ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

DEPARTMENT OF PHYSICS AND ASTRONOMY "A. RIGHI"

SECOND CYCLE DEGREE

PHYSICS

Charged massive vector bosons from the worldline

Supervisor
Prof. Fiorenzo Bastianelli

Co-supervisor
Dr. Filippo Fecit

Defended by
Alessandro Miccichè

Autumn Session / September/ 2025

Academic Year 2024/2025

Abstract

A worldline formulation for charged, massive spin-1 particle is presented through two distinct models. First, a model with bosonic oscillators on the worldline is considered. We extend it to describe massive integer spin particles and, via both Dirac and BRST quantization, the free Proca field theory is reproduced in the spin-1 sector. The coupling of the model to an external electromagnetic field is consistent only for on-shell backgrounds, as determined by the nilpotency of the BRST charge. For such configurations, we perform a path integral quantization of the worldline action for the charged spin-1 particle on the circle. This yields the one-loop effective Lagrangian for a constant electromagnetic field induced by a charged massive vector boson. From the Lagrangian, we quantify vacuum instability by computing the pair production rate for massive vector bosons. Our results confirm previous findings obtained in quantum field theory. Finally, for comparison, we repeat the analysis using the standard $N = 2$ spinning particle model, which contains fermionic worldline degrees of freedom, and obtain identical results.

Contents

Introduction

The Worldline Formalism [1, 2] provides a first-quantized approach to Quantum Field Theory (QFT). Its first appearance can be traced to the early 1950s, in the appendices of Feynman's pioneering papers on Quantum Electrodynamics (QED) [3, 4] and in Schwinger's seminal work on vacuum polarization [5]. In these works, the spinor and scalar QED S-matrix is represented in terms of path integrals of relativistic particle actions. With the introduction of Grassmann variables and the advent of supersymmetry (SUSY), the corresponding Lagrangians were identified as that of a supersymmetric theory defined over the worldline of the particle. The interest in this formalism increased with the development of string theory. Within the study of string scattering amplitudes in the infinite tension limit to recover gauge theories scattering amplitudes, the work by Bern and Kosower [6, 7] provided master formulae for one-loop n-gluon amplitudes which required no knowledge of string theory at the end. An independent derivation of the "Bern-Kosower" rules was provided by Strassler [8] through the path integral on the circle of a suited worldline theory. Subsequently, a systematic formulation of different QFTs based on the quantization of various relativistic particle actions was developed.

In this work, we bridge the gap for a charged, massive vector boson. While massless spin-1 particles cannot exist in a theory with a Lorentz-covariant conserved current [9], and therefore do not admit electromagnetic coupling, charged massive spin-1 particles are consistent, as exemplified by the $W^\pm$ bosons in the Standard Model. Hence, it should be possible to describe them within a worldline approach. We aim to find a worldline representation of this theory and compute the one-loop effective Lagrangian the charged vector boson induces in a constant electromagnetic background. Euler and Heisenberg derived an analogue Lagrangian for electron loops [10], and shortly thereafter Weisskopf extended it to include massive charged scalars [11] (see [12, 13] for a review of Euler and Heisenberg type Lagrangians and [1] for their derivation via worldline techniques). Starting from a quantum field theory of vector electrodynamics, the effective Lagrangian induced by charged, massive spin-1 particles was obtained much later [14]. We reproduce it using worldline techniques. The effective Lagrangian encodes information about vacuum instability, enabling the computation of the pair production rate.

In the first-quantized approach, the worldline theory is a one-dimensional sigma model with values on a target space. The latter is the ordinary spacetime with the addition of auxiliary variables that parametrize the spin degrees of freedom. After quantization, the model reproduces the field equations of a particle of spin s in spacetime. Traditionally, the additional variables are worldline fermions and, as we mentioned, the worldline theory enjoys local $N = 2s$ supersymmetries to ensure unitarity at the quantum level [15–19]. In order to describe interacting QFTs, one couples the worldline theory to background fields. Depending on the spin and type of background, this procedure may break the worldline supersymmetry, leading to an inconsistent quantum theory. In recent works [20–24], BRST quantization has been employed to determine the consistency conditions required for the interacting quantum theory. These conditions restrict the class of admissible backgrounds to on-shell configurations. For such backgrounds, the path integral formulation can be exploited, yielding the full propagator and the one-loop effective action of the corresponding QFT.

We adopt a bosonic worldline model. It contains worldline bosons as additional variables to generate spin degrees of freedom [25–29], and is characterized by a "bosonic supersymmetry"(BUSY). The model analysed in [29] describes massless bosonic particles of any spin. Therefore, we first extend it to accommodate a mass term and describe massive particles. The latter are identified by the field equations encoded in the physicality conditions for quantum states. We focus on the spin $s = 0, 1$ sectors, which are expected to admit a consistent coupling to an external electromagnetic field. Through a BRST analysis, for spin $s = 0$ we find a consistent coupling for any background, while in the $s = 1$ sector, the consistency condition for the coupling, i.e. the nilpotency of the BRST charge, requires the background to satisfy Maxwell's equations. This allows us to compute the one-loop effective action in a constant electromagnetic field through the development of the path integral formulation. As expected, an imaginary part emerges, signalling a non-vanishing pair production rate of massive vector bosons. Our results for the spin-1 sector [30], based on a first-quantized approach, reproduce the findings originally due to Vanyashin and Terent'ev [14]. Also, the one-loop effective Lagrangian for a scalar particle [11] is recovered, thereby providing an additional consistency check on our path integral construction.

The same results are obtained using a fermionic spinning particle model, the $N = 2$ particle. It enjoys two local supersymmetries on the worldline and describes massive or massless $p-$forms. As suggested in [19], the coupling to electromagnetism breaks SUSY. Nonetheless, we consider the massive extension of the model and show, using again BRST techniques, that a coupling to electromagnetism for massive spin-1 excitations is actually feasible, in a way analogous to the bosonic model, leading to identical results.

Our results provide a test of the self-consistency of the Worldline Formalism. We believe our analysis can be extended to more general cases. For instance, one possibility is to extend the particle model to include additional effective coupling to electromagnetism, in order to describe a non-point-like particle, as suggested in [30]. Another is to consider pair creation in a non constant-field, by exploiting worldline instanton techniques [31–35]. Finally, one could study scattering amplitudes within this formalism, as established in [20] and investigated recently in [36].

This thesis is organized into four chapters. In Chapter 1 we review the classical and quantum theory of constrained Hamiltonian system, since worldline theories describe systems of this type. The correspondence between gauge theories and first-class systems is established, and different quantization schemes for constrained systems are discussed. Chapter 2 treats worldline models, both fermionic and bosonic. The general formulation and specific examples are presented for the free case. In particular, we introduce the extended version of the bosonic model to describe massive particles with integer spin. We then address interacting worldline theories and the case of interactions that spoil the algebra of the symmetry group. At the end, we examine the path integral formulation. Chapter 3 contains the main results. We concentrate on the $s = 1$ sector of the bosonic spinning particle. The free field equations are derived, and the path integral is defined after a gauge fixing procedure. To couple the model to a classical Abelian background, BRST quantization is employed, showing that quantum consistency requires the background field to be on-shell, i.e. to satisfy the free Maxwell's equations. Thus, we compute, through its worldline representation, the one-loop effective action in a constant electromagnetic background, extract its imaginary part, and discuss the implications for Schwinger-type pair production of massive spin-1 particles. Lastly, we reproduce the same results using the $N = 2$ fermionic spinning particle. Chapter 4 concludes.

Chapter 1

Gauge systems as constrained Hamiltonian systems

All the fundamental laws of nature appear to be described by gauge theories. A gauge theory is a theory in which physical quantities have a not-unique representation in their mathematical formulation. The mathematical objects describing these quantities can be transformed in a way that depends on the spacetime point in which the transformation acts, without changing the physical quantity they represent. In the Hamiltonian formalism, a characterization of these systems is provided by a particular type of constrained Hamiltonian systems. In order to define a quantum theory that describes only the physical degrees of freedom of the gauge system, specific quantization procedures must be employed. Our discussion here follows [2, 37, 38]

1.1 Constrained Hamiltonian systems

Let us consider a physical system with n degrees of freedom described, in the Lagrangian formulation, by the coordinates $q = (q^1, \ldots, q^n)$ and Lagrangian $L(q, \dot{q})$. The Euler-Lagrange equations read

$$\ddot{q}^j \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} = \frac{\partial L}{\partial q^i} - \dot{q}^j \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} \,, \qquad i = 1, \ldots, n \,. \tag{1.1.1}$$

If the matrix $M_{ij} := \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}$ is not invertible, then there exist $m = n - \mathrm{rank}(M)$ vectors, $k_{(l)} = k_{(l)}(q, \dot{q})$, $l = 1, \ldots, m$, in the kernel of M (for simplicity, we assume $\mathrm{rank}(M)$ a constant function of $(q, \dot{q})$), and the general solution for $\ddot{q}$, which provides the system of second-order differential equations is

$$\ddot{q}(q,\dot{q}) = \ddot{q}_s(q,\dot{q}) + \lambda_{(l)}(t)\; k_{(l)}(q,\dot{q})\,, \tag{1.1.2}$$

where $\ddot{q}_s(q,\dot{q})$ is a solution of the inhomogeneous (1.1.1) and $\lambda_{(l)}(t)$ arbitrary functions of time. Thus, the solution of the equations of motion (1.1.2) contains an arbitrary dependence on time, a feature that typically occurs in systems with gauge symmetries. We stress that this is related to the non-invertibility of the matrix M, i.e.

$$\det\left(\frac{\partial^2 L}{\partial\dot{q}^j\partial\dot{q}^i}\right) = 0\,, \tag{1.1.3}$$

that defines a set of relations between the coordinates and the velocities. A Lagrangian satisfying (1.1.3) is called a *singular Lagrangian*. In particular, if we move to the Hamiltonian description by introducing conjugate momenta $p_i = \frac{\partial L}{\partial\dot{q}^i}$, (1.1.3) implies that these defining relations cannot be inverted to express velocities in terms of momenta. Hence, a set of constraints between the phase space variables (q,p) must hold:

$$c_\alpha(q,p) = 0\,, \tag{1.1.4}$$

where c_α, $\alpha = 1,\ldots,A$, constitutes a set of *primary constraints*, not necessarily independent, that defines the submanifold of the phase space (*primary constraint surface*) on which the dynamics of the system is restricted. The submanifold of the total phase space on which the dynamics occurs is not necessarily a phase space itself, i.e. the restricted symplectic form may fail to be symplectic. In order to describe the Hamiltonian dynamics of a constrained system we must take care of the constraints (1.1.4) when we derive equations of motion in canonical form.

Let's consider the Legendre transform of L,

$$H = p_i\dot{q}^i - L\,. \tag{1.1.5}$$

By varying the phase space coordinates (q,p) subjected to (1.1.4) (we now consider the irreducible case, i.e. independent constraints), we obtain

$$dH = \dot{q}^i\delta p_i - \frac{\partial L}{\partial q^i}\delta q^i = \frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial H}{\partial q^i}\delta q^i\,, \tag{1.1.6}$$

with $\delta p_i, \delta q^i$ tangent to the primary constraint surface, that is, the vector

$$\left(\frac{\partial H}{\partial q^i} + \frac{\partial L}{\partial q^i},\; \frac{\partial H}{\partial p_i} - \dot{q}^i\right)\,, \tag{1.1.7}$$

belongs to the normal space of the primary constraint surface. The canonical equations then read:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} - u^\alpha \frac{\partial c_\alpha}{\partial p_i}\,, \tag{1.1.8}$$

$$\dot{p}_i = \frac{\partial L}{\partial q^i} = -\frac{\partial H}{\partial q^i} + u^\alpha \frac{\partial c_\alpha}{\partial q^i}\,, \tag{1.1.9}$$

along with (1.1.4), where u^α are arbitrary coefficients, which may depend on the phase space point. Such a set of equations describes the dynamics of the constrained system. It can be derived from a Hamiltonian of the form

$$H_T(q,p,u) = H + u^\alpha c_\alpha\,, \tag{1.1.10}$$

called the *total Hamiltonian*, and the action functional

$$S[q,p,u] = \int_{t_1}^{t_2} \left(\dot{q}^i p_i - H_T\right)\,. \tag{1.1.11}$$

The role played by the u's coefficients is exactly that of Lagrange multiplier for the constraints (1.1.4). Notice that the Hamiltonian H_T, defined on the whole phase space, when restricted to the primary constraint surface is equal to H: $H_T \approx H$. The symbol "$\approx$" means that we are considering quantities evaluated on the primary constraint surface and, we say that H_T equals H *weakly*. Given the evolution determined by the Hamiltonian H_T, consistency of the construction requires the primary constraints to be preserved along the physical trajectories. For each of the phase functions c_α, the following must hold:

$$\dot{c}_\alpha = \{c_\alpha, H_T\}_{PB} = \{c_\alpha, H\}_{PB} + u^\beta \{c_\alpha, c_\beta\}_{PB} \approx 0\,, \tag{1.1.12}$$

with Poisson bracket $\{\cdot,\cdot\}_{PB}$ defined as

$$\{F,G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} \tag{1.1.13}$$

for any two phase functions F, G.

The conditions (1.1.12) can either impose a restriction on the u's coefficients or they may define equations that are independent of them, that is, they define a new set of constraints, called *secondary class constraints*. These must be preserved by the Hamiltonian evolution as well, and iterating the consistency requirement above one can find all the constraints that characterize the system. Notice that, except for the primary constraints, all others require the use of the equations of motion to be identified. For this reason, we adopt a distinct nomenclature to differentiate them. However, their role in what follows will be the same. We denote all the constraints by c_β, with $\beta = 1, \ldots, A, A+1, \ldots, B$.

Having found all the constraints, the consistency requirements involving the coefficients u^α fix the latter to be equal to

$$u^\alpha \approx U^\alpha + v^a V^\alpha_a \,, \tag{1.1.14}$$

with U^α a solution of the inhomogeneous (1.1.12) for each c^β, and $v^{(a)} V^\alpha_{(a)}$ a solution, with coefficients $v^{(a)}$ in the basis solutions $V^\alpha_{(a)}$, of the homogeneous $V^\alpha_{(a)} \{c_\beta, c_\alpha\}_{PB} \approx 0$. Hence, we have singled out the arbitrary part of the coefficients u (v^a) from the part fixed by the consistency requirements (U^α). The Hamiltonian is rewritten as

$$H_T = H + U^\alpha c_\alpha + v^a V^\alpha_a c_\alpha \coloneqq H' + v^a c_a \,. \tag{1.1.15}$$

Considering the v coefficients as arbitrary function of time, the general solution of the equations (1.1.8) contains an arbitrary dependence on time, meaning that our mathematical framework has a certain degree of arbitrariness in the description of the physical state. In fact, a physical state is given by the phase space point (q_0, p_0), but the evolution of such a state does not uniquely determine the values (q, p) in the future or in the past, due to the presence of the arbitrary parameters v. On the contrary, we expect the equations of motion uniquely determine the physical state at time $t \neq t_0$ if we know the physical state at t_0, thus, the conclusion is that to a physical state corresponds several phase space points. In particular, let us consider the evolution of the canonical coordinates q, for instance:

$$q(t+\delta t) = q(t) + \dot{q}\; \delta t = q(t) + \left(\{q, H'\}_{PB} + v^a \{q, c_a\}_{PB}\right)\; \delta t \,, \tag{1.1.16}$$

and compare two evolutions with two different values of the v coefficients,

$$q'(t+\delta t) - q(t+\delta t) = \delta t (v'^a - v^a)\, \{q, c_a\}_{PB} \coloneqq \epsilon_a \{q, c_a\}_{PB} \;. \tag{1.1.17}$$

These coordinates, together with their respective conjugate momenta, must represent the same physical state, their difference being proportional to the arbitrary parameters ϵ_a and not being related to different initial conditions. Hence, the transformation generated by the function $v^a c_a$ through the Poisson bracket does not affect the physical state, but connect equivalent phase space points representing the same state. These transformations are called *gauge transformations* and the v coefficients *gauge fields*. The equivalent classes of points connected by gauge transformations are called *gauge orbits*.

To characterize such a system, we notice that the phase functions H' and c_a satisfy

$$\{H', c_\beta\}_{PB} \approx 0 \,, \qquad \{c_a, c_\beta\}_{PB} \approx 0 \,, \tag{1.1.18}$$

for any constraints c_β. A function F whose Poisson bracket with every constraint vanishes weakly is called a *first class function*; otherwise, it is called a *seconds class function.* It follows that the Poisson bracket of two first class function is first class. The constraints c_a are called *first class (primary) constraints* (in particular, they define a complete basis). Therefore, we conclude that first class primary constraints generate gauge transformations (for this reason, we will call them also gauge generators). Since the Poisson bracket of two first class constraints can contain first class secondary constraints, one sees that gauge transformation are generated also by those specific first class secondary constraints. One postulate that *all* first class secondary constraints are gauge generators in physically relevant systems. Thus, every first class constraint is a gauge generator in what follows. With this terminology, a first class function is, in particular, a gauge invariant function, i.e. a function whose Poisson bracket with gauge generators vanishes weakly. Gauge invariant functions will represent physical observables.

Since the most general evolution is generated by a Hamiltonian containing all the gauge generators, in (1.1.15) we need to add also the first class secondary constraints. The *extended Hamiltonian* reads

$$H_E = H' + \lambda^a C_a \tag{1.1.19}$$

where C_a denotes all first class constraints, λ^a the associated gauge fields and now a is an index running over all first class constraints.

Let us consider such a gauge system, with no second class constraints. We can generalize our phase space to contain Grassmann variables, both commuting and anticommuting, collectively denoted by z^A. We introduce the graded Poisson bracket

$$\{F, G\}_{PB} = \frac{\partial_R F}{\partial z^A}\,\Omega^{AB}\,\frac{\partial_L G}{\partial z^B}\,, \tag{1.1.20}$$

where Ω^{AB} is the symplectic constant matrix defining the canonical graded Poisson bracket

$$\{z^A, z^B\}_{PB} = \Omega^{AB}\,, \tag{1.1.21}$$

between the canonical coordinates z^A employed in the following.

From the relations $\{C_a, C_b\}_{PB} \approx 0$, it follows that the first class constraints satisfy

$$\{C_a, C_b\}_{PB} = f_{ab}{}^c C_c\,, \tag{1.1.22}$$

with $f_{ab}{}^c$ functions on the phase space called the *structure functions.* The same applies for the first class Hamiltonian H':

$$\{H', C_a\}_{PB} = h_a{}^b C_b\,. \tag{1.1.23}$$

Given the action principle for the system

$$S[z^A, \lambda^a] = \int dt \left(\frac{1}{2}(\Omega^{-1})_{AB} z^A \dot{z}^B - H'(z) - \lambda^a C_a(z) \right)\,, \tag{1.1.24}$$

we shall verify that a gauge transformation generated by $\epsilon_a C^a$ leaves the action invariant. By varying a generic phase function F by $\delta F = \{F, \epsilon_a C^a\}_{PB}$ and the gauge fields λ^a by $\delta\lambda^a$, we get

$$\delta S = \int dt \left(-\dot{z}^A (\Omega^{-1})_{AB} \delta z^B - \delta H'(z) - \lambda^a \delta C_a(z) - \delta\lambda^a C_a(z) \right)\,. \tag{1.1.25}$$

Now, using $\delta z^A = -\epsilon^a \dot{C}_a, \delta H' = \epsilon^a h_a{}^b C_b, \lambda^a \delta C_a = \epsilon^b \lambda^a f_{ab}{}^c C_c$, this fix how the gauge fields must transform in order to keep the action invariant, up to boundary terms. The gauge transformations of the dynamical variables are:

$$\delta z^A = \left\{z^A, \epsilon_a C^a\right\}_{PB}\,, \tag{1.1.26}$$

$$\delta\lambda^a = \dot{\epsilon}^a - \epsilon^b \lambda^c f_{cb}{}^a - \epsilon^b h_b{}^a\,. \tag{1.1.27}$$

In order to deal with boundary terms, the gauge parameters at the boundaries of the integration region must satisfy specific conditions. We postpone this discussion until we address the relativistic case.

To conclude, let us consider a system that possesses only second class constraints denoted by S_a. By definition, they satisfy $\{S_a, S_b\}_{PB} \not\approx 0$. Let us assume, in addition, that

$$\det \{S_a, S_b\}_{PB} \not\approx 0\,, \tag{1.1.28}$$

i.e. the matrix $N_{ab} := \{S_a, S_b\}_{PB}$ is invertible. This condition makes it possible to define a reduced phase space, which is the submanifold defined by the constraints with a symplectic structure inherited by the symplectic structure on the whole phase space through restriction. The algebra defined over the reduced phase space is given by the Dirac bracket

$$\{F, G\}_{DB} = \{F, G\}_{PB} - \{F, S_a\}_{PB} N^{ab} \{S_b, G\}_{PB} \tag{1.1.29}$$

with $N^{ab} = (N^{-1})_{ab}$. The constraints have vanishing Dirac bracket with any arbitrary phase function L,

$$\{L, S_a\}_{DB} = \{L, S_a\}_{PB} - \{L, S_b\}_{PB} N^{bc} N_{ca} = 0\,. \tag{1.1.30}$$

One can implement the constraints right from the beginning, finding the independent canonical coordinates and working in the resulting phase space with Dirac bracket as usual. This algebra allows ones to carry the ordinary quantization procedure for such systems, differently from the procedure that we will apply to quantize gauge systems.

1.2 Quantization of gauge systems

Canonical quantization requires a well-defined Poisson bracket structure on the classical phase space and a positive-definite Hilbert space built as a representation of the canonical (graded) commutation relations. In the case of gauge systems, these requirements are not both satisfied if one adopts only the usual quantization procedure. To obtain a meaningful quantum theory, one can employ different quantization methods for such systems. Let us discuss them.

Reduced phase space quantization Given the first class constraints C_a, the gauge orbits generated by them foliate the constraint surface. Even if there is no induced Poisson bracket on it, the quotient space, made up by the equivalence class of points lying on the same gauge orbit, is equipped with a symplectic form, i.e. constitutes a phase space, the reduced phase space. Hence, one could consider only functions defined over this space, the gauge invariant functions, and quantize the theory by finding a complete set of these functions, canonical Poisson brackets among them and their irreducible representations. A complete set of gauge invariant functions is a complete set of solutions $\{F_n\}$ to the differential equations:

$$\{F, C_a\}_{PB} \approx 0\,, \tag{1.2.1}$$

for the function F. In general, finding such solutions is far from trivial. To cope with this difficult, one can work in the reduce phase space by defining *gauge fixing conditions*

$$\gamma_a(p, q) \approx 0\,, \tag{1.2.2}$$

which are additional constraints that select a representative from each gauge orbits. This happens provided these conditions satisfy two properties: (i) for each point, there must exist a gauge transformation that maps it to another point satisfying the gauge fixing conditions, (ii) they must fix the gauge completely, i.e.

$$\{\gamma_a, C_b\}_{PB} \not\approx 0\,, \tag{1.2.3}$$

thus, the set of constraints (C_a, γ_a) is second class. Such gauge fixing functions determine a submanifold that intersects each gauge orbit in just one point. The reduced phase space of gauge invariant functions with usual Poisson bracket is obtained as the space defined by the secondary constraints (1.2.3), and a Dirac bracket structure that allows to solve the constraints explicitly to find a coordinate system for the reduced phase space.

The above method can be difficult to implement for technical reasons and not useful when the complete elimination of gauge freedom spoils manifest invariance under symmetries one wants to keep manifest (let us think of Poincaré invariance).

Dirac quantization In this method, the full phase space is retained and the usual Poisson bracket structure is considered. Quantization proceeds as usual, thus, every phase space variable is realized as an operator acting on a Hilbert space $\mathcal{H}$. Actually, this space contains non-physical states as well, since we have constructed it as a representation of all the phase space variables, including non-physical configurations that do not belong to the constraint surface. In particular, the resulting Hilbert space may not be positive-definite. In order to work with physical states only, i.e. gauge invariant states, and a well-defined Hilbert space, one defines the physical Hilbert space $\mathcal{H}_{ph}$ as the set of states, $|\psi\rangle$, invariant under the action of a gauge transformation. Being the constraints C_a the classical gauge generators, their realization as quantum operator $\hat{C}_a$ represents the gauge generators in the quantum theory. Therefore, the definition of a physical state translates into

$$|\psi\rangle \in \mathcal{H}_{ph} \iff \hat{C}_a |\psi\rangle = 0 \,. \tag{1.2.4}$$

Sometimes this condition is too strong and one prefers to define a physical state as a state that, after an infinitesimal gauge transformation, has zero overlap with another physical state, which means the vanishing of the matrix element

$$\langle\chi| \hat{C}_a |\psi\rangle = 0 \,, \tag{1.2.5}$$

for any $|\chi\rangle, |\psi\rangle \in \mathcal{H}_{ph}$. This scheme is also known as Dirac-Gupta-Bleuler quantization.

It is a subtle problem to define a sensible scalar product that provides the right normalization for physical states. If we consider the usual scalar product

$$(\varphi_1, \varphi_2) = \langle\varphi_1|\varphi_2\rangle = \int d^n q \; \varphi_1^*(q) \; \varphi_2(q) \,, \tag{1.2.6}$$

this could be infinite if both states are physical. The solution is to redefine a physical scalar product for physical states with the insertion of a (hermitian) operator $\hat{\mu}$ that has the effect of restricting the integral to the physical degrees of freedom only:

$$(\varphi_1, \varphi_2) = \langle\varphi_1|\,\hat{\mu}\,|\varphi_2\rangle \,, \tag{1.2.7}$$

for any $|\varphi_1\rangle, |\varphi_2\rangle \in \mathcal{H}_{ph}$. The form of $\hat{\mu}$ depends on the explicit expression of the constraints.

BRST quantization The BRST quantization method (named after Becchi, Rouet, Stora and Tyutin) employed in the quantization of gauge systems expressed in Hamiltonian form, is based on the analogous method developed for quantizing gauge theories in their Lagrangian formulation. The latter, through the introduction of Faddeev-Popov ghosts, introduces gauge fixing conditions to define a sensible path integral. The gauge-fixed theory possesses a residual global supersymmetry, namely the BRST symmetry, which now involves the additional ghost variables as well. The generator of this symmetry is nilpotent, allowing physical states to be defined as elements of the cohomology of this operator in the quantum theory. The same idea is applied in the Hamiltonian formulation.

Let us consider a theory with first class constraints C_a and the action (1.1.24). The original phase space is enlarged by including, for each constraint, a pair of conjugate ghost variables (c^a, P_a), c^a the ghost variable and P_a the conjugate ghost momenta. We denote the total phase space, the BRST phase space, by $\mathcal{M}_{BRST}$. The conjugate ghosts have opposite Grassmann parity to that of the constraints with which they are associated. Their Poisson brackets read

$$\left\{P_a, c^b\right\}_{PB} = -\delta_a^b \,. \tag{1.2.8}$$

The total action is given by

$$S[z^A, c^a, P_a, \lambda^a] = \int dt \left(\frac{1}{2}(\Omega^{-1})_{AB} z^A \dot{z}^B + \dot{c}^a P_a - H'(z) - \lambda^a C_a(z)\right) \,. \tag{1.2.9}$$

The ghost-content of a phase function F, defined over the BRST phase space $\mathcal{M}_{BRST}$, is determined by the ghost number

$$\text{gh} : C^\infty(\mathcal{M}_{BRST}) \to \mathbb{R} \,, \tag{1.2.10}$$

with $C^\infty(\mathcal{M}_{BRST})$ the set of phase functions on $\mathcal{M}_{BRST}$. By definition, for the canonical variables we have $\text{gh}(c^a) = 1$, $\text{gh}(P_a) = -1$ and vanishing ghost number for all other variables.

