{[N's]_{q_i}!}e_i^{(N')s}$$

Furthermore using Remark 7 and under the hypothesis $q^N=1$ we can write:

$$\begin{aligned}\frac{[N']_{q_i}!^s}{[N's]_{q_i}!}&=\prod_{p=1}^{s}\frac{[N']_{q_i}!}{\frac{[N'p]_{q_i}!}{[N'(p-1)]_{q_i}!}}=\prod_{p=1}^{s}\frac{[1]_{q_i}[2]_{q_i}\dots[N']_{q_i}}{[N'(p-1)+1]_{q_i}[N'(p-1)+2]_{q_i}\dots[N'p]_{q_i}}\\&=\prod_{p=1}^{s}\frac{[1]_{q_i}[2]_{q_i}\dots[N'-1]_{q_i}}{[N'(p-1)+1]_{q_i}[N'(p-1)+2]_{q_i}\dots[N'(p-1)+N'-1]_{q_i}[p]_{q_i^{N'}}}\\&=\prod_{p=1}^{s}\frac{q_i^{N'^2(p-1)}}{p}=\frac{q_i^{N'^2\frac{s(s-1)}{2}}}{s!}\end{aligned}$$

So we obtain the following relation:

$$e_i^{(N's)}=\frac{q_i^{N'^2\frac{s(s-1)}{2}}}{s!}e_i^{(N')s}$$

We are now ready to state (and prove) Theorem 1.4.3 about the construction of the non deformed enveloping algebra $U(\mathfrak{g})$.

Theorem 1.4.3. *Let $\varepsilon\in\mathbb{C}^*$ be a root of unity of order N, $N>2$. Let $U'_\varepsilon(\mathfrak{g})$ be the subalgebra of the restricted specialization $U_\varepsilon^{res}(\mathfrak{g})$ generated by the elements $e_i^{(N')}, f_i^{(N')}$ for $1\le i\le n$. If N is odd (hence $N=N'$), then the quotient $\bar{U}'_\varepsilon(\mathfrak{g})$ of $U'_\varepsilon(\mathfrak{g})$ by the ideal generated by K_i-1 is isomorphic to the non deformed enveloping algebra $U(\mathfrak{g})$.*

Proof. The proof of this theorem consists in verifying the relations defining the non deformed enveloping algebra.

Using Lemma 1.2.2 we get immediately

$$\begin{aligned}[e_i^{(N')}, f_i^{(N')}] &= \sum_{l=1}^{N'} f_i^{(N'-l)} e_i^{(N'-l)} \prod_{r=1}^{l} \frac{K_i q_i^{1-r} - K_i^{-1} q_i^{r-1}}{q_i^r - q_i^{-r}} \\ &= \sum_{l=1}^{N'} f_i^{(N'-l)} e_i^{(N'-l)} \begin{bmatrix} K_i;0 \\ l \end{bmatrix}_{q_i} = \begin{bmatrix} K_i;0 \\ N' \end{bmatrix}_{q_i}\end{aligned}$$

because in the quotient for $1 \leq l < N'$ we have $\begin{bmatrix} K_i;0 \\ l \end{bmatrix}_{q_i} = 0$.
The formal elements $\begin{bmatrix} K_i;0 \\ N' \end{bmatrix}_{q_i}$ are then supposed to play the role of the elements h_i of $U(\mathfrak{g})$, and indeed we will show that this is the case by calculating their adjoint action.
First of all the relation $[\begin{bmatrix} K_i;0 \\ N' \end{bmatrix}_{q_i}, \begin{bmatrix} K_j;0 \\ N' \end{bmatrix}_{q_i}] = 0$ follows from the commutation relations between the K_i's.
We now need to show that

$$[\begin{bmatrix} K_i;0 \\ N' \end{bmatrix}_{q_i}, e_j^{(N')}] = a_{ij} e_j^{(N')} \tag{1.7}$$

We have:

$$[\begin{bmatrix} K_i;0 \\ N' \end{bmatrix}_{q_i}, e_j^{(N')}] = \begin{bmatrix} K_i;0 \\ N' \end{bmatrix}_{q_i} e_j^{(N')} - e_j^{(N')} \begin{bmatrix} K_i;0 \\ N' \end{bmatrix}_{q_i} = e_j^{(N')} (\begin{bmatrix} K_i;N'a_{ij} \\ N' \end{bmatrix}_{q_i} - \begin{bmatrix} K_i;0 \\ N' \end{bmatrix}_{q_i})$$

We will now rewrite the difference of formal elements $(\begin{bmatrix} K_i;N'a_{ij} \\ N' \end{bmatrix}_{q_i} - \begin{bmatrix} K_i;0 \\ N' \end{bmatrix}_{q_i})$ in the specialization and then we will pass to the quotient.
Thanks to Lemma 1.4.1 we have:

$$\begin{bmatrix} K_i;N'a_{ij} \\ N' \end{bmatrix}_{q_i} = \sum_{0 \leq s \leq N'} (-1)^s q_i^{-N'a_{ij}(N'-s)} \begin{bmatrix} -N'a_{ij}+s-1 \\ s \end{bmatrix}_{q_i} K_i^s \begin{bmatrix} K_i;0 \\ N'-s \end{bmatrix}_{q_i}$$

Now, Remark 8 allows to reduce the summation to the terms for $s = 0$ and $s = N'$, hence we have:

$$\begin{bmatrix} K_i;N'a_{ij} \\ N' \end{bmatrix}_{q_i} = q_i^{-N'^2 a_{ij}} \begin{bmatrix} K_i;0 \\ N' \end{bmatrix}_{q_i} + (-1)^{N'} (-a_{ij}) q_i^{N'(-N'a_{ij}-1)} K_i^{N'}$$

The difference of formal elements, passing to the quotient by the ideal generated by $K_i - 1$, is then

$$(\begin{bmatrix} K_i; N'a_{ij} \\ N' \end{bmatrix}_{q_i} - \begin{bmatrix} K_i; 0 \\ N' \end{bmatrix}_{q_i}) = (q_i^{-N'^2 a_{ij}} - 1)\begin{bmatrix} K_i; 0 \\ N' \end{bmatrix}_{q_i} + (-1)^{N'+1} a_{ij} q_i^{N'(-N'a_{ij}-1)}$$

so finally we get

$$[\begin{bmatrix} K_i; 0 \\ N' \end{bmatrix}_{q_i}, e_j^{(N')}] = ((q_i^{-N'^2 a_{ij}} - 1)\begin{bmatrix} K_i; 0 \\ N' \end{bmatrix}_{q_i} + (-1)^{N'+1} a_{ij} q_i^{N'(-N'a_{ij}-1)}) e_j^{(N')} \tag{1.8}$$

Being N odd (so $N = N'$) $(q_i^{-N'^2 a_{ij}} - 1)\left[\begin{smallmatrix} K_i;0 \\ N' \end{smallmatrix}\right]_{q_i} + (-1)^{N'+1} a_{ij} q_i^{N'(-N'a_{ij}-1)}$ reduces to a_{ij} because $q^{N'} = 1$ so (1.7) is satisfied.

In a similar way it is proved the relation for the $f_i^{(N')}$'s.

Now we want to show that the Serre relations hold; in order to do this, in the following computations we will use Lusztig's formula (1.3) with $m = N'(1 - a_{ij})$ and $n = N'$.
In this case the formula becomes

$$e_i^{(N'(1-a_{ij}))} e_j^{(N')} = \sum_{k=0}^{-N'a_{ij}} C_{N'(1-a_{ij})-k}(q_i) e_i^{(k)} e_j^{(N')} e_i^{(N'(1-a_{ij})-k)} \tag{1.9}$$

with

$$C_s(q_i) = \sum_{l=0}^{N'-1} (-1)^{s+l+1} q_i^{-s(l+1-N')+l} \begin{bmatrix} s \\ l \end{bmatrix}_{q_i}$$

Now we will evaluate formula (1.9) for $q^N = 1$.
Thanks to Lemma 1.4.2 we can use the substitution $k = N'n$ for some $n \in \mathbb{N}$ (the other coefficients are 0) in order to get:

$$\begin{aligned} & e_i^{(N'(1-a_{ij}))} e_j^{(N')} = \\ & \textstyle\sum_{k=0}^{-N'a_{ij}} C_{N'(1-a_{ij})-k}(q_i) e_i^{(k)} e_j^{(N')} e_i^{(N'(1-a_{ij})-k)} = \\ & \textstyle\sum_{n=0}^{-a_{ij}} C_{N'(1-a_{ij}-n)}(q_i) e_i^{(N'n)} e_j^{(N')} e_i^{(N'(1-a_{ij}-n)} = \\ & \textstyle\sum_{n=0}^{-a_{ij}} (-1)^{N'(1-a_{ij}-n)+1} q_i^{N'(1-a_{ij}-n)(N'-1)} e_i^{(N'n)} e_j^{(N')} e_i^{(N'(1-a_{ij}-n))} \end{aligned}$$

Now we apply Remark 9 to finish the computation:

$$\begin{aligned}&(e_i^{(N')})^{1-a_{ij}}e_j^{(N')} = \\ &\frac{(1-a_{ij})!}{q_i^{N'^2\frac{-a_{ij}(1-a_{ij})}{2}}}\sum_{n=0}^{-a_{ij}}(-1)^{N'(1-a_{ij}-n)+1}q_i^{N'(1-a_{ij}-n)(N'-1)}\cdot \\ &\cdot\frac{q_i^{N'^2\frac{(1-a_{ij}-n)(-a_{ij}-n)+n(n-1)}{2}}}{n!(1-a_{ij}-n)!}(e_i^{(N')})^n e_j^{(N')}(e_i^{(N')})^{1-a_{ij}-n} = \\ &\sum_{n=0}^{-a_{ij}}(-1)^{N'(1-a_{ij}-n)+1}q_i^{N'(1-a_{ij}-n)(N'-1)}q_i^{-nN'^2(1-a_{ij}-n)}\binom{1-a_{ij}}{n} \\ &(e_i^{(N')})^n e_j^{(N')}(e_i^{(N')})^{1-a_{ij}-n}\end{aligned} \tag{1.10}$$

We recall that the Serre relation of the non deformed enveloping algebra (up to a factor ± 1) is

$$\sum_{n=0}^{1-a_{ij}}(-1)^{(1-a_{ij}-n)+1}\binom{1-a_{ij}}{n}e_i^n e_j e_i^{1-a_{ij}-n} = 0 \tag{1.11}$$

Being N odd $(-1)^{N'(1-a_{ij}-n)+1}q_i^{N'(1-a_{ij}-n)(N'-1)}q_i^{-nN'^2(1-a_{ij}-n)}$ reduces to $(-1)^{(1-a_{ij}-n)+1}$ because $q^{N'}=1$ so relation (1.11) is satisfied.

A completely analogous computation allows to show the same result also for the Serre relation involving the $f_i^{(N')}$'s. □

Remark 10. We would like to point out that, in general, if N is even, relations (1.8) and (1.10) do not reduce to the correct ones. In particular as for relation (1.8) we note that

1. If N is even and N' is odd the relation is not satisfied.
2. If N is even and N' is even several situations may occur, for example:
   - if $q_i^{N'} = -1$ for all i (which happens if $\mathfrak{g}$ is simply-laced, i.e. all the real roots have the same length, or for some particular non simply-laced algebras) we get exactly a_{ij} because $q_i^{-N'^2 a_{ij}} = 1$ so the coefficient reduces to $(-1)^{N'+1}a_{ij}q_i^{-N'} = a_{ij}$;
   - if there exists i such that $q_i^{N'} = 1$ the relation is not satisfied.

As for relation (1.10) we note that:

1. If N is even and N' is odd then the coefficient reduces to $(-1)^{(n+1)(1-a_{ij}-n)+1}$ if $q_i^{N'} = -1$ and to $(-1)^{(1-a_{ij}-n)+1}$ if $q_i^{N'} = 1$, so in general the Serre relation is not satisfied and depends on the parity of the elements of the Cartan matrix. The condition is verified for example in the case of the affine algebra $A_1^{(1)}$.

2. If N is even and N' is even there are two possibilities:

   - if $q_i^{N'} = -1$ for all i an easy computation shows that we get the correct coefficient;
   - if there exists i such that $q_i^{N'} = 1$ the relation is no more satisfied.

1.5 Shift-invariance of the non-deformed enveloping algebra $U(\mathfrak{g})$

In this section we will show that the action of $U(\mathfrak{g})$ over the L-fold tensor product of a vector space V commutes with the action of the shift operator. Let us first give the definition of the so-called shift operator.

