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Minimizers for a nonlocal anisotropic energy

Supervisor

Prof. Berardo Ruffini

Author

Nicola Zavatta

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Abstract

In this work, we study the equilibrium configurations of a system of interacting particles. We focus on nonlocal interactions of Coulomb type modified with the addition of a generic anisotropic term and consider general confinements, in a bidimensional setting. Using a Γ -convergence argument, we prove that, in the many-particle limit, the equilibrium configurations are given by the minimizers of an energy functional on the space of probability measures. Then, we discuss the existence and uniqueness of the minimizer for such energy, and review a recent result on the explicit characterisation of the minimizer in the case of quadratic confinement. To address the case of general confinements, where the analytic solution is not known, we introduce a novel numerical method for the approximation of the minimizer. Eventually, in the light of the numerical results, we make a conjecture on the shape of the minimizer for quartic confinement.

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CHAPTER 1

Introduction

In this thesis, we study the minimization problem for the energy of a system of interacting particles. We assume that the particles interact with one another in such a way that every particle is influenced by every other particle in the system. Hence, the interaction energy of the system is nonlocal, in that even particles very far apart have a mutual influence. Interactions of this type arise for instance in materials science, when studying the dislocations in a crystalline material. Dislocations are defects in the crystal lattice that result in a change of the lattice structure and are responsible for many of the macroscopic properties of crystalline materials, such as plasticity and fracture toughness.

Given n particles located at $x^1, \dots, x^n \in \mathbb{R}^N$, the interaction energy of the system can be expressed as

$$\frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n W(x^i - x^j),$$

where W is a function that describes the interaction between two particles.

In general, the interaction may depend on the orientation between the particles, being stronger along some directions and weaker along others. In this case, the interaction is said to be anisotropic. In materials science, the arrangement of dislocations in a lattice is frequently observed to be anisotropic, which motivates the study of energies of this type.

In addition to the reciprocal interactions, the particles are subjected to a confinement V that forces them to stay within a bounded region of space. The confinement energy is

$$\frac{1}{n} \sum_{i=1}^n V(x^i).$$

If we assume that the particles have a repulsive behaviour at short range and an attractive one at longer range, it is intuitively clear that there exists a distribution of particles for which the system is in equilibrium. Identifying the distribution of particles with the probability measure

$$\mu_{x^1, \dots, x^n} = \frac{1}{n} \sum_{i=1}^n \delta_{x^i},$$

the equilibrium configurations can be found by computing the minimizers of the energy

$$E(\mu_{x^1, \dots, x^n}) = \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n W(x^i - x^j) + \frac{1}{n} \sum_{i=1}^n V(x^i),$$

under the assumption that:

- $W(x) \rightarrow +\infty$ as $|x| \rightarrow 0$, so that the energy blows up when the particles are too close;
- $V(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$, so that the energy blows up when the particles are too far apart.

The equilibrium configurations will be a trade-off between minimizing the interaction term, which keeps the particles separate from one another, and minimizing the confinement term, which tends to concentrate them in a narrow region.

In this work, we focus on the case $N = 2$, i.e. bidimensional distributions of particles, and consider interactions of the form

$$W(x) = -\ln|x| + \kappa(x),$$

where κ represents the anisotropic part of the interaction, in that $\kappa(x)$ depends only on the angle between x and the horizontal axis.

We will show that, if the number of particles is large, the energy of the system can be expressed by the functional $I: \mathcal{P}(\mathbb{R}^2) \rightarrow \mathbb{R} \cup \{+\infty\}$,

$$I(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} V(x) d\mu(x).$$

In particular, in Chapter 2 we generalise a result by Scagliotti [1] and prove that, under certain assumptions, I is the Γ -limit of the energies

$$I_n(\mu) = \begin{cases} E(\mu_{x^1, \dots, x^n}) & \text{if } \exists x^1, \dots, x^n \in \mathbb{R}^2 : \mu = \mu_{x^1, \dots, x^n} \\ +\infty & \text{otherwise} \end{cases}$$

as $n \rightarrow +\infty$. Therefore, the equilibrium configuration can be found by solving the minimization problem for the functional I .

In Chapter 3 we show that, under mild assumptions, this minimization problem admits a solution. In particular, the minimizer of I is compactly supported and satisfies two Euler-Lagrange equations (in a suitable capacitary sense). To prove that the minimizer is unique, we relate the convexity of the functional I to the sign of the Fourier transform of W . In fact, we show that, if the Fourier transform $\widehat{W}$ is nonnegative on $\mathbb{S}^1$, then I is strictly convex and the minimization problem has a unique solution. Then, we present a characterisation of the minimizers in the case of quadratic confinement, i.e. $V(x) = |x|^2$. The results discussed here follow the approach of a recent work by Mora [2]. We close the chapter with an example from dislocation theory. In particular, we address the minimization problem for the energy

$$I_\alpha(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(-\ln|x-y| + \alpha \frac{(x_1 - y_1)^2}{|x-y|} \right) d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x).$$

This example raises an interesting question on the topology of the equilibrium configuration. In fact, if $|\alpha| < 1$ the minimizer is supported on an ellipse, whereas if $|\alpha| \geq 1$ the support reduces to a segment, that is, the equilibrium configuration has a lower dimension. While it has already been proved that a necessary condition for this loss of dimensionality to occur is that $\widehat{W}$ vanishes somewhere on $\mathbb{S}^1$, a sufficient condition is still unknown. The matter becomes even more complicated if we take into account general confinements.

To shed some light on this question, we aim to compute the equilibrium configuration using numerical methods. In Chapter 4 we introduce a novel method to approximate the minimizer of I , which is based on Ritz method for the solution of variational problems. We approximate the minimizer by its projection onto the set

$$\langle \nu_1, \dots, \nu_n \rangle = \left\{ \mu \in \mathcal{P}(\mathbb{R}^2) : \mu = \sum_{k=1}^n a_k \nu_k, a_k \geq 0 \right\},$$

where $\nu_1, \dots, \nu_n$ are probability measures with compact support. This transforms the original problem into

$$\begin{cases} \min_{a_1, \dots, a_n} \sum_{k=1}^n a_k I(\nu_k) \\ a_k \geq 0 \quad \forall k \in \{1, \dots, n\} \\ \sum_{k=1}^n a_k = 1, \end{cases}$$

which is a finite-dimensional minimization problem that can be solved using numerical algorithms for constrained optimization.

We test the validity of the method by computing the minimizer of I_α and comparing the approximate result with the exact one. Then, we use this method to compute the equilibrium configuration for the energy I_α with a generalised confinement $V(x) = |x|^p$, for several values of α and p .

Finally, in Chapter 5 we conclude by summarising and discussing the main results.

CHAPTER 2

Equilibrium configurations for many-particle systems

In this chapter, we introduce the discrete energy associated with a finite number of particles. We will show that, if the number of particles tends to infinity, the discrete energy Γ -converges to a limit energy, whose minimizers are the limits of the equilibrium configurations for the discrete energy. Much of the material presented in this chapter follows the approach of an earlier work by Scagliotti [1].

2.1. Distributions of particles and the discrete energy

DEFINITION 1. Let $x^1, \dots, x^n \in \mathbb{R}^2$. We define the distribution of n particles located at $x^1, \dots, x^n$ as the probability measure

$$\mu_{x^1, \dots, x^n} = \frac{1}{n} \sum_{i=1}^n \delta_{x^i}. \quad (2.1.1)$$

DEFINITION 2. Let $\mu_{x^1, \dots, x^n}$ be a distribution of n particles, with $n \geq 2$. The energy corresponding to this distribution is

$$E(\mu_{x^1, \dots, x^n}) = \frac{1}{n^2} \sum_{\substack{i, j=1 \\ i \neq j}}^n W(x^i - x^j) + \frac{1}{n} \sum_{i=1}^n V(x^i), \quad (2.1.2)$$

where $W: \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ and $V: \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ are given functions, called *interaction kernel* and *confinement potential*, respectively.

The energy (2.1.2) is nonlocal, in that every particle interacts with every other particle, as can be seen by the interaction term summing over all pairs of particles. Energies of this form are representative, for instance, of the interaction between dislocations in a crystalline material.

In anisotropic media, the interaction between particles depends on their orientation. Hence, we may assume that the interaction kernel has the following form:

$$W(x) := \begin{cases} -\ln|x| + \kappa(x) & \text{if } x \neq 0 \\ +\infty & \text{if } x = 0, \end{cases} \quad (2.1.3)$$

where $\kappa: \mathbb{R}^2 \rightarrow \mathbb{R}$ is an even function with $\kappa(x) = \kappa\left(\frac{x}{|x|}\right)$, that represents the anisotropic behaviour of the interaction and is called *anisotropic kernel*.

In the isotropic case, the interaction reduces to the (2D) Coulomb kernel

$$W_0(x) := \begin{cases} -\ln|x| & \text{if } x \neq 0 \\ +\infty & \text{if } x = 0. \end{cases} \quad (2.1.4)$$

The energy as defined in (2.1.2) is a linear functional on the space of measures of the form (2.1.1). It can be extended to a functional on $\mathcal{P}(\mathbb{R}^2)$, the space of Borel probability measures over $\mathbb{R}^2$, via

$$I_n(\mu) := \begin{cases} E(\mu_{x^1, \dots, x^n}) & \text{if } \exists x^1, \dots, x^n \in \mathbb{R}^2 : \mu = \mu_{x^1, \dots, x^n} \\ +\infty & \text{otherwise.} \end{cases} \quad (2.1.5)$$

The energy I_n is also called *discrete energy*, as it is associated with a discrete distribution of particles.

A distribution of n particles that minimizes I_n , i.e. $\mu \in \mathcal{P}(\mathbb{R}^2)$ such that

$$I_n(\mu) = \min_{\mathcal{P}(\mathbb{R}^2)} I_n,$$

is called an *equilibrium configuration* for n particles.

Under suitable assumptions on the confinement potential, the discrete energies I_n admit an equilibrium configuration.

THEOREM 1. *Let I_n be defined as in (2.1.5), with W as in (2.1.3), $\kappa \in C(\mathbb{S}^1)$ and V lower semi-continuous, bounded from below and strongly coercive, i.e.*

$$\lim_{|x|+|y| \rightarrow +\infty} \left(W(x-y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) = +\infty.$$

Then I_n admits a minimizer on $\mathcal{P}(\mathbb{R}^2)$.

To prove this theorem, we will make use of the following result, which is the semi-continuous equivalent of Weierstrass' extreme value theorem for continuous functions.

THEOREM 2. *Let (X, d) be a metric space, $K \subseteq X$ compact and $f: X \rightarrow \mathbb{R}$ a lower semi-continuous function on K . Then, f is bounded from below on K . In particular, there exists $x_0 \in K$ such that $f(x_0) = \min_{x \in K} f(x)$.*

PROOF. Consider a minimizing sequence $\{x_n\}_n \subseteq K$ such that $f(x_n) \rightarrow \inf_{x \in K} f(x)$. Since K is compact,

$$\lim_{n \rightarrow +\infty} x_n = x_0 \in K$$

and, by lower semi-continuity of f , we have

$$f(x_0) \leq \liminf_{n \rightarrow +\infty} f(x_n) = \lim_{n \rightarrow +\infty} f(x_n) = \inf_{x \in K} f(x),$$

which concludes the proof. $\square$

We can now prove Theorem 1.

PROOF (THEOREM 1). We note that a minimizer of I_n must be of the form (2.1.1). Therefore, minimizing I_n is equivalent to minimizing the function $F: \mathbb{R}^{2n} \rightarrow \mathbb{R} \cup \{+\infty\}$,

$$\begin{aligned} F(x^1, \dots, x^n) &= \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n W(x^i - x^j) + \frac{1}{n} \sum_{i=1}^n V(x^i) \\ &= \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n \left(W(x^i - x^j) + \frac{1}{2}V(x^i) + \frac{1}{2}V(x^j) \right) + \frac{1}{n^2} \sum_{i=1}^n V(x^i). \end{aligned}$$

Clearly, F is lower semi-continuous on $\mathbb{R}^{2n}$, as W and V are continuous and lower semi-continuous on $\mathbb{R}^2$, respectively. Moreover, by Theorem 2 and by the assumptions on V , we have that $(x, y) \mapsto W(x-y) + \frac{1}{2}V(x) + \frac{1}{2}V(y)$ and V are bounded from below. Without loss of generality, we may thus assume that they are non-negative.

Since V is strongly coercive, there exists $K \subseteq \mathbb{R}^2$ compact such that

$$W(x-y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) > n^2(\inf F + 1) \quad \forall (x, y) \notin K \times K.$$

Therefore,

$$\begin{aligned} F(x^1, \dots, x^n) &= \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n \left(W(x^i - x^j) + \frac{1}{2}V(x^i) + \frac{1}{2}V(x^j) \right) + \frac{1}{n^2} \sum_{i=1}^n V(x^i) \\ &\geq \frac{1}{n^2} \left(W(x^i - x^j) + \frac{1}{2}V(x^i) + \frac{1}{2}V(x^j) \right) > \inf F + 1 \quad \forall (x^i, x^j) \notin K \times K, \end{aligned}$$

which implies $F(x^1, \dots, x^n) > \inf F + 1$ for every $(x^1, \dots, x^n) \notin K^n \subseteq \mathbb{R}^{2n}$. By Theorem 2, F has a minimum point $(x_0^1, \dots, x_0^n)$ belonging to the compact set K^n .

Hence, $\mu = \mu_{x_0^1, \dots, x_0^n}$ is a minimizer of I_n .

□

2.2. Γ -convergence of the discrete energy

We will now show that, under suitable assumptions, the family $\{I_n\}_{n \geq 2}$ Γ -converges to the energy functional $I: \mathcal{P}(\mathbb{R}^2) \rightarrow \mathbb{R} \cup \{+\infty\}$,

$$I(\mu) := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x - y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} V(x) d\mu(x). \quad (2.2.1)$$

The energy I can be thought of as the limit of the discrete energies $\{I_n\}_{n \geq 2}$ in the following sense: as the number of particles increases, the equilibrium configurations tend to a limit distribution, which is a minimizer of I .¹

We suppose that the confinement potential satisfies the following hypotheses:

- (H1) V is lower semi-continuous;
- (H2) V is bounded from below;
- (H3) strong coercivity

$$\lim_{|x|+|y| \rightarrow +\infty} \left(W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) = +\infty \quad (2.2.2)$$

- (H4) $\text{dom}(V) := \{x \in \mathbb{R}^2 : V(x) < +\infty\}$ is closed and with non-empty interior;
- (H5) there exists a point $z \in \text{dom}(V)$ such that for every $K \subseteq \text{dom}(V)$ compact and for every $t \in (0, 1)$ the set $z + t(K - z)$ is contained in $\text{dom}(V)$;
- (H6) V is continuous on $\text{dom}(V)$.

Examples of admissible confinements include the power law $V(x) = |x|^p$ with $p > 0$ and the characteristic function of a compact convex set.

Under these assumptions, we can prove the following theorem, that generalises an analogous result by Scagliotti [1].

THEOREM 3. *Let $\{I_n\}_{n \geq 2}$ be the family of functionals defined in (2.1.5), with W as in (2.1.3), $\kappa \in C(\mathbb{S}^1)$ and V satisfying (H1)-(H6). Then, the functional $I: \mathcal{P}(\mathbb{R}^2) \rightarrow \mathbb{R} \cup \{+\infty\}$,*

$$I(\mu) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d\mu(x) d\mu(y) \quad (2.2.3)$$

is well-defined and we have

$$I_n \xrightarrow{\Gamma} I.$$

¹For a short introduction to Γ -convergence, we refer to Appendix B.

Before proving the theorem, we will prove the following useful lemmas.

LEMMA 1. *Let $\{\mu_n\}_n \subseteq \mathcal{P}(\mathbb{R}^2)$. If $\mu_n \xrightarrow{*} \mu$, then $\mu_n \otimes \mu_n \xrightarrow{*} \mu \otimes \mu$. Moreover, suppose that*

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{x^i}$$

and define

$$\mu_n \boxtimes \mu_n := \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n \delta_{(x^i, x^j)}.$$

Then, if $\mu_n \xrightarrow{*} \mu$, we have that $\mu_n \boxtimes \mu_n \xrightarrow{*} \mu \otimes \mu$.

PROOF. Let $\psi \in C_b(\mathbb{R}^2 \times \mathbb{R}^2)$ and $0 < \varepsilon < 1$. By weak convergence, we have

$$\left| \int_{\mathbb{R}^2} \varphi d(\mu_n - \mu) \right| < \varepsilon \quad \forall \varphi \in C_b(\mathbb{R}^2)$$

for n large enough. Since $(x \mapsto \psi(x, y)), (y \mapsto \psi(x, y)) \in C_b(\mathbb{R}^2)$, then

$$\begin{aligned} \left| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi d(\mu_n \otimes \mu_n) - \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi d(\mu \otimes \mu) \right| &= \left| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi d((\mu_n - \mu) \otimes \mu_n) \right. \\ &\quad \left. + \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi d(\mu_n \otimes (\mu_n - \mu)) + \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi d((\mu_n - \mu) \otimes (\mu_n - \mu)) \right| \\ &< \left| 2 \int_{\mathbb{R}^2} \varepsilon d\mu_n + \int_{\mathbb{R}^2} \varepsilon d(\mu_n - \mu) \right| < 3\varepsilon, \end{aligned}$$

which proves the first claim.

Moreover, we note that

$$\mu_n \boxtimes \mu_n = \mu_n \otimes \mu_n - \nu_n, \quad \text{with } \nu_n = \frac{1}{n^2} \sum_{i=1}^n \delta_{(x^i, x^i)},$$

and $\nu_n \xrightarrow{*} 0$. Indeed,

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi d\nu_n = \frac{1}{n^2} \sum_{i=1}^n \varphi(x^i, x^i) \longrightarrow 0 \quad \forall \varphi \in C_b(\mathbb{R}^2 \times \mathbb{R}^2),$$

which concludes the proof. $\square$

LEMMA 2. *Let I be defined as in Theorem 3. Then, for every $\mu \in \mathcal{P}(\mathbb{R}^2)$ there exists a sequence $\{\mu_n\}_n \subseteq \mathcal{P}_c(\mathbb{R}^2)$, whose support is contained in $\text{dom}(V)$, such that*

- $\mu_n \xrightarrow{*} \mu$;
- $I(\mu_n) \leq I(\mu)$ for n large enough.

PROOF. Let $\mu \in \mathcal{P}(\mathbb{R}^2)$ and suppose that it is not compactly supported. We will first prove that it can be approximated by a sequence of measures with compact support that satisfy the thesis.

We consider an exhaustion by compact sets of $\mathbb{R}^2$, i.e. a sequence $\{K_n\}_{n=1}^{+\infty} \subseteq \mathbb{R}^2$ of compact sets such that $K_n \subseteq K_{n+1}$ for every n and $\bigcup_{n=1}^{+\infty} K_n = \mathbb{R}^2$. Without loss of generality, suppose that $\mu(K_1) > 0$. We can then define

$$\mu_n = \frac{1}{\mu(K_n)} \mu|_{K_n} \in \mathcal{P}_c(\mathbb{R}^2) \quad \text{for every } n \geq 1.$$

Let $\varphi \in C_b(\mathbb{R}^2)$, with $|\varphi(x)| \leq M$ for every $x \in \mathbb{R}^2$. We have

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} \varphi d\mu_n - \int_{\mathbb{R}^2} \varphi d\mu \right| &= \left| \int_{K_n} \frac{\varphi}{\mu(K_n)} d\mu - \int_{\mathbb{R}^2} \varphi d\mu \right| \\
&\leq \left| \int_{K_n} \frac{\varphi}{\mu(K_n)} d\mu - \int_{K_n} \varphi d\mu \right| + \left| \int_{K_n^c} \varphi d\mu \right| \\
&\leq \int_{K_n} \left| \left(\frac{1}{\mu(K_n)} - 1 \right) \varphi \right| d\mu + \int_{K_n^c} |\varphi| d\mu \\
&\leq \left(\frac{1}{\mu(K_n)} - 1 \right) M \mu(K_n) + M \mu(K_n^c) \\
&= 2(1 - \mu(K_n))M < \varepsilon
\end{aligned}$$

by choosing n large enough that $\mu(K_n) > 1 - \frac{\varepsilon}{2M}$, with $\varepsilon > 0$. Hence, $\mu_n \xrightarrow{*} \mu$.

As for the second point, we may suppose $I(\mu) < +\infty$, otherwise the conclusion is trivial. We have

$$\begin{aligned}
I(\mu) &= \iint_{K_n \times K_n} \left(W(x-y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d(\mu \otimes \mu)(x, y) \\
&\quad + \iint_{(K_n \times K_n)^c} \left(W(x-y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d(\mu \otimes \mu)(x, y) \\
&\geq \mu(K_n)^2 I(\mu_n) + C(\mu \otimes \mu)((K_n \times K_n)^c) \\
&= \mu(K_n)^2 I(\mu_n) + C(1 - \mu(K_n)^2),
\end{aligned}$$

where in the second line we have used the fact that, by (2.2.2) and $\bigcup_{n=1}^{+\infty} K_n = \mathbb{R}^2$, for every $C > 0$ there exists $\bar{n} \in \mathbb{N}$ such that

$$W(x-y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \geq C \text{ for every } (x, y) \notin K_n \times K_n, \ n > \bar{n}.$$

By choosing $C = I(\mu)$, we obtain

$$I(\mu_n) \leq I(\mu) \text{ for every } n > \bar{n},$$

which proves the second point.

Let now μ be a probability measure with compact support $K \subseteq \text{dom}(V)$ and such that $I(\mu) < +\infty$. Without loss of generality, we may assume that (H5) is fulfilled for $z = 0$ up to a translation. We define the scaling map $\Psi^t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$\Psi^t(x) = tx \text{ for } t \in (0, 1),$$

and consider the push-forward measure

$$(\Psi_*^t \mu)(A) := \mu\left((\Psi^t)^{-1}(A)\right) = \mu\left(\frac{1}{t}A\right) \quad \forall A \in \mathcal{B}(\mathbb{R}^2),$$

which is supported in $tK \subseteq \text{dom}^\circ(V)$.

First we prove that $\Psi_*^t \mu \xrightarrow{*} \mu$ as $t \rightarrow 1$. Let $\varepsilon > 0$, $\varphi \in C_b(\mathbb{R}^2)$ and

$$K' := \overline{\bigcup_{t \in (0, 1)} tK}.$$

We have

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} \varphi(x) d\mu(x) - \int_{\mathbb{R}^2} \varphi(x) d(\Psi_*^t \mu)(x) \right| &= \left| \int_K \varphi(x) d\mu(x) - \int_{tK} \varphi(x) d(\Psi_*^t \mu)(x) \right| \\
&= \left| \int_K \varphi(x) d\mu(x) - \int_K \varphi(\Psi^t(x)) d\mu(x) \right| \\
&= \left| \int_K \varphi(x) - \varphi(tx) d\mu(x) \right| \\
&\leq \int_K |\varphi(x) - \varphi(tx)| d\mu(x).
\end{aligned}$$

By uniform continuity of φ on K' , there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(tx)| < \varepsilon \quad \forall x \in K' : |(1-t)x| < \delta,$$

and by boundedness of K' there exists $C > 0$ such that $|x| < C$. Hence, for $t > \frac{\delta}{C} - 1$ we obtain

$$\int_K |\varphi(x) - \varphi(tx)| d\mu(x) < \int_K \varepsilon d\mu(x) = \varepsilon.$$

We are left to prove that $\lim_{t \rightarrow 1} I(\Psi_*^t \mu) = I(\mu)$. We note that

$$W(t(x-y)) = -\ln|x-y| - \ln t + \kappa(x-y) = W(x-y) - \ln t.$$

Hence,

$$\begin{aligned}
I(\Psi_*^t \mu) &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d(\Psi_*^t \mu \otimes \Psi_*^t \mu)(x, y) + \int_{\mathbb{R}^2} V(x) d(\Psi_*^t \mu)(x) \\
&= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(t(x-y)) d(\mu \otimes \mu)(x, y) + \int_{\mathbb{R}^2} V(tx) d\mu(x) \\
&= \iint_{K \times K} W(x-y) d(\mu \otimes \mu)(x, y) + \int_K V(tx) d\mu(x) - \ln t.
\end{aligned}$$

Since $K' \subseteq \text{dom}(V)$, then $V|_{K'}$ is continuous and $\sup_{K'} V < +\infty$. Therefore, by dominated convergence we conclude that

$$\lim_{t \rightarrow 1} I(\Psi_*^t \mu) = I(\mu).$$

□

We can now prove the theorem.

PROOF (THEOREM 3). (liminf inequality) Let $\mu \in \mathcal{P}(\mathbb{R}^2)$ and $\{\mu_n\}_n \subseteq \mathcal{P}(\mathbb{R}^2)$ such that $\mu_n \xrightarrow{*} \mu$. We need to show that

$$I(\mu) \leq \liminf_{n \rightarrow +\infty} I_n(\mu_n).$$

Without loss of generality, we may assume $\liminf_{n \rightarrow +\infty} I_n(\mu_n) < +\infty$ and $I_n(\mu_n) < +\infty$ for every n . Therefore, for every n there exist $x^1, \dots, x^n \in \text{dom}(V)$ such that

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{x^i}.$$

Moreover, we can assume $V \geq 0$, up to a translation. By definition, we have

$$\begin{aligned} I_n(\mu_n) &= \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n W(x^i - x^j) + \frac{1}{n} \sum_{i=1}^n V(x^i) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W(x - y) d(\mu_n \boxtimes \mu_n)(x, y) + \int_{\mathbb{R}^2} V(x) d\mu_n(x) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d(\mu_n \boxtimes \mu_n)(x, y) + \frac{1}{n^2} \sum_{i=1}^n V(x^i) \\ &\geq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d(\mu_n \boxtimes \mu_n)(x, y). \end{aligned}$$

Lemma 1 ensures that $\mu_n \boxtimes \mu_n \xrightarrow{*} \mu \otimes \mu$ and, since $(x, y) \mapsto W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y)$ is lower semi-continuous and bounded from below, by Lemma 7 we deduce

$$\begin{aligned} I(\mu) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d(\mu \otimes \mu)(x, y) \\ &\leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d(\mu_n \boxtimes \mu_n)(x, y) \leq \liminf_{n \rightarrow +\infty} I_n(\mu_n), \end{aligned}$$

which is the desired inequality.

