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SECOND CYCLE DEGREE

PHYSICS

Dynamical Systems applied to Economic Science: a Sraffian Supermultiplier model with the addition of inventory dynamics

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Abstract

The dynamic study of economic systems is an area of Economics that is now a century old, starting with the first economic growth models of Harrod-Domar. Through the work of economist Nicholas Kaldor and the eclectic Richard M. Goodwin, non-linear relationships have made their appearance in endogenous growth models, which are, we might say, the gateway to complex systems. In this work, we will study a series of models that deal with the mathematisation of the business cycle and the study of its fluctuations. In particular, the absolute protagonist will be the dynamic model of the Sraffian Supermultiplier, belonging precisely to the school of economic thought initiated by the work of Italian economist Piero Sraffa. Although the author will attempt to explain the differences between the various schools of thought and introduce notions of economics, the central theme of the work will be the addition of inventory fluctuations to the Sraffian Supermultiplier model developed by Freitas and Serrano (2015). We firmly believe that this topic deserves detailed investigation within the Sraffian School, not only for its mathematical content, but, above all, for the addition of an element of realism to the study of economic fluctuations and endogenous growth.

Sommario

Lo studio dinamico dei sistemi economici è un'area della Scienza Economica che ha ormai un secolo, a partire dai primi modelli di crescita economica di Harrod-Domar. Attraverso l'opera dell'economista Nicholas Kaldor e attraverso quella dell'eclettico Richard M. Goodwin hanno fatto la loro comparsa nei modelli di crescita endogena le relazioni non lineari, che sono, potremmo dire, la porta di accesso ai Sistemi Complessi. In questo lavoro studieremo una serie di modelli che si occupano della matematizzazione del business cycle e dello studio delle sue fluttuazioni. In particolare, il protagonista assoluto sarà il modello dinamico del Supermoltiplicatore Sraffiano, appartenente appunto alla Scuola di pensiero economico avviata dall'opera dell'economista italiano Piero Sraffa. Sebbene sarà cura dell'autore tentare di spiegare le differenze tra le varie scuole di pensiero e introdurre nozioni di Economia, l'argomento cardine del lavoro sarà l'aggiunta delle fluttuazioni delle giacenze di magazzino (inventories) al modello di Supermoltiplicatore Sraffiano sviluppato da Freitas e Serrano (2015). Crediamo fermamente che questo argomento meriti una investigazione dettagliata all'interno della Scuola Sraffiana, non solo per il contenuto matematico, ma, soprattutto, per l'aggiunta di un elemento di realismo allo studio delle fluttuazioni economiche e della crescita endogena.

Contents

1 Introduction

Economics, as a social science, is characterized by several "Schools of Thought", which are sets of theories and approaches that offer different perspectives on how economic systems function.
These schools have evolved over time, often in response to economic crises or social changes, and they continue to influence the economic policies of governments and institutions worldwide.
It's important to note that, unlike in Physics, the debate often revolves not around cutting-edge theories, but rather the very foundations of knowledge. Often, the debate revolves precisely around causal connections. One school interprets causal connection as meaning that phenomenon A causes phenomenon B, while another school holds the opposite view.
One of the most famous examples is the debate over the importance of supply and demand. For the so-called Orthodox School (Marginalist/Neoclassical), supply-side mechanisms are the most important aspect of market economies. For the Heterodox Schools, however, demand is of great importance and the key mechanism in economic crises.
While Economics is a social science, it is also possible to describe economical systems using mathematical models. In particular, large macroeconomic aggregates can be described by kinetic models similar to those in Physics: dynamical models applied to Economics, and in particular, models of economic growth, will be the main topic of this thesis.
However, we must not delude ourselves: the veil of mathematical description cast over social and economic science cannot eliminate theoretical debates about the underlying foundations. Thus, even in mathematical models, the aspect of different schools of thought recurs, specifically in the hypotheses subsequently described by the model's equations.
For example, some elements used below, such as the multiplier or induced investments, are not universally accepted by all economists, or rather, by all schools of thought. For this reason, a clarification is necessary before we begin. In this thesis, we will not deal with models from the Orthodox School, since the main topic concerns the Sraffian School, which is one of the Heterodox Schools. Beyond that, most of the models will come from the (Post-)Keynesian School.

2 Dynamical systems and bifurcation theory

2.1 Introductory definitions

A very informal definition of a dynamical system is the following: it is a mathematical formalization of the concept of a deterministic process. A state in the future or in the past of many physical, chemical, biological, etc systems may be inferred by knowing the present state and the laws governing the evolution. If these laws do not change in time, then the behaviour of the system is completely determined by its initial conditions. In conclusion, a dynamical system is defined by a set of possible states and the law which settles the evolution of the states in time, see (Kuznetsov, 1995).
More precisely, a dynamical system is defined by a triple $\{T, X, \varphi^t\}$, where T is a number set, X is a state space and φ^t is an evolution operator. Typically $t \in T$ is identified as the time: in discrete-time dynamical systems $T \in \mathbb{Z}$, in continuous-time dynamical systems $T \in \mathbb{R}$. The possible states in state space are identified by the points in a set X. Finally, the evolution operator φ^t is defined for every $t \in T$ as:

$$\varphi^t : X \to X \tag{1}$$

and transforms an initial state $x_0 \in X$ in a state $x_t \in X$ at time t as:

$$x_t = \varphi^t x_0 \tag{2}$$

In addition, the evolution operator has to satisfy the two following properties:

$$x_0 = \varphi^0 x_0 \tag{3}$$

which means that φ^0 is the identity operator and:

$$x_{t+s} = \varphi^{t+s} x_0 = \varphi^t(\varphi^s x_0) \tag{4}$$

which means that the final state x_{t+s} is the same if the system evolves for a time $t + s$ or if it evolves for a time s and then for a time t.
For our purposes we limit our discussion to continuous-time dynamical systems, thus $T \in \mathbb{R}$, in a state space $X \in \mathbb{R}^n$ and the evolution operator is defined by ordinary differential equations. From now on we use the terms dynamical system and ordinary differential equation as synonyms.
If a dynamical system depends on one or more parameters, it can happen that varying these parameters, the qualitative behaviour of the system can change, for example fixed points can be created or destroyed, their stability can change, even periodic solutions arise. These kind of changes are called bifurcations. In the following we will refer to well-behaved real-valued functions, in the sense they are sufficiently smooth and allows all the derivatives we will need.

2.2 Fixed points and stability in 1 dimension

We begin our discussion on fixed points by considering the ODE:

$$\dot{x} = f(x) \tag{5}$$

where D is a subset of $\mathbb{R}$, $f : D \to \mathbb{R}$ is a smooth function and $x \in D$, then a fixed point x^* is defined by:

$$f(x^*) = 0 \tag{6}$$

namely it is an equilibrium solution of Eq(5):

$$x(t) = x^* \tag{7}$$

Note that Eq(5) has an exact general solution obtained by separation of variables:

$$t = t_0 + \int_{x_0}^{x} \frac{dx'}{f(x')} \tag{8}$$

We now focus our attention on the stability of Eq(7), roughly speaking we are asking ourselves: "what happens to equilibrium solutions after a perturbation?".
In order to answer this question we apply a procedure called linearization, thus we define a small deviation $\eta(t) = x(t) - x^* \ll 1$ and check its behaviour:

$$dot\eta = \dot{x} = f(x^* + \eta) \approx f(x^*) + f'(x^*)\eta = f'(x^*)\eta \tag{9}$$

where we neglected the quadratic term in η and supposed $f'(x^*) \neq 0$. If this is the case[1] then Eq(9) has the solution:

$$\eta(t) = \eta(0)e^{f'(x^*)t} \tag{10}$$

which means $\eta(t)$ decays if $f'(x^*) < 0$ (x^* is stable) or grows if $f'(x^*) > 0$ (x^* is unstable).

2.3 Fixed points and stability in 2 dimensions

Consider the system of coupled ODEs:

$$\dot{\vec{X}} = \vec{F}(\vec{X}) \tag{11}$$

where U is a subset of $\mathbb{R}^2$, $\vec{F} : U \to \mathbb{R}^2$ is a smooth vector field defined as $\vec{F}(\vec{X}) = (f(\vec{X}), g(\vec{X}))^T$, $\vec{X} = (x, y) \in U$. Supposing (x^*, y^*) is a fixed point:

$$f(x^*, y^*) = g(x^*, y^*) = 0 \tag{12}$$

[1] If $f'(x^*) = 0$ nothing can be said about stability and problem has to be studied case by case.

then:

$$\begin{aligned} u &= x - x^* \ll 1 \\ v &= y - y^* \ll 1) \end{aligned} \tag{13}$$

define a small deviation from the fixed point and its dynamics is given by:

$$\begin{aligned} \dot{u} = \dot{x} = f(x^* + u, y^* + v) \approx \\ \approx f(x^*, y^*) + u\partial_x f(x^*, y^*) + v\partial_y f(x^*, y^*) = \\ = u\partial_x f(x^*, y^*) + v\partial_y f(x^*, y^*) \end{aligned}$$

and similarly:

$$\dot{v} = u\partial_x g(x^*, y^*) + v\partial_y g(x^*, y^*) \tag{14}$$

in which quadratic terms in u and v are neglected. In this case the deviation evolves according to the linearized system (until quadratic terms are negligible) which can be solved exactly:

$$(\dot{u}, \dot{v})^T = J(x^*, y^*)(u, v)^T \tag{15}$$

where $J(x^*, y^*)$ is the jacobian matrix at the fixed point.
Since we are interested in studying the stability we only care about the behaviour of the system close to fixed points, thus linearization is a good approximation[2].
Due to the fact $J(x^*, y^*)$ is a symmetric matrix, it can always be rewritten in terms of eigenvalues and eigenvectors.
The general solution of a generic 2-dimensional linear system of ODEs:

$$\dot{\vec{X}} = A\vec{X} \tag{16}$$

is given by:

$$\vec{X}(t) = c_1 e^{\lambda_1 t}\vec{w}_1 + c_2 e^{\lambda_2 t}\vec{w}_2 \tag{17}$$

where λ_i is the eigenvalue associated to the eigenvector $\vec{w}_i$, $i = 1, 2$ of matrix A.
In order to check the stability of the fixed point (x^*, y^*), it is sufficient to look at the sign of the real part of the eigenvalues: if at least one of them is positive, then Eq(17) grows exponentially and the fixed point is unstable. On the other hand, if the real parts are all negative then the fixed point is stable. Imaginary parts do not affect the global stability, even though have impact on the trajectories: for example a stable fixed point is a node if $Im(\lambda_{1,2}) = 0$, but it is a focus if $Im(\lambda_{1,2}) \neq 0$.
Since in 2-dimensional the determinant of the jacobian matrix is $\Delta = \lambda_1\lambda_1$ and the trace is $\tau = \lambda_1 + \lambda_1$ the scheme in Figure 1 is useful for classifying ODEs systems by looking only at the sign of the trace and the determinant[3].
An additional tool, which allows us to find fixed points graphically, is represented by

[2]There are particular case where this approximation fails since it isn't enough.
[3]it is often useful for deriving the conditions on a parameter in the bifurcation study.

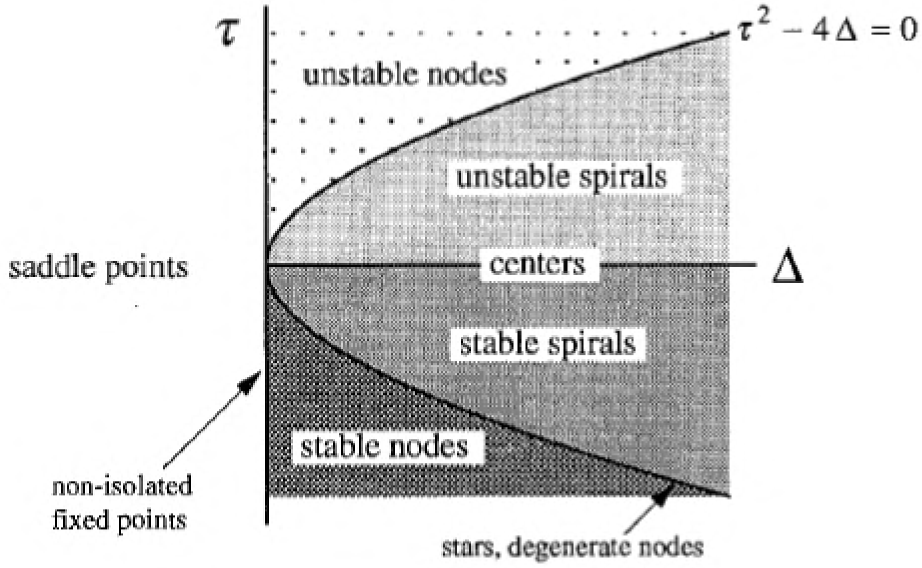


Figure 1: 2-dimensional topology, where Δ is the determinant and τ the trace of the jacobian matrix. The image is taken from (Strogatz, 2024).

the study of nullclines. A nullcline is defined as the curve on which the time derivative of a variable in the dynamical system is equal to zero, namely, for Eq(11), the nullclines are the curves defined by $f(x, y) = 0$ and $g(x, y) = 0$, which can be plotted on phase plane. By definition, the points at which these curves cross are fixed points.
The definitions we introduced in order to study stability or instability of fixed points are pretty general and valid even for higher-dimensional systems.

2.4 Limit cycles

The dynamics in 1 dimension is pretty poor, in fact a trajectory in phase space can only approach, be repelled or stand on a fixed point. In 2 dimensions oscillations and peri-

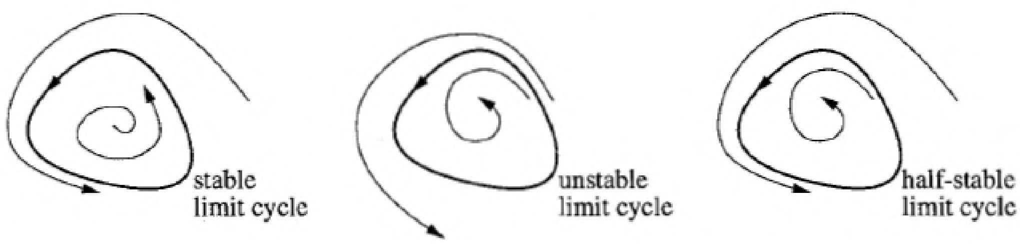


Figure 2: Three kinds of limit cycles. The image is taken from (Strogatz, 2024).

odic orbit can arise, in addition to the already-known behaviour seen in 1 dimension. A limit cycle is an isolated closed trajectory, which means that neighbouring trajectories are not closed: they approach the limit cycle (if the limit cycle is stable) or move away from it (if the limit cycle is unstable). There also exist half-stable limit cycles: trajectories are attracted on one side and repelled on the other. Note that limit cycles occur only in non-linear systems. In linear systems of course closed orbits are possible but they are not isolated: if $\vec{x}(t)$ is a closed orbit, because of linearity also $c\vec{x}(t)$ does, which means $\vec{x}(t)$ is surrounded by a one-parameter family of closed orbits (center).

2.5 Bifurcations in 1 dimension

In this section the main bifurcations in 1 dimension, which occur also in higher dimensional systems, are outlined and a few examples are provided.

2.5.1 Saddle-node bifurcation

The saddle-node bifurcation is the mechanism that manages the creation and destruction of fixed points. As the control parameter is varied, two fixed points can move closer to each other, collide and mutually annihilate at bifurcation point.
The normal form of this kind of bifurcation is given by:

$$\dot{x} = r + x^2 \tag{18}$$

where r is the control parameter. As shown in Figure 3, the two fixed points $x^* = \pm\sqrt{-r}$, one stable and the other unstable, exist if $r < 0$ and approach or move away each other as r varies. If $r < 0$ is increased the fixed points come closer and in $r = 0$ they collide and disappear. On the other hand if $r > 0$ is decreased two fixed points

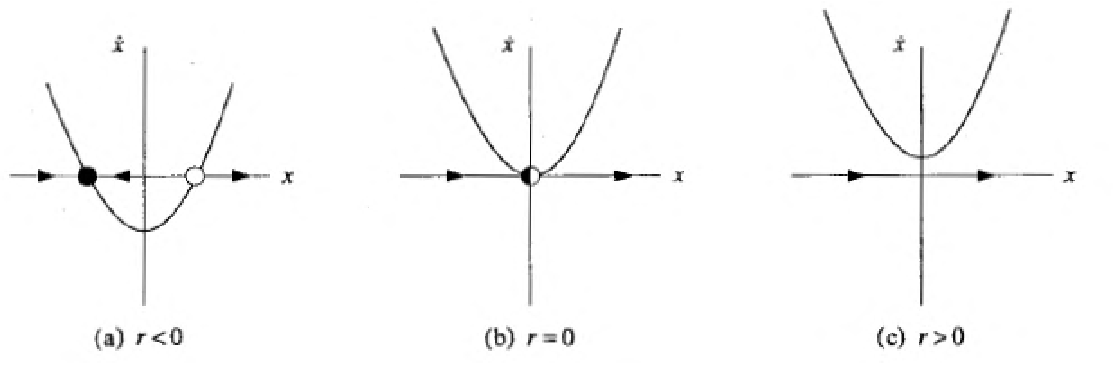


Figure 3: Saddle-node bifurcation. The image is taken from (Strogatz, 2024).

emerge in $r = 0$ and then move away. In $r = 0$ the system changes its qualitative behaviour so this is the bifurcation point. In order to represent this behaviour we can

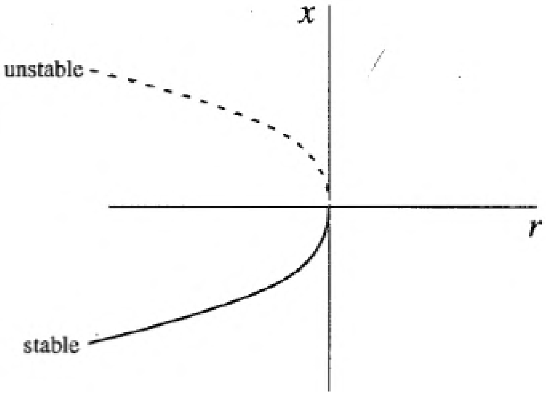


Figure 4: Saddle-node bifurcation diagram. The image is taken from (Strogatz, 2024).

plot the bifurcation diagram which shows the dependence of fixed points as a function of r, as shown in Figure 4.

2.5.2 Transcritical bifurcation

The transcritical bifurcation manages the exchange of stabilities between two fixed point after they collided. This kind of bifurcation typically occurs when a fixed point

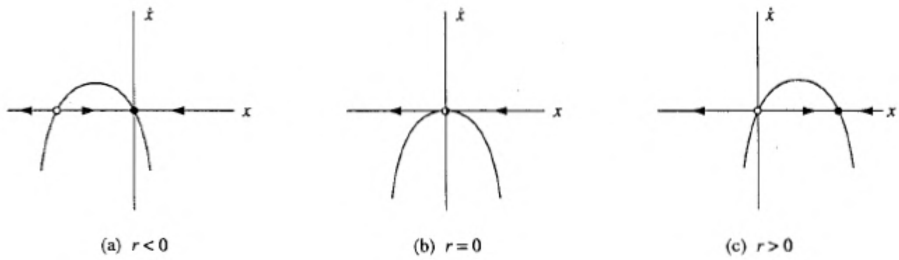


Figure 5: Transcritical bifurcation. The image is taken from (Strogatz, 2024).

always exists and cannot be destroyed. The normal form of this kind of bifurcation is given by:

$$\dot{x} = rx - x^2 \tag{19}$$

In this case the fixed points are $x^* = 0$ and $x^* = r$.
As shown in Figure 5, $x^* = 0$ is a fixed point for all values of r and, if $r < 0$, it is stable and $x^* = r$ is unstable. On the other hand if $r > 0$ then $x^* = 0$ becomes unstable and $x^* = r$ becomes stable. This means the two fixed points at bifurcation point $r = 0$ exchange their stability, as also shown in the bifurcation diagram in Figure 6.

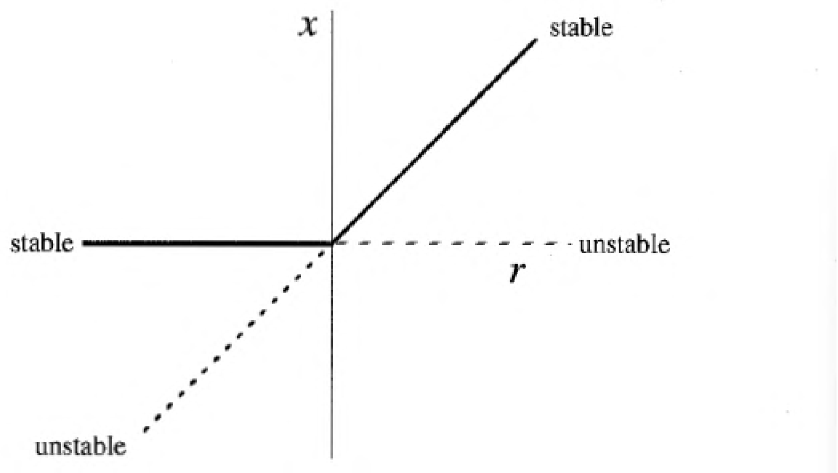


Figure 6: Transcritical bifurcation diagram. The image is taken from (Strogatz, 2024).

2.5.3 Pitchfork bifurcation

The pitchfork bifurcation occurs in systems presenting some kind of symmetry and so fixed points tend to appear and disappear in symmetrical pairs. There exist two kind

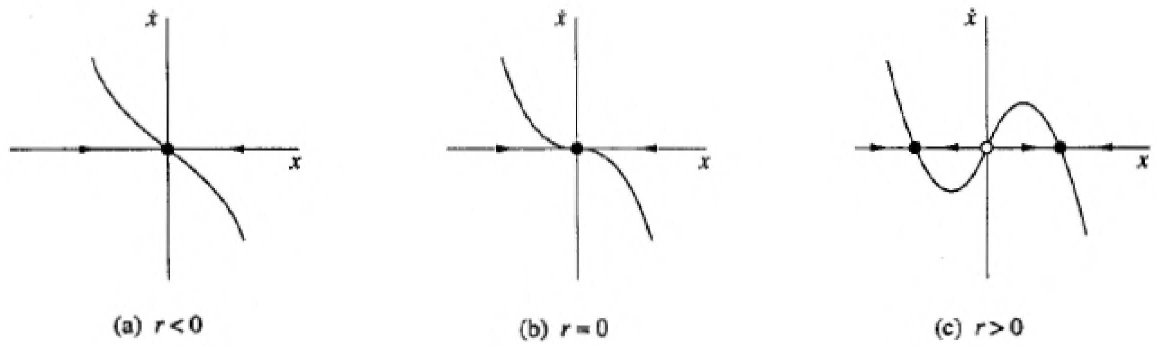


Figure 7: Supercritical pitchfork bifurcation. The image is taken from (Strogatz, 2024).

of pitchfork bifurcation: supercritical and subcritical.
The normal form of the supercritical pitchfork bifurcation is given by:

$$\dot{x} = rx - x^3 \tag{20}$$

whose fixed points are $x^* = 0$ and, if $r > 0$, $x^* = \pm\sqrt{r}$. As shown in Figure 7, $x^* = 0$ is the only existing fixed point if $r \leq 0$ and it is stable. If $r > 0$ then it becomes unstable and the two symmetrical stable fixed points $x^* = \pm\sqrt{r}$ appear, so $r = 0$ is the bifurcation point, as also shown in the bifurcation diagram in Figure 8.

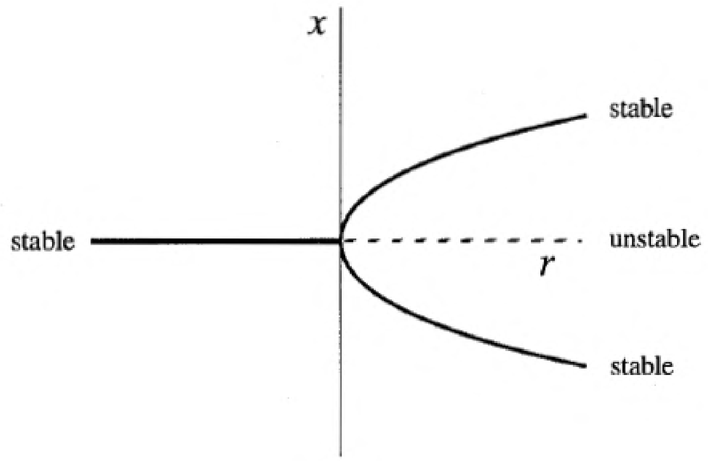


Figure 8: Supercritical pitchfork bifurcation diagram. The image is taken from (Strogatz, 2024).

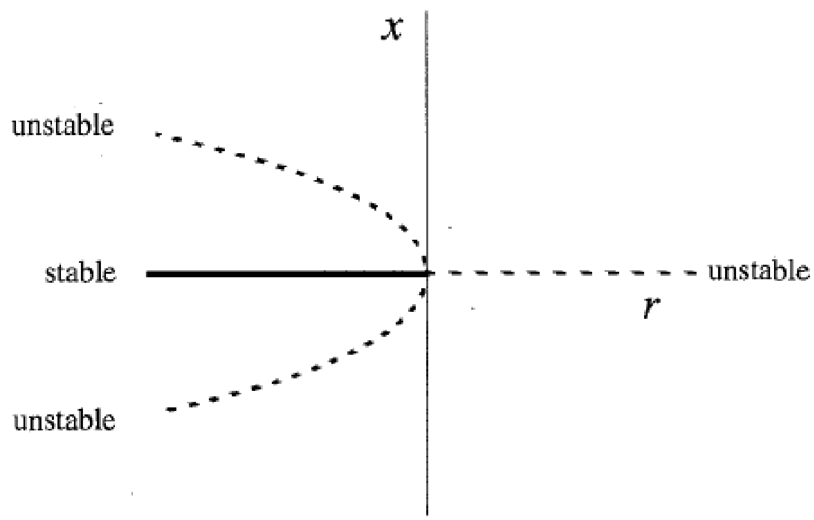


Figure 9: Subcritical pithfork bifurcation diagram. The image is taken from (Strogatz, 2024).

The normal form of the subcritical pitchfork bifurcation is:

$$\dot{x} = rx + x^3 \tag{21}$$

As in the previous case, $x^* = 0$ always exists but now is stable if $r < 0$ and $x^* = \pm\sqrt{-r}$ are unstable. Conversely, if $r \geq 0$ then $x^* = 0$ is the only fixed point and it is unstable, as shown in Figure 9.

2.6 Bifurcations in 2 dimensions

The dynamics in 2 dimensions is richer than in 1 dimension.
In addiction to the bifurcations seen in the previous sections, in fact, new kinds of bifurcations involving limit cycles may arise.

2.6.1 Hopf bifurcation

Given a fixed point, as said in Section 2.3, its stability is regulated by the real part of the eigenvalues λ_1, λ_2 of the jacobian matrix. Since they satisfy a quadratic equation, if the fixed point is stable, they must be real and negative or complex conjugates with negative real part, see Figure 10. In order to destabilize the fixed point, at least one of

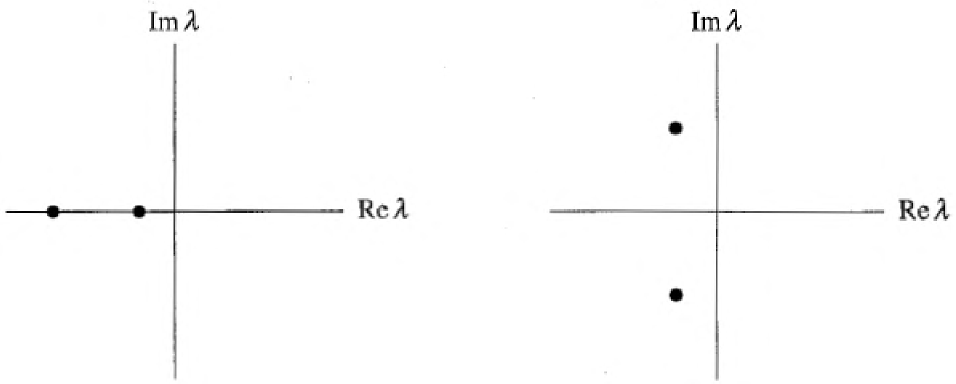


Figure 10: Jacobian eigenvalues of a stable fixed point in complex plain: both real or complex conjugates. The image is taken from (Strogatz, 2024).

the two eigenvalues has to cross the imaginary axis, as the control parameter μ varies. If both complex conjugates eigenvalues do, the system undergoes an Hopf bifurcation. A supercritical Hopf bifurcation occurs when a stable fixed point becomes unstable and is surrounded by a small limit cycle.
The normal form in polar coordinates is given by:

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega \end{cases} \tag{22}$$

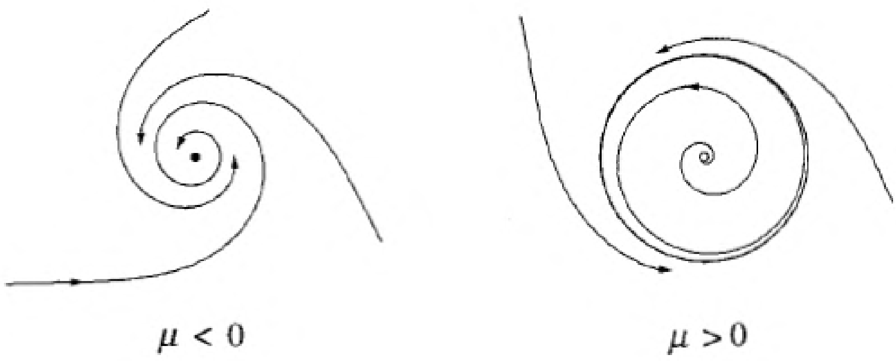


Figure 11: Supercritical Hopf bifurcation phase portrait. The image is taken from (Strogatz, 2024).