Definition 1.15. Let V be a vector space and $L \in \mathbb{N}^+$.
The operator P defined by:

$$P : V^{\otimes L} \to V^{\otimes L}$$
$$(v_1, v_2, ..., v_L) \mapsto (v_2, ..., v_L, v_1)$$

is called shift operator.

The first step in order to prove the shift invariance of $U(\mathfrak{g})$ is to extend a given representation of $U_q(\mathfrak{g})$ over a $\mathbb{C}$-vector space V to a representation over the L-fold tensor product $V^{\otimes L}$. Then, if we consider the restriction of this representation to the algebra $U(\mathfrak{g})$ constructed in the previous section we can prove that the two actions commute. The representation of the algebra over $V^{\otimes L}$ can not be obtained by the tensor product of the given representation L times because this would not be an algebra homomorphism. We can, however, obtain it in a natural way endowing $U_q(\mathfrak{g})$ with a Hopf algebra structure (in fact it is enough to have a bialgebra structure) and using the coproduct, which is an algebra homomorphism in order to associate an element of $U_q(\mathfrak{g})^{\otimes L}$ to an element of $U_q(\mathfrak{g})$. We will describe this construction in detail.

1.5.1 Hopf algebra structure of $U_q(\mathfrak{g})$

In this subsection we want to endow $\mathrm{U_q}(\mathfrak{g})$ with a Hopf algebra structure, so let us first recall the definition of Hopf algebra.

Definition 1.16. Let H be a vector space over the field $\mathbb{K}$. Consider two linear maps $\Delta : H \longrightarrow H \otimes H$ and $\varepsilon : H \longrightarrow \mathbb{K}$.
If the following diagrams commute we say that Δ is coassociative (first diagram) and counital (second diagram).

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle id\otimes\Delta} \\ H \otimes H & \xrightarrow{\Delta\otimes id} & H \otimes H \otimes H \end{array}$$

$$\begin{array}{ccccc} \mathbb{K} \otimes H & \xleftarrow{\varepsilon\otimes id} & H \otimes H & \xrightarrow{id\otimes\varepsilon} & H \otimes \mathbb{K} \\ & {\scriptstyle \cong}\nwarrow & \uparrow{\scriptstyle \Delta} & \nearrow{\scriptstyle \cong} & \\ & & H & & \end{array}$$

Under these hypothesis the triple (H, Δ, ε) is said to be a coalgebra, the map Δ is called comultiplication and the map ε counit.

Definition 1.17. Let H be a vector space over the field $\mathbb{K}$.
Suppose that H is endowed with an algebra structure given by the multiplication μ and the unit η and with a coalgebra structure given by the comultiplication Δ and the counit ε.

If the two structures are compatible, i.e. Δ and ε are morphism of algebras, then $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra.

Definition 1.18. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra.
An endomorphism $S : H \longrightarrow H$ is called an antipode for H if

$$S \star id = id \star S = \eta \circ \varepsilon$$

where the convolution $\star$ of two bialgebra endomorphisms f and g is defined by: $f \star g := \mu \circ (f \otimes g) \circ \Delta$

Definition 1.19. A bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ endowed with an antipode S is called Hopf algebra and it is denoted by $(H, \mu, \eta, \Delta, \varepsilon, S)$.

It is possible to endow $U_q(\mathfrak{g})$ with various comultiplication maps and antipodal maps in order to obtain a Hopf algebra structure. The more convenient maps for our computations are introduced in the following proposition and in order to define them it is necessary to think of $U_q(\mathfrak{g})$ as a subalgebra of a bigger algebra $U'_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm\frac{1}{2}}, K_i^{\pm 1}$ with the usual relations and such that

1. $(K_i^{\pm\frac{1}{2}})^2 = K_i^{\pm 1}$;
2. $[K_i^{\frac{1}{2}}, K_j^{\frac{1}{2}}] = 0$;
3. $K_i^{\frac{1}{2}} K_i^{-\frac{1}{2}} = K_i^{-\frac{1}{2}} K_i^{\frac{1}{2}} = 1$;
4. $K_i^{\frac{1}{2}} e_j K_i^{-\frac{1}{2}} = q_i^{\frac{a_{ij}}{2}} e_j$;
5. $K_i^{\frac{1}{2}} f_j K_i^{-\frac{1}{2}} = q_i^{-\frac{a_{ij}}{2}} f_j$.

The two set of relations are compatible and in fact redundant.

Proposition 1.5.1. *The algebra $U'_q(\mathfrak{g})$ endowed with:*

1. *the comultiplication* $\Delta : U'_q(\mathfrak{g}) \to U'_q(\mathfrak{g}) \otimes U'_q(\mathfrak{g})$
   $\Delta(K_i^{\frac{1}{2}}) = K_i^{\frac{1}{2}} \otimes K_i^{\frac{1}{2}}$
   $\Delta(e_i) = e_i \otimes K_i^{-\frac{1}{2}} + K_i^{\frac{1}{2}} \otimes e_i$
   $\Delta(f_i) = f_i \otimes K_i^{-\frac{1}{2}} + K_i^{\frac{1}{2}} \otimes f_i$
   $\Delta(1) = 1 \otimes 1$

2. *the counit* $\varepsilon : U'_q(\mathfrak{g}) \to \mathbb{Q}(q)$
   $\varepsilon(K_i^{\frac{1}{2}}) = 1 \qquad \varepsilon(e_i) = \varepsilon(f_i) = 0$

3. *the antipode* $S : U'_q(\mathfrak{g}) \to U'_q(\mathfrak{g})$
   $S(K_i^{\frac{1}{2}}) = K_i^{-\frac{1}{2}}$
   $S(e_i) = -q_i^{-1} e_i$
   $S(f_i) = -q_i f_i$

is a Hopf algebra.

Before starting the proof of the proposition we present a useful lemma.

Lemma 1.5.2. *For $n \in \mathbb{N}$ we have:*

$$\Delta(e_i^n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} e_i^k K_i^{\frac{n-k}{2}} \otimes e_i^{n-k} K_i^{-\frac{k}{2}}$$

Proof. The proof is an application of the q-binomial formula A.0.5.
We can write $\Delta(e_i^n) = \Delta(e_i)^n = (e_i \otimes K_i^{-\frac{1}{2}} + K_i^{\frac{1}{2}} \otimes e_i)^n$.
Now we note that $(K_i^{\frac{1}{2}} \otimes e_i)(e_i \otimes K_i^{-\frac{1}{2}}) = K_i^{\frac{1}{2}} e_i \otimes e_i K_i^{-\frac{1}{2}} =$
$= q_i e_i K_i^{\frac{1}{2}} \otimes q_i K_i^{-\frac{1}{2}} e_i = q_i^2 (e_i \otimes K_i^{-\frac{1}{2}})(K_i^{\frac{1}{2}} \otimes e_i)$ therefore, by Proposition A.0.5, we get:

$$\begin{aligned} (e_i \otimes K_i^{-\frac{1}{2}} + K_i^{\frac{1}{2}} \otimes e_i)^n &= \textstyle\sum_{k=0}^n q_i^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} (e_i^k \otimes K_i^{-\frac{k}{2}})(K_i^{\frac{n-k}{2}} \otimes e_i^{n-k}) \\ &= \textstyle\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} q_i^{k(n-k)} e_i^k K_i^{\frac{n-k}{2}} \otimes K_i^{-\frac{k}{2}} e_i^{n-k} \\ &= \textstyle\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} q_i^{k(n-k)} e_i^k K_i^{\frac{n-k}{2}} \otimes q_i^{-k(n-k)} e_i^{n-k} K_i^{-\frac{k}{2}} \\ &= \textstyle\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} e_i^k K_i^{\frac{n-k}{2}} \otimes e_i^{n-k} K_i^{-\frac{k}{2}} \end{aligned}$$

□

Proof of Proposition 1.5.1. The proof consists in verifying the defining properties of a Hopf algebra. We begin by checking that the comultiplication is coassociative.

On the generators we have:

$$\begin{aligned}(id\otimes\Delta)\Delta(K_i^{\frac12}) &= K_i^{\frac12}\otimes K_i^{\frac12}\otimes K_i^{\frac12} = (\Delta\otimes id)\Delta(K_i^{\frac12})\\ (id\otimes\Delta)\Delta(e_i) &= e_i\otimes K_i^{-\frac12}\otimes K_i^{-\frac12} + K_i^{\frac12}\otimes e_i\otimes K_i^{-\frac12} + K_i^{\frac12}\otimes K_i^{\frac12}\otimes e_i\\ &= (\Delta\otimes id)\Delta(e_i)\\ (id\otimes\Delta)\Delta(f_i) &= f_i\otimes K_i^{-\frac12}\otimes K_i^{-\frac12} + K_i^{\frac12}\otimes f_i\otimes K_i^{-\frac12} + K_i^{\frac12}\otimes K_i^{\frac12}\otimes f_i\\ &= (\Delta\otimes id)\Delta(f_i)\end{aligned}$$

Obviously the counitary property is verified.
To prove the compatibility between the algebra and the coalgebra structure we have to show that Δ can be extended to an algebra homomorphism. In particular this means that the relations of the algebra must be satisfied. We have immediately
$[\Delta(K_i^{\frac12}),\Delta(K_j^{\frac12})] = [K_i^{\frac12}\otimes K_i^{\frac12}, K_j^{\frac12}\otimes K_j^{\frac12}] = [K_i^{\frac12},K_j^{\frac12}]\otimes[K_i^{\frac12},K_j^{\frac12}] = 0$
$\Delta(K_i^{\frac12})\Delta(K_i^{-\frac12}) = K_i^{\frac12}K_i^{-\frac12}\otimes K_i^{\frac12}K_i^{-\frac12} = 1\otimes 1$
$\Delta(K_i^{\frac12})\Delta(e_j)\Delta(K_i^{-\frac12}) = K_i^{\frac12}e_jK_i^{-\frac12}\otimes K_j^{-\frac12} + K_j^{\frac12}\otimes K_i^{\frac12}e_jK_i^{-\frac12} = q_i^{a_{ij}}\Delta(e_j)$
and in a similar way we have $\Delta(K_i^{\frac12})\Delta(f_j)\Delta(K_i^{-\frac12}) = q_i^{-a_{ij}}\Delta(f_j)$
Furthermore

$$\begin{aligned}[\Delta(e_i),\Delta(f_j)] &= [e_i\otimes K_i^{-\frac12} + K_i^{\frac12}\otimes e_i, f_j\otimes K_j^{-\frac12} + K_j^{\frac12}\otimes f_j]\\ &= [e_i,f_j]\otimes K_i^{-\frac12}K_j^{-\frac12} + K_i^{\frac12}K_j^{\frac12}\otimes[e_i,f_j]\\ &= \delta_{ij}\left(\tfrac{K_i-K_i^{-1}}{q_i-q_i^{-1}}\otimes K_i^{-1} + K_i\otimes\tfrac{K_i-K_i^{-1}}{q_i-q_i^{-1}}\right)\\ &= \delta_{ij}\left(\tfrac{K_i\otimes K_i}{q_i-q_i^{-1}} - \tfrac{K_i^{-1}\otimes K_i^{-1}}{q_i-q_i^{-1}}\right)\\ &= \delta_{ij}\tfrac{\Delta(K_i)-\Delta(K_i^{-1})}{q_i-q_i^{-1}}\end{aligned}$$

It remains to prove that Δ preserves the Serre relations.
Applying Δ to the first relation it becomes

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}\Delta(e_i)^k\Delta(e_j)\Delta(e_i)^{1-a_{ij}-k}$$

Using Lemma 1.5.2 we get

$$\sum_{k=0}^{1-a_{ij}}\sum_{t=0}^{k}\sum_{s=0}^{1-a_{ij}-k}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}\begin{bmatrix}k\\ t\end{bmatrix}_{q_i}\begin{bmatrix}1-a_{ij}-k\\ s\end{bmatrix}_{q_i}$$
$$(e_i^tK_i^{\frac{k-t}{2}}\otimes e_i^{k-t}K_i^{-\frac{t}{2}})(e_j\otimes K_j^{-\frac12} + K_j^{-\frac12}\otimes e_j)(e_i^sK_i^{\frac{1-a_{ij}-k-s}{2}}\otimes e_i^{1-a_{ij}-k-s}K_i^{-\frac{s}{2}})$$