(limsup inequality) We have to prove that for every $\mu \in \mathcal{P}(\mathbb{R}^2)$ there exists a recovery sequence $\{\mu_n\}_n \subseteq \mathcal{P}(\mathbb{R}^2)$ such that $\mu_n \xrightarrow{*} \mu$ and

$$I(\mu) \geq \limsup_{n \rightarrow +\infty} I_n(\mu_n).$$

Without loss of generality, we may assume that $I(\mu) < +\infty$, otherwise $\mu_n = \mu$ would be a recovery sequence.

By virtue of Lemma 2, it is enough to prove that such a recovery sequence exists for every compactly supported probability measure whose support is contained in the interior of $\text{dom}(V)$. Indeed, let $\mu \in \mathcal{P}(\mathbb{R}^2)$ and consider a sequence $\{\mu_n\}_n \subseteq \mathcal{P}_c(\mathbb{R}^2)$, with support contained in $\text{dom}(V)$, such that $\mu_n \xrightarrow{*} \mu$ and $I(\mu) \geq I(\mu_n)$ for $n > \bar{n}$. Suppose that $\{\mu_m^{(n)}\}_m \subseteq \mathcal{P}(\mathbb{R}^2)$ is a recovery sequence for μ_n , i.e. $\mu_m^{(n)} \xrightarrow{*} \mu_n$ and

$$I(\mu_n) \geq \limsup_{m \rightarrow +\infty} I_m(\mu_m^{(n)}).$$

The diagonal sequence $\{\mu_n^{(n)}\}_n$ satisfies $\mu_n^{(n)} \xrightarrow{*} \mu$ and

$$I(\mu) \geq \limsup_{n \rightarrow +\infty} I_n(\mu_n^{(n)}),$$

therefore it is a recovery sequence for μ .

Step 1 (approximation of μ): Let $\mu \in \mathcal{P}_c(\mathbb{R}^2)$, with support K . We consider a covering of $\mathbb{R}^2$ by squares of side length $2h$,

$$\mathcal{Q}^h := \{[0, 2h]^2 + 2h(i, j) : (i, j) \in \mathbb{Z}^2\},$$

and denote by $\{\tilde{Q}_k^h\}_{k=1, \dots, N_h}$ the squares in $\mathcal{Q}^h$ whose intersection with K is non-empty.

For every $k = 1, \dots, N_h$, we define the square

$$Q_k^h := \left\{ x \in \tilde{Q}_k^h : x + \lambda_1 e_1 + \lambda_2 e_2 \in \tilde{Q}_k^h, 0 \leq \lambda_1, \lambda_2 \leq h \right\}.$$

We approximate μ with the sequence $\{\mu^h\}_h \subseteq \mathcal{P}_c(\mathbb{R}^2)$,

$$\mu^h := \sum_{k=1}^{N_h} \frac{\mu(\tilde{Q}_k^h)}{h^2} \mathcal{L}^2|_{Q_k^h}. \quad (2.2.4)$$

It can be noticed that $\mu^h(\tilde{Q}_k^h) = \mu^h(Q_k^h) = \mu(\tilde{Q}_k^h)$ for every k .

We claim that:

- $\mu^h \xrightarrow{*} \mu$ as $h \rightarrow 0$;
- $I(\mu) \geq \limsup_{h \rightarrow 0} I(\mu^h)$.

For $h < 1$, $\text{supp}(\mu^h) \subseteq \overline{K + B_{\sqrt{2}}(0)} =: K'$. Let $\varphi \in C_b(\mathbb{R}^2)$ and $\varepsilon > 0$. By uniform continuity of φ on K' , there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(y)| < \varepsilon \quad \forall x, y \in K' : |x - y| < \delta.$$

We thus have

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \varphi d\mu^h - \int_{\mathbb{R}^2} \varphi d\mu \right| &= \left| \int_{K'} \varphi d\mu^h - \int_{K'} \varphi d\mu \right| \\ &= \left| \sum_{k=1}^{N_h} \left(\int_{\tilde{Q}_k^h} \varphi(x) d\mu^h(x) - \int_{\tilde{Q}_k^h} \varphi(y) d\mu(y) \right) \right| \\ &= \left| \sum_{k=1}^{N_h} \left(\int_{\tilde{Q}_k^h} \int_{\tilde{Q}_k^h} \frac{\varphi(x)}{\mu(\tilde{Q}_k^h)} d\mu^h(x) d\mu(y) - \int_{\tilde{Q}_k^h} \int_{\tilde{Q}_k^h} \frac{\varphi(y)}{\mu^h(\tilde{Q}_k^h)} d\mu(y) d\mu^h(x) \right) \right| \\ &\leq \sum_{k=1}^{N_h} \frac{1}{\mu(\tilde{Q}_k^h)} \left| \int_{\tilde{Q}_k^h} \int_{\tilde{Q}_k^h} (\varphi(x) - \varphi(y)) d\mu^h(x) d\mu(y) \right| \\ &\leq \sum_{k=1}^{N_h} \frac{1}{\mu(\tilde{Q}_k^h)} \int_{\tilde{Q}_k^h} \int_{\tilde{Q}_k^h} |\varphi(x) - \varphi(y)| d\mu^h(x) d\mu(y) \\ &\leq \sum_{k=1}^{N_h} \frac{1}{\mu(\tilde{Q}_k^h)} \int_{\tilde{Q}_k^h} \int_{\tilde{Q}_k^h} \varepsilon d\mu^h(x) d\mu(y) = \sum_{k=1}^{N_h} \varepsilon \mu(\tilde{Q}_k^h) = \varepsilon, \end{aligned}$$

where in the fourth line we have used $\mu^h(\tilde{Q}_k^h) = \mu(\tilde{Q}_k^h)$ and applied Fubini's theorem, while the last inequality descends from the fact that, for $x, y \in \tilde{Q}_k^h$, $|x - y| \leq 2\sqrt{2}h < \delta$ as $h \rightarrow 0$, therefore by uniform continuity $|\varphi(x) - \varphi(y)| < \varepsilon$.

To prove the second claim, we first notice that $\text{supp}(\mu^h) \subseteq K + B_{2\sqrt{2}h}(0)$, so for h sufficiently small there exists an open set $\Omega \subset\subset \text{dom}(V)$ such that $\text{supp}(\mu^h) \subseteq \Omega$. Hence,

$$\int_{\mathbb{R}^2} V d\mu^h = \int_{\Omega} V d\mu^h \longrightarrow \int_{\Omega} V d\mu = \int_{\mathbb{R}^2} V d\mu,$$

because $\mu^h \xrightarrow{*} \mu$ and $V|_{\Omega} \in C_b(\Omega)$.

Since W is unbounded on $\mathbb{R}^2$, we define its truncation

$$W_M(x) := \min\{W(x), M\}$$

and write

$$\begin{aligned} \limsup_{h \rightarrow 0} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W(x-y) d\mu^h(x) d\mu^h(y) &= \limsup_{h \rightarrow 0} \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (W - W_M)(x-y) d\mu^h(x) d\mu^h(y) \right. \\ &\quad \left. + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_M(x-y) d\mu^h(x) d\mu^h(y) \right) \\ &\leq \limsup_{h \rightarrow 0} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (W - W_M)(x-y) d\mu^h(x) d\mu^h(y) \\ &\quad + \limsup_{h \rightarrow 0} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_M(x-y) d\mu^h(x) d\mu^h(y). \end{aligned}$$

We note that, thanks to the assumption $\kappa \in C(\mathbb{S}^1)$, W_M is continuous and bounded on $\mathbb{R}^2$, therefore by narrow convergence of μ^h we obtain

$$\begin{aligned} \limsup_{h \rightarrow 0} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_M(x-y) d\mu^h(x) d\mu^h(y) &= \lim_{h \rightarrow 0} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_M(x-y) d\mu^h(x) d\mu^h(y) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_M(x-y) d\mu(x) d\mu(y) \\ &\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W(x-y) d\mu(x) d\mu(y). \end{aligned}$$

We will conclude by showing that

$$\limsup_{h \rightarrow 0} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (W - W_M)(x-y) d\mu^h(x) d\mu^h(y) \rightarrow 0 \text{ as } M \rightarrow +\infty.$$

Let $C_1, C_2 \in \mathbb{R}$ be such that

$$-\ln|x-y| + C_1 \leq W(x-y) \leq -\ln|x-y| + C_2.$$

Hence,

$$W(x-y) > M \implies -\ln|x-y| + C_2 > M \iff |x-y| < e^{C_2-M} =: R_M$$

and $R_M \rightarrow 0$ as $M \rightarrow +\infty$. As a result,

$$(W - W_M)(x-y) = 0 \text{ if } |x-y| \geq R_M. \quad (2.2.5)$$

We have

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (W - W_M)(x-y) d\mu^h(x) d\mu^h(y) &= \int_{\Omega} \left(\int_{B_{R_M}(y)} (W - W_M)(x-y) d\mu^h(x) \right) d\mu^h(y) \\ &\leq \int_{\Omega} \left(\int_{B_{R_M}(y)} W(x-y) d\mu^h(x) \right) d\mu^h(y) \\ &= \sum_{i,j=1}^{N_h} \int_{Q_i^h} \left(\int_{Q_j^h \cap B_{R_M}(y)} W(x-y) d\mu^h(x) \right) d\mu^h(y) \\ &= \sum_{i=1}^{N_h} \sum_{\substack{j=1 \\ j \neq i}}^{N_h} \int_{Q_i^h} \left(\int_{Q_j^h \cap B_{R_M}(y)} W(x-y) d\mu^h(x) \right) d\mu^h(y) \\ &\quad + \sum_{i=1}^{N_h} \int_{Q_i^h} \left(\int_{Q_i^h \cap B_{R_M}(y)} W(x-y) d\mu^h(x) \right) d\mu^h(y). \end{aligned}$$

We rewrite the right-hand side of the previous inequality by arranging the squares $\{Q_j^h\}_{j=1,\dots,N_h}$ depending on their distance from Q_i^h , according to the set of indices

$$J_{i,p}^h := \left\{ j \in \{1, \dots, N_h\} : Q_j^h = Q_i^h + 2h(m, n), (m, n) \in \mathbb{Z}^2, \max\{|m|, |n|\} = p \right\},$$

where $p = 1, \dots, P_h$, with $P_h := \left\lceil \frac{R_M}{2h} \right\rceil$. Hence, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (W - W_M)(x - y) d\mu^h(x) d\mu^h(y) &\leq \sum_{i=1}^{N_h} \sum_{p=1}^{P_h} \sum_{j \in J_{i,p}^h} \int_{Q_i^h} \left(\int_{Q_j^h \cap B_{R_M}(y)} W(x - y) d\mu^h(x) \right) d\mu^h(y) \\ &\quad + \sum_{i=1}^{N_h} \int_{Q_i^h} \left(\int_{Q_i^h \cap B_{R_M}(y)} W(x - y) d\mu^h(x) \right) d\mu^h(y). \end{aligned} \quad (2.2.6)$$

If $x \in Q_i^h$ and $y \in Q_j^h$, with $j \in J_{i,p}^h$, then $(2p - 1)h \leq |x - y| \leq (p + 1)2\sqrt{2}h$, so

$$C_1 - \ln((p + 1)2\sqrt{2}h) \leq W(x - y) \leq C_2 - \ln((2p - 1)h).$$

On the other hand, if $x, y \in \tilde{Q}_i^h$, then $|x - y| \leq 2\sqrt{2}h$, so

$$W(x - y) \geq C_1 - \ln(2\sqrt{2}h).$$

Let us fix $i \in \{1, \dots, N_h\}$, $p \in \{1, \dots, P_h\}$ and $j \in J_{i,p}^h$. We have

$$\begin{aligned} \int_{Q_i^h} \left(\int_{Q_j^h \cap B_{R_M}(y)} W(x - y) d\mu^h(x) \right) d\mu^h(y) &\leq \int_{Q_i^h} \left(\int_{Q_j^h \cap B_{R_M}(y)} (C_2 - \ln((2p - 1)h)) d\mu^h(x) \right) d\mu^h(y) \\ &\leq (C_2 - \ln((2p - 1)h)) \mu^h(Q_i^h) \mu^h(Q_j^h) \\ &= \left(C_2 - \ln\left(\frac{2p - 1}{(p + 1)2\sqrt{2}}\right) - \ln((p + 1)2\sqrt{2}h) \right) \mu(\tilde{Q}_i^h) \mu(\tilde{Q}_j^h) \\ &\leq C \mu(\tilde{Q}_i^h) \mu(\tilde{Q}_j^h) + \int_{\tilde{Q}_i^h} \left(\int_{\tilde{Q}_j^h} W(x - y) d\mu(x) \right) d\mu(y), \end{aligned}$$

where C is a constant independent of i, j, p and h . Moreover,

$$\begin{aligned} \int_{Q_i^h} \left(\int_{Q_i^h \cap B_{R_M}(y)} W(x - y) d\mu^h(x) \right) d\mu^h(y) &\leq \int_{Q_i^h} \left(\int_{Q_i^h} (C_2 - \ln|x - y|) d\mu^h(x) \right) d\mu^h(y) \\ &\leq C_2 \mu^h(Q_i^h)^2 - \int_{Q_i^h} \left(\int_{B_{\sqrt{2}h}(y)} \ln|x - y| d\mu^h(x) \right) d\mu^h(y) \\ &= C_2 \mu(\tilde{Q}_i^h)^2 - \frac{\mu(\tilde{Q}_i^h)}{h^2} \int_{Q_i^h} \left(2\pi \int_0^{\sqrt{2}h} \ln(r) r dr \right) d\mu^h(y) \\ &= \mu(\tilde{Q}_i^h)^2 \left(C_2 - \frac{2\pi}{h^2} \int_0^{\sqrt{2}h} \ln(r) r dr \right) \\ &= \mu(\tilde{Q}_i^h)^2 \left(C_2 - \frac{\pi}{h^2} \left[r^2 \ln(r) - \frac{r^2}{2} \right]_0^{\sqrt{2}h} \right) \\ &= C' \mu(\tilde{Q}_i^h)^2 + 2\pi (C_1 - \ln(2\sqrt{2}h)) \mu(\tilde{Q}_i^h)^2 \\ &\leq C' \mu(\tilde{Q}_i^h)^2 + 2\pi \int_{\tilde{Q}_i^h} \left(\int_{\tilde{Q}_i^h} W(x - y) d\mu(x) \right) d\mu(y), \end{aligned}$$

where C' is a constant independent of i and h . By substituting into (2.2.6), we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (W - W_M)(x - y) d\mu^h(x) d\mu^h(y) &\leq \sum_{i=1}^{N_h} \sum_{p=1}^{P_h} \sum_{j \in J_{i,p}^h} \left(C\mu(\tilde{Q}_i^h) \mu(\tilde{Q}_j^h) \right. \\ &\quad \left. + \int_{\tilde{Q}_i^h} \left(\int_{\tilde{Q}_j^h} W(x - y) d\mu(x) \right) d\mu(y) \right) + \sum_{i=1}^{N_h} \left(C'\mu(\tilde{Q}_i^h)^2 + 2\pi \int_{\tilde{Q}_i^h} \left(\int_{\tilde{Q}_i^h} W(x - y) d\mu(x) \right) d\mu(y) \right) \\ &\leq C'' \int_{\Omega} \mu(B_{R_M+2h}(y)) d\mu(y) + 2\pi \int_{\Omega} \int_{B_{R_M+2h}(y)} W(x - y) d\mu(x) d\mu(y). \end{aligned}$$

Hence,

$$\begin{aligned} \limsup_{h \rightarrow 0} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (W - W_M)(x - y) d\mu^h(x) d\mu^h(y) &\leq \lim_{h \rightarrow 0} \left(C'' \int_{\Omega} \mu(B_{R_M+2h}(y)) d\mu(y) \right. \\ &\quad \left. + 2\pi \int_{\Omega} \int_{B_{R_M+2h}(y)} W(x - y) d\mu(x) d\mu(y) \right) \\ &= C'' \int_{\Omega} \mu(B_{R_M}(y)) d\mu(y) \\ &\quad + 2\pi \int_{\Omega} \int_{B_{R_M}(y)} W(x - y) d\mu(x) d\mu(y), \end{aligned}$$

because by assumption $I(\mu) < +\infty$, therefore $(x, y) \mapsto W(x - y) \in L^1(\mathbb{R}^2 \times \mathbb{R}^2, \mu \otimes \mu)$ and we can apply the dominated convergence theorem. Recalling that $R_M \rightarrow 0$ as $M \rightarrow +\infty$, we conclude

$$\lim_{M \rightarrow +\infty} \limsup_{h \rightarrow 0} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (W - W_M)(x - y) d\mu^h(x) d\mu^h(y) = 0.$$

Step 2 (construction of the recovery sequence): it is sufficient to construct a recovery sequence for measures of the form (2.2.4). We approximate μ^h by a sequence

$$\mu_n^h := \sum_{k=1}^{N_h} c_{k,n} \mathcal{L}^2|_{Q_k^h}, \quad (2.2.7)$$

where the coefficients $c_{k,n}$ are such that $\mu_n^h(Q_k^h) \rightarrow \mu^h(Q_k^h)$ as $n \rightarrow +\infty$ and $\sqrt{\mu_n^h(Q_k^h)n} \in \mathbb{N}$ for every $n \in \mathbb{N}$ and every k . This can be obtained by choosing

$$c_{k,n} := \frac{\left[\sqrt{\mu^h(Q_k^h)n} \right]^2}{h^2 n}.$$

The sequence $\{\mu_n^h\}_n$ is narrowly convergent to μ^h . In fact, let $\varphi \in C_b(\mathbb{R}^2)$ and observe that

$$\frac{\mu^h(Q_k^h)}{h^2} + \frac{1 - 2\sqrt{\mu^h(Q_k^h)n}}{h^2 n} = \frac{\left(\sqrt{\mu^h(Q_k^h)n} - 1 \right)^2}{h^2 n} \leq c_{k,n} \leq \frac{\mu^h(Q_k^h)}{h^2}, \quad (2.2.8)$$

therefore by dominated convergence

$$\int_{\mathbb{R}^2} \varphi(x) d\mu_n^h(x) = \sum_{k=1}^{N_h} \int_{Q_k^h} \varphi(x) c_{k,n} dx \longrightarrow \sum_{k=1}^{N_h} \int_{Q_k^h} \varphi(x) \frac{\mu^h(Q_k^h)}{h^2} dx = \int_{\mathbb{R}^2} \varphi(x) d\mu^h(x).$$

We note, however, that in general the measures μ_n^h are not probability measures. In fact,

$$\mu_n^h(\mathbb{R}^2) = \sum_{k=1}^{N_h} \int_{Q_k^h} c_{k,n} dx \leq \sum_{k=1}^{N_h} \int_{Q_k^h} \frac{\mu^h(Q_k^h)}{h^2} dx = 1.$$

We subdivide each square Q_k^h into $Z_{k,n} := \mu_n^h(Q_k^h)n$ subsquares of side length

$$L_{k,n} := \frac{h}{\sqrt{Z_{k,n}}}.$$

The subsquares are denoted by $Q_{k,i}^h$ and their corresponding centres by $w_{k,n}^i$, with $i = 1, \dots, Z_{k,n}$.

For a fixed n , there are in total $\mu_n^h(\mathbb{R}^2)n$ centres $w_{k,n}^i$. We denote $n_r := (1 - \mu_n^h(\mathbb{R}^2))n$. By inequality (2.2.8), we get

$$\begin{aligned} \mu_n^h(\mathbb{R}^2) &= \sum_{k=1}^{N_h} \mu_n^h(Q_k^h) = \sum_{k=1}^{N_h} c_{k,n} h^2 \geq \sum_{k=1}^{N_h} \left(\mu^h(Q_k^h) + \frac{1 - 2\sqrt{\mu^h(Q_k^h)n}}{n} \right) \\ &= 1 + \frac{N_h}{n} - \frac{2}{\sqrt{n}} \sum_{k=1}^{N_h} \sqrt{\mu^h(Q_k^h)n}, \end{aligned}$$

from which we obtain the estimate

$$n_r = \left(1 - \mu_n^h(\mathbb{R}^2)\right)n \leq -N_h + 2\sqrt{n} \sum_{k=1}^{N_h} \sqrt{\mu^h(Q_k^h)n} \leq 2N_h\sqrt{n}. \quad (2.2.9)$$

Similarly to how we defined Q_k^h , for every $k \in \{1, \dots, N_h\}$ we define

$$\hat{Q}_k^h := \left\{ x \in \tilde{Q}_k^h : x - \lambda_1 e_1 - \lambda_2 e_2 \in \tilde{Q}_k^h, 0 \leq \lambda_1, \lambda_2 \leq h \right\}.$$

As above, we subdivide each square $\hat{Q}_k^h$ into $Z_{k,n}$ subsquares of side length $L_{k,n}$ and denote their centres by $\{\hat{w}_{k,n}^i\}_{i=1, \dots, Z_{k,n}}$. We select n_r of these centres at random and denote them by $\{\hat{w}_n^i\}_{i=1, \dots, n_r}$.

Observing that

$$L_{k,n} = \frac{h}{\sqrt{\mu_n^h(Q_k^h)n}} \geq \frac{h}{\sqrt{n}} \quad (2.2.10)$$

and noting that the centres have a distance greater than or equal to $L_{k,n}$ from one another, we get the following estimates:

$$|w_{k,n}^i - w_{k,n}^j| \geq \frac{h}{\sqrt{n}} \text{ if } i \neq j, \quad (2.2.11)$$

$$|w_{k,n}^i - \hat{w}_n^j| \geq \frac{h}{\sqrt{n}} \text{ for every } i, j, \quad (2.2.12)$$

$$|\hat{w}_n^i - \hat{w}_n^j| \geq \frac{h}{\sqrt{n}} \text{ if } i \neq j. \quad (2.2.13)$$

We now define the recovery sequence as follows:

$$\mu_n = \frac{1}{n} \left(\sum_{k=1}^{N_h} \sum_{i=1}^{Z_{k,n}} \delta_{w_{k,n}^i} + \sum_{i=1}^{n_r} \delta_{\hat{w}_n^i} \right).$$

Note that, according to the way $Z_{k,n}$ is defined, the number of dislocations allocated in each square Q_k^h is proportional to the measure $\mu_n^h(Q_k^h)$. As a consequence, the squares whose measure is greater are weighted more by the recovery sequence.

Step 3 (proving the limsup inequality for μ^h): we claim that

- $\mu_n \xrightarrow{*} \mu^h$ as $n \rightarrow +\infty$;
- $I(\mu^h) \geq \limsup_{n \rightarrow +\infty} I_n(\mu_n)$.

As we have observed in step 1, there exists a compact set K such that $\text{supp}(\mu^h) \subseteq K$. By definition of μ_n , this implies $\text{supp}(\mu_n) \subseteq K$. Let $\varphi \in C_b(\mathbb{R}^2)$ such that $|\varphi(x)| \leq M$, fix $\varepsilon > 0$ and let $\delta > 0$ be such that

$$|\varphi(x) - \varphi(y)| < \varepsilon \quad \forall x, y \in K : |x - y| < \delta.$$

We have

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \varphi d\mu^h - \int_{\mathbb{R}^2} \varphi d\mu_n \right| &= \left| \int_K \varphi d\mu^h - \int_K \varphi d\mu_n \right| \\ &= \left| \sum_{k=1}^{N_h} \left(\int_{Q_k^h} \varphi(x) d\mu^h(x) - \int_{Q_k^h} \varphi(y) d\mu_n(y) \right) - \frac{1}{n} \sum_{i=1}^{n_r} \varphi(\hat{w}_n^i) \right| \\ &\leq \sum_{k=1}^{N_h} \left| \int_{Q_k^h} \varphi(x) d\mu^h(x) - \int_{Q_k^h} \varphi(y) d\mu_n(y) \right| + \frac{1}{n} \sum_{i=1}^{n_r} |\varphi(\hat{w}_n^i)| \\ &\leq \sum_{k=1}^{N_h} \left| \int_{Q_k^h} \varphi(x) d\mu^h(x) - \int_{Q_k^h} \varphi(y) d\mu_n(y) \right| + \frac{n_r}{n} M \\ &\leq \sum_{k=1}^{N_h} \left| \int_{Q_k^h} \varphi(x) d\mu^h(x) - \int_{Q_k^h} \varphi(y) d\mu_n(y) \right| + \frac{2N_h M}{\sqrt{n}}, \end{aligned}$$

where the last term comes from inequality (2.2.9) and vanishes for $n \rightarrow +\infty$.

As for the first term of the right-hand side,

$$\begin{aligned} \left| \int_{Q_k^h} \varphi(x) d\mu^h(x) - \int_{Q_k^h} \varphi(y) d\mu_n(y) \right| &= \left| \int_{Q_k^h} \varphi(x) d\mu^h(x) - \frac{1}{n} \sum_{i=1}^{Z_{k,n}} \varphi(w_{k,n}^i) \right| \\ &= \left| \sum_{i=1}^{Z_{k,n}} \left(\int_{Q_{k,i}^h} \varphi(x) d\mu^h(x) - \int_{Q_{k,i}^h} \varphi(w_{k,n}^i) d\mu^h(x) \right. \right. \\ &\quad \left. \left. + \int_{Q_{k,i}^h} \varphi(w_{k,n}^i) d\mu^h(y) - \frac{1}{n} \int_{Q_{k,i}^h} \varphi(w_{k,n}^i) \frac{d\mu^h(y)}{\mu^h(Q_{k,i}^h)} \right) \right| \\ &\leq \sum_{i=1}^{Z_{k,n}} \int_{Q_{k,i}^h} |\varphi(x) - \varphi(w_{k,n}^i)| d\mu^h(x) \\ &\quad + \sum_{i=1}^{Z_{k,n}} \left| \varphi(w_{k,n}^i) \left(\mu^h(Q_{k,i}^h) - \frac{1}{n} \right) \right| \\ &\leq \sum_{i=1}^{Z_{k,n}} \int_{Q_{k,i}^h} \varepsilon d\mu^h(x) + \sum_{i=1}^{Z_{k,n}} \left| \varphi(w_{k,n}^i) \frac{\mu^h(Q_k^h) - \mu_n^h(Q_k^h)}{Z_{k,n}} \right| \\ &\leq \varepsilon \mu^h(Q_k^h) + M \left| \mu^h(Q_k^h) - \mu_n^h(Q_k^h) \right|, \end{aligned}$$

if we choose n large enough that $|x - w_{k,n}^i| < L_{k,n} < \delta$, which is possible because $L_{k,n} \rightarrow 0$ for $n \rightarrow +\infty$. Since $\mu_n^h(Q_k^h) \rightarrow \mu^h(Q_k^h)$ as $n \rightarrow +\infty$, we conclude that $\mu_n \xrightarrow{*} \mu^h$.

