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The bounded cohomology of the transformation groups of Euclidean spaces

Tesi di Laurea in Matematica

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«La matematica è stata il mio apprendistato alla rivoluzione, perché mi ha insegnato a diffidare di verità assolute e autorità indiscutibili. Democrazia e matematica, da un punto di vista politico, si somigliano: come tutti i processi creativi non sopportano di non cambiare mai.»

— Chiara Valerio

Introduction

Bounded cohomology is a functional-analytic variant of ordinary cohomology and it is a powerful tool in the study of manifold geometry, rigidity theory, stable commutator length, and the dynamics of circle actions. Originally introduced in the seventies by Johnson [Joh72] and Trauber in the context of Banach algebras, it was later extended from groups to topological spaces through Gromov in the early eighties. In particular, Gromov's seminal work showcases the relation between bounded cohomology and other topological invariants of manifolds such as the simplicial volume [Gro82].

In this thesis, we aim to study and discuss a result by Fournier-Facio, Monod and Nariman [FFMN24] concerning the bounded acyclicity of transformation groups of Euclidean spaces (a group is said to be *boundedly acyclic* if its bounded cohomology with real coefficients vanishes in all positive degree). To this end, we first explore the foundations of the theory of bounded cohomology beginning with the definition and its main properties and results. Since here we aim to study the bounded cohomology of transformation groups, we will focus on the bounded cohomology of groups, although the bounded cohomology of topological spaces has also been widely studied.

In Chapter 1 we introduce the main players of this thesis. First, we briefly recall the definition of *group cohomology with values in a $R[G]$-module*, where R is a commutative ring and G a group. Group cohomology can be defined in various equivalent ways, some with a more combinatorial flavor via cochain complexes, others with a more topological or algebraic perspective: via classyfing spaces and injective strong resolutions, respectively. For example, given a group G and a $R[G]$-module V, one can define for every $n \in \mathbb{N}$ the $R[G]$-module

$$C^n(G,V) \coloneqq \{f\colon G^{n+1} \to V\}$$

and a suitable differential $d^n\colon C^n(G,V) \to C^{n+1}(G,V)$. In this way, taking the G-invariants, we have a cochain complex

$$0 \to C^0(G,V)^G \xrightarrow{d^0} C^1(G,V)^G \xrightarrow{d^1} C^2(G,V)^G \xrightarrow{d^2} \cdots,$$

whose cohomology is by definition the cohomology of the group G with values in the $R[G]$-module V.

We then focus on *bounded group cohomology*. Bounded cohomology can be defined in principle with values in any normed $R[G]$-module, where G is a group and R a commutative ring, but in this thesis we focus on real coefficients endowed with the trivial G-action. In this case, the absolute value norm on $\mathbb{R}$ allows us to restrict our attention to bounded cochains rather than arbitrary ones and so to define the so-called *bounded cohomology*. If G is additionally a topological group, one can define *continuous bounded cohomology* by considering only those bounded cochains that are continuous [Mon01, BM02]. However, in this thesis, we focus our attention on the discrete case only. More precisely, given a discrete group G we can define the ℓ^∞-norm for elements in $C^n(G,V)$, where V is a *normed $R[G]$-module* (e.g. $V = \mathbb{R}$ with the absolute value) by

$$\|f\|_\infty \coloneqq \sup_{(g_0,\ldots,g_n)\in G^{n+1}} \|f(g_0,\ldots,g_n)\| \in [0,+\infty].$$

We then restrict to the subspace

$$C_b^n(G,V) \coloneqq \{f \in C^n(G,V) \mid \|f\|_\infty < \infty\} \subseteq C^n(G;V),$$

and we can define the cochain complex of the G-invariants, whose cohomology is the bounded cohomology of G with coefficients in V. Analogously, one can define bounded cohomology using a more topological or algebraic approach as for standard cohomology but taking the norm into account; the algebraic approach makes use of *relative* homological algebra.

We then study the functoriality and the main properties of bounded cohomology. Bounded cohomology presents some analogies with standard cohomology but drastically differs in many other aspects. For instance, there is no analogue of the Mayer-Vietoris sequence or excision properties (Section 1.5). This makes the computation of the bounded cohomology hard also for tamed spaces or nice groups. Moreover, it can be readily seen that bounded cohomology and bounded simplicial cohomology are not isomorphic in general (indeed, the latter is not even a homotopy invariant). Thus, to study and compute bouded cohomology more technical and sofisticated tools are needed. We study real bounded cohomology in low degrees and see that it coincides with standard cohomology in degree 0 but it vanishes for every group in degree 1. However, in degree 2 the situation is much more complicated respect to standard cohomology, where a complete characterization in terms of central extensions is available (Theorem 1.18). Fundamental tools for the computation in degree 2 are *quasimorphisms* that are, roughly speaking, homomorphisms up to a "bounded" defect. Quasimorphisms allow us to show that the bounded cohomology in degree 2 of the non-abelian free group with two generators is infinite-dimensional (Theorem 1.21). This computation showcases that bounded cohomology can behave wildly in comparison to standard cohomology: indeed it is not hard to see that the standard cohomology of the non-abelian free group vanishes from degree 2 onwards.

Another tool to study bounded cohomology is the *comparison map* (Definition 1.16). The inclusion of bounded cochains into all cochains $(C_b^\bullet(G;\mathbb{R}), d^\bullet) \hookrightarrow (C^\bullet(G;\mathbb{R}), d^\bullet)$ induces a map in cohomology that is, in general, neither injective nor surjective. Nevertheless, understanding when this map is either injective or surjective plays a crucial role in many computations. For instance, a classical theorem by Matsumoto and Morita (Theorem 1.36) ensures the injectivity of the comparison map if the "UBC-condition" is satisfied.

In Chapter 2, we introduce the class of *amenable groups*, which plays a significant role in various areas of mathematics. They are groups that admit an invariant mean and they encompass a wide range of groups, including finite, abelian, nilpotent and solvable groups. A celebrated theorem by Johnson, which started the interest towards bounded cohomology, says that amenable groups can be characterized as those groups that have vanishing bounded cohomology with values in a certain class of coefficients in positive degree (Theorem 2.12).

So far, we said that bounded cohomology is very hard to compute except for amenable groups. A natural way to proceed in order to better understand its behaviour is to find an example of a boundedly acyclic group which is not amenable. This is the content of Chapter 3. We investigate the relevance of the comparison map, studying a technique widely used to prove the bounded acyclicity of groups. The approach is fairly simple, at least conceptually. One starts with a group that has been proved to have vanishing ordinary cohomology in all positive degrees, uses a certain injectivity criterion for the comparison map (like the one by Matsumoto and Morita mentioned before), and so concludes that the given group is boundedly acyclic. Historically, the first group this technique was applied to is the group $\mathrm{Homeo}_c(\mathbb{R}^n)$ of self-homeomorphisms of the Euclidean space with compact support. Indeed, it was proved to be acyclic by Mather in 1971 (Theorem 3.2), and then to be boundedly acyclic by Matsumoto and Morita using their criterion for the injectivity of the comparison map in 1985 (Theorem 3.4). We point out the compactness hypothesis since the main result of this thesis is the bounded acyclicity of the group $\mathrm{Homeo}(\mathbb{R}^n)$, i.e. without restricting to compacly supported homeomorphisms. The proof of Mather goes as follows: since chains are defined by a finite numbers of elements only, we can assume that every chain is supported in an n-dimensional ball B^n_R of a certain radius R. Than Mather exhibits an element $k \in \mathrm{Homeo}_{\mathrm{c}}(\mathbb{R}^n)$ such that $k^j(\overline{B}^n_R) \cap \overline{B}^n_R = \emptyset$ and $\lim_{j\to\infty} k^j(\overline{B}^n_R) = p$ for some $p \in \mathbb{R}^n$. Namely there exists an homeomorphism that displace the ball infinite many times within a compact support (of course by shrinking it to a point). This provides the key ingredient in the proof of the acyclicity of $\mathrm{Homeo}_{\mathrm{c}}(\mathbb{R}^n)$ and it is evident that without the compactness assumption such a k cannot exist (since the support of $\varphi \in \mathrm{Homeo}_{\mathrm{c}}(\mathbb{R}^n)$ might be the whole $\mathbb{R}^n$). This group was the first known example of a non-amenable boundedly acyclic group and the proof stands as a model for many other results of this type. For instance, *dissipated groups* (Definition 3.8) are those class of groups that allows the existence of an analogous element as the k used in the proof of Mather's Theorem. More generally, a similar approach is taken by Fournier-Facio, Löh, and Moraschini for *pseudo-mitotic groups* (Theorem 3.14) when they computed the bounded cohomology of dissipated groups. In this thesis, we define and analyze these classes of groups and discuss the bounded acyclicity results mentioned above.

In Chapter 4, we show that we obtain bounded acyclicity of the transformation group of Euclidean spaces even without the compactness hypothesis. Although relying on the previous results, this has been proved via a new and different approach. Little is known about the ordinary cohomology of transformation groups of Euclidean spaces, so at this stage, showing the injectivity of the comparison map would not be enough to establish its bounded acyclicity. Considering a group G acting on a set X, Fournier-Facio, Monod and Nariman provide two criteria: one ensures bounded acyclicity (Theorem 4.2), the other ensures standard acyclicity (Theorem 4.3). This represents a new strategy in the study of bounded acyclicity without assuming acyclicity, but in fact provides a new criterion that might well be exploited for investigating standard acyclicity. By verifying the criterion, the authors show that $\mathrm{Diff}(\mathbb{R}^n)$ and thus also $\mathrm{Homeo}(\mathbb{R}^n)$ are boundedly acyclic. Remarkably, this result does not rely on regularity, whereas it is known that $\mathrm{Diff}(\mathbb{R}^n)$ has highly nontrivial standard cohomology.

The criterion goes as follows. One let a group G act on a set X, and extend this action to a poset $\mathcal{P}$ built from this set. Then, one consider the action of G on a G-invariant subposet of $\mathcal{P}$, say $\mathcal{X}$. We ask for three conditions: the poset $\mathcal{X}$ satisfies a certain combinatorial property, called *W-property*, the action of G on $\mathcal{X}$ is transitive and one, or equivalently every, stabilizer in G of an element of $\mathcal{X}$ is (boundedly) acyclic. If these hypotheses hold, the group G is (boundedly) acyclic. The proof relies on the theory of *simplicial sets*, a generalization of the standard simplicial complexes from algebraic topology. We give a brief and intuitive discussion of it in Section 4.1. To show that the group of transformation of the euclidean space satisfies the criterion, we will need to recall some facts about *isotopies* (Definition 4.55) and in particular we will need the *Isotopy Extension Theorem* (Theorem 4.56).

To sum up, in this thesis we explore the broad field of bounded cohomology, with a primary focus on the cohomology of groups. We begin by presenting the foundations of the theory. Then, we introduce amenable groups and we highlight their connection with bounded cohomology. After that, we focus on the (bounded) acyclicity of the group of self-homeomorphisms of Euclidean spaces with compact support. We eventually study the 2024 article by Fournier-Facio, Monod and Nariman that removes the compactness support hypothesis, giving particular attention to the criterion used, which potentially has applicability in various other classes of groups.

Contents

1 Bounded cohomology of groups

In this section we work in the category of (bounded) $R[G]$-modules, where R is an abelian ring (usually $\mathbb{R}$ or $\mathbb{Z}$) and G is a discrete group. Recall that an $R[G]$-module is an R-module endowed with an action of G by R-linear maps, and an $R[G]$-map between $R[G]$-modules is an R-linear map which commutes with the action of G. For simplicity's sake, we refer to this maps as G-maps when R is understood. We review the definition of *standard* and *bounded group cohomology*, focusing on the latter and pointing out analogies and differences. They can be defined in various equivalent ways, sometimes with a more combinatorial flavor, others with a more topological or algebraic perspective.

Later in the chapter, we briefly recall the functorial properties of standard cohomology and examine how they carry over to the bounded cohomology setting. We then define the *comparison map* (Section 1.4), which is a map from bounded cohomology to standard cohomology. The comparison map will play a crucial role in Section 3.2 when computing the bounded cohomology of the group of compactly supported homeomorphisms of $\mathbb{R}^n$ in Theorem 3.4. Next, we explore some properties of bounded cohomology, highlighting that it does not admit a Mayer-Vietoris sequence nor satisfy the excision axiom. Afterward, we study bounded cohomology in low degrees: in particular, we show that it vanishes in degree 1 for real coefficients, while the realm of *quasimorphisms* arises in degree 2. Finally, we consider another invariant, called ℓ^1*-homology*, and investigate the duality properties of bounded cohomology. We also provide a criterion that, among other things, describes properties of the aforementioned comparison maps and will again prove useful for computing the bounded cohomology of the group of compactly supported homeomorphisms of $\mathbb{R}^n$ in Theorem 3.4. Namely, in Theorem 1.36 we prove that a normed chain complex satisfies the so-called q*-Uniform Boundary Condition* for $q \geq 0$ if and only if the comparison map is injective in degree $q+1$. Thus, proving that a group has vanishing bounded cohomology in degree $q+1$ is equivalent to prove that it has vanishing standard cohomology in degree $q+1$ and that the q-Uniform Boundary Condition holds.

1.1 Group cohomology

We start briefly recalling three different definitions of standard group cohomology with values in an $R[G]$-module V. As a source, we refer to the book of Brown [Bro82].

1.1.1 A combinatorial approach to group cohomology

For every $n \in \mathbb{N}$, denote by

$$C^n(G, V) \coloneqq \{f : G^{n+1} \to V\}, \tag{1}$$

and we take $\delta^n : C^n(G, V) \to C^{n+1}(G, V)$ to be the differential defined by

$$\delta^n f(g_0, \dots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \dots, \widehat{g_i}, \dots, g_{n+1}).$$

Observe that $C^n(G, V)$ can be endowed with a structure of $R[G]$-module considering the G-action

$$(g \cdot f)(g_0, ..., g_n) \coloneqq g(f(g^{-1}g_0, ..., g^{-1}g_n)).$$

The cochain complex $(C^\bullet(G, V), \delta^\bullet)$ is usually called the *homogeneous complex*; as we will see below, it is sometimes useful to consider different complexes in the definition of bounded cohomology (e.g. the bar resolution). The complex of the G-invariants $C^\bullet(G,V)^G$ is a subcomplex of $C^\bullet(G,V)$ since δ^n is a G-map for every $n \in \mathbb{N}$. The cohomology of G with coefficients in V is then defined as the *cohomology* of the complex

$$0 \to C^0(G,V)^G \xrightarrow{\delta^0} C^1(G,V)^G \xrightarrow{\delta^1} C^2(G,V)^G \xrightarrow{\delta^2} \cdots .$$

Definition 1.1 (**Group cohomology**)**.** In the notation of before, the cohomology of G with coefficients in V is defined as:

$$H^n(G,V) \coloneqq \frac{Z^n(G,V)}{B^n(G,V)},$$

where $Z^n(G,V) \coloneqq C^n(G,V)^G \cap ker\delta^n$ and $B^n(G,V) \coloneqq \delta^{n-1}(C^{n-1}(G,V)^G)$, with the convention that $B^0(G,V) \coloneqq 0$.

Recall that we say that a group is *acyclic* if its cohomology with integer coefficients vanishes in every positive degree.

Remark 1.2. There is a simple reason for which we look only at the G-invariants: if we tried to define the cohomology directly from the homogeneous complex, we would get the trivial module in every degrees. This can be readily seen by considering the following chain homotopy between the identity and the zero map defined by

$$k^{n+1}: C^{n+1}(G,V) \to C^n(G,V), \quad k^{n+1}f(g_0,\dots,g_n) = f(1,g_0,\dots,g_n).$$

With chain homotopy we intend a completly analogous definition given for other cohomology theories. It is straightforward to see that this implies that the induced maps in cohomology are the same, hence the conclusion.

As said above, it will be in many computations convenient to have other kinds of resolutions: for example, here we describe the *bar resolution*. This construction arises from the observation that elements of $C^n(G;V)$ are determined by the values they take on $(n+1)$-tuples whose first entry is fixed.

Proposition 1.3 (**The bar resolution**)**.** *The complex* $(\overline{C}^\bullet(G;V), \overline{\delta}^\bullet)$ *is canonically isomorphic to the G-invariants of the homogeneous one, where:*

$$\overline{C}^0(G;V) = V, \quad \overline{C}^n(G;V) = C^{n-1}(G;V) = \{f \colon G^n \to V\}$$

and $\overline{\delta}^\bullet : \overline{C}^\bullet(G;V) \to \overline{C}^{\bullet+1}(G;V)$ *is defined by:*

$$\delta^0(v)(g) = g \cdot v - v \quad \textit{for } v \in V = \overline{C}^0(G;V),\ g \in G,$$

and for $f \in \overline{C}^n(G;V)$,

$$\begin{aligned} \delta^n(f)(g_1,\dots,g_{n+1}) = \ & g_1 \cdot f(g_2,\dots,g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1,\dots,g_i g_{i+1},\dots,g_{n+1}) \\ & + (-1)^{n+1} f(g_1,\dots,g_n). \end{aligned}$$

Proof. For every $n \in \mathbb{N}$ we have R-isomorphisms given by

$$C^n(G;V)^G \to \overline{C}^n(G;V), \quad \varphi \mapsto ((g_1,\dots,g_n) \mapsto \varphi(1,g_1,g_1g_2,\dots,g_1\cdots g_n)).$$

Under this isomorphism, the differential $\delta^\bullet \colon C^\bullet(G;V)^G \to C^{\bullet+1}(G;V)^G$ descends to the one defined above, and therefore we have an isomorphism of complexes. $\square$

1.1.2 An algebraic approach to group cohomology

We describe here how group cohomology can be defined with the standard approach of resolutions of modules. In order to do so, we define in particular *relatively injective strong resolutions*. We only provide the essential definitions and summarize the results leading to group cohomology, without proving them; we refer to the book of Frigerio [Fri17, Chapter 4] for the details.

Definition 1.4 (Strongly injective map). Let A, B be $R[G]$-modules and $i\colon A \to B$ a G-map. We say that i is *strongly injective* if there exists an R-linear map $\sigma\colon B \to A$ sucht that $\sigma \circ i = \mathrm{Id}$.

Remark 1.5. We point out that if i is strongly injective then it is injective. Moreover, it is not requested for the map σ to be G-equivariant.

Definition 1.6 (relatively injective module). Let V be $R[G]$-modules. The module V is *relatively injective* if given any other two $R[G]$-modules A, B, a strongly injective map $i\colon A \to B$ and a G-map $\alpha\colon A \to V$, then there exists a map $\beta\colon B \to V$ such that $\beta \circ i = \alpha$.

Remark 1.7. In the case $R = \mathbb{R}$, the definitions of strongly injective and injective maps coincide, so relatively injective $\mathbb{R}[G]$-modules are just injective $\mathbb{R}[G]$-modules.

An augmented complex of $R[G]$-modules of the form

$$0 \to V \xrightarrow{\epsilon} V_0 \xrightarrow{\delta_0} V_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-1}} V_n \xrightarrow{\delta_n} \cdots,$$

where ϵ is an embedding, is said to be a *relatively injective resolution* of V if every V_i is relatively injective. In this context, in order to ensure that a map between two modules can be extended to a map between their resolutions one needs to work with *strong resolutions*.

Definition 1.8 (strong resolution). Let V be an $R[G]$-module. A resolution of V is called *strong* if there exists a *contracting homotopy*, that is, a sequence of R-linear maps $k_i\colon V_i \to V_{i-1}$ such that $\delta_{i-1} \circ k_i + k_{i+1} \circ \delta_i = \mathrm{Id}_{V_i}$ for $i \geq 0$ and $k_0 \circ \varepsilon = \mathrm{Id}_{V_{-1}}$.

Remark 1.9. The maps in the definition of contracting homotopy are not required in general to be G-invariant. Indeed, if they are assumed to be G-invariant, they provide an equivariant cochain homotopy between the zero map and the identity map.

By defining the augmentation $\epsilon\colon V \to C^0(G,V)$ as $\epsilon(v)(g) = v$ for every $v \in V$, $g \in G$, it is readily seen that the augmented homogeneous complex

$$0 \longrightarrow V \xrightarrow{\varepsilon} C^0(\Gamma, V) \xrightarrow{\delta^0} C^1(\Gamma, V) \xrightarrow{\delta^1} C^2(\Gamma, V) \xrightarrow{\delta^2} \ldots$$

is a relatively injective strong resolution of V, as showed in Proposition 4.3 of [Fri17]. Moreover, one can show that any two relatively injective strong resolutions of the same $R[G]$-module are equivalent up to G-homotopy [Mon01, Lemmas 7.2.4 and 7.2.6]. Therefore, group cohomology of an $R[G]$-module can be consistently defined as the cohomology of any relatively injective strong resolution.

1.1.3 A topological approach to group cohomology

Given a group G one can consider a model of its *classifying space* BG, i.e. a pointed connected CW-complex (X, x) together with an isomorphism $\pi_1(X, x) \cong G$ such that the universal covering of X is contractible. This can be performed for every group G, for example constructing a suitable Δ-complex [Hat02, Example 1B.7, p. 87]. It is then a standard result that any two different models are equivalent up to canonical homotopy [Hat02, Section 1.B], and so it is possible to define the group cohomology of a group G as the simplicial cohomology of any representative of its classifying space. In fact the simplicial cochain complex associated to the model of BG constructed by Hatcher [Hat02, Example 1B.7, p. 87] agree with the G-invariants of the homogeneous complex of G.

1.2 Bounded cohomology

By introducing a norm into the framework and considering only bounded cochains, we define the main object of this thesis, namely the *bounded cohomology* of a group. For simplicity we only consider the case of $R = \mathbb{R}$ or $R = \mathbb{Z}$ endowed with the usual absolute value $|\cdot|$. An $R[G]$-module is *normed* if it is endowed with a G-invariant norm, i.e. a map $\|\cdot\| : V \to R$ satisfying for every $v, w \in V$, $g \in G$ and $r \in R$:

- $\|v\| = 0$ if and only if $v = 0$,
- $\|rv\| = |r| \cdot \|v\|$,
- $\|v + w\| \leq \|v\| + \|w\|$,
- $\|g \cdot v\| = \|v\|$.

A G-map between normed $R[G]$-module is an $R[G]$-map which is bounded with respect to the norms. As before, we describe the combinatorial, algebraic and topological approaches for defining bounded cohomology.

1.2.1 A combinatorial approach to bounded cohomology

We endow the space of cochains $C^n(G; V)$ defined in Equation (1) with a norm as follows: for every $f \in C^n(G, V)$, the ℓ^∞-norm is defined by

$$\|f\|_\infty := \sup_{(g_0,\dots,g_n) \in G^{n+1}} \|f(g_0, \dots, g_n)\| \in [0, +\infty].$$

We then restrict to the subspace

$$C_b^n(G, V) := \{f \in C^n(G, V) \mid \|f\|_\infty < \infty\} \subseteq C^n(G; V),$$

which is a normed $R[G]$-submodule of $C^n(G, V)$. The differential of the (possibly unbounded) homogeneous complex restricts to the bounded cochains since it sends bounded cochains to a finite sum of them. In a similar fashion as before, we consider the *space of bounded chain* $Z_b^n(G, V) := C_b^n(G, V)^G \cap \operatorname{Ker} \delta^n$ and the *space of bounded boundaries* $B_b^n(G, V) := \delta^{n-1}(C_b^{n-1}(G, V)^G)$, with the convention that $B_b^0(G, V) := 0$ to define

$$H_b^n(G, V) := \frac{Z_b^n(G, V)}{B_b^n(G, V)}.$$

Definition 1.10 (**Bounded cohomology of a group**)**.** We call $H_b^n(G, V)$ the *bounded cohomology of G with V coefficients.*

Analogously to the classical cohomology setting, as we shall see in the following chapters, it is important to investigate the class of groups for which bounded cohomology vanishes in positive degrees.

Definition 1.11 (**Boundedly acyclic group**)**.** We say that a group is *boundedly acyclic* if its bounded cohomology with trivial real coefficients vanishes in every positive degree.

The norm of the bounded cochains restricts to a norm on $Z_b^n(G, V)$ and descends to a seminorm on the quotient $H_b^n(G, V)$ as follows:

$$\|\alpha\|_\infty := \inf_{\substack{f \in Z_b^n(G,V) \\ [f]=\alpha}} \|f\|_\infty.$$

It is a standard result of functional analysis that this is actually a norm if and only if $B_b^n(G, V)$ is closed in $Z_b^n(G, V)$ [Rud91, Theorem 1.41].

One can also define the *bounded bar resolution* $(\overline{C}_b^\bullet, \overline{\delta}_b^\bullet)$ following step by step what it has been done in Section 1.3 for the standard cohomology. Again the two approach lead to isometrically isomorphic bounded cochain complexes.

1.2.2 An algebraic approach to bounded cohomology

Similarly to the case of standard cohomology, we can study bounded cohomology via the notion of *relatively injective strong resolutions* of normed $R[G]$-modules. The definitions and results are completely analogous, except that now we must keep track of the norm. Referring to the notation introduced in Section 1.1.2, the only differences are the following:

- in Definition 1.4 of *strongly injective* map, it is asked that the morphism $\sigma \colon A \to B$ satisfies $\|\sigma\| \leq 1$;
- in Definition 1.6 of *relatively injective* normed $R[G]$-module, we require that $\|\beta\| \leq \|\alpha\|$;
- we ask for the augmentation ε in the G-complex defined in Section 1.1.2 to be isometric;
- the *contracting homotopy* k in Def 1.8 satisfies $\|k\| \leq 1$ for every $n \in \mathbb{N}$.

With this additional assumptions, one can prove the following:

Theorem 1.12 ([Fri17, Proposition 4.3 and Corollary 4.5])**.** *Let V be a normed $R[G]$-module. Then the followings hold:*

1. *the bounded chain complex*

$$0 \longrightarrow V \xrightarrow{\varepsilon} C_b^0(G; V) \xrightarrow{\delta} C_b^1(G; V) \longrightarrow \cdots \longrightarrow C_b^n(G; V) \longrightarrow \cdots$$

where $\varepsilon \colon V \to C_b^0(G; V)$ is defined by $\varepsilon(v)(G) := v$ for every $v \in V$, $g \in G$ is a relatively injective strong resolution of V;

2. *the cohomology $H_b^n(V^\bullet)$ of any relatively injective strong resolution $(V, V^\bullet, \delta^\bullet)$ of V is canonically isomorphic to the cohomology $H_b^n(G, V)$ defined in Equation 1.10. Moreover, this isomorphism is bi-Lipschitz with respect to the seminorms of $H_b^n(G, V)$ and $H^n(V^\bullet)$.*

The proof of this theorem relies on the usual machinery of homological algebra and can be found in Frigerio's book [Fri17, Proposition 4.3 and Corollary 4.5].

Even if it is not crucial for our purposes, the following fact is worth keeping in mind. The theorem implies that, given a normed relatively injective strong resolution of our $R[G]$-module, the induced seminorms are pairwise equivalent. However, it does not claim (and in general it is not true) that these seminorms coincide. Since in many applications it is important to know the precise value of the *canonical* seminorm, namely the one induced by the standard resolution, the previous result is not as optimal as possible. This is not a big issue since adding some extra hypothesis one actually gets an isometric isomorphism [Fri17, Theorem 4.17].

1.2.3 A topological approach to bounded cohomology:

In this section, let X be a topological space and let $R = \mathbb{R}$ or $R = \mathbb{Z}$. Consider the usual complex of singular n-cochains $(C^\bullet(X,R), \delta^\bullet)$, which we endow with a norm as follows: for every $\varphi \in C^n(X,R)$, we define

$$\|\varphi\|_\infty := \sup\{|\varphi(s)| \mid s \text{ is a singular } n\text{-simplex in } X\} \in [0,\infty].$$

We denote by $C_b^n(X;R)$ the R-module of *bounded* n-cochains, that is, the subspace

$$C_b^n(X;R) = \{\varphi \in C^n(X,R) \mid \|\varphi\|_\infty < \infty\} \subseteq C^n(X;R).$$

The differential $\delta^\bullet$ maps bounded cochains to bounded cochains, thus inducing a differential on the subcomplex of bounded cochains. The cohomology of this subcomplex is called the *bounded cohomology of the topological space* X. The norm on the bounded cochains descends to a seminorm in cohomology.