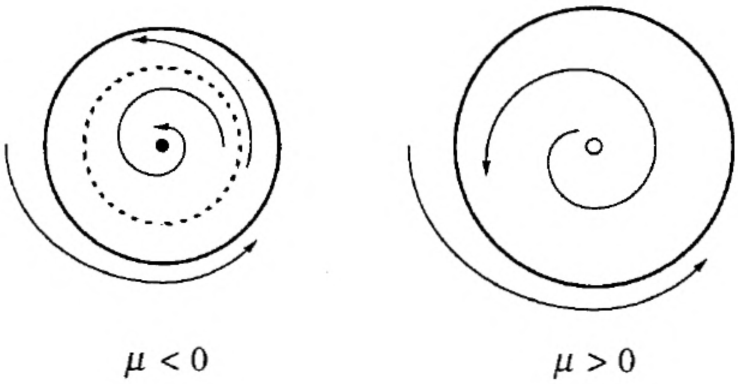


Figure 12: Subcritical Hopf bifurcation phase portrait. The image is taken from (Strogatz, 2024).

After calculation, the eigenvalues of the jacobian are found to be $\lambda = \mu \pm i\omega$, so the bifurcation point is $\mu_c = 0$. As shown in Figure 11 the origin $r = 0$ is stable if $\mu \leq 0$. If $\mu > 0$ it becomes unstable and a stable circular limit cycle of radius $r = \sqrt{\mu}$ appears.
It can be shown that, in general in supercritical Hopf bifurcation, the size of the limit cycle grows continuously from zero of a factor proportional to $\sqrt{\mu - \mu_c}$ and the period of the limit cycle is $T = \frac{2\pi}{Im(\lambda)} + O\left(\mu - \mu_c\right)$, for $\mu - \mu_c$ close to zero.
Instead, in subcritical Hopf bifurcation the trajectories jump to a distant attractor (a fixed point, a limit cycle, infinity).
This means there is not a continuous growth of a limit cycle as in the supercritical case.
The normal form is given by:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega \end{cases} \tag{23}$$

Note that r^3 is now the destabilizing term.
If $\mu < 0$, there is a stable fixed point in the origin surrounded by an unstable limit cycle and an external stable limit cycle, as shown in Figure 12.
As μ increases, the unstable limit cycle shrinks to zero amplitude and makes the origin become unstable at the bifurcation point $\mu = 0$.
This means that there is a trajectory rapidly emerging from the origin which approaches the stable limit cycle after the bifurcation.

2.6.2 Saddle-node bifurcation of cycles

Consider once again Eq(23) we have seen in previous section if $\mu = 0$ a subcritical Hopf bifurcation occurs, anyway another kind of bifurcation may occur too.
As shown in Figure 13, if $\mu_c = -1/4$, an half-stable limit cycle appears and, as μ is increased, it splits into two limit cycles, in a sort of saddle-node bifurcation. This kind of bifurcation is called saddle-node bifurcation of cycles.

2.6.3 Saddle-node on a limit cycle (SNLC) bifurcation

The SNLC bifurcation occurs when two fixed points emerge on a limit cycle in a saddle-node bifurcation.
As an example we consider:

$$\begin{cases} \dot{r} = r - r^3 \\ \dot{\theta} = \mu - \sin\left(\theta\right) \end{cases} \tag{24}$$

As shown in Figure 14, if $\mu > 1$, the trajectories approach the stable limit cycle $r = 1$ surrounding the origin which is always unstable.
As μ decreases, a bottleneck appears at $\theta = \pi/2$ and finally, at bifurcation point $\mu_c = 1$, a fixed point emerges on the limit cycle and the period becomes infinite (it can be shown

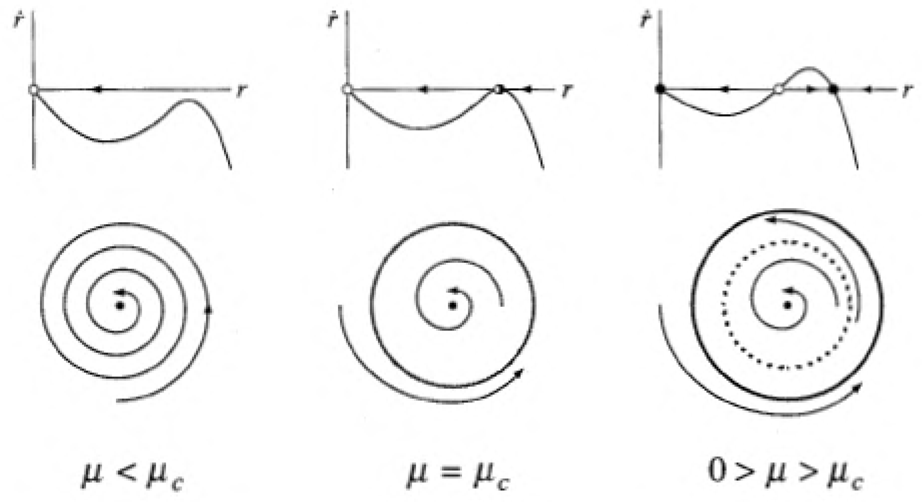


Figure 13: Saddle-node bifurcation of cycles radial plot and phase portrait. The image is taken from (Strogatz, 2024).

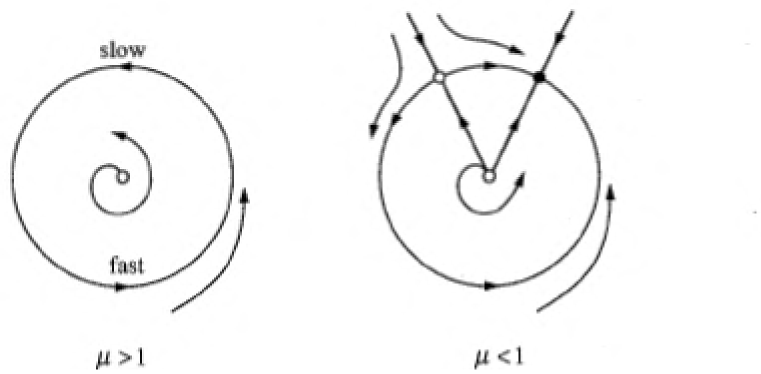


Figure 14: SNLC bifurcation phase portrait. The image is taken from (Strogatz, 2024).

the period is $T = O\left(\frac{1}{\sqrt{\mu-\mu_c}}\right)$ near the bifurcation).
If $\mu < 1$ the fixed point splits into a saddle and node in a saddle-node bifurcation.

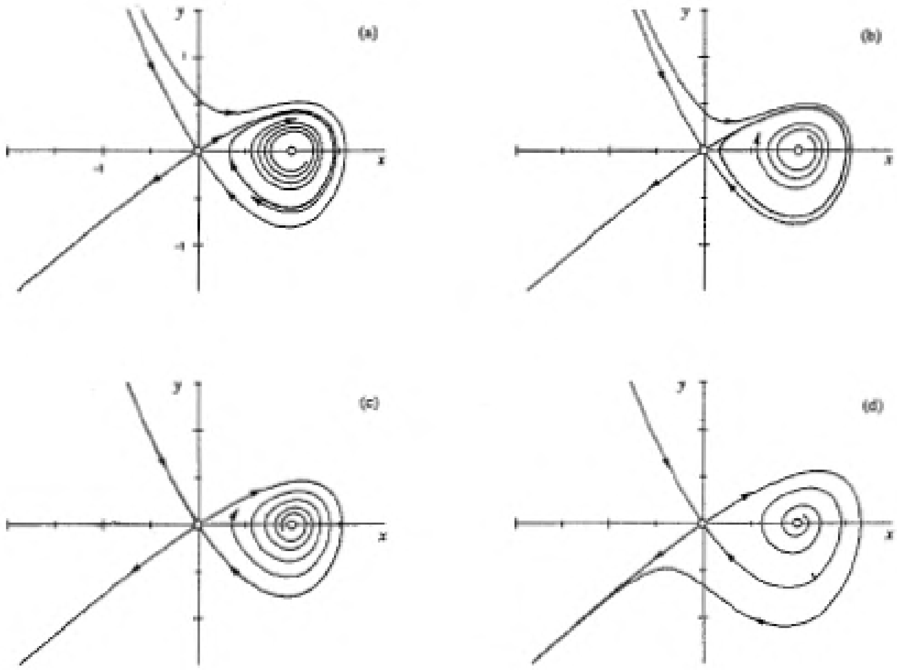


Figure 15: Saddle-homoclinic bifurcation phase portrait for different values of μ. The image is taken from (Strogatz, 2024).

2.6.4 Saddle-homoclinic bifurcation

A saddle-homoclinic bifurcation occurs when a limit cycle and a saddle move closer until they touch at bifurcation point, thus making the orbit homoclinic.
As an example consider:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu y + x - x^2 + xy \end{cases} \tag{25}$$

In Figure 15 the phase plane is plotted for different values of μ: if $\mu < \mu_c \approx -0.8645$, a stable limit cycle surrounds the unstable fixed point on the right and there is a saddle at the origin. As μ increases, the limit cycle moves closer to the saddle until $\mu = \mu_c$, where they touch, and the orbit becomes homoclinic.
Finally, if $\mu > \mu_c$, the limit cycle is destroyed.
We conclude noting that also in this case the period becomes infinite but the asymptotic behaviour is different than in SNLC bifurcation: $T = O\left(\ln\left(\frac{1}{\mu-\mu_c}\right)\right)$ near the

bifurcation point.

3 Examples of dynamical systems in Economics

3.1 The first growth model: the Harrod-Domar model

The Harrod-Domar model is important because it was the first growth model.
Contemporary interest in modern theories of economic growth can be conveniently dated from the (Harrod, 1939) paper (and (Harrod, 1948), years later) followed shortly by Domar' similar, but indipendently derived, contributions, (Domar, 1946) and (Domar, 1947), (see also Hywel, 1975, Chapter 3).
(Cesaratto, 2019)[4] comments on this point:

> Keynes' first biographer was Roy Harrod (1900-1978), who in the late 1930s was also the initiator of modern growth theory. His was an attempt to extend Keynes' ideas to the long term. Keynes had in fact stated that in the short term, when production capacity is given, the degree of utilisation of that capacity depends on aggregate demand. In the long term, economists say, productive capacity (or capital stock) increases. The question is therefore what determines this increase. Harrod came up with a model that was as simple as it was ambiguous, which was neither Keynesian nor marginalist. So much so that both the orthodox and heterodox strands of growth theory departed from him.

The first assumption of the model is that savings S are proportional to national income Y

$$S = sY \tag{26}$$

where s is the average and marginal propensity to save.
The labour force L is assumed to grow at a constant rate n, so it's exogenous

$$\frac{\dot{L}}{L} = n \tag{27}$$

Y is not only the national income but also the output produced. Here, we assume that the capital stock K do not depreciate[5] and there is no technical progress.
A given flow to output Y requires an amount of capital and labour and they are uniquely given[6]. So a stock of capital K is required to produce an output $Y = \frac{K}{\nu_G}$ where ν_G is

[4] The translation from Italian is our.
[5] So the flow of investment is $I = \dot{K}$.
[6] So here it is not possible to produce, as in neoclassical models, a quantity of output through different combinations of capital and labour. Where, on the other hand, the substitution of factors of production applies (e.g. neoclassical models), capitalists will be able to choose the proportion of labour and capital that will minimise their expenditure, depending on the prices of factors of production.

the capital-output ratio or accelerator (here it has two different conceptions, that we'll see later).
However any given flow of output requires $\frac{L}{o}$ units of labour, with o defined as the constant ratio of labour requirements to total output $o = \frac{L}{Y}$. Put another way, if all labour is fully employed then the maximum flow of output, whatever the size of the stock of capital, is $\frac{L}{u}$.
So the production function implied by the Harrod approach is of the fixed proportion variety:

$$Y = \min\left(\frac{K}{\nu_G}, \frac{L}{o}\right) \tag{28}$$

If, at the beginning of time, all labour is fully employed, this assumption implies that, in the absence of technical progress, the maximum rate of growth of national income and output is given by the exogenously determined rate of growth of the labour force. Please note, ν_G also implies that

$$\dot{K} = \nu_G \dot{Y} = I \tag{29}$$

ν_G can be considered the actual increment in the capital stock in any period divided by the actual increment in output, thus, e.g., at the end of year, ν_G could be interpreted as the measured increase in the capital stock during the year divided by the measured increase in income or output. This is the first definition. The second definition is the increment in the capital stock associated with an increment in output that is required by entrepreneurs if, at the end of the period, they are to be satisfied that they have invested the correct amount. We will refer to this interpretation with the symbol ν_{Gw} to distinguish this conception from the first definition.
Now we recall the equilibrium condition of elementary macroeconomics, i.e.aggregate planned investment must equal aggregate planned saving

$$I \equiv S \tag{30}$$

So the output growth rate g is given by

$$g \equiv \frac{\dot{Y}}{Y} = \frac{s}{\nu_G} = \frac{\dot{K}}{K} \equiv g_K \tag{31}$$

so

$$Y(t) = Y(0)\, e^{\frac{s}{\nu_G}t} = K(0)\, e^{\frac{s}{\nu_G}t} = K(t) \tag{32}$$

Both national income and the capital stock must grow at the same constant rate $\frac{s}{\nu_G}$, we call it steady-state growth. Eq(31) is called "fundamental" equation. It can be interpreted in two different ways depending upon with conception of ν_G is employed.

Following (Hywel, 1975), from $\nu_G = \frac{\dot{K}}{\dot{Y}} = \frac{I}{\dot{Y}}$ and Eq(31) we have $\frac{\dot{Y}}{Y}\frac{I}{\dot{Y}} = s = \frac{S}{Y}$ which reduces to accounting identity that investment must equal savings ex-post. If the capital-output ratio is given by the the fist definition than the fundamental equation is a necessary true statement, a "truism"[7]. So if ν_G follows the first definition then the rate of growth of national income *must* equal $\frac{s}{\nu_G}$. Using the symbol g_a for the actual growth rate over any period of time, the fundamental equation, viewed as a truism, can be written as

$$g_a \equiv \frac{s}{\nu_G} \tag{33}$$

The fundamental equation can, however, be given theoretical content if the capital-output ratio is interpreted in the second of the two ways discussed above. Now ν_{Gw} expresses the entrepreneurs' requirements for additions to the capital stock given the growth of income and output, calling $g_w = \frac{\dot{Y}}{Y} = \frac{s}{\nu_{Gw}}$ the rate of growth of output, which will satisfy capitalists that they're investing the correct amount.

$$g_a\nu_G = s = g_w\nu_{Gw} \tag{34}$$

If national income and output happens to grow at the rate g_w then the actual increase in the capital stock associated with the growth of income will equal the increase that entrepreneurs require if they are to be satisfied that the level of the capital stock is exactly appropriate for the production of the current level of national output. Harrod called the rate of growth g_w the warranted rate. There is not, of course, any particular reason why we should expect that economy will actually grow at the warranted rate: the actual growth rate being the outcome of the expectations, decisions and mistakes of a host of different decision-makers.
Now, let's add the level of employment to add more keynesian flavour.
We have already noted that the actual output growth rate could not permanently exceed the labour growth rate because of the assumed constancy of the labour-output ratio $o = \frac{L}{Y}$. Thus $g_a \leq \frac{\dot{L}}{L} = n$. Now, if economy is originally in a situation of full employment, full employment through time would imply that the actual growth rate g_a would equal n. But we have already seen that, for equilibrium steady-state growth, g_a must equal g_w: it's therefore clear that equilibrium steady growth with full employment necessitates that

$$g_a = \frac{s}{\nu_{Gw}} = g_w = n \tag{35}$$

Mrs Robinson has described the mythical state of affairs where economy grows at the constant n rate "The Golden Age".
Hywel (1975):

> It is therefore clear that the Harrod model includes the possibility of equilibrium steady growth at full employment. However, there is clearly no reason to believe that $\frac{s}{\nu_{Gw}}$ will equal $\frac{s}{\nu_G}$ or n. s, ν_G and n are all independently determined.

[7] Of course, one could say that we have used Eq(30) to calculate Eq(31).

> Only a happy accident will generate steady-state growth at full employment in the Harrod model. The propensity to save, s, is determined by the preferences of firms and households in the economy. The rate of growth of the labour force, n, is exogenous and determined simply by the biologically determined birth and death rates. The capital-output ratio, ν_G, is, on our present interpretation, a reflection of the fixity of the technology. If, by coincidence, the actual rate of growth equalled the warranted rate, which itself equalled the rate of growth of the labour force, then steady growth at full employment would occur. But, there is no mechanism in the Harrod model which would ensure the attainment of this Golden Age situation. Although steady state growth at full employment is possible in an Harrod-type model of economic growth, such a Golden Age is highly improbable given the independent constituent variables in the necessary equality of the warranted rate of growth, $\frac{s}{\nu_{Gw}}$, to the natural rate of growth, n. This conclusion is thoroughly keynesian in spirit; there is no reason to believe that full-employment equilibrium growth will be attained. Thus, the "First Harrod Problem" can be interpreted as a dynamic version of the central keynesian allegation that under-employment equilibrium is possible in a capitalist economy.

The First Harrod Problem leads us to the Harrod Stability Problem or Second Harrod Problem. Indeed, Harrod suggested that the warranted rate of growth was fundamental unstable in the sense that divergences of the actual rate of growth, from the warranted rate, would not only not correct themselves but would produce even larger divergences: deviations are cumulative in effect.
It should be clear that there is no reason why entrepreneurs' expectations should be consistent with the warranted rate of growth.
They have no means of knowing $\frac{s}{\nu_G}$ and there would be no reason for them to suppose that a consideration of this expression should enter into their decision making process.
It should also be clearly understood that the Second Harrod Problem is logically independent of the First.

3.2 The Kaldor (1940) model

In this section, we will briefly examine Kaldor's Trade Cycle[8] model. Although it does not directly belong to the line of research followed in this work, we have decided to include it anyway for various reasons.
First of all, it is an example of how economists often reason: in fact, Kaldor (1940) did not originally express the model through equations, but rather in graphic form. Economists, especially at the educational level but not only, often reason through graphs and charts. Furthermore, Kaldor is one of the most important exponents of the Post-Keynesian School, and heterodox in general. Finally, it is one of the first economic models in which

[8]Here, trade cycle is the British synonym for the, now, more common, American term, business cycle. It refers to the study of the alternation of booms and recessions in economic systems.

non-linear relationships appeared, which, as we know, are important in complex systems.
The Kaldor (1940) model represents a milestone in macroeconomic theory due to its ability to generate endogenous cyclical fluctuations, i.e. fluctuations originating from the functioning of the economic system itself rather than from external shocks.
Indeed, the core of this behaviour lies in the non-linear interaction between investment and savings decisions.
Anyway, Kaldor's original model has been extended and studied over the years.
Here, we will briefly present the dynamical model proposed by Chang and Smyth (1971).
Kaldor begins his paper by introducing the terms often used by economists, "ex-ante" and "ex-post". Let's explain them. The former refers to planning and desire, i.e. actions planned with a view to a goal or event. The latter refers to reality, to the result, or how things actually realize. Thus, we can talk about planned investment and actual investment with ex-ante investment and ex-post investment, and the same applies to ex-ante saving and ex-post saving. Let's quote (Kaldor, 1940), then we'll explain it.

> Investment ex-ante is the value of the designed increments of stocks of all kinds (i.e., the value of the net addition to stocks plus the value of the aggregate output of fixed equipment), which differs from Investment ex-post by the value of the undesigned accretion (or decumulation) of stocks. Savings ex-ante is the amount people intend to save - i.e., the amount they actually would save if they correctly forecast their incomes. Hence ex-ante and ex-post Saving can differ only in so far as there is an unexpected change in the amount of income earned. If ex-ante Investment exceeds ex-ante Saving, either ex-post Investment will fall short of ex-ante Investment, or ex-post Saving will exceed ex-ante Saving; and both these discrepancies will induce an expansion in the level of activity. If ex-ante Investment falls short of ex-ante Saving either ex-post Investment will exceed ex-ante Investment, or ex-post Saving will fall short of ex-ante Saving, and both these discrepancies will induce a contraction. This must be so, because a reduction in ex-post Saving as compared with ex-ante Saving will make consumers spend less on consumers' goods, an excess of ex-post Investment over ex-ante. Investment (implying as it does the accretion of unwanted stocks) will cause entrepreneurs to spend less on entrepreneurial goods; while the total of activity is always determined by the sum of consumers' expenditures and entrepreneurs' expenditures. Thus a discrepancy between ex-ante Saving and ex-ante Investment must induce a change in the level of activity which proceeds until the discrepancy is removed.

Let's take an example of what Kaldor is saying: assuming that investment plans (ex-ante) $>$ savings plans (ex-ante). What does this mean? Firms want to inject more money into the economy by investing than households want to take out by saving. In practice, total planned demand is really high. So firms find that they are selling much more than they had predicted: the stocks (inventories) are unexpectedly declining. What is the reaction? To avoid running out of stock and to meet high demand, companies do two things: they increase production and hire more staff. But this means more production and more employment which in turn means more income for everyone (wages and profits increase). So the economy enters a phase of expansion. The exact opposite

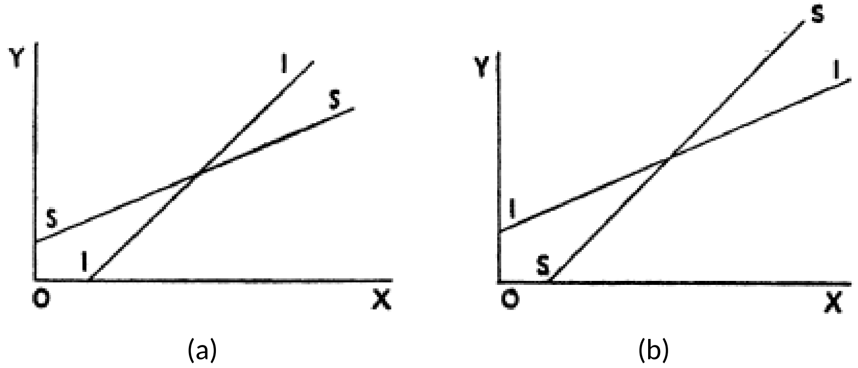


Figure 16: Linear cases: (a) $\frac{dI}{dx} > \frac{dS}{dx}$, only a single position of unstable equilibrium, since above the equilibrium point $I > S$, and thus activity tends to expand, below it $S > I$, and hence it tends to contract, so the economic system would always be rushing either towards a state of hyper-inflation with full employment, or towards a state of complete collapse with zero employment, with no resting-place in between, recorded experience does not bear out such instabilities and (b) $\frac{dS}{dx} > \frac{dI}{dx}$, again, only a single position, but here of stable equilibrium, any disturbance, originating either on the investment side or on the savings side, would be followed by the re-establishment of a new equilibrium, with a stable level of activity, so this assumption fails in the opposite direction: it assumes more stability than the real world appears, in fact, to possess. They're taken from (Kaldor, 1940).

happens if households intend to save more than entrepreneurs intend to invest, leading to a contraction.
Kaldor's reasoning is that this "surprise" (the discrepancy between ex-ante plans and ex-post results) is the signal that prompts firms to change their behaviour, setting the business cycle in motion.
Kaldor denotes the level of economic activity, measured in terms of employment, by x; the level of ex-ante saving is S and I is the level of ex-ante investment. Both are single-valued functions of the level of activity x and they vary positively with x: $\frac{dS}{dx} > 0$ and $\frac{dI}{dx} > 0$.
The $S(x)$ expresses the principle of the multiplier[9] (that the marginal propensity to consume is less than unity) and $I(x)$ denotes the assumption that the demand for capital goods will be greater the greater the level of production.

[9]Here the multiplier is $\frac{1}{1-\frac{dC}{dx}}$ where $\frac{dC}{dx} = 1 - \frac{dS}{dx}$ and $\frac{dC}{dx}$ is the marginal propensity to consume.

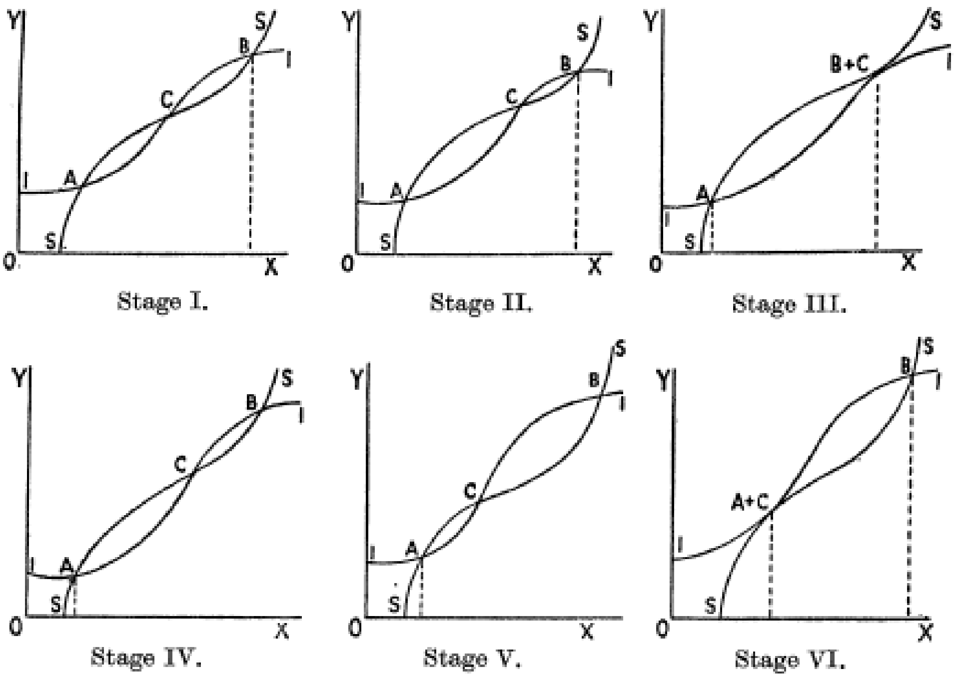


Figure 17: The trade cycles. It's taken from (Kaldor, 1940).

If we regard $S(x)$ and $I(x)$ functions as linear, we have two possibilities. $\frac{dI}{dx} > \frac{dS}{dx}$, in which case, as show by Figure 16a there can be only a single position of unstable equilibrium, since above the equilibrium point $I > S$, and thus activity tends to expand, below it $S > I$, and hence it tends to contract. If the S and I functions were of this character, the economic system would always be rushing either towards a state of hyper-inflation with full employment, or towards a state of complete collapse with zero employment, with no resting-place in between. Since recorded experience does not bear out such dangerous instabilities, this possibility can be dismissed.
The second one also can be dismissed, since $\frac{dS}{dx} > \frac{dI}{dx}$, in which case, as shown in Figure 16b, there will be a single position of stable equilibrium. If the economic system were of this nature, any disturbance, originating either on the investment side or on the savings side, would be followed by the re-establishment of a new equilibrium, with a stable level of activity. Hence this assumption fails in the opposite direction: it assumes more stability than the real world appears, in fact, to possess.
Therefore, $S(x)$ and $I(x)$ functions cannot be linear; furthermore, they must be "short-period" functions.
Figure 17 shows Kaldor's proposal. Now there are three positions of equilibrium (Stage I). $\frac{dS}{dx} > \frac{dI}{dx}$ at the two extreme equilibrium points, A and B, and $\frac{dS}{dx} < \frac{dI}{dx}$ is less than at the equilibrium point C so that A and B are points of stable equilibrium and C is a point of unstable equilibrium. The investment and saving schedules are short-period

functions and will shift as capital stock shifts: a rise in capital stock K will shift the investment function down and the saving function up and a fall in capital stock will shift the investment function up and the saving function down. Thus if the system is in equilibrium at B and capital is being accumulated B is shifted to the left and C to the right (Stage II). When B and C coincide the investment and saving functions are tangent to each other (Stage III) and the system is unstable in a downward direction and employment (or x) falls to A (Stage IV). There the movements in the investment and saving schedules are in the opposite direction provided capital is being decumulated, thus A and C move together (Stage V) until they coincide (Stage VI) and the system is unstable in an upward direction and x or the employment rises until B is reached again, upon which the process repeats itself (Stage I again).
This is the Kaldor's business (or trade) cycle.
To capture the basic spirit of Kaldor's original model, Chang and Smyth (1971) use a differential equation system with general non-linear form.
So ex-ante net investment and saving are functions of income and stock of capital[10]: $I = I(Y,K)$ and $S = S(Y,K)$ where $I_Y \equiv \frac{\partial I}{\partial Y} > 0$, $I_K \equiv \frac{\partial I}{\partial K} < 0$, $S_Y \equiv \frac{\partial S}{\partial Y} > 0$ and $S_K \equiv \frac{\partial S}{\partial K} < 0$.
It would seem reasonable to assume $|I_K| > |S_K|$ so that $I_K - S_K$ is always negative.
Since income will rise if and only if ex ante investment is greater than ex ante saving, the dynamic equation for changes in national income is $\dot{Y} = \alpha [I(Y,K) - S(Y,K)]$ where α is a positive constant which denotes the speed of adjustment.
The accumulation or decumulation of capital can be postulated in different ways, but it is sufficient here to consider the case where the movement is along the I curve, i.e. investment plans are realized. Thus the adjustment in the stock of capital is $\dot{K} = I(Y,K)$.
Together

$$\begin{cases} \dot{Y} = \alpha [I(Y,K) - S(Y,K)] \\ \dot{K} = I(Y,K) \end{cases} \tag{36}$$

Eq(36) constitutes the dynamical system of Kaldor's model.
The jacobian matrix of the dynamical system is

$$J = \begin{bmatrix} \alpha (I_Y - S_Y) & \alpha (I_K - S_K) \\ I_Y & I_K \end{bmatrix} \tag{37}$$

Chang and Smyth (1971) find the limit cycle in Figure 18 and prove it applying the Poincarè-Bendixson Theorem. Kaldor's cycle insight is therefore proven.