One then defines the BRST charge $\mathcal{Q}$ as a phase function such that the following holds:

- $\mathcal{Q}$ is real ;

- $\mathcal{Q}$ is anticommuting;
- $\text{gh}(\mathcal{Q}) = 1$;
- $\mathcal{Q}$ acts as generator of gauge transformations with ghost coordinates c^a as gauge parameters;
- $\{\mathcal{Q}, \mathcal{Q}\}_{PB} = 0$, i.e. $\mathcal{Q}$ is nilpotent.

Such charge $\mathcal{Q}$ can always be constructed. The first four points fix $\mathcal{Q} = c^a C_a + \dots$, and the ghosts are chosen to be real or complex in order to define a real BRST charge. Then, let us decompose $\mathcal{Q}$ as

$$\mathcal{Q} = c^a C_a + q^a_{(1)} P_a + q^{ab}_{(2)} P_a P_b + \dots , \tag{1.2.11}$$

that implies

$$0 = \{\mathcal{Q}, \mathcal{Q}\}_{PB} = \left\{c^a C_a, c^b C_b\right\}_{PB} + 2\left\{q^a_{(1)} P_a, c^b C_b\right\}_{PB} + \left\{q^a_{(1)} P_a, q^b_{(1)} P_b\right\}_{PB} + \dots . \tag{1.2.12}$$

From the first two terms, at order zero in P, we obtain

$$0 = (-1)^{c_a} c^a c^b f^d{}_{ba} C^d - 2q^a_{(1)} C_a , \tag{1.2.13}$$

which fixes $q^a_{(1)}$. If the structure functions are constants, the other terms vanish exactly and the last bracket written in (1.2.12) is zero due to the Jacobi identities for the structure constants. In this case, the nilpotent BRST charge is

$$\mathcal{Q} = c^a C_a + \frac{1}{2}(-1)^{c_d} c^d c^b f^a{}_{bd} P_a . \tag{1.2.14}$$

For non-constant structure functions, higher order terms in P are needed to cancel other pieces and guarantee the nilpotency.

To understand the importance of a nilpotent BRST charge, we need to discuss the concept of cohomology. Given a vector space V, a linear map $\delta : V \to V$, with $\delta^2 = 0$, the operator δ is called a nilpotent operator. Given the kernel of δ,

$$\text{Ker}(\delta) = \{v \in V | \delta v = 0\} , \tag{1.2.15}$$

its elements are called ($\delta-$)closed (or cocycles). In addition, being the image of δ the set

$$\text{Im}(\delta) = \{w \in V | \exists\, z \in V \text{ s.t. } w = \delta z\} , \tag{1.2.16}$$

we call its elements ($\delta-$)exact (or coboundaries). By definition, all exact elements are closed, $\text{Im} \subseteq \text{Ker}(\delta)$. Roughly speaking, the cohomology measures how this inclusion is proper or not. It is defined as the space of equivalent classes

$$H(\delta) = \frac{\mathrm{Ker}(\delta)}{\mathrm{Im}(\delta)}\,, \tag{1.2.17}$$

with equivalence relation given by

$$v \sim v' \qquad \text{if} \qquad v' = v + \delta w\,. \tag{1.2.18}$$

If $\mathrm{Im}(\delta) = \mathrm{Ker}(\delta)$, the cohomology is vanishing.

Now, being the space V the space of phase functions $C^\infty(\mathcal{M}_{BRST})$, and the linear map on this space given by the action of the charge $\mathcal{Q}$ on any phase function F through the Poisson bracket

$$\mathcal{Q}(F) = \{\mathcal{Q}, F\}_{PB}\,, \tag{1.2.19}$$

we notice that, thanks to the Jacobi identities and its last defining property, the BRST charge defines a nilpotent operator on this space:

$$\mathcal{Q}^2(F) = \{\mathcal{Q}, \{\mathcal{Q}, F\}\}_{PB} = 0\,, \qquad \forall\ F \in C^\infty(\mathcal{M}_{BRST})\,. \tag{1.2.20}$$

We can define the cohomology of this operator and divide the space of phase functions into equivalent classes. We identify physical observables with cohomology classes with ghost number zero. Therefore, the function F such that

$$\{\mathcal{Q}, F\}_{PB} = 0\,, \qquad \mathrm{gh}(F) = 0\,, \tag{1.2.21}$$

and the function F', with

$$F' = F + \{\mathcal{Q}, G\}_{PB}\,,\ \ \forall\ G \in C^\infty(\mathcal{M}_{BRST}),\, \mathrm{gh}(G) = -1\,, \tag{1.2.22}$$

represent the same physical observable, $F' \sim F$. The gauge invariant Hamiltonian H', is extended to a BRST invariant Hamiltonian H_B, $\{\mathcal{Q}, H_B\}_{PB} = 0$, which means, also, that the BRST charge is conserved along the evolution generated by H_B. As for any physical observable, there exists equivalent Hamiltonian H'_B,

$$H'_B = H_B + \{\mathcal{Q}, \Psi\}_{PB}\,, \tag{1.2.23}$$

with Ψ an arbitrary function. Different choices of Ψ correspond to different choices of gauge fixing in the theory, which amount to fix the values of unphysical degrees of freedom, as we will see in the next chapters.

Quantizing the theory, one realizes all the dynamical variables, including the ghost, as linear operators. Then, promoting the BRST function (1.2.14)[1] to a hermitian, ghost number one, nilpotent operator $\hat{\mathcal{Q}}$,

$$\hat{\mathcal{Q}}^2 = 0\,, \tag{1.2.24}$$

the same concept of cohomology is defined. By the same argument, a physical observable is a ghost number zero cohomology class in the BRST operator cohomology, meaning a BRST invariant operator $\hat{F}$,

$$[\hat{\mathcal{Q}}, \hat{F}\} = 0\,, \tag{1.2.25}$$

where the graded commutator $[\cdot, \cdot\}$ is the anticommutator if $\hat{F}$ is fermionic, or the commutator otherwise, with the equivalence relation

$$\hat{F}' \sim \hat{F} \qquad \text{if} \qquad \hat{F}' = \hat{F} + [\hat{\mathcal{Q}}, \hat{B}\}\,. \tag{1.2.26}$$

for some operator $\hat{B}$.

Physical states are identified with cohomology classes of the BRST state cohomolgy $H_{st}(\hat{\mathcal{Q}})$, with vanishing ghost number:

$$|\psi\rangle \in \mathcal{H}_{ph} \iff |\psi\rangle \in H_{st}(\hat{\mathcal{Q}}) = \frac{\text{Ker}(\hat{\mathcal{Q}})}{\text{Im}(\hat{\mathcal{Q}})}\,. \tag{1.2.27}$$

Therefore, the following holds:

$$\hat{\mathcal{Q}}\,|\psi\rangle = 0\,, \tag{1.2.28}$$

and

$$|\psi'\rangle \sim |\psi\rangle \qquad \text{if} \qquad |\psi'\rangle = |\psi\rangle + \hat{\mathcal{Q}}\,|\chi\rangle\,, \tag{1.2.29}$$

for every state $|\chi\rangle$ with ghost number -1. Notice that the Hilbert space $\mathcal{H}_{ph}$ has a positive-definite inner product, even if states of the form $\hat{\mathcal{Q}}\,|\chi\rangle$ have vanishing norm due to the nilpotency of $\hat{\mathcal{Q}}$,

$$|\hat{\mathcal{Q}}\,|\chi\rangle|^2 = \langle\chi|\,\mathcal{Q}^2\,|\chi\rangle = 0\,. \tag{1.2.30}$$

Indeed, these elements are identified with the zero vector in $\mathcal{H}_{ph}$. Physical measurable quantities, such as the (modulo squared of) matrix elements between a physical observable $\hat{F}$ and physical states $|\psi\rangle\,, |\phi\rangle$, are gauge invariants,

$$\langle\phi|\,\hat{F}\,|\psi\rangle = \langle\phi'|\,\hat{F}'\,|\psi'\rangle\,. \tag{1.2.31}$$

[1] In general, the existence of a classical BRST charge does not guarantee the existence of a quantum BRST charge, due to possible ordering ambiguities of the quantum operators. However, in the rest of this thesis, such correspondence will always exist.

In fact, one may check, for example, that

$$\begin{aligned} \langle\phi|\,\hat{F}\,|\psi\rangle - \langle\phi|\,\hat{F}\,|\psi'\rangle &= \langle\phi|\,\hat{F}\hat{\mathcal{Q}}\,|\chi\rangle \\ &= \langle\phi|\,[\hat{F},\hat{\mathcal{Q}}\}\,|\chi\rangle \pm \langle\phi|\,\hat{\mathcal{Q}}\hat{F}\,|\chi\rangle \\ &= 0\,, \end{aligned} \tag{1.2.32}$$

since $\langle\phi|$ and $\hat{F}$ are physical.

Time evolution is generated by the equivalence class of the Hamiltonian $\hat{H}_B$ parametrized by the operator $\hat{\Psi}$:

$$\langle\phi|\,\hat{U}_{H_B+\{\mathcal{Q},\Psi\}}(t_f;t_i)\,|\psi\rangle = \langle\phi|\exp\left(-\frac{i}{\hbar}(t_f-t_i)(\hat{H}_B+\{\hat{\mathcal{Q}},\hat{\Psi}\})\right)|\psi\rangle\,. \tag{1.2.33}$$

From these matrix elements we will calculate the gauge-fixed path integral in the next chapters.

Time evolutions generated by equivalent Hamiltonians are equivalent in the following sense. First, the operator $\hat{U}_{H_B}$ is BRST invariant and has ghost number zero, hence time evolution preserves the physical Hilbert space. Then, time evolution operators related to equivalent Hamiltonians $\hat{U}_{H_B}$, $\hat{U}_{H_B+\{\mathcal{Q},\Psi\}}$ are equivalent,

$$\hat{U}_{H_B+\{\mathcal{Q},\Psi\}} = \hat{U}_{H_B} + \{\hat{M},\hat{\mathcal{Q}}\}\,, \tag{1.2.34}$$

for some operator $\hat{M}$. It follows they have the same matrix elements between physical states.

1.3 Example: the relativistic scalar particle

The action of a relativistic scalar particle of mass m is invariant under changes of inertial reference frame, that is under the action of the Poincaré group, in accordance to the principle of special relativity. Given the coordinates $x^\mu = (t, x^i)$[2] in an inertial reference frame, $\mu = 0, \ldots, D-1$, D the dimension of the Minkowski spacetime with metric signature $(-,+,+,\ldots,+)$, we know the action is proportional to the proper time,

$$S[x^\mu] = -m\int\sqrt{-ds^2} = -m\int\sqrt{-dx^\mu dx_\mu}\,. \tag{1.3.1}$$

However, manifest invariance of the action is not a requirement of the theory in principle, rather a useful formulation for some scopes. One could choose to describe the trajectory (or *worldline*) of the particle by employing the time t measured in the inertial reference

[2] We work in natural units.

frame: $x^\mu(t) = (t, x^i(t))$ and the action (1.3.1) becomes

$$S[x^i(t)] = -m \int dt \sqrt{1 - \frac{dx^i}{dt}\frac{dx^i}{dt}}\,, \tag{1.3.2}$$

where the Lorentz symmetry is not manifest but the true dynamical degrees of freedom are shown. This description is correct, but when it comes to describe interactions, the guiding practical principle of manifest covariance is needed, since one has to introduce them by keeping the theory Poincaré invariant. For this reason, one treats space and time on the same footing by the beginning, considering the time coordinate t as a dynamical variable as well, meaning the trajectory of the particle is parametrized by an arbitrary parameter. Having introduced an additional dynamical variable, the physical degrees of freedom should, in principle, be recoverable within this covariant description. Due to the emergency of a new local symmetry for the action, i.e. a gauge symmetry, one can use gauge transformations and gauge fixing to eliminate the unphysical degrees of freedom. Also in this case, it may be more convenient not to remove all unphysical degrees of freedom, selecting a so-called "unitary gauge" and reaching a non-manifest invariant formulation, but instead to choose "covariant gauges" that are preserved by Poincaré transformations. Let us introduce a parameter τ to parametrize the trajectory: $x^\mu(\tau) = (t(\tau), x^i(\tau))$. The action (1.3.1) then reads

$$S[x^\mu(\tau)] = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu}\,, \tag{1.3.3}$$

with $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. This action is Poincaré invariant and, in addition, is invariant under arbitrary reparametrizations of the trajectory

$$\tau \longrightarrow \tau' = \tau'(\tau) \simeq \tau - \xi(\tau)\,, \tag{1.3.4}$$

and

$$x^\mu(\tau) \longrightarrow x'^\mu(\tau') = x^\mu(\tau)\,, \tag{1.3.5}$$

$$x'^\mu(\tau) \simeq x^\mu(\tau) + \xi(\tau)\dot{x}^\mu(\tau)\,, \tag{1.3.6}$$

where "$\simeq$" means infinitesimally. Under such transformations the action varies by a term

$$\delta S[x^\mu(\tau)] = -m \int_{\tau_1}^{\tau_2} d\tau \frac{d}{d\tau}(\xi \sqrt{-\dot{x}^\mu \dot{x}_\mu})\,, \tag{1.3.7}$$

which requires $\xi(\tau_1) = \xi(\tau_2) = 0$, in order for the transformations to constitute a gauge symmetry.

We have shown that gauge systems in Hamiltonian formulation are described by constrained systems with first class constrains as gauge generators. This is the case for the action (1.3.3). In fact, being the momenta

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}}\,, \tag{1.3.8}$$

these are constrained to the submanifold

$$p_\mu p^\mu + m^2 = 0\,. \tag{1.3.9}$$

The Hamiltonian $H = p_u \dot{x}^\mu - L$ vanishes and one can check that no other constraints arise by the consistency requirements. With the above (first class) constraint, the action (1.1.24) reads in this case [3]

$$S_{ph}[x^\mu(\tau), p_\mu(\tau), e(\tau)] = \int d\tau \left(p_\mu \dot{x}^\mu - \frac{e}{2}(p^\mu p_\mu + m^2) \right)\,, \tag{1.3.10}$$

with the gauge field e associated to the constraint $\mathscr{H} := \frac{1}{2}(p^\mu p_\mu + m^2)$. The latter generates gauge transformations of the dynamical variables through the Poisson bracket action (1.1.26)

$$\delta x^\mu = \{x^\mu, \epsilon\mathscr{H}\} = \epsilon p^\mu\,, \tag{1.3.11a}$$

$$\delta p_\mu = \{p_\mu, \epsilon\mathscr{H}\} = 0\,, \tag{1.3.11b}$$

$$\delta e = \dot{\epsilon}\,, \tag{1.3.11c}$$

for a gauge parameter $\epsilon(\tau)$.

Now, let us quantize the theory with the Dirac method. The phase space variables (x^μ, p_μ) are realized as linear operators $(\hat{x}^\mu, \hat{p}_\mu)$ with canonical commutators obtained by the canonical Poisson brackets

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu\,. \tag{1.3.12}$$

The Hilbert space is realized as a representation of the above algebra. The states $|\psi\rangle$ give rise to wave functions $\psi(x^\mu) = \langle x^\mu|\psi\rangle$, which depend on the variables (t, x^i). The evolution of these states is determined by the Hamiltonian H with parameter τ. Being the Hamiltonian vanishing, the Schrödinger equation

$$i\frac{\partial}{\partial \tau}|\psi\rangle = 0\,, \tag{1.3.13}$$

[3] It is worth noting that the phase space formulation of the action accommodates the description of massless particles as well.

tell us that no explicit dependence in τ occurs in $|\psi\rangle$.

The implementation of the constraint $\hat{\mathscr{H}}$ ((1.2.4) and (1.2.5) are equivalent with just one constraint)

$$\left(\hat{p}^{\mu}\hat{p}_{\mu}+m^2\right)|\psi\rangle=0\,, \tag{1.3.14}$$

selects physical states, i.e. gauge invariant states. As a differential equation, the physicality condition reads

$$\left(-\partial_{\mu}\partial^{\mu}+m^2\right)\psi(x)=0\,. \tag{1.3.15}$$

We see how physical states are described by wave functions satisfying the Klein-Gordon equation. They propagate the degrees of freedom of a scalar particle at the quantum level. This is a paradigmatic example of how the first quantization of a relativistic particle produces the equations of motion of the field theory describing the quantum particle.

When we will work out the path integral quantization of theories defined on the worldline, it will be useful to get action expressed in configuration space (and Euclidean time). In this case, from the phase space action (1.3.10) we can pass to the configuration space action by solving for the momenta

$$\frac{\partial S_{ph}}{\partial p_{\mu}}=0\,, \tag{1.3.16}$$

that is

$$p^{\mu}=e^{-1}\dot{x}^{\mu}\,. \tag{1.3.17}$$

By substituting back in (1.3.10) we get

$$S_{\rm co}[x^{\mu}(\tau),e(\tau)]=\int d\tau\,\frac{1}{2}\left(e^{-1}\dot{x}^{\mu}\dot{x}_{\mu}-em^2\right)\,. \tag{1.3.18}$$

The gauge symmetry, from (1.3.11), (1.3.17) is given by

$$\delta x^{\mu}=\xi\dot{x}^{\mu}\,, \tag{1.3.19a}$$

$$\delta e=\frac{d}{d\tau}(\xi e)\,, \tag{1.3.19b}$$

where $\xi=\epsilon e^{-1}$.

The gauge field e is called *einbein* because its squared is equal to the metric defined on the worldline. In fact, we can rewrite (1.3.18)

$$S_{\rm co}[x^{\mu}(\tau),e(\tau)]=\int d\tau\;e\;(e^{-2}\dot{x}^{\mu}\dot{x}_{\mu}-m^2)=\int d\tau\sqrt{-g_{00}}\left((g_{00})^{-1}\dot{x}^{\mu}\dot{x}_{\mu}-m^2\right)\,, \tag{1.3.20}$$

from which we recognize $g_{00}=-e^2$, where g_{00} is the metric on the one dimensional worldline.

From the action (1.3.18), if one solves for e and substitutes it back, one should recover the original action in configuration space (1.3.3). Indeed, by computing:

$$\frac{\partial S_{\text{co}}}{\partial e} = 0 \qquad \Longleftrightarrow \qquad e(x^\mu) = \pm\frac{1}{m}\sqrt{-\dot{x}^\mu \dot{x}_\mu}\,, \tag{1.3.21}$$

choosing the plus sign for the solution, and inserting it into (1.3.18), one obtains

$$S_{\text{co}}[x^\mu(\tau), e(x^\mu(\tau))] = S[x^\mu(\tau)]\,. \tag{1.3.22}$$

Chapter 2

The spinning particle

The correspondence between the action of a relativistic particle and the description of a scalar particle in QFT, obtained from the first-quantized action, can be generalized to accommodate the description of free spin-s particles. This is achieved by starting from suitable worldline actions describing a relativistic particle with additional degrees of freedom in its phase space. These are Grassmann variables that will represent the spin of the particle after quantization. The worldline theory so defined is called *fermionic* (also $O(N)$) or *bosonic* (also $Sp(2N)$) *spinning particle*, according to the additional variables being Grassmann odd or even, respectively. In both case, it will turn out to be a constrained Hamiltonian system of the type (1.1.24). Interacting QFTs quantities are computed from the path integral of the interacting worldline theory. Introducing interactions on the worldline and quantizing it consistently is a subtle task, and its solution depends on the worldline model and type of interaction considered.

The systematic definition of these models constitutes an aspect of the so-called Worldline Formalism, which provides a first-quantized approach to QFT.

This chapter covers both fermionic and bosonic spinning particle, with a survey of specific spin-s cases relevant to the historical development of the subject and to our work in the following chapter. Finally, we address interacting theories and the path integral formulation. The general discussion follows [2], while specific references are provided throughout the sections.

2.1 $O(N)$ spinning particle

2.1.1 Spin-1/2 particle

Let us consider first the description of a massless spin-1/2 particle. The minimal extension of the relativistic particle action (1.3.10) describes a particle with phase space coordinates (x^μ, p_μ, ψ^μ), where ψ^μ are real Grassmann odd variables and will play the role of the supersymmetric partners of x^μ. Adding the symplectic term for the ψ variables, we get

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + \frac{i}{2}\psi^\mu \dot{\psi}_\mu - \frac{e}{2} p^\mu p_\mu \right) , \tag{2.1.1}$$

from which

$$\{x^\mu, p_\nu\}_{PB} = \delta^\mu_\nu \,, \qquad \{\psi^\mu, \psi_\nu\}_{PB} = -i\delta^\mu_\nu \,. \tag{2.1.2}$$

In addition to reparametrization invariance, this system has the following *global* symmetry:

$$\delta x^\mu = i\xi\psi^\mu \,, \tag{2.1.3a}$$

$$\delta\psi^\mu = -\xi p^\mu \,, \tag{2.1.3b}$$

$$\delta p_\mu = 0 \,, \tag{2.1.3c}$$

with ξ a constant parameter, and the conserved fermionic current:

$$Q = p_\mu \psi^\mu \,. \tag{2.1.4}$$

Notice that this is a global $N = 1$ supersymmetric transformation on the worldline phase space. To extend the set of constraints, we gauge this global symmetry. We know this amounts to add the fermionic conserved charge (3.3.2) as a constraint on the worldline action, as it will act as generator of the gauged symmetry (2.1.3). Thus, with $\xi(\tau)$ a local parameter now, the Q constraint, together with the constraint $H = \frac{1}{2}p^\mu p_\mu$ (from now on we denote it by H), generate, through the Poisson brackets, the $N = 1$ local supersymmetry (or supergravity) algebra in (0+1)-dimension

$$\{Q, Q\}_{PB} = -2iH \,. \tag{2.1.5}$$

The other Poisson bracket involving these constraints vanishes. The new action reads

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + \frac{i}{2}\psi^\mu \dot{\psi}_\mu - eH - i\chi Q \right) , \tag{2.1.6}$$

where the gauge fields (e, χ) are called the einbein and the gravitino, as they compose the supergravity multiplet in one dimension. The gauge transformation generated by $V = \epsilon H + i\xi Q$ acts on the phase space variable as

$$\delta x^\mu = \{x^\mu, V\}_{PB} = \epsilon p^\mu + i\xi\psi^\mu \,, \tag{2.1.7}$$

$$\delta\psi^\mu = \{\psi^\mu, V\}_{PB} = -\xi p^\mu \,, \tag{2.1.8}$$

$$\delta p_\mu = \{p_\mu, V\}_{PB} = 0 \,, \tag{2.1.9}$$

while the gauge fields from the algebra (2.1.5) transform as (cf. (1.1.27))

$$\delta e = \dot{\epsilon} + 2i\chi\xi \,, \tag{2.1.10}$$

$$\delta\chi = \dot{\xi} \,. \tag{2.1.11}$$

To see how the equations for spin-1/2 emerge, let us quantize the theory with the Dirac method. The canonical commutation and anticommutation relations read

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu \,, \quad \{\hat{\psi}^\mu, \hat{\psi}^\nu\} = \eta^{\mu\nu} \,, \tag{2.1.12}$$

and the Hilbert space is realized as tensor product of representations of these algebras. From the first term above, we obtain the infinite-dimensional space of square-integrable functions $L^2(\mathbb{R}^D)$. The second term realizes a Clifford algebra, with gamma matrices Γ represented by

$$\hat{\psi}^\mu = \frac{1}{\sqrt{2}}\Gamma^\mu \,, \tag{2.1.13}$$

which gives, indeed,

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \,. \tag{2.1.14}$$

The Clifford algebra is represented on the finite-dimensional space of spinors, with dimension $2^{\lfloor \frac{D}{2} \rfloor}$ [1]: $\mathbb{C}^{2^{\lfloor \frac{D}{2} \rfloor}}$. The total Hilbert space is the space of spinor fields in D dimensions

$$\mathcal{H} = L^2(\mathbb{R}^D) \otimes \mathbb{C}^{2^{\lfloor \frac{D}{2} \rfloor}} \,. \tag{2.1.15}$$

The physical Hilbert space $\mathcal{H}_{ph}$ is composed by states satisfying the conditions

$$\hat{H}\,|\psi\rangle = \frac{1}{2}\hat{p}_\mu\hat{p}^\mu\,|\psi\rangle = 0 \,, \quad \hat{Q}\,|\psi\rangle = \hat{\psi}^\mu\hat{p}_\mu\,|\psi\rangle = 0 \,. \tag{2.1.16}$$

From (2.1.5), these conditions are not independent: $\hat{Q}^2 = \hat{H}$. It therefore suffices for the

[1] The symbol $\lfloor x \rfloor$ indicates the greatest integer less than or equal to x, i.e. its integer part.

$\hat{Q}$ constraint to be satisfied. The latter implements the Dirac massless equations

$$(\Gamma^\mu)_a{}^b \partial_\mu \psi_b(x) = 0\,, \tag{2.1.17}$$

where $\psi(x)$ is the spinorial wave function and a, b are spinorial indices. Finally, the $\hat{H}$ constraint would determine the massless Klein-Gordon equation for each of the components of $\psi(x)$, which is automatically satisfied if the above equations hold, as we know.

We now turn to the description of massive spin-1/2 particle. In this case, the worldline theory can be obtained through the mechanism of dimensional reduction from a higher-dimensional worldline theory (2.1.6). We start from a worldline theory formulated in a $(D+1)-$dimensional spacetime

$$S = \int d\tau \left(p_M \dot{x}^M + \frac{i}{2}\psi^M \dot{\psi}_M - \frac{e}{2} p_M p^M - i\chi p_M \psi^M \right)\,, \tag{2.1.18}$$

with $M = 0, \ldots, D$, and we fix the momenta in the extra dimension to be equal to

$$p_M = m\,, \tag{2.1.19}$$

where m is the mass of the particle being described. The above equation represents an additional first class constraint. The previous action reads

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + m\dot{x}^D + \frac{i}{2}\psi^\mu \dot{\psi}_\mu + \frac{i}{2}\psi^D \dot{\psi}_D - \frac{e}{2}(p_\mu p^\mu + m^2) - i\chi(p_\mu \psi^\mu + m\psi^D) \right)\,. \tag{2.1.20}$$

The second term, being a total derivative in x, does not affect the equations of motion and can thus be discarded. Let us show how this action correctly describes massive Dirac fermions in even dimensions D.