Since simplicial bounded cohomology of spaces is not the same as singular bounded cohomology of a space, it is not as straightforward as in the standard case to identify $H_b^n(G;R)$ with $H_b^n(BG;R)$. However, with some extra work, we can get the desired identification:

Theorem 1.13 ([Fri17, Theorem 5.5])**.** *Let X an aspherical space. Then $H_b^n(X,R)$ is isometrically isomorphic to $H_b^n(\pi_1(X),R)$ for every $n \in \mathbb{N}$.*

What is quite surprising is that the previous theorem still holds without the assumption of asphericity: in the case $R = \mathbb{R}$, the latter isomorphism remains true for every path-connected topological space.

Theorem 1.14 ([Iva17, Theorem 8.3] and [FM23, Theorem 0.2.8])**.** *Let X be a path-connected topological space. Then, for every $n \in \mathbb{N}$, we have that $H_b^n(X,\mathbb{R})$ is canonically isometrically isomorphic to $H_b^n(\pi_1(X),\mathbb{R})$.*

This theorem appeared in Gromov's seminal paper [Gro82, Corollary D and Remark E, page 46] without any assumption on the topology of X. Later, Ivanov proved it for countable CW-complexes (see [Iva87, Theorem 4.1]) and more recently extended the result to every path-connected topological space. Moraschini and Frigerio gave a new proof of this theorem following more closely Gromov's original approach.

1.3 Functoriality

We now briefly study the functorial properties of group cohomology; if not stated otherwise, the same statements hold verbatim for bounded cohomology. The described construction actually defines not only a bunch of (seminormed) modules, but a bifunctor which is controvariant with respect to restriction of scalars in the first entry and covariant in the second one. More precisely:

First entry: let G_1, G_2 be groups, let $\phi \colon G_1 \to G_2$ be a homomorphism, let R be a commutative ring and let V be an $R[G_2]$-module. Then V is in a natural way also an $R[G_1]$-module where G_1 acts on V via ϕ; we denote this module by $\phi^{-1}V$. Note that in the case $V = R$ and G_2 acts on R trivially, we have $\phi^{-1}V = R$. Then, for every $n \in \mathbb{N}$ there are induced maps $\phi^n : C^n(G_2, V) \to C^n(G_1, \phi^{-1}V)$. These induce cochain maps such that $\phi^n(C^n(G_2, V)^{G_2}) \subseteq C^n(G_1, \phi^{-1}V)^{G_1}$ and therefore we have a well defined map in cohomology for every $n \in \mathbb{N}$.

Second entry: let G be a group, let R be a commutative ring, let V_1, V_2 be $R[G]$-modules and let $\alpha : V_1 \to V_2$ be an $R[G]$-map. Then, for every $n \in \mathbb{N}$, there are induced maps $\alpha^n : C^n(G, V_1) \to C^n(G, V_2)$ defined as the postcomposition with α which give rise to well-defined homomorphisms in cohomology.

Observe moreover that if we have a short exact sequence of $R[G]$-modules

$$0 \to V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \to 0,$$

then the induced sequence

$$0 \to C^\bullet(G, V_1)^G \xrightarrow{\alpha^\bullet} C^\bullet(G, V_2)^G \xrightarrow{\beta^\bullet} C^\bullet(G, V_3)^G \to 0$$

is still exact and therefore there exists a long exact sequence of the form

$$0 \to H^0(G, V_1) \to H^0(G, V_2) \to H^0(G, V_3) \to H^1(G, V_1) \to H^1(G, V_2) \to \dots .$$

Remark 1.15. To obtain the same long exact sequence in bounded cohomology, we need to work with short exact sequences of *Banach* $R[G]$-modules (see [MR23, Theorem 2.31] and [Mon01, Proposition 8.2.1]).

1.4 The comparison map

We now introduce a fundamental concept in the theory of bounded cohomology: the *comparison map.*

Definition 1.16 (The comparison map). For every $n \in \mathbb{N}$, the inclusion of bounded n-cochains in possibly unbounded n-cochains $C_b^n(G; V) \hookrightarrow C^n(G; V)$ induces a map

$$comp^n : H_b^n(G, V) \to H^n(G; V),$$

called the *n-th comparison map.* We denote with $\mathrm{EH}_b^n(G, V) := \ker(\mathrm{comp}^n)$ its kernel and call it the *n-th exact bounded cohomology.*

We point out that this map is, in general, neither injective nor surjective. However, we will see that understanding the conditions under which this occurs is one of the key tools in the theory. For instance, bounded cohomology can sometimes be studied by analyzing the kernel and image of the comparison map. Specifically, we have the decomposition

$$H_b^n(G;V) \cong \mathrm{EH}_b^n(G;V) \oplus \mathrm{Im}(\mathrm{comp}^n),$$

where the first summand, $\mathrm{EH}_b^n(G;V)$, is sometimes computable-for example, via *quasi-morphisms* in degree 2, as we shall see in Section 1.6. The second summand is often easier to handle, as it belongs to the realm of standard cohomology. This is the approach we use in Section 1.6 to compute the second bounded cohomology of the free group with two generators.

1.5 Properties of bounded cohomology

In this section, we resume the main properties of bounded cohomology. The main difference with standard cohomology is the lack of any kind of Mayer-Vietoris sequence or excision property, which makes its computation often very difficult. Furthermore, unlike standard cohomology, there is no isomorphism between singular and simplicial bounded cohomology, and the latter is not, in general, a homotopy invariant. For instance, consider a finite simplicial complex, such as the wedge of two circles $S^1 \vee S^1$. Every simplicial cochain is bounded, so simplicial bounded cohomology coincides with standard singular cohomology. However, its singular bounded cohomology is infinite dimensional already in degree 2, as we will show in Theorem 1.21.

Given a couple (X, Y), one can define its *relative bounded cohomology groups* taking bounded cochains which vanish on Y. As said above, one can not hope for an axiomatic cohomology theory this way, since the excision axiom fails. Nevertheless, the other Eilenberg-Steenrod axioms hold, namely:

- **Homotopy invariance:** Maps $g^\bullet, h^\bullet \colon H_b^\bullet(Y, B; R) \to H_b^\bullet(X, A; R)$ induced by homotopic maps are identical.

- **Exactness:** For every pair (X, A) with inclusion maps $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$, the sequence

$$\cdots \xrightarrow{\delta^*} H_b^{n-1}(A;R) \xrightarrow{i^*} H_b^n(X;R) \xrightarrow{j^*} H_b^n(X,A;R) \xrightarrow{\delta^*} H_b^{n+1}(A;R) \xrightarrow{i^*} H_b^{n+1}(X;R) \to \cdots$$

  is exact.

- **Additivity:** For every family $\{X_i\}_{i\in I}$ of spaces, the natural map

$$\bigoplus_{i\in I} H_b^n(X_i;R) \longrightarrow H_b^n\left(\bigsqcup_{i\in I} X_i; R\right)$$

  is an isomorphism.

- **Dimension:** For every singleton space $X = \{*\}$, we have

$$H_b^0(X;R) \cong R \quad \text{and} \quad H_b^n(X;R) = 0 \text{ for } n > 0.$$

The proofs of these facts are quite straightforward: they follow the standard arguments, taking the norms into account and checking that everything that needs to be bounded actually it is [Fri17]. For example, for the dimension axiom: there is only one singular simplex in $\{*\}$, hence $C_b^\bullet(X;R) = C^\bullet(X;R)$ and so bounded and standard cohomologies have to coincide.

We now investigate the behavior of bounded cohomology of groups in low degrees.

In degree 0, from the definitions we obtain $\overline{C}^0(G;R) = \overline{C}_b^0(G;R) = 0$ and $\overline{\delta} = 0$, hence standard and bounded cohomology coincide:

$$H^0(G;R) = H_b^0(G;R) = R.$$

The first concrete difference we can see with standard cohomology is the behaviour of bounded cohomology in degree 1 for real coefficients: it vanishes for every group. This is clearly not the case for standard cohomology; consider for instance the group of the integers $\mathbb{Z}$. Its classifying space is S^1 and so we have $H^1(\mathbb{Z};\mathbb{R}) \cong H^1(S^1;\mathbb{R}) = \mathbb{R}$. In general, it is well known, that $H^1(G;\mathbb{R}) = \mathrm{Hom}(G;\mathbb{R})$ [Bro82, Prop 2.3].

Proposition 1.17 (**First bounded cohomology**)**.** *Let G be a group. Then we have*

$$H_b^1(G;\mathbb{R}) = 0.$$

Proof. Let us consider the bar resolution to compute bounded cohomology. By definition

$$\overline{C}_b^1(G;\mathbb{R}) = \{f\colon G \to \mathbb{R} \text{ bounded}\},$$

and the coboundary operator is given by

$$\overline{\delta} f(g_1, g_2) = f(g_1) + f(g_2) - f(g_1 g_2).$$

Hence $\overline{\delta} f = 0$ if and only if f is a group homomorphism. Moreover, since $B_b^1(G;\mathbb{R}) = 0$, we have

$$H_b^1(G;\mathbb{R}) = Z_b^1(G;\mathbb{R})/B_b^1(G;\mathbb{R}) \cong \{\text{ bounded homomorphisms } G \to \mathbb{R}\,\}.$$

We now want to show that there exist no non-trivial bounded group homomorphisms $G \to \mathbb{R}$. Indeed, if $\phi\colon G \to \mathbb{R}$ is a group homomorphism, then for every $r \in \mathbb{R}$ we have

$$\|\phi\|_\infty = \|r \cdot \phi\|_\infty = |r| \cdot \|\phi\|_\infty.$$

This identity can only hold for all $r \in \mathbb{R}$ if and only if $\|\phi\|_\infty = 0$, hence $\phi = 0$. $\square$

1.6 Bounded group cohomology in degree 2.

Recall that the second group cohomology is fully understood in degree 2:

Theorem 1.18 (**Standard cohomology in degree** 2 [Fri17, Proposition 2.5])**.** *There is a natural bijection:*

$$H^2(G;R) \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \textit{Equivalence classes of central} \\ \textit{extensions of } G \textit{ by } R \end{array} \right\},$$

where we recall that a central extension of G by R is an exact sequence

$$0 \to R \xrightarrow{\iota} G' \xrightarrow{\pi} G \to 0$$

such that $\iota(R)$ is contained in the center of G'.

On the other hand, in the case of bounded cohomology the situation is much more complicated. A fundamental tool for the study of the second bounded cohomology groups are *quasimorphisms*. Roughly speaking, quasimorphisms are homomorphisms up to a "bounded" defect. In this section, we define quasimorphisms and homogeneous quasimorphisms, and we see how they are related to exact bounded cohomology in degree 2. Later, we compute the second bounded cohomology of the free group with two generators with real coefficients, showing that it is infinite-dimensional (Theorem 1.21). As announced, this is a consequence of the lack of any kind of excision properties and it is one of the biggest differences with standard cohomology.

Definition 1.19 (Quasimorphism). A map $f\colon G \to R$ is a *quasimorphism* if there exists a number $D \geq 0$ such that

$$|f(g_1) + f(g_2) - f(g_1 g_2)| \leq D,$$

for every g_1, $g_2 \in G$. The least D for which this inequality is true is called *defect* of f. The space of quasimorphisms is denoted by $Q(G; R)$ and is a normed R-module.

We observe that morphisms and bounded maps are trivially quasimorphisms: the first ones have defect 0, the latter ones with defect bounded by $3\,\|f\|_\infty$ because of the triangle inequality. Moreover, we point out that $\overline{C}_b^1(G; R) \cap \operatorname{Hom}(G, R) = \{0\}$, since bounded homomorphisms are trivial.

Proposition 1.20 (Quasimorphisms and exact cohomology). *There exists a short exact sequence of the form*

$$0 \longrightarrow \overline{C}_b^1(G; R) \oplus Hom(G; R) \xrightarrow{\alpha} Q(G; R) \xrightarrow{\beta} \mathrm{EH}_b^2(G; R) \longrightarrow 0,$$

where the last map is induced by $\overline{\delta}_b^1 : Q(G; R) \to Z_b^2(G; R)$. *Therefore, we have the following isomorphism:*

$$EH_b^2(G; R) \cong \frac{Q(G; R)}{\overline{C}_b^1(G; R) \oplus Hom(G; R)}.$$

Proof. We divide the proof in three steps:

1. **Injectivity of $\boldsymbol{\alpha}$:** It follows from the fact that $\overline{C}_b^1(G; R) \cap \operatorname{Hom}(G, R) = \{0\}$.

2. **Surjectivity of $\boldsymbol{\beta}$:** Let $f \in \overline{C}_b^2$ be a representative of an element in $\mathrm{EH}_b^2(G; R)$. This means that there exists a cochain $\phi \in \overline{C}^1(G; R)$ such that $\overline{\delta}\phi = f$ (since f lies in the kernel of the comparison map). Then, the map

$$\tilde{f}\colon G \to R, \quad g \mapsto \phi(g)$$

   is a quasimorphism since $f = \overline{\delta}\phi$ is bounded, and by construction, $\beta(\tilde{f}) = f$, hence the thesis.

3. **$\boldsymbol{im(\alpha)} = \mathbf{ker}\,(\boldsymbol{\beta})$:** It follows from the definition of the maps.

□

This theorem implies that, in order to prove that $H^2_b(G;\mathbb{R})$ is non-trivial, it is enough to construct quasimorphisms that are not at a bounded distance from any homomorphism because $\mathrm{EH}^2_b(G;\mathbb{R}) \subseteq H^2_b(G;\mathbb{R})$. Moreover, when $H^2(G;R)=0$ we have that

$$H^2_b(G;R) \cong \frac{Q(G;R)}{\overline{C}^1_b(G;R) \oplus \mathrm{Hom}(G;R)}. \tag{2}$$

We use this theorem to compute our first non-trivial bounded cohomology group. We show that $H^2_b(F_2;\mathbb{R})$ is infinite dimensional, where F_2 is the free non-abelian group of rank 2. We observe that $H^2(F_2;\mathbb{R})=0$, since the wedge of two circle is the classifying space of F_2 and $H^2(S^1 \vee S^1;\mathbb{R})=0$. Hence we have $H^n_b(F_2;\mathbb{R}) = EH^n_b(F_2;\mathbb{R})$, so we can completely describe the second bounded cohomology group in terms of quasimorphisms. The third bounded cohomology group is infinite-dimensional, as shown by Soma [Som97, Corollary C], but little is known from degree 4 onwards (only recently Kastenholz announced that also $H^4_b(F_2;\mathbb{R})$ is non-trivial [Kas25]).

We now exhibit a family of non-trivial linearly independent quasimorphisms on F_2. The non-triviality of $H^2_b(F_2;\mathbb{R})$ was first established by Johnson [Joh72]. Subsequently, Brooks [Bro81] constructed an infinite family of quasimorphisms, which were shown to define linearly independent elements in $\mathrm{EH}^2_b(F_2;\mathbb{R})$ by Mitsumatsu [Mit84]. The family of quasimorphisms presented here is due to Rolli [Rol09]; see also [FFa20] for details.

Let s_1, s_2 generators of F_2, and consider the *space of bounded odd sequences*, i.e:

$$\ell^\infty_{\mathrm{odd}}(\mathbb{Z}) = \{u=(u_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}) \mid u_n = -u_{-n} \text{ for every } n \in \mathbb{Z}\}.$$

For every such sequence $u=(u_n)_{n\in\mathbb{Z}}$, we can associate a map $f_u\colon F_2 \to \mathbb{R}$ defined by

$$f_u(s_{i_1}^{n_1} s_{i_2}^{n_2} \cdots s_{i_k}^{n_k}) = \sum_{j=1}^k u_{n_j},$$

where we are using the fact that every element of a free group admits a unique representation as reduced word. One can check then that this map is actually a quasimorphism [Rol09, Prop 2.1]. To conclude, let $\overline{\delta}^1_b$ the map of the last proposition 1.20 that associates to a quasimorphism an element in $H^2_b(G;R)$ via the isomorphism in (2). Then we have:

Theorem 1.21. *The map*

$$\ell^\infty_{\mathrm{odd}}(\mathbb{Z}) \subseteq Q(F_2;\mathbb{R}) \longrightarrow H^2_b(F_2;\mathbb{R}), \qquad (u_n)_{n\in\mathbb{Z}} \longmapsto [\overline{\delta}^1_b(f_u)],$$

is injective. In particular, this shows that $H^2_b(F_2;\mathbb{R})$ is infinite dimensional.

Proof. Let s_1, s_2 be the generators of F_2. Since the map above is linear, it suffices to show that its kernel is trivial. Let $(u_n)_{n\in\mathbb{Z}} \in \ell^\infty_{\mathrm{odd}}(\mathbb{Z})$ such that $[\overline{\delta}^1_b(f_u)] = 0 \in H^2_b(F_2;\mathbb{R})$. Then, by Proposition 1.20, f_u is a trivial quasimorphism. i.e. $f_u = h+b$ where h is a homomorphism and b is bounded. For $i=1$, 2 and $k\in\mathbb{N}$, we have

$$\begin{aligned} k\cdot h(s_i) &= h(s_i^k) \\ &= f_u(s_i^k) - b(s_i^k) \\ &= u_k - b(s_i^k). \end{aligned}$$

The last term is bounded, therefore letting k going to infinity we see that $h(s_i)$ has to be zero for every $i=1$, 2. Therefore, $f_u = b$ is a bounded quasimorphism. Take j, $k\in\mathbb{N}$.

We then have $f_u((s_1^j s_2^j)^k) = \sum_{i=1}^{2k} u_j = (2k)u_j$ for all $j, k \in \mathbb{N}$. Since f_u is bounded, we conclude that $u = 0$.

□

Knowing the bounded cohomology of non-abelian free group of rank 2 is central in the theory because it injects into the bounded cohomology of any group admitting a surjection onto F_2.

Corollary 1.22. *Let G be a group admitting an epimorphism $\phi\colon G \longrightarrow F_2$. Then $H_b^2(G, \mathbb{R})$ is infinite-dimensional.*

Proof. Since $F_2 = \langle s_1, s_2 \rangle$ is free, we can find a right inverse of ϕ. Let's call it φ. Since ϕ is surjective, there exist g_1, $g_2 \in G$ that get mapped to s_1 and s_2 respectively. It suffices to define $\varphi\colon F_2 \longrightarrow G$ as the map that sends s_1 to g_1 and s_2 to g_2, extended by linearity. By functoriality of $H_b^2(\cdot; \mathbb{R})$, the composition

$$H_b^2(F_2; \mathbb{R}) \xrightarrow{H_b^2(\phi)} H_b^2(G; \mathbb{R}) \xrightarrow{H_b^2(\varphi)} H_b^2(F_2; \mathbb{R})$$

is the identity, so $H_b^2(\phi)$ is a surjective homomorphism from an infinite-dimensional vector space to $H_b^2(G; \mathbb{R})$. □

1.7 ℓ^1-homology and duality

In this section, we show that under certain conditions it is possible to obtain the duality between homology and cohomology that, in the standard setting and with real coefficients, is given by the Universal Coefficient Theorem. To this end we introduce the notion of *ℓ^1-homology* and the *q-Uniform Boundary Condition (q-UBC)* for normed chain complexes. Beyond duality, the study of UBC will provide a criterion for the injectivity of the comparison map, as stated in Theorem 1.36.

We will consider the duals of cochain complexes $(C_\bullet, \partial_\bullet)$. Depending on the context, this may refer either to the algebraic or the topological dual. More precisely, we consider the algebraic dual when working with standard homology and cohomology, and the topological dual when working with *ℓ^1-homology* and bounded cohomology.

Definition 1.23 (Banach chain complex). Consider a chain complex of the form

$$0 \longleftarrow C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \longleftarrow \cdots,$$

where C_i is a normed real vector space for all $i \geq 0$. We say that the complex is *Banach* if every C_i is Banach. An analogous definition applies to cochain complexes.

Remark 1.24. Recall that the dual of any normed real vector space is automatically Banach [Rud91, Theorem 4.1], so whenever we work with a chain complex and we dualize it we will obtain a Banach cochain complex.

Let G be a group and R a ring. In a similar way to the definition of group cohomology, one can define *group homology with coefficients in an $R[G]$-module V*. Consider the chain complex $C_n(G; V)$ defined as the $R[G]$-module generated by G^n, with differential maps $\partial_0 = \partial_1 = 0$,

$$\begin{aligned}\partial_n \colon C_n(G;V) &\longrightarrow C_{n-1}(G;V)\\ (g_1,\dots,g_n) &\longmapsto (g_2,\dots,g_n)\\ &\quad+\sum_{i=1}^{n-1}(-1)^i(g_1,\dots,g_ig_{i+1},\dots,g_n)\\ &\quad+(-1)^n(g_1,\dots,g_{n-1})\end{aligned}$$

for $n \geq 2$. One then consider the $R[G]$-module of n-cycles as

$$Z_n(G;V) \coloneqq \ker \partial_n$$

and the $R[G]$-module of n-boundaries as

$$B_n(G;V) \coloneqq \partial_{n+1}(C_{n+1}(G;V)).$$

Note that we are describing the *bar resolution*; to remain consistent, we should keep the notation $\overline{C}_n(G;V)$ from Section 1.3. However, for simplicity, we will denote it as the homogeneous complex.

Definition 1.25 (Group homology). In the notation of before, we define the *group homology of G with coefficients in V* as

$$H_n(G;V) \coloneqq \frac{Z_n(G;V)}{B_n(G;V)}.$$

From now onwards, we denote the chain complexes omitting the coefficients we are working with, implying that we are using real coefficients. Thus, we will write $C_n(G)$ rather then $C_n(G;\mathbb{R})$, or $H_n(G)$ instead of $H_n(G;\mathbb{R})$ and so on.

We can endow $C_n(G;\mathbb{R})$ with the ℓ^1-norm defined by:

$$\left\| \sum a_{g_1,\dots,g_n}(g_1,\dots,g_n) \right\|_1 \coloneqq \sum |a_{g_1,\dots,g_n}|$$

for every n-chain; this is well defined since the sum is finite.

Definition 1.26 (ℓ^1-homology). Let G be a group. Consider the chain complex $(C_\bullet(G);\partial_\bullet)$ equipped with the ℓ^1-norm. Denote by $C_n^{\ell^1}(G)$ the completion of $C_n(G)$ with respect to this norm. The differential ∂_n extends naturally to a map $\partial_n^{\ell^1} : C_n^{\ell^1}(G) \to C_{n-1}^{\ell^1}(G)$, so that $(C_\bullet^{\ell^1}(G);\partial_\bullet^{\ell^1})$ forms a Banach chain complex. The homology $H_n^{\ell^1}(G)$ of this complex is called the *ℓ^1-homology of G with real coefficients*.

From the definition, it follows that the topological dual of both the complete and the possibly non-complete chain complexes gives rise to the bounded cochain complex defined before (the *topological* dual consists of *bounded* functionals, so we get the *bounded* cochains!). Observe that from Remark 1.24, we also obtain that $C_b^n(G;\mathbb{R})$ is always a Banach vector space.

Recall that the Universal Coefficients Theorem implies that, when working with coefficients in a field, taking algebraic duals commute with taking cohomology, i.e. that for a chain complex $(C_\bullet;\partial_\bullet)$, we have $\mathrm{Hom}(H_n(C_\bullet),\mathbb{R}) = H^n(\mathrm{Hom}(C_\bullet,\mathbb{R}))$ (see for example

[Hat02, Theorem 3.1] for details). One way to proceed to prove it is the following: denote with $(C^\bullet; d^\bullet)$ the algebraic dual of $(C_\bullet; \partial_\bullet)$ and consider the *Kronecker pairing*

$$\begin{aligned} H^n(C^\bullet) \times H_n(C_\bullet) &\longrightarrow \mathbb{R}, \\ ([\varphi], [\sigma]) &\longmapsto \varphi(\sigma). \end{aligned}$$

This induces a well-defined map between cohomology and the dual of homology, namely:

$$\begin{aligned} H^n(C^\bullet) &\longrightarrow \mathrm{Hom}(H_n(C_\bullet), \mathbb{R}) \\ [\varphi] &\longmapsto \big([\sigma] \mapsto \varphi(\sigma)\big). \end{aligned}$$

This map is in general surjective with kernel $\mathrm{Ext}(H_{n-1}(C_\bullet); R)$ if we work with coefficients in a ring R. In the case R is a field, as in our case where $R = \mathbb{R}$, this kernel vanishes and we have the result.

In the bounded case we work with topological duals and this result does not hold anymore; nevertheless, it is still possible to define an analogous Kronecker pairing and it turns out that the last map is always surjective. If an additional condition, namely keeping track of the norm of the cocycle and the coboundary (the *q-UBC condition*), is satisfied, Matsumoto and Morita proved that in the Banach case this map is indeed an isomorphism.

Proposition 1.27. *Let $(C_\bullet; \partial_\bullet)$ a normed chain complex. Then the map*

$$H_b^n(C^\bullet) \longrightarrow \mathrm{Hom}(H_n(C_\bullet), \mathbb{R})$$

induced by the Kronecker pairing is surjective.

Proof. Let Z_n, B_n be the space of cycles and boundaries in C_n, respectively, and let $\varphi \in \mathrm{Hom}(H_n(C_\bullet), \mathbb{R})$. The element φ is a bounded functional $Z_n \to \mathbb{R}$ that vanishes on B_n. Such a functional admits a bounded extension to C_n by the Hahn–Banach theorem. This extension defines a class in bounded cohomology that is mapped to φ. □

We give now the precise condition mentioned above.

Definition 1.28 (q-UBC condition). Let $(C_\bullet; \partial_\bullet)$ be a normed chain complex. We say that $(C_\bullet; \partial)$ satisfies the *q-UBC* (q-Uniform Boundary Condition) if

$$\exists K > 0 \ : \ \forall b \in B_q,\ \exists c \in C_{q+1} \text{ such that } \partial_{q+1} c = b \text{ and } \|c\|_1 \leq K \|b\|_1,$$

where B_q is the space of the q-boundaries.

We say that $(C_\bullet; \partial_\bullet)$ satisfies the q-UBC$^{\ell^1}$ if its ℓ^1-chain complex satisfies the q-UBC.

Remark 1.29. Without keeping track of the norm, the map $C_{q+1}/Z_{q+1} \to B_q$ induced by ∂_{q+1} is an isomorphism by the Isomorphism Theorems. This is, in general, no longer true if we work with normed vector spaces, because we have to check continuity.

In our context this map is still a continuous bijection since the differential is linear and bounded (i.e. continuous). However, we might lack the continuity of the inverse. The q-UBC can be seen as a condition ensuring it. In fact, to have continuity, we need that there exists an uniform $K > 0$ such that for every $b \in B_q$, we have:

$$\|\partial_{q+1}^{-1} b\|_1 \leq K \|b\|_1.$$

Equivalently, this means that we can find a preimage $c \in C_{q+1}$ of b such that

$$\|c\|_1 \leq K\|b\|_1,$$

that is exactly the q-UBC condition. Therefore, we could restate the q-UBC as follows:

A normed chain complex $(C_\bullet, \partial_\bullet)$ satisfies the q-UBC if and only if the map

$$\frac{C_{q+1}}{Z_{q+1}} \to B_q$$

induced by ∂_{q+1} is an isomorphism of normed vector spaces.

As announced, the next theorem states that if a Banach normed chain complex satisfies the q-UBC, then an analogous of the Universal Coefficient Theorem holds.

For convenience, we collect here some preliminaries that we need for the proof.

The following is a consequence of the Open Mapping Theorem [Rud91, Theorem 2.11]:

Lemma 1.30 ([Rud91, Lemma 2.12]). *A continuous bijective linear operator between Banach spaces has continuous inverse.*

We will use it this way: the map $C_{q+1}/Z_{q+1} \to B_q$ induced by ∂_{q+1} is a continuous, bijective and linear operator between a Banach space and a normed space. If we know that also B_q is a Banach space, we obtain that the inverse of that map is continuous, hence it is an isomorphism of normed vector spaces. Thus, we have obtained yet another characterization of the q-UBC:

A chain complex $(C_\bullet; \partial_\bullet)$ satisfies the q-UBC if and only if the subspace of q-boundaries is Banach.

We will also need the Closed Range Theorem:

Theorem 1.31 ([Rud91, Theorem 4.14]). *Let X and Y be Banach spaces, let X', Y' be the dual of X and Y respectively and let $T : X \to Y$ be a continuous linear operator with dual $T' : Y' \to X'$. Then the following are equivalent:*

1. $\mathrm{Im}(T)$ *is closed in* Y*;*
2. $\mathrm{Im}(T')$ *is weak*-closed in* X'*;*
3. $\mathrm{Im}(T')$ *is norm-closed in* X'*.*

Lastly, we will use the following observation:

Remark 1.32. Consider the map induced by the Kronecker pairing:

$$\begin{aligned} H_b^n(C^\bullet) &\longrightarrow \mathrm{Hom}(H_n(C_\bullet), \mathbb{R}) \\ [\varphi] &\longmapsto \big([\sigma] \mapsto \varphi(\sigma)\big). \end{aligned}$$

If an element $[\varphi] \in H_b^n(C^\bullet)$ has vanishing seminorm, then it is in the kernel of this map.