[10]No more level of activity x or employment in their model.

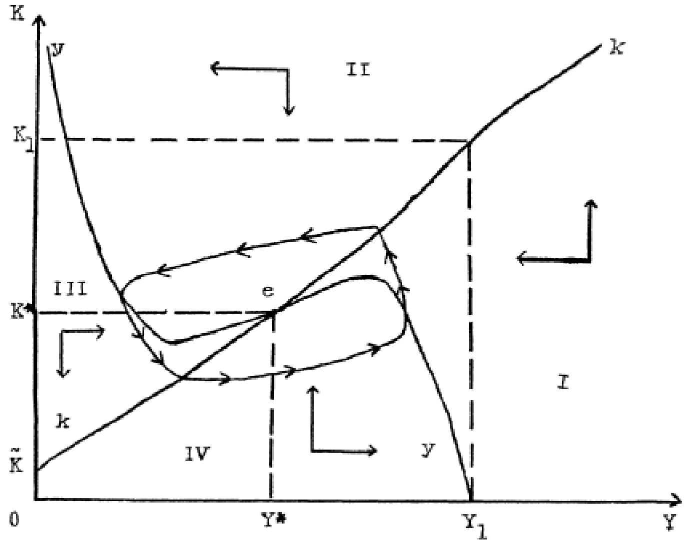


Figure 18: Phase diagram with the limit cycle and nullclines $\dot{Y} = 0$ and $\dot{K} = 0$ of Eq(36). You can also see the invariant set built to apply the Poincarè-Bendixson Theorem. The image is taken from (Chang and Smyth, 1971).

3.3 The Goodwin (1982) model

What we said about the Kaldor (1940) model also applies to the Goodwin (1982) model. Indeed, it is too a milestone in macroeconomic theory due to its ability to generate endogenous cyclical fluctuations.
However, from a mathematical point of view, (Kaldor, 1940) is nothing new: it is the application of the famous Lotka–Volterra predator-prey model to Economics.
In the Author's own words:

> In this form we recognise the Volterra case of prey and predator (Théorie Mathématique de la Lutte pour la Vie. Paris, 1931). To some extent the similarity is purely formal, but not entirely so. It has long seemed to me that Volterra's problem of the symbiosis of two populations - partly complementary, partly hostile - is helpful in the understanding of the dynamical contradictions of capitalism, especially when stated in a more or less Marxian form.

So the main[11] theme of the paper is the Marxian class struggle between capitalists and workers and the Kaleckian disciplinary unemployment, i.e. the capitalists' answer to workers' claim.

[11]On Goodwin important novelty for Economics (see Orlando et al., 2021, Chapter 14) (see also Taylor, 2004, Chapter 9). Economic dynamics, nonlinearities, Chaos and the possible links between Marx, Keynes and Schumpeter are all topics dear to Goodwin and to which he will return several times in his studies (Goodwin, 1991, e.g.).

Since Goodwin (1982) model provides the basis for Keen (1995) model, which we will explain in the next section, we will not derive the model equations here, but will do so directly in the Section 3.4. For now, we will only show the model assumptions and the

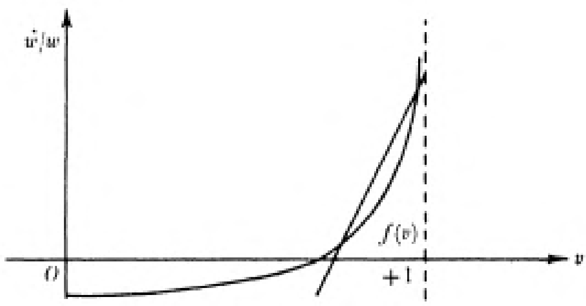


Figure 19: Linear approximation used by Goodwin of the non-linear function $f(\lambda) = \frac{\dot{w}}{w}$ of assumption 7). v stand for λ in our (and Keen) notation. The image is taken from (Goodwin, 1982).

two equations.
These are the assumptions:
1) steady technical progress (disembodied);
2) steady growth in the labour force;
3) only two factors of production, labour and capital (plant and equipment), both homogeneous and non-specific;
4) all quantities real and net[12];
5) all wages consumed, all profits saved and invested;
6) a constant capital-output ratio;
7) a real wage rate which rises in the neighbourhood of full employment.
Assumption 7) may be written as $\frac{\dot{w}}{w} = f(\lambda)$ where w is the real wage and λ is the employment rate.
In Figure 19 the linear approximation proposed by Goodwin: $\frac{\dot{w}}{w} = -t_1 + t_2\lambda$. But in the next section we will also analyse the non-linear case.
The $2D$ system is

$$\begin{cases} \dot{\omega} = \omega\left[-t_1 + t_2\lambda - \alpha\right] \\ \dot{\lambda} = \lambda\left[\frac{1-\omega}{\nu} - \alpha - \beta\right] \end{cases} \tag{38}$$

3.4 From Goodwin to Minsky: the Keen (1995) model

Hyman P. Minsky (1919-1996) was an American Post-Keynesian economist who developed the theory of financial instability. In Minsky's view, rather than stabilising the

[12]In Section 3.4 there will be also depreciation but investment I will still be considered net.

economy and serving it, financial markets are, in fact, promoters of instability and one of the main causes of speculative bubbles and the resulting crises.

Minsky's view matured in the wake of the Great Depression, the Wall Street crash of 1929, which is still considered one of the most serious economic crises in capitalism. Minsky wondered whether such a crisis could happen again (Minsky, 1982), and this led him to develop a cyclical theory of crises. The description of the factors that lead to a crisis stems from a recent crisis, following which, economic operators are reluctant to invest, still mindful of the losses they have suffered. Furthermore, entrepreneurs (capitalists) find difficult to obtain loans from banks, which have had to write off bad debts from their balance sheets. Only investments that are considered extremely safe, including investments in the financial sector, are pursued. Mistrust and caution dominate the economic system.

However, the caution of capitalists means that most activities are successful and debts are repaid: gradually, pessimism gives way to normality and low-risk activities, which therefore offer low returns, give way to less cautious investments. Banks revise their cautious positions, now convinced that capitalists will be able to repay their debts: pessimism gives way to optimism. Increased use of bank credit and investment drives economic growth and profits. The pursuit of ever-greater profits leads to a total abandonment of safe, low-yield investments, and economic operators begin to take on increasingly greater risks. However, banks continue to grant loans, confident in the continued increase in the growth rate of economy. But then operators begin to realise that the debts incurred exceed cash flows and will not be repaid, and that interest on the debts cannot be honoured, so they sell assets to obtain liquidity, but the sale of assets leads to a decline in their value, which further worsens the position of operators in a reinforcing mechanism: euphoria turns to panic and the bubble bursts, leading to an economic crisis and consequent stagnation. Banks are unable to collect their loans and go bankrupt.

In Minsky's view, private debt and financial markets, therefore, play a central role. His idea came back into vogue with the 2007-2008 crisis and the ensuing Great Recession, namely the subprime mortgage and real estate market crisis triggered by the bursting of a real estate bubble. The lack of prudence in granting loans, combined with the belief that house prices would continue to rise and the euphoria of the financial markets in creating financial engineering tools for mortgage securitization, fully followed the pattern proposed by Minsky.

The aim of Keen's study was to translate Minsky's verbal model into a mathematical model. The starting point is the two-equation model originally proposed by Goodwin. The addition of a third equation, which models private debt, leads to Keen model, named "Minsky".

Keen derives his own version of the Goodwin (1982) model, which differs slightly from the one presented in Section 3.3 due to the chosen form of non-linear functions.

Furthermore, we must warn the reader that Keen, being an economist and not a mathe-

matician, does not present a complete examination of fixed points and stability analysis in (Keen, 1995). So in this section we will try to complete the analysis of his model, at the cost of a obviously longer and more pedantic discussion.
The model proposed by Richard M. Goodwin (1913-1996) outlines what economists call redistributive conflict, i.e. the ways in which the social classes involved in production divide up the output, or national income.
There are two classes: workers and capitalists. Redistributive conflict is stylised using a Lotka-Volterra predator-prey model.
To derive the two equations of the model, we must introduce macroeconomic equations and definitions of aggregate quantities.
Labour productivity a is assumed to grow exponentially and exogenously governed by the parameter α due to technological progress

$$a = a_0 e^{\alpha t} \tag{39}$$

The working-age population N is also assumed to grow exponentially with parameter β

$$N = N_0 e^{\beta t} \tag{40}$$

Total output Y is given by productivity multiplied by the number L of workers employed

$$Y = aL \tag{41}$$

The capital-to-output ratio is the fixed accelerator

$$\nu_G = \frac{K}{Y} \tag{42}$$

The fraction of the employed population relative to the total workforce

$$\lambda = \frac{L}{N} \tag{43}$$

and it is one of the two fundamental variables of the model.
The rate of change $\dot{w}$ of real wage w is the product of the wage and a non-linear function $w\left[\cdot\right]$ of employment λ.

$$\frac{dw}{dt} = w\left[\lambda\right] w \tag{44}$$

It's given by:

$$w\left[\lambda\right] = \frac{A}{(B - C\lambda)^2} - D$$

Since this functional form will be used several times, we indicate it more generally as

$$f\left(A, B, C, D, \lambda\right) = w\left[\lambda\right]$$

where A, B, C, D are parameters (see below for numerical values). The function models workers' answer in bargaining wages due to the level of employment: the higher the employment rate λ, the more power workers will have in demanding a wage increase; the higher the unemployment $(1-\lambda)$, the more workers will be inclined to accept the conditions set by employers for fear of remaining unemployed. In particular, workers accept a constant wage when unemployment is 3.6%, accept wage cuts (up to a maximum of 4%) at higher levels of unemployment, and demand wage increases at lower levels of unemployment (which diverge at full employment $\lambda = 1$).
The same type of function, with different parameters, is used to model the response of capitalists to economic conditions in making investment decisions. These are guided by the level of profits Π.

$$f\left(E, F, G, H, \frac{\pi}{\nu_G}\right) = k\left[\frac{\pi}{\nu_G}\right] = \frac{E}{(F - G\frac{\pi}{\nu_G})^2} - H$$

Net[13] investments are a function of profits times the level of output minus the level of capital depreciation given by the depreciation rate δ.

$$I = \frac{dK}{dt} = k\left[\frac{\Pi}{K}\right]Y - \delta K \tag{45}$$

At zero or negative profits (losses), no investments are made ($I = 0$). When profits reach 10%, the investment equals the profits and exceeds the profits at even higher levels of earnings.

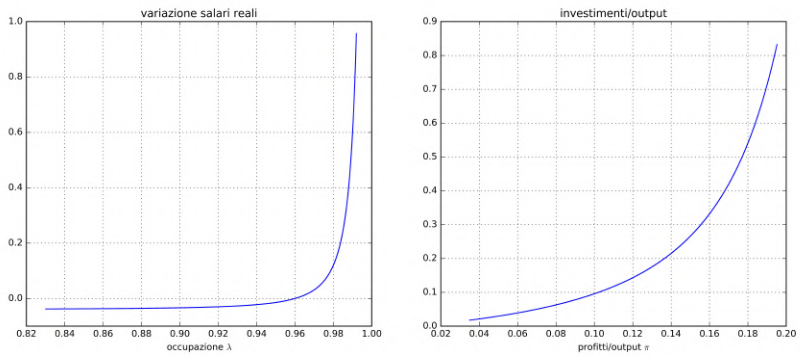


Figure 20: Functions that simulate the behaviour of workers and capitalists in response to the business cycle.

[13]Please note, that here the equation for investment is defined differently than, for example, Eq(164): indeed, what we call gross investment I in $\dot{K} = I - \delta K$, here is given by the function $k\left[\frac{\pi}{\nu_G}\right]$, for this reason we have net investment here.

$$\frac{\Pi}{K} = \frac{\Pi}{Y\nu_G} = \frac{\pi}{\nu_G} \tag{46}$$

where $\frac{\Pi}{K}$ is the rate of profit relative to invested capital.
The profit share of national income $\pi = \frac{\Pi}{Y}$ is residual after workers income

$$\pi = 1 - \omega \tag{47}$$

The wage share of national income is

$$\omega = \frac{W}{Y} = \frac{wL}{aL} = \frac{w}{a} \tag{48}$$

From macroeconomic definitions, we can calculate the time derivatives:

$$\frac{dY}{dt} = \frac{d}{dt}\frac{K}{\nu_G} = \left(\frac{k\left[\frac{\pi}{\nu_G}\right]}{\nu_G} - \delta \right) Y \tag{49}$$

that is the rate of change of output.
The rate of change of employment is

$$\frac{dL}{dt} = \frac{d}{dt}\frac{Y}{a} = \frac{1}{a}\left(\frac{dY}{dt} - \alpha Y \right) \tag{50}$$

From these, we can calculate the derivatives of $\omega = \frac{w}{a}$ and $\lambda = \frac{L}{N}$ to obtain the two predator-prey equations required (λ is the pray, ω is the predator):

$$\begin{cases} \dot{\omega} = \omega\left(w\left[\lambda\right] - \alpha\right) \\ \dot{\lambda} = \lambda\left(\frac{k\left[\frac{\pi}{\nu_G}\right]}{\nu_G} - \alpha - \beta - \delta \right) \end{cases} \tag{51}$$

When the wage share is low, profits are high, and this leads to greater investment, which boosts employment. However, it also gives greater bargaining power to workers who want to increase the share of national income in their hands. This, however, reduces the profits of entrepreneurs, who consequently cut investment, thereby also reducing employment, and the cycle starts again.
However, there is also a simplified version in which a form of the so-called "Say's Law" applies, i.e. it is assumed that capitalists invest all their profits, resulting in substitution.
$k\left[\frac{\pi}{\nu_G}\right] = 1 - \omega$

$$\begin{cases} \dot{\omega} = \omega\left(w\left[\lambda\right] - \alpha\right) \\ \dot{\lambda} = \lambda\left(\frac{1-\omega}{\nu_G} - \alpha - \beta - \delta \right) \end{cases} \tag{52}$$

Here, the values of the parameters used by Keen in the model, assuming them to be given throughout all the study: $\alpha = 0.015$, $\beta = 0.035$, $\delta = 0.02$, $\nu_G = 3$, $A =$

$0.0000641, B = 1, C = 1, D = 0.0400641, E = 0.0175, F = 0.53, G = 6, H = 0.065$.
So the model can be written in this form

$$\begin{cases} \dot{\omega} = \omega g\left(\lambda\right) \\ \dot{\lambda} = \lambda f\left(\omega\right) \end{cases} \tag{53}$$

Although the model is defined on $\mathbb{R}^2$, the physical region of economic interest is $\mathbb{R}_+ \times \mathbb{R}_+$, and λ should not exceed unity, as the working population cannot be greater than the total population.
With reference to Eq(53), let us begin with a general overview of the behaviour of both Goodwin models, which will be analysed in detail later.
In the case of Say's Law, $f(\omega)$ is linear and vanishes for $\bar{\omega}_1 = 0.79$. The behaviour is similar to the Lotka-Volterra model, with the only difference being that in Goodwin model, $g(\lambda)$ is non-linear. In the case of the investment function, $f(\omega)$ also becomes non-linear and has two zeros $0 < \bar{\omega}_{1**} < \bar{\omega}_{1*} < 1$, which are therefore in the economically significant region $[0, 1]$.
The function $g(\lambda)$ has a zero $\bar{\lambda}_1 < 1$ and, limiting ourselves to the physically significant region, we have an attractive point at the origin, a saddle in $\left(\bar{\omega}_{1**}, \bar{\lambda}_1\right)$ and a center in $\left(\bar{\omega}_{1*}, \bar{\lambda}_1\right)$, as shown in the phase portrait in Figure 26.
Considering also the second zero $\bar{\lambda}_{1*}$ of the function $g(\lambda)$, we have $\bar{\lambda}_1 < 1 < \bar{\lambda}_{1*}$, which therefore falls outside the physical region but still influences the dynamics. As shown in Figure 28, in addition to the previous three fixed points, there is a center at $\left(\bar{\omega}_{1**}, \bar{\lambda}_{1*}\right)$ and a saddle at $\left(\bar{\omega}_{1*}, \bar{\lambda}_{1*}\right)$.
Finally, to understand the phase portrait, it should be noted that $f(\omega)$, in the case of the investment function, has a singularity consisting of a pole at $\omega = 0.735$, while $g(\lambda)$ has a pole at $\lambda = 1$.
After this general overview, let us now analyse Goodwin model in detail in the case of Say's Law.
A fixed point is obviously $\left(\bar{\omega}_0, \bar{\lambda}_0\right) = (0, 0)$, while fixed points other than the origin are given by the zeros of the functions $f\left(\omega\right)$ and $g\left(\lambda\right)$: if $f\left(\bar{\omega}_1\right) = g\left(\bar{\lambda}_1\right) = 0$ then $\left(\bar{\omega}_1, \bar{\lambda}_1\right)$ is a fixed point while $(\bar{\omega}_1, 0)$ and $\left(0, \bar{\lambda}_1\right)$ are not.
As already mentioned, according to Say's Law, $f\left(\omega\right)$ is a straight line and has a single zero, while $g\left(\lambda\right)$ has two; however, we only consider the first one, since the second one has $\lambda > 1$.
We therefore have a single fixed point over the origin: from the first equation we have $w\left[\bar{\lambda}_1\right] = \alpha$, i.e. $w^{-1}\left[\alpha\right] = \bar{\lambda}_1$, which involves inverting the function $\frac{A}{(B - C\bar{\lambda}_1)^2} - D = \alpha$, which gives $\bar{\lambda}_{1,*} = \frac{B}{C} \mp \frac{1}{C}\sqrt{\frac{A}{\alpha + D}}$, obtaining the economically meaningful fixed point

$$\left(\bar{\omega}_1, \bar{\lambda}_1\right) = \left(1 - \nu_G(\alpha + \beta + \delta), \frac{B}{C} - \frac{1}{C}\sqrt{\frac{A}{\alpha + D}}\right)$$

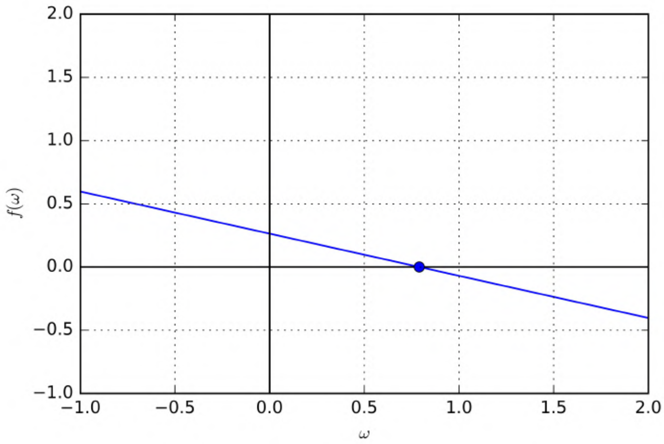


Figure 21: The function $f(\omega)$, in the case where Say's Law applies, is linear and we have only one zero.

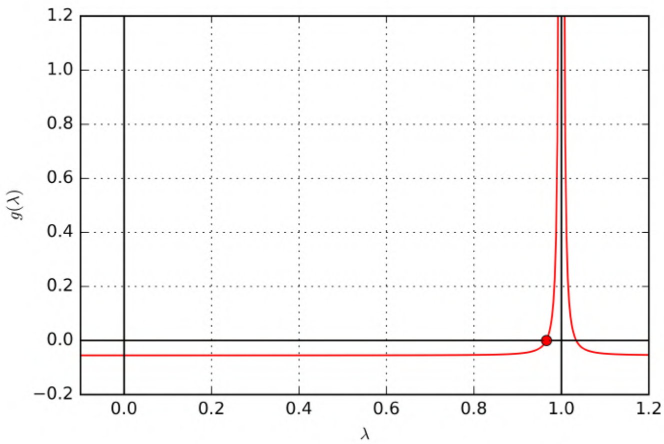


Figure 22: The function $g(\lambda)$, in the case where Say's Law applies, has two zeros, but the second one is greater than 1, so it has no meaning.

The Jacobian matrices are

$$J\left(0,0\right)=\begin{bmatrix}g(0) & 0\\ 0 & f(0)\end{bmatrix} \tag{54}$$

$$J\left(\bar{\omega}_1,\bar{\lambda}_1\right)=\begin{bmatrix}0 & \bar{\omega}_1 g'(\bar{\lambda}_1)\\ \bar{\lambda}_1 f'(\bar{\omega}_1) & 0\end{bmatrix} \tag{55}$$

Since $g(0)<0$, $g'(\bar{\lambda}_1)>0$ and $f(0)>0$, $f'(\bar{\omega}_1)<0$, the origin is unstable and is a saddle, while $\left(\bar{\omega}_1,\bar{\lambda}_1\right)$ is stable but not attractive, i.e. a center (see Appendix 8.2 for detailed calculations).
In Figure 23 we show the phase portrait where we can appreciate the dynamics analogous to the Lotka-Volterra two-population model, while in Figure 24 we show the time graphs of some of the flows. As can be seen from the graphs, in some cases the wage share exceeds the output produced ($\omega>1$). Although the economic interpretation is not straightforward, this behaviour could be understood as an impoverishment of entrepreneurs as profits become negative. Now let us move on to model

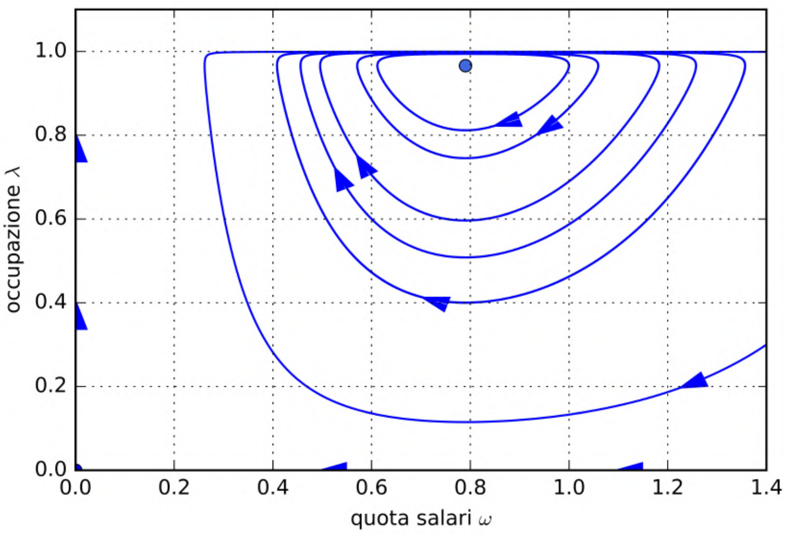
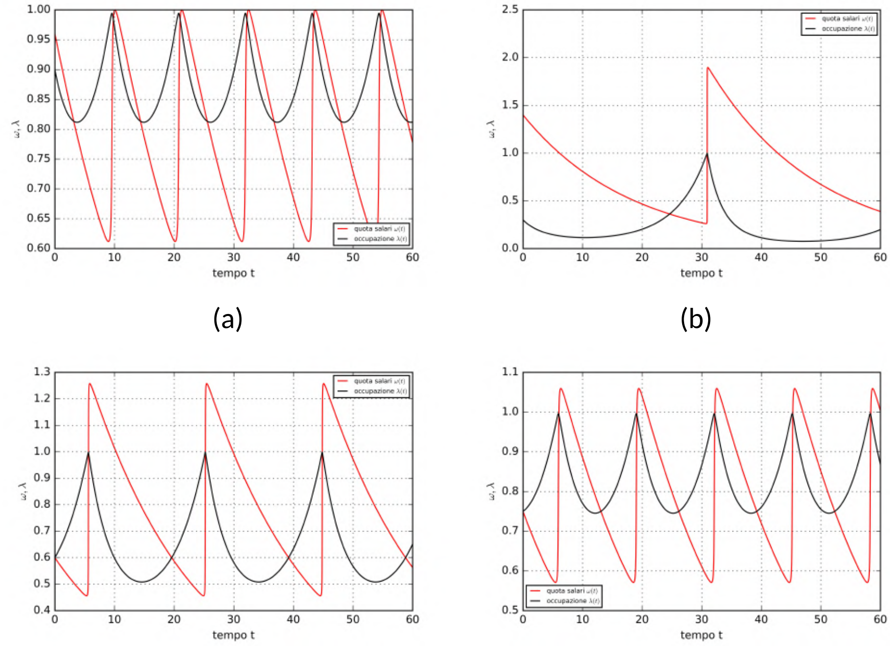


Figure 23: Phase portrait in the case of Say's Law.

in Eq(51) where profits are not automatically invested. Due to the presence of the investment function, $f\left(\omega\right)$ is no longer linear and has two zeros, as can be seen in Figure 25. The two zeros $\bar{\omega}_{1*}$ and $\bar{\omega}_{1**}$ therefore satisfy $k\left[\frac{1-\bar{\omega}_1}{\nu_G}\right]=\nu_G\left(\alpha+\beta+\delta\right)$, i.e. $k^{-1}\left[\nu_G\left(\alpha+\beta+\delta\right)\right]=\frac{\bar{\pi}_1}{\nu_G}=\frac{1-\bar{\omega}_1}{\nu_G}$, from which we obtain the equilibrium profits:

$$\bar{\pi}_1=\frac{\nu_G F}{G}\mp\frac{\nu_G}{G}\sqrt{\frac{E}{H+\nu_G\left(\alpha+\beta+\delta\right)}} \tag{56}$$



(a) (b)

(c) (d)

Figure 24: Time dynamics in the case of Say’s Law.

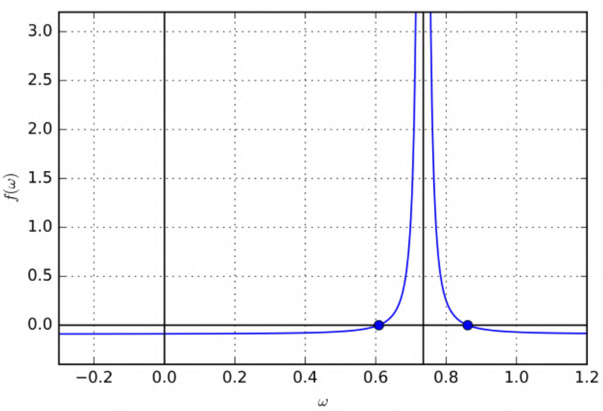


Figure 25: The function $f(\omega)$ in the case of the investment function; unlike the previous model, we have two zeros.

