

SCUOLA DI SCIENZE

Corso di Laurea Magistrale in Matematica

**A Semiclassical Weyl Law
for Quantized Operators on the Torus
Perturbed by Random Potentials**

Tesi di Laurea in Fisica Matematica

Relatore:

Chiar.mo Prof.

Davide Pastorello

Presentata da:

Nicola Bertoni

Correlatore:

Chiar.mo Prof.

Martin Vogel

(CNRS, Université de Strasbourg)

Anno Accademico 2024-2025

Abstract

In this thesis we study the Weyl quantization of periodic symbols on the torus in a semiclassical regime where the parameter $N \rightarrow \infty$. This framework is natural for describing quantum observables that are periodic both in position and momentum. Since the torus is a compact phase space, the corresponding quantizations, when restricted to the space of periodic distributions, yield finite-dimensional operators, i.e., square matrices, whose size grows with N .

Characterizing the spectrum of these operators is challenging, since they are not necessarily self-adjoint. To regularize the spectrum, we introduce small random perturbations in the form of diagonal matrices with compactly supported probability distributions. From a physical perspective, such perturbations can model random potentials in a quantum system.

The main contribution of this work is the establishment of a probabilistic Weyl law for the spectrum of the perturbed operators. Using semiclassical analysis tools, Grushin problems, and techniques from complex analysis and probability theory, we prove that with high probability the eigenvalues of the perturbed operators roughly equidistribute in the range of their principal symbols. Specifically, for relatively compact sets with uniformly Lipschitz boundary, the eigenvalue counting function admits an asymptotic estimate governed by the measure of the set's preimage under the principal symbol.

Our results demonstrate that small random potential perturbations can regularize the spectrum and recover Weyl-type asymptotics in this setting.

Abstract

In questa tesi studiamo la quantizzazione di Weyl di simboli periodici sul toro in un regime semiclassico in cui il parametro $N \rightarrow \infty$. Questo è un contesto naturale in cui descrivere osservabili quantistiche periodiche sia nella posizione che nel momento. Poiché il toro è uno spazio delle fasi compatto, le quantizzazioni corrispondenti, se ristrette allo spazio delle distribuzioni periodiche, sono operatori finito-dimensionali, cioè matrici quadrate, la cui dimensione cresce con N .

Caratterizzare lo spettro di questi operatori può essere complesso, poiché non sono necessariamente autoaggiunti. Al fine di regolarizzarne lo spettro, introduciamo piccole perturbazioni casuali sotto forma di matrici diagonali con distribuzioni di probabilità a supporto compatto. Da un punto di vista fisico, tali perturbazioni possono modellare potenziali casuali in un sistema quantistico.

Il contributo principale di questo lavoro consiste in una legge di Weyl probabilistica per lo spettro degli operatori perturbati. Utilizzando strumenti di analisi semiclassica, problemi di tipo Grushin e tecniche di analisi complessa e teoria della probabilità, dimostriamo che con grande probabilità gli autovalori degli operatori perturbati si equidistribuiscono approssimativamente nell'immagine del loro simbolo principale. Nello specifico, per insiemi in relativamente compatti con bordo uniformemente Lipschitziano, la funzione di conteggio degli autovalori ammette una stima asintotica determinata dalla misura della preimmagine dell'insieme rispetto al simbolo principale.

I nostri risultati mostrano che piccole perturbazioni casuali di tipo potenziale regolarizzano lo spettro e consentono di ottenere asintotiche di tipo Weyl in questo contesto.

Contents

Introduction	iii
Notations	1
1 Semiclassical Calculus and Functional Calculus	3
1.1 Semiclassical Quantization	3
1.2 Quantization of the Torus	10
1.3 Quantization of Symbols on the Torus	12
1.4 Preliminary Results for Functional Calculus	14
1.5 Phase Space Dilation and Logarithmic Estimates	17
2 Grushin Problems and Schur Complement Methods	23
2.1 A General Grushin Problem and the Schur Complement Formula	23
2.2 Grushin Problem for the Unperturbed Operator	27
2.3 Grushin Problem for the Perturbed Operator	31
2.4 Auxiliary Estimates	34
3 Estimates for the Perturbed Operator	37
3.1 Construction of a Potential	37
3.2 Improving Singular Value Bounds via Potential Perturbations	43
3.3 Iterative Construction of a Potential for Controlling the Smallest Singular Value	52
3.3.1 Log-Determinant Estimates for the Perturbed Operator	56
4 Perturbations by Random Potentials	61
4.1 Deterministic Log-Determinant Estimates for Bounded Perturbations	61
4.2 Log-Determinant Estimates under Random Perturbations	63
5 Eigenvalue Counting	75
5.1 Zero Counting Techniques for Holomorphic Functions	75
5.2 Eigenvalue Distribution via Empirical Measure Convergence	82
5.3 Applications with Truncated Gaussian Distribution	91
5.3.1 Application to the Scottish Flag Operator	92

Conclusions	98
Appendix	99
A.1 Useful Results for Semiclassical Calculus	99
A.1.1 Oscillatory Integrals	99
A.1.2 Fourier Conjugation Formula	101
A.2 Stirling's Formula	101
A.3 Mini-max Theorem and Inequalities for Singular Values	103
A.4 Classical Results in Complex and Harmonic Analysis	105
A.4.1 Maximum Principle and Harnack's Inequality	105
A.4.2 Jensen's Formula	106
A.5 General Tools from Analysis	107
Bibliography	111

Introduction

In this thesis we work with pseudodifferential operators in a semiclassical regime where the parameter $N \rightarrow \infty$. In particular, we consider operators obtained as the Weyl quantization of periodic symbols on the torus $\mathbb{T}^{2d}$. Since the torus is a compact phase space, using semiclassical calculus, we show that these quantizations, when restricted to the space of periodic distributions, yield finite-dimensional operators. Thus, they are represented by square matrices whose dimension grows with N .

These operators are not necessarily self-adjoint, and hence their spectral analysis is delicate: the spectrum can be highly unstable. Whereas for self-adjoint operators there are asymptotic Weyl laws describing their eigenvalue distribution, there is no general analogue in the non-selfadjoint setting. Thus, in the framework we are studying, we aim to regularize the spectrum by introducing small random perturbations.

This technique has been used by several authors and has proved particularly effective for establishing probabilistic Weyl-type laws. In particular, we follow ideas from Vogel [Vo20] and Sjöstrand [Sj09]. Vogel works in the same toroidal, finite-dimensional setting considered here and introduces full random matrix perturbations (e.g., with Gaussian entries). By contrast, Sjöstrand studies Weyl quantization on $\mathbb{R}^{2d}$, leading to infinite-dimensional operators, and considers multiplicative random perturbations modeling random potentials.

In this work we adopt Sjöstrand's multiplicative (potential-type) perturbations within the finite-dimensional framework induced by the torus. Consequently, the perturbations are diagonal matrices, in contrast to the full-matrix model considered by Vogel. Although the two cases present similarities, they cannot be approached in the same way and require different arguments and discussions for achieving the desired results.

To set the framework, we first review the core tools of semiclassical analysis and functional calculus, which provide the principal machinery for describing the operators under consideration. We then introduce a Grushin problem: by embedding the original operator into an augmented block system, its spectral properties are captured by a Schur complement, which is often easier to analyze and control. Finally, standard results from complex analysis and probability theory provide the key estimates used in our main arguments.

Beyond its mathematical relevance, this framework admits a clear physical interpretation. The operators introduced above naturally model quantum observables that are periodic in both position and momentum; they arise via "quantization" of periodic symbols representing clas-

sical observables. Moreover, the multiplicative perturbations we consider can model random potentials acting on a quantum system.

Before turning to the technical developments of the subsequent chapters, we present a concrete example illustrating the scope and significance of this work. Fix $N \gg 1$ and consider the two-dimensional torus $\mathbb{T}^2$ as phase space, where the variable x stands for the position and ξ for the momentum. We define the symbol

$$p(x, \xi) := \cos(2\pi x) + i \cos(2\pi \xi), \quad (x, \xi) \in \mathbb{T}^2. \quad (1)$$

The Weyl quantization of p , restricted to the space of periodic distributions, yields a finite-dimensional operator acting on $\mathbb{C}^N$. In particular, it can be represented by the matrix

$$p_N = \begin{pmatrix} \cos(x_1) & \frac{i}{2} & 0 & 0 & \cdots & \frac{i}{2} \\ \frac{i}{2} & \cos(x_2) & \frac{i}{2} & 0 & \cdots & 0 \\ 0 & \frac{i}{2} & \cos(x_3) & \frac{i}{2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{i}{2} & \cos(x_{N-1}) & \frac{i}{2} \\ \frac{i}{2} & 0 & \cdots & 0 & \frac{i}{2} & \cos(x_N) \end{pmatrix}, \quad x_j := \frac{2\pi j}{N}, \quad j = 1, \dots, N, \quad (2)$$

often referred to as the *Scottish flag* operator (a circulant tridiagonal, non-self-adjoint matrix).

Figure 1 displays the spectra obtained from two numerical simulations. The left panel shows the spectrum of the unperturbed operator p_N ; the right panel shows the spectrum of the perturbed matrix $p_N + \delta V_q$, where δ is a small N -dependent parameter and $V_q = \text{diag}(q)$ is a diagonal potential with $q \in \mathbb{C}^N$ sampled from a truncated multivariate Gaussian.

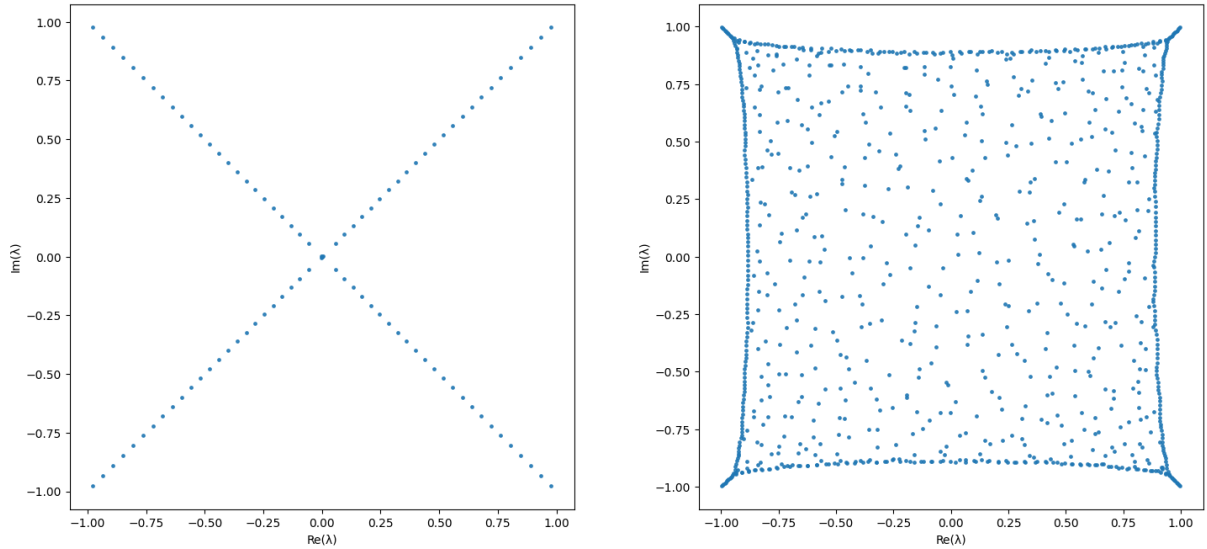


Figure 1: Left: spectrum of p_N with $N = 100$. Right: spectrum of $p_N + \delta V_q$ with $N = 1000$ and $\delta = N^{-3/2}$.

These plots illustrate how a small diagonal (potential-type) random perturbation regular-

izes the spectrum of the Scottish flag operator.

The simulations are consistent with our main result (Theorem 5.5). By applying that theorem to the case of the Scottish flag operator, as shown in Section 5.3, we obtain the following statement.

Theorem 1. Fix a semiclassical parameter $N \gg 1$ and consider p as in (1) and its quantization p_N as in (2). Let $q \in \mathbb{C}^N$ be a random vector distributed as a Gaussian truncated to a ball B , and set $V_q := \text{diag}(q)$. Let D be an open disc contained in the square $[-1, 1]^2$. Then there exists $0 < \delta \ll N^{-1}$ such that the estimate

$$\left| \frac{1}{N} \#(\sigma(p_N + \delta V_q) \cap D) - \lambda(p^{-1}(D)) \right| \leq \mathcal{O}(N^{-\frac{1}{4}} (\ln(N))^3)$$

holds with probability tending to 1 as $N \rightarrow +\infty$. Here, σ denotes the spectrum and λ the Lebesgue measure on the torus.

In particular, in the semiclassical limit, the eigenvalues of $p_N + \delta V_q$ become equidistributed in the image of the symbol $p(\mathbb{T}^2)$ with high probability.

Structure of the Thesis

The thesis is organized into five chapters. Each chapter develops tools and intermediate results, culminating in the main probabilistic Weyl law.

Chapter 1

In the first chapter, we review the core notions of semiclassical analysis and functional calculus. We begin by introducing symbols on the phase space $\mathbb{R}^{2d}$ and their Weyl quantization, viewed as h -dependent pseudodifferential operators defined via oscillatory integrals, where $0 < h \ll 1$. After recording basic properties of this calculus, we pass to the toroidal setting $\mathbb{T}^{2d}$. Restricting the corresponding quantized symbols to distributions that are periodic in both position and momentum yields finite-dimensional operators on $\mathbb{C}^{N^d}$, where $N = (2\pi h)^{-1}$. We then perform a phase-space rescaling and summarize its consequences for symbols and quantizations. The chapter concludes with Theorem 1.25, which provides spectral relations between a symbol and its quantization and will be used repeatedly in the following chapters.

Chapter 2

In the second chapter we focus on the study of Grushin problems. These are very important instruments in spectral theory, as they often simplify the analysis of spectral properties of both finite-dimensional and infinite-dimensional operators.

The basic idea is to embed the original (possibly non-invertible) operator into an augmented, invertible block system; the spectral information of the initial problem is then encoded

in a Schur complement, which is more tractable to analyze. We then establish log-determinant identities and estimates for both the unperturbed operator and for operators perturbed by deterministic potentials. These results will serve as a baseline for the probabilistic analysis in the later chapters.

Chapter 3

In the third chapter we adapt Sjöstrand's strategy in [Sj09, Chapters 5–6] to the toroidal setting. We consider a symbol p satisfying the symmetry assumption $p(x, \xi) = p(x, -\xi)$, and its corresponding Weyl quantization p_N with $N \gg 1$. Assuming a uniform lower bound on a subset of the singular values of p_N , we construct a deterministic diagonal potential by applying some linear-algebraic techniques to the singular vectors of p_N .

The Grushin reduction studied in Chapter 2 yields a bootstrap step that provides a new bound for a larger portion of the singular values of the operator perturbed by this potential. Iterating the procedure, each time updating the perturbation, we obtain a small diagonal perturbation $\delta_0 V_{q_0}$ for which the perturbed operator admits an explicit estimate for its smallest singular value.

As a consequence, we derive a pointwise lower bound for the error functional

$$F(q) := \ln |\det(p_N + \delta V_q - z)| - N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho \right),$$

when evaluated at q_0 . Here p_0 is the principal symbol of p , $V_q := \text{diag}(q)$ and $z \in \mathbb{C}$.

These deterministic constructions form the core original contribution of the thesis and are the key input for the probabilistic analysis.

Chapter 4

In the fourth chapter we equip the diagonal vector q with a probability law supported on a ball in $\mathbb{C}^{N^d}$ containing q_0 (e.g., a truncated Gaussian). After obtaining a uniform upper bound for F over this ball, we apply Jensen's inequality and some measure theory tools to derive a quantitative bound on the probability of the event $(|F(q)| < \varepsilon)$, given an arbitrary $\varepsilon > 0$. This probabilistic control of F is the bridge to the eigenvalue counting statements established in the final chapter.

Chapter 5

In the fifth chapter we prove the asymptotic Weyl law that is the main objective of the thesis, via two different approaches following [Vo20] and [SjVo21].

(i) *Complex-analytic approach.* This approach is based on Theorem 1.2 in [Sj10] concerning zero counting for holomorphic functions of exponential growth. Using the deterministic and probabilistic bounds developed earlier, we construct an appropriate framework where to apply the theorem. This application yields our quantitative main result, Theorem 5.5. Given $\Gamma \Subset \mathbb{C}$

an open set with uniformly Lipschitz boundary, the theorem provides an explicit error bound for

$$\left| \frac{1}{N^d} \#(\sigma(p_N + \delta V_q) \cap \Gamma) - \int_{p_0^{-1}(\Gamma)} d\rho \right|, \quad (3)$$

which holds with probability tending to 1 as $N \rightarrow +\infty$.

(ii) *Measure-theoretic approach.* Adapting Theorem 2.8.3 in [Ta12], we show that the empirical spectral measures of the randomly perturbed operators $p_N + \delta V_q$ converge weakly (in probability) to the push-forward of the Lebesgue measure on the torus under the principal symbol p_0 . Then, by applying Portemanteau's theorem, we obtain the convergence in probability of (3) for a wider class of sets, though without providing an explicit error rate.

Finally, we return to the Scottish flag operator and present the application already shown above, culminating in Theorem 1.

Notations

$\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

$\langle \cdot, \cdot \rangle$ Inner product on a complex Hilbert space (linear in the first argument, conjugate-linear in the second). On real spaces it is the usual scalar product.

$\| \cdot \|_2$ Norm induced by $\langle \cdot, \cdot \rangle$.

$\| \cdot \|_\infty$ Supremum (max) norm on vectors: for $x = (x_j)_j$, $\|x\|_\infty := \sup_j |x_j|$.

$\langle \cdot \rangle := (1 + \| \cdot \|_2^2)^{1/2}$ Japanese brackets.

λ Lebesgue measure on the underlying space.

$D(z, r)$ Open disc in $\mathbb{C}$ with center z and radius r .

$B_d(z, r)$ Open ball in $\mathbb{C}^d$ (with respect to $\| \cdot \|_2$) with center z and radius r .

$S^{d-1} := \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ The unit sphere in $\mathbb{R}^d$.

$\mathcal{D}(\mathcal{M}) := C_c^\infty(\mathcal{M})$ Space of test functions on a smooth manifold $\mathcal{M}$.

$\mathcal{D}'(\mathcal{M})$ Space of distributions on a smooth manifold $\mathcal{M}$.

$\mathcal{S}(\mathbb{R}^d)$ Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.

$\mathcal{S}'(\mathbb{R}^d)$ Space of tempered distributions on $\mathbb{R}^d$.

$\langle \cdot, \cdot \rangle_{X', X}$ Dual pairing between a space X and its dual X' .

$\| \cdot \|$ Operator norm induced by $\| \cdot \|_2$: $\|P\| := \sup_{\|x\|_2=1} \|Px\|_2$.

$M_{r,s}$ Space of complex $r \times s$ matrices.

$\sigma(P)$ Spectrum of a finite-dimensional operator P .

$\text{tr}(P)$ Trace of a finite-dimensional operator P .

$D_x := \frac{1}{i} \partial_x$ Differential operator.

$\bar{\partial} := \partial_{\bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$.

$\{\cdot, \cdot\}$ Poisson brackets.

$\varphi_*(\mu)$ Push-forward of a measure μ by a measurable map φ .

$a \ll b$ There exists a sufficiently large constant $C > 0$ such that $Ca \leq b$.

$f = \mathcal{O}(N)$ There exists a constant $C > 0$ (independent of N) such that $|f| \leq CN$. When we want to emphasize that the constant $C > 0$ depends on some parameter k , then we write C_k , or with the above big- $\mathcal{O}$ notation $\mathcal{O}_k(N)$.

$f = \mathcal{O}(N^{-\infty})$ For every $m \in \mathbb{N}$, there exists a constant $C_m > 0$, depending on m , such that $|f| \leq C_m N^{-m}$.

Chapter 1

Semiclassical Calculus and Functional Calculus

In the first chapter, we review some basic notions and results of semiclassical analysis in $\mathbb{R}^d$, following the exposition in [DiSj99, Ma02, Zw12]. We then turn to the study of Toeplitz quantization on the torus, as developed in [ChZw10] and [NoZw07]. Broadly speaking, this framework can be interpreted as a restriction of semiclassical calculus to periodic symbols and to function spaces consisting of tempered distributions that are periodic both in the spatial variable and in the semiclassical frequency.

1.1 Semiclassical Quantization

We start by giving the definition of an order function.

Definition 1.1. Let $m : \mathbb{R}^{2d} \rightarrow]0, +\infty[$ be a measurable function. We say that m is an order function if there exist constants $C, N_0 > 0$ such that

$$m(\rho) \leq C \langle \rho - \eta \rangle^{N_0} m(\eta), \text{ for all } \rho, \eta \in \mathbb{R}^{2d},$$

where $\langle \rho - \eta \rangle := (1 + \|\rho - \eta\|_2^2)^{\frac{1}{2}}$.

In particular, we consider smooth order functions, that is, functions $m \in C^\infty(\mathbb{R}^{2d},]0, +\infty[)$. As a simple example, we may take

$$m(x, \xi) := \langle \xi \rangle^{M_0}, \quad (x, \xi) \in \mathbb{R}^{2d}, \quad \text{with } 0 < M_0 < +\infty,$$

which depends only on the frequency variable ξ and not on x .

Definition 1.2. Let m be a smooth order function on $\mathbb{R}^{2d}$. The symbol class associated with m is defined as

$$S(m) := \{p \in C^\infty(\mathbb{R}^{2d}); \forall \alpha \in \mathbb{N}^{2d}, \exists C_\alpha > 0 \text{ such that } |\partial_\rho^\alpha p(\rho)| \leq C_\alpha m(\rho), \forall \rho \in \mathbb{R}^{2d}\}. \quad (1.1)$$

A particularly simple case is given by

$$S(1) = \{p \in C^\infty(\mathbb{R}^{2d}); \forall \alpha \in \mathbb{N}^{2d} \sup_{\rho \in \mathbb{R}^{2d}} |\partial_\rho^\alpha p(\rho)| < +\infty\}.$$

Remark 1.3. The Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$ is contained in $S(m)$, and moreover it is dense in $S(m)$ with respect to the topology of $S(\langle(x, \xi)\rangle^\epsilon m)$, for every $\epsilon > 0$.

From now on, we let $h \in]0, 1]$ denote the semiclassical parameter. A symbol $p = p(\rho; h) \in S(m)$ may depend on h ; in this case we assume that the symbol estimates in the definition (1.1) hold uniformly in h . If a symbol $p \in S(m)$ is of the form

$$p(\rho; h) = p_0(\rho) + hr(\rho; h), \quad r \in S(m),$$

then p_0 is called the principal symbol of p . Moreover, we say that p has the asymptotic expansion

$$p \sim p_0 + hp_1 + \dots \text{ in } S(m), \quad p_j \in S(m), \quad (1.2)$$

if $p - \sum_{j=0}^N h^j p_j \in h^{N+1} S(m)$, for all $N \in \mathbb{N}$.

We now define the h -Weyl quantization of a symbol acting on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

Definition 1.4. Given $p \in S(m)$ and $u \in \mathcal{S}(\mathbb{R}^d)$, we set

$$\begin{aligned} Op_h^w(p)(x, hD_x)u(x) &= p^w(x, hD_x)u(x) \\ &:= \frac{1}{(2\pi h)^d} \iint_{\mathbb{R}^{2d}} e^{\frac{i}{h}\langle x-y, \xi \rangle} p\left(\frac{x+y}{2}, \xi; h\right) u(y) dy d\xi, \end{aligned} \quad (1.3)$$

where the ξ -integration is understood in the sense of oscillatory integrals (see Section A.1.1 in the Appendix).

More explicitly, let the operator L be defined as in the Appendix, namely

$$L(\xi, hD_y) := \frac{1}{1 + \|\xi\|_2^2} (1 - h \langle \xi, D_y \rangle).$$

Let $k \in \mathbb{N}$ with $k > d + N_0$, where N_0 is set in Definition 1.1. Using the notion of oscillatory integral extended to $\mathcal{S}$, for $p \in S(m)$ and $u \in \mathcal{S}$, we have

$$p^w(x, hD_x)u(x) = \frac{1}{(2\pi h)^d} \iint_{\mathbb{R}^{2d}} e^{\frac{i}{h}\langle x-y, \xi \rangle} ({}^tL)^k \left(p\left(\frac{x+y}{2}, \xi; h\right) u(y) \right) dy d\xi,$$

where the convergence of the ξ -integral is guaranteed by the fact that

$$\left| ({}^tL)^k \left(p\left(\frac{x+y}{2}, \xi\right) u(y) \right) \right| \leq C_k m\left(\frac{x+y}{2}, \xi\right) \langle \xi \rangle^{-k} \leq \mathcal{O}(\langle \xi \rangle^{N_0-k}).$$

Remark 1.5. If the symbol p belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^{2d})$, then for every $u \in \mathcal{S}(\mathbb{R}^d)$,

$p^w u$ is an absolutely convergent Lebesgue integral. Indeed, for a fixed x , the map

$$(y, \xi) \mapsto p\left(\frac{x+y}{2}, \xi\right) u(y)$$

is Schwartz in (y, ξ) .

More generally, we introduce a family of quantizations depending on a parameter $t \in [0, 1]$:

$$Op_h^t(p)(x, hD_x)u(x) := \frac{1}{(2\pi h)^d} \iint_{\mathbb{R}^{2d}} e^{\frac{i}{h}\langle x-y, \xi \rangle} p(tx + (1-t)y, \xi) u(y) dy d\xi$$

For $t = 0$ and $t = 1$, we obtain the left and the right quantization of p respectively, while the choice $t = \frac{1}{2}$ corresponds to the Weyl quantization, which is of particular interest due to its favorable properties.

By applying integration by parts, we obtain the following theorem.

Theorem 1.6. *If $p \in S(m)$, then*

$$p^w(x, hD_x) : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$$

and

$$p^w(x, hD_x) : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$$

are continuous linear transformations.

Before proving the theorem, we recall the definition of semiclassical Fourier transform:

$$(\mathcal{F}_h u)(\xi) \stackrel{\text{def}}{=} \frac{1}{(2\pi h)^{d/2}} \int_{\mathbb{R}^d} e^{-i\frac{\langle x, \xi \rangle}{h}} u(x) dx, \quad u \in \mathcal{S}(\mathbb{R}^d), \quad (1.4)$$

In particular, $\mathcal{F}_h$ maps $\mathcal{S} \rightarrow \mathcal{S}$ continuously and can be extended to a continuous function $\mathcal{S}' \rightarrow \mathcal{S}'$, mapping $L^2 \rightarrow L^2$ unitarily.

Proof. We prove the result for $h = 1$. Define the auxiliary differential operators

$$L_1 := \frac{1 - \langle \xi, D_y \rangle}{1 + \|\xi\|_2^2}, \quad L_2 := \frac{1 + \langle x - y, D_\xi \rangle}{1 + \|x - y\|_2^2}.$$

A direct computation shows that

$$L_1 \left(e^{i\langle x-y, \xi \rangle} \right) = e^{i\langle x-y, \xi \rangle}, \quad L_2 \left(e^{i\langle x-y, \xi \rangle} \right) = e^{i\langle x-y, \xi \rangle}.$$

Let $u \in \mathcal{S}(\mathbb{R}^d)$ and $k \in \mathbb{N}$ with $k > d + N_0$, where N_0 is set in Definition 1.1. Using the notion of oscillatory integral (see Section A.1.1) extended to $\mathcal{S}$, we obtain,

$$p^w(x, D_x)u(x) = \frac{1}{(2\pi)^d} \iint_{\mathbb{R}^{2d}} e^{i\langle x-y, \xi \rangle} (L_1)^k \left(p\left(\frac{x+y}{2}, \xi\right) u(y) \right) dy d\xi,$$

Since $p \in \mathcal{S}(m)$ and $u \in \mathcal{S}(\mathbb{R}^d)$, there exists $C_k > 0$ such that

$$\left| ({}^tL_1)^k \left(p \left(\frac{x+y}{2}, \xi \right) u(y) \right) \right| \leq C_k \frac{\langle (\frac{x+y}{2}, \xi) \rangle^{N_0}}{\langle \xi \rangle^k}. \quad (1.5)$$

The decay in ξ allows us to integrate by parts using the operator L_2 , providing additional decay in the variable $x - y$. Iterating this procedure, for any integers $n, m \geq 0$, we obtain the following bound for the integrand

$$\left| ({}^tL_2)^n ({}^tL_1)^k \left(p \left(\frac{x+y}{2}, \xi \right) u(y) \right) \right| \leq C_{n,k,m} \frac{\langle (\frac{x+y}{2}, \xi) \rangle^{N_0}}{\langle \xi \rangle^k \langle x-y \rangle^n \langle y \rangle^m}. \quad (1.6)$$

Hence, by choosing the right values for k, m, n it can be shown that $p^w(x, D_x)u(x) \in L^\infty(\mathbb{R}^d)$. Furthermore,

$$x_j p^w(x, D_x)u(x) = \frac{1}{(2\pi)^d} \iint_{\mathbb{R}^{2d}} (D_{\xi_j} + y_j) e^{i\langle x-y, \xi \rangle} p \left(\frac{x+y}{2}, \xi \right) u(y) dy d\xi.$$

Using integration by part and proceeding as above, we can conclude that

$$x^\alpha p^w(x, D_x) : \mathcal{S}(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d), \quad (1.7)$$

for each polynomial x^α .

We recall the Fourier conjugation formula (Theorem A.5 in Appendix A),

$$\mathcal{F}^{-1} p^w(x, D_x) \mathcal{F} = p^w(D_x, -x).$$

Then, for every multiindex β ,

$$D_x^\beta p^w(x, D_x) = \mathcal{F}^{-1} \left(\xi^\beta \mathcal{F} p^w(x, D_x) \right) = \mathcal{F}^{-1} (\xi^\beta p^w(-D_\xi, \xi)) \mathcal{F}. \quad (1.8)$$

From (1.7), we know that for any $n \in \mathbb{N}^*$,

$$\langle x \rangle^{2n} x^\beta p^w(x, D_x) : \mathcal{S}(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d).$$

Thus, applying (1.8), we obtain that for all $u \in \mathcal{S}(\mathbb{R}^d)$ and for all $n \in \mathbb{N}$,

$$\langle \xi \rangle^{2n} \mathcal{F} \left(D_x^\beta p^w(x, D_x) u \right) = \langle \xi \rangle^{2n} \xi^\beta p^w(-D_\xi, \xi) \mathcal{F} u \in L^\infty(\mathbb{R}^d),$$

and hence

$$\partial^\beta p^w(x, D_x) : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{F}^{-1}(\langle \xi \rangle^{-2n} L^\infty(\mathbb{R}^d)).$$

If $2n > d + 1$, it follows that $\mathcal{F}^{-1}(\langle \xi \rangle^{-2n} L^\infty(\mathbb{R}^d)) \subset \mathcal{F}^{-1}(L^1(\mathbb{R}^d)) \subset L^\infty(\mathbb{R}^d)$, and therefore

$$\partial^\beta p^w(x, D_x) : \mathcal{S}(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d).$$

Similarly, it can be shown that $x^\alpha \partial^\beta p^w(x, D_x) : \mathcal{S}(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)$ for all multiindices α, β .

Moreover, the continuity follows from similar arguments using the seminorms. This proves the first statement of the theorem.

Next, we define the modified symbol

$$\tilde{p}(x, \xi) := p(x, -\xi) \in S(m).$$

Then, for all $u, v \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} \langle p^w(x, D_x)u, v \rangle_{S', S} &= \iiint_{\mathbb{R}^{3d}} p\left(\frac{x+y}{2}, \xi\right) e^{-i\langle x-y, \xi \rangle} u(y)v(x) dy d\xi dx \\ &= \iiint_{\mathbb{R}^{3d}} \tilde{p}\left(\frac{x+y}{2}, \eta\right) e^{-i\langle y-x, \eta \rangle} u(y)v(x) dy d\eta dx = \langle u, \tilde{p}^w(x, D_x)v \rangle_{S', S}. \end{aligned}$$

But we have already proved that $\tilde{p}^w(x, D_x)v \in \mathcal{S}(\mathbb{R}^d)$ for all $v \in \mathcal{S}(\mathbb{R}^d)$. Hence, the dual pairing above shows that $p^w(x, D_x)u$ is well defined in $S'(\mathbb{R}^d)$ for every $u \in S'(\mathbb{R}^d)$.

The continuity of p^w on $S'(\mathbb{R}^d)$ follows from the continuity on $\mathcal{S}(\mathbb{R}^d)$ and the definition of the topology on $S'(\mathbb{R}^d)$. $\square$

Remark 1.7. Theorem 4.21 in [Zw12] shows that if the symbol $p \in \mathcal{S}(\mathbb{R}^d)$, then

$$p^w(x, hD_x) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$$

is a bounded operator independently of h .

Now, given two order functions m_1, m_2 on $\mathbb{R}^{2d}$, define the bilinear, continuous h -Moyal product

$$\#_h : S(m_1) \times S(m_2) \rightarrow S(m_1 m_2),$$

given by

$$p \#_h q := e^{ihA(D)}(p(x, \xi)q(y, \eta))|_{y=x, \eta=\xi}, \quad (1.9)$$

where $A(D) := \frac{1}{2} \sigma(D_x, D_\xi; D_y, D_\eta)$, σ denotes the standard symplectic form on $\mathbb{R}^{2d}$, and we refer to [Zw12, Theorem 4.8] for the integral representation of the quantization of quadratic exponentials.

From Theorem 4.17 of [Zw12], the $\#_h$ -product admits the following asymptotic expansion in the symbol class $S(m_1 m_2)$:

$$p \#_h q \sim \sum_{k=0}^{+\infty} \frac{1}{k!} (ihA(D))^k p(x, \xi)q(y, \eta)|_{y=x, \eta=\xi} \in S(m_1 m_2). \quad (1.10)$$

Now we present Theorem 4.11 of [Zw12], concerning the composition of Weyl quantizations of

Schwartz functions.

Theorem 1.8. *Let $p, q \in \mathcal{S}(\mathbb{R}^{2d})$. Then*

$$p^w(x, hD_x)q^w(x, hD_x) = a^w(x, hD_x),$$

where

$$a(x, \xi) = e^{\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)}(p(x, \xi)q(y, \eta))\big|_{y=x, \eta=\xi} = (p \#_h q)(x, \xi). \quad (1.11)$$

Proof. Let $p \in \mathcal{S}(\mathbb{R}^{2d})$. We apply the h -Fourier transform (1.4), and we get

$$\widehat{p}(x^*, \xi^*) = \frac{1}{(2\pi h)^d} \iint_{\mathbb{R}^{2d}} e^{-\frac{i}{h}(\langle x, x^* \rangle + \langle \xi, \xi^* \rangle)} p(x, \xi) dx d\xi,$$

with inversion

$$p(x, \xi) = \frac{1}{(2\pi h)^d} \iint_{\mathbb{R}^{2d}} e^{\frac{i}{h}(\langle x, x^* \rangle + \langle \xi, \xi^* \rangle)} \widehat{p}(x^*, \xi^*) dx^* d\xi^*.$$

Let $r := (x^*, \xi^*) \in \mathbb{R}^{2d}$. By Lemma 4.10 in [Zw12], the Weyl quantization of p admits the following representation

$$p^w(x, hD_x) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} \widehat{p}(r) e^{\frac{i}{h}\ell_r(x, hD)} dr, \quad \text{where } \ell_r(x, \xi) := \langle r, (x, \xi) \rangle. \quad (1.12)$$

Taking also $q \in \mathcal{S}(\mathbb{R}^{2d})$ and $s := (y^*, \eta^*) \in \mathbb{R}^{2d}$, we obtain

$$p^w(x, hD_x) \circ q^w(x, hD_x) = \frac{1}{(2\pi h)^{2d}} \iint_{\mathbb{R}^{4d}} \widehat{p}(r) \widehat{q}(s) e^{\frac{i}{h}\ell_r(x, hD)} e^{\frac{i}{h}\ell_s(x, hD)} dr ds. \quad (1.13)$$

Using the identity (ii) in [Zw12, Theorem 4.7]

$$e^{\frac{i}{h}\ell_r} e^{\frac{i}{h}\ell_s} = e^{\frac{i}{2h}\sigma(r, s)} e^{\frac{i}{h}\ell_{r+s}}, \quad (1.14)$$

and observing that

$$\sigma(r, s) = \{\ell_r, \ell_s\}, \quad (1.15)$$

we get, from (1.13) and (1.12),

$$\begin{aligned} p^w(x, hD_x) \circ q^w(x, hD_x) &= \frac{1}{(2\pi h)^{2d}} \iint_{\mathbb{R}^{4d}} \widehat{p}(r) \widehat{q}(s) e^{\frac{i}{2h}\{\ell_r, \ell_s\}} e^{\frac{i}{h}(\ell_{r+s})(x, hD)} dr ds \\ &= \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} \widehat{a}(t) e^{\frac{i}{h}\ell_t(x, hD)} dt = a^w(x, hD_x), \end{aligned}$$

where

$$\widehat{a}(t) = \frac{1}{(2\pi h)^d} \int_{r+s=t} \widehat{p}(r) \widehat{q}(s) e^{\frac{i}{2h}\{\ell_r, \ell_s\}} dr.$$

Now set

$$\widetilde{a}(x, \xi) = e^{\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)}(p(x, \xi)q(y, \eta))\big|_{y=x, \eta=\xi}.$$

Using the Fourier representations of p and q as in (1.12) together with (1.14), one checks that

$$\begin{aligned}\tilde{a}(x, \xi) &= \frac{1}{(2\pi h)^{2d}} \iint_{\mathbb{R}^{4d}} e^{\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)} e^{\frac{i}{h}(\ell_r(x, \xi) + \ell_s(y, \eta))} e^{\frac{i}{2h}\sigma(r, s)} \Big|_{y=x, \eta=\xi} \hat{p}(r) \hat{q}(s) dr ds \\ &= \frac{1}{(2\pi h)^{2d}} \iint_{\mathbb{R}^{4d}} e^{\frac{i}{h}(\ell_r(x, \xi) + \ell_s(x, \xi))} e^{\frac{i}{2h}\sigma(r, s)} \hat{p}(r) \hat{q}(s) dr ds.\end{aligned}$$

Applying the semiclassical Fourier transform, we obtain

$$\begin{aligned}\widehat{\tilde{a}}(t) &= \frac{1}{(2\pi h)^{2d}} \iint_{\mathbb{R}^{4d}} \left(\frac{1}{(2\pi h)^d} \iint_{\mathbb{R}^{2d}} e^{\frac{i}{h}(\ell_r(x, \xi) + \ell_s(x, \xi) - \ell_t(x, \xi))} dx d\xi \right) e^{\frac{i}{2h}\sigma(r, s)} \hat{p}(r) \hat{q}(s) dr ds \\ &= \frac{1}{(2\pi h)^{2d}} \int_{r+s=t} \hat{p}(r) \hat{q}(s) e^{\frac{i}{2h}\{\ell_r, \ell_s\}} dr,\end{aligned}$$

where we used (1.15) and the fact that the term inside the parenthesis is $\delta_{r+s=t}$ in $\mathcal{S}'(\mathbb{R}^{2d})$. We have shown that $\widehat{\tilde{a}} = \widehat{a}$. Since the Fourier transform is an isomorphism, it follows that $\tilde{a} = a$, which completes the proof. $\square$

Proposition 1.9 (Theorem 4.18 of [Zw12]). *Let $p \in S(m_1)$, $q \in S(m_2)$, where m_1, m_2 are order functions. Then*

$$p^w(x, hD_x) \circ q^w(x, hD_x) = (p \#_h q)^w(x, hD_x),$$

as operators mapping $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$.

Proof. By Theorem 1.8, the statement holds for all $p, q \in \mathcal{S}(\mathbb{R}^{2d})$. The general case follows by density of $\mathcal{S}(\mathbb{R}^d)$ in $S(m)$ (see Remark 1.3), which allows to extend the identity to symbols in $S(m_1)$ and $S(m_2)$. $\square$

Remark 1.10. The bilinear operation $\#_h$ thus endows the set of symbols with an algebra structure, which is faithfully reflected at the operator level by the Weyl quantization.

Definition 1.11. A symbol $p \in S(m)$ is called elliptic if there exists a constant $C > 0$, independent of h , such that

$$|p(x, \xi)| \geq \frac{1}{C} m(x, \xi), \quad (x, \xi) \in \mathbb{R}^{2d}.$$

Remark 1.12. If $p \in S(m)$ is elliptic, then $\frac{1}{p} \in S(\frac{1}{m})$. In particular, applying the preceding results, we obtain

$$p^w(x, hD_x) \circ \left(\frac{1}{p} \right)^w(x, hD_x) = \left(p \#_h \frac{1}{p} \right)^w(x, hD_x),$$

where $p \#_h \frac{1}{p} \in S(1)$. Moreover, the semiclassical expansion (1.10) yields

$$p \#_h \frac{1}{p} \sim 1 + \sum_{k=1}^{+\infty} \frac{1}{k!} (ih)^k (A(D))^k \left(p(x, \xi) \frac{1}{p}(y, \eta) \right) \Big|_{y=x, \eta=\xi},$$

which shows in particular that

$$p\#_h \frac{1}{p} \sim 1 + h S(1) + \mathcal{O}_{S(1)}(h^2).$$

1.2 Quantization of the Torus

In this section, we describe the space of tempered distributions that are simultaneously periodic in both position and frequency variables. Our presentation follows the approach of [ChZw10, NoZw07], where the $\mathbb{Z}^d$ -periodicity was considered, and of [Vo20] for the rescaled torus.

