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QUANTUM MODIFICATIONS TO THE SCHWARZSCHILD RINGDOWN SPECTRUM

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Abstract

In anticipation of a successful theory of quantum gravity, there has been much interest in predicting whether and how such a theory could appreciably influence environments that we can interact with or otherwise measure. The ringdown of a black hole in the aftermath of a violent event such as a merger is thus seen as a potential candidate for revealing the signatures of quantum gravity due to our ability to measure the gravitational waves that emerge. Nevertheless, it is not so far clear how the effects of quantum gravity should change the classical picture of the ringdown, as the discretised energy spectrum of the black hole clashes with the notion of a horizon that absorbs all infalling modes of the metric and matter fields. In this work we propose that the quantum black hole dissipates the energy it cannot absorb with the emission of soft gravitons carrying the energy difference between an infalling particle's energy and the quantum transition energy. We find that the ringdown spectrum is consequently enriched with low frequency components, and particularly that there exists a weak low frequency flux that persists for a timescale far longer than the ringdown itself, due to the leaking cavity properties of the surrounding spacetime.

Introduction

The primary goal of this thesis is to treat the notion of a black hole as an all-absorbing object under the lens of quantum mechanics, the laws of which seeming to deny the possibility of such a system. The issue at hand is posed as the question of, how can one reconcile the classically anticipated properties of a black hole, namely those of a perfect absorber, with the expectation that quantum gravity should impose on it a discrete energy spectrum? In other words: how can the discretised energy levels of a quantum bound state accommodate any infalling object carrying an arbitrary amount of energy?

The stage to study this conceptual problem is set cleanly by the ringdown of a black hole following a perturbation of its spacetime. The weak perturbations of fields in the black hole's vicinity propagate and acquire features that are shaped by the black hole's global structure, the latter acting like a resonant cavity, and pick up oscillating components with a discrete set of frequencies that is strictly determined by the spacetime's geometry; these are the spacetime's quasinormal modes (QNMs). As the quasi- prefix implies, these modes are approximately normal but decaying in time, and are therefore associated with the condition that they dissipate towards infinity and into the horizon, making them solutions to a boundary condition problem on the spacetime's geometry.

Thus every disturbance on the black hole's spacetime, such as those that can arise from a merger event about the final black hole, excites this particular set of frequencies. This setup allows us to formulate the problem as follows: what is the near-horizon behaviour of infalling modes with given frequencies that need not, in general, match the excitation frequency of the black hole?

The answer to this question has been proposed in the literature (see for instance [3]) to come in the form of a modified horizon boundary condition, endowed with a frequency-dependent reflectivity that prevents 'wrong energy' modes from entering the horizon with some width-dependent probability, producing so-called echoes in the ringdown signal. This approach nevertheless comes with its own puzzling ambiguities as to what the physical mechanism could be that may bar arbitrary objects from entering the horizon based on their energy, because the horizon is classically expected to be a surface only of geometrical significance and otherwise vacuous.

In this thesis we explore a novel approach were we suppose instead that the horizon boundary condition remains unchanged, namely that everything that reaches the horizon is indeed absorbed by the black hole, and the difference in energy of the infalling object and a potential excitation of the quantum black hole is emitted as gravitational radiation. This proposal is motivated by the behaviour of more familiar quantum systems such as that of the hydrogen atom, where an analogous central potential created by the proton's charge will accommodate an incoming electron of arbitrary energy to form an atom that will settle into a state with specific energy, emitting the excess via electromagnetic radiation.

The outcome is quantitatively different from the aforementioned, in that the frequencies of the gravitons that emerge from the horizon will correspond to $1/\hbar$ times the energy difference for some transition, rather than carrying the same energy as the infalling object, resulting in a different spectrum. Meanwhile, the emission will occur for every instance of an infalling particle and not with some frequency-dependent probability, as was the case before. We therefore embark on the computation of some qualitative features of the anticipated spectrum using a coherent state model for the quantum black hole (developed in [5]) in order to obtain the relative probability profile for the various potential transitions a given QNM can generate. We furthermore account for backscat-

tering effects in the black hole's spacetime as the gravitons propagate towards a faraway observer. We find that, due to the low transmission rate through the potential barrier, the quantum black hole will slowly but surely dissipate the excess energy in gravitons of the lowermost set of frequencies that result from the potential excitations. This translates to a weak flux of soft gravitons whose duration persists over the decay time of the QNMs themselves.

The thesis is structured as follows. In Chapter 1 we overview QNMs of various spin fields and their presence in the ringdown signal for a generic perturbation, along with some of their defining properties. In Chapter 2 we set up a model for the quantum black hole by following the approach of [5] wherein coherent quantum states are used to reproduce the expectation configuration of a quantised polarisation of the metric field. In Chapters 3 and 4 we present the original parts of this thesis: in the first of these we compute, as a quantum scattering process, the relative probability for a transition of the black hole as a function of the number of levels n; in the second we use this probability profile to derive the expected spectrum of the dissipative processes and additionally account for backscattering on the spacetime's potential barrier.

1 Quasinormal modes of spherically symmetric spacetimes

The quasinormal modes (QNMs) of some field, in the context of black hole physics such as in our case, describe solutions of the linearised field equations in the black hole's exterior spacetime when suitable boundary conditions are applied at the horizon and infinity. In other words, they are field configuration that represent the decaying evolution of some perturbative initial state. They are most notably used to describe the evolution of a small deviation of the metric field from the black hole's background metric, as it settles into a Schwarzschild or Kerr (or charged) black hole after a merger event has taken place.

It is instructive to formulate at first the concept of QNMs for a massless Klein-Gordon scalar field, and then move on to the metric field, as their perturbations turn out to satisfy remarkably similar equations. In any case, the field equations are solved on the background Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{1 - \frac{2GM}{r}} dr^2 + r^2 d\Omega^2 \tag{1.1}$$

where the solid angle element is $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and we will call $f(r) \equiv 1 - \frac{2GM}{r}$.

1.1 Scalar perturbations

A massless Klein-Gordon scalar field $\phi(x^\mu)$ is described by the action

$$S_{KG} = \int d^4x \frac{\sqrt{-g}}{2} \partial_\mu \phi^* \partial^\mu \phi \tag{1.2}$$

where evidently ϕ is inherently coupled to the metric $g_{\mu\nu}$. In the perturbative regime, we split $\phi = \tilde{\phi} + \delta\phi$ and anticipate that $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$, where tilde denotes background quantities; we assume here $\tilde{\phi} = 0$, implying that the field ϕ contains only a small fluctuating part $\delta\phi$.

Solving the field equations for the action (1.2) and the Einstein-Hilbert action, up to linear order in the perturbations $\delta\phi$ and $h_{\mu\nu}$, decouples one from the other and so we may solve for either of them independently. Specifically, the Euler-Lagrange equations for $\delta\phi$ are

$$\partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu \delta\phi) = 0 \tag{1.3}$$

which enforce that any contribution from the metric is zeroth-order, and consequently expand to

$$-\frac{1}{f(r)} \partial_t^2 \delta\phi + \partial_r(r^2 f(r) \partial_r \delta\phi) + \frac{1}{\sin(\theta)} \partial_\theta(\sin(\theta) \partial_\theta \delta\phi) + \frac{1}{\sin(\theta)^2} \partial_\phi^2 \delta\phi = 0, \tag{1.4}$$

where separation of variables is applicable to identify that the angular part can be written on the basis of spherical harmonics $Y_{l,m}(\theta, \phi)$. One then anticipates a solution of the form

$$\delta\phi(x^\mu) = \sum_{lm} \frac{e^{-i\omega t} \psi(r)}{r} Y_{l,m}(\theta, \phi) \tag{1.5}$$

which yields a Schrödinger-like equation for the unknown radial part:

$$\partial_{r^*}^2 \psi + (\omega^2 - V_{eff}(r))\psi = 0. \tag{1.6}$$

Note the use of the tortoise radial coordinate satisfying $dr^* = \frac{1}{f(r)}dr$. The solution for the field is therefore determined by the effective potential

$$V_{eff}(r) = f(r)\left(\frac{l(l+1)}{r^2} + \frac{2GM}{r^3}\right). \tag{1.7}$$

The essence of QNMs stems from the boundary conditions that we apply to this differential equation, as we demand that our solution is only 'ingoing' at the BH horizon and 'outgoing' at radial infinity - in other words, that there is nothing that comes out of the horizon or approaches the BH from far away, as what we want to describe is the dissipation of a perturbation in the BH's proximity. This statement is quantifiable as one looks at the radial equation at these limits, noting that $V_{eff}(r^* \to \pm\infty) = 0$ and so $\psi \sim \exp(\pm i\omega r^*)$. We therefore impose

$$\begin{aligned} \psi(r^* \to -\infty) &\sim e^{-i\omega r^*} \\ \psi(r^* \to +\infty) &\sim e^{+i\omega r^*} \end{aligned} \tag{1.8}$$

which produce respectively the ingoing and outgoing behaviour.

With these conditions in place we now need to solve for the frequencies ω which satisfy them. This is an intricate task that can only be performed approximately with various techniques, which are extensively outlined by Berti, Cardoso and Starinets [2]. The result is a set of discrete, complex-valued frequencies labelled by the multipole index l and the overtone number n. The imaginary part of the solution contributes an exponential decay of the mode in time which models the gradual dissipation it undergoes as it settles down to zero.

1.2 Electromagnetic perturbations

Electromagnetic (EM) perturbations, or vector perturbations, on a flat Schwarzschild background can be treated in a similar fashion. Considering that the background EM field is zero, we investigate the dynamics of a perturbative field A^μ. The part of the action that describes a Maxwell EM field in the absence of a source current is

$$S_{EM} = \frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} \tag{1.9}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday field-strength tensor for the EM four-potential A^μ. The resulting equations of motion are

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0 \Leftrightarrow \nabla_\mu F^{\mu\nu} = 0. \tag{1.10}$$

Because the Faraday tensor is of the order of magnitude of the perturbation A^μ (the background EM field is zero), the perturbations of the metric $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$ once more contribute only terms of at least the order of $\mathcal{O}(h \cdot A)$ and therefore decouple at first order. The covariant derivative consequently should be computed only using the Christoffel symbols of the background Schwarzschild spacetime.

The angular part of the field can again be analysed on the basis of (vector) spherical harmonics. We will not explore the details of the separability of the angular parts here, and will instead use the expression of the four-vector field in terms of even and odd parity parts as in [4]. The field may thus be decomposed as

$$A_\mu = \sum_{lm}\left(\begin{bmatrix} 0 \\ 0 \\ \dfrac{a^{lm}(t,r)}{\sin\theta}\,\partial_\phi Y_{lm} \\ -a^{lm}(t,r)\,\sin\theta\,\partial_\theta Y_{lm} \end{bmatrix} + \begin{bmatrix} f^{lm}(t,r)\,Y_{lm} \\ h^{lm}(t,r)\,Y_{lm} \\ k^{lm}(t,r)\,\partial_\theta Y_{lm} \\ k^{lm}(t,r)\,\partial_\phi Y_{lm} \end{bmatrix}\right), \tag{1.11}$$

allowing for the definition of a single function ψ as a linear combination of the functions $\alpha^{lm}, f^{lm}, h^{lm}, k^{lm}$ which obeys the same differential equation derived for scalar perturbations,

$$\partial^2_{r^*}\psi + (\omega^2 - V_{eff,EM}(r))\psi = 0 \tag{1.12}$$

where the effective potential now takes the form

$$V_{eff,EM}(r) = f(r)\frac{l(l+1)}{r^2}. \tag{1.13}$$

1.3 Metric perturbations

When examining perturbations of the metric field $h_{\mu\nu}$ the task at hand becomes significantly more difficult as one needs to solve the linearised Einstein equations. One such approach to the solution can be found in full detail in Chandrasekhar's book [7]. It is expected, however, that since we are dealing with a vacuum perturbation of the metric, only two independent components will remain after all gauge degrees of freedom are fixed. As Berti et al [2] outline, by working in the Regge-Wheeler gauge, the perturbations of the metric are contained within two independent fields, giving rise to the Zerilli (or polar) perturbations $\Psi_Z(r)$ and the Regge-Wheeler (or axial) perturbations $\Psi_R(r)$. By writing the metric components in the perturbative part:

$$\begin{aligned} h_{(l)\mu\nu} = e^{-i\omega t}&\begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{bmatrix} \left(\sin\theta\frac{\partial}{\partial\theta}\right) Y_{l0}(\theta) \\ +e^{-i\omega t}&\begin{bmatrix} H_0(r)f & H_1(r) & 0 & 0 \\ H_1(r) & H_2(r)/f & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2K(r)\sin^2\theta \end{bmatrix} Y_{l0}(\theta). \end{aligned} \tag{1.14}$$

we can relate them to the Zerilli and Regge-Wheeler fields with the following expressions:

$$\begin{aligned} h_0 &= \frac{i}{\omega}\frac{d}{dr_*}(r\Psi_R) \\ h_1(r) &= \frac{r}{f(r)}\Psi_R \\ K &= \frac{6G^2M^2 + \xi(1+\xi)r^2 + 3GM\xi r}{r^2(3GM+\xi r)}\Psi_Z + \frac{d\Psi_Z}{dr_*} \\ H_1 &= \frac{i\omega\left(3G^2M^2 + 3\xi GMr - \xi r^2\right)}{r(3GM+\xi r)f(r)}\Psi_Z - \frac{i\omega r}{f(r)}\frac{d\Psi_Z}{dr_*} \end{aligned} \tag{1.15}$$

and then

$$\begin{aligned} &\left[(l-1)(l+2) + \frac{6GM}{r}\right]H_0 + \left[\frac{l(l+1)}{\omega r^2}GM - 2i\omega r\right]H_1 \\ &- \left[(l-1)(l+2) + rf' - \frac{4\omega^2r^2 + r^2f'^2}{2f}\right]K = 0, \end{aligned} \tag{1.16}$$

where $\xi = \frac{1}{2}(l-1)(l+2)$. After this decomposition is performed, one finds that the Regge-Wheeler and Zerilli fields are obtained by a Schrödinger equation just like in the scalar field's case. Now the effective potentials are given by the expressions

$$V_R(r) = f(r)\left[\frac{l(l+1)}{r^2} - \frac{6GM}{r^3}\right] \tag{1.17}$$

$$V_Z(r) = \frac{2f(r)}{r^3}\frac{9G^3M^3 + 3\xi^2 GMr^2 + \xi^2(1+\xi)r^3 + 9G^2M^2\xi r}{(3GM+\xi r)^2}. \tag{1.18}$$

The problem is now reduced to an equivalent of the case of the scalar field, once the same boundary conditions, akin to (1.8), are applied to Ψ_R and Ψ_Z.

1.4 Waveform of the perturbations

The preceding discussion has allowed us to establish that perturbations of any tensor field of physical interest will be governed by a differential equation of the form

$$\partial_x^2\psi(x) + (\omega^2 - V(x))\psi(x) = 0, \tag{1.19}$$

covering the region exterior to the horizon with $x = r_* \in \mathbb{R}$, which is itself traced back to a wave equation with a spacial potential

$$-\partial_t^2\phi(t,x) + \partial_x^2\phi(t,x) - V(x)\psi = 0 \tag{1.20}$$

by introducing the time dependence $\phi(t,x) = e^{i\omega t}\psi(x)$.

Reverting back to the nonseparated equation involving time is instructing when it comes to studying the evolution of the wavefunction $\phi(t,x)$ from a known initial state (that is, introducing initial conditions). This is achieved by taking the Laplace transform from a reference time t_0 to infinity:

$$\mathcal{L}[\phi](\omega,x) \equiv \tilde{\phi}(\omega,x) = \int_{t_0}^{\infty} dt\phi(t,x)e^{i\omega t}. \tag{1.21}$$

Note here that the Laplace transform is conventionally defined in terms of a complex, in general, argument s as

$$\mathcal{L}_{Lap}[\phi](s,x) = \int_0^{\infty} dt\phi(t,x)e^{-st}. \tag{1.22}$$

We henceforth work with the aforementioned definition which can alternatively be understood as a Fourier transform of the field weighted with a step function at t_0,

$$\mathcal{L}[\phi(t)](\omega,x) = \mathcal{F}[H(t-t_0)\phi(t)](\omega,x). \tag{1.23}$$

The transform $\mathcal{L}$ has the properties

$$\mathcal{L}[\partial_t\phi] = \phi e^{i\omega t}\Big|_{t_0}^{\infty} - i\omega\mathcal{L}[\phi] = -i\omega\tilde{\phi} - \phi_0 e^{i\omega t_0} \tag{1.24}$$

and consequently

$$\mathcal{L}[\partial_t^2\phi] = -\omega^2\tilde{\phi} - (i\omega\phi_0 + \dot{\phi}_0)e^{i\omega t_0} \tag{1.25}$$

where $\phi_0 = \phi(t_0,x)$ and $\dot{\phi}_0 = \partial_t\phi(t_0,x)$, and we assumed that $\lim_{t\to\infty}\phi(t,x) = 0$. The inverse transform returns our original function computed at a time t provided $t > t_0$:

$$\begin{aligned}\mathcal{L}^{-1}[\tilde{\phi}](t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\tilde{\phi}(\omega,x)\\ &= \frac{1}{2\pi}\int_{t_0}^{\infty} dt'\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}\phi(t',x)\\ &= \int_{t_0}^{\infty} dt'\delta(t-t')\phi(t',x) \overset{t\geq t_0}{=} \phi(t,x).\end{aligned} \tag{1.26}$$

Taking then the Laplace transform of our differential equation yields

$$\partial_x^2\tilde{\phi} + (\omega^2 - V(x))\tilde{\phi} = I(\omega, x, t_0) \tag{1.27}$$

which is our original spacial equation with the addition of the source term

$$I(\omega, x, t_0) = (i\omega\phi_0 + \dot{\phi}_0)e^{i\omega t_0}. \tag{1.28}$$

The source $I(\omega, x, t_0)$ depends only on the initial state and rate of change of the field and formally allows us to prescribe initial data to our differential equation. This initial data can represent, for example, an excitation that was created in the field by external mechanisms (other fields that we have not 'coupled' to ϕ in our simple picture) and whose effect we can therefore fully describe with knowledge of the configuration $\phi_0, \dot{\phi}_0$ they generated at some instance t_0 after which they are no longer present. We will use for instance later on the approximate observation that the initial perturbation to the gravitational field following a black hole-forming merger is localised and centred close to the final black hole's photonsphere, at $r = 1.5R_s$, thus taking $u_0 \sim \delta(r - 1.5R_s)$. We note that any pair of initial data $u_0, \dot{u}_0$ must be a valid field configuration and thus should satisfy the field equation (with external sources or not).

The solution can then in principle be found in terms of the Green function $G(x, x')$ satisfying

$$(\partial_x^2 + \omega^2 - V(x))G_\omega(x, x') = \delta(x, x') \tag{1.29}$$

so that

$$\tilde{\phi}(\omega, x) = \int dx' G_\omega(x, x') I(\omega, x', t_0) \tag{1.30}$$

and therefore

$$\phi(t, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t} \int dx' G_\omega(x, x') I(\omega, x', t_0). \tag{1.31}$$

It should be noted that the equation (1.29) does not have a unique solution, but rather the solution is unique up to adding solutions of the homogeneous equation. This translates to the freedom of fixing the asymptotic behaviour of G_ω by adding or subtracting plane wave solutions of the homogeneous equations which may be ingoing or outgoing. By asymptotic, we refer to the regions $x \to \pm\infty$ where $V(x) \to 0$. We are therefore free to impose ingoing/outgoing boundary conditions of our choice on the Green function in each asymptotic region, thereby imposing the same conditions to our solution $\phi(t, x)$ which is evolved in time by means of the Green function.

Given that we are interested in solutions that are solely outgoing (as they result from a localised excitation 'near' $x \approx 0$ and propagate outwards) our boundary conditions of choice are that the Green function $G_\omega \sim e^{i\omega x}$ as $x \to -\infty$ and $G_\omega \sim e^{-i\omega x}$ as $x \to \infty$. The solution (see the Appendix E for the relevant computations)

$$G_\omega(x, x') = \frac{u_\omega(x')}{W(v_\omega, u_\omega)} v_\omega(x) H(x' - x) + \frac{v_\omega(x')}{W(v_\omega, u_\omega)} u_\omega(x) H(x - x'). \tag{1.32}$$

is expressed in terms of the functions $u(x), v(x)$ which are solutions of the homogeneous equation and carry the required asymptotic behaviour

$$\begin{aligned} v_\omega(x) &= e^{i\omega x}, \quad \text{as} \quad x \to -\infty \\ u_\omega(x) &= e^{-i\omega x}, \quad \text{as} \quad x \to \infty, \end{aligned} \tag{1.33}$$

and their Wronskian $W(v_\omega, u_\omega) = v_\omega \partial_x(u_\omega) - u_\omega \partial_x(v_\omega)$.

