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A study on tame and wild quotient singularities in characteristic p

Tesi di Laurea Magistrale in Geometria Algebrica

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L'algèbre n'est qu'une géométrie écrite,
la géométrie n'est qu'une algèbre figurée.
— Sophie Germain

Abstract

A closed point x on a normal scheme X is a quotient singularity if the corresponding local ring $R = \mathcal{O}_{X,x}$ arises as the ring of invariants by the action of a finite group G on a regular local ring A. If the characteristic of the residue field of R does not divide the order of G, it is called a tame quotient singularity; it is wild otherwise. The aim of this thesis is to discuss the resolution of quotient singularities in three distinct settings: complex toric surfaces, tame cyclic quotient singularities on a normal curve over a Dedekind scheme, wild quotient singularities on a 2-dimensional scheme. While the former two settings can be solved algorithmically and the data of the resolution can be encoded in a so-called Hirzebruch-Jung continued fraction, the latter can be much more unpredictable and difficult to compute explicitly. We will study the dual graph attached to the minimal regular resolution in each case, and see that in the first two cases it is a Dynkin graph of type A_n, while in the wild case it always contains a vertex of valency at least 3.

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Introduction

The topic of this thesis revolves around the resolution of quotient singularities. The setting is the following: consider a normal scheme X of finite type over a base scheme S and a finite group G acting on X via S-automorphisms. For an element $g \in G$, we will denote the corresponding automorphism of schemes σ_g. Then, a *quotient* of X by G is a morphism of schemes $\pi : X \to Y$ satisfying:

1. π is G-*invariant*: for all $g \in G$, $\pi \circ \sigma_g = \pi$;

2. π has the following universal property: for all S-schemes Z with a trivial action of G and all G-invariant morphisms $f : X \to Z$, there is a unique arrow $h : Y \to Z$ making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ & \searrow^{f} & \downarrow{\scriptstyle \exists! h} \\ & & Z \end{array}$$

In the affine case, if $X = \operatorname{Spec} A$, then Y is the spectrum of the ring of invariants A^G, with π being the morphism induced by the inclusion $A^G \hookrightarrow A$. In the non-affine case, a quotient exists if and only if X can be covered by affine open sets $\{U_i\}_i$ that are G-invariant (see [SGA 1], Exposé V, Proposition 1.8). Then, the scheme Y can be obtained by taking the quotient for each U_i and glueing the resulting morphisms. In general, the following holds:

Proposition 1. *In the previous notations:*

1. *Y is an S-scheme of finite type, and π is finite and surjective.*

2. *The fibers of π coincide with the orbits of the G-action on X.*

3. *The topology on Y is the quotient topology.*

4. *There is a natural isomorphism $\mathcal{O}_Y = \pi_*(\mathcal{O}_Y)^G$, where $(\mathcal{O}_Y)^G$ is the sheaf obtained by taking the ring of invariants of the ring of sections for each open sets.*

Proof. [Mus11], A.1. □

In this thesis, we will focus on the 2-dimensional case; then, if Y is non regular, its singular locus is a collection of isolated closed points, each of which is called a *quotient singularity*; a quotient singularity is called *tame* if the characteristic of the residue field does not divide the order of G; it is called *wild* otherwise. We are interested in *resolving* these singularities, i.e. find a birational proper morphism $\rho : \widetilde{Y} \to Y$ where $\widetilde{Y}$ is regular. We refer to such a morphism as a *resolution of singularities*, or *desingularization*. We will also require that ρ is an isomorphism above any regular points of Y; this case is also referred to, in literature, as *desingularization in the strong sense*, but we will never work with resolutions of singularities that do not induce an isomorphism on the regular locus. We say that the resolution of singularities is *minimal* if, whenever $\rho' : Y' \to Y$ is a birational proper morphism and Y' is a regular scheme, there is a unique morphism $Y' \to Y$ making the following diagram commutative:

$$\begin{array}{ccc} Y' & \overset{\exists!}{\dashrightarrow} & \widetilde{Y} \\ & {\scriptstyle \rho'}\searrow & \downarrow{\scriptstyle \rho} \\ & & Y \end{array}$$

We will see that in dimension 2, under the correct hypotheses (for example if S is excellent) a minimal resolution exists, and we will study its exceptional divisor E. More specifically, we are interested in understanding the irreducible components of E and how they intersect; we can then associate to the singularity a graph, called the *dual graph* of the singularity, whose vertices correspond to the irreducible components of E, and an edge between two vertices indicates an intersection between the corresponding components.

Some of the major breakthroughs of the 20th century regarding the subject of the resolution of singularities include Hironaka's proof that in characteristic zero a minimal resolution always exists ([Hir64]), and Lipman's proof of the existence of a desingularization for 2-dimensional schemes under mild hypotheses (for example if the scheme is excellent), regardless of characteristic ([Lip78]). In his classical 1977 paper, Artin ([Art77]) then studied rational double points in positive characteristic, focusing on the cases where $p = 2, 3, 5$: it is indeed in these cases that the algebraic fundamental group (see Section 3.1 for a definition) is not necessarily tame. The topic of wild quotient singularities on surfaces was then revived by Lorenzini, who showed that when a desingularization exists, the corresponding dual graph is a tree ([Lor13]), that for any fixed odd prime p there are infinitely many dual graphs that can arise from the resolution of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities ([Lor14]), and presented examples of wild quotient singularities on products of curves ([Lor18]). In his papers, Lorenzini raised some open questions on the subject matter, prompting further research. For example, Ito and Schröer showed in 2014 that dual graphs of wild quotient singularities always contain at least one node, i.e. a vertex of valency at least 3 ([IS15]), while Obus and Wewers showed in 2019 that a specific type of singularities, called *weakly wild*, can be resolved explicitly ([OW19]). To this day, we still do not know whether a resolution for the wild case exists in higher dimensions; we do know, by Lipman, that it exists for surfaces, but even in the simplest cases it might be very difficult to determine explicitly. Other open problems on wild quotient singularities concern their relation to the so-called *McKay correspondence*, which, in characteristic zero, defines a bijection between conjugacy classes of subgroups of $\mathbf{SL}_2$, du Val singularities, finite Dynkin graphs of type ADE, and Dynkin graphs of affine type. In 2023, Yasuda surveyed a collection of open problems regarding a wild generalization of the McKay correspondence ([Yas23]); following Yasuda's work, Liedtke offered a generalization of the McKay correspondence in positive characteristic ([Lie24]).

The outline of this thesis is the following: the first two chapters revolve around tame quotient singularities, while the third and last chapter will focus on the wild case. More specifically, the work is organized as follows:

Chapter 1. In the first chapter of this thesis, the focus will shift to algebraic varieties in the classical sense, over the ground field $\mathbb{C}$. In particular, the goal of Chapter 1 will be to study normal toric surfaces that arise as a quotient of $\mathbb{C}^n$ by the action of a finite cyclic group. We will see that an affine toric variety can be represented by a *rational convex polyhedral cone*, and that all the data for glueing affine toric varieties into abstract toric varieties can be encoded in a *fan* of cones. Then, if we are working on a non-smooth toric surface, the quotient singularity can be solved by dividing out the corresponding fan into a finer one; this operation geometrically translates to taking a series of blow-ups. This type of resolution is also known as the *Hirzebruch-Jung algorithm* for cyclic quotient singularities, as all the data for the resolution can be encoded in a so-called *Hirzebruch-Jung continued fraction*. The main source for this chapter is the book *Toric varieties* by David Cox, John Little, and Hal Schenck, denoted by the bibliography entry [CLS11]; other valuable sources for this topics are the book *Introduction to toric varieties* by William Fulton ([Ful93]) and the notes *Introduction to toric geometry* by Simon Telen ([Tel22]).

Chapter 2. In the second chapter, we will study the existence and the characteristics of the minimal regular resolution of a normal curve X over a connected Dedekind scheme S, following section 2 of [CES03]. We will see that if either S is excellent or X has smooth generic fiber over S, a minimal regular resolution indeed exists, generalizing a theorem of Lipman on the existence of a desingularization in the strong sense (not necessarily minimal!) on an excellent curve. We will give a formal definition of what we mean by *tame cyclic quotient singularity* and connect back to the case of toric varieties in Theorem 2.22, where we will see how the Hirzebruch-Jung algorithm comes into play to study these singularities.

Chapter 3. In the final chapter of the thesis, we will study the dual graphs of wild quotient singularities. While the first two chapters deal with singularities that are toric in nature, and can thus be simple to resolve algorithmically using tools from logarithmic geometry, wild quotient singularities can be much “stranger”, even though we do not know how much stranger. We will see that the dual graph of a wild quotient singularity always contains a node. This will be proven by showing that a Hirzebruch-Jung singularity, whose dual graph is a tree without nodes, can only occur in the tame case; we will use Grothendieck’s theory of algebraic fundamental groups to prove this result. We will then construct an example of wild quotient singularity whose dual graph has two nodes.

Throughout this thesis, the reader is assumed to be familiar with the basic notions presented in a first course in algebraic geometry: classical algebraic varieties, sheaves, schemes, morphisms of schemes and their properties, as well as some very basic definitions of category theory and homological algebra.

Chapter 1

Resolution of singularities on toric surfaces

In this chapter, we will work with varieties in the sense of classical algebraic geometry, over the base field $\mathbb{C}$. We will cover some basic notions on toric varieties, a subclass of algebraic varieties that can be studied through the lens of combinatorics. In particular, toric surfaces offer a good first example of a simple resolution of quotient singularities, the Hirzebruch-Jung algorithm, which will be generalized to normal curves over a Dedekind scheme in the next chapter.

In presenting the topic of toric varieties, we will follow fairly closely the presentation in [CLS11]. While it may seem disconnected from the other topics presented in the thesis, this chapter constitutes a necessary build-up for the theory in the following chapters. This will require introducing the basic theory of toric varieties; we have tried keeping the presentation at minimum, while still fairly self-contained.

1.1 Affine toric varieties

We will first introduce the notion of an *affine toric variety*. We will see various equivalent ways of constructing affine toric varieties and how these constructions play an important role in studying toric geometry.

1.1.1 Affine toric varieties, lattice points, toric ideals, affine semigroups

In the following, we will use the word *lattice* to mean a free abelian group of finite rank.

Definition 1.1 (Torus). A *torus* is an affine variety isomorphic to $(\mathbb{C}^\times)^n$, from which it inherits the multiplicative group structure.

We will associate to each torus a pair of mutually dual lattices: the lattice of characters and the lattice of one-parameter subgroups.

Definition 1.2 (Character of a torus)**.** Let T be a torus. A *character* of T is a morphism $\chi : T \to \mathbb{C}^\times$ which is a group homomorphism.

It can be proven that all characters of $(\mathbb{C}^\times)^n$ are of the following form: for $\mathbf{m} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, define

$$\chi^{\mathbf{m}}(t_1, \ldots, t_n) = t_1^{a_1} \cdots t_n^{a_n}.$$

Thus, in general, the set of all characters of a torus T is a free abelian group M_T of rank equal to the dimension of T; for $\mathbf{m} \in M_T$, we denote the corresponding character by $\chi^{\mathbf{m}} : T \to \mathbb{C}^\times$. The association $T \longmapsto M_T$ is a contravariant functor from the category of tori (where a morphism of tori is a morphism of varieties between tori which is also a group homomorphism) to the category of lattices, seen as a full subcategory of $\mathbf{Ab}$. This functor maps a morphism of tori $\Phi : T_1 \to T_2$ to a group homomorphism $\widehat{\Phi} : M_{T_2} \to M_{T_1}$ given by composition with Φ on the right. The dual lattice to M, $N := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, can be identified with the group of all one-parameter subgroups of T:

Definition 1.3 (One-parameter subgroup of a torus)**.** A *one-parameter subgroup* of T, or *co-character*, is a morphism $\lambda : \mathbb{C}^\times \to T$ which is a group homomorphism.

Similarly to the case of characters, it can be shown that all one-parameter subgroups of $(\mathbb{C}^\times)^n$ are of the form

$$\lambda^{\mathbf{u}}(t) = (t^{b_1}, \ldots, t^{b_n}),$$

for $\mathbf{u} = (b_1, \ldots, b_n) \in \mathbb{Z}^n$.

Definition 1.4 (Affine toric variety)**.** An *affine toric variety* is an irreducible affine variety V which contains a torus $T_V \simeq (\mathbb{C}^\times)^n$ as a Zariski open set, such that the action of T_V on itself extends to an algebraic action of T_V on V.

For example, the curve $C = \mathrm{V}(x^4 + y^7) \subseteq \mathbb{C}^2$ is an affine toric variety with torus

$$C \setminus \{(0,0)\} = \{(t^7, -t^4) \mid t \in \mathbb{C}^\times\} \simeq \mathbb{C}^\times,$$

where the toric action is given by $s \cdot \left(t^7, -t^4\right) = \left((st)^7, -(st)^4\right)$ for $s \in \mathbb{C}^\times$. This action extends to C by defining $s \cdot (0,0) = (0,0)$.

We will now see three equivalent ways of constructing toric varieties, highlighting their combinatorial nature: through lattice points, through toric ideals, and through affine semigroups. These three ways are not only equivalent to each other, but they also suffice to describe all affine toric varieties.

Toric varieties and lattice points. For a fixed torus T with character lattice M, finite sets of lattice points within M yield toric varieties of rank at most $\dim T = \operatorname{rk} M$. Indeed, fix a torus $T \simeq (\mathbb{C}^\times)^n$ with character lattice M and consider a finite subset of lattice points $\mathscr{A} = \{\mathbf{m}_1, \ldots, \mathbf{m}_s\} \subseteq M$. Define the morphism $\Phi_{\mathscr{A}} : T \to \mathbb{C}^s$ by $\Phi_{\mathscr{A}}(\mathbf{t}) = (\chi^{\mathbf{m}_1}(\mathbf{t}), \ldots, \chi^{\mathbf{m}_s}(\mathbf{t}))$: this induces a morphism of tori $\Phi : T \to (\mathbb{C}^\times)^s$ whose image $T_{\mathscr{A}}$ is a torus and a closed subset of $(\mathbb{C}^\times)^s$ ([CLS11], 1.1.1). Now consider the Zariski closure $Y_{\mathscr{A}}$ of the image of $\Phi_{\mathscr{A}}$ within $\mathbb{C}^s$: it is an affine variety which contains $T_{\mathscr{A}}$ as a Zariski open subset, and it can be shown pretty straightforwardly that $Y_{\mathscr{A}}$ is indeed an affine toric variety with torus $T_{\mathscr{A}}$. Furthermore, consider the following two commutative diagrams, where the diagram on the right is obtained from the diagram on the left by applying the contravariant functor described earlier:

$$\begin{array}{ccc} T & \xrightarrow{\Phi} & (\mathbb{C}^\times)^s \\ & \searrow^{\Phi'} & \uparrow \\ & & T_{\mathscr{A}} \end{array} \qquad \qquad \begin{array}{ccc} M & \xleftarrow{\widehat{\Phi}} & \mathbb{Z}^s \\ & \nwarrow^{\widehat{\Phi'}} & \downarrow \\ & & M_{T_{\mathscr{A}}} \end{array}$$

It is immediate to see that $M_{T_{\mathscr{A}}} \simeq \operatorname{Im} \widehat{\Phi'} = \operatorname{Im} \widehat{\Phi} = \mathbb{Z}\mathscr{A}$. In conclusion, given a finite subset $\mathscr{A} \subseteq M$, we have constructed an affine toric variety $Y_{\mathscr{A}}$, whose dimension is equal to the rank of the sublattice $\mathbb{Z}\mathscr{A} \subseteq M$. Let us see an example:

Example 1.5. The variety $V = \mathrm{V}(xyz - w^2) \subseteq \mathbb{C}^4$ is an affine toric variety with torus

$$T = V \cap (\mathbb{C}^\times)^4 = \{(t_1, t_2, t_1^{-1} t_2^{-1} t_3^2, t_3) \mid t_i \in \mathbb{C}^\times\} \simeq (\mathbb{C}^\times)^3,$$

where the isomorphism $(\mathbb{C}^\times)^3 \simeq T \subseteq \mathbb{C}^4$ is given by $\Psi(t_1, t_2, t_3) = (t_1, t_2, t_1^{-1} t_2^{-1} t_3^2, t_3)$, or, using the previous notation for characters,

$$\Psi = \left(\chi^{\mathbf{e}_1}, \chi^{\mathbf{e}_2}, \chi^{-\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3}, \chi^{\mathbf{e}_3}\right).$$

This prompts us to look at the lattice points $\mathscr{A} = \{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3, \mathbf{e}_3\}$. Indeed, Ψ coincides with the map $\Phi_{\mathscr{A}}$ as defined previously, and indeed $T = \operatorname{Im} \Phi_{\mathscr{A}}$ and V is its Zariski closure in $\mathbb{C}^4$.