Given $\alpha > 0$, we define

$$\mathbb{T}_\alpha^{2d} := \mathbb{R}^{2d} / (\alpha^{-1/2} \mathbb{Z})^{2d}. \quad (1.16)$$

When $\alpha = 1$, we simply write $\mathbb{T}^{2d} = \mathbb{T}_1^{2d}$. Throughout this section, we let $h \ll \alpha \leq 1$, and set $\tilde{h} = \frac{h}{\alpha}$.

We now introduce the space $\mathcal{H}_{h,\alpha}^d \subset \mathcal{S}'(\mathbb{R}^d)$, consisting of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ which are $\alpha^{-1/2} \mathbb{Z}^d$ -periodic both in position and in frequency. More explicitly, we require

$$u(x + \alpha^{-1/2} n) = u(x), \quad (\mathcal{F}_{\tilde{h}} u)(\xi + \alpha^{-1/2} n) = (\mathcal{F}_{\tilde{h}} u)(\xi), \quad \forall n \in \mathbb{Z}^d. \quad (1.17)$$

For $\alpha = 1$ we shall simply write $\mathcal{H}_h^d := \mathcal{H}_{h,1}^d$.

The following lemma characterizes precisely when this space is non-trivial, generalizing the results from the references above.

Lemma 1.13. *Let $h \ll \alpha \leq 1$. Then*

$$\mathcal{H}_{h,\alpha}^d \neq \{0\} \iff h = \frac{1}{2\pi N}, \text{ for some } N \in \mathbb{N}^*. \quad (1.18)$$

In this case $\mathcal{H}_{h,\alpha}^d$ is a finite-dimensional complex vector space of dimension N^d and

$$\mathcal{H}_{h,\alpha}^d = \text{span} \left\{ Q_k^\alpha := \frac{1}{(\alpha^{1/2} N)^{\frac{d}{2}}} \sum_{n \in \mathbb{Z}^d} \delta \left(x - \alpha^{-1/2} \left(n + \frac{k}{N} \right) \right); k \in (\mathbb{Z}/N\mathbb{Z})^d \right\}, \quad (1.19)$$

where δ denotes the Dirac distribution.

Proof. First, for $u \in \mathcal{S}'(\mathbb{R}^d)$, we introduce the notation

$$\hat{u}^{\tilde{h}} = (2\pi\tilde{h})^{d/2} \mathcal{F}_{\tilde{h}} u.$$

The proof relies on the Poisson summation formula (see [Hö83, Section 7.2]), which states that for $\psi \in \mathcal{S}(\mathbb{R}^d)$ and $a \in \mathbb{R} \setminus \{0\}$,

$$\sum_{g \in \mathbb{Z}^d} \hat{\psi}^{\tilde{h}}(ag) = \left(\frac{2\pi\tilde{h}}{a} \right)^d \sum_{g \in \mathbb{Z}^d} \psi \left(\frac{2\pi\tilde{h}}{a} g \right). \quad (1.20)$$

Let $\phi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ be such that $\sum_{g \in \mathbb{Z}^d} \phi(x - \alpha^{-1/2}g) = 1$. Such a function can be built in the following way: choose an open neighborhood $U \supset [-\frac{\alpha^{-1/2}}{2}, \frac{\alpha^{-1/2}}{2}]^d$ and $\psi \in C_c^\infty(U)$ such that $\psi \geq 0$ and $\psi(x) > 0$ for all $x \in [-\frac{\alpha^{-1/2}}{2}, \frac{\alpha^{-1/2}}{2}]^d$; consider the $\alpha^{-1/2}\mathbb{Z}^d$ -periodic sum $\Psi(x) := \sum_{g \in \mathbb{Z}^d} \psi(x - \alpha^{-1/2}g)$ and set $\phi = \frac{\psi}{\Psi}$.

Now, let $u \in \mathcal{H}_{h,\alpha}^d$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$. Since u is $\alpha^{-1/2}\mathbb{Z}^d$ -periodic in position, we obtain

$$\begin{aligned} \langle \hat{u}^{\tilde{h}}, \psi \rangle_{S',S} &= \langle u, \hat{\psi}^{\tilde{h}} \rangle_{S',S} = \langle u, \sum_{g \in \mathbb{Z}^d} \hat{\psi}^{\tilde{h}}(\cdot + \alpha^{-1/2}g) \phi \rangle_{S',S} \\ &= (2\pi\tilde{h}\alpha^{1/2})^d \langle u, \sum_{g \in \mathbb{Z}^d} \psi(2\pi\tilde{h}\alpha^{1/2}g) e^{-2\pi i \alpha^{1/2} \langle g, \cdot \rangle} \phi \rangle_{S',S}, \end{aligned}$$

where in the last step we used the fact that

$$\mathcal{F}_{\tilde{h}}(\psi(y) e^{-\frac{i}{\tilde{h}} \langle y, x \rangle})(\xi) = \hat{\psi}^{\tilde{h}}(\xi + x),$$

and we applied the Poisson summation formula (1.20) to $\psi(y) e^{-\frac{i}{\tilde{h}} \langle y, x \rangle}$ with $a = \alpha^{-1/2}$. This shows that

$$\hat{u}^{\tilde{h}} = (2\pi\tilde{h}\alpha^{1/2})^d \sum_{g \in \mathbb{Z}^d} c_g \delta_{2\pi\tilde{h}\alpha^{1/2}g},$$

with coefficients

$$c_g := \langle u, e^{-2\pi i \alpha^{1/2} \langle g, \cdot \rangle} \phi \rangle_{S',S} = \langle u, e^{-2\pi i \alpha^{1/2} \langle g, \cdot \rangle} \rangle_{\mathbb{T}_\alpha^d}, \quad (1.21)$$

where we see $u \in \mathcal{D}'(\mathbb{T}_\alpha^d)$ as a distribution on $\mathbb{T}_\alpha^d$ and δ_{x_0} denotes the Dirac measure at x_0 . Since by (1.17) $\hat{u}^{\tilde{h}}$ is $\alpha^{-1/2}\mathbb{Z}^d$ -periodic, it follows that

$$\hat{u}^{\tilde{h}} = \delta_{\alpha^{-1/2}g} * \hat{u}^{\tilde{h}}, \quad \forall g \in \mathbb{Z}^d,$$

where $*$ denotes the convolution between distributions. Hence, if we suppose $u \neq 0$, by comparing the supports of both the sides, we obtain the following equality

$$(2\pi\tilde{h}\alpha^{1/2})\mathbb{Z}^d = (\alpha^{-1/2})\mathbb{Z}^d + (2\pi\tilde{h}\alpha^{1/2})\mathbb{Z}^d,$$

and consequently

$$\mathbb{Z}^d = (2\pi\tilde{h}\alpha)^{-1}\mathbb{Z}^d + \mathbb{Z}^d.$$

Hence, necessarily $h = \frac{1}{2\pi N}$ for some $N \in \mathbb{N}^*$.

On the other hand, suppose that $h = \frac{1}{2\pi N}$ for $N \in \mathbb{N}^*$. By condition (1.17) on $\hat{u}^{\tilde{h}}$ and the injectivity of $\mathcal{F}_{\tilde{h}}$, we obtain

$$u(x) = e^{-\frac{i}{\tilde{h}} \alpha^{-1/2} \langle l, x \rangle} u(x), \quad \forall l \in \mathbb{Z}^d.$$

Combining this result with (1.21) yields

$$c_g = c_{g+NI}, \quad \forall l \in \mathbb{Z}^d.$$

Hence, it follows that

$$\begin{aligned} \hat{u}^{\tilde{h}} &= (2\pi\tilde{h}\alpha^{1/2})^d \sum_{j \in (\mathbb{Z}/N\mathbb{Z})^d} c_j \sum_{l \in \mathbb{Z}^d} \delta_{\alpha^{-1/2}(l+N^{-1}j)} \\ &= (2\pi\tilde{h}\alpha^{1/2})^d \sum_{j \in (\mathbb{Z}/N\mathbb{Z})^d} c_j \left(\delta_{\alpha^{-1/2}N^{-1}j} * \sum_{l \in \mathbb{Z}^d} \delta_{l\alpha^{-1/2}} \right). \end{aligned}$$

We observe that for $a \in \mathbb{R} \setminus \{0\}$, if we set $u_a = \sum_{g \in \mathbb{Z}^d} \delta_{ag}$, then

$$\hat{u}_a^{\tilde{h}} = \left(\frac{2\pi\tilde{h}}{a} \right)^d u_{2\pi\tilde{h}/a},$$

as shown in [Hö83, Theorem 7.2.1]. By applying this result and Fourier inversion formula, we get

$$\begin{aligned} u(x) &= (2\pi\tilde{h}\alpha^{1/2})^d (\alpha^{1/2})^d \sum_{j \in (\mathbb{Z}/N\mathbb{Z})^d} c_j \left((2\pi\tilde{h})^{d/2} \mathcal{F}_{\tilde{h}}^{-1}(\delta_{\alpha^{-1/2}N^{-1}j}) \mathcal{F}_{\tilde{h}}^{-1} \left((\alpha^{-1/2})^d u_{\alpha^{-1/2}} \right) \right) \\ &= N^{-d} \sum_{j \in (\mathbb{Z}/N\mathbb{Z})^d} c_j \sum_{l \in \mathbb{Z}^d} \exp \left(\frac{i}{\tilde{h}} N^{-1} \alpha^{-1/2} \langle j, x \rangle \right) \delta(x - 2\pi\tilde{h}\alpha^{1/2}l) \\ &= N^{-d} \sum_{j,k \in (\mathbb{Z}/N\mathbb{Z})^d} c_j \exp \left(2\pi i N^{-1} \langle j, k \rangle \right) \sum_{l' \in \mathbb{Z}^d} \delta(x - \alpha^{-1/2}(l' + N^{-1}k)), \end{aligned}$$

where, in the last equation, we used the identification $l = Nl' + k$, for $l' \in \mathbb{Z}^d$ and $k \in (\mathbb{Z}/N\mathbb{Z})^d$.

Thus, the condition of the lemma is also sufficient and (1.19) follows as well. $\square$

Remark 1.14. We observe that the Fourier transform $\mathcal{F}_{\tilde{h}}$ maps $\mathcal{H}_{h,\alpha}^d$ into $\mathcal{H}_{h,\alpha}^d$, and can be represented in the basis (1.19), by

$$(\mathcal{F}_{\tilde{h}})_{n,m} = \frac{e^{-\frac{2\pi i}{N} \langle n, m \rangle}}{N^{d/2}}, \quad n, m \in (\mathbb{Z}/N\mathbb{Z})^d. \quad (1.22)$$

1.3 Quantization of Symbols on the Torus

Throughout this section we assume $h \ll \alpha \leq 1$ and work on the torus $\mathbb{T}_{\alpha}^{2d}$. A function $m \in C^{\infty}(\mathbb{T}_{\alpha}^{2d},]0, +\infty[)$ is called an order function if there exist constants $C_0, N_0 > 0$, independent of α , such that

$$m(\rho) \leq C_0(1 + \|\rho - \mu\|_{\mathbb{T}_{\alpha}^{2d}}^2)^{N_0/2} m(\mu) =: C_0 \langle \rho - \mu \rangle_{\mathbb{T}_{\alpha}^{2d}}^{N_0} m(\mu), \quad \forall \rho, \mu \in \mathbb{T}_{\alpha}^{2d}, \quad (1.23)$$

where $\|\rho - \mu\|_{\mathbb{T}_\alpha^{2d}}^2 := \inf_{\gamma \in \alpha^{-1/2}\mathbb{Z}^{2d}} \|\rho - \mu + \gamma\|_2^2$. Via the natural projection $\mathbb{R}^{2d} \rightarrow \mathbb{T}_\alpha^{2d}$, we may regard m as a $\alpha^{-1/2}\mathbb{Z}^{2d}$ -periodic function in $C^\infty(\mathbb{R}^{2d},]0, +\infty[)$. In particular, m is an order function on $\mathbb{R}^{2d}$.

With this definition, we introduce the corresponding symbol class:

$$S(m, \alpha) \stackrel{\text{def}}{=} \left\{ a \in C^\infty(\mathbb{T}_\alpha^{2d}); \forall \beta \in \mathbb{N}^{2d} \exists C_\beta > 0 : |\partial_\rho^\beta a(\rho)| \leq C_\beta m(\rho), \forall \rho \in \mathbb{T}_\alpha^{2d} \right\}, \quad (1.24)$$

where the constants $C_\beta > 0$ are independent of α .

Identifying each symbol in $S(m, \alpha)$ with its $\alpha^{-1/2}\mathbb{Z}^{2d}$ -periodic extension in $C^\infty(\mathbb{R}^{2d})$, we have $S(m, \alpha) \subset S(m)$. This identification will be used throughout, and ensures that the quantization procedure described above applies to all $p \in S(m, \alpha)$.

Let $\gamma, \mu \in \alpha^{-1/2}\mathbb{Z}^d$ and consider the unitary operators

$$(\tau_\gamma u)(x) := u(x - \gamma), \quad (M_\mu u)(x) := e^{\frac{i}{h}\langle x, \mu \rangle} u(x),$$

Using the definition of Weyl quantization (1.3) and a direct change of variables, one checks that the following covariance relations are valid for each $p \in S(m, \alpha)$

$$\begin{aligned} \tau_\gamma p^w(x, \tilde{h}D_x) \tau_{-\gamma} &= (p(x - \gamma, \xi))^w(x, \tilde{h}D_x), \\ M_\mu p^w(x, \tilde{h}D_x) M_{-\mu} &= (p(x, \xi - \mu))^w(x, \tilde{h}D_x). \end{aligned}$$

Since $p \in S(m, \alpha)$ is periodic in both x and ξ with respect to $\alpha^{-1/2}\mathbb{Z}^d$, the right-hand sides coincide with $p^w(x, \tilde{h}D_x)$. Thus p^w commutes with all τ_γ and M_μ , and it follows that

$$p^w(x, \tilde{h}D_x) : \mathcal{H}_{h, \alpha}^d \longrightarrow \mathcal{H}_{h, \alpha}^d,$$

where $\mathcal{H}_{h, \alpha}^d \subset \mathcal{S}'(\mathbb{R}^d)$ is the space introduced in the previous section.

Let $h = \frac{1}{2\pi N}$ with $N \in \mathbb{N}^*$ and $\tilde{h} := \frac{h}{\alpha}$. We define the restriction of the operator to $\mathcal{H}_{h, \alpha}^d$ as

$$p_{N, \alpha} \stackrel{\text{def}}{=} p^w(x, \tilde{h}D_x)|_{\mathcal{H}_{h, \alpha}^d} \in \mathcal{L}(\mathcal{H}_{h, \alpha}^d, \mathcal{H}_{h, \alpha}^d). \quad (1.25)$$

When $\alpha = 1$, we simply write $p_N := p_{N, 1}$. In particular, we note that $1_{N, \alpha} = \text{Id}_{\mathcal{H}_{h, \alpha}^d}$.

Recalling the definition (1.9) of the product $\#_h$, we see that if $a, b \in S(m)$ are periodic then $a \#_h b$ is also periodic. Therefore Proposition 1.9 applies to symbols $(a, b) \in S(m_1, \alpha) \times S(m_2, \alpha)$, and yields

$$a_{N, \alpha} \circ b_{N, \alpha} = c_{N, \alpha}, \quad \text{where } c = a \#_{\tilde{h}} b \in S(m_1 m_2, \alpha).$$

The following lemma characterizes the Hilbert space structure of $\mathcal{H}_{h, \alpha}^d$.

Lemma 1.15 (Lemma 11 of [Vo20]). *There exists a unique Hilbert space structure on $\mathcal{H}_{h, \alpha}^d$, determined up to a multiplicative constant, such that every operator $f_{N, \alpha} : \mathcal{H}_{h, \alpha}^d \rightarrow \mathcal{H}_{h, \alpha}^d$ with $f \in C^\infty(\mathbb{T}_\alpha^{2d}; \mathbb{R})$ is self-adjoint. The constant can be chosen so that the basis defined in (1.19) is orthonormal. With this*

choice, the Fourier transform on $\mathcal{H}_{h,\alpha}^d$ defined in (1.22) is unitary.

From now on, we equip $\mathcal{H}_{h,\alpha}^d$ with the inner product that defines this structure, so that the basis (1.19) is orthonormal. Using this basis, we may identify

$$\mathcal{H}_{h,\alpha}^d \cong \ell^2((\mathbb{Z}/N\mathbb{Z})^d) \cong \mathbb{C}^{N^d}. \quad (1.26)$$

From the proof of Lemma 1.15 presented in [Vo20] we can derive the following result which was presented in the case $\alpha = 1$ in [ChZw10, Lemma 2.5].

Proposition 1.16. *Let $f \in S(m, \alpha)$. Then*

$$\mathrm{tr}(f_{N,\alpha}) = (N\alpha)^d \int_{\mathbb{T}_\alpha^{2d}} f(\rho) d\rho + r_N,$$

where for every $k \in \mathbb{N}$, there exists a constant $C_{k,d} > 0$, depending only on k and the dimension d , such that

$$|r_N| \leq C_{k,d} N^{d-k} \alpha^{d-\frac{k}{2}} \sum_{|\beta| \leq \max(2d+1, k)} \|\partial^\beta f\|_{L^1(\mathbb{T}_\alpha^{2d})}. \quad (1.27)$$

We conclude this section with the following proposition.

Proposition 1.17 (Proposition 13 of [Vo20]). *Let $N^{-1} \ll \alpha \leq 1$ and $p \in S(1, \alpha)$. Then, there exists a constant $C > 0$, independent of N and α , such that*

$$\|p_{N,\alpha}\| \leq C.$$

These results are presented as in [Vo20], where full proofs can be found.

The next sections develop the functional calculus for pseudo-differential operators. In the first, we introduce some important preliminary results; in the second, we consider a dilation of the phase space and establish a central estimate for a logarithmic determinant.

1.4 Preliminary Results for Functional Calculus

In this section we collect some auxiliary results that will play a central role in the following section.

Theorem 1.18 (Helffer-Sjöstrand formula). *Let P be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and let $f \in C_c^\infty(\mathbb{R})$. Suppose that $\tilde{f} \in C_c^\infty(\mathbb{C})$ is an almost analytic extension of f satisfying*

$$\tilde{f}|_{\mathbb{R}} = f, \quad (1.28)$$

and

$$\partial_{\bar{z}} \tilde{f}(z) = \mathcal{O}(|\Im(z)|^\infty). \quad (1.29)$$

Then, the following is true

$$f(P) = -\frac{1}{\pi} \int (z - P)^{-1} \partial_{\bar{z}} \tilde{f}(z) \lambda(dz), \quad (1.30)$$

where $\lambda(dz)$ denotes the Lebesgue measure on $\mathbb{C}$. Moreover, the integral in (1.30) converges in the sense of a Riemann integral for functions with values in $\mathcal{L}(\mathcal{H}, \mathcal{H})$.

Proof. We begin by constructing, for $f \in C_c^\infty(\mathbb{R})$ an almost analytic extension satisfying properties (1.28) and (1.29). The existence of such extensions goes back to Hörmander [Hö68] and his idea has subsequently been used by many authors. Here we follow the approach of Mather in [Ma71] and of Jensen and Nakamura in [JeNa94].

Choose $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ in a neighborhood of $\text{supp}(f)$, and let $\chi \in C_c^\infty(\mathbb{R})$ be equal to 1 near 0. Define

$$\tilde{f}(x + iy) := \frac{\psi(x)}{\sqrt{2\pi}} \int e^{i(x+iy)\xi} \chi(y\xi) \hat{f}(\xi) d\xi,$$

where $\hat{f}$ is the Fourier transform of f . Property (1.28) follows from the Fourier inversion formula.

Let $N \in \mathbb{N}$. To verify (1.29), we compute

$$\begin{aligned} \partial_{\bar{z}} \tilde{f}(x + iy) &= \frac{1}{2} \frac{\psi'(x)}{\sqrt{2\pi}} \int e^{i(x+iy)\xi} \chi(y\xi) \hat{f}(\xi) d\xi + \frac{i}{2} \frac{\psi(x)}{\sqrt{2\pi}} \int e^{i(x+iy)\xi} \chi'(y\xi) \xi \hat{f}(\xi) d\xi \\ &= \frac{1}{2} \frac{\psi'(x)}{\sqrt{2\pi}} \iint e^{i(x+iy-\tilde{x})\xi} \chi(y\xi) f(\tilde{x}) d\tilde{x} d\xi + y^N \frac{i}{2} \frac{\psi(x)}{\sqrt{2\pi}} \int e^{i(x+iy)\xi} \chi_N(y\xi) \xi^{N+1} \hat{f}(\xi) d\xi \\ &=: I + II, \end{aligned}$$

where $\chi_N(t) = t^{-N} \chi'(t) \in C_c^\infty(\mathbb{R})$. The second term satisfies the bound

$$|II| \leq C_N |y|^N \|\xi^{N+1} \hat{f}(\xi)\|_{L^1(\mathbb{R})}.$$

For the first term I , note that $x - \tilde{x} \neq 0$ on the support of $\psi'(x)f(\tilde{x})$. Repeated integration by parts yields

$$\begin{aligned} I &= \frac{1}{2\sqrt{2\pi}} \psi'(x) \iint D_{\xi}(e^{i(x-\tilde{x}+iy)\xi}) \frac{\chi(y\xi)}{x - \tilde{x} + iy} f(\tilde{x}) d\tilde{x} d\xi \\ &= \frac{i\psi'(x)}{2\sqrt{2\pi}} \iint e^{i(x-\tilde{x}+iy)\xi} \frac{\chi'(y\xi)y}{x - \tilde{x} + iy} f(\tilde{x}) d\tilde{x} d\xi \\ &= \frac{i\psi'(x)}{2\sqrt{2\pi}} y^N \iint e^{i(x-\tilde{x}+iy)\xi} \xi^N (\xi + i)^2 \frac{\chi_N(y\xi)y}{(x - \tilde{x} + iy)(\xi + i)^2} f(\tilde{x}) d\tilde{x} d\xi \\ &= \frac{i\psi'(x)}{2\sqrt{2\pi}} y^N \iint (i - D_{\tilde{x}})^2 (-D_{\tilde{x}})^N (e^{i(x-\tilde{x}+iy)\xi}) \frac{\chi_N(y\xi)y}{x - \tilde{x} + iy} f(\tilde{x}) \frac{1}{(\xi + i)^2} d\tilde{x} d\xi \\ &= \frac{i\psi'(x)y^N}{2\sqrt{2\pi}} \iint e^{i(x-\tilde{x}+iy)\xi} \frac{\chi_N(y\xi)y}{(\xi + i)^2} (i + D_{\tilde{x}})^2 D_{\tilde{x}}^N \left(\frac{f(\tilde{x})}{x - \tilde{x} + iy} \right) d\tilde{x} d\xi \\ &= \mathcal{O}(|y|^N). \end{aligned}$$

We also observe that if $\tilde{f}_1$ and $\tilde{f}_2$ are two almost analytic extension of f satisfying (1.28) and (1.29), then

$$\tilde{f}_1(z) - \tilde{f}_2(z) = \mathcal{O}(|\Im(z)|^\infty).$$

We conclude by deriving the representation (1.30) following the approach in [Di93]. Define

$$Q := -\frac{1}{\pi} \int (z - P)^{-1} \partial_{\bar{z}} \tilde{f}(z) \lambda(dz) \in \mathcal{L}(\mathcal{H}, \mathcal{H}),$$

and take $u, v \in \mathcal{H}$. We set $E_t = 1_{]-\infty, t]}(P)$ the family of spectral projections associated with P . Stone's formula gives

$$(z - P)^{-1} = \int (z - t)^{-1} dE_t \quad (1.31)$$

and thus

$$\langle (z - P)^{-1} u, v \rangle_{\mathcal{H}} = \int (z - t)^{-1} \langle dE_t u, v \rangle_{\mathcal{H}}.$$

Consequently,

$$\langle Qu, v \rangle_{\mathcal{H}} = -\frac{1}{\pi} \int \partial_{\bar{z}} \tilde{f}(z) \int (z - t)^{-1} \langle dE_t u, v \rangle_{\mathcal{H}} \lambda(dz),$$

and, by Fubini's theorem,

$$\langle Qu, v \rangle_{\mathcal{H}} = \int \left(-\frac{1}{\pi} \int \partial_{\bar{z}} \tilde{f}(z) (z - t)^{-1} \lambda(dz) \right) \langle dE_t u, v \rangle_{\mathcal{H}}.$$

Since $\frac{1}{\pi z}$ is a fundamental solution of $\partial_{\bar{z}}$, the inner integral is equal to

$$\int \tilde{f}(z) \partial_{\bar{z}} \left(-\frac{1}{\pi} (z - t)^{-1} \right) \lambda(dz) = f(t),$$

Therefore,

$$\langle Qu, v \rangle_{\mathcal{H}} = \int f(t) \langle dE_t u, v \rangle_{\mathcal{H}},$$

and hence $Q = f(P)$. □

We next recall Beals' Lemma in the semiclassical setting, following [Zw12]. This result will be crucial in the proof of the main theorem of the next section. Before considering the statement, we need the following definition.

Definition 1.19. If A, B are operators on a Hilbert space $\mathcal{H}$, we define

$$\text{ad}_B(A) := [B, A],$$

where $[\cdot, \cdot]$ is the commutator. In particular ad is called the adjoint action.

Theorem 1.20 (Semiclassical Beals' Theorem, [Zw12, Theorem 8.3]). *Let $A : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ be a continuous linear operator. Then, the following are equivalent:*

1. $A = a^w(x, hD_x)$, for a symbol $a \in S$.

2. For every $N \in \mathbb{N}$ and for all linear functions $l_1(x, \xi), \dots, l_N(x, \xi)$ on $\mathbb{R}^{2d}$, we have

$$\|\text{ad}_{l_1(x, hD_x)} \cdots \text{ad}_{l_N(x, hD_x)} A\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} = \mathcal{O}(h^N).$$

Theorem 1.21 (Theorem 8.6 of [Zw12]). *Let m be an order function and assume for $g := \ln m$ that*

$$|\partial^\alpha g| \leq C_\alpha, \quad \text{for all multiindices } |\alpha| \geq 1.$$

(i) *Then the equation*

$$\begin{cases} \partial_t B(t) = g^w(x, hD_x) B(t), \\ B(0) = I, \end{cases}$$

has a unique solution $B(t) : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ for $t \in \mathbb{R}$.

(ii) *Furthermore, we have*

$$B(t) = b_t^w(x, hD_x)$$

for a symbol

$$b_t \in S(m^t).$$

1.5 Phase Space Dilation and Logarithmic Estimates

We begin by recalling the functional calculus for pseudo-differential operators, as presented in [DiSj99, Section 8], adapted to symbols in the class $S(m, \alpha)$. First, we consider a result for self-adjoint semiclassical pseudo-differential operators, which has been proven in [ChZw10, Lemma 2.8] in the case when $N^{-\rho}C \leq \alpha \ll 1$, $\rho \in]0, 1[$. The following proposition, presented in [Vo20] is an extension of this result, which includes also the case when $\alpha = CN^{-1}$, $C \gg 1$.

Proposition 1.22 (Proposition 14 of [Vo20]). *Let $N^{-1} \ll \alpha \ll 1$ and let $m \geq 1$ be an order function on $\mathbb{T}_\alpha^{2d}$ satisfying (1.23). Let $0 \leq p \in S(m, \alpha)$, with asymptotic expansion $p \sim \sum_{j=0}^{\infty} \tilde{h}^j p_j$ in $S(m, \alpha)$, such that $p + i$ elliptic. We assume $p^w(x, \tilde{h}D_x; h, \alpha)$ is a self-adjoint semiclassical pseudo-differential operator. Then, for every $\psi \in C_c^\infty(\mathbb{R})$, there exists $f \in S(\frac{1}{m}, \alpha)$ such that*

$$\psi(p_{N, \alpha}) = f_{N, \alpha},$$

Moreover, f admits an asymptotic expansion

$$f \sim \sum_{j=0}^{+\infty} \tilde{h}^j f_j(\rho; \alpha) \text{ in } S\left(\frac{1}{m}, \alpha\right), \quad \text{with } f_j \in S\left(\frac{1}{m}, \alpha\right).$$

In particular, its principal symbol satisfies

$$f_0(\rho; \alpha) = \psi(p_0(\rho)),$$

and

$$f_j(\rho; \alpha) = \sum_{\nu=1}^{2j} g_\nu(\rho; \alpha) \psi^{(\nu)}(p_0(\rho)), \quad g_\nu \in S(1, \alpha),$$

for $j \in \mathbb{N}$.

Remark 1.23. In the statement of Proposition 1.22, the functions p, p_j may depend on α , even if not explicitly indicated. However, the constants in the symbol estimates (1.24) are uniform with respect to α .

From now on, we let $h = \frac{1}{2\pi N}$, $N \in \mathbb{N}^*$ and $N^{-1} \ll \alpha \ll 1$. Let $p \in C^\infty(\mathbb{T}^{2d})$ admit an asymptotic expansion $p \sim p_0 + hp_1 + \dots$ in $S(1, 1)$. The corresponding Weyl quantization $P = p^w(x, hD_x; h)$ defines a bounded operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Setting $Q = P^*P$, by [Zw12, Theorem 4.1] we have

$$(p^w(x, hD_x))^* = \bar{p}^w(x, hD_x).$$

Hence, by Proposition 1.9 there exists a symbol $q \in S(1, 1)$ such that

$$Q = q^w(x, hD_x; h), \tag{1.32}$$

where, from (1.10),

$$q = \bar{p} \#_h p \sim q_0 + hq_1 + \dots \quad \text{in } S(1, 1), \quad \text{with } q_0 = |p_0|^2. \tag{1.33}$$

We next introduce the scaling operator

$$(U_\alpha \phi)(x) := \alpha^{d/4} \phi(\alpha^{1/2} x), \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

The operator U_α is a continuous bijection $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$. It extends by duality to $\mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ and on $L^2(\mathbb{R}^d)$ is unitary with adjoint

$$U_\alpha^* = U_\alpha^{-1} = U_{\alpha^{-1}}.$$

Moreover, U_α maps $C^\infty(\mathbb{T}^{2d})$ continuously into $C^\infty(\mathbb{T}_\alpha^{2d})$ and acts unitarily between the spaces $\mathcal{H}_{h,1}^d$ and $\mathcal{H}_{h,\alpha}^d$ with respect to the inner products defined in Lemma 1.15. In particular, recalling (1.18), we have

$$U_\alpha Q_j^1 = Q_j^\alpha.$$

Using U_α we perform the phase space dilation

$$\mathbb{T}^{2d} \ni (x, \xi) = \alpha^{1/2}(\tilde{x}, \tilde{\xi}), \quad (\tilde{x}, \tilde{\xi}) \in \mathbb{T}_\alpha^{2d}.$$

Let $v \in \mathcal{S}(\mathbb{R}^d)$. We compute the conjugation of $q^w(x, hD_x)$ by U_α :

$$(U_\alpha q^w U_\alpha^{-1} v)(\tilde{x}) = \alpha^{d/4} \frac{1}{(2\pi h)^d} \iint e^{\frac{i}{h} \langle \alpha^{1/2} \tilde{x} - y, \zeta \rangle} q\left(\frac{\alpha^{1/2} \tilde{x} + y}{2}, \zeta\right) \alpha^{-d/4} v(\alpha^{-1/2} y) dy d\zeta.$$

With the change of variables $y = \alpha^{1/2} \tilde{y}$, $\zeta = \alpha^{1/2} \tilde{\zeta}$, so that $dy d\zeta = \alpha^d d\tilde{y} d\tilde{\zeta}$, we obtain

$$\begin{aligned} (U_\alpha q^w U_\alpha^{-1} v)(\tilde{x}) &= \frac{\alpha^d}{(2\pi h)^d} \iint e^{\frac{i}{h} \alpha^{1/2} \langle \tilde{x} - \tilde{y}, \alpha^{1/2} \tilde{\zeta} \rangle} q\left(\alpha^{1/2} \frac{\tilde{x} + \tilde{y}}{2}, \alpha^{1/2} \tilde{\zeta}\right) v(\tilde{y}) d\tilde{y} d\tilde{\zeta} \\ &= \frac{1}{(2\pi \tilde{h})^d} \iint e^{\frac{i}{\tilde{h}} \langle \tilde{x} - \tilde{y}, \tilde{\zeta} \rangle} q\left(\alpha^{1/2} \frac{\tilde{x} + \tilde{y}}{2}, \alpha^{1/2} \tilde{\zeta}\right) v(\tilde{y}) d\tilde{y} d\tilde{\zeta}, \end{aligned}$$

where we used the identity $\tilde{h} = h/\alpha$. Hence,

$$q^w(x, hD_x; h) = U_\alpha^{-1} (q(\alpha^{1/2}(\tilde{x}, \tilde{\zeta}); \tilde{h}))^w(\tilde{x}, \tilde{h}D_{\tilde{x}}) U_\alpha. \quad (1.34)$$

Writing

$$\tilde{q}(\tilde{\rho}; h) := q(\alpha^{1/2} \tilde{\rho}; h) \in S(1, \alpha),$$

we conclude from the mapping properties of U_α that

$$q_N = q^w(x, hD_x; h)|_{\mathcal{H}_{h,1}^d} = U_\alpha^{-1} \left(\tilde{q}^w(\tilde{x}, \tilde{h}D_{\tilde{x}}; h)|_{\mathcal{H}_{h,\alpha}^d} \right) U_\alpha = U_\alpha^{-1} \tilde{q}_{N,\alpha} U_\alpha. \quad (1.35)$$

Now we follow the approach of [HaSj08, Section 4] and we introduce an order function adapted to the rescaled symbol $\alpha^{-1} \tilde{q}$. We set

$$m(\rho) := 1 + \frac{q_0(\alpha^{1/2} \rho)}{\alpha} \geq 1, \quad \rho \in \mathbb{T}_\alpha^{2d}. \quad (1.36)$$

Since $q_0 \in S(1, 1)$, it follows that $m \in C^\infty(\mathbb{T}_\alpha^{2d})$.

To verify that m satisfies the order function estimates (1.23), we compute its derivatives. For every multiindices β such that $|\beta| = 1$, using $q_0 = |p_0|^2$, we obtain

$$\partial_\rho^\beta m(\rho) = \frac{(\partial_\rho^\beta q_0)(\alpha^{1/2} \rho)}{\alpha^{1/2}} = \frac{(2\Re(p_0(\alpha^{1/2} \rho)(\partial_\rho^\beta p_0)(\alpha^{1/2} \rho))}{\alpha^{1/2}} \leq C_1 \frac{q_0^{1/2}(\alpha^{1/2} \rho)}{\alpha^{1/2}} \leq C_1 (m(\rho))^{1/2}.$$

For $|\beta| = 2$,

$$\partial_\rho^\beta m(\rho) = (\partial_\rho^\beta q_0)(\alpha^{1/2} \rho) \leq C_2,$$

where the constants $C_1, C_2 > 0$ are independent of α . Applying Taylor's theorem up to second order gives, for all $\rho, \mu \in \mathbb{R}^{2d}$

$$m(\rho) \leq m(\mu) + C_1 (m(\rho))^{1/2} \|\rho - \mu\|_2 + C_2 \|\rho - \mu\|_2^2.$$

Since $m \geq 1$, the mixed term can be absorbed as

$$(m(\rho))^{1/2} \|\rho - \mu\|_2 \leq \frac{1}{2} m(\rho) + C_3 \|\rho - \mu\|_2^2,$$

so that

$$m(\rho) \leq m(\mu) + \frac{1}{2} m(\rho) + C_4 \|\rho - \mu\|_2^2 \implies m(\rho) \leq C m(\mu) \langle \rho - \mu \rangle^2, \quad \forall \rho, \mu \in \mathbb{R}^{2d}.$$

Using the $\alpha^{-1/2} \mathbb{Z}^{2d}$ -translation invariance of m , we obtain that for all $\rho', \mu' \in \mathbb{T}_\alpha^{2d}$ and any $\gamma \in \mathbb{Z}^{2d}$,

$$m(\rho') = m(\rho' + \alpha^{-1/2} \gamma) \leq C m(\mu') \langle \rho' - \mu' - \alpha^{-1/2} \gamma \rangle^2.$$

Taking the infimum over γ shows that m indeed satisfies the order function condition (1.23).

Recalling the asymptotic expansion of q in (1.33), we obtain

$$\begin{aligned} \frac{q(\alpha^{1/2} \rho)}{\alpha} &= \frac{q_0(\alpha^{1/2} \rho)}{\alpha} + \mathcal{O}\left(\frac{h}{\alpha}\right) \leq m(\rho) + \mathcal{O}\left(\frac{h}{\alpha}\right) = \mathcal{O}(1) m(\rho) \\ \partial_\rho^\beta \left(\frac{q(\alpha^{1/2} \rho)}{\alpha} \right) &= \mathcal{O}(1) m^{1/2}(\rho), \quad |\beta| = 1. \end{aligned}$$

Moreover, for derivatives of order greater than 2

$$\partial_\rho^\beta \left(\frac{q(\alpha^{1/2} \rho)}{\alpha} \right) = \mathcal{O}(1) \alpha^{\frac{|\beta|}{2} - 1}, \quad |\beta| \geq 2.$$

Thus, for every multiindex β

$$\partial_\rho^\beta \left(\frac{q(\alpha^{1/2} \rho)}{\alpha} \right) = \mathcal{O}_\beta(1) m(\rho), \tag{1.37}$$

with the constants $\mathcal{O}_\beta(1)$ independent of α . Hence

$$\frac{q(\alpha^{1/2} \rho; h)}{\alpha} \in S(m, \alpha).$$

Introducing the notations

$$\tilde{q}_\nu(\rho) := \alpha^{\nu-1} q_\nu(\alpha^{1/2} \rho), \quad \nu \in \mathbb{N},$$

we therefore have the asymptotic expansion

$$\alpha^{-1} \tilde{q} \sim \sum_{\nu=0}^{+\infty} \tilde{h}^\nu \tilde{q}_\nu, \quad \text{in } S(m, \alpha).$$

Thus, we observe that there exists a constant $C_0 > 0$ such that

$$\sum_{\nu=1}^{+\infty} \tilde{h}^\nu |\tilde{q}_\nu(\rho)| \leq C_0 \tilde{h} m,$$

which implies,

$$|\alpha^{-1}\tilde{q}(\rho) + i| \geq \left| |\tilde{q}_0(\rho) + i| - \sum_{v=1}^{+\infty} \tilde{h}^v |\tilde{q}_v(\rho)| \right| \geq |\tilde{q}_0(\rho) + i| - C_0 m \tilde{h},$$

Hence, since $\tilde{q}_0$ is real, from (1.36) we have

$$|\alpha^{-1}\tilde{q}(\rho) + i| \geq \frac{1}{\sqrt{2}} (\tilde{q}_0(\rho) + 1) - C_0 m \tilde{h} \geq \frac{m(\rho)}{C}.$$

for some constant $C > 0$ independent of α . Therefore $\tilde{q} + i$ is elliptic with respect to the order function m , uniformly in α and $\tilde{q}^w$ is a self-adjoint operator as a consequence of (1.34). Thus, by Proposition 1.22, for every $\psi \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned} \psi(\alpha^{-1}\tilde{q}_{N,\alpha}) &= f_{N,\alpha}, \quad f \in S\left(\frac{1}{m}, \alpha\right), \\ f &\sim \sum_{v=0}^{+\infty} \tilde{h}^v f_v(\rho; \alpha) \text{ in } S\left(\frac{1}{m}, \alpha\right), \quad f_v \in S\left(\frac{1}{m}, \alpha\right), \end{aligned} \quad (1.38)$$

with $f_0(\rho; \alpha) = \psi\left(\frac{q_0(\alpha^{1/2}\rho)}{\alpha}\right)$ and

$$f_v(\rho; \alpha) = \sum_{j=1}^{2v} g_j(\rho; \alpha) \psi^{(j)}\left(\frac{q_0(\alpha^{1/2}\rho)}{\alpha}\right), \quad g_j \in S(1, \alpha).$$

We now recall [HaSj08, Proposition 4.1], adapted to our setting.

Proposition 1.24. *Let $\psi \in C_c^\infty(\mathbb{R})$ and $\tilde{m} \in C^\infty(\mathbb{T}_\alpha^{2d},]0, +\infty[)$ be an order function such that*

$$\tilde{m}(\rho) = 1 \quad \text{whenever} \quad \alpha^{-1}q_0(\alpha^{1/2}\rho) \leq \sup \text{supp } \psi + 1/C,$$

for some $C > 0$ independent of α . Then, (1.38) holds in $S(\tilde{m}, \alpha)$, for $h, \tilde{h}$ sufficiently small.