We can now focus on our solution's properties in terms of the behavior of the Green function. Let us write

$$\phi(t,x) \sim \int_{-\infty}^{\infty} d\omega e^{-i\omega t} G(\omega) I(\omega) \tag{1.34}$$

forgetting about the spacial dependence for a moment. Recalling that the source is $I(\omega) = (i\omega\phi_0 + \dot{\phi}_0)e^{i\omega t_0}$ we can infer that it is regular in ω and, for any physically reasonable initial excitation, it should be zero for $|\omega| \to \infty$. The latter requirement simply means that we do not choose to excite our field with initial data carrying absurdly high frequencies $\mathrm{Re}\,\omega$ or decay rates $\mathrm{Im}\,\omega$.

The integrand therefore contains poles in the complex ω plane that come from poles of the Green function $G(\omega)$, which amount to zeros of the Wronskian $W(v_\omega, u_\omega)$. By applying Cauchy's theorem along a half-circle on the upper lower ω plane (so that $\mathrm{Im}\,\omega < 0$ and so $e^{-i\omega(t-t_0)}$ is decaying when $t > t_0$) we find that the solution contains a superposition of modes carrying the pole frequencies

$$\phi(t,x) = \phi_{\text{prompt \& tail}}(t,x) + \sum_n C_n Res(\omega = \omega_n, e^{-i\omega t} G(\omega) I(\omega)). \tag{1.35}$$

The residue from Cauchy's theorem will thus produce terms that oscillate with the frequencies ω_n which are the zeroes of the Wronskian - these are the quasinormal modes of the system. The frequencies themselves are dependent on the qualities of the background system (and are thus often referred to as resonant frequencies of the spacetime), and the magnitude of their excitation is determined by the initial data, namely the source $I(\omega)$.

This allows us to sketch an idea of what the signal following a generic excitation looks like as a function of the time t (that is, for an observer stationed at fixed x) [1]. The part $\phi_{\text{prompt \& tail}}(t,x)$ of the signal refers to the prompt response and the late-time tails. Prompt response refers to qualities of the signal that mirror the initial excitation which propagates outwards at the maximum speed (light speed) and therefore constitutes the initial part of the signal. For instance, if the initial data is made up of a narrow pulse or a monochromatic wavepacket, the signal will at first mimic these qualities during the prompt response phase. The late time tails describe a power-law decay of the signal after the QNM frequencies die out due to their nonzero imaginary part, and are shown to exist when the Green function has branch cuts in its analytical form.

The QNM frequencies ω_n are present in the signal if the initial perturbation excites them, and decay in time at a rate given by their imaginary part, that is, at a timescale $\tau_n \sim \frac{1}{\mathrm{Im}\,\omega_n}$. In principle one expects at early times to observe a mix of all excited QNMs superimposed on top of each other, but because the decay rates τ_n generally differ amongst themselves, in practice the QNMs with the longest lifetime τ_n are those that will stand out and may therefore actually be measurable. In particular, sorting the ω_n by decreasing lifetime τ_n, primarily the fundamental $n = 0$ and the first overtone $n = 1$ are considered to be distinguishable in a realistic gravitational wave signal.

1.5 Damped oscillator model for asymptotic modes

In this paragraph we conclude our overview of QNMs by highlighting some of their attributes that have been speculated to relate to the quantum nature of gravity. The

[1] Note here that in this discussion we effectively treat t as the retarded time $t_{ret} = t - x$ which is zero when the signal reaches the observer at x. The reason for this is that the Green function additionally contributes an exponential $e^{i\omega x}$ which is necessary for the propagation of a disturbance at a finite speed $c = 1$; we will not show however, for simplicity, the spacial dependence of the solution.

discussion that we touch upon here has been motivated by reasoning in the spirit of Bohr's correspondence principle, which asserts that quantum systems that approach their classical limit feature transitions which correspond to discretised configurations of a classical field, the latter serving as the outlet or the provider of the transferred energy. In our case the various fields that are discretised with QNM frequencies are viewed as outlets for the transitions of a quantum black hole during the last stages of its ringdown.

We will thus go over some of the observations made by Maggiore in [13], where he suggests that the quantum system in question resembles a damped harmonic oscillator. Maggiore emphasises some curious properties of the QNM frequency spectrum, depicted in 1. These include:

- The real part of QNM frequencies for gravitational perturbations features an initial decline, reaches zero, and increases again by tending to some asymptotic value, while the damping rate increases almost linearly. This is unusual for the energy spectrum of oscillations where the least damped mode has the lowest energy, and higher energy modes damp faster.

- The asymptotic value for the real part of the QNM frequency is attained with linearly increasing damping rate. This is not what one would expect in regular systems, as in particular for acoustic oscillations, modes with high frequency and thus small wavelength are expected to wear off quickly as the wavelength approaches the length scale of the material's lattice structure. The asymptotic behaviour of the gravitational $s = l = 2$ modes is given [2] as

$$8\pi GM\omega_n = \ln 3 + 2\pi i\left(n + \frac{1}{2}\right) + \mathcal{O}(n^{-1/2}). \tag{1.36}$$

  Fields of other spins feature similar large-n limits although to different, in general, values. This prevents one from interpreting this limit as characteristic of the quantum states of the black hole, which ought to be the same regardless of the field through which they decay.

For reasons like the aforementioned Maggiore proposes that the spectrum of the quantum black hole resembles that of a damped oscillator $\xi(t)$ satisfying

$$\ddot{\xi} + \gamma_0\dot{\xi} + \omega_0^2\xi = f(t) \tag{1.37}$$

where ω_0, γ_0 are parameters of the system related to its energy and damping rate, and $f(t)$ is an external force. Taking the latter to represent a 'kick' at $t = 0$ we pick $f(t) = \delta(t)$ and so taking the Fourier transform of the oscillator equation we have

$$(-\omega^2 - i\omega\gamma + \omega_0^2)\mathcal{F}[\xi](\omega) = \mathcal{F}[\delta](\omega) = 1 \tag{1.38}$$

and so

$$\xi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega t}}{(\omega - \omega_+)(\omega - \omega_-)} \tag{1.39}$$

where $\omega_\pm = \frac{\gamma_0}{2}i \pm \sqrt{\omega_0^2 - \left(\frac{\gamma_0}{2}\right)^2}$ are the roots that factorise the denominator's quadratic polynomial. The solution is thus found by integrating on the complex plane to be a linear superposition of positive and negative energy modes with frequency

$$\omega_R = \sqrt{\omega_0^2 - \left(\frac{\gamma_0}{2}\right)^2} \tag{1.40}$$

and damped as

$$\omega_I = \frac{\gamma_0}{2}. \tag{1.41}$$

In particular we observe the relation between the system's frequencies

$$\omega_0^2 = \omega_R^2 + \omega_I^2. \tag{1.42}$$

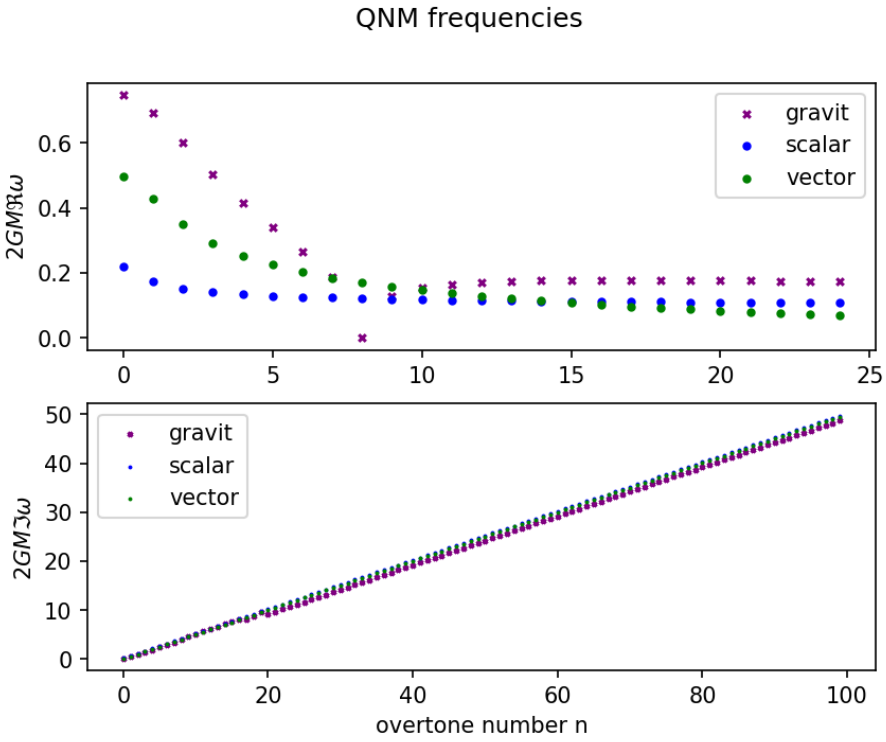

Figure 1: QNM frequencies of $l = s$ as a function of the overtone number n for spin $s = 0, 1, 2$ corresponding respectively to scalar, vector, and gravitational perturbations. QNM frequency values computed by Berti [2].

It is clear that the frequency ω_0 then is dominated by the imaginary part ω_I in the high overtone limit when the oscillation frequencies ω_R, ω_I are taken to be the QNM frequencies with the behaviour (1.36), yielding

$$\omega_0 \sim \omega_I \sim \frac{1}{4GM}\left(n + \frac{1}{2}\right) \quad \text{for} \quad n \gg 1. \tag{1.43}$$

This motivates Maggiore to propose that ω_0 represent the true energy levels of the quantum system, as its asymptotic behaviour resolves the paradoxical features of the real part of the quasinormal frequencies as described previously. Even though we will not adopt this view in the rest of this thesis, it is interesting to note that this linear dependence on n implies that the energy quantum for the transition $n \to n-1$ with high n is equal to

$$\delta M = \frac{\hbar}{4GM} \tag{1.44}$$

which, in the form of the Bekenstein-Mukhanov quantum (2.5), is the choice of quantum that had been originally proposed by Bekenstein.

2 Towards a quantum black hole

2.1 Mass quantisation à la Bekenstein-Mukhanov

Before we move on to a model of a quantum black hole that is founded atop the familiar principles of quantum field theory, we briefly turn to a simpler picture developed by Bekenstein and Mukhanov [1] anticipating the effects of the quantum nature of gravity. In this more general approach, one makes the minimal assumptions necessary to derive a law that governs the resulting quantisation of the black hole's mass.

The primary assumption is that the quantum nature of gravity will manifest as imposing a discretisation on the black hole's mass, or equivalently, its surface area. The authors argue in favour of a law of area quantisation into integer multiples of a fundamental 'tile' of area $\alpha\hbar$, where α is an unknown constant that may be fixed to select a model. At first glance, we thus have

$$A = \alpha\hbar n, \quad n \in \mathbb{N}. \tag{2.1}$$

This already becomes a constraint on the black hole's mass, as $M \sim \sqrt{A} \sim \sqrt{n}$.

The second assumption is that there is a real correspondence between the black hole's physical parameters and its thermodynamical properties, and specifically that the entropy of the $n-th$ excited state is related to its area by

$$S_n = \frac{1}{4\hbar} A_n + S_0 = \frac{\alpha}{4}(n-1) \tag{2.2}$$

where we additionally assumed that the ground state is unique, and thus $S_{n=1} = 0$.

The degeneracy g_n of the $n-th$ level is therefore given as $g_n = \exp(S_n)$. By recognising that g_n also has to be a positive integer, we obtain that

$$\alpha = 4 \ln k, \quad k \in \mathbb{N} \tag{2.3}$$

where the positive integer k is identified with $g_{n=2}$. With this restriction in place, the black hole's mass $M^2 = \frac{A}{16\pi G^2}$ is equal to

$$\left(\frac{M}{m_P}\right)^2 = \frac{\ln k}{4\pi} n \tag{2.4}$$

with m_P denoting the Planck mass, $m_P^2 = \hbar / G$.

If the black hole's mass is sufficiently large, then M can be treated as constant when considering small excitations. Taking for example the minimal transition $\Delta n = 1$, the resulting change in mass is

$$\frac{2M\Delta M}{m_P^2} = \frac{\ln k}{4\pi}$$

or, setting $\Delta M \equiv \hbar\omega$,

$$\omega = \frac{\ln k}{8\pi G M} \tag{2.5}$$

and so we call ω the Bekenstein-Mukhanov quantum that determines the permitted transitions for a black hole of mass M. Evidently, it depends on the unspecified positive integer k, which allows for some freedom in one's choice of model for the quantum black hole.

2.2 Some properties of coherent states

A coherent state $|g(k)\rangle = \hat{D}\,|0\rangle$ is defined from the vacuum state as the action of the exponential operator

$$\hat{D} = e^{\hat{X}} = \exp\left\{ \frac{\int_0^\infty dk k^2}{2\pi^2} (g(k)\hat{\alpha}_k^\dagger - \bar{g}(k)\hat{\alpha}_k) \right\} = \exp\left\{ \frac{\int_0^\infty dk k^2}{2\pi^2} \left(-\frac{\left[\hat{\alpha}_k, \hat{\alpha}_k^\dagger\right]}{2} |g(k)|^2 + g(k)\hat{\alpha}_k^\dagger \right) \right\} \tag{2.6}$$

where the last equality can be demonstrated by virtue of the identity

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}\left[\hat{A},\hat{B}\right]}. \tag{2.7}$$

We notice, to start with, that $\hat{X}^\dagger = -\hat{X} \Leftrightarrow \hat{D}^\dagger = \hat{D}^{-1}$ and thus $\hat{D}$ is unitary.

We will prove a property of $\hat{D}$ that will allow us to see exactly what field configuration it creates. First, the adjoint action of $\hat{D}$ on an annihilation operator, expanded in terms of the infinitesimal generator $\hat{X}$, is

$$\hat{D}^{-1}\hat{\alpha}_k\hat{D} = \mathbb{I}\hat{\alpha}_k + \left[\hat{\alpha}_k, \hat{X}\right] + \frac{1}{2}\left[\hat{X}, \left[\hat{\alpha}_k, \hat{X}\right]\right] + ... \tag{2.8}$$

but

$$\left[\hat{X}, \hat{\alpha}_k\right] = \frac{\int_0^\infty dk' k'^2}{2\pi^2} \left(g(k') \left[\hat{\alpha}_{k'}^\dagger, \hat{\alpha}_k\right] - \bar{g}(k') \left[\hat{\alpha}_{k'}, \hat{\alpha}_k\right] \right) = -g(k)\mathbb{I} \tag{2.9}$$

where we imposed the commutation relation $\left[\hat{\alpha}_k, \hat{\alpha}_{k'}^\dagger\right] = \frac{2\pi^2}{k^2}\delta(k-k')$. Therefore

$$\hat{D}^{-1}\hat{\alpha}_k\hat{D} = \hat{\alpha}_k + g(k)\mathbb{I} \tag{2.10}$$

and all subsequent commutators vanish as $\mathbb{I}$ commutes with every operator. Similarly, we may show that

$$\hat{D}^{-1}\hat{\alpha}_k^\dagger\hat{D} = \hat{\alpha}_k^\dagger + \bar{g}(k)\mathbb{I}. \tag{2.11}$$

The action on the field therefore is

$$\hat{D}^{-1}\hat{\phi}\hat{D} = \hat{\phi} + \frac{\int_0^\infty dk k^2}{2\pi^2} \left(g(k)u_k + \bar{g}(k)\bar{u}_k\right)\mathbb{I} = \hat{\phi} + \phi_{bkg}\mathbb{I}. \tag{2.12}$$

and we see that a classical configuration ϕ_{cl} of the field emerged as a background, with expansion coefficients on the basis u_k given by the function $g(k)$.

In the context of a matrix element of an operator $\hat{O}[\hat{\phi}]$ that contains powers of $\hat{\phi}$, that could for instance be a scattering matrix element, we find that when the coherent state is present at the background of the states initial and final states $|in, g(k)\rangle = \hat{D}\,|in\rangle$ and $|out, g(k)\rangle = \hat{D}\,|out\rangle$, the element becomes

$$\langle out, g(k)|\,\hat{O}[\hat{\phi}]\,|in, g(k)\rangle = \langle in|\,\hat{O}\left[\hat{\phi} + \phi_{bkg}\mathbb{I}\right]\,|in\rangle \tag{2.13}$$

where we made use of the Hermiticity of $\hat{D}$, which allows us to insert the adjoint action at every polynomial term of $\hat{\phi}$ inside $\hat{O}$ as $\hat{D}^{-1}\hat{\phi}\hat{\phi}\hat{D} = \hat{D}^{-1}\hat{\phi}\hat{D}^{-1}\hat{D}\hat{\phi}\hat{D}$, and so on. The field is therefore shifted by the classical configuration at every evaluation of the operator $\hat{O}$.

We will lastly compute the expected number of excitations within the coherent state. The numbering operator reads

$$\hat{N} = \frac{\int_0^\infty dk k^2}{2\pi^2}\hat{N}_k = \frac{\int_0^\infty dk k^2}{2\pi^2}\hat{\alpha}_k^\dagger\hat{\alpha}_k. \tag{2.14}$$

To proceed, we utilise the key characteristic of a coherent state that it is an eigenstate of the annihilation operator $\hat{\alpha}_k$ with eigenvalue determined by $g(k)$. This can be shown by noticing that

$$\left[\hat{\alpha}_k, \hat{D}\right] |0\rangle = \hat{\alpha}_k \hat{D} |0\rangle - \hat{D}\hat{\alpha}_k |0\rangle = \hat{\alpha}_k |g(k)\rangle , \tag{2.15}$$

but

$$\left[\hat{\alpha}_k, \hat{D}\right] = [\hat{\alpha}_k, \mathbb{I}] + \left[\hat{\alpha}_k, \hat{X}\right] + \frac{1}{2}\left[\hat{\alpha}_k, \hat{X}^2\right] + ... = g(k)\hat{D} \tag{2.16}$$

where all n-order terms in $\hat{X}$ reverted to the respective $n-1$-order order term as the commutator $\left[\hat{\alpha}_k, \hat{X}\right]$ is proportional to the identity. Then,

$$\hat{\alpha}_k |g(k)\rangle = g(k) |g(k)\rangle . \tag{2.17}$$

This allows us to compute the expectation value of $\hat{N}_k$ as

$$\langle g(k)| \hat{N}_k |g(k)\rangle = |g(k)|^2 \tag{2.18}$$

and so

$$\langle \hat{N} \rangle = \frac{\int_0^\infty dk k^2}{2\pi^2} |g(k)|^2 \tag{2.19}$$

2.3 The quantum black hole as a coherent state

The tools of quantum field theory allow one to develop the quantum black hole within a familiar and workable framework. Nevertheless, we should be careful with what we mean by a quantum black hole: it is a common expectation that a black hole whose collapse is ultimately brought to a stop by quantum effects in its 'core' will turn out to be a completely regular object, that is, it will not make its spacetime singular. But in our case, lacking a way to describe the collapse of quantum matter effectively and make such conclusions, we will look at a quantum black hole as a quantum object whose expectation value reproduces its classical (and therefore, singular) counterpart.

As is usually assumed for classical black holes, we will suppose that the entirety of its matter, that is sourcing the gravitational field, is accumulated and hidden away at the singularity, and so the field that we will endow with quantum properties is the gravitational field itself. Even though quantising the full metric tensor is known to entail various difficulties (when it comes to, for example, its renormalisability), our approach here will be to quantise a minimal number of the metric tensor's components in order to anticipate quantum effects.

The expectation value of our quantum fields should yield the Schwarzschild metric (1.1):

$$ds^2 = \langle \hat{g}_{\mu\nu} \rangle dx^\mu dx^\nu = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2.$$

Here we define the quantum field $\hat{\phi}$ such that $\hat{f} = 1 + 2\hat{\phi}$ and $\langle \hat{f} \rangle = f(r) \Leftrightarrow \langle \hat{\phi} \rangle = -\frac{GM}{r}$. This expectation value is attained on the state $|M\rangle$ that corresponds to the gravitational field of a Schwarzschild black hole, characterised only by its mass M.

We will subsequently work with the assumption that $|M\rangle$ is a coherent quantum state across all wavenumbers k:

$$|M\rangle = \exp\left\{ \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \left(-\frac{|g(k)|^2}{2} + g_k \hat{\alpha}_k^\dagger \right) \right\} |0\rangle \tag{2.20}$$

where the complex function $g(k) = g_k \exp(i\gamma_k)$ contains the amplitude g_k and the phase γ_k of the coherent state for the wavenumber k: the condition $\langle M| \hat{\phi} |M\rangle = -\frac{M}{r}$ then suffices to determine $g(k)$. Using the result (2.13), we have that

$$\langle M| \hat{\phi} |M\rangle = \langle 0| \hat{\phi} |0\rangle + \phi_{bkg} \langle M| \mathbb{I} |M\rangle = \phi_{bkg} \tag{2.21}$$

and, by the expansion of the field configuration $-\frac{m}{r}$ on the basis of monopole Klein-Gordon modes as performed in Appendix A resulting in (6.6), we get

$$\begin{gathered} -\frac{GM}{r} = \phi_{bkg} \\ \int_0^\infty \left(-\frac{4\pi}{\sqrt{2k^3}} \frac{M}{m_p} e^{ikt} u_k - \frac{4\pi}{\sqrt{2k^3}} \frac{M}{m_p} e^{-ikt} \bar{u}_k \right) = \int_0^\infty dk \left(g(k) u_k + \bar{g}(k) \bar{u}_k \right) \\ g(k) = -\frac{4\pi}{\sqrt{2k^3}} \frac{M}{m_p} e^{ikt} \end{gathered} \tag{2.22}$$

or, simply put, the coherent state functions $g(k)$ are the same as the expansion coefficients of the field configuration that is represented by the state.