Indeed, having vanishing seminorm means that there exists a sequence $\{\varphi_i\}_{i\in\mathbb{N}} \subseteq Z_b^n$ of representatives of $[\varphi]$ such that for every $\varepsilon > 0$ there exists an $\overline{N} \in \mathbb{N}$ with the property that $\|\varphi_N\|_1 \leq \varepsilon$ for all $N \geq \overline{N}$. Therefore, for every $[\sigma] \in H_n(C_\bullet)$ and for every $\varepsilon > 0$, one can find a representative of $[\varphi]$ such that $\varphi(\sigma) < \varepsilon$. This shows that the image of $[\varphi]$ is the zero map.

Theorem 1.33 (**Matsumoto-Morita's theorem** [MM85, Theorem 2.3])**.** *Let $(C_\bullet;\partial_\bullet)$ be a chain complex with dual cochain complex $(C^\bullet;d^\bullet)$. Then the following are equivalent:*

1. *the chain complex $(C_\bullet;\partial_\bullet)$ satisfies the q-UBC^{ℓ^1};*
2. *the seminorm on $H_q^{\ell^1}(C_\bullet)$ is a norm;*
3. *the seminorm on $H_b^{q+1}(C^\bullet)$ is a norm;*
4. *the Kronecker product induces an isomorphism*
$$H_b^{q+1}(C^\bullet)\cong \mathrm{Hom}(H_{q+1}^{\ell^1}(C_\bullet),\mathbb{R}).$$

Proof. Let us fix the notations. We denote with:

- $Z_\bullet^{\ell^1}, B_\bullet^{\ell^1}$ the spaces of cycles and boundaries of $(C_\bullet^{\ell^1},\partial_\bullet^{\ell^1})$;
- $Z_b^\bullet, B_b^\bullet$ the spaces of cycles and boundaries of $(C^\bullet, d^\bullet)$.

(1) $\iff$ (2): Recall that the seminorm on $H_q^{\ell^1}(C_\bullet)$ is a norm if and only if $B_q^{\ell^1}$ is closed in $Z_q^{\ell^1}$ (see, e.g., [Rud91, Theorem 1.41]). Moreover, a subspace of a Banach space is itself Banach if and only if it is closed, since completeness is equivalent to requiring that all limit points lie in the subspace.

By Remark 1.29, condition (1) is equivalent to asking that the map
$$C_{q+1}^{\ell^1}/Z_{q+1}^{\ell^1}\longrightarrow B_q^{\ell^1}$$
induced by $\partial_{q+1}^{\ell^1}$ is an isomorphism. The quotient $C_{q+1}^{\ell^1}/Z_{q+1}^{\ell^1}$ is Banach, since $Z_{q+1}^{\ell^1}$ is the kernel of a linear and continuous map, hence it is closed.

By Lemma 1.30, we have that (1) is equivalent to $B_q^{\ell^1}$ being Banach

Thus, (1) holds if and only if $B_q^{\ell^1}$ is Banach, which is equivalent to condition (2).

(2) $\iff$ (3): Condition (2) holds if and only if $\mathrm{Im}\,\partial_{q+1}^{\ell^1}=B_q^{\ell^1}$ is closed, while condition (3) holds if and only if $\mathrm{Im}\,\partial_{q+1}^{\ell^1}=B_b^{q+1}$ is closed. This two are equivalent by the Closed Range Theorem 1.31.

(1) $\implies$ (4): We proved in 1.27 that the map induced by the Kronecker pairing is always surjective.

Let $f\in Z_b^{q+1}$ be such that $[f]=0\in \mathrm{Hom}(H_{q+1}^{\ell^1}(C_\bullet),\mathbb{R})$. Then $f(Z_{q+1}^{\ell^1})=0$.

By Remark 1.29, the q-UBC is equivalent to asking that the map $C_{q+1}^{\ell^1}/Z_{q+1}^{\ell^1}\longrightarrow B_q^{\ell^1}$ induced by $\partial_{q+1}^{\ell^1}$ is an isomorphism.

Hence f can be written as the composition
$$C_{q+1}^{\ell^1}/Z_{q+1}^{\ell^1}\xrightarrow{\partial_{q+1}^{\ell^1}} B_q^{\ell^1}\xrightarrow{\tilde g}\mathbb{R}.$$

By the Hahn–Banach Theorem, $\tilde g$ admits a continuous extension $g: C_q^{\ell^1}\to\mathbb{R}$. Thus we have
$$f=g\circ\partial_{q+1}^{\ell^1}=d^{q+1}g$$
by definition of the dual differential. Therefore f is a coboundary and hence belongs to the zero class.

(4) $\implies$ (3): By Remark 1.32, if an element has vanishing seminorm then it is in the kernel of the map $H_b^{q+1}(C^\bullet)\to \mathrm{Hom}(H_{q+1}^{\ell^1}(C_\bullet),\mathbb{R})$. This map is an isomorphism by hypothesis, hence its kernel is trivial.

$\square$

As a consequence of Theorem 1.33 we have that $H^2_b(\bullet)$ is always a Banach space, regardless the group or path-connected topological space we are working with.

Corollary 1.34. *Let G be a group and let X be a path-connected topological space. Then $H^2_b(G)$ and $H^2_b(X)$ are Banach spaces.*

Proof. The result about path-connected topological spaces follows from the one about groups by Theorem 1.14.

Thanks to Theorem 1.33, it suffices to prove that 1-UBC$^{\ell^1}$ always holds. Consider the map

$$S\colon G \longrightarrow C^{\ell^1}_2(G), \qquad S(g) = \sum_{k=1}^{\infty} \frac{1}{2}(g^{2^k}, g^{2^k}).$$

We have $\|S(g)\|_1 = 1$ and $\partial^{\ell^1}_2 S(g) = g$ for every $g \in G$. Extending S to a bounded map $S\colon C^{\ell^1}_1(G) \to C^{\ell^1}_2(G)$, we obtain that $C^{\ell^1}_\bullet(G)$ satisfies the 1-UBC, hence the thesis . $\square$

As discussed in Section 1.4, the comparison map is in general neither injective nor surjective. The following theorem gives criteria to ensure its injectivity, and we will use it to prove the bounded acyclicity of $\mathrm{Homeo}_c(\mathbb{R}^n)$ in the next chapter. For the proof, we will need the following observation:

Remark 1.35. If we have a sequence $\{b^{(n)}\}_{n\in\mathbb{N}}$ converging to an element b in a normed space, it is possible to construct an absolutely convergent series $\sum_{k=0}^{+\infty} b_k$ whose sum is b, and which moreover satisfies

$$\sum_{k=0}^{+\infty} \|b_k\| \leq 2\|b\|.$$

Indeed, since $b^{(n)} \to b$, for every $k \in \mathbb{N}$ we can choose an index n_k such that

$$\|b^{(n_k)} - b\| \leq 2^{-(k+2)}\|b\|.$$

Define now

$$b_0 = b^{(n_0)}, \qquad b_k = b^{(n_k)} - b^{(n_{k-1})} \quad \text{for } k \geq 1.$$

Then, by construction,

$$\sum_{k=0}^{N} b_k = b^{(n_N)} \xrightarrow{N\to+\infty} b,$$

so that $\sum_{k=0}^{\infty} b_k = b$.

Moreover, for $k \geq 1$ we have

$$\begin{aligned}
\|b_k\| &= \|b^{(n_k)} - b^{(n_{k-1})}\| \\
&\leq \|b^{(n_k)} - b\| + \|b - b^{(n_{k-1})}\| \\
&\leq 2^{-(k+2)}\|b\| + 2^{-(k+1)}\|b\| \\
&\leq 2 \cdot 2^{-(k+1)}\|b\| \\
&= 2^{-k}\|b\|.
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{\infty} \|b_k\| \leq \|b_0\| + \sum_{k=1}^{\infty} 2^{-k}\|b\| \leq \|b\| + \|b\| = 2\|b\|,$$

after adjusting the choice of n_0 if necessary.

Theorem 1.36 ([MM85, Theorem 2.8]). *Let $(C_\bullet;\partial_\bullet)$ be a chain complex with (algebraic or topological) dual cochain complex $(C^\bullet;d^\bullet)$. Then the following are equivalent:*

1. *$(C_\bullet;\partial_\bullet)$ satisfies the q-UBC;*
2. *$(C_\bullet;\partial_\bullet)$ satisfies the q-UBC^{ℓ^1} and the space of $(q+1)$-cycles of C_{q+1} is dense in the space of $(q+1)$-cycles of $C^{\ell^1}_{q+1}$;*
3. *the comparison map $\mathrm{comp}^{q+1}\colon H^{q+1}_b(C^\bullet)\to H^{q+1}(C^\bullet)$ is injective.*

Proof. Let us fix the notations. We denote with:

- $Z_\bullet, B_\bullet$ the spaces of cycles and boundaries of $(C_\bullet,\partial_\bullet)$;
- $Z^{\ell^1}_\bullet, B^{\ell^1}_\bullet$ the spaces of cycles and boundaries of $(C^{\ell^1}_\bullet,\partial^{\ell^1}_\bullet)$;
- $Z^\bullet_b, B^\bullet_b$ the spaces of cycles and boundaries of $(C^\bullet,d^\bullet)$, the topological dual of the other two chain complexes.

(1) $\implies$ (2): We have to prove the following:

$$\exists K^{\ell^1}>0\ :\ \forall b\in B^{\ell^1}_q,\ \exists c\in C^{\ell^1}_{q+1}\text{ such that }\partial^{\ell^1}_{q+1}c=b\text{ and }\|c\|_1\le K\|b\|_1.$$

Hence, let $b\in B^{\ell^1}_q$. Observe that B_q is the image via a continuous map of a dense subspace, therefore it is dense in its image. In other words, B_q is dense in $B^{\ell^1}_q$. Thus, there exists a sequence $\{b^{(n)}\}_{n\in\mathbb{N}}\subseteq B_q$ converging to b. By Remark 1.35, we can construct an absolutely convergent series $\sum_{n=0}^{+\infty}b_n$ with sum b and such that $\sum_{n=0}^{+\infty}\|b_n\|_1\le 2\|b\|_1$.

The chain complex $(C_\bullet;\partial)$ satisfies the q-UBC, thus for every $n\in\mathbb{N}$ we can find $c^{(n)}\in C_{q+1}$ such that $\partial c^{(n)}=b^{(n)}$ and $\|c^{(n)}\|_1\le K\|b^{(n)}\|_1$ for some fixed $K>0$. Then $\sum_{n=0}^{+\infty}c^{(n)}$ is absolutely convergent, in fact:

$$\sum_{n=0}^{+\infty}\|c^{(n)}\|_1\le\sum_{n=0}^{+\infty}K\|b^{(n)}\|_1\le 2K\|b\|_1.$$

Thus the sum converges to an element $c\in C^{\ell^1}_{q+1}$ such that

$$\partial^{\ell^1}c=\partial^{\ell^1}\lim_{n\to+\infty}c^{(n)}=\lim_{n\to+\infty}\partial c^{(n)}=\lim_{n\to+\infty}b^{(n)}=b.$$

Moreover, we have the estimate $\|c\|_1\le 2K\|b\|_1$, hence $(C^{\ell^1}_\bullet,\partial^{\ell^1})$ satisfies the q-UBC.

Now we need to prove that Z_{q+1} is dense in $Z^{\ell^1}_{q+1}$, that is, every element $z\in Z^{\ell^1}_{q+1}$ is the limit of a sequence in Z_{q+1}.

By the density of C_{q+1} in its completion $C^{\ell^1}_{q+1}$, there exists a sequence $\{c^{(n)}\}_{n\in\mathbb{N}}\subseteq C_{q+1}$ such that

$$\lim_{n\to+\infty}c^{(n)}=z.$$

We now apply the q-UBC to the boundaries $\partial(-c^{(n)})$: for each $n\in\mathbb{N}$, there exists $\tilde{c}^{(n)}\in C_{q+1}$ such that

$$\partial\tilde{c}^{(n)}=\partial(-c^{(n)})\quad\text{and}\quad\|\tilde{c}^{(n)}\|_1\le K\|\partial c^{(n)}\|_1.$$

Then $\partial(\tilde{c}^{(n)}+c^{(n)})=0$, so that we can define $z^{(n)}:=\tilde{c}^{(n)}+c^{(n)}\in Z_{q+1}$. Moreover, we have:

$$\|\tilde{c}^{(n)}\|_1\le K\|\partial c^{(n)}\|_1=K\|\partial^{\ell^1}(c^{(n)}-z)\|_1\le K\|\partial^{\ell^1}\|_1\,\|c^{(n)}-z\|_1\longrightarrow 0\quad\text{as }n\to+\infty.$$

Therefore, $\tilde{c}^{(n)} \to 0$, and it follows that

$$z = \lim_{n\to+\infty} z^{(n)} = \lim_{n\to+\infty} (c^{(n)} + \tilde{c}^{(n)}).$$

(2) $\implies$ (1): Let $b \in B_q$ be a boundary and let $c \in C_{q+1}$ such that $\partial c = b$.

Since $(C^{\ell^1}_\bullet, \partial^{\ell^1})$ satisfies the q-UBC condition, there exist an un uniform constant K^{ℓ^1} and an element $\tilde{c} \in C^{\ell^1}_{q+1}$ such that $\partial^{\ell^1}\tilde{c} = b$ and $\|\tilde{c}\|_1 \leq K^{\ell^1}\|b\|_1$.

We have $c - \tilde{c} \in Z^{\ell^1}_{q+1}$, indeed:

$$\partial^{\ell^1}(c - \tilde{c}) = \partial^{\ell^1} c - \partial^{\ell^1}\tilde{c} = b - b = 0.$$

Therefore, by density of Z_{q+1} in $Z^{\ell^1}_{q+1}$, there exists $z \in Z_{q+1}$ such that $\|(c-\tilde{c})-z\|_1 \leq \|b\|_1$. Then we have $\partial(c - z) = \partial c = b$ and

$$\|c - z\|_1 = \|c - \tilde{c} + \tilde{c} - z\|_1 \leq \|c - \tilde{c} - z\|_1 + \|\tilde{c}\|_1 \leq (K^{\ell^1} + 1)\|b\|_1,$$

thus $(C_\bullet; \partial_\bullet)$ satisfies the q-UBC with constant $K^{\ell^1} + 1$.

(2) $\implies$ (3): Let $f \in Z^{\ell^1}_{q+1}$ be such that $[f] = 0 \in H^{q+1}(C^\bullet)$. By the Universal Coefficient Theorem, we have

$$H^{q+1}(C^\bullet) \cong \mathrm{Hom}(H_{q+1}(C_\bullet), \mathbb{R}),$$

where we are taking the algebraic dual. Thus, f vanishes on Z_{q+1}.

By density of Z_{q+1} in $Z^{\ell^1}_{q+1}$, it follows that f also vanishes on $Z^{\ell^1}_{q+1}$, where f is extended using the Hahn-Banach Theorem, and we are taking the topological dual.

Finally, by Theorem 1.33, this coincides with $H^{q+1}_b(C^\bullet)$, and the thesis follows.

(3) $\implies$ (2): By Theorem 1.33, if we prove that the seminorm on $H^{q+1}_b(C^\bullet)$ is a norm, we have that $(C_\bullet; \partial_\bullet)$ satisfies the q-UBC$^{\ell^1}$. Hence, we reduce to prove that the class of an element $[f]$ with vanishing seminorm is the class of 0. If $[f]$ has vanishing seminorm, then it is in the kernel of the comparison map. This kernel is trivial by hypothesis, hence we have the thesis.

Next, we prove that Z_{q+1} is dense in $Z^{\ell^1}_{q+1}$. Suppose, by contradiction, that this is false. Then by the Hahn–Banach theorem there exists a continuous functional f such that $f|_{Z_{q+1}} = 0$ but $f|_{Z^{\ell^1}_{q+1}} \neq 0$. Since $B_{q+1} \subseteq Z_{q+1}$, the functional f vanishes on B_{q+1}, therefore f defines a cocycle in the topological dual complex, i.e. $f \in Z^{q+1}_b$.

On the other hand, because $f|_{Z_{q+1}} = 0$, the class of f is trivial in the algebraic cohomology:

$$[f] = 0 \in H^{q+1}(C^\bullet),$$

by the Universal Coefficient Theorem. However f does not vanish on $Z^{\ell^1}_{q+1}$, so its class is nonzero in bounded cohomology:

$$[f] \neq 0 \in H^{q+1}_b(C^\bullet).$$

By Theorem 1.33, but this contradicts the injectivity of the comparison map. □

2 Amenability

In this section, we recall the definition, the main properties and examples of amenable group.

Amenable groups were introduced by Neumann [Neu29] in connection with the Banach-Tarski paradox. Intuitively, a group is amenable if it has an *invariant mean*, that is, if it is possible to assign to every subset of G a well-defined "proportion" of the group it occupies.

In the previous chapter, we observed how difficult bounded cohomology is to be computed in general. By contrast, we will see in Theorem 2.12 that amenable groups can be characterized as precisely those groups whose bounded cohomology vanishes in every positive degree, for a large class of coefficients including the real numbers.

This result, due to Johnson, was historically one of the main motivations for the development of bounded cohomology.

Examples of amenable groups include all finite groups and all abelian groups. On the other hand, the prototypical example of a non-amenable group is the non-abelian free group on two generators. Indeed, by Theorem 1.2.1, its second bounded cohomology group with real coefficients is infinite-dimensional, and hence it fails the aforementioned characterization.

In this section we will use the following notations. Let G be a group and consider the space

$$\ell^\infty(G) = C_b^0(G;\mathbb{R}) = \{f\colon G \to \mathbb{R} \mid f \text{ is bounded}\},$$

with the structure of $\mathbb{R}[G]$-module given by $g \cdot f(h) \coloneqq f(g^{-1} \cdot h))$ for all $f \in \ell^\infty(G)$ and $g, h \in G$.

Definition 2.1 (**Amenable group**)**.** A group G is said *amenable* if it admits a left G-invariant mean, i.e. and $\mathbb{R}$-linear map $m\colon \ell^\infty(G) \to \mathbb{R}$ satisfying:

1. *Normalization:* $m(1) = 1$, where 1 denotes the constant functional equal to 1.
2. *Positivity:* $m(f) \geq 0$ for every $f \in \ell^\infty(G)$ such that $f \geq 0$.
3. *Left-invariance:* $m(g \cdot f) = m(f)$ for every $f \in \ell^\infty(G)$ and every $g \in G$.

We denote by $\mathcal{AG}$ the class of all amenable groups.

Note that it is irrelevant whether the mean is left-invariant, right-invariant, or both: one can show that these conditions are all equivalent [Fri17, Chapter 3]. Hence, from now on, we will simply refer to *invariant* means, without specifying the side of invariance and the group G if it is understood from the context.

There exist many other characterization of amenability for discrete group. For instance, one can show that $\mathcal{AG}$ is exactly the class of non-paradoxical group in the context of the Banach-Tarski paradox [Loh17, Theorem 9.2.12]. For our purposes it will be useful the following characterization:

Lemma 2.2 ([Fri17, Lemma 3.2])**.** *Let G be a group. Then the following are equivalent:*

1 there exists an invariant mean on G, i.e. G is amenable,

2 there exists a non-trivial left-invariant $\varphi \in (\ell^\infty(G))'$,

3 there exists a left-invariant finitely additive probability measure on G.

2.1 Example of amenable groups

As a first example, we show that every finite group belongs to this class. In this case we can give a precise formula for the invariant mean: for a finite group G and an element $f \in \ell^\infty(G)$ we average over all the elements of the group, i.e. we set:

$$m(f) = \frac{1}{|G|} \sum_{g \in G} f(g).$$

One can check that this mean is in fact invariant.

Another important and less trivial example of amenable groups is given by the class of abelian group, This result is due to Von Neumann [Neu29, Theorem A], and here we prove it following Frigerio [Fri17, Theorem 3.3], who in turn built on Paterson [Pat88].

For convenience, we recall some results in functional analysis that we will need during the proof.

Theorem 2.3 (**Banach-Alaoglu Theorem** [Rud91, Theorem 3.15])**.** *If X is a normed space then the closed unit ball in the continuous dual space X' (endowed with its usual operator norm) is compact with respect to the weak* topology.*

Theorem 2.4 (**Markov-Kakutani Fixed-Point Theorem** [Rud91, Theorem 5.11])**.** *Let X be a locally convex topological vector space, with a compact convex subset K. Let S be a family of continuous mappings of K to itself which commute and are linear. Then the mappings in S share a fixed point.*

Theorem 2.5 (**Abelian groups are amenable** [Neu29])**.** *Every abelian group is amenable.*

Proof. Let G be a group and let $\ell^\infty(G)'$ its topological dual with the weak* topology. By the Banach-Alouglu Theorem 2.3, the closed unit ball is compact.

We want to find a left-invariant mean. It is not hard to show that a mean exists. For instance, consider the evaluation on the identity of G:

$$\mathrm{ev}\colon \ell^\infty(G) \longrightarrow \mathbb{R}, \quad f \longmapsto f(e_G).$$

What is not immediate is that there exists a mean $\widetilde{\varphi} \in \ell^\infty(G)'$ such that $\widetilde{\varphi}(g \cdot f) = \varphi(f)$ for every $g \in G$ and $f \in \ell^\infty(G)$.

Consider the set of (possibly non-invariant) means:

$$K \coloneqq \{\varphi \in \ell^\infty(G)' \mid \varphi(1) = 1, \, \varphi(f) \geq 0 \text{ for every } f \in \ell^\infty(G)\}.$$

Let $S \coloneqq \{L_g\}_{g \in G}$ be the family of mappings defined by:

$$L_g \colon \ell^\infty(G)' \to \ell^\infty(G)', \quad L_g(\varphi)(f) \coloneqq \varphi(g \cdot f).$$

We want to find a common fixed point for the mappings of S restricted to K. The subset K is closed inside the compact unit ball, hence it is compact. Moreover it is a subvector space of $\ell^\infty(G)'$, thus it is convex. The mappings of S are linear and commute since G is abelian, therefore by Markov-Kakutani Fixed-Point Theorem 2.4, there exists indeed a common fixed point, i.e. a mean $\widetilde{\varphi} \in K$ such that for every $g \in G$ and for every $f \in \ell^\infty(G)$, we have:

$$\widetilde{\varphi}(g \cdot f) = L_g(\widetilde{\varphi})(f) = \widetilde{\varphi}(g \cdot f).$$

This shows that $\widetilde{\varphi}(g \cdot f)$ is a left G-invariant mean, therefore G is amenable. $\square$

From this two class of amenable groups, one can construct many other examples in virtue of the following:

Proposition 2.6 (**Ereditary properties of amenability** [Loh17, Proposition 9.1.9])**.** *The class $\mathcal{AG}$ of amenable groups is closed under taking:*

- *subgroups,*
- *quotients,*
- *extensions,*
- *direct unions.*

Recall that given a group G and a subgroup $H \leq G$ with property $\mathcal{P}$, we say that G is *virtually* $\mathcal{P}$ if the index $[G : H]$ is finite.

If G is amenable, then H is amenable as well, by Proposition 2.6. Conversely, suppose that H is amenable and that $[G : H] < \infty$. Then there is a short exact sequence

$$0 \to H \to G \to G/H \to 0,$$

which means that G is an extension of the finite group G/H by H. Hence, by Proposition 2.6 again, G is amenable.

In conclusion, we have shown the following:

Proposition 2.7. *A group is amenable if and only if it is virtually amenable.*

We denote with $\mathcal{EG}$ the class of *elementary amenable group*, i.e. the smallest class of groups containing all finite and abelian groups and that is closed under taking subgroups, quotients, extensions and direct unions. Thanks to Proposition 2.6, we have $\mathcal{EG} \leq \mathcal{AG}$. One may wonder whether this is an equality, but it turns out that this is not the case: there exist groups which are amenable but not elementary amenable. An example is given by the first Grigorchuk group [Gri84].

An important class of examples of non-amenable groups is given by non-abelian free groups, as we will show combining Theorem 2.12 and Theorem 1.21. A different proof of this fact not involving bounded cohomology can be found in [Loh17, Theorem 9.1.5 and Corollary 9.1.10].

The next examples of non-amenable groups will be of crucial relevance for what we will study later: the group of diffeomorphism of $\mathbb{R}^n$ with or without compact support. Its importance is due to the fact that the bounded cohomology of an amenable group is very simple to compute: it vanishes for a large class of coefficients, as we will see in Theorem 2.12. We will show that $\mathrm{Homeo}_c(\mathbb{R}^n)$ contains a non-abelian free group of rank 2: this implies the non-amenabiity by Proposition 2.6. In order to prove it, we recall the statement of the *ping-pong lemma.*

Theorem 2.8 (**Ping-pong lemma** [Loh17, Theorem 4.3.1])**.** *Let G be a group acting on a set X and let $a, b \in G$. Suppose there exist $A, B \subseteq X$ non-empty subset with B not contained in A such that for every $n \in \mathbb{N}$ we have*

$$a^n \cdot B \subset A, \qquad b^n \cdot A \subset B.$$

Then $\langle a, b\rangle$ is a non-abelian free subgroup of rank 2 .

Example 2.9 (**Non-amenability of $\mathrm{Diff}^r_c(\mathbb{R}^n)$ and $\mathrm{Diff}^r(\mathbb{R}^n)$**)**.** Let $r \in \mathbb{N}$. Since $\mathrm{Homeo}_c(\mathbb{R}^n) \subseteq \mathrm{Diff}^r_{(c)}(\mathbb{R}^n)$, it suffices to show that $\mathrm{Homeo}_c(\mathbb{R}^n)$ is non-amenable.

This follows from the ping-pong lemma 2.8. Indeed, consider the action of $\mathrm{Homeo}_c(\mathbb{R}^n)$ on $\mathbb{R}^n$ and let A, B be non-empty, disjoint open balls of $\mathbb{R}^n$. It is possible to choose elements a, $b \subseteq \mathrm{Homeo}_c(\mathbb{R}^n)$ such that $a^n \cdot B \subseteq A$ and $b^n \cdot A \subseteq B$ for every $n \in \mathbb{N}$. Then $\langle a, b\rangle$ is a non-abelian free group of rank 2 by the ping-pong lemma 2.8.

2.2 Amenability and bounded cohomology

In this section we prove that the amenability of a discrete group can be entirely characterised via its bounded cohomology with coefficients in dual normed $\mathbb{R}[G]$-modules. An $\mathbb{R}[G]$-module V is called *dual* if it is isomorphic to the topological dual of some normed $\mathbb{R}[G]$-module W with the action defined by $g \cdot f(w) \coloneqq f(g^{-1} \cdot w)$. Recall that dual of normed spaces are always Banach (since $\mathbb{R}$ is so), so we can assume we are working with complete spaces.

The proof goes as follow: to prove that an amenable group has vanishing bounded cohomology for every dual $\mathbb{R}[G]$-module one can construct a G-homotopy between the zero map and the identity.

For the other implication, we will rely on a result of Johnson stating that there exists a coclass in degree one for a specific dual $\mathbb{R}[G]$-module that vanishes if and only if G is amenable. More precisely: consider the $\mathbb{R}[G]$-module $V \coloneqq \ell^\infty(G)/\mathbb{R}$, where $\mathbb{R}$ is the $\mathbb{R}[G]$-module of constant functions. Since the latter is closed in $\ell^\infty(G)$, the quotient inherits the structure of Banach $\mathbb{R}[G]$-module. Note that V' is the subspace of elements of $(\ell^\infty(G))'$ that vanish on constant functions. At the cochain level, the Johnson 1-cocycle is defined as the difference of two Dirac deltas. Here for Dirac deltas we mean elements $\delta_g \in (\ell^\infty(G))'$ defined as $\delta_g(f) \coloneqq f(g)$ for all $f \in \ell^\infty(G)$ and a fixed $g \in G$. Namely, we have the following:

Lemma 2.10. *The element $J \in C^1_b(G; V')$ given by*

$$J\colon G^2 \longrightarrow V', \quad (g_0, g_1) \longmapsto \delta_{g_1} - \delta_{g_0}$$

defines a coclass in $H^1_b(G; V')$, called Johnson class.

Proof. We have to prove:

1. $J(g_1, g_2) \in V'$ for every g_1, $g_2 \in G$
2. $J \in Z^1_b(G; V')$.

1) $J(g_1, g_2)$ is in $(\ell^\infty(G))'$ by definition and it is readily that it vanishes on constant functions.