This can be written as

$$\sum_{k=0}^{1-a_{ij}} \sum_{t=0}^{k} \sum_{s=0}^{1-a_{ij}-k} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} \begin{bmatrix} k \\ t \end{bmatrix}_{q_i} \begin{bmatrix} 1-a_{ij}-k \\ s \end{bmatrix}_{q_i} (\Omega_1 + \Omega_2)$$

where

$$\Omega_1 = e_i^t K_i^{\frac{k-t}{2}} e_j e_i^s K_i^{\frac{1-a_{ij}-k-s}{2}} \otimes e_i^{k-t} K_i^{-\frac{t}{2}} K_j^{-\frac{1}{2}} e_i^{1-a_{ij}-k-s} K_i^{-\frac{s}{2}}$$

and

$$\Omega_2 = e_i^t K_i^{\frac{k-t}{2}} K_j^{\frac{1}{2}} e_i^s K_i^{\frac{1-a_{ij}-k-s}{2}} \otimes e_i^{k-t} K_i^{-\frac{t}{2}} e_j e_i^{1-a_{ij}-k-s} K_i^{-\frac{s}{2}}$$

Now we show that

$$\sum_{k=0}^{1-a_{ij}} \sum_{t=0}^{k} \sum_{s=0}^{1-a_{ij}-k} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} \begin{bmatrix} k \\ t \end{bmatrix}_{q_i} \begin{bmatrix} 1-a_{ij}-k \\ s \end{bmatrix}_{q_i} \Omega_1 = 0 \qquad (1.12)$$

First of all we note that Ω_1 can be written in a more compact way thanks to the relation $K_i^{\frac{1}{2}} e_j K_i^{-\frac{1}{2}} = q_i^{\frac{a_{ij}}{2}} e_j$

$\Omega_1 = q_i^{\frac{a_{ij}(k-t)}{2}+s(k-t)-\frac{a_{ij}}{2}(1-a_{ij}-k-s)-t(1-a_{ij}-k-s)} e_i^t e_j e_i^s K_i^{\frac{1-a_{ij}-t-s}{2}} \otimes e_i^{1-a_{ij}-t-s} K_i^{-\frac{t+s}{2}} K_j^{-\frac{1}{2}}$

If we set $m = 1 - a_{ij} - t - s$ and $p = k - t$ we get

$$\begin{aligned} \Omega_1 &= q_i^{\frac{pa_{ij}}{2}+(1-a_{ij}-m-t)p-\frac{a_{ij}}{2}(m-p)-t(m-p)} e_i^t e_j e_i^{1-a_{ij}-m-t} K_i^{\frac{m}{2}} \otimes e_i^m K_i^{m-1+a_{ij}} K_j^{-\frac{1}{2}} \\ &= q_i^{-p(m-1)-\frac{a_{ij}}{2}m-tm} e_i^t e_j e_i^{1-a_{ij}-m-t} K_i^{\frac{m}{2}} \otimes e_i^m K_i^{m-1+a_{ij}} K_j^{-\frac{1}{2}} \end{aligned}$$

hence (1.12) becomes

$$\begin{aligned} \textstyle\sum_{m=0}^{1-a_{ij}} \sum_{t=0}^{1-a_{ij}-m} \sum_{p=0}^{m} \quad & (-1)^{p+t} \left[\begin{smallmatrix} 1-a_{ij} \\ p+t \end{smallmatrix}\right]_{q_i} \left[\begin{smallmatrix} p+t \\ t \end{smallmatrix}\right]_{q_i} \left[\begin{smallmatrix} 1-a_{ij}-p-t \\ 1-a_{ij}-m-t \end{smallmatrix}\right]_{q_i} \\ & q_i^{-p(m-1)} q_i^{-\frac{a_{ij}}{2}m-tm} e_i^t e_j e_i^{1-a_{ij}-m-t} K_i^{\frac{m}{2}} \otimes e_i^m K_i^{m-1+a_{ij}} K_j^{-\frac{1}{2}} \end{aligned}$$

Now we note that the product of q-binomials can be written as follows:

$$\begin{bmatrix} 1-a_{ij} \\ p+t \end{bmatrix}_{q_i} \begin{bmatrix} p+t \\ t \end{bmatrix}_{q_i} \begin{bmatrix} 1-a_{ij}-p-t \\ 1-a_{ij}-m-t \end{bmatrix}_{q_i} = \begin{bmatrix} 1-a_{ij} \\ m+t \end{bmatrix}_{q_i} \begin{bmatrix} m+t \\ t \end{bmatrix}_{q_i} \begin{bmatrix} m \\ p \end{bmatrix}_{q_i}$$

thus we get:

$$\begin{aligned} \textstyle\sum_{m=0}^{1-a_{ij}} \sum_{t=0}^{1-a_{ij}-m} \quad & (\textstyle\sum_{p=0}^{m} (-1)^p \left[\begin{smallmatrix} m \\ p \end{smallmatrix}\right]_{q_i} q_i^{-p(m-1)})(-1)^t \left[\begin{smallmatrix} 1-a_{ij} \\ m+t \end{smallmatrix}\right]_{q_i} \left[\begin{smallmatrix} m+t \\ t \end{smallmatrix}\right]_{q_i} \\ & q_i^{-\frac{a_{ij}}{2}m-tm} e_i^t e_j e_i^{1-a_{ij}-m-t} K_i^{\frac{m}{2}} \otimes e_i^m K_i^{m-1+a_{ij}} K_j^{-\frac{1}{2}} \end{aligned}$$

Now, thanks to Proposition A.0.6, we have that $\sum_{p=0}^{m}(-1)^p \begin{bmatrix} m \\ p \end{bmatrix}_{q_i} q_i^{-p(m-1)} = 0$ for $m > 0$, so in this case the coefficient is always 0. It remains only to prove that we get 0 also for $m = 0$; in this case we have

$$\sum_{t=0}^{1-a_{ij}}(-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{q_i} e_i^t e_j e_i^{1-a_{ij}-t} \otimes K_i^{-1+a_{ij}} K_j^{-\frac{1}{2}}$$

which is 0 because on the left of the tensor product we have exactly the Serre relation.

In a completely analogous way one can prove that $\sum_{k=0}^{1-a_{ij}} \sum_{t=0}^{k} \sum_{s=0}^{1-a_{ij}-k}(-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} \begin{bmatrix} k \\ t \end{bmatrix}_{q_i} \begin{bmatrix} 1-a_{ij}-k \\ s \end{bmatrix}_{q_i} \Omega_2 = 0$.

By similar computations one can show that the relation involving the f_i's is preserved, hence we conclude that Δ is an algebra homomorphism. It is quite immediate to prove that ε is a morphism of algebras and also to show that the antipodal map S is an antihomomorphism. Let us make the explicit computation only for two relations.

We have

$S([e_i, f_i]) = S(\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}) = \frac{K_i^{-1} - K_i}{q_i - q_i^{-1}} = [f_i, e_i] = S(f_i)S(e_i) - S(e_i)S(f_i)$

and

$$\begin{aligned}
&\sum_{k=0}^{1-a_{ij}}(-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} S(e_i^k e_j e_i^{1-a_{ij}-k}) = \\
&\sum_{k=0}^{1-a_{ij}}(-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} S(e_i)^{1-a_{ij}-k} S(e_j) S(e_i)^k = \\
&\sum_{k=0}^{1-a_{ij}}(-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} (-1)^{2-a_{ij}} q_i^{-(2-a_{ij})} e_i^{1-a_{ij}-k} e_j e_i^k = \\
&\sum_{t=0}^{1-a_{ij}}(-1)^{1-a_{ij}-t} \begin{bmatrix} 1-a_{ij} \\ 1-a_{ij}-t \end{bmatrix}_{q_i} (-1)^{2-a_{ij}} q_i^{-(2-a_{ij})} e_i^t e_j e_i^{1-a_{ij}-t} = \\
&-\sum_{t=0}^{1-a_{ij}}(-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{q_i} q_i^{-(2-a_{ij})} e_i^t e_j e_i^{1-a_{ij}-t} = 0
\end{aligned}$$

Now we only need to prove the properties of convolution of the antipode, in particular we need to prove that $S \star id = id \star S = \eta \circ \varepsilon$ where η is the unity of the algebra. On the generators we have:

$(S \star id)(K_i^{\frac{1}{2}}) = S(K_i^{\frac{1}{2}})K_i^{\frac{1}{2}} = 1 = \eta(1) = \eta(\varepsilon(K_i^{\frac{1}{2}}))$

$(S \star id)(e_i) = S(e_i)K_i^{-\frac{1}{2}} + S(K_i^{\frac{1}{2}})e_i = -q_i^{-1} e_i K_i^{-\frac{1}{2}} + K_i^{-\frac{1}{2}} e_i = 0$ remembering that $K_i^{\frac{1}{2}} e_i K_i^{-\frac{1}{2}} = q_i e_i$

$(S \star id)(f_i) = S(f_i)K_i^{-\frac{1}{2}} + S(K_i^{\frac{1}{2}})f_i = -q_i f_i K_i^{-\frac{1}{2}} + K_i^{-\frac{1}{2}} f_i = 0$ remembering that $K_i^{\frac{1}{2}} f_i K_i^{-\frac{1}{2}} = q_i^{-1} f_i$

The proof is the same for $id \star S$.

We can thus conclude that $(U'(\mathfrak{g}), \mu, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra. □

Remark 11. In the following we will always refer to $U_q(\mathfrak{g})$ as a Hopf algebra and we will consider its representations, although due to the given definition, this is a little improper and we should formally work with $U_q'(\mathfrak{g})$. This enlargement can be avoided by defining a different comultiplication (and consequently a different antipodal map). A more classical definition for the comultiplication, which does not require any extension of $U_q(\mathfrak{g})$, is:
$\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$
$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(1) = 1 \otimes 1$
$\Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i^{-1} \otimes f_i$
This possible structure of Hopf algebra has been introduced only for completeness and it will never be used in the thesis.

1.5.2 From a representation over V to a representation over $V^{\otimes L}$

We can now use the Hopf algebra structure of $\mathrm{U_q}(\mathfrak{g})$, in particular its comultiplication, in order to extend a representation of $\mathrm{U_q}(\mathfrak{g})$ over a vector space V to a representation over $V^{\otimes L}$.

Definition 1.20. Consider the Hopf algebra $U_q(\mathfrak{g})$. For each $L \geq 2$ we define recursively the algebra homomorphism $\Delta^{(L)} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})^{\otimes L}$ by:

$$\Delta^{(L)} = (\Delta \otimes id_{L-2})\Delta^{(L-1)} \quad \text{with} \quad \Delta^{(2)} := \Delta$$

Notation 3. We now introduce Sweedler notation which we will use throughout the thesis. Consider the coalgebra (H, Δ, ε) and let $x \in H$. The element $\Delta(x) \in H \otimes H$ is usually written as follows:

$$\Delta(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)}$$

In order to simplify the notation we remove the subscript and agree to write

$$\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}$$

Using this notation, the coassociativity of Δ (i.e. the commutativity of the first diagram in Definition 1.16) is expressed by the following equality:

$$\sum_{(x)}(\sum_{(x^{(1)})}(x^{(1)})^{(1)}\otimes(x^{(1)})^{(2)})\otimes x^{(2)}=\sum_{(x)}x^{(1)}\otimes(\sum_{(x^{(2)})}(x^{(2)})^{(1)}\otimes(x^{(2)})^{(2)}) \quad (1.13)$$

By convention, we identify both sides of (1.13) with

$$\sum_{(x)}x^{(1)}\otimes x^{(2)}\otimes x^{(3)}.$$

When there is no matter of confusion, we will write simply $\sum x^{(1)}\otimes x^{(2)}\otimes x^{(3)}$. This process can be generalized to $\Delta^{(L)}, L\in\mathbb{N}, L\geq 2$ for which we have the following equality:

$$\Delta^{(L)}(x)=\sum x^{(1)}\otimes\ldots\otimes x^{(L-1)}\otimes x^{(L)}.$$

Now let us consider a representation $(U_q(\mathfrak{g}),\rho_V)$ over V. Using the Sweedler notation, we can define the following representation over $V^{\otimes L}$:
for $v_1\otimes v_2\otimes\ldots\otimes v_L\in V^{\otimes L}$ and $x\in U_q(\mathfrak{g})$

$$\begin{aligned} x.(v_1\otimes v_2\otimes\ldots\otimes v_L) &= \Delta^{(L)}(x).(v_1\otimes v_2\otimes\ldots\otimes v_L)\\ &= \sum(x^{(1)}\otimes\ldots\otimes x^{(L)}).(v_1\otimes\ldots\otimes v_L)\\ &= \sum\rho_V(x^{(1)})(v_1)\otimes\ldots\otimes\rho_V(x^{(L)})(v_L)\end{aligned}$$

In order to describe the action of the generators of the non deformed algebra over $V^{\otimes L}$ we prove a preliminary result about the action of e_i, f_i and K_i.