Using the relation (2.19) we may compute the expectation number of excitations in our state as

$$\langle \hat{N} \rangle = \int_0^\infty \frac{dk k^2}{2\pi^2} \frac{8\pi^2}{k^3} \left(\frac{M}{m_P} \right)^2, \tag{2.23}$$

or,

$$\left(\frac{M}{m_P} \right)^2 = \frac{1}{4 \int_0^\infty \frac{dk}{k}} \langle \hat{N} \rangle. \tag{2.24}$$

It is impressive how this result closely resembles the Bekenstein-Mukhanov quantisation law, which was derived using completely different means by assuming that the BH horizon is made up of quanta of area proportional to $\hbar$ and is associated to an entropy that counts the microstates of some abstract quantum system. This similarity between the two descriptions is the reason we will use in the rest of this thesis the coherent BH model to describe quantum processes involving a BH that features quantised energy levels following the Bekenstein-Mukhanov law. This is a working assumption that, even though it is seemingly justified by the aforementioned similarity between the two approaches, cannot go without comment on whether the expressions (2.4) and (2.24) are equivalent.

To start with, the latter expression features an integral on the wavenumber magnitude k that diverges both in the ultraviolet (UV: $k \to \infty$) and in the infrared (IR: $k \to 0$) limit, and must be regularised by placing cutoffs $k_{IR} << k_{UV}$ that restrict us to an intermediate range of length scales. The UV cutoff is perhaps expected as the model has no way to resolve the BH's central singularity and so it should not be valid under some small length scale. The IR cutoff is more peculiar in the sense that the spacetime is taken to be asymptotically flat with no such divergences. We obtain, then,

$$\left(\frac{M}{m_P} \right)^2 = \frac{1}{4 \ln \left(\frac{k_{UV}}{k_{IR}} \right)} \langle \hat{N} \rangle. \tag{2.25}$$

One can thus choose the ratio of the cutoffs to reproduce some desired BM quantisation law

$$\frac{k_{UV}}{k_{IR}} = e^{\frac{\pi}{\ln q}} \tag{2.26}$$

where q is the integer constant of the BM quantisation. It should be noticed that no integer choice of q can provide a several orders of magnitude large ratio of the cutoffs.

The application of these constraints nevertheless means that our coherent state is actually reproducing not the field configuration $\phi_{bkg} = -\frac{GM}{r}$, but rather, since

$$g(k) = -\frac{4\pi}{\sqrt{2k^3}} \frac{M}{m_p} e^{ikt} H(k - k_{IR}) H(k_{UV} - k), \tag{2.27}$$

the configuration obtained by these expansion coefficients is, in the $k_{IR} \to 0$ limit [5],

$$\phi_{bkg,corr}(r) = \frac{2}{\pi} \phi_{bkg}(r) \, \mathrm{Si}\left(\frac{r}{R_{UV}}\right) \tag{2.28}$$

where $R_{UV} = 1/k_{UV}$ and $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt$ is the sine integral function.

The field $\phi_{bkg,corr}$ can loosely be regarded as our model's quantum correction to the Schwarzschild metric, emerging from the presence of some sort of quantum matter in the near centre of the BH (thus invalidating the 'vacuum' condition that necessitates the Schwarzschild metric). This however is not our point presently. In spite of the need to impose cutoffs that seem more or less physically motivated, we recognise that the silver lining in our computation is that the same Bekenstein-Mukhanov-like law as in (2.24) will be obtained as long as the coefficients $g(k)$ are linear in M/m_P, although with a different divergent integral. The possibility therefore is open that an alternative quantisation scheme can match the BM law with lesser need for regularisation.

The primary concern regarding expression (2.24) should nonetheless be the interpretation of the quantity $N \equiv \langle \hat{N} \rangle$. In the case of BM, one deals with mass levels determined by the parameter n that is established to be integer. N, however, is clearly not an integer as it is an expectation value of the occupation number for a state that is not its eigenstate. The coherent state, by expanding the exponential $e^{\hat{X}} |0\rangle = \sum_m \frac{1}{m!} (\hat{X})^m |0\rangle$, is a superposition of an infinite series of states created by the action of various ladder operators. By its property (2.17) that is an eigenstate of $\hat{\alpha}_k$, we have got

$$\hat{\alpha}_k^\dagger \hat{\alpha}_k |g(k)\rangle = g(k) \hat{\alpha}_k^\dagger |g(k)\rangle \tag{2.29}$$

which does not return the state $|g(k)\rangle$ as it's not also an eigenstate of $\hat{\alpha}_k^\dagger$ as well. Therefore $|g(k)\rangle$ is not an eigenstate of $\hat{N}_k$ or $\hat{N}$.

It can nevertheless be argued that, although not an integer, N can be considered to increase in integer increments with the addition of point particles. The latter, roughly speaking, are represented by a delta-narrow configuration, and this is constructed by one mode of each wavenumber, thereby increasing each $\langle \hat{N}_k \rangle$ by one.

3 QNM scattering on the quantum black hole

Having proposed that a black hole, from the point of view of quantum gravity, is characterised by a Bekenstein-Mukhanov-like discretisation of its mass, the necessity emerges to reconcile this picture with our classical notion of a black hole being a 'perfect absorber': how can a black hole take only a discrete set of energies, and yet devour everything that crosses its horizon with no chance of anything escaping? Where exactly can the energy difference between the infalling object's energy and the black hole's energy gaps be accommodated?

Quasinormal modes (QNMs) provide a simpler setup to tackle this issue in a quantifiable manner: they represent a full basis on which a perturbation of a (scalar, vector or metric) field can be decomposed around the black hole's vicinity, so long as it satisfies the conditions that it is made of infalling modes on the horizon and outgoing modes at radial infinity - as we would expect of a perturbation generated near the black hole, for example, after a merger. The energy carried by an isolated quasinormal mode clearly has no relation to the black hole's energy gaps. The behaviour of the perturbative modes near the horizon should therefore encapsule the essence of the aforementioned conundrum.

3.1 Discussion of approaches

There have been attempts in the relevant literature to tackle this issue using heuristic models since a full quantum description of gravity and its interaction with matter fields remains problematic. One such solution bases itself upon the familiar notion of 'width of absorption lines' that is typical of bounded quantum systems, and employs a modified horizon boundary condition, where a reflectivity is introduced that depends on the energy difference between the mode and the black hole's levels (see, for instance, [3]). In other words, each level is assigned a width, that determines the probability of absorption of an incoming excitation.

This approach comes with a few problems. The most practical of those is that it renders the computation of QNM frequencies in the classical picture invalid and calls for their recalculation using the modified, energy-dependent boundary condition on the horizon. Such a task appears difficult, especially given the notorious challenge that is accurately computing classical QNMs.

On the conceptual side of things, the suggestion of reflectivity of a black hole's horizon is no less troubling. If we are looking to accept this solution - of a probability-driven reflection of 'wrong energy' infalling excitations - as the mechanism that resolves the issue at hand, then it should no doubt extend to all types of fields, massless or massive. And even though massless excitations are immune to the concept of a comoving reference frame as they represent particles travelling at light speed, the same is not true for massive fields: a massive excitation, that represents an infalling particle, can always be studied from the local reference frame where the particle is motionless and inert. Then the question arises of, what exactly would such an observer see when the particle gets reflected at the horizon?

From the point of view of the freefalling particle, the horizon is a surface no more special than any surface that comes before or after it, and there is no mechanism that can be responsible for 'kicking' the particle out of its inertial motion. Note here that the notion of a localised particle can be justified in this context since the mode in question can be of arbitrarily high frequency, and so a wavepacket of high frequency modes can in principle be constructed.

Sometimes in the relevant literature it is thus suggested that the horizon carries some structure that interacts with the particle with quantum processes. One such example (developed in [6, 12]) is the use of a membrane-like surface made up of a shell-shaped fluid whose dynamics reproduce the phenomenology of a horizon with partial reflectivity as it interacts with exterior fields. This can be done either by endowing the fluid with quantum properties, or by considering a classical membrane on a quantum spacetime. In any case, the membrane is present to mediate the interaction between the quantum black hole and the exterior fields, and represents the point of view that quantum effects of unclear type occur in the region of the horizon's vicinity.

The effects of this proposition, in any of its forms, have been argued to be observable (though the opposite case has also been made - see [8]) as one expects the reflected modes to emerge, bar some partial backscattering, to infinity. This means that in the context of a black hole's ringdown, the faraway observer will measure the modes that were initially outgoing towards infinity, but they will also receive with some delay modes that were infalling and reflected with some probability. This effect is termed 'echoes' of gravitational waves and is largely expected to be observable evidence for the quantum nature of gravity. An important feature to notice is that echoes, if real, should appear at the same frequency of the initial signal, with some time delay, and significantly lower amplitude, not just because of backscattering in the black hole's field, but majorly because the horizon's reflection coefficient is not one.

3.2 Quantum scattering and dissipation as a resolution

In this thesis we will go after an alternate way of resolving the issue of inconsistency between the energy of the excitation and the quantum system. The mechanism we propose resolves the issue is that of dissipation through gravitational radiation. In order to make our point, we will briefly recall a more familiar example that exhibits similar features as those of our system, namely that of a hydrogen atom.

Consider a lone proton as the source of a Coulomb potential, that acts an an attracting potential to passing electrons. One can throw an electron towards the proton, offering any desired amount of energy in the form of kinetic energy. The electron will no doubt be attracted and captured by the proton and form a final state that is the hydrogen atom. Even though its arbitrary energy will initially be able to be accommodated among the various orbitals of the atom, occupying a superposition of states whose average energy equals what was provided, the system will eventually radiate some of its energy through its time-varying multipoles and settle into an eigenstate of the Hamiltonian, that is, a specific orbital. A key observation here is that the carrier of the excess energy, as the system settles into an eigenstate, is the electromagnetic field, which allows the energy to dissipate as expelled photons.

Note that other potential alternatives, such as excess energy becoming kinetic energy of some form to the atom, can be ruled out by modifying the aforementioned setup - net momentum can, for instance, be made zero by considering two central protons and two infalling electrons with opposite momenta.

Back to our query, the correspondence of roles with the hydrogen atom scenario is clear: the black hole generates the central attracting potential, the infalling excitation (such as a QNM) carries an arbitrary amount of energy, and the final state, a larger black hole, is reached after the excess energy has been dissipated through gravitational radiation, in other words, by emitting soft gravitons.

There are two major differences between this approach and the horizon reflectivity as proposed earlier. First of all, no matter what the infalling particle is (scalar, photon,

graviton, massive spinor, ...) the emitted radiation will be entirely composed of gravitons - the black hole remains a perfect absorber! The infalling observer will not be kicked from the black hole, but they will disturb the spacetime which will radiate this energy before it settles into Schwarzschild once again.

Secondly, the difference persists even if the infalling excitation is also gravitational, such as in the case of QNMs representing gravitational waves after a merger. The reason is that the outgoing gravitons will carry only the excess energy, but they will be emitted at every instance of an infalling particle (roughly speaking, with probability equal to one) - this is in contrast with the partial reflection viewpoint, where the outgoing signal has only a fraction of the original amplitude. We therefore anticipate that the signal that emerges from the horizon to infinity will have the same amplitude as what reached the horizon (excluding backscattering effects, and across all the prominent transitions) and, instead of the frequency of the original excited mode, will contain the energy increments corresponding to the transitions that occured.

Schematically speaking, an infalling mode of frequency ω corresponding to a quantum of energy $E = \hbar\omega$ reaching the horizon will be absorbed and the black hole will undergo the transition from mass M to $M + \delta M$, where $\delta M = n\hbar\omega$ is a multiple of the Bekenstein-Mukhanov quantum, and the emitted gravitons will carry $E' = E - \delta M$. In the following we will therefore attempt to estimate the probability of transition as a function of δM. Knowledge of the relative probability of all transitions we consider possible (we will restrict ourselves to processes of a single outgoing graviton for reasons to be elaborated upon later) and normalising so that the cumulative probability adds up to one will allow us to compute the anticipated spectrum of this mechanism, which we will produce for gravitational excitations as in a realistic ringdown scenario.

3.3 Backreaction on the coherent state

Since our model for the quantum black hole is that of a graviton coherent state, our method of choice for estimating the transition probability will be the QFT scattering amplitude. In particular, the amplitude will be obtained by the scattering matrix element

$$\mathcal{A} = \langle out; M + \delta M | \, \hat{S}[\hat{\phi}] \, | in; M \rangle \tag{3.1}$$

where $|M\rangle$ represents a coherent state of a black hole of mass M and $|in\rangle$ and $|out\rangle$ will be taken to be single graviton states, representing respectively a quantum of a QNM excitation and a graviton that carries the excess energy away. The latter states feature some intricacies due to the nontrivial relation of the Zerilli and Regge-Wheeler fields, whose modes are spherical waves near the horizon, to the metric component $\hat{f} = 1 + 2\hat{\phi}$ that we quantised to produce the coherent state of the background - we will deal with this in the next paragraph. Presently we will treat the issue of transitioning from one coherent state to another, which is effectively the backreaction of the process on the background spacetime, using the formulation developed in [10].

We proceed to evaluate (3.1) by pulling out the coherent state operators. We will denote the functions $g(k)$ that determine the coherent states $|M\rangle$ and $|M' = M + \delta M\rangle$ respectively as $g_i(k)$ and $g_f(k)$. The coherent states thus have the form given in (2.6). Notice that we will show the dependence of $\hat{S}$ on the field $\hat{\phi}$ writing the ladder operators

instead, as in $\hat{S}[\hat{\phi}] = \hat{S}[\hat{\alpha}, \hat{\alpha}^\dagger]$. Then,

$$\begin{aligned}
\mathcal{A} &= \langle out; M + \delta M |\, \hat{S}[\hat{\alpha}, \hat{\alpha}^\dagger] e^{\int dk \left(-\frac{|g_i(k)|^2}{2} + g_i(k)\hat{\alpha}_k^\dagger \right)} \,|in\rangle \\
&= e^{\int dk \left(-\frac{|g_i(k)|^2}{2} \right)} \langle out; M + \delta M |\, \hat{S}[\hat{\alpha}, \hat{\alpha}^\dagger] e^{\int dk \left(g_i(k)\hat{\alpha}_k^\dagger \right)} \,|in\rangle \\
&= e^{\int dk \left(-\frac{|g_i(k)|^2}{2} - \frac{|g_f(k)|^2}{2} \right)} \langle out |\, e^{\int dk (\bar{g}_i(k)\hat{\alpha}_k)} \hat{S}[\hat{\alpha}, \hat{\alpha}^\dagger] e^{\int dk \left(g_i(k)\hat{\alpha}_k^\dagger \right)} \,|in\rangle \\
&= e^{\int dk \left(-\frac{|g_i(k)|^2}{2} - \frac{|g_f(k)|^2}{2} \right)} \langle out |\, e^{\int dk \left(\bar{g}_f(k)\hat{\alpha}_k \right)} e^{\int dk \left(g_i(k)\hat{\alpha}_k^\dagger \right)} \hat{S}[\hat{\alpha} + g_i, \hat{\alpha}^\dagger] \,|in\rangle \\
&= e^{-\frac{1}{2}\int dk \left(|g_i(k)|^2 + |g_f(k)|^2 - 2\bar{g}_f(k) g_i(k) \right)} \cdot \\
&\quad \cdot \langle out |\, e^{\int dk \left(g_i(k)\hat{\alpha}_k^\dagger \right)} \hat{S}[\hat{\alpha} + g_i, \hat{\alpha}^\dagger + \bar{g}_f] e^{\int dk \left(\bar{g}_f(k)\hat{\alpha}_k \right)} \,|in\rangle
\end{aligned} \tag{3.2}$$

and this, in spite of its looks, is a significantly simplified expression for our amplitude. Note that in the last two steps we used properties similar to (2.10), (2.11), modified for our case where the acting operator is $\exp\{\int dk \left(g(k)\alpha_k^\dagger \right)\} \equiv e^{\hat{Y}}$. The adjoint action is

$$\begin{aligned}
e^{-\hat{Y}} \hat{\alpha}_k e^{\hat{Y}} &= \hat{\alpha}_k + [\hat{\alpha}_k, \hat{Y}] + ... = \hat{\alpha}_k + g(k)\mathbb{I} \\
e^{-\hat{Y}} \hat{\alpha}_k^\dagger e^{\hat{Y}} &= \hat{\alpha}_k^\dagger + [\hat{\alpha}_k^\dagger, \hat{Y}] + ... = \hat{\alpha}_k^\dagger
\end{aligned} \tag{3.3}$$

and so only one ladder operator is shifted. This results in an effective background for our S-matrix that is neither the initial nor the final background but a mix of the two, expanded on our basis as

$$\phi_{bkg} = \int dk \left(g_i(k) u_k + \bar{g}_f(k) \bar{u}_k \right). \tag{3.4}$$

Let us evaluate the background-dependent quantities, starting with the prefactor. Observe that, as we're interested in the probability

$$\mathcal{P} = |\mathcal{A}|^2,$$

the prefactor's imaginary phase is irrelevant and we essentially need to compute

$$\left| e^{-\frac{1}{2}\int dk \left(|g_i|^2 + |g_f|^2 - 2\bar{g}_f g_i \right)} \right|^2 = e^{-\int dk |g_f - g_i|^2} \tag{3.5}$$

which is shown by use of $|z - w|^2 = |z|^2 + |w|^2 - 2\,\mathrm{Re}\{z^* w\}$.

Using the result of (2.22), we recall that

$$g(k) = -\frac{4\pi}{\sqrt{2k^3}} \frac{M}{m_P}.$$

Because of the linearity in mass, the difference is thus

$$g_f(k) - g_i(k) = -\frac{4\pi}{\sqrt{2k^3}} \frac{\delta M}{m_P} \tag{3.6}$$

and so the exponential prefactor produces a dependence of the amplitude on δM as

$$\mathcal{P} = e^{-\frac{16}{\pi^2}\left(\int_0^\infty \frac{dk}{k} \right) \frac{\delta M^2}{m_P{}^2}} |\langle ... \rangle|^2. \tag{3.7}$$

We have obtained the remarkable result that the profile of the probability has a simple Gaussian dependence on δM centred around $\delta M = 0$, as is expected, and the exponent is scaled by the same divergent integral that we obtained when computing the expected excitation number $\langle \hat{N} \rangle$.

Notice that this result is independent of the details of the actual process, that is, the $|in\rangle$ and $|out\rangle$ states and the form interaction that drives the process (present in the Lagrangian) that are encapsuled in the amplitude $\langle ... \rangle$. It comes only from our coherent states description, and it could be used as an approximation for the profile of a generic process.

3.4 Single graviton states preparation

We now turn to the evaluation of

$$\langle out| \, e^{\int dk \left(g_i(k) \hat{\alpha}_k^\dagger \right)} \hat{S}[\hat{\phi} + \phi_{bkg}] e^{\int dk \left(\bar{g}_f(k) \hat{\alpha}_k \right)} \, |in\rangle \, . \tag{3.8}$$

The $|in\rangle$ and $|out\rangle$ states could in principle be considered to be single graviton states of momentum k in the sense that they are created by the action of a single $\hat{\alpha}_k^\dagger$ operator from the vacuum (atop the background). Such a mode, however, being a spherical wave solution of the Klein-Gordon equation on flat spacetime, does not resemble any physically sensible perturbation, and certainly not a QNM. In particular we would expect irregular behaviour on the horizon, like the dependence on the variable r^* produces for QNMs. The 'flat background' modes, however, do provide a basis on which any configuration of ϕ can be analysed, as we did in practice by reproducing the Schwarzschild configuration, and can, in principle, repeat to express a perturbative configuration.

To see what is the actual configuration of the field ϕ throughout the scattering, let us first compute the mean background

$$\phi_{bkg} = \int dk \left(g_i(k) u_k + \bar{g}_f(k) \bar{u}_k \right) = \phi_i + \int dk \delta \bar{g}_f(k) \bar{u}_k \tag{3.9}$$

where $\delta g(k) = g_f(k) - g_i(k)$ was evaluated in (3.6). From the form of the modes

$$u_k = u_0 e^{-ikt} \frac{\sin kr}{kr}$$

we can tell that the contribution of δg is real, in addition to the already real field ϕ_i. Thus

$$\phi_{bkg} = \frac{\phi_{bkg} + \bar{\phi}_{bkg}}{2} = \frac{1}{2} \left(\int dk \left(g_i(k) u_k + \bar{g}_f(k) \bar{u}_k + \bar{g}_i(k) \bar{u}_k + g_f(k) u_k \right) \right) = \frac{\phi_i + \bar{\phi}_f}{2} \tag{3.10}$$

and our mean background is ultimately a Schwarzschild spacetime of mass $M_{bkg} = M + \frac{1}{2}\delta M$.