Quantizing the theory, in addition to the algebra (2.1.12), we have the property

$$(\Gamma^5)^2 = 1\,, \tag{2.1.21}$$

where, as before, Γ^D is a realization of ψ^D and we used $\{\hat{\psi}^D, \hat{\psi}^D\} = 1$. Thus, it satisfies the properties of the chirality gamma matrix in D dimensions. The $\hat{Q}$ constraint

$$(\hat{p}_\mu \hat{\psi}^\mu + m\hat{\psi}^D)\,|\psi\rangle = 0\,, \tag{2.1.22}$$

becomes, in terms of the spinorial wave function $\psi(x)$,

$$(-i\Gamma^\mu \partial_\mu + m\Gamma^D)\psi(x)\,, \tag{2.1.23}$$

where we suppressed spinorial indices. By multiplying by Γ^D and working with the equivalent set of gamma matrices $-i\Gamma^D\Gamma^\mu$, we get the Dirac equation for a particle of mass m

$$(\Gamma^\mu\partial_\mu + m)\psi(x) = 0\,, \tag{2.1.24}$$

in its standard form via the redefinition $\Gamma^\mu = -i\gamma^\mu$

$$(-i\gamma^\mu\partial_\mu + m)\psi(x) = 0\,. \tag{2.1.25}$$

2.1.2 Spin-1 particle

Let us start with the massless case. The description of a massless spin-1 particle follows from the same arguments of the previous subsection. We add another set of real Grassmann odd variables to the action (2.1.6), such that our phase space is now composed by the coordinates $(x^\mu, p_\mu, \psi_1^\mu, \psi_2^\mu)$, with

$$\{\psi_i^\mu, \psi_j^\nu\}_{PB} = -i\delta_{ij}\eta^{\mu\nu}\,, \qquad i,j = 1,2\,. \tag{2.1.26}$$

We now gauge the $N=2$ worldline supersymmetry, together with the $SO(2)$ R-symmetry that rotates the fermionic variables, to obtain the worldline theory as the $N=2$ supergravity in (0+1)-dimension. This is a constrained Hamiltonian system whose constraints are given by the generators of the above symmetries. If we introduce the complex fermionic variables

$$\psi^\mu := \frac{1}{\sqrt{2}}(\psi_1^\mu + i\psi_2^\mu)\,, \quad \bar\psi^\mu := \frac{1}{\sqrt{2}}(\psi_1^\mu - i\psi_2^\mu)\,, \tag{2.1.27}$$

with

$$\{\psi^\mu, \bar\psi_\nu\}_{PB} = -i\delta^\mu_\nu\,, \tag{2.1.28}$$

the resulting action reads

$$S = \int d\tau \left(p_\mu\dot x^\mu + i\bar\psi_\mu\dot\psi^\mu - eH - i\bar\chi Q - i\chi\bar Q - aJ\right)\,, \tag{2.1.29}$$

with constraints

$$H = \frac{1}{2}p_\mu p^\mu\,, \quad Q = p_\mu\psi^\mu\,, \quad \bar Q = p_\mu\bar\psi^\mu\,, \quad J = \psi_\mu\bar\psi^\mu\,, \tag{2.1.30}$$

that generate the $N=2$ supergravity algebra

$$\{Q,\bar Q\}_{PB} = -2iH\,, \quad \{J,Q\}_{PB} = -iQ\,, \quad \{J,\bar Q\}_{PB} = i\bar Q\,, \tag{2.1.31}$$

and the set of corresponding gauge fields $(e, \chi, \bar\chi, a)$. Gauging the worldline supersymmetry via the einbein e and the complex conjugate gravitinos $\chi, \bar\chi$, ensures unitarity at the quantum level, as the corresponding constraints eliminate negative norm states. In addition, the field a, which gauges the $U(1)$ R-symmetry in the complex basis, allows projecting the physical Hilbert space onto a subspace containing only a specific number of degrees of freedom. We will see this explicitly shortly.

Under a gauge transformation generated by $V = \epsilon H + i\bar{\xi}Q + i\xi\bar{Q} + \alpha J$, the canonical variables transform according to

$$\delta x^\mu = \{x^\mu, V\}_{PB} = \epsilon p^\mu + i\bar{\xi}\psi^\mu + i\xi\bar{\psi}^\mu\,, \tag{2.1.32a}$$
$$\delta p_\mu = \{p_\mu, V\}_{PB} = 0\,, \tag{2.1.32b}$$
$$\delta \psi^\mu = \{\psi^\mu, V\}_{PB} = -\xi p^\mu + i\alpha\psi^\mu\,, \tag{2.1.32c}$$
$$\delta \bar{\psi}^\mu = \left\{\bar{\psi}^\mu, V\right\}_{PB} = -\bar{\xi} p^\mu - i\alpha\bar{\psi}^\mu\,, \tag{2.1.32d}$$

and the gauge fields as

$$\delta e = \dot{\epsilon} + 2i\bar{\chi}\xi + 2i\chi\bar{\xi}\,, \tag{2.1.33a}$$
$$\delta \chi = \dot{\xi} - ia\xi + i\alpha\chi\,, \tag{2.1.33b}$$
$$\delta \bar{\chi} = \dot{\bar{\xi}} + ia\bar{\xi} - i\alpha\bar{\chi}\,, \tag{2.1.33c}$$
$$\delta a = \dot{\alpha}\,. \tag{2.1.33d}$$

Let us notice that we can add a term of the form

$$c\int d\tau\, a \tag{2.1.34}$$

in the action (2.1.29), since, due to (2.1.33d), this term is gauge invariant. Such a term is called Chern-Simons (CS) term, and the constant c CS coupling. This coupling must take quantized values, as it parametrizes the ordering ambiguities arising in the quantization of the constraint J. Indeed, the CS coupling can be interpreted as a term introduced to compensate for different ordering prescriptions and to allow for non-trivial solutions of the constraint J in the quantized theory.

At the quantum level, we have the canonical commutation and anticommutation relations

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu\,, \quad \{\hat{\psi}^\mu, \hat{\psi}^\dagger_\nu\} = \delta^\mu_\nu\,, \tag{2.1.35}$$

with $\hat{\psi}^\dagger_\nu = \hat{\bar{\psi}}_\nu$, which define the Hilbert space as the representation space

$$\mathcal{H} = L^2(\mathbb{R}) \otimes \mathcal{F}\,, \tag{2.1.36}$$

with $\mathcal{F}$ the Fock space built from the $\hat{\psi}$ creation operators and vacuum state, $|0\rangle$, annihilated by $\hat{\psi}^\dagger$. Hence, a generic state $|\phi\rangle \in \mathcal{H}$ can be written as

$$|\phi\rangle = \sum_{j=0}^{D} \frac{1}{j!} F_{\mu_1 \ldots \mu_j}(x)\, \hat{\psi}^{\mu_1} \ldots \hat{\psi}^{\mu_j} |0\rangle \,. \tag{2.1.37}$$

The functions $F_{\mu_1 \ldots \mu_j}(x)$ are rank-j antisymmetric tensors. For the operator $\hat{J}$ we choose the following antisymmetric quantization prescription:

$$\hat{J} = \frac{1}{2}(\hat{\psi}_\mu \hat{\bar{\psi}}^\mu - \hat{\bar{\psi}}_\mu \hat{\psi}^\mu) = \hat{\psi}_\mu \hat{\bar{\psi}}^\mu - \frac{D}{2} = \hat{N}_\psi - \frac{D}{2} \,, \tag{2.1.38}$$

with $\hat{N}_\psi$ denoting the number operator for the $\hat{\psi}$ creation operators. The presence of the CS term (2.1.34) in the action allows us to choose:

$$c = p + 1 - \frac{D}{2} \,, \tag{2.1.39}$$

and obtain the constraint operator $\hat{J}_c$ in the form

$$\hat{J}_c := \hat{J} - c = \hat{N}_\psi - (p+1) \,. \tag{2.1.40}$$

It should be clear that the relation between the value of the CS coupling c and the number of propagating degrees of freedom, parametrized by p, depends on the quantization scheme adopted.

Let us show that, with this choice, the quantized worldline theory describes a gauge p-form. In particular for $p = 1$ it describes massless spin-1 particles.

Implementing the constraints *à la* Dirac, the physicality condition $\hat{J}_c |\psi\rangle = 0$ selects the subspace of the Hilbert space with occupation number $N_\psi = p + 1$ with elements

$$|\phi\rangle = \frac{1}{(p+1)!} F_{\mu_1 \ldots \mu_{p+1}}(x) \hat{\psi}^{\mu_1} \ldots \hat{\psi}^{\mu_{p+1}} |0\rangle \,, \tag{2.1.41}$$

while the constraint $\hat{Q}^\dagger |\psi\rangle = 0$ gives the following condition for the p-form F:

$$\partial^{\mu_1} F_{\mu_1 \ldots \mu_{p+1}}(x) = 0 \,. \tag{2.1.42}$$

Finally, from $\hat{Q} |\psi\rangle = 0$ one obtains

$$\partial_{[\mu} F_{\nu_1 \ldots \nu_{p+1}]}(x) = 0 \,. \tag{2.1.43}$$

The H constraint is, again, automatically satisfied due to the constraints' algebra. The last two equations are exactly the Maxwell's equations in vacuum for the field strength

$F_{\mu_1...\mu_{p+1}}$, which can be expressed in terms of the p-form gauge potential $A_{\mu_1...\mu_p}$, by solving the Bianchi identity (2.1.43). For $p = 1$, the physicality conditions are the Maxwell's equations in vacuum for a gauge potential A_μ and field strength $F_{\mu\nu}$, i.e. the field equations for a spin-1 particle.

We can describe massive p-forms, and so massive spin-1 particles, by dimensional reduction of the theory (2.1.29) in dimensions $D + 1$, fixing the momentum in the extra dimension equal to the mass of the particle. Doing so, from the action (2.1.29) in $(D+1)$-dimensions, denoting $\theta := \psi^D$, we arrive at

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + i\bar{\psi}_\mu \dot{\psi}^\mu + i\bar{\theta}\dot{\theta} - eH - i\bar{\chi}Q - i\chi\bar{Q} - a(J - c)\right) \tag{2.1.44}$$

with now

$$H = \frac{1}{2}(p_\mu p^\mu + m^2), \quad Q = p_\mu \psi^\mu + m\theta, \quad \bar{Q} = p_\mu \bar{\psi}^\mu + m\bar{\theta}, \quad J = \psi_\mu \bar{\psi}^\mu + \theta\bar{\theta}\,. \tag{2.1.45}$$

Clearly, they form the same algebra (2.1.31).

When we quantize the phase space variables, in addition to (2.1.35), we have

$$\{\hat{\theta}, \hat{\theta}^\dagger\} = 1\,. \tag{2.1.46}$$

Let us omit the hat symbol "^" for operators from now on. The Hilbert space is of the form (2.1.36), but with a Fock space built from creation operator θ as well and a vacuum $|0\rangle$ annihilated also by $\theta^\dagger$. Therefore, a generic state $|\phi\rangle$ now reads

$$|\phi\rangle = \sum_{j=0}^{D} \frac{1}{j!} \left(F_{\mu_1...\mu_j}(x)\, \psi^{\mu_1}...\psi^{\mu_j}\, |0\rangle + imA_{\mu_1...\mu_j}(x)\, \theta\psi^{\mu_1}...\psi^{\mu_j}\, |0\rangle\right)\,, \tag{2.1.47}$$

where the prefactor im has been included for later convenience. Following the antisymmetric quantization prescription for J, we get

$$J = \psi_\mu \bar{\psi}^\mu + \theta\bar{\theta} - \frac{D+1}{2} = N_\psi + N_\theta - \frac{D+1}{2}\,, \tag{2.1.48}$$

that, with a CS coupling of the form

$$c = p + 1 - \frac{D+1}{2}\,, \tag{2.1.49}$$

gives

$$J_c := J - c = N_\psi + N_\theta - (p+1)\,. \tag{2.1.50}$$

Going through quantization *à la* Dirac, the condition $J_c\,|\psi\rangle = 0$ selects the following eigenspace of the occupation number:

$$|\phi\rangle = \frac{1}{(p+1)!} F_{\mu_1 \dots \mu_{p+1}}(x) \psi^{\mu_1} \dots \psi^{\mu_{p+1}} |0\rangle + \frac{im}{p!} A_{\mu_1 \dots \mu_p}(x) \theta \psi^{\mu_1} \dots \psi^{\mu_p} |0\rangle \,, \tag{2.1.51}$$

while the constraint $\hat{Q}|\phi\rangle = 0$ determines the Bianchi identities for F_{p+1} and their solution in terms of A_p

$$\partial_{[\mu} F_{\mu_1 \dots \mu_{p+1}]} = 0 \,, \qquad F_{\mu_1 \dots \mu_{p+1}} = \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]} \,. \tag{2.1.52}$$

Finally, the constraint $\hat{Q}^\dagger |\phi\rangle = 0$ produces the Proca equations and the transversality property for A_p

$$\partial^{\mu_1} F_{\mu_1 \dots \mu_{p+1}} = m^2 A_{\mu_2 \dots \mu_{p+1}} \,, \qquad \partial^{\mu_1} A_{\mu_1 \dots \mu_p} = 0 \,. \tag{7.53}$$

For $p = 1$ in the above equations, we obtain the Proca equations describing a massive spin 1 particle.

2.1.3 Spin-$N/2$ particle

The worldline theories in the previous subsections are particular realizations of the $O(N)$ spinning particle, that is a worldline theory with an $O(N)$ local supersymmetry. The quantization of this model gives the Bargmann-Wigner equations for a spin-$N/2$ in dimension $D = 4$ [39, 40]. In addition to (x^μ, p_μ), one has N families of real fermionic variables ψ_i^μ with

$$\left\{\psi_i^\mu, \psi_j^\nu\right\}_{PB} = -i\eta^{\mu\nu}\delta_{ij} \,, \qquad i,j = 1, \dots, N \,. \tag{2.1.53}$$

The set of constraints is

$$H = \frac{1}{2} p_\mu p^\mu, \quad Q_i = p_\mu \psi_i^\mu, \quad J_{ij} = i\psi_i^\mu \psi_{j\mu} \,, \tag{2.1.54}$$

which generates the $O(N)$ supergravity algebra on the worldline

$$\{Q_i, Q_j\}_{PB} = -2i\,\delta_{ij} H \,, \tag{2.1.55a}$$

$$\{J_{ij}, Q_k\}_{PB} = \delta_{jk} Q_i - \delta_{ik} Q_j \,, \tag{2.1.55b}$$

$$\{J_{ij}, J_{kl}\}_{PB} = \delta_{jk} J_{il} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik} + \delta_{il} J_{jk} \,. \tag{2.1.55c}$$

The $O(N)$ spinning particle action reads

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + \frac{i}{2} \psi_i^\mu \dot{\psi}_{i\mu} - eH - i\chi_i Q_i - \frac{1}{2} a_{ij} J_{ij} \right) \,. \tag{2.1.56}$$

The role of the gauge fields is the same as before: gauging worldline translation, through the einbein e, and local worldline supersymmetries through the gravitons χ_i, ensures unitarity at the quantum level, while gauging the $O(N)$ R-symmetry produces algebraic constraints on the wave function so to obtain irreducible (field) representations for the particles. In this general case, the gauge symmetry generated by $V = \epsilon H + i\xi_i Q_i + \frac{1}{2}\beta_{ij}J_{ij}$ acts on the phase space variables as follows:

$$\delta x^\mu = \{x^\mu, V\}_{PB} = \epsilon p^\mu + i\xi_i\psi_i^\mu\,, \tag{2.1.57a}$$
$$\delta p_\mu = \{p_\mu, V\}_{PB} = 0\,, \tag{2.1.57b}$$
$$\delta\psi_i^\mu = \{\psi_i^\mu, V\}_{PB} = -\xi_i p^\mu + \beta_{ij}\psi_j^\mu\,, \tag{2.1.57c}$$

and on the gauge fields as:

$$\delta e = \dot{\epsilon} + 2i\chi_i\xi_i\,, \tag{2.1.58a}$$
$$\delta\chi_i = \dot{\xi}_i - a_{ij}\xi_j + \beta_{ij}\chi_j\,, \tag{2.1.58b}$$
$$\delta a_{ij} = \dot{\beta}_{ij} + \beta_{im}a_{mj} + \beta_{jm}a_{im}\,. \tag{2.1.58c}$$

By proceeding with quantization *à la* Dirac and finding a representation of the canonical commutation and anticommutation relations, one finds that the wave function is a multispinor field with N spinorial indices $\psi_{\alpha_1\ldots\alpha_N}$. The constraints $\hat{Q}_i^\dagger\,|\psi\rangle = 0$ impose the conditions

$$(\gamma^\mu\partial_\mu)_{\alpha_i}{}^{\alpha_j}\psi_{\alpha_1\ldots\alpha_j\ldots\alpha_N}(x) = 0\,, \qquad i = 1,\ldots,N\,. \tag{2.1.59}$$

The constraints $\hat{J}_{ij}\,|\psi\rangle = 0$ determine algebraic relations required to make the representation $\psi_{\alpha_1\ldots\alpha_N}$ irreducible. As a result, the multispinor field becomes completely symmetric under permutations of its indices. All together, one obtains the Bargmann-Wigner equations for a massless spin-$N/2$ particle in $D = 4$ dimensions [41, 42].

2.2 $Sp(2)$ spinning particle

2.2.1 Massless integer spin particles

This worldline model is defined by the usual set of phase space variables (x^μ, p_μ) for a relativistic particle moving in a Minkowski target spacetime, augmented by an additional pair of complex bosonic variables $(\alpha^\mu, \bar{\alpha}^\mu)$, with $\bar{\alpha}^\mu = \alpha^{\mu *}$ [25–29]. As for the fermionic case, these additional variables are needed in order to account for the spin degrees of freedom. The symplectic term

$$S = \int \mathrm{d}\tau\,(p_\mu\dot{x}^\mu - i\bar{\alpha}_\mu\dot{\alpha}^\mu) \tag{2.2.1}$$

defines the phase space symplectic structure and fixes the Poisson brackets to

$$\{x^\mu, p_\nu\}_{\rm PB} = \delta^\mu_\nu \ , \quad \{\alpha^\mu, \bar\alpha^\nu\}_{\rm PB} = i\eta^{\mu\nu} \ . \tag{2.2.2}$$

As it stands, the model is not unitary, as upon quantization negative norm states will be generated by the $(x^0, p^0, \alpha^0, \bar\alpha^0)$ variables. Moreover, the model, as we shall discuss, contains particle excitations of any integer spin, and one needs to eliminate some further degrees of freedom to describe a single particle with fixed spin, or a finite multiplet of particles with fixed maximum spin. Both problems can be addressed by gauging suitable constraints. The gauged worldline action we are interested in is given by

$$S = \int \mathrm{d}\tau \ \Big[p_\mu \dot x^\mu - i\bar\alpha_\mu \dot\alpha^\mu - eH - \bar u L - u\bar L - aJ\Big] \ , \tag{2.2.3}$$

where we introduced the worldline gauge multiplet $(e, \bar u, u, a)$ acting as a set of Lagrange multipliers that enforces the constraints

$$H = \frac{1}{2}p^\mu p_\mu \ , \quad L = \alpha^\mu p_\mu \ , \quad \bar L = \bar\alpha^\mu p_\mu \ , \quad J = \alpha^\mu \bar\alpha_\mu \ . \tag{2.2.4}$$

The latter satisfy a first class algebra

$$\{L, \bar L\}_{\rm PB} = 2iH \ , \quad \{J, L\}_{\rm PB} = -iL \ , \quad \{J, \bar L\}_{\rm PB} = i\bar L \ . \tag{2.2.5}$$

The role played by the constraint J is analogous to the one played in the fermionic case: here, it is a $U(1)$ generator which rotates the bosonic oscillators by a phase; its gauging is optional as far as unitarity is concerned. However, upon quantization, it projects the Hilbert space onto the physical subspace with a specific occupation number, describing the degrees of freedom of a particle with maximal spin s. For this to occur, one must add a CS term on the worldline with the CS coupling fine-tuned according to the desired value of s. The $(H, L, \bar L)$ constraints remove the negative-norm states, as usual, and must be gauged to make the model consistent with unitarity. The Hamiltonian constraint H corresponds to the mass-shell condition for massless particles, and generates τ-reparametrization in phase space, while the remaining pair, L and $\bar L$ generates "bosonic" supersymmetry (BUSY):[2].

[2]The algebra of $(H, L, \bar L)$ can be obtained from the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra of the Virasoro algebra

$$\{L_0, L_1\}_{PB} = -iL_1 \, , \quad \{L_0, L_{-1}\}_{PB} = iL_{-1} \, , \quad \{L_1, L_{-1}\}_{PB} = 2iL_0 \, ,$$

as a contraction

$$H = \frac{L_0}{\alpha'} \, , \quad L = \frac{L_1}{\sqrt{\alpha'}} \, , \quad \bar L = \frac{L_{-1}}{\sqrt{\alpha'}} \, ,$$

in the string tensionless limit $\alpha' \to \infty$ [43, 44].

The action of a generic gauge transformation $V = \epsilon H + \bar{\xi} L + \xi \bar{L} + \phi J$, with gauge parameters $(\epsilon, \bar{\xi}, \xi, \phi)$, on the phase space variables is:

$$\delta x^\mu = \{x^\mu, V\}_{PB} = \epsilon p^\mu + \xi \bar{\alpha}^\mu + \bar{\xi} \alpha^\mu \ , \tag{2.2.6a}$$

$$\delta p_\mu = \{p_\mu, V\}_{PB} = 0 \ , \tag{2.2.6b}$$

$$\delta \alpha^\mu = \{\alpha^\mu, V\}_{PB} = i\xi p^\mu + i\phi \alpha^\mu \ , \tag{2.2.6c}$$

$$\delta \bar{\alpha}^\mu = \{\bar{\alpha}^\mu, V\}_{PB} - i\bar{\xi} p^\mu - i\phi \bar{\alpha}^\mu \ , \tag{2.2.6d}$$

and, in order for the action (2.2.3) to be invariant, the gauge fields must transform as follows:

$$\delta e = \dot{\epsilon} + 2iu\bar{\xi} - 2i\bar{u}\xi \ , \tag{2.2.7a}$$

$$\delta u = \dot{\xi} - ia\xi + i\phi u \ , \tag{2.2.7b}$$

$$\delta \bar{u} = \dot{\bar{\xi}} + ia\bar{\xi} - i\phi \bar{u} \ , \tag{2.2.7c}$$

$$\delta a = \dot{\phi} \ . \tag{2.2.7d}$$

The need for the worldline constraints to enforce unitarity remains somewhat obscure up to this point. To review and clarify this claim, it may be beneficial to perform a brief light-cone analysis. Despite the loss of manifest covariance, a light-cone analysis allows for a direct calculation of the number of propagating physical degrees of freedom. It is a well-known method, employed in many worldline models, see e.g. [45–47]. We define light-cone coordinates by

$$x^\mu = (x^+, x^-, x^a) \ , \quad \text{with} \quad x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{D-1}) \ , \tag{2.2.8}$$

where x^a, $a = 1, \ldots, D-2$, are the transverse directions. The line element reads $\mathrm{d}s^2 = -2\mathrm{d}x^+\mathrm{d}x^- + \mathrm{d}x^a\mathrm{d}x^a$, whence, for any vector V^μ, $V^+ = -V_-$ and $V^- = -V_+$.

The guiding idea behind the light-cone analysis is to remove negative-norm states by implementing a gauge fixing that isolates the physical degrees of freedom, which in turn lead to a manifestly positive-norm Hilbert space upon quantization. To do that, let us first assume motion with $p^+ \neq 0$ and consider the Hamiltonian constraint

$$H = \frac{1}{2}p^\mu p_\mu = -p^+p^- + \frac{1}{2}p^a p^a = 0 \ . \tag{2.2.9}$$

This symmetry is gauge-fixed by imposing the light-cone gauge

$$x^+ = \tau \ . \tag{2.2.10}$$

Correspondingly, the Hamiltonian constraint is solved for the momentum p^-, conjugate to x^+,

$$p^- = \frac{1}{2p^+} p^a p^a \ . \tag{2.2.11}$$

At this point, the remaining independent phase space variables are (x^-, p^+) and (x^a, p^a). A Hilbert space can be constructed by quantizing these independent variables to obtain a positive-definite Hilbert space.

On top of these variables, there are also the bosonic oscillators, which may as well lead to negative norms. That this does not happen is again made explicit by completing the light-cone gauge fixing. The gauge symmetries generated by L and $\bar{L}$, see (2.2.6c) and (2.2.6d), are fixed by setting

$$\alpha^+ = 0 \ , \quad \bar{\alpha}^+ = 0 \ , \tag{2.2.12}$$

while the constraints $L = \bar{L} = 0$ are solved explicitly by expressing the variables conjugate to (2.2.12) in terms of the remaining independent variables

$$\bar{\alpha}^- = \frac{1}{p^+} \bar{\alpha}^a p_a \ , \quad \alpha^- = \frac{1}{p^+} \alpha^a p_a \ . \tag{2.2.13}$$

The conjugated pairs $(\bar{\alpha}^-, \alpha^+)$ and $(\bar{\alpha}^+, \alpha^-)$ are thus eliminated as independent phase space coordinates, highlighting the fact that the only independent physical oscillators are the transverse ones $(\bar{\alpha}^a, \alpha^a)$. They produce states with positive norm upon quantization, as can be inferred by promoting their Poisson brackets to commutation relations

$$[\bar{\alpha}^a, \alpha^b] = \delta^{ab} \ , \tag{2.2.14}$$

which are realized on a Fock space, where α^a act as creation operators while $\bar{\alpha}^a$ as destruction operators, thus yielding a unitary spectrum of massless particles that decompose into irreducible representations of the little group $SO(D-2)$.

The gauge fixing functions (2.2.10), (2.2.12), together with the constraints $(H, L, \bar{L})$ form a set of second class constraints that allows one to solve for the conjugated variables (2.2.11), (2.2.13). Accordingly, the space parametrized by the independent variables $(x^-, p^+), (x^a, p^a), (\bar{\alpha}^a, \alpha^a)$ is the reduced phase space. The Lagrangian restricted to this space, i.e. the (partially) gauge-fixed Lagrangian, is

$$\mathcal{L} = p_- \dot{x}^- + p_a \dot{x}^a - \frac{1}{2p^+} p^a p^a - i \bar{\alpha}_a \dot{\alpha}^a - a\, \alpha^b \bar{\alpha}_b \ , \tag{2.2.15}$$

It only remains to address the further constraint related to the worldline gauge field $a(\tau)$, but this has no relevance to unitarity.

As previously discussed, a non-covariant gauge, such as the light-cone gauge, spoils the manifest covariance of the theory, and one prefers to work in a covariant formulation when adding interactions.

Hence, it is useful to proceed with covariant quantization techniques. The canonical commutation and anticommutation relation reads

$$[x^\mu, p_\nu] = i\delta^\mu_\nu \ , \quad [\bar{\alpha}^\mu, \alpha^\nu] = \eta^{\mu\nu} \ , \tag{2.2.16}$$

and the first class algebra becomes

$$[\bar{L}, L] = 2H \ , \quad [J_c, L] = L \ , \quad [J_c, \bar{L}] = -\bar{L} \ . \tag{2.2.17}$$

We have defined as usual $J_c = J - c$, after having introduced the CS coupling c in the action (2.2.3). The quantum operator J_c is defined by a symmetric quantization prescription

$$J_c = \frac{1}{2}(\alpha_\mu \bar{\alpha}^\mu + \bar{\alpha}^\mu \alpha_\mu) - c = \alpha_\mu \bar{\alpha}^\mu + \frac{D}{2} - c = N_\alpha - s \ , \tag{2.2.18}$$

with $N_\alpha \coloneqq \alpha_\mu \bar{\alpha}^\mu$ the occupation number operator for the α-oscillators and the following CS coupling c related to the real number s

$$c = \frac{D}{2} + s \ . \tag{2.2.19}$$

Let us observe once again that the relation between the CS coupling c and the value of the spin s depends on the quantization prescription employed.