To prove the second claim, we consider

$$I_n(\mu_n) = \frac{1}{n^2} \left(\sum_{k=1}^{N_h} \sum_{\substack{i,j=1 \\ i \neq j}}^{Z_{k,n}} W(w_{k,n}^i - w_{k,n}^j) + \sum_{\substack{k,l=1 \\ k \neq l}}^{N_h} \sum_{i=1}^{Z_{k,n}} \sum_{j=1}^{Z_{l,n}} W(w_{k,n}^i - w_{l,n}^j) \right. \\ \left. + 2 \sum_{k=1}^{N_h} \sum_{i=1}^{Z_{k,n}} \sum_{j=1}^{n_r} W(w_{k,n}^i - \hat{w}_n^j) + \sum_{\substack{i,j=1 \\ i \neq j}}^{n_r} W(\hat{w}_n^i - \hat{w}_n^j) \right) + \frac{1}{n} \left(\sum_{k=1}^{N_h} \sum_{i=1}^{Z_{k,n}} V(w_{k,n}^i) + \sum_{i=1}^{n_r} V(\hat{w}_n^i) \right).$$

For the confinement term, we have

$$\frac{1}{n} \left(\sum_{k=1}^{N_h} \sum_{i=1}^{Z_{k,n}} V(w_{k,n}^i) + \sum_{i=1}^{n_r} V(\hat{w}_n^i) \right) = \int_K V(x) d\mu_n(x) \longrightarrow \int_K V(x) d\mu^h(x),$$

because $V|_K \in C_b(K)$ and μ_n converges narrowly to μ^h .

We can thus focus on the interaction terms. We have

$$\frac{1}{n^2} \sum_{i=1}^{Z_{k,n}} \sum_{j=1}^{Z_{l,n}} W(w_{k,n}^i - w_{l,n}^j) = \iint_{Q_k^h \times Q_l^h} W(x - y) d(\mu_n \otimes \mu_n)(x, y) \longrightarrow \iint_{Q_k^h \times Q_l^h} W(x - y) d(\mu^h \otimes \mu^h)(x, y),$$

because $(x, y) \mapsto W(x - y) \in C_b(Q_k^h \times Q_l^h)$ if $k \neq l$ and $\mu_n \otimes \mu_n \xrightarrow{*} \mu^h \otimes \mu^h$ by Lemma 1.

As for the other terms, we get

$$\begin{aligned} \frac{2}{n^2} \sum_{k=1}^{N_h} \sum_{i=1}^{Z_{k,n}} \sum_{j=1}^{n_r} W(w_{k,n}^i - \hat{w}_n^j) &\leq \frac{2}{n^2} \sum_{k=1}^{N_h} \sum_{i=1}^{Z_{k,n}} \sum_{j=1}^{n_r} (C_2 - \ln |w_{k,n}^i - \hat{w}_n^j|) \\ &\leq \frac{2}{n^2} \sum_{k=1}^{N_h} \sum_{i=1}^{Z_{k,n}} \sum_{j=1}^{n_r} \left(C_2 - \ln \left(\frac{h}{\sqrt{n}} \right) \right) \\ &= \frac{2}{n^2} N_h Z_{k,n} n_r \left(C_2 + \ln \left(\frac{\sqrt{n}}{h} \right) \right) \\ &\leq \frac{4N_h^2}{\sqrt{n}} \left(C_2 + \ln \left(\frac{\sqrt{n}}{h} \right) \right) \longrightarrow 0, \end{aligned}$$

where the second inequality descends from estimate (2.2.12), while the last inequality is a consequence of (2.2.9). Furthermore,

$$\begin{aligned} \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n_r} W(\hat{w}_n^i - \hat{w}_n^j) &\leq \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n_r} (C_2 - \ln |\hat{w}_n^i - \hat{w}_n^j|) \leq \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n_r} \left(C_2 - \ln \left(\frac{h}{\sqrt{n}} \right) \right) \\ &\leq \frac{n_r^2}{n^2} \left(C_2 + \ln \left(\frac{\sqrt{n}}{h} \right) \right) \leq \frac{4N_h^2}{n} \left(C_2 + \ln \left(\frac{\sqrt{n}}{h} \right) \right) \longrightarrow 0, \end{aligned}$$

where again we have used inequality (2.2.9) and estimate (2.2.13).

Similarly to step 1, we consider the truncation

$$W_M(x) = \min\{W(x), M\}$$

and rewrite

$$\frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{Z_{k,n}} W(w_{k,n}^i - w_{k,n}^j) = \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{Z_{k,n}} W_M(w_{k,n}^i - w_{k,n}^j) + \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{Z_{k,n}} (W - W_M)(w_{k,n}^i - w_{k,n}^j).$$

By the boundedness of W_M and Lemma 1, we obtain

$$\begin{aligned} \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{Z_{k,n}} W_M(w_{k,n}^i - w_{k,n}^j) &= \int_{Q_k^h} \int_{Q_k^h} W_M(x - y) d(\mu_n \boxtimes \mu_n)(x, y) \\ &\longrightarrow \int_{Q_k^h} \int_{Q_k^h} W_M(x - y) d(\mu^h \otimes \mu^h)(x, y) \\ &\leq \int_{Q_k^h} \int_{Q_k^h} W(x - y) d(\mu^h \otimes \mu^h)(x, y). \end{aligned}$$

The claim is proved if we show that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{Z_{k,n}} (W - W_M)(w_{k,n}^i - w_{k,n}^j) \rightarrow 0 \text{ as } M \rightarrow +\infty \quad (2.2.14)$$

for every $k \in \{1, \dots, N_h\}$.

Thanks to (2.2.5), we can write

$$\frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{Z_{k,n}} (W - W_M)(w_{k,n}^i - w_{k,n}^j) \leq \frac{1}{n^2} \sum_{i=1}^{Z_{k,n}} \sum_{j \in J_{k,n}^i} W(w_{k,n}^i - w_{k,n}^j),$$

with $J_{k,n}^i := \{j \in \{1, \dots, Z_{k,n}\} : |w_{k,n}^i - w_{k,n}^j| < R_M, j \neq i\}$.

We arrange the points $w_{k,n}^j$ according to their distance from $w_{k,n}^i$ by defining

$$J_{k,n}^i(p) := \{j \in \{1, \dots, Z_{k,n}\} : w_{k,n}^j = w_{k,n}^i + (l, m)L_{k,n}, (l, m) \in \mathbb{Z}^2, \max\{|l|, |m|\} = p\},$$

for $p = 1, \dots, P_{k,n}$, with $P_{k,n} := \left\lceil \frac{R_M}{L_{k,n}} \right\rceil$.

For every $i \in \{1, \dots, Z_{k,n}\}$, $j \in J_{k,n}^i(p)$, we have $|w_{k,n}^i - w_{k,n}^j| \geq pL_{k,n}$, which gives the following inequality:

$$W(w_{k,n}^i - w_{k,n}^j) \leq C_2 - \ln |w_{k,n}^i - w_{k,n}^j| \leq C_2 - \ln(pL_{k,n}). \quad (2.2.15)$$

Moreover, we can estimate

$$\text{card}(J_{k,n}^i(p)) \leq (2p+1)^2 - (2(p-1)+1)^2 = 8p, \quad (2.2.16)$$

by observing that the points $w_{k,n}^j$ with $j \in J_{k,n}^i(p)$ are contained in the difference of two squares of centre $w_{k,n}^i$ and side lengths equal to $2p+1$ and $2(p-1)+1$, respectively.

We thus get

$$\begin{aligned}
\frac{1}{n^2} \sum_{i=1}^{Z_{k,n}} \sum_{j \in J_{k,n}^i} W(w_{k,n}^i - w_{k,n}^j) &= \frac{1}{n^2} \sum_{i=1}^{Z_{k,n}} \sum_{p=1}^{P_{k,n}} \sum_{j \in J_{k,n}^i(p)} W(w_{k,n}^i - w_{k,n}^j) \\
&\leq \frac{1}{n^2} \sum_{i=1}^{Z_{k,n}} \sum_{p=1}^{P_{k,n}} \sum_{j \in J_{k,n}^i(p)} (C_2 - \ln(pL_{k,n})) \\
&= \frac{1}{n^2} \sum_{i=1}^{Z_{k,n}} \sum_{p=1}^{P_{k,n}} \text{card}(J_{k,n}^i(p)) (C_2 - \ln(pL_{k,n})) \\
&\leq \frac{1}{n^2} \sum_{i=1}^{Z_{k,n}} \sum_{p=1}^{P_{k,n}} 8p(C_2 - \ln(pL_{k,n})) \\
&= \frac{8}{n^2} \sum_{i=1}^{Z_{k,n}} \sum_{p=1}^{P_{k,n}} \left(pC_2 - \frac{1}{L_{k,n}} pL_{k,n} \ln(pL_{k,n}) \right) \\
&\leq \frac{8}{n^2} Z_{k,n} \left(\frac{P_{k,n}(P_{k,n} + 1)}{2} C_2 + \frac{1}{eL_{k,n}} P_{k,n} \right),
\end{aligned}$$

where we have used the estimates (2.2.15) and (2.2.16), while the last inequality comes from the fact that $-x \ln(x) \leq 1/e$.

Finally, we obtain

$$\begin{aligned}
\frac{8}{n^2} Z_{k,n} \left(\frac{P_{k,n}(P_{k,n} + 1)}{2} C_2 + \frac{P_{k,n}}{eL_{k,n}} \right) &\leq \frac{4}{n} \left(\frac{R_M}{L_{k,n}} + 1 \right) \left(\frac{R_M}{L_{k,n}} + 2 \right) C_2 + \frac{8}{neL_{k,n}} \left(\frac{R_M}{L_{k,n}} + 1 \right) \\
&= \frac{4}{n} \left(\frac{R_M(C_2 R_M + 2e^{-1})}{(L_{k,n})^2} + \frac{3C_2 R_M + 2e^{-1}}{L_{k,n}} + 2C_2 \right) \\
&\leq 4 \left(\frac{R_M(C_2 R_M + 2e^{-1})}{h^2} + \frac{3C_2 R_M + 2e^{-1}}{h\sqrt{n}} + \frac{2C_2}{n} \right) \\
&\rightarrow 4 \frac{R_M(C_2 R_M + 2e^{-1})}{h^2} \text{ as } n \rightarrow +\infty,
\end{aligned}$$

where the last inequality comes from estimate (2.2.10). Since $R_M \rightarrow 0$ as $M \rightarrow +\infty$, claim (2.2.14) is proved.

Hence, $I(\mu^h) \geq \limsup_{n \rightarrow +\infty} I_n(\mu_n)$, which concludes the proof. $\square$

THEOREM 4. *Let $\{I_n\}_{n \geq 2}$ be the family of functionals defined in (2.1.5). If V satisfies assumptions (H1)-(H3), then I_n is coercive with respect to the narrow topology for every $n \geq 2$. Moreover, the sequence $\{I_n\}_{n \geq 2}$ is equi-coercive.*

PROOF. We first show that I_n is coercive for every $n \geq 2$, i.e. $\{\mu \in \mathcal{P}(\mathbb{R}^2) : I_n(\mu) \leq t\}$ is compact (with respect to the narrow topology) for every $t \in \mathbb{R}$.

By Theorem 12, it is enough to prove that there exists a compact set K_t containing the support of every measure μ such that $I_n(\mu) \leq t$, as this implies that $\{\mu \in \mathcal{P}(\mathbb{R}^2) : I_n(\mu) \leq t\}$ is tight and hence compact.

We fix $n \geq 2$ and $t \in \mathbb{R}$ and suppose

$$I_n(\mu) = \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n \left(W(x^i - x^j) + \frac{1}{2}V(x^i) + \frac{1}{2}V(x^j) \right) + \frac{1}{n^2} \sum_{i=1}^n V(x^i) \leq t.$$

Without loss of generality, by Theorem 2 we may assume $(x, y) \mapsto W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y)$ and V to be non-negative. We observe that, by (2.2.2), there exists a compact $K_t \subseteq \mathbb{R}^2$ such that

$$W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) > \frac{n^2}{n-1}t \quad \forall (x, y) \in (K_t \times K_t)^c.$$

By contradiction, suppose that the support of μ is not contained in K_t , e.g.

$$\mu = \frac{1}{n} \sum_{i=1}^n \delta_{x^i}, \quad x^1 \notin K_t.$$

We thus have

$$\begin{aligned} I_n(\mu) &= \frac{1}{n^2} \sum_{j=2}^n \left(W(x^1 - x^j) + \frac{1}{2}V(x^1) + \frac{1}{2}V(x^j) \right) \\ &\quad + \frac{1}{n^2} \sum_{i=2}^n \sum_{j \neq i} \left(W(x^i - x^j) + \frac{1}{2}V(x^i) + \frac{1}{2}V(x^j) \right) + \frac{1}{n^2} \sum_{i=1}^n V(x^i) \\ &> \frac{1}{n^2} \sum_{j=2}^n \frac{n^2}{n-1}t = t, \end{aligned}$$

which gives a contradiction.

To prove equi-coercivity, we will show that, for every $t \in \mathbb{R}$, the set

$$X_t = \bigcup_{n \geq 2} \{ \mu \in \mathcal{P}(\mathbb{R}^2) : I_n(\mu) \leq t \}$$

is relatively compact.

If there exists $N \geq 2$ such that

$$X_t \subseteq \bigcup_{n=2}^N \{ \mu \in \mathcal{P}(\mathbb{R}^2) : I_n(\mu) \leq t \},$$

then the claim follows from the previous result, because X_t is contained in a finite union of compact sets.

Otherwise, there exists a sequence $\{\mu_k\}_k \subseteq X_t$ such that $\mu_k \in \{ \mu \in \mathcal{P}(\mathbb{R}^2) : I_{n_k}(\mu) \leq t \}$, with $\{n_k\}_k$ strictly increasing. Again, by Prokhorov's theorem it suffices to show that $\{\mu_k\}_k$ has a tight subsequence.

For every $M > 0$, there exists a compact $K \subseteq \mathbb{R}^2$ such that

$$\inf_{(x,y) \in (K \times K)^c} W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) > M.$$

Therefore we have

$$\begin{aligned}
t \geq I_{n_k}(\mu_k) &= \frac{1}{n_k^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n_k} \left(W(x^i - x^j) + \frac{1}{2}V(x^i) + \frac{1}{2}V(x^j) \right) + \frac{1}{n_k^2} \sum_{i=1}^{n_k} V(x^i) \\
&\geq \frac{1}{n_k^2} \sum_{\substack{(i,j) \in I_k \\ i \neq j}} \left(W(x^i - x^j) + \frac{1}{2}V(x^i) + \frac{1}{2}V(x^j) \right) \\
&\geq M(\mu_k \boxtimes \mu_k)((K \times K)^c) \geq M \left((\mu_k \otimes \mu_k)((K \times K)^c) - \frac{1}{n_k} \right) \\
&= M \left(1 - (\mu_k(K))^2 - \frac{1}{n_k} \right) = M \left((1 - \mu_k(K))(1 + \mu_k(K)) - \frac{1}{n_k} \right) \\
&\geq M \left(\mu_k(K^c) - \frac{1}{n_k} \right),
\end{aligned}$$

where $\mu_k = \frac{1}{n_k} \sum_{i=1}^{n_k} \delta_{x^i}$ and $I_k := \{(i, j) : (x^i, x^j) \in (K \times K)^c\}$.

Choosing $M > 0$ and $\bar{k} \in \mathbb{N}$ such that $\frac{t}{M} = \frac{\varepsilon}{2}$ and $\frac{1}{n_k} < \frac{\varepsilon}{2}$, we obtain

$$\mu_k(K^c) \leq \frac{t}{M} + \frac{1}{n_k} < \varepsilon \quad \forall k \geq \bar{k},$$

hence the subsequence $\{\mu_k\}_{k \geq \bar{k}}$ is tight. $\square$

Since the family $\{I_n\}_{n \geq 2}$ is equi-coercive and Γ -converges to I , by the fundamental theorem of Γ -convergence (Theorem 14) we deduce that I has a minimum. In particular,

$$\min_{\mathcal{P}(\mathbb{R}^2)} I = \lim_{n \rightarrow +\infty} \min_{\mathcal{P}(\mathbb{R}^2)} I_n. \quad (2.2.17)$$

The minimizers of I can be found as the limit of those of the discrete energies I_n in that, if $\{\mu_n\}_n$ is a relatively compact sequence such that

$$\lim_{n \rightarrow +\infty} I_n(\mu_n) = \lim_{n \rightarrow +\infty} \min_{\mu \in \mathcal{P}(\mathbb{R}^2)} I_n(\mu),$$

then any limit point of $\{\mu_n\}_n$ is a minimizer of I .

CHAPTER 3

Existence and uniqueness of the minimizer and its characterisation

In this chapter, we focus on the minimization problem for the energy

$$I(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} V(x) d\mu(x).$$

We first review some recent results concerning the existence and uniqueness of the minimizer (see e.g. [2–4]) and then we provide a characterisation for the case $V(x) = |x|^2$.

We will start by giving some useful definitions.

DEFINITION 3. *Let $s \in \mathbb{R}$. The fractional Sobolev space of order s on $\mathbb{S}^1$ is*

$$H^s(\mathbb{S}^1) := \left\{ u \in \mathcal{D}'(\mathbb{S}^1) : \sum_{k \in \mathbb{Z}} (1+k^2)^s |\hat{u}_k|^2 < +\infty \right\}.$$

Here, $\mathcal{D}'(\mathbb{S}^1)$ denotes the space of distributions on $\mathbb{S}^1$ (which can be identified with the space of periodic distributions on $\mathbb{R}^2$) and $\{\hat{u}_k\}_{k \in \mathbb{Z}}$ is the sequence of Fourier coefficients of u . The space $H^s(\mathbb{S}^1)$, endowed with the inner product

$$\langle u, v \rangle_{H^s} := \sum_{k \in \mathbb{Z}} (1+k^2)^s \hat{u}_k \overline{\hat{v}_k}$$

and the norm

$$\|u\|_{H^s} := \sum_{k \in \mathbb{Z}} (1+k^2)^s |\hat{u}_k|^2 = \sqrt{\langle u, u \rangle_{H^s}},$$

is a Hilbert space. In particular, $H^0(\mathbb{S}^1) = L^2(\mathbb{S}^1)$.

Provided that the order s is high enough, the space $H^s(\mathbb{S}^1)$ can be embedded into the space $C^k(\mathbb{S}^1)$ of k -times continuously differentiable functions.

THEOREM 5 (Sobolev embedding). *Let $k \in \mathbb{Z}_+$, $s > k + \frac{1}{2}$. Then, the embedding*

$$H^s(\mathbb{S}^1) \hookrightarrow C^k(\mathbb{S}^1)$$

is continuous, i.e. there exists a constant $C > 0$ such that $\|u\|_{C^k} \leq C \|u\|_{H^s}$ for every $u \in H^s(\mathbb{S}^1)$.

Moreover, if $s > t$, we have the compact embedding

$$H^s(\mathbb{S}^1) \hookrightarrow\hookrightarrow H^t(\mathbb{S}^1).$$

We recall that the interaction kernel is of the form

$$W(x) = W_0(x) + \kappa(x), \tag{3.0.1}$$

where W_0 is the Coulomb kernel defined in (2.1.4). For reasons that will become more clear later, we assume that the anisotropic kernel κ is of class $H^s(\mathbb{S}^1)$, with $s > 3/2$.

Owing to Theorem 5, $H^s(\mathbb{S}^1)$ is continuously embedded into $C(\mathbb{S}^1)$ for $s > 1/2$, therefore κ is continuous on $\mathbb{S}^1$, up to modifications on a set of zero measure. Hence, κ is bounded on $\mathbb{R}^2 \setminus \{0\}$ and there exist two constants $C_1, C_2 \in \mathbb{R}$ such that

$$W_0(x) + C_1 \leq W(x) \leq W_0(x) + C_2 \quad \forall x \in \mathbb{R}^2. \tag{3.0.2}$$

We will now state the following notion of capacity.

DEFINITION 4. Let $K \subseteq \mathbb{R}^2$ compact. We define the logarithmic capacity of K as

$$\text{cap}(K) := \Phi \left(\inf_{\mu \in \mathcal{P}(K)} \iint_{K \times K} W_0(x-y) d\mu(x) d\mu(y) \right),$$

where

$$\Phi(t) := \begin{cases} e^{-t} & \text{if } t \in \mathbb{R} \\ 0 & \text{if } t = +\infty. \end{cases}$$

For any Borel set $U \subseteq \mathbb{R}^2$, we define its logarithmic capacity

$$\text{cap}(U) := \sup \{ \text{cap}(K) : K \subseteq U \text{ compact} \}.$$

A property is said to hold quasi everywhere (q.e.) if it holds up to sets of zero capacity.

The capacity satisfies the following property.

LEMMA 3. Let $U \subseteq \mathbb{R}^2$ be a Borel set such that $\text{cap}(U) = 0$. Then $\mu(U) = 0$ for every $\mu \in \mathcal{P}(\mathbb{R}^2)$ with compact support and such that

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\mu(x) d\mu(y) < +\infty. \quad (3.0.3)$$

Moreover, the countable union of sets of zero capacity has zero capacity.

PROOF. By contradiction, assume that $\mu(U) > 0$ for a Borel set $U \subseteq \mathbb{R}^2$ such that $\text{cap}(U) = 0$. Then there exists a compact $K \subseteq U$ such that $\mu(K) > 0$. We define

$$\nu := \frac{1}{\mu(K)} \mu|_K.$$

Because μ has compact support, there exists $C > 0$ such that

$$W(x-y) \geq -C \quad \forall (x, y) \in (\text{supp } \mu)^2,$$

which gives

$$\begin{aligned} \iint_{K \times K} W(x-y) d\nu(x) d\nu(y) &= \frac{1}{\mu(K)^2} \iint_{K \times K} W(x-y) d\mu(x) d\mu(y) \\ &\leq \frac{1}{\mu(K)^2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\mu(x) d\mu(y) + C \left(\frac{1}{\mu(K)^2} - 1 \right) < +\infty. \end{aligned}$$

Recalling (3.0.2), we have

$$\iint_{K \times K} W_0(x-y) d\nu(x) d\nu(y) \leq \iint_{K \times K} W(x-y) d\nu(x) d\nu(y) - C_1 < +\infty.$$

Hence, $\text{cap}(K) > 0$, which contradicts the assumption $\text{cap}(U) = 0$.

To prove the second claim, we consider $V = \bigcup_{n \in \mathbb{N}} V_n$, with $\text{cap}(V_n) = 0$ for every n , and suppose by contradiction that $\text{cap}(V) > 0$. Then, there exist $K \subseteq V$ compact and $\mu \in \mathcal{P}(K)$ such that

$$\iint_{K \times K} W_0(x-y) d\mu(x) d\mu(y) < +\infty.$$

Since $\mu(K) = 1$, there exist $n_0 \in \mathbb{N}$ and a compact $K_{n_0} \subseteq K \cap V_{n_0}$ for which $\mu(K_{n_0}) > 0$. As above, we define

$$\nu := \frac{1}{\mu(K_{n_0})} \mu|_{K_{n_0}}$$

and observe that

$$\iint_{K_{n_0} \times K_{n_0}} W_0(x-y) d\nu(x) d\nu(y) \leq \iint_{K \times K} W_0(x-y) d\mu(x) d\mu(y) + C < +\infty,$$

where $C = \inf \{W_0(x - y) : (x, y) \in (\text{supp } \mu)^2\}$.

Therefore, we obtain $\text{cap}(K_{n_0}) > 0$, contradicting the assumption $\text{cap}(V_{n_0}) = 0$. $\square$

In particular, if a property holds quasi everywhere, it also holds μ -almost everywhere for all measures $\mu \in \mathcal{P}_c(\mathbb{R}^2)$ satisfying (3.0.3). Notably, this is the class of measures of relevance for the minimization of I .

In the following, we assume that the confinement potential V is lower semi-continuous, bounded from below and satisfies

$$\lim_{|x| \rightarrow +\infty} \left(W_0(x) + \frac{1}{2}V(x) \right) = +\infty \quad (3.0.4)$$

and

$$\text{cap}(\{x \in \mathbb{R}^2 : V(x) < +\infty\}) > 0. \quad (3.0.5)$$

3.1. Existence of minimizers

Under the assumptions above, the functional $I: \mathcal{P}(\mathbb{R}^2) \rightarrow \mathbb{R} \cup \{+\infty\}$

$$I(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d\mu(x)d\mu(y) \quad (3.1.1)$$

is well-defined. Indeed, the function $(x, y) \mapsto W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y)$ is lower semi-continuous. Since a lower semi-continuous function is bounded from below on a compact set, by (3.0.2) and (3.0.4) we deduce that the integrand is bounded from below by a constant $c_0 < 0$. Therefore,

$$I(\mu) \geq \iint_{\mathbb{R}^2 \times \mathbb{R}^2} c_0 d\mu(x)d\mu(y) = c_0 \quad \forall \mu \in \mathcal{P}(\mathbb{R}^2)$$

and $\inf I > -\infty$. Moreover, by applying Tonelli's theorem one can see that the definitions (2.2.1) and (3.1.1) are equivalent.