Lemma 1.5.3. *Let $L\in\mathbb{N}, L\geq 2$. In $\mathrm{U_q}(\mathfrak{g})^{\otimes L}$, for $1\leq n\leq L$ we set:*

- $K_i(n) := 1\otimes\ldots\otimes 1\otimes \underset{n\text{-th place}}{K_i}\otimes 1\otimes\ldots\otimes 1;$
- $E_i(n) := K_i^{\frac{1}{2}}\otimes\ldots\otimes K_i^{\frac{1}{2}}\otimes \underset{n\text{-th place}}{e_i}\otimes K_i^{-\frac{1}{2}}\otimes\ldots\otimes K_i^{-\frac{1}{2}};$
- $F_i(n) := K_i^{\frac{1}{2}}\otimes\ldots\otimes K_i^{\frac{1}{2}}\otimes \underset{n\text{-th place}}{f_i}\otimes K_i^{-\frac{1}{2}}\otimes\ldots\otimes K_i^{-\frac{1}{2}}.$

Then we have:

- $\Delta^{(L)}(K_i) = \bar{K}_i = \prod_{n=1}^{L} K_i(n)$;
- $\Delta^{(L)}(e_i) = E_i = \sum_{n=1}^{L} E_i(n)$;
- $\Delta^{(L)}(f_i) = F_i = \sum_{n=1}^{L} F_i(n)$.

Proof. The first relation is immediate because $\Delta(K_i) = K_i \otimes K_i$. The second is easily obtained by induction on $L \geq 2$; if $L = 2$ it is clearly true by definition of comultiplication. Assuming it is true for $L-1$ we have
$\Delta^{(L)}(e_i) = (\Delta \otimes id_{L-2})\Delta^{(L-1)}(e_i) = (\Delta \otimes id_{L-2}) \sum_{n=1}^{L-1} E_i(n) = \Delta(e_i) \otimes K_i^{-\frac{1}{2}} \otimes \ldots \otimes K_i^{-\frac{1}{2}} + \sum_{n=3}^{L} E_i(n) = \sum_{n=1}^{L} E_i(n)$
The third relation can be proved in the same way. □

The following proposition can be proved by induction on L using Lemma 1.5.2:

Proposition 1.5.4. *The following relation holds:*

$$\Delta^{(L)}(e_i^{(N)}) = E_i^{(N)} = \sum_{0=n_0 \leq n_1 \ldots \leq n_L = N} \bigotimes_{l=1}^{L} e_i^{(n_l - n_{l-1})} K_i^{\frac{N-n_l-n_{l-1}}{2}}$$

for $L \in \mathbb{N}$, $L \geq 2$ and $N \in \mathbb{N}$.

A similar formula describes the action of the $f_i^{(N)}$'s.

1.5.3 Shift-invariance

In this subsection we will prove the invariance of $U(\mathfrak{g})$ with respect to the shift operator. Before stating this result we introduce two technical lemmas which will be used in its proof.

Lemma 1.5.5. *Let $L \in \mathbb{N}, L \geq 2$, $n \in \mathbb{N}$, $1 \leq n \leq L$. We have:*

1. $PE_i(n)P^{-1} = E_i(n-1)K_i(L)$ *if* $n > 1$
   $PE_i(1)P^{-1} = E_i(L)\prod_{j=1}^{L-1} K_i^{-1}(j)$ *if* $n = 1$
2. $PE_iP^{-1} = E_iK_i(L) + E_i(L)(\bar{K}_i^{-1} - 1)K_i(L)$

Proof. The first relation is immediate: let $v_1 \otimes v_2 \otimes ... \otimes v_L \in V^{\otimes L}$. For $n > 1$ we have:
$PE_i(n)P^{-1}(v_1 \otimes v_2 \otimes ... \otimes v_L) = P(K_i^{\frac{1}{2}} v_L \otimes K_i^{\frac{1}{2}} v_1 \otimes ... \otimes e_i v_{n-1} \otimes ... \otimes K_i^{-\frac{1}{2}} v_{L-1}) =$
$K_i^{\frac{1}{2}} v_1 \otimes ... \otimes e_i v_{n-1} \otimes ... \otimes K_i^{-\frac{1}{2}} v_{L-1} \otimes K_i^{\frac{1}{2}} v_L = E_i(n-1)K_i(L)$
The case $n = 1$ follows from similar computations.
We will now deduce the second relation from the first.

$$\begin{aligned} PE_iP^{-1} &= \sum_{n=1}^{L} PE_i(n)P^{-1} \\ &= \sum_{n=1}^{L-1} E_i(n)K_i(L) + E_i(L) \prod_{j=1}^{L-1} K_i^{-1}(j) \\ &= E_iK_i(L) + E_i(L) \prod_{j=1}^{L-1} K_i^{-1}(j) - E_i(L)K_i(L) \\ &= E_iK_i(L) + E_i(L)((\prod_{j=1}^{L-1} K_i^{-1}(j))K_i^{-1}(L) - 1)K_i(L) \\ &= E_iK_i(L) + E_i(L)(\bar{K}_i^{-1} - 1)K_i(L) \end{aligned}$$

□

We shall now extend Lemma 1.5.5 to the powers of E_i:

Lemma 1.5.6. *Let $m \in \mathbb{N}^+$.*
We have

$$PE_i^m P^{-1} = \sum_{n=0}^{m} E_i^{m-n} E_i(L)^n q_i^{n(m-1)} \begin{bmatrix} m \\ n \end{bmatrix}_{q_i} K_i(L)^m \prod_{l=0}^{n-1} (q_i^{-2l} \bar{K}_i^{-1} - 1)$$

where the product is meant to be equal to 1 if $n = 0$.

Proof. The proof is by induction on $m \geq 1$. If $m = 1$, we get formula 2. in Lemma 1.5.5.
Let us now suppose that it is true for m and let us prove it for $m + 1$.

$$\begin{aligned} PE_i^{m+1}P^{-1} &= PE_i^m P^{-1} PE_i P^{-1} \\ &= (\sum_{n=0}^{m} E_i^{m-n} E_i(L)^n q_i^{n(m-1)} \begin{bmatrix} m \\ n \end{bmatrix}_{q_i} K_i(L)^m \prod_{l=0}^{n-1} (q_i^{-2l} \bar{K}_i^{-1} - 1)) \\ &\quad (E_iK_i(L) + E_i(L)(\bar{K}_i^{-1} - 1)K_i(L)) \end{aligned} \tag{1.14}$$

Note that
$(\bar{K}_i^{-1} - 1)E_i(j) = \bar{K}_i^{-1} E_i(j) - E_i(j) = E_i(j)(q_i^{-2} \bar{K}_i^{-1} - 1)$ hence

$(q_i^{-2l}\bar{K}_i^{-1} - 1)E_i(j) = E_i(j)(q_i^{-2(l+1)}\bar{K}_i^{-1} - 1)$
From this we get:

$$\prod_{l=0}^{n-1}(q_i^{-2l}\bar{K}_i^{-1} - 1)E_i = E_i\prod_{l=0}^{n-1}(q_i^{-2(l+1)}\bar{K}_i^{-1} - 1) = E_i\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1) \quad (1.15)$$

Besides, we have:

$$\begin{aligned} K_i(L)^m E_i &= (E_i(1) + ... + E_i(L-1))K_i(L)^m + q_i^{2m}E_i(L)K_i(L)^m \\ &= E_iK_i(L)^m + (q_i^{2m} - 1)E_i(L)K_i(L)^m \end{aligned} \quad (1.16)$$

and

$$E_i(L)^nE_i = q_i^{2n}E_iE_i(L)^n + E_i(L)^{n+1}(1 - q_i^{2n}) \quad (1.17)$$

Now, applying (1.15), (1.16) and (1.17) to (1.14) we get

$$\begin{aligned}
&PE_i^{m+1}P^{-1} = \\
&\textstyle\sum_{n=0}^{m} E_i^{m-n}E_i(L)^nq_i^{n(m-1)}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}[E_iK_i(L)^m + (q_i^{2m} - 1)E_i(L)K_i(L)^m]K_i(L) \\
&\textstyle\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1)+ \\
&\textstyle\sum_{n=0}^{m} E_i^{m-n}E_i(L)^{n+1}q_i^{n(m-1)}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}q_i^{2m}(\bar{K}_i^{-1} - 1)K_i(L)^{m+1}\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1) = \\
&\textstyle\sum_{n=0}^{m} E_i^{m-n}E_i(L)^nE_iq_i^{n(m-1)}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}K_i(L)^{m+1}\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1)+ \\
&\textstyle\sum_{n=0}^{m} E_i^{m-n}E_i(L)^{n+1}q_i^{n(m-1)}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}(q_i^{2m} - 1)K_i(L)^{m+1}\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1)+ \\
&\textstyle\sum_{n=0}^{m} E_i^{m-n}E_i(L)^{n+1}q_i^{n(m-1)}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}q_i^{2m}K_i(L)^{m+1}(\bar{K}_i^{-1} - 1)\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1) = \\
&\textstyle\sum_{n=0}^{m} E_i^{m-n}E_i(L)^nE_iq_i^{n(m-1)}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}K_i(L)^{m+1}\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1)+ \\
&\textstyle\sum_{n=0}^{m} E_i^{m-n}E_i(L)^{n+1}q_i^{n(m-1)}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}(q_i^{2m}\bar{K}_i^{-1} - 1)K_i(L)^{m+1}\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1) = \\
&\textstyle\sum_{n=0}^{m} E_i^{(m+1)-n}E_i(L)^nq_i^{2n}q_i^{n(m-1)}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}K_i(L)^{m+1}\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1)+ \\
&\textstyle\sum_{n=0}^{m} E_i^{m-n}E_i(L)^{n+1}(1 - q_i^{2n})q_i^{n(m-1)}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}K_i(L)^{m+1}\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1)+ \\
&\textstyle\sum_{n=0}^{m} E_i^{m-n}E_i(L)^{n+1}q_i^{n(m-1)}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}(q_i^{2m}\bar{K}_i^{-1} - 1)K_i(L)^{m+1}\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1) = \\
&\textstyle\sum_{n=0}^{m} E_i^{(m+1)-n}E_i(L)^nq_i^{nm}q_i^{n}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}K_i(L)^{m+1}\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1)+ \\
&\textstyle\sum_{n=0}^{m} E_i^{m-n}E_i(L)^{n+1}(q_i^{2m}\bar{K}_i^{-1} - q_i^{2n})q_i^{n(m-1)}\begin{bmatrix} m \\ n\end{bmatrix}_{q_i}K_i(L)^{m+1}\prod_{l=1}^{n}(q_i^{-2l}\bar{K}_i^{-1} - 1) =
\end{aligned}$$

$$\sum_{n=0}^{m} E_i^{(m+1)-n} E_i(L)^n q_i^{nm} q_i^n \left[{m \atop n}\right]_{q_i} K_i(L)^{m+1} \prod_{l=1}^{n} (q_i^{-2l} \bar{K}_i^{-1} - 1) +$$
$$\sum_{n=1}^{m+1} E_i^{(m+1)-n} E_i(L)^n (q_i^{2m} \bar{K}_i^{-1} - q_i^{2(n-1)}) q_i^{(n-1)(m-1)} \left[{m \atop n-1}\right]_{q_i} K_i(L)^{m+1}$$
$$\prod_{l=1}^{n-1} (q_i^{-2l} \bar{K}_i^{-1} - 1) =$$

$$E_i^{m+1} K_i(L)^{m+1} + E_i(L)^{m+1} q_i^{m(m-1)} K_i(L)^{m+1} \prod_{l=0}^{n-1} (q_i^{-2l} \bar{K}_i^{-1} - 1) +$$
$$\sum_{n=1}^{m} E_i^{(m+1)-n} E_i(L)^n q_i^{nm} \Gamma K_i(L)^{m+1} \prod_{l=1}^{n-1} (q_i^{-2l} \bar{K}_i^{-1} - 1)$$

where in the last equality we set

$$\Gamma := q_i^n \begin{bmatrix} m \\ n \end{bmatrix}_{q_i} (q_i^{-2n} \bar{K}_i^{-1} - 1) + q_i^{-n-m+1} \begin{bmatrix} m \\ n-1 \end{bmatrix}_{q_i} (q_i^{2m} \bar{K}_i^{-1} - q_i^{2n-2})$$

Note that, using Pascal identity (Proposition A.0.4-4), we get:

$$\begin{aligned} \Gamma &= [q_i^n \left[{m \atop n}\right]_{q_i} q_i^{-2n} + q_i^{-n-m+1} \left[{m \atop n-1}\right]_{q_i} q_i^{2m}] \bar{K}_i^{-1} - [q_i^n \left[{m \atop n}\right]_{q_i} + q_i^{-n-m+1} \left[{m \atop n-1}\right]_{q_i} q_i^{2n-2}] \\ &= [q_i^{-n} \left[{m \atop n}\right]_{q_i} + q_i^{-n+(m+1)} \left[{m \atop n-1}\right]_{q_i}] \bar{K}_i^{-1} - [q_i^n \left[{m \atop n}\right]_{q_i} + q_i^{n-(m+1)} \left[{m \atop n-1}\right]_{q_i}] \\ &= \left[{m+1 \atop n}\right]_{q_i} (\bar{K}_i^{-1} - 1) \end{aligned}$$

So finally we obtain the formula for $m+1$

$$\sum_{n=0}^{m+1} E_i^{(m+1)-n} E_i(L)^n q_i^{nm} \begin{bmatrix} m+1 \\ n \end{bmatrix}_{q_i} K_i(L)^{m+1} \prod_{l=0}^{n-1} (q_i^{-2l} \bar{K}_i^{-1} - 1)$$

and the proof is completed.