From (2.2.16), the Hilbert space $\mathcal{H}$ is the tensor product space

$$\mathcal{H} = L^2(\mathbb{R}) \otimes \mathcal{F} \, , \tag{2.2.20}$$

with $\mathcal{F}$ the Fock space with vacuum defined by

$$\bar{\alpha}^\mu \ |0\rangle = 0 \ . \tag{2.2.21}$$

The decomposition of a generic state $|\Phi\rangle$ is thus written in terms of coefficients corresponding to rank-s symmetric tensors

$$|\Phi\rangle = \sum_{r=0}^{\infty} |\Phi^{(r)}\rangle = \sum_{r=0}^{\infty} \frac{1}{r!} \, \Phi^{(r)}_{\mu_1 \dots \mu_r}(x) \otimes \alpha^{\mu_1} \dots \alpha^{\mu_r} \ |0\rangle \ . \tag{2.2.22}$$

We know quantization may proceed either following a procedure *à la* Dirac or by using BRST techniques. We start with the former method, leaving the BRST analysis for the next chapter, when it will be useful for dealing with the interacting worldline theory (in the massive case).

The physical Hilbert space is composed by states $|\varphi\rangle$, $|\chi\rangle$ such that

$$\langle\chi|(H, L, \bar{L}, J_c)|\varphi\rangle = 0\ . \tag{2.2.23}$$

This can be satisfied by requiring

$$H\,|\varphi\rangle = \bar{L}\,|\varphi\rangle = J_c\,|\varphi\rangle = 0 \tag{2.2.24}$$

for any physical state $|\varphi\rangle$, since then also $\langle\varphi|\bar{L} = 0$, as $\bar{L}$ is the hermitian conjugate of L.

First, since the operator J_c counts the occupation number of the α-oscillators, shifted by $-s$, we see that the J_c constraint selects precisely states with occupation number s, which must be a nonnegative integer. Hence, physical states are contained in states of the form

$$|\varphi\rangle = \frac{1}{s!}\,\varphi^{(s)}_{\mu_1\ldots\mu_s}(x)\alpha^{\mu_1}\ldots\alpha^{\mu_s}\,|0\rangle\ . \tag{2.2.25}$$

Since the operator p_u acts as $-i\partial/\partial^\mu$ on the above state, imposing the constraints H and $\bar{L}$:

$$H\,|\varphi\rangle = -\frac{1}{2s!}\,\Box\varphi^{(s)}_{\mu_1\ldots\mu_s}(x)\alpha^{\mu_1}\ldots\alpha^{\mu_s}\,|0\rangle = 0\,, \tag{2.2.26}$$

$$\bar{L}\,|\varphi\rangle = -\frac{i}{(s-1)!}\partial^\nu\varphi^{(s)}_{\nu\mu_2\ldots\mu_s}(x)\alpha^{\mu_2}\ldots\alpha^{\mu_s}\,|0\rangle = 0\,, \tag{2.2.27}$$

with $\Box = \partial^\mu\partial_\mu$, we obtain the following conditions for the wave function of a physical state:

$$\Box\varphi^{(s)}_{\mu_1\ldots\mu_s}(x) = 0\,, \tag{2.2.28}$$

$$\partial^\nu\varphi^{(s)}_{\nu\mu_2\ldots\mu_s}(x) = 0\,. \tag{2.2.29}$$

This wave function propagates the degrees of freedom of a rank-s symmetric tensor in dimensions $D-1$.

The conditions (2.2.24) defines a physical state as an equivalence class

$$|\varphi\rangle \sim |\varphi\rangle + |\varphi_{\text{null}}\rangle\ , \tag{2.2.30}$$

where $|\varphi_{\text{null}}\rangle$ is a null state of the form

$$|\varphi_{\text{null}}\rangle = L\,|\rho\rangle\ , \qquad \text{with} \quad H\,|\xi\rangle = \bar{L}\,|\rho\rangle = (J_c+1)\,|\rho\rangle = 0\ . \tag{2.2.31}$$

These null states are physical, but have vanishing overlap with any other physical state, therefore, they have zero norm. They give rise to redundancies or "residual" gauge symmetries of the state $|\varphi\rangle$. The null state, from (2.2.31), reads

$$|\varphi_{\text{null}}\rangle = -\frac{i}{(s-1)!}\partial_{(\mu_1}\rho^{(s-1)}_{\mu_2\ldots\mu_s}\alpha^{\mu_1}\ldots\alpha^{\mu_s}|0\rangle \,, \tag{2.2.32}$$

with

$$\begin{aligned} \Box\rho^{(s-1)}_{\mu_1\ldots\mu_{s-1}}(x) &= 0\,, \\ \partial^\nu\rho^{(s-1)}_{\nu\mu_2\ldots\mu_{s-1}}(x) &= 0\,. \end{aligned} \tag{2.2.33}$$

With this gauge symmetry, the physical degrees of freedom of the tensor $\varphi^{(s)}$ correspond to that of a rank-s symmetric tensor in $D-2$ dimension, i.e. it belongs to the rank-s symmetric representation of the little group $SO(D-2)$. Notice that it is not traceless, thus $\varphi^{(s)}$ does not provide an irreducible representation of the Lorentz group for a spin-s massless particle in $D-$dimensions. Instead, it propagates a reducible multiplet consisting of massless particles of spin $s, s-2, s-4, \ldots, 0$ or $s, s-2, s-4, \ldots, 1$ for even or odd s, respectively. For $s = 0, 1$ the representations are irreducible.

2.2.2 Mass from dimensional reduction

We have extended the bosonic spinning particle model to describe massive particles (so to study massive spin-1 particle in the next chapter) through dimensional reduction [30].

Thus, we consider the theory (2.2.3) in $(D+1)$-dimensions and gauge the direction x^D by imposing the first class constraint

$$p_D = m\,, \tag{2.2.34}$$

with m the mass of the particle. We further define $(\beta, \bar\beta) := (\alpha^D, \bar\alpha^D)$, which inherit the following Poisson bracket

$$\{\beta, \bar\beta\}_{\text{PB}} = i\,. \tag{2.2.35}$$

The constraints (2.2.4) get modified by the presence of the mass:

$$H = \frac{1}{2}(p^\mu p_\mu + m^2)\,, \quad L = \alpha^\mu p_\mu + \beta m\,, \quad \bar L = \bar\alpha^\mu p_\mu + \bar\beta m\,, \quad J_c = \alpha^\mu\bar\alpha_\mu + \beta\bar\beta - c\,. \tag{2.2.36}$$

We notice they still satisfy the first class algebra (2.2.5). The gauge transformations (2.2.6) are enriched by

$$\delta\beta = i\xi m + i\phi\beta \,, \tag{2.2.37a}$$

$$\delta\bar{\beta} = -i\bar{\xi} m - i\phi\bar{\beta} \,. \tag{2.2.37b}$$

Upon covariant quantization, in addition to (2.2.16), we have

$$[\bar{\beta}, \beta] = 1 \,. \tag{2.2.38}$$

With the same quantization prescription, the constraint operator J_c is

$$J_c = \frac{1}{2}(\alpha_\mu \bar{\alpha}^\mu + \bar{\alpha}^\mu \alpha_\mu + \beta\bar{\beta} + \bar{\beta}\beta) - c = \alpha_\mu \bar{\alpha}^\mu + \beta\bar{\beta} + \frac{D+1}{2} - c = N_\alpha + N_\beta - s \,, \tag{2.2.39}$$

with the CS coupling c taking into account the additional β-oscillator,

$$c = \frac{D+1}{2} + s \,. \tag{2.2.40}$$

Accordingly, the Hilbert space is of the type (2.2.20) but now with the Fock space built out from the β-oscillator too, and a vacuum state such that

$$(\bar{\alpha}^\mu, \bar{\beta}) \; |0\rangle = 0 \,. \tag{2.2.41}$$

A generic element of this space is given by

$$|\Phi\rangle = \sum_{r,p=0}^{\infty} |\Phi^{(r,p)}\rangle = \sum_{r,p=0}^{\infty} \frac{1}{r!p!} \, \Phi^{(r,p)}_{\mu_1 \dots \mu_r}(x) \otimes \alpha^{\mu_1} \dots \alpha^{\mu_r} \beta^p \, |0\rangle \;. \tag{2.2.42}$$

Defining physical states as in (2.2.23), so (2.2.24) holds, and imposing J_c, we write a generic state at occupation number s as

$$|\varphi\rangle = \sum_{\substack{r,p=0 \\ r+p=s}}^{s} \frac{1}{r!p!} \, \varphi^{(r,p)}_{\mu_1 \dots \mu_r}(x) \otimes \alpha^{\mu_1} \dots \alpha^{\mu_r} \beta^p \, |0\rangle \;, \tag{2.2.43}$$

i.e. an element of the eigenspace of $N = N_\alpha + N_\beta$ with eigenvalue $n = s$. The constraint H determines the massive Klein-Gordon equation

$$(\Box - m^2)\varphi^{(r,p)}_{\mu_1 \dots \mu_r} = 0 \,, \qquad \forall\, r, p \in \{0, \dots, s\}, r + p = s \,, \tag{2.2.44}$$

while the constraint $\bar{L}$, due to the presence of $\bar{\beta}$, gives rise to a set of first order differential equation involving $\varphi^{(r)(p)}$ and $\varphi^{(r-1)(p+1)}$:

$$-i\partial^{\nu}\varphi^{(r)(p)}_{\nu\mu_1\ldots\mu_{r-1}} + m\varphi^{(r-1)(p+1)}_{\mu_1\ldots\mu_{r-1}} = 0\,, \tag{2.2.45}$$

with $r \in \{1,\ldots,s\}, p \in \{0,\ldots,s\}$ and $r+p=s$. These fields equations exhibit a gauge symmetry due to the existence of null states, as happens for the massless case (2.2.31). Let us consider the null state $L\,|\rho\rangle$, with $|\rho\rangle$ a state of the form (2.2.43) with $r+p=s-1$, and a set of tensors $\{\rho^{(r)(p)}\}$ that, due to $H\,|\rho\rangle = \bar{L}\,|\rho\rangle = 0$, satisfies the equations

$$(\Box - m^2)\rho^{(r,p)}_{\mu_1\ldots\mu_r} = 0\,, \tag{2.2.46}$$

$$-i\partial^{\nu}\rho^{(r)(p)}_{\nu\mu_1\ldots\mu_{r-1}} + m\rho^{(r-1)(p+1)}_{\mu_1\ldots\mu_{r-1}} = 0\,, \tag{2.2.47}$$

with $r \in \{1,\ldots,s-1\}, p \in \{0,\ldots,s-1\}, r+p=s-1$. The gauge transformation reads, in components,

$$\delta\varphi^{(r)(p)}_{\mu_1\ldots\mu_r} = -i\,r\,\partial_{(\mu_1}\rho^{(r-1)(p)}_{\mu 2\ldots\mu_r)} + m\,p\,\rho^{(r)(p-1)}_{\mu 1\ldots\mu_r}\,, \qquad \forall r,p \in \{0,\ldots,s\}, r+p=s. \tag{2.2.48}$$

If we write the system of equations (2.2.45) explicitly:

$$-i\partial^{\nu}\varphi^{(s)(0)}_{\nu\mu_1\ldots\mu_{s-1}} + m\varphi^{(s-1)(1)}_{\mu_1\ldots\mu_{s-1}} = 0 \tag{2.2.49}$$

$$-i\partial^{\nu}\varphi^{(s-1)(1)}_{\nu\mu_1\ldots\mu_{s-2}} + m\varphi^{(s-2)(2)}_{\mu_1\ldots\mu_{s-2}} = 0 \tag{2.2.50}$$

$$\vdots \tag{2.2.51}$$

$$-i\partial^{\nu}\varphi^{(1)(s-1)}_{\nu} + m\varphi^{(0)(s)} = 0 \tag{2.2.52}$$

we see that we can get rid of all the tensors labelled by a non-zero value of p by moving to a gauge in which $\varphi^{(s-1)(1)}$ vanishes. In this gauge, the tensor $\varphi^{(s)(0)}$ satisfies the Fierz-Pauli equation for a massive spin-s particle, without the traceless condition. Hence, it would propagate a reducible massive multiplet, as in the massless case. Such a gauge can be reached through the transformation

$$\delta\varphi^{(s-1)(1)}_{\mu_1\ldots\mu_{s-1}} = -i\,(s-1)\,\partial_{(\mu_1}\rho^{(s-2)(1)}_{\mu 2\ldots\mu_{s-1})} + m\,\rho^{(s-1)(0)}_{\mu 1\ldots\mu_{s-1}} = -\varphi^{(s-1)(1)}_{\mu_1\ldots\mu_{s-1}}\,. \tag{2.2.53}$$

To solve this equation, one must take into account the dependence between $\rho^{(s-2)(1)}$ and $\rho^{(s-1)(0)}$, from (2.2.47),

$$-i\partial^{\nu}\rho^{(s-1)(0)}_{\nu\mu_1\ldots\mu_{s-2}} + m\rho^{(s-2)(1)}_{\mu_1\ldots\mu_{s-2}} = 0\,. \tag{2.2.54}$$

This makes the equation for the gauge condition not algebraic (i.e. we cannot set $\rho^{(s-2)(1)}$ to zero in (2.2.53)). Taking the symmetrized gradient of the above expression and using the massive Klein-Gordon equation for $\rho^{(s-1)(0)}$, we can write

$$-i\partial_{(\mu_1}\rho^{(s-2)(1)}_{\mu_2\dots\mu_{s-1})} = \frac{1}{m}\frac{s}{s-1}\partial^\nu\partial_{(\nu}\rho^{(s-1)(0)}_{\mu_1\dots\mu_{s-1})} - m\frac{1}{s-1}\rho^{(s-1)(0)}_{\mu_1\dots\mu_{s-1}}\,, \tag{2.2.55}$$

thus, the gauge condition (2.2.53) becomes

$$\frac{1}{m}\,s\,\partial^\nu\partial_{(\nu}\rho^{(s-1)(0)}_{\mu_1\dots\mu_{s-1})} = -\varphi^{(s-1)(1)}_{\mu_1\dots\mu_{s-1}}\,. \tag{2.2.56}$$

Provided the above differential equations admit a solution, at least locally, one finds the following for the only non-vanishing $\varphi^{(s)(0)}$:

$$(\Box - m^2)\varphi^{(s)(0)}_{\mu_1\dots\mu_s}(x) = 0\,, \tag{2.2.57}$$

$$\partial^\nu\varphi^{(s)(0)}_{\nu\mu_2\dots\mu_s}(x) = 0\,. \tag{2.2.58}$$

It forms a rank-s symmetric representation of the little group $SO(D-1)$. Since it is not traceless, it propagates the degrees of freedom of a multiplet containing massive particles of spin $s, s-2, s-4, \dots, 0$ or $s, s-2, s-4, \dots, 1$ for even or odd s, respectively.

2.3 Interactions

So far we have discussed worldline theories that, upon quantization, describe the propagation of free spin$-s$ particles. To make contact with interacting QFTs, one couples the worldline to background fields, obtaining an interacting worldline theory. This theory will represent the interaction of the particle with either other quantum particles or an external potential determined by the experimental apparatus. As discussed previously, the way a specific interaction is introduced is constrained both by the symmetry one wants to preserve manifestly on the worldline, the Lorentz symmetry, and by the symmetries that the relativistic field equations should enjoy in order to describe that interaction.

Let us suppose we want to describe the interaction with a scalar particle, then, we will introduce a scalar potential, on the other hand the interaction with a gauge boson will require a gauge potential. Coupling the particle with the graviton requires the introduction of a background metric on the target space time.

Let us consider the free worldline theory (1.1.24). Then, consider a potential V, of generic type. The interacting worldline theory is the constrained Hamiltonian system with "covariantized" constraints $\tilde{C}_a$ containing the potential,

$$S[z^A, \lambda^a; V] = \int dt \left(\frac{1}{2}(\Omega^{-1})_{AB}z^A\dot{z}^B - H'(z) - \lambda^a\tilde{C}_a(z;V);\right)\,. \tag{2.3.1}$$

Thus, the quantized theory, through the implementation of the above constraints, will reproduce the relativistic field equations for a particle coupled to the potential V.

Let us consider the coupling of a spin$-s$ particle, described by the $O(N)$ spinning particle action (2.1.56), with $N = 2s$, with a $U(1)$ background gauge field $A_\mu(x)$ [19]. We introduce the covariant momenta

$$\pi_\mu = p_\mu - qA_\mu \,, \tag{2.3.2}$$

where q is the charge of the particle, in the constraints Q_i:

$$Q_i \to \tilde{Q}_i = \pi_\mu \psi_i^\mu \,. \tag{2.3.3}$$

We define the Hamiltonian as the N-extendend generalization of the minimal Hamiltonian H_{min}:

$$H \to \tilde{H} = H_{min} + i\frac{q}{2}F_{\mu\nu}\psi_i^\mu\psi_i^\nu = \frac{1}{2}\pi_\mu\pi^\mu + i\frac{q}{2}F_{\mu\nu}\psi_i^\mu\psi_i^\nu \,, \tag{2.3.4}$$

with $F_{\mu\nu}$ the field strength given by

$$\{\pi_\mu, \pi_\nu\}_{PB} = q(\partial_\mu A_\nu - \partial_\nu A_\mu) = qF_{\mu\nu} \,. \tag{2.3.5}$$

Substituting these constraints into (2.1.56), eliminating the momenta p_μ through their equations of motion

$$p^\mu = e^{-1}(\dot{x}^\mu - i\psi_i^\mu\chi_i) + qA_\mu \,, \tag{2.3.6}$$

the interacting $O(N)$ action in configuration space becomes

$$S_{\text{co}} = S_{\text{co}}^{\text{free}} + S_{\text{co}}^{\text{int}} \,, \tag{2.3.7}$$

where $S_{\text{co}}^{\text{free}}$ the free action (2.1.56) in configuration space,

$$S_{\text{co}}^{\text{free}} = \int \mathrm{d}\tau \left[\frac{1}{2}e^{-1}\left(\dot{x}^\mu - i\chi_i\psi_i^\mu\right)^2 + \frac{i}{2}\psi_{i\mu}\dot{\psi}_i^\mu - \frac{i}{2}a_{ij}\psi_i^\mu\psi_{j\mu} \right] , \tag{2.3.8}$$

and $S_{\text{co}}^{\text{int}}$ the additional term due to the interacting part,

$$S_{\text{co}}^{\text{int}} = q\int \mathrm{d}\tau \left(\dot{x}^\mu A_\mu - i\frac{e}{2}F_{\mu\nu}\psi_i^\mu\psi_i^\nu \right) . \tag{2.3.9}$$

Now, we may ask: is this action still gauge invariant under the transformations in Subsection 2.1.3 ? This is equivalent to ask if the system is still a first class constrained Hamiltonian system and the constraints' algebra is preserved. Let us answer the question and consider, for instance, the transformation generated by $V = \xi_i\tilde{Q}_i$. In configuration space, it acts on the fields through

$$\delta x^\mu = i\xi_i\psi_i^\mu\,, \tag{2.3.10}$$
$$\delta\psi_i^\mu = -e^{-1}(\dot{x}^\mu - i\psi_i^\mu\chi_i)\xi_i\,. \tag{2.3.11}$$

so it acts as a supersymmetry transformation does in the free case. We keep the transformation of the gauge fields to be the same as in (2.1.58). The action $S_{\rm co}^{\rm free}$ is invariant up to boundary terms, while the action $S_{\rm co}^{\rm int}$ transforms, up to a boundary term, as

$$\delta S_{\rm co}^{\rm int} = \int {\rm d}\tau\,\left(\xi_i\psi_j^\mu(\chi_i\psi_j^\nu - \chi_j\psi_i^\nu)F_{\mu\nu} + \frac{e}{2}\xi_i\psi_j^\mu\psi_j^\nu\psi_i^\sigma\partial_\sigma F_{\mu\nu}\right)\,. \tag{2.3.12}$$

The second term vanishes due to the Bianchi identity $\partial_{[\sigma}F_{\mu\nu]} = 0$. The first term vanishes only for $N = 0, 1$, while for $N \geqslant 2$ it requires $F_{\mu\nu} = 0$. In deriving this variation, we assumed the constraints' algebra (2.1.55) remains unchanged for the new $\tilde{C}_a = (\tilde{H}, \tilde{Q}_i, J_{ij})$, since we used the variation (2.1.58) for their gauge fields. If the assumption is right, then the above variation vanishes. We see that for the $N = 1$ model (and, trivially, $N = 0$) this is the case, while for $N \geqslant 2$ the covariantized constraints $\tilde{C}_a$ break the SUSY algebra (2.1.55). However, this does not exclude, a priori, that they form a different algebra. One can check, by computing their Poisson brackets, that it does not happen, hence, for $N \geqslant 2$, the covariantized constraints $(\tilde{H}, \tilde{Q}_i, J_{ij})$ are no longer first class constraints. Equivalently, the interacting worldline is not a gauge system any more. The analysis for the massive case furnishes the same results.

This renders the quantization of these models more complicated, as one has to deal with second class constraints, which are difficult to solve explicitly in order to work in the reduced phase space. Physically, we noticed how worldline theories led to relativistic field equations by applying the physicality conditions for first class constraints. For this reason, the role of these latter is fundamental in the first-quantized approach to QFT.

Such a problem also arises when the $O(N)$ spinning particle is coupled to a gravitational background and $N > 2$ [40] (see [48, 49] for the coupling of the $O(2)$ spinning particle).

To cope with situations where the interaction breaks worldline SUSY or BUSY, BRST quantization techniques have been employed. The first important progress was made in [20] for the $O(2)$ spinning particle coupled to a Yang-Mills background and then the same strategy was applied in the recent works [21–24] for different N and interactions. These methods enable the construction of a consistent quantum theory for the interacting worldline, allowing one to describe the coupling of the particle with the background potential. We will discuss it in the next chapter, where we will investigate the behaviour of the bosonic spinning particle when it couples to a background $U(1)$ gauge field.

2.4 Path integral representations

The path integral quantization of worldline theories permits to compute propagators and one-loop effective actions in the presence of a background potential. Precisely the representation of the propagator and one-loop effective action in QED and scalar QED in terms of relativistic particles' path integral, worked independently by Feynman and Schwinger, posed the basis for a systematic formulation of QFT in terms of relativistic particle's actions.

Let us consider the action (2.3.1), under the assumption that the interaction has been consistently introduced via one of the approaches discussed in the previous section. Then, denoting by $\hat{\phi}(x)$ the field operator associated to the described particle, worldline theories show the following relation between the propagator in the presence of the background V and the path integral over a worldline being an open curve I in spacetime:

$$\langle\Omega|\, T\{\hat{\phi}^{\dagger}(x'')\, \hat{\phi}(x')\}\, |\Omega\rangle \sim Z_I = \int_{BC(I)} \frac{\mathrm{D}z\mathrm{D}\lambda}{\mathrm{Vol(Gauge)}} \mathrm{e}^{iS[z,\lambda;V]}\,, \tag{2.4.1}$$

with boundary conditions (BC) $I(\tau_i) = x', I(\tau_f) = x''$, x the spacetime coordinates (remember $z^A = (x^\mu, \dots)$). The symbol "$\sim$" indicates that the path integral is known up to an infinite normalization factor.

In addition, the one-loop effective action in the presence of the background V can be obtained by the path integral over a closed curve (a loop, sometimes called the worldloop) L,

$$\Gamma_{\text{1-loop}}[\phi; V] \sim Z_L = \int_{BC(L)} \frac{\mathrm{D}z\mathrm{D}\lambda}{\mathrm{Vol(Gauge)}} \mathrm{e}^{iS[z,\lambda;V]}\,, \tag{2.4.2}$$

with $L(\tau_i) = L(\tau_f)$, i.e. periodic boundary conditions for the x.

Since the theory on the worldline is a gauge theory, it is necessary to formally divide by the volume of gauge equivalent configurations, i.e., practically, to gauge fix the path integral. This is achieved with the Faddeev-Popov method or using BRST techniques. The result is that one fixes the values of the gauge fields through their gauge transformations. Values that cannot be globally (i.e. in all the worldline) reached through a gauge transformation are called moduli. The moduli space depends on the chosen topology for the worldline. Formally, for each path integral measure over the gauge fields λ, we can split it into an integration over gauge equivalent configurations, parametrized by the set of gauge parameters $\epsilon = (\epsilon_1, \dots, \epsilon_a)$, and gauge inequivalent configurations, parametrized by the modulus μ_λ:

$$\mathrm{D}^a\lambda = \mathrm{d}^a\epsilon\, \mathrm{d}\mu_\lambda\, J(\mu_\lambda)\,, \tag{2.4.3}$$

with $J(\mu_\lambda)$ the Jacobian from the change of integration variable. At the end, one is left with an integration over the moduli space of the gauge fields. A moduli-dependent term, $f(\mu_\lambda)$, can appear from the integration of the ghosts. The path integral over the phase space variables is

$$Z_{\mathcal{T}} \sim \int \mathrm{d}\mu_\lambda J(\mu_\lambda) f(\mu_\lambda) \int_{BC(\mathcal{T})} \mathrm{D}z \, \mathrm{e}^{iS[z,\mu_\lambda;V]} \,, \tag{2.4.4}$$

with $\mathcal{T} \in \{I, L\}$ indicating the only two possible topologies for the worldline, the open or closed curve, i.e. the line and the circle. Boundary conditions for the fields depends on the topology.

Let us see explicitly how this works for a massive scalar particle coupled to a background $U(1)$ gauge field A_μ, for instance. Working in Euclidean metric, the Euclidean full propagator is

$$\begin{aligned} D^{x'x''}[A] &:= \langle \Omega | \mathcal{T} \hat{\phi}^\dagger(x'') \, \hat{\phi}(x') \, |\Omega\rangle = \langle x''| \left[-(\partial - iqA)^2 + m^2 \right]^{-1} |x'\rangle \\ &= \int_0^\infty \mathrm{d}T \, \mathrm{e}^{-m^2 T} \, \langle x''| \, e^{-(-(\partial - iqA)^2)T} \, |x'\rangle \,. \end{aligned} \tag{2.4.5}$$

By using the Euclidean version of the path integral

$$\langle x''| \, \mathrm{e}^{-T\hat{H}} \, |x'\rangle = \int_{x(0)=x'}^{x(T)=x''} \mathrm{D}x \, \mathrm{e}^{-\int_0^T d\tau \left(\frac{m}{2} \dot{x}^2 + V(x(\tau)) \right)} \,, \tag{2.4.6}$$

with $\hat{H} = -\frac{1}{2m}\Box + V(x)$, we rewrite

$$D^{x'x''}[A] = \int_0^\infty \mathrm{d}T \, \mathrm{e}^{-m^2 T} \int_{x(0)=x'}^{x(T)=x''} \mathrm{D}x \, \mathrm{e}^{-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} - iq \, \dot{x} \cdot A(x) \right)} \,. \tag{2.4.7}$$

This is called the Fock-Schwinger proper time representation of the propagator and T is the so-called Schwinger proper time. In this representation, the propagator can be obtained from a first-quantized picture within the Worldline Formalism.