Toric varieties and toric ideals. Consider an affine toric variety V with torus T and let $\mathrm{I}(V)$ be its vanishing ideal. We will see that there is a very special class of ideals whose zero loci are toric varieties:

Definition 1.6 (Toric ideal)**.** A prime ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is a *toric ideal* if it is of the form $I = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}^s, \mathbf{u} - \mathbf{v} \in L \rangle$ for some lattice $L \subseteq \mathbb{Z}^s$.

Let us look at the toric variety $Y_{\mathscr{A}}$ defined earlier, for $\mathscr{A} = \{\mathbf{m}_1, \ldots, \mathbf{m}_s\}$. We have an induced morphism of lattices $\widehat{\Phi} : \mathbb{Z}^s \to M$ whose matrix has the elements of

$\mathscr{A}$ as columns. Let $L = \ker \widehat{\Phi}$: for any $\ell = (\ell_1, \dots, \ell_s) \in L$, define $\ell_+ = \sum_{\ell_i > 0} \ell_i \mathbf{e}_i$, $\ell_- = -\sum_{\ell_i < 0} \ell_i \mathbf{e}_i$: then clearly $\ell = \ell_+ - \ell_-$. Observe that any binomial of the form $\mathbf{x} = \mathbf{x}^{\ell_+} - \mathbf{x}^{\ell_-}$ vanishes on $Y_{\mathscr{A}}$, since for any $\Phi_{\mathscr{A}}(\mathbf{t}) = (\chi^{\mathbf{m}_1}(\mathbf{t}), \dots, \chi^{\mathbf{m}_s}(\mathbf{t}))$ we have

$$(\Phi_{\mathscr{A}}(\mathbf{t}))^{\ell_+} - (\Phi_{\mathscr{A}}(\mathbf{t}))^{\ell_-} = \prod_{\ell_i > 0} \chi^{\ell_i \mathbf{m}_i}(\mathbf{t}) - \prod_{\ell_i < 0} \chi^{\ell_i \mathbf{m}_i}(\mathbf{t}) = \chi^{\sum\limits_{\ell_i>0} \ell_i \mathbf{m}_i}(\mathbf{t}) - \chi^{\sum\limits_{\ell_i<0} \ell_i \mathbf{m}_i}(\mathbf{t}) = 0,$$

since ℓ satisfies $\sum_{i=1}^s \ell_i \mathbf{m}_i = 0 \iff \sum_{\ell_i > 0} \ell_i \mathbf{m}_i = \sum_{\ell_i < 0} \ell_i \mathbf{m}_i$. Thus, the ideal $I = \left\langle \mathbf{x}^{\ell_+} - \mathbf{x}^{\ell_-}, \ell \in L \right\rangle$ satisfies $I \subseteq \mathrm{I}(Y_{\mathscr{A}})$. It can be shown ([CLS11], 1.1.9) that this inclusion is actually an equality:

$$\mathrm{I}(Y_{\mathscr{A}}) = \left\langle \mathbf{x}^{\ell_+} - \mathbf{x}^{\ell_-}, \ell \in \ker \widehat{\Phi} \right\rangle = \left\langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}^s, \mathbf{u} - \mathbf{v} \in \ker \widehat{\Phi} \right\rangle.$$

On the other hand, given a toric ideal $I = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}^s, \mathbf{u} - \mathbf{v} \in L \rangle$ for some lattice $L \subseteq \mathbb{Z}^s$, we can recover a toric variety of the form $Y_{\mathscr{A}}$ as follows: as earlier, fix a torus T with character lattice M, and consider a homomorphism $\Phi : \mathbb{Z}^s \to M$ whose kernel is L. Set set $\mathbf{m}_1 = \widehat{\Phi}(\mathbf{e}_1), \dots, \mathbf{m}_s = \widehat{\Phi}(\mathbf{e}_s)$: the set $\mathscr{A} = \{\mathbf{m}_1, \dots, \mathbf{m}_s\}$ yields the toric variety $Y_{\mathscr{A}}$ as per the previous paragraph, with $\mathrm{I}(Y_{\mathscr{A}}) = I$. Thus, the zero locus of a toric ideal is a toric variety.

Example 1.7. Consider the toric variety $V = \mathrm{V}(xyz - w^2)$ from the previous example. The ideal $I = (xyz - w^2)$ is a toric ideal corresponding (by the previous description) to the lattice $L = \mathbb{Z}\{(1, 1, 1, -2)\}$, which arises as the kernel of the matrix

$$\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & -\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3 & \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

Toric varieties and affine semigroups. Let us again consider the affine toric variety $Y_{\mathscr{A}}$ described in the previous two paragraphs. We will see that its coordinate ring arises from an affine semigroup:

Definition 1.8 (Affine semigroup)**.** An *affine semigroup* is a set S endowed with a binary operation $+$ which satisfies:

(i) $(\mathsf{S}, +)$ is a commutative monoid;

(ii) there is a finite subset $\mathscr{S} \subseteq \mathsf{S}$ such that any element of S can be written as an $\mathbb{N}$-linear combination of elements of $\mathscr{S}$;

(iii) S can be embedded into a lattice M.

Given an affine semigroup S, the *semigroup algebra* $\mathbb{C}[\mathsf{S}]$ is the set of all finite formal sums $\sum_{\mathbf{m} \in \mathsf{S}} a_{\mathbf{m}} \xi^{\mathbf{m}}$, with $a_{\mathbf{m}} \in \mathbb{C}$ and multiplication defined by $\xi^{\mathbf{m}} \cdot \xi^{\mathbf{n}} = \xi^{\mathbf{m}+\mathbf{n}}$.

Example 1.9. Let us see a few examples of affine semigroups and their corresponding semigroup algebras:

1. $\mathbb{N}^n$ is an affine semigroup with $\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \ldots, x_n]$;
2. any lattice is an affine semigroup; in particular, $\mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[x_1{}^{\pm}, \ldots, x_n{}^{\pm}]$;
3. the lattice points $\mathscr{A} = \{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3, \mathbf{e}_3\}$ from the previous example generate the affine semigroup $\mathbb{N}\mathscr{A}$, which yields the semigroup algebra $\mathbb{C}[\mathbb{N}\mathscr{A}] = \mathbb{C}[s, t, u^2 s^{-1} t^{-1}, u] \simeq \mathbb{C}[x, y, z, w]/(xyz - w^2)$: by no coincidence, this is exactly the coordinate ring of the affine toric variety $Y_{\mathscr{A}} = \mathrm{V}(xyz - w^2)$.

We have the following result:

Proposition 1.10. *Let* S *be an affine semigroup.*

1. *The semigroup algebra* $\mathbb{C}[\mathsf{S}]$ *is an integral domain;*
2. *The affine variety* $\operatorname{Specm} \mathbb{C}[\mathsf{S}]$ *is a toric variety whose torus has character lattice* $\mathbb{Z}\mathsf{S}$.

Proof. Let $\mathsf{S} \hookrightarrow M$ be an embedding of S into a lattice M; then we have an embedding of $\mathbb{C}[\mathsf{S}]$ into $\mathbb{C}[M] = \mathbb{C}[x_1{}^{\pm}, \ldots, x_n{}^{\pm}]$; since the latter is an integral domain, so is $\mathbb{C}[\mathsf{S}]$. This proves (1). Now let $\mathscr{A} \subseteq M$ such that $\mathsf{S} \simeq \mathbb{N}\mathscr{A}$. We have a map $\Phi_{\mathscr{A}} : T \to \mathbb{C}^s$, where T is a torus with character lattice M; this induces a morphism $\Phi^*_{\mathscr{A}} : \mathbb{C}[x_1, \ldots, x_s] \to \mathbb{C}[M]$ between their coordinate rings, whose image is $\mathbb{C}[\mathsf{S}]$. Thus we have a short exact sequence

$$0 \longrightarrow \mathrm{I}(Y_{\mathscr{A}}) \longrightarrow \mathbb{C}[x_1, \ldots, x_s] \longrightarrow \mathbb{C}[\mathsf{S}] \longrightarrow 0,$$

yielding $\mathbb{C}[\mathsf{S}] \simeq \mathbb{C}[x_1, \ldots, x_s]/\,\mathrm{I}(Y_{\mathscr{A}})$. This proves $\operatorname{Specm} \mathbb{C}[\mathsf{S}] \simeq Y_{\mathscr{A}}$, thus it is a toric variety whose torus has character lattice $\mathbb{Z}\mathscr{A} = \mathbb{Z}\mathsf{S}$. □

Thus, we have seen that toric varieties can arise from lattice points, toric ideals, and affine semigroups, and that these constructions are all connected with each other. More is actually true: all toric varieties arise in these equivalent ways.

Theorem 1.11. *Let* V *be an affine variety. The following are equivalent:*

1. V *is an affine toric variety;*
2. $V \simeq Y_{\mathscr{A}}$ *for some finite subset* $\mathscr{A}$ *of a lattice* M*;*
3. V *is the zero locus of a toric ideal;*
4. *the coordinate ring of* V *is the semigroup algebra* $\mathbb{C}[\mathsf{S}]$ *for some affine semigroup* S.

Proof. See [CLS11], 1.1.17. □

1.1.2 Affine toric varieties arising from polyhedral cones

In the previous subsection, we have looked at the toric variety $V = V(xyz - w^2)$ from multiple angles. In particular, we have seen that it arises from the lattice points $\mathscr{A} = \{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3, \mathbf{e}_3\} \subseteq \mathbb{Z}^3$ which yield the affine semigroup $\mathbb{N}\mathscr{A}$. If we embed $\mathbb{N}\mathscr{A}$ into $M_T \simeq \mathbb{Z}^3$ and consider its image through the functor $- \otimes_{\mathbb{Z}} \mathbb{R}$, we obtain a polyhedral cone within $\mathbb{R}^3$, as pictured in Figure 1.1. We will see that all the lattice points lying in the polyhedral cone are elements of S, and that the cone determines V uniquely.

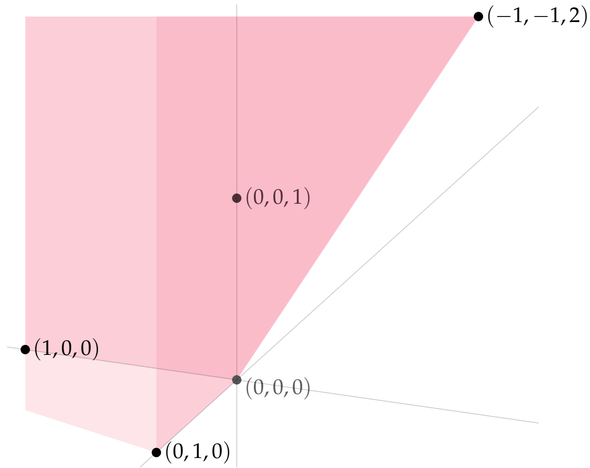


Figure 1.1: The cone generated by the lattice points corresponding to the affine toric variety $V(xyz - w^2)$.

Notice that V is a normal variety: we will see that the corresponding polyhedral cone is *strongly convex*. This is not a coincidence: *normal toric varieties arise from strongly convex polyhedral cones*. To make sense of this statement, let us introduce some basic notions on cones.

Let M, N be a pair of dual lattices, and let $M_{\mathbb{R}}$, $N_{\mathbb{R}}$ be the corresponding dual real vector spaces, obtained by applying the functor $- \otimes_{\mathbb{Z}} \mathbb{R}$. For $\mathbf{m} \in M_{\mathbb{R}}$, $\mathbf{n} \in N_{\mathbb{R}}$, denote the usual pairing by $\langle m, n \rangle$.

Definition 1.12 (Rational convex polyhedral cone)**.** A *rational convex polyhedral cone* in $N_{\mathbb{R}}$ is a subset of the form

$$\sigma = \text{Cone}(S) = \left\{ \sum_{\mathbf{u} \in S} \lambda_{\mathbf{u}} \mathbf{u} \mid \lambda_{\mathbf{u}} \geq 0 \right\} \subseteq \mathbb{N}_{\mathbb{R}},$$

where $S \subseteq N$ is finite. The *dimension* $\dim \sigma$ of σ is the dimension of Span(σ).

Let σ be a (rational convex polyhedral) cone (we will often omit subsets of {rational, convex, polyhedral} for the sake of simplicity), and consider its dual cone within $M_{\mathbb{R}}$:

$$\sigma^\vee = \{\mathbf{m} \in M_{\mathbb{R}} \mid \langle \mathbf{m}, \mathbf{u} \rangle \geq 0 \ \forall \mathbf{u} \in \sigma\}.$$

Then $\sigma^\vee$ is a rational convex polyhedral cone in $M_{\mathbb{R}}$ and $(\sigma^\vee)^\vee = \sigma$. Now, for $\mathbf{m} \in \sigma^\vee$, consider the hyperplane

$$H_{\mathbf{m}} = \{\mathbf{u} \in N_{\mathbb{R}} \mid \langle \mathbf{m}, \mathbf{u} \rangle = 0\} \subseteq N_{\mathbb{R}}.$$

Given a cone σ, we call each $\tau = \sigma \cap H_{\mathbf{m}}$ a *face* of σ. In this case, we write $\tau \preceq \sigma$. Note that $\sigma = \sigma \cap H_0$, thus σ is a face of itself; if $\tau \preceq \sigma$ and $\tau \neq \sigma$, we say that τ is a *proper face* and we write $\tau \prec \sigma$. Clearly, a face of a cone is still a cone.

The following results on cones and their faces hold:

Proposition 1.13. *Let σ, τ, τ' be cones. Then:*

1. *if $\tau \preceq \sigma$ and $\tau' \preceq \sigma$, then $\tau \cap \tau' \preceq \sigma$;*
2. *if $\tau' \preceq \tau$ and $\tau \preceq \sigma$, then $\tau' \preceq \sigma$;*
3. *if $\tau \preceq \sigma$ and $v, w \in \sigma$, then $v + w \in \tau$ implies $v, w \in \tau$;*
4. *there is an inclusion reversing bijection between faces of σ and faces of $\sigma^\vee$.*

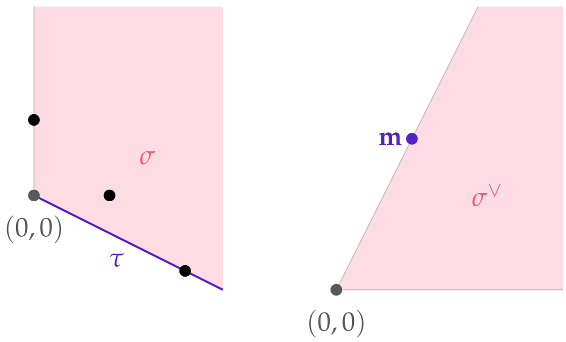


Figure 1.2: A rational cone σ and its dual: highlighted in purple are a point $\mathbf{m} \in \sigma^\vee$ and the corresponding face $\tau \preceq \sigma$, while the bold black points correspond to the generators of the cone.