□

We can now present the theorem about the shift-invariance of $U(\mathfrak{g})$.

Theorem 1.5.7. *Let $(U_q(\mathfrak{g}), \rho_V)$ be a representation of $\mathrm{U_q}(\mathfrak{g})$ over V such that, when specializing q at an odd N-th root of unity ε, we have $\rho_V(K_i) = 1, \rho_V(e_i^N) = \rho_V(f_i^N) = 0$. Extend ρ_V to a representation of $\mathrm{U_q}(\mathfrak{g})$ over $V^{\otimes L}$ as shown in the previous paragraph. Then the shift operator $P : V^{\otimes L} \to V^{\otimes L}$ commutes with the action of the algebra $U(\mathfrak{g})$, obtained by specializing $U_q(\mathfrak{g})$ at ε as described in Section 1.4.*

Proof. In order to prove the theorem it is enough to prove the statement for the generators of the algebra, i.e. to show that:

1. $[P, \Delta^{(L)}(\left[{K_i;0 \atop N}\right])] = 0$

2. $[P, E_i^{(N)}] = 0$

3. $[P, F_i^{(N)}] = 0$

where we recall that $E_i = \Delta^{(L)}(e_i)$, $F_i = \Delta^{(L)}(f_i)$. For the first relation the problem reduces to show that $P\bar{K}_i P^{-1} = \bar{K}_i$ and this is clearly true due to the form of the $\bar{K}_i$'s calculated in Lemma 1.5.3.
Let us now compute $PE_i^{(N)}P^{-1}$ using Lemma 1.5.6:

$$\begin{aligned} PE_i^{(N)}P^{-1} &= \frac{1}{[N]_{q_i}!}\sum_{n=0}^{N} E_i^{N-n} E_i(L)^n q_i^{n(N-1)} \begin{bmatrix} N \\ n \end{bmatrix}_{q_i} K_i(L)^N \prod_{l=0}^{n-1}(q_i^{-2l}\bar{K}_i^{-1} - 1) \\ &= \sum_{n=0}^{N} E_i^{(N-n)} E_i(L)^{(n)} q_i^{n(N-1)} K_i(L)^N \prod_{l=0}^{n-1}(q_i^{-2l}\bar{K}_i^{-1} - 1) \end{aligned}$$

Now we note that specializing q at ε and passing to the quotient with respect to the ideal generated by $K_i - 1$ each summand contains a zero factor with the exception of $n = 0$ because in this case it yields 1 by definition, so the sum reduces exactly to $PE_i^{(N)}P^{-1} = E_i^{(N)}$ and the invariance is proved.
Completely analogous computations show the invariance of the $F_i^{(N)}$'s.

□

1.6 R-matrix and Yang-Baxter equation

In the first part of this section we present some results of the theory of Hopf algebras so that in the second part we can apply these propositions to the special case of quantum groups, which we showed to be endowed with this structure.
Throughout this section H will denote the Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$.

Definition 1.21. The Hopf algebra H is said:

- *commutative* if $\mu(a \otimes b) = \mu(b \otimes a) \qquad \forall a, b \in H$
- *cocommutative* if $\Delta(a) = \tau \circ \Delta(a) := \Delta^{op}(a) \qquad \forall a \in H$
  where τ is the flip operator $\tau : H \otimes H \to H \otimes H, \tau(a \otimes b) = b \otimes a$

In general, Hopf algebras are not either commutative or cocommutative; an example of this is evidently given by quantum groups. We introduce now another property, which relates Δ and Δ^{op} in a weaker way.

Definition 1.22. The Hopf algebra H is said *almost cocommutative* if there exists an invertible element $R \in H \otimes H$ such that

$$\Delta^{op}(a) = R\Delta(a)R^{-1} \tag{1.18}$$

for all $a \in H$.

We arrive now to the main definition of the section.

Definition 1.23. An almost cocommutative Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S, R)$ is said *quasi-triangular* (or braided) if the following two properties hold:

$$(\Delta \otimes id_H)(R) = R_{13}R_{23} \tag{1.19}$$

$$(id_H \otimes \Delta)(R) = R_{13}R_{12} \tag{1.20}$$

where, if $R = \sum_i a_i \otimes b_i$, $R_{12} = \sum_i a_i \otimes b_i \otimes 1$, $R_{13} = \sum_i a_i \otimes 1 \otimes b_i$ and $R_{23} = \sum_i 1 \otimes a_i \otimes b_i$.
In this case R is called *universal R-matrix.*
If, in addition, $R_{21} := \tau(R) = R^{-1}$, H is said *triangular.*

Proposition 1.6.1. *Let H be a quasi-triangular Hopf algebra with universal R-matrix R. Then R is a solution of the classical Yang-Baxter equation:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Proof. Let $R = \sum_i a_i \otimes b_i$. We have

$$\begin{aligned}
R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes id_H)(R) && \text{by equation (1.19)} \\
&= R_{12}(\textstyle\sum_i R^{-1}\Delta^{op}(a_i)R \otimes b_i) && \text{due to the almost cocommutativity} \\
&= (\Delta^{op} \otimes id_H)(R)R_{12} \\
&= (\tau \otimes id_H)(\Delta \otimes id_H)(R)R_{12} \\
&= (\tau \otimes id_H)(R_{13}R_{23})R_{12} && \text{by equation (1.19)} \\
&= R_{23}R_{13}R_{12}
\end{aligned}$$

□

Proposition 1.6.2. *Let H be a quasi-triangular Hopf algebra with universal R-matrix R. Then $(\varepsilon \otimes id_H)(R) = 1 = (id_H \otimes \varepsilon)(R)$. In particular, if the antipode is invertible, then:*

1. $(H, \mu, \eta, \Delta^{op}, \varepsilon, S^{-1}, \tau(R))$ *is a quasi-triangular algebra*

2. $(S \otimes id_H)(R) = R^{-1} = (id_H \otimes S^{-1})(R)$

3. $(S \otimes S)(R) = R$

Proof. The relation $(\varepsilon \otimes id_H)(R) = 1$ can be proved using (1.19).
We recall that due to the counity property of Hopf algebras we have $(\varepsilon \otimes id_H)\Delta = id_H$.
Then $R = (\varepsilon \otimes id_H \otimes id_H)(\Delta \otimes id_H)(R) = (\varepsilon \otimes id_H \otimes id_H)(R_{13}R_{23}) = (\varepsilon \otimes id_H)(R)\varepsilon(1)R = (\varepsilon \otimes id_H)(R)R$
In an analogous way, using (1.20), one can prove the second equality.

In order to prove 1. we have to show that the properties defining a quasi-triangular algebra are verified. We begin by showing that $(H, \mu, \eta, \Delta^{op}, \varepsilon, S^{-1})$ is a Hopf algebra: in order to prove the coassociativity we note that
$(\Delta^{op} \otimes id_H)\Delta^{op} = \tau_{13}(id_H \otimes \Delta)\Delta = \tau_{13}(\Delta \otimes id_H)\Delta = (id_H \otimes \Delta^{op})\Delta^{op}$
The counity and the compatibility between the algebra and coalgebra structures are evident.
We now need to verify the property of the antipodal map S. Since S is an antimorphism of algebras we immediately have $S^{-1} \star id_H = \mu(S^{-1} \otimes id_H)\Delta^{op} = S^{-1}\mu(S \otimes id_H)\Delta = S^{-1}\mu\varepsilon = \mu\varepsilon$ Furthermore:

- for $x \in H$, $(\Delta^{op})^{op}(x) = \tau\Delta^{op}(x) = \tau(R\Delta(x)R) = \tau(R)\Delta^{op}(x)\tau(R)$

- $(\Delta^{op} \otimes id_H)(\tau(R)) = \tau_{13}(id_H \otimes \Delta)(R) = \tau_{13}(R_{13}R_{12}) = \tau_{13}(R_{13})\tau_{23}(R_{23})$

- $(id_H \otimes \Delta^{op})(\tau(R)) = \tau_{13}(\Delta \otimes id_H)(R) = \tau_{13}(R_{13}R_{23}) = \tau_{13}(R_{13})\tau_{12}(R_{12})$

This concludes the proof of 1.

In order to prove relation 2. we recall that $(S \star id_H) = \eta\varepsilon$.
We have:

$$(\mu \otimes id_H)(S \otimes id_H \otimes id_H)(\Delta \otimes id_H)(R) = (\eta\varepsilon \otimes id_H)(R) = 1 \otimes 1$$

but also

$$\begin{aligned}(\mu \otimes id_H)(S \otimes id_H \otimes id_H)(\Delta \otimes id_H)(R) &= (\mu \otimes id_H)(S \otimes id_H \otimes id_H)(R_{13}R_{23}) \\ &= (S \otimes id_H)(R)R\end{aligned}$$

So, finally we get $1 \otimes 1 = (S \otimes id_H)(R)R$
This equality in the quasi-triangular algebra constructed in 1. can be written as follows
$(S^{-1} \otimes id_H)(\tau(R)) = \tau(R^{-1})$ hence applying the flip operator to both sides, we get immediately $(id_H \otimes S^{-1})(R) = R^{-1}$

In order to prove relation 3. we note that

$$\begin{aligned}(S \otimes S)(R) &= (id_H \otimes S)(S \otimes id_H)(R) \\ &= (id_H \otimes S)(R^{-1}) \\ &= (id_H \otimes S)(id_H \otimes S^{-1})(R) \\ &= (id_H \otimes id_H)(R) \\ &= R\end{aligned}$$

□

We now give the definition of R-matrix and quantum Yang-Baxter equation. These concepts are strictly related to the Hopf algebra theory that we introduced and will be immediately related also to the quantum groups theory.

Definition 1.24. Let V be a vector space and $R \in End(V \otimes V)$.
The *quantum Yang-Baxter equation* is

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

A solution R of the quantum Yang-Baxter equation is called *R-matrix*.

The notation has a meaning analogous to that already introduced in Proposition 1.6.1; in this case, however, the elements appearing in both sides of the equation lie in $End(V^{\otimes 3})$ so, for example, we have that $R_{23} = id_V \otimes R$ and so on.

We recall from the introduction that our final purpose is to show a particular symmetry between vertex models and quantum groups when the quantum parameter is specialized at a root of unity. In order to show this symmetry we have, of course, to relate quantum groups and vertex models: this will be done using the R-matrix and the quantum Yang-Baxter equation. In the last part of this section we will show how to associate a solution of the quantum Yang-Baxter equation to a representation of the affine quantum universal enveloping algebra $\mathrm{U_q}(\hat{\mathfrak{g}})$. In the following chapter we will finally construct a vertex model for a given R-matrix.

We will consider in particular evaluation representations of affine quantum enveloping algebras as defined in Subsection 1.2.1, so, in the following, $\mathring{\mathfrak{g}}$ will be a simple finite dimensional Lie algebra with Cartan matrix A of rank n, $\hat{\mathfrak{g}}$ its affinisation with Cartan matrix A' and $\tilde{\mathfrak{g}}$ the extension with a derivation element.