Proof. The proof of Proposition 4.1 in [HaSj08] relies on the Helffer-Sjöstrand formula (Theorem 1.18) and standard semiclassical calculus. It translates directly to our setting using the notions discussed above. $\square$

We now state the main result concerning log-determinant estimates, which will play a crucial role later in this dissertation. The proof is presented in [Vo20, Proposition 17]

Theorem 1.25. *Let $N \in \mathbb{N}^*$ with $N^{-1} \ll \alpha \ll 1$, and let the symbol q be as in (1.33). Assume that there exists $\kappa \in]0, 1]$ such that the t -dependent volume*

$$V(t) := \lambda\left(\{\rho \in \mathbb{T}^{2d}; q_0(\rho) \leq t\}\right) = \mathcal{O}(t^\kappa), \quad 0 \leq t \ll 1. \quad (1.39)$$

Then, for every $\psi \in C_c^\infty(\mathbb{R})$,

$$\text{tr } \psi\left(\frac{\tilde{q}_{N,\alpha}}{\alpha}\right) = N^d \left(\int \psi\left(\frac{q_0}{\alpha}\right) dV(q_0) + \mathcal{O}(N\alpha)^{-1} \alpha^\kappa \right). \quad (1.40)$$

Moreover, taking $\chi \in C_c^\infty([0, +\infty[, [0, +\infty[)$ with $\chi(0) > 0$, we have

$$\ln \det \left(q_N + \alpha \chi \left(\frac{q_N}{\alpha} \right) \right) = N^d \left(\int_{\mathbb{T}^{2d}} \ln q_0(\rho) d\rho + \mathcal{O} \left(\alpha^\kappa \ln \left(\frac{1}{\alpha} \right) \right) \right). \quad (1.41)$$

Remark 1.26. Considering the same assumptions of Theorem 1.25, the equality (1.40) is particularly useful for estimating the number of eigenvalues of q_N in the interval $[0, \alpha]$. Indeed, let $\psi \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $\psi \equiv 1$ on $[0, 1]$ and $\psi \equiv 0$ outside $[-\frac{1}{2}, \frac{3}{2}]$. Thus, in the right-hand side of (1.40) we obtain

$$\begin{aligned} N^d \left(\int \psi \left(\frac{q_0}{\alpha} \right) dV(q_0) + \mathcal{O}(N\alpha)^{-1} \alpha^\kappa \right) &\leq N^d \left(\int \mathbb{1}_{[-\frac{1}{2}, \frac{3}{2}]} \left(\frac{q_0}{\alpha} \right) dV(q_0) + \mathcal{O}(N\alpha)^{-1} \alpha^\kappa \right) \\ &= N^d \left(\int_0^{\frac{3}{2}\alpha} dV(q_0) + \mathcal{O}(N\alpha)^{-1} \alpha^\kappa \right) \\ &= N^d \left(V \left(\frac{3}{2}\alpha \right) + \mathcal{O}(N\alpha)^{-1} \alpha^\kappa \right) \\ &= \mathcal{O}(N^d \alpha^\kappa), \end{aligned}$$

by assumption (1.39). Furthermore, the left-hand side of (1.40) is bounded from below in the following way

$$\begin{aligned} \operatorname{tr} \left(\psi \left(\frac{\tilde{q}_{N,\alpha}}{\alpha} \right) \right) &\geq \operatorname{tr} \left(\mathbb{1}_{[0,1]} \left(\frac{\tilde{q}_{N,\alpha}}{\alpha} \right) \right) = \operatorname{tr} \left(\mathbb{1}_{[0,1]} \left(\frac{q_N}{\alpha} \right) \right) \\ &= \operatorname{tr} \left(\mathbb{1}_{[0,\alpha]} (q_N) \right) = \sum_{\zeta \in \sigma(\mathbb{1}_{[0,\alpha]}(q_N))} \zeta \\ &= \#\{\zeta \in \sigma(q_N) : \zeta \in [0, \alpha]\}, \end{aligned}$$

where we used the fact that the spectrum is invariant under the unitary conjugation (1.35). Therefore,

$$\#\{\zeta \in \sigma(q_N) : \zeta \in [0, \alpha]\} = \mathcal{O}(N^d \alpha^\kappa). \quad (1.42)$$

Chapter 2

Grushin Problems and Schur Complement Methods

In this chapter we set up a Grushin problem for the operator

$$P(z) := p_{N,\alpha} - z, \quad z \in \mathbb{C},$$

following the definition (1.25) in Chapter 1. We then extend the analysis to the case where $P(z)$ is perturbed by a small potential, formulating a modified Grushin problem. The results obtained here will later be applied to the analysis of random perturbations of $P(z)$.

2.1 A General Grushin Problem and the Schur Complement Formula

We begin by giving a short overview on Grushin problems. For more general details see for instance [SjZw07].

In a general finite-dimensional setting, we consider an operator

$$P : \mathcal{H} \longrightarrow \mathcal{H},$$

where $\mathcal{H}$ is a Hilbert space with $\dim \mathcal{H} < +\infty$. A priori we don't know if P is invertible and we would like to study and provide an estimate for $\det P$. To address this, we introduce the operator

$$\mathcal{P} := \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H} \oplus \mathcal{H}_- \longrightarrow \mathcal{H} \oplus \mathcal{H}_+, \quad (2.1)$$

where $\mathcal{H}_+$ and $\mathcal{H}_-$ are finite-dimensional Hilbert spaces, and

$$R_+ : \mathcal{H} \rightarrow \mathcal{H}_+, \quad R_- : \mathcal{H}_- \rightarrow \mathcal{H},$$

are suitably chosen operators such that $\mathcal{P}$ is bijective. If $\dim \mathcal{H}_+ = \dim \mathcal{H}_- < +\infty$, we may

write

$$\mathcal{P}^{-1} = \mathcal{E} := \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}. \quad (2.2)$$

We now turn to the question of the invertibility of P . A natural tool in this setting is the Schur complement formula, which relates the invertibility of P to that of the finite-dimensional block E_{-+} . This connection plays a central role in the analysis of the Grushin problem and allows us to extract useful information about P .

Proposition 2.1 (Schur Complement Formula). *Let $A \in M_{r,r}$, $C \in M_{r,s}$, $R \in M_{s,r}$, and $B \in M_{s,s}$. Consider the block matrix*

$$\mathcal{A} := \begin{pmatrix} A & C \\ R & B \end{pmatrix} \in M_{r+s,r+s}.$$

Suppose A is invertible. We define the Schur complement of A as

$$S := B - RA^{-1}C.$$

Under this assumption, the following determinant factorization holds:

$$\det(\mathcal{A}) = \det(A) \det(S). \quad (2.3)$$

Moreover, if both A and S are nonsingular, then $\mathcal{A}$ is bijective and

$$\mathcal{A}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}CS^{-1}RA^{-1} & -A^{-1}CS^{-1} \\ -S^{-1}RA^{-1} & S^{-1} \end{pmatrix}. \quad (2.4)$$

Remark 2.2. In the same setting of Proposition 2.1, if A is invertible, from (2.3) we obtain the equivalence

$$\mathcal{A} \text{ is invertible} \Leftrightarrow S \text{ is invertible}.$$

Proof. We begin by noting that

$$E_1 \mathcal{A} E_2 = L,$$

where

$$L := \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}, \quad E_1 := \begin{pmatrix} I & 0 \\ -RA^{-1} & I \end{pmatrix}, \quad E_2 := \begin{pmatrix} I & -A^{-1}C \\ 0 & I \end{pmatrix}.$$

Since E_1 and E_2 are block triangular with identity blocks on the diagonal, we have $\det(E_1) = \det(E_2) = 1$. Thus,

$$\det \mathcal{A} = \det(L) = \det(A) \det(S).$$

Finally, the inverse formula (2.4) follows from a straightforward verification. $\square$

We now apply Proposition 2.1 to the setting introduced in (2.1) and (2.2). If P is invertible, its Schur complement is

$$S = -R_+ P^{-1} R_-.$$

We recall that $R_{\pm}$ were chosen so that $\mathcal{P}$ is invertible. Thus, Remark 2.2 implies that S is invertible and, by (2.2) and (2.4), we obtain

$$E_{-+} = S^{-1} = (-R_+ P^{-1} R_-)^{-1}. \quad (2.5)$$

Corollary 2.3. *Let $A \in M_{r,r}$, $C \in M_{r,s}$, $R \in M_{s,r}$, and $B \in M_{s,s}$. Suppose the block matrix*

$$\mathcal{A} := \begin{pmatrix} A & C \\ R & B \end{pmatrix} \in M_{r+s, r+s},$$

to be invertible with inverse

$$\mathcal{M} = \begin{pmatrix} M & M_+ \\ M_- & M_{-+} \end{pmatrix} \in M_{r+s, r+s}.$$

Then the following factorization holds:

$$\det(\mathcal{A}) \det(M_{-+}) = \det(A). \quad (2.6)$$

In particular,

$$A \text{ is invertible} \Leftrightarrow M_{-+} \text{ is invertible}.$$

Proof. Since A is a square matrix,

$$\dim \ker A = \dim \operatorname{coker} A.$$

Choose a complement Y of $\operatorname{Im} A$ in $\mathbb{C}^r$, so that

$$\mathbb{C}^r = \operatorname{Im} A \oplus Y, \quad \dim Y = \dim \ker A.$$

Fix an isomorphism $T : \ker A \rightarrow Y$, and let $\Pi_{\ker A}$ denote the projection of $\mathbb{C}^r$ onto $\ker A$. We then define the operator

$$Q := T \circ \Pi_{\ker A} : \mathbb{C}^r \rightarrow \mathbb{C}^r.$$

By construction, for every $\varepsilon > 0$, the perturbed operator

$$A + \varepsilon Q$$

is invertible. Applying Proposition 2.1 to the perturbed block matrix

$$\mathcal{A}^\varepsilon := \begin{pmatrix} A + \varepsilon Q & C \\ R & B \end{pmatrix},$$

we obtain, from (2.3),

$$\det(\mathcal{A}^\varepsilon) = \det(A + \varepsilon Q) \det(S^\varepsilon), \quad (2.7)$$

where $S^\varepsilon := B - R(A + \varepsilon Q)^{-1}C$. Moreover,

$$\mathcal{A}^\varepsilon \mathcal{M} = \mathcal{A} \mathcal{M} + \varepsilon \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \mathcal{M} = I_{r+s} + \varepsilon \begin{pmatrix} QM & QM_+ \\ 0 & 0 \end{pmatrix}.$$

For ε sufficiently small, we may assume that

$$\left\| \varepsilon \begin{pmatrix} QM & QM_+ \\ 0 & 0 \end{pmatrix} \right\| < 1.$$

Hence, by the Neumann series, $\mathcal{A}^\varepsilon \mathcal{M}$ is bijective with inverse

$$\begin{aligned} (\mathcal{A}^\varepsilon \mathcal{M})^{-1} &= I_{r+s} + \sum_{k=1}^{+\infty} (-\varepsilon)^k \begin{pmatrix} QM & QM_+ \\ 0 & 0 \end{pmatrix}^k \\ &= I_{r+s} + \sum_{k=1}^{+\infty} (-\varepsilon)^k \begin{pmatrix} (QM)^k & (QM)^{k-1}QM_+ \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Consequently, $\mathcal{A}^\varepsilon$ is bijective with inverse

$$\begin{aligned} \mathcal{M}^\varepsilon := \mathcal{M}(\mathcal{A}^\varepsilon \mathcal{M})^{-1} &= \mathcal{M} + \sum_{k=1}^{+\infty} (-1)^k \begin{pmatrix} M(\varepsilon QM)^k & (\varepsilon MQ)^k M_+ \\ M_-(\varepsilon QM)^k & M_-(\varepsilon QM)^{k-1}(\varepsilon Q)M_+ \end{pmatrix} \\ &=: \begin{pmatrix} M^\varepsilon & M_+^\varepsilon \\ M_-^\varepsilon & M_{-+}^\varepsilon \end{pmatrix}, \end{aligned} \tag{2.8}$$

where to identify M_+^ε we used

$$M(QM)^{k-1}Q = (MQ)^k, \quad \forall k \geq 1,$$

which follows from the associativity of matrix multiplication. Consequently, we observe that

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{M}^\varepsilon - \mathcal{M}\| = 0. \tag{2.9}$$

Moreover, from (2.7) we know that the Schur complement S^ε is invertible, and thus by (2.4)

$$\det(\mathcal{A}^\varepsilon) \det(M_{-+}^\varepsilon) = \det(A + \varepsilon Q). \tag{2.10}$$

Since this equality holds for all sufficiently small $\varepsilon > 0$, taking the limit as $\varepsilon \rightarrow 0$ and using the continuity of the determinant yields

$$\det(\mathcal{A}) \det(M_{-+}) = \det(A),$$

where we used (2.9). □

Now, we can also apply Corollary 2.3 to the setting introduced in (2.1) and (2.2). Thus,

without requiring the invertibility of P , we obtain that

$$\det(\mathcal{P}) \det(E_{-+}) = \det(P). \quad (2.11)$$

2.2 Grushin Problem for the Unperturbed Operator

In this section, we construct a Grushin problem for the operator obtained by restricting the Weyl quantization of a symbol p on the torus to the space of periodic distributions, making use of its left and right singular vectors.

We begin by fixing

$$h = \frac{1}{2\pi N}, \quad N \in \mathbb{N}^*, \quad \text{and} \quad N^{-1} \ll \alpha \ll 1, \quad (2.12)$$

and we consider a symbol $p \in C^\infty(\mathbb{T}^{2d})$ admitting a semiclassical expansion

$$p \sim p_0 + hp_1 + \dots \quad \text{in } S(1,1). \quad (2.13)$$

We recall the identification $\mathcal{H}_h^d \cong \ell^2((\mathbb{Z}/N\mathbb{Z})^d) \cong \mathbb{C}^{N^d}$ from (1.26), and define

$$P := p_N : \mathbb{C}^{N^d} \rightarrow \mathbb{C}^{N^d}, \quad (2.14)$$

as in (1.25). We fix $z \in \mathbb{C}$ and consider the operators

$$Q(z) := (P - z)^*(P - z), \quad Q'(z) := (P - z)(P - z)^*. \quad (2.15)$$

Let

$$0 \leq t_1^2 \leq t_2^2 \leq \dots \leq t_{N^d}^2$$

denote the eigenvalues of $Q(z)$ with corresponding orthonormal eigenvectors $e_1, \dots, e_{N^d} \in \mathcal{H}_h^d$. Since P is a square matrix, we have $\text{rank}(P - z) = \text{rank}(P - z)^*$, and consequently

$$\dim \ker(P - z) = \dim \ker(P - z)^*.$$

Observing that $\ker Q(z) = \ker(P - z)$ and $\ker Q'(z) = \ker(P - z)^*$, it follows that

$$\dim \ker Q(z) = \dim \ker Q'(z) =: N_0.$$

Let $\{f_1, \dots, f_{N_0}\}$ be an orthonormal basis of $\ker(P - z)^* = \ker Q'(z)$, and define

$$f_i := t_i^{-1}(P - z)e_i, \quad N_0 < i \leq N^d. \quad (2.16)$$

These vectors are well-defined since $t_i > 0$ for all $i > N_0$. For such an i , we have

$$Q'(z)f_i = t_i^{-1}(P - z)Q(z)e_i = t_i^{-1}(P - z)t_i^2 e_i = t_i^2 f_i.$$

Hence, $Q(z)$ and $Q'(z)$ share the same spectrum $\{t_i^2\}_{i=1}^{N^d}$. Furthermore, we observe that

$$\begin{aligned} \langle f_i, f_j \rangle &= \langle (P - z)^* f_i, t_j^{-1} e_j \rangle = \langle t_i e_i, t_j^{-1} e_j \rangle = \delta_{ij}, \quad i, j > N_0, \\ \langle f_i, f_j \rangle &= \langle (P - z)^* f_i, t_j^{-1} e_j \rangle = \langle 0, t_j^{-1} e_j \rangle = 0, \quad i \leq N_0, j > N_0. \end{aligned}$$

Thus, $\{f_i\}_{i=1}^{N^d}$ form an orthonormal basis of $\mathcal{H}_h^d$ consisting of eigenvectors of $Q'(z)$ with eigenvalues $\{t_i^2\}_{i=1}^{N^d}$. Moreover, we record the relations

$$(P - z)^* f_i = t_i e_i, \quad (P - z)e_i = t_i f_i, \quad i = 1, \dots, N^d. \quad (2.17)$$

Applying (1.32) and (1.33) to $Q(z)$, we obtain

$$Q(z) = q^w(x, hD_x; h), \quad q \sim q_0 + hq_1 + \dots \quad \text{in } S(1, 1), \quad \text{with } q_0 = |p_0 - z|^2. \quad (2.18)$$

Let α be as in (2.12). Fix $M > 0$ so that

$$0 \leq t_1^2 \leq t_2^2 \leq \dots \leq t_M^2 \leq \alpha < t_{M+1}^2, \quad (2.19)$$

and let $\{\delta_i\}_{i=1}^M$ be an orthonormal basis of $\mathbb{C}^M$. Assuming that the principal symbol q_0 of $Q(z)$ satisfies (1.39), Theorem 1.25 and Remark (1.26) yield

$$M = \mathcal{O}(N^d \alpha^\kappa). \quad (2.20)$$

Remark 2.4. From the proof of Theorem 1.25 in [Vo20, Proposition 17] it follows that, if z varies in a compact set $K \Subset \mathbb{C}$, and the condition (1.39) holds uniformly for $z \in K$, then the estimate (2.20) holds uniformly in $z \in K$.

We now define the operators

$$R_+ : \mathcal{H}_h^d \longrightarrow \mathbb{C}^M : \quad u \longmapsto \sum_{i=1}^M \langle u, e_i \rangle \delta_i, \quad (2.21)$$

and

$$R_- : \mathbb{C}^M \longrightarrow \mathcal{H}_h^d : \quad u_- \longmapsto \sum_{i=1}^M u_-(i) f_i, \quad (2.22)$$

where $u_-(i) := \langle u_-, \delta_i \rangle$. The associated Grushin problem is given by

$$\mathcal{P}(z) := \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H}_h^d \oplus \mathbb{C}^M \longrightarrow \mathcal{H}_h^d \oplus \mathbb{C}^M. \quad (2.23)$$

Our aim is to show that $\mathcal{P}(z)$ is bijective and derive an explicit formula for its inverse, which we denote by $\mathcal{E}(z)$. Given $(v, v_+) \in \mathcal{H}_h^d \oplus \mathbb{C}^M$, we wish to solve

$$\mathcal{P}(z) \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}. \quad (2.24)$$

We write u and v with respect to the bases $\{e_j\}_{j=1}^{N^d}$ and $\{f_j\}_{j=1}^{N^d}$, respectively:

$$u = \sum_{j=1}^{N^d} u_j e_j, \quad v = \sum_{j=1}^{N^d} v_j f_j.$$

Similarly, we write u_- and v_+ with respect to the basis $\{\delta_i\}_{i=1}^M$, with coefficients $\{u_-(i)\}_{i=1}^M$ and $\{v_+(i)\}_{i=1}^M$ respectively. Substituting into (2.24), we obtain

$$\begin{cases} \sum_{i=1}^{N^d} t_i u_i f_i + \sum_{j=1}^M u_-(j) f_j = \sum_{k=1}^{N^d} v_k f_k, \\ \sum_{i=1}^M u_i \delta_i = \sum_{j=1}^M v_+(j) \delta_j. \end{cases}$$

By linear independence of $\{f_j\}_{j=1}^{N^d}$ and $\{\delta_j\}_{j=1}^M$, this system is equivalent to

$$\begin{cases} t_j u_j = v_j, & j = M+1, \dots, N^d, \\ \begin{pmatrix} t_i & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_i \\ u_-(i) \end{pmatrix} = \begin{pmatrix} v_i \\ v_+(i) \end{pmatrix}, & i = 1, \dots, M. \end{cases} \quad (2.25)$$

Since

$$\begin{pmatrix} t_i & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -t_i \end{pmatrix}, \quad i = 1, \dots, M, \quad (2.26)$$

and $t_j > 0$, for all $j \geq M+1$, we deduce that $\mathcal{P}(z)$ is bijective with inverse

$$\mathcal{P}^{-1}(z) = \mathcal{E}(z) := \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}, \quad (2.27)$$

where

$$E(z) = \sum_{i=M+1}^{N^d} \frac{1}{t_i} e_i f_i^*, \quad E_+(z) = \sum_{i=1}^M e_i \delta_i^*, \quad (2.28)$$

$$E_-(z) = \sum_{i=1}^M \delta_i f_i^*, \quad E_{-+}(z) = - \sum_{i=1}^M t_i \delta_i \delta_i^*. \quad (2.29)$$

Indeed, with these definitions, we have

$$\mathcal{E}(z) \begin{pmatrix} v \\ v_+ \end{pmatrix} = \begin{pmatrix} E(z)v + E_+(z)v_+ \\ E_-(z)v + E_{-+}(z)v_+ \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^M v_+(i)e_i + \sum_{i=M+1}^{N^d} \frac{1}{t_i} v_i e_i \\ \sum_{i=1}^M (v_i - t_i v_+(i)) \delta_i \end{pmatrix} = \begin{pmatrix} u \\ u_- \end{pmatrix},$$

where the last equality follows from (2.25) and (2.26).

We now provide some estimates on the norms of the matrices defined above, which will be useful later on. For the matrix $E(z)$, take $u \in \mathcal{H}_h^d \cong \mathbb{C}^{N^d}$ with $\|u\|_2 = 1$. Then, by applying Parseval's identity with respect to the orthonormal basis $\{f_i\}_i$, we obtain

$$\|E(z)u\|_2^2 = \left\| \sum_{i=M+1}^{N^d} \frac{1}{t_i} \langle u, f_i \rangle e_i \right\|_2^2 = \sum_{i=M+1}^{N^d} \frac{1}{t_i^2} |\langle f_i, u \rangle|^2 \leq \frac{1}{t_{M+1}^2} \sum_{i=M+1}^{N^d} |\langle f_i, u \rangle|^2 \leq \frac{1}{t_{M+1}^2}.$$

Hence,

$$\|E(z)\| \leq \frac{1}{t_{M+1}}. \quad (2.30)$$

A similar argument applied to the remaining matrices yields the bounds

$$\|E_{\pm}(z)\| = 1, \quad \|E_{-+}(z)\| \leq t_M. \quad (2.31)$$

Now, we turn to the computation of $|\det \mathcal{P}(z)|$. To this end, we consider $\mathcal{P}(z)$ expressed in the bases

$$\{\tilde{e}_{M+1}, \dots, \tilde{e}_{N^d}, \tilde{e}_1, \dots, \tilde{e}_M, \tilde{\delta}_1, \dots, \tilde{\delta}_M\}, \quad \{\tilde{f}_{M+1}, \dots, \tilde{f}_{N^d}, \tilde{f}_1, \tilde{\delta}_1, \dots, \tilde{f}_M, \tilde{\delta}_M\},$$

where

$$\tilde{e}_i := \begin{pmatrix} e_i \\ \mathbf{0} \end{pmatrix}, \quad \tilde{f}_i := \begin{pmatrix} f_i \\ \mathbf{0} \end{pmatrix}, \quad \tilde{\delta}_i := \begin{pmatrix} \mathbf{0} \\ \delta_i \end{pmatrix} \in \mathcal{H}_h^d \oplus \mathbb{C}^M.$$

In this representation, $\mathcal{P}(z)$ takes the block-diagonal form

$$\begin{bmatrix} t_{M+1} & & & & & \\ & \ddots & & & & \\ & & t_{N^d} & & & \\ & & & \begin{bmatrix} t_1 & 1 \\ 1 & 0 \end{bmatrix} & & \\ & & & & \ddots & \\ & & & & & \begin{bmatrix} t_M & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}.$$

Hence, since $t_M^2 \leq \alpha < t_{M+1}^2$, it follows that

$$|\det(\mathcal{P}(z))|^2 = \prod_{i=M+1}^{N^d} t_i^2 = \alpha^{-M} \prod_{i=1}^{N^d} \mathbb{1}_{\alpha}(t_i^2), \quad (2.32)$$

where $\mathbb{1}_\alpha(x) := \max(x, \alpha)$. Hence,

$$|\det(\mathcal{P}(z))|^2 = \alpha^{-M} \det(\mathbb{1}_\alpha(Q(z))). \quad (2.33)$$

Since we don't know if $P - z$ is invertible, we can apply Corollary 2.1, specifically result (2.11), and we obtain

$$\ln |\det(P - z)| = \ln |\det(\mathcal{P}(z))| + \ln |\det(E_{-+}(z))|.$$

Next, we estimate $\ln |\det \mathcal{P}(z)|$. Let $\chi \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function supported in $[0, 2]$, with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $[0, 1]$. For every $x \geq 0$, we have the inequalities

$$x + \frac{\alpha}{4} \chi\left(\frac{4x}{\alpha}\right) \leq \mathbb{1}_\alpha(x) \leq x + \alpha \chi\left(\frac{x}{\alpha}\right). \quad (2.34)$$

As shown in (2.18), the principal symbol of the operator $Q(z)$ is given by $|p_0 - z|^2$. By combining Theorem 1.25 with the estimates (2.33), (2.34), and (2.20), we obtain

$$\begin{aligned} \ln |\det \mathcal{P}(z)|^2 &= \ln |\det(\mathbb{1}_\alpha(Q(z)))| + M \ln \left(\frac{1}{\alpha}\right) \\ &= N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z|^2 d\rho + \mathcal{O}\left(\alpha^\kappa \ln \left(\frac{1}{\alpha}\right)\right) \right). \end{aligned} \quad (2.35)$$

In particular, the second equality follows from the fact that

$$\ln \left| \det \left(Q(z) + \frac{\alpha}{4} \chi \left(\frac{4Q(z)}{\alpha} \right) \right) \right| \leq \ln |\det \mathbb{1}_\alpha(Q(z))| \leq \ln \left| \det \left(Q(z) + \alpha \chi \left(\frac{Q(z)}{\alpha} \right) \right) \right|,$$

together with the application of (1.41) to both the sides.

From Remark 2.4, the equality (2.35) holds uniformly for all $z \in K$, for any $K \Subset \mathbb{C}$.

2.3 Grushin Problem for the Perturbed Operator

We begin this section by considering a deterministic linear perturbation $Q : \mathcal{H}_h^d \rightarrow \mathcal{H}_h^d$ (i.e. an $N^d \times N^d$ matrix since $\mathcal{H}_h^d \simeq \mathbb{C}^{N^d}$ from (1.26)). We study the properties of the operators perturbed by Q through the associated Grushin problem; a randomized setting will be addressed later in the dissertation.

We fix $N \in \mathbb{N}^*$ and we recall assumption (2.12) on h and α . Our goal is to analyze the spectrum of

$$P^\delta := P + \delta Q, \quad \text{with } 0 \leq \delta \ll 1,$$

where $P = p_N$ is defined as in (2.14). We proceed as in Section 2.2 and set up a Grushin problem for the perturbed operator.

Fix $z \in \mathbb{C}$. Let the operators $R_{\pm}$ be as in (2.21) and (2.22) and set

$$\mathcal{P}^{\delta}(z) := \begin{pmatrix} P^{\delta} - z & R_{-} \\ R_{+} & 0 \end{pmatrix} : \mathcal{H}_h^d \oplus \mathbb{C}^M \longrightarrow \mathcal{H}_h^d \oplus \mathbb{C}^M,$$

where M is defined by the condition (2.19). We set α as in (2.12) and we suppose that $\delta \geq 0$ is such that

$$\delta \alpha^{-1/2} \|Q\| \leq \frac{1}{2}. \quad (2.36)$$

Now we aim to establish the invertibility of $\mathcal{P}^{\delta}(z)$ and to derive an explicit expression for its inverse. The argument follows the same strategy used in the proof of Corollary 2.3. We use the same notations adopted in the previous section.

Let $\mathcal{E}(z) = \mathcal{P}^{-1}(z)$, as defined in (2.23) and (2.27). We compute

$$\mathcal{P}^{\delta}(z) \mathcal{E}(z) = I + \delta \begin{pmatrix} QE(z) & QE_{+}(z) \\ 0 & 0 \end{pmatrix}.$$

Hence, applying (2.30), (2.31) and (2.36), we get

$$\begin{aligned} \left\| -\delta \begin{pmatrix} QE(z) & QE_{+}(z) \\ 0 & 0 \end{pmatrix} \right\| &\leq \delta \|Q\| (\|E(z)\| + \|E_{+}(z)\|) \leq \delta \alpha^{-1/2} \|Q\| + \delta \|Q\| \\ &\leq \frac{1}{2} (1 + \alpha^{1/2}) < 1. \end{aligned}$$

Therefore, by the Neumann series, we know that $\mathcal{P}^{\delta}(z) \mathcal{E}(z)$ is bijective with inverse

$$(\mathcal{P}^{\delta}(z) \mathcal{E}(z))^{-1} = I + \sum_{k=1}^{+\infty} (-\delta)^k \begin{pmatrix} (QE(z))^k & (QE(z))^{k-1} QE_{+}(z) \\ 0 & 0 \end{pmatrix},$$

whose norm is $\leq \mathcal{O}(1)$. Consequently, $\mathcal{P}^{\delta}(z)$ is bijective with inverse

$$\mathcal{E}^{\delta}(z) = \mathcal{E}(z) + \sum_{k=1}^{+\infty} (-1)^k \begin{pmatrix} E(\delta QE)^k & (\delta EQ)^k E_{+} \\ E_{-}(\delta QE)^k & E_{-}(\delta QE)^{k-1} (\delta Q) E_{+} \end{pmatrix} =: \begin{pmatrix} E^{\delta}(z) & E_{+}^{\delta}(z) \\ E_{-}^{\delta}(z) & E_{-+}^{\delta}(z) \end{pmatrix}. \quad (2.37)$$

Hence, by (2.30) and (2.36), we obtain the following inequality

$$\begin{aligned} \|E^{\delta}(z)\| &= \left\| E \left(I + \sum_{k=1}^{+\infty} (-1)^k (\delta QE)^k \right) \right\| \leq \|E\| \left\| I + \sum_{k=1}^{+\infty} (-1)^k (\delta QE)^k \right\| \\ &\leq \frac{1}{t_{M+1}} \left(\|I\| + \sum_{k=1}^{+\infty} \|\delta QE\|^k \right) \\ &= \frac{1}{t_{M+1}} \left(\frac{1}{1 - \|\delta QE\|} \right) \leq \frac{2}{t_{M+1}}. \end{aligned} \quad (2.38)$$

Using also (2.31), we get these results

$$\|E_+^\delta(z)\| = \left\| \left(I + \sum_{k=1}^{+\infty} (-1)^k (\delta EQ)^k \right) E_+ \right\| \leq \left(\sum_{k=0}^{+\infty} \|\delta EQ\|^k \right) \|E_+\| \leq 2; \quad (2.39)$$

$$\|E_-^\delta(z)\| = \left\| E_- \left(I + \sum_{k=1}^{+\infty} (-1)^k (\delta QE)^k \right) \right\| \leq 2; \quad (2.40)$$

$$\begin{aligned} \|E_{-+}^\delta(z) - E_{-+}\| &= \left\| \sum_{k=1}^{+\infty} (-1)^k E_- (\delta QE)^{k-1} (\delta Q) E_+ \right\| \\ &\leq \|E_-\| \left(\sum_{k=1}^{+\infty} \|\delta QE\|^{k-1} \right) \|\delta Q\| \|E_+\| \\ &\leq 2\delta \|Q\| \leq \alpha^{1/2}, \end{aligned} \quad (2.41)$$

Applying Corollary 2.3 to $\mathcal{P}^\delta(z)$ and $\mathcal{E}^\delta(z)$, we obtain

$$\ln |\det(P^\delta - z)| = \ln |\det(\mathcal{P}^\delta(z))| + \ln |\det(E_{-+}^\delta(z))|. \quad (2.42)$$

By Jacobi's formula (see Proposition A.16 in the Appendix),

$$\frac{d}{d\delta} \det(\mathcal{P}^\delta(z)) = \text{tr} \left(\text{adj}(\mathcal{P}^\delta(z)) \frac{d}{d\delta} \mathcal{P}^\delta(z) \right), \quad (2.43)$$

where $\text{adj}(\mathcal{P}^\delta(z))$ denotes the adjugate of $\mathcal{P}^\delta(z)$. Therefore, by taking the logarithms, we get

$$\begin{aligned} \frac{d}{d\delta} (\ln |\det(\mathcal{P}^\delta(z))|) &= \Re \left(\frac{1}{\det(\mathcal{P}^\delta(z))} \text{tr} \left(\text{adj}(\mathcal{P}^\delta(z)) \frac{d}{d\delta} \mathcal{P}^\delta(z) \right) \right) \\ &= \Re \left(\frac{1}{\det(\mathcal{P}^\delta(z))} \text{tr} \left(\det(\mathcal{P}^\delta(z)) \mathcal{E}^\delta(z) \frac{d}{d\delta} \begin{pmatrix} P^\delta - z & R_- \\ R_+ & 0 \end{pmatrix} \right) \right) \\ &= \Re(\text{tr}(E^\delta(z)Q)). \end{aligned} \quad (2.44)$$

Remark 2.5. Let $A, B \in M_{N,N}$; the following inequality is true

$$|\text{tr}(AB)| \leq \|A\| \|B\|_{\text{tr}},$$

where the trace-norm of B is defined as $\|B\|_{\text{tr}} := \text{tr}((B^*B)^{1/2})$.

Thus, using this result and applying (2.38) and (2.44), we obtain

$$\begin{aligned} \left| \ln |\det(\mathcal{E}^\delta(z))| - \ln |\det(\mathcal{E}(z))| \right| &= \left| \ln |\det(\mathcal{P}^\delta(z))| - \ln |\det(\mathcal{P}(z))| \right| \\ &= \left| \int_0^\delta \frac{d}{d\tau} (\ln |\det(\mathcal{P}^\tau(z))|) d\tau \right| \\ &= \left| \Re \int_0^\delta \text{tr}(E^\tau(z)Q) d\tau \right| \leq \mathcal{O}(\delta \alpha^{-1/2} \|Q\|_{\text{tr}}). \end{aligned} \quad (2.45)$$

Hence, using (2.35) and (2.45), we get

$$\begin{aligned} \ln |\det(\mathcal{P}^\delta(z))| &\leq \ln |\det(\mathcal{P}(z))| + \mathcal{O}(\delta \alpha^{-1/2} \|Q\|_{\text{tr}}) \\ &= N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho + \mathcal{O} \left(\alpha^\kappa \ln \left(\frac{1}{\alpha} \right) \right) + \mathcal{O}(\delta N^{-d} \alpha^{-1/2} \|Q\|_{\text{tr}}) \right). \end{aligned} \quad (2.46)$$

Moreover, under the assumption (2.36), by (2.31) and (2.41), we have

$$\|E_{-+}^\delta(z)\| \leq \alpha^{1/2} + t_M = \mathcal{O}(\alpha^{1/2}),$$

which, in view of (2.20), yields the following upper bound

$$\ln |\det E_{-+}^\delta(z)| \leq M \ln(\|E_{-+}^\delta(z)\|) \leq \mathcal{O}(N^d \alpha^\kappa) |\ln \alpha|. \quad (2.47)$$

2.4 Auxiliary Estimates

We conclude this chapter with a general estimate on the singular values arising in Grushin problems.

Lemma 2.6. *Let $\mathcal{H}$ be an N -dimensional complex Hilbert space, and let $N \geq M > 0$. Suppose that*

$$\mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H} \oplus \mathbb{C}^M \rightarrow \mathcal{H} \oplus \mathbb{C}^M$$

is a bijective matrix of linear operators, with inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}.$$

*Let $0 \leq t_1(P) \leq \dots \leq t_N(P)$ denote the eigenvalues of $(P^*P)^{1/2}$, and let $0 \leq t_1(E_{-+}) \leq \dots \leq t_M(E_{-+})$ denote the eigenvalues of $(E_{-+}^*E_{-+})^{1/2}$. Then, for $m = 1, \dots, M$, we have*

$$\frac{t_m(E_{-+})}{\|E\| t_m(E_{-+}) + \|E_-\| \|E_+\|} \leq t_m(P) \leq \|R_+\| \|R_-\| t_m(E_{-+}). \quad (2.48)$$

Proof. By Corollary 2.3 we know that P is invertible if and only if E_{-+} is invertible, and in that case, by formula (2.4) in Theorem 2.1, the following two are true

$$P^{-1} = E - E_+ E_{-+}^{-1} E_-, \quad E_{-+}^{-1} = -R_+ P^{-1} R_-. \quad (2.49)$$

We adopt a decreasing notation for the singular values. In particular, we denote the singular values of P as $0 \leq s_N(P) \leq s_{N-1}(P) \leq \dots \leq s_1(P)$ and the singular values of E_{-+} as $0 \leq$

$s_M(E_{-+}) \leq s_{M-1}(E_{-+}) \leq \dots \leq s_1(E_{-+})$. Thus,

$$\begin{aligned} s_n(P) &= t_{N-n+1}(P), \quad n = 1, \dots, N \\ s_m(E_{-+}) &= t_{M-m+1}(E_{-+}), \quad m = 1, \dots, M. \end{aligned}$$

Assume P is invertible. Then, we have

$$s_n(P^{-1}) = \frac{1}{t_n(P)}, \quad n = 1, \dots, N, \quad (2.50)$$

and similarly,

$$s_m(E_{-+}^{-1}) = \frac{1}{t_m(E_{-+})}, \quad m = 1, \dots, M. \quad (2.51)$$

Recall also that $s_1(A) = \|A\|$ for every matrix A . Since Ky Fan's inequalities (Corollary A.9 in the Appendix) apply to trace-class operators on a Hilbert space, we use them together with the identities (2.49). Applying once (A.7) and twice (A.8), it follows that

$$\begin{aligned} s_m(P^{-1}) &= s_m(E - E_+ E_{-+}^{-1} E_-) \leq s_1(E) + s_m(-E_+ E_{-+}^{-1}) s_1(E_-) \\ &\leq \|E\| + \|E_+\| \|E_-\| s_m(E_{-+}^{-1}), \quad m = 1, \dots, M. \end{aligned} \quad (2.52)$$

Therefore, invoking (2.50) and (2.51) yields

$$t_m(P) = \frac{1}{s_m(P^{-1})} \geq \frac{1}{\|E\| + \|E_+\| \|E_-\| \frac{1}{t_m(E_{-+})}} = \frac{t_m(E_{-+})}{\|E\| t_m(E_{-+}) + \|E_+\| \|E_-\|}, \quad m = 1, \dots, M,$$

which is the desired lower bound. For the upper bound, from the second identity in (2.49), we get

$$s_m(E_{-+}^{-1}) = s_m(R_+ P^{-1} R_-) \leq s_1(R_+) s_m(P^{-1}) s_1(R_-) = \|R_+\| \|R_-\| s_m(P^{-1}).$$

Taking the reciprocals and using (2.50) and (2.51),

$$t_m(P) \leq \|R_+\| \|R_-\| t_m(E_{-+}), \quad m = 1, \dots, M.$$

Assume now that P is not invertible. We proceed as in the proof of Corollary 2.3. We introduce the perturbed operator $P^\varepsilon := P + \varepsilon X$, where $\|X\| \leq 1$ and $0 < \varepsilon \ll 1$, so that P^ε is bijective. By the same Neumann series argument used before, the associated Grushin problem $\mathcal{P}^\varepsilon$ remains invertible, with inverse $\mathcal{E}^\varepsilon$ defined as in (2.8). Consequently, we may apply (2.48) to this perturbed problem.

Moreover, since

$$\|P^\varepsilon - P\| \rightarrow 0, \quad \|\mathcal{E}^\varepsilon - \mathcal{E}\| \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,$$

from Corollary A.10 in the Appendix, the singular values of P^ε and of E_{-+}^ε depend continuously on ε . Thus, it follows that (2.48) also holds in the case where P fails to be invertible. This

completes the proof.

□

Chapter 3

Estimates for the Perturbed Operator

In this chapter we construct a suitable perturbation for an operator P defined as in (2.14) in order to obtain quantitative lower bounds for the small singular values of the perturbed operator. Our approach adapts the method used by Sjöstrand in [Sj09]) to the toroidal phase space $\mathbb{T}^{2d}$. Although the setting is periodic, the underlying ideas carry over with minor modifications and yield bounds comparable to those in the Euclidean case.

We proceed in two stages. First, using an elementary linear–algebraic argument, we build a preliminary perturbation that already furnishes effective lower bounds for a larger set of the singular values of the perturbed operator. Second, following Sjöstrand’s iterative scheme, we propagate the estimate down to the bottom of the singular-value scale, thereby controlling even the lowest singular value of the perturbed operator.

3.1 Construction of a Potential

In this section we construct a diagonal potential from two families of linearly independent vectors by selecting an appropriate set of coordinate indices. We then apply this construction to the framework of the previous chapters, obtaining a potential that will serve as the initial perturbation for the iterative perturbation scheme developed in the following sections.

We fix $N \in \mathbb{N}^*$ and consider the following results.