Perturbations on top of the mean background can therefore be treated as in regular Schwarzschild perturbation theory, with their radial dependence satisfying a Schrödinger-like equation with an effective potential. Specifically for gravitational perturbations, we recall from Chapter 1 that their two independent degrees of freedom, encoded in the Zerilli (polar) and Regge-Wheeler (axial) functions $\Psi_Z(r)$ and $\Psi_R(r)$, are governed by the potentials (1.17) and (1.18). Notice here that our quantum field ϕ is associated to the metric components g_{00} and g_{11}, which are, from relations (1.16), obtained from the Zerilli field Ψ_Z in the Regge-Wheeler gauge employed here. We will therefore only concern ourselves in the following computations with Zerilli modes.

In order to evaluate the scattering amplitude, we need to define what our asymptotic (far from the scattering region) states are. To do that, we take on the point of view that quantum effects should occur near the horizon:

- Far from the horizon, we expect particles (representing all excitations) to freefall towards the black hole without experiencing any quantum effects. Since the quantum effect in our case is absorption by the black hole, it stands to reason that it is unlikely to occur anywhere far from the horizon.
- Inside the horizon, if any quantum processes of this type occur, they are not of our interest as any graviton emitted will have no chance of escaping the black hole and it will therefore be impossible for it to help dissipate the ingoing particle's excess energy. It is therefore not relevant to the issue at hand.
- The conclusion is that the $|in\rangle$ and $|out\rangle$ states, that each represent a single excitation of the Zerilli field, are prepared at some distance out of the horizon that is significantly larger than the Planck scale $l_P = \sqrt{\hbar}$ but also close enough to the horizon so we can reasonably expect to approximate $r \approx 2Gm \Leftrightarrow V_{eff} \approx 0$, which allows us to express Zerilli modes as spherical waves $\sim \frac{1}{r}\exp(-ik(t \pm r^*))$. This is perfectly achievable as the Planck scale is a lot smaller than the spacetime's length scale, the Schwarzschild radius $R_S = 2GM_{bkg}$.

In the following we therefore keep in mind that our 'asymptotic states' refer to Zerilli modes (or more appropriately, wavepackets thereof) that are located at rome distance L from the horizon, such that $\sqrt{\hbar} \ll L \ll R_s$, and at a time in the past or future from a reference $t_0 = 0$ that is large enough for a massless mode to travel there. In terms of the tortoise variable $r^* = r + 2GM_{bkg}\ln\left(\frac{r}{2GM_{bkg}} - 1\right)$, spanning the exterior region, we similarly denote $L^* = r^*(R_s + L)$ the spot of preparation.

The ingoing state is especially interesting to us when taken to represent a QNM of some given frequency. As mentioned before, a QNM is not one of the modes that we obtained upon quantising on the flat background, but is certainly expressible in terms of them. We could, for instance, create our desired ingoing state as

$$|in\rangle = \hat{\alpha}^\dagger_{(\psi)p}|0\rangle = \int dk\psi(p,k)\hat{\alpha}^\dagger_k|0\rangle \tag{3.11}$$

with $\psi(p,k)$ being some suitable profile function. The modes created by $\hat{\alpha}_k$ are nevertheless not suited to performing calculations on the Schwarzschild background, as will be evident when we employ the LSZ formula shortly. It is simpler to observe that a perturbative configuration can be expanded in any complete set of modes, and one such basis is that of the solution of the Klein-Gordon equation on the curved background. Especially in our region of interest where $r^* \in (-\infty, L^*)$ where $V(r) \approx 0$, these modes are the spherical waves (as elaborated upon in the Appendices A and C)

$$\tilde{u}_{k\sigma lm} \sim e^{-ik(t-\sigma r^*)}Y_{l,m}(\theta,\phi)R_l(r) \tag{3.12}$$

for a solution R_l of the l radial equation that is of the form of a Bessel function - hereon we will restrict ourselves to the $l = 2, m = 0$ modes matching the fundamental mode of a gravitational perturbation,

$$\tilde{u}_{k\sigma} \sim \frac{e^{-ik(t-\sigma r^*)}}{r}Y_2(\theta). \tag{3.13}$$

Such a change of basis of our modes is a Bogoljubov transformation and corresponds to defining the new creation operators, giving a new meaning to our previous suggestion that was

$$\hat{\tilde{\alpha}}_p^\dagger = \int dk \psi(p,k) \hat{\alpha}_k^\dagger \tag{3.14}$$

where the function $\psi(p,k)$ is now our transformation coefficient. Because there is no inclusion of negative energy modes (no contribution from annihilation operators) the two bases share the same vacuum state and so the new creation operator can construct states on top of the old ones, that is, the coherent state.

We can now write our asymptotic, single-excitation, near-horizon states

$$\begin{aligned} |in\rangle &= \hat{\tilde{\alpha}}^\dagger_{\omega,-1} |0\rangle \\ |out\rangle &= \hat{\tilde{\alpha}}^\dagger_{\omega',1} |0\rangle \end{aligned} \tag{3.15}$$

where we recall that $E = \hbar\omega$ is the ingoing QNM's energy, $E' = \hbar\omega' = E - \delta M$ is the outgoing graviton's energy, and the labels $\sigma = \pm 1$ indicate whether the mode is ingoing or outgoing.

3.5 Computation of the amplitude

We have finally set the stage to proceed with the amplitude's calculations using the standard tools of perturbative QFT. Since the process we're interested in is a single graviton to single graviton procedure, one would regard the result as a sort of correction to the graviton propagator on our background. This is nevertheless not the case as our process explicitly violates energy conservation (between the two gravitons) and this would not be achieved by the Schwarzschild propagator, the latter being a static background.

The time evolution of the system is driven by the action (6.16) of our field expanded in terms of the Zerilli perturbation, as explained in Appendix C. The Lagrangian is Klein-Gordon-like and quadratic in the field, and therefore we can expect to obtain vertices with two legs when taking Wick contractions of our fields. We use this observation to drop the exponentials with the annihilation operators acting on our asymptotic states in (3.8): because we need an even number of field insertions to have nonvanishing Wick contractions, and the interaction Lagrangian will produce even insertions of fields, our asymptotic states need to have an even number of particles as well.

We therefore distinguish between four possibilities, which emerge as the four terms (schematically):

$$\langle 1| e^{\bar{g}_f \hat{\alpha}^\dagger} \hat{S} e^{g_i \hat{\alpha}} |1\rangle = \langle 1| \hat{S} |1\rangle + g_i \langle 1| \hat{S} |0\rangle + \bar{g}_f \langle 0| \hat{S} |1\rangle + \bar{g}_f g_i \langle 0| \hat{S} |0\rangle \tag{3.16}$$

where all higher terms vanish due to the action of annihilation operators on the vacuum. Remembering that $\hat{S}$ will contribute an even number of fields, the second and third terms are seen to be zero. The fourth term will consist of loop diagrams and can be ignored by choosing to restrict ourselves to tree-level calculations, as we will do. We can nevertheless observe additionally that to first order in our interaction Lagrangian, where two insertions will be made, the diagram will be that of a loop with a single vertex and will thus vanish as a dimensionless integral.

Before we proceed to apply the LSZ formula, we write the action that governs our field. This is the Klein-Gordon action with an additional potential term as in (6.16), with

the potential (6.20) which takes the form

$$U(r) = \frac{2}{r^3}\frac{9G^3M_{bkg}^3 + 3\xi^2 GM_{bkg}r^2 + \xi^2(1+\xi)r^3 + 9G^2M_{bkg}^2\xi r}{(3GM_{bkg}+\xi r)^2} - \frac{2GM_{bkg}}{r^3} - \frac{2(\xi+1)}{r^2} \tag{3.17}$$

where $\xi = \frac{1}{2}(l-1)(l+2)$.

A preliminary attempt

It is, in principle, our goal to treat the process as caused by an interaction that is perturbative in δM. The motivation for that is that we want to restrict the use of a full Schwarzschild propagator to the background and expand the perturbative part on top of it, thinking that a process with a nonzero δM cannot be caused by the background. We will see how this is sensible - and, sadly, fails - shortly. For now we will implement our second approximation, that is, that $r \approx 2GM_{bkg}$ for the entire duration of our process. Then, considering $\xi = 2$ for $l = 2$ gravitational perturbations,

$$U \approx \frac{1}{4G^2M_{bkg}^2}\left(\frac{189}{49} - 7\right). \tag{3.18}$$

The expansion in terms of δM can certainly be done as $\frac{1}{M_{bkg}^2} \approx \frac{1}{M^2}\left(1 - \frac{\delta M}{M}\right)$. We are nevertheless faced with the glaring issue that the potential is constant in our regime of interest.

Let us recall here that the $r \approx 2Gm$ approximation has been justified on the basis that our modes are $\exp(ikr^*)$ waves. This implies that whereas our potential, varying with r, has a Fourier transform with support on the wavenumbers of roughly the $\sim 1/R_S$ order of magnitude, the waves of the rapidly varying r^* variable carry much higher wavenumbers when contrasted with r waves. Pictorially, $r*$ waves are 'squished' near the horizon on the r axis, and conversely r modes, that construct $U(r)$, become infinitely elongated on the r^* axis as $r^* \to -\infty$. For this reason the radial dependence of the potential would not alter the results we are about to obtain (see the last-but-one line of (3.29)).

Regardless, we have been able to split our Lagrangian as $L = L_0 + L_{int}$ so that $L_{int} = \sqrt{-g_0}U_{int}\phi^2 = -\sin\theta\left(\frac{189}{49} - 7\right)\frac{\delta M}{M}\phi^2$. Then we may write the LSZ formula for our amplitude as

$$\left\langle 1_q\right| \hat{S}[\hat{\phi}_{bkg}] \left|1_p\right\rangle = \int d^4x\sqrt{-g}\int d^4x'\sqrt{-g}u_p(x)\bar{u}_q(x')\mathcal{K}_x\mathcal{K}_{x'}\langle \hat{T}\hat{\phi}_x\hat{\phi}_{x'}e^{i\int d^4y\hat{L}_{int}(y)}\rangle. \tag{3.19}$$

The formula is derived in Appendix D up to modifications to the standard expression accounting for the curved background.

The correlator is expanded in Wick contractions, yielding two-point functions, as we anticipated when we kicked the exponential insertions of annihilation operators. We remember that for the amplitude to be nonzero we need an even number of fields inside the correlator, which will be made as pairs into two-point correlators by the Wick contractions.

The two-point function is the Green's function of the equations of motion as we will demonstrate using the path integral formalism. Taking the partition function

$$Z[J] = \int D\phi \exp\{iS[\phi] + iJ\cdot\phi\} \tag{3.20}$$

where it is implied that $J \cdot \phi = \int d^4x \sqrt{-g} J(x)\phi(x)$, we expand around a variation of the field $\phi'(x) = \phi(x) + \epsilon(x)$ for which $\int D\phi' = \int D\phi$

$$Z[J] = \int D\phi \exp\{iS[\phi] + iJ \cdot \phi\} \left(1 + i\left(\frac{\delta S}{\delta \phi} + J\right) \cdot \epsilon\right) \tag{3.21}$$

and take functional derivatives with respect to $\epsilon(x)$ and $J(y)$ to obtain

$$\int D\phi \exp\{iS[\phi] + iJ \cdot \phi\} \left(\delta(x,y) + i\phi(y)\left(\frac{\delta S}{\delta \phi(x)} + J(x)\right)\right). \tag{3.22}$$

For $J = 0$ we then have that

$$\langle \phi_x \frac{\delta S}{\delta \phi_y} \rangle = i\delta_{xy} \tag{3.23}$$

where he highlight that the delta function satisfies the distribution relation

$$\int d^4x \sqrt{-g} \delta(x,y) = 1 \tag{3.24}$$

throughout this derivation.

In our case the action is

$$\begin{aligned} S[\phi] &= \frac{1}{2}\int d^4x \sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + U(x)\phi^2\right) \\ &= \frac{1}{2}\int d^4x \sqrt{-g}\left(-\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\partial_\nu\phi)\phi + U(x)\phi^2\right) \\ &= \frac{1}{2}\int d^4x \sqrt{-g}\left(-\phi\nabla^2\phi + \phi U(x)\phi\right) \\ &= \int d^4x \sqrt{-g}\int d^4y \sqrt{-g}\phi(x)\mathcal{K}(x,y)\phi(y) \end{aligned} \tag{3.25}$$

with $\mathcal{K}$ being the kinetic operator

$$\mathcal{K}(x,y) = \delta(x,y)(-\nabla_y^2 + U(y)). \tag{3.26}$$

With this rewriting we have that

$$\frac{\delta S}{\delta \phi(x)} = \int d^4y \sqrt{-g}\mathcal{K}(x,y)\phi(y) = \frac{1}{2}(\nabla_x^2 + U(x))\phi(x) \equiv \mathcal{K}_x\phi(x) \tag{3.27}$$

and so, from the two point function identity (3.23),

$$\mathcal{K}_y\langle\phi_x\phi_y\rangle = i\delta_{xy} \tag{3.28}$$

implying that the two-point function is the Green's function $\langle\phi_x\phi_y\rangle = G(x,y)$ of the equations of motion $\mathcal{K}_x$.

The first order term in δM which is responsible for our interaction is thus

$$\begin{aligned} & i\int d^4y\sqrt{-g}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g}u_p(x)\bar{u}_q(x')\mathcal{K}_x\mathcal{K}_{x'}U_{int}(y)\langle\hat{T}\hat{\phi}_x\hat{\phi}_{x'}\hat{\phi}_y\hat{\phi}_y\rangle \\ &= -i\int d^4y\sqrt{-g}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g}u_p(x)\bar{u}_q(x')\mathcal{K}_x\mathcal{K}_{x'}U_{int}(y)G(x,y)G(y,x') \\ &= -i\int d^4y\sqrt{-g}\int d^4x\int d^4x' u_p(x)\bar{u}_q(x')U_{int}(y)\delta(x,y)\delta(y,x') \\ &= -i\int d^4y\sqrt{-g}u_p(y)\bar{u}_q(y)U_{int}(y) \end{aligned} \tag{3.29}$$

that is, analogous to the Fourier transform of the interaction potential on the r^* (quadrupolar, as $l = 2$ has been imposed) spherical wave basis. The potential, however, is constant as we justified previously: its radial dependence is very slow compared to the modes, and its time dependence is nonexistent. The same goes for the volume element factor $\sqrt{-g}$ which, following the integration of the solid angle, is a function of r only. Therefore the integral amounts to four-momentum conservation between the p and q gravitons (as a regular constant interaction vertex does). But because $p \neq q$, specifically with $E_q = E_p - \delta M$ and $q_r = -p_r$, the amplitude vanishes.

3.6 A revisited setup and the final amplitude

It is surely odd that the amplitude of our process would be killed by what seems to be four-momentum . After all, the missing four-momentum between the particles p and q has been transferred to the background coherent state, contributing an increase to the black hole's mass.

The reason that our amplitude vanished can be traced back to our primary assumptions from the fact that the missing four-momentum needed to be encoded in the potential $U(r)$ that was computed on our background spacetime, which in turn was a merge of our initial and final coherent states. The fact that ϕ_{bkg} was itself a Schwarzschild spacetime of intermediate mass means that we cannot expect to get a process that could not happen in a usual Schwarzschild spacetime anyway. It is thus a matter of revisiting our definition of initial and final states.

In our setup, we assumed that, upon the absorption of the ingoing particle, the entire spacetime jumps to a state higher by δM. That is true only in the far future, and even though this can naively be seen as the $t \to \infty$ limit, it is not what we considered to be future infinity in the preparation of our states. In fact, we took the asymptotic states to represent waves in the vicinity of the horizon. This implies that the transfer of mass to the black hole has not taken place yet; the mass needs to propagate along lightlike geodesics to the centre of the black hole.

At first approximation, this could be modelled as Vaidya-type addition to our initial metric, where the matter would fall along a $v = t + r^* = const.$ null shell. This however would not suffice to resolve our problem as the exterior spacetime of the null shell will be identical to the Schwarzschild spacetime of the total mass. The missing ingredient is the existence of a wavepacket carrying the elusive four-momentum, following precisely the trajectory of the null shell. This wavepacket will be highly concentrated in the r direction, representing the thinness of the shell, but because it is an r^* mode it can carry the precise difference $k^\mu = p^\mu - q^\mu$ as usual with negligible uncertainty. In effect, this can be interpreted as a fluctuation of the mass seen in the region near the horizon by the factor $\exp(-i\delta M(t - r^*))$.

Weighing the mass δM by the aforementioned factor, always to first order in δM, immediately resolves our problem as four-momentum conservation is identically satisfied and the integral of the LSZ reduction is equal to one, times multiplicative factors. Squaring the amplitude to obtain the probability as $\mathcal{P} = |\mathcal{A}|^2$ we obtain, ignoring multiplicative constants that will be reproduced by our subsequent normalisations, the probability profile

$$\mathcal{P}(n) \sim n^2 e^{-\frac{16}{\pi^2}\left(\int_0^\infty \frac{dk}{k}\right)\frac{\hbar^2\varpi^2}{m_p{}^2}n^2} \tag{3.30}$$

as a function of n excitations of the Bekenstein-Mukhanov quantum ϖ.

We may revert back to expressing energy in terms of the incident QNM and the

emitted graviton's frequency as

$$\mathcal{P}(\omega_{out}) \sim n^2 \exp\left\{ -\frac{16}{\pi^2} \left(\int_0^\infty \frac{dk}{k} \right) \left(\frac{\hbar\omega_{out} - h\omega_{QNM}}{m_P} \right)^2 \right\} \tag{3.31}$$

keeping in mind that $\hbar\omega_{out} = \hbar\omega_{QNM} - n\hbar\varpi$. Recalling our regularisation scheme for the divergent integral, as in (2.25) we have

$$\mathcal{P}(\omega_{out}) \sim n^2 \exp\left\{ -\frac{16}{\pi^2} \ln\left(\frac{k_{UV}}{k_{IR}} \right) \left(\frac{\hbar\omega_{out} - h\omega_{QNM}}{m_P} \right)^2 \right\}. \tag{3.32}$$

In general the exponent depends on the cutoffs as $\sim \ln\left(\frac{k_{UV}}{k_{IR}}\right)^{-1}$ due to the presence of $\varpi^2 \sim \ln\left(\frac{k_{UV}}{k_{IR}}\right)^{-2}$ as is inferred from (2.26). Neglecting to regularise as done above, or effectively at the limit $\frac{k_{UV}}{k_{IR}} \to \infty$, we obtain a zero exponent, making the probability profile dependent solely on the factor n^2 representing the perturbative expansion.

4 The modified ringdown spectrum

Having modeled the dissipation of energy from the black hole due to quantum effects, we will now turn our attention to estimating its impact on the gravitational wave spectrum during a ringdown. The reasoning will be the following: anticipating that the ringdown will contain various excited QNMs, and particularly the fundamental mode and the first overtone, we compute the various possible transitions for each QNM frequency. Every transition will be characterised by the levels n that it excites the BH and the energy $\hbar\omega_{QNM} - n\hbar\varpi$ that dissipates - the number n allows us to compute the relative probability of each process by use of relation (3.32). After obtaining the relative amplitudes of the spectral lines in this fashion, we will examine whether and how these gravitons may propagate towards and observer far from the BH.

4.1 Spectral lines for QNMs and transitions

We will rely on data provided by Berti et al [2] for QNM frequencies of $l = s = 2$ gravitational perturbations, which can be found at Berti's website. We do not work with higher multipole frequencies on the basis that they are generally characterised by higher decay rates and expected to be less excited by merger perturbations, for instance, which are largely quadrupolar. One could nevertheless extend the same analysis to higher multipole QNMs, accounting for a physical excitation profile across the various multipoles.

In our case the excitation profile for the modes of the $l = 2$ fundamental and overtones will be highly approximate based on the expectation that the signal is dominated by the fundamental mode $(n = 0)$ and the first overtone $(n = 1)$. We point, for instance, to Khera, Metidieri et al [11] where it is observed that, in simulations of ringdown after BH collisions, the signal is better reproduced by the inclusion of nonlinear modes containing the $n = 0, n = 1$ frequencies rather than moving to higher n overtones. This in particular was studied by constructing and fitting signals using 3 or 4 frequencies at most to prevent overfitting, picking out the most prominent modes in the simulated signal as the best match.

In our case we do not presently need something more than a qualitative excitation profile. For that reason we assume that the signal is made up as follows:

$$\psi_{obs}(t, x = x_{obs}) = A\cos(\omega_0 t)e^{-\gamma_0 t} + \frac{A}{2}\cos(\omega_1 t)e^{-\gamma_1 t} + \frac{A}{10}\sum_{n\geq 2} e^{-n}\cos(\omega_n t)e^{-\gamma_n t} \quad (4.1)$$

for an unspecified 'global' amplitude A - hence we will refer to our spectral curves as unnormalised amplitude curves. The frequencies ω_n are the real gravitational QNM frequencies sorted by increasing decay rate γ_n, where $\omega_{QNM,n} = \omega_n - i\gamma_n$. For overtones of $n = 2$ we adopted an exponential decay with n in the amplitude profile.