Therefore, in addition to the origin, the two fixed points are

$$(\bar{\omega}_{1*},\bar{\lambda}_1)=\left(1-\frac{\nu_G F}{G}+\frac{\nu_G}{G}\sqrt{\frac{E}{H+\nu_G\left(\alpha+\beta+\delta\right)}},\frac{B}{C}-\frac{1}{C}\sqrt{\frac{A}{\alpha+D}}\right)$$

$$(\bar{\omega}_{1**},\bar{\lambda}_1)=\left(1-\frac{\nu_G F}{G}-\frac{\nu_G}{G}\sqrt{\frac{E}{H+\nu_G\left(\alpha+\beta+\delta\right)}},\frac{B}{C}-\frac{1}{C}\sqrt{\frac{A}{\alpha+D}}\right)$$

We have that $0<\bar{\omega}_{1**}<\bar{\omega}_{1*}$ (since $\bar{\omega}_{1*}\approx 0.86$ and $\bar{\omega}_{1**}\approx 0.61$), as we can see, the two fixed points share the same λ.
With reference to Eq(54) and Eq(55) in the current case we have $g(0)<0$, $g'(\bar{\lambda}_1)>0$ but $f(0)<0$ and also $f'(\bar{\omega}_{1**})>0$ and $f'(\bar{\omega}_{1*})<0$, therefore $(0,0)$ is a stable fixed point, $(\bar{\omega}_{1**},\bar{\lambda}_1)$ is a saddle point and $(\bar{\omega}_{1*},\bar{\lambda}_1)$ is a center (see Appendix 8.3 for more details).
In the phase portrait in Figure 26, it is possible to appreciate the attractive origin, the saddle and the center with its cycles, while Figure 27 shows the time graphs of some of the flows. It should also be noted that $f(\omega)$ has a pole at $\omega_{pole}=1-\frac{\nu_G F}{G}=0.735$ and $g(\lambda)$ at $\lambda_{pole}=\frac{B}{C}=1$. As can be seen, for example, from graph 27c, the dynamics along them are problematic and difficult to interpret economically, as such sudden changes in macroeconomic variables are unlikely. The stable fixed point at the origin is also difficult to interpret, given that the Lotka-Volterra model, and consequently Goodwin model, are structurally unstable, so it is reasonable to expect changes in dynamics as the model changes. the fact that the orbits converge at the origin, i.e. at the collapse of the economy, with the dynamics shown in Figure 27b cannot even be explained as a conflict in income redistribution, as we do not have oscillations of ω and λ.
Although it is not relevant to the economic model, Figure 28 also shows the two fixed points given by the solution $\bar{\lambda}_{1*}>1$ (shown in Figure 22), which is obviously absurd since workers cannot exceed the population: the situation is reversed with respect to the lower portion, we have that $(\bar{\omega}_{1**},\bar{\lambda}_{1*})$ is a center and $(\bar{\omega}_{1*},\bar{\lambda}_{1*})$ is a saddle (the vector field has also been plotted in a stylised manner in the graph). Finally, in Figure 29, we can see a comparison between the two versions of Goodwin model for the economically meaningful part. The two centers and a cycle obtained from the same initial conditions $(\omega_0,\lambda_0)=(0.96,0.9)$ are represented: it can be seen that the system with the investment function has a less extensive cycle than the other, due to the presence of the saddle in the other portion of the graph. Let us now move on to the part of the model added by Keen.
To model financial instability, we need to introduce a new social class, in addition to workers and capitalists: bankers. They lend money to entrepreneurs to make investments, and their income B consists of the interest paid on the debt D incurred by

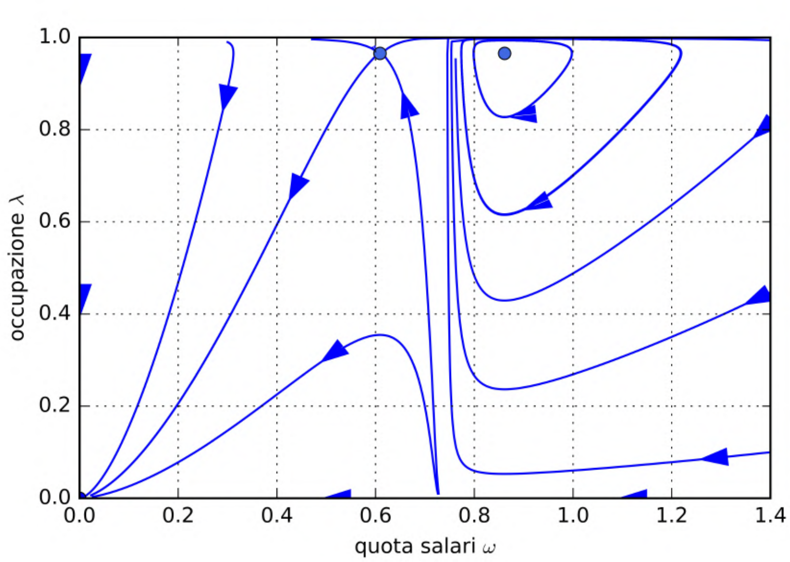
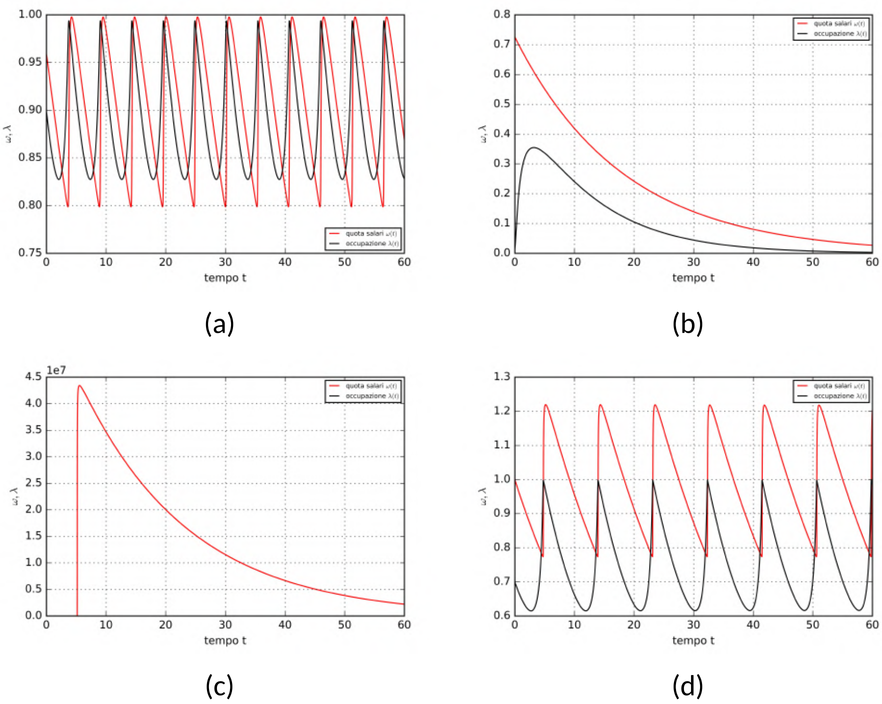


Figure 26: Phase portrait in the case of the investment function.



(a) (b) (c) (d)

Figure 27: Time dynamics in the case of Say's Law.

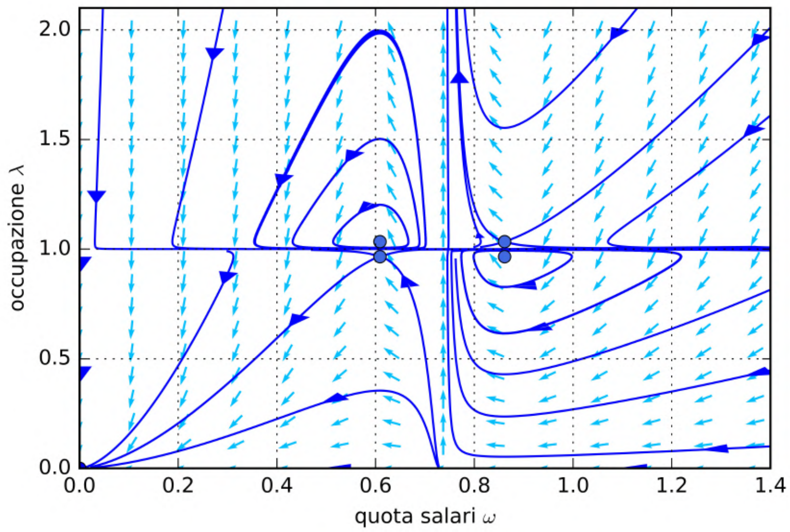


Figure 28: Phase portrait in the case of the investment function, where the second zero of the function $g(\lambda)$ is also shown.

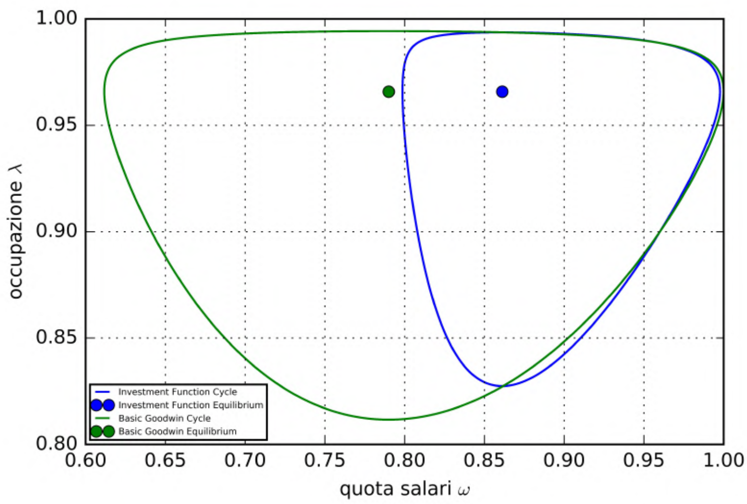


Figure 29: The centers and cycles of the two Goodwin models compared, for the same initial conditions $(\omega_0, \lambda_0) = (0.96, 0.9)$.

capitalists; the debt share of national income is

$$b = \frac{B}{Y} = \frac{rD}{Y} = rd \tag{57}$$

where r is the interest rate set by the Central Bank, i.e. a parameter.
The rate of change of debt is due to the interest on the outstanding debt rD, the increase in investments I and is reduced by the increase in profits Π

$$\frac{dD}{dt} = rD + I - \Pi \tag{58}$$

With the presence of bankers, capitalists' profits are now residual compared to national income, 100%, after the income of workers and the income of the financial sector has been removed.

$$\pi = 1 - \omega - b \tag{59}$$

Moving on to $d = \frac{D}{Y}$ and deriving with respect to time, using definitions Eq(58) and Eq(49), we find the third equation added to Goodwin model

$$\dot{d} = \frac{d}{dt}\left(\frac{D}{Y}\right) = b - \pi + (\nu_G - d)\left(\frac{k\left[\frac{\pi}{\nu_G}\right]}{\nu_G} - \delta\right) \tag{60}$$

that is, the rate of change of private debt ratio. In it, we can still express b and π.
So the Keen (1995) is:

$$\begin{cases} \dot{\omega} = \omega\left(w\left[\lambda\right] - \alpha\right) \\ \dot{\lambda} = \lambda\left(\frac{k\left[\frac{1-\omega-rd}{\nu_G}\right]}{\nu_G} - \alpha - \beta - \delta\right) \\ \dot{d} = 2rd - 1 + \omega + (\nu_G - d)\left(\frac{k\left[\frac{1-\omega-rd}{\nu_G}\right]}{\nu_G} - \delta\right) \end{cases} \tag{61}$$

Let us begin with a general overview of the equilibrium points of Eq(61). Compared to Goodwin model, the second equation is now better expressed as $\dot{\lambda} = \lambda f(\pi)$, the zeros are still given by Eq(56), but now π includes both ω and d. From these two zeros, two fixed points can be obtained, only one of which has economic significance and is stable. The main competitor in attracting the orbits of this stable fixed point is another stable fixed point $(0, 0, \pm\infty)$, whose $+\infty$ version symbolises the collapse of the economy due to a speculative bubble. However, it is also possible to set $\dot{\lambda}$ to zero by taking $\lambda = 0$ which, together with $\omega = 0$, leads to a triplet of fixed points $\left(0, 0, \bar{d}_0\right)$ whose $\bar{d}_0$ are zeros of the third equation of the model. These three fixed points have no economic interpretation, but the second of them is stable and the other two are unstable. Finally, again $\omega = 0$ and a choice of d such as to set, simultaneously, the second and third equations to zero, provide, as λ varies, a straight line of unstable fixed points that exist only

when a very specific condition on the interest rate is met: these too have no economic significance. Let us now go into detail.
By setting the derivatives equal to zero, we find that the first equation is not affected by the transition from the two-dimensional to the three-dimensional system and therefore the equilibrium value of λ is decoupled from the other two equations.
From the second equation, we find the profit equilibrium condition already encountered $\frac{\bar{\pi}_1}{\nu_G} = k^{-1}\left[\nu_G\left(\alpha+\beta+\delta\right)\right]$ and through this and the third equation we have the equilibrium debt. Finally, the equilibrium wages can be found by inverting $\bar{\pi}_1 = 1-\bar{\omega}_1 - r\bar{d}_1$; therefore, the equilibrium point is

$$\begin{cases} \bar{\omega}_1 = 1-\bar{\pi}_1 - r\frac{\bar{\pi}_1-\nu_G(\alpha+\beta)}{r-(\alpha+\beta)} \\ \bar{\lambda}_1 = \frac{B}{C} - \frac{1}{C}\sqrt{\frac{A}{\alpha+D}} \\ \bar{d}_1 = \frac{\bar{\pi}_1-\nu_G(\alpha+\beta)}{r-(\alpha+\beta)} \\ \bar{\pi}_1 = \frac{\nu_G F}{G} \mp \frac{\nu_G}{G}\sqrt{\frac{E}{H+\nu_G(\alpha+\beta+\delta)}} \end{cases}$$

This fixed point $\left(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1\right)$ corresponds to a finite level of debt and an employment rate that is not zero and less than 1, so it is economically meaningful and desirable.
We also note another interesting aspect, common to both Goodwin and Keen models: substituting $\bar{\pi}_1$ in Eq(49), we find that the growth rate of the economy at this desirable equilibrium point is given by

$$\frac{\dot{Y}}{Y} = \frac{k\left[\frac{\bar{\pi}_1}{\nu_G}\right]}{\nu_G} - \delta = \frac{\nu_G\left(\alpha+\beta+\delta\right)}{\nu_G} - \delta = \alpha+\beta \tag{62}$$

So, once equilibrium has been reached, the growth of the economic system is determined by the productivity and labour force growth rates.
Studying its stability (for details, see Appendix 8.4), we find that $\left(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1\right)$ is stable if the following conditions are simultaneously satisfied:

$$\alpha+\beta > 2r - r\frac{\nu_G - \bar{d}_1}{\nu_G}\left(\frac{\frac{2EG}{\nu_G}}{\left(F-G\frac{\bar{\pi}_1}{\nu_G}\right)^3}\right) \tag{63}$$

$$\alpha+\beta > r \tag{64}$$

$$r\left[\left(\nu_G - \frac{\bar{\pi}_1-\nu_G\left(\alpha+\beta\right)}{r-\left(\alpha+\beta\right)}\right)\left(\frac{\frac{2EG}{\nu_G}}{\left(F-G\frac{\bar{\pi}_1}{\nu_G}\right)^3}\right) - \nu_G\right] > 0 \tag{65}$$

Numerical simulations show that usually $\bar{\pi}_1$ in Eq(56) with a positive sign produces a negative $\bar{d}_1$ and the corresponding fixed point does not satisfy either Eq(64) or Eq(65)

and it is therefore unstable.
There is also another fixed point that models financial instability: the fixed point $\left(\bar{\omega}_2, \bar{\lambda}_2, \bar{d}_2\right) = (0, 0, \pm\infty)$. In this case, the stability conditions are not stringent and the equilibrium is practically always stable. However, please refer to Appendix 8.4 for further information.
A third fixed point is given by $\left(\bar{\omega}_0, \bar{\lambda}_0, \bar{d}_0\right) = \left(0, 0, \bar{d}_0\right)$, where $\bar{d}_0$ are the solutions of the third equation of the model:

$$2r\bar{d}_0 - 1 + (\nu_G - \bar{d}_0)\left(\frac{k\left[\frac{1-r\bar{d}_0}{\nu_G}\right]}{\nu_G} - \delta\right) = 0 \tag{66}$$

Finally, there is a fourth fixed point that is obtained by setting $\omega = 0$ and choosing π to set the second equation to zero

$$1 - rd = \bar{\pi}_1 = \nu_G k^{-1}\left[\nu_G\left(\alpha + \beta + \delta\right)\right]$$

In this way, we have that $\dot{\omega} = \dot{\lambda} = 0$ regardless of the values assumed by λ. However, in order for $\dot{d} = 0$ as well, we must have $d = \bar{d}_1$, which must also simultaneously satisfy the condition written above. This leads us to an extremely specific condition for the model parameters:

$$\frac{1 - \bar{\pi}_1}{r} = \frac{\bar{\pi}_1 - \nu_G(\alpha + \beta)}{r - (\alpha + \beta)} \tag{67}$$

The fixed point $\left(0, \lambda, \bar{d}_1\right)$ is therefore structurally unstable, as a small variation in the model parameters could cause it to disappear. If we consider the interest rate as the only free parameter, then from (67) we can obtain the two r values for which this point exists

$$r = \frac{(\alpha + \beta)\left(\bar{\pi}_1 - 1\right)}{2\bar{\pi}_1 - 1 - \nu_G\left(\alpha + \beta\right)} \tag{68}$$

These are $r \approx 4.9\%$ for $\bar{\pi}_1$ with negative sign and $r \approx 8.3\%$ for the positive sign.
Let us now begin to examine the phase space by choosing a low interest rate $r = 2.5\%$. As can be seen from Figure 30, most orbits diverge, both going to $-\infty$ (e.g. Figure 31d) and $+\infty$. there is a small area where focus form that converge to the stable fixed point (in Figure 31a and Figure 31b) $\left(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1\right)$ (as we will analyse in more detail below) and then there are three points $\left(0, 0, \bar{d}_0\right)$ solutions of Eq(66) of which the second, the central one, is stable 31c, while the other two are unstable.
Let us now consider the case of a high interest rate, where we choose one of the Eq(68) and obtain the phase portrait in Figure 32, where we can see the straight line with the fixed points $\left(0, \lambda, \bar{d}_1\right)$ which, having two negative eigenvalues and one positive eigenvalue, are unstable. The focuses now diverge to $+\infty$, see Figure 33b, while the second of the points $\left(0, 0, \bar{d}_0\right)$ is still stable, in Figure 33a. Furthermore, the third of these

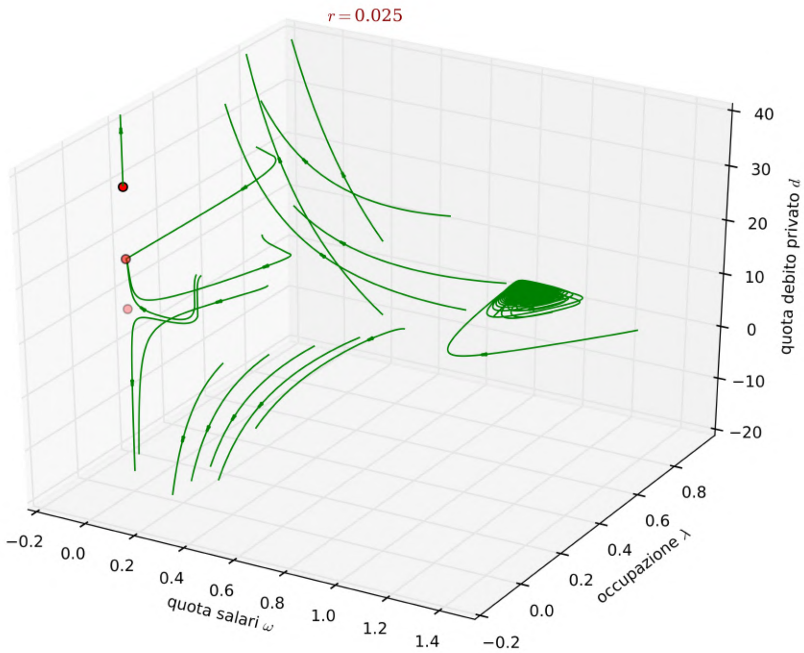
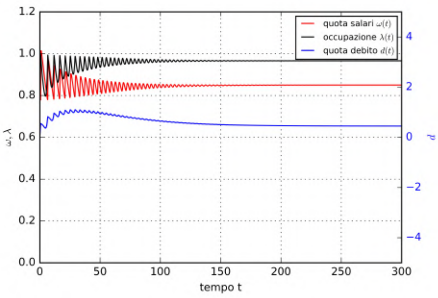
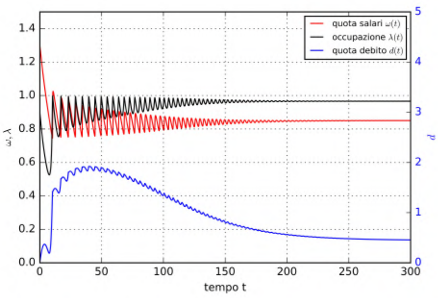


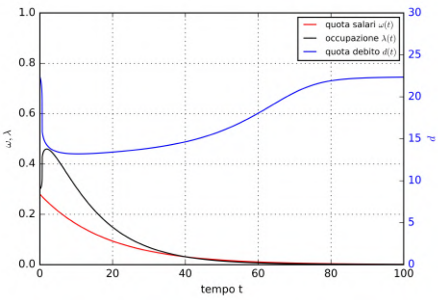
Figure 30: Phase portrait of the Minsky model at low interest rate r.



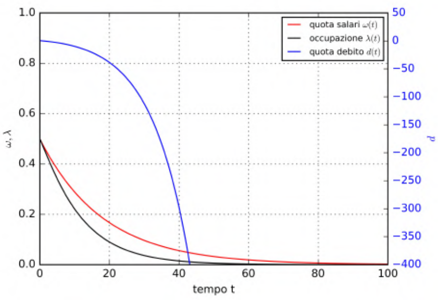
(a)



(b)



(c)



(d)

Figure 31: Time dynamics of the Minsky model at low interest rate r.

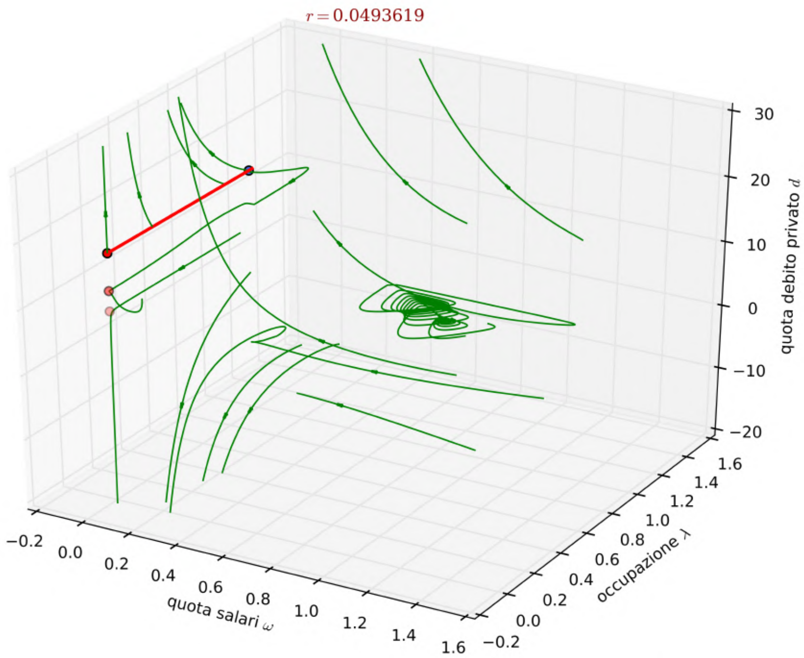
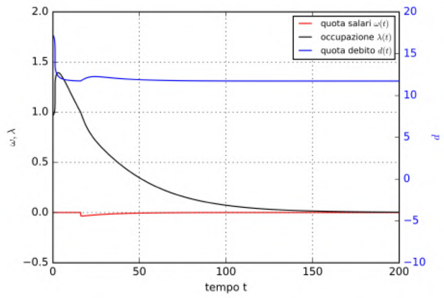
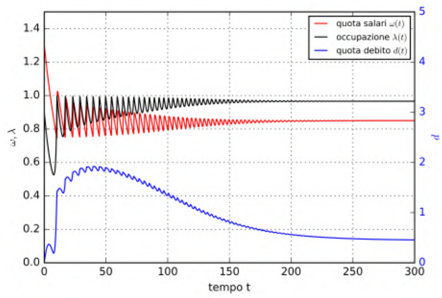


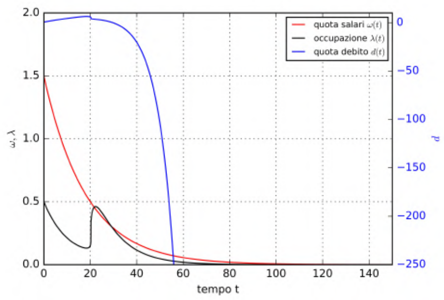
Figure 32: Phase portrait of the Minsky model at high r.



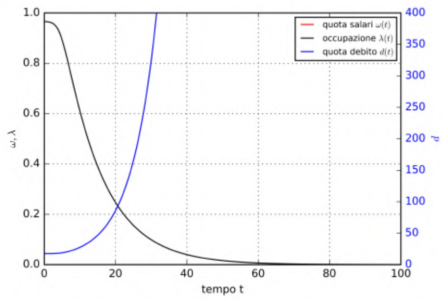
(a)



(b)



(c)



(d)

Figure 33: Time diagrams of the Minsky model at high interest rate r.

points lies on the line, as does the fixed point $(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1)$, which therefore becomes unstable, Figure 33d: indeed, it no longer satisfies Eq(65). It is not possible to assign an economical interpretation to the three points $(0, 0, \bar{d}_0)$ because an economy without employment or wages makes no sense, but with a non-zero debt level, which is also very high, as well as the fact that the second of these is stable and therefore attractive. Furthermore, the points $(0, \lambda, \bar{d}_1)$ also make no sense, where, in addition to the criticality of such a high level of debt, workers are not paid because the wage share is zero. In general, all trajectories that go to $-\infty$ and those that go directly to $+\infty$ without oscillating do not seem to make sense either.
Therefore, the portion of the phase space that is economically meaningful seems to be limited to that where there are focuses. For reasons related to scale, this part appeared confusing in the phase portraits shown: let us therefore explore the two cases numerically, using the same initial conditions $(\omega_0, \lambda_0, d_0) = (0.96, 0.9, 0)$.
We start with a low interest rate ($r = 2.5\%$) and obtain convergence to the finite equilibrium, which in this case is $(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1) = (0.85, 0.97, 0.44)$.
At low interest rates, increased investment leads to higher employment and wages, but also to increased debt and therefore higher interest payments, which reduces profits, subsequent investment and, consequently, economic booms, slowing down the rush to take on further debt. All this therefore results in dampening capitalists' investments to a sustainable level that favours convergence towards the finite equilibrium that characterises good economical stability and a healthy economy: this achieves a division of output between the three social classes (workers, capitalists, bankers) that ensures balanced growth. The time dynamic is illustrated in Figures 34 and in Figure 35, in Figure 36 we have the phase diagram.

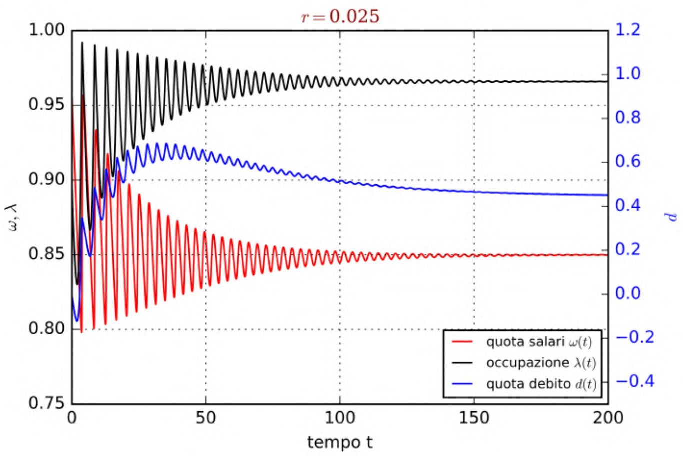


Figure 34: Time dynamic for initial condition $(\omega_0, \lambda_0, d_0) = (0.96, 0.9, 0)$ at low interest rate r.

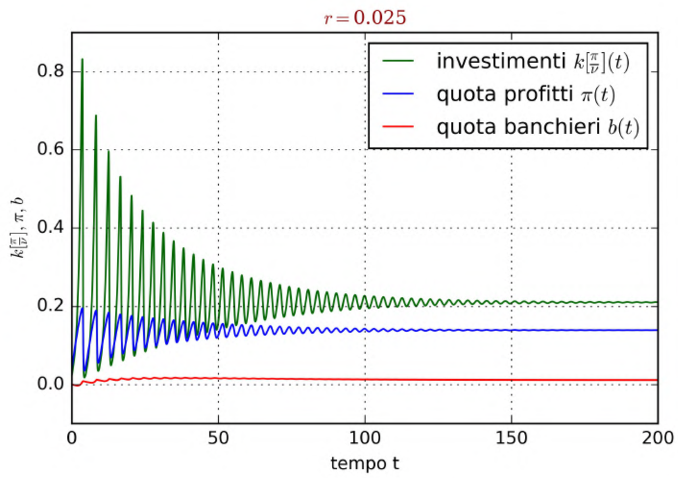
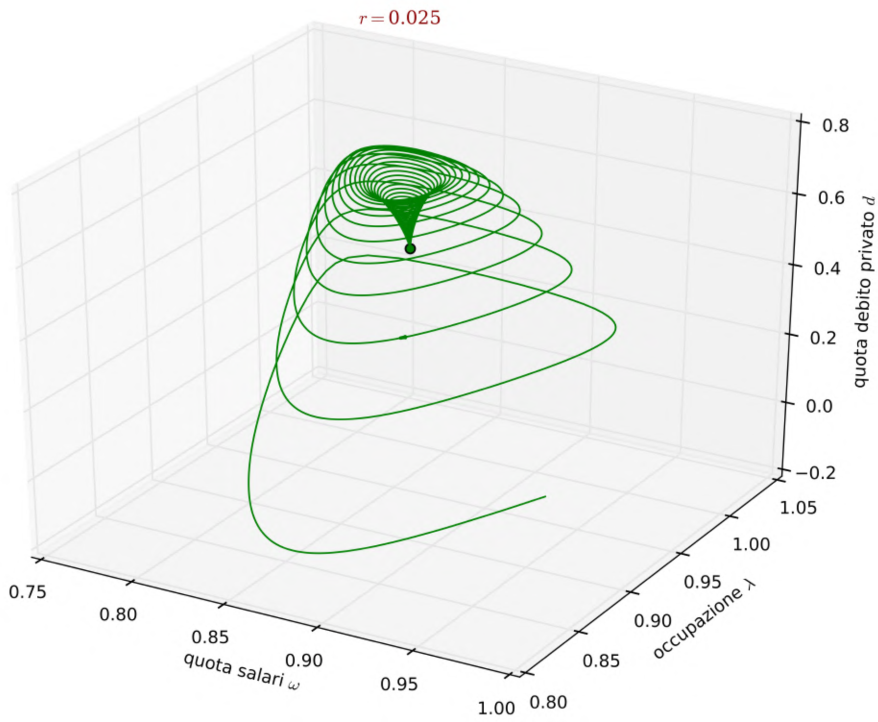
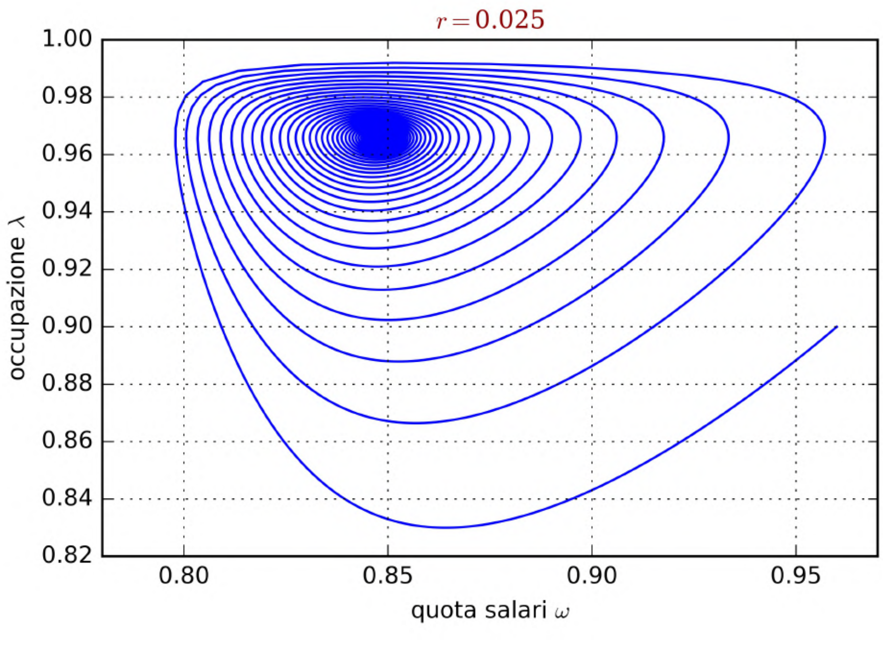


Figure 35: Time dynamic at low interest rate r.