Proposition 3.1. *Let $M \in \mathbb{N}$, $1 \leq M \leq N^d$ and let $e_1, \dots, e_M \in \mathbb{C}^{N^d}$ be linearly independent vectors, such that $e_i = (e_i(n))_{n=1}^{N^d}$, for each $i = 1, \dots, M$. Under these assumptions we can find M different points $n_1, \dots, n_M \in \{1, \dots, N^d\}$ such that the vectors $\vec{e}(n_1), \dots, \vec{e}(n_M)$ are linearly independent in $\mathbb{C}^M$, where*

$$\vec{e}(n) := \begin{pmatrix} e_1(n) \\ e_2(n) \\ \vdots \\ e_M(n) \end{pmatrix}, \quad \forall n \in \{1, \dots, N^d\}. \quad (3.1)$$

Proof. We define the set $E \subset \mathbb{C}^M$ as $E := \text{Span}\{\vec{e}(n); n = 1, \dots, N^d\}$ and we claim that $E = \mathbb{C}^M$. Indeed, if we suppose that this isn’t true, there would exist $\mathbf{0} \neq \lambda = (\lambda_1, \dots, \lambda_M) \in \mathbb{C}^M$ such

that $\lambda \in E^\perp$, i.e.

$$0 = \langle \lambda, \vec{e}(n) \rangle = \sum_{j=1}^M \lambda_j e_j(n), \quad \forall n \in \{1, \dots, N^d\},$$

and consequently

$$\mathbf{0} = \sum_{j=1}^M \lambda_j e_j.$$

But this implies that $e_1, \dots, e_M$ are linearly dependent vectors, which is in contradiction with the hypothesis. Hence, we know that $E = \mathbb{C}^M$, and we can find $n_1, \dots, n_M \in \{1, \dots, N^d\}$ such that $\vec{e}(n_1), \dots, \vec{e}(n_M)$ form a basis in $\mathbb{C}^M$, and consequently they are linearly independent. $\square$

Proposition 3.2. *Let $M \in \mathbb{N}$, $1 \leq M \leq N^d$ and let $\{e_1, \dots, e_M\}, \{f_1, \dots, f_M\}$ be two linearly independent families in $\mathbb{C}^{N^d}$. For both of them, we adopt the notations introduced in Proposition 3.1. We assume that we can find M points $n_1, \dots, n_M \in \{1, \dots, N^d\}$ such that both $\{\vec{e}(n_i)\}_{i=1}^M$ and $\{\vec{f}(n_i)\}_{i=1}^M$ are linearly independent families in $\mathbb{C}^M$. We define*

$$\begin{aligned} B : \mathbb{C}^M &\longrightarrow \mathbb{C}^M \\ u &\longmapsto \sum_{j=1}^M \langle u, \vec{f}(n_j) \rangle \vec{e}(n_j). \end{aligned} \tag{3.2}$$

Then B is bijective.

Proof. Let $u \in \text{Ker } B$. Since $\vec{e}(n_1), \dots, \vec{e}(n_M)$ form a basis in $\mathbb{C}^M$, by the definition of B we have

$$\langle u, \vec{f}(n_j) \rangle = 0, \quad \text{for all } j = 1, \dots, M.$$

Since $\vec{f}(n_1), \dots, \vec{f}(n_M)$ form a basis in $\mathbb{C}^M$, it follows that $u = \mathbf{0}$. Hence, B is bijective. $\square$

We observe that, under the same assumptions of Proposition 3.2, the matrix associated with B , expressed in the canonical basis, is given by:

$$(B)_{j,k} = \sum_{v=1}^M e_j(n_v) \overline{f_k(n_v)}, \quad j, k = 1, \dots, M. \tag{3.3}$$

Hence, if we define the matrices

$$E := (\vec{e}(n_1), \dots, \vec{e}(n_M)) \in \mathbb{C}^{M \times M} \tag{3.4}$$

$$F := (\vec{f}(n_1), \dots, \vec{f}(n_M)) \in \mathbb{C}^{M \times M}, \tag{3.5}$$

then we have

$$B = EF^*. \tag{3.6}$$

Lemma 3.3. *Let $M \in \mathbb{N}$, $1 \leq M \leq N^d$ and let $e_1, \dots, e_M$ be as in Proposition 3.1. Let $L \subset \mathbb{C}^M$ be a*

linear subspace of dimension $S - 1$, for some $1 \leq S \leq M$. Then, there exists $n \in \{1, \dots, N^d\}$ such that

$$\text{dist}(\vec{e}(n), L)^2 \geq \frac{1}{N^d} \text{tr}((I - \pi_L) \mathcal{E}_{N^d}), \quad (3.7)$$

where $\mathcal{E}_{N^d} := (\langle e_j, e_k \rangle)_{1 \leq j, k \leq M}$ and π_L is the orthogonal projection from $\mathbb{C}^M$ onto L .

Proof. Let $\{v_1, \dots, v_M\}$ be an orthonormal basis of $\mathbb{C}^M$ chosen so that $L = \text{span}\{v_1, \dots, v_{S-1}\}$ ($L = \{0\}$ when $S = 1$). We write

$$v_l = \begin{pmatrix} v_{1,l} \\ \vdots \\ v_{M,l} \end{pmatrix} \in \mathbb{C}^M \quad \forall l = 1, \dots, M.$$

By definition we have, for all $n \in \{1, \dots, N^d\}$,

$$\begin{aligned} \text{dist}(\vec{e}(n), L)^2 &= \sum_{l=S}^M |\langle \vec{e}(n), v_l \rangle|^2 = \sum_{l=S}^M \left| \sum_{j=1}^M e_j(n) \overline{v_{j,l}} \right|^2 \\ &= \sum_{l=S}^M \left(\sum_{j=1}^M e_j(n) \overline{v_{j,l}} \right) \left(\sum_{k=1}^M \overline{e_k(n) \overline{v_{k,l}}} \right) = \sum_{l=S}^M \sum_{j,k=1}^M \overline{v_{j,l}} e_j(n) \overline{e_k(n)} v_{k,l}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{n=1}^{N^d} \text{dist}(\vec{e}(n), L)^2 &= \sum_{n=1}^{N^d} \sum_{l=S}^M \sum_{j,k=1}^M \overline{v_{j,l}} e_j(n) \overline{e_k(n)} v_{k,l} \\ &= \sum_{l=S}^M \sum_{j,k=1}^M \overline{v_{j,l}} (\mathcal{E}_{N^d})_{j,k} v_{k,l} = \sum_{l=S}^M \langle \mathcal{E}_{N^d} v_l, v_l \rangle = \text{tr}((I - \pi_L) \mathcal{E}_{N^d}). \end{aligned}$$

Now, by estimating the summation, we obtain

$$\text{tr}((I - \pi_L) \mathcal{E}_{N^d}) = \sum_{n=1}^{N^d} \text{dist}(\vec{e}(n), L)^2 \leq N^d \max_{n=1, \dots, N^d} \text{dist}(\vec{e}(n), L)^2. \quad (3.8)$$

Thus, we can find an $n \in \{1, \dots, N^d\}$ which satisfies (3.7). $\square$

Remark 3.4. For the purposes of the upcoming application, we consider the assumption that

$$\{e_1, \dots, e_M\} \text{ is an orthonormal family in } \mathbb{C}^{N^d}. \quad (3.9)$$

Then $\mathcal{E}_{N^d} = I$ and (3.8) simplifies to

$$\max_{n=1, \dots, N^d} \text{dist}(\vec{e}(n), L)^2 \geq \frac{M - S + 1}{N^d}.$$

In the general case, we let $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_M$ denote the eigenvalues of $\mathcal{E}_{N^d}$. Then we

have

$$\inf_{\dim L=S-1} \operatorname{tr}((1 - \pi_L)\mathcal{E}_{N^d}) = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{M-S+1} =: E_S. \quad (3.10)$$

Indeed, the mini-max principle (Theorem A.6) implies that for $k = 1, \dots, M$

$$\varepsilon_k = \inf_{\dim L=k} \sup_{\substack{v \in L \\ \|v\|_2=1}} \langle \mathcal{E}_{N^d} v, v \rangle.$$

Thus, for a general subspace L of dimension $S - 1$, the eigenvalues of $(1 - \pi_L)\mathcal{E}_{N^d}(1 - \pi_L)$ are $\varepsilon'_1 \leq \dots \leq \varepsilon'_{M-S+1}$, where $\varepsilon'_j \geq \varepsilon_j$, for $j = 1, \dots, M - S + 1$.

Now let $\{e_1, \dots, e_M\}$ and $\{f_1, \dots, f_M\}$ be two linearly independent families in $\mathbb{C}^{N^d}$, and consider the notation (3.1) for $\vec{e}(n)$ and $\vec{f}(n)$. Applying Lemma 3.3 together with (3.10) to the first family, we successively choose indices

$$n_1, \dots, n_M \in \{1, \dots, N^d\} \quad (3.11)$$

such that

$$\begin{aligned} \|\vec{e}(n_1)\|^2 &\geq \frac{1}{N^d} \operatorname{tr}((I - \pi_{\{0\}})\mathcal{E}_{N^d}) \geq \frac{1}{N^d} \inf_{\dim L=0} \operatorname{tr}((I - \pi_L)\mathcal{E}_{N^d}) = \frac{E_1}{N^d}, \\ \operatorname{dist}(\vec{e}(n_2), \mathbb{C}\vec{e}(n_1))^2 &\geq \frac{1}{N^d} \inf_{\dim L=1} \operatorname{tr}((I - \pi_L)\mathcal{E}_{N^d}) = \frac{E_2}{N^d}, \\ &\vdots \\ \operatorname{dist}(\vec{e}(n_S), \mathbb{C}\vec{e}(n_1) \oplus \dots \oplus \mathbb{C}\vec{e}(n_{S-1}))^2 &\geq \frac{1}{N^d} \inf_{\dim L=S-1} \operatorname{tr}((I - \pi_L)\mathcal{E}_{N^d}) = \frac{E_S}{N^d}, \\ &\vdots \\ \operatorname{dist}(\vec{e}(n_M), \mathbb{C}\vec{e}(n_1) \oplus \dots \oplus \mathbb{C}\vec{e}(n_{M-1}))^2 &\geq \frac{1}{N^d} \inf_{\dim L=M-1} \operatorname{tr}((I - \pi_L)\mathcal{E}_{N^d}) = \frac{\varepsilon_1}{N^d}. \end{aligned}$$

Let $\{v_1, v_2, \dots, v_M\}$ be the Gram-Schmidt orthonormalization of the basis $\{\vec{e}(n_j)\}_{j=1}^M$, so that

$$\begin{aligned} \vec{e}(n_1) &= c_1 v_1, \quad \text{where } |c_1| \geq \left(\frac{E_1}{N^d}\right)^{\frac{1}{2}}, \\ \vec{e}(n_S) &\equiv c_S v_S \bmod (v_1, \dots, v_{S-1}), \quad \text{where } |c_S| \geq \left(\frac{E_S}{N^d}\right)^{\frac{1}{2}}, \quad \text{for } S = 2, \dots, M \end{aligned} \quad (3.12)$$

i.e.

$$\vec{e}(n_S) = c_S v_S + \sum_{j=1}^{S-1} \lambda_j v_j, \quad \text{for } S = 2, \dots, M,$$

so that $\vec{e}(n_S) - \sum_{j=1}^{S-1} \lambda_j v_j$ is orthogonal to the vectors $v_1, \dots, v_{S-1}$ and $\|v_S\|_2 = 1$.

We consider the matrix E defined as in (3.4). Expressing the vectors $\vec{e}(n_1), \vec{e}(n_2), \dots, \vec{e}(n_M)$ in the basis $v_1, \dots, v_M$ does not affect its determinant. Indeed, with this change of basis the

matrix E becomes upper triangular with diagonal entries $c_1, \dots, c_M$. Hence,

$$|\det E| = |c_1 \cdots c_M|,$$

and (3.12) implies that

$$|\det E| \geq \frac{(E_1 E_2 \cdots E_M)^{1/2}}{(N^d)^{M/2}}. \quad (3.13)$$

We now consider the matrix B defined as in (3.2) for the indices in (3.11). Suppose that

$$f_j := \bar{e}_j, \quad j = 1, \dots, M. \quad (3.14)$$

Then, recalling definitions (3.4) and (3.5), we have $F^* = E^t$, and from (3.6),

$$B = EE^t. \quad (3.15)$$

Hence, from (3.13) and (3.15), we get

$$|\det B| \geq \frac{E_1 E_2 \cdots E_M}{N^{Md}}. \quad (3.16)$$

Moreover, under the assumption (3.9), this result simplifies to

$$|\det B| \geq \frac{M!}{N^{Md}}.$$

We are interested in estimating the singular values $s_1(B) \geq s_2(B) \geq \dots \geq s_M(B)$ of the matrix B . Writing $s_j := s_j(B)$, we have, for every $k = \{1, \dots, M\}$

$$s_1^M \geq s_1^{k-1} s_k^{M-k+1} \geq \prod_{j=1}^M s_j = |\det B|. \quad (3.17)$$

We recall that $s_1 = \|B\|$, and combining (3.16) and (3.17), we obtain the following.

Proposition 3.5. *Under the assumption (3.14), the following two inequalities are true*

$$s_1 \geq \frac{(E_1 E_2 \cdots E_M)^{\frac{1}{M}}}{N^d}, \quad (3.18)$$

$$s_k \geq s_1 \left(\prod_{j=1}^M \left(\frac{E_j}{s_1 N^d} \right) \right)^{\frac{1}{M-k+1}}, \quad \text{for } k = 1, \dots, M. \quad (3.19)$$

Proof. The first inequality is an immediate consequence of (3.16) and (3.17).

By using the same results we now derive the second one. Letting $k \in \{1, \dots, M\}$, we have

$$s_1^{k-1} s_k^{M-k+1} \geq |\det B| \geq \prod_{j=1}^M \left(\frac{E_j}{N^d} \right),$$

and hence

$$s_k \geq \left(\left(\frac{1}{s_1} \right)^{k-1} \prod_{j=1}^M \left(\frac{E_j}{N^d} \right) \right)^{\frac{1}{M-k+1}} = \left((s_1)^{M-k+1} \prod_{j=1}^M \left(\frac{E_j}{s_1 N^d} \right) \right)^{\frac{1}{M-k+1}} = s_1 \left(\prod_{j=1}^M \left(\frac{E_j}{s_1 N^d} \right) \right)^{\frac{1}{M-k+1}}.$$

□

Now, we would like to express the matrix B as a product involving a diagonal potential. We consider the indices $n_1, \dots, n_M \in \{1, \dots, N^d\}$ chosen in (3.11). Recalling (3.3), we obtain for every $l, m = 1, \dots, M$,

$$(B)_{l,m} = \sum_{j=1}^M e_l(n_j) \overline{f_m(n_j)} = \sum_{n=1}^{N^d} e_l(n) \overline{f_m(n)} q(n), \quad (3.20)$$

where $q \in \{0, 1\}^{N^d}$ is defined as

$$q(n) := \begin{cases} 1, & \text{if } \exists j : n = n_j \\ 0, & \text{else} \end{cases}, \quad (3.21)$$

for every $n = 1, \dots, N^d$. In particular, q has M non-zero entries.

We now adopt the same framework of Chapter 2. Let $\{e_i\}_{i=1}^{N^d}$ and $\{f_i\}_{i=1}^{N^d}$ denote the orthonormal eigenbases of $Q(z)$ and $Q'(z)$, respectively, defined previously (see (2.15) and (2.16)). Let M be the number of eigenvalues of $Q(z), Q'(z)$ in the interval $[0, \alpha]$: We consider the matrices E_+ and E_- as introduced in (2.28) and (2.29), and we recall their properties as stated in (2.31). With these choices, by (3.20), we have

$$B^t = E_- V_q E_+, \quad (3.22)$$

where

$$V_q := \text{diag} \left(q(n); n = 1, \dots, N^d \right) \in \mathbb{C}^{N^d \times N^d} \quad (3.23)$$

is in the form of a potential. We note that this diagonal matrix is entirely determined by the choice of the indices $n_1, \dots, n_M$, which in turn depends on the eigenvectors $\{e_i\}_{i=1, \dots, M}$. Moreover, we observe that

$$\|V_q\| = \|q\|_\infty = 1. \quad (3.24)$$

This construction will play a key role in the following sections, where we will use it to identify a suitable perturbation for the operator under consideration.

3.2 Improving Singular Value Bounds via Potential Perturbations

In this section we adapt the work of Sjöstrand in [Sj09, Chapters 5–6] to the discrete case. From now on, we adopt the same setting and notation as in Chapter 2. We set

$$h = \frac{1}{2\pi N}, \quad N \in \mathbb{N}^*, \quad \text{and} \quad N^{-1} \ll \alpha \ll 1,$$

Let

$$p \in C^\infty(\mathbb{T}^{2d}) \quad \text{admit the asymptotic expansion} \quad p \sim p_0 + hp_1 + \dots \quad \text{in } S(1,1), \quad (3.25)$$

and let $P := p_N$ be the restriction of its Weyl quantization to $\mathcal{H}_h^d$ as in (1.25). By the identification (1.26), P defines a linear operator

$$P : \mathbb{C}^{N^d} \rightarrow \mathbb{C}^{N^d}.$$

Fix $z \in \mathbb{C}$. Our goal is to construct a small perturbation δV_q in the form of a potential, that yields suitable lower bounds on the smallest singular values of the operator $P + \delta V_q - z$. We also assume the symmetry condition

$$p(x, \xi) = p(x, -\xi), \quad (3.26)$$

which is equivalent to requiring that

$$P^* = \Gamma P \Gamma, \quad (3.27)$$

where $\Gamma u := \bar{u}$ denotes the antilinear operator of complex conjugation. We observe that condition (3.27) remains valid under the addition of any diagonal matrix to P .

As before, we introduce the volume

$$V_z(t) := \lambda \left(\left\{ \rho \in \mathbb{T}^{2d} ; |p_0(\rho) - z|^2 \leq t \right\} \right),$$

and assume that, for the given value of z ,

$$V_z(t) = \mathcal{O}(t^\kappa), \quad 0 \leq t \ll 1, \quad (3.28)$$

for some exponent $\kappa \in]0, 1]$.

Remark 3.6. Condition (3.28) is employed throughout the literature in this framework, including in [ChZw10, HaSj08, Vo20]. In particular, it is important for controlling the number of small eigenvalues of $(P - z)^*(P - z)$ for a fixed $z \in \mathbb{C}$.

As observed in [ChZw10], if p is real analytic, then (3.28) always holds for some $\kappa > 0$. Similarly, if p is real analytic and $p(\mathbb{T}^{2d}) \subset \mathbb{C}$ has non-empty interior, then for $z \in \mathbb{C}$,

$$dp|_{p^{-1}(z)} \neq 0 \Rightarrow (3.28) \text{ holds with } \kappa > \frac{1}{2}.$$

Moreover, for $p \in C^\infty(\mathbb{T}^{2d})$ and $z \in \mathbb{C}$, if

$$dp, d\bar{p} \text{ are linearly independent at every point of } p^{-1}(z) \Rightarrow (3.28) \text{ holds with } \kappa = 1. \quad (3.29)$$

In particular, we observe that dp and $d\bar{p}$ are linearly independent at a point ρ when

$$\{p, \bar{p}\}(\rho) \neq 0,$$

where $\{a, b\} := \partial_{\bar{z}} a \partial_x b - \partial_x a \partial_{\bar{z}} b$ denotes the Poisson brackets on the torus $\mathbb{T}^{2d}$. The two conditions are equivalent when the dimension $d = 1$, but in dimension $d > 1$ this is not true in general. Furthermore, condition (3.29) does not hold when $z \in \partial(p_0(\mathbb{T}^{2d}))$, but it was observed in [HaSj08, Example 12.1] that for $z \in \mathbb{C}$, if

$$\forall \rho \in p^{-1}(z) : \{p, \bar{p}\}(\rho) \neq 0 \text{ or } \{p, \{p, \bar{p}\}\}(\rho) \neq 0, \text{ then (3.28) holds with } \kappa = \frac{3}{4}.$$

Under the assumptions stated above, Theorem 1.25, and in particular its consequence (1.42), yields the following.

Corollary 3.7. *For $N^{-1} \ll \alpha \ll 1$, if (3.28) holds, the number $M(\alpha)$ of eigenvalues of $(P - z)^*(P - z)$ in $[0, \alpha]$ satisfies*

$$M(\alpha) = \mathcal{O}(\alpha^\kappa N^d).$$

Now, we consider a small arbitrary perturbation applied to the operator P . Let $q_0 \in \mathbb{C}^{N^d}$ and $V_{q_0} := \text{diag}(q_0(j); j = 1, \dots, N^d)$ its associated diagonal matrix. Let $\delta_0 \geq 0$ such that

$$\|\delta_0 V_{q_0}\| = \delta_0 \|q_0\|_\infty \ll N^{-1}, \quad (3.30)$$

and consider the perturbed operator $P_0 := P + \delta_0 V_{q_0}$. We observe that thanks to the hypothesis (3.30), Corollary 3.7 still applies after replacing P with P_0 . Indeed, applying the mini-max theorem (see Theorem A.6 in the Appendix) to the j -th singular value of $P - z$, denoted $t_j(P - z)$, we obtain the following

$$\begin{aligned} t_j(P - z) &= \sqrt{\min_{V \subset \mathbb{C}^{N^d}, \dim V = j} \max_{\psi \in V: \|\psi\|_2 = 1} \|(P - z)\psi\|_2^2} \\ &= \min_{V \subset \mathbb{C}^{N^d}, \dim V = j} \max_{\psi \in V: \|\psi\|_2 = 1} \|(P_0 - z)\psi + (-\delta_0 V_{q_0})\psi\|_2 \\ &\leq t_j(P_0 - z) + \delta_0 \|V_{q_0}\|. \end{aligned} \quad (3.31)$$

This result can alternatively be derived by applying Ky Fan inequalities (Corollary A.9).

Let α be as in the assumptions of Corollary 3.7, and fix an arbitrary constant $C > 0$. Let $M(C\alpha)$ be defined as in the corollary and set $j \geq M(C\alpha) + 1$. By (3.30), there exists a sufficiently large constant $C_1 > 0$ such that

$$\|\delta_0 V_{q_0}\| \leq \frac{N^{-1}}{C_1}.$$

Under these conditions, we obtain:

$$C\alpha < t_j(P - z) \leq t_j(P_0 - z) + \|\delta_0 V_{q_0}\| \leq t_j(P_0 - z) + \frac{N^{-1}}{C_1},$$

which implies

$$t_j(P_0 - z) > C\alpha - \frac{N^{-1}}{C_1}.$$

Furthermore, the assumption $N^{-1} \ll \alpha$ implies

$$\frac{N^{-1}}{\alpha} \leq \frac{1}{C_2},$$

for another large constant $C_2 > 0$. Hence, for a suitable choice of the constants,

$$t_j(P_0 - z) > C\alpha \left(1 - \frac{1}{CC_1C_2}\right) > \frac{C\alpha}{2},$$

for all $j \geq M(C\alpha) + 1$. Therefore, if we let $M_0(\frac{C\alpha}{2})$ be the number of singular values of $P_0 - z$ lying in the interval $[0, \frac{C\alpha}{2}]$, we have shown that

$$M_0\left(\frac{C\alpha}{2}\right) \leq M(C\alpha).$$

This establishes the validity of Corollary 3.7 for the perturbed operator P_0 .

Now, we fix $\tau_0 \in]0, (CN^{-1})^{1/2}[$ for a new constant $C > 0$, and denote by

$$0 \leq t_1(P_0 - z) \leq t_2(P_0 - z) \leq \dots \leq t_M(P_0 - z) < \tau_0,$$

all the singular values of $P_0 - z$ lying in the interval $[0, \tau_0[$. Following the previous notation, we have $M = M_0(\tau_0^2)$. Thus, if we take $\alpha = CN^{-1}$, we obtain

$$M \leq M(\alpha) = \mathcal{O}(\alpha^\kappa N^d) = \mathcal{O}(N^{d-\kappa}). \quad (3.32)$$

We fix $\theta \in]0, \frac{1}{4}[$ and recall that M is defined by the condition

$$t_M(P_0 - z) < \tau_0 \leq t_{M+1}(P_0 - z). \quad (3.33)$$

We also fix a constant $\epsilon_0 > 0$. Then we obtain the following.

Proposition 3.8. *Under the assumptions described above, the following statements hold:*

1. Suppose M is sufficiently large. Then there exists a vector $q \in \{0, 1\}^{N^d}$, and the corresponding potential $V_q := \text{diag}(q(n); n = 1, \dots, N^d)$ with $\|V_q\| = 1$ such that, setting

$$P^\delta := P_0 + \delta V_q, \quad \delta := \frac{\tau_0}{C} N^{-d},$$

for some sufficiently large constant $C > 0$, the following estimates hold:

$$\begin{aligned} t_\nu(P^\delta - z) &\geq t_\nu(P_0 - z) - \frac{\tau_0 N^{-d}}{C} \\ &\geq \left(1 - \frac{N^{-d}}{C}\right) t_\nu(P_0 - z), \quad \text{for every } \nu > M, \end{aligned} \quad (3.34)$$

$$t_\nu(P^\delta - z) \geq \tau_0 N^{-M_1}, \quad \text{for } [M(1 - \theta)] + 1 \leq \nu \leq M, \quad (3.35)$$

where

$$M_1 := 2d + \epsilon_0, \quad (3.36)$$

and $[a] := \max(\mathbb{Z} \cap (-\infty, a])$ denotes the integer part of the real number a .

2. If $M = \mathcal{O}(1)$, the same conclusion holds, provided that estimate (3.35) is replaced by

$$t_M(P^\delta - z) \geq \tau_0 N^{-M_1}. \quad (3.37)$$

Remark 3.9. This result is particularly useful in the analysis of the operator $P_0 - z$. Indeed, if $P_0 - z$ has many singular values lying in a small interval, one can construct a suitable perturbation that provides a uniform lower bound for a portion of the corresponding singular values of the perturbed operator, a property that is not guaranteed for $P_0 - z$ itself.

Proof. 1. We start by supposing M sufficiently large, i.e. $M \gg 1$, as in the hypotheses of the proposition.

Let $\{e_1, \dots, e_M\}$ be an orthonormal family of eigenvectors of the operator $(P_0 - z)^*(P_0 - z)$, associated with its first M eigenvalues, analogously to those we introduced in Section 2.2 for the unperturbed operator P . In particular, for each $j = 1, \dots, M$,

$$(P_0 - z)^*(P_0 - z)e_j = t_j^2(P_0 - z)e_j.$$

Using the symmetry assumption (3.27), we see that a corresponding orthonormal family of eigenvectors of $(P_0 - z)(P_0 - z)^*$ associated to the same eigenvalues, is given by

$$\tilde{f}_j := \Gamma e_j = \bar{e}_j, \quad j = 1, \dots, M. \quad (3.38)$$

If the non-zero t_j are not all distinct, it is not immediately clear if we can obtain $\tilde{f}_j = f_j$, where f_j are defined in the same way as we did in Section 2.2 for P (see (2.16)). However, we know that the two orthonormal families $\{f_1, \dots, f_M\}$ and $\{\tilde{f}_1, \dots, \tilde{f}_M\}$ generate the same vector space F_M . If we let E_M be the subspace generated by $\{e_1, \dots, e_M\}$, we know that the two operators

$$(P_0 - z)|_{E_M} : E_M \rightarrow F_M \quad \text{and} \quad (P_0 - z)^*|_{F_M} : F_M \rightarrow E_M,$$

are well-defined and have the same singular values $0 \leq t_1 \leq \dots \leq t_M$.

We now consider a Grushin problem for $P_0 - z$, following similar steps as in Section 2.2. In analogy with that construction, we define the operators R_-^0 and R_+^0 as in (2.21), (2.22), but in this case we use $\tilde{f}_j$ in place of f_j . This leads to the same structural results.

We adopt the same notations for the associated matrices as in the unperturbed case, with the addition of a superscript 0 to indicate their dependence on the initial perturbation $\delta_0 V_{q_0}$: namely, $\mathcal{P}^0, \mathcal{E}^0, E^0, E_{\pm}^0, E_{-+}^0$.

Since the properties stated in (2.17) do not necessarily hold in this setting, we cannot assert that $E_{-+}^0 = \text{diag}(t_j; j = 1, \dots, M)$. However, we do know that the singular values of E_{-+}^0 satisfy

$$t_j(E_{-+}^0) = t_j(P_0 - z), \quad j = 1, \dots, M.$$

We also adopt the notation

$$s_j(E_{-+}^0) := t_{M-j+1}(E_{-+}^0),$$

so that, equivalently, $s_j(E_{-+}^0) = t_{M-j+1}(P_0 - z)$, for $j = 1, \dots, M$.

We now divide the remainder of the proof into two separate cases. Before proceeding, we recall that:

- M is assumed to be sufficiently large,
- $\theta \in]0, \frac{1}{4}[$ has been fixed,
- M_1 is defined as in (3.36).

Moreover, since z is fixed throughout, we may assume without loss of generality that $z = 0$.

CASE 1:

In the first case, we assume that a subset of the singular values of E_{-+}^0 , and thus of the operator P_0 , satisfy the following lower bound:

$$s_j(E_{-+}^0) \geq \tau_0 N^{-M_1}, \quad \text{for } 1 \leq j \leq M - [(1 - \theta)M].$$

Under this assumption, we can take the trivial perturbation given by $q = \mathbf{0} \in \mathbb{C}^{N^d}$, so that $P^\delta = P_0$. Then, we have

$$t_{M-j+1}(P^\delta) = s_j(E_{-+}^0) \geq \tau_0 N^{-M_1} \quad \text{for } 1 \leq j \leq M - [(1 - \theta)M],$$

which proves (3.35). Moreover,

$$t_\nu(P^\delta) = t_\nu(P_0) \geq \left(1 - \frac{N^{-d}}{C}\right) t_\nu(P_0), \quad \text{for } \nu > M,$$

which proves (3.34).

CASE 2:

We suppose there exists j , $1 \leq j \leq M - [(1 - \theta)M]$ such that

$$s_j(E_{-+}^0) < \tau_0 N^{-M_1}. \quad (3.39)$$

We now recall the results obtained in Section 3.1 and apply them under our current assumptions with $\tilde{f}_j = \bar{e}_j$. In particular, from (3.15), (3.21) and (3.22), we know that there exist a matrix $E \in \mathbb{C}^M$ depending on the eigenvectors $\{e_i\}_{i=1,\dots,M}$ and a potential V_q , determined by a vector $q \in \{0, 1\}^{N^d}$ with exactly M non-zero entries equal to 1, such that

$$EE^t = E_-^0 V_q E_+^0.$$

The matrix V_q is the perturbation we intend to apply to the operator P_0 in order to derive the conclusions of the theorem. Accordingly, we define the perturbed operator

$$P^\delta := P_0 + \delta V_q,$$

with $\delta \geq 0$, whose precise value will be determined later. In particular, we observe that

$$\|V_q\| = 1. \quad (3.40)$$

We now set up a new Grushin problem for the perturbed operator, following the same procedure as in Section 2.3, with $Q = V_q$. We therefore consider the matrix

$$\mathcal{P}^\delta := \begin{pmatrix} P^\delta & R_-^0 \\ R_+^0 & 0 \end{pmatrix}, \quad (3.41)$$

and verify that condition (2.36) holds in our framework:

$$\delta \|V_q\| \tau_0^{-1} \leq \frac{1}{2} \Leftrightarrow \delta \leq \frac{\tau_0}{2\|V_q\|} = \frac{\tau_0}{2}.$$

Hence, for any

$$0 \leq \delta \leq \frac{\tau_0}{2}, \quad (3.42)$$

the Grushin problem (3.41) is well-posed, and its inverse has the form

$$\mathcal{E}^\delta = \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix}, \quad (3.43)$$

where the blocks are defined analogously to (2.37) and they satisfy the same norm estimates.

Our next goal is to estimate the singular values of P^δ . Recalling Lemma 2.6 and applying it

to the perturbed Grushin system (3.41), we obtain for all $\nu = 1, \dots, M$

$$\frac{t_\nu(E_{-+}^\delta)}{\|E^\delta\| t_\nu(E_{-+}^\delta) + \|E_-^\delta\| \|E_+^\delta\|} \leq t_\nu(P^\delta) \leq \|R_+^0\| \|R_-^0\| t_\nu(E_{-+}^\delta).$$

Using (2.38), (2.39), (2.40), (2.41), (2.31) and recalling (3.33), we infer

$$\begin{aligned} \|E^\delta\| t_\nu(E_{-+}^\delta) + \|E_-^\delta\| \|E_+^\delta\| &\leq \frac{2}{t_{M+1}(P_0)} (\|E_{-+}^\delta - E_{-+}\| + \|E_{-+}\|) + 4 \\ &\leq \frac{2}{t_{M+1}(P_0)} (2\delta \|V_q\| + t_M) + 4 \\ &\leq 6 + \frac{4}{t_{M+1}(P_0)} \delta \|V_q\| = \mathcal{O}(1), \end{aligned}$$

where we assumed (3.42) to be true. Moreover, it is easy to obtain

$$\|R_+^0\| \|R_-^0\| \leq \mathcal{O}(1).$$

Hence, there exists a constant $\tilde{C} > 0$ such that

$$\frac{1}{\tilde{C}} t_\nu(E_{-+}^\delta) \leq t_\nu(P^\delta) \leq \tilde{C} t_\nu(E_{-+}^\delta), \quad \nu = 1, 2, \dots, M. \quad (3.44)$$

From (2.37), the lower right block of (3.43) is defined as

$$E_{-+}^\delta = E_{-+}^0 - \delta E_-^0 V_q E_+^0 + \sum_{k=2}^{+\infty} (-1)^k E_-^0 (\delta V_q E^0)^{k-1} (\delta V_q) E_+^0. \quad (3.45)$$

Applying estimates (2.30) and (2.31) with hypothesis (3.33) to our setting yields

$$\|E_\pm^0\| = 1, \quad \|E^0\| \leq \frac{1}{t_{M+1}(P_0)} \leq \frac{1}{\tau_0}. \quad (3.46)$$

By Ky Fan's inequalities (Corollary A.9 in the Appendix), we derive the following estimates for the singular values of the sum of bounded linear operators A , B , and C :

$$s_\nu(A + B) \geq s_{\nu+k-1}(A) - s_k(B), \quad (3.47)$$

and

$$s_\nu(A + B + C) \geq s_{\nu+k+\ell-2}(A) - s_k(B) - s_\ell(C), \quad (3.48)$$

for $\nu, l, k = 1, 2, \dots$

We fix $1 \leq \nu \leq M - [(1 - \theta)M]$ and recall that

$$1 \leq j \leq M - [(1 - \theta)M]$$

where j is defined in the assumption (3.39). Thus $\nu + j - 1 \leq M$ and applying (3.48) with $\ell = 1$

and $k = j$ to the decomposition (3.45) yields

$$s_\nu(E_{-+}^\delta) \geq s_{\nu+j-1}(-\delta E_-^0 V_q E_+^0) - s_j(E_{-+}^0) - s_1 \left(\sum_{k=2}^{+\infty} (-1)^k E_-^0 (\delta V_q E^0)^{k-1} (\delta V_q) E_+^0 \right). \quad (3.49)$$

We now estimate each term separately. For the second term, hypothesis (3.39) implies

$$-s_j(E_{-+}^0) > -\tau_0 N^{-M_1}. \quad (3.50)$$

For the third term in (3.49), we recall (3.40), (3.46) and we obtain

$$\begin{aligned} s_1 \left(\sum_{k=2}^{+\infty} (-1)^k E_-^0 (\delta V_q E^0)^{k-1} (\delta V_q) E_+^0 \right) &= \left\| \sum_{k=2}^{+\infty} (-1)^k E_-^0 (\delta V_q E^0)^{k-1} (\delta V_q) E_+^0 \right\| \\ &\leq \sum_{k=2}^{+\infty} \delta^k \left\| E_-^0 (V_q E^0)^{k-1} V_q E_+^0 \right\| \\ &\leq \sum_{k=2}^{+\infty} \delta^k \|V_q\|^k \|E^0\|^{k-1} \\ &= \delta^2 \|V_q\|^2 \|E^0\| \sum_{k=0}^{+\infty} (\delta \|V_q\| \|E^0\|)^k \\ &\leq 2\delta^2 \tau_0^{-1}, \end{aligned} \quad (3.51)$$

where we used the fact that

$$\delta \|V_q\| \|E^0\| \leq \frac{\tau_0}{2} \|E^0\| \leq \frac{1}{2},$$

ensuring the convergence of the series.

Finally, we turn to the first term in (3.49). From (3.38), we can invoke Proposition 3.5 and rewrite (3.19) as

$$s_k(E_-^0 V_q E_+^0) \geq s_1(E_-^0 V_q E_+^0)^{-\frac{k-1}{M-k+1}} \left(\prod_{\nu=1}^M \frac{E_\nu}{N^d} \right)^{\frac{1}{M-k+1}}, \quad k = 1, \dots, M.$$

Under our orthogonality assumptions, this becomes

$$\begin{aligned} s_k(E_-^0 V_q E_+^0) &\geq s_1(E_-^0 V_q E_+^0)^{-\frac{k-1}{M-k+1}} \left(\frac{M!}{N^{Md}} \right)^{\frac{1}{M-k+1}} \\ &= s_1(E_-^0 V_q E_+^0)^{-\frac{k-1}{M-k+1}} (M!)^{\frac{1}{M-k+1}} N^{-\frac{Md}{M-k+1}}, \quad k = 1, \dots, M. \end{aligned} \quad (3.52)$$

Moreover, inequality (3.18) becomes

$$s_1(E_-^0 V_q E_+^0) \geq (M!)^{1/M} N^{-d}.$$

Substituting it into (3.52) yields

$$\begin{aligned} s_k(E_-^0 V_q E_+^0) &\geq \left((M!)^{\frac{1}{M}} N^{-d} \right)^{-\frac{k-1}{M-k+1}} (M!)^{\frac{1}{M-k+1}} N^{-\frac{Md}{M-k+1}} \\ &= (M!)^{\frac{1-k+M}{M(M-k+1)}} N^{-d\frac{M-k+1}{M-k+1}} = (M!)^{\frac{1}{M}} N^{-d}, \quad k = 1, \dots, M. \end{aligned} \quad (3.53)$$

We recall Stirling's formula (A.4) for $M \gg 1$

$$M! = (1 + o(1)) \sqrt{2\pi M} M^M e^{-M},$$

and we observe that

$$\frac{d}{dM} (\ln((2\pi M)^{\frac{1}{2M}})) = \frac{1}{2M^2} (1 - \ln(2\pi M)).$$

Therefore, the function $\ln((2\pi M)^{\frac{1}{2M}})$ is decreasing for all $M \geq 1$, and so is $(2\pi M)^{\frac{1}{2M}}$. Hence, since $(2\pi M)^{\frac{1}{2M}} \rightarrow 1$ as $M \rightarrow +\infty$, we conclude that

$$(2\pi M)^{\frac{1}{2M}} \geq 1, \quad \text{for all } M \geq 1. \quad (3.54)$$

Thus, for M large enough, applying (3.53), (A.5) and (3.54) we get

$$s_{\nu+j-1}(-\delta E_-^0 V_q E_+^0) > \frac{\delta M (2\pi M)^{\frac{1}{2M}}}{e} N^{-d} \geq \frac{\delta M}{e} N^{-d} \geq \frac{\delta N^{-d}}{e}, \quad (3.55)$$

Hence, combining (3.49), (3.50), (3.51), and (3.55) yields the following inequality

$$s_\nu(E_{-+}^\delta) \geq \frac{\delta N^{-d}}{e} - \tau_0 N^{-M_1} - 2\delta^2 \tau_0^{-1}, \quad 1 \leq \nu \leq M - [(1 - \theta)M]. \quad (3.56)$$

We recall that $\theta < \frac{1}{4}$ and we fix $1 \leq \nu \leq M - [(1 - \theta)M]$. We choose

$$\delta = \frac{\tau_0}{C} N^{-d}, \quad (3.57)$$

where $C > 0$ is a sufficiently large constant. Hence, condition (3.42) on δ is respected and substituting this value in (3.56), we obtain

$$s_\nu(E_{-+}^\delta) \geq \frac{\tau_0 N^{-2d}}{eC} - \tau_0 N^{-2d-\epsilon_0} - 2\frac{\tau_0}{C^2} N^{-2d} = \delta N^{-d} \left(\frac{1}{e} - CN^{-\epsilon_0} - \frac{2}{C} \right).$$

Then, for N sufficiently large and C chosen large enough, there exists a new constant $C_0 > 0$, such that

$$s_\nu(E_{-+}^\delta) \geq \frac{\delta}{C_0} N^{-d}. \quad (3.58)$$

We recall that $M_1 = 2d + \epsilon_0$. Thus, by possibly taking a larger N , we obtain

$$s_\nu(E_{-+}^\delta) \geq \tilde{C} \tau_0 N^{-M_1},$$

with the same constant $\tilde{C}$ that appears in (3.44). Hence, from (3.44) it follows that

$$t_\nu(P^\delta) \geq \frac{t_\nu(E_{-+}^\delta)}{\tilde{C}} \geq \tau_0 N^{-M_1}, \quad \text{for } 1 + [(1 - \theta)M] \leq \nu \leq M,$$

and therefore (3.35) is proved. On the other hand, if $\nu > M$, applying (3.47) with $k = 1$ yields

$$\begin{aligned} t_\nu(P^\delta) &= s_{N^d - \nu + 1}(P_0 + \delta V_q) \geq t_\nu(P_0) - s_1(\delta V_q) = t_\nu(P_0) - \frac{\tau_0 N^{-d}}{C} \\ &\geq t_\nu(P_0) \left(1 - \frac{N^{-d}}{C}\right), \end{aligned}$$

where we used the fact that $t_\nu(P_0) \geq \tau_0$ in this case.

2. Finally, we consider the case where $M = \mathcal{O}(1)$.

In this situation, the argument proceeds as before, except that it now reduces to two sub-cases concerning only the M -th singular value of E_{-+}^0 . Carrying out the proof in this way yields inequality (3.58) with $\nu = 1$, thereby establishing (3.37). $\square$

Remark 3.10. From the preceding proof, we note that Proposition 3.8 remains valid provided that M satisfies only the condition

$$t_{M+1}(P_0 - z) \geq \tau_0.$$

Hence, it is not necessary that M coincides with the exact number of singular values of $P_0 - z$ lying in the interval $[0, \tau_0]$, as required in the hypothesis (3.33).