In the following we will show spectral plots with spectral lines generated on frequencies of interest as Lorentzian profile lines. The Lorentzian line for a frequency ω_0 with full width at half-maximum $FWHM$ is given as

$$L(\omega) = \frac{1}{1 + 4\left(\frac{\omega - \omega_0}{FWHM}\right)^2} \quad (4.2)$$

and can be seen as being proportional to the real part of the $t_0 = 0$ Laplace transform of

a signal with a fixed oscillation frequency and decay rate,

$$\mathcal{L}[e^{-i\omega_0 t}e^{-\gamma t}](\omega) = \int_0^\infty dt e^{i(\omega-\omega_0)t-\gamma t} = \frac{1/\gamma}{1+\left(\frac{\omega-\omega_0}{\gamma}\right)^2} + i\frac{(\omega-\omega_0)/\gamma}{1+\left(\frac{\omega-\omega_0}{\gamma}\right)^2}. \tag{4.3}$$

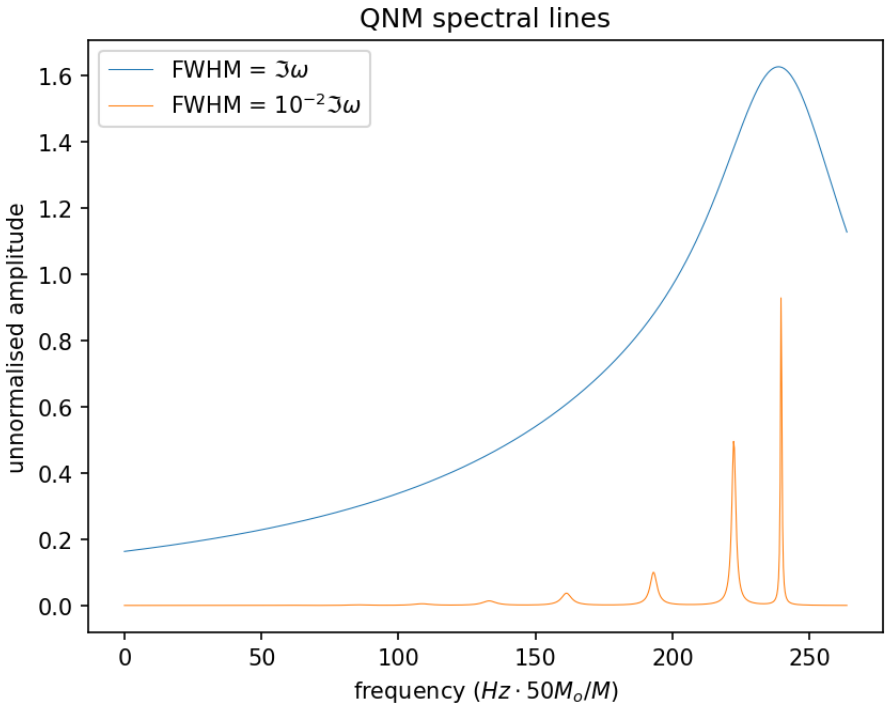

Figure 2: Spectrum of the QNM frequencies calculated by Berti et al [2] with the excitation profile adopted in (4.1). Spectral lines are plotted as Lorentzian lines, with full width a half maximum $FWHM$ depending inversely on the decay rate given as the imaginary part of the QNM frequency. In subsequent plots we will use narrow lines with $FWHM = 10^{-2}\,\mathrm{Im}\,\omega_{QNM}$, keeping in mind that this is for illustrative reasons more than it is based on physical grounds.

We proceed by making a choice of a Bekenstein-Mukhanov quantum ϖ by fixing the integer constant k in the expression (2.5)

$$\varpi = \frac{\ln k}{8\pi GM}. \tag{4.4}$$

For the plots that follow we have made the choice of $k = 2$. This choice is encouraged in the original paper [1], though it feels peculiar from the perspective of our coherent states model, with the respective result (2.24), due to the awkward choice of the cutoffs ratio k_{UV}/k_{IR} that should inevitably be imposed.

With these caveats in place, the process is straightforward: for every $\omega = \mathrm{Re}\,\omega_{QNM}$ in our dataset, we compile a list of all the possible transition levels n such that $\omega - n\varpi > 0$, and compute the relative probability that corresponds to each of those transitions as a function of n. We use two models based on our scattering picture, one by using the

relation (3.32) derived for $\mathcal{P}(n)$, namely

$$\mathcal{P}_{exp,sq}(n) = n^2 \exp\left(-\ln(k) \frac{n^2}{4\pi^3 \left(\frac{M}{m_P}\right)^2} \right), \tag{4.5}$$

and the other by including only the exponential factor that characterises our initial and final coherent states regardless of the specifics of the scattering problem and the mediating interaction,

$$\mathcal{P}_{exp}(n) = \exp\left(-\ln(k) \frac{n^2}{4\pi^3 \left(\frac{M}{m_P}\right)^2} \right). \tag{4.6}$$

We remember here that the only difference between the two that did not amount to a normalising constant was the n^2 factor that came from the perturbative expansion of the scattering amplitude in terms of $\delta M/M = n\hbar\omega/M \sim n$.

It is insightful to note that the Planck mass m_P is very large compared to any amount of energy that a quantum in our regime can carry, and so we will obtain that $\delta M/M \approx 0$ in the original (3.32), which is here seen as $m_P/M \approx 0$. In any case, the exponential factor is almost unity, which greatly simplifies the behaviour of $\mathcal{P}$.

In the second and simpler case, $\mathcal{P}_{exp}$ is constant and so we describe a uniform probability distribution between the available transitions: a given QNM depending on its frequency can excite a set of transitions $\{n\}$, and each of those has a $1/\max(n)$ probability of occurring. We therefore expect to obtain quanta with frequencies $\omega_{QNM} - n\omega$ at an equal rate across n, and thus with equal amplitude. The total amplitude of these n spectral lines is only normalised so that their collective amplitudes sum up to the amplitude of the QNM that generated them, based on the principle that we should obtain 1 outgoing graviton for every 1 infalling quantum as we posited in our simple description of the quantum process.

In the first case, $\mathcal{P}_{exp,sq}$ is dominated by the factor of n^2, resulting in an uneven probability distribution among the possible transitions. Every transition n then has a probability $n^2/\sum_{n'} \mathcal{P}_{exp}(n')$, and amplitudes are computed accordingly with the same reasoning as before. The final spectra of near-horizon emission from the quantum processes are depicted in figure 3.

4.2 Propagation in the near-potential region

The spectrum we derived is constructed for gravitons that are outgoing near the horizon, based on the QNM content of a perturbation that reaches the horizon. In order to predict what a faraway observer would see in terms of gravitational waves, we should take into account the scattering of the outgoing gravitons in the vicinity of the black hole. The scattering occurs in the region near the bulk of the effective potential roughly around $r_* = 0$.

The frequency of the gravitons is not altered as they reach the faraway observer because the effective potential $V_Z(r)$ is zero both at the horizon and at infinity. The amplitude, on the other hand, is attenuated by some grey body factor that we will try to estimate. Our end goal is to be able to relate the amplitude of the original ringdown signal a distant observer receives to the amplitude of the delayed dissipative gravitons that emerge after the initial pulse reaches the horizon and the radiated gravitons travel outwards. If the original signal takes a time t_{travel} to reach the distant observer from its

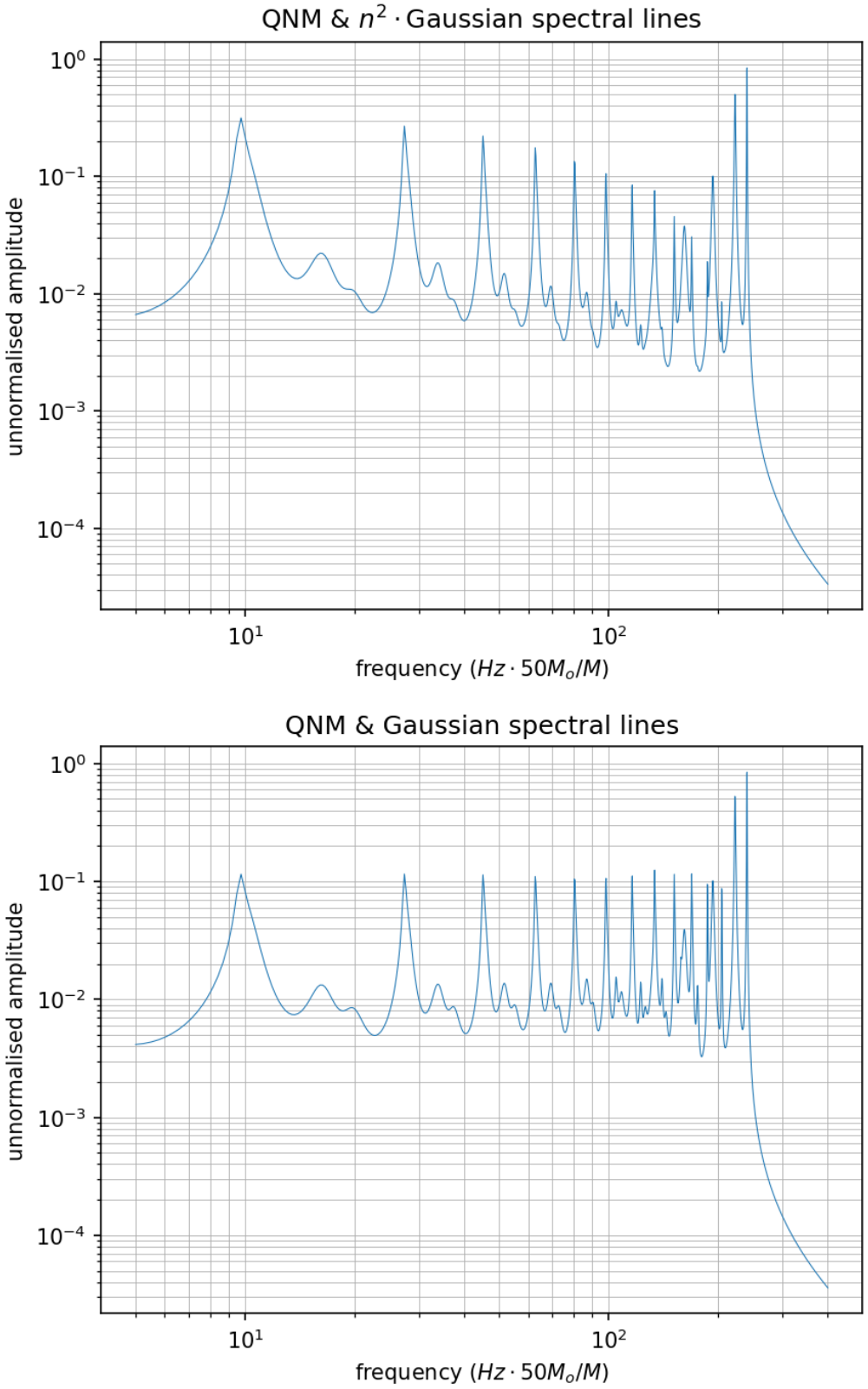

Figure 3: Emission spectra generated with the relative probability models (4.5) (up) and (4.6) (down), for the QNM excitation profile depicted in Figure 2. Note that all the frequencies $f(Hz)$ depend inversely on the mass M of the central black hole.

initial location and a time t_{fall} to reach the region of quantum effects $R_s + L$, then the delay between the reception of the original pulse and the dissipative radiation will be $t_{travel} + 2t_{fall} - t_{travel} = 2t_{fall}$. Because of the bound $L > 0$ which defines the quantum effects region slightly outside of the horizon, t_{fall} will be finite albeit possibly quite large depending on choice of L.

We first model the motion of the original pulse traveling partially outwards and partially inwards from some initially localised perturbation. This serves both as a test of the numerics involved and as a check of the amplitudes of the ingoing and outgoing pulses - because we initialise the configuration as a narrow, δ-like function near the photosphere, which is in the vicinity of the potential's peak, we find that the pulses have comparable amplitude. Note here that the location of our initial data matters a lot, as, if for instance they are positioned far inside the potential's peak, every outgoing signal (initial pulse or subsequent) will be attenuated by one application of the potential's grey body factor T, whereas if they are positioned outside, the first pulse will not be affected while the subsequent signals will pass through the barrier twice, experiencing a T^2 reduction in amplitude. Initialising the data near the potential peak means that both pulses are affected roughly equally as they climb down the two sides of the potential.

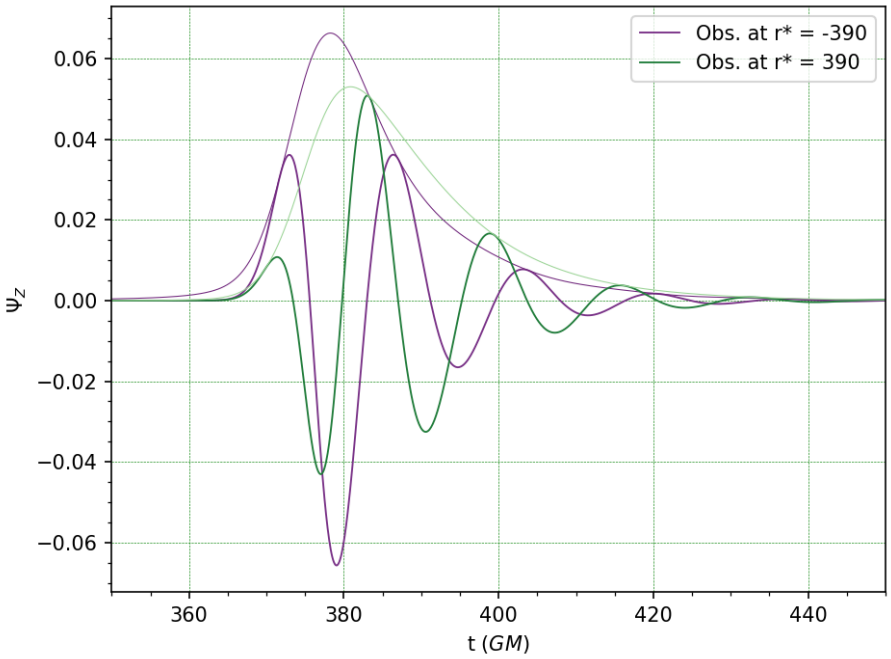

Figure 4: The waveform as seen by two observers stationed at constant $x = \pm 390$, obtained as a constant x slice of the simulation data of figure 5. The envelope of each signal is traced with a thin line of the same colour. The signal features and initial fast increase because of the prompt response of the pulse that travels at c, and an oscillatory part with an exponential amplitude decrease, which we find to be consistent with the $n = 0$ QNM frequency's real and imaginary part.

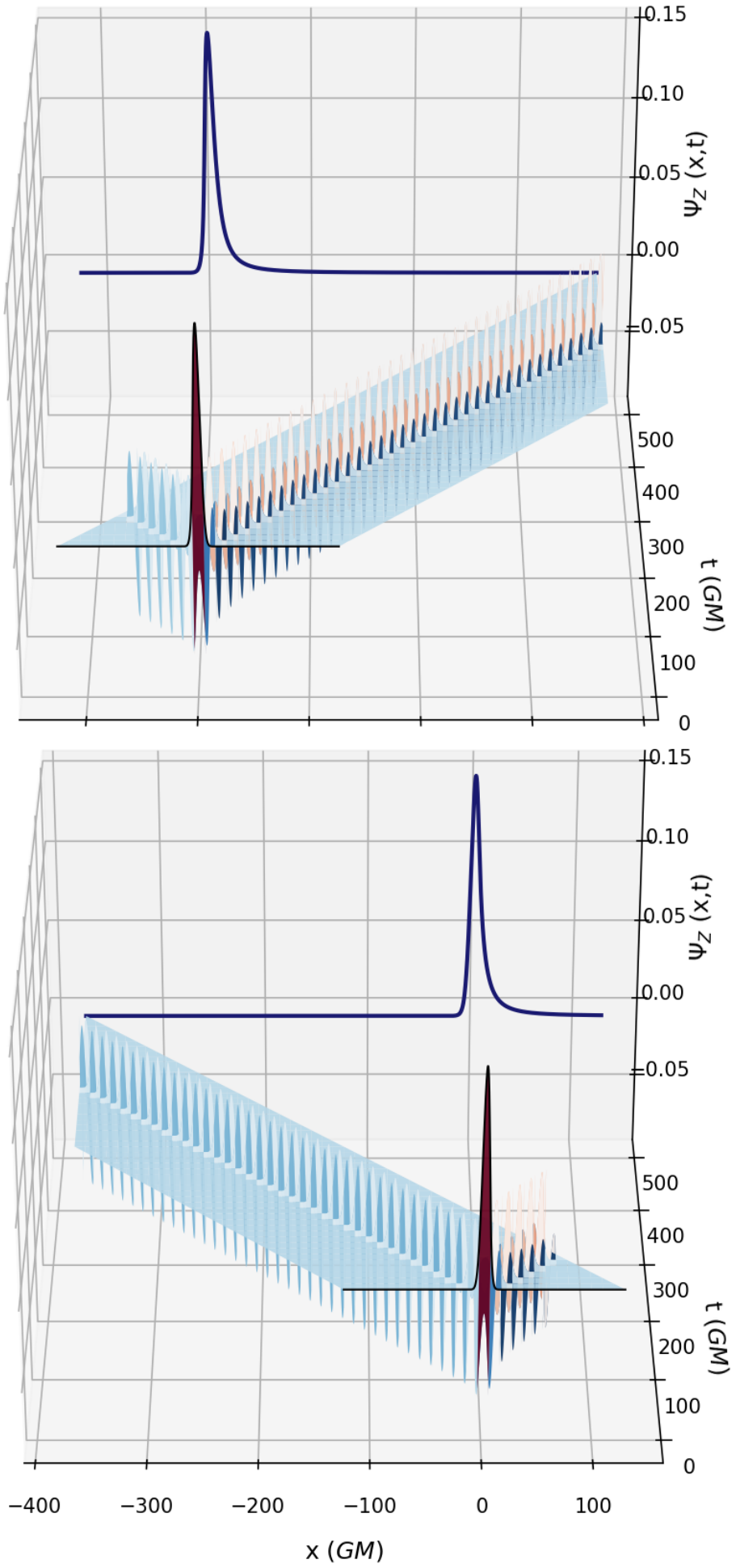

Figure 5: Simulation of an initially localised configuration for the Zerilli field. We track both the outgoing (up) and the ingoing (down) pulses originating from the same initial data, evolving on the background potential $V_Z(r)$ (thick blue line).

In order to describe the numerics, we write a dimensionless version of the Zerilli mode equation. Starting with the potential, we define a dimensionless variable $x = r/GM$, and so

$$V_Z(r) = \frac{2}{r^3}\left(1 - \frac{2GM}{r}\right)\frac{9 + 3\xi^2 GMr^2 + \xi^2(1+\xi)r^3 + 9G^2M^2\xi r}{(3GM + \xi r)^2} = \frac{1}{(GM)^2}\tilde{V}_Z(x) \quad (4.7)$$

where $\xi = \frac{1}{2}(l-1)(l+2)$ and

$$\tilde{V}_Z(x) = \frac{2}{x^3}\left(1 - \frac{2}{x}\right)\frac{9 + 3\xi^2 x^2 + \xi^2(1+\xi)x^3 + 9\xi x}{(3 + \xi x)^2}. \quad (4.8)$$

The equation

$$\partial^2_{r_*}\psi + (\omega^2 - V_Z(r))\psi = 0 \quad (4.9)$$

becomes

$$\partial^2_x\psi + (w^2 - \tilde{V}_Z(x))\psi = 0 \quad (4.10)$$

with $w = GM\omega$. We note that w is thus half the value provided in Berti's catalogue as $2GM\omega$.

To perform the numerical simulation we pick a square region of x, t ranges both equal to some value L, up to which we wish to propagate the initial disturbance. We additionally set a length $l < L$ of a window on which we define the initial data. In terms of the grid that underlies the integration region, we have an $N \times N$ square grid corresponding to the side length L and a window of $n < N$ points corresponding to the length l. Because signals propagate at light speed $|dx/dt| = 1$, at every iteration where we evolve the initial data by dt we also slide the window forward or backward by $|dx| = dt$ (or, by a single grid point) depending on whether we intend to keep track of the left- or right-moving signal. The data on grid points outside of the window is set to NaN in order to reduce the computational load of the integration and illustration, and thus the purpose of the sliding window is to make the integration more efficient. In the plots depicted here we choose a range of $N = 500$ and window size of $n = N/2$.

The time evolution of the initial data is performed with a nested RK4 algorithm. This consists by separating our differential equation to a nested system of two first order equations,

$$\begin{aligned} \partial_t\psi &= u \equiv f_\psi(\psi, u, t) \\ \partial_t u &= \partial^2_x\psi - \tilde{V}_Z(x)\psi \equiv f_u(\psi, u, t). \end{aligned} \quad (4.11)$$

Note that the value of $\partial^2_x\psi$ may always be calculated at every point x in a constant t slice.