2) We have to show that $J \in Z^1_b(G; V')$ and that J is G-invariant. Let $f \in V'$; then J is a cocycle since:

$$\begin{aligned}
\delta J(g_0, g_1, g_2)(f) &= J(g_1, g_2)(f) - J(g_0, g_2)(f) + J(g_0, g_1)(f) \\
&= (\delta_{g_2} - \delta_{g_1})(f) - (\delta_{g_2} - \delta_{g_0})(f) + (\delta_{g_1} - \delta_{g_0})(f) \\
&= f(g_2) - f(g_1) - f(g_2) + f(g_0) + f(g_1) - f(g_0) \\
&= 0.
\end{aligned}$$

Moreover J is G-invariant. Indeed, note that G acts on the Dirac delta via $g \cdot \delta_{g_0} = \delta_{gg_0}$. Then we have

$$\begin{aligned}
(g \cdot J)(g_0, g_1) &= g(J(g^{-1}g_0, g^{-1}g_1)) \\
&= g(\delta_{g^{-1}g_0} - \delta_{g^{-1}g_1}) \\
&= \delta_{gg^{-1}g_0} - \delta_{gg^{-1}g_1} \\
&= \delta_{g_0} - \delta_{g_1} \\
&= J(g_0, g_1).
\end{aligned}$$

□

Lemma 2.11. *If the Johnson class $[J] \in H_b^1(G; V')$ vanishes, then G is amenable.*

Proof. Recall that by lemma 2.2 it suffices to show that there exists a non-trivial invariant element $\phi \in (\ell^\infty(G))'$.

If $[J] = 0$, then J is a coboundary, that is: there exists $\psi \in C_b^0(G; (\ell^\infty(G)/\mathbb{R})')^G$ such that $J = \delta\psi$. For every $g \in G$, let $\hat{\psi}(g) \in (\ell^\infty(G))'$ be the pullback via the canonical projection $\pi\colon \ell^\infty(G) \twoheadrightarrow \ell^\infty(G)/\mathbb{R}$, i.e. the map $\psi(g) \circ \pi$.

We claim that the element $\varphi \in (\ell^\infty(G))'$, $\varphi := \delta_1 - \hat{\psi}(1)$ is the desired functional, namely it is non-trivial and it is G-invariant. The non-triviality can be seen for example evaluating φ in the constant function equal to 1 and using that $\hat{\psi}$ vanishes on constant functions, obtaining $\varphi(1) = \delta_1(1) - \hat{\psi}(1) = \delta_1(1) - 0 = 1$. For the G-invariance we have to prove that $g \cdot \varphi = \varphi$. We already know that $g \cdot \delta_1 = \delta_g$; noting that $\hat{\psi}$ inherits the G-invariance from ψ, we have

$$\hat{\psi}(g) = (g \cdot \hat{\psi})(g) = g \cdot (\hat{\psi}(1)), \quad \text{for every } g \in G. \tag{3}$$

Since $J = \delta\psi$, it follows that

$$\delta_{g_1} - \delta_{g_0} = \hat{\psi}(g_1) - \hat{\psi}(g_0), \quad \text{for every } g_0, g_1 \in G;$$

in particular

$$\delta_g - \hat{\psi}(g_1) = \delta_1 - \hat{\psi}(1), \quad \text{for every } g \in G. \tag{4}$$

Hence:

$$\begin{aligned}
g \cdot \varphi &= g \cdot (\delta_1 - \hat{\psi}(1)) \\
&= (g \cdot \delta_1) - g \cdot (\hat{\psi}(1)) \\
&\overset{(3)}{=} \delta_g - \hat{\psi}(g) \\
&\overset{(4)}{=} \delta_1 - \hat{\psi}(1) \\
&= \varphi.
\end{aligned}$$

□

Now we are ready to state and prove the aforementioned theorem:

Theorem 2.12 (**Johnson's characterization of amenability**)**.** *The following are equivalent:*

1. *G is amenable.*

2. *$H_b^n(G; V) = 0$ for every dual normed $\mathbb{R}[G]$-module V and for every $n \in \mathbb{N}_{>0}$.*

3. $H_b^1(G;V) = 0$ *for every dual normed* $\mathbb{R}[G]$*-module* V.

Proof. (1) $\Rightarrow$ (2): Recall that the bounded cohomology of G is defined as the cohomology of the complex:

$$0 \longrightarrow C_b^0(G,V)^G \xrightarrow{\delta^0} C_b^1(G,V)^G \xrightarrow{\delta^1} C_b^2(G,V)^G \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^{n-1}} C_b^n(G,V)^G \xrightarrow{\delta^n} \cdots .$$

A standard tool in (co)homology theory to show that an object has vanishing (co)homology is to find an homotopy between the zero map and the identity map as explained in Section 1.2.1. More precisely, we will find maps

$$k^{n+1} \colon C_b^{n+1}(G;V)^G \to C_b^n(G;V)^G$$

such that $k^{n+1} \circ \delta^n + \delta^{n+1} \circ k^n = \mathrm{Id}$ using amenability, i.e. averaging over elements of G. Let m be an invariant mean on G and $f \in C_b^{n+1}(G;V)^G$. Let W be the normed $\mathbb{R}[G]$-module such that $V \cong W'$; we will see $f(g_0, \dots g_{n+1})$ as an element of W', and we will construct $k^{n+1}(f)$ as a function $G^{n+1} \to W'$. Which means: we have to look for an expression for $k^{n+1}(f)(g_0, \dots g_n)(w)$, where $g_0, \dots g_n \in G$, $w \in W$.

For all such $g_0, \dots g_n \in G$, $w \in W$, define $f_w \in \ell^\infty(G)$, $f_w(g) = f(g, g_0, \dots g_n)(w)$. Then we can average via m this element and obtain $k^{n+1}(f)(g_0, \dots g_n)(w) \coloneqq m(f_w)$. To resume: the image of f is given by the following composition:

$$\begin{aligned} G^{n+1} &\longrightarrow \Big(W \longrightarrow \ell^\infty(G) \longrightarrow \mathbb{R}\Big). \\ (g_0, \dots, g_n) &\longmapsto \Big(w \longmapsto f_w \longmapsto m(f_w)\Big) \end{aligned}$$

By the definitions and some calculation we obtain what is left to prove, i.e.:

- f_w is indeed an element of $\ell^\infty(G)$ and not just a function $G \to \mathbb{R}$,
- the function $(w \mapsto m(f_w))$ is continuous and bounded,
- the map j^{n+1} is bounded and G-equivariant,
- it holds $k^{n+1} \circ \delta^n + \delta^{n+1} \circ k^n = \mathrm{Id}$.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): If $H_b^1(G;V) = 0$ for every dual normed $\mathbb{R}[G]$-module V, then in particular $H_b^1(G;((\ell^\infty(G))') = 0$, so the Johnson class vanishes and we conclude with Lemma 2.11. $\square$

Using this theorem we obtain a proof of the non-amenability of F_2: the free group with two or more generator has bounded cohomology with real coefficients non-trivial, in fact infinite-dimensional, hence it is non-amenable (Theorem 1.21). Note that $\mathbb{R}$ is a dual $\mathbb{R}[G]$-module since $\mathbb{R} = \{*\}'$, where the singleton is endowed with the trivial G-action. Moreover, since by Proposition 2.6 amenability is preserved by taking subgroups, if a group contains a non-abelian free group then it is non-amenable.

A natural question is what happens when we drop the assumption that the module is dual and consider all normed $\mathbb{R}[G]$-modules for which the bounded cohomology vanishes. It turns out that this condition characterizes finite groups, as the following theorem shows:

Theorem 2.13 (Characterization of finite groups). *The following are equivalent:*

1. *G is finite.*

2. *$H_b^n(G;V)=0$ for every normed $\mathbb{R}[G]$-module V and for every $n\in\mathbb{N}_{>0}$.*

3. *$H_b^1(G;V)=0$ for every normed $\mathbb{R}[G]$-module V.*

Proof. The proof is analogous to the one given for the Johnson's characterization of amenability: (1) $\Rightarrow$ (2) can be showed by finding a suitable homotopy, (2) $\Rightarrow$ (3) is obvious and (3) $\Rightarrow$ (1) is done by proving that the vanishing of the so-called *characteristic coclass* in a proper Banach (non-necessarily dual) G-module implies the finiteness of G. For further details we refer to [Fri17, Theorem 3.2]. □

An important result concerning amenability and bounded cohomology of topological space is the following Gromov's Mapping Theorem.

Theorem 2.14 (**Gromov's Mapping Theorem** [Iva17, Theorem 8.4] and [FM23, Theorem 5.0.1])**.** *Let X, Y be path-connected topological spaces and let $f\colon X\to Y$ be a continuous map inducing a surjective map at level of fundamental groups with amenable kernel. Then the induced map in bounded cohomology*

$$H_b^n(f)\colon H_b^n(Y)\to H_b^n(X)$$

is an isometric isomorphism for every $n\in\mathbb{N}$.

This theorem, as Theorem 1.14, appeared first in Gromov's seminal paper [Gro82, Section 3.1, page 40]. Ivanov gave a completely different proof for spaces that are homotopy equivalent to countable CW-complex [Iva87] and later for every path-connected topological space. Recently, Moraschini and Frigerio proved it following more closely Gromov's original approach.

3 The bounded cohomology of transformation groups of euclidean spaces: the compact case

So far we have seen how bounded cohomology can behave wildly, and we have pointed out several times how hard it is to be computed. For amenable groups things are actually very easy for a large class of coefficients: it vanishes in every positive degree (see Section 2.12). A natural way to proceed in order to better understand the behaviour of this invariant is to find an example of bounded acyclic group which is not amenable. Historically, the first group with this property that was discovered is the group $\mathrm{Homeo}_c(\mathbb{R}^n)$ of *homeomorphisms of the Euclidean space with compact support*. Relying on a result of Mather who proved the acyclity of this group [Mat71], Matsumoto and Morita were able to prove also its bounded acyclicity by finding a criterion which ensures the injectivity of the comparison map [MM85]. Thus, bounded cohomology injects into the trivial group, therefore it has to vanish.

Later in this chapter, we introduce the notion of *dissipated groups* and show that $\mathrm{Homeo}_c(\mathbb{R}^n)$ belongs to this class. Dissipated groups form a subclass of binate groups, which have been proven much more recently to be boundedly acyclic by Fournier-Facio, Löh and Moraschini [FFLM23].

3.1 Standard cohomology of $\mathrm{Homeo}_c(\mathbb{R}^n)$

In this section we follow what was done by Mather in 1971 in his article [Mat71] to prove that the group $\mathrm{Homeo}_c(\mathbb{R}^n)$ is acyclic, meaning that its cohomology with integer coefficients vanishes in every positive degree. This actually implies that, thanks to the Universal Coefficient Theorem [Hat02, Section 3.1], all homology and cohomology groups with coefficients in any R-module is trivial. In this section we omit the coefficient group, implying that we are working with integer coefficients.

The proof of Mather goes as follows: since chains are defined by a finite number of elements only, we can assume that every chain is supported on an n-dimensional ball $B^n_R \subset \mathbb{R}^n$ of a certain radius $R \geq 0$. Then Mather exhibits an element $k \in \mathrm{Homeo}_c(\mathbb{R}^n)$ such that $k^j(B^n_R) \cap B^n_R = \emptyset$ and $\lim_{j\to\infty} k^j(B^n_R) = p$ for some $p \in \mathbb{R}^n$. Namely, there exists a homeomorphism that displaces the ball infinitely many times within a compact support (of course by shrinking it to a point). This provides the key ingredient in the proof of the acyclicity of $\mathrm{Homeo}_c(\mathbb{R}^n)$, and it is evident that without the compactness assumption on the support of φ such a k cannot exist (since the support of $\varphi \in \mathrm{Homeo}_c(\mathbb{R}^n)$ might be the whole $\mathbb{R}^n$).

From now on, we will denote for brevity by G_n the group $\mathrm{Homeo}_c(\mathbb{R}^n)$. Recall that the *support* of a homeomorphism $\varphi \in \mathbb{R}^n$ is defined as

$$\mathrm{supp}(\varphi) \coloneqq \overline{\{x \in \mathbb{R}^n \mid \varphi(x) \neq x\}}.$$

The support of a chain $\sigma = \sum_{i=1}^q a_i\,(g_1, \dots, g_q) \in C_q(G_n)$ is

$$\mathrm{supp}(\sigma) = \bigcup_{i=1}^q \mathrm{supp}(g_i).$$

Denote by $B_1(0)$ the open unit ball in $\mathbb{R}^n$, and let

$$G^0_n \coloneqq \{\varphi \in G_n \mid \mathrm{supp}(\varphi) \subseteq B_1(0)\}.$$

The inclusion $i\colon G^0_n \to G_n$ induces the map i_* in cohomology, which turns out to be an isomorphism.

Lemma 3.1. *The map induced by the inclusion $i_*\colon H_q(G^0_n) \to H_q(G_n)$ is an isomorphism.*

Proof. We prove before the surjectivity and then the injectivity of the map i_*.

Surjectivity: Let $h \in H_q(G_n)$ and let $c \in C_q(G_n)$ a cycle representing h; the element c has the form

$$c = \sum_{i=1}^q a_i\,(g_1, \dots, g_q),$$

with $g_i \in G_n$ for every $i = 1, 2, \dots, q$. We look for an element $h' \in H_q(G^0_n)$ such that $i_*h' = h$. The support of c is compact, therefore there exists $\varphi \in G_n$ such that $\varphi(\mathrm{supp}(c)) \subseteq B_1(0)$. Consider the inner automorphism

$$\begin{aligned} I_\varphi\colon G_n &\to G_n \\ g &\mapsto \varphi g \varphi^{-1}. \end{aligned}$$

We claim that $[(I_\varphi)_*(h)]$ is the desired element h'. We have to prove that:

1. $[(I_\varphi)_*(h)] \in H_q(G^0_n)$,

2. $h = i_*[(I_\varphi)_*(h)]$.

The first statement follows from $\operatorname{supp} I_\varphi c = \varphi \operatorname{supp} c \subseteq B_1(0)$, while the second follows from the fact that inner automorphisms induce the identity in homology. More precisely, $(I_\varphi)_* h = h$ and $(I_\varphi)_* h = [I_\varphi c]$. This gives the thesis.

Injectivity: Let $h \in H_q(G_n^0)$ such that $i_* h = 0$ and let $c \in C_q(G_n^0)$ be a cycle representing h. Since $i_* h = 0 \in H_q(G_n)$, there exists $c' \in C_{q+1}(G_n)$ such that $dc' = i_* c$. Because c has compact support contained in $B_1(0)$, there exist $\varphi \in G_n$ and a neighborhood $U(\operatorname{supp} c)$ of $\operatorname{supp} c$ such that $\varphi\big|_{U(\operatorname{supp} c)} = \operatorname{Id}$ and $\varphi(\operatorname{supp} c') \subseteq B_1(0)$. We have that:

1. $I_\varphi c' \in C_{q+1}(G_n^0)$;
2. $d(I_\varphi c') = I_\varphi(dc') = I_\varphi c = c$.

Thus we obtain $h = [c] = [d(I_\varphi c')] = [0]$. □

Mather's theorem states that any homology class of G_n vanishes, and to prove it we will work with the assumption that a general homeomorphism has support contained in $B_1(0)$. The lemma shows exactly this: the precise support of a homeomorphism is not so important as long as it is compact, since we can always reduce to the case where it is contained in the unit ball.

Theorem 3.2 (Mather). *Let G_n the group of homeomorphisms of $\mathbb{R}^n$ with compact support. Then we have:*

$$H_q(G_n; \mathbb{Z}) = 0$$

for every $q > 0$.

Proof. We will divide the proof in three parts. In the first one we will use a homeomorphism that displaces the ball infinitely many times within a compact support (of course by shrinking it to a point) to define the conjugate homomorphisms $\phi_0, \phi_1 \colon G_n^0 \to G_n$. In the second part we show a relation between the expression of ϕ_0 and ϕ_1 that we will use to conclude the thesis by induction in the third part.

1. There exists an element $k \in G_n$ such that:

   (i) $k^j(\overline{B_1(0)}) \cap \overline{B_1(0)} = \emptyset$ for every $j > 0$;

   (ii) there exists a point $p \in \mathbb{R}^n$ such that $k^j\big(\overline{B_1(0)}\big) \xrightarrow[j\to\infty]{} p$;

   (iii) $k^j(\overline{B_1(0)}) \subseteq \overline{B_R(0)}$ for every $j > 0$ and R sufficiently big.

   Define now, for every $g \in G_n^0$ and $i = 0, 1$, the maps $\phi_i \colon G_n^0 \to G_n$ by

   $$\phi_i(g)(x) \;=\; \begin{cases} k^j g k^{-j}(x), & \text{if } x \in k^j(\overline{B_1(0)}) \text{ and } j \geq i, \\ x, & \text{if } x \notin \bigcup_{j\geq i} k^j(\overline{B_1(0)}). \end{cases}$$

   By item (i), $k^j(\overline{B_1(0)}) \bigcap k^{j'}(\overline{B_1(0)}) = \emptyset$ if $j \neq j'$, hence ϕ_i are well defined.

   Furthermore, item (iii) ensures that the support of $\phi_i(g)$ is compact, so that $\phi_i(g) \in G_n$ for every $g \in G_n^0$, for $i = 0, 1$.

   Moreover, ϕ_0 and ϕ_1 are conjugate homomorphism since $\phi_1(g) = k\phi_0(g)k^{-1}$ and therefore they induce the same map in homology.

2. Let η the map defined by

$$\begin{aligned} \eta\colon G_n^0 \times G_n^0 &\longrightarrow G_n \\ (g,h) &\mapsto g\,\phi_1(h). \end{aligned}$$

In order to prove that η is a homomorphism, we need to show that g commutes with $\phi_1(h)$. This follows from the fact that $\operatorname{supp} g \bigcap \operatorname{supp} \phi_1(h) = \emptyset$ since $\operatorname{supp} g \subseteq B_1(0)$ and $\operatorname{supp} \phi_1(h) \subseteq \bigcup_{j\geq 1} k^j(\overline{B_1(0)}) \cup p$.

Let Δ be the diagonal homomorphism, i.e.:

$$\begin{aligned} \Delta\colon G_n^0 &\longrightarrow G_n^0 \times G_n^0, \\ g &\longmapsto (g,g). \end{aligned}$$

A computation shows that $\phi_0 = \eta\Delta$.

3. Now we are ready to prove the thesis by induction: for $1 \leq r \leq q$, we prove that $H_r(G_n^0) = 0$.

Base step: recall that for $r = 1$, the first homology of a group coincides with its abelianization [Bro82, Section 3.3]. Explicitly, we have $H_1(G_n^0) \cong G_n^0/[G_n^0, G_n^0]$, so it is enough to show that the commutator subgroup equals the entire group, i.e., that G_n^0 is a *perfect group*. This fact is established in the next proposition (Proposition 3.3), which we state separately for clarity.

Inductive step: assume the statement true until degree $q-1$. Recall that the Künneth formula for real coefficients boils down to:

$$H_q\Big(G_n^0 \times G_n^0\Big) \;\cong\; \bigoplus_{i+j=q} H_i(G_n^0) \otimes H_j(G_n^0);$$

the only possibly non-null coefficients occur for $i = q$ and $j = 0$, and conversely, hence we have:

$$H_q\Big(G_n^0 \times G_n^0\Big) \cong H_q(G_n^0) \oplus H_q(G_n^0).$$

Take $h \in H_q(G_n^0)$. Noting that $\Delta_* h = h \oplus h$ and recalling that $(\phi_0)_* = (\phi_1)_*$, since the differ by a conjugated, we have:

$$(\phi_0)_* h = \eta_* \Delta_* h = i_* h + (\phi_1)_* h = i_* h + (\phi_0)_* h,$$

so we obtain $i_* h = 0$ and by Lemma 3.1 it follows that $h = 0$, hence the thesis.

□

Proposition 3.3. *The group* $G_n^0 := \{g \in \mathrm{Homeo}_c(\mathbb{R}^n) \mid \operatorname{supp}(g) \subseteq B_1(0)\}$ *is perfect, i.e.*

$$G_n^0 = [G_n^0, G_n^0].$$

Proof. Let $g \in G_n^0$; we have to prove that g can be written as product of commutators. Since $\operatorname{supp}(g)$ is compact and $B_1(0)$ is open, there exists $h_g \in G_n^0$ such that

i. $h_g^j(\operatorname{supp}(g)) \cap \operatorname{supp}(g) = \emptyset$ for every $j > 0$,

ii. there exists a point $p \in B_1(0)$ such that $h_g^j\Big(\operatorname{supp}(g)\Big) \xrightarrow[j\to\infty]{} p$.

Note that this is similar but it is not the same map as before: for instance, we take it with support contained in the unit ball.

Define a map $a_g \in \mathrm{Homeo}_c(\mathbb{R}^n)$ by

$$a_g(x) := \begin{cases} h_g^n\, g\, h_g^{-n}(x), & \text{if } x \in h_g^n(\mathrm{supp}(g)) \text{ for some } n \geq 1, \\ x, & \text{if } x \notin \bigcup_{n\geq 1} h_g^n(\mathrm{supp}(g)). \end{cases}$$

Then $\mathrm{supp}(a_g) \subset B_1(0)$ and $[h_g^{-1}, a_g] = g$. □

3.2 Bounded cohomology of $\mathrm{Homeo}_c(\mathbb{R}^n)$

Combining the standard acyclicity of $\mathrm{Homeo}_c(\mathbb{R}^n)$ proved in Mather's Theorem 3.2 and Matsumoto-Morita's Theorem 1.33, we obtain the bounded acyclicity of $\mathrm{Homeo}_c(\mathbb{R}^n)$. As we already mentioned, to prove it we will show that the chain complex of $\mathrm{Homeo}_c(\mathbb{R}^n)$ satisfies the q-UBC condition for every $q \geq 1$. This then implies that the comparison map injects into the trivial group, hence the thesis.

Theorem 3.4 ([MM85, Theorem 3.1])**.** *For every $q \geq 1$, we have*

$$H_b^q(\mathrm{Homeo}_c(\mathbb{R}^n)) = H_q^{\ell^1}(\mathrm{Homeo}_c(\mathbb{R}^n)) = 0.$$

Proof. Let us fix the notations:

- let G be the group $\mathrm{Homeo}_c(\mathbb{R}^n)$;
- let $C_\bullet$, $Z_\bullet$ and $B_\bullet$ be respectively the chain complex, the cycles and the boundaries of G;
- let $G^i := \{g \in G \mid \mathrm{supp}(g) \subseteq \mathrm{int}(jB_i^n(0))\}$, where $jB_i^n(0) \subseteq \mathbb{R}^n$ is the n-ball centered at 0 with ray j;
- let $C_\bullet^i$, $Z_\bullet^i$ and $B_\bullet^i$ be respectively the chain complex, the cycles and the boundaries of G^i.

By Theorem 3.2, we have $Z_\bullet = B_\bullet$ and $Z_\bullet^i = B_\bullet^i$.

To continue, we need the existence of certain bounded linear operators. We state this fact as a lemma and we prove it later.

Lemma 3.5. *In the notation above, there exist bounded linear operators $S_q \colon B_q^1 \to C_{q+1}$ such that $\partial_{q+1} S_q = i_*$, where $i : G_n^1 \to \mathrm{Homeo}_c(\mathbb{R}^n)$ is the inclusion.*

We show now that G^1 satisfies q-UBC (Definition 1.28), i.e.:

$$\exists K > 0 \ : \ \forall b \in B_q^1,\ \exists c \in C_{q+1}^1 \text{ such that } \partial_{q+1} c = b \text{ and } \|c\|_1 \leq K\|b\|_1.$$

Let $b \in B_q^1$ and choose $\varphi \in G$ such that:

1. $\varphi|_{\mathrm{supp}(b)} = \mathrm{id}$;
2. $\mathrm{supp}\, S_q(b) \overset{\varphi}{\hookrightarrow} \mathrm{int}\, B_0^n(i)$.

Consider the inner automorphism

$$G \xrightarrow{I_\varphi} G, \qquad I_\varphi(g) = \varphi g \varphi^{-1}.$$

We claim that $c \coloneqq (I_\varphi)_*(S_q(b))$ is the desired element. Indeed by Equation (2) we have that $(I_\varphi)_*(S_q(b)) \in C^1_{q+1}$. Moreover its norm satisfies

$$\|(I_\varphi)_*(S_q(b))\| = \|(S_q(b))\| \leq \|S_q\|\|b\|$$

and finally, using (1) we have

$$\partial_{q+1}(I_\varphi)_*(S_q(b)) = (I_\varphi)_*\partial_{q+1}S_q(b) = (I_\varphi)_* i_*(b) = b. \qquad \square$$

One of the main ingredients of the proof is the existence of bounded linear operators $S_q\colon B^1_q \to C_{q+1}$ such that $\partial_{q+1}S_q = i_*$, where $i\colon G^1_n \to \mathrm{Homeo}_c(\mathbb{R}^n)$ is the inclusion. These operators work as follows: S_q takes a q-boundary $b \in B^1_q$ and produces a $(q+1)$-chain $S_q(b) \in C_{q+1}$ such that its boundary is b, considered inside C_q. Moreover, S_q is bounded, meaning that there exists a uniform bound on the norms.

To clarify what is going on, let us consider an analogous map in simplicial homology. Take the standard 2-simplex with vertices $\{v_0, v_1, v_2\}$, corresponding to a triangle if we look at its geometric realization. The 1-cycle $b = [v_0, v_1] + [v_1, v_2] - [v_2, v_0]$ is a boundary, and S_1 would send it to the 2-chain $[v_0, v_1, v_2]$.

To prove Lemma 3.5, we need some preliminaries. We refer to [Mac63, Chapter VIII, Section 8] for details.

Recall that, given groups G, H, there exist chain maps

$$\alpha\colon C_\bullet(G \times H) \longrightarrow C_\bullet(G) \otimes C_\bullet(H),$$
$$\beta\colon C_\bullet(G) \otimes C_\bullet(H) \longrightarrow C_\bullet(G \times H).$$

entirely determined up to chain homotopy by naturality and their definitions on 0-chains,

$$\alpha(x, y) = x \otimes y, \qquad \beta(x \otimes y) = (x, y).$$

By Eilenberg-Zilber Theorem [Mac63, Theorem 8.1], α and β are mutually inverse, hence they define a natural chain equivalence

$$C_\bullet(G \times H) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} C_\bullet(G) \otimes C_\bullet(H).$$

They are functorial and if we endow the chain complexes with a norm, they are bounded linear.

During the proof we denote with the same letters this maps for G and G^i, for every $i \geq 1$.

One can also give an explicit formula for these maps. We need the following representation of α:

Theorem 3.6 ([Mac63, Theorem 8.5]). *For any group G, H, a natural chain transformation $\alpha\colon C_\bullet(G \times H) \to C_\bullet(G) \to C_\bullet(H)$ for the Eilenberg-Zilber theorem is given in degree n by:*

$$\alpha(x_0, \dots x_n; y_0, \dots, y_n) = \sum_{p=0}^{n} (x_0, \dots, x_p) \otimes (y_p, \dots, y_n)$$

This chain transformation is known as the Alexander-Whitney map.

Intuitively, we are taking an n-simplex in $C_\bullet(G \times H)$, which is a pair of n-simplices, one in $C_n(G)$ and the other in $C_n(H)$. The Alexander–Whitney map is then defined as a sum over all possible ways of decomposing these simplices into pairs of (p, q)-simplices with $p + q = n$.

Proof of Lemma 3.5. Let us denote by $i^1 \colon G^1 \to G^2$, $i^2 \colon G^2 \to G^3$, $i^2 \circ i^1 = i^{1,3} \colon G^1 \to G^3$ and $i \colon G^1 \to G$ the inclusions. We prove by induction the following statement:

For every $j \geq 0$ and for $i = 1, 2$, there exist bounded linear operators $S^i_q \colon B^i_q \to C^{i+1}_{q+1}$ and $S_q \colon B_q \to C_{q+1}$ such that $\partial_{q+1} S^{(i)}_q = i^{(i)}_*$.

The base step is trivial: the space of 0-coboundaries is trivial, therefore we can define $S^{(i)}_0$ as the map sending 0 in 0. This maps satisfy the inductive hypothesis.

For the inductive step, suppose we have already defined $S^1_j \colon B^1_j \to C^2_{j+1}$ and $S^2_j \colon B^2_j \to C^3_{j+1}$ for $0 \leq j \leq q-1$. We want to construct: $S_q \colon B^1_j \to C_{j+1}$.

Recall that by acyclicity boundaries and cycles coincide, hence the operators are defined also on cycles.