We refer to the term

$$I_W(\mu) := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x - y) d\mu(x)d\mu(y)$$

as interaction energy, while the term

$$I_V(\mu) := \int_{\mathbb{R}^2} V(x) d\mu(x)$$

is called confinement energy.

The minimization problem for I admits a solution. We have the following result.

THEOREM 6 (existence of minimizers). *The energy I admits a minimizer $\mu \in \mathcal{P}(\mathbb{R}^2)$ such that $I(\mu) < +\infty$. Moreover, μ has compact support and satisfies the following Euler-Lagrange equations: there exists $c \in \mathbb{R}$ such that*

$$(W * \mu)(x) + \frac{1}{2}V(x) = c \quad \text{for } \mu\text{-a.e. } x \in \text{supp } \mu, \quad (3.1.2)$$

$$(W * \mu)(x) + \frac{1}{2}V(x) \geq c \quad \text{for q.e. } x \in \mathbb{R}^2. \quad (3.1.3)$$

PROOF. We first prove that $\inf I < +\infty$. We write

$$\text{dom}(V) = \{x \in \mathbb{R}^2 : V(x) < +\infty\} = \bigcup_{n \in \mathbb{Z}} K_n,$$

with

$$K_n := \{x \in \mathbb{R}^2 : V(x) \leq n\}.$$

By assumption (3.0.5) and Lemma 3, we have $\text{cap}(K_{n_0}) > 0$ for some $n_0 \in \mathbb{Z}$. Therefore, there exists $\mu_0 \in \mathcal{P}(K_{n_0})$ such that

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} W_0(x - y) d\mu_0(x) d\mu_0(y) < +\infty,$$

which implies

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x - y) d\mu_0(x) d\mu_0(y) < +\infty.$$

Moreover,

$$\int_{\mathbb{R}^2} V(x) d\mu_0(x) \leq n_0,$$

therefore $I(\mu_0) < +\infty$.

The existence of a minimizer can be proved using the direct method of calculus of variations, i.e. showing that I is lower semi-continuous and that any minimizing sequence has a converging subsequence. Let $\{\mu_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(\mathbb{R}^2)$ be a minimizing sequence, i.e. $I(\mu_n) \rightarrow \inf I$. Without loss of generality, we may suppose that $I(\mu_n) \leq C$ for every $n \in \mathbb{N}$, for some $C > 0$. By (3.0.2) and assumption (3.0.4), for every $M > 0$ there exists a compact $K_M \subseteq \mathbb{R}^2$ such that

$$W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \geq M \quad \forall (x, y) \notin K_M \times K_M.$$

Recalling that $(x, y) \mapsto W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y)$ is bounded from below by $c_0 < 0$, we have

$$\begin{aligned} C \geq I(\mu_n) &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d\mu_n(x) d\mu_n(y) \\ &\geq M(\mu_n \otimes \mu_n)((K_M \times K_M)^c) + c_0(\mu_n \otimes \mu_n)(K_M \times K_M) \\ &\geq M\mu_n(K_M^c)^2 + c_0. \end{aligned}$$

Therefore, the sequence $\{\mu_n\}_n$ is tight and, by Theorem 12, it has a subsequence (we still denote it $\{\mu_n\}_n$) that converges narrowly to some $\mu_0 \in \mathcal{P}(\mathbb{R}^2)$. By applying Lemma 7, we can show that I is lower semi-continuous:

$$\begin{aligned} I(\mu_0) &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \liminf_{n \rightarrow +\infty} \left(W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d\mu_n(x) d\mu_n(y) \\ &\leq \liminf_{n \rightarrow +\infty} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d\mu_n(x) d\mu_n(y) = \liminf_{n \rightarrow +\infty} I(\mu_n). \end{aligned}$$

Hence, μ_0 is a minimizer.

We will now prove that, if μ is a minimizer of, then $\mu \in \mathcal{P}_c(\mathbb{R}^2)$. By (3.0.4), there exists a compact $K \subseteq \mathbb{R}^2$ such that

$$W(x - y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) > I(\mu) \quad \forall (x, y) \in (K \times K)^c.$$

Without loss of generality, we may assume that $\mu(K) > 0$. By contradiction, suppose that $\text{supp } \mu \not\subseteq K$, which implies $(\mu \otimes \mu)((K \times K)^c) > 0$. We can construct a measure

$$\nu := \frac{1}{\mu(K)} \mu|_K.$$

Therefore, we obtain

$$\begin{aligned}
I(\nu) &= \frac{1}{\mu(K)^2} \iint_{K \times K} \left(W(x-y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d\mu(x)d\mu(y) \\
&= \frac{1}{\mu(K)^2} \left(I(\mu) - \iint_{(K \times K)^c} \left(W(x-y) + \frac{1}{2}V(x) + \frac{1}{2}V(y) \right) d\mu(x)d\mu(y) \right) \\
&< \frac{1}{\mu(K)^2} I(\mu)(1 - (\mu \otimes \mu)((K \times K)^c)) = I(\mu),
\end{aligned}$$

which contradicts the minimality of μ .

We are left to prove (3.1.2) and (3.1.3). Let $\nu \in \mathcal{P}(\mathbb{R}^2)$ be a measure with compact support and such that $I(\nu) < +\infty$. By minimality of μ , for every $\varepsilon \in (0, 1)$ we have

$$I(\mu) \leq I((1 - \varepsilon)\mu + \varepsilon\nu),$$

which implies

$$\begin{aligned}
0 &\leq -2\varepsilon \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\mu(x)d\mu(y) + 2\varepsilon \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\mu(x)d\nu(y) \\
&\quad + \varepsilon \int_{\mathbb{R}^2} V(x) d(\nu - \mu)(x) + O(\varepsilon^2) \\
&= -2\varepsilon \int_{\mathbb{R}^2} W * \mu d\mu + 2\varepsilon \int_{\mathbb{R}^2} W * \mu d\nu + 2\varepsilon \int_{\mathbb{R}^2} \frac{1}{2}V d(\nu - \mu) + O(\varepsilon^2).
\end{aligned}$$

Dividing by 2ε and letting $\varepsilon \rightarrow 0^+$, we obtain

$$\int_{\mathbb{R}^2} \left(W * \mu + \frac{1}{2}V \right) d\nu \geq \int_{\mathbb{R}^2} \left(W * \mu + \frac{1}{2}V \right) d\mu =: c. \quad (3.1.4)$$

Let $\Phi := W * \mu + \frac{1}{2}V$ and assume by contradiction that

$$\text{cap}(\{x \in \mathbb{R}^2 : \Phi(x) < c\}) > 0.$$

We note that $\{x \in \mathbb{R}^2 : \Phi(x) < c\}$ is open because Φ is lower semi-continuous, so its capacity is well-defined. Therefore, there must exist a compact $K \subseteq \mathbb{R}^2$ and $\mu_0 \in \mathcal{P}(K)$ such that

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\mu_0(x)d\mu_0(y) < +\infty.$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{1}{2}V(x) d\mu_0(x) &< \int_{\mathbb{R}^2} (c - (W * \mu)(x)) d\mu_0(x) \\
&= c - \iint_{(\text{supp } \mu) \times K} W(x-y) d\mu(y)d\mu_0(x) < +\infty,
\end{aligned} \quad (3.1.5)$$

because $(x, y) \mapsto W(x-y)$ is bounded from below on the compact set $(\text{supp } \mu) \times K$. Hence, μ_0 has finite energy and must satisfy (3.1.4), which gives a contradiction with (3.1.5). This shows that $W * \mu + \frac{1}{2}V \geq c$ quasi everywhere. By Lemma 3, this inequality is also true μ -almost everywhere, and by definition of c we deduce that $W * \mu + \frac{1}{2}V = c$ μ -almost everywhere, which concludes the proof. $\square$

We notice that, while we have assumed $\kappa \in C(\mathbb{S}^1)$ thanks to the continuous embedding $H^s(\mathbb{S}^1) \hookrightarrow C(\mathbb{S}^1)$, Theorem 6 holds for more general interaction kernels. In particular, the only regularity assumptions on κ that were required to prove the existence of a minimizer are lower semi-continuity and boundedness.

However, more regularity is needed to ensure that the minimizer is unique, as we shall see in the following section.

3.2. Uniqueness of the minimizer

Uniqueness of the minimizer is guaranteed if the energy functional is strictly convex. On the other hand, strict convexity is ensured if the Fourier transform of W (provided that it exists) satisfies the sign condition $\widehat{W}(\xi) \geq 0$ for $\xi \neq 0$.

Given a function $\varphi \in \mathcal{S}(\mathbb{R}^2)$, the Schwartz space of rapidly decreasing functions, its Fourier transform (denoted $\mathcal{F}\varphi$ or $\widehat{\varphi}$) is defined by

$$\widehat{\varphi}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \varphi(x) e^{-i\langle x, \xi \rangle} dx \quad \forall \xi \in \mathbb{R}^2.$$

The Fourier transform $\widehat{u}$ of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^2)$ can thus be defined by duality as

$$\langle \widehat{u} | \varphi \rangle = \langle u | \widehat{\varphi} \rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^2).$$

The Coulomb kernel W_0 is locally integrable and has sublinear growth, therefore it defines a tempered distribution and its Fourier transform is well-defined. Observing that $-\Delta W_0 = 2\pi\delta_0$, we can compute

$$-\widehat{\Delta W_0}(\xi) = |\xi|^2 \widehat{W_0}(\xi) = 2\pi \widehat{\delta_0}(\xi) = 1,$$

which yields $\widehat{W_0}(\xi) = \frac{1}{|\xi|^2}$. Since this function is not integrable at 0, it does not define a tempered distribution. To take into account the singularity at 0, we can interpret $\widehat{W_0}$ in a distributional sense as

$$\langle \widehat{W_0} | \varphi \rangle = c_0 \varphi(0) + \int_{|\xi| \leq 1} \frac{1}{|\xi|^2} (\varphi(\xi) - \varphi(0)) d\xi + \int_{|\xi| > 1} \frac{1}{|\xi|^2} \varphi(\xi) d\xi \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^2), \quad (3.2.1)$$

where $c_0 = \frac{1}{2\pi}(\gamma + \ln \pi)$, γ being the Euler-Mascheroni constant.

In order to compute the Fourier transform of the anisotropic kernel, we can consider its Fourier expansion. Since $\kappa(z) = \kappa\left(\frac{z}{|z|}\right)$ for every $z \in \mathbb{R}^2$, κ is fully determined by the values that it takes on $\mathbb{S}^1$. By parameterizing $z \in \mathbb{S}^1$ via $z = e^{i\theta}$, with $\theta \in [0, 2\pi]$, we can write κ as a Fourier series of variable θ . Moreover, because κ is even, the series contains only the even terms. In particular,

$$\kappa(z) = \kappa(e^{i\theta}) = a_0 + \sum_{n=1}^{+\infty} (a_{2n} \cos(2n\theta) + b_{2n} \sin(2n\theta)), \quad (3.2.2)$$

with $\{a_{2n}\}_{n \in \mathbb{N}}, \{b_{2n}\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$. We have

$$\cos(n\theta) = \operatorname{Re} \frac{z^n}{|z|^n} = \frac{\phi_n(z)}{|z|^n}, \quad \sin(n\theta) = \operatorname{Im} \frac{z^n}{|z|^n} = \frac{\psi_n(z)}{|z|^n},$$

where ϕ_n, ψ_n are homogeneous polynomials of degree n . In fact, using binomial expansion, for $z = x + iy$ we obtain

$$\begin{aligned} \phi_n(z) &= \operatorname{Re} (x + iy)^n = \operatorname{Re} \sum_{k=0}^n \binom{n}{k} i^k x^{n-k} y^k = \sum_{\substack{k=0 \\ k \text{ even}}}^n \binom{n}{k} i^k x^{n-k} y^k \\ \psi_n(z) &= \operatorname{Im} (x + iy)^n = \operatorname{Im} \sum_{k=0}^n \binom{n}{k} i^k x^{n-k} y^k = \sum_{\substack{k=0 \\ k \text{ odd}}}^n \binom{n}{k} i^{k-1} x^{n-k} y^k. \end{aligned}$$

Hence, we can write

$$\kappa(z) = a_0 + \sum_{n=1}^{+\infty} \left(a_{2n} \frac{\phi_{2n}(z)}{|z|^{2n}} + b_{2n} \frac{\psi_{2n}(z)}{|z|^{2n}} \right) \quad \forall z \in \mathbb{R}^2.$$

Since the addition of a constant to κ does not affect the minimization problem, we assume $a_0 = 0$. A straightforward computation shows that ϕ_n, ψ_n are harmonic, i.e. $\Delta\phi_n = 0$ and $\Delta\psi_n = 0$. The Fourier transform of κ can thus be computed according to the following lemma.

LEMMA 4. *Let $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a harmonic homogeneous polynomial function of degree n . Then,*

$$\left(\mathcal{F} \frac{\phi(\cdot)}{|\cdot|^n} \right)(\xi) = c_n \frac{\phi(\xi)}{|\xi|^{n+2}},$$

where c_n is a constant. In particular, $c_{2n} = (-1)^n 2n$.

The function $\frac{\phi(\xi)}{|\xi|^{n+2}}$ is not integrable at 0, because it behaves like $\frac{1}{|\xi|^2}$ close to 0. Therefore, the Fourier transform should be interpreted in a distributional sense as

$$\langle \mathcal{F} \frac{\phi(\cdot)}{|\cdot|^n} | \varphi \rangle = c_n \left(\int_{|\xi| \leq 1} \frac{\phi(\xi)}{|\xi|^{n+2}} (\varphi(\xi) - \varphi(0)) d\xi + \int_{|\xi| > 1} \frac{\phi(\xi)}{|\xi|^{n+2}} \varphi(\xi) d\xi \right) \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^2).$$

Hence, the Fourier transform of κ (to be interpreted in a distributional sense as above) is given by

$$\widehat{\kappa}(\xi) = \sum_{n=1}^{+\infty} \left((-1)^n 2na_{2n} \frac{\phi_{2n}(\xi)}{|\xi|^{2n+2}} + (-1)^n 2nb_{2n} \frac{\psi_{2n}(\xi)}{|\xi|^{2n+2}} \right). \quad (3.2.3)$$

This expression makes sense if $\{2na_{2n}\}_{n \in \mathbb{N}}, \{2nb_{2n}\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$, i.e. if $\kappa \in H^1(\mathbb{S}^1)$.

By (3.2.1) and (3.2.3), we obtain

$$\widehat{W}(\xi) = c_0 \delta_0 + \frac{1}{|\xi|^2} + \frac{\widehat{\kappa}\left(\frac{\xi}{|\xi|}\right)}{|\xi|^2} = c_0 \delta_0 + \frac{\widehat{\Psi}(\xi)}{|\xi|^2}, \quad (3.2.4)$$

where $\widehat{\Psi}(\xi) := 1 + \widehat{\kappa}\left(\frac{\xi}{|\xi|}\right)$ and the formula should be interpreted again in a distributional sense. We note that $\widehat{\Psi}$ is real, as it is the Fourier transform of an even function. In addition, it is even and homogeneous.

To prove the uniqueness of the minimizer, we will require $\widehat{\Psi}$ to be continuous on $\mathbb{S}^1$, therefore we need $\widehat{\kappa} \in C(\mathbb{S}^1)$. Because the embedding $H^s(\mathbb{S}^1) \hookrightarrow C(\mathbb{S}^1)$ is continuous if $s > 1/2$, this is true if $\widehat{\kappa} \in H^s(\mathbb{S}^1)$ with $s > 1/2$. By (3.2.3) and writing $\xi = e^{i\vartheta}$, we can express $\widehat{\kappa}$ on $\mathbb{S}^1$ as a Fourier series of variable ϑ :

$$\widehat{\kappa}(e^{i\vartheta}) = \sum_{n=1}^{+\infty} ((-1)^n 2na_{2n} \cos(2n\vartheta) + (-1)^n 2nb_{2n} \sin(2n\vartheta)).$$

Therefore, we can determine the Fourier coefficients of $\widehat{\kappa}$ and compute

$$\|\widehat{\kappa}\|_{H^s} = \sum_{n=1}^{+\infty} (1 + 4n^2)^s 4n^2 (a_{2n}^2 + b_{2n}^2).$$

On the other hand, by (3.2.2) we have

$$\|\kappa\|_{H^{s+1}} = \sum_{n=1}^{+\infty} (1 + 4n^2)^{s+1} (a_{2n}^2 + b_{2n}^2).$$

Comparing the two expressions, we observe that

$$\|\widehat{\kappa}\|_{H^s} \leq \|\kappa\|_{H^{s+1}}.$$

Hence, $\widehat{\kappa} \in H^s(\mathbb{S}^1)$ with $s > 1/2$ if $\kappa \in H^s(\mathbb{S}^1)$ with $s > 3/2$, which justifies our initial assumption.

LEMMA 5. *Let $\widehat{\Psi} \geq 0$ on $\mathbb{S}^1$ and $\mu_0, \mu_1 \in \mathcal{P}_c(\mathbb{R}^2)$ with finite interaction energy. Then, letting $\nu := \mu_0 - \mu_1$, we have*

$$\int_{\mathbb{R}^2} (W * \nu)(x) d\nu(x) = 2\pi \int_{\mathbb{R}^2} \frac{\widehat{\Psi}(\xi)}{|\xi|^2} |\widehat{\nu}(\xi)|^2 d\xi \quad (3.2.5)$$

In particular,

$$\int_{\mathbb{R}^2} (W * \nu)(x) d\nu(x) = 0 \iff \mu_0 = \mu_1. \quad (3.2.6)$$

PROOF. Formally, the idea is to apply Plancherel theorem:

$$\int_{\mathbb{R}^2} (W * \nu)(x) d\nu(x) = \int_{\mathbb{R}^2} (\widehat{W * \nu})(\xi) \overline{\widehat{\nu}(\xi)} d\xi.$$

Then, since

$$\widehat{W * \nu} = 2\pi \widehat{W} \widehat{\nu},$$

we can write

$$\int_{\mathbb{R}^2} (W * \nu)(x) d\nu(x) = 2\pi \int_{\mathbb{R}^2} \widehat{W}(\xi) \widehat{\nu}(\xi) \overline{\widehat{\nu}(\xi)} d\xi = 2\pi \int_{\mathbb{R}^2} \widehat{W}(\xi) |\widehat{\nu}(\xi)|^2 d\xi.$$

By (3.2.4), we have

$$\int_{\mathbb{R}^2} \widehat{W}(\xi) |\widehat{\nu}(\xi)|^2 d\xi = c_0 \int_{\mathbb{R}^2} |\widehat{\nu}(0)|^2 d\xi + \int_{\mathbb{R}^2} \frac{\widehat{\Psi}(\xi)}{|\xi|^2} |\widehat{\nu}(\xi)|^2 d\xi,$$

and, because

$$\widehat{\nu}(0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\nu = \frac{1}{2\pi} (\mu_0(\mathbb{R}^2) - \mu_1(\mathbb{R}^2)) = 0,$$

we obtain

$$\int_{\mathbb{R}^2} (W * \nu)(x) d\nu(x) = 2\pi \int_{\mathbb{R}^2} \frac{\widehat{\Psi}(\xi)}{|\xi|^2} |\widehat{\nu}(\xi)|^2 d\xi.$$

Unfortunately, W and ν are not regular enough to apply Plancherel theorem. To circumvent this difficulty, we will prove (3.2.5) by approximation.

Let $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^2} \varphi(\frac{x}{\varepsilon})$ be a mollifier supported in $B_\varepsilon(0)$, with $\varepsilon > 0$. We consider

$$\nu_\varepsilon(x) := (\nu * \varphi_\varepsilon)(x) = \int_{\mathbb{R}^2} \varphi_\varepsilon(x - y) d\nu(y) \in C_c^\infty(\mathbb{R}^2) \subseteq \mathcal{S}(\mathbb{R}^2).$$

Therefore, $\widehat{\nu}_\varepsilon \in \mathcal{S}(\mathbb{R}^2)$ as well, and

$$\begin{aligned} \widehat{\nu}_\varepsilon(0) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \varphi_\varepsilon(x - y) d\nu(y) \right) dx \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \varphi_\varepsilon(x - y) dx \right) d\nu(y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\nu = 0. \end{aligned}$$

We observe that $W * \nu_\varepsilon \in C^\infty(\mathbb{R}^2)$. Moreover, because $W \in \mathcal{S}'(\mathbb{R}^2)$ and $\nu_\varepsilon \in C_c^\infty(\mathbb{R}^2)$ (thus ν_ε belongs also to the space $\mathcal{E}'(\mathbb{R}^2)$ of distributions with compact support), we have¹

$$\widehat{W * \nu_\varepsilon} = 2\pi \widehat{W} \widehat{\nu}_\varepsilon.$$

Since $\widehat{\nu}_\varepsilon(0) = 0$, this implies

$$(\widehat{W * \nu_\varepsilon})(\xi) = 2\pi \left(c_0 \delta_0 + \frac{\widehat{\Psi}(\xi)}{|\xi|^2} \right) \widehat{\nu}_\varepsilon(\xi) = 2\pi \frac{\widehat{\Psi}(\xi)}{|\xi|^2} \widehat{\nu}_\varepsilon(\xi) \in L^1(\mathbb{R}^2).$$

¹For this result, see for instance Theorem 6.1 in [5].

In fact, using polar coordinates and recalling that $\widehat{\nu}_\varepsilon \in \mathcal{S}(\mathbb{R}^2)$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |(\widehat{W * \nu_\varepsilon})(\xi)| d\xi &= 2\pi \int_0^{2\pi} \int_0^{+\infty} \frac{|\widehat{\Psi}(re^{i\theta})|}{r^2} |\widehat{\nu}_\varepsilon(re^{i\theta})| r dr d\theta \\ &= 2\pi \int_0^{2\pi} |\widehat{\Psi}(e^{i\theta})| \left(\int_0^{+\infty} \frac{|\widehat{\nu}_\varepsilon(re^{i\theta})|}{r} dr \right) d\theta < +\infty. \end{aligned}$$

We can now apply Plancherel theorem to $W * \nu_\varepsilon$ and ν_ε and obtain an approximate version of (3.2.5):

$$\int_{\mathbb{R}^2} W * \nu_\varepsilon d\nu_\varepsilon = \int_{\mathbb{R}^2} (W * \nu_\varepsilon)(x) \nu_\varepsilon(x) dx = \int_{\mathbb{R}^2} (\widehat{W * \nu_\varepsilon})(\xi) \overline{\widehat{\nu}_\varepsilon(\xi)} d\xi = 2\pi \int_{\mathbb{R}^2} \frac{\widehat{\Psi}(\xi)}{|\xi|^2} |\widehat{\nu}_\varepsilon(\xi)|^2 d\xi.$$

Letting $\varepsilon \rightarrow 0^+$, on the right-hand side we have

$$\widehat{\nu}_\varepsilon(\xi) = 2\pi \widehat{\nu}(\xi) \widehat{\varphi}(\varepsilon\xi) \longrightarrow 2\pi \widehat{\nu}(\xi) \widehat{\varphi}(0) = \widehat{\nu}(\xi) \quad \forall \xi \in \mathbb{R}^2,$$

so $\frac{\widehat{\Psi}(\xi)}{|\xi|^2} |\widehat{\nu}_\varepsilon(\xi)|^2$ converges to $\frac{\widehat{\Psi}(\xi)}{|\xi|^2} |\widehat{\nu}(\xi)|^2$ for a.e. $\xi \in \mathbb{R}^2$, as $\varepsilon \rightarrow 0^+$. Since

$$|\widehat{\varphi}(\varepsilon\xi)| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \varphi(x) dx = \frac{1}{2\pi} \|\varphi\|_{L^1},$$

by dominated convergence we obtain

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^2} \frac{\widehat{\Psi}(\xi)}{|\xi|^2} |\widehat{\nu}_\varepsilon(\xi)|^2 d\xi = \int_{\mathbb{R}^2} \frac{\widehat{\Psi}(\xi)}{|\xi|^2} |\widehat{\nu}(\xi)|^2 d\xi.$$

For the left-hand side, a direct computation shows that

$$\begin{aligned} \int_{\mathbb{R}^2} (W * \nu_\varepsilon)(x) \nu_\varepsilon(x) dx &= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} W(y) \left(\int_{\mathbb{R}^2} \varphi_\varepsilon(x - y - z) d\nu(z) \right) dy \right) \left(\int_{\mathbb{R}^2} \varphi_\varepsilon(x - t) d\nu(t) \right) dx \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W(y) \left(\int_{\mathbb{R}^2} \varphi_\varepsilon(y + z - x) \varphi_\varepsilon(x - t) dx \right) dy d\nu(z) d\nu(t) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W(y) \left(\int_{\mathbb{R}^2} \varphi_\varepsilon(y + z + t - x) \varphi_\varepsilon(x) dx \right) dy d\nu(z) d\nu(t) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} W(y) (\varphi_\varepsilon * \varphi_\varepsilon)(y + z + t) dy \right) d\nu(z) d\nu(t) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} W(y) (\varphi_\varepsilon * \varphi_\varepsilon)(-z - t - y) dy \right) d\nu(z) d\nu(t) \\ &= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} (W * \varphi_\varepsilon * \varphi_\varepsilon)(z + t) d\nu(z) \right) d\nu(t) \\ &= \int_{\mathbb{R}^2} (W * \varphi_\varepsilon * \varphi_\varepsilon * \nu)(t) d\nu(t) = \int_{\mathbb{R}^2} (W * \varphi_\varepsilon * \varphi_\varepsilon) * \nu d\nu \\ &= \int_{\mathbb{R}^2} (W * \psi_\varepsilon) * \nu d\nu, \end{aligned}$$

with $\psi_\varepsilon := \varphi_\varepsilon * \varphi_\varepsilon$ radial, non-negative, compactly supported in $B_{2\varepsilon}(0)$ and integrating to one. The function $(x, y) \mapsto (W * \psi_\varepsilon)(x - y)$ is continuous, hence bounded from below on compact sets, and even. Since $\text{supp } \mu_0, \text{supp } \mu_1$ are compact, by Tonelli's theorem we

can thus write

$$\begin{aligned} \int_{\mathbb{R}^2} (W * \psi_\varepsilon) * \nu \, d\nu &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (W * \psi_\varepsilon)(x - y) \, d\mu_0(y) d\mu_0(x) \\ &\quad + \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (W * \psi_\varepsilon)(x - y) \, d\mu_1(y) d\mu_1(x) \\ &\quad - 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (W * \psi_\varepsilon)(x - y) \, d\mu_0(y) d\mu_1(x). \end{aligned} \quad (3.2.7)$$