3.3 Iterative Construction of a Potential for Controlling the Smallest Singular Value

We now adopt the same hypotheses considered at the beginning of the previous section. We aim to iterate the construction of the potential considered in Proposition 3.8, starting from the unperturbed operator P . Our goal is to determine a new potential V_{q_0} , with $q_0 \in \mathbb{C}^{N^d}$, and to establish a lower bound for the smallest singular value of the perturbed operator

$$P + \delta V_{q_0} - z, \quad z \in \mathbb{C},$$

where $\delta \geq 0$.

Let p be defined as in (3.25). We fix $z \in \mathbb{C}$ that satisfies (3.28) for some $\kappa \in]0, 1]$, and set $P^{(0)} := P$. Let $\tau_0^{(0)} \in]0, (CN^{-1})^{1/2}]$, for some constant $C > 0$, and denote by

$$M^{(0)} = M^{(0)}(\tau_0^{(0)}), \tag{3.59}$$

the number of singular values of $P^{(0)} - z$ lying in the interval $[0, \tau_0^{(0)}[$.

We assume that $M^{(0)}$ is sufficiently large and we apply Proposition 3.8 in this framework. Consequently, there exists a vector $q^{(1)} \in \{0, 1\}^{N^d}$ with exactly $M^{(0)}$ non-zero entries, defining the potential $V_{q^{(1)}}$ with $\|V_{q^{(1)}}\| = 1$, and a coefficient

$$\delta^{(1)} := \frac{\tau_0^{(0)}}{C} N^{-d}$$

for some sufficiently large constant $C > 0$, such that the singular values of

$$P^{(1)} := P^{(0)} + \delta^{(1)} V_{q^{(1)}}$$

satisfy (3.34) and (3.35) with the corresponding updated notations. In particular, setting

$$\tau_0^{(1)} := \tau_0^{(0)} N^{-M_1}, \quad M^{(1)} := [M^{(0)}(1 - \theta)],$$

with M_1 defined as in (3.36) and $\theta \in]0, \frac{1}{4}[$, we obtain

$$t_\nu(P^{(1)} - z) \geq \tau_0^{(1)}, \quad \text{for } M^{(1)} + 1 \leq \nu \leq M^{(0)}.$$

Thanks to Remark 3.10, we can apply Proposition 3.8 once more, replacing $(P^{(0)}, M^{(0)}, \tau_0^{(0)})$ with $(P^{(1)}, M^{(1)}, \tau_0^{(1)})$, while keeping the same value of M_1 . To proceed, we must ensure that the perturbation remains sufficiently small, so that condition (3.30) holds:

$$\|\delta^{(1)} V_{q^{(1)}}\| = \frac{\tau_0^{(0)}}{C} N^{-d} \ll N^{-1}.$$

Thus, we iterate this procedure, applying at each step k -th the first case of Proposition 3.8, with the initial conditions given by the outcome of the previous step. In this way, we construct a sequence

$$(P^{(k)}, M^{(k)}, \tau_0^{(k)}), \quad k = 0, 1, \dots, k(M),$$

where the final index $k(M)$ is chosen so that $M^{(k(M))}$ remains of the order of a large constant, ensuring that the first case can still be applied. Therefore at each $(k + 1)$ -th step of the iteration, there exists $q^{(k+1)} \in \{0, 1\}^{N^d}$, with exactly $M^{(k)}$ non-zero entries, such that, introducing the notations

$$\tau_0^{(k+1)} := \tau_0^{(k)} N^{-M_1}, \tag{3.60}$$

$$M^{(k+1)} := [M^{(k)}(1 - \theta)], \tag{3.61}$$

$$P^{(k+1)} := P^{(k)} + \delta^{(k+1)} V_{q^{(k+1)}}, \tag{3.62}$$

where

$$\delta^{(k+1)} = \frac{\tau_0^{(k)}}{C} N^{-d}, \quad \text{and } \|V_{q^{(k+1)}}\| = 1, \tag{3.63}$$

the following bounds are true

$$t_\nu(P^{(k+1)} - z) \geq \tau_0^{(k+1)}, \quad M^{(k+1)} + 1 \leq \nu \leq M^{(k)}, \quad (3.64)$$

$$t_\nu(P^{(k+1)} - z) \geq t_\nu(P^{(k)} - z) - \frac{\tau_0^{(k)} N^{-d}}{C}, \quad \nu > M^{(k)}. \quad (3.65)$$

We observe that (3.64) guarantees that, at each iteration step, the assumption required in Proposition 3.8 holds in the weaker form described in Remark 3.10. From the iteration, we also deduce that

$$\tau_0^{(k)} = \tau_0^{(0)} N^{-M_1 k}, \quad (3.66)$$

and therefore

$$\delta^{(k+1)} = \frac{\tau_0^{(0)} N^{-M_1 k}}{C} N^{-d} = \delta^{(1)} N^{-M_1 k}. \quad (3.67)$$

Hence, it follows that

$$P^{(k+1)} = P^{(k)} + \delta^{(k+1)} V_{q^{(k+1)}} = P^{(0)} + \sum_{j=1}^{k+1} \delta^{(j)} V_{q^{(j)}} = P + \delta^{(1)} \sum_{j=0}^k N^{-M_1 j} V_{q^{(j+1)}}.$$

Taking the norm of the perturbative term, we find by (3.63)

$$\left\| \delta^{(1)} \sum_{j=0}^k N^{-M_1 j} V_{q^{(j+1)}} \right\| \leq \delta^{(1)} \sum_{j=0}^k N^{-M_1 j} \|V_{q^{(j+1)}}\| < 2\delta^{(1)} \ll N^{-1}.$$

This bound shows that the hypothesis (3.30) is satisfied for every k . Therefore, the application of Proposition 3.8 at each step of the iteration is justified.

We observe that $M^{(k)}$ decays exponentially fast in k . In particular the following inequality is true:

$$M^{(k)} \leq (1 - \theta)^k M^{(0)}. \quad (3.68)$$

We want the condition on k that $(1 - \theta)^k M^{(0)} \geq C$ to be valid, with a constant $1 \ll C < M^{(0)}$, which is equivalent to

$$k \leq \frac{\ln\left(\frac{M^{(0)}}{C}\right)}{\ln\left(\frac{1}{1-\theta}\right)} =: k_0. \quad (3.69)$$

In this way we ensure that $M^{(k)} \geq \tilde{C}$ for all $k \leq k_0$, with a slightly smaller constant $\tilde{C} \gg 1$. Hence, this iterative process continues until we reach $k = k_0$. Beyond this point, the iteration proceeds by decreasing $M^{(k)}$ by one unit at each step until 1 is reached. Thus, for these steps, we can apply (3.37).

Now, given $\nu > M^{(0)} > M^{(1)} > \dots > M^{(k)}$, we use (3.66) and iterate (3.65) to obtain

$$\begin{aligned} t_\nu(P^{(k)} - z) &\geq t_\nu(P^{(0)} - z) - \tau_0^{(0)} \frac{N^{-d}}{C} (1 + N^{-M_1} + N^{-2M_1} + \dots + N^{-M_1(k-1)}) \\ &\geq t_\nu(P^{(0)} - z) - \tau_0^{(0)} \mathcal{O}\left(\frac{N^{-d}}{C}\right). \end{aligned}$$

Moreover, for $1 \leq \nu \leq M^{(0)}$, let $l = l(M^{(0)})$ denote the unique iteration step such that

$$M^{(l)} + 1 \leq \nu \leq M^{(l-1)},$$

which is $\nu = M^{(l-1)}$, whenever $\nu \leq \tilde{C}$. Then, applying (3.64) or (3.37), we obtain

$$t_\nu(P^{(l)} - z) \geq \tau_0^{(l)}.$$

Hence, if $k > l$, from (3.64) and (3.65) we get

$$\begin{aligned} t_\nu(P^{(k)} - z) &\geq t_\nu(P^{(l)} - z) - \frac{N^{-d}\tau_0^{(l)}}{C} \left(N^{-M_1(k-l-1)} + N^{-M_1(k-l-2)} + \dots + N^{-M_1} + 1 \right) \\ &\geq t_\nu(P^{(l)} - z) - \tau_0^{(l)} \mathcal{O}\left(\frac{N^{-d}}{C}\right) \geq \tau_0^{(l)} \left(1 - \mathcal{O}\left(\frac{N^{-d}}{C}\right)\right). \end{aligned} \quad (3.70)$$

In this way, we obtain the following result.

Proposition 3.11. *Let P defined as in Section 3.2 and fix $z \in \mathbb{C}$ that satisfies (3.28). Let $M_1 = 2d + \epsilon_0$, where $\epsilon_0 > 0$. Fix $0 < \tau_0 \leq \sqrt{C_0 N^{-1}}$, $C_0 > 0$ and let $M^{(0)} = \mathcal{O}(N^{d-\kappa})$ be the number of singular values of $P - z$ in $[0, \tau_0]$. Let $0 < \theta < \frac{1}{4}$ and let $M(\theta) \gg 1$ be a sufficiently large constant.*

We set $M^{(k)}$, $1 \leq k \leq k_1$ iteratively as follows. As long as $M^{(k)} \geq M(\theta)$, we set $M^{(k+1)} = [(1 - \theta)M^{(k)}]$. Let $k_0 \geq 0$ be the last k value obtained in this way. For $k > k_0$ we put $M^{(k+1)} = M^{(k)} - 1$, until we reach the value k_1 such that $M^{(k_1)} = 1$. We set $\tau_0^{(k)} = \tau_0 N^{-kM_1}$, for $1 \leq k \leq k_1 + 1$.

Then there exists $q_0 \in \mathbb{C}^{N^d}$ with its corresponding potential matrix $V_{q_0} \in \mathbb{C}^{N^d \times N^d}$, such that

$$\|q_0\|_\infty = \mathcal{O}(1), \quad \|q_0\|_2 = \mathcal{O}(N^{\frac{d-\kappa}{2}}). \quad (3.71)$$

Moreover, if we define $\delta = \frac{\tau_0}{C} N^{-d}$, for $C > 0$ sufficiently large and $P^\delta = P + \delta V_{q_0}$, then we have the following estimates on the singular values of $P^\delta - z$:

- *If $\nu > M^{(0)}$, we have*

$$t_\nu(P^\delta - z) \geq \left(1 - \frac{N^{-d}}{C}\right) t_\nu(P - z). \quad (3.72)$$

- *If $M^{(k)} + 1 \leq \nu \leq M^{(k-1)}$, with $1 \leq k \leq k_1$, then*

$$t_\nu(P^\delta - z) \geq (1 - \mathcal{O}(N^{-d})) \tau_0^{(k)}. \quad (3.73)$$

- *Finally, for $\nu = M^{(k_1)} = 1$, we have*

$$t_1(P^\delta - z) \geq \tau_0^{(k_1+1)} \geq (1 - \mathcal{O}(N^{-d})) \tau_0^{(k_1+1)}. \quad (3.74)$$

Proof. The perturbation is obtained as a result of the iteration procedure as it follows

$$\delta V_{q_0} := \sum_{k=1}^{k_1} \delta^{(k)} V_{q^{(k)}} = \sum_{k=1}^{k_1} \delta^{(1)} N^{-M_1(k-1)} V_{q^{(k)}} = \delta^{(1)} \left(\sum_{k=0}^{k_1-1} N^{-M_1 k} V_{q^{(k+1)}} \right),$$

where we applied (3.67). Thus,

$$q_0 := \sum_{k=0}^{k_1-1} N^{-M_1 k} q^{(k+1)} \in \mathbb{C}^{N^d},$$

and

$$\delta := \delta^{(1)} = \frac{\tau_0}{C} N^{-d}.$$

As a consequence we have

$$\|q_0\|_2 \leq \sum_{k=0}^{k_1-1} N^{-M_1 k} \|q^{(k+1)}\|_2 = \sum_{k=0}^{k_1-1} N^{-M_1 k} \sqrt{M^{(k)}} \leq \sqrt{M^{(0)}} \sum_{k=0}^{k_1-1} \left(N^{-M_1} \sqrt{1-\theta} \right)^k = \mathcal{O}(N^{\frac{d}{2}-\frac{\kappa}{2}}), \quad (3.75)$$

and

$$\|q_0\|_\infty \leq \sum_{k=0}^{k_1-1} N^{-M_1 k} \|q^{(k+1)}\|_\infty < 2.$$

We have already proved the first two inequalities (3.72) and (3.73) previously. The bound (3.74) is obtained by (3.37) as it follows

$$t_1(P^\delta - z) = t_{M^{(k_1)}}(P^{(k_1+1)} - z) \geq \tau_0^{(k_1+1)}.$$

□

3.3.1 Log-Determinant Estimates for the Perturbed Operator

We now proceed under the same assumptions of Proposition 3.11. In particular, we consider the perturbed operator $P^\delta(q_0) := P + \delta V_{q_0}$ arising from that result and we fix $z \in \mathbb{C}$. Our goal is to establish estimates for $\det(P^\delta(q_0) - z)$ which will be crucial in the next chapters.

We let $C_0 > 0$ and consider

$$M := M(C_0 N^{-1}) = \mathcal{O}(N^{d-\kappa}), \quad (3.76)$$

the number of singular values of $P^\delta(q_0) - z$ in the interval $[0, (C_0 N^{-1})^{1/2}]$. In particular we are working with $\alpha = C_0 N^{-1}$, so that the hypothesis of Theorem 1.25 is satisfied.

We set up the M -dimensional Grushin problem for $P^\delta(q_0) - z$ as in Section 2.3. First, we verify that the hypothesis (2.36) holds in a suitable ball. Fix $R > 0$ and assume

$$\|q_0\|_2 < \frac{R}{2}.$$

The condition (2.36) is equivalent to

$$\delta \alpha^{-1/2} \|q_0\|_\infty \leq \frac{1}{2}.$$

Recalling that $\delta = \frac{\tau_0 N^{-d}}{C}$ as in Proposition 3.11, a sufficient requirement is therefore

$$\delta \alpha^{-\frac{1}{2}} \frac{R}{2} \leq \frac{1}{2} \iff R \leq \frac{\sqrt{C_0 N^{-1}}}{\delta} = C_1 \tau_0^{-1} N^{d-\frac{1}{2}}, \quad (3.77)$$

where $C_1 := \sqrt{C_0} C$. Combining this with the bound $\|q_0\|_2 = \mathcal{O}(N^{\frac{d-\kappa}{2}})$, and using $0 < \tau_0 \leq \sqrt{C_0} N^{-1/2}$, we choose $R > 0$ and two constants $C_2, C_3 > 0$ such that

$$C_2 N^{\frac{d-\kappa}{2}} \leq R \leq C_3 N^{d-1} \leq C_1 \tau_0^{-1} N^{d-\frac{1}{2}}, \quad (3.78)$$

ensuring that $\|q_0\|_2 < \frac{R}{2}$ and the validity of (2.36).

Under this hypothesis the perturbed Grushin problem is well defined. We keep the same notations as in Section 2.3, now emphasizing the dependence on q_0 . In particular, we denote the matrices of the problem as $\mathcal{P}^\delta(q_0)$, $R_\pm(q_0)$, and the inverse matrix of $\mathcal{P}^\delta(q_0)$ is

$$\mathcal{E}^\delta(q_0) = \begin{pmatrix} E^\delta(q_0) & E_-^\delta(q_0) \\ E_+^\delta(q_0) & E_{-+}^\delta(q_0) \end{pmatrix}.$$

Our goal is to analyze the spectrum of the operator $P^\delta(q_0)$. To this end, we recall formula (2.42):

$$\ln |\det(P^\delta(q_0) - z)| = \ln |\det \mathcal{P}^\delta(q_0)| + \ln |\det E_{-+}^\delta(q_0)|, \quad (3.79)$$

and we study the two terms separately. For the first term, by first applying (2.45) and then (2.35), we obtain

$$\begin{aligned} \ln |\det \mathcal{P}^\delta(q_0)| &\geq \ln |\det \mathcal{P}(z)| - \mathcal{O}(\delta N^{1/2} \|V_{q_0}\|_{\text{tr}}) \\ &= N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho + \mathcal{O}(N^{-\kappa} \ln(N)) \right) - \mathcal{O}(\delta N^{1/2} \|V_{q_0}\|_{\text{tr}}). \end{aligned}$$

Hence, by recalling that $\|V_{q_0}\|_{\text{tr}} \leq N^{d/2} \|q_0\|_2$, it follows from (3.78) that

$$\ln |\det \mathcal{P}^\delta(q_0)| \geq N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho - \mathcal{O}(N^{-\frac{d}{2}}) \right). \quad (3.80)$$

On the other hand, for the second term we observe that

$$|\det(E_{-+}^\delta(q_0))| = \prod_{\nu=1}^{M^{(0)}} t_\nu(E_{-+}^\delta(q_0)) \prod_{\nu=M^{(0)}+1}^M t_\nu(E_{-+}^\delta(q_0)), \quad (3.81)$$

where M is defined in (3.76) and $M^{(0)}$ in (3.59). Therefore we aim to obtain a good lower bound

for this quantity.

From (3.61), we deduce that, for $1 \leq k \leq k_0$,

$$\begin{aligned} M^{(k-1)} - M^{(k)} &= M^{(k-1)} - [(1 - \theta)M^{(k-1)}] \\ &\leq M^{(k-1)} - (1 - \theta)M^{(k-1)} + 1 \\ &\leq 1 + (1 - \theta)^{k-1}\theta M^{(0)}. \end{aligned} \quad (3.82)$$

We recall inequalities (3.44). Then, applying Proposition 3.11, we obtain for $1 \leq k \leq k_0$,

$$\prod_{\nu=1+M^{(k)}}^{M^{(k-1)}} t_\nu(E_{-+}^\delta(q_0)) \geq \prod_{\nu=1+M^{(k)}}^{M^{(k-1)}} \frac{1}{\tilde{C}} (1 - \mathcal{O}(N^{-d})) \tau_0 N^{-kM_1} \geq \left(\frac{1}{\tilde{C}} \tau_0 N^{-kM_1} \right)^{1+(1-\theta)^{k-1}\theta M^{(0)}}. \quad (3.83)$$

for a new constant $\tilde{C}' > 0$. On the other hand, if $k_0 < k \leq k_1$, then $M^{(k-1)} - M^{(k)} = 1$. Applying the same arguments, we obtain

$$t_{M^{(k-1)}}(E_{-+}^\delta(q_0)) \geq \frac{1}{\tilde{C}'} \tau_0 N^{-kM_1}. \quad (3.84)$$

Hence, by putting together (3.83) and (3.84), we get

$$\begin{aligned} \ln \left(\prod_{\nu=1}^{M^{(0)}} t_\nu(E_{-+}^\delta(q_0)) \right) &= \sum_{k=1}^{k_0} \ln \left(\prod_{\nu=1+M^{(k)}}^{M^{(k-1)}} t_\nu(E_{-+}^\delta(q_0)) \right) + \sum_{k=k_0+1}^{k_1+1} \ln \left(t_{M^{(k-1)}}(E_{-+}^\delta(q_0)) \right) \\ &\geq \sum_{k=1}^{k_0} \ln \left(\left(\frac{1}{\tilde{C}'} \tau_0 N^{-kM_1} \right)^{1+(1-\theta)^{k-1}\theta M^{(0)}} \right) + \sum_{k=k_0+1}^{k_1+1} \ln \left(\frac{1}{\tilde{C}'} \tau_0 N^{-kM_1} \right) \\ &= - \sum_{k=1}^{k_0} (1 + (1 - \theta)^{k-1}\theta M^{(0)}) \left(\ln(\tilde{C}') + \ln \left(\frac{1}{\tau_0} \right) + kM_1 \ln(N) \right) \\ &\quad - \sum_{k=k_0+1}^{k_1+1} \left(\ln(\tilde{C}') + \ln \left(\frac{1}{\tau_0} \right) + kM_1 \ln(N) \right). \end{aligned}$$

Since $(1 + (1 - \theta)^{k-1}\theta M^{(0)}) \geq 1$ for $k_0 + 1 \leq k \leq k_1 + 1$, we obtain

$$\ln \left(\prod_{\nu=1}^{M^{(0)}} t_\nu(E_{-+}^\delta(q_0)) \right) \geq - \sum_{k=1}^{k_1+1} (1 + (1 - \theta)^{k-1}\theta M^{(0)}) \left(\ln(\tilde{C}') + \ln \left(\frac{1}{\tau_0} \right) + kM_1 \ln(N) \right). \quad (3.85)$$

We recall from (3.32) that $M^{(0)} = \mathcal{O}(N^{d-\kappa})$, and we consider the constant $M(\theta) \gg 1$ specified in the assumptions of Proposition 3.11. Hence, by definition (3.69), we obtain

$$k_1 + 1 = \frac{\ln \left(\frac{M^{(0)}}{M(\theta)} \right)}{\ln \left(\frac{1}{1-\theta} \right)} + M(\theta) + 1 = \mathcal{O}(\ln(M^{(0)})) = (d - \kappa) \mathcal{O}(\ln(N)) = \mathcal{O}(1) \ln(N).$$

Hence, by applying this result in (3.85), we obtain

$$\begin{aligned} \ln \left(\prod_{\nu=1}^{M^{(0)}} t_{\nu}(E_{-+}^{\delta}(q_0)) \right) &\geq - \sum_{k=1}^{k_1+1} (1 + (1-\theta)^{k-1} \theta M^{(0)}) \left(\ln(\tilde{C}') + \ln \left(\frac{1}{\tau_0} \right) + \mathcal{O}(1) (\ln(N))^2 \right) \\ &\geq - \left(\mathcal{O}(1) \ln(N) + \mathcal{O}(N^{d-\kappa}) (1 - (1-\theta)^{k_1+1}) \right) \\ &\quad \cdot \left(\ln(\tilde{C}') + \ln \left(\frac{1}{\tau_0} \right) + \mathcal{O}(1) (\ln(N))^2 \right). \end{aligned} \quad (3.86)$$

From (3.72), the remaining factor in (3.81) can be bounded in the following way

$$\prod_{\nu=M^{(0)}+1}^M t_{\nu}(E_{-+}^{\delta}(q_0)) \geq \prod_{\nu=M^{(0)}+1}^M \frac{t_{\nu}(P-z)}{\tilde{C}'} \geq \left(\frac{\tau_0}{\tilde{C}'} \right)^{M-M^{(0)}} \quad (3.87)$$

Hence, applying the logarithm, we have

$$\ln \left(\prod_{\nu=M^{(0)}+1}^M t_{\nu}(E_{-+}^{\delta}(q_0)) \right) \geq -M \left(\ln \left(\frac{1}{\tau_0} \right) + \ln(\tilde{C}') \right) \geq -\mathcal{O}(N^{d-\kappa}) \ln \left(\left(\frac{1}{\tau_0} \right) + \ln(\tilde{C}') \right). \quad (3.88)$$

Now we combine together (3.86) and (3.88) in order to find an estimate for $\ln |\det E_{-+}^{\delta}(q_0)|$. Since $\ln(\tilde{C}')$ is a constant, it is absorbed into the big- $\mathcal{O}$. Hence, there exists a constant $C_1 > 0$ such that

$$\ln |\det E_{-+}^{\delta}(q_0)| \geq -C_1 N^d \left(N^{-d} \ln(N) + N^{-\kappa} \right) \left(\ln \left(\frac{1}{\tau_0} \right) + (\ln(N))^2 \right), \quad (3.89)$$

Finally, combining (3.79), (3.80), and (3.89), we deduce the existence of another constant $C_2 > 0$ such that

$$\begin{aligned} \ln |\det(P^{\delta}(q_0) - z)| &= \ln |\det(\mathcal{P}^{\delta}(q_0))| + \ln |\det(E_{-+}^{\delta}(q_0))| \\ &\geq N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho \right) \\ &\quad - N^d C_2 \left(N^{-\frac{d}{2}} + \left(N^{-d} \ln(N) + N^{-\kappa} \right) \left(\ln \left(\frac{1}{\tau_0} \right) + (\ln(N))^2 \right) \right). \end{aligned}$$

These computations lead to the following result.

Proposition 3.12. *Under the assumptions of Proposition 3.11, for the potential V_{q_0} constructed therein, we have*

$$\begin{aligned} \ln |\det(P^{\delta}(q_0) - z)| &- N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho \right) \\ &\geq -N^d \mathcal{O} \left(N^{-\frac{d}{2}} + \left(N^{-d} \ln(N) + N^{-\kappa} \right) \left(\ln \left(\frac{1}{\tau_0} \right) + (\ln(N))^2 \right) \right). \end{aligned} \quad (3.90)$$

Chapter 4

Perturbations by Random Potentials

In this chapter we first record some estimates derived from the computations in Chapter 3. We then introduce a probability measure for the vectors $q \in \mathbb{C}^{N^d}$ that generate the perturbation potentials V_q . Finally, using complex analysis and measure theory techniques, we obtain a key probabilistic bound that will be used in the last chapter.

Throughout the chapter we work under the following standing assumptions:

- $p \in C^\infty(\mathbb{T}^{2d})$ satisfies (3.25) and $P := p_N$ satisfies (3.27).
- $h = \frac{1}{2\pi N}$ with $N \in \mathbb{N}$, $N \gg 1$ and $\tau_0 \in [0, (C_0 N^{-1})^{1/2}[$ with $C_0 > 0$.
- $z \in \mathbb{C}$ such that (3.28) holds for some $\kappa \in]0, 1]$.
- $q_0 \in \mathbb{C}^{N^d}$ and $\delta = \frac{\tau_0}{C} N^{-d}$ (with $C > 0$ independent of N) from Proposition 3.11.
- We retain the notation of Chapter 3 for all matrices associated with the Grushin problems $\mathcal{P}$ and $\mathcal{P}^\delta$, with $\alpha = C_0 N^{-1}$.

4.1 Deterministic Log-Determinant Estimates for Bounded Perturbations

In this section we consider a general small perturbation given by a potential matrix V_q , assuming that q lies in a fixed ball in $\mathbb{C}^{N^d}$. Our goal is to derive a deterministic upper bound for $\ln |\det(P^\delta(q) - z)|$ by using the results we have established previously.

First, we consider $R > 0$ satisfying (3.78), so that

$$\|q_0\|_2 < \frac{R}{2}. \quad (4.1)$$

In particular, we choose the constants in (3.78) so that condition (2.36) is still verified in the ball $B_{N^d}(0, 3R)$.

Let $q \in \mathbb{C}^{N^d}$ be an arbitrary vector such that

$$\|q\|_2 < 3R.$$

We observe that for its corresponding potential matrix V_q , the following estimate is true

$$\|V_q\|_{\text{tr}} \leq N^{d/2} \|q\|_2 < \widetilde{C}_1 N^{\frac{3d-1}{2}} \tau_0^{-1},$$

for a new constant $\widetilde{C}_1 > 0$. Hence, by applying (2.46) with $\alpha = C_0 N^{-1}$, we obtain

$$\ln |\det \mathcal{P}^\delta(q)| \leq N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho + \mathcal{O}(N^{-\kappa} \ln(N)) + \mathcal{O}(N^{-\frac{d}{2}}) \right), \quad (4.2)$$

where we used the fact that $\delta = \frac{\tau_0}{C} N^{-d}$. Now, recalling results (2.42) and (2.47), we achieve the following result

$$\begin{aligned} \ln |\det(P^\delta(q) - z)| &= \ln |\det(\mathcal{P}^\delta(q))| + \ln |\det(E_{-+}^\delta(q))| \\ &\leq N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho + \mathcal{O}(N^{-\kappa} \ln(N)) + \mathcal{O}(N^{-\frac{d}{2}}) \right) \\ &\leq N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho \right) \\ &\quad + N^d \mathcal{O} \left(N^{-\frac{d}{2}} + \left(N^{-d} \ln(N) + N^{-\kappa} \right) \left(\ln \left(\frac{1}{\tau_0} \right) + (\ln(N))^2 \right) \right). \end{aligned} \quad (4.3)$$

Thus, we let $C'' > 0$ a sufficiently large constant, depending only on the bounds obtained in (3.90) and (4.3), and set

$$\varepsilon_0(N) := C'' \left(N^{-\frac{d}{2}} + \left(N^{-d} \ln(N) + N^{-\kappa} \right) \left(\ln \left(\frac{1}{\tau_0} \right) + (\ln(N))^2 \right) \right). \quad (4.4)$$

Then, for every $q \in \mathbb{C}^{N^d}$ satisfying

$$\|q\|_2 < 3R,$$

we obtain the upper bound

$$\ln |\det(P^\delta(q) - z)| - N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho \right) \leq N^d \varepsilon_0(N). \quad (4.5)$$

Moreover, from Proposition 3.12, there exists $q_0 \in \mathbb{C}^{N^d}$ with

$$\|q_0\|_2 < \frac{R}{2},$$

such that the following lower bound holds:

$$\ln |\det(P^\delta(q_0) - z)| - N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho \right) \geq -N^d \varepsilon_0(N). \quad (4.6)$$

These two estimates will be crucial in the next section, where we combine them with tools from complex analysis and measure theory.

Remark 4.1. If τ_0 is not too small, then

$$\varepsilon_0(N) \xrightarrow{N \rightarrow +\infty} 0. \quad (4.7)$$

For instance, if we consider $\tau_0 \geq e^{-N^{\frac{\kappa}{2}}}$, then for every $m \geq \kappa$,

$$N^{-m} \ln \left(\frac{1}{\tau_0} \right) \leq N^{\frac{\kappa}{2}-m}.$$

This implies that for each $0 \leq l < \frac{\kappa}{2}$, the following is true

$$\lim_{N \rightarrow +\infty} N^l \varepsilon_0(N) = 0. \quad (4.8)$$

Moreover, without any further assumption on τ_0 , for every $l \geq \kappa$, we have

$$\lim_{N \rightarrow +\infty} N^l \varepsilon_0(N) = +\infty.$$

4.2 Log-Determinant Estimates under Random Perturbations

In this section we follow the same approach and the same techniques used by Sjöstrand in [Sj09, Chapter 8] and we adapt his work to our discrete framework.

In particular, we introduce a probability distribution on $\mathbb{C}^{N^d}$ supported on a ball centered at the origin. Within this setting, we study holomorphic functions naturally associated with our problem, which are bounded on the ball and satisfy an additional lower bound at one point. Using tools from complex analysis and measure theory we derive probabilistic estimates on these functions, which will be applied to the spectral analysis of the perturbed operator in the following chapter.

First of all, we fix $z \in \mathbb{C}$ and $\tau_0 \in]0, (C_0 N^{-1})^{1/2}]$ with a constant $C_0 > 0$ as in Chapter 3. Then we define the function

$$F : \mathbb{C}^{N^d} \longrightarrow \mathbb{C} \\ q \longmapsto \det(P^\delta(q) - z) \exp \left(-N^d \int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho \right), \quad (4.9)$$

where $P^\delta(q) := P + \delta V_q$, with $V_q = \text{diag}(q(n); n = 1, \dots, N^d) \in \mathbb{C}^{N^d \times N^d}$ and $\delta = \frac{\tau_0 N^{-d}}{C}$, defined in the same way as in the previous chapter. Since the determinant is a polynomial, F is a holomorphic function in q . We observe that if $F(q) = 0$ for some $q \in \mathbb{C}^{N^d}$, then z is an eigenvalue for the perturbed operator $P + \delta V_q$. Moreover, from (4.5) and (4.6), we know that

$$\ln |F(q)| \leq \varepsilon_0(N) N^d \quad \text{if } \|q\|_2 < 3R, \quad (4.10)$$

$$\ln |F(q_0)| \geq -\varepsilon_0(N) N^d, \quad \text{for some } q_0 \in B_{N^d} \left(0, \frac{R}{2} \right), \quad (4.11)$$

where

$$\varepsilon_0(N) := C'' \left(N^{-\frac{d}{2}} + \left(N^{-d} \ln(N) + N^{-\kappa} \right) \left(\ln \left(\frac{1}{\tau_0} \right) + (\ln(N))^2 \right) \right), \quad (4.12)$$

for a suitable constant $C'' > 0$ and $R > 0$ satisfies (3.78).

We take $q_1 \in \mathbb{C}^{N^d}$ such that $\|q_1\|_2 = R$ and we define the function

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ w &\longmapsto F(q_0 + wq_1). \end{aligned} \quad (4.13)$$

Clearly, f is well-defined and holomorphic. We observe that

$$f(0) = F(q_0).$$

We would like to restrict the argument of f so that the corresponding argument of F is in the ball $B_{N^d}(0, R)$. Hence we consider the following equivalences

$$\begin{aligned} \|q_0 + wq_1\|_2 < R &\Leftrightarrow \|q_0\|_2^2 + \|wq_1\|_2^2 + 2\Re(\langle q_0, wq_1 \rangle) < R^2 \\ &\Leftrightarrow |w|^2 R^2 + 2\Re(\langle q_0, wq_1 \rangle) < R^2 - \|q_0\|_2^2 \\ &\Leftrightarrow |w|^2 + 2\Re \left(\overline{w} \left\langle \frac{q_0}{R}, \frac{q_1}{R} \right\rangle \right) < 1 - \left\| \frac{q_0}{R} \right\|_2^2 \\ &\Leftrightarrow \left| w + \left\langle \frac{q_0}{R}, \frac{q_1}{R} \right\rangle \right|^2 < 1 - \left\| \frac{q_0}{R} \right\|_2^2 + \left| \left\langle \frac{q_0}{R}, \frac{q_1}{R} \right\rangle \right|^2 =: r_0^2. \end{aligned} \quad (4.14)$$

Thus, we are interested in restricting the domain of the function f to the disc

$$D_{q_0, q_1} := D \left(- \left\langle \frac{q_0}{R}, \frac{q_1}{R} \right\rangle, r_0 \right).$$

Moreover, we have the following estimates for r_0 :

$$r_0^2 \leq 1 - \left\| \frac{q_0}{R} \right\|_2^2 + \left\| \frac{q_0}{R} \right\|_2^2 \left\| \frac{q_1}{R} \right\|_2^2 = 1 - \left\| \frac{q_0}{R} \right\|_2^2 + \left\| \frac{q_0}{R} \right\|_2^2 = 1 \quad (4.15)$$

$$r_0^2 \geq 1 - \left\| \frac{q_0}{R} \right\|_2^2 > 1 - \frac{1}{4} = \frac{3}{4}, \quad (4.16)$$

which imply

$$\frac{\sqrt{3}}{2} \leq r_0 \leq 1.$$

From (4.15), we also have the following estimate for the center of the disc $w_0 := - \left\langle \frac{q_0}{R}, \frac{q_1}{R} \right\rangle$,

$$|w_0|^2 = \left| \left\langle \frac{q_0}{R}, \frac{q_1}{R} \right\rangle \right|^2 = r_0^2 + \left\| \frac{q_0}{R} \right\|_2^2 - 1 < r_0^2 - \frac{3}{4} \leq \frac{1}{4} r_0^2. \quad (4.17)$$

So far, we know that

$$\begin{aligned} \ln |f(0)| &\geq -\varepsilon_0(N)N^d \\ \ln |f(w)| &\leq \varepsilon_0(N)N^d, \quad \text{if } w \in D_{q_0, q_1}. \end{aligned} \quad (4.18)$$

In particular, since (4.10) holds in a larger ball, we may assume that the last estimate still holds in the disc $D(w_0, 3r_0)$. Indeed, following the same steps as in (4.14),

$$\|q_0 + wq_1\|_2 < 3R \Leftrightarrow |w - w_0|^2 < 8 + r_0^2,$$

and $9r_0^2 \leq 8 + r_0^2$.

We know that f is analytic in a neighborhood of $|w| \leq \frac{5}{2}r_0$. Let the points $w_1, \dots, w_{m_1} \in \mathbb{C}$ be its zeros in $|w| < \frac{5}{2}r_0$ counted with multiplicity, listed in non-decreasing order of their modulus. Moreover, we set

$$m := \# \left\{ w \in D\left(w_0, \frac{3}{2}r_0\right); f(w) = 0 \right\} \leq m_1,$$

where the inequality is true because $D(w_0, \frac{3}{2}r_0) \subset D(0, 2r_0)$ by (4.17).

We observe that $\ln |f(0)| \geq -\varepsilon_0(N)N^d$ implies $f(0) \neq 0$. Hence all the hypotheses of Jensen's formula (see Theorem A.15 in the Appendix) for f in the disc $\overline{D}(0, \frac{5}{2}r_0)$ are satisfied. From (A.12) we obtain

$$\ln |f(0)| = - \sum_{j=1}^{m_1} \ln \left(\frac{5r_0}{2|w_j|} \right) + \frac{1}{2\pi} \int_0^{2\pi} \ln \left| f\left(\frac{5}{2}r_0 e^{i\theta}\right) \right| d\theta.$$

We notice that $\overline{D}(0, \frac{5}{2}r_0) \subset D(w_0, 3r_0)$. Indeed, if $w \in \mathbb{C}$, $|w| \leq \frac{5}{2}r_0$, then by (4.17)

$$|w - w_0| \leq |w| + |w_0| < \frac{5}{2}r_0 + \frac{1}{2}r_0 = 3r_0.$$

Consequently, by applying inequalities (4.18), with the second one extended to $D(w_0, 3r_0)$, we obtain

$$\sum_{j=1}^{m_1} \ln \left(\frac{5r_0}{2|w_j|} \right) = -\ln |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} \ln \left| f\left(\frac{5}{2}r_0 e^{i\theta}\right) \right| d\theta \leq 2\varepsilon_0(N)N^d.$$

We observe from (4.17) that the following inclusions are true

$$\overline{D}(0, r_0) \subset D\left(w_0, \frac{3}{2}r_0\right) \subset D(0, 2r_0).$$

We suppose that $m < m_1$ and we observe that for all $m+1 \leq j \leq m_1$, we have $r_0 < |w_j| < \frac{5}{2}r_0$, which implies

$$0 < \ln \left(\frac{5r_0}{2|w_j|} \right) < \ln \left(\frac{5}{2} \right).$$

Instead, if we take $1 \leq j \leq m$, we have $0 < |w_j| < 2r_0$, and then

$$0 < \ln\left(\frac{5}{4}\right) < \ln\left(\frac{5r_0}{2|w_j|}\right).$$

Thus, we can write the following

$$m \ln\left(\frac{5}{4}\right) < \sum_{j=1}^m \ln\left(\frac{5r_0}{2|w_j|}\right) \leq \sum_{j=1}^{m_1} \ln\left(\frac{5r_0}{2|w_j|}\right) \leq 2\varepsilon_0(N)N^d,$$

which implies

$$m \leq \frac{2\varepsilon_0(N)N^d}{\ln\left(\frac{5}{4}\right)} = \mathcal{O}(\varepsilon_0(N)N^d). \quad (4.19)$$

The case $m = m_1$ yields the same result trivially.