The RK4 algorithm for advancing a step dt is as follows:

$$\begin{aligned} \psi(t+dt) &= \psi(t) + \frac{dt}{6}(f_1 + 2f_2 + 2f_3 + f_4) \\ u(t+dt) &= u(t) + \frac{dt}{6}(g_1 + 2g_2 + 2g_3 + g_4) \end{aligned} \quad (4.12)$$

where

$$
\begin{aligned}
f_1 &= f_\psi(\psi, u, t) \\
g_1 &= f_u(\psi, u, t) \\
f_2 &= f_\psi\left(\psi + \frac{dt}{2} f_1, u + \frac{dt}{2} g_1, t + \frac{dt}{2}\right) \\
g_2 &= f_u\left(\psi + \frac{dt}{2} f_1, u + \frac{dt}{2} g_1, t + \frac{dt}{2}\right) \\
f_3 &= f_\psi\left(\psi + \frac{dt}{2} f_2, u + \frac{dt}{2} g_2, t + \frac{dt}{2}\right) \\
g_3 &= f_u\left(\psi + \frac{dt}{2} f_2, u + \frac{dt}{2} g_2, t + \frac{dt}{2}\right) \\
f_4 &= f_\psi\left(\psi + dt f_3, u + dt g_3, t + dt\right) \\
g_4 &= f_u\left(\psi + dt f_3, u + dt g_3, t + dt\right)
\end{aligned}
\tag{4.13}
$$

with $\psi = \psi(t), u = u(t)$. This is essentially a regular RK4 integration performed pointwise in x for our fields - so a total of n RK4 integrations carried out in parallel.

After the integration is performed, we obtain the waveform for a fixed x observer and examine the signal. Taking the observer to be sufficiently distant from $x = 0$, we expect the QNM content of the signal to contain primarily the least damped modes and especially the fundamental. In order to verify this, we obtain the frequency and the rate of decay of the signal using its Hilbert transform $\mathcal{H}$ (see properties and details in Appendix F). If $s(t) = \psi(t, x = x_{obs})$ is the signal at fixed x, we obtain its envelope

$$R(t) = |s(t) + i\mathcal{H}[s](t)| \tag{4.14}$$

as defined in (6.66), and depicted in figure 4. We find the peak of the outgoing signal as the maximum of $R(t)$ at some time t_{max} and perform a fit of the envelope function $R(t)$ to an exponentially decaying function Ae^{-Bt} to obtain the decay rate B. We additionally isolate the phase of the signal (after the peak) as

$$\phi(t) = \arg(s(t) + i\mathcal{H}[s](t)). \tag{4.15}$$

Expecting that it approximately contains a single oscillation frequency after the other QNMs have diminished, we perform a linear fit $Ct + D$ and obtain the oscillation frequency C. Contrasting our values for $B = 0.37303$ and $C = 0.08913$ with the $\mathrm{Re}\, w = 0.37365$ and $\mathrm{Im}\, w = 0.08895$ for the fundamental gravitational QNM as provided by Berti, we have an agreement in several decimal places. The simulation therefore illustrates the presence of QNMs in the signal following an arbitrary pulse, and this consistency means that the numerical procedure is sufficiently reliable for our qualitative analysis.

4.3 Spacetime grey-body factor

Our discussion has up to this point led to some estimates of the spectrum of gravitational radiation that we postulated should emerge from the BH horizon. Nevertheless this cannot be identified with the gravitational spectrum measured by a faraway observer, because the near-horizon region of the BH induces backscattering on modes that propagate through it, an effect mirrored in the fact that the effective potential of the wave equation is localised but peaked around $r_* \sim 0$.

Because the effective potential is zero asymptotically in $r_* \to \pm\infty$, a wave that is transmitted through the potential barrier will have the same frequency as the incident wave - and so the spectral lines that we computed will not change their location. Their amplitude however will be attenuated by a (so called) grey-body factor that will in general depend on the wave's frequency. This is in essence a transmission factor akin to those computed for the scattering of a stationary wave on a potential barrier.

It is nontrivial to extend the property of stationarity to our case due to the time decay that QNMs feature. Namely, even though in standard barrier scattering problems one assumes that the incident wave represents a 'beam' that does not deplete in time and so the system at hand becomes time-independent, here one must recall that the outgoing radiation's amplitude is immediately connected to the horizon-infalling QNMs' amplitude due to the 1-to-1 infalling particle-to-graviton correspondence, and the decay rate of most QNMs is quick enough to make such an assumption dubious. We will thus turn our attention to addressing some ways to compute the transmission amplitude for our outgoing gravitational radiation.

In its simplest form, the problem resembles as described above the usual scattering on a potential barrier, where one considers a wave incident on the potential barrier (right-moving from the left), a reflected wave (left-moving on the left side) and a transmitted wave (right-moving on the right side). These waves all make up the beam's wavefunction (in the locally normalisable sense), which can be written at the asymptotic left and right regions as

$$\begin{aligned} \psi(x \to -\infty) &= e^{-i\omega x} + re^{i\omega x} \\ \psi(x \to \infty) &= te^{-i\omega x} \end{aligned} \tag{4.16}$$

where we note that the system is time independent and the time oscillation is encoded in the separation constant ω. The coefficients r, t are such that ψ is a solution of its wave equation with the effective potential. The barrier's transmission factor is given simply as

$$T(\omega) = \left| \frac{A_{transm}}{A_{in}} \right|^2 = |t|^2. \tag{4.17}$$

We can numerically compute $T(\omega)$ by picking the value of $\psi(x_0)$ and its first spacial derivative $\psi'(x_0)$ at an arbitrary point x_0 and integrating its wave equation across both sides of the potential. There, we can analyse the wavefunction in right- and left-moving modes and infer the respective amplitudes. If the initial conditions are chosen so that the wave on one side is only moving away from the barrier (transmitted), then the above-mentioned setup is recreated.

We may then move away from the stationary problem by consider a time-evolving wavepacket that represents an outgoing graviton. This is a configuration that can in practice be initialised far inside the potential peak and be evolved numerically until it scatters with the potential barrier and the reflected and transmitted wavepackets are created, have separated and are distinguishable from each other. For this purpose we use the RK4 integration algorithm described in the previous paragraph, appropriately choosing the initial data: the field configuration of a monochromatic wavepacket of frequency ω is represented with a sinusoid with a Gaussian envelope

$$\psi_\omega(x, t=0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-x_0}{\sqrt{2}\sigma}\right)^2} \cos(\omega(x - x_0)) \tag{4.18}$$

with our choice of parameters being $\sigma = 40, x_0 = -250$. We note that this configuration is simply a narrow Gaussian in momentum space, as the inverse Fourier transform of

the function

$$\hat{f}(\omega) = e^{-\alpha(\omega-\omega_0)^2} \tag{4.19}$$

is

$$f(x) = \frac{1}{2\sqrt{\pi a}} e^{-\frac{x^2}{4a}} e^{i\omega_0 x} \tag{4.20}$$

and we infer that, as expected, a narrow configuration in momentum space (large a) translates to a wide configuration in regular space, justifying our choice for $\sigma \gg 1$.

In order to create an outgoing wavepacket, we pick the initial time derivative so that, in the asymptotic regime, the configuration solves the right-moving wave equation

$$\partial_t \psi_\omega(x,t)\Big|_0 + \partial_x \psi_\omega(x,t)\Big|_0 = 0 \Leftrightarrow \dot{\psi}_\omega(x,0) = -\partial_x \psi_\omega(x,0) \tag{4.21}$$

or explicitly

$$\dot{\psi}_\omega(x,t=0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-x_0}{\sqrt{2}\sigma}\right)^2} \left(\omega \sin(\omega(x-x_0)) + \frac{x-x_0}{\sigma^2}\cos(\omega(x-x_0))\right). \tag{4.22}$$

The resulting field evolution via the RK4 algorithm is depicted in figure 7 for a wavepacket of $\omega = 0.4$, close to the frequency of the fundamental gravitational QNM at $w = 0.3737$.

One can repeat this procedure sampling enough values of the frequency ω and measuring the transmitted amplitude in relation to the incident amplitude to obtain the transmission factor. We plot the latter in figure 6 for both of the aforementioned methods.

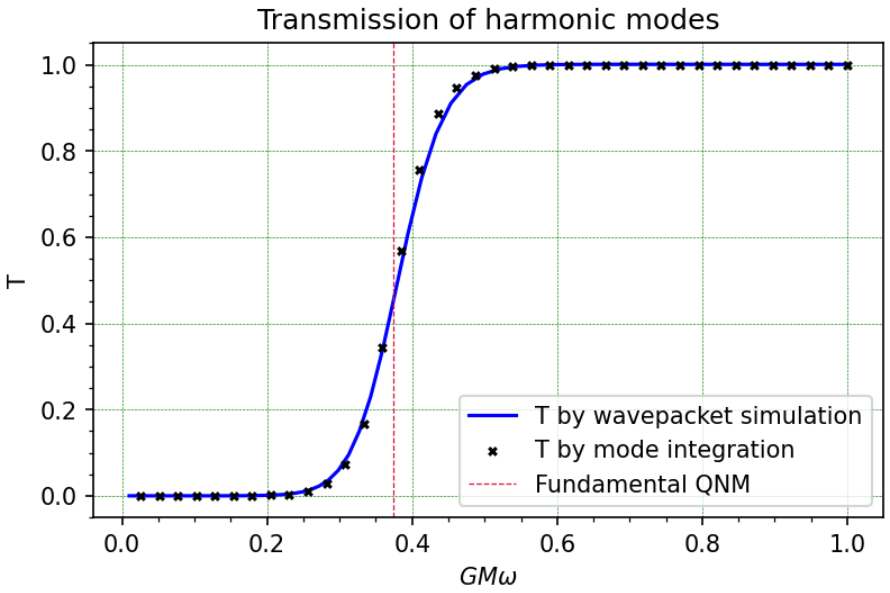

Figure 6: The transmission factor for the Zerilli potential obtained by integrating a stationary mode across the barrier (black markers), and by simulating wavepackets scattering on the barrier (blue line). The transmission factor T in both cases is defined as the squared modulus of the ratio of the transmitted wave amplitude to the incident wave amplitude.

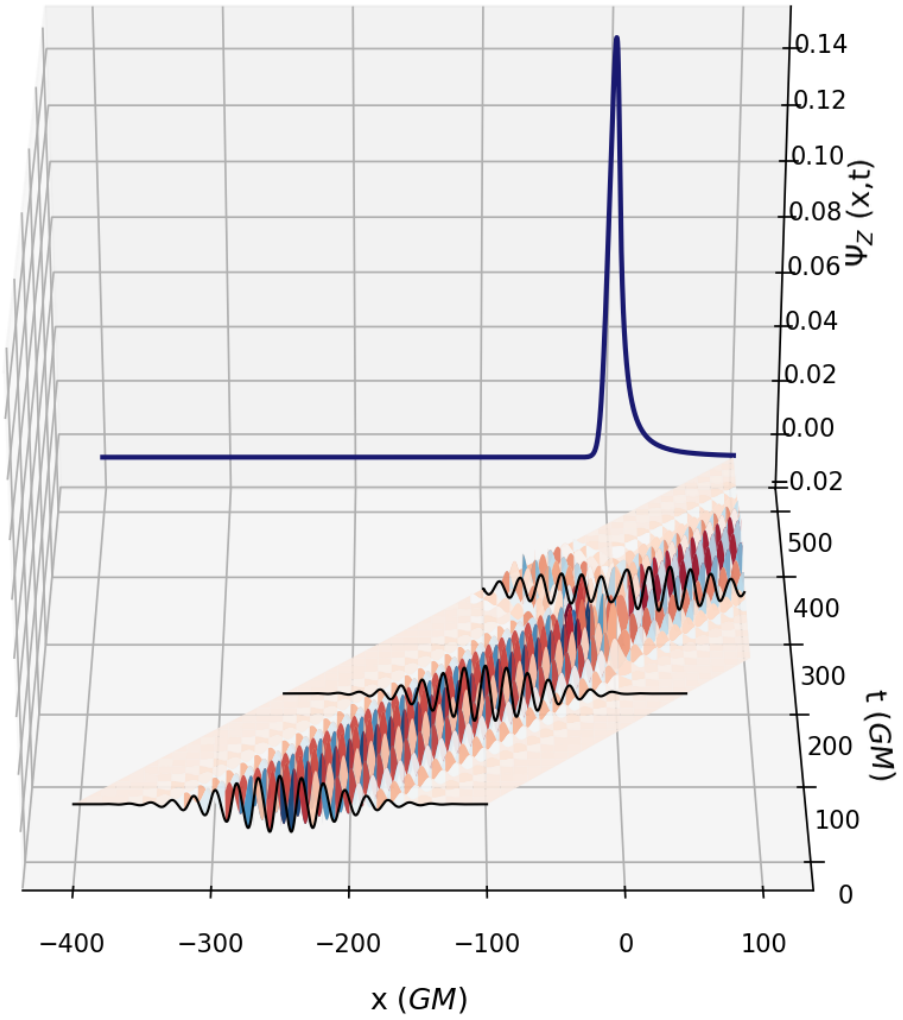

Figure 7: Scattering of an incident monochromatic wavepacket of frequency $w = 0.4$ with the initial data prescribed as in (4.18) and (4.22) on the Zerilli potential barrier (thick blue line), initialised at $x_0 = -250$ with $\sigma = 40$. Black lines represent constant t sections of the field.

4.4 The black hole as a leaking cavity

The computation of the transmitted amplitude as a function of the outgoing wavepacket's frequency, which we take to model the graviton emerging from the scattering, shows that the fraction of the emitted radiation that escapes sharply reduces as the energy of the quanta becomes smaller. Most of the transitions which take place and offer a large number of BM quanta to the black hole therefore emit gravitons that will most likely not emerge to the outside of the black hole's photosphere. The emitted spectrum, weighted by the transmission factor as a function of the frequency, is depicted in Figure 8.

It is crucial to observe, nevertheless, that the energy carried by these modes will not simply disappear - it will, however, scatter against the potential wall and head back towards the horizon almost in its entirety. Then the issue comes of what will happen to it as it approaches the horizon once again. Certainly the same process as before will follow, with the particles that constitute the mode becoming absorbed and re-emitting some increment of energy.

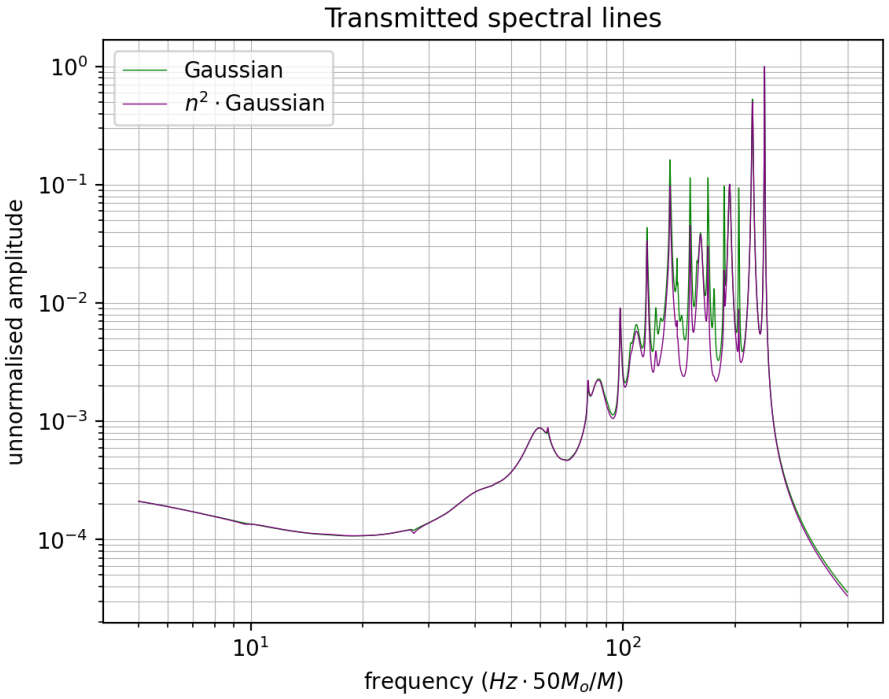

Figure 8: The emitted spectrum for the different probability models of Figure 3 weighted by the spacetime's transmission factor illustrated in Figure 6. In both cases the spectrum is heavily concentrated around the high frequencies while the lower components have been significantly attenuated with respect to they horizon spectra. The cavity properties of the spacetime mean, however, that the leftmost set of peaks (see Figure 3) will persist for an amount of time substantially longer than the QNMs' decay time.

Suppose that the original QNM had a frequency ω_{QNM}, so that a potential emitted graviton will have energy $\epsilon = \hbar(\omega_{QNM} - n\varpi) > 0$ for some transition of n levels. The graviton, particularly for large n, has a high chance of being backscattered on the vicinity of the photosphere (where the peak of the effective potential is located) and heading back inside. Upon reaching the horizon again, it will induce the same process and so a graviton will be emitted with energy $\epsilon' = \epsilon - \hbar n'\varpi = \hbar(\omega_{QNM} - (n + n')\varpi) > 0$. The key thing to notice is that this process yields the same set of outgoing energies as the initial one, namely $\hbar(\omega_{QNM} - N\varpi)$ for an appropriate integer N. Because, however, these gravitons will most likely be blocked by the potential barrier once more due to their low frequency, the horizon scattering will repeat for a large number of times until the particle escapes, each time lowering its energy. It is our conclusion then that the soft gravitons will in their majority escape with a small flux in unit time, but with a very long duration compared to their expected decay time, and carrying the least possible energy ϵ_{out} so that $\epsilon_{out} = \hbar(\omega_{QNM} - N\varpi) > 0$. This energy corresponds to the leftmost frequency peaks seen in 3.

This behaviour is analogous to confinement of the soft modes in a cavity with minimal leaking. Since the modes have to escape eventually, being barred from falling inside the horizon in their entirety, this weak signal will persist for a large amount of time as is needed for the disturbance to dissipate substantially. We may quantify these statements with a back-of-the-envelope computation of the relevant timescales: consider first that

the time it takes for a lightlike signal to propagate from a point at $r = R_S + \alpha l_P$, where α is a parameter, to the photonsphere $R = 3/2R_S$ is given by integrating the null line element to obtain

$$t(R) = \frac{1}{c}\left(R - R_S - \alpha l_P + R_S \ln\left(\frac{R - R_S}{\alpha l_P}\right)\right). \tag{4.23}$$

We may take the initial emission point to lie at a Planck length distance from the horizon ($\alpha = 1$) or several orders of magnitude away as discussed in the context of preparation of asymptotic states ($\alpha = 10^{10}$, for instance) - the resulting times, for a black hole of $M = 50M_\odot$, will be respectively $46ms$ and $34.5ms$. At the same time, the decay timescale for the $s = l = 2$ fundamental mode of a black hole of the same mass will be around $3ms$. We can already infer that the time it will take for the signal to perform a round trip inside the photonsphere-horizon cavity is significantly larger than the decay timescale no matter the choice of α.

Take again the example of the fundamental gravitational mode, for which we know that $2GM\omega_0 = 0.747$, while the Bekenstein-Mukhanov quantum, as we took it in previous computations, is $2GM\varpi = 0.055$ - this allows for 13 possible transitions. Each of those will result in an emitted graviton of frequency $\omega_n = \omega_0 - n\varpi$, and the latter will propagate as a wavepacket towards the potential barrier where it will transmit with some probability given by the (frequency dependent) transmission factor T_n. We note, for the mass choice $M = 50M_\odot$, that the transitions with $n = 1, n = 2$ and $n = 13$ respectively have transmission factors around $0.3, 0.1$ and 10^{-8}.

Even for the case of the highest transmission, this is achieved for about 30% of the emerging flux - the rest will therefore reach the horizon once more and its energy will be redistributed to frequencies with even lower transmission rates. It will take no more than a few tens of round trips (so a few $100ms$) for most of the energy carried by dissipative gravitons to be transferred to the lowest frequency ω_{13} with the lowest possible transmission, 10^{-8}. Assuming then that the energy content E_j of the gravitons at the $j - th$ trip is

$$E_j = (1 - T_{13})E_{j-1} \tag{4.24}$$

this behaviour gives us the approximate expression of E_j as

$$E_j = E_0 e^{-T_{13} j} \tag{4.25}$$

and so the energy dissipation timescale t_d will correspond to $j \sim 1/T_{13} \sim 10^8$, leading to

$$t_d \sim 10^8 t_{\text{round trip}} \sim 10^7 s \sim 0.3\, yrs. \tag{4.26}$$

We conclude that the ω_{13} flux, though weak, will persist for a timescale several orders of magnitude longer than the decay time of the QNMs.

5 Conclusions and discussion

The takeaway message of this work, irrespective of any computational details, boils down to a few simple points:

- The behaviour of the quantum black hole need not reflect its classical properties, and it is thus not possible for us to state that one model is preferable to another on such a basis. We are, however, motivated to keep the property of perfect absorption (instead of partial reflection) accompanied with radiative dissipation because it makes our system analogous to other, familiar quantum systems, such as the hydrogen atom as a result of an electron getting captured in a positive Coulomb potential.

- The all-absorbing nature of the black hole and the expectation that it may occupy discretised energy states can be reconciled by conjecturing that the energy increment between an infalling object's energy and the induced transition is expelled as gravitational radiation.