We observe that $(W * \psi_\varepsilon)(x) \rightarrow W(x)$ for every $x \in \mathbb{R}^2$, as $\varepsilon \rightarrow 0^+$, because W is continuous. By (3.0.2), we have

$$(W * \psi_\varepsilon)(x) \leq (W_0 * \psi_\varepsilon)(x) + C_2 \quad \forall x \in \mathbb{R}^2.$$

Because $W_0 * \psi_\varepsilon$ is harmonic and W_0 is superharmonic in $\mathbb{R}^2$, then for every $\overline{B_{2\varepsilon}(x)} \subseteq \mathbb{R}^2$ we have²

$$\begin{aligned} (W_0 * \psi_\varepsilon)(x) &= \frac{1}{4\pi\varepsilon} \int_{\partial B_{2\varepsilon}(x)} (W_0 * \psi_\varepsilon)(y) \, d\sigma(y) = \frac{1}{4\pi\varepsilon} \int_{\partial B_{2\varepsilon}(0)} (W_0 * \psi_\varepsilon)(y - x) \, d\sigma(y) \\ &= \frac{1}{4\pi\varepsilon} \int_{\partial B_{2\varepsilon}(0)} \left(\int_{B_{2\varepsilon}(0)} W_0(y - x - z) \psi_\varepsilon(z) \, dz \right) d\sigma(y) \\ &= \frac{1}{4\pi\varepsilon} \int_{\partial B_{2\varepsilon}(0)} \left(\int_0^{2\varepsilon} \int_{\partial B_\rho(0)} W_0(z - (y - x)) \psi_\varepsilon(z) \, d\sigma(z) d\rho \right) d\sigma(y) \\ &= \frac{1}{2\varepsilon} \int_{\partial B_{2\varepsilon}(0)} \int_0^{2\varepsilon} \left(\frac{1}{2\pi\rho} \int_{\partial B_\rho(y-x)} W_0(z) \, d\sigma(z) \right) \tilde{\psi}_\varepsilon(\rho) \rho \, d\rho d\sigma(y) \\ &\leq \frac{1}{2\varepsilon} \int_{\partial B_{2\varepsilon}(0)} \int_0^{2\varepsilon} W_0(y - x) \tilde{\psi}_\varepsilon(\rho) \rho \, d\rho d\sigma(y) \\ &= 2\pi \int_0^{2\varepsilon} \left(\frac{1}{4\pi\varepsilon} \int_{\partial B_{2\varepsilon}(x)} W_0(y) \, d\sigma(y) \right) \tilde{\psi}_\varepsilon(\rho) \rho \, d\rho \\ &\leq 2\pi \int_0^{2\varepsilon} W_0(x) \tilde{\psi}_\varepsilon(\rho) \rho \, d\rho = \int_{B_{2\varepsilon}(0)} W_0(x) \psi_\varepsilon(z) \, dz = W_0(x), \end{aligned}$$

where we have used the fact that W_0 and ψ_ε are radial and we denote $\tilde{\psi}_\varepsilon(|z|) := \psi_\varepsilon(z)$. Hence, it follows that

$$(W_0 * \psi_\varepsilon)(x) \leq W_0(x) \leq W(x) - C_1 \quad \forall x \in \mathbb{R}^2.$$

Combining the two inequalities above, we obtain the upper bound

$$(W * \psi_\varepsilon)(x) \leq W(x) - C_1 + C_2 \quad \forall x \in \mathbb{R}^2.$$

By hypothesis, μ_0, μ_1 have finite interaction energy, therefore by dominated convergence applied to each term on the right-hand side of (3.2.7) we obtain

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^2} (W * \psi_\varepsilon) * \nu \, d\nu = \int_{\mathbb{R}^2} W * \nu \, d\nu,$$

²We recall that a function $\varphi: \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$, with $\Omega \subseteq \mathbb{R}^n$ open, is superharmonic if:

- φ is lower semi-continuous;
- for every closed ball $\overline{B_r(x)} \subseteq \Omega$, it satisfies the inequality

$$\varphi(x) \geq \frac{1}{\sigma(\partial B_r(x))} \int_{\partial B_r(x)} \varphi(y) \, d\sigma(y),$$

where σ denotes the surface measure on the n -sphere.

If $\varphi \in C^2(\Omega)$, the equality holds if and only if φ is harmonic, i.e. $\Delta\varphi = 0$ on Ω .

which proves (3.2.5).

To prove the last claim, assume that

$$0 = \int_{\mathbb{R}^2} (W * \nu)(x) d\nu(x) = 2\pi \int_{\mathbb{R}^2} \frac{\widehat{\Psi}(\xi)}{|\xi|^2} |\widehat{\nu}(\xi)|^2 d\xi.$$

We observe that $\widehat{\Psi}(\xi_0) > 0$ for some $\xi_0 \in \mathbb{S}^1$, otherwise $\widehat{W} = c_0 \delta_0$, which would imply W to be constant. Since $\widehat{\Psi}$ is continuous on $\mathbb{S}^1$, there exist $\eta > 0$, $r > 0$ such that $\widehat{\Psi}(\xi) > \eta$ for every $\xi \in B_r(\xi_0)$. Therefore, we should have $\widehat{\nu} = 0$ on $B_r(\xi_0)$. On the other hand, since $\nu \in \mathcal{E}'(\mathbb{R}^2)$, by Paley-Wiener-Schwartz theorem, its Fourier transform $\widehat{\nu}$ is an entire function. Hence, $\widehat{\nu} = 0$ on the whole $\mathbb{R}^2$, which implies $\nu = 0$. $\square$

We are now able to prove the main result of this section.

THEOREM 7 (uniqueness of the minimizer). *If $\widehat{\Psi} \geq 0$ on $\mathbb{S}^1$, the functional I is strictly convex on the class of compactly supported measures with finite interaction energy.*

In particular, the minimizer is unique. Moreover, a measure $\mu \in \mathcal{P}_c(\mathbb{R}^2)$, with $I(\mu) < +\infty$, minimizes I if and only if it satisfies the Euler-Lagrange equations (3.1.2)-(3.1.3).

PROOF. Let $\mu_0, \mu_1 \in \mathcal{P}_c(\mathbb{R}^2)$, with $I_W(\mu_0), I_W(\mu_1) < +\infty$. By Lemma 5, we have

$$\int_{\mathbb{R}^2} W * (\mu_0 - \mu_1) d(\mu_0 - \mu_1) \geq 0,$$

which implies

$$\int_{\mathbb{R}^2} W * \mu_0 d\mu_0 + \int_{\mathbb{R}^2} W * \mu_1 d\mu_1 \geq 2 \int_{\mathbb{R}^2} W * \mu_0 d\mu_1.$$

For $t \in [0, 1]$, we define $\mu_t := (1 - t)\mu_0 + t\mu_1 \in \mathcal{P}_c(\mathbb{R}^2)$. Therefore,

$$\begin{aligned} \int_{\mathbb{R}^2} W * \mu_t d\mu_t &= (1 - t)^2 \int_{\mathbb{R}^2} W * \mu_0 d\mu_0 + t^2 \int_{\mathbb{R}^2} W * \mu_1 d\mu_1 + 2t(1 - t) \int_{\mathbb{R}^2} W * \mu_0 d\mu_1 \\ &\leq (1 - t) \int_{\mathbb{R}^2} W * \mu_0 d\mu_0 + t \int_{\mathbb{R}^2} W * \mu_1 d\mu_1, \end{aligned}$$

where, by (3.2.6), the equality holds if and only if $\mu_0 = \mu_1$. Moreover,

$$\int_{\mathbb{R}^2} V d\mu_t = (1 - t) \int_{\mathbb{R}^2} V d\mu_0 + t \int_{\mathbb{R}^2} V d\mu_1,$$

hence we deduce that I is strictly convex.

This implies that the minimizer is unique. In fact, suppose by contradiction that μ_0, μ_1 are both minimizers of I , with $\mu_0 \neq \mu_1$ and $I(\mu_0) = I(\mu_1) = m < +\infty$ (which is guaranteed by Theorem 6). Considering $\mu_t = (1 - t)\mu_0 + t\mu_1$ with $t \in [0, 1]$ as above, we obtain

$$I(\mu_t) < (1 - t)I(\mu_0) + tI(\mu_1) = m,$$

which contradicts the minimality of μ_0, μ_1 .

Finally, we prove that if a compactly supported measure with finite energy satisfies the Euler-Lagrange equations (3.1.2)-(3.1.3), then it is the minimizer of I . By Theorem 6, there exists a minimizer μ that satisfies (3.1.2)-(3.1.3) for some constant $c_\mu \in \mathbb{R}$. Assume that there exists another measure $\nu \in \mathcal{P}_c(\mathbb{R}^2)$ with $I(\nu) < +\infty$, $\nu \neq \mu$, that satisfies the Euler-Lagrange equations for a constant $c_\nu \in \mathbb{R}$. Letting $\mu_t := (1 - t)\mu + t\nu$ for $t \in (0, 1)$, we have

$$\begin{aligned} (W * \mu_t + \frac{1}{2}V)(x) &= \int_{\mathbb{R}^2} W(x - y) + \frac{1}{2}V(x) d\mu_t(y) \\ &= (1 - t) \int_{\mathbb{R}^2} W(x - y) + \frac{1}{2}V(x) d\mu(y) + t \int_{\mathbb{R}^2} W(x - y) + \frac{1}{2}V(x) d\nu(y) \\ &= (1 - t)c_\mu + tc_\nu. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^2} \left(W * \mu_t + \frac{1}{2}V \right) d\mu_t &= \int_{\mathbb{R}^2} (1-t)c_\mu + tc_\nu d\mu_t = (1-t)c_\mu + tc_\nu \\ &= (1-t) \int_{\mathbb{R}^2} \left(W * \mu + \frac{1}{2}V \right) d\mu + t \int_{\mathbb{R}^2} \left(W * \nu + \frac{1}{2}V \right) d\nu. \end{aligned}$$

On the other hand, by strict convexity

$$\int_{\mathbb{R}^2} \left(W * \mu_t + \frac{1}{2}V \right) d\mu_t < (1-t) \int_{\mathbb{R}^2} \left(W * \mu + \frac{1}{2}V \right) d\mu + t \int_{\mathbb{R}^2} \left(W * \nu + \frac{1}{2}V \right) d\nu,$$

which gives a contradiction. $\square$

3.3. Characterisation of minimizers for quadratic confinement

In this section, we will discuss the case of quadratic confinement, i.e. $V(x) = |x|^2$. This confinement potential clearly satisfies the assumptions established in the beginning, being continuous, non-negative and fulfilling (3.0.4)-(3.0.5), and therefore guarantees the existence and uniqueness of the minimizer. In addition, with this particular choice of the confinement, we can also give an explicit characterisation of the shape of the minimizer.

We consider two distinct cases: isotropic and anisotropic energy.

3.3.1. Isotropic energy. First we consider the case of isotropic energy, that is, the interaction energy does not depend on the direction (i.e. the energy functional is invariant of rotations). This corresponds to $\kappa = 0$, and the energy functional reduces to

$$I(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W_0(x-y) d\mu(x)d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x). \quad (3.3.1)$$

The minimizer $\mu_0 \in \mathcal{P}_c(\mathbb{R}^2)$ must also be invariant of rotations. Otherwise, we would obtain another minimizer by considering the push-forward of μ_0 through a rotation $R \in SO(2)$, i.e.

$$\mu_1(A) := \mu_0(R^T A) \quad \forall A \in \mathcal{B}(\mathbb{R}^2),$$

which would contradict the uniqueness of the minimizer.

By the Euler-Lagrange equation (3.1.2), we have that μ_0 must satisfy

$$(W_0 * \mu_0)(x) + \frac{1}{2}|x|^2 = c \quad \text{for } \mu_0\text{-a.e. } x \in \text{supp } \mu_0,$$

which, recalling that $\Delta W_0 = -2\pi\delta_0$, implies

$$\Delta W_0 * \mu_0 + 2 = -2\pi\mu_0 + 2 = 0 \quad \text{on } \text{supp } \mu_0.$$

Hence, the minimizer of I must be constant on its support, i.e. it is a uniform probability measure. Indeed, we have the following result.

THEOREM 8. *The minimizer of the isotropic energy (3.3.1) is given by the measure*

$$\mu_0 = \frac{1}{\pi} \chi_{B_1(0)} \mathcal{L}^2, \quad (3.3.2)$$

i.e. the uniform probability distribution on the unit ball. This is also called circle law.

PROOF. We need to show that μ_0 satisfies the Euler-Lagrange equations. By a change of coordinates, we can write

$$(W_0 * \mu_0)(x) = \frac{1}{\pi} \int_{B_1(0)} W_0(x-y) dy = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} W_0(x - re^{i\theta}) r d\theta dr.$$

Then, we have

$$\frac{1}{2\pi} \int_0^{2\pi} W_0(x - re^{i\theta}) d\theta = \begin{cases} -\ln|x| & \text{if } r \leq |x| \\ -\ln r & \text{if } r > |x|. \end{cases} \quad (3.3.3)$$

In fact, since W_0 is harmonic on $\mathbb{R}^2 \setminus \{0\}$, if $r < |x|$, by the mean value property of harmonic functions we have

$$-\ln |x| = W_0(x) = \frac{1}{2\pi r} \int_{\partial B_r(x)} W_0(y) d\sigma(y) = \frac{1}{2\pi} \int_0^{2\pi} W_0(x - re^{i\theta}) d\theta.$$

Analogously, denoting $x = |x|e^{i\theta_1}$ with $\theta_1 \in [0, 2\pi]$, if $r > |x|$, we obtain

$$\begin{aligned} -\ln r = W_0(re^{i\theta_1}) &= \frac{1}{2\pi|x|} \int_{\partial B_{|x|}(re^{i\theta_1})} W_0(y) d\sigma(y) = \frac{1}{2\pi} \int_0^{2\pi} W_0(re^{i\theta_1} - |x|e^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} W_0(re^{-i\theta} - |x|e^{-i\theta_1}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} W_0(\overline{|x|e^{i\theta_1} - re^{i\theta}}) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} W_0(\overline{x - re^{i\theta}}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} W_0(x - re^{i\theta}) d\theta. \end{aligned}$$

If $r = |x|$, we apply dominated convergence:

$$\frac{1}{2\pi} \int_0^{2\pi} W_0(x - re^{i\theta}) d\theta = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi} \int_0^{2\pi} W_0(x - (r - \varepsilon)e^{i\theta}) d\theta = -\ln |x|.$$

Using (3.3.3), we compute

$$\frac{1}{\pi} \int_0^1 \left(\int_0^{2\pi} W_0(x - re^{i\theta}) d\theta \right) r dr = \begin{cases} -2 \left(\int_0^x \ln |x|r dr + \int_x^1 \ln(r)r dr \right) & \text{if } |x| \leq 1 \\ -2 \int_0^1 \ln |x|r dr & \text{if } |x| > 1, \end{cases}$$

which yields

$$(W_0 * \mu_0)(x) = \begin{cases} \frac{1}{2}(1 - |x|^2) & \text{if } |x| \leq 1 \\ -\ln |x| & \text{if } |x| > 1. \end{cases}$$

Hence, we obtain

$$(W_0 * \mu_0)(x) + \frac{1}{2}|x|^2 = \frac{1}{2} \quad \text{if } x \in B_1(0)$$

and

$$(W_0 * \mu_0)(x) + \frac{1}{2}|x|^2 = -\ln |x| + \frac{1}{2}|x|^2 \geq \frac{1}{2} \quad \text{if } x \notin B_1(0),$$

therefore μ_0 satisfies the Euler-Lagrange equations (3.1.2)-(3.1.3) with $c = \frac{1}{2}$, thus being the unique minimizer of I . $\square$

3.3.2. Anisotropic energy. Now we consider the more general case of anisotropic energy, i.e. $\kappa \neq 0$. The energy functional is

$$I(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (W_0(x - y) + \kappa(x - y)) d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x), \quad (3.3.4)$$

with κ of class $H^s(\mathbb{S}^1)$, $s > 3/2$, as assumed in the beginning.

Perhaps not surprisingly, the shape of the minimizer can be related to the sign of the Fourier transform $\widehat{\Psi}$, which already played a central role in the uniqueness of the minimizer. We first suppose that $\widehat{\Psi}$ is strictly positive.

THEOREM 9. *Let $\widehat{\Psi} > 0$ on $\mathbb{S}^1$. Then, the minimizer of the anisotropic energy (3.3.4) is given by the measure*

$$\mu = \frac{1}{|E|} \chi_E \mathcal{L}^2, \quad (3.3.5)$$

where E is the elliptic domain $E = RE_0$, with

$$E_0 = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \leq 1 \right\}, \quad a_1^2 + a_2^2 = 2 \text{ and } R \in SO(2).$$

PROOF. As we did in the isotropic case, we want to show that μ satisfies the Euler-Lagrange equations (3.1.2)-(3.1.3). To this end, we need to compute $W * \mu$. The idea is to retrieve its expression by calculating the inverse Fourier transform of $\widehat{W * \mu}$. Since $\widehat{W}$ is not a function, but rather a tempered distribution, we apply this reasoning to the Fourier transform of $\nabla(W * \mu)$ (note that the gradient exists as $W * \mu \in C^1(\mathbb{R}^2)$). In fact,

$$\nabla(\widehat{W * \mu})(\xi) = i\xi(\widehat{W * \mu})(\xi) = 2\pi i\xi \widehat{W}(\xi) \widehat{\mu}(\xi),$$

and

$$i\xi \widehat{W}(\xi) = i\xi \left(c_0 \delta_0 + \frac{\widehat{\Psi}(\xi)}{|\xi|^2} \right) = i\xi \frac{\widehat{\Psi}(\xi)}{|\xi|^2},$$

because the factor ξ cancels out the singular part of $\widehat{W}$. Therefore, we have

$$\nabla(\widehat{W * \mu})(\xi) = 2\pi i\xi \frac{\widehat{\Psi}(\xi)}{|\xi|^2} \widehat{\mu}(\xi).$$

To compute the Fourier transform $\widehat{\mu}$ we reason as follows. Writing $a = (a_1, a_2) \in \mathbb{R}^2$ and

$$D(a) := \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix},$$

we have $E = RD(a)B_1(0)$, therefore

$$\chi_E(x) = \chi_{B_1(0)}((RD(a))^{-1}x). \quad (3.3.6)$$

The Fourier transform of $\chi_{B_1(0)}$ is given by

$$\widehat{\chi_{B_1(0)}}(\xi) = \frac{J_1(|\xi|)}{|\xi|}, \quad (3.3.7)$$

where J_1 is the Bessel function of first kind of order 1, which can be expressed as³

$$J_1(|\xi|) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(k+1)!} \left(\frac{|\xi|}{2} \right)^{2k+1}.$$

From this expression, we observe that

$$J_1(|\xi|) \simeq \frac{1}{2}|\xi| \quad \text{as } |\xi| \rightarrow 0.$$

Moreover,

$$|J_1(|\xi|)| \leq C|\xi|^{-\frac{1}{2}} \quad \text{as } |\xi| \rightarrow +\infty.$$

By (3.3.6) and (3.3.7), we have

$$\begin{aligned} \widehat{\chi_E}(\xi) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_E(x) e^{-i\langle x, \xi \rangle} dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_{B_1(0)}(x) e^{-i\langle x, D(a)R^T \xi \rangle} |\det(D(a))| dx \\ &= \frac{|E|}{\pi} \widehat{\chi_{B_1(0)}}(D(a)R^T \xi) = \frac{|E|}{\pi} \frac{J_1(|D(a)R^T \xi|)}{|D(a)R^T \xi|}, \end{aligned}$$

therefore we obtain

$$\widehat{\mu}(\xi) = \frac{1}{\pi} \frac{J_1(|D(a)R^T \xi|)}{|D(a)R^T \xi|}.$$

³See formula 9.1.10 in [6].

Hence,

$$\nabla(\widehat{W * \mu})(\xi) = 2i\xi \frac{\widehat{\Psi}(\xi)}{|\xi|^2} \frac{J_1(|D(a)R^T \xi|)}{|D(a)R^T \xi|},$$

which belongs to $L^1(\mathbb{R}^2)$. In fact, due to the properties of $\widehat{\Psi}$ and J_1 , we have

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla(\widehat{W * \mu})(\xi)| d\xi &= 2 \int_0^{2\pi} \int_0^{+\infty} \frac{|\widehat{\Psi}(re^{i\theta})|}{r} \frac{|J_1(|D(a)R^T re^{i\theta}|)|}{|D(a)R^T re^{i\theta}|} r dr d\theta \\ &= 2 \int_0^{2\pi} |\widehat{\Psi}(e^{i\theta})| \left(\int_0^{+\infty} \frac{|J_1(r|D(a)R^T e^{i\theta}|)|}{r|D(a)R^T e^{i\theta}|} dr \right) d\theta < +\infty. \end{aligned}$$

We can thus apply the Fourier inversion formula, which gives

$$\nabla(W * \mu)(x) = \int_{\mathbb{R}^2} i\xi \frac{\widehat{\Psi}(\xi)}{|\xi|^2} \widehat{\mu}(\xi) e^{i\langle x, \xi \rangle} d\xi = - \int_{\mathbb{R}^2} \xi \frac{\widehat{\Psi}(\xi)}{|\xi|^2} \widehat{\mu}(\xi) \sin(\langle x, \xi \rangle) d\xi,$$

where the last equality descends from the fact that $\widehat{\Psi}$ and $\widehat{\mu}$ are even. By the change of variables $\xi = rz$, with $(r, z) \in [0, +\infty) \times \mathbb{S}^1$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} \xi \frac{\widehat{\Psi}(\xi)}{|\xi|^2} \widehat{\mu}(\xi) \sin(\langle x, \xi \rangle) d\xi &= \int_{\mathbb{S}^1} \int_0^{+\infty} z \widehat{\Psi}(rz) \widehat{\mu}(rz) \sin(r\langle x, z \rangle) dr d\sigma(z) \\ &= \frac{1}{\pi} \int_{\mathbb{S}^1} \int_0^{+\infty} z \widehat{\Psi}(z) \frac{J_1(r|D(a)R^T z|)}{r|D(a)R^T z|} \sin(r\langle x, z \rangle) dr d\sigma(z), \end{aligned}$$

therefore

$$\nabla(W * \mu)(x) = -\frac{1}{\pi} \int_{\mathbb{S}^1} \frac{z \widehat{\Psi}(z)}{|D(a)R^T z|} \left(\int_0^{+\infty} \frac{J_1(\rho)}{\rho} \sin(\rho\alpha(x, z)) d\rho \right) d\sigma(z), \quad (3.3.8)$$

where $\rho := r|D(a)R^T z|$ and $\alpha(x, z) := \frac{\langle x, z \rangle}{|D(a)R^T z|}$.

The integral on the right-hand side can be computed using

$$\int_0^{+\infty} \frac{J_1(\rho)}{\rho} \sin(\rho\alpha) d\rho = \begin{cases} \alpha & \text{if } 0 \leq \alpha \leq 1 \\ \frac{1}{\alpha + \sqrt{\alpha^2 - 1}} & \text{if } \alpha > 1 \end{cases} \quad (3.3.9)$$

and observing that $|\alpha(x, z)| \leq 1$ for every $x \in E$, because

$$|\langle x, z \rangle| = | \langle (RD(a))^{-1}x, D(a)R^T z \rangle | \leq |(RD(a))^{-1}x| |D(a)R^T z| \leq |D(a)R^T z|.$$

Hence, we conclude that, for every $x \in E$,

$$\nabla(W * \mu)(x) = -\frac{1}{\pi} \int_{\mathbb{S}^1} \frac{\widehat{\Psi}(z)}{|D(a)R^T z|} \alpha(x, z) z d\sigma(z) = -\frac{1}{\pi} \int_{\mathbb{S}^1} \frac{\widehat{\Psi}(z)}{|D(a)R^T z|^2} \langle x, z \rangle z d\sigma(z).$$

From this expression, we note that $\nabla(W * \mu)$ is a linear homogeneous polynomial. Therefore, up to a constant, $W * \mu$ is a quadratic homogeneous polynomial inside E .