The construction is based on the fact that $\mathrm{U_q}(\tilde{\mathfrak{g}})$ is not only a Hopf algebra, but a quasi-triangular Hopf algebra, i.e. it admits a universal R-matrix. This result was first proved by Drinfeld in [4] (see also [3]), where the term quantum groups was introduced to refer to all these structures of quantum enveloping algebras, and an explicit computation of the universal R-matrix for quantum untwisted affine Lie algebras can be found in [12]. In fact universal R-matrices have been explicitly calculated for all the members of the classical series and for some exceptional algebras. Furthermore for $\mathrm{U_q}(\tilde{\mathfrak{g}})$ the universal R-matrix is uniquely determined up to a scalar factor (see [7], [13]). The idea is now to "specialize" to a R-matrix using an evaluation representation. This process is not immediate and it is based on the particular structure of the universal R-matrix $\tilde{R} \in \mathrm{U_q}(\tilde{\mathfrak{g}})^{\hat{\otimes} 2}$. We consider only finite dimensional representations, so, Proposition 1.2.4 assures that the central element acts as 0. Now the explicit expression of $\tilde{R}$ reveals that, for these representations, $\tilde{R}$ is no more dependent on the derivation element, so, removing this factor, it is possible to define the truncated R-matrix R and consider this as the universal R-matrix of $\mathrm{U_q}(\hat{\mathfrak{g}})$.

Proposition 1.6.3. *Let R be the universal R-matrix associated to $U_q(\hat{\mathfrak{g}})$, ρ_V an evaluation representation and $\rho_{V(u)}$ the family of evaluation representations given in Definition 1.10.*
We set

$$R(u) := (\rho_{V(u)} \otimes \rho_V)(R) = (\rho_V \otimes \rho_{V(-u)})(R) \in End(V \otimes V)$$

and

$$\Delta_u := (\rho_{V(u)} \otimes \rho_V) \circ \Delta$$

Then for all $a \in U_q(\hat{\mathfrak{g}})$ the intertwining property is satisfied:

$$\Delta_u^{op}(a) R(u) = R(u) \Delta_u(a)$$

Remark 12. The equality appearing in the definition of $R(u)$ holds due to the homogeneity properties of R, but it does not hold in general.

Proof. The proof immediately follows from the definitions of $R(u)$ and Δ_u and relation (1.18). We have:

$$\begin{aligned}
\Delta_u^{op}(a) R(u) &= ((\rho_{V(u)} \otimes \rho_V)\Delta^{op}(a))((\rho_{V(u)} \otimes \rho_V)(R)) \\
&= (\rho_{V(u)} \otimes \rho_V)(\Delta^{op}(a) R) \\
&= (\rho_{V(u)} \otimes \rho_V)(R \Delta(a)) \\
&= R(u) \Delta_u(a)
\end{aligned}$$

□

Theorem 1.6.4. *$R(u)$ is a R-matrix, i.e. it satisfies the following quantum Yang-Baxter equation:*

$$R_{12}(u) R_{13}(u+v) R_{23}(v) = R_{23}(v) R_{13}(u+v) R_{12}(u).$$

Proof. Let $R = \sum_i a_i \otimes b_i$.
We note that

$$(\Delta_u \otimes id_{End(V)})(id_{\mathrm{U_q}(\hat{\mathfrak{g}})} \otimes \rho_{V(-v)})(R) = R_{13}(u+v)R_{23}(v) \tag{1.21}$$

Indeed we have:

$$\begin{aligned}
&(\Delta_u \otimes id_{End(V)})(id_{\mathrm{U_q}(\hat{\mathfrak{g}})} \otimes \rho_{V(-v)})(R) = \\
&((\rho_{V(u)} \otimes \rho_V) \circ \Delta \otimes id_{End(V)})(id_{\mathrm{U_q}(\hat{\mathfrak{g}})} \otimes \rho_{V(-v)})(R) = \\
&(\rho_{V(u)} \otimes \rho_V \otimes \rho_{V(-v)})(\Delta \otimes id_{U_q(\hat{\mathfrak{g}})})(R) = \\
&(\rho_{V(u)} \otimes \rho_V \otimes \rho_{V(-v)})(R_{13}R_{23}) = \\
&R_{13}(u+v)R_{23}(v)
\end{aligned}$$

Using formula (1.21) we have immediately

$$\begin{aligned}
R_{12}(u)R_{13}(u+v)R_{23}(v) &= R_{12}(u)(\Delta_u \otimes id_{End(V)})(id_{U_q(\hat{\mathfrak{g}})} \otimes \rho_{V(-v)})(R) \\
&= R_{12}(u)(\textstyle\sum_i (R^{-1}(u)\Delta_u^{op}(a_i)R(u) \otimes \rho_{V(-v)}(b_i))) \\
&= (\textstyle\sum_i \Delta_u^{op}(a_i) \otimes \rho_{V(-v)}(b_i))R_{12}(u) \\
&= (\Delta_u^{op} \otimes \rho_{V(-v)})(R)R_{12}(u) \\
&= (\tau \otimes id_{End(V)})(\Delta_u \otimes \rho_{V(-v)})(R)R_{12}(u) \\
&= (\tau \otimes id_{End(V)})(R_{13}(u+v)R_{23}(v))R_{12}(u) \\
&= R_{23}(v)R_{13}(u+v)R_{12}(u)
\end{aligned}$$

□

We now introduce a permuted version of the R-matrix which will be important in the following since, as we will show, it commutes with the action of the quantum group.

Definition 1.25. Let $\tau_V : V \otimes V \longrightarrow V \otimes V$ be the flip operator.
We denote by $\mathscr{R}(u) \in End(V \otimes V)$ the endomorphism defined as follows:

$$\begin{aligned}
\mathscr{R}(u):\ & V \otimes V \longrightarrow V \otimes V \\
& v \otimes w \mapsto \tau(R(u)(v \otimes w))
\end{aligned}$$

We will refer to $\mathscr{R}(u)$ as the permuted R-matrix.

Proposition 1.6.5. *The endomorphism* $\mathscr{R}(u) : V(u) \otimes V \longrightarrow V \otimes V(u)$ *is an isomorphism of representations of* $\mathrm{U_q}(\hat{\mathfrak{g}})$ *hence it is a so-called intertwiner operator.*

Proof. For $x \in U_q(\hat{\mathfrak{g}})$ let $\Delta(x) = \sum_j x_j \otimes y_j$.
Let R be the universal R-matrix of $U_q(\hat{\mathfrak{g}})$, $R = \sum_i a_i \otimes b_i$.
The action of $x \in \mathrm{U_q}(\hat{\mathfrak{g}})$ over $V(u) \otimes V$ is given by $\Delta_u(x)$, while for the representation over $V \otimes V(u)$, we introduce the notation $\Delta_{u,1} = (\rho_V \otimes \rho_{V(u)})\Delta(x)$.
For $v, w \in V$ we have

$$\begin{aligned}
\mathscr{R}(u)\Delta_u(x)(v \otimes w) &= \tau_V[(R(u)\Delta_u(x))(v \otimes w)] \\
&= \tau_V[(\Delta_u^{op}(x)R(u))(v \otimes w)] \\
&= \tau_V[((\rho_{V(u)} \otimes \rho_V)(\Delta^{op}(x)R))(v \otimes w)] \\
&= \tau_V[\textstyle\sum_{i,j}((\rho_{V(u)}(y_j a_i))(v) \otimes (\rho_V(x_j b_i))(w))] \\
&= \textstyle\sum_{i,j}(\rho_V(x_j b_i))(w) \otimes (\rho_{V(u)}(y_j a_i))(v) \\
&= (\rho_V(x_j b_i) \otimes \rho_{V(u)}(y_j a_i))(w \otimes v) \\
&= \Delta_{u,1}(x)(\rho_V(b_i) \otimes \rho_{V(u)}(a_i))(w \otimes v) \\
&= \Delta_{u,1}(x)(\rho_V(b_i)(w) \otimes \rho_{V(u)}(a_i)(v)) \\
&= \Delta_{u,1}(x)\tau_V(\rho_{V(u)}(a_i)(v) \otimes \rho_V(b_i)(w)) \\
&= \Delta_{u,1}(x)\tau_V(R(u)(v \otimes w)) \\
&= \Delta_{u,1}(x)\mathscr{R}(u)(v \otimes w)
\end{aligned}$$

□

This proposition can be generalized to a representation over $V^{\otimes L}$.

Notation 4. For each $0 \leq L' \leq L - 1$ we set:

$$\Delta_{u,L'}^{(L)} := (\rho_V^{L'-1} \otimes \rho_{V(u)} \otimes \rho_V^{\otimes L-L'})\Delta^{(L)}$$

Proposition 1.6.6. *For all* $j = 1, ..., L-1$, $x \in U(\hat{\mathfrak{g}})$, *the endomorphism*

$$\mathscr{R}_{jj+1}(u) : V_1 \otimes ... \otimes V_j(u) \otimes V_{j+1} \otimes ... \otimes V_L \longrightarrow V_1 \otimes ... \otimes V_j \otimes V_{j+1}(u) \otimes ... \otimes V_L$$

is an isomorphism of representations of $\mathrm{U_q}(\tilde{\mathfrak{g}})$. *In particular we have*

$$\Delta_{u,j}^{(L)}(x)\mathscr{R}_{jj+1}(u) = \mathscr{R}_{jj+1}(u)\Delta_{u,j+1}^{(L)}(x)$$

Proof. The proof follows from Proposition 1.6.5 and the coassociativity of the comultiplication. In particular we have

$$\Delta^{(L)}_{u,j}(x)\mathscr{R}_{jj+1}(u) = (\rho_V^{\otimes j-1} \otimes \rho_{V(u)} \otimes \rho_V^{\otimes L-j})(\Delta \otimes id^{\otimes L-2})...(\Delta \otimes id^{\otimes L-j-2})$$
$$(\Delta \otimes id^{\otimes L-j-3})...(\Delta \otimes id)\Delta(x)\mathscr{R}_{jj+1}(u) =$$
$$(\rho_V^{\otimes j-1} \otimes \rho_{V(u)} \otimes \rho_V^{\otimes L-j})(\Delta \otimes id^{\otimes L-2})...(id \otimes \Delta \otimes id^{\otimes L-j-1})(\Delta \otimes id^{\otimes L-j-3})$$
$$...\Delta(x)\mathscr{R}_{jj+1}(u) =$$
$$\mathscr{R}_{jj+1}(u)(\rho_V^{\otimes j} \otimes \rho_{V(u)} \otimes \rho_V^{\otimes L-j-1})(\Delta \otimes id^{\otimes L-2})...(id \otimes \Delta \otimes id^{\otimes L-j-1})$$
$$(\Delta \otimes id^{\otimes L-j-3})...\Delta(x) =$$
$$\mathscr{R}_{jj+1}(u)\Delta^{(L)}_{u,j+1}(x)$$

□

Chapter 2

Vertex models

In this second chapter we present a physical application of the previous results, in particular we associate a vertex model to $U_q(\hat{\mathfrak{g}})$ and show that some symmetries occur between the model and the algebra $U(\hat{\mathfrak{g}})$ obtained by specializing the quantum parameter q at an odd root of unity.

2.1 Definition

A vertex model is a model of statistical physics obtained considering a lattice (here we will restrict to 2-dimensional lattices) in every vertex of which we imagine to have an atom. The idea is to study the thermodynamic energy of the system starting from the interaction energy of its atoms. In this type of models the interaction energy of the atoms depends only on the bonds to the nearest atoms and on eventual external electric or magnetic fields, but in our analysis we will ignore external fields.
Let us now consider a 2-dimensional lattice, consisting of a $M \times L$ array of atoms. In this situation the interaction energy of every atom depends only on the four bonds with the atoms to which it is directly connected. A physical study of this model usually starts by considering a finite lattice and later it is extended to infinite lattices by a limit operation ($M, L \to \infty$), this is an idea of the so-called passage to the thermodynamic limit.

A configuration (or state) of the model is given by setting a state for every bond because this implies that the interaction energy of every atom is then fixed and the model is completely determined. The possible states of the bonds are restricted to the elements of a finite set $\{1, ..., n\}$. Sometimes it is useful to impose also constraints on the possible combination of states of the bonds of an atom (e.g. in the so-called six vertex model, bonds have 2 possible states but only 6 over the $2^4 = 16$ possible combinations among the 4 bonds of a vertex are possible).
Besides, since we will work with finite lattices, we will also have to fix boundary conditions. We will use periodic boundary condition, so we impose the opposite bonds of the lattice (i.e. the 2 bonds at the extremities of the same row or column) to be in the same state.