- This process implies that the spectrum of gravitational waves during the black hole's ringdown in enriched by the presence of the gravitons with frequencies corresponding to these energy increments as $E = \hbar\omega$. The spectrum's profile for the near-horizon waves can be sketched by knowing the relative probability for the various transitions a given infalling QNM can induce.

- The black hole's near-horizon spacetime acts like a cavity that traps many of these low frequency waves, which are led to repetitively return to the horizon and redistribute their energy to even lower frequencies, leading most of the energy to be deposited on the lowest of the dissipative frequencies corresponding to a given QNM. We thus anticipate that the ringdown spectrum for a distant observer will feature a weak flux of this frequency, which will persist in time far longer than the decay of the QNMs.

One should note that this last observation is mostly independent of our relative probability profile $\mathcal{P}(n)$ (so long as the latter is not strongly favouring the smallest n) and therefore also independent of our way of modelling the quantum black hole, provided that the Bekenstein-Mukhanov quantisation scheme is reproduced. It is nevertheless important to state some caveats about our quantum model.

Our construction of the quantum black hole was performed on a quantised scalar field that represented a single polarisation of the metric tensor field - which, in principle, should be fully quantised in a proper treatment of quantum gravity. One can look to extending this construction to include the Regge-Wheeler field to the quantised degrees of freedom, although the resulting spectrum will not differ substantially as the QNMs of Regge-Wheeler and Zerilli perturbations are isospectral [7], so the primary difference would be around their excitation profile following some initial disturbance.

This model comes with some problems of its own, as, for instance, we saw that it must be regularised in order to prevent divergences in quantities such as the expected occupation number $\langle M|\,\hat{N}\,|M\rangle$ in the coherent state $|M\rangle$. This regularisation is achieved using both IR and UV cutoffs on the wavenumbers k of the modes of the quantum field, and though the UV cutoff can be understood as a threshold on the validity of our model for very short distances (where, for example, a quantum matter core of the black hole could be present), it is not clear what the origin of the IR cutoff could be, given that

the large distance limit of our model is chosen to be a flat spacetime (our model has no knowledge of the true asymptotics of our universe such as de Sitter curvature).

Furthermore, the correspondence with the Bekenstein-Mukhanov quantisation scheme was done on the basis that the $(M/m_P)^2 \sim N$ pattern, where N is an integer, is reproduced. We obtained however that, aside from the regularised divergent prefactor, the same expression is derived with $N = \langle M| \hat{N} |M\rangle$ within our model. This raises the concern that N is no longer an integer, although it can be rationalised to increase in integer increments with the addition of δ-localised particles. This relation between the two expressions seems like an impressive similarity, in spite of these differences, though it is an interesting question to ask where they come from and whether the resemblances are purely coincidental. In any case, we have relied on this weakly constructed correspondence to study quantum scattering for transitions of Bekenstein-Mukhanov quanta.

Pertaining to the scattering itself, it is clear that more rigor can be put into getting a result that is accurate to an extent further than merely performing a qualitative study of the relative probability profile, as we did here. Some factors that were accordingly overlooked are the inclusion of various numbers of in- and outgoing gravitons, and thus the inclusion of more vertex types in the Lagrangian, the consistent expansion of the Lagrangian up to some desired perturbative order in the field fluctuation ϕ and in the mass increase δM, the inclusion of higher l modes, and a more explicit remedy of the vanishing of the scattering amplitude by energy/momentum conservation, which we here attributed to a missing ingoing mode to transfer the absorbed energy inwards and imposed conservation *a posteriori*. Lastly, more work can be done on studying the propagation of the graviton radiation in the spacetime surrounding the horizon, referred to here as the black hole's cavity, to compute the form of the time-dependent spectrum of the leaking graviton flux.

These points of improvement would be useful as one attempts to estimate the modifications to the ringdown spectrum of a Schwarzschild (and, even better, Kerr) black hole that could arise from quantum gravity effects. The signature of the latter in our case would be a weak low frequency flux with very slow decay. For a ringing black hole of $M = 50M_\odot$, where the fundamental gravitational QNM lies at roughly 250 Hz, this signal would be located at (see Figure 3) near 10 Hz - these frequencies depend on the black hole's mass as $1/M$. This frequency range nears the threshold where ground-based detectors are heavily influenced by seismic noise (roughly < 10 Hz), and the signal is orders of magnitude weaker than that of the elusive QNMs themselves. One can nevertheless explore different quantisation schemes than the ones used here (an obvious starting point could be different choices of the Bekenstein-Mukhanov parameter) where, if one selects a transition quantum ϖ such that the difference $\omega_{QNM} - \varpi$ is smaller, the frequencies of interest will be even smaller and potentially into the frequency range of different detector types like LISA (currently < 0.1 Hz). The key point is that the value of the dissipation frequency is completely determined by the choice of the quantisation scheme, and the latter is significantly unconstrained as we already saw for the case of the Bekenstein-Mukhanov model (featuring an arbitrary integer parameter) and the coherent state model (featuring an arbitrary regularisation scheme). As such there may exist avenues for making possible the detection of such a signature of quantum gravity by gravitational wave observation of strong black hole merger events.

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Availability

All computational material utilised in this work, along with specifics regarding the numerics, will be uploaded in the following GitHub repository: KonTopal / Thesis_Quantum_Schwarzschild .

Contact the author at kon.topaloglou@gmail.com .

AI use declaration

AI has been used in the creation of this thesis primarily for refining code used in the numerics; additionally for sanity checks on various results. It was not used to generate research data, perform any type of analysis, or produce academic arguments or conclusions. All content was independently developed and reviewed by the author, who takes full responsibility for the final work.

6 Appendices

A Klein-Gordon modes on a flat background

In the present section we will treat the modes of a Klein-Gordon field on a flat background, and will subsequently use the quantum Klein-Gordon field to construct the Schwarzschild configuration as a coherent state.

The action of the Klein-Gordon field ϕ is defined as in (1.2)

$$S_{KG} = \int d^4x \frac{\sqrt{-g}}{2} \partial_\mu \phi^* \partial^\mu \phi$$

yielding the equations of motion (1.3)

$$\partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = 0.$$

Taking a flat background $g_{\mu\nu} = \eta_{\mu\nu}$ and separating the angular part as $\phi(x^\mu) = \tilde{\phi}(t,r)\Omega(\theta,\phi)$, we find that $\Omega(\theta,\phi)$ can be found on the basis of spherical harmonics $Y_{l,m}(\theta,\phi)$ and produces the constraint that the constant of separation be equal to $l(l+1)$, leaving us with

$$-\partial_t^2 \tilde{\phi} + \frac{1}{r^2}\partial_r(r^2 \partial_r \tilde{\phi}) = \frac{l(l+1)}{r^2}\tilde{\phi}.$$

We will now restrict ourselves to monopole modes $l = 0$ and therefore $m = 0$ in order to construct a spherically symmetric configuration. Note here that even though monopole modes are usually discarded in gravitational perturbation theory as unphysical (as a monopole fluctation represents a fluctuation of the source's mass) here they are necessary to create a spherically symmetric configuration, that represents a source of mass M, from the vacuum. Separating again as $\tilde{\phi}(t,r) = T(t)R(r)$,

$$\begin{aligned} \partial_t^2 T &= \lambda^2 T \\ \partial_r(r^2 \partial_r R) &= \lambda^2 r^2 R. \end{aligned} \tag{6.1}$$

The first of these equations can readily be solved as $T(t) = T_0 \exp(\pm\lambda t)$, where we restrict to those solutions with negative exponent by anticipating positive energy modes, and set $\lambda = ik$, yielding

$$T(t) = T_0 \exp(-ikt).$$

For the radial equation we introduce the dimensionless variable $x = \lambda r$ and $y(x) = xR(x)$, and so

$$\partial_r(r^2\partial_r R) = \lambda^2 r^2 R \Leftrightarrow \partial_x(x^2 \partial_x R) = x^2 R \Leftrightarrow \partial_x^2 y = y$$

solved by $y(x) = y_0 \exp(\pm x)$. We therefore find a general mode of the form

$$u_{k\sigma} = u_0 \frac{e^{-ik(t-\sigma r)}}{kr}. \tag{6.2}$$

Here, $\sigma = \pm 1$ denotes an ingoing or outgoing mode.

Then, the field is constructed by summing over the basis modes weighted by a coefficient. In the case of the wavenumber k, we choose sum over all three-dimensional momentum vectors $\vec{k}$ with magnitude $|\vec{k}| = k$, and so due to spherical symmetry (independence of the modes from the orientation of $\vec{k}$) the integral is simplified as

$$\int \frac{d^3k}{(2\pi)^3} u(k) = \int d\Omega_k \int \frac{dk k^2}{(2\pi)^3} u(k) = \int \frac{dk k^2}{2\pi^2} u(k)$$

and so we write

$$\phi = \sum_\sigma \int \frac{dkk^2}{2\pi^2}(g_{k\sigma}u_{k\sigma} + \bar{g}_{k\sigma}\bar{u}_{k\sigma}) = \int \frac{dkk^2}{2\pi^2}\left(u_0\frac{e^{-ikt}}{kr}\left(g_{k1}e^{-ikr} + g_{k-1}e^{ikr}\right) + (conj.)\right)$$

where $g_{k\sigma}$ are expansion coefficients in the u basis.

The normalisation of the modes can be fixed with the computation of the inner product. Taking, for example, two outgoing modes with $\sigma = 1$,

$$\begin{aligned}(u_p, u_q) &= -i\int_{\Sigma:t=t_0} d^3x t^\mu \sqrt{g_\Sigma}(u_p\partial_\mu u_q^* - u_q^*\partial_\mu u_p*) \\ &= 4\pi|u_0|^2\frac{p+q}{pq}e^{-i(p-q)t}\int_0^\infty dre^{-(p-q)r} \\ &= 4\pi^2|u_0|^2\frac{2}{p}\delta(p-q).\end{aligned}$$

We computed the radial integral using the result (6.13) from Appendix B. We discarded the term containing the principal value, expecting that the inner product satisfies $(u_p, u_q) = (u_q, u_p)^*$ as can be shown from the definition - and the principal value is an odd function in $k = p - q$.

Furthermore, imposing that the field ϕ is regular at the origin we set $g_{k1} = -g_{k-1}$:

$$\phi = \int \frac{dkk^2}{2\pi^2}\frac{\sin(kr)}{kr}\left(u_0 g_k e^{-ikt} + \bar{u}_0\bar{g}_k e^{ikt}\right).$$

With the normalisation choice of

$$(u_p, u_q) = \frac{4\pi^2}{p^2}\delta(p-q) \tag{6.3}$$

we obtain

$$\phi = \int \frac{dkk^2}{2\pi^2}\sqrt{\frac{1}{2k}}(g_k e^{-ikt} + \bar{g}_k e^{ikt})\frac{\sin kr}{kr}.$$

We will now reproduce the desired field configuration $\phi_{cl} = -\frac{m}{r}$ using the set of modes developed above. Observing that the configuration is both static and real, we select expansion coefficients of the form $g_k = |g_k|\exp(ikt)$, canceling out the time exponential. We then test for a solution of the form $|g_k| = ck^{-3/2}$:

$$\phi_{cl} = \int \frac{dkk^2}{2\pi^2}\frac{1}{\sqrt{2k}}2ck^{-3/2}\frac{\sin(kr)}{kr} = \frac{c}{\sqrt{2}\pi^2 r}\int_0^\infty dx\frac{\sin x}{x} = \frac{\sqrt{2}c}{4\pi r} \tag{6.4}$$

and therefore $c = -\frac{4\pi}{\sqrt{2}}m$ leading to

$$g_k = -\frac{4\pi}{\sqrt{2k^3}}me^{ikt}. \tag{6.5}$$

Our last step is to rescale the field $\phi \to m_p\phi$, where $Gm_P = \sqrt{\hbar}$ is the Planck mass, anticipating that it ought to have the correct mass dimension so that the action has mass dimension zero. Then,

$$g_k = -\frac{4\pi}{\sqrt{2k^3}}\frac{m}{m_p}e^{ikt}. \tag{6.6}$$

B Fourier transform of the Heaviside step function

The Fourier transform of the Heaviside step function

$$H(x) = \begin{cases} 0 & , x < 0 \\ 1 & , x > 0 \end{cases} \tag{6.7}$$

is found as the integral

$$\tilde{H}(k) = \int_{-\infty}^{\infty} H(x) e^{ikx} dx = \int_{0}^{\infty} e^{ikx} dx. \tag{6.8}$$

Notice that we omitted to define $H(0)$, which may be set to some finite value for the function to satisfy a certain criterion of continuity - but this will not concern us here as the integration limit represents an infinitesimal approach to $x = 0$ from the positive side (in other words, $\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} H(x) \exp(ikx) dx = 0$).

The aforementioned integral can only be computed with the help of a regulator as the limit $\lim_{x \to \infty} \exp(ikx)$ does not exist. We therefore compute instead

$$\begin{gathered} \int_0^\infty e^{(ik-\epsilon)x} dx \equiv \int_0^\infty e^{ax} dx = \frac{1}{a} e^{ax}|_0^\infty \\ = \frac{1}{a} \lim_{x \to \infty} e^{ikx} e^{-\epsilon x} - \frac{1}{a} \lim_{x \to 0^+} e^{ikx} e^{-\epsilon x} = \frac{-1}{ik - \epsilon} = \frac{i}{k + i\epsilon}. \end{gathered} \tag{6.9}$$

Here it is understood that $\epsilon \in \mathbb{R}^+$ and we're interested in the limit $\epsilon \to 0^+$.

This limit can be evaluated in the distributional sense, that is, by acting upon another function $f(k)$ as the integral

$$\int_{-\infty}^{\infty} \frac{i f(k)}{k + i\epsilon} dk = i \int_{-\infty}^{\infty} \frac{k f(k)}{k^2 + \epsilon^2} dk + \epsilon \int_{-\infty}^{\infty} \frac{f(k)}{k^2 + \epsilon^2} dk. \tag{6.10}$$

We promote k to a variable in the complex plane for the evaluation of these integrals, with poles at $k = \pm i\epsilon$, by closing our curve with a semicircle of radius $L \to \infty$, on the $\mathrm{Im}\{k\} > 0$ plane. It is important to note that our subsequent solution demands of $f(z)$ to satisfy reasonable convergence properties at infinity, so that the integrand vanishes along the semicircle.

The first of these integrals, in the $\epsilon \to 0^+$ limit, is written as the Cauchy principal value distribution of $1/k$,

$$\int_{-\infty}^{\infty} p.v.(k) f(k) dk = \int_{-\infty}^{\infty} \frac{1}{k} f(k) dk \tag{6.11}$$

whereas the second integral is evaluated by the pole at $k = i\epsilon$ to be

$$\epsilon \int_{-\infty}^{\infty} \frac{f(k)}{k^2 + \epsilon^2} dk = 2\pi i \epsilon \lim_{z \to i\epsilon} \frac{z - i\epsilon}{z^2 + \epsilon^2} f(z) = \pi f(i\epsilon). \tag{6.12}$$

We conclude that, as a distribution acting upon a function $f(z)$ that decays at infinity sufficiently fast, the Fourier transform of $H(x)$ is

$$\tilde{H}(k) = i p.v.(k) + \pi \delta(k) \tag{6.13}$$

C Lagrangian for Zerilli modes

Gravitational perturbations that are represented by Zerilli modes (see (1.14) to (1.16) for the decomposition) satisfy the radial Schrödinger-like equation

$$\partial_{r^*}^2 \psi_Z + (\omega^2 - V_Z(r))\psi_Z = 0 \tag{6.14}$$

and are thus governed by the effective potential $V_Z(r)$ in (1.18).

In order to handle Zerilli modes as quantum states we seek to write the action of our field, which in our case is the Einstein-Hilbert action, in terms of this field. Taking advantage of the perturbative character of Zerilli modes, we will expand the EH action about a Schwarzschild background of mass m on top of which the perturbations occur. A generic perturbation of the metric can be written in terms of the fields Ψ_R and Ψ_Z that encode all of the metric components' fluctuations in the Regge-Wheeler gauge. Since we are interested only in Zerilli-type perturbations, we will consider $\Psi_R = 0$.

The Langrangian of the EH action is $L = \sqrt{-g}R$ where R is the metric configuration's Ricci scalar. Expanding in a series of powers of Ψ_Z,

$$S_{EH}[g] = \int d^4x \sqrt{-g}R = \sum_{n=0}^{\infty} S^{(n)} \tag{6.15}$$

where $S^{(n)}$ is the term that contains n insertions of the field Ψ_Z. Evidently $S^{(0)} = 0$ because the Ricci scalar of a Schwarzschild spacetime is zero, and $S^{(1)} = 0$ because the action is stationary about a solution of the Einstein field equations, such as the Schwarzschild metric. Therefore the most prominent contributor is the $n = 2$ term, and we discard all others on the basis that Ψ_Z is perturbative.

The action $S^{(2)}$, being second order to the Zerilli field, is the one that will reproduce the equations of motion (6.14). We therefore observe that it will take a Klein-Gordon-like form with an additional potential $U(x)$. Let the scalar field $\phi(x)$ represent a perturbation corresponding to the Zerilli field. Consider then

$$S^{(2)} = \int d^4x \frac{\sqrt{-g_{(0)}}}{2} \left(\partial_\mu \phi \partial^\mu \phi + U(r)\phi^2\right) \tag{6.16}$$

on the mass m Schwarzschild background $g_{(0)\mu\nu}$. After the separation of variables (see also the similar computation of Appendix A) in the form

$$\phi_{lm}(x^\mu) \sim \frac{\psi(r)}{r} T(t) Y_{l,m}(\theta, \phi) \tag{6.17}$$

the Euler-Lagrange equations for ϕ become

$$-\frac{1}{f(r)}\partial_t^2 \left(\frac{\psi}{r}T\right) + \frac{1}{r^2 f(r)}\partial_{r^*}\left(r^2 \partial_{r^*}\left(\frac{\psi}{r}T\right)\right) - U(r)\frac{\psi}{r}T = \frac{l(l+1)}{r^2}\frac{\psi}{r}T \tag{6.18}$$

and with the positive energy solution $T(t) = \exp(-i\omega t)$ we obtain the radial equation

$$\partial_{r^*}^2 \psi + \omega^2 \psi - f(r)\left(\frac{f'(r)}{r} + U(r) + \frac{l(l+1)}{r^2}\right) = 0 \tag{6.19}$$

and so we identify the Lagrangian that furnishes the Zerilli radial equation (6.14) with the potential $V_Z(r)$ as the Klein-Gordon Lagrangian with the potential $U(r)$

$$U(r) = \frac{V_Z(r)}{f(r)} - \frac{f'(r)}{r} - \frac{l(l+1)}{r^2} \tag{6.20}$$

D The LSZ reduction formula on a curved background

In our derivation of the LSZ formula we will follow, though with some brevity, the steps outlined in the notes [14]. The goal is to obtain a useful relation that allows the computation of a scattering amplitude for a quantum system whose free evolution is known and a perturbative interaction is included. For illustrative clarity we will restrict ourselves to the case of a single particle in the $|in\rangle$ and the $|out\rangle$ state, although the extension to an arbitrary number is immediate and clear.

Suppose our asymptotic states consist of wavepackets of our particle centred strongly around some four-momentum that we label p^μ for the initial and q^μ for the final state. If a function $\psi(k,p)$ gives us the profile of our wavepacket across wavenumbers k centred at p, which we could take to be a Gaussian $(1/\sqrt{2\pi}\sigma)\exp\left(-(k-p)^2/2\sigma^2\right)$ with the limit $\sigma \to 0$ in mind, then we can create our states as

$$\left|p_\psi(t)\right\rangle = \hat{\alpha}^\dagger_{(\psi)p}(t)\left|\Omega\right\rangle = \int dk \psi(k,p)\hat{\alpha}^\dagger_k(t)\left|\Omega\right\rangle \tag{6.21}$$

where $|\Omega\rangle$ is the interacting vacuum state, that is, the vacuum state $|0\rangle$ at a reference time t_0 evolved with the interaction Hamiltonian only, $|\Omega\rangle = e^{i\hat{H}_{int}(t-t_0)}|0\rangle$. The evolution of the operators, on the other hand, is performed with the non-interacting part $\hat{\alpha}^\dagger_k(t) = e^{i\hat{H}_{int}(t-t_0)}\hat{\alpha}^\dagger_k(t_0)e^{-i\hat{H}_{int}(t-t_0)}$.