Definition 1.14 (Strongly convex cone)**.** A cone $\sigma \subseteq N_{\mathbb{R}}$ is *strongly convex* if any of the following equivalent conditions hold:

1. $\sigma \cap (-\sigma) = \{0\}$;
2. $\{0\} \preceq \sigma$;
3. $\dim \sigma^\vee = n$.

In other words, a strongly convex polyhedral cone does not contain any line through the origin.

If σ is a cone, the one-dimensional faces of σ are called *rays* of σ. If σ is strongly convex and $\rho \preceq \sigma$ is a ray, then there is a unique element $\mathbf{r}_\rho \in \rho$ such that $\rho = \mathbb{N} \cdot \mathbf{r}_\rho$: the set of all such elements for all the rays of σ is the set of *minimal generators* of σ.

Definition 1.15 (Smooth cone)**.** A strongly convex cone $\sigma \subseteq N_\mathbb{R}$ is *smooth* if its ray generators form part of a $\mathbb{Z}$-basis of N.

Let us now see how to obtain an affine toric variety from a rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$: the set $\mathsf{S}_\sigma = \sigma^\vee \cap M$ is an affine semigroup by a result known as Gordan's lemma ([CLS11] 1.2.17), which states that $\sigma^\vee \cap M = \mathbb{N}\mathscr{A}$ for some finite set $\mathscr{A} \subseteq M$. Then we have the following immediate corollary:

Proposition 1.16. *Let $\sigma \subseteq N_\mathbb{R}$ be a rational polyhedral cone, S_σ the corresponding affine semigroup. Then $U_\sigma = \operatorname{Specm} \mathbb{C}[\mathsf{S}_\sigma]$ is an affine toric variety. Moreover, $\dim U_\sigma = n$ if and only if σ is strongly convex.*

Example 1.17. Consider the strongly convex polyhedral cone $\sigma = \operatorname{Cone}(\mathbf{e}_1, \ldots, \mathbf{e}_r) \subseteq \mathbb{R}^n$, with $r < n$. Then $\sigma^\vee = \operatorname{Cone}(\mathbf{e}_1, \ldots, \mathbf{e}_r, \pm\mathbf{e}_{r+1}, \ldots, \pm\mathbf{e}_n)$. Thus

$$\mathsf{S}_\sigma \simeq \mathbb{C}\left[x_1, \ldots, x_r, x_{r+1}^\pm, \ldots, x_n^\pm\right] \Longrightarrow U_\sigma \simeq \mathbb{C}^r \times (\mathbb{C}^\times)^{n-r}.$$

To conclude this subsection, we will now briefly look at how faces of a cone σ correspond to affine open subsets of U_σ.

Theorem 1.18. *Let σ be a strongly convex polyhedral cone, $\tau \preceq \sigma$ with $\tau = H_\mathbf{m} \cap \sigma$ for $\mathbf{m} \in \sigma^\vee \cap M$. Then:*

1. *The semigroup algebra $\mathbb{C}[\mathsf{S}_\tau]$ is the localization $\mathbb{C}[\mathsf{S}_\sigma]_{\chi^\mathbf{m}}$;*
2. *We have an inclusion $U_\tau \hookrightarrow U_\sigma$ with U_τ being isomorphic to the open subset $(U_\sigma)_{\chi^\mathbf{m}}$;*
3. *If σ, σ' are cones which intersect at a common face τ, we have inclusions $U_\sigma \supseteq U_\tau \subseteq U_{\sigma'}$ given by $(U_\sigma)_{\chi^\mathbf{m}} \simeq U_\tau \simeq (U_{\sigma'})_{\chi^{-\mathbf{m}}}$, where $\mathbf{m} \in \sigma^\vee \cap (-\sigma')^\vee$ is such that $\tau = \sigma \cap H_\mathbf{m} = \sigma' \cap H_\mathbf{m}$.*

Proof. [CLS11], 1.2.13, 1.3.16. □

Example 1.19. Consider the strongly convex cone $\sigma = \operatorname{Cone}(\mathbf{e}_2, 2\mathbf{e}_1 - \mathbf{e}_2) \subseteq \mathbb{Z}^2$; its dual cone, as computed with `Macaulay2`, is $\sigma^\vee = \operatorname{Cone}(\mathbf{e}_1, \mathbf{e}_1 + 2\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2)$ (pictured in Figure 1.2), which yields the affine toric variety

$$U_\sigma = \operatorname{Specm} \mathbb{C}[u, ut^2, ut] = \operatorname{Specm} \frac{\mathbb{C}[x,y,z]}{(xy - z^2)}.$$

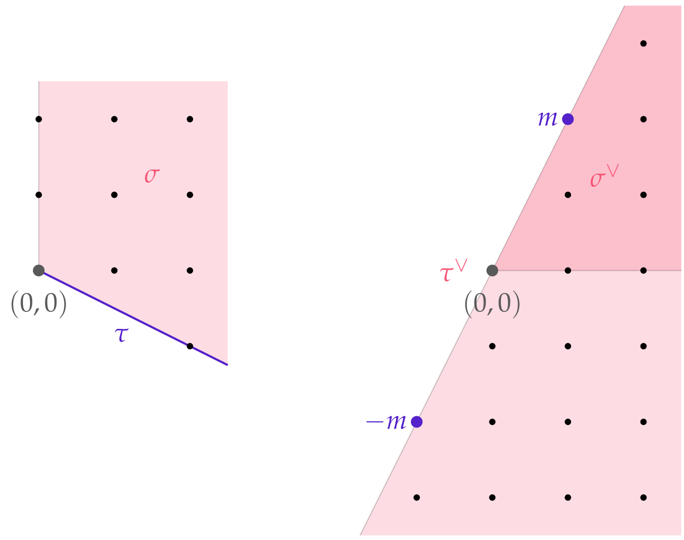


Figure 1.3: The strongly convex cone $\sigma = \text{Cone}(\mathbf{e}_2, 2\mathbf{e}_1 - \mathbf{e}_2)$, its face $\tau = \text{Cone}(2\mathbf{e}_1 - \mathbf{e}_2)$, and their duals. The semigroup algebra $\mathbb{C}[\mathsf{S}_\tau]$ is the localization $\mathbb{C}[\mathsf{S}_\sigma]_{\xi^m}$.

The element $\mathbf{m} = \mathbf{e}_1 + 2\mathbf{e}_2 \in \sigma^\vee$ yields the face $\tau = \text{Cone}(2\mathbf{e}_1 - \mathbf{e}_2) \preceq \sigma$; the corresponding toric variety is

$$U_\tau = \text{Specm}\left(\frac{\mathbb{C}[x,y,z]}{(xy - z^2)}\right)_y,$$

which is isomorphic to the affine open set $U_\sigma \setminus \{y = 0\}$.

1.1.3 Description of the torus action

Let $V = \text{Specm}\,\mathbb{C}[\mathsf{S}]$ be an affine toric variety with torus T: in this subsection, we will give an intrinsic description of the torus action $T \times V \to V$. A fundamental tool in achieving this task will be characterizing the points of V via (affine) semigroup homomorphisms $\mathsf{S} \to \mathbb{C}$, where by *semigroup homomorphism* we mean a map between commutative monoids that preserves the binary operation and the $\mathbb{N}$-action.

Proposition 1.20. *Let* $V = \text{Specm}\,\mathbb{C}[\mathsf{S}]$ *be a toric variety. The following data are equivalent:*

1. *points* $p \in V$*;*
2. *maximal ideals* $\mathfrak{m}_p$ *of* $\text{Specm}\,\mathbb{C}[\mathsf{S}]$*;*
3. *semigroup homomorphisms* $\mathsf{S} \to \mathbb{C}$.

The correspondence between points and maximal ideals is standard, while the correspondence with semigroup homomorphisms works as follows: to a point $p \in V$, associate the semigroup homomorphism $m \longmapsto \xi^m(p)$; on the other hand, given a semigroup homomorphism $\gamma : \mathsf{S} \to \mathbb{C}$, consider the kernel $\mathfrak{m}_p$ of the induced $\mathbb{C}$-algebras homomorphism $\gamma^* : \mathbb{C}[\mathsf{S}] \to \mathbb{C}$: $\mathfrak{m}_p$ is a maximal ideal corresponding to a point $p \in V$. The reader may refer to [CLS11], 1.3.1 for a detailed proof.

This description of points allows us to look at the torus action intrinsically, without needing to embed the variety into an affine space. First observe that the coordinate ring of $T \times V$ is the tensor product $\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[\mathsf{S}]$; then, since the action $T \times V \to V$ is the extension of the action $T \times T \to T$, we have the commutative diagrams

$$\begin{array}{ccc} T \times T & \longrightarrow & T \\ \downarrow & & \downarrow \\ T \times V & \longrightarrow & V \end{array} \quad \xrightarrow{\text{Specm}(\cdot)} \quad \begin{array}{ccc} \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M] & \longleftarrow & \mathbb{C}[M] \\ \uparrow & & \uparrow \\ \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[\mathsf{S}] & \longleftarrow & \mathbb{C}[\mathsf{S}] \end{array} .$$

It follows that the action of T on V is given by the algebraic map $\xi^m \longmapsto \xi^m \otimes \xi^m$, just like the action of T on itself. Thus, we have the following result:

Proposition 1.21. *In the previous notation, an element $t \in T$ acts on V by*

$$\underbrace{\big(m \longmapsto \gamma(m)\big)}_{\substack{\text{point on } V, \\ \text{seen as a semigroup homomorphism}}} \longmapsto \big(m \longmapsto \xi^m(t)\gamma(m)\big).$$

Let us now conclude this subsection by giving a series of results regarding the existence of fixed points for the torus action. Given an affine semigroup S, we say that S is *pointed* if its only invertible element is 0, i.e. $\mathsf{S} \cap (-\mathsf{S}) = \{0\}$.

Proposition 1.22. *Consider an affine toric variety V with dense torus T.*

1. *If we write $V = \operatorname{Specm} \mathbb{C}[\mathsf{S}]$ for some affine semigroup S, then the torus action on V has a fixed point if and only if S is pointed, in which case the unique fixed point corresponds to the semigroup homomorphism*

$$\gamma : m \longmapsto \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. *If we write $V = Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ for a set of generators $\mathscr{A} \subseteq \mathsf{S}$, then the torus action on V has a fixed point if and only if $0 \in Y_{\mathscr{A}}$, in which case the unique fixed point is 0.*

3. *If we write $V = U_\sigma$ for some strongly convex rational cone $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$, then the torus action on V has a fixed point if and only if σ has dimension n, in which case the unique fixed point corresponds to the maximal ideal $\langle \xi^m \mid m \in \mathsf{S}_\sigma \smallsetminus \{0\} \rangle$.*

Proof. [CLS11], 1.3.2, 1.3.3. □

In each case, the unique fixed point is called the *distinguished point* of the variety, and we denote it by γ_σ if $V = U_\sigma$ for a strongly convex rational cone σ of dimension n.

1.1.4 Normality and smoothness

It follows from Example 1.17 that any affine toric variety arising from a smooth cone is isomorphic to the product of an affine space with a torus, and is thus smooth. We will see in this subsection that the converse also holds and that, in general, properties of polyhedral cones are linked to geometric properties of the toric varieties they yield.

Definition 1.23. An affine semigroup $\mathsf{S} \subseteq M = \mathbb{Z}\mathsf{S}$ is *saturated* if $k\mathbf{m} \in \mathsf{S}$ for $\mathbf{m} \in M, k \in \mathbb{N}_{>0}$ implies $\mathbf{m} \in \mathsf{S}$.

Theorem 1.24. *Let V be an affine toric variety with torus $T \simeq (\mathbb{C}^\times)^n$, character lattice M and co-character lattice N. The following are equivalent:*

1. *V is a normal toric variety.*
2. *$V = \mathrm{Specm}(\mathbb{C}[\mathsf{S}])$ for some saturated affine semigroup S.*
3. *$V \simeq U_\sigma = \mathrm{Specm}(\mathbb{C}[\mathsf{S}_\sigma])$ for some strongly convex rational cone $\sigma \subseteq N_\mathbb{R}$.*

Proof. Recall that $V = \mathrm{Specm}(\mathbb{C}[\mathsf{S}])$ for some affine semigroup S, and that the character lattice of V is $\mathbb{Z}\mathsf{S} = M$.
(1) ⇒ (2) Since V is normal, its coordinate ring $\mathbb{C}[\mathsf{S}]$ is integrally closed in its field of fractions. Let $k\mathbf{m} \in \mathsf{S}$ for some $\mathbf{m} \in M, k \in \mathbb{N}_{>0}$: then $\xi^{k\mathbf{m}} \in \mathbb{C}[\mathsf{S}]$. The monic equation $x^k - \xi^{k\mathbf{m}} = 0$ is satisfied by $\xi^{\mathbf{m}} \in \mathbb{C}[M] = \mathrm{Frac}\,\mathbb{C}[\mathsf{S}]$, whence by normality $\xi^{\mathbf{m}} \in \mathbb{C}[\mathsf{S}]$ and $\mathbf{m} \in \mathsf{S}$. Thus S is saturated.
(2) ⇒ (3) Let S be saturated: we need to find a strongly convex rational cone $\sigma \subseteq N$ such that $\mathsf{S} = \mathsf{S}_\sigma$. Let $\mathscr{A} = \{\mathbf{m}_1, \dots, \mathbf{m}_s\} \subseteq M$ be a finite generating set for S and let $\sigma^\vee = \mathrm{Cone}(\mathscr{A}) \subseteq M_\mathbb{R}$. Clearly $\mathsf{S} \subseteq \sigma^\vee \cap M$; on the other hand, the $\mathbb{Q}$-vector space $\sigma^\vee \cap (M \otimes_\mathbb{Z} \mathbb{Q})$ is generated over $\mathbb{Q}$ by $\mathscr{A}$, thus any $\mathbf{m} \in \sigma^\vee \cap M$ can be written as $\mathbf{m} = \sum_{i=1}^s \lambda_i \mathbf{m}_i$ with $\lambda_i \in \mathbb{Q}_{>0}$. Clearing the denominators we can find a natural number $k \in \mathbb{N}_{>0}$ such that $k\mathbf{m} \in \mathsf{S}$: since S is saturated, it follows $\mathbf{m} \in \mathsf{S}$ thus $\mathsf{S} = \sigma^\vee \cap M = \mathsf{S}_\sigma$.
(3) ⇒ (1) See [CLS11] 1.3.5. □

Example 1.25. A standard example of non-normal variety is the cusp $V = \mathrm{V}(x^3 - y^2)$. This is the toric variety corresponding to the lattice points $\mathscr{A} = \{2, 3\} \subseteq \mathbb{Z}$, with affine semigroup $\mathbb{N}\mathscr{A} = \{2, 3, 4, 5, \ldots\}$, which is not saturated as it does not contain 1 (but it contains all of its other nonzero natural multiples).

Example 1.26. Consider the toric variety $V = \mathrm{V}(xyz - w^2)$ from the first subsection; we have $V = U_\sigma$ for $\sigma = \mathrm{Cone}(\mathbf{e}_3, 2\mathbf{e}_1 + \mathbf{e}_3, 2\mathbf{e}_2 + \mathbf{e}_3)$ (pictured in Figure 1.4), with $\sigma^\vee$ being the polyhedral cone in Figure 1.1. Since σ is strongly convex, as it contains no line through the origin (i.e. $\sigma \cap (-\sigma) = \{0\}$), V is normal.