Consider the diagonal morphism $\Delta \colon G^1 \to G^1 \times G^1$, defined by $\Delta g = (g, g)$.

Define

$$Z'_q(C^1 \otimes C^1) \coloneqq Z_q(C^1 \otimes C^1) \cap \sum_{i=1}^{q-1} C^1_i \otimes C^1_{q-i},$$

that is, the q-cycles of $C^1 \otimes C^1$ whose components lie strictly in the "interior", i.e., they have no part in $C^1_0 \otimes C^1_q$ or $C^1_q \otimes C^1_0$.

Let us define the bounded and linear map

$$D \colon B^1_q \longrightarrow Z'_q(C^1 \otimes C^1), \quad D(z) = \alpha\, \Delta_* z - (z \otimes 1 + 1 \otimes z).$$

The image of D is a cycle. Indeed, the maps α and Δ_* are chain maps, therefore the commute with the differential. Since $x \in B^1_q$, we have $\partial z = 0$, thus we obtain:

$$\begin{aligned} \partial D(z) &= \partial(\alpha \Delta_* z) - \partial(z \otimes 1 + 1 \otimes z) \\ &= \alpha \Delta_*(\partial z) - (\partial z \otimes 1 + 1 \otimes \partial z) \\ &= 0 - (0 + 0) = 0. \end{aligned}$$

Moreover, the image of D lies in $\sum_{i=1}^{q-1} C^1_i \otimes C^1_{q-i}$ by definition of the Alexander-Whitney map as given in Theorem 3.6: the addendum $(z \otimes 1 + 1 \otimes z)$ is exactly the component outside the "interior".

Define now the chain complexes

- Z^1 by $(Z^1)_q \coloneqq Z^1_q$ with trivial differential;
- $\overline{B}^1$ by $(\overline{B}^1)_q \coloneqq B^1_{q-1}$ with trivial differential.

Thus, we have a short exact sequence:

$$0 \longrightarrow C^1 \otimes Z^1 \to C^1 \otimes C^1 \xrightarrow{1 \otimes \partial} C^1 \otimes \overline{B}^1 \longrightarrow 0$$

Connecting the rows, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & (C^1 \otimes Z^1)_{q+1} \arrow[r] \arrow[d, "\partial"] & (C^1 \otimes C^1)_{q+1} \arrow[r, "1\otimes\partial"] \arrow[d, "\partial"] & (C^1 \otimes \overline{B}^1)_{q+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & (C^1 \otimes Z^1)_{q} \arrow[r] \arrow[d, "\partial"] & (C^1 \otimes C^1)_{q} \arrow[r, "1\otimes\partial"] \arrow[d, "\partial"] & (C^1 \otimes \overline{B}^1)_{q} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & (C^1 \otimes Z^1)_{q-1} \arrow[r] & (C^1 \otimes C^1)_{q-1} \arrow[r, "1\otimes\partial"] & (C^1 \otimes \overline{B}^1)_{q-1} \arrow[r] & 0,
\end{tikzcd}$$

where the horizonatal maps from the second and the third columns are induced by the identity and the inclusion.

We can write similar diagrams for G^2 and G^3 and connect them via i^1_* and i^3_*.

Since $\partial D(z) \in Z'_q(C^1 \otimes C^1)$ for every $z \in B^1_q$, we have that

$$(1 \otimes \partial)D(z) \in Z'_q(C^1 \otimes \overline{B}^1) = (Z^1 \otimes \overline{B}^1)'_q.$$

In other words, $(1\otimes\partial)D(z)$ lives in the interior degrees, where we have already defined S^1 by inductive hypothesis. Hence we can consider

$$(S^1 \otimes S^1)(1 \otimes \partial)D(z) = (S^1 \otimes S^1\partial)D(z) \in (C^2 \otimes C^2)_{q+1}.$$

Let us consider now the following element:

$$u \coloneqq (i^1_* \otimes i^1_*)D(z) - \partial(S^1 \otimes S^1\partial)D(z) \in (C^2 \otimes C^2)'_q.$$

We are pushing $D(z)$ in $C^2 \otimes C^2$, hence only interior degrees are involved. We want to apply S^2 on u. Our goal is to apply S^2 to u. Recall that, by the inductive hypothesis, S^2 has already been defined on cycles up to degree $q-1$. Therefore, it suffices to show that u is a cycle lying in interior degrees. We look for a lift of u in $(C^2 \otimes Z^2)'_q$, i.e.: we want to show that u is in the image of the firt map in the short exact sequence

$$0 \longrightarrow (C^2 \otimes Z^2)_q \to (C^2 \otimes C^2)_q \xrightarrow{1\otimes\partial} (C^2 \otimes \overline{B}^2)_q \longrightarrow 0.$$

Equivalently, we need to show that $u \in \operatorname{Ker}(1 \otimes \partial)$. This is true since $\partial D(z) = 0$. Moreover, u is a q-cycle of this chain complex. i.e. $\partial u = 0$, therefore:

$$u \in Z'_q(C^2 \otimes Z^2) = (Z^2 \otimes Z^2)'_q$$

Thus, by inductive hypothesis we can consider the element

$$(S^2 \otimes (i^2_* - S^2\partial))u \in (C^3 \otimes C^3)_{q+1}$$

Let us compute its differential. We have:

$$\begin{aligned}
\partial(S^2 \otimes (i^2_* - S^2\partial))u &= (\partial S^2 \otimes (i^2_* - S^2\partial))u + (-1)^{q+1}(S^2 \otimes \partial(i^2_* - S^2\partial))u \\
&= (i^2_* \otimes i^2_*)u + (-1)^{q+1}(S^2 \otimes (\partial i^2_* - \partial i^2_*))u \\
&= (i^2_* \otimes i^2_*)u
\end{aligned}$$

The equality for the first addendum follows by this: decompose u as follows

$$u = \sum_k a_k \otimes b_k.$$

The equality $(1 \otimes \partial)u = 0$ means that $\partial b_k = 0$, thus for every k we obtain

$$\begin{aligned}(\partial S^2 \otimes (i^2_* - S^2\partial))(a_k \otimes b_k) &= i^2_* a_k \otimes (i^2_* b_k - S^2 \partial b_k) \\ &= i^2_* a_k \otimes (i^2_* b_k - 0) \\ &= (i^2_* \otimes i^2_*)(a_k \otimes b_k).\end{aligned}$$

We work now in the same notation of the proof of Theorem 3.2; for details about well definition of the involved morphisms we refer to the comment therein. There exists an element $k \in G$ such that

- $k^p\Big(3\overline{B^n_i(0)}\Big) \cap k^q\Big(3\overline{B^n_i(0)}\Big) = \emptyset$ for every $p \neq q \in \mathbb{N}$;
- $k^j\Big(3\overline{B^n_i(0)}\Big) \xrightarrow[j\to\infty]{} p$ for some $p \in \mathbb{R}^n$.

Define now, for every $g \in G^3$ and $i = 0, 1$, the maps $\phi_i \colon G^3 \to G$ by

$$\phi_i(g)(x) \;=\; \begin{cases} k^j g k^{-j}(x), & \text{if } x \in k^j(3\overline{B^n_i(0)}) \text{ and } j \geq i, \\ x, & \text{if } x \notin \bigcup_{j\geq i} k^j(3\overline{B^n_i(0)}). \end{cases}$$

These morhpisms are well defined and conjugates. We denote with the same letters their restriction to G^1.

Let η the map defined by

$$\begin{aligned}\eta \colon G^3 \otimes G^3 &\longrightarrow G \\ (g, h) &\mapsto g\,\phi_1(h).\end{aligned}$$

A computation shows $\phi_0 = \eta\Delta$

Define $E \colon Z'_q(C^1 \otimes C^1) \longrightarrow (C^3 \otimes C^3)_{q+1}$ by

$$E \coloneqq \Big(i^2_* S^1 \;\otimes\; i^2_* S^2\partial\Big) \;+\; \Big(S^2 \;\otimes\; (i^2_* - S^2\partial)\Big)\Big(i^1_* - \partial(S^1 \otimes S^1\partial)\Big),$$

we have $(i^{1,3} \otimes i^{1,3})D(z) = \partial(ED(z))$.

Moreover, a calculation shows:

$$(i^{1,3} \otimes i^{1,3})\alpha\Delta_* z = (i^{1,3} \otimes i^{1,3})z \otimes 1 + \partial(ED(z)) + (i^{1,3} \otimes i^{1,3})(1 \otimes z)$$

and applying $\eta_*\beta$, by naturality of the cross product this becomes

$$\eta_*(i^{1,3} \times i^{1,3})_*\beta\alpha\Delta_* z = i_* z + \eta_*\beta\partial(ED(z)) + \phi_{1*} z.$$

Recall that there exists a bounded chain homotopy between $\beta\alpha$ and the identity, i.e. there exists a bounded linear map $h \colon C_\bullet(G^1 \times G^1) \to C_{\bullet+1}(G^1 \times G^1)$ such that $\beta\alpha - \mathrm{Id} = \partial h + h\partial$. Thus the equality becomes

$$\phi_{0*} z + \eta_*(i^{1,3} \times i^{1,3})_*\partial h\Delta_* z = i_* z + \eta_*\beta\delta ED(z) + \phi_{1*} z.$$

Since ϕ_1 and ϕ_0 are conjugated, there exists a bounded chain homotopy $H \colon C^1_\bullet \to C^1_{\bullet+1}$ such that $\phi_1 - \phi_0 = H\partial + \partial H$.

Thus, if we define

$$S_q \coloneqq \eta_*(i^{1,3} \times i^{1,3})_* h\Delta_* - H - \eta_*\beta ED$$

we obtain $i_* = \partial_{q+1} S_q$, and this concludes the proof since S_q is sum and composition of bounded linear maps.

□

Roughly, the idea is this. We want to construct inductively S_q using operators coming from lower degrees. We decompose the problem via the Alexander-Whitney map and we deal with the single components. The "extern" ones are the easier, for example it suffices to subtract $1 \otimes z$ and $z \otimes 1$. For the "intern" components one has to be careful using already constructed operators and homotopies. Everything can be done since the Eilenberg-Zilber construction is bounded.

3.3 Dissipated groups

In this section we will see that what was shown for $\mathrm{Homeo}_c(\mathbb{R}^n)$ is just an instance of a more general behaviour. Recall that the key ingredient in Mather's proof of the acyclicity of $\mathrm{Homeo}_c(\mathbb{R}^n)$ was the existence of a certain homeomorphism that displaces the ball infinitely many times within compact support. The class of *dissipated groups* consists exactly of those groups that admit a similar element. Hence, it will be clear that what was done by Mather can be replicated in this context and thus the same proof leads to the acyclicity of dissipated groups.

More in general, we will see the notion of *pseudo-mitotic groups*, that are a further generalization of dissipated groups. Recently, this class of group was proved to be boundedly acyclic, as we will state in Theorem 3.14.

Lastly, Theorem 3.15 states that many groups of compactly supported diffeomorphisms are boundedly acyclic. We will apply this result in Lemma 4.54, as a part of the proof that the group $\mathrm{Diff}^r(\mathbb{R}^n)$ is boundedly acyclic.

Recall that an action of a group G on a space X is called *faithful* if $g \cdot x = x$ for every $x \in X$ implies that g is the identity element.

Recall moreover that a *directed union of sets* is defined as follows. Let $(I, \leq)$ be an index set with a partial *directed* order $\leq$ on it, i.e. we require that for every $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Consider a family of sets $\{X_i\}_{i\in I}$ such that whenever $i \leq j$, we have $X_i \subseteq X_j$. The directed union of $\{X_i\}_{i\in I}$ is the set

$$X \coloneqq \bigcup_{i \in I} X_i$$

Similarly, one can define the *directed union of groups*, which is a group again.

Definition 3.7 (**Boundedly supported group**)**.** Let G be a group acting faithfully on a directed union of sets $X = \bigcup_{i\in I} X_i$. For every $i \in I$, let

$$G_i \coloneqq \{g \in G \mid g \text{ is supported in } X_i\}.$$

We say that G is *boundedly supported* if G is the directed union of the subgroups G_i. If this happens, each of the X_i is called *bounded set*.

Definition 3.8 (**Dissipated group**)**.** Let G be a dissipated group acting on X and let $(G_i, X_i)_{i\in I}$ be as before. A *dissipator* for G_i is an element $\rho_i \in G$ such that:

1. $\rho_i^k(X_i) \cap X_i = \emptyset \quad$ for all $k \geq 1$;
2. For all $g \in G_i$, the bijection of X defined by

$$\varphi_i(g)(x) \;=\; \begin{cases} \rho_i^k g \rho_i^{-k}(x), & \text{if } x \in \rho_i^k(X_i) \text{ for some } k \geq 1, \\ x, & \text{otherwise,} \end{cases}$$

belongs to G.

If there exists a dissipator for every G_i, we say that G is *dissipated.*

Remark 3.9. The properties of being boundedly supported and dissipated are invariant under isomorphism. Indeed, let G and H isomorphic groups via a map $f\colon G \to H$. If G acts on a space X, then we can define an action of H on X via $h \cdot x := f^{-1}(h) \cdot x$ for every $h \in H$.

Suppose that G is boundedly supported with $X = \bigcup_{i\in I} X_i$ and $G = \bigcup_{i\in I} G_i$ as in Definition 3.7. Then $(f^{-1}(G_i), X_i)_{i\in I}$ satifies the condition for H to be boundedly supported.

Analogously, considering the isomorphic image of ρ_i via f in Definition 3.8, we obtain that if G is dissipated then H is dissipated too.

With this result, it follows that the group $\mathrm{Homeo}_c(\mathbb{R}^n)$ is dissipated: consider the isomorphic group $G_n^0 := \{\phi \in \mathrm{Homeo}_c(\mathbb{R}^n) \mid \mathrm{supp}(\phi) \subseteq B_1(0)\}$; then, what was done in Proposition 3.3 proves exactly that this group is dissipated.

Example 3.10 (Examples of dissipated groups). The following are some examples of dissipated groups.

- Let C be the Cantor set embedded in $[0, 1]$. The group of homeomorphisms of C that are the identity around 0 and 1 is dissipated.

- Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}$ with euclidean topology and let G be the group of homeomorphisms of $\mathbb{K}$ with support contained in some interval. Then G is dissipated [SV90, Theorem 1.13].

- The group of bijections of $\mathbb{Q}$ with support contained in some open interval is dissipated [SV87, Theorem 3.2].

Following verbatim what was done to prove Mather's Theorem in 3.2, we obtain the following:

Proposition 3.11. *Dissipated groups are acyclic, i.e. if G is dissipated, then we have*

$$H_n(G; \mathbb{Z}) = 0,$$

for every $n \geq 1$.

This result can be seen as a particular case of a more general phenomenon. Namely, dissipated groups fall into the class of *pseudo-mitotic groups*, that have been shown to be acyclic indipendently by Varadarajan [Var85, Theorem 1.7] and Berrick [Ber89]. Berrick in fact uses a different but equivalent definition of *binate group*. For more details about the equivalence of the definitions, see for instance [BV94, Remark 2.3].

Definition 3.12 (Pseudo-mitotic group). Let G a group and H a subgroup. We say that H admits a *pseudo-mitosis* in G if there exist ϕ_0, $\phi_1\colon H \to G$ and $g \in G$ such that for every h, $h' \in H$ it holds:

1. $\phi_0(h) = h\phi_1(h)$;

2. $[h, \phi_1(h')] = e$;

3. $\phi_1(h) = g^{-1}\phi_0(h)g$.

A group G is called *pseudo-mitotic* if every finitely-generated subgroup of G admits a *pseudo-mitosis* in G.

The definition can be roughly stated as follows. Let H be a subgroup of G. Suppose we have a homomorphism

$$\phi_1 \colon H \to G$$

such that the image $\phi_1(H)$ commutes with H inside G. This homomorphism allows us to define another homomorphism

$$\phi_0 \colon H \to G, \quad \phi_0(h) := h\,\phi_1(h).$$

If there exists an element $g \in G$ such that ϕ_0 and ϕ_1 are conjugate by g in G, i.e.,

$$\phi_0(h) = g\,\phi_1(h)\,g^{-1} \quad \text{for all } h \in H,$$

then we say that H has a *pseudo-mitosis* in G.

As said before, we have this result:

Proposition 3.13 ([Ber02, Section 3.1.6] and [SV90, Theorem 1.5])**.** *Dissipated groups are pseudo-mitotic.*

Fournier-Facio, Löh and Moraschini proved the following:

Theorem 3.14 ([FFLM23, Theorem 3.5])**.** *Pseudo-mitotic groups are boundedly acyclic.*

Therefore, we obtain another proof of the fact that $\mathrm{Homeo}_c(\mathbb{R}^n)$ is boundedly acyclic.

One can prove with similar but essentially new techniques the same result for a wider class of space:

Theorem 3.15 ([MN23, Theorem 4.2 and Corollary 4.3])**.** *Let $n \in \mathbb{N}^*$, $r \in \mathbb{N}^* \cup \{+\infty\}$, let M be a closed C^r-manifold and let Z be a manifold C^r-diffeomorfic to $M \times \mathbb{R}^n$. Then the groups $\mathrm{Homeo}_c(Z)$ and $\mathrm{Diff}^r_c(Z)$ are boundedly acyclic.*

Moreover, any (possibly infinite) product of such groups is boundedly acyclic.

4 The bounded cohomology of transformation groups of euclidean spaces: the general case

In this section we show that the main result of the previous chapter, i.e. that $\mathrm{Homeo}_c(\mathbb{R}^n)$ is boundedly acyclic, also holds without the assumption of compact support. As we have seen, compact support was a crucial and necessary hypothesis for the technique previously employed. We now proceed as follows: in the first part, we state and prove a different criterion that ensures bounded acyclicity; in the second part we show how it can be applied to the group $\mathrm{Diff}^r(\mathbb{R}^n)$. Moreover, we see the statement of an analogous criterion that ensures the standard acyclicity of a group and that can be proved with similar ideas.

As we shall see, this new criterion is, at least in principle, independent of the previous result on bounded acyclicity. In other words, one could, conceptually, use this technique to prove the bounded acyclicity of $\mathrm{Diff}^r(\mathbb{R}^n)$ without assuming the bounded acyclicity of $\mathrm{Diff}^r_c(\mathbb{R}^n)$. Nevertheless, when applying the criterion to a given group G, one of the hypotheses requires certain subgroups of G to be boundedly acyclic. In our case, these subgroups turn out to be precisely the groups of compactly supported diffeomorphisms (or products of them). Thus, while the criterion itself is logically independent from the results obtained so far, its application in this setting ultimately relies on them.

These criteria and almost all the content of this section is taken from the paper by Fournier-Facio, Monod and Nariman [FFMN24].

4.1 A criterion for (bounded) acyclicity

In this section, we present the aforementioned criteria, which guarantee that a group G satisfying certain hypotheses is (boundedly) acyclic.

Suppose that G is acting on a set X. We aim to extend this action to a poset $\mathcal{P}$ built from this set. To this end, we consider the set of sequences of elements of X, partially ordered by the *subsequence relation*: a sequence is declared smaller than another whenever it appears as a subsequence of it. We consider the induced action of G on any *G-poset of sequences in X*, i.e. any G-invariant subposet $\mathcal{X}$ of $\mathcal{P}$.

We ask for three conditions:

1. the poset $\mathcal{X}$ satisfies a certain combinatorial property, called *W-property*;
2. the action of G on $\mathcal{X}$ is transitive;
3. one, or equivalently every, stabilizer in G of an element of $\mathcal{X}$ is (boundedly) acyclic.

If these conditions hold, the group G is then (boundedly) acyclic.

The proof goes roughly as follows. When a group G acts on a *simplicial set* $X_\bullet$, it is possible to prove that under certain hypotheses the (bounded) cohomology of G is isomorphic to the (bounded) cohomology of the orbit complex $X_\bullet/G$.

Intuitively, simplicial sets are a combinatorial way to encode informations of an object, as simplicial complexes do. For example, one can treat the homology of a well-behaved topological space X by choosing a simplicial complex K whose geometric realization $|K|$ is homotopy equivalent to X. This allows us to compute the homology of X by working with the simplicial chain complex of K, avoiding the more cumbersome singular chain complex built from all continuous maps from standard simplices into X. Simplicial sets are conceptually analogous to simplicial complexes but allows more freedom in contructing them. We shall see that simplicial sets naturally arise when constructing (co)chain complexes to define homology and (bounded) cohomology. For example, one could define the homology of a group G by taking its *nerve*, a simplicial set whose geometric realization is the classisfying space of G. In the same spirit, we will see how to define bounded cohomology of posets and of monoids.

One of the hypotheses that are required to prove the isomorphism aformentioned between the (bounded) cohomology of G and of the orbit complex $X_\bullet/G$ is that the simplicial set $X_\bullet$ is (boundedly) acyclic. This holds whenever we take the associated simplicial set of a poset satisfying the W-property. Another condition is that the stabilizer of the action of G on $X_\bullet$ is (boundedly) acyclic. This is analogous to what we ask in the hypothesis of our criterion. Lastly, there must be finitely many isomorphism classes of such stabilizers, and this follows from the transitivity of the action of G on $\mathcal{X}$.

To resume, at this point we are in the following situation. The group G acts on a set X and we define a certain poset $\mathcal{X}$ from it. We take the simplicial set $\mathcal{X}_\bullet$ associated to $\mathcal{X}$, and we see that it satisfies the hypothesis for which $H^n_{(b)}(G) \cong H^n_{(b)}(\mathcal{X}_\bullet/G)$.

Now, we show that, regardless the specific group, if G satisfies our hypothesis we obtain an isomorphism

$$H^n_{(b)}(\mathcal{X}_\bullet/G) \cong H^n_{(b)}(\mathrm{Emb}_<(\mathbb{N}^*)_\bullet),$$

for all $n \geq 0$, where $\mathrm{Emb}_<(\mathbb{N}^*)$ is the monoid of *orientation-preserving* of $\mathbb{N}^*$. Here again the transitivity of the action of G on $\mathcal{X}$ will be crucial.

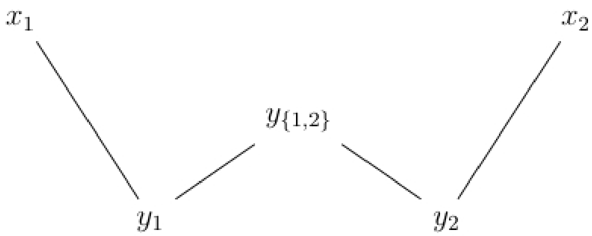
At this point, we have proved that every group satisfying our hypothesis has the same (bounded) cohomology. Thus, it suffices to find a single group satisfying our hypothesis that is boundedly acyclic. To this aim, we use the *group of countably supported bijections of an uncountable set*, that is binate and hence (boundedly) acyclic by Theorem 3.14.

Recall that a *poset* is a pair $\mathcal{P} = (X, \leq)$ where X is a set and $\leq$ is a *partial order* on X i.e. a reflexive, transitive and antisymmetric binary relation.

Definition 4.1 (**W-property**)**.** Let $\mathcal{P}$ be a poset. We say that $\mathcal{P}$ satisfies the W property if the following holds: for every $\mathcal{Q} \subseteq \mathcal{P}$ finite subposet of $\mathcal{P}$ with minimal elements $x^1, x^2, \dots, x^k$, for every $I \subseteq \{1, 2, \dots, k\}$, there exists $y^I \in \mathcal{P}$ such that:

1. if $I \subseteq J$, then $y^I \leq y^J$;
2. if $x \in \mathcal{Q}$ and $x^i \leq x$ for every $i \in I$, then $y^I \leq x$

The following picture exhibits the property in the case $\mathcal{Q} = \{x_1, x_2\}$ and justifies the name.



Source: [FFMN24].

Let G be a group acting on a set X. Consider the poset $\mathcal{P}$ of all sequences $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ of pairwise distinct elements of X, where we say that $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{x}$ is a subsequence of $\mathbf{y}$. Observe that we have an induced order-preserving action of G on such a poset. We call *G-poset of sequences in X* any G-invariant subposet $\mathcal{X}$ of $\mathcal{P}$.

We can now state the main theorem of this section. As said, it is a criterion to show the bounded acyclicity of a group G satisfying certain hypothesis.

Theorem 4.2 (**Criterion for bounded acyclicity** [FFMN24, Theorem 2.2])**.** *Let G be a group acting on a set X and let $\mathcal{X}$ be a G-poset of sequences in X. Suppose that:*

1. *the poset $\mathcal{X}$ satisfies the W-property;*
2. *the induced action of G on $\mathcal{X}$ is transitive;*
3. *the stabilizer of some (or, equivalently, every) sequence $\mathbf{x} \in \mathcal{X}$ is boundedly acyclic;*

Then, the group G is boundedly acyclic.

Remarkably, the same ideas can be applied to prove an analogous criterion for the acyclicity of a group. For completeness we also state this result but we prove the bounded cohomology version only. The proof of the next result follows the same ideas and can be found in the paper of Forunier-Facio, Monod and Nariman [FFMN24, Section 4.5].

Theorem 4.3 (**Criterion for acyclicity** [FFMN24, Theorem 2.3])**.** *Let G be a group acting on a set X and let $\mathcal{X}$ be a G-poset of sequences in X. Suppose that:*

1. *the poset $\mathcal{X}$ satisfies the W-property;*

2. *the induced action of G on $\mathcal{X}$ is transitive;*

3. *the stabilizer of some (or, equivalently, every) sequence $\mathbf{x} \in \mathcal{X}$ is acyclic;*

Then, the group G is acyclic.

We introduce a framework to produce posets satisfying the W-property. Recall that a *preorder* $\lesssim$ is a reflexive and transitive binary relation and that an equivalence relation $\cong$ is *contained in* $\lesssim$ if whenever $x \cong y$, then $x \lesssim y$.

Recall that, given a set endowed with a preorder $(X, \lesssim)$, we say that a sequence $(x_i)_{i\in I}$ of elements of X is *cofinal* if for every $x \in X$ there exists some x_j in the sequence such that $x \lesssim x_j$.

For example, cofinal sequences in $(\mathbb{R}, \leq)$ are precisely those that are unbounded from above (they do not need to be convergent).

Let $(X, \lesssim, \cong)$ be a set endowed with an equivalence relation $\cong$ contained in a preorder relation $\lesssim$. Let $\mathcal{X}$ be the set

$$\mathcal{X} \coloneqq \{\text{cofinal increasing sequences of pairwise inequivelent elements of } X\}.$$

Explicitly, an element of $\mathcal{X}$ is a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

1. **cofinal:** for all $x \in X$, there exists $n \in \mathbb{N}$ such that $x \lesssim x_n$;

2. **increasing:** if $n \leq m$, then $x_n \lesssim x_m$;

3. **pairwise distinct elements:** if $n \neq m$, then $x_n \ncong x_m$.

Observe that if $(x_n)_{n\in\mathbb{N}} \in \mathcal{X}$ and we take a subsequence, the resulting sequence is still an element of $\mathcal{X}$. In other words:

Lemma 4.4. *The set $\mathcal{X}$ is closed under taking subsequences.*

We give a poset structure to $\mathcal{X}$ via the *subsequence partial order*. That means, we define a partial order as follows: given two sequences $\mathbf{x}$, $\mathbf{y} \in \mathcal{X}$, we say $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{x}$ is a subsequence of $\mathbf{y}$.

Proposition 4.5 (Criterion for the W-property). *Let $(X, \lesssim, \cong)$ be a set endowed with an equivalence relation $\cong$ contained in a preorder relation $\lesssim$ and let $\mathcal{X}$ be the poset of cofinal increasing sequences of pairwise inequivalent elements of X with the subsequence partial order. Then, the poset $\mathcal{X}$ satisfies the W-property.*

Proof. Let $\mathcal{Q} \subseteq \mathcal{X}$ be a finite subposet with minimal elements $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$. We have to check that for every $I \subseteq \{1, 2, \ldots, k\}$, there exists $\mathbf{y}^I \in \mathcal{X}$ such that:

1. if $I \subseteq J$, then $\mathbf{y}^I \leq \mathbf{y}^J$;

2. if $\mathbf{x} \in \mathcal{Q}$ and $\mathbf{x}^i \leq \mathbf{x}$ for every $i \in I$, then $\mathbf{y}^I \leq \mathbf{x}$

We construct the various $\mathbf{y}^I$ as subsequences of a single sequence $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$. Denote with $\mathbf{x}^j$ the sequence $(x^j_n)_{n\in\mathbb{N}}$.

We construct a sequence $\mathbf{y}$ such that y_i is an element of $\mathbf{x}^j$ for $j \equiv i$ mod k by induction.