Instead, at high interest rates (4.6% or higher), the race for debt is not slowed down in time by convergence to finite equilibrium, and the debt accumulated at each cycle due to the high interest rate ultimately proves unsustainable, causing profits to collapse and, consequently, investment and employment: the system thus converges to the equilibrium $\left(\bar{\omega}_2, \bar{\lambda}_2, \bar{d}_2\right)$, i.e. to the collapse of the economy. Let us look in detail at the economical meaning of this dynamic.
The initial level of wages and the absence of debt and therefore of bankers' share lead to high investments financed through borrowing. The emergence of debt provides a share for bankers, and investments generate jobs and therefore wage claims that increase the wage share of output. The combination of these two factors reduces profits and leads to a fall in investment, which reverses the growth in the workers' share. In this first phase of the dynamic, profits exceed investments, leading to the partial but not total repayment of bank debt, reducing the bankers' share. However, the cycle is not dampened and does not converge to a good equilibrium due to the formation of a focus between wages and employment, where the higher interest rate continuously increases the bankers' share rather than converging it to a fixed value.
The increase in the bankers' share causes a fall in investment and therefore in employment, as well as a slightly more marked decline in wages. There is a renewed increase in profits and investment, leading to higher debt and a higher share of bankers, but instead of easing, the cycle becomes more intense with more marked falls in the wage share and equivalent surges in investment (due to the non-linearity of the investment function) and therefore in debt, leading to more abrupt wage demands. The cycle repeats itself each time at a higher level of debt and becomes increasingly drastic until, eventually, the increase in debt lead to such a high share for bankers that profits fall



(a)



(b)

Figure 36: Phase portrait for initial condition $(\omega_0, \lambda_0, d_0) = (0.96, 0.9, 0)$ at low interest rate r.

and remain below zero. Then the system collapses towards zero employment, wages and profits: this is the debt-induced collapse from which, without some kind of intervention, such as a change in the rules or a debt moratorium, there is no escape.
Figure 37 shows the time dynamic of employment, wage share and capitalists' debt, and Figure 38 shows the phase diagram before the collapse. In this case, the finite equilibrium was obviously not reached, and private debt and the bankers' share go to $+\infty$. Minsky's contribution to the development of the theory of financial instability

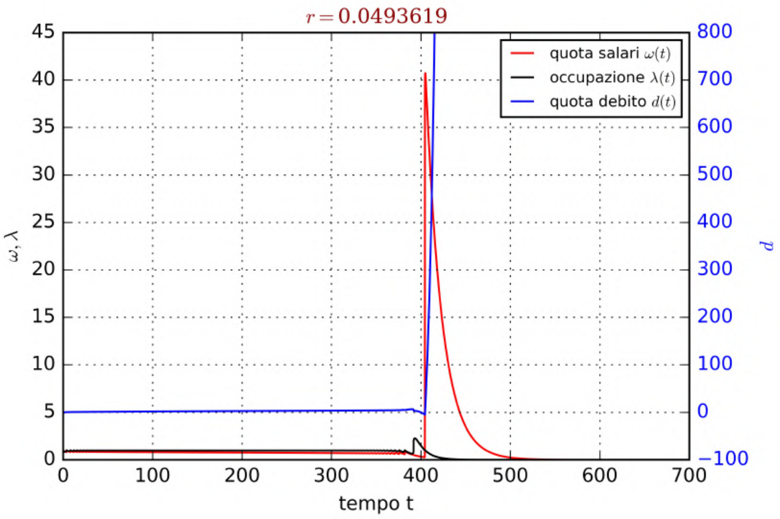
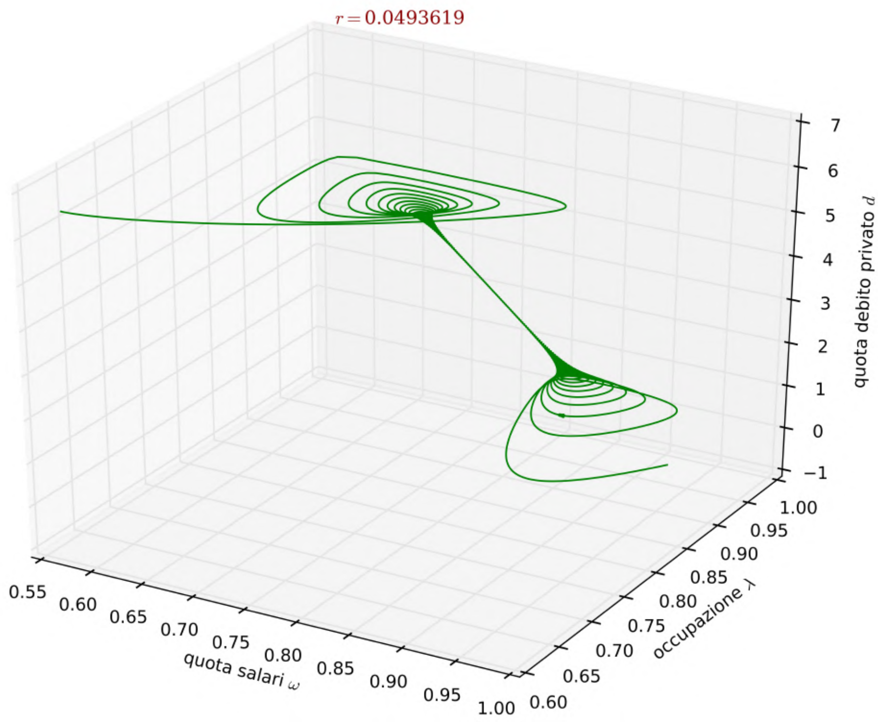
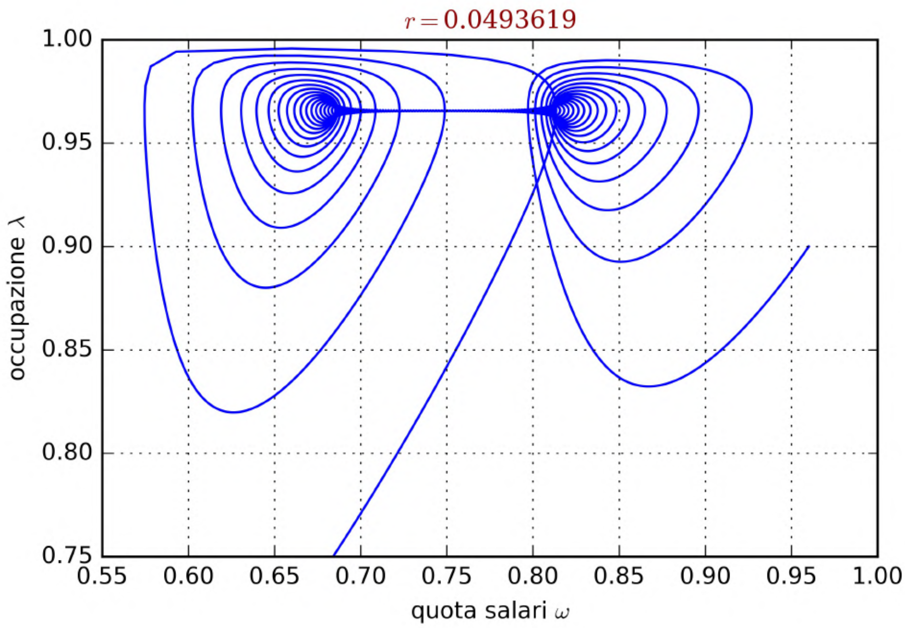


Figure 37: Time dynamic for the initial condition $(\omega_0, \lambda_0, d_0) = (0.96, 0.9, 0)$ at high interest rate r.

was to conceive of great depressions as one of the possible states in which capitalist economies could find themselves.
The addition of the banking class, and therefore debt, to Goodwin extremely simplified model - which only models the "stylized facts" of the business cycle - is sufficient to demonstrate the validity of Minsky's hypothesis. Indeed, simply inserting plausible interest rates and capitalists' profit expectations (what Keynes called "animal spirit") during economic booms is sufficient to explain the accumulation of debt beyond what the system can sustain. Furthermore, the simulated crashes that occur in the model are similar to those that occur in real economies. The difference lies in the fact that, after the collapse, capitalists' indebtedness continues to grow improbably forever, whereas in the real world this is obviously impossible: some capitalists go bankrupt, many creditors are forced to cancel their debts and suffer capital losses.
In cases where divergent equilibrium is achieved, the long period of apparent stability is an illusion, and the crisis, when it occurs, is sudden and violent. This is a warning for real-world economic policies: crises occur too rapidly to be prevented by changes



(a)



(b)

Figure 38: Phase portrait for initial condition $(\omega_0, \lambda_0, d_0) = (0.96, 0.9, 0)$ at high interest rate r.

in discretionary policies; consequently, (private) debt crises should be prevented, they cannot be avoided.
For example, conventional government policies responding to rising inflation due to overheating economies during economic booms - such as raising interest rates with the intention of curbing investment and thus dampening booms - might not simply affect the incentive to invest, but also the level of debt: if debt is already high, raising the interest rate could even turn the economic boom into a crisis. On the other hand, the resulting attempt to revitalize the economy by reducing the interest rate, and thus stimulate investment, in accordance with conventional economic analysis, should force the system back along the phase diagram toward the stable region of the focuses.
However, in the presence of excessively large accumulated debt and a depressed economy, government action may be too weak and too late.
Consequently, as Minsky suggests, crises should primarily be avoided through active support for the economy and financial regulation. The goal of stabilization policies is therefore not to prevent economic cycles, which are endemic in a complex system, but to prevent the system's collapse.
The chaotic dynamics explored in Keen model suggest that periods of relative tranquillity in a financial economy may actually be nothing more than the calm before the storm. This aspect of the model resonates with economic reality: it recalls the so-called "Great Moderation", a period from the 1980s to 2007 characterized by the decline in employment volatility, wages, and consequently inflation. This period was long hailed as beneficial and a bringer of stability, until it resulted in the bursting of the housing bubble and the Great Recession. Indeed, one economic variable that was not adequately considered in assessments of the Great Moderation was the increase in private debt.
Since the dynamics of the stable fixed point that goes to infinity, seen in the $(\omega - \lambda)$ plane, show increasingly smaller cycles and seem to suggest a convergence towards a good economic stability, Keen (1995) convinced himself that his model[14] was capable of studying and predicting the Great Moderation and the subsequent crash.

[14]To be completely honest, in our opinion, Keen model also shows some details that are not entirely compatible with the Minskian tale that it would like to tell.

4 Growth models with inventories

4.1 The Franke (1996) model

Here we'll discuss the work made by (Franke, 1996). This is a growth model in which inventories[15] play a role in the business cycle. This was a novelty, as theoretical studies of macroeconomic inventory cycles are rare and they take into account neither long-run growth nor capacity effects from investment in fixed capital goods.
Through the Author's words:

> The aim of the paper has been to contribute to a macroeconomic theory of inventory fluctuations, which has to take sustained growth into account and must not neglect the concurrent accumulation of fixed capital.

All variables are in real terms and they are divided by the capital stock K to deflate growth. So

$$u_F = \frac{Y}{K} \tag{69}$$

the output capital ratio is proportional to the utilization of productive capacity and therefore characterizes total economic activity. We assume a constant ratio u_n relating normal output to fixed capital, a normal utilization of productive capacity.
V stands for inventories of finished goods and its ratio is

$$v_F = \frac{V}{K} \tag{70}$$

Eventually, Y_e are the expected sales and are divided by the capital stock

$$z_F = \frac{Y_e}{K} \tag{71}$$

Let total output Y and all other flow magnitudes be measured per year.
In continuous time, the accounting identity for the change in inventories is given by the difference between production Y and final sales (or effective demand) Y_d

$$\dot{V} = Y - Y_d \tag{72}$$

There is an "inventory accelerator" $f_d > 0$: it is a certain desired ratio of inventories to expected sales Y_e, above which firms based stock management

$$V_d = f_d Y_e \tag{73}$$

[15] As regards the inventories, the inspiration comes from an old paper, (Metzler, 1941), that was the first study on the subject.

Since desired and inventories generally differ, firms seek to close this gap gradually with speed $\lambda_F > 0$, i.e. the stock of inventories would reach its target level V_d in $\frac{1}{\lambda_F}$ years. But firms also account for the growth of economy, doing this in view of a "normal" trend rate of growth g, which is a constant here. All together leads to desired change in inventories J_d:

$$J_d = gV_d + \lambda_F (V_d - V) \tag{74}$$

Firms decide on production before actual sales are known and the decision are not modified within the short period, so inventories absorb any excess supply, similarly an excess of demand over current production is served from the existing stock. The buffer-stock aspect is represented by this equation

$$Y = J_d + Y_e \tag{75}$$

Regarding net investments I in fixed capital, we have an unspecified (for now) function $h(u_F)$, which, at equilibrium, has to give u_n since firms strive for a level of normal utilization when the system grows at the normal rate of growth g

$$\frac{I}{K} = h(u_F) \qquad h(u_n) = g \qquad \frac{\partial h}{\partial u_F} > 0 \tag{76}$$

We postulated that total demand can be represent as a function of the output-capital ratio u_F. Moreover, the requirement that the marginal propensity to spend c be less than unity is converted to the notion that $\frac{Y_d}{Y}$ depends negatively on u_F. So the excess demand function is

$$\frac{Y_d - Y}{Y} = e(u_F) \qquad e(u_n) = -\frac{gf_d}{1 + gf_d} \qquad \frac{\partial e}{\partial u_F} < 0 \tag{77}$$

to obtain the equilibrium equation (77) (the second one) we use together Eq(73), (74), (75):

$$Y = J_d + Y_e = gV_d + \lambda_F (V_d - V) + Y_e = gf_d Y_e + \lambda_F (V_d - V) + Y_e$$

but at equilibrium we necessary have $V = V_d$ and $Y_d = Y_e$ so

$$Y = (1 + gf_d) Y_e = (1 + gf_d) Y_d$$

or

$$Y - Y_d = gf_d Y_d$$

where we use $Y_d = \frac{Y}{1+gf_d}$ for Y_d on the right side to obtain the second equation of Eq(77).
How does Y_e change? Author adopt adaptive expectations to determine the formation of firms' sale expectations

$$\dot{Y}_e = gY_e + \sigma (Y_d - Y_e) \tag{78}$$

At this point, Franke specifies a linear form of the two demand functions in Eq(76) and (77)

$$h(u_F) = g + \frac{\beta_h(u_F - u_n)}{u_n} \tag{79}$$

and

$$e(u_F) = -\frac{gf_d}{1 + gf_d} - \frac{\beta_e(u_F - u_n)}{u_n} \tag{80}$$

The parameters, β_h, β_e, g and f_d are estimated from a regression on quarterly data of US non-financial corporate business sector.
Using Eq(73), (74), (75) and division by K, we have a relationship between utilization u_F and those two that will soon become the state variables of the two differential equations that define the Franke model, sales expectation z_F and inventories v_F:

$$u_F = u_F(v_F, z_F) = (1 + gf_d) z_F + \lambda_F(z_F)(f_d z_F - v_F) \tag{81}$$

$\lambda_F(z_F)$ is a continuous and piecewise differentiable function of z_F replacing the stock-adjustment speed λ_F in Eq(74). Afterwards, we will see why Franke made this decision.
Using Eq(72) and (77) we obtain

$$\frac{\dot{V}}{V} = \frac{\dot{V}}{Y}\frac{Y}{K}\frac{K}{V} = -e(u_F)\frac{u_F}{v_F} \tag{82}$$

Logarithmic differentiation of v_F gives $\frac{\dot{v}_F}{v_F} = \frac{\dot{V}}{V} - \frac{\dot{K}}{K}$, where from Eq(76) $\frac{\dot{K}}{K} = h(u_F)$ [16].
Using Eq(81) we obtain our first differential equation

$$\dot{v}_F = -u_F(v_F, z_F)\, e(u_F(v_F, z_F)) - v_F h(u_F(v_F, z_F)) \tag{83}$$

To derive the equation for $\dot{z}_F$ we observe that

$$\frac{Y_d}{Y_e} = \left[1 + \frac{Y_d - Y}{Y}\right]\frac{Y}{K}\frac{K}{Y_e} = [1 + e(u_F)]\frac{u_F}{z_F} \tag{84}$$

Division of Eq(78) by Y_e and logarithmic differentiation of z_F leads to $\frac{\dot{z}_F}{z_F} = \frac{\dot{Y}_e}{Y_e} - \frac{\dot{K}}{K} = g + \sigma\left[(1 + e(u_F))\frac{u_F}{z_F} - 1\right] - h(u_F)$.
Last differential equation is

$$\dot{z}_F = z_F[g - h(u_F(v_F, z_F))] + \sigma\{[1 + e(u_F(v_F, z_F))]\, u_F(v_F, z_F) - z_F\} \tag{85}$$

To summarise, the $2D$ system is

$$\begin{cases} \dot{v}_F = -u_F(v_F, z_F)\, e(u_F(v_F, z_F)) - v_F h(u_F(v_F, z_F)) \\ \dot{z}_F = z_F[g - h(u_F(v_F, z_F))] + \sigma\{[1 + e(u_F(v_F, z_F))]\, u_F(v_F, z_F) - z_F\} \end{cases} \tag{86}$$

[16] Please note that in this model the investment is considered net so $\dot{K} = I$, if it were gross we would have depreciation δ as in Eq(164): $\dot{K} = I - \delta K$

From Eq(76), (77) and (81), with respect to the given value of normal utilization u_n, the steady-state position of the economy is

$$(\bar{v}_F, \bar{z}_F) = \left(\frac{f_d u_n}{1 + g f_d}, \frac{u_n}{1 + g f_d} \right)$$

Franke shows that the local stability analysis of this stationary point rests on two sign conditions, called E and H

$$\begin{cases} E \equiv 1 + e\left(u_n\right) + u_n \frac{\partial e}{\partial u_F}\Big|_{u_F = u_n} > 0 \\ H \equiv - \left[e\left(u_n\right) + u_n \frac{\partial e}{\partial u_F}\Big|_{u_F = u_n} + \bar{v}_F \frac{\partial h}{\partial u_F}\Big|_{u_F = u_n} \right] > 0 \end{cases} \tag{87}$$

which give a relationship for the critical value L of the stock-adjustment speed, $(\bar{v}_F, \bar{z}_F)$ is locally asymptotically stable if

$$\bar{\lambda}_F \equiv \lambda_F\left(\bar{z}_F\right) < L \equiv \frac{(1 + g f_d)}{E f_d} u_n \left| \frac{\partial e}{\partial u_F}\Big|_{u_F = u_n} \right| \tag{88}$$

Using $\bar{\lambda}_F$, he also finds a critical value σ_0 for σ that gives local asymptotic stability $\sigma < \sigma_0$.
The critical value is found to be $L = 0.71$. It does not appear very far-fetched to assume that, in the course of one year, the underlying time unit, firms intend to close at least three-quarters of the gap between desired and actual inventory in Eq(74). So fast adaptive expectations of sales, i.e. high σ, render the steady state unstable.
At $\sigma = \sigma_0$ the jacobian matrix of the process Eq(86) possesses a pair of purely imaginary eigenvalues (the determinant is positive), so they are complex conjugate. This is warning sign for oscillatory motions of the economy. As σ rises beyond σ_0 a Hopf bifurcation occurs since the complex eigenvalues are crossing the imaginary axis.
Author finds that while equilibrium is locally repelling, it is attractive in the outer regions of the space. Through his words:

> It then turned out that the steady state of the model is typically repelling. Furthermore, tendencies for oscillatory motions could be recognized. To prevent the economy from continuously spiralling outwards in the phase space, we concentrated on the propagation effects of inventory investment and postulated that the stock-adjustment speed towards the desired level of inventories is sufficiently flexible when expected sales deviate too much from normal. It was demonstrated that this non-linearity bends the spirals inwards in the outer regions of the state space. As a consequence, persistent and bounded oscillations come into being. More specifically, all trajectories converge to a periodic orbit. By means of numerical simulations it was shown that the model can be calibrated such as to generate cyclical behaviour that is qualitatively and quantitatively compatible with empirical time series.

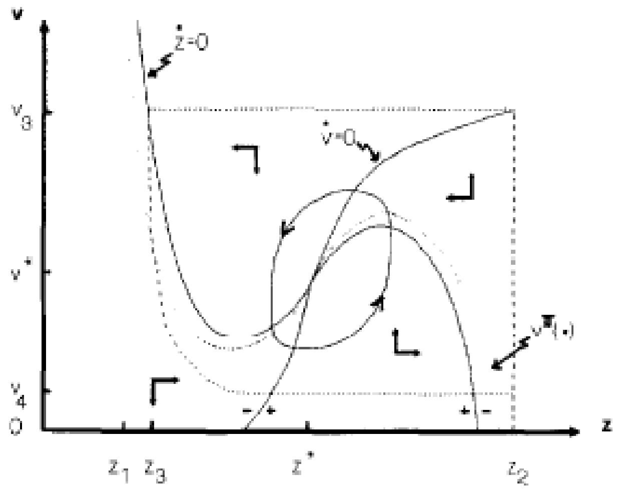


Figure 39: Phase diagram with the limit cycle and nullclines $\dot{v}_F = 0$ and $\dot{z}_F = 0$ of Eq(86). You can also see the invariant set built to apply the Poincarè-Bendixson Theorem. The image is taken from (Franke, 1996). Please note that our notation is slightly different from that of the original paper, e.g. we use v_F and z_F instead of v and z and $(\bar{v}_F, \bar{z}_F)$ instead of (v^*, z^*) for the fixed point. We will explain the reason for this choice in the next section.

So here is the explanation of what we left pending earlier: $\lambda_F(z_F)$, a continuous and piecewise differentiable function of z_F, has replaced the previous definition of the stock-adjustment speed λ_F in Eq(74) to ensure the convergence of the trajectories to the periodic orbit. See Figure 39 for the phase portrait.
The most important qualitative result of the model, mentioned above, is that, compared to empirical data, the loops have the correct counter-clockwise orientation.
At the end of the paper, Author suggests further developments of the model for the future:

> In a second step, the present model or a more advanced version of it may be incorporated into another macroeconomic model which, for example, studies inflation, income distribution and/or financial markets.

Indeed, this will be the path taken years later by Grasselli and Hguyen-Huu, whom we will discuss in the next section (Section 4.2).

4.2 The Grasselli and Nguyen-Huu (2018) model

The Franke (1996) model is the main inspiration for the work of (Grasselli and Nguyen-Huu, 2018). To this end, in the previous section, we did not follow Franke's original notation but we attempted to merge it with that used in (Grasselli and Nguyen-Huu,

2018), in order to show the similarities. With regard to notation, we have also tried to avoid the use of certain letters (or to distinguish them by adding subscripts[17]) that will be assigned to important variables in the following sections.
The other important inspiration comes from the tradition of Goodwin's class struggle model and from Keen's Minsky model. So the Grasselli and Nguyen-Huu (2018) is a Post-Keynesian model with many features: depreciation rate $\delta(u)$ expressed as a function of capital utilization, productivity and workforce exogenous growth, nominal prices, inventories, bargaining power, Kaleckian disciplinary unemployment and wage share dynamics, many behavioural rules with also capitalists' forecast for the long-run growth, 5D and stock-flow consistent model.
Unfortunately, as often happens, the more elegant and complete a model is, the more complicated it is analytically and heuristically.
Quoting the Authors:

> Global analysis of such high-dimensional nonlinear system is beyond the scope of current techniques, and even local analysis of the interior equilibrium proves to be laborious and not very illuminating.

and, once again:

> The present model is nevertheless highly complex. It needs the specification of at least fourteen parameters in addition to three behavioural functions. The exploration of other possible equilibrium points is considerably involved, and any local stability analysis will reveal to be cumbersome and non-intuitive exercise. In order to build intuition about the system, we follow the strategy of considering the lower-dimensional subsystems that arise in some limiting cases for the model parameters and behavioural functions. We start with a few special cases corresponding to known models in the literature.

Indeed, we'll show that under certain conditions, it reduces to the models already discussed in the previous sections. Furthermore, this model will be important also for the Section 6.1, as it formed the basis for the model proposed in this thesis.
Authors consider a three-sector closed economy consisting of firms, banks and households. The firm sector produces one homogeneous good used both for consumption and investment. We have accelerator ν or constant technical capital/capacity output ratio:

$$Y_K = \frac{K}{\nu} \tag{89}$$

where the total stock of capital K in real term determines potential or capacity output Y_K. Anyway Y_K isn't the actual output Y produced by firms, which is assumed to

[17] As in the case of the letter z, we used z_F for Eq(71) to avoid conflict with $z = \frac{Z}{Y}$, the share of the autonomous component of demand.

consist of expected sales Y_e plus planned inventory changes I_p

$$Y = Y_e + I_p \tag{90}$$

In turn, Eq(90) determines capacity utilization

$$u = \frac{Y}{Y_K} \tag{91}$$

Finally, capital is assumed change according the famous equation

$$\dot{K} = I_k + \delta\left(u\right) K \tag{92}$$

where I_k denotes capital investment in real terms and $\delta\left(u\right)$ is a depreciation rate expressed as a function of capital utilization u.
The effective demand Y_d is the total sales demand

$$Y_d = C + I_k \tag{93}$$

where C denotes the total real consumption that, now, it isn't only given by households but also by banks.
Denote the stock of inventories by V, its change $\dot{V}$ is the investment in inventory, which consists of both planned I_p and unplanned I_u change in inventory held by firms

$$\dot{V} = I_p + I_u = Y - Y_d \tag{94}$$

Since the difference between output and demand determines actual changes in the level of inventory, substituting Eq(90) into (94), we see that unplanned changes in inventories are given by

$$I_u = \dot{V} - I_p = (Y - Y_d) - (Y - Y_e) = Y_e - Y_d \tag{95}$$

and therefore accommodate any surprises in actual sales compared to the expected one.
Finally total real investment I in economy is given by

$$I = Y - C = Y - I_d + I_k = I_p + I_u + I_k \tag{96}$$

W is the nominal wage bill (total expenditure paid by capitalists in wages), N the total workforce and ℓ the number of employed workers, a the productivity per worker, λ the employment rate and w the wage rate.

$$a = \frac{Y}{\ell} \tag{97}$$

$$\lambda = \frac{\ell}{N} = \frac{Y}{aN} \tag{98}$$

$$w = \frac{W}{\ell} \tag{99}$$

whereas the unit cost of production c is given by

$$c = \frac{W}{Y} = \frac{w}{a} \tag{100}$$

We will postulate throughout that productivity and workforce growth rates are exogenously given

$$\frac{\dot{a}}{a} = \alpha \tag{101}$$

$$\frac{\dot{N}}{N} = \beta \tag{102}$$

Denoting the unit price level for the homogeneous good by p, the nominal sales should be given by $pY_d = pC + pI_k$. To obtain nominal output Y_n we can't simply use $Y_n = pY$ since we have to account for inventory changes:

$$Y_n = pC + pI_k + c\dot{V} = pY_d + c\dot{V} = pY_d + \frac{d\left(cV\right)}{dt} - \dot{c}V \tag{103}$$

so $Y_n = pY$ is true if and only if either $p = c$ or $\dot{V} = 0$.
The net profit for firms, after paying wages, interest on debt, and accounting for consumption of fixed capital (a.k.a. depreciation), is given by

$$\Pi = Y_n - W - rD - p\delta\left(u\right)K \tag{104}$$

The debt change for the firm sector is

$$\dot{D} = p\left(I_k - \delta\left(u\right)K\right) + c\dot{V} - \Pi = pI_k + c\dot{V} - \Pi_p \tag{105}$$

where $\Pi_p = Y_n - W - rD$ denotes the pre-depreciation profit.
We will also use the ratios:

$$y_e = \frac{Y_e}{Y} \tag{106}$$

$$y_d = \frac{Y_d}{Y} \tag{107}$$

$$v = \frac{V}{Y} \tag{108}$$

$$\omega = \frac{W}{pY} = \frac{c}{p} = \frac{w}{pa} \tag{109}$$

$$d = \frac{D}{pY} \tag{110}$$

where we use p, the unit price level for the homogeneous good, to obtain nominal quantities.
Authors clarify the logical sequence:

> We now specify the behavioural rules for firms, banks, and households. Namely, for given values of the state variables, firms decide the level of capital investment I_k, planned changes in inventory I_p, and expected sales Y_e, whereas banks and households decide the level of consumption C_b and C_h. This in turn determines capital by Eq(92), output by Eq(90), utilization by Eq(91), sales demand by Eq(93), and unplanned changes in inventory by Eq(95). Consequently, since productivity and workforce growth are exogenous, the level of output Y in turn gives the number of employed workers ℓ and the employment rate λ by Eq(98). Further specification of the dynamics for the nominal wage rate w and prices p then completes the model.