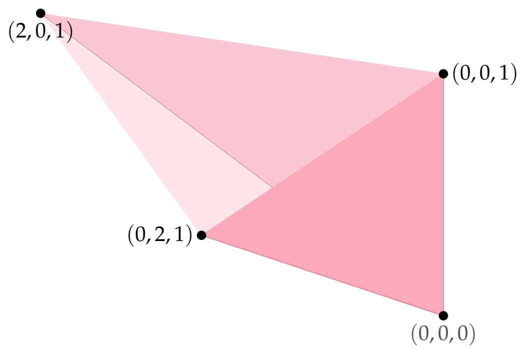


Figure 1.4: The cone $\sigma = \mathrm{Cone}(\mathbf{e}_3, 2\mathbf{e}_1 + \mathbf{e}_3, 2\mathbf{e}_2 + \mathbf{e}_3)$, which yields the toric variety $U_\sigma = \mathrm{V}(xyz - w^2)$.

Similarly to normality, smoothness can also be characterized in terms of polyhedral cones. Since smooth implies normal, it is clear that a smooth variety will be one that arises from a strongly convex cone. More specifically, we have the following result, whose proof is omitted (the reader can refer to [CLS11], 1.3.12).

Proposition 1.27. *Let V be an affine toric variety. Then V is smooth if and only if $V = U_\sigma$ for some smooth strongly convex rational cone σ.*

Proof. [CLS11], 1.3.12. □

1.2 Abstract toric varieties

Just like in the case of classical algebraic geometry, we can extend our class of objects by introducing the notion of an *abstract* toric variety, which will amount to a glueing of affine toric varieties. In this way, we can use the tools of toric geometry on more complicated objects, like projective spaces, or non-affine subsets of affine toric varieties. This will prove valuable in the study of quotient singularities on affine toric varieties, since the regular locus of an affine variety need not be affine.

1.2.1 Glueing affine toric varieties

Consider the following definition:

Definition 1.28 (Toric variety)**.** A *toric variety* is an irreducible abstract algebraic variety X containing a torus $T \simeq (\mathbb{C}^\times)^n$ as an open subset, such that the action of T on itself extends to an algebraic action of T on X.

Similarly to the case of classical algebraic varieties, we can construct (abstract) toric varieties by glueing affine toric varieties. We will restrict to the case of normal affine varieties: we know by the previous section that normal affine toric varieties arise from strongly convex polyhedral cones. It turns out that the glueing data for a collection of normal affine toric varieties $\{U_\sigma\}_\sigma$ can be encoded in a *fan*.

As before, let N, M be a pair of dual lattices.

Definition 1.29 (Fan)**.** A *fan* in $N_\mathbb{R}$ is a finite collection Σ of cones $\sigma \subseteq N_\mathbb{R}$, such that:

1. each $\sigma \in \Sigma$ is strongly convex;
2. $\forall \sigma \in \Sigma, \tau \preceq \sigma \implies \tau \in \Sigma$;
3. $\forall \sigma_1, \sigma_2 \in \Sigma, \sigma_1 \succeq \sigma_1 \cap \sigma_2 \preceq \sigma_2$.

Given a fan Σ:

- its *support* is $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subseteq N_\mathbb{R}$;
- $\Sigma(r)$ is the set of all r-dimensional cones of Σ.

Given a fan Σ, consider the collection of affine toric varieties

$$\{U_\sigma\}_{\sigma \in \Sigma} = \{\operatorname{Specm} \mathbb{C}[S_\sigma]\}_{\sigma \in \Sigma} .$$

Recall that a face $\tau \preceq \sigma$ is given by $\tau = \sigma \cap H_\mathbf{m}$ for some $\mathbf{m} \in \sigma^\vee$ and that the following hold:

1. $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\xi^\mathbf{m}}$;
2. if $\tau = \sigma_1 \cap \sigma_2$, then $\left(U_{\sigma_1}\right)_{\xi^\mathbf{m}} \simeq U_\tau \simeq \left(U_{\sigma_2}\right)_{\xi^{-\mathbf{m}}}$.

Then, if $\sigma_1, \sigma_2 \in \Sigma$, $\tau = \sigma_1 \cap \sigma_2$, we have a gluing function

$$g_{\sigma_2,\sigma_1} : \left(U_{\sigma_1}\right)_{\xi^\mathbf{m}} \simeq \left(U_{\sigma_2}\right)_{\xi^{-\mathbf{m}}} .$$

These isomorphisms verify the compatibility conditions for gluing the affine toric varieties U_σ, and thus yield a toric variety X_Σ associated to the fan Σ.

Theorem 1.30. *In the previous notations, the variety X_Σ is a normal separated toric variety. Moreover, any normal separated toric variety arises in this way.*

Proof. [CLS11], 3.1.5, 3.1.8. □

Example 1.31. Consider a strongly convex rational cone $\sigma \subseteq N_{\mathbb{R}}$ and the fan Σ consisting of σ and all of its faces. Then $X_\Sigma = U_\sigma$.

Example 1.32. Let us see how the projective plane $\mathbb{P}^2$ arises as an abstract toric variety. Consider the fan $\Sigma \subseteq N_{\mathbb{R}} = \mathbb{R}^2$ in Figure 1.5, consisting of the cones $\sigma_0 = \text{Cone}(\mathbf{e}_1, \mathbf{e}_2)$, $\sigma_1 = \text{Cone}(-\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2)$, $\sigma_2 = \text{Cone}(\mathbf{e}_1, -\mathbf{e}_1 - \mathbf{e}_2)$ and their intersections. By computing the dual cones for each cone in the fan, we obtain the affine toric varieties

$$\begin{aligned} U_{\sigma_0} &= \operatorname{Specm} \mathbb{C}[S_{\sigma_0}] = \mathbb{C}[x, y]; \\ U_{\sigma_1} &= \operatorname{Specm} \mathbb{C}[S_{\sigma_1}] = \mathbb{C}[x^{-1}, x^{-1}y]; \\ U_{\sigma_2} &= \operatorname{Specm} \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[xy^{-1}, y^{-1}], \end{aligned}$$

which cover X_Σ. The glueing morphisms are given by

$$\begin{aligned} g_{1,0} &: (U_{\sigma_0})_x \simeq (U_{\sigma_1})_{x^{-1}} \\ g_{2,0} &: (U_{\sigma_0})_y \simeq (U_{\sigma_2})_{y^{-1}} \\ g_{1,2} &: (U_{\sigma_2})_{xy^{-1}} \simeq (U_{\sigma_1})_{x^{-1}y}. \end{aligned}$$

With a change of coordinates $x \longmapsto \frac{x_1}{x_0}$, $y \longmapsto \frac{x_2}{x_0}$, we see immediately that X_Σ is indeed the projective plane, with the affine open sets U_{σ_i} corresponding to the usual affine open sets U_i covering $\mathbb{P}^2$.

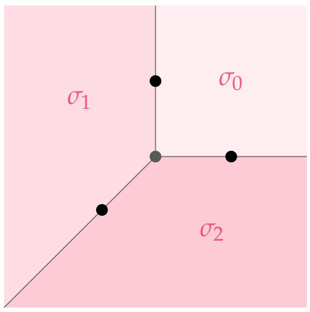


Figure 1.5: The fan Σ corresponding to the projective plane $\mathbb{P}^2$.

Just like with cones, properties of fans correspond to algebraic properties of the toric variety they yield.

Definition 1.33 (Smooth and simplicial fans)**.** Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan. We say that Σ is *smooth* if all its cone are smooth, while it is *simplicial* if each $\sigma \in \Sigma$ is simplicial, i.e. its minimal generators are linearly independent over $\mathbb{R}$.

Theorem 1.34. *In the previous notations:*

1. *X_Σ is smooth if and only if Σ is smooth.*

2. *X_Σ is an orbifold (i.e. it has finite quotient singularities) if and only if Σ is simplicial.*

Proof. The first statement follows immediately from the corresponding property of affine toric varieties and cones, since smoothness is a local property. As for the second statement, let us look at X_Σ locally: suppose that $\sigma \subseteq N_\mathbb{R}$ is a simplicial cone of dimension n with minimal generators $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Then $N' = \sum_i \mathbb{Z}\mathbf{u}_i$ is a sublattice of N of finite index: by [CLS11], 1.3.18, this implies that the group $G = N/N'$ acts on the smooth variety $U_{\sigma,N'}$ with quotient $U_{\sigma,N'}/G \simeq U_{\sigma,N}$. This means that any toric variety arising from a simplicial fan locally looks like a quotient of a smooth variety by a finite group, so it has only finite quotient singularities. For the other implication, see [CLS11], 11.4.8. □

1.2.2 The orbit-cone correspondence

Let us now introduce the last ingredient we need in order to start talking about the resolution of singularities on toric varieties: the orbit-cone correspondence. It is a major result in toric geometry which, given a toric variety X_Σ with dense torus T, describes a bijection between orbits of the T-action on X_Σ and the cones of Σ. Before stating the theorem, let us outline how the correspondence works through an example.

Example 1.35. Consider the projective plane $\mathbb{P}^2$: as seen earlier, it is a toric variety arising from the fan in Figure 1.5, with dense torus $T = \{[1 : s : t] \,|\, s, t \in \mathbb{C}^\times\} \simeq (\mathbb{C}^\times)^2$. The action of T on $\mathbb{P}^2$ is given by multiplication of homogeneous coordinates:

$$[1 : s : t] : [x_0 : x_1 : x_2] \longmapsto [x_0 : x_1 s : x_2 t].$$

What do the orbits of this action look like? Clearly, a point $[x_0 : x_1 : x_2]$ with $x_i \neq 0$ for some i cannot lie in the same orbit as one where $x_i = 0$. Vice versa, two points lying in the same hyperplanes $\mathrm{V}(x_i)$ must lie in the same orbit. Thus, it is immediate to see that the action of T on $\mathbb{P}^2$ has the following seven orbits:

$$O_1 = \{[x_0 : x_1 : x_2] \,|\, x_i \neq 0 \text{ for all } i\}, \qquad O_2 = \{[x_0 : x_1 : 0] \,|\, x_0, x_1 \neq 0\},$$
$$O_3 = \{[x_0 : 0 : x_2] \,|\, x_0, x_2 \neq 0\}, \qquad O_4 = \{[0 : x_1 : x_2] \,|\, x_1, x_2 \neq 0\},$$
$$O_5 = \{[x_0 : 0 : 0] \,|\, x_0 \neq 0\}, \qquad O_6 = \{[0 : x_1 : 0] \,|\, x_1 \neq 0\},$$
$$O_7 = \{[0 : 0 : x_2] \,|\, x_2 \neq 0\}.$$

Consider now for an element $\mathbf{u} = (a, b) \in N \simeq \mathbb{Z}^2$ the corresponding co-character $\lambda^\mathbf{u}(t) = [1 : t^a : t^b] \in T$. Recall that $\mathbb{P}^2$ arises from a fan whose support is all of $N_\mathbb{R}$: we observe that the limit $\lim_{t \to 0} \lambda^\mathbf{u}(t)$ is different depending on where $\mathbf{u}$ lies. The limits are summarized in Figure 1.6. We immediately see that two co-characters yield the same limit if and only if they lie in the relative interior of the same cone of Σ; thus, we have a bijection between cones and limits of co-characters. Moreover, each of the possible values for $\lim_{t\to 0} \lambda^{\mathbf{u}}(t)$ belongs to one and only one of the seven orbits described earlier, so we also have a correspondence between the limits and the orbits of the T-action. Thus, we have a correspondence between cones and orbits:

$$\sigma \text{ corresponds to } O \iff \lim_{t\to 0} \lambda^{\mathbf{u}}(t) \in O \text{ for any } \mathbf{u} \text{ in the relative interior of } \sigma.$$

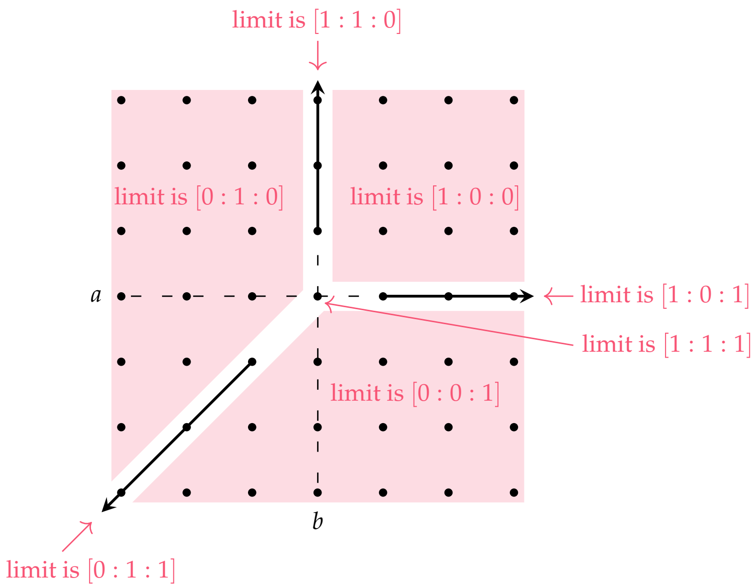


Figure 1.6: The value of $\lim_{t\to 0} \lambda^{\mathbf{u}}(t)$ for different values of $\mathbf{u} = (a, b) \in N$.

This correspondence extends to all toric varieties:

Theorem 1.36 (Orbit-cone correspondence)**.** *Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan and X_Σ the corresponding toric variety, with torus T.*

1. *There is a bijection*

$$\begin{aligned} \{\text{cones } \sigma \in \Sigma\} &\longleftrightarrow \{T\text{-orbits in } X_\Sigma\} \\ \sigma &\longmapsto O(\sigma) \simeq \operatorname{Hom}_{\mathbb{Z}}(\sigma^\vee \cap M, \mathbb{C}^\times). \end{aligned}$$

2. *If* $\dim N_{\mathbb{R}} = n$, *for any* $\sigma \in \Sigma$ *we have* $\dim O(\sigma) = n - \dim \sigma$.

3. *The affine open subset U_σ is given by the union*

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau).$$

4. *$\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, in which case $\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$, where the closure $\overline{O(\tau)}$ is both in the classical and Zariski topologies.*

Proof. [CLS11], 3.2.6. □

Observe that we already knew by 1.22 that each cone σ of maximal dimension corresponds to a fixed point γ_σ (i.e., a zero-dimensional orbit), which is called the *distinguished point* of σ.

1.2.3 Refinements of fans

Blowing up a toric variety X_Σ at a point can be represented by dividing the cones of Σ into smaller cones, creating a finer fan. Before we expand on this idea, let us give some preliminary notions on toric morphisms.

Definition 1.37 (Toric morphism)**.** Let X_{Σ_1}, X_{Σ_2} be normal toric varieties arising from fans $\Sigma_1 \subseteq (N_1)_\mathbb{R}$ and $\Sigma_2 \subseteq (N_2)_\mathbb{R}$. A morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is *toric* if it maps $\phi(T_1) \subseteq T_2$ and $\phi_{|T_1} : T_1 \to T_2$ is a group homomorphism, where T_i is the dense torus of X_{Σ_i}.