Base step: for $n = 1$, let $y_1 \coloneqq x^1_1$.

Inductive step: to simplify the notation, we show how to choose y_2 and see that the actual inductive step is almost identical, but with more indices to take into account. We choose y_2 among the elements of $\mathbf{x}^2$. Since $\mathcal{Q}$ is a set of cofinal, increasing, pairwise inequivalent sequences, we can select $q \in \mathbb{N}$ such that for every $\mathbf{x} \in \mathcal{Q}$ and for every $r > q$, we have $x_r \ncong y_1$.

Indeed, it may happen that for a given $\mathbf{x} \in \mathcal{Q}$, we have $x_j \cong y_1$ for $j \in \mathbb{N}$, but then we must have $x_r \ncong y_1$ for all $r > j$, since otherwise we would have two equivalent distinct elements. For each $\mathbf{x} \in \mathcal{Q}$, denote by $j_{\mathbf{x}}$ the index j such that $x_j \cong y_1$, and set $j_{\mathbf{x}} = 0$ if $x_r \ncong y_1$ for all $r \geq 0$. Finally, define

$$q \coloneqq \max_{\mathbf{x} \in \mathcal{Q}} j_{\mathbf{x}}.$$

The maximum is achieved since $\mathcal{Q}$ is finite.

Since $\mathbf{x}^2$ is increasing and cofinal, there exists $p > q$ such that $y_1 \lesssim x_p^2$. We set $y_2 \coloneqq x_p^2$. This way, (y_1, y_2) is increasing and $y_1 \ncong y_2$.

The actual inductive step is analogous. Suppose to have defined $y_1, \dots, y_n$, with $y_i \in \mathbf{x}^j$ for $j \equiv i \bmod k$. We choose y_{n+1} among elements of $\mathbf{x}^{j+1}$ or $\mathbf{x}^1$ if $j \equiv 0 \bmod k$. We can again follow the same procedure of before.

This way, $\mathbf{y}$ is by definition an increasing sequence of pairwise distinct elements, and it is cofinal because it shares a subsequence with a cofinal sequence (namely, each of the $\mathbf{x}^i$ for $0 \leq i \leq k$). Therefore $\mathbf{y} \in \mathcal{X}$.

Take $I \subseteq \{1, \dots, k\}$ and define $\mathbf{y}^I$ as the subsequence of $\mathbf{y}$ that selects only those indices that are congruent to an element of I modulo k. For instance, if $I = \{1, \dots, k\}$, then $\mathbf{y}^I = \mathbf{y}$.

Then by definition if $I \subseteq J$ then $\mathbf{y}^I \leq \mathbf{y}^J$.

By construction, it is clear that the second condition of the W-property holds. Indeed, let $\mathbf{x} \in \mathcal{Q}$ with $\mathbf{x}^i \leq \mathbf{x}$ for all $i \in I$. We are constructing $\mathbf{y}$ by interweaving elements of $\mathbf{x}^i$, hence $\mathbf{y} \leq \mathbf{x}$. Since we define $\mathbf{y}^I$ by taking a subsequence of $\mathbf{y}$, we have $\mathbf{y}^I \leq \mathbf{y} \leq \mathbf{x}$. □

4.1.1 Bounded cohomology of simplicial sets

We recall here the notion of *simplicial set* and of its bounded cohomology. For an introduction on theses contructions, we refer to the survey of Frigerio [Fri23], of Ivanov [Iva20], of Li, Moraschini and Raptis [LMR25] and for more details to the book of May [May92].

We start by briefly recalling what a *simplicial complex* is.

Definition 4.6 (Simplicial complex). An *abstract simplicial complex* (which, for simplicity, we will just call a *simplicial complex*) is a pair $K = (V, S)$ consisting of a set V of *vertices* and a set S of *simplices*, where each element of S is a finite subset of V. An element $\sigma \in S$ is called an *n-simplex* if it contains exactly $n+1$ vertices. Moreover, we require the following conditions:

1. Every vertex $v \in V$ determines a 0-simplex, i.e. $\{v\} \in S$;
2. If $\sigma \in S$ and $\sigma' \subseteq \sigma$, then $\sigma' \in S$.

For an n-simplex $\sigma \in S$, its subsets are called *faces*, and in particular the subsets that are $(n-1)$-simplices are called its *boundary faces*.

Given two simplicial complexes $K_1 = (V_1, S_1)$ and $K_2 = (V_2, S_2)$, a *simplicial map* from K_1 to K_2 is a function $f\colon V_1 \to V_2$ such that $f(\sigma) \in S_2$ for every $\sigma \in S_1$.

Example 4.7 (Standard n-simplex). In the notation of before, let $V := \{0, 1, \dots, n\}$ and S consisting of every non-empty subset of V. Then (V, S) is called *standard n-simplex*, and we denote it Δ^n.

To a simplicial complex, which is purely a combinatorial object, one can always associate a topological space, its *geometric realization*.

Definition 4.8 (Geometric realization). The *geometric realization* of a simplicial complex $K = (V, S)$ is a topological space $|K|$ defined as follows. Let I^V be the set of all functions $t \colon V \to I$, so that each element $t \in I^V$ is determined by a family of real numbers $(t_v)_{v \in V}$ with $t_v \in [0, 1]$.

For each n-simplex $\sigma = \{v_0, \dots, v_n\} \in S$, we define

$$|\sigma| := \left\{ t \in I^V \;\middle|\; \sum_{v \in \sigma} t_v = 1 \text{ and } t_v = 0 \text{ for all } v \notin \sigma \right\}.$$

This set sits inside the finite-dimensional vector space $\mathbb{R}^\sigma \cong \mathbb{R}^{n+1}$, and we endow it with the subspace topology inherited from this vector space.

As a set, the geometric realization of K is then defined by

$$|K| := \bigcup_{\sigma \in S} |\sigma| \;\subseteq\; I^V.$$

If K is finite, then $|K|$ is contained in the finite-dimensional vector space $\mathbb{R}^V$, and we endow it with the corresponding subspace topology. In the general case, we define the topology on $|K|$ by declaring a set $U \subseteq |K|$ to be open if and only if $U \cap |\sigma|$ is open in $|\sigma|$ for every $\sigma \in S$.

A *triangulation* of a topological space X is a homeomorphism between X and the geometric realization of a simplicial complex.

For example, the geometric realizations of the standard simplices are as follows:

- the standard 0-simplex: a single point;
- the standard 1-simplex: a line segment;
- the standard 2-simplex: a triangle;

When defining *simplicial homology*, one typically requires the set of vertices V to be totally ordered and considers only those n-simplices corresponding to strictly increasing (and hence non-repeating) $(n+1)$-tuples, leading to the notion of *ordered simplicial complexes*. One then defines appropriate face maps which allows the construction of simplicial homology. Namely, for every $n \geq 1$ and $0 \leq i \leq n$, we define the face map d_i from the set of n-simplices to the set of $(n-1)$-simplices as the map that takes an n-simplex and removes the i-th vertex (for more details see, for instance, [Fri23, Section 2.3]). We denote ordered simplices using square brackets. For example, the *ordered standard n-simplex* is the ordered simplicial complex whose set of vertices is $V = \{0, 1, \dots, n\}$ and whose k-simplices are the $(k+1)$-tuples of vertices of the form $[i_0, \dots, i_k]$ with $0 \leq i_0 < i_1 < \dots < i_k \leq n$.

While simplicial complexes provide a combinatorial way to build topological spaces from simplices, they are somewhat rigid. For instance, it is not straightforward to define the product of two simplicial complexes or to find triangulations of even relatively simple topological spaces (see, for example, [Hat02] for more details). We will not discuss these issues in depth, but they typically arise when working in algebraic topology (see, for example, Friedman's essay on simplicial sets, [Fri23, Chapter 5]). To overcome these limitations and to have a more flexible framework suitable for our purposes, it is useful to work with *simplicial sets*. Simplicial sets generalize simplicial complexes by encoding the combinatorial structure through *face* and *degeneracy maps*.

Definition 4.9 (Simplicial set). A *simplicial set* is a collection of sets $X_\bullet = \{X_0, X_1, X_2, \dots\}$ together with maps between them: *face maps* $d_{n,i}\colon X_n \to X_{n-1}$ for every $n \geq 1$ and for every $0 \leq i \leq n$ and *degeneracy maps* $s_{n,i}\colon X_n \to X_{n+1}$ for every $n \geq 0$ and for every $0 \leq i \leq n$. If n is understood, we simply write d_i and s_i for face and degeneracy maps, respectively. Moreover, we ask for this functions to satisfy some compatibility conditions:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i && \text{if } i < j,\\
d_i s_j &= s_{j-1} d_i && \text{if } i < j,\\
d_j s_j &= d_{j+1} s_j = \mathrm{id}, &&\\
d_i s_j &= s_j d_{i-1} && \text{if } i > j+1,\\
s_i s_j &= s_{j+1} s_i && \text{if } i \leq j.
\end{aligned}$$

A *morphism between simplicial sets* $(X_\bullet, d_\bullet, s_\bullet)$ and $(Y_\bullet, d_\bullet, s_\bullet)$ consists of maps $f_n\colon X_n \to Y_n$ that commute with face and degeneracy maps.

For $n \geq 0$, the elements of X_n are called *n-simplices*, and the elements of X_0 are called *vertices*. An n-simplex can be represented as an $(n+1)$-tuple of vertices, where repetitions are allowed. The face map $d_i\colon X_n \to X_{n-1}$ takes an n-simplex and returns its i-th face, that is, the $(n-1)$-simplex obtained by removing the i-th vertex. On the other hand, $s_i\colon X_n \to X_{n+1}$ takes an n-simplex and gives us back the $(n+1)$-simplex obtained by duplicating the i-th vertex. Sometimes, one has to be careful with the precise definition of the face and degeneracy maps; see, for example, how they are defined in the construction of the *nerve of a group* 4.18.

Thus, contrary to what happens when working with simplicial complex, here we allow simplices with repeated vertices. Whenever a simplex has repeated vertices, we call it *degenerate*. Notice that degenerate simplices are those that can be written as $s_i(x)$ for some n-dimensional simplex x.

Example 4.10. Any simplicial complex can be regarded as a simplicial set, where the n-simplices are the same and the face and degeneracy maps are defined as in Definition 4.9: the face map $d_i\colon X_n \to X_{n-1}$ removes the i-th vertex of an n-simplex, while the degeneracy map $s_i\colon X_n \to X_{n+1}$ duplicates the i-th vertex of an n-simplex.

Example 4.11. Consider the standard 2-simplex with vertices $\{v_0, v_1, v_2\}$. Define two vertex maps $\pi, \widetilde{\pi}$ by

$$\pi(v_0) = v_0, \quad \pi(v_1) = v_1, \quad \pi(v_2) = v_1,$$

and

$$\widetilde{\pi}(v_0) = v_0, \quad \widetilde{\pi}(v_1) = v_1, \quad \widetilde{\pi}(v_2) = v_0.$$

Both maps collapse the geometric realization of the 2-simplex onto its face with vertices $\{v_0, v_1\}$. If we work in the context of simplicial complexes, the images $\pi(\{v_0, v_1, v_2\})$ and $\widetilde{\pi}(\{v_0, v_1, v_2\})$ coincide (they are both the edge $\{v_0, v_1\}$), so the two simplicial maps send the 2-simplex to the same 1-simplex.

If, instead, we work with simplicial sets (where simplices are ordered tuples and repetition are allowed), then the induced maps on the 2-simplex $[v_0, v_1, v_2]$ are

$$\pi([v_0, v_1, v_2]) = [v_0, v_1, v_1], \qquad \widetilde{\pi}([v_0, v_1, v_2]) = [v_0, v_1, v_0],$$

Thus the images of the two maps are different for simplicial sets.

Remark 4.12. We have been slightly sloppy when talking about vertex maps, maps between simplicial complexes, and maps between simplicial sets. When working with simplicial complexes there is no ambiguity: any simplicial map is completely determined by its action on the vertices.

By contrast, this is no longer true for simplicial sets: one also has to specify what happens to simplices of higher degree (see Example 4.13). The image of degenerate simplices, however, is completely determined by the compatibility relation

$$f(s_i(x)) = s_i(f(x)).$$

Similarly, we do not have to specify the image of all faces explicitly, because they are determined by the identities

$$f(d_i(x)) = d_i(f(x)).$$

Thus, we do no not have to specify every single value of a simplicial map, but still it does not suffice, in general, to declare only the image of the vertices.

Nevertheless, ordered simplicial complexes are in fact completely determined by their vertices; thus, when such a complex is regarded as a simplicial set, specifying the image of each vertex is sufficient to determine the entire simplicial map. This is because simplicial maps must commute with the face and degeneracy maps, which uniquely determines the images of all higher-dimensional and degenerate simplices from the images of the vertices. This is the case of Example 4.11 where we are working with ordered simplicial complexes regarded as simplicial sets. For instance,

$$\pi[v_0, v_0, v_1, v_2, v_2] = \pi(s_4 s_0[v_0, v_1, v_2]) = s_4 s_0 \pi[v_0, v_1, v_2] = s_4 s_0[v_0, v_1, v_1] = [v_0, v_0, v_1, v_1, v_1].$$

Here, the action of π on the degenerate simplex $[v_0, v_0, v_1, v_2, v_2]$ is entirely determined by its action on the non-degenerate simplex $[v_0, v_1, v_2]$.

Example 4.13. Consider the standard 1-simplex Δ^1 with vertices $\{v_0, v_1\}$, and the simplicial set X with a single vertex w and two 1-simplices: one given by the degenerate simplex $[w, w]$, and a "loop" σ that starts and ends at w.

Maps $\Delta^1 \to X$ are not determined solely by their action on vertices: both v_0 and v_1 must be sent to w, but the 1-simplex $[v_0, v_1]$ can be sent either to the degenerate simplex $[w, w]$ or to the non-degenerate loop σ, yielding two distinct well-defined simplicial maps.

As with simplicial complexes, it is possible to associate to a simplicial set X, that is a purely combinatorial object, a topological space $|X|$ called *geometric realization of* X.

Let $[n] = \{0, 1, \ldots, n\}$ denote the finite ordered sets. Define the maps

$$D_i \colon [n] \to [n+1], \quad i = 0, \ldots, n,$$

as the order-preserving maps that skip i:

$$D_i([0, \ldots, n]) = [0, \ldots, \hat{i}, \ldots, n+1].$$

Define moreover

$$S_i\colon [n+1] \to [n], \quad i = 0, \dots, n,$$

as the order-preserving maps that duplicate i:

$$S_i([0, \dots, n+1]) = [0, \dots, i, i, \dots, n].$$

Definition 4.14 (Geometric realization). Let $X_\bullet$ be a simplicial set, let $|\Delta^n|$ be the geometric realization of the standard n-simplex and let D_i, S_i as before. Endow X_n with the discrete topology for every $n \geq 0$ and $|\Delta^n|$ with the Eucliden topology. The geometric realization $|X|$ of a simplicial set $X_\bullet$ is given by

$$|X| = \coprod_{n=0}^{\infty} X_n \times |\Delta^n| \Big/ \sim,$$

where $\sim$ is the equivalence relation generated by

$$(x, D_i(p)) \sim (d_i(x), p) \quad \text{for } x \in X_{n+1},\ p \in |\Delta^n|,$$

and

$$(x, S_i(p)) \sim (s_i(x), p) \quad \text{for } x \in X_{n-1},\ p \in |\Delta^n|.$$

Example 4.15. Let $n \in \mathbb{N}$. The sphere S^n can be obtained as the geometric realization of a simplicial set as follows.

For every $m \geq 0$, consider the set of m-simplices in X_m given by elements of the form $[i_0, \dots, i_m]$, where $0 \leq i_0 \leq \cdots \leq i_m \leq n$.

The face maps d_i remove the i-th element, and the degeneracy maps s_i duplicate the i-th vertex.

Another simplicial set yielding the same geometric realization is the following. The only non-degenerate simplices are the 0-simplex $[0]$ and the n-simplex $[0, \dots, n]$; all other simplices in $\widetilde{X}_m$ for $1 \leq m \leq n-1$ are of the form $[0, \dots, 0]$. Face and degeneracy maps are defined as before.

One can check that the geometric realizations $|X|$ and $|\widetilde{X}|$ are indeed (homeomorphic to) the n-sphere.

Definition 4.16 (Connected simplicial complex). Let $X_\bullet$ be a simplicial set. We say that $X_\bullet$ is *connected* if its geometric realization $|X|$ is connected.

Equivalently, $X_\bullet$ is connected if and only if for every vertex $x_1, x_2 \in X_0$, there exists a path of 1-simplices connecting them, that is: the graph $|X_1|$ with set of vertices $\{x_i \mid x_i \in X_0\}$ and set of edges $\{[x_i, x_j] \mid [x_i, x_j] \in X_1\}$ is connected.

Definition 4.17 (Simplicial action). Let G be a group and $X_\bullet$ a simplicial set. We say that G acts *simplicially* on $X_\bullet$ if every $g \in G$ acts by simplicial maps $X_\bullet \to X_\bullet$.

The next example will be of fundamental relevance for what we will do later: given a group G, one can often associate a simplicial set to it, its *nerve*.

Example 4.18 (Nerve of a group). The *nerve* of a group G is the simplicial set $N(G)_\bullet$, defined as follows:

- the n-simplices are $N(G)_n = G^n$, with G^0 being the trivial group $\{e\}$;

- the face maps $d_i : G^n \to G^{n-1}$ are given by

$$d_0(g_1, \ldots, g_n) = (g_2, \ldots, g_n), \quad d_n(g_1, \ldots, g_n) = (g_1, \ldots, g_{n-1}),$$

  and for $0 < i < n$,

$$d_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_n);$$

- the degeneracy maps $s_i : G^n \to G^{n+1}$ are

$$s_0(g_1, \ldots, g_n) = (e, g_1, \ldots, g_n), \quad s_n(g_1, \ldots, g_n) = (g_1, \ldots, g_n, e)$$

  and for $0 < i < n$,

$$s_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_n);$$

The action of G on itself induces a natural simplicial action on $N(G)_\bullet$.

Remark 4.19. There exists an intermediate construction between simplicial complexes and simplicial sets: *semisimplicial sets*, also called *Delta sets* or Δ-*sets*, defined similarly to simplicial sets but without taking into account the degeneracy maps, thus not allowing repeated vertices. This construction yields to the definition of *homology of a semisimplicial set*: as we shall see in Section 4.23, we will construct a differential $\partial_\bullet$ using only the face maps. Nevertheless, some technical problems arise when not allowing degeneracies. For example, only using face maps when defining the nerve of a group lead to degeneracies that should be treated in a different way when working with semisimplicial sets.

Consider for instance the group $G = \mathbb{Z}$ and the 3-simplex $[2, 3, 6] \in N(\mathbb{Z})_3$. Then we have

$$d_1[2, 3, 6] = [6, 6],$$

that is a degenerate simplex! Thus, it is more natural to work with simplicial sets for our purposes.

The notion of nerve can be defined not only for groups, but also for many other mathematical objects. For instance, one can construct nerves for *monoids* and *posets*. The nerve construction provides a systematic way to associate a simplicial set to these structures, encoding their combinatorial and algebraic properties.

Example 4.20 (Nerve of a monoid). Recall that a *monoid* is a triple $(M, \cdot, e)$, where M is a set, $\cdot$ is a binary associative operation, and $e \in M$ is the identity element. In other words, a monoid is like a group but without the requirement that every element has an inverse.

Some examples, besides groups, are:

- the set of positive integers $\mathbb{N}$ with addition;
- the set of square matrices of fixed size with matrix multiplication;
- the set of ordered preserving embeddings of $\mathbb{N}^*$, as we will see in Section 4.40.

The construction presented in Example 4.18 applies verbatim in this context and leads to the definition of *nerve of a monoid*.

Example 4.21 (Nerve of a poset). Let $(\mathcal{P}, \leq)$ be a poset. We can associate to it a simplicial set $N(\mathcal{P})_\bullet$ called *nerve* of $\mathcal{P}$ as follows:

- the set of n-simplices $N(\mathcal{P})_n$ is given by all totally ordered chains of length $n+1$

$$\{x_0 \leq \cdots \leq x_n\}, \qquad x_i \in \mathcal{P};$$

- the i-th face map
$$d_i : N(\mathcal{P})_n \longrightarrow N(\mathcal{P})_{n-1}$$
is defined by deleting the i-th element of the chain:
$$d_i\{x_0 \leq \cdots \leq x_n\} = \{x_0 \leq \cdots \leq \widehat{x_i} \leq \cdots \leq x_n\},$$
where $\widehat{x_i}$ means that x_i is omitted;

- the i-th degeneracy map
$$s_i : N(\mathcal{P})_n \longrightarrow N(\mathcal{P})_{n+1}$$
is defined by duplicating the i-th element of the chain:
$$s_i\{x_0 \leq \cdots \leq x_i \leq \cdots \leq x_n\} = \{x_0 \leq \cdots \leq x_i \leq x_i \leq \cdots \leq x_n\}.$$

In this way, $N(\mathcal{P})_\bullet$ is a well-defined simplicial set. For simplicity, we denote the nerve simply by $\mathcal{P}_\bullet$.

Lemma 4.22. *Let $\mathcal{X}$ be a poset satisfying the W-property. Then, the simplicial set $\mathcal{X}_\bullet$ is connected.*

Proof. We show that every vertex $x_1, x_2 \in X_0$ can be connected by a path in X_1.

Consider the case where x_1 and x_2 are comparable. Without loss of generality, suppose $x_1 \leq x_2$. Then the 1-simplex $\{x_1 \leq x_2\}$ connects them.

Now assume that x_1 and x_2 are incomparable. Then both are minimal elements of the finite subposet $\mathcal{Q} \coloneqq \{x_1, x_2\}$.

By the W-property 4.1, there exist elements $y_{\{1\}}, y_{\{2\}}, y_{\{1,2\}}$ with

$$y_{\{1\}} \leq x_1, \qquad y_{\{2\}} \leq x_2, \qquad y_{\{1\}}, y_{\{2\}} \leq y_{\{1,2\}}.$$

Hence we obtain a path connecting x_1 and x_2. □

As mentioned, we can define the *homology of a simplicial set* by defining a chain complex with a suitable differential.

Definition 4.23 (Homology of a simplicial set). Let $(X_\bullet, d_\bullet, s_\bullet)$ be a simplicial set and let R be a ring. For every $n \geq 0$, let $C_n(X; R)$ be the free R-module generated by X_n. Define the boundary operator

$$\partial_n \colon C_n(X; R) \longrightarrow C_{n-1}(X; R), \qquad \partial_n \sigma \coloneqq \sum_{i=0}^{n} (-1)^i d_i \sigma.$$

This gives a chain complex $(C_\bullet(X; R), \partial_\bullet)$ called the *simplicial chain complex of $X_\bullet$*. The *homology of the simplicial set $X_\bullet$* is defined as

$$H_n(X_\bullet; R) \coloneqq H_n(C_\bullet(X; R)).$$

When working with real coefficients, we can endow the simplicial chain complex with the ℓ^1-norm via

$$\|\sum_k a_k\sigma_k\| := \sum_k |a_k|$$

for every $\sum_k a_k\sigma_k \in C_n(X_\bullet;\mathbb{R})$.

From now on, we will omit the coefficients implying that we are working with real ones.

For convenience, we will work with *reduced* homology of simplicial sets. This means that we take the reduced chain complex $\widetilde{C}_\bullet(X_\bullet)$, defined as usual in positive degrees, and we set $\widetilde{C}_{-1}(X_\bullet) := \mathbb{R}$, with the differential $\partial_0\colon \widetilde{C}_0(X_\bullet) \to \widetilde{C}_{-1}(X_\bullet)$ sending every vertex to $1 \in \mathbb{R}$. The *reduced homology* $\widetilde{H}_\bullet(X_\bullet)$ is the homology of this chain complex.

Definition 4.24 (**Bounded homotopy**)**.** Let $f, g\colon X_\bullet \to Y_\bullet$ be two simplicial maps between simplicial sets. A *bounded homotopy* between f and g is a sequence of bounded linear maps $h_p\colon \widetilde{C}_p(X_\bullet) \to \widetilde{C}_{p+1}(Y_\bullet)$ for all $p \geq 0$ such that, for all $p \in \mathbb{N}^*$,

$$d_1h_0 = f_0 - g_0, \qquad d_{p+1}h_p + h_{p-1}d_p = f_p - g_p.$$

We say that f and g are *boundedly homotopic* if there exists a bounded homotopy between them.

Two simplicial sets are *boundedly homotopy equivalent* if there exist simplicial maps $f\colon X_\bullet \to Y_\bullet$ and $g\colon Y_\bullet \to X_\bullet$ such that $f\circ g$ and $g \circ f$ are both boundedly homotopic to the respective identity maps.

Definition 4.25 (**Bounded cohomology of a simplicial set**)**.** Given a simplicial set $(X_\bullet, d_\bullet, s_\bullet)$, we can define its bounded cohomology $H_b^\bullet(X_\bullet)$ with real coefficients as the cohomology of the cochain complex

$$0 \longrightarrow \ell^\infty(X_0) \xrightarrow{d^0} \ell^\infty(X_1) \xrightarrow{d^1} \ell^\infty(X_2) \xrightarrow{d^2} \cdots$$

where the differentials $d^n\colon \ell^\infty(X_n) \to \ell^\infty(X_{n+1})$ are obtained by dualizing the face maps.

A simplicial set is said *boundedly acyclic* if its bounded cohomology with real coefficients vanishes for every positive degree.

Example 4.26. Let G be a group acting simplicially on its nerve $N(G)_\bullet$ as in Example 4.18. Then the bounded cohomology of G is precisely the bounded cohomology of the simplicial sets of orbits $N(G)_\bullet/G$.

More generally, whenever we have a group G acting simplicially on a simplicial set $X_\bullet$ satisfying certain additional hypotheses, we have that the simplicial set of orbits computes the bounded cohomology of G. More precisely, we have the following:

Theorem 4.27 ([MN23, Theorem 3.3])**.** *Let G be a group acting on a connected, boundedly acyclic simplicial set $X_\bullet$. Suppose that, for all $p \in \mathbb{N}$, we have:*

1. *the stabilizer of X_p is boundedly acyclic;*
2. *there are only finitely many isomorphism classes of such stabilizers.*

Then, we have an isomorphism $H_b^n(G) \cong H_b^n(X_\bullet/G)$.

Definition 4.28 (**Uniform acyclity**)**.** Let $X_\bullet$ be a simplicial set. We say that $X_\bullet$ is *uniformly acyclic* if it is $\mathbb{R}$-acyclic (i.e. $\widetilde{H}_n(X_\bullet; \mathbb{R}) = 0$ for all $n \geq 0$) and it satisfies the q-UBC condition (Definition 1.28) for all $q \geq 0$.

By Theorem 1.36,the following lemma is straightforward:

Lemma 4.29. *If a simplicial set $X_\bullet$ is uniformly acyclic, then it is boundedly acyclic.*

Example 4.30 (**Cone over a simplicial set**)**.** Let $(X_\bullet, d_\bullet, s_\bullet)$ be a simplicial set. The *cone over* $X_\bullet$ is the simplicial set $CX_\bullet$ obtained by adding a new vertex $v \notin X_0$ and connecting every simplex of $X_\bullet$ to v.

More precisely, an n-simplex of $CX_\bullet$ is a tuple of the form

$$(x, \underbrace{v, \dots, v}_{q \text{ times}}), \quad x \in X_{n-q},\ 0 \leq q \leq n.$$

If $x \in X_{n-q}$ is non-degenerate, then the non-degenerate simplices of $CX_\bullet$ are:

- those with $q = 0$, i.e., the original simplices of $X_\bullet$;
- those with $q = 1$, i.e., simplices obtained by connecting x to the new vertex v.

The face maps $d_i \colon CX_n \to CX_{n-1}$ are defined by

$$d_i(x, \underbrace{v, \dots, v}_{q}) = \begin{cases} (x, \underbrace{v, \dots, v}_{q-1}), & 0 \leq i < q, \\ (d_{i-q}x, \underbrace{v, \dots, v}_{q}), & q \leq i \leq n. \end{cases}$$

Degeneracy maps $s_i \colon CX_n \to CX_{n+1}$ are defined similarly by inserting a repeated vertex either in x or in the copies of v. Precisely:

$$s_i\Big(x, \underbrace{v, \dots, v}_{q}\Big) = \begin{cases} \Big(x, \underbrace{v, \dots, v}_{q+1}\Big), & 0 \leq i < q, \\ \Big(s_{i-q}x, \underbrace{v, \dots, v}_{q}\Big), & q \leq i \leq n. \end{cases}$$

Analogously to what happens in the standard cohomology setting, one can prove that a *cone over a simplex* is uniformly acyclic.