We need to find $a_1, a_2 > 0$ and $R \in SO(2)$ that satisfy the Euler-Lagrange equation (3.1.2). This is equivalent to satisfying

$$\nabla(W * \mu)(x) + x = 0 \quad \forall x \in E.$$

Therefore, we need to solve the following system of three equations:

$$\frac{1}{\pi} \int_{\mathbb{S}^1} \frac{\widehat{\Psi}(z)}{|D(a)R^T z|^2} z_j z_k d\sigma(z) = \delta_{jk} \quad \text{for } j, k = 1, 2, \quad (3.3.10)$$

where δ_{jk} denotes the Kronecker delta. We observe that

$$|D(a)R^T z|^2 = \langle D(a)R^T z, D(a)R^T z \rangle = \langle Mz, z \rangle, \text{ with } M := RD(a)^2 R^T \in SPD(2),$$

where $SPD(2)$ denotes the set of 2×2 symmetric positive-definite matrices. Therefore, solving with respect to a_1, a_2, R is equivalent to finding $M \in SPD(2)$ that satisfies

$$\frac{1}{\pi} \int_{\mathbb{S}^1} \frac{\widehat{\Psi}(z)}{\langle Mz, z \rangle} z_j z_k d\sigma(z) = \delta_{jk} \quad \text{for } j, k = 1, 2. \quad (3.3.11)$$

Denoting by M_{jk} the elements of M and multiplying both sides of the equation by M_{jk} , we obtain

$$\text{tr}(M) = M_{11} + M_{22} = \frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(z) d\sigma(z) = \frac{1}{\pi} \int_{\mathbb{S}^1} (1 + \widehat{\kappa}(z)) d\sigma(z) = 2$$

and, since $\text{tr}(M) = a_1^2 + a_2^2$, the semi-axes of E_0 must satisfy

$$a_1^2 + a_2^2 = 2.$$

We define $f: SPD(2) \rightarrow \mathbb{R}$,

$$f(M) := -\frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(z) \ln(\langle Mz, z \rangle) d\sigma(z) + \text{tr}(M).$$

We note that $M_0 \in SPD(2)$ is a solution of (3.3.11) if it is a critical point of f , i.e. $\nabla_M f(M_0) = 0$, with $\nabla_M = (\frac{\partial}{\partial M_{11}}, \frac{\partial}{\partial M_{12}}, \frac{\partial}{\partial M_{22}})$. We will show that f has a minimum in the open set $SPD(2)$, therefore it also has a critical point, which is thus a solution of (3.3.11). For a fixed M and for $t > 0$, we consider

$$f(tM) = -\frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(z) \ln(\langle Mz, z \rangle) d\sigma(z) - 2 \ln(t) + t \text{tr}(M)$$

as a function of t and observe that it is minimized when $t = \frac{2}{\text{tr}(M)}$. Therefore, minimizing f on $SPD(2)$ is equivalent to minimizing it on the subset

$$\mathcal{M} = \{M \in SPD(2) : \text{tr}(M) = 2\}.$$

By the spectral theorem, every matrix $M \in \mathcal{M}$ can be decomposed as

$$M = QD(b)Q^T,$$

with $Q \in SO(2)$ and $b = (\beta, 2 - \beta)$, $\beta \in (0, 2)$. Hence, for $M \in \mathcal{M}$, we can write

$$\begin{aligned} f(M) &= -\frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(z) \ln(\langle QD(b)Q^T z, z \rangle) d\sigma(z) + 2 \\ &= -\frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(Qz) \ln(\beta z_1^2 + (2 - \beta)z_2^2) d\sigma(z) + 2. \end{aligned}$$

We define $\psi: [0, 2] \times SO(2) \rightarrow \mathbb{R}$,

$$\psi(\beta, Q) := -\frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(Qz) \ln(\beta z_1^2 + (2 - \beta)z_2^2) d\sigma(z),$$

and look for a minimizer in $(0, 2) \times SO(2)$. The function ψ is continuous, therefore it admits a minimizer (β_0, Q_0) in the compact set $[0, 2] \times SO(2)$. We need to prove that $\beta_0 \neq 0, 2$.

By assumption, there exists $C_0 > 0$ such that $\widehat{\Psi}(\xi) \geq C_0$ for every $\xi \in \mathbb{S}^1$. Hence,

$$\begin{aligned} \frac{\partial}{\partial \beta} \psi(\beta, Q_0) &= -\frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(Q_0 z) \frac{z_1^2 - z_2^2}{\beta z_1^2 + (2 - \beta) z_2^2} d\sigma(z) \\ &= \frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(Q_0 z) \frac{2z_2^2 - 1}{\beta z_1^2 + (2 - \beta) z_2^2} d\sigma(z) \\ &\leq \frac{1}{\pi} \left(\int_{\mathbb{S}^1} \widehat{\Psi}(Q_0 z) \frac{2}{2 - \beta} d\sigma(z) - \int_{\mathbb{S}^1} C_0 \frac{1}{\beta z_1^2 + (2 - \beta) z_2^2} d\sigma(z) \right) \\ &= \frac{4}{2 - \beta} - \frac{1}{\pi} \int_{\mathbb{S}^1} \frac{C_0}{\beta z_1^2 + (2 - \beta) z_2^2} d\sigma(z). \end{aligned}$$

By Fatou's lemma, we have

$$\begin{aligned} \liminf_{\beta \rightarrow 0^+} \frac{1}{\pi} \int_{\mathbb{S}^1} \frac{C_0}{\beta z_1^2 + (2 - \beta) z_2^2} d\sigma(z) &\geq \frac{1}{\pi} \int_{\mathbb{S}^1} \frac{C_0}{2z_2^2} d\sigma(z) \\ &= \frac{C_0}{2\pi} \int_0^{2\pi} \frac{1}{\sin^2(\theta)} d\theta = \frac{C_0}{2\pi} [-\cot(\theta)]_0^{2\pi} = +\infty, \end{aligned}$$

therefore $\frac{\partial}{\partial \beta} \psi(\beta, Q_0) \rightarrow -\infty$ as $\beta \rightarrow 0^+$. We deduce that there exists $\delta > 0$ such that $\frac{\partial}{\partial \beta} \psi(\beta, Q_0) < 0$ for every $\beta \in (0, \delta)$. Hence, $\beta_0 \neq 0$. Repeating the same reasoning for $\beta \rightarrow 2^-$, we can show that $\beta_0 \neq 2$ as well. This proves the existence of a critical point for f , that is, a solution of (3.3.11).

We are left to show that μ satisfies the second Euler-Lagrange equation (3.1.3). This is true if $(W * \mu)(x) + \frac{1}{2}|x|^2$ increases in the outward normal direction to ∂E , i.e.

$$\langle \nabla(W * \mu)(x) + x, x \rangle = \langle \nabla(W * \mu)(x), x \rangle + |x|^2 \geq 0 \quad \forall x \in \mathbb{R}^2 \setminus E.$$

Every $x \in \mathbb{R}^2 \setminus E$ can be written as $x = tx_0$ for some $x_0 \in E$ and $t > 0$. By the first Euler-Lagrange equation, we have

$$\langle \nabla(W * \mu)(x_0), x_0 \rangle + |x_0|^2 \geq 0,$$

which yields

$$-\frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(z) \alpha^2(x_0, z) d\sigma(z) + |x_0|^2 = 0.$$

Multiplying by t^2 , we obtain

$$-\frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(z) \alpha^2(x, z) d\sigma(z) + |x|^2 = 0. \quad (3.3.12)$$

On the other hand, by (3.3.8) and (3.3.9), we find

$$\begin{aligned} \langle \nabla(W * \mu)(x), x \rangle &= -\frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(z) \alpha^2(x, z) \chi_{[0,1]}(|\alpha(x, z)|) d\sigma(z) \\ &\quad - \frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(z) \frac{|\alpha(x, z)|}{|\alpha(x, z)| + \sqrt{\alpha^2(x, z) - 1}} \chi_{(1,+\infty)}(|\alpha(x, z)|) d\sigma(z). \end{aligned}$$

Together with (3.3.12), this implies

$$\begin{aligned} \langle \nabla(W * \mu)(x), x \rangle + |x|^2 &= \frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(z) \alpha^2(x, z) \chi_{(1,+\infty)}(|\alpha(x, z)|) d\sigma(z) \\ &\quad - \frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(z) \frac{|\alpha(x, z)|}{|\alpha(x, z)| + \sqrt{\alpha^2(x, z) - 1}} \chi_{(1,+\infty)}(|\alpha(x, z)|) d\sigma(z) \\ &= \frac{1}{\pi} \int_{\mathbb{S}^1} \widehat{\Psi}(z) |\alpha(x, z)| \sqrt{\alpha^2(x, z) - 1} \chi_{(1,+\infty)}(|\alpha(x, z)|) d\sigma(z) \geq 0, \end{aligned}$$

which concludes the proof. $\square$

Now, we will consider the more general case of a non-negative $\widehat{\Psi}$. Remarkably, we will observe that in the degenerate case, i.e. if $\widehat{\Psi}(\xi) = 0$ for some $\xi \in \mathbb{S}^1$, the shape of the minimizer may change radically.

THEOREM 10. *Let $\widehat{\Psi} \geq 0$ on $\mathbb{S}^1$. Then, the minimizer of the anisotropic energy is either as in (3.3.5) or is the push-forward of the so-called semicircle law,*

$$\frac{1}{\pi} \delta_0(x_1) \otimes \sqrt{2 - x_2^2} \mathcal{H}^1|_{[-\sqrt{2}, \sqrt{2}]}(x_2), \quad (3.3.13)$$

through a rotation map $\rho(x) = Rx$, with $R \in SO(2)$ such that $\widehat{\Psi}(Re_1) = 0$.⁴

PROOF. Let $\varepsilon > 0$. We consider the following approximation of I :

$$I_\varepsilon(\mu) := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W_\varepsilon(x - y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x),$$

where W_ε is the approximation of W given by

$$W_\varepsilon(x) := (1 + \varepsilon)W_0(x) + \kappa(x).$$

The corresponding Fourier transform is

$$\widehat{W}_\varepsilon(\xi) = c_\varepsilon \delta_0 + \frac{\widehat{\Psi}_\varepsilon(\xi)}{|\xi|^2},$$

where $\widehat{\Psi}_\varepsilon := \widehat{\Psi} + \varepsilon > 0$ on $\mathbb{S}^1$. Hence, by the previous theorem, I_ε admits a unique minimizer μ_ε of the form

$$\mu_\varepsilon = \frac{1}{|E_\varepsilon|} \chi_{E_\varepsilon} \mathcal{L}^2,$$

where $E_\varepsilon = R_\varepsilon E_{0,\varepsilon}$, with

$$E_{0,\varepsilon} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{a_{1,\varepsilon}^2} + \frac{x_2^2}{a_{2,\varepsilon}^2} \leq 1 \right\}, \quad a_{1,\varepsilon}^2 + a_{2,\varepsilon}^2 = 2 \text{ and } R_\varepsilon \in SO(2).$$

For every $\varepsilon > 0$, we have that $\text{supp } \mu_\varepsilon \subseteq \overline{B_{\sqrt{2}}(0)}$, therefore the sequence $\{\mu_\varepsilon\}_{\varepsilon>0}$ is tight. By Prokhorov's theorem, $\{\mu_\varepsilon\}_{\varepsilon>0}$ converges narrowly to a measure $\mu_0 \in \mathcal{P}_c(\mathbb{R}^2)$, up to subsequences. We will show that μ_0 is the minimizer of I .

Since W_0 is bounded from below on $\text{supp } \mu_\varepsilon$ by a constant $c_0 < 0$, we have

$$I_\varepsilon(\mu_\varepsilon) = I(\mu_\varepsilon) + \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varepsilon W_0(x - y) d\mu_\varepsilon(x) d\mu_\varepsilon(y) \geq I(\mu_\varepsilon) + c_0 \varepsilon,$$

therefore, by lower semi-continuity of I ,

$$\liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon(\mu_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0^+} I(\mu_\varepsilon) \geq I(\mu_0).$$

Moreover, by minimality of μ_ε , one has that

$$\limsup_{\varepsilon \rightarrow 0^+} I_\varepsilon(\mu_\varepsilon) \leq \lim_{\varepsilon \rightarrow 0^+} I_\varepsilon(\mu) = I(\mu)$$

for every $\mu \in \mathcal{P}_c(\mathbb{R}^2)$ such that $\iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x - y) d\mu(x) d\mu(y) < +\infty$. Because this is the class of admissible minimizers for I , then μ_0 is the minimizer of I on $\mathcal{P}(\mathbb{R}^2)$.

To deduce the shape of μ_0 , we observe that, up to subsequences, $a_{1,\varepsilon} \rightarrow a_1$, $a_{2,\varepsilon} \rightarrow a_2$ and $R_\varepsilon \rightarrow R$ as $\varepsilon \rightarrow 0^+$. Also, $a_1, a_2 \geq 0$, $a_1^2 + a_2^2 = 2$ and $R \in SO(2)$. We have two cases. If both a_1 and a_2 are strictly positive, then

$$\mu_0 = \frac{1}{|E|} \chi_E \mathcal{L}^2,$$

⁴Here, $\mathcal{H}^1|_{[-\sqrt{2}, \sqrt{2}]}$ denotes the restriction of the 1-dimensional Hausdorff measure to the interval $[-\sqrt{2}, \sqrt{2}]$.

with $E = RE_0$ and

$$E_0 = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \leq 1 \right\}.$$

If either $a_1 = 0$ or $a_2 = 0$, then the ellipse E_0 becomes degenerate. Suppose that $a_1 = 0$ and $a_2 = \sqrt{2}$. For every $\varphi \in C_b(\mathbb{R}^2)$, we have

$$\begin{aligned} \frac{1}{|E_\varepsilon|} \int_{\mathbb{R}^2} \varphi(x) \chi_{E_{0,\varepsilon}}(x) dx &= \frac{1}{\pi a_{1,\varepsilon} a_{2,\varepsilon}} \iint \left\{ |x_2| \leq a_{2,\varepsilon}, |x_1| \leq a_{1,\varepsilon} \sqrt{1 - \frac{x_2^2}{a_{2,\varepsilon}^2}} \right\} \varphi(x_1, x_2) dx_1 dx_2 \\ &= \frac{1}{\pi a_{1,\varepsilon} a_{2,\varepsilon}} \iint_{\{|x_2| \leq a_{2,\varepsilon}, |x_1| \leq a_{1,\varepsilon}\}} \varphi \left(\sqrt{1 - \frac{x_2^2}{a_{2,\varepsilon}^2}} x_1, x_2 \right) \sqrt{1 - \frac{x_2^2}{a_{2,\varepsilon}^2}} dx_1 dx_2, \end{aligned}$$

where in the last line we have used the change of variables $(x_1, x_2) \mapsto \left(\left(1 - \frac{x_2^2}{a_{2,\varepsilon}^2}\right)^{-\frac{1}{2}} x_1, x_2 \right)$.

Passing to the limit for $\varepsilon \rightarrow 0^+$, we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi a_{1,\varepsilon} a_{2,\varepsilon}} \iint_{\{|x_2| \leq a_{2,\varepsilon}, |x_1| \leq a_{1,\varepsilon}\}} \varphi \left(\sqrt{1 - \frac{x_2^2}{a_{2,\varepsilon}^2}} x_1, x_2 \right) \sqrt{1 - \frac{x_2^2}{a_{2,\varepsilon}^2}} dx_1 dx_2 \\ = \iint_{\mathbb{R}^2} \lim_{\varepsilon \rightarrow 0^+} \left(\frac{1}{\pi a_{1,\varepsilon} a_{2,\varepsilon}} \varphi \left(\sqrt{1 - \frac{x_2^2}{a_{2,\varepsilon}^2}} x_1, x_2 \right) \sqrt{1 - \frac{x_2^2}{a_{2,\varepsilon}^2}} \chi_{[-a_{1,\varepsilon}, a_{1,\varepsilon}]}(x_1) \chi_{[-a_{2,\varepsilon}, a_{2,\varepsilon}]}(x_2) \right) dx_1 dx_2 \\ = \int_{\mathbb{R}} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi a_{1,\varepsilon} a_{2,\varepsilon}} \left(\int_{\mathbb{R}} \varphi \left(\sqrt{1 - \frac{x_2^2}{a_{2,\varepsilon}^2}} x_1, x_2 \right) \chi_{[-a_{1,\varepsilon}, a_{1,\varepsilon}]}(x_1) dx_1 \right) \sqrt{1 - \frac{x_2^2}{a_{2,\varepsilon}^2}} \chi_{[-a_{2,\varepsilon}, a_{2,\varepsilon}]}(x_2) dx_2 \\ = \int_{\mathbb{R}} \frac{1}{\sqrt{2}\pi} \left(\int_{\mathbb{R}} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \varphi \left(\sqrt{1 - \frac{x_2^2}{2}} x_1, x_2 \right) \chi_{[-\varepsilon, \varepsilon]}(x_1) dx_1 \right) \sqrt{1 - \frac{x_2^2}{2}} \chi_{[-\sqrt{2}, \sqrt{2}]}(x_2) dx_2. \end{aligned}$$

We note that, denoting by $F \in C^1(\mathbb{R})$ a primitive of the function $x_1 \mapsto \varphi \left(\sqrt{1 - \frac{x_2^2}{2}} x_1, x_2 \right)$, one has

$$\begin{aligned} \int_{\mathbb{R}} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \varphi \left(\sqrt{1 - \frac{x_2^2}{2}} x_1, x_2 \right) \chi_{[-\varepsilon, \varepsilon]}(x_1) dx_1 &= \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} \frac{1}{\varepsilon} \varphi \left(\sqrt{1 - \frac{x_2^2}{2}} x_1, x_2 \right) \chi_{[-\varepsilon, \varepsilon]}(x_1) dx_1 \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{F(\varepsilon) - F(-\varepsilon)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} 2F'(\varepsilon) \\ &= \lim_{\varepsilon \rightarrow 0^+} 2\varphi \left(\sqrt{1 - \frac{x_2^2}{2}} \varepsilon, x_2 \right) = 2\varphi(0, x_2). \end{aligned}$$

Therefore, we conclude that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{|E_\varepsilon|} \int_{\mathbb{R}^2} \varphi(x) \chi_{E_{0,\varepsilon}}(x) dx = \frac{1}{\pi} \int_{\mathbb{R}} \varphi(0, x_2) \sqrt{2 - x_2^2} \chi_{[-\sqrt{2}, \sqrt{2}]}(x_2) dx_2,$$

that is, the measure $\frac{1}{|E_\varepsilon|} \chi_{E_{0,\varepsilon}} \mathcal{L}^2$ converges narrowly to the measure

$$\mu_s = \frac{1}{\pi} \delta_0 \otimes \sqrt{2 - x_2^2} \mathcal{H}^1|_{[-\sqrt{2}, \sqrt{2}]}.$$

Hence, upon defining the rotation maps $\rho_\varepsilon(x) := R_\varepsilon x$ and $\rho(x) := Rx$, we have

$$\mu_\varepsilon = \frac{1}{|E_\varepsilon|} \chi_{R_\varepsilon E_{0,\varepsilon}} \mathcal{L}^2 = \rho_{\varepsilon*} \left(\frac{1}{|E_\varepsilon|} \chi_{E_{0,\varepsilon}} \mathcal{L}^2 \right) \xrightarrow{*} \rho_*(\mu_s) = \mu_0.$$

Similarly, if $a_1 = \sqrt{2}$ and $a_2 = 0$, we have that μ_0 is the push-forward of μ_s through a rotation map $\rho(x) = RJx$, where $J \in SO(2)$ corresponds to a rotation of $\pi/2$.

Finally, we prove that $\widehat{\Psi}(Re_1) = 0$. Without loss of generality, assume that $a_1 = 0$ and $a_2 = \sqrt{2}$. By (3.3.10) and denoting $a_\varepsilon = (a_{1,\varepsilon}, a_{2,\varepsilon})$, we obtain, for $j = 1, 2$,

$$\frac{1}{\pi} \int_{\mathbb{S}^1} \frac{\widehat{\Psi}_\varepsilon(z)}{|D(a_\varepsilon)R_\varepsilon^T z|^2} |\langle z, e_j \rangle|^2 d\sigma(z) = \frac{1}{\pi} \int_{\mathbb{S}^1} \frac{\widehat{\Psi}(R_\varepsilon z) + \varepsilon}{a_{1,\varepsilon}^2 z_1^2 + a_{2,\varepsilon}^2 z_2^2} |\langle R_\varepsilon z, e_j \rangle|^2 d\sigma(z) = 1.$$

Therefore, summing over j and applying Fatou's lemma, we find

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{\widehat{\Psi}(Rz)}{z_2^2} d\sigma(z) \leq \liminf_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_{\mathbb{S}^1} \frac{\widehat{\Psi}(R_\varepsilon z) + \varepsilon}{a_{1,\varepsilon}^2 z_1^2 + a_{2,\varepsilon}^2 z_2^2} d\sigma(z) = 2.$$

On the other hand,

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{\widehat{\Psi}(Rz)}{z_2^2} d\sigma(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\widehat{\Psi}(Re^{it})}{\sin^2(t)} dt,$$

and the integral converges only if $\widehat{\Psi}(Re^{it}) = 0$ for $t = 0$ and $t = \pi$. Hence, we conclude that $\widehat{\Psi}(Re_1) = 0$. $\square$

Interestingly, we observe that if $\widehat{\Psi}$ vanishes somewhere on $\mathbb{S}^1$, the minimizer may exhibit a loss of dimensionality, in that its support reduces to a segment.

However, this condition is only necessary. In fact, one can show examples of κ for which $\widehat{\Psi}(\xi) = 0$ for some $\xi \in \mathbb{S}^1$, while the minimizer is still supported on an ellipse. At present, the question of under which conditions the loss of dimensionality occurs is still open. We might conjecture that the minimizer exhibits a loss of dimensionality only if the region where $\widehat{\Psi}$ is sufficiently large. In the next chapter, we will propose a numerical method to further investigate this question.

3.4. An example: the dislocation energy

We conclude this chapter by showing a concrete example of anisotropic energy coming from material science. For a linear elastic material (e.g. a metal), the interaction energy of a 2-dimensional system of dislocations is given by

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} (W_0(x - y) + \kappa(x - y)) d\mu(x) d\mu(y),$$

with

$$\kappa(x) = -\frac{1}{4} \frac{a+b}{a} \ln \left(\frac{x_1^2 + (a+b)^2 x_2^2}{|x|^2} \right) + \frac{1}{4} \frac{b-a}{a} \ln \left(\frac{x_1^2 + (b-a)^2 x_2^2}{|x|^2} \right),$$

where $0 < a < b$ are constants depending on the material.⁵ In particular, if the material is isotropic, i.e. its mechanical properties do not depend on the orientation, then $a \rightarrow 0^+$ and $b = 1$, therefore the anisotropic kernel reduces to

$$\kappa(x) = \frac{x_1^2}{|x|^2}.$$

More generally, we can consider an interaction kernel of the form

$$W_\alpha(x) = -\ln|x| + \alpha \frac{x_1^2}{|x|^2},$$

where the parameter $\alpha \in \mathbb{R}$ represents the degree of anisotropy of the interaction. Note that, if $\alpha = 0$, we retrieve the Coulomb kernel W_0 . If we assume that the confinement potential is quadratic, the energy becomes

$$I_\alpha(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W_\alpha(x - y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x). \quad (3.4.1)$$

⁵For an in-depth discussion of dislocations in anisotropic materials see Chapters 13-3 and 13-4 in [7].

Since

$$\kappa_\alpha(x) := \alpha \frac{x_1^2}{|x|^2} = \frac{\alpha}{2} \frac{x_1^2 - x_2^2}{|x|^2} + \frac{\alpha}{2},$$

by Lemma 4 its Fourier transform is given by

$$\widehat{\kappa_\alpha}(\xi) = -\alpha \frac{\xi_1^2 - \xi_2^2}{|\xi|^4} + \pi\alpha\delta_0.$$

Therefore,

$$\widehat{W_\alpha}(\xi) = (c_0 + \pi\alpha)\delta_0 + \frac{\widehat{\Psi_\alpha}(\xi)}{|\xi|^2},$$

where

$$\widehat{\Psi_\alpha}(\xi) = 1 + \widehat{\kappa_\alpha}\left(\frac{\xi}{|\xi|}\right) = (1 - \alpha) \frac{\xi_1^2}{|\xi|^2} + (1 + \alpha) \frac{\xi_2^2}{|\xi|^2}.$$

We note that the sign of $\widehat{\Psi_\alpha}$ depends on the value of α . If $\alpha < 1$, then $\widehat{\Psi_\alpha} > 0$ on $\mathbb{S}^1$. Theorem 9 thus ensures that the minimizer of (3.4.1) is of the form (3.3.5). The ellipse E_α can be found noticing that W_α is symmetric with respect to the coordinate axes x_1 and x_2 , i.e. $W_\alpha(-x_1, x_2) = W_\alpha(x_1, x_2) = W_\alpha(x_1, -x_2)$ for every $(x_1, x_2) \in \mathbb{R}^2$. Therefore, the minimizer must respect this symmetry, which implies that the rotation matrix R is equal to the identity and

$$E_\alpha = E_{0,\alpha} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{a_{1,\alpha}^2} + \frac{x_2^2}{a_{2,\alpha}^2} \leq 1 \right\}.$$

The semiaxes $a_{1,\alpha}, a_{2,\alpha}$ can be computed solving (3.3.10), which gives

$$\frac{1}{\pi} \int_{\mathbb{S}^1} \frac{(1 - \alpha)z_1^2 + (1 + \alpha)z_2^2}{a_{1,\alpha}^2 z_1^2 + a_{2,\alpha}^2 z_2^2} z_j^2 d\sigma(z) = 1 \quad \text{for } j = 1, 2.$$

A simple computation shows that $a_{1,\alpha} = \sqrt{1 - \alpha}$, $a_{2,\alpha} = \sqrt{1 + \alpha}$ is a solution of the system. In fact, since the minimizer is unique, it is the only solution. Hence, if $|\alpha| < 1$, the minimizer is

$$\mu_\alpha = \frac{1}{|E_\alpha|} \chi_{E_\alpha} \mathcal{L}^2, \quad (3.4.2)$$

with

$$E_\alpha = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{1 - \alpha} + \frac{x_2^2}{1 + \alpha} \leq 1 \right\}.$$

If $\alpha = 1$, then $\widehat{\Psi_\alpha} \geq 0$ on $\mathbb{S}^1$. Reasoning as in the proof of Theorem 10, we observe that the minimizer of I_1 is the limit of the minimizer of I_α as $\alpha \rightarrow 1^-$. This results in the semicircle law

$$\mu_1 = \frac{1}{\pi} \delta_0 \otimes \sqrt{2 - x_2^2} \mathcal{H}^1|_{[-\sqrt{2}, \sqrt{2}]}. \quad (3.4.3)$$

If $\alpha = -1$, we can apply the same reasoning for $\alpha \rightarrow -1^+$, which yields

$$\mu_{-1} = \frac{1}{\pi} \sqrt{2 - x_1^2} \mathcal{H}^1|_{[-\sqrt{2}, \sqrt{2}]} \otimes \delta_0, \quad (3.4.4)$$

that is the semicircle law on the horizontal axis.