By generalizing [CLS11], 1.3.14, we have that all toric morphisms are equivariant with respect to the toric action, meaning that the following diagram is commutative:

$$\begin{array}{ccc} T_1 \times X_{\Sigma_1} & \longrightarrow & X_{\Sigma_1} \\ {\scriptstyle \phi_{|T_1} \times \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ T_2 \times X_{\Sigma_2} & \longrightarrow & X_{\Sigma_2} \end{array} .$$

Definition 1.38 (Compatible map with respect to a pair of fans)**.** Let N_1, N_2 be a pair of lattices, with Σ_i a fan in $(N_i)_\mathbb{R}$, for $i = 1, 2$. A $\mathbb{Z}$-linear map $\bar{\phi} : N_1 \to N_2$ is *compatible* with the fans Σ_1, Σ_2 if for any cone $\sigma_1 \in \Sigma_1$, there is a cone $\sigma_2 \in \Sigma_2$ such that $\bar{\phi}_\mathbb{R}(\sigma_1) \subseteq \sigma_2$.

Theorem 1.39. *Let N_1, N_2 be lattices and Σ_i a fan in $(N_i)_\mathbb{R}$ for $i = 1, 2$.*

1. *If $\bar{\phi} : N_1 \to N_2$ is a $\mathbb{Z}$-linear map which is compatible with Σ_1 and Σ_2, then there is a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ such that $\phi_{|T_1}$ coincides with the map $\bar{\phi} \otimes 1 : N_1 \otimes_\mathbb{Z} \mathbb{C}^\times \to N_2 \otimes_\mathbb{Z} \mathbb{C}^\times$.*

2. *On the other hand, if $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism, then ϕ induces a $\mathbb{Z}$-linear map $\bar{\phi} : N_1 \to N_2$ which is compatible with the fans Σ_1 and Σ_2.*

Proof. [CLS11], 3.3.4. □

We now give the following definition:

Definition 1.40 (Refinement of a fan)**.** Given two fans $\Sigma, \Sigma' \subseteq N$, we say that Σ' *refines* Σ if $|\Sigma| = |\Sigma'|$ and each cone in Σ' is contained in a cone of Σ.

Here is an example of refinement that will come useful in a moment:

Definition 1.41. Let Σ be a fan in $N_\mathbb{R}$ and let $\sigma = \mathrm{Cone}(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be a smooth cone in Σ. For $\mathbf{u}_0 := \mathbf{u}_1 + \ldots + \mathbf{u}_n$, let $\Sigma'(\sigma)$ be the set of all cones generated by subsets of $\{\mathbf{u}_0, \ldots, \mathbf{u}_n\}$ that do not contain $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. The *star subdivision of* Σ *along* σ is the fan

$$\Sigma^\star(\sigma) = (\Sigma \smallsetminus \{\sigma\}) \cup \Sigma'(\sigma).$$

In other words, the star subdivision of Σ along σ is obtained by "replacing" σ in Σ with $\Sigma'(\sigma)$.

Now observe that if Σ' refines Σ, then the identity map $\overline{\phi} : N \to N$ is compatible with the fans Σ and Σ', yielding a toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$. In particular, the star subdivision satisfies the following theorem:

Theorem 1.42. *In the previous notations, $\Sigma^\star(\sigma)$ is a refinement of Σ and the associated toric morphism*

$$\phi : X_{\Sigma^\star(\sigma)} \to X_\Sigma$$

is the blow-up of X_Σ at the distinguished point γ_σ corresponding to the cone σ.

Proof. [CLS11], 3.3.15. □

We can also consider a more general type of star subdivision: given a fan Σ and a primitive element $\mathbf{v} \in |\Sigma| \cap N_\mathbb{R}$ (where *primitive* means that $\frac{1}{k}\mathbf{v} \notin N_R$ for all $k > 1$), define $\Sigma^\star(\mathbf{v})$ as the following set of cones:

- all $\sigma \in \Sigma$ that do not contain $\mathbf{v}$;
- $\mathrm{Cone}(\mathbf{v}, \tau)$ for all $\tau \in \Sigma$ that do not contain $\mathbf{v}$ and such that $\{\mathbf{v}\} \cup \tau \subseteq \sigma$ for some $\sigma \in \Sigma$.

We call $\Sigma^\star(\mathbf{v})$ the *star subdivision* of Σ at $\mathbf{v}$; it is a refinement of Σ ([CLS11], 11.1.3.) and, thus, it induces a morphism $X_{\Sigma^\star(\mathbf{v})} \to X_\Sigma$.

Star subdivisions turn out to be extremely important in the process of solving a singularity on a toric surface. Recall the following definition:

Definition 1.43 (Resolution of singularities)**.** A proper morphism $\varphi : X \to Y$ is a *resolution of singularities* if Y is a smooth surface and φ induces an isomorphism

$$Y \smallsetminus \varphi^{-1}(X_{\mathrm{sing}}) \simeq X \smallsetminus X_{\mathrm{sing}},$$

where X_{sing} is the singular locus of X.

We have the following theorem:

Theorem 1.44. *Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan. Then there exists a refinement Σ' of Σ satisfying the following properties:*

1. *Σ' is smooth;*
2. *Σ' contains every smooth cone of Σ;*
3. *Σ' is obtained by a sequence of star subdivisions;*
4. *The induced morphism $\phi : X_{\Sigma'} \to X_{\Sigma}$ is a projective resolution of singularities.*

Proof. [CLS11], 11.1.9. □

1.2.4 Resolution of quotient singularities on toric surfaces

Consider the toric surface $V = \mathrm{V}(xy - z^2)$ from Example 1.19: it is a non-smooth variety, corresponding to the polyhedral cone $\sigma = \mathrm{Cone}\,(\mathbf{e}_2, 2\mathbf{e}_1 - \mathbf{e}_2) \subseteq \mathbb{Z}^2$, whose singular locus is the origin. It is a known fact of classical algebraic geometry that V arises as a quotient of $\mathbb{C}^2$ by the action of the cyclic group $\{-1, 1\} \simeq \mathbb{Z}/2\mathbb{Z}$, given by the algebraic map $-1 : (u, v) \longmapsto (-u, -v)$.

More generally, for a cone $\sigma = \mathrm{Cone}\,(\mathbf{e}_2, m\mathbf{e}_1 - \mathbf{e}_2) \subseteq N_{\mathbb{R}}$, the corresponding toric variety U_σ arises as the quotient of $\mathbb{C}^2$ by the action of the cyclic group μ_m of m-th roots of unity. Indeed, observe that $\sigma^\vee = \mathrm{Cone}\,(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \dots, \mathbf{e}_1 + m\mathbf{e}_2)$, so that $\mathbb{C}[\mathrm{S}_\sigma] = \mathbb{C}[x, xy, \dots, xy^m] = \mathbb{C}[u^m, u^{m-1}v, \dots, v^m]$ for $u = x^{\frac{1}{m}}$, $v = x^{\frac{1}{m}}y$. Consider the sublattice N' generated by the ray generators $\mathbf{e}_2$ and $m\mathbf{e}_1 - \mathbf{e}_2$, and let σ' be the same cone as σ, but seen within N': it is clear that σ' is smooth, so it yields a smooth toric variety $U_{\sigma'}$ isomorphic to $\mathbb{C}^2$. More specifically, $N' = \langle m\mathbf{e}_1, \mathbf{e}_2 \rangle_{\mathbb{Z}}$, whence its dual $M' \supseteq M$ is spanned over $\mathbb{Z}$ by $\left\{\frac{1}{m}\mathbf{e}_1, \mathbf{e}_2\right\}$. Since the dual cone $(\sigma')^\vee \subseteq M'$ is given by $(\sigma')^\vee = \mathrm{Cone}\left(\frac{1}{m}\mathbf{e}_1, \frac{1}{m}\mathbf{e}_1 + \mathbf{e}_2\right)$, we have $\mathbb{C}[\mathrm{S}_{\sigma'}] = \mathbb{C}[x^{\frac{1}{m}}, x^{\frac{1}{m}}y] = \mathbb{C}[u, v]$. At this point, it is clear that $\mathbb{C}[\mathrm{S}_\sigma]$ is the ring of invariants of $\mathbb{C}[\mathrm{S}_{\sigma'}]$ under the action of $\mu_m = \langle \zeta \rangle$ given by

$$\zeta \cdot p(u, v) = p(\zeta u, \zeta v).$$

Thus, the inclusion $\mathbb{C}[\mathrm{S}_\sigma] = \mathbb{C}[u^m, u^{m-1}v, \dots, v^m] \hookrightarrow \mathbb{C}[u, v] = \mathbb{C}[\mathrm{S}_{\sigma'}]$ yields a morphism $U_{\sigma'} \to U_\sigma$ which is a quotient of $U_{\sigma'} \simeq \mathbb{C}^2$ by the action of μ_m.

In general, any normal affine toric surface can be written as U_σ for a cone of the form $\sigma = \mathrm{Cone}(\mathbf{e}_2, m\mathbf{e}_1 - k\mathbf{e}_2)$. This is a consequence of the following proposition, whose proof follows from the following modified version of the Euclidean algorithm: *for any pair of integers n and m, with $m > 0$, there are unique integers h and k such that $n = mh - k$, with $0 \leq k < m$.*

Proposition 1.45. *Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{Z}^2$ be a strongly convex two-dimensional cone. Then there is a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of N such that $\sigma = \mathrm{Cone}(\mathbf{e}_2, m\mathbf{e}_1 - k\mathbf{e}_2)$, $0 \leq k < m$ and $\gcd(m,k) = 1$. The integers m and k are essentially unique in the following sense: if $\sigma = \mathrm{Cone}(\mathbf{e}_2, m\mathbf{e}_1 - k\mathbf{e}_2)$ and $\widetilde{\sigma} = \mathrm{Cone}(\mathbf{e}'_2, \widetilde{m}\mathbf{e}'_1 - \widetilde{k}\mathbf{e}'_2)$ are lattice equivalent, i.e. there is a $\mathbb{Z}$-linear bijection $\varphi : N \to N$ mapping one cone into the other, then $\widetilde{d} = d$ and either $\widetilde{k} = k$ or $\widetilde{k}k \equiv 1 \mod d$.*

Proof. [CLS11], 10.1.1, 10.1.3. □

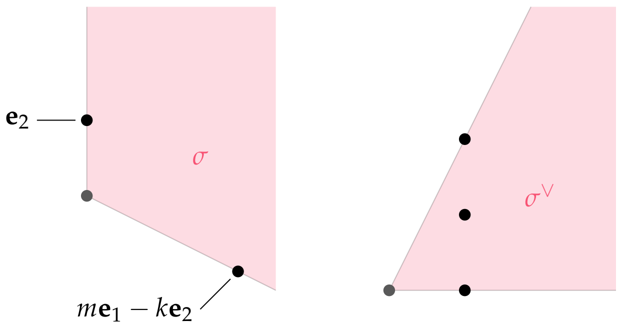


Figure 1.7: The cone $\sigma = \mathrm{Cone}(m\mathbf{e}_1 - k\mathbf{e}_2)$ and its dual.

We call the integers m and k *parameters* of the cone, and $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a *normal basis for N relative to σ*.

Writing a normal affine toric surface in normal form can be very useful to study its quotient singularities. Indeed, consider the sublattice N' of N generated by $\mathbf{u}_1 = \mathbf{e}_2$ and $\mathbf{u}_2 = m\mathbf{e}_1 - k\mathbf{e}_2$. Similarly to the case where $k = 1$ seen at the beginning of this subsection, the group $G = N/N'$ acts on $U_{\sigma,N'} \simeq \mathbb{C}^2$, and U_σ arises as a quotient of $\mathbb{C}^2$ by this action. More specifically, by writing σ in normal form we find

$$N' = \mathbb{Z}\mathbf{e}_2 \oplus \mathbb{Z}(m\mathbf{e}_1 - k\mathbf{e}_2) = \mathbb{Z}(m\mathbf{e}_1) \oplus \mathbb{Z}\mathbf{e}_2 \Longrightarrow N/N' \simeq \mathbb{Z}/m\mathbb{Z}.$$

The following proposition describes the action of $G = N/N' \simeq \mu_m$ on $\mathbb{C}^2$.

Proposition 1.46. *Let M' be the dual lattice of N' and let $\mathbf{m}_1, \mathbf{m}_2 \in M'$ the dual elements to $\mathbf{u}_1$ and $\mathbf{u}_2$. Using coordinates $x = \xi^{\mathbf{m}_1}$, $y = \xi^{\mathbf{m}_2}$ of $\mathbb{C}^2$, the action of the cyclic group $\mu_m = \langle \zeta \rangle$ on $\mathbb{C}[x,y]$, where ζ is a primitive m-th root of unity, is given by*

$$\zeta \cdot p(x,y) = p(\zeta x, \zeta^k y).$$

Moreover, $U_\sigma \simeq \mathbb{C}^2/\mu_m$ with respect to this action.

Proof. By [CLS11], 1.3.18, for $\mathbf{m} \in \sigma^\vee \cap M'$ and $\mathbf{u} \in N$ the group $N/N' \simeq \mathbb{Z}/m\mathbb{Z}$ acts on the coordinate ring of $\mathbb{C}^2$ via

$$(\mathbf{u} + N') \cdot \xi^{\mathbf{m}} = e^{2\pi i \langle \mathbf{m}, \mathbf{u} \rangle} \xi^{\mathbf{m}}.$$

Fix $\mathbf{u} = \mathbf{e}_1$: we have an isomorphism $\mu_m \simeq N/N'$ via $e^{2\pi i \frac{j}{m}} \longmapsto j\mathbf{e}_1 + N'$. Since $\langle \mathbf{m}_1, \mathbf{e}_1 \rangle = \frac{1}{m}$ and $\langle \mathbf{m}_2, \mathbf{e}_2 \rangle = \frac{k}{m}$, we have for $\zeta = e^{\frac{2\pi i}{m}}$

$$\zeta \cdot p(x, y) = p\left(e^{\frac{2\pi i}{m}} x, e^{\frac{2k\pi i}{m}} y\right) = p(\zeta x, \zeta^k y).$$

□

In general, consider a (non necessarily affine) normal toric surface X. The surface obtained by removing the fixed points of the torus action is smooth, and since there are only finitely many such points (by the orbit-cone correspondence), this means that there are only finitely many singularities. Moreover, since any 2-dimensional cone is simplicial, X is an orbifold, meaning that each of these singularities arises as a quotient singularity. Since X locally looks like an affine normal surface, we know by the previous discussion that it locally arises as a quotient of $\mathbb{C}^2$ by a finite cyclic group G. This will help us work out an algorithm for solving these quotient singularities.

Given a fan Σ yielding $X = X_\Sigma$, we will construct a resolution of singularities by dividing the cones of Σ into smaller cones, to obtain a new fan Σ' whose cones are all smooth, like we saw in the previous subsection. While the general construction can be complicated, in the case of toric surfaces the resolution can be found with a simple algorithm. Before stating the theorem on the resolution of singularities on toric surfaces, let us look at how it works through an example.

Example 1.47. Consider a cone σ with parameters $m = 7$ and $k = 3$. The action of $G = N'/N \simeq \mathbb{Z}/7\mathbb{Z}$ on $\mathbb{C}[x, y]$ is given by

$$\zeta \cdot p(x, y) = p(\zeta x, \zeta^3 y),$$

for a primitive 7-th root of unity ζ. The ring of invariants by this action is

$$\mathbb{C}[x, y]^G = \mathbb{C}[x^7, x^4 y, x y^2, y^7] \simeq \frac{\mathbb{C}[s, t, u, v]}{(t^2 - su, u^4 - tv, tu^3 - sv)},$$

where the relations between the generators have been computed using `Macaulay2`. Thus, we can identify the toric surface U_σ with the affine variety

$$\mathrm{V}(t^2 - su, u^4 - tv, tu^3 - sv) \subseteq \mathbb{C}^4.$$

By applying the Jacobian criterion, we see that this variety is smooth outside the origin, with the origin being its only singular point. Indeed, the corresponding cone $\sigma = \text{Cone}(\mathbf{e}_2, 7\mathbf{e}_1 - 3\mathbf{e}_2)$ is not smooth. Let us divide σ into a fan of smooth cones by inserting three new rays:

- $\rho_1 = \text{Cone}(\mathbf{e}_1)$;
- $\rho_2 = \text{Cone}(3\mathbf{e}_1 - \mathbf{e}_2)$;
- $\rho_3 = \text{Cone}(5\mathbf{e}_1 - 2\mathbf{e}_2)$.