Writing our amplitude as

$$\langle out\,|in\rangle = \langle q_\psi(\infty)\left|p_\psi(-\infty)\right\rangle = \int dk \int dk' \psi(k,p)\bar{\psi}(k',q)\left\langle\Omega\right|\hat{\alpha}_{k'}(\infty)\hat{\alpha}^\dagger_k(-\infty)\left|\Omega\right\rangle \tag{6.22}$$

we make use of our assumption that the final state does not contain the momenta of the initial state in order to introduce creation operators that can commute all the way to the other side: because $\hat{\alpha}_k(-\infty)|\Omega\rangle = 0$ we have that

$$\begin{aligned}\langle out\,|in\rangle &= \int dk \int dk' \psi(k,p)\bar{\psi}(k',q)\left\langle\Omega\right|\left(\hat{\alpha}_{k'}(\infty) - \hat{\alpha}_{k'}(-\infty)\right)\hat{\alpha}^\dagger_k(-\infty)\left|\Omega\right\rangle \\ &= \int dk' \bar{\psi}(k',q)\left\langle\Omega\right|\left(\hat{\alpha}_{k'}(\infty) - \hat{\alpha}_{k'}(-\infty)\right)\left|in\right\rangle .\end{aligned} \tag{6.23}$$

Recalling that the ladder operators are the projections of the field operator onto the basis of the mode functions u_k with the Klein-Gordon inner product

$$(f,g) = -i\int_{\Sigma(t)} d\Sigma^\mu (f\partial_\mu \bar{g} - \bar{g}\partial_\mu f), \tag{6.24}$$

such that

$$\hat{\alpha}_k = (\hat{\phi}, u_k), \tag{6.25}$$

we write

$$\begin{aligned}\langle out\,|in\rangle &= \int dk \bar{\psi}(k,q)\left\langle\Omega\right|\left((\hat{\phi},u_k)|_\infty - (\hat{\phi},u_k)|_{-\infty}\right)\left|in\right\rangle \\ &= \left\langle\Omega\right|\left((\hat{\phi},\Psi_q(x))|_\infty - (\hat{\phi},\Psi_q(x))|_{-\infty}\right)\left|in\right\rangle \\ &= -i\left\langle\Omega\right|\left(\int_{\Sigma(\infty)} d\Sigma^\mu - \int_{\Sigma(-\infty)} d\Sigma^\mu\right)\left(\hat{\phi}\partial_\mu\bar{\Psi}_q(x) - \bar{\Psi}_q(x)\partial_\mu\hat{\phi}\right)\left|in\right\rangle\end{aligned} \tag{6.26}$$

where we defined $\Psi_q(x) = \int dk \psi(k,q)u_k$. Assuming that the wavepacket is sufficiently localised, we consider the surface S that together with $\Sigma(\pm\infty)$ bounds a closed four-volume of the manifold V, large enough so that $\Psi(x)$ and its derivative approximately

vanish on S. Hence the integral can be written by Gauss' law on the interior four-volume

$$\langle out\,|in\rangle = -i\,\langle\Omega|\int_V d^4x\sqrt{-g}\nabla^\mu(\hat{\phi}\partial_\mu\bar{\Psi}_q(x) - \bar{\Psi}_q(x)\partial_\mu\hat{\phi})\,|in\rangle\,. \tag{6.27}$$

Expanding the derivatives and writing $\partial_\mu f = \nabla_\mu f$ for a scalar f,

$$\langle out\,|in\rangle = -i\,\langle\Omega|\int_V d^4x\sqrt{-g}(\hat{\phi}\nabla^2\bar{\Psi}_q - \bar{\Psi}_q\nabla^2\hat{\phi})\,|in\rangle\,. \tag{6.28}$$

At this point we recall that the basis modes u_k are solutions of our field equation, whether these are purely Klein-Gordon or something else. In our case, following the discussion in the Appendix C, we are considering a Klein-Gordon field with a spatially dependent potential that yields the equations of motion

$$\nabla^2\phi(x) = U(r)\phi(x). \tag{6.29}$$

The property of being a solution of the equations of motion extends to the functions $\Psi_p(x)$ as

$$(\nabla^2 - U(x))\Psi_p(x) = \int dk\psi(k,p)(\nabla^2 - U(x))u_k(x) = 0 \tag{6.30}$$

and so we get that

$$\begin{aligned}\langle out\,|in\rangle &= i\,\langle\Omega|\int_V d^4x\sqrt{-g}(\hat{\phi}U(x)\bar{\Psi}_q + \bar{\Psi}_q\nabla^2\hat{\phi})\,|in\rangle \\ &= i\,\langle\Omega|\int_V d^4x\sqrt{-g}\bar{\Psi}_q(U(x)+\nabla^2)\hat{\phi}\,|in\rangle\,.\end{aligned} \tag{6.31}$$

Now we take the limit of our profile function being δ-narrow around the wavepacket frequence, and so $\Psi_p(x) = \int dk\delta(k-p)u_k(x) = u_p(x)$, yielding

$$\langle out\,|in\rangle = i\,\langle\Omega|\int_V d^4x\sqrt{-g}\bar{u}_q(U(x)+\nabla^2)\hat{\phi}\,|in\rangle\,. \tag{6.32}$$

The process is repeated for an arbitrary number of fields in the in- and out states and the result generalises directly. Here, for the in state creation operator we similarly obtain

$$\langle out\,|in\rangle = i\int_V d^4x\sqrt{-g}\int_V d^4y\sqrt{-g}\;\bar{u}_q(x)u_p(y)\,(U(x)+\nabla_x^2)\,(U(y)+\nabla_y^2)\;\langle\Omega|\,\hat{\phi}(x)\hat{\phi}(y)\,|\Omega\rangle \tag{6.33}$$

or more compactly

$$\langle out\,|in\rangle = \int_V d^4x\sqrt{-g}\int_V d^4y\sqrt{-g}\;\bar{u}_q(x)u_p(y)\,\mathcal{K}_x\mathcal{K}_y\;\langle\Omega|\,\hat{\phi}_x\hat{\phi}_y\,|\Omega\rangle \tag{6.34}$$

with $\mathcal{K}_x$ denoting the equations of motion operator with respect to the variable x.

E Properties of the Wronskian

The Wronskian of two functions $f(x), g(x)$ is an antisymmetric bilinear map defined as

$$W(f,g) = fg' - gf' \tag{6.35}$$

with $f' = \partial_x f(x)$. It is a convenient tool to study solutions of a differential equation of 2nd degree (generalised versions of the preceding formula exist for higher order differential equations).

We will focus here on homogeneous differential equations of the form $y'' + a(x)y' + b(x)y = 0$. We notice:

$$\begin{aligned} W'(f,g) &= f'g' + fg'' - f'g' - f''g \\ &= f(ag' + bg) - g(af' + bf) = -a(fg' - gf') = -aW(f,g) \end{aligned} \tag{6.36}$$

and therefore, if the value $W(f,g)|_{x_0} = W_0$ is known, then

$$W(f,g)(x) = W_0 e^{-\int_{x_0}^{x} a(x')dx'}. \tag{6.37}$$

This property is called Abel's identity. Restricting to $a(x) = 0$, we find that

$$W(f,g)(x) = W_0 \tag{6.38}$$

or that the Wronskian is constant for two given functions and may be evaluated anywhere equivalently. We will remain within the bounds of this restriction in the following.

We will henceforth follow some of the steps outlined in [9]. Let $y_1(x), y_2(x)$ be two linearly independent solutions. Because our equation is of 2nd order, there can only be a maximum of two linearly independent solutions that can be used as a basis to write a general solution. We may always then switch to a new basis $C(x), S(x)$ which satisfies $W(C,S) = 1$ (while, trivially, $W(C,C) = W(S,S) = 0$). Writing, for instance, the transformation

$$\begin{pmatrix} C \\ S \end{pmatrix} = \mathbb{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{6.39}$$

we may identify the four elements of $\mathbb{A}$ by demanding that at a certain point x_0 we have $C(x_0) = S'(x_0) = 1$ and $C'(x_0) = S(x_0) = 0$, a total of four conditions that ensure $W(C,S)(x) = 1$ everywhere since W is constant. Specifically, $W(C,S) \equiv 1 = W(\alpha y_1 + \beta y_2, \gamma y_1 + \delta y_2) = (\alpha\gamma - \beta\delta)W(y_1, y_2)$ and so $\det \mathbb{A} = 1/W(y_1, y_2)$ which is satisfied by

$$\mathbb{A} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \tag{6.40}$$

Writing a general solution on this basis will give us

$$y(x) = AC(x) + BS(x) \tag{6.41}$$

where $A = W(y,S)$ and $B = W(C,y)$ as we find by projecting with W onto the basis.

Such transformations therefore allow us to switch between pairs of linearly independent solutions that constitute a basis for our field. A general transformation

$$\begin{pmatrix} C_2 \\ S_2 \end{pmatrix} = R \begin{pmatrix} C_1 \\ S_1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} C_2 = R_{11}C_1 + R_{12}S_1 \\ S_2 = R_{21}C_1 + R_{22}S_1 \end{pmatrix} \tag{6.42}$$

will therefore contain the matrix elements

$$R = \begin{pmatrix} W(C_2, S_1) & -W(C_2, C_1) \\ W(S_2, S_1) & -W(S_2, C_1) \end{pmatrix}. \tag{6.43}$$

We are especially interested in picking out basis functions that take a simple form in asymptotic regions of our system. To illustrate this we will bring our general differential equation (with $a(x) = 0$ imposed) to the form

$$y'' + (\omega^2 - V(x))y = 0. \tag{6.44}$$

Let us assume that the potential is zero in two distinct regions which we consider to be asymptotic. This can for example be the case with a localised potential that is nonzero near $x = 0$ and so there exist two asymptotic regions $x \to \pm\infty$ where it holds that $V(x \to \pm\infty) \to 0$. In these regions we can find two simple solutions that make up a basis, that is, $e^{\pm i\omega x}$. Labeling the regions in minus/plus infinity as 1 and 2 respectively, we therefore obtain two pairs of bases, which behave as

$$\begin{pmatrix} C_1 = e^{-i\omega x} \\ S_1 = e^{i\omega x} \end{pmatrix} \text{ at } x \to -\infty \text{ , and } \begin{pmatrix} C_2 = e^{-i\omega x} \\ S_2 = e^{i\omega x} \end{pmatrix} \text{ at } x \to +\infty. \tag{6.45}$$

It is important to note that these expressions are valid in the respective asymptotic region we defined them in, but do not in general maintain their form in the other asymptotic region, and hence the need to utilise two sets of bases. Their form in the adjacent region can always be found by integrating the differential equation from a reference point within their initial domain. Nevertheless, as we established already, we can relate the bases at any point using the Wronskian of the basis modes as in (6.43) - the Wronskian can additionally be evaluated at any point because it is constant.

Application to scattering problems

This construction allows us to evaluate quantities such as the transmission and reflection ratios in a scattering problem described by the potential $V(x)$. The scattering setup requires that the field $y(x)$, a solution of our differential equation, has an ingoing and outgoing part in one region, and only an outgoing part in the other. Hence we set its component on the basis mode S_2 equal to zero, requiring that $y(x) = AC_2(x)$ which implies $y(x) = Ae^{-i\omega x}$ as $x \to +\infty$. Because of the relation of the basis through the Wronskian we may also write $y(x) = AC_2(x) = AW(C_2, S_1)C_1(x) - AW(C_2, C_1)S_1(x)$ or $y(x) = AW(C_2, S_1)e^{-i\omega x} - AW(C_2, C_1)e^{i\omega x}$. We therefore identify the transmission as the squared amplitude ratio of the incident to the transmitted wave

$$T = \frac{|A|^2}{|AW(C_2, S_1)|^2} = \frac{1}{|W(C_2, S_1)|^2}. \tag{6.46}$$

One can treat in this case the example of the delta barrier potential such that $-\frac{1}{2}y'' + V_0\delta(x)y = Ey$. The two asymptotic regions therefore extend to the vicinity of $x = 0$, becoming $x \in (-\infty, 0)$ and $(0, \infty)$, and the two basis are related by matching conditions on $x = 0$. These are, firstly, continuity of the field $y(x)$, and, secondly, a jump of the derivative coming from the integration of the differential equation in the vicinity of the discontinuity:

$$-\frac{1}{2}\int_{-\epsilon}^{\epsilon} dx y'' + V_0 \int_{-\epsilon}^{\epsilon} dx \delta(x) y = E \int_{-\epsilon}^{\epsilon} dx y \tag{6.47}$$

yielding, in the $\epsilon \to 0^+$ limit,

$$2V_0 y(0) = y'(0^+) - y'(0^-). \tag{6.48}$$

Then, by writing $C_1 = \alpha e^{ikx} + \beta e^{-ikx}$ for $x > 0$, with $k = \sqrt{2E}$, we find $\alpha = 1 - (V_0/k)i$ and $\beta = (V_0/k)i$ and thus $C_1 = (1 - (V_0/k))C_2 + (V_0/k)iS_2$. Similarly we can find $S_1 = -(V_0/k)iC_2 + (1 - (V_0/k))S_2$. We finally evaluate $W(C_1, S_2) = 1 - (V_0/k)i$, giving us the transmission coefficient

$$T = \frac{1}{1 + \frac{V_0^2}{2E}}. \tag{6.49}$$

Computation of the Green function

The Wronskian is particularly useful to us because its analytical expression is related to the QNM frequencies, as the latter are its poles. The key fact behind this observation is that the Green function of the differential operator, that is used to construct a general solution, directly depends on the Wronskian of the asymptotic solutions. Here we will demonstrate how the Green function can be constructed using a set of asymptotic solution, chosen to satisfy a boundary condition of interest.

The Green function $G(x,\xi)$ satisfies the relation

$$D_x G(x,x') = \delta(x-\xi) \tag{6.50}$$

where D_x is the differential operator, in our case $D_x = \partial_x^2 + \omega^2 - V(x)$. The main idea behind this construction is that, for a given ξ, in the regions $x \neq \xi$ the Green function satisfies the homogeneous equation

$$D_x G(x,\xi) = 0, \quad x \neq \xi. \tag{6.51}$$

We can therefore express the functional form of $G(x,\xi)$ in these regions as a linear combination of the asymptotic solutions. This allows us to consider a family potential Green functions depending on our choice of the coefficients on the asymptotic basis. We choose our boundary conditions by selecting these coefficients: in the problem at hand, if we are looking for an 'outgoing-only' solution, we can impose that $G \sim e^{i\omega x}$ for $x < 0$ and $G \sim e^{-i\omega x}$ for $x > 0$. We therefore write

$$G(x,\xi) = A(\xi)e^{i\omega x}H(\xi - x) + B(\xi)e^{-i\omega x}H(x-\xi) \tag{6.52}$$

where we should remember that, because the asymptotic behaviour on x is imposed for given ξ, the multiplicative factors $A(\xi), B(\xi)$ may depend only on ξ.

The specification of the factors comes from imposing matching conditions on $x = \xi$, that is, continuity of G and jump of G'. By the expression

$$G(x,\xi) = A(\xi)S_1(x)H(\xi - x) + B(\xi)C_2(x)H(x-\xi) \tag{6.53}$$

with C, S denoting the right and left moving solution, by imposing continuity we find $B(\xi) = \frac{S_1(\xi)}{C_2(\xi)}A(\xi)$, and so

$$G(x,\xi) = A(\xi)\left(S_1(x)H(\xi - x) + \frac{S_1(\xi)}{C_2(\xi)}C_2(x)H(x-\xi)\right). \tag{6.54}$$

Further, the jump condition depends on the differential operator. For our example, integrating in the vicinity of $x = \xi$ yields

$$1 = \int_{-\epsilon}^{\epsilon} dx D_x G(x,\xi) = G'(\xi^+,\xi) - G'(\xi^-,\xi) \tag{6.55}$$

which leads to the condition

$$A(\xi) = \frac{1}{\frac{S_1(\xi)}{C_2(\xi)}C_2'(\xi) - S_1'(\xi)} = \frac{C_2(\xi)}{W(S_1,C_2)}. \tag{6.56}$$

We therefore obtain obtain the Green function

$$G(x,\xi) = \frac{C_2(\xi)}{W(S_1,C_2)}S_1(x)H(\xi - x) + \frac{S_1(\xi)}{W(S_1,C_2)}C_2(x)H(x-\xi). \tag{6.57}$$

F Signal Hilbert transform

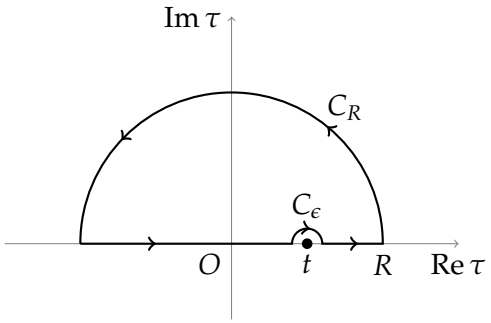
The Hilbert transform of a signal $s(t)$ is defined as the principal value

$$\mathcal{H}[s](t) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} d\tau \frac{s(\tau)}{t-\tau}. \tag{6.58}$$

By definition of the Cauchy principal value this is equal to

$$\mathcal{H}[s](t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \left(\int_{-\infty}^{t-\epsilon} d\tau \frac{s(\tau)}{t-\tau} + \int_{t+\epsilon}^{\infty} d\tau \frac{s(\tau)}{t-\tau} \right). \tag{6.59}$$

It is instructive to check what the transformation returns for a simple wave $s(t) = e^{i\omega t}$ for $\omega > 0$. In order to calculate the two integrals, we construct a closed contour on the complex τ plane, where our original integrals lie on the real line, joined together by a semicircle C_ϵ of radius ϵ on the upper plane that avoids the pole at $\tau = t$, and then by a large semicircle C_R of radius R again on the upper plane. We pick the latter to lie on the upper complex plane because when ω is positive the exponential $e^{i\omega\tau} \sim e^{-\omega R \sin\theta}$ decays to 0 as $R \to \infty$, having assumed the parameterisation $C_R : \tau = Re^{i\theta}, \theta \in [0, \pi]$.



We consequently have that

$$\lim_{R \to \infty} \int_{C_R} d\tau \frac{e^{i\omega\tau}}{t-\tau} = 0 \tag{6.60}$$

and therefore

$$\begin{aligned} \mathcal{H}\left[e^{i\omega\tau}\right](t) &= \frac{1}{\pi} \lim_{\epsilon \to 0} \left(\int_{-\infty}^{t-\epsilon} d\tau \frac{e^{i\omega\tau}}{t-\tau} + \int_{t+\epsilon}^{\infty} d\tau \frac{e^{i\omega\tau}}{t-\tau} \right) \\ &= -\frac{1}{\pi} \lim_{\epsilon \to 0} \int_{C_\epsilon} d\tau \frac{e^{i\omega\tau}}{t-\tau} \end{aligned} \tag{6.61}$$

and, assuming the parameterisation $C_\epsilon : \tau = t + \epsilon e^{i\theta}, \theta \in [0, \pi]$ and thus $d\tau = i\epsilon e^{i\theta} d\theta$, we have got that

$$\begin{aligned} \mathcal{H}\left[e^{i\omega\tau}\right](t) &= -\frac{i\epsilon}{\pi} \lim_{\epsilon \to 0} \int_{\pi}^{0} d\theta e^{i\theta} \frac{e^{i\omega\tau}}{-\epsilon e^{i\theta}} = \frac{i}{\pi} \lim_{\epsilon \to 0} \int_{\pi}^{0} d\theta e^{i\omega\tau} \\ = \frac{i}{\pi} \lim_{\epsilon \to 0} \int_{\pi}^{0} d\theta e^{i\omega(\epsilon e^{i\theta} + t)} &= \frac{i}{\pi} e^{i\omega t} \lim_{\epsilon \to 0} \int_{\pi}^{0} d\theta = -ie^{i\omega t} = e^{i(\omega t - \frac{\pi}{2})} \end{aligned} \tag{6.62}$$

so we conclude that the Hilbert transform induces a phase difference of a quarter of a cycle.

Consider that an arbitrary real signal can be expressed in a basis of sine and cosine functions, which in turn are expressible in terms of periodic exponentials of the same frequency. Applying a Hilbert transform therefore shifts the phase of every sine and cosine by $-\frac{\pi}{2}$, thereby turning $\mathcal{H}[\cos(\omega t)] = -\sin(\omega t)$ and $\mathcal{H}[\sin(\omega t)] = \cos(\omega t)$. This means that for a sinusoidal signal such as $f(t) = A\cos(\omega t)$ we may construct its complex form as

$$\hat{f}(t) = Ae^{-i\omega t} = A\cos(\omega t) - iA\sin(\omega t) = f(t) + i\mathcal{H}[f](t) \tag{6.63}$$

which allows us to relate its amplitude to the signal and its Hilbert transform:

$$A = |\hat{f}(t)| = |f(t) + i\mathcal{H}[f](t)|. \tag{6.64}$$

For the signal $s(t)$ we can therefore construct

$$\begin{aligned} \hat{s}(t) &= s(t) + i\mathcal{H}[s](t) \\ &= \sum_n (A_n \cos(\omega_n t) + B_n \sin(\omega_n t)) + i\sum_n (-A_n \sin(\omega_n t) + B_n \cos(\omega_n t)) \\ &= \sum_n \left(A_n e^{-i\omega_n t} + B_n e^{i\omega_n t} \right) \end{aligned} \tag{6.65}$$

and define its envelope as

$$R(t) = |\hat{s}(t)|. \tag{6.66}$$

Note that the envelope $R(t)$ is not simply the sum of the individual sinusoidal amplitudes A_n, B_n because the magnitude operation is not linear, but rather represents an evolving 'strength' of the signal across all frequencies. For an illustration of an envelope function see figure 4.

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