Lemma 4.31 ([KS23, Theorem 7.18])**.** *Let $X_\bullet$ be a simplicial set and $CX_\bullet$ the cone over it. Then $CX_\bullet$ is uniformly acyclic with constants for the UBC bounded by* 1.

Definition 4.32 (**Uniformly acyclic map**)**.** Let $X_\bullet, Y_\bullet$ simplicial sets and $f \colon X_\bullet \to Y_\bullet$ a simplicial map. We say that f is *uniformly acyclic* if for every $q \geq 0$, there exists a constant $K_q^f > 0$ such that for every cycle $z \in \widetilde{C}_q(X_\bullet)$, there exists $c \in \widetilde{C}_{q+1}(Y)$ such that $\partial_{q+1} c = f_q(z)$ and $\|c\| \leq K_q^f \|z\|$.

If it is clear from the context, we will denote the constant K_q^f simply by K.

A *cycle* $z \in \widetilde{C}_q(X_\bullet)$ is a *finite linear combination* of q-simplices in X_q, that is,

$$z = \sum_{i=1}^{N} a_i \, \sigma_i, \qquad \sigma_i \in X_q,\ a_i \in \mathbb{R},$$

such that $\partial z = 0$.

Notice that even if $X_\bullet$ is infinite, each chain involves only finitely many simplices. Therefore, to check whether the UBC holds for $X_\bullet$, it suffices to verify it on all *finite subsimplicial sets* $Y_\bullet \subseteq X_\bullet$ containing the simplices appearing in the cycle. More precisely, it holds the following:

Lemma 4.33. *Let $X_\bullet$ be a simplicial set such that every finite subsimplicial set $Y_\bullet \subseteq X_\bullet$ is uniformly acyclic with constants for the UBC bounded by $K > 0$. Then $X_\bullet$ is uniformly acyclic with constants for the UBC bounded by K.*

Lemma 4.34. *Let $X_\bullet, Y_\bullet$ be simplicial sets and let $f\colon X_\bullet \to Y_\bullet$ be a simplicial map that factors through the simplicial cone $CX_\bullet$ of $X_\bullet$, i.e. there exists a simplicial map $\tilde{f}\colon CX_\bullet \to Y_\bullet$ such that the following diagram commutes:*

$$\begin{tikzcd} X_\bullet \arrow[r, hook, "i"] \arrow[dr, "f"'] & CX_\bullet \arrow[d, "\tilde{f}"] \\ & Y_\bullet \end{tikzcd}$$

where $i\colon X_\bullet \hookrightarrow CX_\bullet$ is the canonical inclusion. Then f is uniformly acyclic with constant bounded by 1.

Lemma 4.35 ([FFMN24, Lemma 4.9])**.** *Let $X_\bullet, Y_\bullet$ be simplicial sets, and let $f, g\colon X_\bullet \to Y_\bullet$ be simplicial maps. Suppose that f and g are boundedly homotopic via a family of maps $\{h_p\}_{p\geq 0}$.*

Then f is uniformly acyclic if and only if g is uniformly acyclic. Moreover, the constants satisfy

$$K_p^g \leq K_p^f + \|h_p\|$$

for all $p \geq 0$.

Definition 4.36 (**Carrier**)**.** Let $X_\bullet$ and $Y_\bullet$ be simplicial sets. A *carrier* ϕ is a map that sends simplices $\sigma \in X_p$ to subsimplicial sets of $Y_\bullet$ and such that $\phi(\sigma) \subseteq \phi(\tau)$ whenever $\sigma \subseteq \tau$ (i.e. σ is a face of τ).

A carrier ϕ is called *uniformly acyclic* if $\phi(\sigma)$ is a uniformly acyclic simplicial set for every $\sigma \in X_p$. Moreover, we ask for the constants $K_p^{\phi(\sigma)}$ to be bounded in every degree by a constant K_p that does not depend on σ.

If $f\colon X_\bullet \to Y_\bullet$ is a simplicial map and ϕ is a carrier, we say that ϕ is a *carrier for f* if $f(\sigma) \subseteq \phi(\sigma)$ for every $\sigma \in X_\bullet$.

Carriers are a useful tool to prove that maps between simplicial sets are boundedly homotopic, as shown by the following proposition.

Lemma 4.37 ([FFMN24, Lemma 4.11])**.** *Let $X_\bullet$ and $Y_\bullet$ be simplicial sets, and let $f, g\colon X_\bullet \to Y_\bullet$ be simplicial maps.*

If there exists a uniformly acyclic common carrier ϕ for f and g, then f and g are boundedly homotopic.

Moreover, if $\{K_p\}_{p\in\mathbb{N}}$ are the constants witnessing the uniform acyclicity of ϕ, and if $1 \leq K_p \leq K_{p+1}$ for every p, then the bounded homotopy $\{h_p\}_{p\in\mathbb{N}}$ satisfies

$$\|h_p\| \leq 2(p+1)K_p^p \qquad \textit{for every } p \in \mathbb{N}.$$

When working with posets, there are relatively simple conditions that allow one to apply this result. In particular, we have the following bounded version of Quillen's Order Homotopy Theorem [Bjo95, Theorem 10.11].

Theorem 4.38 (Bounded Version of Quillen's Order Homotopy Theorem). *Let $X_\bullet$ be a simplicial set, let $(\mathcal{P}, \leq)$ be a poset, and let $f, g\colon X_\bullet \to \mathcal{P}_\bullet$ be simplicial maps, where $\mathcal{P}_\bullet$ is the nerve of $\mathcal{P}$ as in Example 4.21.*

If $f(x) \leq g(x)$ for every vertex $x \in X_0$, then there exists a bounded homotopy $\{h_p\}_{p\in\mathbb{N}}$ between f and g such that

$$\|h_p\| \leq 2(p+1) \qquad \textit{for all } p \in \mathbb{N}.$$

Proof. We want to apply Lemma 4.37, so we look for a common uniformly acyclic carrier ϕ for f and g.

Let ϕ be the map that associate to $\sigma \in X_p$ the subsimplicial set generated by $f(\sigma) \cup g(\sigma)$ for all $p \in \mathbb{N}$. Since f and g are simplicial maps, if $\sigma \subseteq \tau$ then $\phi(\sigma) \subseteq \phi(\tau)$ and clearly $f(\sigma), g(\sigma) \subseteq \phi(\sigma)$, thus ϕ is a carrier. We have to prove the uniform acyclicity, i.e. that $\phi(\sigma)$ is uniformly acyclic for every $\sigma \in X_p$, for every $p \in \mathbb{N}$. Let v be the minimal element of $f(\sigma)$. Then v is lower or equal to every vertex of $\phi(\sigma)$, that is: $\phi(\sigma)$ is a cone. Thus, by Lemma 4.31, $\phi(\sigma)$ is uniformly acyclic with constants bounded by 1, hence we have the thesis by Lemma 4.37. □

The reason why we introduced posets and their bounded cohomology lies in the next theorem. To apply Theorem 4.27, we need the simplicial set we are working with to be boundedly acyclic. It turns out that this condition is satisfied if we work with the nerve of a poset satisfying the W-property.

Theorem 4.39. *Let $\mathcal{P}$ be a poset, and let $\mathcal{P}_\bullet$ be its nerve. If $\mathcal{P}$ satisfies the W-property, then $\mathcal{P}_\bullet$ is acyclic and uniformly acyclic.*

Recall that we defined uniform acyclicity using real coefficients. Therefore, uniform acyclicity does not imply (integral) acyclicity, which is defined with integer coefficients.

Proof. We prove here the uniform acyclity using the Bounded Version of Quillen's Order Homotopy Theorem 4.38. The acyclity follows from the same arguments, without taking track of the norms and using the Quillen's Order Homotopy Theorem [Bjo95, Theorem 10.11].

Let $\mathcal{Q}$ be a finite subposet of $\mathcal{P}$. Let $\{x_1, \dots x_k\}$ be the minimal elements of $\mathcal{Q}$ and let y^I be the element whose existence comes from the W-property for all $I \subseteq \{x_1, \dots, x_k\}$. We want to show that the inclusion $i\colon \mathcal{Q} \to \mathcal{P}$ is uniformly acyclic to apply Lemma 4.33. Define the map

$$f\colon Q \longrightarrow P, \qquad x \longmapsto y_{I_x},$$

where

$$I_x = \{\, i \in \{1, \dots, k\} \mid x_i \leq x \,\}.$$

By Theorem 4.38, the map f is boundedly homotopic to the inclusion i via $\{h_p\}_{p\in\mathbb{N}}$ such that $\|h_p\| \leq 2(p+1)$ for all $p \in \mathbb{N}$. In fact, by definition of the W-property 4.1, we have:

1. the map f is order preserving since $f(y^I) \leq f(y^J)$ whenever $I \subseteq J$;
2. $y_{I_x} \leq x$ for every $x \in \mathcal{Q}$.

Thus, we obtain $f(x) \leq x$ for every $x \in \mathcal{Q}$.

Moreover, every element of $f(\mathcal{Q})$ is dominated by the element $y_{\{1,\dots,k\}}$, that means: $f(\mathcal{Q}_\bullet)$ is contained in a cone. Therefore, by Lemma 4.34, we obtain that f is uniformly acyclic with constant bounded by 1. Thus by Lemma 4.35, this shows that i is boundedly acyclic with constant bounded by $1 + 2(p+1)$, which is independent of $\mathcal{Q}$. We conclude by applying Lemma 4.33. □

4.1.2 The orbit complex and the embedding monoid

In this section we work in the setting of Theorem 4.2, namely the setting of the criterion we want to prove. We have a group action of G on a set X and a G-poset $\mathcal{X}$ of sequences in X. This poset satisfies the W-property, thus by Theorem 4.39 it is acyclic and uniformly acyclic. The induced action of G on $\mathcal{X}$ is transitive and every stabilizer is (boundedly) acyclic. In this section we show that the (bounded) cohomology of G is the same as the one of the *embedding monoid* $\mathrm{Emb}_<(\mathbb{N}^*)$. To prove it, we rely on Theorem 4.27 which states that, under certain hypothesis, we have isomorphisms $H_b^n(G) \cong H_b^n(\mathcal{X}_\bullet/G)$ for every $n \in \mathbb{N}$. Thus, working with the orbit complex $\mathcal{X}_\bullet/G$, we will show that it is isomorphic to the nerve of the *embedding monoid*.

Definition 4.40 (Embedding monoid). We define the *embedding monoid* $\mathrm{Emb}_<(\mathbb{N}^*)$ as the monoid of orientation-preserving embeddings $\mathbb{N}^* \to \mathbb{N}^*$, where the operation is given by the right action of $\mathrm{Emb}_<(\mathbb{N}^*)$ on $\mathbb{N}^*$. Precisely, given $\eta_1, \eta_2 \in \mathrm{Emb}_<(\mathbb{N}^*)$, their product $\eta_1\eta_2$ is the embedding obtained applying η_1 and then η_2.

Recall that, by Example 4.20, it is always possible to associate a simplicial set to a monoid by taking its nerve. Let us denote the nerve of $\mathrm{Emb}_<(\mathbb{N}^*)$ by $\mathrm{Emb}_<(\mathbb{N}^*)_\bullet$.

We want to define a simplicial map

$$I_\bullet \colon \mathcal{X}_\bullet \to \mathrm{Emb}_<(\mathbb{N}^*)_\bullet$$

and show that it induces a simplicial isomorphism

$$I_\bullet \colon \mathcal{X}_\bullet/G \xrightarrow{\cong} \mathrm{Emb}_<(\mathbb{N}^*)_\bullet.$$

Let $p \in \mathbb{N}$. A p-simplex in $\mathcal{X}_p$ is a chain of $p+1$ sequences of pairwise distinct elements of X of the form $\{x_0 \leq \cdots \leq x_p\}$, where $x_i \leq x_{i+1}$ means that x_i is a subsequence of x_{i+1}.

Define the *index map*

$$\iota_i \colon x_i \to \mathbb{N}^*, \quad x_i^{(j)} \mapsto j,$$

which associates to each element of the sequence $x_i = (x_i^{(j)})_{j \in \mathbb{N}^*}$ its index. Since x_{i-1} is a subsequence of x_i, the sequence x_{i-1} is entirely determined by which indices it takes from x_i. That is, once x_i is given, we can recover x_{i-1} in terms of an order-preserving embedding $\eta_i \in \mathrm{Emb}_<(\mathbb{N}^*)$ such that the following diagram commutes

$$\begin{tikzcd} x_{i-1} \arrow[r, hook] \arrow[d, "\iota_{i-1}"] & x_i \arrow[d, "\iota_i"] \\ \mathbb{N}^* \arrow[r, hook, "\eta_i"] & \mathbb{N}^*. \end{tikzcd}$$

Example 4.41. Let $X = \mathbb{R}$. Consider the sequence $x_i = (1, 2, 3, \dots)$ of positive integers. The map ι_i sends $n \in x_i$ to $n \in \mathbb{N}^*$. Consider the order-preserving embedding

$$\eta_i \colon \mathbb{N}^* \to \mathbb{N}^*, \quad m \mapsto 2m.$$

The sequence x_{i-1} is the sequence $(2, 4, 6, \dots)$ of even positive numbers.

Proceeding inductively, we obtain that a p-simplex $\{x_0 \leq \cdots \leq x_p\}$ can be given specifying the greatest sequence x_p and prescribing p order-preserving embeddings of $\mathbb{N}^*$ such that the following diagram commute:

$$\begin{tikzcd}
x_0 \arrow[r, hook] \arrow[d, "\iota_0"'] & x_1 \arrow[r, hook] \arrow[d, "\iota_1"'] & \cdots \arrow[r, hook] & x_{p-1} \arrow[r, hook] \arrow[d, "\iota_{p-1}"'] & x_p \arrow[d, "\iota_p"] \\
\mathbb{N}^* \arrow[r, hook, "\eta_1"] & \mathbb{N}^* \arrow[r, hook, "\eta_2"] & \cdots \arrow[r, hook] & \mathbb{N}^* \arrow[r, hook, "\eta_p"] & \mathbb{N}^*.
\end{tikzcd}$$

We define the map I_p as

$$I_p \colon \mathcal{X}_p \longrightarrow \mathrm{Emb}_<(\mathbb{N}^*)_p, \qquad \{x_0 \leq \cdots \leq x_p\} \longmapsto (\eta_1, \dots, \eta_p).$$

This map is clearly not a bijection: as we already observed, one would also need to specify the greatest element of each sequence in order to fully recover it. Nevertheless, this map descends to a map on the quotient $\mathcal{X}_p/G \to \mathrm{Emb}_<(\mathbb{N}^*)_p$ and we prove that here we obtain a bijection. We still denote this induced map by I_p.

Lemma 4.42. *The map $I_p \colon \mathcal{X}_p \to \mathrm{Emb}_<(\mathbb{N}^*)_p$ induces a bijection $\mathcal{X}_p/G \to \mathrm{Emb}_<(\mathbb{N}^*)_p$ for every $p \geq 0$.*

Proof. To prove that I_p descends to a map on the quotient, we have to check that I_p is constant on the orbits, i.e. that for every $\{x_0 \leq \cdots \leq x_p\} \in \mathcal{X}_p$ and for every $g \in G$, we have $I_p(g \cdot \{x_0 \leq \cdots \leq x_p\}) = I_p(\{x_0 \leq \cdots \leq x_p\})$. This is true since the action of G is order-preserving.

To prove that $I_p \colon \mathcal{X}_p/G \to \mathrm{Emb}_<(\mathbb{N}^*)_p$ is a bijection, we have to prove

1. **Surjectivity:** Let $(\eta_1, \dots, \eta_p) \in \mathrm{Emb}_<(\mathbb{N}^*)_p$. Take any sequence x_p and define x_{p-1} as the subsequence of x_p determined by the indices prescribed by η_p. Proceeding inductively, we construct a sequence $\{x_0 \leq \cdots \leq x_p\}$ whose image under I_p is, by construction, $(\eta_1, \dots, \eta_p) \in \mathrm{Emb}_<(\mathbb{N}^*)_p$.

2. **Injectivity:** Suppose that $\{x_0 \leq \cdots \leq x_p\}$ and $\{y_0 \leq \cdots \leq y_p\} \in \mathcal{X}_p$ have the same image under I_p. By the transitivity of G on $\mathcal{X}$, there exists $g \in G$ such that $g \cdot x_p = y_p$. The indices defining y_{p-1} as a subsequence of y_p are the same defining $g \cdot x_{p-1}$ as a subsequence of $g \cdot x_p = y_p$ since, again, the action is orientation-preserving. Thus, $g \cdot x_{p-1} = y_{p-1}$. Proceeding inductively, we obtain

$$g \cdot \{x_0 \leq \cdots \leq x_p\} = \{y_0 \leq \cdots \leq y_p\},$$

hence this elements represent the same equivalence class. □

We have now bijections in every degree of the simplicial sets $\mathcal{X}_\bullet/G$ and $\mathrm{Emb}_<(\mathbb{N}^*)_\bullet$. To assemble them into a single simplicial map, it remains to check that they commute with the face maps and the degeneracy maps.

Lemma 4.43. *The map*

$$I_\bullet\colon \mathcal{X}_\bullet/G \longrightarrow \mathrm{Emb}_<(\mathbb{N}^*)_\bullet$$

is a simplicial map.

Proof. We have to show that for every $p \geq 1$ and every $0 \leq i \leq p$,

$$I_{p-1} \circ d_i = d_i \circ I_p.$$

and that for every $p \geq 0$ and every $0 \leq i \leq p$,

$$I_{p+1} \circ s_i = s_i \circ I_p.$$

Consider a p-simplex $\{x_0 \leq \cdots \leq x_p\} \in \mathcal{X}_p$. We split into three cases:

1. **Case $i = 0$:** For the face maps we have:

$$\begin{aligned} I_{p-1}\Big(d_0(\{x_0 \leq \cdots \leq x_p\})\Big) &= I_{p-1}(\{x_1 \leq \cdots \leq x_p\}) \\ &= (\eta_2, \ldots, \eta_p) \\ &= d_0(\eta_1, \ldots, \eta_p) = d_0\Big(I_p(\{x_0 \leq \cdots \leq x_p\})\Big), \end{aligned}$$

while for the degeneracy maps:

$$\begin{aligned} I_{p+1}\Big(s_0(\{x_0 \leq \cdots \leq x_p\})\Big) &= I_{p+1}(\{x_0 \leq x_0 \leq \cdots \leq x_p\})\Big) \\ &= (\eta_1, e, \ldots \eta_p) \\ &= s_0\Big(I_p(\{x_0 \leq \cdots \leq x_p\})\Big). \end{aligned}$$

2. **Case $i = p$:** For the face maps we have:

$$\begin{aligned} I_{p-1}\Big(d_p(\{x_0 \leq \cdots \leq x_p\})\Big) &= I_{p-1}(\{x_0 \leq \cdots \leq x_{p-1}\}) \\ &= (\eta_1, \ldots, \eta_{p-1}) \\ &= d_p(\eta_1, \ldots, \eta_p) = d_p\Big(I_p(\{x_0 \leq \cdots \leq x_p\})\Big), \end{aligned}$$

while for the degeneracy maps:

$$\begin{aligned} I_{p+1}\Big(s_p(\{x_0 \leq \cdots \leq x_p\})\Big) &= I_{p+1}(\{x_0 \leq \cdots \leq x_p \leq x_p\}) \\ &= (\eta_1, \ldots, \eta_{p-1}, \eta_p, e) \\ &= s_p(\eta_1, \ldots, \eta_{p-1}, \eta_p) = s_p\Big(I_p(\{x_0 \leq \cdots \leq x_p\})\Big). \end{aligned}$$

3. **Case $0 < i < p$:**

$$\begin{aligned} I_{p-1}\Big(d_i(\{x_0 \leq \cdots \leq x_p\})\Big) &= I_{p-1}(\{x_0 \leq \cdots \leq x_{i-1} \leq x_{i+1} \leq \cdots \leq x_p\}) \\ &= (\eta_1, \ldots, \eta_{i-1}, \eta_i\eta_{i+1}, \eta_{i+2}, \ldots, \eta_p) \\ &= d_i(\eta_1, \ldots, \eta_p) = d_i\Big(I_p(\{x_0 \leq \cdots \leq x_p\})\Big). \end{aligned}$$

while for the degeneracy maps:

$$\begin{aligned}
I_{p+1}\Big(s_i(\{x_0 \le \cdots \le x_i \le \cdots \le x_p\})\Big) &= I_{p+1}(\{x_0 \le \cdots \le x_i \le x_i \le \cdots \le x_p\}) \\
&= (\eta_1, \ldots, \eta_i, e, \eta_{i+1}, \ldots, \eta_p) \\
&= s_i(\eta_1, \ldots, \eta_i, \ldots, \eta_p) \\
&= s_i\Big(I_p(\{x_0 \le \cdots \le x_p\})\Big). \quad \square
\end{aligned}$$

To summarize, we have obtained:

Lemma 4.44. *The simplicial sets $\mathcal{X}_\bullet/G$ and $\mathrm{Emb}_<(\mathbb{N}^*)_\bullet$ are isomorphic. In particular, there is an isomorphism in bounded cohomology:*

$$H_b^n(\mathcal{X}_\bullet/G) \cong H_b^n(\mathrm{Emb}_<(\mathbb{N}^*)_\bullet), \quad \textit{for all } n \ge 0.$$

As we announced, we obtain that in this setting, whenever the group we are working with satisfies our hypothesis, its bounded cohomology is the same as the bounded cohomology of the embedding monoid. Precisely:

Proposition 4.45. *Let G be a group acting on a set X and let $\mathcal{X}$ be a G-poset of sequences in X. Suppose that:*

1. *the poset $\mathcal{X}$ satisfies the W-property;*
2. *the induced action of G on $\mathcal{X}$ is transitive;*
3. *the stabilizer of some (or, equivalently, every) sequence $\mathbf{x} \in \mathcal{X}$ is boundedly acyclic.*

Then, we have isomorphisms in all degrees $n \ge 0$

$$H_b^n(G) \cong H_b^n(\mathrm{Emb}_<(\mathbb{N}^*)_\bullet).$$

Proof. By Lemma 4.44, we have $H_b^n(\mathcal{X}_\bullet/G) \cong H_b^n(\mathrm{Emb}_<(\mathbb{N}^*)_\bullet)$ for all $n \ge 0$. Hence, it suffices to prove that $H_b^n(G) \cong H_b^n(\mathcal{X}_\bullet/G)$ for all $n \ge 0$. To this end we verify the hypotheses of Theorem 4.27:

1. **$\mathcal{X}_\bullet$ is boundedly acyclic and connected.**

   The poset set $\mathcal{X}$ satisfies the W-property; hence, by Theorem 4.39, its nerve $\mathcal{X}_\bullet$ is uniformly acyclic. Then, by Lemma 4.29, it is boundedly acyclic. Moreover, the simplicial set $\mathcal{X}_\bullet$ is connected by Lemma 4.22.

2. **The stabilizer of each p-simplex is boundedly acyclic.**

   The stabilizer of a chain $\{x_0 \le \cdots \le x_p\}$ coincides with the stabilizer of x_p, since x_i are subsequences of x_p for $0 \le i \le p$. By hypothesis, the stabilizer of x_p is boundedly acyclic.

3. **There exists only finitely many isomorphism classes of such stabilizers.**

   By transitivity of the action, all stabilizers are conjugate, so there is only one isomorphism class. $\square$

The last statement justify why in the hypothesis of our criterion 4.2 it is equivalent to require either that just one or every orbit are boundedly acyclic: when we look at the action on the resulting simplicial set, they are all conjugated.

4.1.3 Proof of the criterion

Recall the setting we are working with. We have a group G acting on a set X, and there exists a G-poset $\mathcal{X}$ of sequences in X such that the following properties hold:

1. the poset $\mathcal{X}$ satisfies the W-property;
2. the induced action of G on $\mathcal{X}$ is transitive;
3. the stabilizer of some (equivalently, every) sequence $\mathbf{x} \in \mathcal{X}$ is boundedly acyclic.

Let $\mathrm{Emb}_<(\mathbb{N}^*)$ be the monoid of orientation-preserving embeddings $\mathbb{N}^* \to \mathbb{N}^*$. Its bounded cohomology is defined as the bounded cohomology of its nerve, which is a simplicial set.

In Proposition 4.45 we proved that, under these assumptions, the bounded cohomology of the group G and of $\mathrm{Emb}_<(\mathbb{N}^*)$ coincide. This means that every group satisfying our hypotheses has the same bounded cohomology, hence to prove the criterion it suffices to find one explicit boundedly acyclic group fitting into this framework.

In this section, we show that taken an uncountable set X and the group G of countably supported bijections, then G satisfies our hypothesis and it is binate, thus boundedly acyclic by Theorem 3.14.

Theorem 4.46. *Let G be the group of countably supported bijections of an uncountable set X. Then G is binate and, in particular, it is boundedly acyclic*

Proof. We have to show that for every finitely generated subgroup H of G, there exist two homomorphisms $\phi_0, \phi_1 \colon H \to G$ and an element $g \in G$ such that for every h, h' it holds:

1. $\phi_0(h) = h\phi_1(h)$;
2. $[h, \phi_1(h')] = e$;
3. $\phi_1(h) = g^{-1}\phi_0(h)g$.

Since every element of H has countable support and H is finitely generated, there exists a countable subset $Y_0 \subseteq X$ such that H is supported on Y_0. Moreover, by uncountability of X, we can choose pairwise disjoint countable sets Y_i disjoint from Y_0 for every $i \in \mathbb{Z}^*$.

For every $i \in \mathbb{Z}$, fix bijections $\pi_i \colon Y_{i-1} \longrightarrow Y_i$, and let

$$\Pi_i := \pi_i \circ \pi_{i-1} \circ \cdots \circ \pi_1 \colon Y_0 \longrightarrow Y_i.$$

Now define $\phi_1 \colon H \longrightarrow G$ by setting, for all $h \in H$ and $x \in X$,

$$\phi_1(h)(x) := \begin{cases} \Pi_i \, h \, \Pi_i^{-1}(x), & \text{if } x \in Y_i, i \geq 1 \\ x, & \text{otherwise.} \end{cases}$$

For every $h \in H$, $\phi_1(h)$ is supported on $\bigcup_{i\geq 1} Y_i$. Hence, $\phi_1(h)$ is an element of G. We then set $\phi_0(h) := h\phi_1(h)$ for every $h \in H$.

Consider the following element $g \in G$: for every $x \in X$, we define

$$g(x) := \begin{cases} \pi_i(x), & \text{if } x \in Y_{i-1}, \\ x, & \text{otherwise.} \end{cases}$$

Then g is indeed an element of G because it is supported on $\bigcup_{i\in\mathbb{Z}} Y_i$.

Since the sets Y_i are pairwise disjoint, we can prove that $[h, \phi_1(h')] = e$ for every $h, h' \in H$.

Indeed, let $x \in X$ and let us consider three cases:

- If $x \notin \bigcup_{i\geq 0} Y_i$: neither h nor $\phi_1(h')$ (nor their inverses) move x, so
$$h\phi_1(h')(x) = \phi_1(h')h(x) = x.$$
- If $x \in Y_0$: $\phi_1(h')$ acts trivially on Y_0, hence
$$h\phi_1(h')(x) = h(x) = \phi_1(h')h(x).$$
- If $x \in Y_i$ with $i \geq 1$: h acts trivially on Y_i, so
$$h\phi_1(h')(x) = \phi_1(h')(x) = \phi_1(h')h(x).$$

Thus h and $\phi_1(h')$ commute for every $x \in X$, and hence $[h, \phi_1(h')] = e$. An analogous computation shows that $\phi_1(h) = g^{-1}\phi_0(h)g$, thus we have the thesis. □

It is left to prove that the group of countably supported bijections of an uncountable set satisfies our hypotheses. To do so, we will have to find a suitable set on which G acts on and use Proposition 4.5 to show that the associated poset of sequences satisfies the W-property. Once this is done, the proof of the criterion in Theorem 4.2 is concluded.

Lemma 4.47. *The group G of countably supported bijections of an uncountable set X satisfies the hypotheses of Theorem 4.2. Namely G acts on a set X, and there exists a G-poset $\mathcal{X}$ of sequences in X such that the following properties hold:*

1. *the poset $\mathcal{X}$ satisfies the W-property;*
2. *the induced action of G on $\mathcal{X}$ is transitive;*
3. *the stabilizer of some (equivalently, every) sequence $\mathbf{x} \in \mathcal{X}$ is boundedly acyclic.*

Proof. The group G acts naturally on X. Consider the trivial preorder $\lesssim$ on X (i.e. $x \lesssim y$ for all $x, y \in X$) and the trivial equivalence $\cong$ (i.e. $x \cong y$ if and only if $x = y$). Then the poset $\mathcal{X}$ of all sequences of pairwise distinct elements of X is a G-poset, and it satisfies the W-property by Lemma 4.5.