REMARK. We could arrive at the same result also noticing that

$$\kappa_{-\alpha}(x) = -\alpha \frac{x_1^2}{|x|^2} = \alpha \frac{x_2^2}{|x|^2} - \alpha.$$

Therefore, since the addition of a constant to κ does not affect the minimization problem, the minimizer of $I_{-\alpha}$ can be obtained from that of I_α just by swapping the roles of x_1 and x_2 , i.e. by a rotation of $\pi/2$.

If $\alpha > 1$, we observe that

$$I_\alpha(\mu) \geq I_1(\mu) \quad \forall \mu \in \mathcal{P}(\mathbb{R}^2).$$

Also, because $\kappa(x) = 0$ for every $x \in \text{supp } \mu_1$, and by minimality of μ_1 and uniqueness of the minimizer, we have

$$I_\alpha(\mu_1) = I_1(\mu_1) < I_1(\mu) \leq I_\alpha(\mu) \quad \forall \mu \neq \mu_1,$$

hence μ_1 is the minimizer of I_α for every $\alpha > 1$. Finally, owing to the above remark, we deduce that μ_{-1} is the minimizer of I_α for every $\alpha < -1$.

CHAPTER 4

A numerical method for the approximation of the minimizer

As we have seen in the previous chapter, the minimization problem for the anisotropic energy poses some interesting questions. Even if we limit ourselves to the case of quadratic confinement, determining the shape of the minimizer is not obvious. In particular, we have observed that, if $\widehat{\Psi} = 0$ somewhere on $\mathbb{S}^1$, then the minimizer could either be an ellipse law of the form (3.3.5) or the semicircle law (3.3.13) (up to rotations). Ideally, we would like to devise a criterion to determine a priori which of the two cases occurs. Currently, such a criterion does not exist. If we consider more general confinements, such as $V(x) = |x|^p$ with $p > 0$, we may expect things to become even more complicated, and a characterisation of the minimizer is missing. To the author's knowledge, even basic questions regarding the topology of the minimizer are unanswered.

We might gain more insight into these matters by computing the minimizer using numerical methods. In this chapter, we will introduce a novel technique, based on Ritz method, for the approximation of the energy functional. We will test the method by solving the minimization problem for the dislocation energy with quadratic confinement and comparing the approximate solution with the exact one. Then, we will compute the minimizer of the dislocation energy with generalised confinements.

4.1. Modified Ritz method

4.1.1. Ritz method. The Ritz method is a technique to compute the approximate solution of variational problems. In its standard form, the method considers the minimization problem for a functional $I: X \rightarrow \mathbb{R}$,

$$I(u) = \int_{\Omega} F(\nabla u(x), u(x), x) dx, \quad (4.1.1)$$

where $\Omega \subseteq \mathbb{R}^d$ is bounded and $F: \mathbb{R}^{2d+1} \rightarrow \mathbb{R}$ is a fixed function (sufficiently regular, e.g. $F \in C(\mathbb{R}^{2d+1})$). The function space X is assumed to be large enough to ensure the existence and uniqueness of the minimizer (classically, $X = C^1(\Omega)$ or $X = H^1(\Omega)$). Also, u is usually required to satisfy some boundary condition, such as

$$u(x) = g(x) \quad \forall x \in \partial\Omega. \quad (4.1.2)$$

The idea is to project the infinite-dimensional space X onto a subspace V of dimension $n \in \mathbb{N}$, and search for a minimizer in V . Therefore, we approximate u by a linear combination

$$\tilde{u}(x) = \sum_{h=1}^n a_h \varphi_h(x),$$

where $\{\varphi_h\}_{h=1}^n$ is a complete basis of V and the coefficients $\{a_h\}_{h=1}^n \subseteq \mathbb{R}$ are to be computed. The functional (4.1.2) can then be rewritten as a function $\tilde{I}: \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\tilde{I}(a_1, \dots, a_n) = \int_{\Omega} F\left(\sum_{h=1}^n a_h \nabla \varphi_h(x), \sum_{h=1}^n a_h \varphi_h(x), x\right) dx. \quad (4.1.3)$$

To enforce the boundary condition (4.1.2), we require that

$$\sum_{h=1}^n a_h \varphi_h(x) = g(x) \quad \forall x \in \partial\Omega. \quad (4.1.4)$$

Note that this condition can be fulfilled easily if $g = 0$, in which case we can choose the basis functions φ_h in such a way that

$$\varphi_h(x) = 0 \quad \forall x \in \partial\Omega$$

for every $h \in \{1, \dots, n\}$. Also, if $\Omega = [a, b]$, condition (4.1.2) reduces to

$$\sum_{h=1}^n a_h \varphi_h(a) = g(a), \quad \sum_{h=1}^n a_h \varphi_h(b) = g(b).$$

For more complicated cases, the boundary condition can be relaxed to hold only at a finite number of points $\{x_1, \dots, x_m\} \subseteq \partial\Omega$, with $m < n$, i.e.

$$\sum_{k=1}^n a_k \varphi_k(x_i) = g(x_i) \quad \forall i \in \{1, \dots, m\}. \quad (4.1.5)$$

The approximate variational problem can be solved by minimizing the function (4.1.3) subject to constraint (4.1.4). This can be done by expressing $a_{n-m+1}, \dots, a_n$ in terms of $a_1, \dots, a_{n-m}$ using (4.1.5), and then solving

$$\frac{\partial \tilde{I}}{\partial a_k} = 0 \quad \text{for } k = 1, \dots, n - m.$$

This yields the values of $a_1, \dots, a_{n-m}$, which are substituted back in (4.1.5) to compute the remaining $a_{n-m+1}, \dots, a_n$.

The accuracy and complexity of Ritz method depend largely on the choice of the basis functions $\{\varphi_h\}_{h=1}^n$. Choosing suitable basis functions for a given problem represents a critical step in the implementation of Ritz method, and can be rather difficult, particularly if the domain Ω does not possess a simple geometry. In this case, it is often convenient to subdivide Ω into smaller subdomains and consider separate sets of basis functions on each subdomain (which is equivalent to considering basis functions that are supported only on a small subset of Ω). This approach leads to the finite element method, which is widely applied for the solution of partial differential equations.

4.1.2. Modified Ritz method. In the spirit of Ritz method, our idea is to solve the minimization problem for the energy functional $I: \mathcal{P}(\mathbb{R}^2) \rightarrow \mathbb{R} \cup \{+\infty\}$,

$$I(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x - y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} V(x) d\mu(x),$$

by approximating μ with a linear combination

$$\tilde{\mu} = \sum_{k=1}^n a_k \nu_k,$$

where the measures $\{\nu_k\}_{k=1}^n$ should be taken in a suitable subset of $\mathcal{P}(\mathbb{R}^2)$ and the coefficients $\{a_k\}_{k=1}^n \subseteq \mathbb{R}$ are to be determined.

The original problem then transforms into that of minimizing the function

$$\begin{aligned} \tilde{I}(a_1, \dots, a_n) &:= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\left(\sum_{h=1}^n a_h \nu_h(x)\right) d\left(\sum_{k=1}^n a_k \nu_k(y)\right) \\ &\quad + \int_{\mathbb{R}^2} V(x) d\left(\sum_{k=1}^n a_k \nu_k(x)\right) \\ &= \sum_{h=1}^n \sum_{k=1}^n a_h a_k \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\nu_h(x) d\nu_k(y) + \sum_{h=1}^n a_h \int_{\mathbb{R}^2} V(x) d\nu_h(x). \end{aligned} \quad (4.1.6)$$

We note that, in general, evaluating the integrals

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\nu_h(x) d\nu_k(y) \quad \text{and} \quad \int_{\mathbb{R}^2} V(x) d\nu_h(x)$$

is rather complicated from the numerical point of view. In fact, both the domain of integration and the integrand functions are unbounded, which makes integration techniques such as Gaussian quadrature unsuitable.

However, we have already seen that the minimizer of I must be compactly supported; therefore, we can restrict the minimization problem to the class $\mathcal{P}_c(\mathbb{R}^2)$ of probability measures with compact support. The minimizer μ can thus be approximated by its projection onto

$$\langle \nu_1, \dots, \nu_n \rangle := \left\{ \mu \in \mathcal{P}_c(\mathbb{R}^2) : \mu = \sum_{k=1}^n a_k \nu_k, a_k \geq 0 \right\},$$

with $\{\nu_k\}_{k=1}^n \subseteq \mathcal{P}_c(\mathbb{R}^2)$. This yields the finite-dimensional constrained minimization problem

$$\min_{\mu \in \langle \nu_1, \dots, \nu_n \rangle} I(\mu) = \min_{(a_1, \dots, a_n) \in \mathcal{A}} \tilde{I}(a_1, \dots, a_n), \quad (4.1.7)$$

where

$$\mathcal{A} := \left\{ a \in \mathbb{R}^n : \sum_{k=1}^n a_k = 1, a_k \geq 0 \forall k \in \{1, \dots, n\} \right\}.$$

By analogy with Ritz method, the measures $\{\nu_k\}_{k=1}^n$ will be called *basis measures*.

With a suitable choice of the basis measures, the integrals in (4.1.6) can be evaluated numerically, which gives the values

$$M_{hk} = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\nu_h(x) d\nu_k(y), \quad b_h = \int_{\mathbb{R}^2} V(x) d\nu_h(x). \quad (4.1.8)$$

Therefore, denoting

$$a := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad M := \begin{pmatrix} M_{11} & \dots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \dots & M_{nn} \end{pmatrix}, \quad b := \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

the approximate energy (4.1.6) becomes

$$\tilde{I}(a) = a^T M a + a^T b. \quad (4.1.9)$$

We can thus rewrite (4.1.7) as

$$\begin{cases} \min_{a \in \mathbb{R}^n} a^T M a + a^T b \\ a_k \geq 0 \forall k \in \{1, \dots, n\} \\ \sum_{k=1}^n a_k = 1. \end{cases} \quad (4.1.10)$$

This is a minimization problem for a quadratic function with inequality constraints, and can be solved using numerical optimization algorithms.

4.2. Implementation of modified Ritz method

The method above lends itself to numerical computing. In particular, the matrix formulation (4.1.9) makes it possible to leverage the existing, highly optimized linear algebra packages. Here, we implement the modified Ritz method using MATLAB.

4.2.1. Choice of the basis measures. Without loss of generality, assume that the support of the minimizer is contained in $\Omega = [-R, R]^2$. On this domain, we consider a quadrangular grid with $N \times N$ elements. The elements are given by the squares

$$Q_{i,j} := \left\{ x \in \mathbb{R}^2 : x \in [0, l_N]^2 + (i-1, j-1)l_N \right\},$$

where $l_N = \frac{2R}{N}$ is the element side length and $i, j = 1, \dots, N$. We denote the centre of each square by

$$x_{i,j} := \left(i - \frac{1}{2}, j - \frac{1}{2} \right) l_N.$$

Rearranging the elements as $\{Q_h\}_{h=1,\dots,n} := \{Q_{i,j}\}_{i,j=1,\dots,N}$, with $n = N^2$, we can define a set of basis measures by considering the uniform probability measure on each square, i.e.

$$\nu_h := \frac{1}{l_N^2} \chi_{Q_h} \mathcal{L}^2. \quad (4.2.1)$$

Alternatively, if we let $\{x_h\}_{h=1,\dots,n} := \{x_{i,j}\}_{i,j=1,\dots,N}$, we could construct another set of basis measures by considering the Dirac measure at each node, that is,

$$\nu_h := \delta_{x_h}. \quad (4.2.2)$$

With either choice of the basis measures, we can compute the matrix coefficients M_{hk} , b_h . For the uniform basis measures, the integrals in (4.1.8) are evaluated using 4-point Gaussian quadrature, that is,

$$\begin{aligned} M_{hk} &= \frac{1}{l_N^4} \int_{Q_h} \int_{Q_k} W(x-y) dx dy = \frac{1}{l_N^4} \int_Q \int_Q W(\varphi_h(s) - \varphi_k(t)) \det(J_{\varphi_h}(s) J_{\varphi_k}(t)) ds dt \\ &= \frac{1}{l_N^4} \sum_{i,j=1}^4 \sum_{l,m=1}^4 w_i w_j w_l w_m W(\varphi_h(\xi_i, \xi_j) - \varphi_k(\xi_l, \xi_m)) \det(J_{\varphi_h}(\xi_i, \xi_j) J_{\varphi_k}(\xi_l, \xi_m)), \\ b_h &= \frac{1}{l_N^2} \int_{Q_h} V(x) dx = \frac{1}{l_N^2} \int_Q V(\varphi_h(s)) \det(J_{\varphi_h}(s)) ds \\ &= \sum_{i,j=1}^4 w_i w_j V(\xi_i, \xi_j) \det(J_{\varphi_h}(\xi_i, \xi_j)), \end{aligned}$$

where $\varphi_h: Q \rightarrow Q_h$ is a mapping from the unit square $Q := [-1, 1]^2$ onto Q_h (with positive Jacobian determinant) and $\{\xi_i\}_{i=1}^4, \{w_i\}_{i=1}^4$ are the Gaussian points and weights, respectively. To avoid the singularity at $W(0, 0)$, if $h = k$ we consider

$$\int_{Q_h} \int_{Q_h} W(x-y) dx dy = \int_{Q_h} \int_{Q_h^-} W(x-y) dx dy + \int_{Q_h} \int_{Q_h^+} W(x-y) dx dy,$$

with $Q_h^- := \{x \in Q_h : x_1 < \varepsilon\}$, $Q_h^+ := Q_h \setminus Q_h^-$, $\varepsilon > 0$, and apply the quadrature rule above separately to each integral on the right-hand side.

For the Dirac basis measures, the integrals are computed by evaluating the integrand at the centre of each square, which gives

$$M_{hk} = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x - y) d\delta_{x_h}(x) d\delta_{x_k}(y) = W(x_h - x_k),$$

$$b_h = \int_{\mathbb{R}^2} V(x) d\delta_{x_h}(x) = V(x_h).$$

We can then assemble the matrices M , b and compute the approximate energy (4.1.9). Notably, the choice of Dirac basis measures is more efficient from a computational point of view, as the calculation of each M_{hk} requires a single function evaluation, while using the quadrature formula employed for the uniform basis measures the same computation requires 256 function evaluations.

4.2.2. Minimization algorithm. The constrained minimization problem (4.1.10) is solved using the Interior Point algorithm implemented by the `fmincon` function in MATLAB. We give here a sketch of the method¹.

Let us consider the following minimization problem:

$$\begin{cases} \min_{x \in \Omega} f(x) \\ g(x) \leq 0 \\ h(x) = 0, \end{cases} \quad (4.2.3)$$

where $\Omega \subseteq \mathbb{R}^n$, $f: \Omega \rightarrow \mathbb{R}$, $g: \Omega \rightarrow \mathbb{R}^m$ and $h: \Omega \rightarrow \mathbb{R}^r$.

For $\mu > 0$, we define the approximate problem

$$\begin{cases} \min_{x,s} f_\mu(x,s), \text{ with } f_\mu(x,s) := f(x) - \mu \sum_{i=1}^m \ln(s_i) \\ g(x) + s = 0 \\ h(x) = 0, \end{cases} \quad (4.2.4)$$

where $s_i \geq 0$ for every $i = 1, \dots, m$. We note that this problem is simpler than the original one, as it only contains equality constraints. Since the minimum of f_μ tends to that of f as $\mu \rightarrow 0$, we can approximate the solution of (4.2.3) by solving a sequence of problems (4.2.4). Each of these is solved by either Newton's method or the conjugate gradient method. If the approximate problem is solved with sufficient accuracy, the algorithm cuts μ by a given factor (1/5 or 1/100 depending on the speed of convergence). The process is iterated until a stopping criterion is reached.

REMARK. It should be noted that, in general, the domain $\Omega = [-R, R]$ where we search for a minimizer is not known a priori (even though it is guaranteed to exist as the minimizer is compactly supported). For a quadratic confinement, we have seen that the support of the minimizer is contained in an ellipse whose semiaxes satisfy $a_1^2 + a_2^2 = 2$, therefore it is sufficient to take $R = \sqrt{2}$. For general confinements, one would need to estimate the size of the support. In the absence of such an estimate, we could follow a trial-and-error procedure, starting with a large value of R and a coarse grid to guess the general shape of the minimizer, and then refining the grid in the regions where the minimizer is supported.

¹For a detailed description of the algorithm, see MATLAB documentation [8].

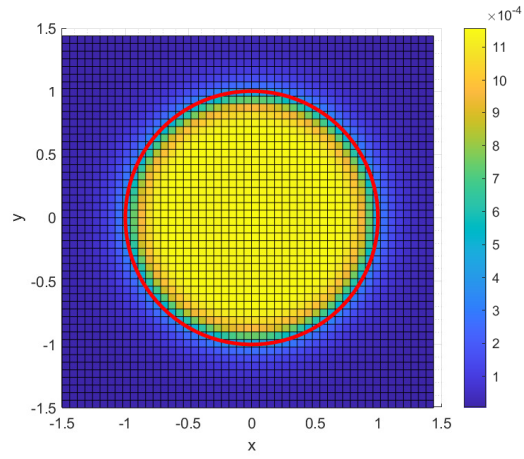
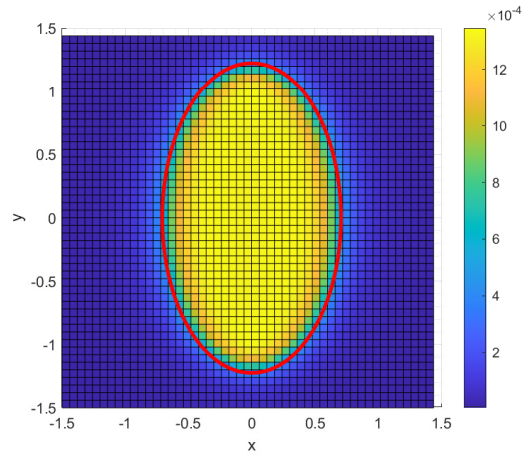
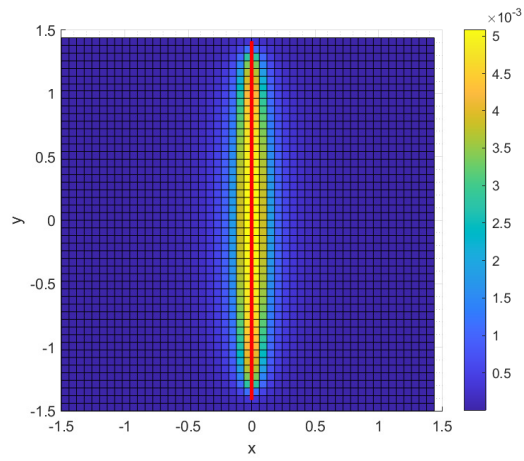

 $\alpha = 0$

 $\alpha = 0.5$

 $\alpha = 1$

FIGURE 1. Minimizer computed for $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$, in the case $p = 2$, using uniform basis measures. The red line is the boundary of the support of the exact minimizer.

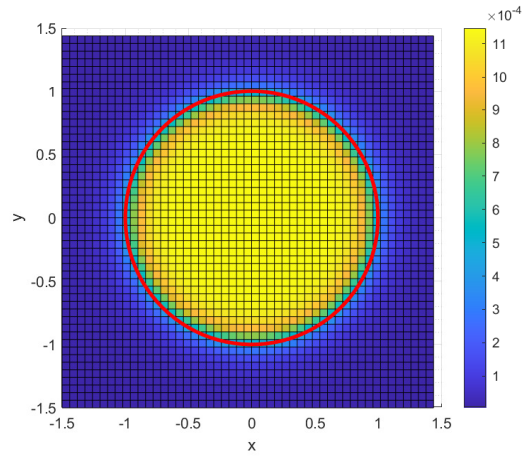
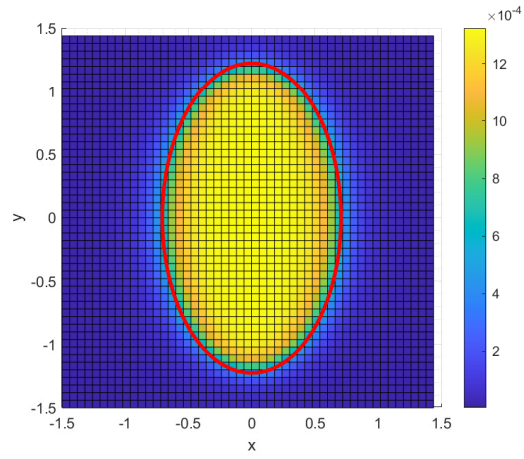
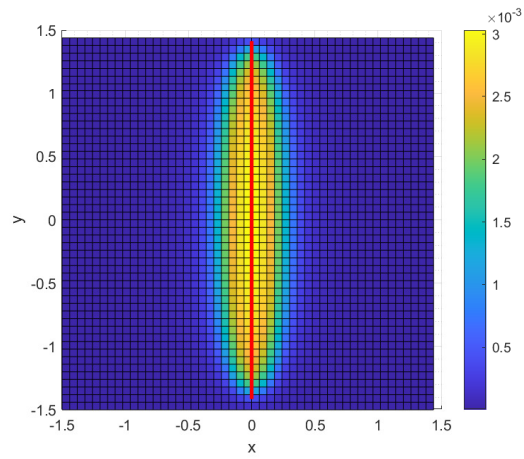

 $\alpha = 0$

 $\alpha = 0.5$

 $\alpha = 1$

FIGURE 2. Minimizer computed for $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$, in the case $p = 2$, using Dirac basis measures. The red line is the boundary of the support of the exact minimizer.

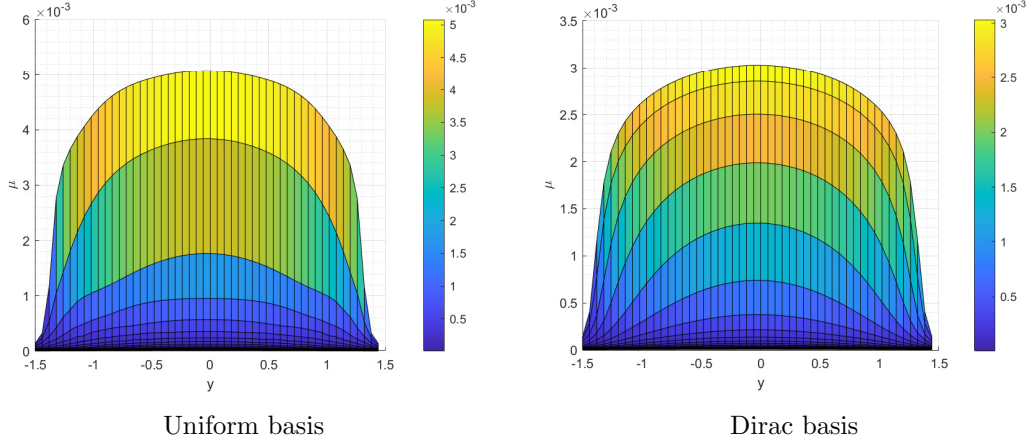


FIGURE 3. Density profile computed with uniform and Dirac basis measures for $\alpha = 1$, in the case $p = 2$.

4.3. Numerical results

We test the modified Ritz method by solving the minimization problem for the dislocation energy discussed in Section 3.4. Here, we consider a more general power-law confinement $V(x) = |x|^p$, with $p > 1$. The dislocation energy functional becomes

$$I_\alpha(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(-\ln|x-y| + \alpha \frac{(x_1 - y_1)^2}{|x-y|^2} \right) d\mu(x)d\mu(y) + \int_{\mathbb{R}^2} |x|^p d\mu(x).$$

4.3.1. Quadratic confinement. First, we consider the case $p = 2$, i.e. quadratic confinement. As we have seen in the previous chapter, in this case we can find an exact expression for the minimizer, whose shape depends strongly on the value of α .

Figures 1 and 2 show the minimizer computed using the uniform basis measures and the Dirac basis measures, respectively, for different values of α . We observe that for $\alpha = 0$ and $\alpha = 0.5$ the modified Ritz method yields a good approximation of the exact minimizer (whose boundary is shown in red in the figures). In particular, the approximate minimizer has almost constant density inside its support and vanishes close to the boundary, which is a discretized version of (3.4.2). For $\alpha = 1$, the results differ slightly depending on the choice of the basis measures. With the uniform basis, the approximate minimizer is more concentrated around the support of the exact minimizer (3.4.3), that is, the vertical segment of endpoints $(0, -\sqrt{2})$, $(0, \sqrt{2})$, while with the Dirac basis it is more spread out. In both cases, the density of the approximate minimizer is maximum at $y = 0$ and decreases towards $y = \pm\sqrt{2}$, as can be seen in Figure 3. This is similar to the profile given by the semicircle law.

4.3.2. General confinement. We now consider a general exponent $p \neq 2$. In particular, we investigate the cases $p = 4$ and $p = 1.5$.