Now, since f is holomorphic, we consider the following factorization on the disc $D(w_0, \frac{3}{2}r_0)$

$$f(w) = \prod_{j=1}^m (w - w_j)t(w), \quad \text{in } D\left(w_0, \frac{3}{2}r_0\right), \quad (4.20)$$

with $t(w) \neq 0$ and holomorphic on $D(w_0, \frac{3}{2}r_0)$. Thus, there exists a function $g(w)$ holomorphic on such disc, such that

$$t(w) = e^{g(w)}, \quad \text{in } D\left(w_0, \frac{3}{2}r_0\right). \quad (4.21)$$

We define $r_j^0 := |w_j - w_0| < \frac{3}{2}r_0$, for $j = 1, \dots, m$, and we observe that

$$\prod_{j=1}^m |w - w_j| \geq \prod_{j=1}^m ||w - w_0| - r_j^0|. \quad (4.22)$$

We suppose that there exists $\tau \in]\frac{4}{3}r_0, \frac{3}{2}r_0[$ such that

$$\prod_{j=1}^m |\tau - r_j^0| \geq e^{-mC_3}, \quad \text{for a constant } C_3 > 0. \quad (4.23)$$

Under this assumption, we take $w \in \mathbb{C}, |w - w_0| = \tau$, and we obtain from (4.19), (4.20) and (4.22),

$$|t(w)| = \frac{|f(w)|}{\prod_{j=1}^m |w - w_j|} \leq \frac{e^{\varepsilon_0(N)N^d}}{\prod_{j=1}^m |\tau - r_j^0|} \leq e^{\varepsilon_0(N)N^d + mC_3} = e^{\mathcal{O}(\varepsilon_0(N)N^d)}. \quad (4.24)$$

Now, by the maximum principle (see Theorem A.11 in the Appendix) we know that the bound (4.24) for the function $|t|$ is true on the whole closed disc $\overline{D}(w_0, \tau)$. Thus, by (4.21), we obtain

$$\Re(g(w)) = \ln(e^{\Re(g(w))}) = \ln|t(w)| \leq \mathcal{O}(\varepsilon_0(N)N^d), \quad \text{in } \overline{D}(w_0, \tau). \quad (4.25)$$

We aim to prove the same bound for the modulus of $\Re(g(w))$. To this end, we introduce the

auxiliary function

$$l(w) := C_4(\varepsilon_0(N)N^d) - \Re(g(w)),$$

where $C_4 > 0$ is the constant which defines the bound (4.25). We know that l is harmonic and non-negative on $D(w_0, \tau)$ and that $D(w_0, \frac{4}{3}r_0) \Subset D(w_0, \tau)$. Hence, by applying Harnack's inequality (see Theorem A.12 in the Appendix), we know that there exists a constant $C_5 = C_5(D(w_0, \frac{4}{3}r_0)) > 0$ depending on the smaller disc, such that

$$\sup_{w \in D(w_0, \frac{4}{3}r_0)} l(w) \leq C_5 \inf_{w \in D(w_0, \frac{4}{3}r_0)} l(w). \quad (4.26)$$

Since

$$|w_j| \leq r_j^0 + |w_0| < \frac{3}{2}r_0 + \frac{1}{2}r_0 \leq 2, \quad j = 1, \dots, m,$$

from (4.18) and (4.19), we also know that

$$\begin{aligned} \Re(g(0)) &= \ln |t(0)| = \ln \left(\frac{|f(0)|}{\prod_{j=1}^m |w_j|} \right) > \ln \left(\frac{e^{-\varepsilon_0(N)N^d}}{2^m} \right) \\ &= -\varepsilon_0(N)N^d - m \ln(2) \geq -C_6 \varepsilon_0(N)N^d, \end{aligned}$$

for a suitable constant $C_6 > 0$. Then

$$l(0) = C_4(\varepsilon_0(N)N^d) - \Re(g(0)) \leq (C_4 + C_6)(\varepsilon_0(N)N^d). \quad (4.27)$$

Thus, by (4.26) and (4.27) the following inequalities hold

$$\sup_{w \in D(w_0, \frac{4}{3}r_0)} l(w) \leq C_5 \inf_{w \in D(w_0, \frac{4}{3}r_0)} l(w) \leq C_5 l(0) \leq C_5(C_4 + C_6)(\varepsilon_0(N)N^d).$$

Finally, if we define $\tilde{C} := C_5(C_4 + C_6) > 0$, we get

$$-\inf_{D(w_0, \frac{4}{3}r_0)} \Re(g(w)) = \sup_{D(w_0, \frac{4}{3}r_0)} -\Re(g(w)) \leq (\tilde{C} - C_4)(\varepsilon_0(N)N^d).$$

and then

$$\Re(g(w)) \geq -(\tilde{C} - C_4)(\varepsilon_0(N)N^d), \quad w \in D\left(w_0, \frac{4}{3}r_0\right).$$

This result, together with (4.25), implies

$$|\Re(g(w))| \leq \mathcal{O}(\varepsilon_0(N)N^d), \quad \text{on } D\left(w_0, \frac{4}{3}r_0\right).$$

To conclude, it remains to verify that assumption (4.23) holds. To this end, we observe that for $0 < a < b$ fixed, the function

$$G(x) := -\int_a^b \ln |t - x| dt, \quad x \in]0, b[$$

has a maximum point at $x_0 := \frac{a+b}{2}$. Indeed, using the fact that $\int \ln(u) du = u \ln(u) - u$, a direct computation gives

$$G(x) = \begin{cases} -(b-x) \ln(b-x) + (a-x) \ln(a-x) + (b-a), & x \in]0, a[\\ -(b-a) \ln(b-a) + (b-a), & x = a, \\ -(b-x) \ln(b-x) - (x-a) \ln(x-a) + (b-a), & x \in]a, b[\end{cases}$$

In particular, G is continuous on $]0, b[$ and differentiable in $]0, b[\setminus \{a\}$ with

$$G'(x) = -\ln\left(\frac{|x-a|}{b-x}\right), \quad x \in]0, b[\setminus \{a\}.$$

We observe that

$$\begin{cases} G'(x) > 0, & x \in]0, x_0[\setminus \{a\} \\ G'(x_0) = 0 \\ G'(x) < 0, & x \in]x_0, b[\end{cases}$$

Thus, from the continuity of G , x_0 is a maximum point for G in $]0, b[$. Therefore,

$$G(x) \leq G(x_0) = -\int_a^b \ln|t-x_0| dt = -2 \int_0^{\frac{b-a}{2}} \ln|t| dt.$$

This implies that for all $j = 1, \dots, m$

$$-\int_{\frac{4}{3}r_0}^{\frac{3}{2}r_0} \ln|t-r_j^0| dt \leq -2 \int_0^{\frac{1}{12}r_0} \ln|t| dt = \frac{r_0}{6} \left(1 - \ln\left(\frac{r_0}{12}\right)\right) =: C'_3 > 0,$$

Hence, if we define the function $\tilde{G}(t) := -\sum_{j=1}^m \ln|t-r_j^0| \in L^1_{\text{loc}}(\mathbb{R})$, we observe that

$$\int_{\frac{4}{3}r_0}^{\frac{3}{2}r_0} \tilde{G}(t) dt \leq C'_3 m.$$

Thus, there exists $\tau \in]\frac{4}{3}r_0, \frac{3}{2}r_0[$ such that

$$\tilde{G}(\tau) \leq \frac{6C'_3 m}{r_0},$$

which is equivalent to

$$\prod_{j=1}^m |\tau - r_j^0| \geq e^{-C_3 m}, \quad (4.28)$$

where $C_3 := \frac{6C'_3}{r_0}$. In conclusion, we have found that

$$f(w) = \prod_{j=1}^m (w - w_j) e^{g(w)}, \quad \text{in } D\left(w_0, \frac{4}{3}r_0\right),$$

with

$$m = \mathcal{O}(\varepsilon_0(N)N^d), \quad |\Re(g(w))| \leq \mathcal{O}(\varepsilon_0(N)N^d), \quad \text{in } D\left(w_0, \frac{4}{3}r_0\right). \quad (4.29)$$

Now, we fix $0 < \varepsilon \ll 1$, and we define

$$\Omega(\varepsilon) := \{r \in [0, r_0(q_1)[: \exists w \in D_{q_0, q_1} \text{ such that } |w| = r \text{ and } |f(w)| < \varepsilon\}, \quad (4.30)$$

where $r_0 = r_0(q_1)$ is defined as in (4.14). We set $r_j := |w_j|$, $j = 1, \dots, m$ and we take $r \in \Omega(\varepsilon)$ with w its corresponding point in D_{q_0, q_1} . Then

$$\begin{aligned} \prod_{j=1}^m |r - r_j| &= \prod_{j=1}^m ||w| - |w_j|| \leq \prod_{j=1}^m |w - w_j| = \left| \prod_{j=1}^m (w - w_j) \right| \\ &= |f(w)| e^{-\Re(g(w))} < \varepsilon e^{-\Re(g(w))} \leq \varepsilon \exp(\mathcal{O}(\varepsilon_0(N)N^d)). \end{aligned} \quad (4.31)$$

This implies that at least one of the factors from the left-hand side is bounded by the term $(\varepsilon \exp(\mathcal{O}(\varepsilon_0(N)N^d)))^{\frac{1}{m}}$, and consequently

$$\Omega(\varepsilon) \subset \bigcup_{j=1}^m]r_j - (\varepsilon \exp(\mathcal{O}(\varepsilon_0(N)N^d)))^{\frac{1}{m}}, r_j + (\varepsilon \exp(\mathcal{O}(\varepsilon_0(N)N^d)))^{\frac{1}{m}}[. \quad (4.32)$$

Hence, the Lebesgue measure of the set $\Omega(\varepsilon)$ is bounded in the following way

$$\lambda(\Omega(\varepsilon)) \leq \sum_{j=1}^m \lambda\left(]r_j - (\varepsilon e^{\mathcal{O}(\varepsilon_0(N)N^d)})^{\frac{1}{m}}, r_j + (\varepsilon e^{\mathcal{O}(\varepsilon_0(N)N^d)})^{\frac{1}{m}}[\right) = 2m \left(\varepsilon e^{\mathcal{O}(\varepsilon_0(N)N^d)} \right)^{\frac{1}{m}}, \quad (4.33)$$

where we used the sub-additivity of the measure.

By examining the intervals described in (4.32), we see that it is desirable for the right-hand side in (4.31) to be less than or equal to 1. Under this condition, if m is large, then each interval covering $\Omega(\varepsilon)$ in (4.32) becomes sufficiently small. As a result, their union does not cover the entire interval $[0, r_0(q_1)[$, and the estimate in (4.33) remains non-trivial and informative. Moreover, we observe that under the same assumption that the last term in (4.31) is ≤ 1 , the bound obtained in (4.33) increases with m . Indeed, for $0 < a \leq 1$ and $m > 0$,

$$\frac{d}{dm} \left(2m a^{\frac{1}{m}} \right) = 2a^{\frac{1}{m}} \left(1 - \frac{\ln(a)}{m} \right) > 0,$$

so the function $m \mapsto 2m a^{\frac{1}{m}}$ is increasing in m .

By recalling the definition of f in (4.13) and the results in (4.29), from (4.33) we obtain the following result.

Proposition 4.2. *Let $q_1 \in \mathbb{C}^{N^d}$ such that $\|q_1\|_2 = R$ and assume that $\varepsilon > 0$ is small enough so that*

the last member of (4.31) is ≤ 1 . Then the following is true:

$$\begin{aligned} & \lambda(\{r \in [0, r_0(q_1)] : \|q_0 + rq_1\|_2 < R, |F(q_0 + rq_1)| < \epsilon\}) \\ & \leq \varepsilon_0(N)N^d \exp\left(\mathcal{O}(1) + \frac{N^{-d}}{\mathcal{O}(1)\varepsilon_0(N)} \ln(\epsilon)\right), \end{aligned} \quad (4.34)$$

where the symbol $\mathcal{O}(1)$ in a denominator indicates a bounded positive quantity.

Remark 4.3. We observe that (4.34) holds because

$$\{r \in [0, r_0(q_1)] : \|q_0 + rq_1\|_2 < R, |F(q_0 + rq_1)| < \epsilon\} \subset \Omega(\epsilon),$$

as a consequence of (4.14).

An interesting choice we can make for ϵ could be $\epsilon = \exp(-\varepsilon_0(N)N^{d+\alpha})$, for a small constant $\alpha > 0$. Then the upper bound in (4.34) becomes

$$\varepsilon_0(N)N^d \exp\left(\mathcal{O}(1) - \frac{N^\alpha}{\mathcal{O}(1)}\right).$$

We now shift to a probabilistic setting by introducing a new measure on a ball in $\mathbb{C}^{N^d}$. Concretely, this means that the vector defining the potential is no longer deterministic but instead treated as a random vector. Specifically, we consider the probability space given by equipping the ball $B_{N^d}(0, R)$ with a probability measure of the form

$$\mathbb{P}(dq) = C(N)e^{\Phi(q)}\lambda(dq), \quad (4.35)$$

where $\lambda(dq)$ is the Lebesgue measure on $\mathbb{C}^{N^d}$ and Φ is a C^1 function that depends on the semi-classical parameter N that satisfies

$$\|\nabla\Phi(q)\|_2 = \mathcal{O}(N^{M_1}), \quad \forall q \in \mathbb{C}^{N^d} \quad (4.36)$$

for a fixed constant $M_1 > 0$ and

$$C(N) := \frac{1}{\int_{B_{N^d}(0, R)} e^{\Phi(q)}\lambda(dq)}$$

is the normalization constant. Now we consider the following change of coordinates

$$(w_1, r) \mapsto q = q_0 + rRw_1, \quad (4.37)$$

where $w_1 \in \mathbb{S}^{2N^d-1}$, $0 \leq r < r_0(w_1)$, and $\frac{\sqrt{3}}{2} \leq r_0(w_1) \leq 1$. We get

$$\mathbb{P}(dq) = \tilde{C}(N)e^{\phi(r)}r^{2N^d-1}dr\sigma(dw_1), \quad (4.38)$$

where $\phi(r) := \Phi(q_0 + rRw_1)$, $\sigma(dw_1)$ denotes the spherical measure on $\mathbb{S}^{2N^d-1}$ and $\tilde{C}(N) :=$

$R^{2N^d}C(N)$. We observe that for all $r \in [0, r_0(w_1)[$,

$$|\phi'(r)| = |\langle \nabla \Phi(q_0 + rRw_1), Rw_1 \rangle| \leq R \|\nabla \Phi(q_0 + rRw_1)\|_2 \|w_1\|_2 = \mathcal{O}(N^{d-1}N^{M_1}),$$

and hence

$$\phi'(r) = \mathcal{O}(N^{M_2}), \quad (4.39)$$

where $M_2 := M_1 + d - 1$.

We fix a direction $w_1 \in \mathbb{S}^{2N^d-1}$ and consider the probability measure on $[0, r_0(w_1)]$ of the form

$$\mu(dr) = \widehat{C}(N) e^{\phi(r)} r^{2N^d-1} dr, \quad (4.40)$$

where $\widehat{C}(N)$ is the normalization constant. We define the function

$$\psi(r) := \phi(r) + (2N^d - 1) \ln(r),$$

on $[0, r_0(w_1)]$ and we obtain

$$\mu(dr) = \widehat{C}(N) \exp(\psi(r)) dr.$$

Furthermore, we define $\widetilde{r}_0 := C_7^{-1} \min(1, N^{d-M_2})$, where $C_7 > 0$ is chosen large enough so that $2\widetilde{r}_0 \leq r_0(w_1)$. For any $0 \leq r \leq 2\widetilde{r}_0$, applying (4.39), yields a constant $\widetilde{C}_7 > 0$ such that

$$\psi'(r) = \phi'(r) + \frac{2N^d - 1}{r} \geq -\widetilde{C}_7 N^{M_2} + \frac{N^d}{2\widetilde{r}_0} \geq -\widetilde{C}_7 N^{M_2} + \frac{C_7}{2} N^{M_2}.$$

Choosing $C_7 > 0$ large enough, the last term is non-negative. Consequently $\psi(r)$ is an increasing function in the interval $[0, 2\widetilde{r}_0]$.

Now, we introduce a new measure on $[0, r_0(w_1)]$ given by

$$\widetilde{\mu}(dr) := \widehat{C}(N) e^{\phi(r_{\max})} r_{\max}^{2N^d-1} dr,$$

where $r_{\max}(r) := \max(r, \widetilde{r}_0)$. This new measure is not normalized and it's just a truncation of μ which will be useful for obtaining some estimates. Since $r_{\max}(r) = r$, whenever $r \geq \widetilde{r}_0$, if we restrict the two measures to the interval $[\widetilde{r}_0, r_0(w_1)]$, we have

$$\mu|_{[\widetilde{r}_0, r_0(w_1)]} = \widetilde{\mu}|_{[\widetilde{r}_0, r_0(w_1)]}.$$

Moreover, since $r_{\max}(r) \geq r$ and $\psi(r)$ is an increasing function on $[0, 2\widetilde{r}_0]$, we deduce that

$$\widetilde{\mu} \geq \mu.$$

Hence, we obtain

$$\mu([\widetilde{r}_0, 2\widetilde{r}_0]) = \widetilde{\mu}([\widetilde{r}_0, 2\widetilde{r}_0]) = \int_{\widetilde{r}_0}^{2\widetilde{r}_0} \widehat{C}(N) e^{\psi(r_{\max})} dr \geq \int_{\widetilde{r}_0}^{2\widetilde{r}_0} \widehat{C}(N) e^{\psi(\widetilde{r}_0)} dr = \widehat{C}(N) e^{\psi(\widetilde{r}_0)} \widetilde{r}_0 = \widetilde{\mu}([0, \widetilde{r}_0]).$$

This implies

$$\begin{aligned}\tilde{\mu}([0, r_0(w_1)]) &= \tilde{\mu}([0, \tilde{r}_0]) + \tilde{\mu}([\tilde{r}_0, 2\tilde{r}_0]) + \tilde{\mu}([2\tilde{r}_0, r_0(w_1)]) \\ &\leq 2\mu([\tilde{r}_0, 2\tilde{r}_0]) + \mu([2\tilde{r}_0, r_0(w_1)]),\end{aligned}\tag{4.41}$$

and then, since μ is normalized,

$$\tilde{\mu}([0, r_0(w_1)]) \leq \mathcal{O}(1).\tag{4.42}$$

Now we simplify the notation by defining $\tilde{\psi}(r) := \psi(r_{\max})$, so that

$$\tilde{\mu}(dr) = \widehat{C}(N)e^{\tilde{\psi}(r)}dr.$$

It is easy to see that, unlike ψ , the function $\tilde{\psi}$ has no singularity at 0. Hence, by continuity, it is bounded on the interval $[0, r_0(w_1)]$ by a constant depending on N . We compute

$$\tilde{\psi}'(r) = \begin{cases} \phi'(r) + \frac{2N^d-1}{r}, & \text{if } \tilde{r}_0 < r \leq r_0(w_1) \\ 0, & \text{if } 0 \leq r < \tilde{r}_0 \end{cases},$$

which is clearly not continuous in $\tilde{r}_0$. Consequently, if $\tilde{r}_0 < r \leq r_0(w_1)$, we have

$$\begin{aligned}|\tilde{\psi}'(r)| &\leq |\phi'(r)| + \left| \frac{2N^d-1}{\tilde{r}_0} \right| \leq \widetilde{C}_7 N^{M_2} + \frac{2N^d}{C_7^{-1} \min(1, N^{d-M_2})} \\ &\leq \widetilde{C}_7 N^{M_2} + C_7 \max(2N^d, 2N^{M_2}),\end{aligned}$$

and hence

$$\tilde{\psi}'(r) = \mathcal{O}(\max(N^d, N^{M_2})) = \mathcal{O}(N^{M_3}), \quad \forall r \in]\tilde{r}_0, r_0(w_1)],\tag{4.43}$$

where

$$M_3 := M_2 + d.\tag{4.44}$$

Remark 4.4. We observe that this is reason for the truncation of the measure μ : we couldn't have obtained the same result with ψ , because its derivative $\psi'(r)$ goes to $+\infty$ as $r \rightarrow 0^+$.

Since we know that $\frac{\sqrt{3}}{2} \leq r_0(w_1) \leq 1$, we can decompose the interval $[0, r_0(w_1)]$ into $\sim N^{M_3}$ intervals of Lebesgue measure $\sim N^{-M_3}$. If I is one of these intervals, we observe that there exists a constant $C_8 > 0$ such that

$$\frac{\lambda(dr)}{C_8 \lambda(I)} \leq \frac{\tilde{\mu}(dr)}{\tilde{\mu}(I)} \leq \frac{C_8 \lambda(dr)}{\lambda(I)}, \quad \text{on } I.\tag{4.45}$$

To prove this, it suffices to work with closed intervals in I , as they generate the Borel σ -algebra

on I . Let $[a, b] \subset I$; we then have

$$\frac{\tilde{\mu}([a, b])}{\tilde{\mu}(I)} = \frac{\int_{[a, b]} \widehat{C}(N) e^{\tilde{\psi}(r)} dr}{\int_I \widehat{C}(N) e^{\tilde{\psi}(r)} dr} \leq \frac{(\sup_{r \in I} e^{\tilde{\psi}(r)}) \lambda([a, b])}{(\inf_{r' \in I} e^{\tilde{\psi}(r')}) \lambda(I)}. \quad (4.46)$$

Now, if $I \subset [0, \tilde{r}_0]$, then $\tilde{\psi}$ is constant on the interval and the inequalities (4.45) are trivial. Instead, if $I \subset [\tilde{r}_0, r_0(w_1)]$, then $\tilde{\psi}$ is differentiable on I . Thus, as a consequence of (4.43) and by the mean value theorem we know that for all $r_1, r_2 \in I, r_1 < r_2$, there exists $r_3 \in]r_1, r_2[$ such that

$$|\tilde{\psi}(r_1) - \tilde{\psi}(r_2)| \leq |\tilde{\psi}'(r_3)| \lambda(I) \leq \mathcal{O}(N^{M_3}) \mathcal{O}(N^{-M_3}) = \mathcal{O}(1), \quad (4.47)$$

Lastly, if there exists $\varepsilon_1 > 0$ such that $I \cap]\tilde{r}_0 - \varepsilon_1, \tilde{r}_0 + \varepsilon_1[\neq \emptyset$, we obtain (4.47) in the same way, by adding and subtracting $\tilde{\psi}(\tilde{r}_0)$ in the left-hand side. Consequently,

$$\frac{\sup_{r \in I} e^{\tilde{\psi}(r)}}{\inf_{r' \in I} e^{\tilde{\psi}(r')}} = \sup_{(r, r') \in I^2} e^{\tilde{\psi}(r) - \tilde{\psi}(r')} \leq e^{\mathcal{O}(1)} =: C_8, \quad (4.48)$$

where $C_8 > 0$ is a constant. Hence, by putting together (4.46) and (4.48), we obtain

$$\frac{\tilde{\mu}([a, b])}{\tilde{\mu}(I)} \leq \frac{C_8 \lambda([a, b])}{\lambda(I)}.$$

The inequality on the left-hand side of (4.45) is then derived in a similar way with the same constant C_8 .

Our goal is now to estimate the measure $\tilde{\mu}$ of the set $\Omega(\epsilon)$. To this end, we first state the following remark.

Remark 4.5. Within the same framework established above, there exists a value

$$r(w_1) \in \left] \frac{r_0(w_1)}{2}, r_0(w_1) \right],$$

such that

$$r \in [0, r_0(w_1)], \|q_0 + rRw_1\|_2 < R \Leftrightarrow r \in [0, r(w_1)].$$

Hence, we let I be one of the intervals we defined above such that $I \cap [0, r(w_1)] \neq \emptyset$. By applying this remark, together with Proposition 4.2 and the inequalities (4.45), if ϵ is small enough so that the right-hand side of (4.31) is ≤ 1 , we get

$$\begin{aligned} & \tilde{\mu}(\{r \in I \cap [0, r(w_1)] : |F(q_0 + rRw_1)| < \epsilon\}) / \tilde{\mu}(I) \\ & \leq \frac{C_8}{\lambda(I)} \lambda(\{r \in [0, r(w_1)] : |F(q_0 + rRw_1)| < \epsilon\}) \\ & \leq \frac{\mathcal{O}(1)}{\lambda(I)} \varepsilon_0(N) N^d \exp\left(\frac{N^{-d}}{\mathcal{O}(1) \varepsilon_0(N)} \ln(\epsilon)\right) \\ & = \mathcal{O}(1) N^{M_3+d} \varepsilon_0(N) \exp\left(\frac{N^{-d}}{\mathcal{O}(1) \varepsilon_0(N)} \ln(\epsilon)\right). \end{aligned} \quad (4.49)$$

Multiplying both sides of (4.49) by $\tilde{\mu}(I)$ and summing over all such intervals I , we obtain

$$\begin{aligned}
& \tilde{\mu}(\{r \in [0, r(w_1)] : |F(q_0 + rRw_1)| < \epsilon\}) \\
&= \tilde{\mu}\left(\bigcup_I \{r \in I \cap [0, r(w_1)] : |F(q_0 + rRw_1)| < \epsilon\}\right) \\
&\leq \sum_I \frac{\tilde{\mu}(\{r \in I \cap [0, r(w_1)] : |F(q_0 + rRw_1)| < \epsilon\})}{\tilde{\mu}(I)} \tilde{\mu}(I) \\
&\leq \mathcal{O}(1) N^{M_3+d} \varepsilon_0(N) \exp\left(\frac{N^{-d}}{\mathcal{O}(1)\varepsilon_0(N)} \ln(\epsilon)\right) \left(\sum_I \tilde{\mu}(I)\right). \tag{4.50}
\end{aligned}$$

Finally, we aim to eliminate the dependence on $\tilde{\mu}$ in the last term. Using the same approach as in the proof of (4.45), we obtain a similar result for the ratio $\frac{\tilde{\mu}(I)}{\tilde{\mu}([0, r_0(w_1)])}$. Thus, by recalling (4.42) we obtain

$$\sum_I \tilde{\mu}(I) \leq \mathcal{O}(1) \sum_I \frac{\lambda(I)}{\lambda([0, r_0(w_1)])} = \mathcal{O}(1). \tag{4.51}$$

We recall that $\mu \leq \tilde{\mu}$, and hence we have the same estimates by replacing $\tilde{\mu}$ with μ . Then from the definitions of $\mathbb{P}(dq)$ in (4.38) and $\mu(dr)$ in (4.40), we obtain the following.

Proposition 4.6. *Let $\epsilon > 0$ be small enough so that the right-hand side of (4.31) is ≤ 1 . Then*

$$\mathbb{P}(|F(q)| < \epsilon) \leq \mathcal{O}(1) N^{M_3+d} \varepsilon_0(N) \exp\left(\frac{N^{-d}}{\mathcal{O}(1)\varepsilon_0(N)} \ln(\epsilon)\right). \tag{4.52}$$

Proof. We observe that this result is just a consequence of (4.50) and (4.51) as it follows

$$\begin{aligned}
\mathbb{P}(|F(q)| < \epsilon) &= \iint_{B_{N^d}(0, R)} \mathbb{1}_{(|F(q)| < \epsilon)} \mathbb{P}(dq) \\
&= \int_{\mathbb{S}^{2N^d-1}} \int_{[0, r(w_1)]} \mathbb{1}_{(|F(q_0 + rRw_1)| < \epsilon)} \mu(dr) \sigma(dw_1) \\
&\leq \int_{\mathbb{S}^{2N^d-1}} \tilde{\mu}(\{r \in [0, r(w_1)] : |F(q_0 + rRw_1)| < \epsilon\}) \sigma(dw_1) \\
&\leq \mathcal{O}(1) N^{M_3+d} \varepsilon_0(N) \exp\left(\frac{N^{-d}}{\mathcal{O}(1)\varepsilon_0(N)} \ln(\epsilon)\right).
\end{aligned}$$

□

Remark 4.7. From definition (4.9), we have

$$\ln(|F(q)|) = \ln|\det(P^\delta(q) - z)| - N^d \int_{\mathbb{T}^{2d}} \ln|p_0(\rho) - z| d\rho.$$

Thus, taking $\epsilon = \exp(-\varepsilon_0(N)N^{d+\alpha})$ for a small constant $\alpha > 0$ as before, result (4.52) implies that the inequality

$$\ln|\det(P^\delta(q) - z)| - N^d \int_{\mathbb{T}^{2d}} \ln|p_0(\rho) - z| d\rho < -\varepsilon_0(N)N^{d+\alpha}$$

is true with probability $\leq \mathcal{O}(1)\varepsilon_0(N)N^{M_3+d} \exp\left(\frac{-N^\alpha}{\mathcal{O}(1)}\right)$.

Chapter 5

Eigenvalue Counting

In this final chapter, we take the last steps toward establishing a Weyl law for randomly perturbed operators. We apply two different techniques: first, we obtain probabilistic estimates for the number of eigenvalues of the perturbed operator lying in a given region of the complex plane, under suitable assumptions on its boundary; then, we show that the corresponding eigenvalue counting measures converge weakly as $N \rightarrow +\infty$.

5.1 Zero Counting Techniques for Holomorphic Functions and Applications to Spectral Estimates

We begin by deriving a quantitative result that controls the spectrum of the perturbed operator in its intersection with a prescribed domain in $\mathbb{C}$. The main tool is a complex-analytic argument: we reduce the problem to counting the zeros of holomorphic functions subject to explicit exponential growth bounds. Afterwards, we will apply this result to the determinant of randomly perturbed operators.

The key result we rely on is Theorem 1.2 from [Sj10], which we will apply without going through its proof. Before stating it, we introduce a specific notion of Lipschitz boundary, which is required in its formulation.

Let $\Gamma \Subset \mathbb{C}$ be an open set and let $\gamma := \partial\Gamma$ be its boundary. Consider $r : \gamma \rightarrow]0, \infty[$ a Lipschitz function with a Lipschitz modulus $\leq \frac{1}{2}$:

$$|r(x) - r(y)| \leq \frac{1}{2}|x - y|, \quad \forall x, y \in \gamma. \quad (5.1)$$

We assume that the boundary γ is Lipschitz with respect to the weight r in the following precise sense: we suppose that there exists a constant $C_0 > 0$ such that for every $x \in \gamma$, there exist new affine coordinates $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$ of the form $\tilde{y} = U(y - x)$, for $y \in \mathbb{C} \cong \mathbb{R}^2$, being the old coordinates, where $U = U_x$ is an orthogonal matrix, such that the intersection of Γ and the rectangle

$$R_x := \{y \in \mathbb{C} : |\tilde{y}_1| < r(x), |\tilde{y}_2| < C_0 r(x)\},$$

corresponds to

$$\{y \in R_x : |\tilde{y}_1| < r(x), \tilde{y}_2 > f_x(\tilde{y}_1)\}. \quad (5.2)$$

Here $f_x(\tilde{y}_1)$ is a Lipschitz function defined on $[-r(x), r(x)]$ with Lipschitz modulus $\leq C_0$.

We notice that the assumption (5.2) remains valid if we decrease the weight r . Moreover, it will be convenient to extend the function r to all the complex plane $\mathbb{C}$, by defining

$$r(x) = \inf_{y \in \gamma} \left(r(y) + \frac{1}{2}|x - y| \right). \quad (5.3)$$

The extended function is still Lipschitz with modulus $\leq \frac{1}{2}$.

Theorem 5.1 (Theorem 1.2 of [Sj10]). *Let $\Gamma \Subset \mathbb{C}$ be a simply connected set with a Lipschitz boundary γ with an associated Lipschitz weight r as in (5.1), (5.2) and (5.3). We put $\tilde{\gamma}_r := \bigcup_{x \in \gamma} D(x, r(x))$. Consider M points $z_j^0 \in \gamma$, with $j \in \mathbb{Z}/M\mathbb{Z}$, distributed along the boundary in the positively oriented sense so that*

$$\frac{r(z_j^0)}{4} \leq |z_{j+1}^0 - z_j^0| \leq \frac{r(z_j^0)}{2}. \quad (5.4)$$

Let $0 < h \leq 1$ and let ϕ be a continuous subharmonic function on $\tilde{\gamma}_r$ with a distribution extension to $\Gamma \cup \tilde{\gamma}_r$, denoted by the same symbol. Then there exists a constant $C_1 > 0$ depending only on the constant C_0 and there exist M points $\tilde{z}_j \in D\left(z_j^0, \frac{r(z_j^0)}{2C_1}\right)$, such that if u is a holomorphic function on $\Gamma \cup \tilde{\gamma}_r$, which satisfies

$$h \ln |u(z)| \leq \phi(z), \quad \text{on } \tilde{\gamma}_r, \quad (5.5)$$

$$h \ln |u(\tilde{z}_j)| \geq \phi(\tilde{z}_j) - \epsilon_j, \quad \text{for } j = 1, \dots, M, \quad (5.6)$$

where $\epsilon_j \geq 0$, then the number of zeros of u in Γ satisfies the following inequality

$$\left| \#(u^{-1}(0) \cap \Gamma) - \frac{1}{2\pi h} \mu(\Gamma) \right| \leq \frac{C_2}{h} \left(\mu(\tilde{\gamma}_r) + \sum_{j=1}^M \epsilon_j \right) \quad (5.7)$$

for a constant $C_2 > 0$ which only depends on C_0 and on C_1 . In particular, we have taken $\mu := \Delta\phi \in \mathcal{D}'(\Gamma \cup \tilde{\gamma}_r)$, which is a positive measure on $\tilde{\gamma}_r$ so that $\mu(\Gamma)$ and $\mu(\tilde{\gamma}_r)$ are well-defined.

Remark 5.2. This theorem is a direct consequence of Theorem 1.1 in [Sj10], which states that result (5.7) is valid for every $z_j \in D(z_j^0, \frac{r(z_j^0)}{2C_1})$, with the addition of the term

$$\sum_{j=1}^M \int_{D(z_j, \frac{r(z_j)}{4C_1})} \left| \ln \left(\frac{|w - z_j|}{r(z_j)} \right) \right| \mu(dw), \quad (5.8)$$

to the right-hand side. Indeed, we observe that the average of $\left| \ln \left(\frac{|w - z_j|}{r(z_j)} \right) \right|$ with respect to the Lebesgue measure $\lambda(dz_j)$ over the disc $D(z_j^0, \frac{r(z_j^0)}{2C_1})$ is $\mathcal{O}(1)$. Thus, integrating each term of the

sum (5.8) on the disc $D(z_j^0, \frac{r(z_j^0)}{2C_1})$, we obtain

$$\begin{aligned} \int_{D(z_j^0, \frac{r(z_j^0)}{2C_1})} \int_{D(z_j, \frac{r(z_j)}{4C_1})} \left| \ln \frac{|w - z_j|}{r(z_j)} \right| \mu(dw) \lambda(dz_j) &\leq \int_{D(z_j^0, \frac{r(z_j^0)}{2C_1})} \int_{D(z_j^0, \frac{r(z_j^0)}{C_1})} \left| \ln \frac{|w - z_j|}{r(z_j)} \right| \mu(dw) \lambda(dz_j) \\ &= \int_{D(z_j^0, \frac{r(z_j^0)}{C_1})} \int_{D(z_j^0, \frac{r(z_j^0)}{2C_1})} \left| \ln \frac{|w - z_j|}{r(z_j)} \right| \lambda(dz_j) \mu(dw) \\ &= \int_{D(z_j^0, \frac{r(z_j^0)}{C_1})} \mathcal{O}(1) \mu(dw) = \mathcal{O}(1) \end{aligned}$$

and then by the mean value theorem there exist M points $\tilde{z}_j$ as in Theorem 5.1.

Now we apply Theorem 5.1 to our specific setting. Using the results of the previous chapters, we verify that all the required hypotheses are satisfied. From this point on, we work under the following standing assumptions.

- Let $p \in S(1, 1)$ satisfy the asymptotic expansion (3.25), and set

$$h = \frac{1}{2\pi N}, \quad 1 \ll N \in \mathbb{N}, \quad P := p_N,$$

with P verifying the symmetry assumption (3.27).

- Let $\Gamma \Subset \mathbb{C}$ be a relatively compact, simply connected open set, independent of N , with uniformly Lipschitz boundary $\gamma = \partial\Gamma$ as defined above. We denote by $r_0 > 0$ its associated weight. Furthermore, assume that there exists $\kappa \in]0, 1]$ such that the volume condition (3.28) holds uniformly for all

$$z \in \tilde{\gamma}_r := \gamma + D(0, r), \quad 0 \leq r \leq r_0.$$

- Fix $\tau_0 \in]0, (C_0 N^{-1})^{1/2}]$, with $C_0 > 0$. Define the perturbed operator

$$P^\delta(q) := P + \delta V_q, \quad \delta = \frac{\tau_0}{C} N^{-d}, \quad C > 0 \quad (5.9)$$

where $V_q = \text{diag}(q(n); n = 1, \dots, N^d)$, $q \in \mathbb{C}^{N^d}$.

- Let $\mathbb{P}$ be a probability measure on the ball $B_{N^d}(0, R)$, defined as in (4.35) and (4.36), with radius $R > 0$ which satisfies (3.78).
- Finally, we recall the definition of $\varepsilon_0(N)$ given in (4.4) and we take $\alpha = C_0 N^{-1}$.

We consider the definition of logarithmic potential of a measure, and some of its main properties.

Definition 5.3 (Logarithmic potential). Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. We define the logarithmic potential of μ as

$$U_\mu(z) := - \int_{\mathbb{C}} \ln |z - x| \mu(dx), \quad \text{for } z \in \mathbb{C}. \quad (5.10)$$

We observe that for every finite Borel measure μ on $\mathbb{C}$ with compact support, we have $U_\mu \in L^1_{\text{loc}}(\mathbb{C}, \lambda(dz))$, and hence

$$U_\mu(z) < +\infty, \text{ for a.e. } z \in \mathbb{C}. \quad (5.11)$$

Indeed, let $K \subset \mathbb{C}$ be a compact set and μ a measure as above. Then, there exists $R_1 > 0$ such that $\text{supp } \mu \subset D(0, R_1)$, and for every $x \in \text{supp } \mu$, by performing a change of variable, we obtain

$$\int_K |\ln |z - x|| dz = \int_{K-x} |\ln |w|| dw \leq \int_{K+D(0, R_1)} |\ln |w|| dw \leq R_2, \quad (5.12)$$

with $R_2 > 0$ a uniform constant with respect to x . Therefore, by Fubini-Tonelli theorem, we have:

$$\begin{aligned} \int_K |U_\mu(z)| dz &= \int_K \left| \int_{\mathbb{C}} \ln |x - z| \mu(dx) \right| dz \leq \int_K \int_{\mathbb{C}} |\ln |x - z|| \mu(dx) dz \\ &= \int_{\mathbb{C}} \int_K |\ln |x - z|| dz \mu(dx) \leq \int_{\mathbb{C}} R_2 \mu(dx) < +\infty. \end{aligned} \quad (5.13)$$

Hence $U_\mu \in L^1_{\text{loc}}(\mathbb{C}, \lambda(dz)) \subset \mathcal{D}'(\mathbb{C})$.

Another important property of the logarithmic potential is obtained by applying the Laplacian. In particular, fixing a measure μ as before, we have

$$\Delta U_\mu = -2\pi\mu, \quad \text{in } \mathcal{D}'(\mathbb{C}). \quad (5.14)$$

Indeed, since $\frac{\ln |z|}{2\pi}$ is a fundamental solution of Δ_z , for every $\varphi \in C_c^\infty(\mathbb{C})$,

$$\begin{aligned} \langle \Delta U_\mu, \varphi \rangle_{\mathcal{D}', \mathcal{D}} &= \int_{\mathbb{C}} \left(- \int_{\mathbb{C}} \ln |x - z| \mu(dx) \right) \Delta_z \varphi(z) dz \\ &= \int_{\mathbb{C}} \left(- \int_{\mathbb{C}} \ln |x - z| \Delta_z \varphi(z) dz \right) \mu(dx) = \int_{\mathbb{C}} -\langle 2\pi\delta_x, \varphi \rangle_{\mathcal{D}', \mathcal{D}} \mu(dx) \\ &= \int_{\mathbb{C}} -2\pi\varphi(x) \mu(dx) = \langle -2\pi\mu, \varphi \rangle_{\mathcal{D}', \mathcal{D}}, \end{aligned}$$

which proves the statement (5.14).

We now consider $(p_0)_*(\lambda)$, the push-forward of the Lebesgue measure $\lambda(d\rho)$ on $\mathbb{T}^{2d}$ under the principal symbol p_0 of p . Since p_0 is continuous and $\mathbb{T}^{2d}$ is compact, the image $\text{Im}(p_0)$ is compact, and consequently $(p_0)_*(\lambda)$ is a finite Borel measure supported on $\text{Im}(p_0)$. Accordingly, we set

$$\phi(z) := -U_{(p_0)_*(\lambda)}(z) = \int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho, \quad z \in \mathbb{C}.$$

By (5.13) and (5.14), we know that $\phi \in L^1_{\text{loc}}(\mathbb{C}) \subset \mathcal{D}'(\mathbb{C})$ and that its Laplacian

$$\Delta\phi = 2\pi(p_0)_*(\lambda), \quad \text{in } \mathcal{D}'(\mathbb{C}). \quad (5.15)$$

Hence, the positivity of λ implies that ϕ is subharmonic.

We now aim to prove that ϕ is continuous in a neighborhood of γ , specifically in $\tilde{\gamma}_r$. To do so, we invoke Theorem 1 from [Ar60], without reproducing its proof.

Theorem 5.4 (Theorem 1 of [Ar60]). *Let m be a finite, positive Borel measure with compact support and let U_m be its logarithmic potential*

$$U_m(z) = \int \ln \frac{1}{|z - \zeta|} m(d\zeta), \quad z \in \mathbb{C}.$$

Then U_m is continuous at a point $z_0 \in \mathbb{C}$ if and only if

$$\lim_{r \rightarrow 0} \left\{ \limsup_{z \rightarrow z_0} \int_0^r \frac{m_t(z)}{t} dt \right\} = 0, \quad (5.16)$$

where $m_t(z) := m(D(z, t))$. Moreover, when U_m is continuous at z_0 , the limit actually exists in the bracketed expression, and in fact

$$\lim_{z \rightarrow z_0} \int_0^r \frac{m_t(z)}{t} dt = \int_0^r \frac{m_t(z_0)}{t} dt.$$

Now, let $z_0 \in \tilde{\gamma}_r$ and consider $z \in \text{neigh}(z_0) \subset \tilde{\gamma}_r$. Since ϕ is the negative logarithmic potential of $(p_0)_*(\lambda)$, we consider, for $r \geq 0$

$$\int_0^r \frac{(p_0)_*(\lambda(d\rho))(D(z, t))}{t} dt = \int_0^r \frac{\lambda(\{\rho \in \mathbb{T}^{2d} : |p_0(\rho) - z|^2 < t^2\})}{t} dt$$

Hence, if we take $0 \leq r \ll 1$, by applying the bound (3.28) uniform on $\tilde{\gamma}_r$, we obtain

$$\int_0^r \frac{(p_0)_*(\lambda(d\rho))(D(z, t))}{t} dt = \int_0^r \frac{\mathcal{O}(t^{2\kappa})}{t} dt \leq \frac{C}{2\kappa} r^{2\kappa},$$

for a constant $C > 0$. Since this result is true uniformly on a neighborhood of z_0 , it follows that

$$\lim_{r \rightarrow 0} \left\{ \limsup_{z \rightarrow z_0} \int_0^r \frac{(p_0)_*(\lambda(d\rho))(D(z, t))}{t} dt \right\} \leq \lim_{r \rightarrow 0} \frac{C}{2\kappa} r^{2\kappa} = 0.$$

Therefore, by Theorem 5.4, ϕ is continuous in $\tilde{\gamma}_r$.

Now we define, for $q \in B_{N^d}(0, R)$,

$$u(z) := \det(P^\delta(q) - z), \quad z \in \mathbb{C},$$

and we observe that u is a holomorphic function on $\mathbb{C}$.