We consider the Euclidean, massive version of the action (2.3.7) for $N = 0$:

$$S[x^\mu, e; A] = \int_0^1 \mathrm{d}\tau \left(\frac{1}{2} e^{-1} \dot{x}^\mu \dot{x}_\mu + \frac{1}{2} e m^2 - iqA_\mu(x) \dot{x}^\mu \right) \,, \tag{2.4.8}$$

where we have chosen to parametrize the worldline with the parameter $\tau \in [0, 1]$. Let us compute the (Euclidean) path integral of the above action with boundary conditions $x(0) = x', x(1) = x''$. From the general formula for the gauge-fixed path integral (2.4.4), we have to find the modulus for the gauge field e, with gauge symmetry:

$$\delta e = \dot{\epsilon} \,. \tag{2.4.9}$$

This transformation leaves the length of the worldline invariant:

$$\mathfrak{L}[e] = \int_0^1 \mathrm{d}\tau\, e(\tau)\,, \tag{2.4.10}$$

as $\epsilon(0) = \epsilon(1) = 0$. Hence, for each einbein $e(\tau)$ such that the length is $\mathfrak{L}[e] = 2T$, with T an arbitrary positive value for e positive, the einbein $e' = e + \delta e$ will determine the same length for the worldline. We can fix such gauge equivalent einbein fields to have a constant value $e(\tau) = 2T$, by solving the differential equation for $\epsilon(\tau)$ in (2.4.9). Thus, gauge inequivalent einbein fields are parametrized by the modulus T, the length of the worldline. After gauge fixing the action with $e(\tau) = 2T$, the path integral of the relativistic particle described by the action (2.4.8) reads:

$$Z_I[A] \sim \int_0^\infty \mathrm{d}T\, \mathrm{e}^{-m^2 T} \int_{x(0)=x'}^{x(1)=x''} \mathrm{D}x\, \mathrm{e}^{-\int_0^1 d\tau \left(\frac{\dot{x}^2}{4} - iq\,\dot{x}\cdot A(x)\right)}\,, \tag{2.4.11}$$

being exactly equal to the full propagator (2.4.7) (after the rescaling $\tau \to \tau/T$) if we choose a normalization factor equal to 1.

The same can be observed for the Euclidean one-loop effective action $\Gamma_{\text{1-loop}}[A]$,

$$\mathrm{e}^{-\Gamma_{\text{1-loop}}[A]} = \int \mathrm{D}\phi \mathrm{D}\phi^* \mathrm{e}^{-S[\phi,\phi^*;A]} = \mathrm{Det}^{-1}\left(-(\partial - iqA)^2 + m^2\right)\,, \tag{2.4.12}$$

from which

$$\Gamma_{\text{1-loop}}[A] = -\ln \mathrm{Det}^{-1}\left(-(\partial - iqA)^2 + m^2\right) = \mathrm{Tr}\ln\left(-(\partial - iqA)^2 + m^2\right)\,. \tag{2.4.13}$$

By using the integral representation of the logarithm

$$\ln\frac{a}{b} = -\int_0^\infty \frac{dT}{T}\left(\mathrm{e}^{-aT} - \mathrm{e}^{-bT}\right)\,, \tag{2.4.14}$$

we obtain the Fock-Schwinger proper time representation of the one-loop effective action,

$$\Gamma_{\text{1-loop}}[A] = -\int_0^\infty \frac{\mathrm{d}T}{T}\,\mathrm{Tr}\,\mathrm{e}^{-T\left(-(\partial - iqA)^2 + m^2\right)} = -\int_0^\infty \frac{\mathrm{d}T}{T}\int \mathrm{d}^D x\,\langle x|\, e^{-T(-(\partial - iqA)^2 + m^2)}\,|x\rangle\,. \tag{2.4.15}$$

Then, from

$$\mathrm{Tr}\,\mathrm{e}^{-T\hat{H}} = \int \mathrm{d}^D x\,\langle x|\, e^{-T\hat{H}}\,|x\rangle = \int_{x(0)=x(T)} \mathrm{D}x\,\mathrm{e}^{-\int_0^T \mathrm{d}\tau\left(\frac{m}{2}\dot{x}^2 + V(x(\tau))\right)}\,, \tag{2.4.16}$$

one gets the final expression

$$\Gamma_{\text{1-loop}}[A] = -\int_0^\infty \frac{\mathrm{d}T}{T} \mathrm{e}^{-m^2 T} \int_{x(0)=x(T)} \mathrm{D}x \, \mathrm{e}^{-\int_0^T \mathrm{d}\tau \left(\frac{\dot{x}^2}{4} - iq\,\dot{x}\cdot A(x)\right)} \,. \tag{2.4.17}$$

The above effective action can be obtained from the path integral of the system (2.4.8) over a worldline with the topology of the circle, with periodic boundary conditions $x(0) = x(1)$. The gauge-fixed path integral now reads

$$Z_L[A] \sim \int_0^\infty \frac{\mathrm{d}T}{T} \, \mathrm{e}^{-m^2 T} \int_{x(0)=x(1)} \mathrm{D}x \, \mathrm{e}^{-\int_0^1 d\tau \left(\frac{\dot{x}^2}{4} - iq\,\dot{x}\cdot A(x)\right)} \,. \tag{2.4.18}$$

The measure in the modulus space of e, $1/T$, comes from the fact that the condition for (2.4.9) to be a gauge symmetry now is $\epsilon(0) = \epsilon(1)$. This allows the existence of killing vectors ϵ' such that $\delta e = 0$. Thus, they are constant vectors $\epsilon' = k$. In the gauge orbit $e = 2T$, these vectors correspond to the generators of time translation in τ (remember $\delta x^\mu = e^{-1}\epsilon \dot{x}^\mu$). Thus, we need to divide by the volume of these transformations, which represent a residual gauge symmetry. Their volume is the length of the worldline. If we fix the overall normalization factor in (2.4.18) to be -1, then it reproduces the effective action induced by a charged massive scalar in the presence of an Abelian background (2.4.17).

Chapter 3

Charged massive vector bosons

In this chapter, we investigate massive spin-1 particles coupled to electromagnetism. We shall describe this interaction through the worldline techniques discussed in the previous chapter. The formulation is first developed in terms of the bosonic spinning particle and then, for completeness, within the fermionic model. In both models, consistency conditions for the electromagnetic background emerge. For a particular class of backgrounds satisfying these conditions, we derive the one-loop effective action for the electromagnetic field induced by the spin-1 particle and compute the particle-antiparticle pair production rate in a constant electric field. The material presented in the following sections is based on [30].

3.1 Free massive vector bosons from the $Sp(2)$ particle

We aim to describe massive spin-1 particles from the bosonic spinning particle action discussed in Section 2.2. Hence, we impose the operator constraint J_c in (2.2.39) with CS coupling given by (2.2.40) with $s = 1$. A generic state at occupation number $s = 1$ (2.2.43) is given by

$$|\psi\rangle = W_\mu(x)\alpha^\mu |0\rangle - i\varphi(x)\beta |0\rangle \, , \tag{3.1.1}$$

where we have factored out $-i$ for convenience. The remaining physicality conditions (2.2.44) and (2.2.45) translate into the following set of equations:

$$(\Box - m^2)W_\mu = 0 \, , \tag{3.1.2}$$

$$(\Box - m^2)\varphi = 0 \, , \tag{3.1.3}$$

$$\partial^\mu W_\mu + m\varphi = 0 \, . \tag{3.1.4}$$

The gauge symmetries (2.2.48) related to the null state $L\,|\rho\rangle$, with $|\rho\rangle = \rho(x)\,|0\rangle$, are given by

$$\delta W_\mu = \partial_\mu \rho \,, \quad \delta\varphi = -m\rho \,. \tag{3.1.5}$$

Using the gauge symmetry to set $\varphi(x) = 0$, we recover the standard Fierz-Pauli equations for a massive spin-1 field $W_\mu(x)$.

3.2 Counting degrees of freedom

The degrees of freedom propagated by the massive model, for different values of the CS coupling, can be computed through the one-loop effective action obtained by path integrating the free action on worldlines with the topology of a circle S^1. After fixing the overall normalization to match the scalar case, we will get the number of degrees of freedom in the other sectors of the worldline theory.

3.2.1 Gauge fixing

We first need to define a finite path integral for the worldline action. Let us consider the following path integral

$$Z_{S^1} \sim \int_{PBC} \frac{DGDX}{\mathrm{Vol(Gauge)}}\, \mathrm{e}^{iS[z,\lambda]} \,, \tag{3.2.1}$$

where $\lambda = (e, \bar{u}, u, a)$ denotes the gauge fields, whereas $z = (x^\mu, p_\mu, \alpha^\mu, \bar{\alpha}_\mu, \beta, \bar{\beta})$ collectively denotes all dynamical variables parametrizing the phase space. Periodic boundary conditions (PBC) are understood to implement the path integral on the circle. It will be useful to explicitly rewrite the action, namely the one in (2.2.3) with the constraints in (2.2.36),

$$\begin{aligned} S[z,\lambda] = \int_0^1 \mathrm{d}\tau \Big[& p_\mu \dot{x}^\mu - i\bar{\alpha}_\mu \dot{\alpha}^\mu - \frac{e}{2}(p^\mu p_\mu + m^2) - \bar{u}(\alpha^\mu p_\mu + \beta m) \\ & - u(\bar{\alpha}^\mu p_\mu + \bar{\beta} m) - a(\alpha^\mu \bar{\alpha}_\mu + \beta\bar{\beta} - c) \Big] \,. \end{aligned} \tag{3.2.2}$$

We prefer to work in the Euclidean version of the theory, so that we first pass to configuration space by eliminating the momenta p_μ:

$$\frac{\partial S}{\partial p_\mu} = 0 \,, \iff p_\mu = e^{-1}(\dot{x}\mu - u\bar{\alpha}^\mu - \bar{u}\alpha^\mu) \,, \tag{3.2.3}$$

Wick rotate the action with $\tau \to -i\tau$, taking into account also the rotation of the gauge field $a \to ia$, and get the Euclidean worldline action

$$S_E[z,\lambda] = \int_0^1 \mathrm{d}\tau \bigg[\frac{1}{2e}(\dot{x}^\mu + iu\bar{\alpha}^\mu + i\bar{u}\alpha^\mu)^2 + \alpha_\mu \dot{\bar{\alpha}}^\mu + \beta\dot{\bar{\beta}} + \frac{e}{2}m^2 \\ + u\bar{\beta}m + \bar{u}\beta m + ia(\alpha^\mu\bar{\alpha}_\mu + \beta\bar{\beta} - c)\bigg] . \tag{3.2.4}$$

The transformation rules for phase space variables and gauge fields in Euclidean time are obtained by Wick rotating the gauge parameters $\epsilon \to -i\epsilon, \xi \to -i\xi, \bar{\xi} \to -i\bar{\xi}$. We report the transformation of the gauge fields, useful to study the gauge fixing:

$$\delta e = \dot{\epsilon} + 2u\bar{\xi} - 2\bar{u}\xi \ , \tag{3.2.5a}$$
$$\delta u = \dot{\xi} - ia\xi + i\phi u \ , \tag{3.2.5b}$$
$$\delta \bar{u} = \dot{\bar{\xi}} + ia\bar{\xi} - i\phi\bar{u} \ , \tag{3.2.5c}$$
$$\delta a = \dot{\phi} \ . \tag{3.2.5d}$$

The Wick rotation of the gauge field a is needed in order to maintain the $U(1)$ group compact.

As we know, the overcounting in (3.2.1) from summing over gauge equivalent configurations is formally taken into account by dividing by the volume of the gauge group. We will work the BRST procedure to define this path integral. To do so, it is useful to notice that we can gauge fix the worldline gauge fields to constant moduli

$$\lambda = (e, \bar{u}, u, a) \quad \to \quad \hat{\lambda} = (2T, 0, 0, \theta) \ . \tag{3.2.6}$$

We have already seen T, the modulus related to the einbein $e(\tau)$, corresponding to the gauge-invariant worldline length. On the other hand, the gauge fields $(u, \bar{u})$ can be gauge-fixed to zero. The modulus θ is associated with the worldline $U(1)$ gauge field $a(\tau)$ and parametrizes the gauge invariant Wilson loop. It is responsible for the reduction of the Hilbert space to a given spin sector. Let us see how these moduli emerge.

From the finite $U(1)$ transformation generated by ϕJ

$$u(\tau) \to u'(\tau) = \mathrm{e}^{i\phi(\tau)}u(\tau) \, ,$$
$$\bar{u}(\tau) \to \bar{u}'(\tau) = \mathrm{e}^{-i\phi(\tau)}\bar{u}(\tau) \, ,$$
$$a(\tau) \to a'(\tau) = a(\tau) + \dot{\phi}(\tau) \, ,$$

since periodic boundary conditions for $u, \bar{u}$ hold, the gauge transformations $g = \mathrm{e}^{i\phi(\tau)}$ must be periodic functions on $[0,1]$, that is

$$\phi(1) = \phi(0) + 2\pi n\,, \qquad n \in \mathbb{N}\,. \tag{3.2.7}$$

We can define the gauge invariant Wilson loop

$$w[a] = \mathrm{e}^{i\int_0^1 \mathrm{d}\tau\, a}\,. \tag{3.2.8}$$

Then, we can transform each field $a(\tau)$ into a constant field $a'(\tau) = \theta$ through

$$\phi(\tau) = \phi(0) + \theta\,\tau - \int_0^\tau \mathrm{d}\tau'\, a(\tau')\,, \tag{3.2.9}$$

but this constant cannot be arbitrary, but, from (3.2.7), it is fixed to be

$$\theta = \int_0^1 \mathrm{d}\tau\, a(\tau) + 2\pi n\,. \tag{3.2.10}$$

Being the Wilson loop w defined on the gauge slices of $a(\tau)$, and, from the above equation, $w[\theta] = w[a]$, the variable θ is a modulus in the space of the field $a(\tau)$, as it parametrizes gauge inequivalent configurations. From $w[\theta] = w[\theta + 2\pi n]$, we can take $[0, 2\pi]$ as the fundamental region of the moduli space. The killing vectors ϕ_0, such that $\delta a = \dot{\phi}_0 = 0$, are constant vectors $\phi_0 = k$.

Moving to the field u (equivalently for $\bar{u}$), the gauge transformation generated by $\xi\bar{L}$ can be employed to set it to zero, by solving the first order linear differential equation in ξ with $a = \theta$ (3.2.5b),

$$0 = \dot{\xi} - i\theta\xi + u\,. \tag{3.2.11}$$

In fact, no constraint arises in the constant values the field u can take. There is not a gauge invariant quantity for such field as long as θ is different from zero. In that case, the quantity $\int_0^1 \mathrm{d}\tau\, \xi(\tau)$ is gauge invariant due to the periodic boundary conditions $\xi(1) = \xi(0)$. We will see the value $\theta = 0$ can be handled through a limiting procedure, as in the standard $N = 2$ particle case [48, 49]. Therefore, $(u, \bar{u})$ carry no moduli, they are "pure gauge".

Finally, in Section 2.4, we have seen how the space of gauge fields e shows moduli T associated to the length of the worldline, and the killing vectors correspond, in each gauge slice, to the generators of time translation.

Having examined the fields space, we construct a finite path integral by gauge fixing the action and splitting the functional integration measure over these fields in moduli- and gauge-dependent measures. To obtain a BRST invariant gauge-fixed action, one introduces a set of anticommuting ghost, anticommuting antighost, and commuting auxiliary field for each gauge field by

$$\begin{aligned}
&\epsilon \to (c, b, \varpi_\epsilon)\,, && \epsilon = \Lambda c\,, && \delta_B b = i\Lambda\varpi_\epsilon\,, && \delta_B\varpi_\epsilon = 0\,,\\
&\xi \to (\mathcal{C}, \bar{\mathcal{B}}, \varpi_\xi)\,, && \xi = \Lambda\mathcal{C}\,, && \delta_B\bar{\mathcal{B}} = i\Lambda\varpi_\xi\,, && \delta_B\varpi_\xi = 0\,,\\
&\bar\xi \to (\bar{\mathcal{C}}, \mathcal{B}, \varpi_{\bar\xi})\,, && \bar\xi = \Lambda\bar{\mathcal{C}}\,, && \delta_B\mathcal{B} = i\Lambda\varpi_{\bar\xi}\,, && \delta_B\varpi_{\bar\xi} = 0\,,\\
&\phi \to (f, g, \varpi_\phi)\,, && \phi = \Lambda f\,, && \delta_B g = i\Lambda\varpi_\phi\,, && \delta_B\varpi_\phi = 0\,,
\end{aligned} \tag{3.2.12}$$

where $\delta_B := \Lambda\mathcal{S}$ indicates the BRST transformation, Λ is an anticommuting parameter and $\mathcal{S}$ the Slavnov operator. The nilpotency of the BRST transformation requires $\mathcal{S}^2$. Thus, each gauge parameter is replaced by the constant Λ times the associated ghost, and the BRST transformation for the gauge fields are obtained from (3.2.5) and the above replacement for the parameters. Through the gauge fermion:

$$\Psi = \int_0^1 \mathrm{d}\tau \big(b\,F_1(c,\varpi_\epsilon) + \bar{\mathcal{B}}\,F_2(\mathcal{C},\varpi_\xi) + \mathcal{B}\,F_3(\bar{\mathcal{C}},\varpi_{\bar\xi}) + gF_4(f,\varpi_\phi)\big)\,, \tag{3.2.13}$$

with F_1, F_2, F_3, F_4 arbitrary functions that parametrize the choice of the gauge fixing, the gauge-fixed action reads

$$S = S_E[z,\lambda] + \frac{\delta}{\delta\Lambda}\Psi\,, \tag{3.2.14}$$

where $\frac{\delta}{\delta\Lambda}$ indicates the BRST transformation with the parameter Λ factorized and removed from the left. With this action, the path integral becomes

$$Z_{S^1} \sim \int_{PBC} \mathrm{D}z\mathrm{D}\lambda\mathrm{D}G\,\mathrm{e}^{-S}\,, \tag{3.2.15}$$

where G denotes the set of all ghosts, antighosts and auxiliary fields.

Each gauge field can be decomposed in a modulus-dependent part and gauge-dependent part, infinitesimally given by

$$\lambda^i = \mu_{\lambda^i} + \delta\lambda^i|_{\lambda^i=\mu_{\lambda^i}}\,, \tag{3.2.16}$$

and the measure correspondingly splits into

$$\mathrm{D}^a\lambda = \mathrm{D}'^a\epsilon\,\mathrm{d}\mu_\lambda\,J(\mu_\lambda)\,, \tag{3.2.17}$$

where a prime indicates that the integration is not carried over the killing vectors. We choose a function Ψ such that the gauge fields are set equal to their moduli and the variations due to gauge transformations are set to zero:

$$\Psi = \int_0^1 d\tau\,\big(b(2T-e) - \bar{\mathcal{B}}u - \mathcal{B}\bar u + g(\theta - a)\big)\,. \tag{3.2.18}$$

In fact, if one computes $\mathcal{S}\Psi$, then expresses the gauge fields as

$$e = 2T + \delta e|_{e=2T}\,, \quad u = \delta u|_{u=0}\,, \quad \bar{u} = \delta\bar{u}|_{\bar{u}=0}\,, \quad a = \theta + \delta a|_{a=\theta}\,, \tag{3.2.19}$$

and integrates over the ϖ variables, this produces delta functions in the variations of the fields $\delta(\delta e|_{e=2T}), \ldots, \delta(\delta a|_{a=\theta})$, that, after integration of the gauge parameters from which these variations depend, $\int \mathrm{D}'\epsilon \ldots \mathrm{D}'\phi$, fixes the values of the gauge fields as in (3.2.6) in the total action (3.2.14). At this point, the path integral reads

$$Z_{S^1} \sim \int_0^\infty \mathrm{d}T J(T) \int_0^{2\pi} \mathrm{d}\theta J(\theta) \int_{PBC} \mathrm{D}z\mathrm{D}\bar{G}\, \mathrm{e}^{-S}\,, \tag{3.2.20}$$

with $\bar{G}$ denoting the set of ghosts and antighosts, the action

$$S = S_E[z,\hat{\lambda}] + \int_0^1 \mathrm{d}\tau \left(b\dot{c} + \bar{\mathcal{B}}\left(\frac{d}{d\tau} - i\phi\right)\mathcal{C} + \mathcal{B}\left(\frac{d}{d\tau} + i\phi\right)\bar{\mathcal{C}} + g\dot{f} \right)\,, \tag{3.2.21}$$

and, explicitly,

$$S_{\mathrm{E}}[z,\hat{\lambda}] = \int_0^1 \mathrm{d}\tau \left[\frac{1}{4T}\dot{x}^2 + \alpha_\mu(\partial_\tau + i\theta)\bar{\alpha}^\mu + \beta(\partial_\tau + i\theta)\bar{\beta} + m^2 T - ic\theta \right]\,. \tag{3.2.22}$$

From the integration of the ghosts associated to the BUSY parameters, we obtain the modulus-dependent functional determinants (the others can be reabsorbed into the overall normalization of the path integral)

$$\mathrm{Det}(\partial_\tau + i\theta)\mathrm{Det}(\partial_\tau - i\theta)\,, \tag{3.2.23}$$

evaluated with periodic boundary conditions (as the ghost and antighost inherit the boundary conditions of the gauge parameters they are associated with). For the functional determinants arising from the integration measures, the modulus-dependent one is that for T [50, 51]

$$J(T) = T^{-1}\,. \tag{3.2.24}$$

The final form of the path integral is conveniently written

$$Z_{S^1} \sim \int_0^\infty \frac{\mathrm{d}T}{T} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \mathrm{Det}(\partial_\tau + i\theta)\mathrm{Det}(\partial_\tau - i\theta) \int_{PBC} \mathrm{D}z\, \mathrm{e}^{-S_E[z,\hat{\lambda}]}\,. \tag{3.2.25}$$

The path integral over the coordinates x^μ can be computed by decomposing the space of closed loops in a constant path $\bar{x}^\mu$ (zero mode of the free kinetic operator) and fluctuations

$t^\mu(\tau)$ satisfying Dirichlet boundary conditions (DBC)

$$x^\mu(\tau) = \bar{x}^\mu + t^\mu(\tau) \ , \quad \text{with} \quad t^\mu(0) = t^\mu(1) = 0 \ , \tag{3.2.26}$$

such that

$$\int_{x(0)=x(1)} \mathrm{D}x \, \mathrm{e}^{-\frac{1}{2}\int_0^1 \mathrm{d}\tau \, x^\mu \left(-\frac{1}{2T}\partial_\tau^2 \delta_{\mu\nu}\right) x^\nu} = \int \mathrm{d}^D \bar{x} \int_{t(0)=t(1)=0} \mathrm{D}t \, \mathrm{e}^{-\frac{1}{2}\int_0^1 \mathrm{d}\tau \, t^\mu \left(-\frac{1}{2T}\partial_\tau^2 \delta_{\mu\nu}\right) t^\nu} \tag{3.2.27}$$

$$= \int \mathrm{d}^D \bar{x} \frac{1}{(4\pi T)^{\frac{D}{2}}} \ , \tag{3.2.28}$$

where, in the last line, we used the result of the free path integral of a non-relativistic particle.

Path integrating over the bosonic oscillator brings down the functional determinants $\mathrm{Det}(\partial_\tau + i\theta)^{-D-1}$, evaluated with periodic boundary conditions.

3.2.2 Degrees of freedom

It is useful to recast the final expression for the path integral over the circle, i.e. the free one-loop effective action Γ, as

$$\Gamma \equiv Z_{S^1} = -\int_0^\infty \frac{\mathrm{d}T}{T} \mathrm{e}^{-m^2 T} \int \frac{\mathrm{d}^D \bar{x}}{(4\pi T)^{\frac{D}{2}}} \ \mathrm{DoF}(c, D) \ , \tag{3.2.29}$$

where we denoted by $\mathrm{DoF}(c, D)$ the number of (complex) degrees of freedom that acquires the expression

$$\mathrm{DoF}(c, D) = k \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \, \mathrm{e}^{ic\theta} \ \mathrm{Det}\left(\partial_\tau - i\theta\right) \mathrm{Det}\left(\partial_\tau + i\theta\right) \left[\mathrm{Det}(\partial_\tau + i\theta)\right]^{-D-1} \ , \tag{3.2.30}$$

with k an overall normalization to be fixed later on. The value $\mathrm{DoF} = 1$ corresponds to a complex scalar, as seen by comparing with QFT expression.

The determinants need to be regularized, as they are infinite in principle. We regularize them by dividing for the determinant of the "free operator" without the zero mode, so it is infinite too. We compute them as infinite product of their eigenvalues. They act on periodic functions F on $[0, 1]$, hence, a generic function can be expanded in the basis

$$F(\tau) = \sum_{n \in \mathbb{Z}} F_n \, \mathrm{e}^{i2\pi n\tau} \ . \tag{3.2.31}$$

Therefore, the regularized determinants

$$\mathrm{Det}(\partial_\tau \pm i\theta) \longrightarrow \frac{\mathrm{Det}(\partial_\tau \pm i\theta)}{\mathrm{Det}'(\partial_\tau)} \tag{3.2.32}$$

are easily computed:

$$\begin{aligned}
\frac{\mathrm{Det}(\partial_\tau \pm i\theta)}{\mathrm{Det}'(\partial_\tau)} &= \frac{\prod_{n\in\mathbb{Z}} i(2\pi n \pm \theta)}{\prod_{n\in\mathbb{Z}\setminus\{0\}} i2\pi n} \\
&= \pm i\theta \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1 \pm \frac{\theta}{2\pi n}\right) \\
&= \pm 2i \sin\left(\frac{\theta}{2}\right) .
\end{aligned} \tag{3.2.33}$$

Setting the CS coupling to[1]

$$c = \frac{D-1}{2} + s\,, \tag{3.2.34}$$

and fixing $k = -1$ as overall normalization, we find the following expression for the number of degrees of freedom

$$\mathrm{DoF}(s, D) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, \mathrm{e}^{i\left(\frac{D-1}{2}+s\right)\theta} \left(2i \sin\frac{\theta}{2}\right)^{1-D} . \tag{3.2.35}$$

To evaluate it, we find it more convenient to recast it in terms of the Wilson loop variable $w = \mathrm{e}^{-i\theta}$, so that

$$\mathrm{DoF}(s, D) = \oint_{\gamma_-} \frac{\mathrm{d}w}{-2\pi i} \frac{1}{w^{s+1}} \frac{1}{(1-w)^{D-1}}\,, \tag{3.2.36}$$

where γ_- indicates the clockwise oriented contour. The singular point $\theta = 0$ is mapped to the pole $w = 1$. Our prescription to deal with this pole is to deform the contour to exclude it in such a way to take care only of the pole $w = 0$ (cf. Fig. 3.1). Thus, from the Residue Theorem,

$$\mathrm{DoF}(s, D) = \frac{1}{s!} \frac{\mathrm{d}^s}{\mathrm{d}w^s} \frac{1}{(1-w)^{D-1}}\bigg|_{w=0} = \frac{(D-1)D\cdots(D+s-2)}{s!}\,, \tag{3.2.37}$$

which indeed describes the degrees of freedom of a reducible (for $s \geqslant 1$) representation of the little group $SO(D-1)$ as carried by a symmetric tensor with s indices. This confirms that the massive bosonic spinning particle propagates, for a given value of s, the degrees of freedom of a multiplet of massive particles of decreasing spin $s, s-2, s-4, \cdots, 0$ for

[1] The shift from the value given in (2.2.19) is due to the contribution of the ghost fields. For convenience, we now indicate the degrees of freedom by $\mathrm{DoF}(s, D)$, which highlights the dependence on the value of the spin s. This should not cause any confusion.