It is assumed that firms forecast the long-run growth rate of the economy to be a function $g_e\left(u,\pi_e\right)$ of utilization u and pre-depreciation expected profitability π_e defined as

$$\pi_e = \frac{Y_{ne} - W - rD}{pY} \tag{111}$$

where

$$Y_{ne} = pY_e + cI_p \tag{112}$$

denotes the expected nominal output.
Inserting Eq(106), (109) and (110) in Eq(111), expected profitability can also be expressed as

$$\pi_e = y_e\left(1-\omega\right) - rd \tag{113}$$

In addition to taking into account the long-run growth rate $g_e\left(u,\pi_e\right)$, firms adjust their short-term expectations based to the observed demand, leading to the following dynamics for expected demand:

$$\dot{Y}_e = g_e\left(u,\pi_e\right)Y_e + \eta_e\left(Y_d - Y_e\right) \tag{114}$$

which is analogous to Franke's Eq(78). $\eta_e \geq 0$ is the speed of short-term adjustments to observed demand.
As Franke, we assume that firms aim to maintain inventories at a desired level

$$V_d = f_d Y_e \tag{115}$$

for a fixed proportion $0 \leq f_d \leq 1$.
While this means that the long-term growth rate of desired inventory level should be

$g_e(u, \pi_e)$, we assume again that firms adjust their short-term expectations based on the observed level of inventory for a constant $\eta_d \geq 0$, representing the speed of short-term adjustments to observed inventory

$$I_p = g_e(u, \pi_e) V_d + \eta_d (V_d - V) \tag{116}$$

which is analogous to Eq(74).
We further assume that firms' investment is given by

$$I_k = \frac{k(u, \pi_e)}{\nu} K \tag{117}$$

for an unspecified function, as all the others, $k(\cdot, \cdot)$ capturing the effects of both capacity utilization and expected profits. Based on $\dot{K}$ definition in Eq(92), this leads to the dynamics for capital

$$\frac{\dot{K}}{K} = \frac{k(u, \pi_e)}{\nu} - \delta(u) \tag{118}$$

The total consumption is given by assumption by

$$C = \theta(\omega, d) Y \tag{119}$$

Consumption of households and banks is assumed to be given by constant fractions of income and wealth, namely,

$$pC_h = c_{ih}(W + r_m M) + c_{wh} M \tag{120}$$

$$pC_b = c_{ib}(rD - r_m M) + c_{wb}(D - M) \tag{121}$$

for M stands for household deposits and r_m for interest rate on them. Under the simplifying assumption that $c_{ih} = c_{ib} = c_i$ and $c_{wh} = c_{wb} = c_w$, we have

$$pC = c_1 W + c_2 D \tag{122}$$

with $c_1 = c_i$ and $c_2 = c_w + c_i r$.
We see that Eq(122) is an example of Eq(119) with

$$\theta(\omega, d) = c_1 \omega + c_2 d \tag{123}$$

Nominal demand pY_d is given by

$$pY_d = pC + pI_k = p\theta(\omega, d) Y + p\frac{k(u, \pi_e)}{\nu} K \tag{124}$$

from which

$$y_d = y_d(\omega, d, y_e, u) = \frac{Y_d}{Y} = \theta(\omega, d) + \frac{k(u, \pi_e)}{u} \tag{125}$$

Authors say:

For the price dynamics, we follow the Post-Keynesian tradition and assume that the long-run equilibrium price is given by a constant markup $m \geq 1$ times unit labour cost c,whereas observe prices converge to this through a lagged adjustment with speed $\eta_p > 0$. A second component with adjustment speed $\eta_q > 0$ is added to that dynamics to take into account short-term considerations regarding unplanned changes in inventory volumes:

$$\begin{aligned}\frac{\dot{p}}{p} &= \eta_p\left(m\frac{c}{p}-1\right)-\eta_q\frac{Y_e-Y_d}{Y} = \\ &= \eta_p\left(m\omega-1\right)+\eta_q\left(y_d-y_e\right) \equiv \\ &\equiv i\left(\omega, y_d, y_e\right)\end{aligned} \tag{126}$$

for $Y_e - Y_d = I_u$, remember Eq(95).
Workers are supposed to bargain for wages based on the current state of the labour market, but also take into account the observed inflation rates with a degree of money illusion $0 \leq \gamma \leq 1$, with $\gamma = 1$ corresponding to the case where inflation is fully incorporated in workers' bargaining

$$\frac{\dot{w}}{w} = \Phi\left(\lambda\right)+\gamma\frac{\dot{p}}{p} \tag{127}$$

Combining Eq(115) and Eq(116), we see that output is given by

$$Y = Y_e + I_p = \left[f_d\left(g_e\left(u,\pi_e\right)+\eta_d\right)+1\right]Y_e - \eta_d V \tag{128}$$

so we can calculate also

$$v = \frac{V}{Y} = \frac{\left[1+f_d\left(g_e\left(u,\pi_e\right)+\eta_d\right)\right]y_e-1}{\eta_d} \tag{129}$$

Differentiating Eq(128) and using Eq(114) and Eq(94) we obtain the output growth rate

$$\begin{aligned}\frac{\dot{Y}}{Y} &= \left[1+f_d\left(g_e\left(u,\pi_e\right)+\eta_d\right)\right]\left[y_e g_e\left(u,\pi_e\right)+\right. \\ &\left.+\,\eta_e\left(y_d-y_e\right)\right]+\eta_d\left(y_d-1\right) \equiv \\ &\equiv g\left(u,\pi_e,y_d,y_e\right)\end{aligned} \tag{130}$$

Here, we have an issue since in this equation there isn't $\frac{dg_e(u,\pi_e)}{dt}$. Instead this term should come from

$$\frac{\dot{I}_p}{Y} = \frac{dg_e\left(u,\pi_e\right)}{dt}\frac{V_d}{Y}+g_e\left(u,\pi_e\right)\frac{\dot{V}_d}{Y}+\eta_d\left(\frac{\dot{V}_d}{Y}-\frac{\dot{V}}{Y}\right)$$

but since it does not appear in the Authors' equations, it seems that Grasselli and Nguyen-Huu (2018) implicitly choose to set $\dot{g}_e\left(u,\pi_e\right) \approx 0$, this is a significant simplification, since that term would have brought with it, in the final equations that we will

soon derive, terms like $\dot{u}$ and $\dot{\pi}_e$.
Anyway, moving forward, taking $\ln$ of $\omega = \frac{w}{pa}$ we have

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{w}}{w} - \frac{\dot{a}}{a} - \frac{\dot{p}}{p} = \Phi(\lambda) - \alpha - (1-\gamma)\, i(\omega, y_d, y_e) \tag{131}$$

Again, logarithmic differentiation of $\lambda = \frac{Y}{aN}$ with Eq(101), (102) and (130)

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} - \frac{\dot{N}}{N} = g(u, \pi_e, y_d, y_e) - \alpha - \beta \tag{132}$$

For the debt ratio $d = \frac{D}{pY}$, using Eq(105) we find that

$$\begin{aligned} \frac{\dot{d}}{d} &= \frac{\dot{D}}{D} - \frac{\dot{p}}{p} - \frac{\dot{Y}}{Y} = \\ &= \frac{pI_k + c\dot{V} - (Y_n - W - rD)}{D} - i(\omega, y_d, y_e) - g(u, \pi_e, y_d, y_e) = \\ &= \frac{W + rD - pC}{D} - i(\omega, y_d, y_e) - g(u, \pi_e, y_d, y_e) = \\ &= \frac{\omega + rd - \theta(\omega, d)}{D} - i(\omega, y_d, y_e) - g(u, \pi_e, y_d, y_e) \end{aligned} \tag{133}$$

Similarly, for the expected sales ratio $y_e = \frac{Y_e}{Y}$ we use Eq(114) to obtain

$$\frac{\dot{y_e}}{y_e} = \frac{\dot{Y_e}}{Y_e} - \frac{\dot{Y}}{Y} = g_e(u, \pi_e) + \eta_e\left(\frac{y_d}{y_e} - 1\right) - g(u, \pi_e, y_d, y_e) \tag{134}$$

Finally, for the capacity utilization $u = \frac{\nu Y}{K}$, using Eq(118) we find

$$\frac{\dot{u}}{u} = \frac{\dot{Y}}{Y} - \frac{\dot{K}}{K} = g(u, \pi_e, y_d, y_e) - \frac{k(u, \pi_e)}{\nu} + \delta(u) \tag{135}$$

Since y_d is expressed in Eq(125) as a function of (ω, d, π_e, u) and π_e is given in Eq(113) as a function of (ω, d, y_e), we see that the model can be completely characterized by the state variables $(\omega, \lambda, d, y_e, u)$ satisfying the following system of 5 ordinary differential equations:

$$\begin{cases} \dot{\omega} = \omega\left[\Phi(\lambda) - \alpha - (1-\gamma)\, i(\omega, y_d, y_e)\right] \\ \dot{\lambda} = \lambda\left[g(u, \pi_e, y_d, y_e) - \alpha - \beta\right] \\ \dot{d} = d\left[r - g(u, \pi_e, y_d, y_e) - i(\omega, y_d, y_e)\right] + \omega - \theta(\omega, d) \\ \dot{y_e} = y_e\left[g_e(u, \pi_e) - g(u, \pi_e, y_d, y_e)\right] + \eta_e(y_d - y_e) \\ \dot{u} = u\left[g(u, \pi_e, y_d, y_e) - \frac{k(u, \pi_e)}{\nu} + \delta(u)\right] \end{cases} \tag{136}$$

where $i(\omega, y_d, y_e)$ is given by Eq(126) and $g(u, \pi_e, y_d, y_e)$ by Eq(130).
To obtain the equilibrium point $\left(\bar{\omega}, \bar{\lambda}, \bar{d}, \bar{y}_e, \bar{u}\right)$, observe that the second differential equation in Eq(136) requires that

$$g(\bar{u}, \bar{\pi}_e, \bar{y}_d, \bar{y}_e) = \alpha + \beta \tag{137}$$

which inserted in the forth equation leads to $\bar{y}_d = \bar{y}_e$ and

$$g_e(\bar{u}, \bar{\pi}_e) = \alpha + \beta \tag{138}$$

Therefore, using this and Eq(137) in (130) gives

$$\bar{y}_d = \bar{y}_e = \frac{1}{1 + f_d(\alpha + \beta)} \tag{139}$$

Inserting the latter into Eq(129) implies that $\bar{v} = f_d \bar{y}_e$, so that the equilibrium level of inventory is the desired level $V_d = f_d \bar{y}_e Y$.
Substituting $\bar{y}_d = \bar{y}_e$ into Eq(126) leads to an equilibrium inflation of the form

$$i(\bar{\omega}, \bar{y}_d, \bar{y}_e) = i(\bar{\omega}) = \eta_p(m\bar{\omega} - 1) \tag{140}$$

Using the third differential equation we have

$$\bar{d} = \frac{\bar{\omega} - \theta\left(\bar{\omega}, \bar{d}\right)}{\alpha + \beta + i(\bar{\omega}) - r} \tag{141}$$

Moving to the last equation, we obtain that the investment function at equilibrium satisfies

$$k(\bar{u}, \bar{\pi}_e) = \nu\left[\alpha + \beta + \delta(\bar{u})\right] \tag{142}$$

which, inserted in Eq(125), gives

$$\bar{u} = \frac{\nu\left[\alpha + \beta + \delta(\bar{u})\right]\left[1 + f_d(\alpha + \beta)\right]}{1 - \theta\left(\bar{\omega}, \bar{d}\right)\left[1 + f_d(\alpha + \beta)\right]} \tag{143}$$

We can then obtain the values of $\left(\bar{\omega}, \bar{d}\right)$ by solving Eq(141) and Eq(142) with $\bar{\pi}_e$ defined from Eq(113). Finally, returning to the first equation in Eq(136) we find the equilibrium employment rate by solving

$$\Phi\left(\bar{\lambda}\right) = \alpha + (1 - \gamma)\, i(\bar{\omega}) \tag{144}$$

Of course, the Authors admit:

> We therefore see that existence and uniqueness of the interior equilibrium depends on properties of the functions k and θ, which need to be asserted in specific realizations of the model.

Now let's move on to simplified cases. Indeed, Grasselli and Nguyen-Huu (2018) derive models already seen in the Section 3 from the Eq(136) as special cases through precise choices of parameters.
The simplest special case consists of the model proposed in (Goodwin, 1982) (Section 3.3 and Section 3.4). The original Goodwin model is formulated in real terms, which we can easily reproduce by setting $\eta_p = \eta_q = \gamma = 0$, meaning that the rate of inflation is zero, and setting unitary prices $p = 1$.
Like most models, it also makes no reference to inventories, thereby implicitly assuming that output equals demand. We can recover this from the general model by assuming that $f_d = \eta_d = 0$, meaning that there is no desired inventory level, i.e. $V_d = 0$, or planned investment in inventory, i.e. $I_p = 0$, and that $\eta_e \to \infty$, meaning that firms have perfect forecast of demand and set $Y_e = Y_d = Y$ at all times.
In addition, Goodwin adopts a constant capital-to-output ratio $\nu_G = \frac{K}{Y}$, which we can recover by setting $u \equiv 1$, so $Y \equiv Y_K$. Finally, although not explicitly mentioned in (Goodwin, 1982), we adopt a constant depreciation rate $\delta(u) = \delta > 0$.
The only explicit assumption of the Goodwin model regarding the behaviour of firms is that investment is equal to profits (remember Say's Law in Section 3.4), which in the present setting corresponds to

$$k(u, \pi_e) = \pi_e = 1 - \omega - rd \tag{145}$$

since $y_e = \frac{Y_e}{Y} = 1$ in Eq(113).
The model is also silent about banks, but it follows from Eq(105) and the investment rule above (recalling that $\dot{V} = 0$) that $\dot{D} = 0$ at all times, so we assume for simplicity that $d = D_0 = 0$.
Regarding households, the assumption is that all wages are consumed, namely $c_{ih} = c_1 = 1$ in the notation of Eq(120). For consistency, we set $c_2 = r$, even though this is not relevant with $D = 0$.
For the growth rate, observe that we can no longer obtain it by simply differentiating Eq(128), since Eq(114) is degenerate in the limit case $\eta_e \to \infty$.
Instead, since $u = 1$, we can use the fact that $Y = \frac{K}{\nu}$ to obtain

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{1-\omega}{\nu} - \delta \tag{146}$$

With these parameter choices, the system in Eq(136) reduces to the form

$$\begin{cases} \dot{\omega} = \omega \left[\Phi(\lambda) - \alpha\right] \\ \dot{\lambda} = \lambda \left[\frac{1-\omega}{\nu} - \alpha - \beta - \delta\right] \end{cases} \tag{147}$$

discussed also in (Grasselli and Costa Lima, 2012).
The solutions are closed periodic orbits around the non-hyperbolic equilibrium point (center)

$$\left(\bar{\omega}, \bar{\lambda}\right) = \left(1 - \nu(\alpha + \beta + \gamma), \Phi^{-1}(\alpha)\right) \tag{148}$$

which is a special case of Eq(142) and Eq(143).
Now it's the turn of the model developed by (Keen, 1995). It's based on the same assumptions of the Goodwin model. So again we have $\eta_p = \eta_q = \gamma = 0$, $p = 1$, $f_d = \eta_d = V_d = I_p = 0$, $\eta_e \to \infty$, $Y_e = Y_d = Y$, $u = 1$, $\delta(u) = \delta$.
The innovation in the model is the presence of debt and that investment is now given by

$$k(u, \pi_e) = k(\pi_e) = k(1 - \omega - rd) \tag{149}$$

where we used the fact that $y_e = \frac{Y_e}{Y} = 1$ in Eq(113). Moreover, the identity $Y_d = Y$ implies that

$$C = Y_d - I_k = [1 - k(\pi_e)]\, Y \tag{150}$$

i.e. $\theta(\omega, d) = 1 - k(1 - \omega - rd)$ in Eq(119), so in the absence of either price or quantity adjustments, total consumption plays the role of an accommodating variable in the model.
Since Eq(114) is degenerate, we again use $Y = \frac{K}{\nu}$ to obtain the growth rate of the economy as

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{k(\pi_e)}{\nu} - \delta \tag{151}$$

So Eq(136) now becomes

$$\begin{cases} \dot{\omega} = \omega\left[\Phi(\lambda) - \alpha\right] \\ \dot{\lambda} = \lambda\left[\frac{k(\pi_e)}{\nu} - \alpha - \beta - \delta\right] \\ \dot{d} = d\left[r - \frac{k(\pi_e)}{\nu} - \delta\right] + \omega - 1 + k(\pi_e) \end{cases} \tag{152}$$

where $\pi_e = 1 - \omega - rd$.
Eq(141), (142) and (144) reduce to

$$\bar{d} = \frac{\omega - 1 + \nu(\alpha + \beta + \delta)}{\alpha + \beta - r} \tag{153}$$

$$k(\bar{\pi}_e) = \nu(\alpha + \beta + \delta) \tag{154}$$

$$\Phi\left(\bar{\lambda}\right) = \alpha \tag{155}$$

from where we obtain the equilibrium point $\left(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1\right)$ found in (Grasselli and Costa Lima, 2012), which is shown to be locally stable provided the investment function $k(\cdot)$ is sufficiently increasing at equilibrium, but does not exceed the amount of net profits by too much.
There is also another equilibrium $\left(\bar{\omega}_2, \bar{\lambda}_2, \bar{d}_2\right) = (0, 0, +\infty)$ which is a "bad" equilibrium that economically represents the bursting of the financial bubble caused by debt deflation and it is locally asymptotically stable provided

$$\lim_{\pi_e \to -\infty} < \nu(r + \delta) \tag{156}$$

Finally, it's the turn of (Franke, 1996). The Franke model is also formulated in real terms, so we maintain the choice of $\eta_p = \eta_q = \gamma = 0$ and $p = 1$ and normalizes all variables by dividing them by K instead of Y

$$u_F \equiv \frac{Y}{K} = \frac{u}{\nu} \tag{157}$$

$$z_F \equiv \frac{Y_e}{K} = y_e u_F \tag{158}$$

$$v_F \equiv \frac{V}{K} = v u_F \tag{159}$$

Crucially, the model in (Franke, 1996) implicitly assumes a constant wage share ω, so that the first equation is simply $\dot{\omega} = 0$. The second equation then decouples from the rest of the system and simply provides the employment rate along the solution path, in particular leading to a constant employment rate at equilibrium. As with the Goodwin model, the Franke model is also silent about banks, implicitly assuming that firms can raise the necessary funds for investment through retained profits and savings from households, which we reproduce here by setting $\dot{d} = 0$ in Eq(136).

The behaviour of firms, on the other hand, is almost identical to the one adopted here, provided we take

$$g_e(u, \pi_e) = \alpha + \beta \tag{160}$$

as the long-run growth rate of expected sales.

For the investment function, we recover the assumption in (Franke, 1996) by setting

$$k(u, \pi_e) = \nu h(u_F) \tag{161}$$

for an increasing function $h(\cdot)$.

Regarding effective demand, instead of modelling consumption and investment separately, the assumption is that demand in excess of output is given directly in terms of utilization, which we can reproduce in our model by setting

$$y_d = e(u_F) + 1 \tag{162}$$

for a decreasing function $e(\cdot)$.

With these choices, it is a simple exercise to verify that the fourth and fifth equations in Eq(136) are equivalent to equations for v_F and z_F in (Franke, 1996), with equilibrium values given by

$$(\bar{v}_F, \bar{z}_F) = \left(\frac{f_d \bar{u}_F}{1 + f_d(\alpha + \beta)}, \frac{\bar{u}_F}{1 + f_d(\alpha + \beta)} \right) = (\bar{v}\bar{u}_F, \bar{y}_e \bar{u}_F) \tag{163}$$

It is shown in (Franke, 1996), as we have seen, that this equilibrium is locally asymptotically stable provided the speed of adjustment of inventories η_d is sufficiently small.

For η_d above a certain threshold, however, local stability can only be asserted when the speed of adjustment of expected sales η_e is sufficiently small.
The Franke's main innovation consists of adopting a variable speed of adjustment $\eta_d = \eta_d\left(z_F\right)$ and investigating its effect on the stability of the equilibrium.
It is then shown that even in the unstable case, namely when both $\eta_d\left(\bar{z}_F\right)$ and η_e are large enough that the equilibrium is locally repelling, global stability can be achieved provided $\eta_d\left(z_F\right)$ decreases fast enough away from the equilibrium.

5 The Sraffian Supermultiplier

5.1 A brief review of the Sraffian Supermultiplier literature

The term “Supermultiplier” was coined by Hicks (1950) as an extension of the more famous Keynesian Multiplier, it has later acquired the broader meaning of a formal theoretical apparatus, based on the principle of effective demand, that takes into account both the multiplier and accelerator effects. It became “Sraffian” with Serrano (1995). Years later, the paradigm is strengthened with Cesaratto et al. (2003) where Authors strongly reject the neo-Schumpeterian notion of autonomous investment, in favour of the view that, in the long period, all investment is induced. The neoclassical claim that market mechanisms will restore full employment, whenever workers are displaced by technical change, and the argument of automatic compensation are rejected as well. Following Post-Keynesian lines, the ultimate engines of growth are located by Authors in exports, government spending and autonomous consumption: components which all share the status of autonomous components of effective demand, e.g. they are neither financed by wage income nor can create capacity. Finally, paper claims that technical change plays a role in the accumulation process through its effects on consumption and the material requirements, but it is seen to depend upon income distribution, exchange rate policy, bank liquidity and other circumstances. Cesaratto examines more in depth the link between the Supermultiplier and the Endogenous Money Theories with a focus on Initial and Final Finance in (Cesaratto, 2016a), and again in (Cesaratto and Di Bucchianico, 2021). In (Cesaratto and Pariboni, 2022), the Authors demonstrate, once again, the compatibility between the Keynesian and Sraffian Schools.

The next step is the aforementioned paper, (Freitas and Serrano, 2015). Its main contribution will be to change the mathematical shape of the models for the following years: for the first time, a dynamical model of Sraffian Supermultiplier sees the light, in the form of ODE; its stability conditions are also discussed. On henceforth, the research on topic comes alive. Without claiming to be complete, we shall cite some papers, but the main differences are in the incarnation of the autonomous component of demand and in the number of components (one or two): although Serrano and the other Authors referred to the components of autonomous demand, the dynamical model in Freitas and Serrano (2015) had only one component to keep things simple. So, for example, Morlin (2022) studies economic growth, fiscal policy rules aimed at debt stability and open economies, in a model with two autonomous components: public expenditure and exports (imports are instead proportional to the level of income). We’ll see it in Section 5.4.

Two components are also protagonists in (Di Bucchianico et al., 2024): workers’ credit-financed consumption and rentiers’ consumption which is the earnings given by the interest rate calculated to extant debt accumulated by workers. Instead Morlin and

Pariboni (2024) introduce, into the (Freitas and Serrano, 2015) landscape, the conflict inflation getting a long-period autonomous demand-led model with endogenous distribution. We'll see it in Section 5.3.
Serrano et al. (2018), in contrast to other works, study Harrodian instability not with a continuous time model (ODE), but with a map, a discrete time model. They find a Keynesian stability condition: marginal propensity to invest have to remain lower than the marginal propensity to save.
Of course, critics exist and Sraffian Supermultiplier advocates react in (Serrano et al., 2023), (Serrano et al., 2024a) and (Serrano et al., 2024b)[18]. Precisely regarding the criticism, we would like to point out that one of the most interesting ones concerns instability through the study of a map version model in (Thompson, 2024).

5.2 The Freitas and Serrano (2015) model

Here, we will derive Freitas and Serrano (2015) model, as you will see, calculations are really simple and derivation is straightforward. First of all, let's start by mentioning the assumptions from the paper. The system is a closed capitalist economy without a government sector. The only method of production in use required a fixed combination of a homogeneous labor input with homogeneous fixed capital to produce a single homogeneous output[19]. Natural resources are supposed to be abundant; also constant returns to scale and no technological progress are assumed, as well as no labour scarcity. All variables are measured in real terms. Moreover, output, income, profits, investment and savings are all in gross terms. Time is continuous. The level of aggregate gross fixed investment I is given by the equation

$$\dot{K}_t = I_t - \delta K_t \tag{164}$$

where δ is exogenously given depreciation rate of the capital stock K. The last, really important, assumption of the model is this equation, often called "flexible accelerator"

$$\dot{h}_t = h_t \gamma \left(u_t - u_n\right) \tag{165}$$

where h_t[20] is a variable and it's the investment share in aggregate output or the marginal propensity to invest

$$h_t = \frac{I_t}{Y_t} \tag{166}$$

[18]They are free unpublished discussion papers available on `https://www.ie.ufrj.br/publicacoes-j/textos-para-discussao.html`.

[19]The firm sector produces one homogeneous good used both for consumption and investment, e.g. the corn metaphor: cereal grain can be used both for consumption and for planting, as an investment (new capital-goods), for future harvests.

[20]Here, for a variable Ξ, the notation Ξ_t doesn't mean a map or discrete time, $\Xi_t \equiv \Xi(t)$.
From now on, we will use this notation, which differs from that used in the previous Sections, in order to be consistent with the notation used by Sraffian Authors.

u_t is the other main variable of the model, it's the actual degree of capacity utilization

$$u_t = \frac{Y_t}{Y_{K_t}} \tag{167}$$

$\gamma > 0$ is a parameter that measures the reaction of the growth rate of the marginal propensity to invest $\frac{\dot{h}_t}{h_t}$ to the deviation of the actual degree of utilization u_t from the normal or planned level u_n (with $0 < u_n < 1$). Eq(165) is a hypothesis about the behaviour[21] of capitalists, but it's also a version of flexible accelerator investment function. Since this is a $2D$ model, one half of the model is assumed by hypothesis, let's derive the remaining part.

The accelerator ν (not the constant one[22]) is the technical capital/capacity output ratio:

$$\nu = \frac{K_t}{Y_{K_t}} \tag{168}$$

where Y_{K_t} is the capacity output of the economy at time t and K_t is the level of capital stock installed in the economy at that time. Since ν is a constant (a.k.a. parameter), to be consistent, it's time derivative should be zero

$$\begin{aligned} 0 = \frac{d\nu}{dt} = \frac{d}{dt}\left(\frac{K_t}{Y_{K_t}}\right) &= \frac{\dot{K}_t}{Y_{K_t}} - \frac{\dot{Y}_{K_t}}{Y_{K_t}}\frac{K_t}{Y_{K_t}} = \frac{I_t - \delta K_t}{Y_{K_t}} - g_{K_t}\nu \\ &= \frac{I_t}{Y_t}\frac{Y_t}{Y_{K_t}} - \nu\delta - \nu g_{K_t} = h_t u_t - \nu\delta - \nu g_{K_t} \end{aligned}$$

which, finally, gives the rate of capital accumulation

$$g_{K_t} = \frac{h_t u_t}{\nu} - \delta \tag{169}$$

but g_{K_t} is also the rate of growth of capacity output, so $g_{K_t} = \frac{\dot{K}_t}{K_t} = \frac{\dot{Y}_{K_t}}{Y_{K_t}}$.
To clarify, h_t is not a constant, as ν, so it's derivative is not zero, i.e., if we take $\frac{d}{dt}\ln h_t$, we find

$$\frac{\dot{h}_t}{h_t} = \frac{\dot{I}_t}{I_t} - \frac{\dot{Y}_t}{Y_t} = g_{I_t} - g_t$$

which corresponds to Eq(9) in (Freitas and Serrano, 2015).
This approach derives the equations of motion directly from the macroeconomic definitions, so to get the last main equation it is sufficient to evaluate the time derivative

[21] They want to stay as close to the normal level as they can, so they change the level of investment accordingly.

[22] Beware, it is also possible to define it as $\nu_G = \frac{K_t}{Y_t}$ (e.g. Goodwin (1982) adopts a fixed capital-to-output ratio) but here we will use only the one with Y_{K_t}. However, one could always recover the constant one by setting $u_t = 1\ \forall t$.

of $\ln u_t = \ln Y_t - \ln Y_{K_t}$

$$\dot{u}_t = u_t \left(g_t - g_{K_t}\right) \tag{170}$$

Now, the last piece is the growth rate of output or demand g_t calculation. It comes from $Y_t = C_t + I_t + Z_t$ where Z_t is the autonomous consumption financed by credit. Consumption C_t is instead induced, i.e., $C_t = cY_t$ where $c = \varphi\omega$[23] is the marginal propensity to consume, φ is the marginal propensity to consume out of wages and ω is the wage share of output (constant, following Classical and Sraffian hypothesis), so

$$Y_t = \frac{Z_t}{1 - c - h_t} \tag{171}$$

where $(1 - c - h_t)^{-1}$ is the Sraffian Supermultiplier, which is a variable as h_t. Taking $\ln$ and time derivative of Eq(171), the growth rate of aggregate demand is

$$g_t = \frac{\dot{Z}_t}{Z_t} - \frac{0 - 0 - \dot{h}_t}{1 - c - h_t} = g_Z + \frac{h_t \gamma \left(u_t - u_n\right)}{1 - c - h_t} \tag{172}$$

where g_Z is a parameter, i.e., the growth rate of autonomous component of demand is exogenously given.

Serrano and Freitas firmly clarify the difference between $s = 1 - c$, which is the aggregate marginal propensity to save, and the actual saving ratio, the average propensity to save, $\frac{S_t}{Y_t} = s - \frac{Z_t}{Y_t} = sf_t = h_t$ where $f_t = \frac{I_t}{I_t + Z_t} = \frac{S_t}{Y_t} s^{-1}$ is called "the fraction". $c + h_t$ is the marginal propensity to spend and it's lower than one, the case $c + h_t = 1$ is the Say's Law, which is rejected.