Let us now introduce the main functions used to study this model in order to show the connection between $U_q(\hat{\mathfrak{g}})$ and vertex models.
The main function associated to a model is the partition function. This function is very important and characterizes the model because it allows to calculate all the main thermodynamic variables of the system, such as total energy, free energy, entropy. The main difficulty in the study of a system is, in fact, to calculate an analytic form of this function. If this is possible one says that the model is exactly solvable.

Definition 2.1. The *partition function* of a vertex model is:

$$Z = \sum_{\text{states}} e^{-\beta\varepsilon(\text{state})}$$

where

- $\beta := \frac{1}{kT}$ with k Bolzmann's constant and T temperature of the system
- $\varepsilon(\text{state})$ is the total energy of the model in a fixed state, obtained as the sum of the interaction energy of each atom

Remark 13. We will label i, j, k, l the 4 bonds of a vertex as in the following picture and denote the energy of the vertex by ε_{ij}^{kl}, expressing in this way its dependence on the bonds.

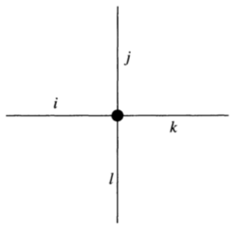

Figure 2.1: A vertex and its 4 bonds

Note that the summands in the definition of the partition function can be written in terms of the ε_{ij}^{kl} in the following way:

$$e^{-\beta\varepsilon(\text{state})} = \prod_{\text{vertices}} e^{-\beta\varepsilon_{ij}^{kl}} = \prod_{\text{vertices}} R_{ij}^{kl}$$

where we set $R_{ij}^{kl} := e^{-\beta\varepsilon_{ij}^{kl}}$. The quantities R_{ij}^{kl} are called *Boltzmann weights.*

2.2 Transfer matrix

The purpose of this section is to define the transfer matrix and to show its relation with the partition function just introduced. The transfer matrix encodes information about the energy of the model in every allowed configuration. The notation is quite heavy, so we start by considering just the first row (i.e. a $1 \times L$ model) and later we will extend the argument to the other rows.
Let V be an n-dimensional vector space with basis $\{v_1, ..., v_n\}$ and consider the endomorphism $R \in End(V \otimes V)$ given by

$$R(v_i \otimes v_j) = \sum_{k,l} R_{ij}^{kl} v_k \otimes v_l$$

This endomorphism can be expressed by a matrix whose elements are all the possible Boltzmann weights. Now, we want to express the partition function in terms of R.

Figure 2.2: The $1 \times L$ lattice

If we consider a $1 \times L$ lattice with fixed boundary conditions (not necessarily periodic) and use the notation of Figure 2.2, then the partition function is

$$Z^{i'_1 l_1 \dots l_L}_{i_1 j_1 \dots j_L} = \sum_{k_1 \dots k_{L-1}} R^{k_1 l_1}_{i_1 j_1} R^{k_2 l_2}_{k_1 j_2} \dots R^{i'_1 l_L}_{k_{L-1} j_L}$$

Now we define another endomorphism $T \in End(V \otimes V^{\otimes L})$ whose matrix elements are the partition functions for one row imposing all the possible boundary conditions. We remark that we need $L+1$ vector spaces. The first is called auxiliary space and encodes the information about the boundary state of the row while the other L are for the boundary of the columns.
The endomorphism will be naturally defined by

$T(v_{i_1} \otimes v_{j_1} \otimes \dots \otimes v_{j_L}) =$
$\sum_{i'_1, l_1, \dots, l_L} Z^{i'_1 l_1 \dots l_L}_{i_1 j_1 \dots j_L} v_{i'_1} \otimes v_{l_1} \otimes \dots \otimes v_{l_L} =$
$\sum_{i'_1, l_1, \dots, l_L, k_1 \dots k_{L-1}} R^{k_1 l_1}_{i_1 j_1} R^{k_2 l_2}_{k_1 j_2} \dots R^{i'_1 l_L}_{k_{L-1} j_L} v_{i'_1} \otimes v_{l_1} \otimes \dots \otimes v_{l_L}$

Now we remark that T can be written in terms of R in the following way.

$$T = R_{0L} R_{0L-1} \dots R_{01} \tag{2.1}$$

where we make use of the usual notation and label 0 the auxiliary vector space.
In fact we have

$$\begin{aligned} R_{0L} \dots R_{02} R_{01} \quad & (v_{i_1} \otimes v_{j_1} \otimes \dots \otimes v_{j_L}) = \\ & = R_{0L} \dots R_{02} \textstyle\sum_{k_1, l_1} R^{k_1 l_1}_{i_1 j_1} (v_{k_1} \otimes v_{l_1} \otimes v_{j_2} \otimes \dots \otimes v_{j_L}) \\ & = R_{0L} \dots R_{03} \textstyle\sum_{k_1, k_2, l_1, l_2} R^{k_1 l_1}_{i_1 j_1} R^{k_2 l_2}_{k_1 j_2} (v_{k_1} \otimes v_{l_1} \otimes v_{l_2} \otimes v_{j_3} \otimes \dots \otimes v_{j_L}) \\ & = \dots \\ & = T(v_{i'_1} \otimes v_{l_1} \otimes \dots \otimes v_{l_L}) \end{aligned}$$

Now we impose the periodicity of the boundary condition on the row (i.e. $i_1 = i'_1$) and define the endomorphism $\mathscr{T} \in End(V^{\otimes L})$ in the following way:

$$\mathscr{T}(v_{j_1} \otimes ... \otimes v_{j_N}) = \sum_{l_1 ... l_L} (\mathrm{Tr}_V(T))^{l_1 ... l_L}_{j_1 ... j_L} (v_{l_1} \otimes ... \otimes v_{l_L})$$

where Tr_V is the partial trace operator applied to the auxiliary space. $\mathscr{T}$ is called transfer matrix and its elements are the sum, over all the possible states of the boundary bond of the row ($i_1 = i'_1$), of the partition functions of a $1 \times L$ lattice with fixed boundary states (periodic on the row).

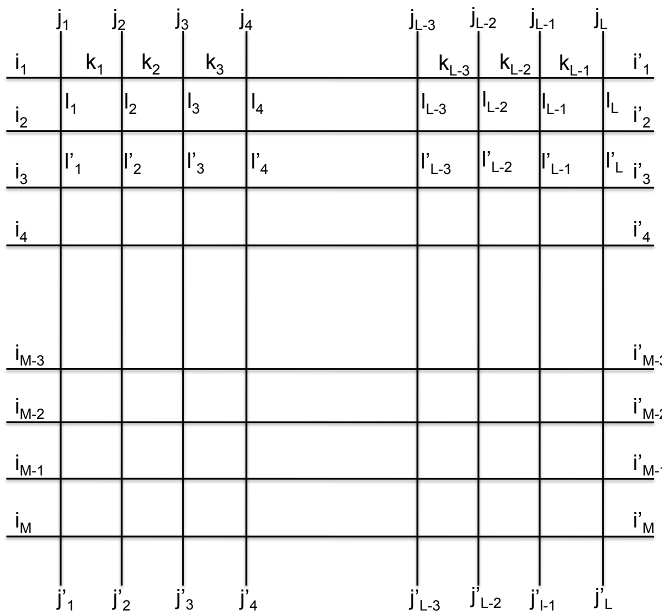

Figure 2.3: The $M \times L$ lattice

The generalization of this process to M rows is quite easy. First of all we remark that for fixed boundary conditions on the columns the partition function of a $2 \times L$ lattice with periodic boundary condition on the 2 rows can be written, using the notation of Figure 2.3 and considering the first two rows, as

$$\sum_{l_1 ... l_L} (\mathrm{Tr}_V(T))^{l_1 ... l_L}_{j_1 ... j_L} (\mathrm{Tr}_V(T))^{l'_1 ... l'_L}_{l_1 ... l_L}$$

and this is clearly one of the elements of $(\mathrm{Tr}_V(T))^2$. The other elements are obtained for all the possible states of the boundary bonds of the columns. For the case with M rows we simply iterate the process and we obtain that the partition functions for for periodic boundary condition on the rows and fixed boundary states on the columns are the elements of $\mathscr{T}^M$.
We can now easily impose the condition of periodicity for the boundary of the columns in order to obtain the classical partition function; it is in fact enough to evaluate the ordinary trace of $\mathscr{T}^M$.
We proved in this way the following theorem.

Theorem 2.2.1. *If Z is the partition function of a 2-dimensional $M \times L$ vertex model with periodic boundary condition and $\mathscr{T}$ is its transfer matrix, then*

$$Z = \mathrm{Tr}_{V^{\otimes L}}(\mathscr{T}^M)$$

2.3 Vertex models and $U_q(\hat{\mathfrak{g}})$

We can now easily associate a $M \times L$ vertex model to every evaluation representation of a quantum affine enveloping algebra.
Let $R(u) \in End(V \otimes V)$ be the R-matrix associated to an evaluation representation of $U_q(\hat{\mathfrak{g}})$; we interpret its elements as Boltzmann weights so we define the transfer matrix of our model as follows:

$$\mathscr{T}(u) := \mathrm{Tr}_V \, R_{0L}(u) R_{0L-1} ... R_{01}(u)$$

following the construction of the previous section.
Theorem 2.2.1 immediately implies that the partition function can be defined as

$$Z = \mathrm{Tr}_{V^{\otimes L}}(\mathscr{T}(u)^M)$$

The model is therefore completely determined. The dependence of the R-matrix on the variable u, which is mathematically due to the parametrization of the automorphism used for the evaluation representation, can be physically interpreted as a dependence of the Boltzmann weights on an external parameter. This helps to understand the physical meaning of the following definition.

Definition 2.2. Let $R(\cdot) \in End(V \otimes V)$ be an endomorphism depending on an external parameter. The associated vertex model is integrable if for each value of the parameters α, β there exists γ such that

$$R_{12}(\gamma)R_{13}(\alpha)R_{23}(\beta) = R_{23}(\beta)R_{13}(\alpha)R_{12}(\gamma)$$

Proposition 2.3.1. *The vertex model associated to an affine quantum enveloping algebra is always integrable.*

Proof. The proof immediately follows from Theorem 1.6.4. We have indeed that $R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)$ and for every possible choice of v and $u+v$ we obtain u as their difference, so the condition of integrability is satisfied. □

The integrability property will be fundamental for the symmetries we want to prove. In particular its importance is due to Proposition 2.3.3, which proves a commutativity property of the transfer matrix.
We first need a lemma.

Lemma 2.3.2. *Let $R(\cdot) \in End(V \otimes V)$ be an endomorphism depending on an external parameter and consider the associated vertex model with transfer matrix $\mathscr{T}(\alpha) = \mathrm{Tr}_V\, T(\alpha)$ and $T(\alpha) = R_{0L}(\alpha)...R_{01}(\alpha)$. If the model is integrable the following equality between elements of $End(V_0 \otimes V^{\otimes L} \otimes V_{\bar{0}})$ holds:*

$$R_{\bar{0}0}(\gamma)(id_{V_0} \otimes T(\alpha))(T(\beta) \otimes id_{V_{\bar{0}}}) = (T_0(\beta) \otimes id_{V_{\bar{0}}})(id_{V_0} \otimes T(\alpha))R_{\bar{0}0}(\gamma)$$