Since Γ is relatively compact and its boundary γ is locally Lipschitz with respect to the weight r_0 , there exist l points $x_1, \dots, x_l \in \gamma$ such that $\gamma \subset \bigcup_{j=1}^l A_{x_j}$, where

$$A_{x_j} := \{y \in R_{x_j} : \tilde{y} = U_{x_j}(y - x_j), |\tilde{y}_1| < r_0, \tilde{y}_2 = f_{x_j}(\tilde{y}_1)\}, \quad j = 1, \dots, l.$$

Here the rectangles R_{x_j} , the Lipschitz functions f_{x_j} and the orthogonal matrices U_{x_j} are defined as in the start of this section. We parametrize each patch with the function

$$\Phi_j(t) := x_j - U_{x_j}^{-1}((t, f_{x_j}(t))), \quad t \in]-r_0, r_0[.$$

By Rademacher's theorem $f'_{x_j}(t)$ exists for a.e. $t \in]-r_0, r_0[$ and its modulus is bounded by the corresponding Lipschitz constant L_j of f_{x_j} . Thus, since U_{x_j} is orthogonal,

$$\|\Phi'_j(t)\|_2 = \|(1, f'_{x_j}(t))\|_2 = \sqrt{1 + |f'_{x_j}(t)|^2} \leq \sqrt{1 + L_j^2}, \quad \text{for a.e. } t \in]-r_0, r_0[.$$

Let $\mathcal{H}^1_{|\gamma}$ denote the arc length on γ . By subadditivity,

$$\mathcal{H}^1(\gamma) \leq \sum_{j=1}^l \int_{A_{x_j}} d\mathcal{H}^1(y) = \sum_{j=1}^l \int_{-r_0}^{r_0} \|\Phi'_j(t)\|_2 dt \leq \sum_{j=1}^l 2r_0 \sqrt{1 + L_j^2} < +\infty,$$

and consequently, by the hypotheses of N -independence, $\mathcal{H}^1(\gamma) = \mathcal{O}(1)$.

Now we take a smaller, possibly N -dependent weight $0 < r \ll 1$ in the definition of Lipschitz boundary and we choose N_0 points

$$z_j^0 \in \gamma, \quad j = 1, 2, \dots, N_0,$$

satisfying the conditions (5.4). By definition of arc length, we have

$$N_0 \frac{r}{4} \leq \sum_{j=1}^{N_0} |z_{j+1}^0 - z_j^0| \leq \mathcal{H}^1(\gamma) = \mathcal{O}(1),$$

and thus $N_0 = \mathcal{O}(r^{-1})$. Applying Proposition 4.6, we know that for every fixed $z \in \tilde{\gamma}_r$, the inequality

$$|\det(P^\delta(q) - z))| \geq \epsilon \exp \left(N^d \int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho \right), \quad (5.17)$$

is true with a probability $\geq 1 - \mathcal{O}(1)N^{M_3+d}\epsilon_0(N) \exp \left(-\frac{N^{-d}}{\mathcal{O}(1)\epsilon_0(N)} \ln \left(\frac{1}{\epsilon} \right) \right)$, provided that $\epsilon > 0$ is small enough so that the right-hand side of (4.31) is ≤ 1 .

We define $\tilde{\epsilon} := N^{-d} \ln \left(\frac{1}{\epsilon} \right)$, so that $\epsilon = \exp(-\tilde{\epsilon}N^d)$. We observe that the condition on ϵ that the right-hand side of (4.31) is ≤ 1 holds if

$$\tilde{\epsilon} \geq C\epsilon_0(N),$$

for some large constant $C > 0$, i.e. $\tilde{\epsilon} \gg \epsilon_0(N)$. Thus, we can rephrase (5.17) as it follows.

For every $z \in \tilde{\gamma}_r$, with probability

$$\geq 1 - \mathcal{O}(1)N^{M_3+d}\epsilon_0(N) \exp \left(-\frac{\tilde{\epsilon}}{C\epsilon_0(N)} \right),$$

we have

$$|\det(P^\delta(q) - z)| \geq \exp \left(N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho - \tilde{\epsilon} \right) \right),$$

which is equivalent to write

$$N^{-d} \ln |u(z)| \geq \phi(z) - \tilde{\epsilon}.$$

Now, we apply Theorem 5.1 and we take $\tilde{z}_1, \dots, \tilde{z}_{N_0} \in \tilde{\gamma}_r$ as in its statement. We apply what we have just found to each of these points and we obtain

$$\begin{aligned} \mathbb{P} \left(\bigcap_{j=1}^{N_0} \left(N^{-d} \ln |u(\tilde{z}_j)| \geq \phi(\tilde{z}_j) - \tilde{\epsilon} \right) \right) &= 1 - \mathbb{P} \left(\bigcup_{j=1}^{N_0} \left(N^{-d} \ln |u(\tilde{z}_j)| \geq \phi(\tilde{z}_j) - \tilde{\epsilon} \right)^c \right) \\ &\geq 1 - \sum_{j=1}^{N_0} \mathbb{P} \left(N^{-d} \ln |u(\tilde{z}_j)| < \phi(\tilde{z}_j) - \tilde{\epsilon} \right) \\ &\geq 1 - N_0 \mathcal{O}(1) \varepsilon_0(N) N^{M_3+d} e^{\left(-\frac{\tilde{\epsilon}}{C\varepsilon_0(N)} \right)}. \end{aligned}$$

Furthermore, we recall that inequality (4.5) obtained in Section 4.1

$$\ln |\det(P^\delta(q) - z)| - N^d \left(\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho \right) \leq N^d \varepsilon_0(N),$$

is true for all $q \in B_{N^d}(0, 3R)$ and for every $z \in \tilde{\gamma}_r$.

Hence, in the probability space $(B_{N^d}(0, R), \mathbb{P}(dq))$ that we are considering, the following inequality

$$N^{-d} \ln |u(z)| \leq \phi(z) + \varepsilon_0(N) \leq \phi(z) + \frac{\tilde{\epsilon}}{C},$$

is true with probability equal to 1 and for all $z \in \tilde{\gamma}_r$.

Finally, we define the function $\tilde{\phi}(z) := \phi(z) + \frac{\tilde{\epsilon}}{C}$, which still satisfies the same hypotheses of ϕ . Then with probability

$$\geq 1 - \frac{\mathcal{O}(1)}{r} \varepsilon_0(N) N^{M_3+d} e^{\left(-\frac{\tilde{\epsilon}}{C\varepsilon_0(N)} \right)},$$

the following two are true

$$\begin{aligned} N^{-d} \ln |u(z)| &\leq \tilde{\phi}(z), \quad \text{for all } z \in \tilde{\gamma}_r \\ N^{-d} \ln |u(\tilde{z}_j)| &\geq \tilde{\phi}(\tilde{z}_j) - \tilde{\epsilon} \left(1 + \frac{1}{C} \right), \quad \text{for } j = 1, \dots, N_0. \end{aligned}$$

Thus we can apply Theorem 5.1 and obtain the following result.

Theorem 5.5. *Under all the previous hypotheses and notations, there exists a constant $C_2 > 0$ such that*

$$\left| \#(\sigma(P^\delta) \cap \Gamma) - N^d \int_{p_0^{-1}(\Gamma)} d\rho \right| \leq C_2 N^d \left(2\pi \int_{p_0^{-1}(\gamma + D(0, r))} d\rho + \mathcal{O} \left(\frac{1}{r} \right) \tilde{\epsilon} \right), \quad (5.18)$$

with probability $\geq 1 - \frac{\mathcal{O}(1)}{r} \varepsilon_0(N) N^{M_3+d} e^{\left(-\frac{\tilde{\varepsilon}}{C\varepsilon_0(N)}\right)}$.

Proof. We recall (5.15) and we observe that the left-hand side in the inequality (5.7) of Theorem 5.1, in our context, becomes

$$\begin{aligned} \left| \#(u^{-1}(0) \cap \Gamma) - \frac{N^d}{2\pi} \mu(\Gamma) \right| &= \left| \#(\sigma(P^\delta) \cap \Gamma) - \frac{2\pi N^d}{2\pi} (p_0)_*(\lambda)(\Gamma) \right| \\ &= \left| \#(\sigma(P^\delta) \cap \Gamma) - N^d \int_{p_0^{-1}(\Gamma)} d\rho \right|. \end{aligned}$$

On the other hand, the right-hand side is given by

$$C_2 N^d \left(\mu(\tilde{\gamma}_r) + \sum_{j=1}^{N_0} \tilde{\varepsilon} \left(1 + \frac{1}{C} \right) \right) = C_2 N^d \left(2\pi \int_{p_0^{-1}(\gamma + D(0,r))} d\rho + \mathcal{O}\left(\frac{1}{r}\right) \tilde{\varepsilon} \left(1 + \frac{1}{C} \right) \right).$$

Then, by putting together these two equalities, (5.18) is proved. $\square$

5.2 Eigenvalue Distribution via Empirical Measure Convergence

In this section, we work within the same framework and use the same definitions as in the previous one. Our goal remains to understand the distribution of eigenvalues of the randomly perturbed operator $P^\delta(q)$, considered on the probability space $(B_{N^d}(0, R), \mathbb{P}(dq))$.

We now adopt a qualitative perspective based on weak convergence of empirical measures, rather than providing explicit error estimates. We begin by defining what we mean by empirical measure.

Definition 5.6 (Empirical measure). Let $P \in \mathbb{C}^{M \times M}$ be a linear operator, with spectrum $\sigma(P)$. We define the empirical measure of its eigenvalues as the probability measure

$$\mu := \frac{1}{M} \sum_{\zeta \in \sigma(P)} \delta_\zeta.$$

Hence, the empirical measure for the perturbed operator $P_N^\delta = P^\delta(q)$ is given by

$$\zeta_N = \frac{1}{N^d} \sum_{\zeta \in \sigma(P_N^\delta)} \delta_\zeta.$$

As in Section 5.1 we consider the following measure

$$\tilde{\zeta} := (p_0)_*(\lambda(d\rho)), \tag{5.19}$$

which is the push-forward of the Lebesgue measure $\lambda(d\rho)$ on $\mathbb{T}^{2d}$ through the principal symbol p_0 of p .

By Definition 5.3, the logarithmic potentials of the measures ζ_N, ζ are given respectively by

$$\begin{aligned} U_{\zeta_N}(z) &= -\frac{1}{N^d} \sum_{\zeta \in \sigma(P_N^\delta)} \int_{\mathbb{C}} \ln |z - x| \delta_\zeta(dx) \\ &= -\frac{1}{N^d} \ln \left(\prod_{\zeta \in \sigma(P_N^\delta)} |z - \zeta| \right) = -\frac{1}{N^d} \ln \left| \det(P_N^\delta - z) \right|. \end{aligned} \quad (5.20)$$

$$U_\zeta(z) = -\int_{\mathbb{C}} \ln |z - x| ((p_0)_*(\lambda))(dx) = -\int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho. \quad (5.21)$$

An additional property of the logarithmic potentials that we will need, is the following: let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of finite Borel measures with compact support, and suppose that the supports of all μ_n are contained in a fixed compact set. Then, the almost sure convergence of the associated logarithmic potentials $U_{\mu_n}(z) \rightarrow U_\mu(z)$, where μ is a finite Borel measure with compact support, implies the weak convergence of the measures $\mu_n \rightharpoonup \mu$.

We have considered this result in the setting of random measures, as stated in [Ta12, Theorem 2.8.3], and adapted it to our framework. Before presenting the result, we introduce the definition of random measure.

Definition 5.7 (Random measure). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let (X, Σ) be a measurable space. A random measure on (X, Σ) is a map

$$\omega \mapsto \mu_\omega$$

from Ω to the set of all possible measures on (X, Σ) , such that for every $E \in \Sigma$, the function

$$\omega \mapsto \mu_\omega(E)$$

is measurable with respect to $\mathcal{F}$. Equivalently, a random measure is a function

$$\mu : \Omega \times \Sigma \rightarrow [0, +\infty]$$

such that for each fixed $\omega \in \Omega$, the map $E \mapsto \mu(\omega, E)$ defines a measure on (X, Σ) , and for each fixed $E \in \Sigma$, the map $\omega \mapsto \mu(\omega, E)$ is measurable.

We denote $\mathcal{M}(\mathbb{C})$ the space of all finite Borel measures on $\mathbb{C}$ with compact support.

Here we present the main result we are interested in.

Theorem 5.8. *Let $K, K' \Subset \mathbb{C}$ be open, relatively compact sets with $\overline{K} \subset K'$, and let $\{\mu_n\}_{n \in \mathbb{N}}, \mu$ be random measures in $\mathcal{M}(\mathbb{C})$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that all the measures μ_n are almost surely uniformly bounded in total mass by the same constant and that almost surely*

$$\text{supp } \mu, \text{supp } \mu_n \subset K, \quad \text{for } n \text{ sufficiently large.} \quad (5.22)$$

Suppose that for almost every $z \in K'$, the limit

$$U_{\mu_n}(z) \rightarrow U_\mu(z), \quad n \rightarrow +\infty, \quad (5.23)$$

is true almost surely (respectively in probability). Then,

$$\mu_n \rightharpoonup \mu, \quad n \rightarrow +\infty, \quad \text{weakly} \quad (5.24)$$

almost surely (respectively in probability).

Proof. We begin with the proof in the case of almost sure convergence. The case of convergence in probability will follow similarly.

First of all, we notice that the assumption that for a.e. $z \in K'$ the limit (5.23) holds almost surely is equivalent to assume that almost surely (5.23) holds for a.e. $z \in K'$. Indeed, we define the set

$$E := \{(z, \omega) \in K' \times \Omega : U_{\mu_n(\omega)}(z) \rightarrow U_\mu(\omega)(z)\} \subset K' \times \Omega,$$

and we observe that the first of the two assumptions is equivalent to say that for a.e. $z \in K'$

$$\int_{\Omega} \mathbb{1}_E(z, \omega) \mathbb{P}(d\omega) = 1,$$

and hence

$$\int_{K'} \left(\int_{\Omega} \mathbb{1}_E(z, \omega) \mathbb{P}(d\omega) \right) \lambda(dz) = \lambda(K').$$

Furthermore, the second assumption is equivalent to say that almost surely (for a.e. $\omega \in \Omega$)

$$\int_{K'} \mathbb{1}_E(z, \omega) \lambda(dz) = \lambda(K'),$$

and hence

$$\int_{\Omega} \left(\int_{K'} \mathbb{1}_E(z, \omega) \lambda(dz) \right) \mathbb{P}(d\omega) = \lambda(K').$$

Applying Tonelli theorem confirms the equivalence of the two formulations.

Now, we observe that $\ln |\cdot - x| \in L^2(K')$ uniformly in $x \in K'$. Indeed, using the same argument we used for showing (5.12), we have, for all $x \in K'$

$$\int_{K'} |\ln |z - x||^2 \lambda(dz) \leq \int_{K' - x} |\ln |w||^2 \lambda(dw) \leq \int_{K' + K'} |\ln |w||^2 \lambda(dw) \leq C,$$

with a constant $C > 0$ that does not depend on x . Then, by Minkowski's inequality, for almost

every $\omega \in \Omega$ and for all $n \geq 1$, we obtain

$$\begin{aligned} \left(\int_{K'} |U_{\mu_n(\omega)}(z)|^2 \lambda(dz) \right)^{1/2} &= \left(\int_{K'} \left| \int_{\mathbb{C}} -\ln |z-x| \mu_n(\omega)(dx) \right|^2 \lambda(dz) \right)^{1/2} \\ &\leq \int_{\mathbb{C}} \left(\int_{K'} |\ln |z-x||^2 \lambda(dz) \right)^{1/2} \mu_n(\omega)(dx) \\ &\leq \int_{\mathbb{C}} C^{\frac{1}{2}} \mu_n(\omega)(dx) \leq \tilde{C}, \end{aligned}$$

where $\tilde{C} > 0$ is a constant that does not depend on n , as guaranteed by the hypotheses of the theorem. By applying a similar argument also for U_{μ} , we conclude that for all $n \geq 1$ and for almost every $\omega \in \Omega$, $U_{\mu(\omega)}, U_{\mu_n(\omega)} \in L^2(K')$ uniformly.

Now, combining this result with (5.22) and with the equivalence between the statements proved above, we obtain that there exists a measurable set $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$, such that for each $\omega \in \Omega'$, the following are true

1. $U_{\mu_n(\omega)}(z) \rightarrow U_{\mu(\omega)}(z)$ as $n \rightarrow +\infty$, for a.e. $z \in K'$,
2. there exists $n_0 \geq 1$ such that $\text{supp } \mu_n(\omega), \text{supp } \mu(\omega) \subset K$, for all $n \geq n_0$,
3. there exists a constant $C_{K', \Omega'} > 0$, depending only on K' and Ω' , such that

$$\|U_{\mu_n(\omega)}\|_{L^2(K')}, \|U_{\mu(\omega)}\|_{L^2(K')} \leq C_{K', \Omega'}, \quad (5.25)$$

for all $n \geq 1$.

In order to prove the statement (5.24), it is sufficient to show that for all $\omega \in \Omega'$ and for any $\varphi \in C_c^\infty(K', \mathbb{R})$ with support in K' ,

$$\langle \mu_n(\omega), \varphi \rangle_{\mathcal{D}', \mathcal{D}} \rightarrow \langle \mu(\omega), \varphi \rangle_{\mathcal{D}', \mathcal{D}}, \quad \text{for } n \rightarrow +\infty. \quad (5.26)$$

Fix $\omega \in \Omega'$, let $M > 0$ and define

$$g_n^M(z) := \min(|U_{\mu_n(\omega)}(z) - U_{\mu(\omega)}(z)|, M), \quad (5.27)$$

for $z \in K'$ and $n \geq n_0$. We know that $g_n^M(z) \xrightarrow{n \rightarrow +\infty} 0$ for a.e. $z \in K'$ and that $|g_n^M| \leq M \in L^1(K')$. Hence, by the dominated convergence theorem, we have

$$g_n^M \xrightarrow{n \rightarrow \infty} 0, \quad \text{in } L^1(K'), \text{ for any } M > 0.$$

We now use Markov's inequality (see Theorem A.17 in the Appendix) and the uniform $L^2(K')$ -

bounds from (3) to estimate the truncation error.

$$\begin{aligned}
\|g_n^M - |U_{\mu_n(\omega)} - U_{\mu(\omega)}|\|_{L^1(K')} &= \int_{\{z \in K' : |U_{\mu_n(\omega)}(z) - U_{\mu(\omega)}(z)| \geq M\}} (|U_{\mu_n(\omega)}(z) - U_{\mu(\omega)}(z)| - M) \lambda(dz) \\
&\leq \left(\int_{\{z \in K' : |U_{\mu_n(\omega)}(z) - U_{\mu(\omega)}(z)| \geq M\}} \lambda(dz) \right)^{1/2} \|U_{\mu_n(\omega)} - U_{\mu(\omega)}\|_{L^2(K')} \\
&\leq 2C_{K', \Omega'} \left(\lambda \left(\{z \in K' : |U_{\mu_n(\omega)}(z) - U_{\mu(\omega)}(z)|^2 \geq M^2\} \right) \right)^{1/2} \\
&\leq \frac{4C_{K', \Omega'}^2}{M}.
\end{aligned} \tag{5.28}$$

Hence, for each $\omega \in \Omega'$ and for all $M > 0$, we obtain

$$\|U_{\mu_n(\omega)} - U_{\mu(\omega)}\|_{L^1(K')} \leq \|g_n^M - |U_{\mu_n(\omega)} - U_{\mu(\omega)}|\|_{L^1(K')} + \|g_n^M\|_{L^1(K')},$$

which implies

$$\lim_{n \rightarrow +\infty} \|U_{\mu_n(\omega)} - U_{\mu(\omega)}\|_{L^1(K')} \leq \frac{4C_{K', \Omega'}^2}{M}.$$

Thus, by taking the limit for $M \rightarrow +\infty$, we obtain that for every $\omega \in \Omega'$

$$U_{\mu_n(\omega)} \rightarrow U_{\mu(\omega)} \quad \text{in } L^1(K'), \quad \text{for } n \rightarrow +\infty.$$

Consequently, almost surely

$$U_{\mu_n} \xrightarrow[n \rightarrow \infty]{} U_{\mu}, \quad \text{in } \mathcal{D}'(K'),$$

and thus (5.24) holds. Indeed, by (5.14) we have

$$\Delta_z U_{\mu_n} = -2\pi\mu_n, \quad \Delta_z U_{\mu} = -2\pi\mu, \quad \text{in } \mathcal{D}'(\mathbb{C}). \tag{5.29}$$

Now, we want to prove the same result but we consider the convergence in probability instead of the almost sure convergence.

We first assume that for almost every $z \in K'$, and for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|U_{\mu_n}(z) - U_{\mu}(z)| > \varepsilon) = 0.$$

In this framework, we can apply the same arguments as before and obtain that there exists a measurable set $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$ properties (2) and (3) still hold. We define $g_n^M(z)$ on K' as in (5.27), recalling also its dependence on $\omega \in \Omega$. Fix $M > 0$. For every $\varepsilon > 0$ and for a.e. $z \in K'$:

- If $M > \varepsilon$, then

$$\mathbb{P}(|g_n^M(z)| > \varepsilon) = \mathbb{P}(|U_{\mu_n}(z) - U_{\mu}(z)| > \varepsilon) \xrightarrow[n \rightarrow +\infty]{} 0.$$

- If $M \leq \varepsilon$, then $|g_n^M(z)| \leq M \leq \varepsilon$, and hence $\mathbb{P}(|g_n^M(z)| > \varepsilon) = 0$, for all $n \geq 1$.

Therefore $g_n^M(z) \xrightarrow{n \rightarrow +\infty} 0$ in probability for a.e. $z \in K'$ and $|g_n^M(z)| \leq M \in L^1(\Omega, \mathbb{P})$. Fix $\varepsilon > 0$. Then, for almost every $z \in K'$, we have

$$\begin{aligned} \int_{\Omega} |g_n^M(z)| \mathbb{P}(d\omega) &= \int_{\{|g_n^M(z)| > \varepsilon\}} |g_n^M(z)| \mathbb{P}(d\omega) + \int_{\{|g_n^M(z)| \leq \varepsilon\}} |g_n^M(z)| \mathbb{P}(d\omega) \\ &\leq M \mathbb{P}(|g_n^M(z)| > \varepsilon) + \varepsilon \mathbb{P}(|g_n^M(z)| \leq \varepsilon) \leq M \mathbb{P}(|g_n^M(z)| > \varepsilon) + \varepsilon. \end{aligned}$$

Thus, since $g_n^M(z) \xrightarrow{n \rightarrow +\infty} 0$ in probability, we obtain

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |g_n^M(z)| \mathbb{P}(d\omega) \leq \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |g_n^M(z)| \mathbb{P}(d\omega) = 0, \quad \text{i.e.} \quad g_n^M(z) \xrightarrow{n \rightarrow +\infty} 0, \quad \text{in } L^1(\Omega, \mathbb{P}),$$

for almost every $z \in K'$. Moreover,

$$\left| \int_{\Omega} |g_n^M(z)| \mathbb{P}(d\omega) \right| \leq M \in L^1(K'), \quad \text{for all } z \in K'.$$

Thus, by Tonelli theorem and by the dominated convergence theorem,

$$\int_{\Omega} \left(\int_{K'} |g_n^M(z)| dz \right) \mathbb{P}(d\omega) = \int_{K'} \left(\int_{\Omega} |g_n^M(z)| \mathbb{P}(d\omega) \right) dz \xrightarrow{n \rightarrow +\infty} 0.$$

Finally, by Markov's inequality (Theorem A.17), for any $\varepsilon > 0$

$$\mathbb{P} \left(\int_{K'} |g_n^M(z)| dz > \varepsilon \right) \leq \frac{\int_{\Omega} \left(\int_{K'} |g_n^M(z)| dz \right) \mathbb{P}(d\omega)}{\varepsilon} \xrightarrow{n \rightarrow +\infty} 0,$$

which proves that

$$g_n^M \xrightarrow{n \rightarrow +\infty} 0 \quad \text{in } L^1(K'), \quad \text{in probability,} \quad (5.30)$$

for any $M > 0$.

We observe that also in this case, inequality (5.28) for $\|g_n^M - |U_{\mu_n} - U_{\mu}|\|_{L^1(K')}$ is still valid almost surely. Hence, for every $\varepsilon > 0$ and for each $M > 0$, we have

$$\begin{aligned} \mathbb{P}(\|U_{\mu_n} - U_{\mu}\|_{L^1(K')} > \varepsilon) &\leq \mathbb{P}(\|g_n^M - |U_{\mu_n} - U_{\mu}|\|_{L^1(K')} + \|g_n^M\|_{L^1(K')} > \varepsilon) \\ &\leq \mathbb{P} \left(\frac{4C_{K', \Omega'}^2}{M} + \|g_n^M\|_{L^1(K')} > \varepsilon \right) \\ &= \mathbb{P} \left(\|g_n^M\|_{L^1(K')} > \varepsilon - \frac{4C_{K', \Omega'}^2}{M} \right). \end{aligned}$$

Thus, taking $M = \frac{8C_{K',\Omega'}^2}{\varepsilon}$, from (5.30) we obtain

$$\mathbb{P}(\|U_{\mu_n} - U_\mu\|_{L^1(K')} > \varepsilon) \leq \mathbb{P}\left(\|g_n^M\|_{L^1(K')} > \frac{\varepsilon}{2}\right) \xrightarrow{n \rightarrow +\infty} 0,$$

which proves that

$$U_{\mu_n} \xrightarrow{n \rightarrow +\infty} U_\mu \quad \text{in } L^1(K'), \text{ in probability.}$$

Finally, we let $\varphi \in C_c^\infty(K')$; for every $\varepsilon > 0$ we have

$$\mathbb{P}(|\langle U_{\mu_n} - U_\mu, \varphi \rangle_{\mathcal{D}', \mathcal{D}}| > \varepsilon) \leq \mathbb{P}(\|U_{\mu_n} - U_\mu\|_{L^1(K')} \sup_{z \in K'} |\varphi(z)| > \varepsilon) \xrightarrow{n \rightarrow \infty} 0,$$

which means that $U_{\mu_n} \xrightarrow{n \rightarrow +\infty} U_\mu$ in $\mathcal{D}'(K')$ in probability. Hence, from (5.29),

$$\mu_n \xrightarrow{n \rightarrow +\infty} \mu \quad \text{in probability.}$$

□

Finally, we apply Theorem 5.8 to the measures ξ_N, ξ in the semiclassical limit $N \rightarrow +\infty$. Before, we need to show that its hypotheses hold.

We observe that since ξ_N are all probability measures, the hypothesis of the uniform bound on the total mass is satisfied. Recall that the support of $\xi = (p_0)_*(\lambda)$ is the compact set $\text{Im}(p_0)$. Thus, we have

$$\text{supp } \xi = p_0(\mathbb{T}^{2d}) \subset \overline{D}\left(0, \|p_0\|_{L^\infty(\mathbb{T}^{2d})}\right).$$

Moreover, for each N , the support of the empirical measure ξ_N is given by

$$\text{supp } \xi_N = \sigma(P_N^\delta) \subset \overline{D}\left(0, \|P_N^\delta\|\right).$$

By Proposition 1.17, there exists a constant $C_1 > 0$ that bounds the norm of p_N uniformly in N . Hence,

$$\|P_N^\delta\| \leq \|P\| + \delta\|V_q\| \leq C_1 + \delta\|q\|_\infty.$$

Since the probability measure $\mathbb{P}$ we are considering is defined on the space $B_{N^d}(0, R)$ and since $\|q\|_\infty \leq \|q\|_2$ for all $q \in \mathbb{C}^{N^d}$, we obtain

$$\text{supp } \xi_N \subset \overline{D}(0, C_1 + \delta R), \tag{5.31}$$

where, from (3.78) and (5.9),

$$\delta R = \mathcal{O}(N^{-1/2}) \ll 1, \quad \text{for } N \gg 1.$$

Fix $\eta_0 > 0$ and define

$$R_* := \max\left\{\|p_0\|_{L^\infty(\mathbb{T}^{2d})}, C_1 + \delta R\right\}, \quad K := D(0, R_* + \eta_0) \Subset \mathbb{C}.$$

Then

$$\text{supp } \xi, \text{ supp } \xi_N \subset K,$$

for all $N \gg 1$. We consider an open relatively compact set $K' \Subset \mathbb{C}$ such that $\bar{K} \subset K'$. If we show that for a.e. $z \in K'$, the convergence

$$U_{\xi_N}(z) \xrightarrow{N \rightarrow +\infty} U_{\xi}(z),$$

is true in probability, then the hypotheses of Theorem 5.8 are satisfied. Fix $z \in K'$. By (5.20) and (5.21) we have

$$U_{\xi}(z) - U_{\xi_N}(z) = N^{-d} \left(\ln \left| \det(P_N^{\delta} - z) \right| - N^d \int_{\mathbb{T}^{2d}} \ln |p_0(\rho) - z| d\rho \right) = N^{-d} \ln |F(q)|,$$

where F is defined as in (4.9) and depends explicitly on the random vector q . Hence, for every $\tilde{\epsilon} > 0$, we have

$$\mathbb{P} \left(|(U_{\xi}(z) - U_{\xi_N}(z))| > \tilde{\epsilon} \right) = \mathbb{P} \left(N^{-d} \ln |F(q)| > \tilde{\epsilon} \right) + \mathbb{P} \left(N^{-d} \ln |F(q)| < -\tilde{\epsilon} \right). \quad (5.32)$$

Recalling Remark 4.1, we suppose τ_0 is not too small. For instance, we assume

$$\tau_0 \geq \exp(-N^{\frac{\kappa}{2}}).$$

We fix $\tilde{\epsilon} > 0$. Thus, from (4.8), there exists $N_1 = N_1(\tilde{\epsilon}) \geq 0$ such that

$$\varepsilon_0(N) \leq \tilde{\epsilon}, \quad \text{for all } N \geq N_1.$$

Applying (4.5), for every $N \geq N_1$ we have

$$\mathbb{P} \left(N^{-d} \ln |F(q)| > \tilde{\epsilon} \right) \leq \mathbb{P} \left(N^{-d} \ln |F(q)| > \varepsilon_0(N) \right) = 0. \quad (5.33)$$

Moreover, let $0 < \alpha < \frac{\kappa}{2}$ and define

$$\epsilon_1(N) := \exp(-\varepsilon_0(N)N^{d+\alpha}).$$

With this definition, we observe that for $N \gg 1$,

$$\epsilon_1 \exp(\mathcal{O}(\varepsilon_0(N)N^d)) = \exp(\mathcal{O}(\varepsilon_0(N)N^d) - \varepsilon_0(N)N^{d+\alpha}) \leq 1,$$

and hence the hypothesis of Proposition 4.6 is satisfied with ϵ_1 . By applying the subsequent Remark 4.7, we obtain

$$\mathbb{P} \left(N^{-d} \ln |F(q)| < -\varepsilon_0(N)N^{\alpha} \right) \leq \mathcal{O}(1)\varepsilon_0(N)N^{M_3+d} \exp \left(\frac{-N^{\alpha}}{\mathcal{O}(1)} \right). \quad (5.34)$$

Since $0 < \alpha < \frac{\kappa}{2}$, we know from Remark 4.1 that $\varepsilon_0(N)N^\alpha \xrightarrow{N \rightarrow +\infty} 0$. Thus, there exists $N_2 \geq 0$ such that

$$\varepsilon_0(N)N^\alpha \leq \tilde{\varepsilon}, \quad \text{for all } N \geq N_2.$$

Now, we recall the following standard asymptotic: for any $p, q \geq 0$ and $r > 0$,

$$x^p (\ln(x))^q e^{-x^r} \rightarrow 0, \quad \text{as } x \rightarrow +\infty. \quad (5.35)$$

Thus, recalling the definition (4.4), the right-hand side of (5.34) converges to 0 as $N \rightarrow +\infty$. Fixing $\eta > 0$, there exists $N_3 = N_3(\eta) \geq 0$ such that

$$\mathcal{O}(1)\varepsilon_0(N)N^{M_3+d} \exp\left(\frac{-N^\alpha}{\mathcal{O}(1)}\right) \leq \eta, \quad \text{for every } N \geq N_3.$$

Consequently, for all $N \geq \max(N_2, N_3)$, we obtain

$$\mathbb{P}\left(N^{-d} \ln |F(q)| < -\tilde{\varepsilon}\right) \leq \mathbb{P}\left(N^{-d} \ln |F(q)| < -\varepsilon_0(N)N^\alpha\right) \leq \eta. \quad (5.36)$$

Finally, applying both (5.33) and (5.36), we obtain that for every $\tilde{\varepsilon} > 0$

$$\lim_{N \rightarrow +\infty} \mathbb{P}\left(|U_{\tilde{\zeta}}(z) - U_{\tilde{\zeta}_N}(z)| > \tilde{\varepsilon}\right) = 0.$$

We observe that this result is true for all $z \in K'$. Hence, we can invoke Theorem 5.8, which yields the following convergence in probability:

$$\frac{1}{N^d} \sum_{\zeta \in \sigma(P_N^\delta)} \delta_\zeta \rightharpoonup (p_0)_*(\lambda(d\rho)), \quad \text{as } N \rightarrow +\infty. \quad (5.37)$$

Remark 5.9. We observe that the empirical measure $\tilde{\zeta}_N$ "counts" the eigenvalues of P_N^δ over measurable subsets of $\mathbb{C}$. More precisely, for any measurable set $\Gamma \subset \mathbb{C}$,

$$\tilde{\zeta}_N(\Gamma) = \frac{1}{N^d} \sum_{\zeta \in \sigma(P_N^\delta)} \delta_\zeta(\Gamma) = \frac{1}{N^d} \#\{\zeta \in \sigma(P_N^\delta) \cap \Gamma\}. \quad (5.38)$$

We now consider an adapted version of the Portemanteau theorem (see Theorem A.18 in the Appendix) in a probabilistic setting, as presented in Corollary A.19. In particular, we apply it to (5.37), taking as underlying metric space the compact set K , so that $C_b(K) = C_c(K)$. Thus, we obtain a quantitative formulation for the number of eigenvalues as a consequence of weak convergence.

In particular, for every measurable set $\Gamma \subset \mathbb{C}$ such that

$$\lambda(p_0^{-1}(\partial\Gamma)) = 0,$$

we obtain, for all $\varepsilon > 0$,

$$\mathbb{P} \left(\left| \frac{1}{N^d} \# \{ \zeta \in \sigma(P_N^\delta) \cap \Gamma \} - \int_{p_0^{-1}(\Gamma)} d\rho \right| > \varepsilon \right) \longrightarrow 0, \quad \text{as } N \rightarrow +\infty. \quad (5.39)$$

Remark 5.10. The result in (5.39) is not substantially different from the quantitative estimate established earlier in (5.18). The main distinction lies in the fact that Theorem 5.5 provides an explicit error bound, but only for a specific class of sets Γ , whereas (5.39) yields convergence in probability without an explicit rate, yet applies to a broader family of measurable sets.

5.3 Applications with Truncated Gaussian Distribution

In the final section of the thesis we consider a practical example of the results we found previously. In particular we apply Theorem 5.5 to a simple framework, in order to obtain some explicit quantitative bounds for the spectrum of the perturbed operator.

We consider a framework where the dimension $d = 1$, so that we work with symbols on $\mathbb{T}^2$ and with the corresponding restricted Weyl quantizations acting on $\mathbb{C}^N$, with $N = \frac{1}{2\pi h}$. Additionally, we consider $\tau_0 = C_0 N^{-m}$ for $M > 0$ and $C_0 > 0$, so that, by definition (4.4), we have

$$\begin{aligned} \varepsilon_0(N) &:= \mathcal{O} \left(\left(N^{-\frac{1}{2}} + \left(N^{-1} \ln(N) + N^{-\kappa} \right) \left(\mathcal{O}(m \ln(N)) + (\ln(N))^2 \right) \right) \right) \\ &= \mathcal{O} \left(N^{-1} (\ln(N))^3 + N^{-\kappa} (\ln(N))^2 \right), \quad \text{for } N \gg 1, \end{aligned} \quad (5.40)$$

where the last equality holds because $\kappa \in]0, 1]$.

By Proposition 3.11 we define $\delta = \frac{N^{-m-1}}{C}$, for a sufficiently large constant $C > 0$. Moreover, for the perturbation V_q that we want to apply, we consider a probabilistic setting given by a Gaussian distribution truncated to the ball $B_N(0, R)$, where

$$C_1 N^{\frac{1-\kappa}{2}} \leq R \leq C_2 N^{m+\frac{1}{2}},$$

for two constants $C_1, C_2 > 0$ as in (3.78). Specifically, we define

$$\mathbb{P}(dq) := C(N) e^{\Phi(q)},$$

where

$$\Phi(q) := -\frac{1}{2} \|q\|_2^2, \quad q \in B_N(0, R); \quad C(N) := \frac{1}{\int_{B_N(0, R)} \exp(\Phi(q)) dq}.$$

We observe that

$$\|\nabla \Phi(q)\|_2 = \| -q \|_2 < R = \mathcal{O}(N^{m+\frac{1}{2}}),$$

and thus, by following the definitions in Chapter 4, we obtain

$$M_1 := m + \frac{1}{2}, \quad M_2 := m + \frac{1}{2}, \quad M_3 := m + \frac{3}{2}.$$

5.3.1 Application to the Scottish Flag Operator

We consider the symbol whose quantization yields the "Scottish flag" operator, called in this way for the particular geometry of its spectrum, which recalls a Scottish flag.

In particular we define

$$p(x, \xi) := \cos(2\pi x) + i \cos(2\pi \xi), \quad (x, \xi) \in \mathbb{T}^2.$$

We note that

$$p(x, \xi) = p_0(x, \xi) \in S(1, 1),$$

and satisfies the symmetry assumption $p(x, \xi) = p(x, -\xi)$.

We recall definitions (1.3) and (1.25) and, as shown in [ChZw10] we observe that the Weyl quantizations of functions depending only on x or ξ restricted to $\mathcal{H}_h^1$ are given by

$$\begin{aligned} C^\infty(\mathbb{T}^2) \ni f = f(x) &\mapsto f_N = \text{diag}(f(l/N); l = 0, 1, \dots, N-1) \\ C^\infty(\mathbb{T}^2) \ni g = g(\xi) &\mapsto g_N = \mathcal{F}_N^* \text{diag}(g(l/N); l = 0, 1, \dots, N-1) \mathcal{F}_N, \end{aligned}$$

where $\mathcal{F}_N$ is the discrete Fourier transform, defined as in (1.22). Thus, we obtain

$$p_N = \begin{pmatrix} \cos(x_1) & \frac{i}{2} & 0 & 0 & \cdots & \frac{i}{2} \\ \frac{i}{2} & \cos(x_2) & \frac{i}{2} & 0 & \cdots & 0 \\ 0 & \frac{i}{2} & \cos(x_3) & \frac{i}{2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{i}{2} & \cos(x_{N-1}) & \frac{i}{2} \\ \frac{i}{2} & 0 & \cdots & 0 & \frac{i}{2} & \cos(x_N) \end{pmatrix}, \quad x_j := \frac{2\pi j}{N}, \quad j = 1, \dots, N.$$

Write $p_N = D + \frac{i}{2}T$, where $D := \text{diag}(\cos(x_i); i = 1, \dots, N)$ and T is the real symmetric matrix with ones on the super- and sub-diagonals and on the entries $(1, N)$ and $(N, 1)$. If (u, μ) is an eigencouple of p_N , with $\|u\|_2 = 1$, then

$$\mu = \langle p_N u, u \rangle = \langle Du, u \rangle + \frac{i}{2} \langle Tu, u \rangle.$$

It follows that

$$\Re(\mu) = \sum_{j=1}^N |u_j|^2 \cos(x_j) \in [-1, 1],$$

and

$$\Im(\mu) = \left\langle \frac{1}{2} Tu, u \right\rangle \in \left[-\frac{1}{2} \|T\|, \frac{1}{2} \|T\| \right] = [-1, 1].$$

The last equality holds from the fact that each column and row of T contains exactly two ones, and thus,

$$\|T\| \leq \sqrt{\|T\|_1 \|T\|_\infty} = 2,$$

where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are the induced max-column-sum and max-row-sum norms, respectively. Therefore

$$\sigma(p_N) \subset [-1, 1]^2.$$

We consider $z_0 \in]-1, 1[^2$ and $R_0 > 0$ such that

$$\Gamma := D(z_0, R_0) \subsetneq [-1, 1]^2. \quad (5.41)$$

The set Γ is open, simply connected, relatively compact with uniform Lipschitz boundary, following the definition introduced in Section 5.1. For each $z \in \partial\Gamma = \partial D(z_0, R_0)$, we consider affine coordinates $(\tilde{y}_1, \tilde{y}_2)$ translating z to the origin and rotating so that the tangent to the disc in z is horizontal. Thus, if we denote $y = (y_1, y_2)$ the old coordinates, taking $0 < r_0 < R_0$, the local description of Γ in the rectangle $\{y \in \mathbb{C} : (\tilde{y}_1, \tilde{y}_2) \in]-r_0, r_0[\times]-r_0, r_0[\}$ is

$$\{y \in \mathbb{C} : \tilde{y}_1 \in]-r_0, r_0[, \tilde{y}_2 > f_z(\tilde{y}_1)\},$$

where the function $f_z(\tilde{y}_1) := R_0 - \sqrt{R_0^2 - \tilde{y}_1^2}$ is Lipschitz in $[-r_0, r_0]$ with modulus

$$\sup_{|\tilde{y}_1| \leq r_0} |f'_z(\tilde{y}_1)| = \frac{r_0}{\sqrt{R_0^2 - r_0^2}} \leq 1, \quad \text{whenever } r_0 \leq \frac{R_0}{\sqrt{2}}.$$

Let $z = \Re(z) + i\Im(z) \in \mathbb{C}$ and $0 \leq t \ll 1$. We compute

$$\begin{aligned} \lambda(\{(x, \xi) \in \mathbb{T}^2 : |p_0(x, \xi) - z|^2 \leq t\}) &= \iint_{\mathbb{T}^2} \mathbb{1}_{\{(\cos(2\pi x) - \Re(z))^2 + (\cos(2\pi \xi) - \Im(z))^2 \leq t\}} dx d\xi \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \mathbb{1}_{\{(\cos(\theta) - \Re(z))^2 + (\cos(\phi) - \Im(z))^2 \leq t\}} d\theta d\phi, \end{aligned} \quad (5.42)$$

where we used the change of variables $\theta = 2\pi x$, $\phi = 2\pi \xi$. By making a second change of coordinates $u = \cos(\theta)$, $v = \cos(\phi)$, we obtain that the last term in (5.42) is equal to

$$\begin{aligned} &\frac{4}{(2\pi)^2} \int_0^\pi \int_0^\pi \mathbb{1}_{\{(\cos(\theta) - \Re(z))^2 + (\cos(\phi) - \Im(z))^2 \leq t\}} d\theta d\phi \\ &= \frac{1}{\pi^2} \int_{-1}^1 \int_{-1}^1 \mathbb{1}_{\{(u - \Re(z))^2 + (v - \Im(z))^2 \leq t\}} \frac{1}{\sqrt{(1-u^2)(1-v^2)}} du dv. \end{aligned} \quad (5.43)$$

Now we take a smaller, possibly N -dependent weight $0 < r \ll 1$ in the definition of Lipschitz boundary. In particular, we choose r sufficiently small so that there exists $\eta > 0$ with

$$\tilde{\gamma}_r := \partial D(z_0, R_0) + D(0, r) \subset [-1 + \eta, 1 - \eta]^2. \quad (5.44)$$

Hence, taking t sufficiently small, for all $z \in \tilde{\gamma}_r$, we have

$$\{(u, v) \in [-1, 1] : (u - \Re(z))^2 + (v - \Im(z))^2 \leq t\} \subset [-1 + \delta, 1 - \delta]^2,$$

for a constant $\delta > 0$ dependent on η and t . For example, if we consider $t = (\frac{\eta}{2})^2$, we can take $\delta = \frac{\eta}{2}$, since the center $(\Re(z), \Im(z))$ is at distance $\geq \eta$ from the boundary of $[-1, 1]^2$ and the radius is $\sqrt{t} = \frac{\eta}{2}$.