Finally the $2D$ model is:

$$\begin{cases} \dot{u}_t = u_t \left[g_Z + \frac{\gamma h_t (u_t - u_n)}{1 - c - h_t} - \frac{h_t u_t}{\nu} + \delta \right] \\ \dot{h}_t = \gamma h_t \left(u_t - u_n\right) \end{cases} \tag{173}$$

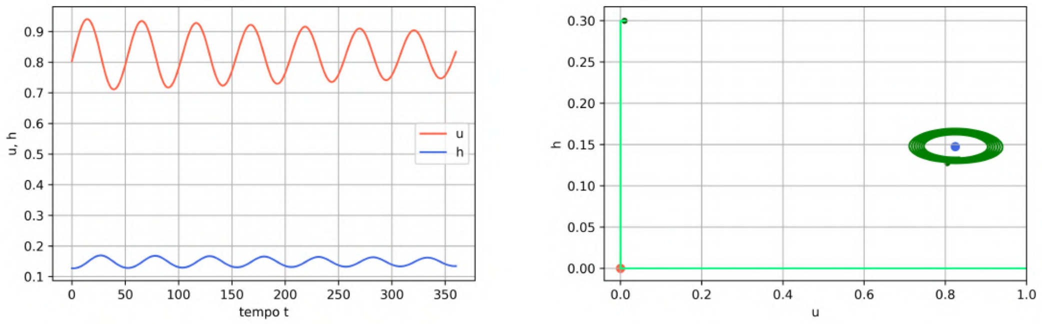
Regarding fixed points, the origin is obviously the trivial solution; the economically meaningful equilibrium is achieved by setting $u \neq 0$ and $g = g_K$: $\left(\bar{u}, \bar{h}\right) = \left(u_n, \frac{\nu(g_Z + \delta)}{u_n}\right)$. So at equilibrium $g = g_Z = g_K$. I will not study the stability conditions and the bifurcations here, I'll just say that, for parameters in Section 8.1 Appendix, topologically speaking, the point $(0, 0)$ is a saddle and $\left(\bar{u}, \bar{h}\right)$ is a stable focus (a.k.a. spiral).

From jacobian linearization, Authors find this condition for the stability

$$g_Z < g_{max} = \frac{s}{\nu} u_n - \delta - \gamma u_n \tag{174}$$

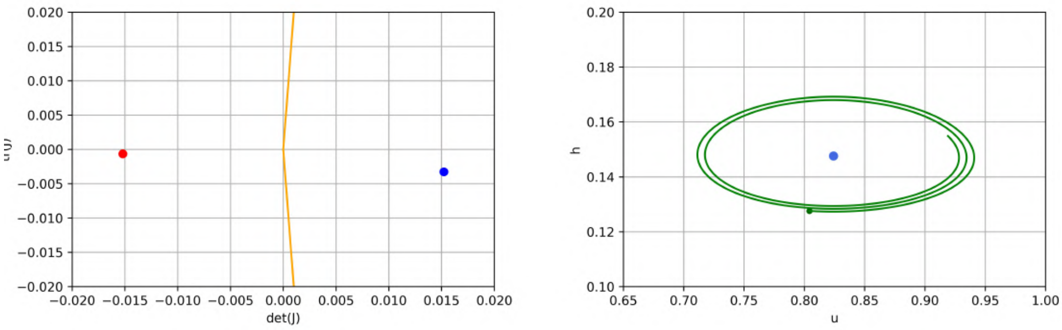
In Figure 40 and 41 you can see a phase portrait of the phase space and the really slow secular convergence to equilibrium stable point.

[23] $0 < c < 1$ since $0 < \varphi < 1$ and $0 < \omega < 1$.



(a) (b)

Figure 40: (a) the secular convergence to the spiral point and (b) the phase portrait with both fixed points, initial condition are also marked with smaller dot.



(a) (b)

Figure 41: (a) determinant Δ and trace τ of the jacobian calculated in the fixed points and in orange the function $\tau^2 - 4\Delta = 0$ which divides spirals from nodes for the economically meaningful equilibrium point (e.g. see Strogatz, 2024, Chapter 5) and (b) zoom of the focus point in Figure 40b.

5.3 The Morlin and Pariboni (2024) model

As an example of an extension of the previous model, again with a single component of autonomous demand, we study here the Morlin and Pariboni (2024) model. It's has some similarities with the Goodwin (1982) class struggle model.
The money wages w_t have their growth rate given by

$$g_{w_t} \equiv \frac{\dot{w}_t}{w_t} = \alpha_1 \pi_{int} + \alpha_2 \left(\omega_{W_t} - \omega_t\right) \tag{175}$$

This equation means that workers push for money wage increases to preserve their purchasing power after price increases, so the rate of growth of money wages depends on the inflation rate π_{in} since changes in the cost-of-living affect wage negotiations. This effect is nevertheless incomplete, so that $\alpha_1 < 1$ but positive. Workers also have an ideal wage share ω_W level in mind, which is their target, so they try to close the aspiration gap $\omega_W - \omega$ with the actual real wage share ω_t. However, workers' ability to meet their targets depends on their bargaining power: hence, α_2 is the sensitivity of the rate of change in money wages to the workers' aspiration gap.
Inflation rate is the rate of growth of prices p $\left(\pi_{in} \equiv g_p \equiv \frac{\dot{p}}{p}\right)$ and this process depends upon the capitalists' pricing decisions. So also capitalists have their aspiration gap with targeted wage share ω_K and coefficient λ_2. Moreover, capitalists partially pass through labour cost increases into final prices according to another coefficient λ_1, positive and smaller than one.

$$g_{p_t} \equiv \pi_{int} = \lambda_1 g_{w_t} + \lambda_2 \left(\omega_t - \omega_{K_t}\right) \tag{176}$$

The rate of growth of real wage is the difference between the two rates $g_{\omega_t} = g_{w_t} - g_{p_t}$. Putting Eq(175) in Eq(176) we have

$$\pi_{int} = \frac{\alpha_2 \lambda_1 \left(\omega_{W_t} - \omega_t\right) + \lambda_2 \left(\omega_t - \omega_{K_t}\right)}{1 - \alpha_1 \lambda_1}$$

and, finally, we find the first differential equation of the model

$$\begin{aligned} g_{\omega_t} \equiv \frac{\dot{\omega}_t}{\omega_t} = g_{w_t} - g_{p_t} = \pi_{int}\left(\alpha_1 - 1\right) + \alpha_2 \left(\omega_{W_t} - \omega_t\right) = \\ = \frac{\omega_{K_t}\left(1 - \alpha_1\right)\lambda_2 + \omega_{W_t}\left(1 - \lambda_1\right)\alpha_2 - \omega_t\left[\left(1 - \alpha_1\right)\lambda_2 + \left(1 - \lambda_1\right)\alpha_2\right]}{1 - \alpha_1 \lambda_1} \end{aligned} \tag{177}$$

Authors also endogenize workers' income claim: workers' target for the wage share depends on an autonomous component, expressing institutional and political factor θ_0 and a second term that expresses the effect of unemployment rate j

$$\omega_{W_t} = \theta_0 - \theta_1 j_t \tag{178}$$

The unemployment rate evolves according to the difference between the growth of the labour force g_N and the growth of the labour demand g_L. The latter is determined by the output growth rate, so $g_L = g$. Labour supply follows an exogenous demographic trend β_0 with an added endogenous component, according to the coefficient β_1, that captures the entrance into and exit from the labour force and migratory movements in time of low unemployment. So we have another linear relationship:

$$g_{N_t} = \beta_0 - \beta_1 j_t \tag{179}$$

As in (Freitas and Serrano, 2015) we have Eq(165), (171) and (170), but now the propensity to consume c can vary in time since an increase in the wage share leads workers to spend a greater portion of their income on consumption (it's a natural tendency to do so): $c_t = \varphi\omega_t$.
Using the same mathematical procedure as before, we obtain the analogue of Eq(172)

$$g_t = g_Z + \frac{\varphi\dot{\omega}_t + \gamma h_t \left(u_t - u_n\right)}{1 - \varphi\omega_t - h_t} \tag{180}$$

To obtain the rate of chance in the unemployment rate, we reason as follows. Since, N is the total population of workers, the labour force, and L is employed one, the unemployment rate j is $j = \frac{N-L}{N}$. Now taking, as usual, $\ln$ and time derivative of the employment rate $1 - j = \frac{L}{N}$ we find $\frac{0-\dot{j}}{1-j} = \frac{\dot{L}}{L} - \frac{\dot{N}}{N}$ which is the growth rate of j

$$g_{j_t} = \frac{1-j_t}{j_t}\left(g_{N_t} - g_{L_t}\right) = \frac{1-j_t}{j_t}\left[\beta_0 - \beta_1 j_t - g_Z - \frac{\varphi\dot{\omega}_t + \gamma h_t \left(u_t - u_n\right)}{1 - \varphi\omega_t - h_t}\right] \tag{181}$$

So our $4D$ system is:

$$\begin{cases} \dot{h}_t = \gamma h_t \left(u_t - u_n\right) \\ \dot{u}_t = u_t \left[g_Z + \frac{\varphi\dot{\omega}_t + \gamma h_t (u_t - u_n)}{1-\varphi\omega_t - h_t} - \frac{h_t u_t}{\nu} + \delta\right] \\ \dot{j}_t = \left(1 - j_t\right)\left[\beta_0 - \beta_1 j_t - g_Z - \frac{\varphi\dot{\omega}_t + \gamma h_t (u_t - u_n)}{1-\varphi\omega_t - h_t}\right] \\ \dot{\omega}_t = \frac{\omega_t \omega_{K_t}(1-\alpha_1)\lambda_2 + \omega_t \omega_{W_t}(1-\lambda_1)\alpha_2 - \omega_t^2[(1-\alpha_1)\lambda_2 + (1-\lambda_1)\alpha_2]}{1-\alpha_1\lambda_1} \end{cases} \tag{182}$$

This model combines conflict inflation (class struggle) and demand-led growth, following the supermultiplier approach: growth and distribution become interconnected processes linked through the unemployment rate. The equilibrium between inflation and distribution resulting from the conflict inflation implies an equilibrium wage share $\dot{\omega}_t = 0$ and the convergence to the growth rate of autonomous demand $g = g_Z$.
Beside the trivial solution $(0, 0, 1, 0)$, the economically meaningful equilibrium is

$$\left(\bar{h}, \bar{u}, \bar{j}, \bar{\omega}\right) = \left(\frac{\nu\left(g_Z + \delta\right)}{u_n}, u_n, \frac{\beta_0 - g_Z}{\beta_1}, \frac{\omega_K \lambda_2 \left(1 - \alpha_1\right) + \alpha_2 \left(1 - \lambda_1\right)\left(\theta_0 - \theta_1 \bar{j}\right)}{\lambda_2 \left(1 - \alpha_1\right) + \alpha_2 \left(1 - \lambda_1\right)}\right)$$

In equilibrium, the inflation rate is given by

$$\bar{\pi}_{in} = \frac{\alpha_2 \lambda_2 \left(\theta_0 - \theta_1 \bar{j} - \omega_K\right)}{\alpha_2 \left(1 - \lambda_1\right) + \lambda_2 \left(1 - \alpha_1\right)} \tag{183}$$

Authors find this stability condition:

$$\varphi \bar{\omega} + \bar{h} + \gamma \nu < 1 \tag{184}$$

To have an $\bar{j}$ within the valid range $1 < \bar{j} \leq 0$ we also need $\beta_0 \geq g_Z$ and $\beta_1 > \beta_0 - g_Z$.

5.4 The Morlin (2022) model

As an example of a two-component model of autonomous demand, we see Morlin's paper. He studied growth in an open economy with government with a Sraffian Supermultiplier model with two autonomous expenditures: exports and public expenditure. Both expenditures are autonomous once they are neither finances out of the current income nor directly caused by production decisions. Exports depend on foreign demand, not the domestic demand, and government counts with several degrees of freedom to run deficits and expand public expenditures independent of the current income and taxation. According to (Cesaratto, 2016b), this is particularly true for countries issuing public deb in their sovereign currency. So, now we have also the government sector G, export expenditure X and import M, hence Y equation here becomes

$$Y_t = C_t + I_t + G_t + X_t - M_t \tag{185}$$

To build the Supermultiplier we have to replace the equations for induced components, but the only differences compared to previous models are in $C_t = c\left(1 - \tau\right) Y_t$ and $M_t = mY_t$, where τ is a constant income tax rate, m is a constant propensity to import and the propensity to consume c is constant again. So Eq(171) here becomes:

$$Y_t = \frac{Z_t}{1 - c\left(1 - \tau\right) - h_t + m} \tag{186}$$

where $Z_t = X_t + G_t$.
Eq(165) and (170) are, once again, two of our differential equations. Taking $\ln$ and time derivative of Eq(186) we find the growth rate of aggregate demand

$$g_t = \frac{\gamma h_t \left(u_t - u_n\right)}{1 - c\left(1 - \tau\right) - h_t + m} + g_Z \tag{187}$$

Since Z is the sum of G and X, g_Z is given by the average of the growth rates of exports and public expenditures, weighted by the share of each expenditures on Z

$$g_Z = \sigma_t g_G + \left(1 - \sigma_t\right) g_X \tag{188}$$

where $\sigma_t = \frac{G_t}{Z_t}$. Taking time derivative of $\ln \sigma_t = \ln G_t - \ln Z_t$ we obtain $\dot{\sigma}_t = \sigma_t (g_G - g_Z)$. So σ, which is the share of government expenditure G in autonomous demand Z, changes whenever the growth rates of exports and public expenditure differ. Using Eq(188) we find the third differential equation of the model

$$\dot{\sigma}_t = \sigma_t (1 - \sigma_t)(g_G - g_X) \tag{189}$$

If public expenditure grows faster (slower) than exports, then σ_t continuously increases (decreases). So one of the expenditures keeps growing faster than the other, until σ converges to one of the extreme positions: either $\sigma = 0$ or $\sigma = 1$. In that case, one of the autonomous expenditures dominates the explanation of the growth of autonomous demand. However, as modern economies usually present positive demand from exports and public expenditures, Author argues that this condition does not usually hold. Taking this into account, a constant growth rate g_Z for the total autonomous demand can only be obtained if both expenditures constantly grow at the same rate, which is highly unlikely. So a stable fixed point will need $g_Z = g_G = g_X$.
A second alternative requires that the growth rate of each autonomous component varies in time, exactly compensating from movements in the other and from changes in σ to keep a constant value for g_Z. Anyway, Author does not explore this possibility.
Our $3D$ system is

$$\begin{cases} \dot{u}_t = u_t \left[\sigma_t g_G + (1 - \sigma_t) g_X + \frac{\gamma h_t (u_t - u_n)}{1 - c(1-\tau) - h_t + m} - \frac{h_t u_t}{\nu} + \delta \right] \\ \dot{h}_t = \gamma h_t (u_t - u_n) \\ \dot{\sigma}_t = \sigma_t (1 - \sigma_t)(g_G - g_X) \end{cases} \tag{190}$$

Beside the trivial solution, the economically meaningful one is

$$\left(\bar{u}, \bar{h}, \bar{\sigma}\right) = \left(u_n, \frac{\nu (g_Z + \delta)}{u_n}, \sigma\right)$$

Indeed, since $g_Z = g_G = g_X$, the equilibrium can be achieved at any level of σ.
The stability analysis (see Morlin (2022) Appendix) is expressed as follow:

$$\gamma\nu + c(1 - \tau) + \frac{\nu (g_Z + \delta)}{u_n} - m < 1 \tag{191}$$

Author concludes that the simple inclusion of another autonomous expenditure does not affect the local stability of the (Freitas and Serrano, 2015) supermultiplier growth model.
He also proposes an extension of his model from 3 to 5 equations. So let's add the public debt B which changes according to government expenditures G, taxes (proportional to the level of income Y) and the debt service iB, where i is the interest rate

$$\dot{B}_t = G_t - \tau Y_t + iB_t \tag{192}$$

The relevant economic variable for the analysis of public debt is the ratio between debt and income $b_t = \frac{B_t}{Y_t}$. Taking time derivative and using Eq(192), we have

$$\dot{b}_t = \sigma_t z_t - \tau + i b_t - g_t B_t \tag{193}$$

of course, $z_t = \frac{Z_t}{Y_t}$. The term $\sigma_t z_t - \tau$ corresponds to the primary government deficit (or surplus) to output ratio.
Moving on to the balance of payment, we assume that it's composed of the trade balance, factor income balance and capital account. The factor income balance is solely composed of external debt service payments $R_t = rD_t$, r is the international interest rate (it's fixed) and D the external debt. Hence, trade balance plus external debt service R constitute the current account and we assume that capital flows F are just sufficient to cover the deficit in the current account

$$F_t = M_t - X_t + R_t \tag{194}$$

Here the result of the balance of payment is equal to zero so that the country neither accumulate nor loses reserves.
The change in external debt will be given by net entrance of capital to finance the current account deficit, thus

$$\dot{D}_t = F_t = M_t - X_t + rD_t \tag{195}$$

Defining $d_t = \frac{D_t}{X_t}$, we have

$$\dot{d}_t = \frac{m}{(1-\sigma_t)\, z_t} - 1 + (r - g_X)\, d_t \tag{196}$$

Foreign debt stability requires an export growth rate g_X larger than the international interest rate r

$$r < g_X \tag{197}$$

Eq(196) of course shows that the path of foreign debt to exports ratio diverges as σ approaches 1: an increase share of government in autonomous demand implies an increase in imports, owing to the increase in domestic demand, without a counterpart in exports.
Now, our system becomes $5D$

$$\begin{cases} \dot{u}_t = u_t \left[\sigma_t g_G + (1-\sigma_t)\, g_X + \frac{\gamma h_t (u_t - u_n)}{1 - c(1-\tau) - h_t + m} - \frac{h_t u_t}{\nu} + \delta\right] \\ \dot{h}_t = \gamma h_t \left(u_t - u_n\right) \\ \dot{\sigma}_t = \sigma_t \left(1 - \sigma_t\right)\left(g_G - g_X\right) \\ \dot{b}_t = \sigma_t z_t - \tau + i b_t - g_t B_t \\ \dot{d}_t = \frac{m}{(1-\sigma_t) z_t} - 1 + (r - g_X)\, d_t \end{cases} \tag{198}$$

With these additions to the model, the equilibrium becomes

$$\left(\bar{u}, \bar{h}, \bar{\sigma}, \bar{b}, \bar{d}\right) = \left(u_n, \frac{\nu\left(g_Z+\delta\right)}{u_n}, \sigma, \frac{\sigma\bar{z}-\tau}{g_Z-i}, \left(\frac{m}{\left(1-\sigma\right)\bar{z}}-1\right)\left(\frac{1}{g_X-r}\right)\right)$$

where $\bar{z}$ is given by the inverse of Eq(186):

$$\bar{z} = 1 - c\left(1-\tau\right) - \frac{\nu\left(g_Z+\delta\right)}{u_n} + m$$

6 Adding inventory dynamics to the Sraffian Supermultiplier

6.1 Adding inventories

Now we will try to add inventory cycle to the Freitas and Serrano (2015) model previously seen (Section 5.2), but before we start, we would like to say that our attempt could not even exist without the work done by (Grasselli and Nguyen-Huu, 2018) (Section 4.2).

To avoid their problems, we took the road of simplification, but in addition, we decided to add the autonomous component of demand Z and to make their model compatible with (Freitas and Serrano, 2015): so, although we based our work on (Grasselli and Nguyen-Huu, 2018) paper for how to deal with inventories, our model, also, shows some own peculiarities.

Following Sraffian lines, we decided, in particular, to turn off the Goodwin engine and to keep ω[24] constant: so it will be a parameter exogenously determined and not a variable.

In the previous model, output produced Y and demand for goods $C+I+Z$, and therefore income Y, were equal by construction (i.e. $Y \equiv Y_d$ in Serrano-Freitas model), but the existence of inventories relegates this to a special case.

So the actual output produced by firms Y_t is assumed to consist of expected sales $Y_{e,t}$ plus planned inventory changes $I_{p,t}$

$$Y_t = Y_{e,t} + I_{p,t} \tag{199}$$

The total sales demand or effective demand $Y_{d,t}$ is

$$Y_{d,t} = C_t + I_t + Z_t \tag{200}$$

The difference between output and demand determines actual changes $\dot{V}_t$ in the level of inventory V_t held by firms/capitalists

$$\dot{V}_t = I_{p,t} + I_{u,t} = Y_t - Y_{d,t} \tag{201}$$

where $I_{u,t}$ are the unplanned changes in inventories (anyway it won't be important for the model). So here as accounting identity we have

$$Y_t = \dot{V}_t + C_t + I_t + Z_t \tag{202}$$

[24]we also set $\varphi = 1$, so workers spend their entire wage, and therefore $\omega = c = \frac{C_t}{Y_t}$.

and the Supermultiplier becomes

$$Y_t = \frac{\dot{V}_t + Z_t}{1 - \omega - h_t} = \frac{Z_t}{y_{d,t} - \omega - h_t} \tag{203}$$

where we define

$$y_{d,t} = \frac{Y_{d,t}}{Y_t} = \omega + h_t + z_t \tag{204}$$

z_t is simply the share of the autonomous component of the demand

$$z_t = \frac{Z_t}{Y_t} \tag{205}$$

and

$$y_{e,t} = \frac{Y_{e,t}}{Y_t} \tag{206}$$

Furthermore the capitalists are supposed to close the gap between the actual demand and the expected sales[25]

$$\dot{Y}_{e,t} = \eta_e \left(Y_{d,t} - Y_{e,t}\right) \tag{207}$$

for a constant $\eta_e \geq 0$ representing the speed of short-term adjustment to observed demand. As for the capacity utilization, we assume that firms aim to maintain inventories at a desired level

$$V_{d,t} = f_d Y_{e,t} \tag{208}$$

for a fixed proportion $0 \leq f_d \leq 1$.
We further assume that capitalists adjust their short-term expectations based on the observed level of inventory

$$I_{p,t} = \eta_d \left(V_{d,t} - V_t\right) \tag{209}$$

for a constant $\eta_d \geq 0$, representing the speed of short-term adjustments to observed inventory.
Since $\dot{V}_{d,t} = f_d \dot{Y}_{e,t}$ and $\dot{I}_{p,t} = \eta_d \left(\dot{V}_{d,t} - \dot{V}_t\right)$, using Eq(201) and (207), it gives

$$\frac{\dot{I}_{p,t}}{Y_t} = \eta_d f_d \frac{\dot{Y}_{e,t}}{Y_t} + \eta_d \left(y_{d,t} - 1\right)$$

[25]As we have seen, (Franke, 1996) and (Grasselli and Nguyen-Huu, 2018) added a term concerning the output growth rate to Eq(207) and to Eq(209). However, in Franke (1996) model this was a constant, while in Grasselli and Nguyen-Huu (2018) model there was a capitalists' expected growth rate $g_e\left(u, \pi_e\right)$, whose derivative, strangely enough, seemed to disappear in the calculations.
In our case, we could have included g_t or g_Z, but we preferred to take the path of simplicity and assume that capitalists have no expectations regarding growth.

Now we can't use Eq(203) to get g_t as in (Freitas and Serrano, 2015), since we would find an equation with $\ddot{V}_t$, but we can use Eq(199):

$$g_t = \frac{\dot{Y}_{e,t}}{Y_t} + \frac{\dot{I}_{p,t}}{Y_t} = \eta_e \left(1 + \eta_d f_d\right) \left(y_{d,t} - y_{e,t}\right) + \eta_d \left(y_{d,t} - 1\right) \tag{210}$$

Finally, calculating time derivative of $\ln z_t = \ln Z_t - \ln Y_t$ and of $\ln y_{e,t} = \ln Y_{e,t} - \ln Y_t$[26] we ends the model with four ODE system which, we must admit, is not much simpler than the $5D$ model of (Grasselli and Nguyen-Huu, 2018), despite the efforts made to simplify it:

$$\begin{cases} \dot{z}_t = z_t \left(g_Z - g_t\right) \\ \dot{y}_{e,t} = \eta_e \left(y_{d,t} - y_{e,t}\right) - g_t y_{e,t} \\ \dot{u}_t = u_t \left(g_t - \frac{h_t u_t}{\nu} + \delta\right) \\ \dot{h}_t = \gamma h_t \left(u_t - u_n\right) \\ g_t = \eta_e \left(1 + \eta_d f_d\right) \left(y_{d,t} - y_{e,t}\right) + \eta_d \left(y_{d,t} - 1\right) \\ y_{d,t} = \omega + h_t + z_t \end{cases} \tag{211}$$

6.2 Dimensionless group and fixed points

It's also possible reduce the number of parameters when the model is expressed in dimensionless form (e.g. see Boccara, 2010, pag.27,28).
In Section 8.1 Appendix we perform the dimensional analysis, but for what concerns us here, it is enough to know that all the variables of Eq(211) are already dimensionless by construction, except time.
To define the new time scale τ, we could use any parameter or combination of them, however to simplify the equations, it is better to use η_e, i.e., $\tau = t\eta_e$ and $\frac{d}{d\tau} = \frac{1}{\eta_e}\frac{d}{dt}$.
So Eq(211) system now becomes

$$\begin{cases} \frac{dz_\tau}{d\tau} = z_\tau \left(g_{eZ} - g_{e,\tau}\right) \\ \frac{dy_{e,\tau}}{d\tau} = \left(y_{d,\tau} - y_{e,\tau}\right) - g_{e,\tau} y_{e,\tau} \\ \frac{du_\tau}{d\tau} = u_\tau \left(g_{e,\tau} - \frac{h_\tau u_\tau}{\nu_e} + \delta_e\right) \\ \frac{dh_\tau}{d\tau} = \gamma_e h_\tau \left(u_\tau - u_n\right) \\ g_{e,\tau} = \frac{g_\tau}{\eta_e} = M_e \left(y_{d,\tau} - y_{e,\tau}\right) + \eta_{ed} \left(y_{d,\tau} - 1\right) \\ y_{d,\tau} = \omega + h_\tau + z_\tau \end{cases} \tag{212}$$

where we have defined $M_e = (1 + \eta_d f_d)$, $g_{eZ} = \frac{g_Z}{\eta_e}$, $\nu_e = \eta_e \nu$, $\delta_e = \frac{\delta}{\eta_e}$, $\gamma_e = \frac{\gamma}{\eta_e}$, $\eta_{ed} = \frac{\eta_d}{\eta_e}$.
With this choice we went from 9 to 7 parameters: ω, M_e, g_{eZ}, ν_e, δ_e, γ_e, η_{ed}.

[26] Beware that $Y_{e,t}$ and $y_{e,t}$ are unknown, but we know $Y_{e,t}$ definition and Eq(207). It's enough.