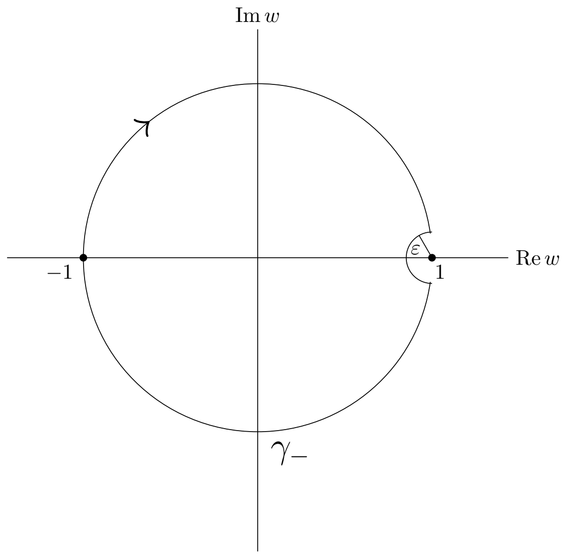

Figure 3.1: Regulated contour of the integral in the Wilson loop variable w. The pole at $w = 1$ is excluded through a limiting procedure, pictorially represented by the semicircle of radius $\epsilon \to 0$.

even s, and $s, s-2, \cdots, 1$ for odd s, as discussed in Subsection 2.2.2.

3.3 Coupling to electromagnetism

So far, we have quantized the free theory through the Dirac method. On the other side, the (Hamiltonian) BRST quantization is especially well-suited for analysing the conditions required for consistent background interactions, when these spoil the first class nature of the constraints. For this reason, we quantize the free particle in this framework and then examine its interaction with an electromagnetic background.

3.3.1 Free BRST analysis

Following the discussion in Section 1.2, we proceed with the BRST quantization focusing only on the subalgebra of (2.2.17) generated by $(H, L, \bar{L})$. The constraint J_c is treated on different footings: it is imposed as a constraint on the BRST Hilbert space, *à la* Dirac, defining a restricted Hilbert space where the cohomology of the BRST operator will be analysed.

The Hilbert space is enlarged to realize the fermionic ghost-antighost pairs of operators

$$\{b, c\} = 1\ , \quad \{\mathcal{B}, \bar{\mathcal{C}}\} = 1\ , \quad \{\bar{\mathcal{B}}, \mathcal{C}\} = 1\ , \tag{3.3.1}$$

associated with the $(H, L, \bar{L})$ constraints, respectively. We assign them the following ghost numbers: $\mathrm{gh}(c, \bar{\mathcal{C}}, \mathcal{C}) = +1$, $\mathrm{gh}(b, \bar{\mathcal{B}}, \mathcal{B}) = -1$. The BRST charge associated with a first class system is readily constructed from (1.2.14). In the present case, it takes the form

$$\mathcal{Q} = cH + \bar{\mathcal{C}}L + \mathcal{C}\bar{L} - 2\mathcal{C}\bar{\mathcal{C}}b \, . \tag{3.3.2}$$

It is an anticommuting, ghost number +1, nilpotent operator by construction. It is hermitian provided that

$$c^\dagger = c \, , \quad b^\dagger = b \, , \quad \mathcal{C}^\dagger = \bar{\mathcal{C}} \, , \quad \mathcal{B}^\dagger = \bar{\mathcal{B}} \, . \tag{3.3.3}$$

The "matter" sector Hilbert space with elements as in (2.2.42) is extended to the BRST Hilbert space $\mathcal{H}_{\mathrm{BRST}}$ by a tensor product with the ghost sector, associated with the $(c, b, \mathcal{B}, \bar{\mathcal{C}}, \mathcal{C}, \bar{\mathcal{B}})$ operators. The latter is constructed as a Fock space on the ghost vacuum defined by

$$(b, \bar{\mathcal{C}}, \bar{\mathcal{B}}) \; |0\rangle_{\mathrm{gh}} = 0 \, . \tag{3.3.4}$$

Since all ghosts are Grassmann odd, $\mathcal{H}_{\mathrm{gh}}$ is finite dimensional. A generic state $|\Phi\rangle$ in the BRST-extended Hilbert $\mathcal{H}_{\mathrm{BRST}}$ space reads

$$|\Phi\rangle = \sum_{s,p=0}^{\infty} \sum_{q,r,t=0}^{1} c^q \mathcal{C}^r \mathcal{B}^t \, |\Phi^{(s,p)(q,r,t)}\rangle \tag{3.3.5}$$

where

$$|\Phi^{(s,p)(q,r,t)}\rangle = \frac{1}{s!p!} \, \Phi^{(s,p)(q,r,t)}_{\mu_1 \ldots \mu_s}(x) \, \alpha^{\mu_1} \ldots \alpha^{\mu_s} \beta^p \, |0\rangle \; , \tag{3.3.6}$$

with $|0\rangle$ now denoting the full BRST vacuum. With this choice, the conjugate momenta act as derivatives:

$$p_\mu = -i\partial_\mu \, , \quad \bar{\alpha}^\mu = \partial_{\alpha^\mu} \, , \quad \bar{\beta} = \partial_\beta \, , \quad b = \partial_c \, , \quad \bar{\mathcal{C}} = \partial_{\mathcal{B}} \, , \quad \bar{\mathcal{B}} = \partial_{\mathcal{C}} \, . \tag{3.3.7}$$

We now introduce a couple of operators, G and $\mathcal{J}_s$, to further restrict the full BRST Hilbert space. These are the ghost number operator

$$G = cb + \mathcal{C}\bar{\mathcal{B}} - \mathcal{B}\bar{\mathcal{C}} \, , \qquad [G, \mathcal{Q}] = \mathcal{Q} \, , \tag{3.3.8}$$

and the (shifted) occupation number operator

$$\mathcal{J}_s = \alpha_\mu \bar{\alpha}^\mu + \beta\bar{\beta} + \mathcal{C}\bar{\mathcal{B}} + \mathcal{B}\bar{\mathcal{C}} - s \, , \quad [\mathcal{Q}, \mathcal{J}_s] = 0 \, . \tag{3.3.9}$$

They commute between themselves, $[G, \mathcal{J}_s] = 0$. The occupation number operator $\mathcal{J}_s$ is defined in such a way to commute with the BRST charge operator, and it is derived from the quantization of the constraint operator J_c (2.2.36) with the addition of the above ghost operators. With the usual antisymmetric quantization's prescription for fermionic operators and Weyl prescription for the bosonic ones, the effect is to shift the CS coupling in (2.2.40) as [2]

$$c = \frac{D-1}{2} + s\,. \tag{3.3.10}$$

The ghost number operator grades the BRST Hilbert space according to the ghost number, and the commutator $[G, \mathcal{Q}] = \mathcal{Q}$ manifests that the BRST charge has ghost number 1. The occupation number operator also grades the BRST Hilbert space according to its eigenvalues and can be used as a constraint to project the Hilbert space onto the subspace with fixed occupation number s. In fact, the Hilbert space decomposes into the direct sum

$$\mathcal{H}_{BRST} = \bigoplus_{s=0}^{\infty} \bigoplus_{g=-1}^{2} \mathcal{H}_{s,g}\,, \tag{3.3.11}$$

with $\mathcal{H}_{s,g}$ the eigenspace of the occupation number operator $\mathcal{J}_s$ with zero eigenvalue, and of the ghost number operator with eigenvalue g. The BRST cohomology can be studied at fixed values of s, since $\mathcal{J}_s$ map physical states into physical states due to $[\mathcal{Q}, \mathcal{J}_s] = 0$ and $[G, \mathcal{J}_s] = 0$. Therefore, physical states are identified as elements of the BRST cohomology

$$\mathcal{Q}\,|\Phi\rangle = 0\,, \qquad |\Phi\rangle \sim |\Phi\rangle + \mathcal{Q}\,|\Lambda\rangle \tag{3.3.12}$$

restricted to the subspace with vanishing eigenvalues of the ghost number and shifted occupation number operators, i.e.

$$G\,|\Phi\rangle = \mathcal{J}_s\,|\Phi\rangle = 0\,. \tag{3.3.13}$$

Our interest lies in the first-quantized description of a massive spin-1 particle, thus we choose $s = 1$. From (3.3.5), an arbitrary wave function at zero ghost number and with $s = 1$ is then given by

$$|\psi\rangle = W_\mu(x)\alpha^\mu\,|0\rangle - i\varphi(x)\beta\,|0\rangle + f(x)c\mathcal{B}\,|0\rangle\,, \tag{3.3.14}$$

where the complex fields $W_\mu(x)$, $\varphi(x)$, and $f(x)$ must be further constrained by Eq. (3.3.12) to represent the physical states of the theory. From the closure equation, i.e.,

[2] This value has already been used in the path integral construction (see footnote 1), which evidently involves a regularization consistent with this ordering prescription.

the first one in (3.3.12), we obtain:

$$\left(\Box - m^2\right) W_\mu - 2i\partial_\mu f = 0 \,, \tag{3.3.15a}$$
$$\left(\Box - m^2\right) \varphi + 2imf = 0 \,, \tag{3.3.15b}$$
$$\partial_\mu W^\mu + m\varphi - 2if = 0 \,, \tag{3.3.15c}$$

which, upon eliminating the auxiliary field $f(x)$, represent the field equations of the Proca field in the Stückelberg formulation:

$$\left(\Box - m^2\right) W_\mu - \partial_\mu \partial^\nu W_\nu - m\partial_\mu\varphi = 0 \,, \tag{3.3.16a}$$
$$\Box\varphi + m\partial_\mu W^\mu = 0 \,. \tag{3.3.16b}$$

The above equations enjoy a gauge symmetry, which, from (3.3.12), reads

$$\delta \left|\psi\right\rangle = Q\left|\Lambda\right\rangle \,, \quad \text{with} \quad \left|\Lambda\right\rangle = i\rho(x)\mathcal{B}\left|0\right\rangle \,, \tag{3.3.17}$$

i.e.

$$\delta W_\mu = \partial_\mu\rho \,, \quad \delta\varphi = -m\rho \,, \tag{3.3.18}$$

which is the well-known Stückelberg gauge symmetry. The presence of the Stückelberg scalar φ restores the $U(1)$ gauge symmetry [52] originally broken due to the introduction of the mass. In the so-called *unitary gauge*, namely setting the Stückelberg field to zero, one reduces the field equations to the standard Fierz-Pauli system for the massive spin-1 field $W_\mu(x)$.

Taking the massless limit produces from (3.3.16) a pair of decoupled equations: one for a free-propagating massless vector field $W_\mu(x)$ and one for a massless scalar field $\varphi(x)$. This is tantamount to the fact that the theory of massive spin-1 does not suffer from the so-called "vDVZ discontinuity", differently from the massive spin 2 case [53, 54].

Finally, let us notice that the wave function (3.3.14) can be interpreted as a spacetime Batalin-Vilkovisky (BV) "string field" displaying only the classical fields out of the minimal BV spectrum of the Proca theory, along with an auxiliary field.[3] The Grassmann parities and ghost numbers of the field components are all equal to zero.

[3] The complete minimal BV spectrum is obtained by relaxing the condition $G\left|\psi\right\rangle = 0$, see for instance [23, 24].

3.3.2 Consistent electromagnetic coupling

The coupling of the worldline to an Abelian background field $A_\mu(x)$ in spacetime (with coupling constant q) is achieved by covariantizing the BUSY constraints as follows:

$$L \to \alpha^\mu \pi_\mu + \beta m \ , \quad \bar{L} \to \bar{\alpha}^\mu \pi_\mu + \bar{\beta} m \ , \tag{3.3.19}$$

where the covariantized momentum π_μ with coupling constant q is defined as usual by

$$\pi_\mu = p_\mu - qA_\mu \ . \tag{3.3.20}$$

It becomes the covariant derivative in the coordinate representation, $\pi_\mu = -i(\partial_\mu - iqA_\mu) = -iD_\mu$. The new constraints do not form a first class algebra any more: while the BUSY charges do commute into a possibly deformed Hamiltonian

$$[\bar{L}, L] = \pi^2 + m^2 + \bar{\alpha}^\mu \alpha^\nu \tilde{F}_{\mu\nu} =: H_{-1/2} \ , \tag{3.3.21}$$

where we have denoted $[\pi_\mu, \pi_\nu] = -[D_\mu, D_\nu] = iqF_{\mu\nu} =: \tilde{F}_{\mu\nu}$, we find that the remaining commutators read

$$[L, H_{1/2}] = i\alpha^\mu \partial^\nu \tilde{F}_{\nu\mu} + \frac{3}{2}\alpha^\mu \tilde{F}_{\mu\nu}\pi^\nu + \frac{i}{2}\bar{\alpha}^\nu \alpha^\rho \alpha^\mu \partial_\mu \tilde{F}_{\rho\nu} \ , \tag{3.3.22}$$

$$[\bar{L}, H_{1/2}] = i\bar{\alpha}^\mu \partial^\nu \tilde{F}_{\nu\mu} + \frac{3}{2}\bar{\alpha}^\mu \tilde{F}_{\mu\nu}\pi^\nu + \frac{i}{2}\bar{\alpha}^\nu \alpha^\rho \bar{\alpha}^\mu \partial_\mu \tilde{F}_{\rho\nu} \ , \tag{3.3.23}$$

with $\partial_\rho \tilde{F}_{\mu\nu} = i[\pi_\rho, \tilde{F}_{\mu\nu}]$, and they do not allow for a suitable redefinition of the constraints to form a first class algebra. Since the definition of a nilpotent BRST charge (1.2.14) relies on the latter, we expect that the corresponding BRST charge, defined from (3.3.2) by substituting the covariantized constraints, fails to be nilpotent. This indicates an inconsistency of the interacting worldline theory at the quantum level.

We aim to "deform" the covariantized BRST operator, in order to define a nilpotent one. With the BUSY charges given as in (3.3.19), we make the ansatz

$$\mathcal{Q}_A = cH_\kappa + \bar{\mathcal{C}}L + \mathcal{C}\bar{L} - 2\mathcal{C}\bar{\mathcal{C}}b \tag{3.3.24}$$

$$=: cH_\kappa + S^\mu \pi_\mu + \bar{\mathcal{C}}\beta m + \mathcal{C}m\bar{\beta} - Mb \ , \tag{3.3.25}$$

with a deformed Hamiltonian

$$H_\kappa = \frac{1}{2}\left(\pi^2 + m^2 + 2\kappa \alpha^\mu \bar{\alpha}^\nu \tilde{F}_{\mu\nu}\right) \tag{3.3.26}$$

that contains a non-minimal coupling with constant κ to be conveniently fixed, and the shorthand notation

$$S^\mu \coloneqq \alpha^\mu \bar{\mathcal{C}} + \bar{\alpha}^\mu \mathcal{C} \ , \qquad M \coloneqq 2\mathcal{C}\bar{\mathcal{C}} \ . \tag{3.3.27}$$

Thus, let us compute:

$$\begin{aligned} Q_A^2 &= c[H_\kappa, S^\mu \pi_\mu] - M H_\kappa + S^\mu S^\nu \pi_\mu \pi_\nu + \frac{M}{2} m^2 && (3.3.28)\\ &= c[H_\kappa, S^\mu \pi_\mu] - \frac{M}{2}\big(\pi_\mu \pi^\mu + 2\kappa \alpha^\mu \bar{\alpha}^\nu \tilde{F}_{\mu\nu}\big) + \frac{M}{2}\big(\eta^{\mu\nu} + \alpha^\nu \bar{\alpha}^\mu - \alpha^\mu \bar{\alpha}^\nu\big)\pi^\mu \pi^\nu \ , && (3.3.29) \end{aligned}$$

where we used $S^\mu S^\nu = M/2\big(\eta^{\mu\nu} + \alpha^\nu \bar{\alpha}^\mu - \alpha^\mu \bar{\alpha}^\nu\big)$. Denoting $S^{\mu\nu} \coloneqq \alpha^\mu \bar{\alpha}^\nu - \alpha^\nu \bar{\alpha}^\mu$, we obtain

$$\mathcal{Q}_A^2 = c[H_\kappa, S^\mu \pi_\mu] - \frac{2\kappa + 1}{4} M S^{\mu\nu} \tilde{F}_{\mu\nu} \ , \tag{3.3.30}$$

In general, this is not zero, except for the trivial case of vanishing field strength, which manifests the inconsistency of coupling massive spin s particles, with generic s, to an electromagnetic background. This is also the case for massless particles, as already discussed in [29]. Even if we cannot define a nilpotent BRST charge on the whole Hilbert space, we can restrict its action to a specific subspace and study the cohomology there. In fact the deformed BRST charge still commutes with the occupation number (3.3.9). Restricting the occupation number to be $s \leqslant 1$, the nilpotency condition simplifies: if there are two or more annihilation operators in $\mathcal{Q}_A^2$, they annihilate the physical wave function for $s = 0, 1$. Thus, the squared BRST charge in this subspace reads

$$\begin{aligned} \mathcal{Q}_A^2\big|_{s=0,1} &= c[H_\kappa, S^\mu \pi_\mu]\big|_{s=0,1} \\ &= -\frac{ic}{2}\Big(\partial_\mu \tilde{F}^{\mu\nu} S_\nu + 2i(1-\kappa)\tilde{F}^{\mu\nu}\pi_\mu S_\nu - \kappa\, S^\nu S^{\alpha\beta}\partial_\nu \tilde{F}_{\alpha\beta}\Big)\big|_{s=0,1} \ . \end{aligned} \tag{3.3.31}$$

For the $s = 0$ sector, this expression is automatically zero regardless of any condition on the background electromagnetic field, as this operator contains destruction operators sitting on the right that annihilate the $s = 0$ wave function (recall the expressions for the operators S^μ and $S^{\mu\nu}$). Physically, this expresses the fact that spinless particles can be consistently coupled to off-shell Abelian background fields. As for the massive spin-1 sector (3.3.14), using

$$S^\nu S^{\rho\sigma}\big|_{s=1} = \mathcal{C}(\eta^{\nu\rho}\bar{\alpha}^\sigma - \eta^{\nu\sigma}\bar{\alpha}^\rho) \ , \tag{3.3.32}$$

the previous equation further simplifies to

$$\mathcal{Q}_A^2\big|_{s=1} = \frac{q}{2} c(\alpha^\nu \bar{\mathcal{C}} - \bar{\alpha}^\nu \mathcal{C})\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) \ , \tag{3.3.33}$$

having set $\kappa = 1$ to achieve this result: then, nilpotency of the deformed BRST charge requires the background $A_\mu(x)$ to be on-shell, i.e. it satisfies Maxwell's equations

$$\partial^\mu F_{\mu\nu} = \Box A_\nu - \partial_\nu(\partial^\mu A_\mu) \stackrel{!}{=} 0 \, . \tag{3.3.34}$$

This is enough to prove the consistency of the coupling, since, provided the background satisfies the Maxwell's equations, in the subspace with $s = 1$ we can define a nilpotent BRST charge for the interacting theory. Then, the usual cohomology of this operator can be exploited to define physical states and observables.

Let us notice that the mass does not obstruct the nilpotency as it does not enter in the squared BRST charge, namely, it does not seem to carry substantial differences with respect to the massless case. Therefore, with worldline techniques, coupling massless spin-1 particle to an Abelian $U(1)$ background seems to not show obstacles. However, from QFT, we know that massless charged spin-1 particles are inconsistent due to the breaking of their own gauge invariance by the electromagnetic coupling. How the no-go theorem about massless charged particles [9] appears from a worldline perspective is at the moment unclear to us.

For general s, despite the mass m does not explicitly enter in the BRST algebra, it may obstruct the nilpotency for higher-spin particles, starting from the spin-2 case as discussed in [24] for the gravitational coupling.

3.4 Effective action in electromagnetic background

Let us employ the worldline model to compute the one-loop effective action induced by a charged spin-1 particle in a constant electromagnetic background.

The worldline representation of the effective action is derived by following the same approach as in the free case (cf. Section 3.2), which in particular determines the overall normalization of the path integral. Now, the constraint's algebra of the interacting worldline theory is not first class and the classical system is not a gauge system. However, we have seen that a BRST charge can be defined in the $s = 0, 1$ subspaces, thus, an appropriate way of thinking about the gauge-fixed path integral is to consider it as describing a quantum BRST system from the beginning, not as being derived from a gauge invariant classical theory.

We treat the $s = 0$ and $s = 1$ cases simultaneously. This approach allows for a direct comparison, with the spinless case serving as a check on the novel spin-1 contribution within the first-quantized framework.

As the interacting worldline action, we take the covariantized version of the free action in Euclidean configuration space (3.2.4) with covariantized constraints (3.3.19) and deformed Hamiltonian H_1 (3.3.26),

$$S_E[z,\lambda;A] = \int_0^1 d\tau \Bigg[\frac{1}{2e}\left(\dot{x} + i\bar{u}\alpha + iu\bar{\alpha}\right)^2 + \alpha^\mu \dot{\bar{\alpha}}_\mu + \beta\dot{\bar{\beta}} + \frac{e}{2}\left(m^2 + 2iq\alpha^\mu\bar{\alpha}^\nu F_{\mu\nu}\right) \\ + u\bar{\beta}m + \bar{u}\beta m + ia(J-c) - iqA^\mu \dot{x}_\mu \Bigg] . \tag{3.4.1}$$

The gauge-fixed version (cf. (3.2.6)), factoring out the $m^2T - ic\theta$ constant term, reads

$$S_\mathrm{E}[z,\hat{\lambda};A] = \int \mathrm{d}\tau \left[\frac{\dot{x}^2}{4T} - iqA^\mu \dot{x}_\mu + \alpha^\mu \left(\delta_{\mu\nu}\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + i\theta\right) + 2iqTF_{\mu\nu}\right)\bar{\alpha}^\nu + \beta\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + i\theta\right)\bar{\beta}\right] \tag{3.4.2}$$

We restrict our analysis to four spacetime dimensions and consider a constant electromagnetic field as the on-shell background. Under these conditions, we derive the one-loop effective action of the Euler-Heisenberg type induced by a massive spin-1 particle. This effective action is given by the path integral on the circle of the gauge-fixed action and takes the form:

$$\Gamma[A] = \int_0^\infty \frac{\mathrm{d}T}{T}\mathrm{e}^{-m^2T}\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{ic\theta}\,\mathrm{Det}\left(\partial_\tau - i\theta\right)\mathrm{Det}\left(\partial_\tau + i\theta\right)\int_{\mathrm{PBC}} DX\,\mathrm{e}^{-S_\mathrm{E}[X,\hat{\lambda};A]} , \tag{3.4.3}$$

with measure in moduli space and determinants already fixed by the free case, and the CS coupling fixed to $c = \frac{3}{2} + s$. Recalling the coordinate split in (3.2.26), we use the Fock-Schwinger gauge around $\bar{x}$ for the background field, i.e.

$$(x - \bar{x})^\mu A_\mu(x) = 0 , \tag{3.4.4}$$

to express derivatives of the gauge potential at the point $\bar{x}$ in terms of derivatives of the field strength tensor

$$A_\mu(\bar{x} + t) = \frac{1}{2}t^\nu F_{\nu\mu}(\bar{x}) + \dots , \tag{3.4.5}$$

where the higher-derivative terms hidden inside the dots vanish since we focus on the constant electromagnetic background case. Then, the path integral becomes Gaussian, and it simplifies to

$$\Gamma[A] = \int \mathrm{d}^4\bar{x}\int_0^\infty \frac{\mathrm{d}T}{T}\frac{\mathrm{e}^{-m^2T}}{(4\pi T)^2}\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\,\mathrm{e}^{i\left(\frac{3}{2}+s\right)\theta}\,4\sin^2\left(\frac{\theta}{2}\right)\int_{\mathrm{DBC}} Dt\,\mathrm{e}^{-S_t[z,\hat{\lambda};A]} \\ \int_{\mathrm{PBC}} D\alpha D\bar{\alpha}\,\mathrm{e}^{-S_\alpha[z,\hat{\lambda};A]}\int_{\mathrm{PBC}} D\beta D\bar{\beta}\,\mathrm{e}^{-S_\beta[z,\hat{\lambda}]} , \tag{3.4.6}$$

where we factored out the normalization of the free particle path integral, and where we defined

$$S_t[z,\hat{\lambda};A] = \int \mathrm{d}\tau \, \frac{1}{2} t^\mu \Delta^{(t)}_{\mu\nu} t^\nu \ , \quad \text{with} \quad \Delta^{(t)}_{\mu\nu} = -\frac{1}{2T}\delta_{\mu\nu}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - iqF_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\tau} \ , \tag{3.4.7}$$

$$S_\alpha[z,\hat{\lambda};A] = \int \mathrm{d}\tau \, \alpha^\mu \Delta^{(\alpha)}_{\mu\nu} \bar{\alpha}^\nu \ , \quad \text{with} \quad \Delta^{(\alpha)}_{\mu\nu} = \delta_{\mu\nu}\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + i\theta\right) + 2iqTF_{\mu\nu} \ , \tag{3.4.8}$$

$$S_\beta[z,\hat{\lambda}] = \int \mathrm{d}\tau \beta \Delta^{(\beta)} \bar{\beta} \ , \quad \text{with} \quad \Delta^{(\beta)} = \frac{\mathrm{d}}{\mathrm{d}\tau} + i\theta \ , \tag{3.4.9}$$

in order to highlight the three differential operators whose functional determinants have to be computed as a result of the path integration over the variables $z(\tau)$. The determinant of the last operator above was computed in (3.2.33), while the first two involves differential operators acting now on the space of four-vector fields with boundary conditions as indicated in (3.4.6). The result for the determinants are:

$$\mathrm{Det}\left(\Delta^{(t)}\right) = \det\left(\frac{\sin(qT\mathbf{F})}{qT\mathbf{F}}\right) \ , \tag{3.4.10}$$

$$\mathrm{Det}\left(\Delta^{(\alpha)}\right) = \det\left[2i\sin\left(\frac{\theta}{2}\mathbf{1} + qT\mathbf{F}\right)\right] \ , \tag{3.4.11}$$

$$\mathrm{Det}\left(\Delta^{(\beta)}\right) = 2i\sin\left(\frac{\theta}{2}\right) \ . \tag{3.4.12}$$

To derive these results, let $\mathbf{1}$ denote the identity matrix and $\mathbf{F}$ the Euclidean field strength tensor with components

$$F_{4i} = -iE_i, \quad F_{ij} = \epsilon_{ijk}B_k, \quad i,j = 1,2,3 \, . \tag{3.4.13}$$