Under these assumptions, we get a uniform bound on the integrand function

$$\frac{1}{\sqrt{(1-u^2)(1-v^2)}},$$

over the plane $\{(u - \Re(z))^2 + (v - \Im(z))^2 \leq t\}$. Thus, from (5.42) and (5.43), there exists $C_1 > 0$ such that

$$\begin{aligned} \lambda(\{(x, \xi) \in \mathbb{T}^2 : |p_0(x, \xi) - z|^2 \leq t\}) &\leq C_1 \int_{-1}^1 \int_{-1}^1 \mathbb{1}_{\{(u - \Re(z))^2 + (v - \Im(z))^2 \leq t\}} du dv \\ &= C_1 \lambda(D(z, \sqrt{t})) = \mathcal{O}(t). \end{aligned}$$

Therefore the volume condition (3.28) in the hypotheses of Theorem 5.5 holds with $\kappa = 1$. Applying the theorem, for any $\tilde{\epsilon} \geq C_3 \varepsilon_0(N)$ (with $C_3 > 0$ large enough), there exists a constant $C_2 > 0$ such that

$$\left| \#(\sigma(p_N + \delta V_q) \cap \Gamma) - N \lambda(p^{-1}(\Gamma)) \right| \leq C_2 N \left(2\pi \lambda(p^{-1}(\partial\Gamma + D(0, r))) + \mathcal{O}\left(\frac{1}{r}\right) \tilde{\epsilon} \right),$$

with probability at least

$$1 - \frac{\mathcal{O}(1)}{r} \varepsilon_0(N) N^{m+\frac{5}{2}} \exp\left(-\frac{\tilde{\epsilon}}{C_3 \varepsilon_0(N)}\right). \quad (5.45)$$

By (5.40), we have

$$\varepsilon_0(N) = \mathcal{O}\left(N^{-1} (\ln(N))^3\right).$$

For instance, choosing

$$1 \gg \tilde{\epsilon} = \varepsilon_0(N) \sqrt{N} \gg \varepsilon_0(N),$$

(5.45) becomes

$$1 - \frac{\mathcal{O}(1)}{r} \varepsilon_0(N) N^{m+\frac{5}{2}} \exp\left(-\frac{\sqrt{N}}{C_3}\right),$$

which tends to 1 as $N \rightarrow +\infty$, since the exponential dominates any polynomial or logarithmic factor.

Moreover, with the change of variables $(u, v) = (\cos(2\pi x), \cos(2\pi \xi))$, we obtain

$$\lambda(p^{-1}(\partial D(z_0, R_0) + D(0, r))) = \int_{-1}^1 \int_{-1}^1 \frac{1}{\pi^2} \mathbb{1}_{C_{z_0, R_0, r}}(u, v) \frac{1}{\sqrt{(1-u^2)(1-v^2)}} du dv, \quad (5.46)$$

where we define the annulus

$$C_{z_0, R_0, r} = \{(u, v) : (R_0 - r)^2 < (u - \Re(z_0))^2 + (v - \Im(z_0))^2 < (R_0 + r)^2\}.$$

If r is small enough so that $C_{z_0, R_0, r} \subset \tilde{\gamma}_r \subset [-1 + \delta, 1 - \delta]^2$, with $\delta > 0$, then the weight is uniformly bounded, and hence

$$\lambda \left(p^{-1}(\partial D(z_0, R_0) + D(0, r)) \right) \leq \mathcal{O}(1) \lambda(C_{z_0, R_0, r}) = \mathcal{O}(r).$$

Thus, for a new constant $\widetilde{C}_2 > 0$, we obtain

$$\left| \frac{1}{N} \#(\sigma(p_N + \delta V_q) \cap D(z_0, R_0)) - \lambda \left(p^{-1}(D(z_0, R_0)) \right) \right| \leq \widetilde{C}_2 \left(r + \frac{N^{-\frac{1}{2}} (\ln(N))^3}{r} \right),$$

with probability arbitrarily close to 1 as $N \rightarrow +\infty$.

Remark 5.11. Choosing $N^{-\frac{1}{2}} \ll r \ll 1$, makes the right-hand size tend to 0 as $N \rightarrow +\infty$. For example, if we take $r = N^{-\frac{1}{4}}$, the right-hand size is equal to

$$\widetilde{C}_2 N^{-\frac{1}{4}} \left(1 + (\ln(N))^3 \right) \longrightarrow 0, \quad \text{as } N \rightarrow +\infty.$$

Thus, for N sufficiently large, with high probability the following is true

$$\frac{1}{N} \#(\sigma(p_N + \delta V_q) \cap D(z_0, R_0)) \sim \lambda \left(p^{-1}(D(z_0, R_0)) \right),$$

which is the desired probabilistic Weyl law in this model.

Conclusions

In this work we have achieved our main objective by proving an original result: a probabilistic Weyl law for the spectrum of finite-dimensional operators arising from the Weyl quantization of symbols on the torus $\mathbb{T}^{2d}$, under random potential perturbations. Specifically, we have shown how the eigenvalues asymptotically equidistribute in the semiclassical limit $N \rightarrow +\infty$.

This result extends the framework studied by M. Vogel in [Vo20], where he considered random full-matrix perturbations applied to the same class of operators. Using similar tools and ideas, we have established an analogous theorem for random diagonal matrix perturbations. In our context, we have followed the approach of J. Sjöstrand [Sj09], who studied multiplicative random perturbations for quantizations on $\mathbb{R}^{2d}$, and we have adapted these ideas to the finite-dimensional setting arising from the compact torus.

The final application in Section 5.3 provides a concrete example of our results. In particular, we have considered the Scottish flag operator, whose spectrum is not regularly distributed but concentrates along the diagonals of the square $[-1, 1]^2 \subset \mathbb{C}$. By adding a small diagonal perturbation given by a Gaussian distribution truncated to a ball, we have obtained a regularization of the spectrum and an asymptotic eigenvalue law in the semiclassical limit $N \rightarrow +\infty$.

An important aspect of the framework considered is its possible physical interpretation. In a quantum-mechanical setting, these operators can describe periodic quantum observables, and the associated perturbations can model random potentials acting on a quantum system. Moreover, these results highlight how random perturbations serve as an effective tool for spectral regularization of non-selfadjoint pseudodifferential operators, and they naturally suggest several directions for further research. In particular, we outline the following:

- **Relaxation of Distributional Assumptions.** One generalization can be obtained by removing the compact-support assumptions on the distribution of the random perturbations. This would allow a wider range of admissible models and avoid, for instance, the truncation used in our application. To preserve high-probability concentration estimates, it would be natural to assume suitable tail-decay conditions on the distribution.
- **Generalization of the Phase Space.** A second direction is to seek an analogous probabilistic Weyl law on a more general phase space, given by a compact Kähler manifold, via Berezin-Toeplitz quantization. The torus is a particular compact phase space where periodicity holds; analogous results may be attainable in other geometric frameworks.

- **Broadening of the Perturbation Class.** Lastly, one further possibility is to extend the class of random perturbations beyond diagonal and fully random matrices, and to characterize the minimal structural or correlation assumptions under which the probabilistic Weyl law persists.

Appendix

This appendix gathers auxiliary definitions and results used throughout the dissertation. Given the heterogeneous material, we organize it into separate sections.

A.1 Useful Results for Semiclassical Calculus

In this section, we collect notions and statements essential for the first chapter of the thesis, including the Schwartz kernel theorem, the definition of oscillatory integrals and the Fourier conjugation formula.

Theorem A.1 (Schwartz Kernel Theorem). *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open sets. Then every continuous linear operator*

$$A : C_c^\infty(X) \rightarrow \mathcal{D}'(Y)$$

can be represented uniquely by a distribution $K_A \in \mathcal{D}'(Y \times X)$ such that for all test functions $\phi \in C_c^\infty(X)$ and $\psi \in C_c^\infty(Y)$,

$$\langle A\phi, \psi \rangle_{\mathcal{D}'(Y), \mathcal{D}(Y)} = \langle K_A, \psi \otimes \phi \rangle_{\mathcal{D}'(Y \times X), \mathcal{D}(Y \times X)},$$

where $\psi \otimes \phi(y, x) := \psi(y)\phi(x)$. K_A is called the Schwartz kernel of A .

A.1.1 Oscillatory Integrals

We introduce the notion of oscillatory integral following the approach of [Ma02, Section 2.4]. In particular, we work on the vector space $\mathbb{R}^{3n} = \mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\xi^n$ and we consider the phase function $e^{\frac{i}{h}\langle x-y, \xi \rangle}$, with $h \in]0, 1]$. The following construction applies also for other phase functions.

Let $a = a(x, y, \xi) \in S(\langle \xi \rangle^m)$, with $m \in \mathbb{R}$, in the sense of Definition 1.2. We wish to give a meaning to the possibly divergent integral

$$I(a) := \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi \rangle} a(x, y, \xi) d\xi.$$

First, we notice that if $m < -n$, then the integral is absolutely convergent and is therefore well-defined. To define it when $m \geq -n$, we are going to interpret it as the distribution kernel of an operator.

Let $u \in C_c^\infty(\mathbb{R}^n)$ and assume $m < -n$. Then, we may define the operator

$$A_a u(x; h) := \iint_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y, \xi \rangle} a(x, y, \xi) u(y) dy d\xi,$$

which is an absolutely convergent integral. If we introduce the differential operator

$$L(\xi, hD_y) := \frac{1}{1 + \|\xi\|_2^2} (1 - h \langle \xi, D_y \rangle),$$

we have

$$L \left(e^{\frac{i}{h}\langle x-y, \xi \rangle} \right) = e^{\frac{i}{h}\langle x-y, \xi \rangle}. \quad (\text{A.1})$$

It is this particular property, based on the oscillatory character of the phase function, that will allow us to give a sense to $I(a)$. In particular, we observe that $L^k(e^{\frac{i}{h}\langle x-y, \xi \rangle}) = e^{\frac{i}{h}\langle x-y, \xi \rangle}$, for all $k \in \mathbb{N}^*$. Integrating by parts k times, with tL denoting the formal adjoint of L with respect to y , we obtain

$$A_a u(x; h) = \iint_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y, \xi \rangle} ({}^tL(\xi, hD_y))^k (au) dy d\xi =: I_k u(x),$$

where

$$({}^tL(\xi, hD_y))^k (au) = \left(\frac{1 + h \langle \xi, D_y \rangle}{1 + \|\xi\|_2^2} \right)^k (au) = \mathcal{O}(\langle \xi \rangle^{m-k}),$$

uniformly as $\|\xi\|_2 \rightarrow +\infty$. As a consequence, the integral $I_k u$ is absolutely convergent whenever $m - k < -n$. Moreover, since L satisfies (A.1), repeated integration by parts shows that $I_{k+\ell} u = I_k u$ for all $\ell \geq 0$. Therefore, for any $m \in \mathbb{R}$ and $a \in S(\langle \xi \rangle^m)$, we define, for $u \in C_c^\infty(\mathbb{R}^n)$

$$A_a u(x, h) = \iint_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y, \xi \rangle} ({}^tL(\xi, hD_y))^k (au) dy d\xi, \quad (\text{A.2})$$

where k is a nonnegative integer such that $k > n + m$. The integral is absolutely convergent and the right-hand side is independent of the particular choice of k . Thus, we obtain the following theorem, whose proof is presented in [Ma02].

Theorem A.2 (Theorem 2.4.3 of [Ma02]). A_a defines a continuous linear operator from $C_c^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

As a consequence, by applying Theorem A.1, it follows that the operator A_a admits a unique distribution kernel in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$. This motivates the following definition.

Definition A.3. Given $a \in S(\langle \xi \rangle^m)$, $m \in \mathbb{R}$, the oscillatory integral associated to a with the phase function $e^{\frac{i}{h}\langle x-y, \xi \rangle}$ is the distribution

$$I(a) = \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi \rangle} a(x, y, \xi) d\xi \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n).$$

It is the Schwartz kernel of A_a .

Remark A.4. This construction can be extended to a broader class of symbols. In particular, analogous results hold for the class $S(m)$ defined in (1.1), where m is an order function as in Definition 1.1.

A.1.2 Fourier Conjugation Formula

Theorem A.5 (Fourier conjugation formula). *Let m be an order function as in Definition 1.1 and let $p \in S(m)$ (defined as in (1.1)). Recalling definitions (1.3) and (1.4), the following equivalence holds*

$$\mathcal{F}_h^{-1} p^w(x, hD_x) \mathcal{F}_h = p^w(hD_x, -x).$$

Proof. We note that the distribution kernel of $\mathcal{F}_h^{-1} p^w(x, hD_x) \mathcal{F}_h$ is

$$K_h(x, y) = \frac{1}{(2\pi h)^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{\frac{i}{h}(\langle x', x \rangle + \langle x' - y', \zeta \rangle - \langle y', y \rangle)} p\left(\frac{x' + y'}{2}, \zeta\right) dy' dx' d\zeta.$$

With the change of variables $x' = x'$, $z = \frac{x' + y'}{2}$, we obtain

$$K_h(x, y) = \frac{2^d}{(2\pi h)^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{\frac{i}{h}\Phi(x', z, \zeta, y, x)} p(z, \zeta) dx' dz d\zeta,$$

where

$$\Phi(x', z, \zeta, y, x) := 2\left(\langle x', \zeta + \frac{x+y}{2} \rangle - \langle z, y + \zeta \rangle\right).$$

Moreover,

$$\frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} e^{\frac{2i}{h}\langle x', \zeta + \frac{x+y}{2} \rangle} dx' = 2^{-d} \delta\left(\zeta + \frac{x+y}{2}\right).$$

Hence,

$$K_h(x, y) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} e^{\frac{i}{h}\langle x-y, z \rangle} p\left(z, -\frac{x+y}{2}\right) dz.$$

which is the distribution kernel of $\tilde{p}^w(x, hD_x)$, where $\tilde{p}(x, \xi) := p(\xi, -x)$. $\square$

A.2 Stirling's Formula

We consider a proof of Stirling's formula that follows the approach of Tao in [Ta12, Section 1.2].

We start by interpreting the factorial through the Gamma function, which is defined in the following way

$$\begin{aligned} \Gamma : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto \int_0^{+\infty} t^{x-1} e^{-t} dt. \end{aligned}$$

Repeated integration by parts yields the identity:

$$n! = \Gamma(n+1) = \int_0^{+\infty} t^n e^{-t} dt.$$

Thus, estimating $n!$ reduces to estimating the integral above. Elementary calculus shows that the integrand $t^n e^{-t}$ attains its maximum at $t = n$, which suggests the substitution $t = n + s$. This gives:

$$n! = \int_{-n}^{+\infty} (n+s)^n e^{-n-s} ds = n^n e^{-n} \int_{-n}^{+\infty} \left(1 + \frac{s}{n}\right)^n e^{-s} ds,$$

where in the last equality we factored out the terms $n^n e^{-n}$.

Combining the integrand into a single exponential, we obtain

$$n! = n^n e^{-n} \int_{-n}^{+\infty} \exp\left(n \ln\left(1 + \frac{s}{n}\right) - s\right) ds.$$

Using a Taylor expansion, we observe that

$$n \ln\left(1 + \frac{s}{n}\right) = s - \frac{s^2}{2n} + \dots,$$

which suggests the heuristic approximation

$$\exp\left(n \ln\left(1 + \frac{s}{n}\right) - s\right) \approx \exp\left(-\frac{s^2}{2n}\right).$$

To make this rigorous, we first scale s by $\sqrt{n}$ to eliminate the denominator n : we make a change of variable in the integral given by $s = \sqrt{n}x$ and we get

$$n! = \sqrt{n} n^n e^{-n} \int_{-\sqrt{n}}^{+\infty} \exp\left(n \ln\left(1 + \frac{x}{\sqrt{n}}\right) - \sqrt{n}x\right) dx.$$

The Taylor expansion assures us that for fixed x , the integrand converges pointwise:

$$\exp\left(n \ln\left(1 + \frac{x}{\sqrt{n}}\right) - \sqrt{n}x\right) \rightarrow \exp\left(-\frac{x^2}{2}\right) \quad (\text{A.3})$$

in the limit $n \rightarrow +\infty$. More precisely, since the function $n \ln\left(1 + \frac{x}{\sqrt{n}}\right)$ vanishes at the origin with first derivative $\sqrt{n}$ and has the second derivative equal to $-\frac{1}{(1+x/\sqrt{n})^2} < 0$, by applying two times the fundamental theorem of calculus we obtain

$$n \ln\left(1 + \frac{x}{\sqrt{n}}\right) - \sqrt{n}x = -\int_0^x \frac{(x-y)}{(1+y/\sqrt{n})^2} dy.$$

Then, it is easy to see that the following uniform bound holds when $|x| \leq \sqrt{n}$

$$n \ln\left(1 + \frac{x}{\sqrt{n}}\right) - \sqrt{n}x \leq -cx^2,$$

for some $c > 0$. Moreover, for $|x| > \sqrt{n}$, we have

$$n \ln \left(1 + \frac{x}{\sqrt{n}} \right) - \sqrt{n}x \leq -cx\sqrt{n}.$$

These bounds ensure that the integrands are dominated by an absolutely integrable function. By the pointwise convergence (A.3) and the Lebesgue dominated convergence theorem, we conclude

$$\int_{-\sqrt{n}}^{+\infty} \exp \left(n \ln \left(1 + \frac{x}{\sqrt{n}} \right) - \sqrt{n}x \right) dx \xrightarrow{n \rightarrow +\infty} \int_{-\infty}^{+\infty} \exp \left(-\frac{x^2}{2} \right) dx.$$

We know that

$$\int_{-\infty}^{+\infty} \exp \left(-\frac{x^2}{2} \right) dx = \sqrt{2\pi},$$

leading to Stirling's formula:

$$n! = (1 + o(1)) \sqrt{2\pi n} n^n e^{-n}, \quad (\text{A.4})$$

which means that

$$\lim_{n \rightarrow +\infty} \frac{n!}{\sqrt{2\pi n} n^n e^{-n}} = 1.$$

Furthermore, by Robbins' refinement of Stirling's formula [Ro55], we have

$$n! = \sqrt{2\pi n} n^n e^{-n} e^{r_n}, \quad (\text{A.5})$$

where

$$0 < \frac{1}{12n+1} < r_n < \frac{1}{12n}.$$

A.3 Mini-max Theorem and Inequalities for Singular Values

In this section we recall a few fundamental results from spectral theory which we use extensively in Chapter 3. We begin with the Courant–Fischer min–max principle in the formulation of [Ta12]; a proof is omitted.

Theorem A.6 (Courant-Fischer mini-max theorem, [Ta12, Theorem 1.3.2]). *Let A be an $n \times n$ Hermitian matrix. For each $1 \leq i \leq n$, the i -th eigenvalue of A , $\lambda_i(A)$, admits the following characterizations:*

$$\lambda_i(A) = \sup_{\dim(V)=i} \inf_{v \in V: \|v\|_2=1} v^* A v,$$

and

$$\lambda_i(A) = \inf_{\dim(V)=n-i+1} \sup_{v \in V: \|v\|_2=1} v^* A v.$$

Here V ranges over all subspaces of $\mathbb{C}^n$ with the specified dimension.

We next collect some standard inequalities for singular values. Although in this thesis we

are primarily concerned with finite-dimensional operators, we present a more general framework, following the approach in Chapter 2 of [GoKr78].

Let $\mathcal{H}$ denote a separable Hilbert space and $\mathcal{S}_\infty$ the space of all linear completely continuous operators acting on $\mathcal{H}$, while $\mathcal{B}$ the space of all bounded linear operators on $\mathcal{H}$.

Let $A \in \mathcal{S}_\infty$; then $H = (A^*A)^{1/2} \in \mathcal{S}_\infty$. We define the singular values of A as the eigenvalues of H , and we enumerate them in decreasing order in the following way

$$s_1(A) \geq s_2(A) \geq \cdots \geq 0.$$

Definition A.7. Let $A \in \mathcal{B}$. The dimension of A is defined as

$$r(A) := \dim(\overline{\text{Im}(A)}) \leq +\infty.$$

Theorem A.8 (Dž. E. Allahverdiev, [GoKr78, Theorem 2.1]). *Let $A \in \mathcal{S}_\infty$ a linearly continuous operator. Then for any $n = 0, 1, 2, \dots$*

$$s_{n+1}(A) = \min_{K \in \mathcal{R}_n} \|A - K\|, \quad (\text{A.6})$$

where $\mathcal{R}_n$ is the set of all finite-dimensional operators of dimension $\leq n$.

Consequently, we obtain the following corollaries.

Corollary A.9 (K. Fan, [GoKr78, Corollary 2.2]). *Let $A, B \in \mathcal{S}_\infty$. Then the following inequalities are true*

$$s_{m+n-1}(A+B) \leq s_m(A) + s_n(B), \quad \text{for } m, n = 1, 2, \dots; \quad (\text{A.7})$$

$$s_{m+n-1}(AB) \leq s_m(A)s_n(B), \quad \text{for } m, n = 1, 2, \dots \quad (\text{A.8})$$

Proof. Let the $(m-1)$ -dimensional operator K_1 and the $(n-1)$ -dimensional operator K_2 be such that

$$s_m(A) = \|A - K_1\|, \quad \text{and } s_n(B) = \|B - K_2\|.$$

Then

$$\begin{aligned} s_{m+n-1}(A+B) &\leq \|A+B - (K_1 + K_2)\| \\ &\leq \|A - K_1\| + \|B - K_2\| = s_m(A) + s_n(B). \end{aligned}$$

Moreover, since the dimension of the operator $AK_2 + K_1(B - K_2)$ does not exceed $m+n-2$, and $(A - K_1)(B - K_2) = AB - AK_2 - K_1(B - K_2)$, we obtain

$$s_{m+n-1}(AB) \leq \|A - K_1\| \|B - K_2\| = s_m(A)s_n(B).$$

□

Corollary A.10 (Corollary 2.3 of [GoKr78]). *For any operators $A, B \in \mathcal{S}_\infty$,*

$$|s_n(A) - s_n(B)| \leq \|A - B\|, \quad n = 1, 2, \dots \quad (\text{A.9})$$

Proof. Let $n \in \mathbb{N}$. By Theorem A.8, we have

$$\begin{aligned} s_{n+1}(A) &= \min_{K \in \mathcal{R}_n} \|A - K\| = \min_{K \in \mathcal{R}_n} \|B - K + A - B\| \\ &\leq \min_{K \in \mathcal{R}_n} \|B - K\| + \|A - B\| = s_{n+1}(B) + \|A - B\|. \end{aligned}$$

Interchanging the roles of the operators A and B , we obtain

$$s_{n+1}(B) \leq s_{n+1}(A) + \|A - B\|,$$

from which (A.9) follows. □

A.4 Classical Results in Complex and Harmonic Analysis

In this section we collect classical results from complex and harmonic analysis. These theorems admit many equivalent formulations and broad generalizations; we state them in versions that best suit our context and adapt them accordingly, without considering all the proofs. These results play an important role in Chapter 4 and Chapter 5.

A.4.1 Maximum Principle and Harnack's Inequality

We begin by presenting the Maximum Principle, as stated in [Ah79].

Theorem A.11 (The maximum principle, [Ah79, Theorem 12']). *Let $E \subset \mathbb{C}$ be a closed bounded set. If $f(z)$ is defined and continuous on E and holomorphic on the interior of E , then the maximum of $|f(z)|$ on E is assumed on the boundary of E , i.e.*

$$\max_{z \in E} |f(z)| = \max_{z \in \partial E} |f(z)|.$$

Next, we recall a version of Harnack's inequality, adapted from [Ev10] and specialized to the case of the Laplacian:

Theorem A.12 (Harnack's Inequality). *Let $U \subset \mathbb{C}$ be an open set, and suppose $u \geq 0$ is a C^2 solution of*

$$\Delta u = 0 \quad \text{in } U.$$

If $V \Subset U$ is a connected set, then there exists a constant $C > 0$, depending only on V , such that

$$\sup_V u \leq C \inf_V u.$$

A.4.2 Jensen's Formula

Here, we recall Jensen's formula for holomorphic functions of one complex variable, a fundamental identity that follows from Poisson's formula. We use it in Chapter 5.

We start by considering Poisson's formula.

Theorem A.13 (Poisson's Formula, [Ah79, Theorem 22]). *Let $r > 0$ and let u be a harmonic function in $|z| < r$ and continuous on $|z| \leq r$. Then for every a with $|a| < r$,*

$$u(a) = \frac{1}{2\pi} \int_{|z|=r} \frac{r^2 - |a|^2}{|z - a|^2} u(z) d\theta. \quad (\text{A.10})$$

Remark A.14. Setting $a = 0$ in (A.10), gives the mean-value property

$$u(0) = \frac{1}{2\pi} \int_{|z|=r} \frac{r^2}{|z|^2} u(z) d\theta = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta. \quad (\text{A.11})$$

We now state Jensen's formula, which follows directly from Theorem A.13.

Theorem A.15 (Jensen's formula). *Let $r > 0$ and let f be a holomorphic function on a neighborhood of the closed disc $|z| \leq r$. Assume $f(0) \neq 0$ and let $a_1, a_2, \dots, a_n$ be the zeros of f in $|z| < r$, counted with multiplicity. Then the following equality is true*

$$\ln |f(0)| = - \sum_{i=1}^n \ln \left(\frac{r}{|a_i|} \right) + \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| d\theta. \quad (\text{A.12})$$

Proof. Step 1 (No zeros in $\overline{D}(0, r)$). If f has no zeros in $|z| \leq r$, then $\ln |f|$ is harmonic on $|z| < r$ and continuous in $|z| \leq r$. Thus, we can apply Theorem A.13 obtaining

$$\ln |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| d\theta. \quad (\text{A.13})$$

Step 2 (Zeros in $|z| = r$). Equality (A.13) remains valid if f has zeros in the circle $|z| = r$. Write

$$f(z) = \prod_{i=1}^m (z - re^{i\theta_i}) g(z),$$

where g is holomorphic and non-vanishing on $|z| \leq r$ and $\{re^{i\theta_i}\}_{i=1, \dots, m}$ are the zeros of f in the circle. Then we can apply (A.13) to g . Moreover we observe that for every $i \in \{1, \dots, m\}$, we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \ln |re^{i\theta} - re^{i\theta_i}| d\theta &= \ln(r) + \frac{1}{2\pi} \int_0^{2\pi} \ln |e^{i\theta} - e^{i\theta_i}| d\theta = \ln(r) + \frac{1}{2\pi} \int_{-\theta_i}^{2\pi-\theta_i} \ln |e^{i\phi} - 1| d\phi \\ &= \ln(r) + \frac{1}{2\pi} \int_0^{2\pi} \ln |e^{i\phi} - 1| d\phi, \end{aligned}$$

where we used the change of variables $\phi = \theta - \theta_i$. By recalling the classical identity

$$\int_0^\pi \ln |\sin(x)| dx = -\pi \ln(2),$$

and considering the change of variables $x = \frac{\phi}{2}$, we obtain

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \ln |e^{i\phi} - 1| d\phi &= \frac{1}{2\pi} \int_0^{2\pi} \ln(2) + \ln \left| \sin \left(\frac{\phi}{2} \right) \right| d\phi \\ &= \ln(2) + \frac{1}{\pi} \int_0^{\pi} \ln |\sin(x)| dx = 0, \end{aligned}$$

implying

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |re^{i\theta} - re^{i\theta_i}| d\theta = \ln(r),$$

for $i = 1, \dots, m$. Hence (A.13) is still valid for f .

Step 3 (General case: zeros in $|z| < r$). Suppose f has zeros in $|z| < r$ as in the hypotheses of the theorem and define

$$F(z) := f(z) \prod_{i=1}^n \frac{r^2 - \bar{a}_i z}{r(z - a_i)}.$$

Then F is holomorphic and free from zeros in $|z| < r$. For $|z| = r$ we have

$$|F(z)| = |f(z)| \prod_{i=1}^n \frac{|z\bar{z} - \bar{a}_i z|}{|r(z - a_i)|} = |f(z)| \prod_{i=1}^n \frac{r|\bar{z} - \bar{a}_i|}{r|z - a_i|} = |f(z)|.$$

Hence, by applying Step 2 to F , we get

$$\ln |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| d\theta. \quad (\text{A.14})$$

Finally, substituting the value of $F(0) = f(0) \prod_{i=1}^n \frac{r^2}{-ra_i}$ in (A.14), we obtain

$$\ln |f(0)| = - \sum_{i=1}^n \ln \left(\frac{r}{|a_i|} \right) + \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| d\theta.$$

□

A.5 General Tools from Analysis

In this section we collect some general analytic results used throughout the dissertation. We record the statements without proofs.

Proposition A.16 (Jacobi's formula). *Let $A : \mathbb{R} \rightarrow M_n$ be a differentiable map, where M_n denotes the space of complex $n \times n$ matrices. Then*

$$\frac{d}{dt} \det(A(t)) = \text{tr} \left(\text{adj}(A(t)) \frac{dA(t)}{dt} \right),$$

where $\text{adj}(A)$ denotes the adjugate of A , i.e. the transpose of its cofactor matrix

$$C = ((-1)^{i+j} M_{i,j})_{i,j=1}^n,$$

with $M_{i,j}$ the (i,j) -minor of A .

We now recall Markov's inequality, a classical result in measure theory. In particular, we consider an adaptation of [Ta11, Lemma 1.3.15] for functions of complex vectors.

Theorem A.17 (Markov's inequality). *Let $f : \mathbb{C}^D \rightarrow [0, +\infty]$ be measurable with respect to the Lebesgue measure λ and let $K \subset \mathbb{C}^D$ be an arbitrary subset. Then for any $0 < M < +\infty$, the following is true*

$$\lambda(\{z \in K : f(z) \geq M\}) \leq \frac{1}{M} \int_K f(z) \lambda(dz). \quad (\text{A.15})$$

Proof. First, we consider the trivial point-wise inequality

$$M \mathbb{1}_{\{z \in K : f(z) \geq M\}}(z) \leq f(z), \quad \forall z \in K.$$

Hence, by integrating over K both the sides of the inequality, we obtain

$$M \lambda(\{z \in K : f(z) \geq M\}) \leq \int_K f(z) \lambda(dz),$$

which proves (A.15). $\square$

We now state a version of the Portemanteau theorem (see [Du02]), adapted to the setting of finite measures. This result provides equivalent characterizations of the weak convergence $\mu_n \Rightarrow \mu$, whose definition is recalled in the statement.

Theorem A.18 (Portemanteu, [Du02, Theorem 11.1.1]). *Let (M, d) be a metric space and let μ and $\{\mu_n\}_{n \in \mathbb{N}}$ be finite Borel measures on (M, d) . Assume that $\{\mu_n\}_{n \in \mathbb{N}}$ are uniformly bounded in total mass. Then the following are equivalent*

1. $\mu_n \Rightarrow \mu$, for $n \rightarrow +\infty$, i.e.

$$\int f d\mu_n \xrightarrow{n \rightarrow +\infty} \int f d\mu \quad \text{for every } f \in C_b(M),$$

where $C_b(M)$ denotes the space of all bounded continuous real-valued functions on M .

2. For every open set $U \subset M$,

$$\liminf_{n \rightarrow +\infty} \mu_n(U) \geq \mu(U)$$

3. For every closed set $F \subset M$,

$$\limsup_{n \rightarrow +\infty} \mu_n(F) \leq \mu(F)$$

4. For every Borel set $A \subset M$ with $\mu(\partial A) = 0$ (a continuity set of μ),

$$\lim_{n \rightarrow +\infty} \mu_n(A) = \mu(A).$$

Corollary A.19. *Let (M, d) be a separable metric space. Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of random finite Borel measures on (M, d) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let μ be a finite Borel measure on (M, d) . Suppose that μ_n are almost surely uniformly bounded in total mass. If*

$$\mu_n \xrightarrow[n \rightarrow +\infty]{\Rightarrow} \mu, \quad \text{in probability,}$$

then, for every Borel set $A \subset M$ such that $\mu(\partial A) = 0$ (a continuity set of μ), one has

$$\mu_n(A) \xrightarrow[n \rightarrow +\infty]{} \mu(A) \quad \text{in probability.}$$

Proof. We view $\{\mu_n\}_{n \in \mathbb{N}}$ as a sequence of random variables that take value in the space of all finite Borel measures on (M, d) , denoted by $\mathcal{M}_f((M, d))$. This space, with the topology induced by the $\Rightarrow$ -convergence is metrizable, for example using the Prohorov metric, as defined in [Kl14, Chapter 13]. Hence, considering this structure, we can apply [Ka21, Lemma 5.2]. From any subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$ we can extract a further subsequence $\{\mu_{n_{j_k}}\}$ such that

$$\mu_{n_{j_k}} \xrightarrow[n \rightarrow +\infty]{\Rightarrow} \mu \quad \text{almost surely.}$$

Fix a Borel set $A \subset M$ with $\mu(\partial A) = 0$. For almost every $\omega \in \Omega$, the sequence of deterministic measures

$$v_k := \mu_{n_{j_k}}(\omega)$$

satisfies $v_k \Rightarrow \mu$. Hence, by Portemanteau theorem (Theorem A.18),

$$v_k(A) \longrightarrow \mu(A), \quad \text{i.e.} \quad \mu_{n_{j_k}}(\omega)(A) \longrightarrow \mu(A).$$

Therefore $\mu_{n_{j_k}}(A) \rightarrow \mu(A)$ almost surely. We have shown that every subsequence of $\mu_n(A)$ admits a further subsequence converging almost surely to $\mu(A)$.

Consequently, applying again [Ka21, Lemma 5.2], the entire sequence $\mu_n(A)$ converges to $\mu(A)$ in probability. $\square$

Bibliography

- [Ah79] L. Ahlfors. *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*, 3rd ed., McGraw-Hill, 1979.
- [Ar60] M. G. Arsove. *Continuous Potentials and Linear Mass Distributions*, SIAM Review 2 (1960), no. 3, 177–184.
- [Bi99] P. Billingsley. *Convergence of Probability Measures*, 2nd ed., Wiley Series in Probability and Statistics, Wiley, 1999.
- [ChZw10] T.J. Christiansen and M. Zworski. *Probabilistic Weyl Laws for Quantized Tori*, Communications in Mathematical Physics 299 (2010), 305–334.
- [Di93] M. Dimassi. *Développements asymptotiques des perturbations lentes de l'opérateur de Schrödinger périodique*, Communications in Partial Differential Equations 18 (1993), no. 5–6, 771–803.
- [DiSj99] M. Dimassi and J. Sjöstrand. *Spectral Asymptotics in the Semi-Classical Limit*, London Mathematical Society Lecture Note Series, vol. 268, Cambridge University Press, 1999.
- [Du02] R. M. Dudley. *Real Analysis and Probability*, Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, 2002.
- [Ev10] L. C. Evans. *Partial Differential Equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, 2010.
- [GoKr78] I. Gohberg and M. G. Kreĭn. *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space*, Translations of Mathematical Monographs, vol. 18, American Mathematical Society, 1969.
- [Ha06] M. Hager. *Instabilité spectrale semiclassique pour des opérateurs non-autoadjoints I: un modèle*, Annales de la Faculté des Sciences de Toulouse (6) 15 (2006), no. 2, 243–280.
- [HaSj08] M. Hager and J. Sjöstrand. *Eigenvalue asymptotics for randomly perturbed non-selfadjoint operators*, Mathematische Annalen 342 (2008), 177–243.
- [Hö68] L. Hörmander, *Fourier Integral Operators: Lectures at the Nordic Summer School of Mathematics*, 1969.

- [Hö83] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Grundlehren der mathematischen Wissenschaften, vol. 256, Springer-Verlag, 1983.
- [JeNa94] A. Jensen and S. Nakamura. *Mapping properties of functions of Schrödinger operators between L^p -spaces and Besov spaces*, Advanced Studies in Pure Mathematics 23 (1994), 187–209.
- [Ka21] O. Kallenberg. *Foundations of Modern Probability*, 3rd ed., Springer, 2021.
- [Kl14] F. Klenke. *Probability Theory: A Comprehensive Course*, 2nd ed, Springer, 2014.
- [Le18] Y. Le Floch. *A Brief Introduction to Berezin–Toeplitz Operators on Compact Kähler Manifolds*, CRM Short Courses, Springer Cham, 2018.
- [Ma02] A. Martinez. *An Introduction to Semiclassical and Microlocal Analysis*, Universitext, Springer, 2002.
- [Ma71] J. N. Mather. *On Nirenberg’s proof of Malgrange’s preparation theorem*, in *Proceedings of Liverpool Singularities, Symposium I*, Lecture Notes in Mathematics, vol. 192, Springer, 1971, 116–120.
- [Me23] C. D. Meyer. *Matrix Analysis and Applied Linear Algebra*, 2nd ed., Society for Industrial and Applied Mathematics, 2023.
- [NoZw07] S. Nonnenmacher and M. Zworski. *Distribution of resonances for open quantum maps*, Communications in Mathematical Physics 269 (2007), 311–365.
- [Ro55] H. Robbins. *A remark on Stirling’s formula*. The American Mathematical Monthly 62 (1955), no. 1, 26–29.
- [Sj10] J. Sjöstrand. *Counting zeros of holomorphic functions of exponential growth*, Journal of Pseudo-Differential Operators and Applications 1 (2010), no. 1, 75–100.
- [Sj09] J. Sjöstrand. *Eigenvalue distribution for non-self-adjoint operators with small multiplicative random perturbations*, Annales de la Faculté des Sciences de Toulouse (6), vol. 18 (2009), no. 4, 739–795.
- [SjVo21] J. Sjöstrand and M. Vogel. *Toeplitz band matrices with small random perturbations*, Indagationes Mathematicae 32 (2021), 275–322
- [SjZw07] J. Sjöstrand and M. Zworski. *Elementary linear algebra for advanced spectral problems*, Annales de l’Institut Fourier 57 (2007), 2095–2141.
- [StSh03] E. M. Stein and R. Shakarchi. *Complex Analysis*, Princeton Lectures in Analysis, vol. 2, Princeton University Press, 2003.
- [Tr06] F. Trèves. *Topological Vector Spaces, Distributions and Kernels*, Dover Publications, 2006.

- [Vo20] M. Vogel. *Almost sure Weyl law for quantized tori*, Communications in Mathematical Physics 378 (2020), 1539–1585.
- [Ta11] T. Tao. *An Introduction to Measure Theory*, Graduate Studies in Mathematics, vol. 126, American Mathematical Society, 2011.
- [Ta12] T. Tao. *Topics in Random Matrix Theory*, Graduate Studies in Mathematics, vol. 132, American Mathematical Society, 2012.
- [TaVu10] T. Tao and V. Vu. *Smooth analysis of the condition number and the least singular value*, Mathematics of Computation 79 (2010), no. 272, 2333–2352.
- [Zw12] M. Zworski. *Semiclassical Analysis*, Graduate Studies in Mathematics, vol. 138, American Mathematical Society, 2012.