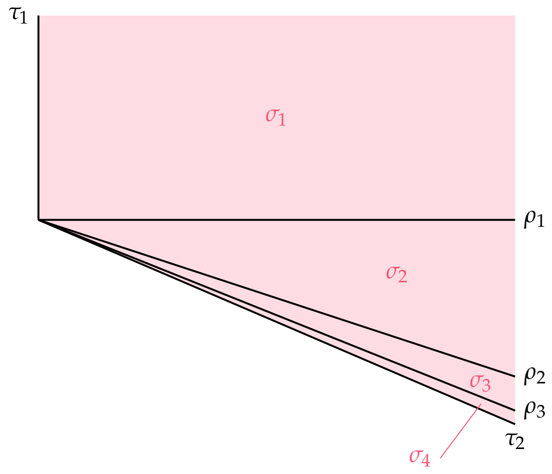


Figure 1.8: The smooth fan Σ refining σ, obtained by adding the rays ρ_1, ρ_2, ρ_3.

In this way, we have divided σ into a smooth fan Σ, pictured in Figure 1.8, which corresponds to a smooth toric surface X_Σ. Since the identity map on N is compatible with the fans Σ and σ, by the previous theorem we have a toric morphism $\phi : X_\Sigma \to U_\sigma$, which is proper by [CLS11], 3.4.11. Let us now use the orbit-cone correspondence to show that ϕ is indeed a resolution of singularities. Let γ_σ be the distinguished point of U_σ, i.e. the unique fixed point of the torus action on U_σ. Then $U_\sigma \smallsetminus \{\gamma_\sigma\}$ is a smooth abstract toric variety with no fixed points, which means (by the orbit-cone correspondence) that its fan consists of the rays of σ, i.e. $\tau_1 = \text{Cone}(\mathbf{e}_2)$ and $\tau_2 = \text{Cone}(7\mathbf{e}_1 - 3\mathbf{e}_2)$, along with the origin. The toric morphism ϕ is obtained by gluing the local toric morphisms $\phi_{\sigma_i} : U_{\sigma_i} \to U_\sigma$ arising from the inclusions $\sigma_i \subseteq \sigma$; for each of these maps, the preimage of the distinguished point must be a union of orbits of U_{σ_i}. In particular, ϕ maps $O(\tau_1)$ and $O(\tau_2)$ into themselves, while $\phi^{-1}(\gamma_\sigma) = \overline{O(\rho_1)} \cup \overline{O(\rho_2)} \cup \overline{O(\rho_3)}$. It follows that $X_\Sigma \smallsetminus \phi^{-1}(\gamma_\sigma)$ has the same fan as $U_\sigma \smallsetminus \{\gamma_\sigma\}$, so ϕ restricts to an isomorphism

$$X_\Sigma \smallsetminus \phi^{-1}(\gamma_\sigma) \simeq U_\sigma \smallsetminus \{\gamma_\sigma\}.$$

Thus, it is a resolution of singularities.
Each orbit $\overline{O(\rho_i)}$ is isomorphic to the projective line $\mathbb{P}^1$, which is the toric variety arising from the one-dimensional fan $\Sigma = \{(-\infty, 0], \{0\}, [0, +\infty)\} \subseteq \mathbb{R}$ (see

[CLS11], 3.1.11). We call the subset $E = \phi^{-1}(\gamma_\sigma)$ the *exceptional divisor* on the surface. By point (4) of Theorem 1.36, we know that the closed subsets $\overline{O(\rho_1)}$ and $\overline{O(\rho_2)}$ intersect transversally on the distinguished point of X_Σ corresponding to the cone σ_2, while $\overline{O(\rho_2)}$ and $\overline{O(\rho_3)}$ intersect transversally on the distinguished point corresponding to the cone σ_3. Thus, the exceptional divisor E looks like three projective lines intersecting as in Figure 1.9. We associate to the singularity a graph, called the *dual graph*, whose vertices represent the irreducible components of the exceptional divisor, and two vertices are connected by an edge if the corresponding components intersect. In this case, the dual graph is the Dynkin graph A_3: we say that it is a *du Val singularity* of type A_3. Further discussion on exceptional divisors and dual graphs will occur in the next chapters.

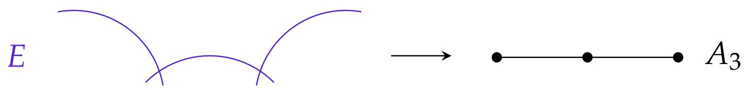


Figure 1.9: The exceptional divisor E of the resolution of singularities $X_\Sigma \to U_\sigma$ and its dual graph A_3.

The previous discussion can be extended to abstract toric surfaces with more than one singular point. Recall that given two fans $\Sigma, \Sigma' \subseteq N$, we say that Σ' *refines* Σ if $|\Sigma| = |\Sigma'|$ and each cone in Σ' is contained in a cone of Σ. In this case, the identity map $\overline{\phi} : N \to N$ is compatible with the fans Σ and Σ', so it yields a toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$.

Theorem 1.48. *Let X_Σ be a normal toric surface. There is a smooth fan Σ' that refines Σ, such that the associated toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$ is a toric resolution of singularities.*

Proof. [CLS11], 10.1.10. □

We have yet to explain how one can refine a non-smooth strongly convex cone σ into a smooth fan, like we did in the previous example. By Proposition 1.45, we may assume that $\sigma = \text{Cone}(\mathbf{e}_2, m\mathbf{e}_1 - k\mathbf{e}_2)$. The first step is to refine σ into a fan containing the cones

$$\sigma' = \text{Cone}(\mathbf{e}_1, \mathbf{e}_2), \quad \sigma'' = \text{Cone}(\mathbf{e}_1, m\mathbf{e}_1 - k\mathbf{e}_2).$$

The first cone is clearly smooth, but second need not be. Observe that σ'' is not written in normal form: it will have parameters k, k_1, with

$$m = b_1 k - k_1, \quad b_1 \geq 2, 0 \leq k_1 < k.$$

Write σ'' in normal form as $\sigma'' = \text{Cone}(\mathbf{e}'_2, k\mathbf{e}'_1 - k_1\mathbf{e}'_2)$, for $\mathbf{e}'_2 = \mathbf{e}_1$ and $\mathbf{e}'_1$ such that $k\mathbf{e}'_1 - k_1\mathbf{e}'_2 = m\mathbf{e}_1 - k\mathbf{e}_2$. We can re-iterate the procedure on σ'', by adding a new ray which divides σ'' into a smooth cone and a second cone, which may not be smooth, with parameters k_1, k_2 satisfying $k = b_2 k_1 - k_2$. Repeating this operation yields a modified Euclidean algorithm which generates the parameters of the cones of the smooth fan Σ:

$$\begin{aligned} m &= b_1 k - k_1; \\ k &= b_2 k_1 - k_2; \\ &\vdots \\ k_{r-3} &= b_{r-1} k_{r-2} - k_{r-1}; \\ k_{r-2} &= b_r k_{r-1}. \end{aligned} \tag{1.49}$$

The algorithm terminates in a finite number of steps when $k_r = 0$; in each step, we require that b_i must satisfy $b_i \geq 2$.

The equations in 1.49 can be rewritten as

$$\begin{aligned} m/k &= b_1 - k_1/k; \\ k/k_1 &= b_2 - k_2/k_1; \\ &\vdots \\ k_{r-3}/k_{r-2} &= b_{r-1} - k_{r-1}/k_{r_2}; \\ k_{r-2}/k_{r-1} &= b_r, \end{aligned}$$

yielding the continued fraction

$$m/k = b_1 - \frac{1}{b_2 - \frac{1}{\cdots - \frac{1}{b_r}}}.$$

This is called the *Hirzebruch-Jung continued fraction expansion of* m/k. We use the notation

$$m/k = [\![b_1, \ldots, b_r]\!].$$

We call the integers b_i *partial quotients* of the continued fraction, while the truncated fractions $[\![b_1, \ldots, b_i]\!]$ are called *convergents*.

The Hirzebruch-Jung continued fraction expansion satisfies the following proposition:

Proposition 1.50. *Let* $m, k > 0$ *be integers with* $\gcd(m, k) = 1$ *and let* $m/k = [\![b_1, \ldots, b_r]\!]$. *We define the sequences* $(P_i)_i$ *and* $(Q_i)_i$ *recursively as follows: set* $P_0 = 1$, $Q_0 = 0$, $P_1 = b_1$, $Q_1 = 1$; *for each* $2 \leq i \leq r$, *let*

$$P_i = b_i P_{i-1} - P_{i-2}; \quad Q_i = b_i Q_{i-1} - Q_{i-2}.$$

The following hold:

1. *$(P_i)_i$ and $(Q_i)_i$ are increasing sequences of integers;*
2. *$P_i/Q_i = [\![b_1, \dots, b_i]\!]$ for all $1 \le i \le r$;*
3. *$P_{i-1}Q_i - P_iQ_{i-1} = 1$ for all $1 \le i \le r$;*
4. *the convergents P_i/Q_i form a strictly decreasing sequence*

$$\frac{m}{k} = \frac{P_r}{Q_r} < \frac{P_{r-1}}{Q_{r-1}} < \cdots < \frac{P_1}{Q_1}.$$

Proof. [CLS11], 10.2.2. □

We use the previous proposition to finally state the following theorem on the resolution of a quotient singularity of a normal affine toric surface.

Theorem 1.51. *Let $\sigma = \mathrm{Cone}(\mathbf{e}_2, m\mathbf{e}_1 - k\mathbf{e}_2) \subseteq \mathbb{R}^2$ be a strongly convex 2-dimensional cone written in normal form. Set $\mathbf{u}_0 = \mathbf{e}_2$ and let $(P_i)_i$, $(Q_i)_i$ be the sequences of integers satisfying the previous proposition. Define the vectors*

$$\mathbf{u}_i = P_{i-1}\mathbf{e}_1 - Q_{i-1}\mathbf{e}_2, \quad 1 \le i \le r+1.$$

The cones $\sigma_i = \mathrm{Cone}(\mathbf{u}_{i-1}, \mathbf{u}_i)$ with $1 \le i \le r+1$ satisfy the following:

1. *Each σ_i is smooth and $\mathbf{u}_{i-1}, \mathbf{u}_i$ are its ray generators.*
2. *For all indices i, $\sigma_{i+1} \cap \sigma_i = \mathrm{Cone}(\mathbf{u}_i)$.*
3. *The fan Σ consisting of the σ_i and their faces is a smooth refinement of σ; in particular, $\sigma_1 \cup \ldots \cup \sigma_{r+1} = \sigma$.*
4. *The toric morphism $\phi : X_\Sigma \to U_\sigma$ induced by the refinement is a resolution of singularities.*

Proof. [CLS11], 10.2.3. □

Chapter 2

Hirzebruch-Jung resolution of tame cyclic quotient singularities

In this chapter, we move from algebraic $\mathbb{C}$-varieties in the classical sense to the language of schemes. We will study so-called *tame cyclic quotient singularities*, i.e. singularities on a variety over a discrete valuation ring W, arising from a quotient by the action of a finite cyclic group G, of order coprime to the characteristic exponent of the residue field at the singular point. By *characteristic exponent* of a field κ, we mean its characteristic if $\operatorname{char}\kappa = p > 0$, and 1 if $\operatorname{char}\kappa = 0$. We will work in the 2-dimensional case and assume that any point with nontrivial stabilizer is a closed point having the same residue field as W. Note that we are not requiring W to be equicharacteristic, as it may as well be of mixed characteristic. While the setup now may not look too similar to toric surfaces, and a lot more work will be required in this setting even in the 2-dimensional case, in the end we will find an explicit algorithm for the minimal resolution of singularities, which is of toric nature. The link with toric geometry is a first hint at how the case of tame quotient singularities is, as the name suggests, better behaved than the wild case, which will be examined in the next chapter, where methods from toric geometry may not be useful to resolve the singularities.

Let us fix some notation: throughout this chapter, S will indicate a Dedekind scheme, i.e. a normal locally Noetherian scheme of dimension 1. An *S-curve* is a flat, separated, finitely presented S-scheme whose fibers over S have pure dimension 1 (by flatness, this can be checked only on the generic fibers). Given a connected regular proper curve X over a discrete valuation ring W with residue field κ, and given two distinct irreducible and reduced divisors D and D' lying in the closed fiber of X, their *intersection multiplicity* is defined as

$$D.D' := \dim_\kappa \mathrm{H}^0(D \cap D', \mathcal{O}) = \sum_{d \in D \cap D'} \dim_\kappa \mathcal{O}_{D \cap D', d}.$$

2.1 An example of Hirzebruch-Jung resolution

Before studying the theory on the resolution of tame cyclic quotient singularities, let us work on an example reminiscing of Example 1.47 in the previous chapter. Recall that in that case, we applied the Hirzebruch-Jung algorithm to solve a cyclic quotient singularity on an affine toric variety arising from a cone of parameters $m = 7, k = 3$.

Let W be a complete discrete valuation ring with algebraically closed residue field κ of positive characteristic $p \neq 7$ and fraction field K. Let t be the uniformizer of W and consider the regular domain

$$A = \frac{W[x,y]}{(xy^2 - t)}.$$

Let $G = \mu_7(\kappa) = \langle \zeta \rangle$ act W-algebraically on A via $\zeta \cdot x = \zeta x, \zeta \cdot y = \zeta^3 y$, let $B := A^G$ be the ring of G-invariants, and let $Z := \operatorname{Spec} B$: since the action of G is free away from $x = y = t = 0$, Z is normal and regular away from the image point $z \in Z$ of the ideal (x, y, t). To understand what B looks like, observe that a monomial $x^a y^b$ is G-invariant if and only if $a + 3b = 7h$ for some $h \in \mathbb{N}$. This is equivalent to asking that $b \leq \frac{7}{3}h$. Let us change variables by defining $u = x^7$, $v = y/x^3$: then, $x^a y^b = u^h v^b$ and for all $0 \leq j \leq \frac{7}{3}i$, $u^i v^j \in W[x,y]^G$. Thus,

$$W[x,y]^G = \bigoplus_{0 \leq j \leq \frac{7}{3}i} W u^i v^j.$$

Since $xy^2 = uv^2$, we have the following expression of B:

$$B = \frac{\bigoplus_{0 \leq j \leq \frac{7}{3}i} W u^i v^j}{(uv^2 - t)}. \tag{2.1}$$

In this notation, the singular point z corresponds to the maximal ideal (u, uv) of B. We want to show that Z (or, rather, the morphism $Z \to \operatorname{Spec} W$) is a W-curve: it is flat because B is torsion-free over a principal ideal domain; it is separated because Z and $\operatorname{Spec} W$ are both affine; it is finitely presented because B is of finite type over a Noetherian ring; all is left to show is that the fibers of Z over $\operatorname{Spec} W$ all have pure dimension 1.