Then, for the first functional determinant (3.4.7), since we factored out the normalization of the free particle path integral, we have to compute

$$\frac{\mathrm{Det}'\left(-\frac{1}{2T}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathbf{1} - iq\mathbf{F}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)}{\mathrm{Det}'\left(-\frac{1}{2T}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathbf{1}\right)} = \frac{\mathrm{Det}'\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{1} + 2iqT\mathbf{F}\right)}{\mathrm{Det}'\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{1}\right)} \ , \tag{3.4.14}$$

hence

$$\frac{\mathrm{Det}'\big(-\frac{1}{2T}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathbf{1}-iq\mathbf{F}\frac{\mathrm{d}}{\mathrm{d}\tau}\big)}{\mathrm{Det}'\left(-\frac{1}{2T}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathbf{1}\right)} = \det\left(\prod_{n\in\mathbb{Z}\setminus\{0\}}\frac{2\pi in\mathbf{1}+2iqT\mathbf{F}}{2\pi in}\right) = \det\left(\prod_{n>0}\left(\mathbf{1}-\frac{(qT\mathbf{F})^2}{\pi^2n^2}\right)\right) = \det\left(\frac{\sin(qT\mathbf{F})}{qT\mathbf{F}}\right)\,, \tag{3.4.15}$$

where we have taken into account the Dirichlet boundary conditions that exclude the zero mode. The second functional determinant (3.4.8), regularized,

$$\mathrm{Det}\left(\left(\frac{\mathrm{d}}{\mathrm{d}\tau}+i\theta\right)\mathbf{1}+2iqT\mathbf{F}\right)\longrightarrow\frac{\mathrm{Det}\left(\left(\frac{\mathrm{d}}{\mathrm{d}\tau}+i\theta\right)\mathbf{1}+2iqT\mathbf{F}\right)}{\mathrm{Det}'\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{1}\right)}\,, \tag{3.4.16}$$

is computed as

$$\frac{\mathrm{Det}\left(\left(\frac{\mathrm{d}}{\mathrm{d}\tau}+i\theta\right)\mathbf{1}+2iqT\mathbf{F}\right)}{\mathrm{Det}'\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{1}\right)} = \det\left(\frac{\prod_{n\in\mathbb{Z}} i(2\pi n+\theta)\mathbf{1}+2iqT\mathbf{F}}{\prod_{n\in\mathbb{Z}\setminus\{0\}} i2\pi n}\right) = \det\left(i(\theta\mathbf{1}+2qT\mathbf{F})\prod_{n>0}\left(\mathbf{1}-\frac{(\theta\mathbf{1}+2qT\mathbf{F})^2}{4\pi^2n^2}\right)\right) = \det\left[2i\sin\left(\frac{\theta}{2}\mathbf{1}+qT\mathbf{F}\right)\right]. \tag{3.4.17}$$

At this point, our final expression is

$$\boxed{\Gamma[A]=\int\mathrm{d}^4\bar{x}\int_0^\infty\frac{\mathrm{d}T}{T}\frac{\mathrm{e}^{-m^2T}}{(4\pi T)^2}\det{}^{-1/2}\left(\frac{\sin(qT\mathbf{F})}{qT\mathbf{F}}\right)I_s(T,A)}\,, \tag{3.4.18}$$

where all that is left to do is to perform the modular integration in θ for a given value of spin s

$$I_s(T,A)=\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi i}\,\mathrm{e}^{i\left(\frac{3}{2}+s\right)\theta}\,2\sin\left(\frac{\theta}{2}\right)\,\det{}^{-1}\left[2i\sin\left(\frac{\theta}{2}\mathbf{1}+qT\mathbf{F}\right)\right]. \tag{3.4.19}$$

It is convenient to recast the determinants above by diagonalizing the field strength $\mathbf{F}$. Its eigenvalues are

$$\lambda_1=K_-\,,\quad \lambda_2=iK_+\,,\quad \lambda_3=-K_-\,,\quad \lambda_4=-iK_+\,, \tag{3.4.20}$$

having defined $K_\pm = \sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2} \pm \mathcal{F}}$ in terms of the Maxwell invariants

$$\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{\vec{B}^2-\vec{E}^2}{2}\ , \quad \mathcal{G} = -\frac{i}{4}\tilde{F}_{\mu\nu}F^{\mu\nu} = \vec{E}\cdot\vec{B}\ . \tag{3.4.21}$$

Therefore, we find

$$\begin{aligned}&\det{}^{-1}\left[2i\sin\left(\frac{\theta}{2}\mathbf{1}+qT\mathbf{F}\right)\right]\\ &= \frac{1}{2i}\left[\sin\left(\frac{\theta}{2}-qTK_-\right)2i\sin\left(\frac{\theta}{2}-iqTK_+\right)2i\sin\left(\frac{\theta}{2}+qTK_-\right)2i\sin\left(\frac{\theta}{2}+iqTK_+\right)\right]^{-1}\\ &= \frac{1}{4}\left[\cos^2(\theta)-\cos(\theta)(\mathcal{K}_++\mathcal{K}_-)+\mathcal{K}_+\mathcal{K}_-\right]^{-1}\ ,\end{aligned} \tag{3.4.22}$$

and

$$\det{}^{-1/2}\left(\frac{\sin(qT\mathbf{F})}{qT\mathbf{F}}\right) = \frac{(qT\mathcal{K}_-)(qT\mathcal{K}_+)}{\sin(qT\mathcal{K}_-)\sin(iqT\mathcal{K}_+)} = \frac{q^2T^2\mathcal{K}_-\mathcal{K}_+}{\sin(\mathcal{K}_-)\sinh(\mathcal{K}_+)}\ , \tag{3.4.23}$$

where $\mathcal{K}_+ = \cosh(2qTK_+)$ and $\mathcal{K}_- = \cos(2qTK_-)$. The modular integration in the Wilson variable $w = \mathrm{e}^{-i\phi}$ is then

$$I_s(T,A) = \oint_{\gamma_-}\frac{\mathrm{d}w}{-2\pi i}\frac{1}{w^{s+1}}\frac{w-1}{(1+w^2-2w\mathcal{K}_+)(1+w^2-2w\mathcal{K}_-)}\ . \tag{3.4.24}$$

We now have all the ingredients to investigate the effective action $\Gamma[A] = \int \mathrm{d}^4\bar{x}\,\mathcal{L}[A]$ for spin $s = 0, 1$. In particular:

(i) the scalar case $s = 0$, which corresponds to scalar QED, comes from the simple pole at $w = 0$,

$$I_0(T,A) = \mathrm{Res}\left[\frac{1}{w}\frac{w-1}{(1+w^2-2w\mathcal{K}_+)(1+w^2-2w\mathcal{K}_-)}\right]_{w=0} = -1\ , \tag{3.4.25}$$

hence it correctly reproduces the celebrated Weisskopf Lagrangian [11]

$$\boxed{\mathcal{L}_{s=0}[A] = -\int_0^\infty\frac{\mathrm{d}T}{T}\frac{\mathrm{e}^{-m^2T}}{(4\pi T)^2}\frac{q^2T^2K_-K_+}{\sinh(qTK_+)\sin(qTK_-)}}\ ; \tag{3.4.26}$$

(ii) the massive spin-1 case instead arises from the pole of order two at $w = 0$,

$$I_1(T,A) = \mathrm{Res}\left[\frac{1}{w^2}\frac{w-1}{(1+w^2-2w\mathcal{K}_+)(1+w^2-2w\mathcal{K}_-)}\right]_{w=0} \tag{3.4.27}$$

$$= \frac{\mathrm{d}}{\mathrm{d}w}\frac{w-1}{(1+w^2-2w\mathcal{K}_+)(1+w^2-2w\mathcal{K}_-)}\bigg|_{w=0} = 1-2(\mathcal{K}_+ + \mathcal{K}_-)\ , \tag{3.4.28}$$

leading to

$$\boxed{\mathcal{L}_{s=1}[A] = \int_0^\infty \frac{\mathrm{d}T}{T}\frac{\mathrm{e}^{-m^2T}}{(4\pi T)^2}\frac{q^2T^2K_-K_+}{\sinh(qTK_+)\sin(qTK_-)}\left[1-2\cosh(2qTK_+)-2\cos(2qTK_-)\right]}\ . \tag{3.4.29}$$

This last expression corresponds to the Heisenberg-Euler effective Lagrangian for a massive charged vector boson in a constant electromagnetic background. It was originally derived in 1965 by Vanyashin and Terent'ev, starting from a quantum field theory of vector electrodynamics [14]. In contrast, our derivation employs a self-consistent first-quantized approach, which offers a more direct and transparent computation than the conventional second-quantized formalism. This constitutes the main result we set out to obtain using the worldline method.

This approach offers a natural framework for exploring possible extensions. For instance, one could interpret our final expression as the result of a locally constant field approximation [55] and investigate corrections by systematically including higher-order terms in (3.4.5). This would likely involve following a procedure similar to that of [56] for performing perturbative corrections from the worldline, ultimately leading to the determination of the generalized heat kernel coefficients computed in [57, 58]. We leave this analysis to future work.

Let us report the perturbative expression given by an expansion in the particle's electric charge q

$$\mathcal{L}_{s=1}[A] = \int_0^\infty \frac{\mathrm{d}T}{T}\frac{\mathrm{e}^{-m^2T}}{(4\pi T)^2}\bigg(-3+\frac{7}{4}q^2T^2\,\mathrm{tr}[F_{\mu\nu}^2] + \frac{5}{32}q^4T^4\,\mathrm{tr}^2[F_{\mu\nu}^2] - \frac{27}{40}q^4T^4\,\mathrm{tr}[F_{\mu\nu}^4] + \mathcal{O}(q^6)\bigg)\ . \tag{3.4.30}$$

The first two terms give divergent contributions, the first one being an infinite vacuum energy, while the second one corresponds to the one-loop divergence in the photon self-energy, and they should be renormalized away. On the other hand, the last two terms are finite and give rise to the quartic interaction's contributions once integrated in the proper time. Thus, the leading terms of the renormalized effective (Euclidean) Lagrangian, with

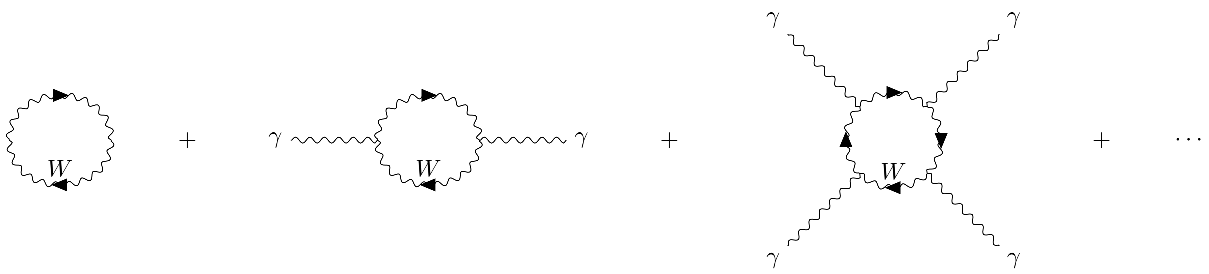

Figure 3.2: Feynman diagrams representing the first terms of the expansion, in the coupling constant, of the one-loop effective action induced by the vector boson W in a constant electromagnetic background

the tree-level Maxwell term included, are expressed as

$$\mathcal{L}^{\text{ren}}_{s=1}[A] = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^4}{16\pi^2 m^4}\left(\frac{5}{32}(F_{\mu\nu}F^{\nu\mu})^2 - \frac{27}{40}F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}\right) + \cdots \tag{3.4.31}$$

which shows the leading vertices for the scattering of light by light.

An overall minus sign arises upon continuation back to Minkowski spacetime. Inserting this sign, the Lagrangian in Minkowski spacetime can be written in the more explicit form

$$\begin{aligned}\mathcal{L}^{\text{ren}}_{s=1}[A] &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^4}{16\pi^2 m^4}\left(-\frac{5}{32}(F_{\mu\nu}F^{\nu\mu})^2 + \frac{27}{40}F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}\right) + \cdots \\ &= \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \frac{\alpha^2}{40m^4}\left(29(\vec{E}^2 - \vec{B}^2)^2 + 108(\vec{E}\cdot\vec{B})^2\right) + \cdots\end{aligned} \tag{3.4.32}$$

where, for ease of comparison with the literature, we have introduced the fine-structure constant $\alpha = \frac{q^2}{4\pi}$ in natural units, and used the relations

$$F_{\mu\nu}F^{\mu\nu} = 2(\vec{B}^2 - \vec{E}^2)\ , \quad F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu} = 2(\vec{E}^2 - \vec{B}^2)^2 + 4(\vec{E}\cdot\vec{B})^2\ , \tag{3.4.33}$$

to obtain the second line.

It may be interesting to compare this result with the more widely known results for the spin-0 and spin-$\frac{1}{2}$ cases, which we include here for convenience:

$$\begin{aligned}\mathcal{L}^{\text{ren}}_{s=0}[A] &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^4}{16\pi^2 m^4}\left(\frac{1}{288}(F_{\mu\nu}F^{\nu\mu})^2 + \frac{1}{360}F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}\right) + \cdots \\ &= \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \frac{\alpha^2}{360m^4}\left(7(\vec{E}^2 - \vec{B}^2)^2 + 4(\vec{E}\cdot\vec{B})^2\right) + \cdots\end{aligned} \tag{3.4.34}$$

and:

$$\begin{aligned}\mathcal{L}^{\text{ren}}_{s=\frac{1}{2}}[A] &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^4}{16\pi^2 m^4}\left(-\frac{1}{32}(F_{\mu\nu}F^{\nu\mu})^2 + \frac{7}{90}F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}\right) + \cdots \\ &= \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \frac{2\alpha^2}{45m^4}\left((\vec{E}^2 - \vec{B}^2)^2 + 7(\vec{E}\cdot\vec{B})^2\right) + \cdots .\end{aligned} \tag{3.4.35}$$

They arise from the Weisskopf and Euler–Heisenberg effective Lagrangians, respectively.

3.5 Pair production

It is well-known that if the effective action in the presence of a classical background field assumes a non-vanishing imaginary contribution, this has the physical interpretation of an instability of the quantum field theory vacuum. In turn, this signals the appearance of states with a non-vanishing number of particles, namely, a production of particle-antiparticle pairs takes place. This is the essence of the so-called "Schwinger effect" [5]. Quantitatively, the Minkowskian effective action Γ_{M} is related to the vacuum persistence amplitude by

$$\langle 0_{\text{out}}|0_{\text{in}}\rangle = \mathrm{e}^{i\Gamma_{\text{M}}} , \tag{3.5.1}$$

and thus to the vacuum persistence probability

$$|\langle 0_{\text{out}}|0_{\text{in}}\rangle|^2 = \mathrm{e}^{-2\,\mathrm{Im}\,\Gamma_{\text{M}}} , \tag{3.5.2}$$

from which the pair production probability is given by

$$P_{\text{pair}} := 1 - \mathrm{e}^{-2\,\mathrm{Im}\,\Gamma_{\text{M}}} \approx 2\,\mathrm{Im}\,\Gamma_{\text{M}} . \tag{3.5.3}$$

In this section, we compute the rate for the Schwinger pair production of massive charged spin-1 particles in a constant external electric field $\vec{E}$. The Euclidean effective action, from (3.4.29), with $K_+ = 0$[4] and $K_- = E$, with E being the modulus of the electric field, reduces to

$$\Gamma[A] = \int \mathrm{d}^4\bar{x} \int_0^\infty \frac{\mathrm{d}T}{T}\frac{\mathrm{e}^{-m^2T}}{(4\pi T)^2}\frac{qTE}{\sin(qTE)}\left[-1 - 2\cos(2qTE)\right] . \tag{3.5.4}$$

Apparently, it is a real quantity, but the presence of poles in the T-integral signals that this is not the case. To extract its imaginary part, we go back to Minkowski space via a Wick rotation, using $T \to iT$, $\mathcal{L} \to -\mathcal{L}$, to obtain the Minkowskian effective Lagrangian

$$\mathcal{L}[A] = \int_0^\infty \frac{\mathrm{d}T}{T}\frac{\mathrm{e}^{-im^2T}}{(4\pi T)^2}\left(-3\frac{iqTE}{\sin(iqTE)} + 4(iqTE)\sin(iqTE)\right) . \tag{3.5.5}$$

[4]Let us observe that $K_+ = 0$ is a removable singularity of the effective Lagrangian (3.4.29).

For certain values of proper time, the integral develops poles in the T-plane, which in turn produce an imaginary part of the Minkowskian effective action. In fact, let us consider the complex conjugated Lagrangian

$$\mathcal{L}^*[A] = \int_0^\infty \frac{\mathrm{d}T}{T}\frac{\mathrm{e}^{im^2T}}{(4\pi T)^2}\left(-3\frac{iqTE}{\sin(iqTE)} + 4(iqTE)\sin(iqTE)\right) \tag{3.5.6}$$

$$= -\int_{-\infty}^0 \frac{\mathrm{d}T}{T}\frac{\mathrm{e}^{-im^2T}}{(4\pi T)^2}\left(-3\frac{iqTE}{\sin(iqTE)} + 4(iqTE)\sin(iqTE)\right) , \tag{3.5.7}$$

where in the second line we have changed variable $T \to -T$. The imaginary part is then given by

$$\operatorname{Im}\mathcal{L}[A] = \frac{\mathcal{L}[A]-\mathcal{L}^*[A]}{2i} = \frac{1}{2i}\int_{-\infty}^{+\infty} \frac{\mathrm{d}T}{T}\frac{\mathrm{e}^{-im^2T}}{(4\pi T)^2}\left(-3\frac{iqTE}{\sin(iqTE)} + 4(iqTE)\sin(iqTE)\right) . \tag{3.5.8}$$

Closing the contour in the lower half-plane, it is determined by the residues at the poles of the first integrand function, located at

$$T = -i\frac{\pi n}{qE} , \quad 0 < n \in \mathbb{N} . \tag{3.5.9}$$

Let us notice that they correspond to the zero modes of the differential operator $\Delta^{(t)}_{\mu\nu}$ (3.4.7) in Minkowski spacetime except for the value $n = 0$, which indicates a UV divergence as discussed at the end of the previous section. For these values of the proper time T, this operator has a non-trivial kernel, hence a zero functional determinant.

Evaluating the following residues

$$\operatorname{Res}\left[-3\frac{\mathrm{e}^{-im^2T}}{T(4\pi T)^2}\frac{iqTE}{\sin(iqTE)}\right]\Bigg|_{T=-i\frac{\pi n}{qE}} = \frac{3}{16\pi^4}(qE)^2(-1)^n\,\frac{\mathrm{e}^{-\frac{m^2\pi n}{qE}}}{n^2} , \tag{3.5.10}$$

the final result is

$$\operatorname{Im}\mathcal{L}[A] = \frac{3}{16\pi^3}(qE)^2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\mathrm{e}^{-\frac{m^2\pi n}{qE}}}{n^2} . \tag{3.5.11}$$

In conclusion, the rate for massive spin-1 particle-antiparticle pair production in the presence of a constant electric field per unit of volume and time $\mathcal{P} := P/\Delta V\Delta\mathcal{T}$ can be written as

$$\mathcal{P}_{\text{pair}} \approx 2\operatorname{Im}\mathcal{L}[A] = \frac{3}{8\pi^3}(qE)^2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\mathrm{e}^{-\frac{m^2\pi n}{qE}}}{n^2} , \tag{3.5.12}$$

or, using the polylogarithm of order 2 [5], in compact form

$$\mathcal{P}_{\text{pair}} \approx -\frac{3}{8\pi^3}(qE)^2\,\text{Li}_2\left(-\mathrm{e}^{-\frac{m^2\pi}{qE}}\right)\ . \tag{3.5.13}$$

As already noted in [14], this probability corresponds to three times the probability of the production of pairs of scalar particles with mass m.

3.6 $O(2)$ spinning particle analysis

With the aim of completeness and to compare with our previous method, we present here an alternative first-quantized derivation of the same results. We make use of the massive version of the spinning particle model with $N = 2$ (cf. Subsection 2.1.2), which contains fermionic oscillators.

Worldline action For convenience, let us rewrite the main formulae. The action is

$$S = \int \mathrm{d}\tau \left[p_\mu \dot{x}^\mu + i\bar{\psi}_\mu \dot{\psi}^\mu + i\bar{\theta}\dot{\theta} - eH - i\bar{\chi}Q - i\chi\bar{Q} - aJ_c\right]\ , \tag{3.6.1}$$

with first class constraints

$$H = \frac{1}{2}(p^\mu p_\mu + m^2)\ , \quad Q = \psi^\mu p_\mu + \theta m\ , \quad \bar{Q} = \bar{\psi}^\mu p_\mu + \bar{\theta} m\ , \quad J_c = \psi^\mu \bar{\psi}_\mu + \theta\bar{\theta} - c\ . \tag{3.6.2}$$

The main difference with respect to the bosonic theory (2.2.3) consists of the presence of fermionic oscillators employed to describe the spin degrees of freedom: their Poisson brackets read

$$\{\psi^\mu, \bar{\psi}^\nu\}_{\text{PB}} = -i\eta^{\mu\nu}\ , \quad \{\theta, \bar{\theta}\}_{\text{PB}} = -i\ , \tag{3.6.3}$$

and will be translated into anticommutation relations upon quantization.

The constraints' algebra, the $N = 2$ supersymmetry algebra in $(0+1)$-dimension, reads

$$\{\bar{Q}, Q\}_{\text{PB}} = -2iH\ , \quad \{Q, J_c\}_{\text{PB}} = iQ\ , \quad \{\bar{Q}, J_c\}_{\text{PB}} = -i\bar{Q}\ . \tag{3.6.4}$$

[5] The polylogarithm function of order s is defined by

$$\text{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}\ .$$

Under a gauge transformation generated via Poisson bracket by

$$V = \epsilon H + i\bar{\xi}Q + i\xi\bar{Q} + \alpha J_c \,, \tag{3.6.5}$$

the phase space variables transform according to

$$\delta x^\mu = \epsilon\, p^\mu + i\xi\, \bar{\psi}^\mu + i\bar{\xi}\, \psi^\mu \,, \tag{3.6.6a}$$
$$\delta p_\mu = 0, \tag{3.6.6b}$$
$$\delta \psi^\mu = -\,\xi\, p^\mu + i\alpha\, \psi^\mu \,, \tag{3.6.6c}$$
$$\delta \bar{\psi}^\mu = -\,\bar{\xi}\, p^\mu - i\alpha\, \bar{\psi}^\mu \,, \tag{3.6.6d}$$
$$\delta \theta = -\,\xi\, m + i\alpha\theta \,, \tag{3.6.6e}$$
$$\delta \bar{\theta} = -\,\bar{\xi}\, m - i\alpha\, \bar{\theta} \,, \tag{3.6.6f}$$

while the gauge fields

$$\delta e = \dot{\epsilon} + 2i\, \bar{\chi}\, \xi + 2i\, \chi\, \bar{\xi} \,, \tag{3.6.7a}$$
$$\delta \chi = \dot{\xi} - ia\xi + i\alpha\chi \,, \tag{3.6.7b}$$
$$\delta \bar{\chi} = \dot{\bar{\xi}} + ia\bar{\xi} - i\alpha\bar{\chi} \,, \tag{3.6.7c}$$
$$\delta a = \dot{\alpha} \,. \tag{3.6.7d}$$

Worldloop path integral and DOF To construct the path integral over the circle and compute the number of degrees of freedom propagated in the loop, we choose antiperiodic boundary conditions (ABC) for the fermionic fields and periodic boundary conditions for the bosonic ones. Similarly to the bosonic case, the gauge symmetries (3.6.7) with the chosen boundary conditions allow us to set

$$\lambda = (e, \bar{\chi}, \chi, a) \;\to\; \hat{\lambda} = (2T, 0, 0, \phi) \,. \tag{3.6.8}$$

After Wick rotating, the gauge-fixed Euclidean action in configuration space is

$$S_E[z, \hat{\lambda}] = \int \mathrm{d}\tau \left[\frac{1}{4T}\dot{x}^2 + \bar{\psi}^\mu \left(\frac{d}{d\tau}\delta_{\mu\nu} - i\phi\, \delta_{\mu\nu} \right) \psi^\nu + \bar{\theta} \left(\frac{d}{d\tau} - i\phi \right) \theta - i\phi c \right] , \tag{3.6.9}$$

and, after integration, the path integral is written as

$$\Gamma = -\int_0^\infty \frac{\mathrm{d}T}{T} \mathrm{e}^{-m^2 T} \int \frac{\mathrm{d}^D \bar{x}}{(4\pi T)^{D/2}}\, \mathrm{DoF}(p, D) \,, \tag{3.6.10}$$

with the number of degrees of freedom given by

$$\mathrm{DoF}(p, D) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi}\, \mathrm{e}^{i\left(\frac{1-D}{2}+p\right)\phi} \left(2\cos\frac{\phi}{2}\right)^{D-1} . \tag{3.6.11}$$

In the above expression we have set the quantized CS coupling to $c = \frac{1-D}{2} + p$. The cosines in this expression arise from the integration over the fermionic phase space variables and from the Faddeev-Popov determinants associated with the SUSY ghosts, which are now bosonic. Its calculation leads to

$$\mathrm{DoF}(p, D) = \binom{D-1}{p} . \tag{3.6.12}$$

It corresponds to the number of degrees of freedom of a massive p-form in D spacetime dimensions.

BRST quantization Upon quantization, the "matter" Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{\mathbb{D}}) \otimes \mathcal{F}$, with $\mathcal{F}$ the Fock space with vacuum $|0\rangle$ defined by

$$(\bar\psi^\mu, \bar\theta)\, |0\rangle = 0\,, \tag{3.6.13}$$

consists of the states

$$\begin{aligned} |\Phi\rangle &= \sum_{j=0}^{D} (|\Phi_j\rangle + |\Phi_j^{(\theta)}\rangle) \\ &= \sum_{j=0}^{D} \left(\frac{1}{j!}\, \Phi_{\mu_1 \dots \mu_j}(x)\, \psi^{\mu_1} \dots \psi^{\mu_j}\, |0\rangle + \frac{1}{j!}\, \Phi^{(\theta)}_{\mu_1 \dots \mu_j}(x)\, \theta\, \psi^{\mu_1} \dots \psi^{\mu_j}\, |0\rangle \right) \end{aligned} \tag{3.6.14}$$

with $\Phi_{\mu_1 \dots \mu_j}(x)$ and $\Phi^{(\theta)}_{\mu_1 \dots \mu_j}(x)$ rank-j antisymmetric tensors. Proceeding with BRST quantization along the lines of [21–24] to build a positive-definite Hilbert space, one enlarges the phase space with the ghost pairs

$$\{b, c\} = 1\,, \quad [B, \bar C] = 1\,, \quad [\bar B, C] = 1\,, \tag{3.6.15}$$

associated with $(H, Q, \bar Q)$ respectively. Note that the pairs associated with the SUSY charges are now bosonic. Their ghost number assignments are $\mathrm{gh}(c, \bar C, C) = 1$ and $\mathrm{gh}(b, B, \bar B) = -1$. From these operators, the full Hilbert space $\mathcal{H}_{\mathrm{BRST}}$ is then constructed as described in Subsection 3.3.1 for the bosonic case. The nilpotent BRST charge is

$$\mathcal{Q} = cH + \bar C Q + C \bar Q - 2C\bar C b\,. \tag{3.6.16}$$

Once again, the ghost number operator $G = cb + C\bar{B} + B\bar{C}$ and the occupation number operator $\mathcal{J}_p = \psi_\mu\bar{\psi}^\mu + \theta\bar{\theta} + C\bar{B} - B\bar{C} - p$ are introduced, with the choice for the CS coupling

$$c = -\frac{D+1}{2} + p + 1\,. \tag{3.6.17}$$

They satisfy

$$[G, \mathcal{J}_p] = 0\ , \quad [G, \mathcal{Q}] = \mathcal{Q}\ , \quad [\mathcal{J}_p, \mathcal{Q}] = 0\ , \tag{3.6.18}$$

and are used to define physical states as those states in the cohomology of $\mathcal{Q}$ with vanishing ghost number and occupation number (as measured by $\mathcal{J}_p$). The physical states at $p = 1$ are contained in the wave function

$$|\psi\rangle = W_\mu(x)\psi^\mu\,|0\rangle - i\varphi(x)\theta\,|0\rangle + f(x)cB\,|0\rangle\ , \tag{3.6.19}$$

where, requiring $|\psi\rangle$ to be Grassmann-odd, the Grassmann parities and ghost numbers of the component fields $W_\mu(x), \varphi(x), f(x)$ are all vanishing. The field equations, obtained by computing $\mathcal{Q}\,|\psi\rangle = 0$, are found to be

$$(\Box - m^2)W_\mu + 2i\partial_\mu f = 0\ , \tag{3.6.20a}$$

$$(\Box - m^2)\varphi - 2imf = 0\ , \tag{3.6.20b}$$

$$\partial^\mu W_\mu + m\varphi + 2if = 0\ , \tag{3.6.20c}$$

while the gauge symmetries, arising from $\delta\,|\psi\rangle = \mathcal{Q}\,|\Lambda\rangle$, $|\Lambda\rangle = -i\rho(x)B\,|0\rangle$, are given by

$$\delta W_\mu = \partial_\mu\rho\ , \quad \delta\varphi = -m\rho\ , \quad \delta f = \frac{i}{2}(\Box - m^2)\rho\,. \tag{3.6.21}$$

As in the bosonic case, the auxiliary field $f(x)$ can be eliminated and subsequently the gauge symmetry of $\varphi(x)$ may be used to set it to zero. Hence, the cohomology at $p = 1$ coincides with the one obtained from the bosonic worldline model, with $W_\mu(x)$ being the massive spin-1 field.