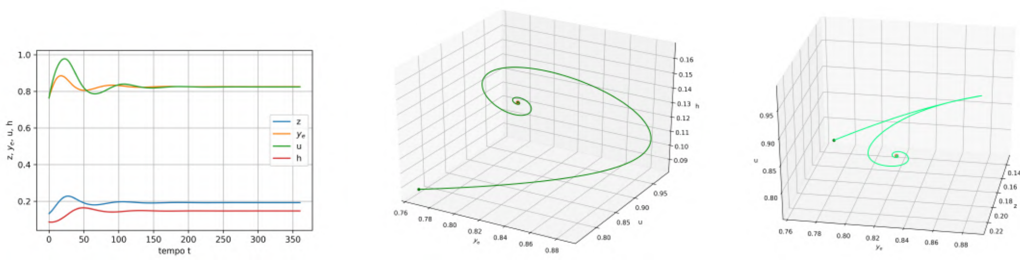
Later, we will also use the shorthand $M_{ed} = \eta_{ed} + M_e$.
The first fixed point requires, from $\frac{dz}{d\tau} = 0$ and $z \neq 0$, an equilibrium growth rate $\bar{g}_{e,1} = g_{eZ}$.
To have $\frac{dy_e}{d\tau} = 0$ and $y_e \neq 0$, it's needed $y_e = \frac{y_d}{1+g_e}$, so $y_e = y_d$ isn't a solution, as one could expect. $\bar{y}_e = \frac{\bar{y}_d}{1+\bar{g}_e}$ and $\bar{z} = \bar{y}_d - \bar{h} - \omega$, further $\frac{du}{d\tau} = 0$ and $u \neq 0$ requires $\bar{h} = \frac{\nu_e(g_e+\delta_e)}{u}$.
So we need only $\bar{y}_d$ that we can find solving $\bar{g}_e = g_{eZ}$ with the other equilibrium variables: $\bar{y}_{d,1} = \frac{(\eta_{ed}+g_{eZ})(1+g_{eZ})}{M_e g_{eZ}+\eta_{ed}(1+g_{eZ})}$.
The first fixed point is:

$$(\bar{z}_1, \bar{y}_{e,1}, \bar{u}_1, \bar{h}_1) = \left(\frac{(\eta_{ed} + g_{eZ})(1 + g_{eZ})}{M_e g_{eZ} + \eta_{ed}(1 + g_{eZ})} - \frac{\nu_e (g_{eZ} + \delta_e)}{u_n} - \omega, \frac{(\eta_{ed} + g_{eZ})(1 + g_{eZ})}{M_e g_{eZ} + \eta_{ed}(1 + g_{eZ})} \frac{1}{1 + g_{eZ}}, u_n, \frac{\nu_e (g_{eZ} + \delta_e)}{u_n} \right)$$

For the sake of completeness, in the case of the system in Eq(211), the point becomes

$$(\bar{z}_1, \bar{y}_{e,1}, \bar{u}_1, \bar{h}_1) = \left(\frac{(\eta_d + g_Z)(\eta_e + g_Z)}{g_Z \eta_e (1 + \eta_d f_d) + \eta_d (\eta_e + g_Z)} - \frac{\nu (g_Z + \delta)}{u_n} - \omega, \frac{(\eta_d + g_Z)(\eta_e + g_Z)}{g_Z \eta_e (1 + \eta_d f_d) + \eta_d (\eta_e + g_Z)} \frac{\eta_e}{\eta_e + g_Z}, u_n, \frac{\nu (g_Z + \delta)}{u_n} \right)$$

For the second fixed point we set $\bar{z}_2 = 0$, thus the reason for having $\bar{g}_e = g_{eZ}$ disap-



(a) (b) (c)

Figure 42: Numerical integration of the Eq(211) system with numerical values of Section 8.1 Appendix. (a) You can see the convergence to the fixed point $(\bar{z}_1, \bar{y}_{e,1}, \bar{u}_1, \bar{h}_1)$, the economically meaningful equilibrium point and (b) the same dynamics but in the phase space and (c) the phase portrait with z, instead of h. Since the system is $4D$, obviously, we can't represent it in only one $3D$ graph, so it is split in two.

pears, this needs to solve a quadratic equation for $\bar{g}_{e,2}$, i.e. $\bar{g}_{e,2}^2 O + \bar{g}_{e,2} P + Q = 0$[27] but, unfortunately, it has not solutions in $\mathbb{R}$ since the discriminant is negative.
Finally, for the third fixed point, we have $\bar{h}_3 = \bar{u}_3 = \bar{z}_3 = 0$, $\bar{y}_{d,3} = \omega$ and $\bar{y}_{e,3} = \frac{\bar{y}_{d,3}}{1+\bar{g}_{e,3}}$, replacing in Eq(210) divided by η_e, it gives another quadratic equation

$$\bar{g}_{e,3}^2 + \bar{g}_{e,3}\left[1 - \omega M_e + \eta_{ed}(1-\omega)\right] + \eta_{ed}(1-\omega) = 0$$

[27] It has not solution for the parameters in Section 8.1 Appendix, anyway it isn't easy to study. $O = 1 - \frac{\nu_e}{u_n}(\eta_{ed} + M_e)$, $P = 1 + \eta_{ed} - \left(\omega + \frac{\delta_e \nu_e}{u_n}\right)(\eta_{ed} + M_e) - \frac{\nu_e \eta_{ed}}{u_n}$, $Q = \eta_{ed}\left(1 - \omega - \frac{\delta_e \nu_e}{u_n}\right)$.

without real solutions.
So also the third fixed point doesn't exist for these values.
Moving on the jacobian matrix is

$$J\left(\bar{z},\bar{y}_e,\bar{u},\bar{h}\right)=\begin{bmatrix} g_{eZ}-\bar{g}_e-\bar{z}\left(\eta_{ed}+M_e\right) & \bar{z}M_e & 0 & -\bar{z}\left(\eta_{ed}+M_e\right)\\ 1-\bar{y}_e\left(\eta_{ed}+M_e\right) & \bar{y}_eM_e-1-\bar{g}_e & 0 & 1-\bar{y}_e\left(\eta_{ed}+M_e\right)\\ \bar{u}\left(\eta_{ed}+M_e\right) & -\bar{u}M_e & \bar{g}_e-2\frac{\bar{h}\bar{u}}{\nu_e}+\delta_e & \bar{u}\left(\eta_{ed}+M_e\right)-\frac{\bar{u}^2}{\nu_e}\\ 0 & 0 & \gamma_e\bar{h} & \gamma_e\left(\bar{u}-u_n\right)\end{bmatrix} \tag{213}$$

and in the economically meaningful equilibrium point

$$J\left(\bar{z}_1,\bar{y}_{e,1},\bar{u}_1,\bar{h}_1\right)=\begin{bmatrix} -\bar{z}_1\left(\eta_{ed}+M_e\right) & \bar{z}_1M_e & 0 & -\bar{z}_1\left(\eta_{ed}+M_e\right)\\ 1-\bar{y}_{e,1}\left(\eta_{ed}+M_e\right) & \bar{y}_{e,1}M_e-1-g_{eZ} & 0 & 1-\bar{y}_{e,1}\left(\eta_{ed}+M_e\right)\\ u_n\left(\eta_{ed}+M_e\right) & -u_nM_e & g_{eZ}-2\frac{\bar{h}_1u_n}{\nu_e}+\delta_e & u_n\left(\eta_{ed}+M_e\right)-\frac{u_n^2}{\nu_e}\\ 0 & 0 & \gamma_e\bar{h}_1 & 0\end{bmatrix} \tag{214}$$

The determinant of this matrix implies

$$0=(-1)^{4+3}\gamma_e\bar{h}_1\det\begin{bmatrix} -\bar{z}_1\left(\eta_{ed}+M_e\right)-\lambda & \bar{z}_1M_e & -\bar{z}_1\left(\eta_{ed}+M_e\right)\\ 1-\bar{y}_{e,1}\left(\eta_{ed}+M_e\right) & \bar{y}_{e,1}M_e-1-g_{eZ}-\lambda & 1-\bar{y}_{e,1}\left(\eta_{ed}+M_e\right)\\ u_n\left(\eta_{ed}+M_e\right) & -u_nM_e & u_n\left(\eta_{ed}+M_e\right)-\frac{u_n^2}{\nu_e}\end{bmatrix}+$$
$$+(-1)^{4+4}(-\lambda)\det\begin{bmatrix} -\bar{z}_1\left(\eta_{ed}+M_e\right)-\lambda & \bar{z}_1M_e & 0\\ 1-\bar{y}_{e,1}\left(\eta_{ed}+M_e\right) & \bar{y}_{e,1}M_e-1-g_{eZ}-\lambda & 0\\ u_n\left(\eta_{ed}+M_e\right) & -u_nM_e & g_{eZ}-2\frac{\bar{h}_1u_n}{\nu_e}+\delta_e-\lambda\end{bmatrix}$$

The eigenvalue equation for the first fixed point gives:

$$\begin{aligned}0=\lambda^4&+\lambda^3\left[2\frac{\bar{h}_1u_n}{\nu_e}-\delta_e+1-\bar{y}_{e,1}M_e+\bar{z}_1M_{ed}\right]+\lambda^2\left[\left(\bar{y}_{e,1}M_e-1-g_{eZ}\right)\right.\\ &\left(-2\frac{\bar{h}_1u_n}{\nu_e}+g_{eZ}+\delta_e\right)-\bar{z}_1M_{ed}\left(-2\frac{\bar{h}_1u_n}{\nu_e}+g_{eZ}+\delta_e\right)+\bar{z}_1M_{ed}\left(1+g_{eZ}\right)-\\ &\left.-\bar{z}_1M_e+\gamma_e\frac{\bar{h}_1u_n^2}{\nu_e}-u_n\bar{h}_1\gamma_eM_{ed}\right]+\lambda\left[\gamma_e\frac{\bar{h}_1u_n^2}{\nu_e}\left(1+g_{eZ}-\bar{y}_{e,1}M_e\right)-\right.\\ &-\gamma_e\bar{h}_1u_nM_{ed}\left(1+g_{eZ}\right)+\gamma_e\frac{\bar{h}_1u_n^2}{\nu_e}\bar{z}_1M_{ed}+\gamma_e\bar{h}_1u_nM_e-\left(\bar{z}_1\eta_{ed}+\bar{z}_1M_{ed}g_{eZ}\right)\\ &\left.\left(-2\frac{\bar{h}_1u_n}{\nu_e}+g_{eZ}+\delta_e\right)\right]+\gamma_e\frac{\bar{h}_1u_n^2}{\nu_e}\bar{z}_1\left[M_{ed}\left(1+g_{eZ}\right)-M_e\right]\end{aligned}$$

Routh-Hurwitz stability criterion for a fourth-order polynomial

$$P(\lambda)=a_4\lambda^4+a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0=0$$

requires all coefficients must be positive, i.e., $a_i > 0$ for all i, $0 < a_2 a_1 - a_3 a_0$ and $0 < a_3 a_2 a_1 - a_4 a_1^2 - a_3^2 a_0$.
Unfortunately, such conditions are not simple to check, here.
Moving on to numerical simulations, the four eigenvalues are:

$$(-0.024 + i0.078, -0.024 - i0.078, -0.098 + i0.074, -0.098 - i0.074)$$

They all have negative real part so the fixed point is a stable economically meaningful equilibrium point.
Moreover, the fact that two eigenvalues are complex conjugate suggests that it could be a focus, e.g., a fact that seems to be confirmed by the Figure 42.

7 Conclusions and further developments

In this thesis, we discussed some of the most important models of the Heterodox Schools. In particular, we focused on endogenous growth models.
Then, We moved on to the main topic: the version of the Keynesian Multiplier proposed by the School founded by Piero Sraffa. In particular, the recent version of the Sraffian Supermultiplier was discussed. The research proposal for this thesis was to add the inventory dynamics to the Sraffian Supermultiplier.
As we have just seen, the model seems promising, however it presents mathematical complications. For this reason, it seems unwise to further complicate an already complicated model, nevertheless, one can easily imagine additions to the model, which we will discuss in this section.

7.1 Class struggle, workers' bargaining power and Goodwin cycles

Just as Morlin and Pariboni (2024) studied conflict inflation and endogenous distribution, even though they are Sraffian authors; one could reactivate here the class struggle Goodwin engine that we had decided to ignore before.
Goodwin (1982), and Keen (1995) and Grasselli and Nguyen-Huu (2018) following him, model their investment function and bargaining power on the disciplinary role of unemployment and the "scarcity of hands": here ω becomes a variable and $\dot{\omega}_t$ depends on λ_t, so when the employment rate increases, also the strength of the workers increases and therefore the wage claim, but capitalists react to the decrease of their profits $(1-\omega_t)$ cutting investment and therefore the employment. Grasselli and Nguyen-Huu (2018) don't use h variable, but in our case, Eq(165) should also depend on the profit share $\pi_t = 1-\omega_t$ since $\dot{h}_t$ must decrease. The price to pay would be to add two more equations to the model, since we need the employment rate λ_t and the wage share ω_t.

7.2 Labor force

The issue here is that in Goodwin (1982) and other related papers, there isn't a limit for $\dot{\lambda}_t$ and Y_t, so $\lambda_t > 1$ and $Y_t > Y_{max} = aN$[28] are potential outcomes of the model. They use $\frac{\dot{a}_t}{a_t} = \alpha$ and $\frac{\dot{N}_t}{N_t} = \beta$ so both grow exponentially fast: this often is enough to avoid issues. For us, the problem still remains[29].

[28]Here N is the total population and a the labour productivity. N is assumed fixed, so the population doesn't grow in our model, and so does α.

[29]We don't find very interesting to study the dynamics if there is exponential growth, plus it must be added that in the real world countries have problems with both productivity and population growth, e.g., $\alpha = \beta \approx 0$.
Someone could use something like $Y_t = \min(aL_t, aN)$ where L_t is the number of workers employed

A solution could be to use the logistic equation with carrying capacity aN, i.e., $\frac{\dot{Y}_t}{Y_t} = g_t\left(1-\frac{Y_t}{aN}\right) = g_t\left(1-\lambda\right) = \frac{\dot{\lambda}_t}{\lambda_t}$. But here an issue arises with the Sraffian Supermultiplier: if g_Z is given, Z_t always increases, so when output will reach its aN limit, we think that Z_t to continuing to grow, it will absorb all the other components and, eventually, the model will stop making sense. Again, one would try $\frac{\dot{Z}_t}{Z_t} = g_Z\left(1-\frac{Z_t}{aN}\right)$, although, it must be said, there is no real economical reason to have such a behaviour. Maybe that's why Serrano and Freitas don't consider the employment rate: it seems to us that exogenous g_z stops making sense in conditions of full employment. Anyway, continuing to follow this proposal, then one should derive g_t from these equations and the Freitas and Serrano (2015) model would become this $4D$ model:

$$\begin{cases} \dot{\lambda}_t = g_t\lambda\left(1-\lambda\right) \\ \dot{z}_t = z_t\left[g_Z - g_t + \lambda_t\left(g_t - g_Z z_t\right)\right] \\ \dot{u}_t = u_t\left[g_t\left(1-\lambda_t\right) - \frac{h_t u_t}{\nu} + \delta\right] \\ \dot{h}_t = \gamma h_t\left(u_t - u_n\right) \\ g_t = \frac{1}{1-\lambda_t}\left[g_Z\left(1-z_t\lambda_t\right) + \frac{\gamma h_t\left(u_t-u_n\right)}{1-c-h_t}\right] \end{cases} \tag{215}$$

at time t, but such approach complicates the derivatives for the jacobian matrix.

8 Appendix

8.1 Stocks and Flows. Dimensional analysis, benchmark parameters

A flow is how much a stock varies in time. Since, in this work, we have defined quantity as a number (i.d. how many goods the output is made of[30]) they are dimensionless.
So stocks should be dimensionless, while flows have dimensions of $[Time]^{-1}$.
Of course, one must also choose the time scale and define the size of the model parameters accordingly. One could choose, for example, quarters or years, we decided to choose year as time scale.
K_t, V_t and $V_{d,t}$ are stocks; I_t, Y_t, $Y_{K,t}$, C_t, Z_t, $Y_{d,t}$, $Y_{e,t}$, $I_{p,t}$, $\dot{V}_t$ are flows.
Time derivative of a flow has dimension $[Time]^{-2}$, e.g. acceleration in Physics.
Growth rates are $[Time]^{-1}$ since a growth rate is a ratio between the derivative of a flow and the flow itself.
Ratios are dimensionless, e.g. h_t is a ratio between two flows.
In this model we have 9 parameters. The accelerator ν is a $[Time]$ since it's a ratio between a stock and a flow. f_d is also a $[Time]$ since Eq(208) is an equality between a stock and a flow, so f_d must transform a flow in a stock. All other parameters are $[Time]^{-1}$, e.g. δ is the inverse of a time since in Eq(164) $\dot{K}_t$ and I_t are flows and so the stock of capital K_t must become a flow. A summary can be found in the Table 1.

8.2 Calculations for Goodwin model with Say's Law

The jacobian matrix for the Goodwin model with Say's Law is

$$J(\omega,\lambda) = \begin{bmatrix} w[\lambda]-\alpha & \omega w'[\lambda] \\ -\frac{\lambda}{\nu_G} & \frac{1-\omega}{\nu_G}-\alpha-\beta-\delta \end{bmatrix} \tag{216}$$

where the derivative of the function with respect to employment is $w'[\lambda] = \frac{2AC}{(B-C\lambda)^3}$.
We introduce the notation $w_0 = w[0] = \frac{A}{B^2} - D < 0$, where $w_0 = -0.04$ depends solely on the value of the parameters chosen for the model (we said that workers were willing to accept a maximum reduction in their wages of 4%).
For the first fixed point, we obtain

$$J_0(0,0) = \begin{bmatrix} w_0-\alpha & 0 \\ 0 & \frac{1}{\nu_G}-\alpha-\beta-\delta \end{bmatrix} = \begin{bmatrix} -0.055 & 0 \\ 0 & 0.26\bar{3} \end{bmatrix} \tag{217}$$

[30] Since we did not want to make a particular choice of the representative good in my model. But, for example, in the corn metaphor, the output could be composed of a number of grains, or, more likely, a unit of measurement of the mass, e.g. in tons or kg. However, now, we prefer to keep this quantity dimensionless, so it will be, simply, a number.

Symbol	Status	Dimension	Benchmark value or initial condition
γ	parameter	$[Time]^{-1}$	0.1500
u_n	parameter	$[Time]^{-1}$	0.8242
ν	parameter	$[Time]$	0.9890
δ	parameter	$[Time]^{-1}$	0.0840
ω	parameter	/	0.7000
g_Z	parameter	$[Time]^{-1}$	0.0390
η_e	parameter	$[Time]^{-1}$	0.1500
η_d	parameter	$[Time]^{-1}$	0.1500
f_d	parameter	$[Time]$	0.1500
z_t	variable	/	0.1329
$y_{e,t}$	variable	/	0.7658
u_t	variable	/	0.7642
h_t	variable	/	0.0876

Table 1: Summary of the main variables and parameters. There are also the benchmark values for the parameters and the initial conditions for the variable. Some values come from Di Bucchianico et al. (2024).

where $detJ_0 = -0.014 < 0$ and $trJ_0 = 0.208 > 0$ and since the eigenvalues are one positive and the other negative, the origin is a saddle point.
Instead, for the second fixed point

$$J_1\left(\bar{\omega}_1, \bar{\lambda}_1\right) = \begin{bmatrix} 0 & \bar{\omega}_1 w'\left[\bar{\lambda}_1\right] \\ -\frac{\bar{\lambda}_1}{\nu_G} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2.550 \\ -0.322 & 0 \end{bmatrix} \tag{218}$$

where $detJ_1 = 0.821 > 0$ and obviously $trJ_1 = 0$, the eigenvalues are purely imaginary complex conjugates $\lambda_{1,2} = \pm 0.906i$. The fixed point, having a real part of the eigenvalues equal to zero, is a non-hyperbolic point and is a center: therefore, for initial conditions different from it, cycles will form around it.

8.3 Calculations for Goodwin model without Say's Law

The jacobian matrix of Goodwin model in the case of the investment function is

$$J(\omega, \lambda) = \begin{bmatrix} w[\lambda] - \alpha & \omega w'[\lambda] \\ \frac{\lambda}{\nu_G} k'\left[\frac{\pi}{\nu_G}\right] & \frac{k\left[\frac{\pi}{\nu_G}\right]}{\nu_G} - \alpha - \beta - \delta \end{bmatrix} = \begin{bmatrix} w[\lambda] - \alpha & \omega \frac{2AC}{(B-C\lambda)^3} \\ -\frac{\lambda}{\nu_G} \frac{\frac{2EG}{\nu_G}}{(F - G\frac{1-\omega}{\nu_G})^3} & \frac{k\left[\frac{1-\omega}{\nu_G}\right]}{\nu_G} - \alpha - \beta - \delta \end{bmatrix} \tag{219}$$

In the same way, let us now introduce the notation $k_m = k\left[\frac{\bar{\pi}_{max}}{\nu_G}\right] = k\left[\frac{1-0}{\nu_G}\right] = \frac{E}{(F - G\frac{1}{\nu_G})^2} - H = -0.057 < 0$.

For the fixed point $\left(\bar{\omega}_0, \bar{\lambda}_0\right)$ we have

$$J_0\left(0,0\right) = \begin{bmatrix} w_0 - \alpha & 0 \\ 0 & \frac{k_m}{\nu_G} - \alpha - \beta - \delta \end{bmatrix} = \begin{bmatrix} -0.055 & 0 \\ 0 & -0.089 \end{bmatrix} \tag{220}$$

where $detJ_0 = 0.005 > 0$ e $trJ_0 = -0.144 < 0$, unlike the previous case, now both eigenvalues are negative, so the origin is a stable fixed point. This is due to the presence of the inversion function, particularly in the second eigenvalue: since $k_m < 0$ due to the chosen parameters, also $\frac{k_m}{\nu_G} - \alpha - \beta - \delta < 0$.
Instead, for $\left(\bar{\omega}_{1**}, \bar{\lambda}_1\right)$ we have

$$J_1\left(\bar{\omega}_{1**}, \bar{\lambda}_1\right) = \begin{bmatrix} 0 & \bar{\omega}_{1**} w'\left[\bar{\lambda}_1\right] \\ \frac{\bar{\lambda}_1}{\nu_G} k'\left[\frac{1-\bar{\omega}_{1**}}{\nu_G}\right] & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1.965 \\ 1.404 & 0 \end{bmatrix} \tag{221}$$

It has $detJ_1 = -2.759 < 0$ and $trJ_1 = 0$, the eigenvalues are real with opposite signs $\lambda_{1,2} = \pm 1.661$, therefore, the fixed point is a saddle.
Finally, for the fixed point $\left(\bar{\omega}_{1*}, \bar{\lambda}_1\right)$

$$J_2\left(\bar{\omega}_{1*}, \bar{\lambda}_1\right) = \begin{bmatrix} 0 & \bar{\omega}_{1*} w'\left[\bar{\lambda}_1\right] \\ \frac{\bar{\lambda}_1}{\nu_G} k'\left[\frac{1-\bar{\omega}_{1*}}{\nu_G}\right] & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2.779 \\ -1.404 & 0 \end{bmatrix} \tag{222}$$

It has $detJ_2 = 3.902 > 0$ e $trJ_2 = 0$, the eigenvalues are purely imaginary complex conjugates $\lambda_{1,2} = \pm 1.975i$. Since the real part of the eigenvalues is zero, the fixed point is a non-hyperbolic point and is a center.

8.4 Calculations for Keen model

The jacobian matrix of the Keen model is

$$J\left(\omega, \lambda, d\right) = \begin{bmatrix} w\left[\lambda\right] - \alpha & \omega\left(\frac{2AC}{(B-C\lambda)^3}\right) & 0 \\ -\frac{\lambda}{\nu_G}\left(\frac{\frac{2EG}{\nu_G}}{\left(F-G\frac{\pi}{\nu_G}\right)^3}\right) & \frac{k\left[\frac{\pi}{\nu_G}\right]}{\nu_G} - \alpha - \beta - \delta & -\frac{\lambda}{\nu_G}\left(\frac{r\frac{2EG}{\nu_G}}{\left(F-G\frac{\pi}{\nu_G}\right)^3}\right) \\ 1 - \frac{\nu_G - d}{\nu_G}\left(\frac{\frac{2EG}{\nu_G}}{\left(F-G\frac{\pi}{\nu_G}\right)^3}\right) & 0 & 2r - \left(\frac{k\left[\frac{\pi}{\nu_G}\right]}{\nu_G} - \delta\right) - \frac{\nu_G - d}{\nu_G}\left(\frac{r\frac{2EG}{\nu_G}}{\left(F-G\frac{\pi}{\nu_G}\right)^3}\right) \end{bmatrix} \tag{223}$$

Let us, therefore, introduce a notation that will facilitate our subsequent calculations:

$$\left.\frac{\partial\dot{\omega}}{\partial\lambda}\right|_{\left(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1\right)} = \bar{\omega}_1\left(\frac{2AC}{\left(B-C\bar{\lambda}_1\right)^3}\right) = K_0$$

$$\left.\frac{\partial\dot{\lambda}}{\partial\omega}\right|_{\left(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1\right)} = -\frac{\bar{\lambda}_1}{\nu_G}\left(\frac{\frac{2EG}{\nu_G}}{\left(F-G\frac{\bar{\pi}_1}{\nu_G}\right)^3}\right) = -K_1$$

$$\left.\frac{\partial\dot{\lambda}}{\partial d}\right|_{\left(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1\right)} = -\frac{\bar{\lambda}_1}{\nu_G}\left(\frac{r\frac{2EG}{\nu_G}}{\left(F-G\frac{\bar{\pi}_1}{\nu_G}\right)^3}\right) = -rK_1$$

$\frac{\partial \dot{d}}{\partial \omega}\Big|_{\left(\bar{\omega}_1,\bar{\lambda}_1,\bar{d}_1\right)} = 1 - \frac{\nu_G - \bar{d}_1}{\nu_G}\left(\frac{\frac{2EG}{\nu_G}}{\left(F - G\frac{\bar{\pi}_1}{\nu_G}\right)^3}\right) = K_2$

$\frac{\partial \dot{d}}{\partial d}\Big|_{\left(\bar{\omega}_1,\bar{\lambda}_1,\bar{d}_1\right)} = 2r - \left(\frac{k\left[\frac{\bar{\pi}_1}{\nu_G}\right]}{\nu_G} - \delta\right) - \frac{\nu_G - \bar{d}_1}{\nu_G}\left(\frac{r\frac{2EG}{\nu_G}}{\left(F - G\frac{\bar{\pi}_1}{\nu_G}\right)^3}\right) = r + rK_2 - (\alpha + \beta)$

where $K_0 > 0$ and $K_1 > 0$.
So, for the point $\left(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1\right)$ we have

$$J\left(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1\right) = \begin{bmatrix} 0 & K_0 & 0 \\ -K_1 & 0 & -rK_1 \\ K_2 & 0 & r + rK_2 - (\alpha+\beta) \end{bmatrix} \tag{224}$$

Moving on to the characteristic equation

$$\lambda^3 + \left[(\alpha+\beta) - r - rK_2\right]\lambda^2 + K_0K_1\lambda + (\alpha+\beta)K_0K_1 - rK_0K_1 = 0$$

according to the Routh-Hurwitz criterion, see (Grasselli and Costa Lima, 2012), to have negative real parts of the all roots of a cubic polynomial of the form $p(y) = a_3y^3 + a_2y^2 + a_1y + a_0$ it's sufficient that $a_n > 0\,\forall n$ and that $a_2a_1 > a_3a_0$.
We therefore have the following conditions:
$\alpha + \beta - r - rK_2 > 0$
$K_0K_1 > 0$
$K_0K_1(\alpha + \beta - r) > 0$
$K_0K_1(\alpha + \beta - r - rK_2) > K_0K_1(\alpha + \beta - r)$
In our case, we already know that $K_0 > 0$ and $K_1 > 0$ and that $\alpha > 0$ and $\beta > 0$, so the second condition is automatically verified. Furthermore, the first condition requires $\alpha + \beta > r(1 + K_2)$ to be satisfied, and the third requires $\alpha + \beta > r$. For the fourth condition, however, it is sufficient that

$$rK_2 < 0$$

Expressing, $rK_2 < 0$ we have $r - r\frac{\nu_G - \bar{d}_1}{\nu_G}\left(\frac{\frac{2EG}{\nu_G}}{\left(F - G\frac{\bar{\pi}_1}{\nu_G}\right)^3}\right) < 0$

that is $r\left[\left(\nu_G - \bar{d}_1\right)\left(\frac{\frac{2EG}{\nu_G}}{\left(F - G\frac{\bar{\pi}_1}{\nu_G}\right)^3}\right) - \nu_G\right] > 0$

and replacing $\bar{d}_1$ as well, we finally get

$$r\left[\left(\nu_G - \frac{\bar{\pi}_1 - \nu_G\left(\alpha+\beta\right)}{r - (\alpha+\beta)}\right)\left(\frac{\frac{2EG}{\nu_G}}{\left(F - G\frac{\bar{\pi}_1}{\nu_G}\right)^3}\right) - \nu_G\right] > 0$$

If this and the other conditions stated above hold, the fixed point $\left(\bar{\omega}_1, \bar{\lambda}_1, \bar{d}_1\right)$ is stable.

Instead, to study the stability of $\left(\bar{\omega}_2, \bar{\lambda}_2, \bar{d}_2\right) = (0, 0, \pm\infty)$ we must perform the coordinate change $d = \frac{1}{u}$ so that u tends to zero when d tends to infinity and therefore the fixed point of interest becomes $\left(\bar{\omega}_2, \bar{\lambda}_2, \bar{u}_2\right) = (0, 0, 0)$. We then rewrite the system of Eq(61)

$$\begin{cases} \dot{\omega} = \omega\left(w\left[\lambda\right] - \alpha\right) \\ \dot{\lambda} = \lambda\left(\frac{k\left[\frac{1-\omega-\frac{r}{u}}{\nu_G}\right]}{\nu_G} - \alpha - \beta - \delta\right) \\ \dot{u} = u^2 - \omega u^2 - 2ru - \left(\nu_G u^2 - u\right)\left(\frac{k\left[\frac{1-\omega-\frac{r}{u}}{\nu_G}\right]}{\nu_G} - \delta\right) \end{cases} \tag{225}$$

It is easy to verify that $\left(\bar{\omega}_2, \bar{\lambda}_2, \bar{u}_2\right)$ is a fixed point.
Let's move on to the jacobian matrix

$$J\left(\omega, \lambda, u\right) = \begin{bmatrix} w\left[\lambda\right] - \alpha & \omega\left(\frac{2AC}{(B-C\lambda)^3}\right) & 0 \\ -\frac{\lambda}{\nu_G}\left(\frac{\frac{2EG}{\nu_G}}{\left(F-G\frac{\pi}{\nu_G}\right)^3}\right) & \frac{k\left[\frac{\pi}{\nu_G}\right]}{\nu_G} - \alpha - \beta - \delta & \frac{r\lambda}{\nu_G u^2}\left(\frac{\frac{2EG}{\nu_G}}{\left(F-G\frac{\pi}{\nu_G}\right)^3}\right) \\ \frac{\nu_G u^2 - u}{\nu_G}\left(\frac{\frac{2EG}{\nu_G}}{\left(F-G\frac{\pi}{\nu_G}\right)^3}\right) - u^2 & 0 & U \end{bmatrix} \tag{226}$$

dove $U = 2u - 2\omega u - 2r - \left(2\nu_G u - 1\right)\left(\frac{k\left[\frac{\pi}{\nu_G}\right]}{\nu_G} - \delta\right) - \frac{\nu_G u^2 - u}{\nu_G}\left(\frac{\frac{r}{u^2}\frac{2EG}{\nu_G}}{\left(F-G\frac{\pi}{\nu_G}\right)^3}\right)$.
In the case of the fixed point $\left(\bar{\omega}_2, \bar{\lambda}_2, \bar{u}_2\right) = (0, 0, 0)$ it becomes

$$J\left(\bar{\omega}_2, \bar{\lambda}_2, \bar{u}_2\right) = \begin{bmatrix} w_0 - \alpha & 0 & 0 \\ 0 & \frac{k_0 - \nu_G(\alpha+\beta+\delta)}{\nu_G} & 0 \\ 0 & 0 & \frac{k_0 - \nu_G(2r+\delta)}{\nu_G} \end{bmatrix} \tag{227}$$

where it was necessary to introduce the notation

$$k_0 = \lim_{\pi \to -\infty} k\left[\frac{\pi}{\nu_G}\right] = -H$$

since $\pi = 1 - \omega - \frac{r}{u}$ tends to $-\infty$ when $u \to 0$, as in the case of this fixed point. The equilibrium is stable if the eigenvalues are all negative, i.e. if The equilibrium is stable if the eigenvalues are all negative, i.e. if
$w_0 < \alpha$
$k_0 < \nu_G\left(\alpha + \beta + \delta\right)$
$k_0 < \nu_G\left(2r + \delta\right)$
The first two conditions are automatically satisfied by the choice of parameters, while the third implies

$$r > \frac{1}{2}\left(\frac{k_0}{\nu_G} - \delta\right)$$

or, in our model, $r > -0.021$, which is not a limiting condition at all considering a positive interest rate.
Consequently this fixed point is stable and competes with the one seen previously.

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