First note that we have a W-algebra injection

$$\bigoplus_{0 \leq j \leq \frac{7}{3}i} W u^i v^j \hookrightarrow W[x,y],$$

which is finite because $x^7 = y$ and $y^7 = u^3v^7$, meaning that $W[x,y]$ is finitely generated over the left hand side by $x, x^2, \dots, x^7, y, y^2, \dots, y^7$. By the going-up theorem, it follows that the left hand side has dimension 3. Passing to the quotient yields a finite surjection $\operatorname{Spec} A \to \operatorname{Spec} B$. Let η be the generic point of $\operatorname{Spec} W$: the quotient by the action of G commutes with taking the generic fiber by [SGA 1], Proposition 1.9:

$$(\operatorname{Spec} B)_\eta = (\operatorname{Spec} A)_\eta / G.$$

Since

$$(\operatorname{Spec} A)_\eta = \operatorname{Spec}\left(\frac{W[x,y]}{(xy^2 - t)} \otimes_W K\right) = \operatorname{Spec} \frac{K[x,y]}{(xy^2-t)},$$

it follows that the generic fiber of Z is

$$(\operatorname{Spec} B)_\eta = \operatorname{Spec}\left(\frac{\bigoplus_{0 \le j \le \frac{7}{3}i} K u^i v^j}{(uv^2 - t)}\right)$$

and has dimension 1. By flatness, we conclude that Z is a W-curve.

Let us now look at the closed fiber of Z: if ξ is the closed point of $\operatorname{Spec} W$, the closed fiber Z_ξ of Z is given by

$$Z_\xi = \operatorname{Spec}\left(B \otimes_W W/(t)\right) = \operatorname{Spec} \frac{\bigoplus_{0 \le j \le \frac{7}{3}i} \kappa u^i v^j}{(uv^2)}.$$

It is immediate to see that Z_ξ has two distinct irreducible components that intersect transversally at the singular point z; we will call them the x-axis and the y-axis.

Like in the case of toric varieties, we will now compute a series of blow-ups along the singular locus to obtain a minimal resolution of singularities. Let $Z' = \operatorname{Bl}_{(u,uv)} Z$ be the blow-up of Z at (u, uv): then, Z' is covered by the affine open sets $D_+(u)$ and $D_+(uv)$, where

$$D_+(u) = \operatorname{Spec} B\left[\frac{uv}{u}\right] = \operatorname{Spec} B[v] = \operatorname{Spec} \frac{W[u,v]}{(uv^2 - t)};$$

$$D_+(uv) = \operatorname{Spec} B\left[\frac{u}{uv}\right] = \operatorname{Spec} B\left[\frac{1}{v}\right] = \operatorname{Spec} \frac{\bigoplus_{i \ge 0, j \le \frac{7}{3}i} W u^i v^j}{(uv^2 - t)}.$$

It is apparent that $D_+(u)$ is regular and connected, but $D_+(uv)$ need not be; recall that this is exactly what happens in the toric case when we divide the cone $\sigma = \operatorname{Cone}(\mathbf{e}_2, m\mathbf{e}_1 - k\mathbf{e}_2)$ into the smooth cone $\sigma' = \operatorname{Cone}(\mathbf{e}_1, \mathbf{e}_2)$ and the not necessarily smooth cone $\sigma'' = \operatorname{Cone}(\mathbf{e}_1, m\mathbf{e}_1 - k\mathbf{e}_2)$. Now, just like in Chapter 1, we will use the modified Euler algorithm to write $D_+(uv)$ in an easier way. Let $7 = 3b_1 - k_1$ for $b_1 \ge 2$ and $0 \le k_1 < 3$: then $b_1 = 3, k_1 = 2$. Thus, the condition $0 \le j \le \frac{7}{3}i$ now becomes $0 \le i \le \frac{3}{2}(\frac{7+2}{3}i - j) = \frac{3}{2}(3i - j)$. Let $j' = i$ and $i' = 3i - j$: recall that in the toric case, we wrote σ'' in normal form by performing substitutions $\mathbf{e}_2' = \mathbf{e}_1$ and $\mathbf{e}_1' = \frac{m+k_1}{k}\mathbf{e}_1 - \mathbf{e}_2$!

With these substitutions, we get $u^i v^j = u'^{i'} v'^{j'}$ for $u' = 1/v$ and $v' = uv^3$; thus, we can write $D_+(uv)$ as

$$D_+(uv) = \operatorname{Spec} \frac{\bigoplus_{0 \le j' \le \frac{3}{2} i'} W u'^{i'} v'^{j'}}{(u'v' - t)}.$$

Notice how similar this is to 2.1. Let us compute the closed fiber Z'_ξ of the blow-up $Z' \to Z$: it is the union of the affine open sets

$$Z'_\xi \cap D_+(u) = \operatorname{Spec} \frac{\kappa[u,v]}{(uv^2)};$$
$$Z'_\xi \cap D_+(uv) = \operatorname{Spec} \frac{\kappa[u',v']}{(u'v')}.$$

The first open set has two irreducible components intersecting transversally, corresponding to the zero loci of u and v respectively; the second open set also has two irreducible components, corresponding to the zero loci of u' and v' respectively. Using the glueing data $u' = 1/v$, we infer that the closed fiber of Z' has three irreducible components: the v'-axis D_1 in $D_+(uv)$ with multiplicity 1, the u-axis D_2 in $D_+(u)$ with multiplicity 2, and the exceptional divisor $E \simeq \mathbb{P}^1_\kappa$ obtained by glueing the v-axis in $D_+(u)$ to the $(1/v)$-axis in $D_+(uv)$, with multiplicity 1. The uniformizer t has Weil divisor

$$\operatorname{div}_{Z'}(t) = D_1 + 2D_2 + E.$$

Since $\operatorname{div}_{Z'}(t)$ is principal, $\operatorname{div}_{Z'}(t).E = 0$, yielding

$$0 = \operatorname{div}_{Z'}(t).E = 1 + 2 + (E.E) \iff (E.E) = -3 = -b_1.$$

Now, let us go back to our series of blow-ups. We have seen that Z' is the union of the affine open sets $D_+(u)$ and $D_+(uv)$, the first of which is regular. Much like the original case, let us compute the blow-up $Z'' = \operatorname{Bl}_{(u',u'v')} Z'$ of Z' along the point corresponding to the ideal $(u', u'v')$ of $C := B[1/v]$. If E' is the exceptional divisor of the blow-up, then $Z'' \setminus E'$ is isomorphic to $Z' \setminus \{(u', u'v')\}$. Thus, we only need to look at Z'' around E', for example by considering the preimage of $D_+(uv)$, which is given by the union of the affine open sets

$$D_+(u') = \operatorname{Spec} C[v'] = \operatorname{Spec} \frac{W[u',v']}{(u'v'-t)};$$
$$D_+(u'v') = \operatorname{Spec} C\left[\frac{1}{v'}\right] = \operatorname{Spec} \frac{\bigoplus_{i' \ge 0, j' \le \frac{3}{2} i'} W u'^{i'} v'^{j'}}{(u'v' - t)}.$$

Like in the previous case, $D_+(u')$ is regular, but $D_+(u'v')$ may not be. To write $D_+(u'v')$ in "normal form", let $3 = 2b_2 - k_2$, for $b_2 \geq 2$ and $0 \leq k_2 < k_1$: we get $b_2 = 2, k_2 = 1$. If $u'' = 1/v'$, $v'' = u'v'^2$, $j'' = i'$ and $i'' = 2i' - j'$, then

$$D_+(u'v') = \operatorname{Spec} \frac{\bigoplus_{0 \leq j'' \leq 2i''} Wu''^{i''}v''^{j''}}{(u''v'' - t)}.$$

In the closed fiber of Z'', the blow-up left D_2 untouched, while the strict transform E_1 of E now plays the same role that D_2 played in the closed fiber of Z': following the same steps as before, we see that the closed fiber has four irreducible components: D_2 (rather, its strict transform, which is isomorphic to D_2), E_1, the new exceptional divisor E', which is isomorphic to the projective line over κ, and the v''-axis D_1'; D_2 with multiplicity 2, all the others with multiplicity 1. Thus, the uniformizer has divisor

$$\operatorname{div}_{Z''}(t) = D_1' + E' + E_1 + 2D_2.$$

Like before, we compute $0 = \operatorname{div}_{Z''}(t).E' = 1 + (E'.E') + 1 + 0$, yielding

$$(E'.E') = -2 = -b_2.$$

We will now blow-up Z'' along the closed point corresponding to the ideal $(u'', u''v'')$ of $D_+(u'v')$; like before, we are only interested in looking at what happens around the exceptional divisor E_3. If $D := C[1/v']$, E has a neighborhood isomorphic to the union of the affine open sets

$$D_+(u'') = \operatorname{Spec} D[v''] = \operatorname{Spec} \frac{W[u'', v'']}{(u''v'' - t)},$$

which is regular, and

$$D_+(u''v'') = \operatorname{Spec} D\left[\frac{1}{v''}\right] = \operatorname{Spec} \frac{\bigoplus_{i'' \geq 0, j'' \leq 2i''} Wu''^{i''}v''^{j''}}{(u''v'' - t)}.$$

Write $2 = b_3 - k_3$ for $b_3 \geq 2$ and $0 \leq k_3 < 1$: we have $b_3 = 2$ and $k_3 = 0$. By setting $u''' = 1/v''$, $v''' = u''v''^2$, $j''' = i''$ and $i''' = 2i'' - j''$, we get

$$D\left[\frac{1}{v''}\right] = \frac{W[u''', v''']}{(u'''v''' - t)}.$$

Thus, the blow-up $Z''' = \operatorname{Bl}_{(u'', u''v'')} Z''$ is regular. Its closed fiber has 5 irreducible components: (the strict transforms of) D_2 and E_1, which were left untouched by the blow-up, with multiplicities 2 and 1; the strict transform E_2 of E', with multiplicity 1; the exceptional divisor of the blow-up E_3, which is isormophic to the projective line over κ, with multiplicity 1; the v''-axis D_1'', with multiplicity 1. The divisor of t is $\operatorname{div}_{Z'''}(t) = D_1'' + E_3 + E_2 + E_1 + 2D_2$, whence $(E_3.E_3) = -2 = -b_3$.

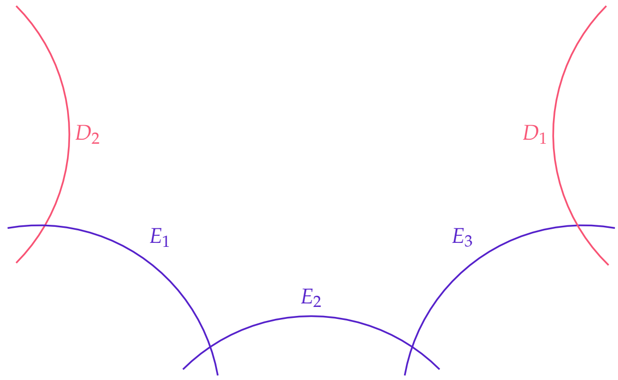


Figure 2.1: The closed fiber of the minimal regular resolution of Z.

Notice that the morphism $Z''' \to Z$ is a minimal resolution of singularities, because all the intermediate blow-ups had exceptional divisor of self intersection strictly smaller than -1: only in the last step did we finally get a regular scheme. In particular, the self intersection coefficients satisfy

$$\frac{7}{3} = 3 - \frac{1}{2 - \frac{1}{2}},$$

as we expected by the dissertation in the previous chapter.

The closed fiber of the minimal regular resolution is represented in Figure 2.1.

2.2 Tame cyclic quotient singularities

In this section, we will explore some of the theory on tame quotient singularities, following the dissertation in section 2 of [CES03]. We will first study some general theory on the resolution of curves, refining a theorem of Lipman (Theorem 2.7) to a theorem on the existence of a minimal regular resolution on an S-curve X, where either S is excellent (as in the original hypotheses of Lipman's theorem) or $X_{/S}$ has smooth generic fiber (Theorem 2.9). At the end of the first subsection we will finally give a formal definition of what we mean by "tame cyclic quotient singularity"; we will then generalize the example in the previous section, proving a theorem on the resolution of tame cyclic quotient singularities using the Hirzebruch-Jung algorithm (Theorem 2.22), where we will show that the numerical data for a tame cyclic quotient singularity at a point is intrinsic even in the absence of an explicit cyclic group action.

2.2.1 Resolution of singularities on excellent schemes

To motivate why we need to work on excellent schemes when tackling questions of desingularization, consider the following result:

Theorem 2.2. *Let X be a universally catenary locally Noetherian scheme such that all integral schemes that are finite over X admit a desingularization. Then X is excellent.*

Proof. [EGA IV], 7.9.5. □

Indeed, under the hypothesis of excellence, Lipman ([Lip78]) proved a theorem on the existence of a desingularization in the strong sense (Theorem 2.7 below). Before we get to the theory of desingularization, let us recall some basic facts and definitions on excellent rings and schemes.

Definition 2.3 ((Universally) catenary ring/scheme)**.** A Noetherian ring R is *catenary* if for any triplet of prime ideals $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{m}$, the following equality is satisfied:

$$\mathrm{ht}(\mathfrak{m}/\mathfrak{q}) = \mathrm{ht}(\mathfrak{m}/\mathfrak{p}) + \mathrm{ht}(\mathfrak{m}/\mathfrak{q}).$$

We say that R is *universally catenary* if every finitely generated algebra over R is catenary. Similarly, we say that a locally Noetherian scheme X is *catenary* if its local rings are catenary, and that it is *universally catenary* if $\mathbb{A}^n_X$ is catenary for all $n \geq 0$.

Definition 2.4 (Excellent ring/scheme)**.** Let $(R, \mathfrak{m})$ be a local ring, $\widehat{R}$ its completion for the $\mathfrak{m}$-adic topology. We call the fibers of the canonical morphism $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$ the *formal fibers* of R. We say that R is *excellent* if it satisfies the following conditions:

i) $\operatorname{Spec} R$ is universally catenary;

ii) For every prime $\mathfrak{p} \subseteq R$, the formal fibers of $A_{\mathfrak{p}}$ are geometrically regular;

iii) For every finitely generated R-algebra A, the set of regular points of $\operatorname{Spec} A$ is open in $\operatorname{Spec} A$.

A locally Noetherian scheme X is *excellent* if it has an affine open covering $\{U_i\}_i$ such that all $\mathcal{O}_X(U_i)$ are excellent.

Excellent schemes satisfy the following theorem:

Theorem 2.5. *The following facts are true.*

1. *Any complete local Noetherian ring is excellent; in particular, fields are excellent.*
2. *For a Noetherian local ring to be excellent, it is sufficient that it satisfies (i) and (ii) of the previous definition.*
3. *Any scheme that is locally of finite type over a locally Noetherian excellent scheme is excellent. In particular, open and closed subschemes of excellent schemes are excellent.*

4. *If X is excellent and integral, the normalization morphism $X' \to X$ is finite.*

Proof. [Liu02], 8.2.39. □

Corollary 2.6. 1. *Any algebraic variety over a field is excellent.*

2. *If R is a regular local ring, for it to be excellent it is sufficient that* $\mathrm{Frac}(\widehat{R})$ *be separable over* $\mathrm{Frac}(R)$.

3. *Any Dedekind domain of characteristic zero is excellent.*

Proof. [Liu02], 8.2.40. □

Now let S be an excellent connected Dedekind scheme and X a normal S-curve. Consider the following chain of morphisms:

$$\ldots \to X_{n+1} \to X_n \to \ldots \to X_1 \to X,$$

where $X_1 \to X$ is the normalization of X and for all indices $i \geq 1$, the map $X_{i+1} \to X_i$ is the composition of the blow-up of X_i along its singular locus (endowed with the reduced scheme structure) and its normalization. The following result is a corollary of a theorem of Lipman ([Lip69], 2.1), which ensures that this process produces a strong desingularization of X in a finite number of steps.