Proof. If we set $T_0 = R_{0L}...R_{01}$ and $T_{\bar{0}} = R_{\bar{0}L}...R_{\bar{0}1}$, the relation to prove can be written as $R_{\bar{0}0}(\gamma)T_{\bar{0}}(\alpha)T_0(\beta) = T_0(\beta)T_{\bar{0}}(\alpha)R_{\bar{0}0}(\gamma)$.
Hence we have the following chain of equalities:

$$\begin{aligned}
&R_{\bar{0}0}(\gamma)T_{\bar{0}}(\alpha)T_0(\beta) = \\
&R_{\bar{0}0}(\gamma)R_{\bar{0}L}(\alpha)...R_{\bar{0}1}(\alpha)R_{0L}(\beta)...R_{01}(\beta) = \\
&[R_{\bar{0}0}(\gamma)R_{\bar{0}L}(\alpha)R_{0L}(\beta)]R_{\bar{0}L-1}(\alpha)...R_{\bar{0}1}(\alpha)R_{0L-1}(\beta)...R_{01}(\beta) = \\
&[R_{0L}(\beta)R_{\bar{0}L}(\alpha)R_{\bar{0}0}(\gamma)]R_{\bar{0}L-1}(\alpha)...R_{\bar{0}1}(\alpha)R_{0L-1}(\beta)...R_{01}(\beta) = \\
&R_{0L}(\beta)R_{\bar{0}L}(\alpha)[R_{\bar{0}0}(\gamma)R_{\bar{0}L-1}(\alpha)R_{0L-1}(\beta)]...R_{\bar{0}1}(\alpha)R_{0L-2}(\beta)...R_{01}(\beta) = \\
&R_{0L}(\beta)R_{\bar{0}L}(\alpha)[R_{0L-1}(\beta)R_{\bar{0}L-1}(\alpha)R_{\bar{0}0}(\gamma)]...R_{\bar{0}1}(\alpha)R_{0L-2}(\beta)...R_{01}(\beta) = \\
&R_{0L}(\beta)R_{0L-1}(\beta)R_{\bar{0}L}(\alpha)R_{\bar{0}L-1}(\alpha)R_{\bar{0}0}(\gamma)...R_{\bar{0}1}(\alpha)R_{0L-2}(\beta)...R_{01}(\beta) = \\
&... = \\
&R_{0L}(\beta)...R_{01}(\beta)R_{\bar{0}L}(\alpha)...R_{\bar{0}1}(\alpha)R_{\bar{0}0}(\gamma) = \\
&T_0(\beta)T_{\bar{0}}(\alpha)R_{\bar{0}0}(\gamma)
\end{aligned}$$

where we use that R_{ij} commutes with R_{kl} if they act over different spaces (i.e. $i \neq k \neq l \neq j$) and the integrability of the model (brackets have been added to show when the property has been applied). □

Proposition 2.3.3. *Consider an integrable vertex model for which $R(\gamma) \in End(V \otimes V)$ is invertible for all γ. Then the transfer matrices obtained for different values of the spectral parameter commute.*

$$[\mathscr{T}(\alpha), \mathscr{T}(\beta)] = 0 \qquad \forall \alpha, \beta$$

Proof. From Lemma 2.3.2 we have

$$(id_{V_0} \otimes T(\alpha))(T(\beta) \otimes id_{V_{\bar{0}}}) = R_{\bar{0}0}(\gamma)^{-1}(T_0(\beta) \otimes id_{V_{\bar{0}}})(id_{V_0} \otimes T(\alpha))R_{\bar{0}0}(\gamma)$$

Evaluating the partial trace $\mathrm{Tr}_{V_0 \otimes V_{\bar{0}}}$ we get

$$\mathrm{Tr}_{V_{\bar{0}}}(T(\alpha))\, \mathrm{Tr}_{V_0}(T(\beta)) = \mathrm{Tr}_{V_0 \otimes V_{\bar{0}}}(R_{\bar{0}0}(\gamma))^{-1}\, \mathrm{Tr}_{V_0}(T(\beta))\, \mathrm{Tr}_{V_{\bar{0}}}(T(\alpha))\, \mathrm{Tr}_{V_0 \otimes V_{\bar{0}}}(R_{\bar{0}0}(\gamma))$$

Remarking that $\mathrm{Tr}_{V_0 \otimes V_{\bar{0}}}(R_{\bar{0}0}(\gamma))$ is a diagonal matrix and so commute, we finally obtain

$$\mathscr{T}(\alpha)\mathscr{T}(\beta) = \mathscr{T}(\beta)\mathscr{T}(\alpha)$$

□

2.4 $U(\hat{\mathfrak{g}})$ invariance of the vertex model

In this section we will finally show the symmetries that arise between a representation of $U_q(\hat{\mathfrak{g}})$ and the associated vertex model when specializing the quantum parameter at an odd root of unity. In particular we will prove that the vertex model, expressed trough the transfer matrix, is invariant with respect to the action of the non deformed enveloping algebra constructed from $U_q(\hat{\mathfrak{g}})$ when specializing q at an odd root of unity.
This idea is formalized in the following theorem.

Theorem 2.4.1. *Let $\mathring{\mathfrak{g}}$ be a simple finite dimensional Lie algebra and let $U_q(\hat{\mathfrak{g}})$ be the associated quantum affine algebra. Let $U(\hat{\mathfrak{g}})$ be the non deformed enveloping algebra obtained when specializing q at a root of unity ε of odd order N (see Theorem 1.4.3). Consider an evaluation representation $(U_q(\hat{\mathfrak{g}}), \rho_V)$ over a finite dimensional vector space V such that, when specializing the parameter q at ε, $\rho_V(e_i^N) = \rho_V(f_i^N) = 0$ and $\rho_V(K_i) = 1$ and extend it to a representation over $V^{\otimes L}$ using the comultiplication of the Hopf algebra structure of the quantum group. Then the transfer matrix $\mathscr{T}(u) \in End(V^{\otimes L})$ of the integrable vertex model associated to the representation ρ_V (thanks to the universal R-matrix of the quantum group) is $U(\hat{\mathfrak{g}})$-invariant.*

Before proving the theorem we state some preliminary results.

Remark 14. For affine quantum groups it is always possible to normalize the R-matrix so that

$$\lim_{u\to 0} R(u) = \tau$$

We will work under this assumption.

Lemma 2.4.2. *Shift operator and transfer matrix are linked by the following relation*

$$\mathscr{T}(0) = P^{-1}.$$

Proof. We recall that $\mathscr{T}(u) = \mathrm{Tr}_{V_0}\, R_{0L}(u)R_{0L-1}(u)...R_{01}(u)$.
Thanks to Remark 14 for $u = 0$ we have

$$\mathscr{T}(0) = \mathrm{Tr}_{V_0}\, \tau_{0L}\tau_{0L-1}...\tau_{01} = P^{-1}.$$

□

Definition 2.3. The boost operator of the integrable vertex model with transfer matrix $\mathscr{T}$ is implicitly defined by

$$-\frac{\partial}{\partial u}\mathscr{T}(u) = [K, \mathscr{T}(u)]$$

Proposition 2.4.3. *The boost operator can be explicitly written in terms of the R-matrix as follows (see [16]):*

$$K = \sum_{n\in\mathbb{N}}\sum_{j=1}^{L}(j+nL)\frac{\partial}{\partial u}\mathscr{R}_{jj+1}(u)|_{u=0}$$

Proof of Theorem 2.4.1. First of all we note that the adjoint action of the boost operator allows us to show that

$$\mathscr{T}(u+v) = e^{-vK}\mathscr{T}(u)e^{vK}$$

Indeed we have

$$\frac{d}{dv}e^{-vK}\mathscr{T}(u)e^{vK} = -Ke^{-vK}\mathscr{T}(u)e^{vK} + e^{-vK}\mathscr{T}(u)e^{vK}K = -[K, e^{-vK}\mathscr{T}(u)e^{vK}],$$

hence for $v = 0$ we get the relation of the definition.
Now we evaluate the relation for $u = 0$ and apply Lemma 2.4.2. We obtain

$$\mathscr{T}(v) = e^{-vK}P^{-1}e^{vK}$$

In order to prove the invariance of the transfer matrix under the action of $U(\hat{\mathfrak{g}})$ it is therefore enough to show the invariance of the shift and of the boost operator. In Theorem 1.5.7 we already proved that the action of $U(\hat{\mathfrak{g}})$ commutes with the shift operator so it remains to show the invariance of the boost operator.
The explicit expression for K in Proposition 2.4.3 reduces the problem to show that the permuted R-matrix $\mathscr{R}_{jj+1}(u)$ is invariant with respect to the action of $U(\hat{\mathfrak{g}})$. Now Proposition 1.6.6 proves this for $j = 1, ..., L-1$ and is still valid for $U(\hat{\mathfrak{g}})$ because the hypothesis on the representation allows to

define the R-matrix for the specialization. It remains therefore to prove only the case $j = L$, under the usual condition of periodicity $L + 1 \equiv 1$.
We start noticing that $P\mathscr{R}_{L1}(u)P^{-1} = \mathscr{R}_{L-1L}(u)$.
In fact, for $v_1 \otimes ... \otimes v_L \in V^{\otimes L}$ we have

$$\begin{aligned} P\mathscr{R}_{L1}(u)P^{-1}(v_1 \otimes ... \otimes v_L) &= P\mathscr{R}_{L1}(u)(v_L \otimes v_1 \otimes ... \otimes v_{L-1}) \\ &= P \sum v_L'' \otimes v_1 \otimes ... \otimes v_{L-1}' \\ &= \sum v_1 \otimes ... \otimes v_{L-1}' \otimes v_L'' \\ &= \mathscr{R}_{L-1L}(v_1 \otimes ... \otimes v_{L-1} \otimes v_L) \end{aligned}$$

Using this remark, the shift invariance and the case $j = L - 1$, we have

$$\begin{aligned} \Delta^{(L)}_{u,L}(x)\mathscr{R}_{L1}(u) &= P^{-1}\Delta^{(L)}_{u,L-1}(x)P\mathscr{R}_{L1}(u) \\ &= P^{-1}(\Delta^{(L)}_{u,L-1}(x)P\mathscr{R}_{L1}(u)P^{-1})P \\ &= P^{-1}(\Delta^{(L)}_{u,L-1}(x)\mathscr{R}_{L-1L}(u))P \\ &= P^{-1}(\mathscr{R}_{L-1L}(u)\Delta^{(L)}_{u,L}(x))P \\ &= (P^{-1}\mathscr{R}_{L-1L}(u)P)(P^{-1}\Delta^{(L)}_{u,L}(x)P) \\ &= \mathscr{R}_{L1}(u)\Delta^{(L)}_{u,1}(x) \end{aligned}$$

So we proved that the action of $U(\hat{\mathfrak{g}})$ commutes with the boost operator and consequently the invariance of the transfer matrix. □

Appendix A

q-integers and the q-binomial formula

We briefly recall the main definitions concerning q-integers.

Definition A.1. Let $q \in \mathbb{C}$. For $n \in \mathbb{N}$ we set

$$[n]_q = q^{-n+1} + q^{-n+3} + ... + q^{n-3} + q^{n-1} = \frac{q^n - q^{-n}}{q - q^{-1}}$$

We immediately note that, if $q = 1, [n]_q = n$ (in this case, of course, we can not have the last equality because it does not make sense).

Definition A.2. In analogy with the classical definition, we define

1. the q-factorial of $n \in \mathbb{N}$:

$$\begin{cases} [0]_q! = 1 \\ [n]_q! = [1]_q[2]_q...[2]_q = \frac{(q-q^{-1})(q^2-q^{-2})...(q^n-q^{-n})}{(q-q^{-1})^n} & \text{if } n > 0 \end{cases}$$

2. the q-binomial coefficient: for $n \in \mathbb{N}, 0 \leq k \leq n$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

Proposition A.0.4. *Let $n, m \in \mathbb{N}, 0 \leq k \leq n$. We have:*

1. $\begin{bmatrix} n \\ k \end{bmatrix}_q \in \mathbb{Z}[q, q^{-1}]$ *and for $q = 1$ we get the usual binomial coefficient;*

2. $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$;

3. $[m+n] = q^{-n}[m] + q^m[n]$;

4. *the q-deformation of the Pascal identity is*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{-k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = q^{k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{k-n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

Proof. The proof of 2. is immediate, 3. and 4. follow from easy computations and 1. follows from 3. and 4. using induction on n. □

The following proposition, like in the classical case, can be proved by induction on n.

Proposition A.0.5. *If a, b are such that $ba = q^2 ab$ then the q-binomial formula is the following:*

$$(a+b)^n = \sum_{k=0}^{n} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k b^{n-k}$$

Proposition A.0.6. *Let $r \in \mathbb{N}^+$. Then*

$$\sum_{k=0}^{r} (-1)^k \begin{bmatrix} r \\ k \end{bmatrix}_q q^{-k(r-1)} = 0$$

Proof. Using the q-Pascal identity, if $r > 0$, we can split the summation in $\sum_{k=0}^{r-1}(-1)^k q^{-rk} \begin{bmatrix} r-1 \\ k \end{bmatrix}_q + \sum_{k=1}^{r}(-1)^k q^{-r(k-1)} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}_q =$
$\sum_{k=0}^{r-1}(-1)^k q^{-rk} \begin{bmatrix} r-1 \\ k \end{bmatrix}_q - \sum_{k=0}^{r-1}(-1)^k q^{-rk} \begin{bmatrix} r-1 \\ k \end{bmatrix}_q = 0$

□

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