Figures 4 and 5 show the approximate minimizer computed with the uniform and Dirac basis measures for $p = 4$. Compared to the case $p = 2$ (represented by the red line), we observe some remarkable differences. For $\alpha = 0$ and $\alpha = 0.5$, the minimizer is contained in a smaller region, which can be intuitively explained by the increased confinement if $|x| > 1$. Moreover, the density is concentrated in an elliptical annulus instead of being uniformly distributed inside an ellipse. For $\alpha = 1$, the minimizer is also concentrated in a thin region and exhibits two distinct peaks at the upper and lower extremities. Both choices of the basis measures yield almost identical approximations of the minimizer, the

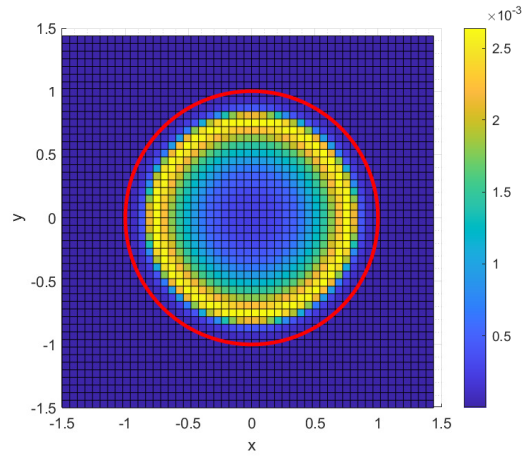
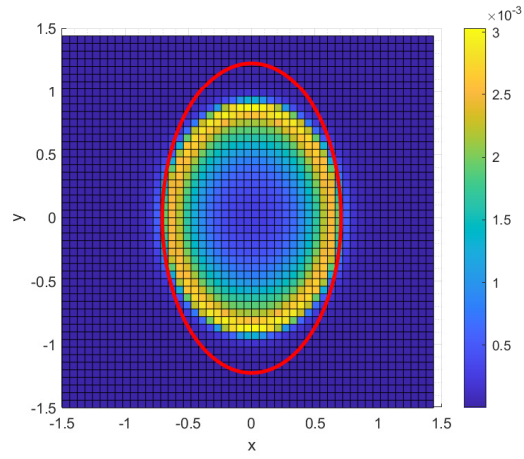
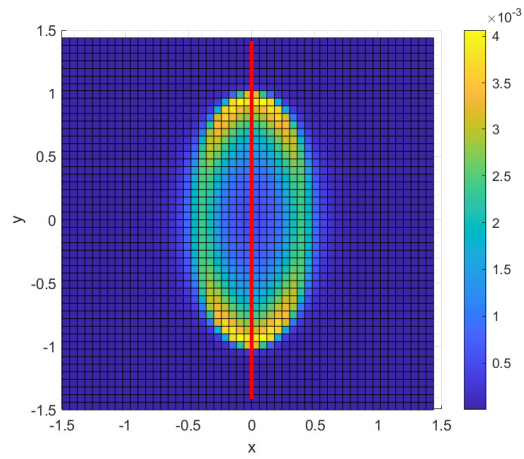

 $\alpha = 0$

 $\alpha = 0.5$

 $\alpha = 1$

FIGURE 4. Minimizer computed for $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$, in the case $p = 4$, using uniform basis measures. The red line is the boundary of the support of the exact minimizer.

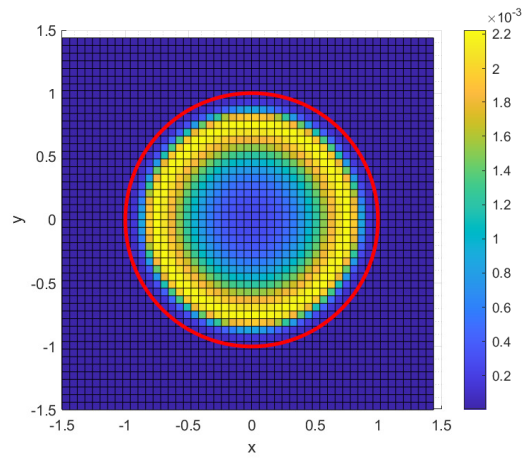
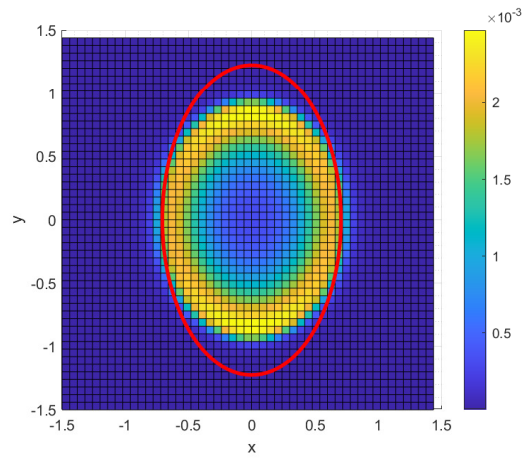
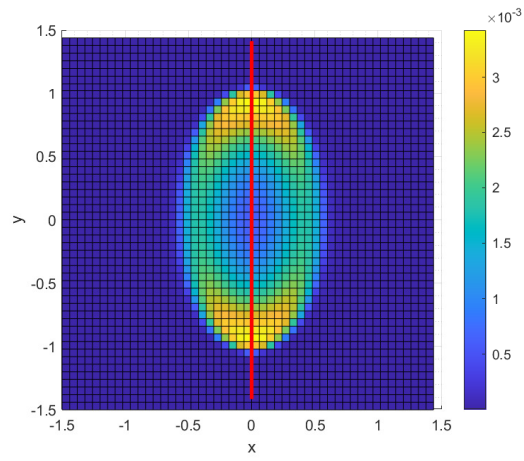

 $\alpha = 0$

 $\alpha = 0.5$

 $\alpha = 1$

FIGURE 5. Minimizer computed for $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$, in the case $p = 4$, using Dirac basis measures. The red line is the boundary of the support of the exact minimizer.

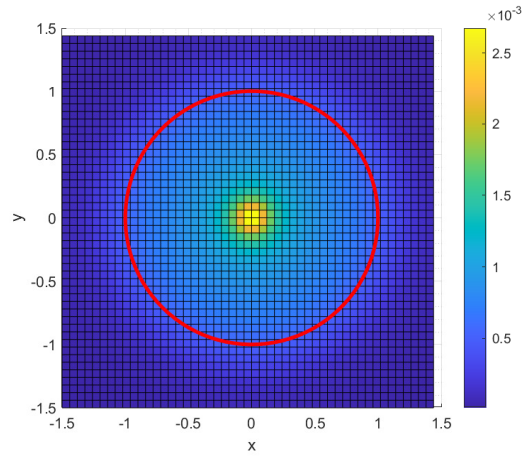
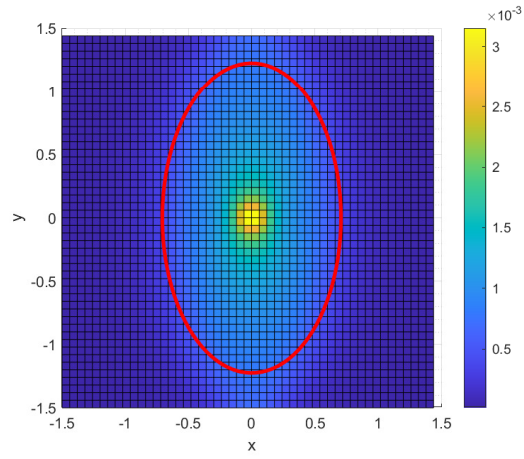
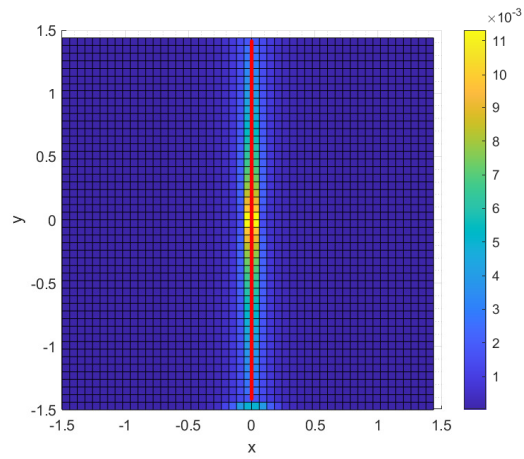
 $\alpha = 0$  $\alpha = 0.5$  $\alpha = 1$

FIGURE 6. Minimizer computed for $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$, in the case $p = 1.5$, using uniform basis measures. The red line is the boundary of the support of the exact minimizer.

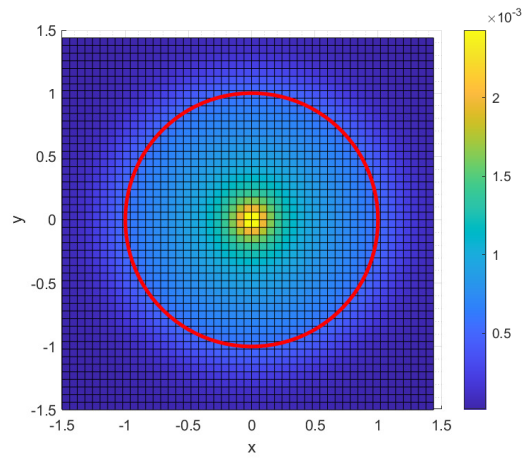
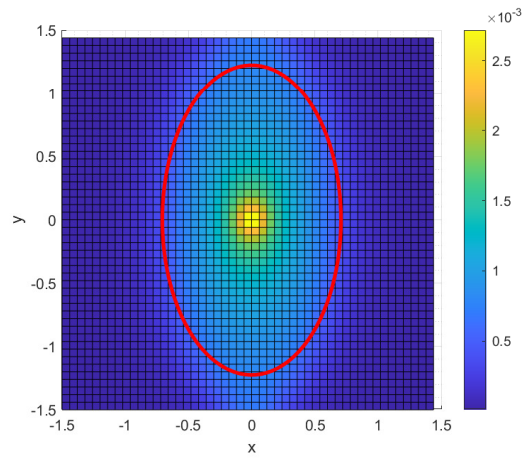
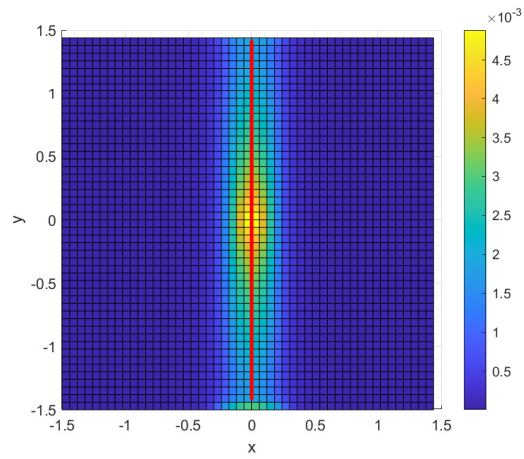

 $\alpha = 0$

 $\alpha = 0.5$

 $\alpha = 1$

FIGURE 7. Minimizer computed for $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$, in the case $p = 1.5$, using Dirac basis measures. The red line is the boundary of the support of the exact minimizer.

density obtained with the Dirac basis being slightly more spread out, similarly to what was found in the case $p = 2$.

These results seem to suggest that the topology of the minimizer might differ significantly from that observed in the case of quadratic confinement. In particular, it is not clear whether the support of the minimizer is simply connected or not.

Finally, we consider the case $p = 1.5$. The minimizers calculated with the two sets of basis measures are shown in Figures 6 and 7. For all values of α , the minimizer exhibits a strong peak at the origin and decreases radially. The direction of maximum decrease depends on the degree of anisotropy; in particular, for $\alpha = 1$ the density decreases very fast in the horizontal direction and is concentrated along a vertical segment.

For all the cases considered here, the results are almost unaffected by the choice of the basis measures. On the other hand, using the Dirac basis yields a slight computational advantage and results in a shorter computational time.

The minimizers computed for $p = \{1.5, 4\}$ are in agreement with those found by Scagliotti [1] using a different numerical method. Interestingly, for $p = 4$ and $\alpha = 1$, Scagliotti observes that the support of the minimizer seems to become one-dimensional. Our results tend to support this hypothesis and even open up the question of whether the support of the minimizer is a simply connected set. Moreover, the annular shape observed for $p = 4$ and $\alpha \in \{0, 0.5\}$ is consistent with a recent remark by Mora [2], who suggests that for $p \geq 2$ and $\widehat{W}$ strictly positive the minimizer should retain full dimensionality. These questions about the shape of the minimizer need to be investigated in more detail and could become the subject of further research using analytical tools.

CHAPTER 5

Conclusions

In this work, we analysed the minimization problem for the energy

$$I(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W(x-y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} V(x) d\mu(x),$$

with interaction kernel

$$W(x) = -\ln|x| + \kappa(x).$$

We proved that, in the rather general case where $\kappa \in C(\mathbb{S}^1)$, the energy I arises as the Γ -limit of a sequence of discrete energies $\{I_n\}_{n \geq 2}$. Therefore, the minimizer of I is the limit of the equilibrium configurations for the energies I_n , as $n \rightarrow +\infty$.

Under suitable assumptions, I admits a minimizer μ , which is compactly supported and satisfies the Euler-Lagrange equations

$$(W * \mu)(x) + \frac{1}{2}V(x) = c \quad \text{for } \mu\text{-a.e. } x \in \text{supp } \mu,$$

$$(W * \mu)(x) + \frac{1}{2}V(x) \geq c \quad \text{for q.e. } x \in \mathbb{R}^2,$$

for some constant $c \in \mathbb{R}$. In addition, if W is regular enough and its Fourier transform $\widehat{W}$ is nonnegative on $\mathbb{S}^1$, then the minimizer is unique.

In the case of quadratic confinement, the minimizer can be characterised as follows:

- If $\widehat{W} > 0$ on $\mathbb{S}^1$, the minimizer is

$$\mu = \frac{1}{|E|} \chi_E \mathcal{L}^2,$$

where E is the domain contained in an ellipse with semiaxes a_1, a_2 such that $a_1^2 + a_2^2 = 2$.

- If $\widehat{W} \geq 0$ on $\mathbb{S}^1$, the minimizer is either as above or the push-forward of

$$\frac{1}{\pi} \delta_0 \otimes \sqrt{2 - x_2^2} \mathcal{H}^1|_{[-\sqrt{2}, \sqrt{2}]}$$

through a rotation map.

Notably, if $\widehat{W}$ vanishes somewhere on $\mathbb{S}^1$, the topology of the minimizer may change substantially, passing from a measure with 2-dimensional support to one supported on a segment. However, this condition is only necessary. One could conjecture that the loss of dimensionality occurs only if the region where $\widehat{W}$ vanishes is large enough.

To solve the minimization problem with general confinements, we developed a novel numerical method. The accuracy of the method was assessed by computing the approximate minimizer of

$$I_\alpha(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(-\ln|x-y| + \alpha \frac{(x_1 - y_1)^2}{|x-y|} \right) d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x).$$

The numerical results are in agreement with the exact ones. Finally, we computed the minimizer of I_α with generalised confinement $V(x) = |x|^p$, where the analytic solution is still unknown, for various values of α and p . The results show that rather complicated

patterns could arise. In fact, for $p = 4$ the topology of the minimizer seems to change, with the support becoming not simply connected.

Rather than marking an endpoint, these findings suggest some paths for future developments. In particular, the patterns emerging from the numerical results should be examined in more detail through the lens of analytical techniques.

APPENDIX A

Fundamentals on measures

Let (X, d) be a separable metric space. We denote by $\mathcal{B}(X)$ the Borel σ -algebra on X and by $\mathcal{P}(X)$ the family of probability measures on X . The *support* of a measure $\mu \in \mathcal{P}(X)$ is defined as

$$\text{supp } \mu := \{x \in X : \mu(U) > 0 \text{ for every neighborhood } U \text{ of } x\}.$$

We denote by $\mathcal{P}_c(X)$ the space of probability measures with compact support.

A sequence $\{\mu_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X)$ is said to converge *narrowly* (or *weakly*) to $\mu \in \mathcal{P}(X)$ if

$$\lim_{n \rightarrow +\infty} \int_X \varphi d\mu_n = \int_X \varphi d\mu$$

for every continuous and bounded function $\varphi \in C_b(X)$. This will be denoted by $\mu_n \xrightarrow{*} \mu$.

The topology of narrow convergence is generated by the following basis of open sets:

$$\{U_{\varphi_1, \dots, \varphi_n}(\mu, \delta) : \varphi_1, \dots, \varphi_n \in C_b(X), n \in \mathbb{N}, \mu \in \mathcal{P}(X), \delta > 0\},$$

where

$$U_{\varphi_1, \dots, \varphi_n}(\mu, \delta) := \left\{ \nu \in \mathcal{P}(X) : \left| \int_X \varphi_i d\nu - \int_X \varphi_i d\mu \right| < \delta, i = 1, \dots, n \right\}.$$

The space $\mathcal{P}(X)$ can be endowed with the Kantorovich-Rubinshtein norm

$$\|\mu\|_0 := \sup \left\{ \int_X \varphi d\mu : \varphi \in \text{Lip}_1(X), \sup_{x \in X} |\varphi(x)| \leq 1 \right\},$$

where

$$\text{Lip}_1(X) := \{f : X \rightarrow \mathbb{R} : |f(x) - f(y)| \leq d(x, y), \forall x, y \in X\}.$$

This norm induces the so-called Kantorovich-Rubinshtein metric

$$d_0(\mu, \nu) := \|\mu - \nu\|_0.$$

The space $\mathcal{P}(X)$ endowed with the narrow topology is metrizable using this metric.

THEOREM 11. *Let (X, d) be a separable metric space. Then the narrow topology on $\mathcal{P}(X)$ is generated by the Kantorovich-Rubinshtein metric d_0 . Moreover, if (X, d) is complete, then the space $(\mathcal{P}(X), d_0)$ is also complete.*

A proof of this theorem can be found in [9] (Theorems 8.3.2 and 8.10.43).

Since the narrow topology is metrizable, compactness is equivalent to sequential compactness. The following result is useful for characterising compact subsets of $\mathcal{P}(X)$.

THEOREM 12 (Prokhorov). *If a set $\mathcal{K} \subseteq \mathcal{P}(X)$ is tight, i.e.*

$$\forall \epsilon > 0 \quad \exists K_\epsilon \subseteq X \text{ compact s.t. } \mu(X \setminus K_\epsilon) \leq \epsilon \quad \forall \mu \in \mathcal{K},$$

then $\mathcal{K}$ is relatively compact in $\mathcal{P}(X)$. Conversely, if X is a Polish space, i.e. a separable, completely metrizable space, then every relatively compact subset of $\mathcal{P}(X)$ is tight.

LEMMA 6. *Tightness is equivalent to the following condition: there exists a coercive function $F: X \rightarrow [0, +\infty]$, i.e. a function whose sublevel sets $\{x \in X : F(x) \leq c\}$ are compact in X , such that*

$$\sup_{\mu \in \mathcal{K}} \int_X F d\mu < +\infty.$$

We have the following useful characterisation of narrow convergence in terms of lower semi-continuous functions.

LEMMA 7. *Let (X, d) be a metric space and $\mu \in \mathcal{P}(X)$. Then, a sequence $\{\mu_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X)$ is narrowly convergent to μ if and only if*

$$\int_X \psi d\mu \leq \liminf_{n \rightarrow +\infty} \int_X \psi d\mu_n$$

for every function $\psi: X \rightarrow \mathbb{R}$ that is lower semi-continuous and bounded from below.

PROOF. Suppose $\mu_n \xrightarrow{*} \mu$ and let $\psi \in L_b(X)$, the space of lower semi-continuous functions that are bounded from below. We can approximate ψ by a sequence of bounded continuous functions $\{\psi_k\}_{k \in \mathbb{N}}$ such that $\inf \psi \leq \psi_k \leq \psi_{k+1} \nearrow \psi$. Consider for instance

$$\psi_k(x) := \min\{\tilde{\psi}_k(x), k\},$$

with

$$\tilde{\psi}_k(x) = \inf_{y \in X} \{\psi(y) + kd(x, y)\}.$$

For every $k \in \mathbb{N}$,

$$\liminf_{n \rightarrow +\infty} \int_X \psi d\mu_n \geq \liminf_{n \rightarrow +\infty} \int_X \psi_k d\mu_n = \int_X \psi_k d\mu,$$

so by monotone convergence we deduce $\liminf_{n \rightarrow +\infty} \int_X \psi d\mu_n \geq \int_X \psi d\mu$.

Conversely, suppose $\int_X \psi d\mu \leq \liminf_{n \rightarrow +\infty} \int_X \psi d\mu_n$ for every $\psi \in L_b(X)$ and let $\varphi \in C_b(X)$. Since $C_b(X) \subseteq L_b(X)$, then $\pm\varphi \in L_b(X)$ and we have

$$\begin{aligned} \int_X \varphi d\mu &\leq \liminf_{n \rightarrow +\infty} \int_X \varphi d\mu_n \\ \int_X -\varphi d\mu &\leq \liminf_{n \rightarrow +\infty} \int_X -\varphi d\mu_n, \end{aligned}$$

which implies

$$\limsup_{n \rightarrow +\infty} \int_X \varphi d\mu_n \leq \int_X \varphi d\mu \leq \liminf_{n \rightarrow +\infty} \int_X \varphi d\mu_n.$$

Hence, $\lim_{n \rightarrow +\infty} \int_X \varphi d\mu_n = \int_X \varphi d\mu$. □

We conclude this section by defining the Fourier transform of a probability measure. Given a measure $\mu \in \mathcal{P}(\mathbb{R}^n)$, we define its Fourier transform by

$$\hat{\mu}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\langle x, \xi \rangle} d\mu(x) \quad \forall \xi \in \mathbb{R}^n.$$

We observe that $\hat{\mu}$ is well-defined, because $e^{-i\langle \cdot, \xi \rangle} \in L^1(\mathbb{R}^n, \mu)$ for every $\xi \in \mathbb{R}^n$.

APPENDIX B

Introduction to Γ -convergence

Let X be a topological space and $\{F_n\}_{n \in \mathbb{N}}$ a sequence of functions $F_n: X \rightarrow \overline{\mathbb{R}}$, with $\overline{\mathbb{R}} = [-\infty, +\infty]$. We define the Γ -lower limit and Γ -upper limit of $\{F_n\}_n$ at x in X as

$$\begin{aligned}\Gamma\text{-}\liminf_{n \rightarrow +\infty} F_n(x) &= \sup_{U \in \mathcal{N}(x)} \liminf_{n \rightarrow +\infty} \inf_{y \in U} F_n(y), \\ \Gamma\text{-}\limsup_{n \rightarrow +\infty} F_n(x) &= \sup_{U \in \mathcal{N}(x)} \limsup_{n \rightarrow +\infty} \inf_{y \in U} F_n(y),\end{aligned}$$

where $\mathcal{N}(x)$ denotes the collection of all open neighbourhoods of x in X . We say that $\{F_n\}_n$ Γ -converges to $F: X \rightarrow \overline{\mathbb{R}}$ at $x \in X$ as $n \rightarrow +\infty$ if

$$F(x) = \Gamma\text{-}\liminf_{n \rightarrow +\infty} F_n(x) = \Gamma\text{-}\limsup_{n \rightarrow +\infty} F_n(x).$$

In this case, $F(x)$ is called the Γ -limit of F_n at x , which is written

$$F(x) = \Gamma\text{-}\lim_{n \rightarrow \infty} F_n(x).$$

If this holds true for every x in X , we say that $\{F_n\}_n$ Γ -converges to F on X and F is called the Γ -limit of F_n (on the whole X).

We use the notation $F_n \xrightarrow{\Gamma} F$.

REMARK. Γ -convergence is independent of pointwise convergence, as shown by the following examples.

We consider $X = \mathbb{R}$ (with the usual Euclidean topology).

(a) If $F_n(x) = nxe^{-2n^2x^2}$, then $\{F_n(x)\}_n$ Γ -converges to

$$F(x) = \begin{cases} -\frac{1}{2}e^{-\frac{1}{2}}, & x = 0 \\ 0, & x \neq 0, \end{cases}$$

while it converges pointwise to 0.

(b) If

$$F_n(x) = \begin{cases} nxe^{-2n^2x^2}, & \text{if } n \text{ is even} \\ 2nxe^{-2n^2x^2}, & \text{if } n \text{ is odd,} \end{cases}$$

then $\{F_n(x)\}_n$ converges pointwise to 0 $\forall x \in \mathbb{R}$, but

$$\Gamma\text{-}\liminf_{n \rightarrow +\infty} F_n(x) = \begin{cases} -e^{-\frac{1}{2}}, & x = 0 \\ 0, & x \neq 0, \end{cases}$$

$$\Gamma\text{-}\limsup_{n \rightarrow +\infty} F_n(x) = \begin{cases} -\frac{1}{2}e^{-\frac{1}{2}}, & x = 0 \\ 0, & x \neq 0, \end{cases}$$

so $\{F_n(x)\}_n$ does not Γ -converge in $x = 0$.

(c) If $F_n(x) = \sin(nx)$, then $\{F_n(x)\}_n$ Γ -converges to $F(x) = -1 \forall x \in \mathbb{R}$, but it does not converge pointwise.

For metric spaces (in fact for first-countable topological spaces), we can give the following sequential characterisation of Γ -convergence.

THEOREM 13. *Let X be a first-countable topological space and $\{F_n\}_{n \in \mathbb{N}}$ a sequence of functions $F_n: X \rightarrow \overline{\mathbb{R}}$. Then, the Γ -convergence of $\{F_n\}_n$ to $F: X \rightarrow \overline{\mathbb{R}}$ at $x \in X$ is equivalent to the following conditions:*

(i) (liminf inequality) *for every sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ converging to x ,*

$$F(x) \leq \liminf_{n \rightarrow +\infty} F_n(x_n);$$

(ii) (limsup inequality) *there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ converging to x such that*

$$F(x) \geq \limsup_{n \rightarrow +\infty} F_n(x_n).$$

Equivalently, (i) holds and

(ii)' (existence of a recovery sequence) *there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ converging to x such that*

$$F(x) = \lim_{n \rightarrow +\infty} F_n(x_n).$$

DEFINITION 5. *Given a topological space X and a sequence of functions $F_n: X \rightarrow \overline{\mathbb{R}}$, we say that $\{F_n\}_n$ is equi-coercive if for every $t \in \mathbb{R}$ there exists a compact set K_t such that $\{x \in X : F_n(x) \leq t\} \subseteq K_t$ for every $n \in \mathbb{N}$.*

THEOREM 14 (Fundamental theorem of Γ -convergence). *Let (X, d) be a metric space, $\{F_n\}_n$ an equi-coercive sequence of functions $F_n: X \rightarrow \overline{\mathbb{R}}$ and $F = \Gamma\text{-}\lim_{n \rightarrow +\infty} F_n$. Then*

$$\min_X F = \lim_{n \rightarrow +\infty} \inf_X F_n.$$

Moreover, if $\{x_n\}_n$ is a relatively compact sequence such that

$$\lim_{n \rightarrow +\infty} F_n(x_n) = \lim_{n \rightarrow +\infty} \inf_{x \in X} F_n(x),$$

then every limit point of $\{x_n\}_n$ is a minimum point for F .

For a proof of this theorem, one can refer to Theorem 7.8 and Corollary 7.20 in [10].

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