Since sequences are countable, it is always possible to find a bijection between any two of them. In other words, the action is transitive.

Let $x = (x_n)_{n\in\mathbb{N}}$. The stabilizer G_x of x in G consists of all bijections that fix every element of x, that is, the subgroup of G whose support is contained in $X \setminus \bigcup_{n\in\mathbb{N}} x_n$. The complement $X \setminus \bigcup_{n\in\mathbb{N}} x_n$ is still uncountable, so G_x is itself the group of countably supported bijections of an uncountable set. Hence, by Lemma 4.46, G_x is boundedly acyclic. □

4.2 Verifying the criterion

In this section we show that the group $\mathrm{Diff}^r_+(\mathbb{R}^n)$ of *orientation-preserving* C^r-diffeomorphisms satisfies the hypotheses of Theorem 4.2, and hence it is boundedly acyclic. The proof will use the action of G on *fat spheres*, i.e., C^r-diffeomorphic images of germs of embeddings of spheres. This allows us to define the $\mathrm{Diff}^r_+(\mathbb{R}^n)$-poset of *fat sequences*. This poset will satisfy the W-property, all the stabilizers of G on this poset will be boundedly acyclic, and the action of G on this poset will be transitive.

The result for $\mathrm{Diff}^r_+(\mathbb{R}^n)$ also implies that the group of possibly non-orientation-preserving diffeomorphisms is boundedly acyclic. Indeed, recall that a topological group is a topological space with a structure of a group where the maps corresponding to the sum and the inverse are continuous. The group $\mathrm{Diff}^r(\mathbb{R}^n)$ with the strong topology is a topological group (see [Hir76, Chapter 2]). Taking the Jacobian and then the determinant gives rise to a continuous map

$$\mathrm{Diff}^r(\mathbb{R}^n) \to \mathrm{GL}_n(\mathbb{R}) \to \{+1, -1\},$$

where $\mathrm{GL}_n(\mathbb{R})$ has the induced topology from $\mathbb{R}^{n^2}$ and $\{+1, -1\}$ has the discrete topology. From this, we see that the connected components of $\mathrm{Diff}^r(\mathbb{R}^n)$ are the preimage of $+1$ and -1 respectively, namely: the group of orientation-preserving and orientation-reversing diffeomorphisms. The connected component containing the identity is always a (closed and normal) subgroup of a topological group whose index corresponds to the number of connected components [Bou89, Section 3.2, Proposition 7]). In this case, this component is the group $\mathrm{Diff}^r_+(\mathbb{R}^n)$ of orientation-preserving diffeomorphism. Thus, it has finite index equal to 2 and we have that the result for this group implies the result for the entire $\mathrm{Diff}^r(\mathbb{R}^n)$ by the following:

Theorem 4.48 ([Mon01, Corollary 8.8.5])**.** *Let G be a group and $H \leq G$ a finite-index subgroup of G. Then, for all $n \geq 0$, we have an isometric isomorphism*

$$H^n_b(G;\mathbb{R}) \cong H^n_b(H;\mathbb{R})^{G/H}$$

between the bounded cohomology of G and the G/H-invariants of the bounded cohomology of H.

To resume: if we want to prove that a group is boundedly acyclic, we can use instead a finite-index subgroup. Therefore, instead of working with $\mathrm{Diff}^r(\mathbb{R}^n)$ we will use its finite-index subgroup $\mathrm{Diff}^r_+(\mathbb{R}^n)$, that from now on we will denote by G. This will be useful when proving the transitivity of the action of G on the poset of *fat sequences*. In particular, we will use the fact that any orientation-preserving diffeomorphism is *isotopic* to the identity [Mil65, Lemma 6.2].

Definition 4.49 (**Model of the fat sphere**)**.** The *model of the fat sphere* is the germ at $S^{n-1} \times \{0\}$ of the orientation-preserving C^r-embedding

$$\Sigma\colon S^{n-1} \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \longrightarrow \mathbb{R}^n, \quad (x, \rho) \longmapsto (1+\rho)\, x.$$

We define:

- the *core* as the embedding $\dot{\Sigma}\colon S^{n-1} \to \mathbb{R}^n$ obtained by restricting Σ to $S^{n-1} \times \{0\}$;
- the *bounded component* of $\mathbb{R}^n \setminus \mathrm{Im}\,\dot{\Sigma}$ as the ball $\Sigma_b \coloneqq B^n_1(0)$;

- the *unbounded component* of $\mathbb{R}^n \setminus \operatorname{Im} \dot{\Sigma}$ as the complement $\Sigma_u \coloneqq \mathbb{R}^n \setminus \overline{B_1^n(0)}$.

Given an orientation-preserving C^r-diffeomorphism Σ and $g \in G$, we can consider the orientation-preserving C^r-diffeomorphism $g.\Sigma \coloneqq g \circ \Sigma$.

Definition 4.50 (**Fat sphere**). A *fat sphere* is the germ at $S^{n-1} \times \{0\}$ of an orientation-preserving C^r-embedding $S\colon S^{n-1} \times (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}^n$ that can be obtained as $g.\Sigma$ for some $g \in G$. We define:

- the *core* of S, denoted by $\dot{S}$, as the embedding $\dot{S}\colon S^{n-1} \to \mathbb{R}^n$ obtained by restricting S to $S^{n-1} \times \{0\}$. Note that if $S = g \cdot \Sigma$ then $\dot{S} = g \cdot \dot{\Sigma}$;
- the *bounded component* of $\mathbb{R}^n \setminus \operatorname{Im} \dot{S}$, denoted by S_b, as $S_b = g \cdot \Sigma_b$;
- the *unbounded component* of $\mathbb{R}^n \setminus \operatorname{Im} \dot{S}$, denoted by S_u, as $S_u = g \cdot \Sigma_u$.

Intuitively, one can think at the model of the fat sphere is a germ of the embedding that sends the cilinder $S^{n-1} \times (-\frac{1}{2}, \frac{1}{2})$ to an annullus in $\mathbb{R}^n$, "squeezing" it. A fat sphere can be pictured as a C^r-deformation of this annullus.

Observe that we have a natural action of G on the set of fat spheres. Indeed, this set is the one we use to satisfy the hypotheses of Theorem 4.2. Moreover, note that this action is transitive by definition of fat spheres.

Definition 4.51. Let $B \subseteq \mathbb{R}^n$ be a set. We say that a fat sphere S *englobes* B if $B \subseteq S_b$.

We define a poset of sequences of fat spheres as prescribed by Proposition 4.5, so that it satisfies automatically the W-property. Let S and T be fat spheres and define the equivalence relation $\cong$ and the preorder $\lesssim$ as follows: we say $S \cong T$ if $\dot{S} = \dot{T}$ and $S \lesssim T$ if either $\dot{S} = \dot{T}$ or T englobes $\dot{S}$.

Clearly $\cong$ is contained in $\lesssim$, thus the poset $\mathcal{F}$ of cofinal increasing sequences of pairwise inequivalent fat spheres with the subsequence partial order relation satisfies the W-property by Proposition 4.5.

We can describe more explicitly the poset $\mathcal{F}$ as follows:

Definition 4.52 (**Fat sequence**). We say that a sequence of fat spheres $(S_i)_{i \in \mathbb{N}}$ is *concentric going to infinity* if

- S_i englobes $\operatorname{Im} \dot{S}_{i-1}$ for all $i \geq 1$;
- for every compact set $K \subseteq \mathbb{R}^n$, there exists $N >> 0$ such that S_N englobes K.

A *fat sequence* is a concentric going to infinity sequence of fat spheres.

The poset $\mathcal{F}$ is the set of fat sequences with the following partial order relation: given two sequences $\mathbf{S} = (S_i)_{i \in \mathbb{N}}$ and $\mathbf{T} = (T_i)_{i \in \mathbb{N}}$, we say that $\mathbf{S} \leq \mathbf{T}$ if $\mathbf{S}$ is a subsequence of $\mathbf{T}$. As we shall see, having concentric fat spheres that does not intersect will be crucial for our purposes, in particular for the proof of the transitivity of the action.

For convenience, we fix a particular fat sequence $\mathbf{\Sigma}$ which we will use to simplify some steps in the proof of the remaining items of Theorem 4.2. Namely, we will prove that the stabilizer $G_{\mathbf{\Sigma}}$ of $\mathbf{\Sigma}$ in G is boundedly acyclic and that the action of G on fat sequences is transitive by showing that every fat sequence $\mathbf{S}$ can be written as $g.\mathbf{\Sigma}$ for some $g \in G$.

Definition 4.53 (**Model fat sequence**)**.** Let Σ be the model fat sphere. For every $i \in \mathbb{N}^*$ define Σ_i to be the image of Σ via the homothety of ray i. The sequence $\mathbf{\Sigma} := (\Sigma_i)_{i\in\mathbb{N}^*}$ is a fat sequence, and we call it the *model fat sequence*.

Observe that for every $i \in \mathbb{N}^*$ we have that $\operatorname{Im}\dot{\Sigma}_i$ is iS^n, the n-sphere of radius i around the origin.

Lemma 4.54. *The stabilizer $G_{\mathbf{\Sigma}}$ of $\mathbf{\Sigma}$ in G is C^r-diffeomorphic to the infinite product $\mathrm{Diff}^r_c(\mathbb{R}^n) \times \mathrm{Diff}^r_c(S^{n-1} \times \mathbb{R})^{\mathbb{N}}$. In particular, it is boundedly acyclic.*

Proof. An element $g \in G$ lies in the stabilizer of $\mathbf{\Sigma}$ if it is the identity in a neighborhood of $\operatorname{Im}\dot{\Sigma}_i = iS^n$ for all $i \in \mathbb{N}^*$ (so that $g.\Sigma_i$ and Σ_i define the same germ).

Thus, we have that the stabilizer is isomorphic to:

$$\mathrm{Diff}^r_c(B^n_i(0)) \times \prod_{i\geq 2} \mathrm{Diff}_c(B^n_i(0) \setminus \overline{B^n_{i-1}(0)}).$$

Since $B^n_i(0)$ is C^r-diffeomorphic to $\mathbb{R}^n$ and $\mathrm{Diff}_c(B^n_i(0) \setminus \overline{B^n_{i-1}(0)})$ is C^r-diffeomorphic to $S^{n-1} \times \mathbb{R}$, we have the thesis.

The bounded acyclity now follows from Theorem 3.15. $\square$

We are left to show that the action is transitive. The proof of this fact is a bit tangled, but the idea goes as follows. The action of the group $\mathrm{Diff}^r_+(\mathbb{R}^n)$ on fat spheres is transitive by definition of fat sphere. What we have to prove is that for every fat sequence $\mathbf{S} = (S_i)_{i\in\mathbb{N}^*}$, there exists an element $g \in G$ such that for every $i \in \mathbb{N}^*$ we have $g.\Sigma_i = S_i$, where $\mathbf{\Sigma} = (\Sigma_i)_{i\in\mathbb{N}^*}$ is the model fat sequence. Fat spheres are defined as diffeomorphic images of germs of embeddings of (compact) spheres. Moreover this spheres are concentric going to infinity, meaning that this compact sets are disjoint. We have different embeddings g_i for $i \in \mathbb{N}^*$, each of these fixing the corresponding fat sphere S_i. We want to find a global embedding g agreeing with all these g_i's. To this aim, we need to recall some facts about *isotopies* and in particular we will use the *Isotopy Extension Theorem*,

Two embeddings (that is, injective immersions which are homeomorphic into their images) $f, g\colon V \hookrightarrow M$ are *isotopic* if one can be deformed to the other through embeddings. Such a deformation is called *isotopy between f and g*. In other words: an isotopy is a 1-parameter family of embeddings.

A useful property of isotopies is given by the Isotopy Extension Theorem, that can be applied to show that an embedding can be extended to a larger domain under certain hypothesis.

More precisely we have the following:

Definition 4.55 (**Isotopy**)**.** Let V and M be manifolds. An *isotopy* from V to M is a map $F\colon V \times I \to M$ such that, for every $t \in Y$, the map

$$\begin{aligned} F_t : V &\longrightarrow M, \\ x &\longmapsto F(x,t). \end{aligned}$$

is an embedding.

We say that two embeddings $f, g\colon V \to M$ are *isotopic embeddings* if there exists an isotopy F from V to M such that $f(x) = F(x,0)$ and $g(x) = F(x,1)$

An isotopy $F\colon M \times I \to M$ such that F_t is a diffeomorphism for all $t \in I$ is called *diffeotopy.* Moreover, given a diffeotopy $F\colon M \times I \to M$ such that $F_0 = \mathrm{Id}|_M$ and

$f\colon V \to M$ embedding, we have that $F(f(v), t)$ is a diffeotopy between f and $F \circ f$. Sometimes such diffeotopy is called *ambient isotopy*.

The *support of an isotopy* $F\colon V \times I \to M$ is defined as the closure of

$$\{x \in V \mid F(x,t) \neq F(x,0) \text{ for some } t \in I\}$$

Theorem 4.56 (**Isotopy Extension Theorem** [Hir76, Theorem 1.3, Chapter 8])**.** *Let M be a manifold, let $V \subseteq M$ be a compact submanifold and let $F\colon V \times I \to M$ be an isotopy. If either* $\operatorname{Im} F \subseteq \partial M$ *or* $\operatorname{Im} F \subseteq M \setminus \partial M$*, then F extends to a compactly supportated ambient isotopy of M.*

A corollary is the following: if we want to extend the domain of definition of an embedding, we can show that it is isotopic to another embedding that is already known to be extendable.

Theorem 4.57 ([Hir76, Theorem 1.5, Chapter 8])**.** *Let N, M be manifolds, let $V \subseteq N$ be a compact submanifold and let $f_0, f_1 \hookrightarrow V \to M$ isotopic embeddings in $M \setminus \partial M$. Then, if f_0 can be extended to an embedding $N \hookrightarrow M$, so does f_1.*

We are now ready to conclude the proof of Theorem 4.2 by showing the transitivity of the action of G on $\mathcal{F}$.

Lemma 4.58. *The action of G on $\mathcal{F}$ is transitive.*

Proof. The action of G on the fat spheres is transitive by definition. What we have to prove is that given two fat sequences $\mathbf{S} = (S_i)_{i\in\mathbb{N}}$ and $\mathbf{T} = (T_i)_{i\in\mathbb{N}}$, there exists a C^r-diffeomorphism such that

$$g.\mathbf{S} = (g.S_0, g.S_1, \dots) = (T_0, T_1, \dots) = \mathbf{T}.$$

In other words, given two fat sequences, we have to find an element that works simultaneously for all the fat spheres of the sequences.

Let $\mathbf{\Sigma}$ be the model fat sequence and $\mathbf{S} = (S_i)_{i\in\mathbb{N}^*}$ any other fat sequence. By transitivity of the action on fat spheres, for every $i \in \mathbb{N}$ there exists an element $g_i \in G$ such that $g_i.\Sigma_i = S_i$. Moreover, one can choose these elements such that for some $\varepsilon_i \in (0, 1/4)$, the sets $\Sigma_i(S^{n-1} \times (-\varepsilon_i, \varepsilon_i))$ and the sets $g_i.\Sigma_i(S^{n-1} \times (-\varepsilon_i, \varepsilon_i))$ are pairwise disjoint.

Goal: show that there exists $g \in G$ such that for every $i \in \mathbb{N}^*$ we have

$$g|_{\Sigma_i(S^{n-1}\times(-\varepsilon_i,\varepsilon_i))} = g_i|_{\Sigma_i(S^{n-1}\times(-\varepsilon_i,\varepsilon_i))}.$$

If we prove it, then the germs of the embeddings are the same and we have $g.\mathbf{\Sigma} = \mathbf{S}$.

We construct by induction a family of maps $h_k \in G$ for all $k \geq 1$ such that:

1. $h_k(\Sigma_i) = S_i$ for all $i \leq k$;
2. $h_k \equiv h_{k+1}$ in an open neighborhood of the ball $B^n_k(0) = (\Sigma_k)_b$.

Then we define g as

$$g(x) := \lim_{k\to+\infty} h_k(x), \quad \text{for every } x \in \mathbb{R}^n.$$

This is well-defined since, for each $x \in \mathbb{R}^n$, there exists $K \in \mathbb{N}$ such that $h_k(x)$ assumes the same value for all $k \geq K$. Therefore, the limit is eventually constant and coincides with $h_K(x)$. Moreover, the first item ensures that g is indeed the element we are looking for.

Base step: for $k = 1$, let $h_1 = g_1$.

Inductive step: suppose h_k is given. We construct h_{k+1} via an element $\gamma \in G$ whose support does not intersect an open neighborhood of $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$. Then, we will define $h_{k+1} = \gamma \circ h_k$, so that the second item of our construction will be satisfied.

We have two different embeddings $\Sigma_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}]) \hookrightarrow \mathbb{R}^n$ given by g_{k+1} and h_k, respectevely. Both are global C^r-diffeomorphisms, and here we look at their restrictions, which we denote with the same name. Recall that every element of $\mathrm{Diff}^r_+(\mathbb{R}^n)$ is isotopic to the identity (see for instance [Mil65, Chapter 6, Lemma 2]), thus g_{k+1} and h_k are isotopic embeddings.

We need to check that this isotopy does not intersects $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$, so that the support of γ will not intersect it.

To this end, we need the following version of the Annullus Theorem.

Lemma 4.59 ([Hir76, Section 8], [Pal60] and [EK71])**.** *Let W_i be the region between two consecutive cores of the fat spheres, namely:*

$$\dot{S}_i \coloneqq g_i \circ \Sigma_i\Big(S^{n-1} \times \{0\}\Big) \quad \text{and} \quad \dot{S}_{i+1} \coloneqq g_{i+1} \circ \Sigma_{i+1}\Big(S^{n-1} \times \{0\}\Big).$$

Then the manifold W_i is C^r-diffeomorphic to $S^{n-1} \times [0,1]$.

We can isotope $g_{k+1} \circ \Sigma_{k+1}(S^{n-1} \times \{0\})$ to a neighborhood of $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$ by pushing along the region between them, since the latter is C^r-diffeomorphic to the annullus $S^{n-1} \times [0,1]$ by Lemma 4.59. Analogously, we can do the same for $h_k \circ \Sigma_{k+1}(S^{n-1} \times \{0\})$ and $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$: the region between them is again C^r-diffeomorphic to an annullus.

Therefore, we see that there exists an isotopy between $g_{k+1} \circ \Sigma_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}])$ and $h_k \circ \Sigma_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}])$ that does not intersect $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$. This exactly means that the region that we wanted not to modify is indeed untouched.

The sets involved are compact, so this isotopy is compactly supported. Thus, by the Isotopy Extension Theorem 4.56, we can extend the isotopy to the whole $\mathrm{int}(B^n_R(0))$ for R sufficiently big. This isotopy provides the element γ we are looking for. Indeed, there exists $\gamma \in \mathrm{Diff}^r_c(\mathrm{int}(B^n_R(0))$ satisfying:

1. the support of γ does not intersect a neighborhood of $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$;
2. γ sends $h_k \circ \Sigma_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}])$ to $g_{k+1} \circ \Sigma_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}])$

We define now $h_{k+1} \coloneqq \gamma \circ h_k$; by the two properties above, we see that h_{k+1} satisfies the two conditions (1) and (2), therefore we have the thesis. □

Proof of Lemma 4.59. To simplify the notations, let us denote by $S(r)$ the n-sphere around the origin with ray r.

By radial dilation, we can send $\Sigma_i\Big(S^{n-1} \times \{0\}\Big)$ to $\Sigma_{i+1}\Big(S^{n-1} \times \{0\}\Big)$, so we can find an element $f \in G$ that shrinks the cores of the fat spheres. Precisely, we have an element $f \in G$ such that $f(\dot{S}_{i+1}) = \dot{S}_i$ for all $i \geq 1$. For such an element, we have $S' \coloneqq f(S(1)) \subseteq B^n_1(0)$. Concretely, what we are doing is a change of coordinates to reduce the problem to show that the region between S' and $S(1)$ is C^r-diffeomorphic to $S^{n-1} \times [0,1]$.

Recall that f is isotopic to the identity since it is an element of $\mathrm{Diff}^r_+(\mathbb{R}^n)$ [Mil65, Lemma 6.2]. Thus, the sphere $S(1)$ is isotopic to $S(1/2)$. Since we are dealing with compact sets, we can assume that this isotopy is compactedly supported in $B^n_R(0)$ for a large enough $R > 0$. Then the Isotopy Extension Theorem 4.56 says that we can extend this isotopy to a compactly supported isotopy on the entire $B^n_R(0)$. In other words, we have a path of C^r-diffeomorphism of $B^n_R(0)$ that are the identity on $\partial B^n_R(0) = S(R)$. Thus, we found an element $g \in \mathrm{Diff}^r_c(B^n_R(0))$ such that $g(S') = S(1/2)$.

The region between S' and $S(1)$ is C^r-diffeomorphic to the region between S' and $S(k)$, since we are just adding the annulus between $S(1)$ and $S(k)$. Moreover, the map g defines a C^r-diffeomorphism from the region between S' and $S(k)$ onto the region between $S(1/2)$ and $S(k)$, which is C^r-diffeomorphic to $S^{n-1} \times [0, 1]$. Therefore, the claim follows. $\square$

References

[Ber02] A. J. Berrick, *A topologist's view of perfect and acyclic groups*, in *Invitations to Geometry and Topology*, Oxford Graduate Texts in Mathematics, vol. 7, Oxford Univ. Press, Oxford, 2002, pp. 1–28. 43

[Ber89] A. J. Berrick, *Universal groups, binate groups and acyclicity*, in *Group Theory, Proceedings of the Singapore Conference*, de Gruyter, Berlin, 1989, pp. 253–266. 42

[Bjo95] A. Björner, *Topological methods*, in *Handbook of Combinatorics*, Vol. 1, Elsevier Science B.V., Amsterdam, 1995, pp. 1819–1872. 57

[BM02] M. Burger and N. Monod, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. **12** (2002), 219–280. 2

[Bou89] N. Bourbaki, *General Topology, Chapters 1–4*, English translation of the 1966 French ed., Springer, 1989. 64

[Bro81] R. Brooks, *Some remarks on bounded cohomology*, in *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference*, Annals of Mathematics Studies, vol. 97, Princeton Univ. Press, Princeton, 1981, pp. 53–63. 17

[Bro82] K. S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics, vol. 87, Springer, 1982. 7, 15, 35

[BV94] A. J. Berrick and K. Varadarajan, *Binate towers of groups*, Arch. Math. (Basel) **62** (1994), no. 2, 97–111. 42

[EK71] R. D. Edwards and R. C. Kirby, *Deformations of spaces of embeddings*, Ann. of Math. (2) **93** (1971), 63–88. 68

[FFa20] F. Fournier-Facio, *Infinite sums of Brooks quasimorphisms and cup products in bounded cohomology*, master's thesis, ETH Zürich, 2020, preprint, arXiv:2002.10323. 17

[FFLM23] F. Fournier-Facio, C. Löh and M. Moraschini, *Bounded cohomology and binate groups*, J. Aust. Math. Soc. **115** (2023), no. 2, 204–239. 32, 43

[FFMN24] F. Fournier-Facio, N. Monod, S. Nariman and A. Kupers, *The bounded cohomology of transformation groups of Euclidean spaces and discs*, preprint, arXiv:2405.20395. 2, 43, 45, 56

[FM23] R. Frigerio and M. Moraschini, *Gromov's theory of multicomplexes with applications to bounded cohomology and simplicial volume*, Mem. Amer. Math. Soc., Vol. 283, 2023, 153 pp. doi:10.1090/memo/1402. 12, 32

[Fri17] R. Frigerio, *Bounded Cohomology of Discrete Groups*, Mathematical Surveys and Monographs, vol. 227, Amer. Math. Soc., 2017. 9, 11, 12, 15, 26, 27, 32

[Fri23] G. Friedman, *An elementary illustrated introduction to simplicial sets*, preprint, arXiv:0809.4221v8 [math.AT]. 47, 48, 49

[Gri84] R. I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985. 28

[Gro82] M. Gromov, *Volume and bounded cohomology*, Publ. Math. IHÉS **56** (1982), 5–99. 2, 12, 32

[Hat02] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, Cambridge, 2002. 10, 20, 33, 48

[Hir76] M. W. Hirsch, *Differential Topology*, Springer-Verlag, New York, 1976. 64, 67, 68

[Iva17] N. V. Ivanov, *Notes on the bounded cohomology theory*, preprint, arXiv:1708.05150. 12, 32

[Iva20] N. V. Ivanov, *Simplicial sets, Postnikov systems, and bounded cohomology*, preprint, arXiv:2012.00947. 47

[Iva87] N. V. Ivanov, *Foundations of the theory of bounded cohomology*, J. Soviet Math. **37** (1987), 1090–1114. 12, 32

[Joh72] B. E. Johnson, *Cohomology in Banach Algebras*, Mem. Amer. Math. Soc., no. 127, 1972. 2, 17

[Kas25] T. Kastenholz, *Non-vanishing of the fourth bounded cohomology of free groups and codimension 2 subspaces*, preprint, arXiv:2503.22511. 17

[KS23] T. Kastenholz and R. J. Sroka, *Simplicial bounded cohomology and stability*, preprint, arXiv:2309.05024. 55

[LMR25] K. Li, M. Moraschini and G. Raptis, *The Serre spectral sequence in bounded cohomology*, preprint, arXiv:2503.22505. 47

[Loh17] C. Löh, *Geometric Group Theory: An Introduction*, Universitext, First edition, Springer, Cham, 2017. doi:10.1007/978-3-319-72254-2. 26, 28

[Mac63] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, vol. 114, Springer-Verlag, 1963. 37

[Mat71] J. N. Mather, *The vanishing of the homology of certain groups of homeomorphisms*, Topology **10** (1971), pp. 297–298. 32, 33

[May92] J. P. May, *Simplicial Objects in Algebraic Topology*, Univ. of Chicago Press, Chicago, 1992. 47

[Mil65] J. W. Milnor, *Topology from the Differentiable Viewpoint*, Univ. Press of Virginia, Charlottesville, 1965. 64, 68

[Mit84] Y. Mitsumatsu, *Bounded cohomology and ℓ^1-homology of surfaces*, Topology **23** (1984), no. 4, pp. 465–471. 17

[MM85] S. Matsumoto and S. Morita, *Bounded cohomology of certain groups of homeomorphisms*, Proc. Amer. Math. Soc. **94** (1985), no. 3, pp. 539–544. 22, 24, 32, 36

[MN23] N. Monod and S. Nariman, *Bounded and unbounded cohomology of homeomorphism and diffeomorphism groups*, Invent. Math. **232** (2023), no. 3, pp. 1439–1475. MR 4588567. 43, 54

[Mon01] N. Monod, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Mathematics, vol. 1758, Springer-Verlag, Berlin, 2001. 2, 9, 13, 64

[MR23] M. Moraschini and G. Raptis, *Amenability and acyclicity in bounded cohomology*, Rev. Mat. Iberoam. **39** (2023), no. 6, pp. 2371–2404. 13

[Neu29] J. von Neumann, *Zur allgemeinen Theorie des Maßes*, Fund. Math. **13** (1929), no. 1, pp. 73–116. 26, 27

[Pal60] R. S. Palais, *Local triviality of the restriction map for embeddings*, Comment. Math. Helv. **34** (1960), pp. 305–312. 68

[Pat88] A. L. T. Paterson, *Amenability*, Mathematical Surveys and Monographs, no. 29, Amer. Math. Soc., Providence, RI, 1988. 27

[Rol09] P. Rolli, *Quasi-morphisms on free groups*, preprint, arXiv:0911.4234. 17

[Rud91] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991. 11, 18, 21, 22, 27

[Som97] T. Soma, *Bounded cohomology of closed surfaces*, Topology **36** (1997), pp. 1221–1246. 17

[SV87] P. Sankaran and K. Varadarajan, *Some remarks on pseudo-mitotic groups*, Indian J. Math. **29** (1987), pp. 283–293. 42

[SV90] P. Sankaran and K. Varadarajan, *Acyclicity of certain homeomorphism groups*, Canad. J. Math. **42** (1990), pp. 80–94. 42, 43

[Var85] K. Varadarajan, *Pseudo-mitotic groups*, J. Pure Appl. Algebra **37** (1985), pp. 205–213. 42