Interacting theory The coupling with the background field $A_\mu(x)$ is realized by

$$Q \to \psi^\mu\pi_\mu + \theta m\ , \quad \bar{Q} \to \bar{\psi}^\mu\pi_\mu + \bar{\theta}m\ , \quad H \to \frac{1}{2}\left(\pi^2 + m^2 + 2iqF_{\mu\nu}\psi^\mu\bar{\psi}^\nu\right)\ . \tag{3.6.22}$$

The squared deformed BRST charge now reads

$$\begin{aligned}\mathcal{Q}_A^2 = &-\frac{i}{2}c\left(\bar{C}\psi^\rho - C\bar{\psi}^\rho\right)\partial^\mu\tilde{F}_{\mu\rho} - ic\,\bar{C}\psi^\mu\psi^\rho\bar{\psi}^\nu\partial_\rho\tilde{F}_{\mu\nu} + ic\,C\psi^\mu\bar{\psi}^\nu\bar{\psi}^\rho\partial_\rho\tilde{F}_{\mu\nu}\\ &+ \bar{C}^2\psi^\mu\psi^\nu\tilde{F}_{\mu\nu} + C^2\bar{\psi}^\mu\bar{\psi}^\nu\tilde{F}_{\mu\nu} - C\bar{C}\psi^\mu\bar{\psi}^\nu\tilde{F}_{\mu\nu}\,.\end{aligned} \tag{3.6.23}$$

In general, it is not zero. However, when its action is restricted to the subspace $p = 1$, see (3.6.19), all but the first term vanish

$$\mathcal{Q}_A^2\Big|_{p=1} = -\frac{i}{2}c\left(\bar{C}\psi^\rho - C\bar{\psi}^\rho\right)\partial^\mu \tilde{F}_{\mu\rho}\ . \tag{3.6.24}$$

Once again, we find that nilpotency is achieved if the background electromagnetic field A_μ is on-shell.

Massive $p-$forms with $p > 1$ do not admit electromagnetic coupling within this framework.

Effective action in electromagnetic background The path integral for the interacting theory in the $p = 0, 1$ sectors is constructed analogously to the free theory. Considering the constraints (3.6.22), the gauge-fixed action is

$$\begin{aligned} S_E[z,\hat{\lambda};A] = \int d\tau \Bigg[&\frac{1}{4T}\dot{x}^2 - iqA^\mu \dot{x}_\mu + \bar{\psi}^\mu \left(\frac{d}{d\tau}\delta_{\mu\nu} - i\phi\,\delta_{\mu\nu} + 2iqTF_{\mu\nu}\right)\psi^\nu \\ &+ \bar{\theta}\left(\frac{d}{d\tau} - i\phi\right)\theta - i\phi c\Bigg]\ . \end{aligned} \tag{3.6.25}$$

Thus, taking a constant background electromagnetic field in the Fock-Schwinger gauge (3.4.4) and setting $D = 4$, the path integral reads

$$\begin{aligned} \Gamma[A] = -\int \mathrm{d}^4\bar{x} \int_0^\infty \frac{\mathrm{d}T}{T}\mathrm{e}^{-m^2T}\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\,\mathrm{e}^{i\left(-\frac{3}{2}+p\right)\phi}\,\frac{1}{4}\cos^{-2}\left(\frac{\phi}{2}\right)\int_{\mathrm{DBC}} Dt\,\mathrm{e}^{-S_t[z,\hat{\lambda};A]} \\ \int_{\mathrm{ABC}} D\bar{\psi}D\psi\,\mathrm{e}^{-S_\psi[z,\hat{\lambda};A]}\int_{\mathrm{ABC}} D\bar{\theta}D\theta\,\mathrm{e}^{-S_\theta[z,\hat{\lambda}]}\ , \end{aligned} \tag{3.6.26}$$

where

$$S_t[z,\hat{\lambda};A] = \int \mathrm{d}\tau\,\frac{1}{2}t^\mu \Delta^{(t)}_{\mu\nu}t^\nu\ , \quad \text{with} \quad \Delta^{(t)}_{\mu\nu} = -\frac{1}{2T}\delta_{\mu\nu}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - iqF_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\tau}\ , \tag{3.6.27}$$

$$S_\psi[z,\hat{\lambda};A] = \int \mathrm{d}\tau\,\bar{\psi}^\mu \Delta^{(\psi)}_{\mu\nu}\psi^\nu\ , \quad \text{with} \quad \Delta^{(\psi)}_{\mu\nu} = \delta_{\mu\nu}\left(\frac{\mathrm{d}}{\mathrm{d}\tau} - i\phi\right) + 2iqTF_{\mu\nu}\ , \tag{3.6.28}$$

$$S_\theta[z,\hat{\lambda}] = \int \mathrm{d}\tau\,\bar{\theta}\Delta^{(\theta)}_{\mu\nu}\theta\ , \quad \text{with} \quad \Delta^{(\theta)}_{\mu\nu} = \frac{\mathrm{d}}{\mathrm{d}\tau} - i\phi\ . \tag{3.6.29}$$

Evaluating the functional determinants, we get

$$\Gamma[A] = -\int \mathrm{d}^4\bar{x}\int_0^\infty \frac{\mathrm{d}T}{T}\frac{\mathrm{e}^{-m^2T}}{(4\pi T)^2}\,\mathrm{det}^{-1/2}\left(\frac{\sin(qT\mathbf{F})}{qT\mathbf{F}}\right) I_p(T,A)\ , \tag{3.6.30}$$

with

$$I_p(T,A) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi}\, \mathrm{e}^{i\left(-\frac{3}{2}+p\right)\phi}\, 8\cos^{-1}\left(\frac{\phi}{2}\right) \det\left[2\cos\left(-\frac{\phi}{2}\mathbf{1} + qT\mathbf{F}\right)\right] . \tag{3.6.31}$$

Diagonalizing the field strength $F_{\mu\nu}$, cf. (3.4.20)–(3.4.21), the modular integration in the Wilson variable $z = \mathrm{e}^{-i\phi}$ becomes

$$I_p(T,A) = \oint_{\gamma_-} \frac{\mathrm{d}z}{-2\pi i} \frac{1}{z^{p+1}} \frac{(1+z^2)^2 + 2z(1+z^2)(\mathcal{K}_+ + \mathcal{K}_-) + 4z^2\mathcal{K}_+\mathcal{K}_-}{z+1} , \tag{3.6.32}$$

where $\mathcal{K}_+ = \cosh(2qTK_+)$ and $\mathcal{K}_- = \cos(2qTK_-)$. Deforming the contour to avoid the pole in $z=-1$, we find:

(i) for the $p = 0$ case

$$I_0(T,A) = \operatorname{Res}\left[\frac{1}{z}\frac{(1+z^2)^2 + 2z(1+z^2)(\mathcal{K}_+ + \mathcal{K}_-) + 4z^2\mathcal{K}_+\mathcal{K}_-}{z+1}\right]_{z=0} = 1 ; \tag{3.6.33}$$

(ii) for the massive 1-form case, $p = 1$

$$\begin{aligned} I_1(T,A) &= \operatorname{Res}\left[\frac{1}{z^2}\frac{(1+z^2)^2 + 2z(1+z^2)(\mathcal{K}_+ + \mathcal{K}_-) + 4z^2\mathcal{K}_+\mathcal{K}_-}{z+1}\right]_{z=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}z}\frac{(1+z^2)^2 + 2z(1+z^2)(\mathcal{K}_+ + \mathcal{K}_-) + 4z^2\mathcal{K}_+\mathcal{K}_-}{z+1}\Bigg|_{z=0} = -1 + 2(\mathcal{K}_+ + \mathcal{K}_-) . \end{aligned} \tag{3.6.34}$$

Thus, we have reobtained the same results found in previous sections. In particular, setting $p = 0$ we get the usual Weisskopf effective Lagrangian, whereas for $p = 1$ we obtain (3.4.29).

Chapter 4

Conclusions

In this thesis, we have described a charged, massive spin-1 particle through worldline actions with bosonic or fermionic oscillators. The actions enjoy the gauge symmetries necessary to ensure unitarity at the quantum level, thereby leading to a positive-definite Hilbert space for physical states. Precisely the physicality conditions encode the field equations for particles of a given spin, in the different sectors of the theory's spectrum. The projection onto the subspace containing the relevant degrees of freedom is achieved by gauging the oscillator number operator, appropriately shifted by a Chern-Simons coupling. Using a BRST analysis, we derived consistency conditions for the coupling of the spin-1 sector to an electromagnetic field: in both the models, the field must satisfy the vacuum Maxwell's equations. For such configurations, the path integral of the interacting worldline action can be constructed. By contrast, the spin-0 sector admits coupling to arbitrary electromagnetic fields without requiring any conditions. As for the other sectors, corresponding to higher-spin particles (in the bosonic model) and massive p-forms with $p > 1$ (in the fermionic model), they do not admit electromagnetic couplings within our worldline framework.

The one-loop effective action induced by the charged, massive spin-1 particle in a constant electromagnetic background is obtained by computing the worldline path integral on the circle. From the effective Lagrangian, we derived the production rate for spin-1 particle-antiparticle pairs in a constant electric field. Our results, derived entirely within a first-quantized framework and independent of any second-quantized formalism, fully agree with the QFT result originally obtained in [14].

The work presented in this thesis and collected in [30] can be extended to include effective interactions that account for the potential non-point-like nature of the particle under consideration, as already explored in [30]. Also, more general field configurations can be considered by employing worldline instanton techniques [31–35].

Our findings contribute to the programme of constructing worldline formulations of QFT, based solely on first quantization principles, as has been successfully carried out for Yang-Mills [20, 36], gravity [21, 22, 24, 59–62], and scalar theories [63].

Acknowledgement

I would like to express my gratitude to my supervisor, Prof. Fiorenzo Bastianelli, for his guidance throughout my thesis' activity. With his deep and clear understanding of the subject, he introduced me to the realm of the Worldline Formalism. Thanks to his constant presence, I learned from our discussions and his way of thinking, and our work over these months has been a true collaboration.

I would like to thank my Co-supervisor, Dr. Filippo Fecit, for having been part of the work. His availability and knowledge greatly facilitated my efforts. He readily engaged with the questions I posed, providing suggestions and clarifications. I sincerely appreciated his mentorship throughout these months.

Finally, I am grateful to both of them for their support and advice whenever I shared my thoughts about future steps.

Bibliography

[1] C. Schubert, 'Perturbative quantum field theory in the string inspired formalism,' *Phys. Rept.*, vol. 355, pp. 73–234, 2001. DOI: 10.1016/S0370-1573(01)00013-8 arXiv: hep-th/0101036.

[2] F. Bastianelli and C. Schubert, *Worldline Path Integrals and Quantum Field Theory.* Cambridge University Press, 2025, in preparation.

[3] R. P. Feynman, 'Mathematical formulation of the quantum theory of electromagnetic interaction,' *Phys. Rev.*, vol. 80, pp. 440–457, 3 Nov. 1950. DOI: 10.1103/PhysRev.80.440

[4] R. P. Feynman, 'An operator calculus having applications in quantum electrodynamics,' *Phys. Rev.*, vol. 84, pp. 108–128, 1 Oct. 1951. DOI: 10.1103/PhysRev.84.108

[5] J. S. Schwinger, 'On gauge invariance and vacuum polarization,' *Phys. Rev.*, vol. 82, K. A. Milton, Ed., pp. 664–679, 1951. DOI: 10.1103/PhysRev.82.664

[6] Z. Bern and D. A. Kosower, 'Efficient calculation of one-loop qcd amplitudes,' *Phys. Rev. Lett.*, vol. 66, pp. 1669–1672, 13 Apr. 1991. DOI: 10.1103/PhysRevLett.66.1669

[7] Z. Bern, 'A compact representation of the one-loop n-gluon amplitude,' *Physics Letters B*, vol. 296, no. 1, pp. 85–94, 1992, ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(92)90807-G

[8] M. J. Strassler, 'Field theory without feynman diagrams: One-loop effective actions,' *Nuclear Physics B*, vol. 385, no. 1, pp. 145–184, 1992, ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(92)90098-V

[9] S. Weinberg and E. Witten, 'Limits on Massless Particles,' *Phys. Lett. B*, vol. 96, pp. 59–62, 1980. DOI: 10.1016/0370-2693(80)90212-9

[10] W. Heisenberg and H. Euler, 'Consequences of Dirac's theory of positrons,' *Z. Phys.*, vol. 98, no. 11-12, pp. 714–732, 1936. DOI: 10.1007/BF01343663 arXiv: physics/0605038.

[11] V. Weisskopf, 'The electrodynamics of the vacuum based on the quantum theory of the electron,' *Kong. Dan. Vid. Sel. Mat. Fys. Med.*, vol. 14N6, no. 6, pp. 1–39, 1936.

[12] G. V. Dunne, 'Heisenberg-Euler effective Lagrangians: Basics and extensions,' in *From fields to strings: Circumnavigating theoretical physics. Ian Kogan memorial collection (3 volume set)*, M. Shifman, A. Vainshtein and J. Wheater, Eds. Jun. 2004, pp. 445–522. DOI: 10.1142/9789812775344_0014 arXiv: hep-th/0406216.

[13] G. V. Dunne, 'The Heisenberg-Euler Effective Action: 75 years on,' *Int. J. Mod. Phys. A*, vol. 27, M. Asorey, M. Bordag and E. Elizalde, Eds., p. 1 260 004, 2012. DOI: 10.1142/S0217751X12600044 arXiv: 1202.1557 [hep-th].

[14] V. S. Vanyashin and M. V. Terentev, 'The Vacuum Polarization of a Charged Vector Field,' *Zh. Eksp. Teor. Fiz.*, vol. 48, no. 2, pp. 565–573, 1965.

[15] F. A. Berezin and M. S. Marinov, 'Particle Spin Dynamics as the Grassmann Variant of Classical Mechanics,' *Annals Phys.*, vol. 104, p. 336, 1977. DOI: 10.1016/0003-4916(77)90335-9

[16] A. Barducci, R. Casalbuoni and L. Lusanna, 'Supersymmetries and the Pseudoclassical Relativistic electron,' *Nuovo Cim. A*, vol. 35, p. 377, 1976. DOI: 10.1007/BF02730291

[17] L. Brink, S. Deser, B. Zumino, P. Di Vecchia and P. S. Howe, 'Local Supersymmetry for Spinning Particles,' *Phys. Lett. B*, vol. 64, A. Salam and E. Sezgin, Eds., p. 435, 1976, [Erratum: Phys.Lett.B 68, 488 (1977)]. DOI: 10.1016/0370-2693(76)90115-5

[18] V. D. Gershun and V. I. Tkach, 'Classical and quantum dynamics of particles with arbitrary spins,' *JETP Lett.*, vol. 29, pp. 288–291, 1979.

[19] P. S. Howe, S. Penati, M. Pernici and P. K. Townsend, 'A Particle Mechanics Description of Antisymmetric Tensor Fields,' *Class. Quant. Grav.*, vol. 6, p. 1125, 1989. DOI: 10.1088/0264-9381/6/8/012

[20] P. Dai, Y.-t. Huang and W. Siegel, 'Worldgraph Approach to Yang-Mills Amplitudes from N=2 Spinning Particle,' *JHEP*, vol. 10, p. 027, 2008. DOI: 10.1088/1126-6708/2008/10/027 arXiv: 0807.0391 [hep-th].

[21] R. Bonezzi, A. Meyer and I. Sachs, 'Einstein gravity from the $\mathcal{N} = 4$ spinning particle,' *JHEP*, vol. 10, p. 025, 2018. DOI: 10.1007/JHEP10(2018)025 arXiv: 1807.07989 [hep-th].

[22] R. Bonezzi, A. Meyer and I. Sachs, 'A Worldline Theory for Supergravity,' *JHEP*, vol. 06, p. 103, 2020. DOI: 10.1007/JHEP06(2020)103 arXiv: 2004.06129 [hep-th].

[23] M. Carosi and I. Sachs, 'Proca theory from the spinning worldline,' *JHEP*, vol. 01, p. 135, 2022. DOI: 10.1007/JHEP01(2022)135 arXiv: 2110.10573 [hep-th].

[24] F. Fecit, 'Massive gravity from a first-quantized perspective,' *Eur. Phys. J. C*, vol. 84, no. 4, p. 445, 2024. DOI: 10.1140/epjc/s10052-024-12799-2 arXiv: 2312.15428 [hep-th].

[25] M. Henneaux and C. Teitelboim, 'First and second quantized point particles of any spin,' in *2nd Meeting on Quantum Mechanics of Fundamental Systems (CECS)*, 1987.

[26] G. Barnich, M. Grigoriev, A. Semikhatov and I. Tipunin, 'Parent field theory and unfolding in BRST first-quantized terms,' *Commun. Math. Phys.*, vol. 260, pp. 147–181, 2005. DOI: 10.1007/s00220-005-1408-4 arXiv: hep-th/0406192.

[27] K. Hallowell and A. Waldron, 'Supersymmetric Quantum Mechanics and Super-Lichnerowicz Algebras,' *Commun. Math. Phys.*, vol. 278, pp. 775–801, 2008. DOI: 10.1007/s00220-007-0393-1 arXiv: hep-th/0702033.

[28] F. Bastianelli, O. Corradini and A. Waldron, 'Detours and Paths: BRST Complexes and Worldline Formalism,' *JHEP*, vol. 05, p. 017, 2009. DOI: 10.1088/1126-6708/2009/05/017 arXiv: 0902.0530 [hep-th].

[29] R. Bonezzi, 'Yang-Mills theory from the worldline,' *Phys. Rev. D*, vol. 110, no. 6, p. 065 022, 2024. DOI: 10.1103/PhysRevD.110.065022 arXiv: 2406.19045 [hep-th].

[30] F. Bastianelli, F. Fecit and A. Miccichè, 'Pair production of massive charged vector bosons from the worldline,' Jul. 2025, submitted to JHEP for pubblication. arXiv: 2507.15943 [hep-th].

[31] I. K. Affleck, O. Alvarez and N. S. Manton, 'Pair Production at Strong Coupling in Weak External Fields,' *Nucl. Phys. B*, vol. 197, pp. 509–519, 1982. DOI: 10.1016/0550-3213(82)90455-2

[32] G. V. Dunne and C. Schubert, 'Worldline instantons and pair production in inhomogeneous fields,' *Phys. Rev. D*, vol. 72, p. 105 004, 2005. DOI: 10.1103/PhysRevD.72.105004 arXiv: hep-th/0507174.

[33] G. V. Dunne, Q.-h. Wang, H. Gies and C. Schubert, 'Worldline instantons. II. The Fluctuation prefactor,' *Phys. Rev. D*, vol. 73, p. 065 028, 2006. DOI: 10.1103/PhysRevD.73.065028 arXiv: hep-th/0602176.

[34] O. Amat, L.-N. Hu, A. Sawut, M. Mohamedsedik, M. A. Bake and B. S. Xie, 'Schwinger pair production rate and time for some space-dependent electromagnetic fields via worldline instantons formalism,' Apr. 2022. DOI: 10.1140/epjd/s10053-022-00519-y arXiv: 2204.07263 [hep-ph].

[35] G. Degli Esposti and G. Torgrimsson, 'Worldline instantons for the momentum spectrum of Schwinger pair production in spacetime dependent fields,' *Phys. Rev. D*, vol. 107, no. 5, p. 056 019, 2023. DOI: 10.1103/PhysRevD.107.056019 arXiv: 2212.11578 [hep-ph].

[36] F. Bastianelli, R. Bonezzi, O. Corradini and F. Fecit, 'Gluon amplitudes in first quantization,' Aug. 2025. arXiv: 2508.05486 [hep-th].

[37] P. Dirac, *Lectures on Quantum Mechanics* (Dover Books on Physics). Dover Publications, 2013, ISBN: 9780486320281.

[38] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*. Princeton University Press, 1992, ISBN: 9780691087757.

[39] V. D. Gershun and V. I. Tkach, 'Classical and quantum dynamics of particles with arbitrary spins,' *JETP Lett.*, vol. 29, pp. 288–291, 5 1979. [Online]. Available: http://jetpletters.ru/ps/0/article_22085.shtml

[40] P. S. Howe, S. Penati, M. Pernici and P. K. Townsend, 'Wave Equations for Arbitrary Spin From Quantization of the Extended Supersymmetric Spinning Particle,' *Phys. Lett. B*, vol. 215, pp. 555–558, 1988. DOI: 10.1016/0370-2693(88)91358-5

[41] P. A. M. Dirac, 'Relativistic wave equations,' *Proceedings of the Royal Society of London. Series A - Mathematical and Physical Sciences*, vol. 155, no. 886, pp. 447–459, 1936. DOI: 10.1098/rspa.1936.0111

[42] V. Bargmann and E. P. Wigner, 'Group theoretical discussion of relativistic wave equations,' *Proceedings of the National Academy of Sciences*, vol. 34, no. 5, pp. 211–223, 1948. DOI: 10.1073/pnas.34.5.211

[43] A. K. H. Bengtsson, 'A Unified Action for Higher Spin Gauge Bosons From Covariant String Theory,' *Phys. Lett. B*, vol. 182, pp. 321–325, 1986. DOI: 10.1016/0370-2693(86)90100-0

[44] N. Bouatta, G. Compere and A. Sagnotti, *An introduction to free higher-spin fields*, 2004. arXiv: hep-th/0409068 [hep-th].

[45] W. Siegel, *Introduction to string field theory*. 1988, vol. 8. arXiv: hep-th/0107094.

[46] F. Bastianelli, R. Bonezzi, O. Corradini and E. Latini, 'Massive and massless higher spinning particles in odd dimensions,' *JHEP*, vol. 09, p. 158, 2014. DOI: 10.1007/JHEP09(2014)158 arXiv: 1407.4950 [hep-th].

[47] F. Bastianelli, R. Bonezzi, O. Corradini and E. Latini, 'Spinning particles and higher spin field equations,' *J. Phys. Conf. Ser.*, vol. 1208, no. 1, A. Bashir and C. Schubert, Eds., p. 012 006, 2019. DOI: 10.1088/1742-6596/1208/1/012006 arXiv: 1504.02683 [hep-th].

[48] F. Bastianelli, P. Benincasa and S. Giombi, 'Worldline approach to vector and antisymmetric tensor fields,' *JHEP*, vol. 04, p. 010, 2005. DOI: 10.1088/1126-6708/2005/04/010 arXiv: hep-th/0503155.

[49] F. Bastianelli, P. Benincasa and S. Giombi, 'Worldline approach to vector and antisymmetric tensor fields. II.,' *JHEP*, vol. 10, p. 114, 2005. DOI: 10.1088/1126-6708/2005/10/114 arXiv: hep-th/0510010.

[50] A. G. Cohen, G. W. Moore, P. C. Nelson and J. Polchinski, 'An Off-Shell Propagator for String Theory,' *Nucl. Phys. B*, vol. 267, pp. 143–157, 1986. DOI: 10.1016/0550-3213(86)90148-3

[51] A. M. Polyakov, *Gauge Fields and Strings*. London: Taylor & Francis, 1987, ISBN: 978-1-351-44609-9, 978-3-7186-0393-0, 978-0-203-75508-2. DOI: 10.1201/9780203755082

[52] E. C. G. Stueckelberg, 'Theory of the radiation of photons of small arbitrary mass,' *Helv. Phys. Acta*, vol. 30, pp. 209–215, 1957.

[53] H. van Dam and M. J. G. Veltman, 'Massive and massless Yang-Mills and gravitational fields,' *Nucl. Phys. B*, vol. 22, pp. 397–411, 1970. DOI: 10.1016/0550-3213(70)90416-5

[54] V. I. Zakharov, 'Linearized gravitation theory and the graviton mass,' *JETP Lett.*, vol. 12, p. 312, 1970.

[55] A. Fedotov et al., 'Advances in QED with intense background fields,' *Phys. Rept.*, vol. 1010, pp. 1–138, 2023. DOI: 10.1016/j.physrep.2023.01.003 arXiv: 2203.00019 [hep-ph].

[56] F. Fecit, S. A. Franchino-Viñas and F. D. Mazzitelli, 'Resummed effective actions and heat kernels: the Worldline approach and Yukawa assisted pair creation,' *JHEP*, vol. 07, p. 041, 2025. DOI: 10.1007/JHEP07(2025)041 arXiv: 2501.17094 [hep-th].

[57] S. A. Franchino-Viñas, C. García-Pérez, F. D. Mazzitelli, V. Vitagliano and U. W. Haimovichi, 'Resummed heat kernel and effective action for Yukawa and QED,' *Phys. Lett. B*, vol. 854, p. 138 684, 2024. DOI: 10.1016/j.physletb.2024.138684 arXiv: 2312.16303 [hep-th].

[58] S. A. Franchino-Viñas, 'Comment on 'Index-free heat kernel coefficients',' *Class. Quant. Grav.*, vol. 41, no. 12, p. 128 001, 2024. DOI: 10.1088/1361-6382/ad4949 arXiv: 2401.01296 [hep-th].

[59] F. Bastianelli, R. Bonezzi, O. Corradini and E. Latini, 'One-loop quantum gravity from the $\mathcal{N} = 4$ spinning particle,' *JHEP*, vol. 11, p. 124, 2019. DOI: 10.1007/JHEP11(2019)124 arXiv: 1909.05750 [hep-th].

[60] F. Bastianelli, R. Bonezzi and M. Melis, 'Gauge-invariant coefficients in perturbative quantum gravity,' *Eur. Phys. J. C*, vol. 82, no. 12, p. 1139, 2022. DOI: 10.1140/epjc/s10052-022-11119-w arXiv: 2206.13287 [hep-th].

[61] F. Bastianelli, F. Comberiati, F. Fecit and F. Ori, 'Six-dimensional one-loop divergences in quantum gravity from the $\mathcal{N} = 4$ spinning particle,' *JHEP*, vol. 10, p. 152, 2023. DOI: 10.1007/JHEP10(2023)152 arXiv: 2307.09353 [hep-th].

[62] F. Fecit, 'Worldline path integral for the massive graviton,' *Eur. Phys. J. C*, vol. 84, no. 3, p. 339, 2024. DOI: 10.1140/epjc/s10052-024-12707-8 arXiv: 2402.13766 [hep-th].

[63] R. Bonezzi and M. F. Kallimani, 'Worldline geometries for scattering amplitudes,' *JHEP*, vol. 06, p. 167, 2025. DOI: 10.1007/JHEP06(2025)167 arXiv: 2502.18030 [hep-th].