Theorem 2.7. *If S is an excellent Dedekind scheme and $X \to S$ an S-curve, then the sequence above is finite. In particular, X admits a desingularization in the strong sense.*

The following result establishes the existence of a *minimal* resolution of singularities (in the strong sense) for an S-curve X when we assume that either S is excellent, or that the generic fiber of X over S is smooth. First recall the following definition:

Definition 2.8. A prime divisor E on a regular fibered surface $X \to S$ is called a (-1)-*curve* if there exists a regular fibered surface $Y \to S$ and an S-morphism $f : X \to Y$ such that E reduces to a point through f and $f : X \setminus E \to Y \setminus f(E)$ is an isormorphism.

By Castelnuovo's theorem ([Liu02], 9.3.8), such a divisor is isomorphic to a projective line over the field $k = \mathrm{H}^0(E, \mathcal{O}_E)$. When searching for a resolution of singularities $\pi : X^{\mathrm{reg}} \to X$, we will require that there be no (-1)-curves in the fibers of π. Since (-1)-curves can be blown down to a point, this will amount to a condition of minimality.

Theorem 2.9 (Existence of a minimal resolution)**.** *Let S be a connected Dedekind scheme, and let $X \to S$ be a normal S-curve. If either S is excellent or the generic fiber of X over S is smooth, there exists a birational proper morphism $\pi : X^{\mathrm{reg}} \to X$ such that X^{reg}*

is a regular S-curve and there are no (-1)-curves in the fibers of π. Such a morphism is unique up to unique isomorphism; moreover, if $\pi' : X' \to X$ is another birational proper morphism with X' a regular S-curve, then π' factors uniquely through π. Formation of X^{reg} is compatible with base change to $\operatorname{Spec} \mathcal{O}_{S,s}$ *and* $\operatorname{Spec} \widehat{\mathcal{O}_{S,s}}$ *for any closed point $s \in S$.*

Before proving the theorem, let us recall some important results that we will use in the proof.

Theorem 2.10 (Nagata's compactification theorem)**.** *Let S be a Noetherian scheme and $X \to S$ a separated S-scheme of finite type. Then there is a compactification of X over S, i.e. an open immersion $X \hookrightarrow \overline{X}$ with schematically dense image, such that $\overline{X}$ is a proper S-scheme.*

Proof. [Lüt93], 3.2. □

Theorem 2.11 (Factorization theorem for birational morphisms)**.** *Let $f : X' \to X$ be a proper birational morphism between regular integral Noetherian schemes of dimension 2. Then f factors as a sequence of blow-ups at closed points.*

Proof. [Stacks], tag 0C5R. □

Theorem 2.12 (Stein factorization)**.** *Let S be a locally Noetherian scheme, $X \to S$ a proper S-scheme. Then there exists a factorization*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & S' \\ & \searrow \quad \swarrow_{\pi} & \\ & S & \end{array}$$

where π is finite and f proper with geometrically connected fibers.

Proof. [Stacks], tag 03H0. □

We also state an elementary result in intersection theory:

Theorem 2.13. *Let X be a connected regular proper curve over a discrete valuation ring with residue field k, let $x \in X$ be a closed point in the closed fiber and consider the blow-up $\pi : X' = \mathrm{Bl}_x(X) \to X$, with exceptional divisor $E \simeq \mathbb{P}^1_{\kappa(x)}$. Let C_1, C_2 be two (not necessarily distinct) effective divisors supported in the closed fiber of X, with each C_i passing through x, and let C_i' be their strict transforms under π. We have*

$$\pi^{-1}(C_i) = C_i' + m_i E,$$

where m_i is the multiplicity of C_i at x. Moreover, $m_i = C_i'.E/[\kappa(x) : k]$ and

$$C_1.C_2 = C_1'.C_2' + m_1 m_2 [\kappa(x) : k].$$

Proof. [CES03], 2.1.2. □

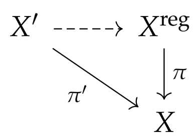
The proof of Theorem 2.9 will work as follows: first, we prove the theorem in the case where S is excellent. We will use Nagata's compactification theorem to reduce to the case where X is proper over S, which will simplify the proof of the uniqueness of the minimal resolution. Then, we have the existence of the resolution by Theorem 2.7 and we will use the Factorization for birational morphisms and Stein factorization theorems to prove uniqueness. The case where S is not necessarily excellent but X has smooth generic fiber will be proven by reducing to the assumption that S is the spectrum of a local ring; then, we will base-change to its completion, which is excellent, and reduce back to the previous case. This last step of the proof will require the following lemma:

Lemma 2.14. *Let R be a discrete valuation ring with fraction field K and let X be an R-scheme which is locally of finite type over R and has regular generic fiber. Let $R \to R'$ be a local extension of discrete valuation rings (where* local *means that $\mathfrak{m}_R R' = \mathfrak{m}_R$) such that the residue field extension $k \to k'$ is separable, and assume that either the fraction field extension $K \to K'$ is separable or that X has smooth generic fiber (in both cases, $X_{/K'}$ is regular). Then for any $x' \in X' := X \times_R R'$ lying over $x \in X$, the local ring $\mathcal{O}_{X',x'}$ is regular (resp. normal) if and only if $\mathcal{O}_{X,x}$ is regular (resp. normal).*

Proof. [CES03], 2.1.1. □

Proof of Theorem 2.9. First assume that S is excellent. Since we will want to use the Factorization theorem for birational morphisms, let us assume that X is proper. This can be done without loss of generality by the following argument: by Nagata's compactification theorem, there is a schematically dense open immersion $X \hookrightarrow \overline{X}$ where $\overline{X}$ is proper over S; since $\overline{X}$ is of finite type over an excellent scheme, it is itself excellent, so its normalization is finite. Thus, by replacing $\overline{X}$ with its normalization if necessary, we may assume that $\overline{X}$ is a normal S-curve (where flatness follows from [Liu02], 4.3.9). Up to solving any singularity in $\overline{X} \setminus X$, we can also assume that the singular locus of $\overline{X}$ coincides with that of X: thus, proving the existence and uniqueness of the resolution on $\overline{X}$ will imply the same results on X. Hence, we may assume that X is proper over S. Up to reducing to the (disjoint, by normality) connected components of X, we may also assume that X is connected. By Theorem 2.7, there is a birational proper (recall that blow-ups are proper and the normalization is finite) morphism $X_1 \to X$ with X_1 a regular proper S-curve. If X_1 has any (-1)-curves in the fiber of X over some closed point, then it suffices to blow them down to a point to obtain the regular X-scheme X^{reg} with no (-1)-curves in its fibers that we wanted.

Suppose now that $\pi' : X' \to X$ is a different birational proper morphism with X' a regular S-curve. Then we have a birational map $X' \dashrightarrow X^{\text{reg}}$, as follows:

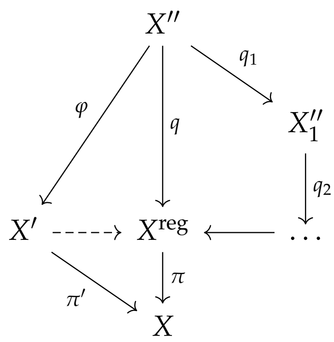


If X is regular, then obviously $X = X^{\text{reg}}$ and we immediately get the existence and uniqueness of the factorization of π' through $\pi = \text{id} : X^{\text{reg}} \to X$. Suppose then that X is not regular. By [Liu02], 9.2.7, blowing-up X' at certain closed points yields a birational morphism from the blow-up X'' to X^{reg}; thus, if we replace X' with X'' for the moment, we obtain a birational morphism $X'' \to X^{\text{reg}}$. We will now show that given a tower of birational proper morphisms

$$X'' \to X^{\text{reg}} \to X$$

with X'' and X^{reg} regular and no (-1)-curves in the fibers of X^{reg} over X, then any (-1)-curve in a fiber of $X'' \to X$ must be contracted by $X'' \to X^{\text{reg}}$. From this, it will follow that the birational map $X' \dashrightarrow X^{\text{reg}}$ extends to a morphism even without blowing up X'.

By the Factorization theorem, the morphism $q : X'' \to X^{\text{reg}}$ factors as a composition of blow-ups of X^{reg} at closed points. Thus, we have the following diagram:



where the q_i are the blow-ups of the factorization. By Stein's factorization theorem, we may assume that the S-curves X, X^{reg} and X' (that share the same generic fiber over S) have geometrically connected fibers. Now let C be a (-1)-curve in a fiber of π''. We may assume that C meets the exceptional divisor E of the first blow-down map $q_1 : X'' \to X''_1$. If $C = E$, then clearly C is contracted by q. Suppose that $C \neq E$. Then $q_1(C)$ is an irreducible divisor on X'_1 with strict transform C. By 2.13, we have

$$q_1(C).q_1(C) = \underbrace{C.C}_{=-1} + (C.E[\kappa(P).\kappa(s)])^2,$$

for suitable points $P \in X_1''$, $s \in S$. Thus, $q_1(C)$ has non negative self-intersection number, which must then be zero. By [Liu02], 9.1.23, this implies that the divisor $q_1(C)$ is a multiple of the closed fiber of X_1'' over s. Since $\left(X_1''\right)_s$ is geometrically connected by Stein factorization, we conclude that $q_1(C)$ is the entire closed fiber $\left(X_1''\right)_s$, which is thus irreducible. The morphism $X_1'' \to X$ is birational, proper and surjective; by surjectivity, the image of $\left(X_1''\right)_s$ cannot be a single point, so it has to be a 1-dimensional subscheme of X. By repeating this argument for the other closed fibers, we find that $X_1'' \to X$ is quasi-finite, as its fibers are finite. By surjectivity and Zariski's main theorem, it follows that $X_1'' \to X$ is an isomorphism, so X is regular. We had assumed that X is not regular, so the proof of the existence of a birational morphism $X' \to X^{\mathrm{reg}}$ is concluded. So we have shown that any birational proper morphism $\pi' : X' \to X$ with X' regular factors through π. Uniqueness of this factorization follows trivially from [Liu02], 3.3.11, since two such maps will coincide on the generic fiber, and must thus coincide on the whole scheme.
Now fix a closed point $s \in S$ and let S' be either $\operatorname{Spec} \mathcal{O}_{S,s}$ or $\operatorname{Spec} \widehat{\mathcal{O}_{S,s}}$. By the previous Lemma, the base change $X^{\mathrm{reg}}_{/S'}$ is regular, and the induced morphism over the normal curve $X_{/S'}$ proper and birational. Since base-change did not create new (-1)-curves in the fibers of $X^{\mathrm{reg}}_{/S'}$ over $X_{/S'}$, we are done.
Now suppose S is not necessarily excellent, but X has smooth generic fiber over S. In this case, only a finite number of fibers of X over S may be non-smooth, so we can restrict to the local case where S is the spectrum of a discrete valuation ring R. By the previous lemma, $X_{/\widehat{R}}$ is a normal $\widehat{R}$-curve, and since $\widehat{R}$ is excellent, by the previous dissertation we have a minimal regular resolution

$$\widehat{\pi} : \left(X_{/\widehat{R}}\right)^{\mathrm{reg}} \to X_{/\widehat{R}}.$$

By [Lip78], Remark C at page 155, $\widehat{\pi}$ is obtained as a blow-up of $X_{/\widehat{R}}$ along a zero-dimensional subscheme $\widehat{Z}$ supported in the singular locus of $X_{/\widehat{R}}$, so $\widehat{Z}$ is supported in the closed fiber. Since $\widehat{Z}$ is artinian, it lies in some infinitesimal fiber of $X_{/\widehat{R}}$. The base-change $X \times_R \operatorname{Spec} \widehat{R} \to X$ induces isomorphisms on the n-th infinitesimal fibers for all natural n, so there is a unique zero-dimensional closed subscheme Z in X whose base-change in $X_{/\widehat{R}}$ is $\widehat{Z}$. Since $\widehat{R}$ is faithfully flat over R ([Stacks], tag 00MC) and blow-ups commute with flat base-change ([Stacks], tag 085S), we have an isomorphism

$$(\mathrm{Bl}_Z(X))_{/\widehat{R}} \simeq \mathrm{Bl}_{\widehat{Z}}\left(X_{/\widehat{R}}\right) = \left(X_{/\widehat{R}}\right)^{\mathrm{reg}},$$

so $\mathrm{Bl}_Z(X)$ is regular by the previous lemma. Since the extension $R \to \widehat{R}$ is residually trivial, there are no (-1)-curves in the fibers of $\mathrm{Bl}_Z(X)$ over X. Thus, we have the existence of a regular resolution $\pi : X^{\mathrm{reg}} \to X$. For uniqueness of π, its universal factorization property, and compatibility of the construction with certain base-changes, we can carry over the results to X by base-changing with $\widehat{R}$, using again the previous lemma. □

We can also consider a regular resolution along a finite subset of closed points. This allows us to compute the minimal resolution along the singular locus iteratively, one point at a time.

Definition 2.15. Let S be a connected Dedekind scheme, and let $X \to S$ be a normal S-curve. Assume that either S is excellent or the generic fiber of X over S is smooth. Let $\Sigma \subseteq X$ be a finite set of closed points in closed fibers over S and let $U \subseteq X$ be an open neighborhood containing Σ, such that $U \cap (X_{\text{sing}} \smallsetminus \Sigma) = \emptyset$. The *minimal regular resolution along* Σ is the morphism $\pi_\Sigma : X(\Sigma) \to X$ obtained by glueing $X \smallsetminus \Sigma$ to the part of X^{reg} lying over U.

The minimal regular resolution along Σ is compatible with base-change to a complete local ring on S, and it is uniquely characterized among normal S-curves that are proper and birational X-schemes by the following conditions:

i) π_Σ is an isomorphism over $X \smallsetminus \Sigma$;

ii) $X(\Sigma)$ is regular at points over Σ;

iii) $X(\Sigma)$ has no (-1)-curves in its fibers over Σ.

Thus, we have the following corollary, stating that construction of $X(\Sigma)$ is étale-local on X. Recall that an *étale morphism* is a morphism of schemes that is flat and unramified; if X is a scheme, an open set for the étale topology on X is an étale morphism $U \to X$. For more details, see [Mil13] or [Stacks].

Corollary 2.16. *Let $X_{/S}$ be a normal S-curve over a connected Dedekind scheme S, assume that either S is excellent or $X_{/S}$ has smooth generic fiber over S, and let $\Sigma \subseteq X$ be a finite set of closed points in closed fibers over S. Let $X' \to X$ be étale (implying that X' is an S-curve) and let $\Sigma \subseteq X'$ be the preimage of Σ in X'. Let $X(\Sigma) \to X$ be the minimal regular resolution along Σ and suppose that X' is residually trivial over X (for example if S is local with separably closed residue field). Then the base change $X(\Sigma) \times_X X' \to X'$ is the minimal regular resolution along Σ'.*

Proof. By [Stacks], tag 03PC, (4), base changes of étale morphisms are étale. Thus, $X(\Sigma) \times_X X'$ is étale over $X(\Sigma)$, so it is an S-curve that is regular along the preimage of Σ'. Moreover, it is proper and birational over X' (because these properties as well are stable under base change) and it is an isomorphism over $X' \smallsetminus \Sigma'$. Lastly, the morphism $X(\Sigma) \times_X X' \to X'$ has no (-1)-curves in the fibers over Σ, because $X' \to X$ is residually trivial over Σ. □

We now state a fundamental result in invariant theory. This is known as the Chevalley-Shephard-Todd theorem on the complex numbers, and was generalized by Serre to an arbitrary field. It gives a necessary and sufficient condition for the regularity of the ring of invariants of a Noetherian regular local ring by the action of a finite group G. The proof of the theorem can be found in [Wat76]; we give a slightly different statement that allows more relaxed hypotheses on the order of G. For more details, see [CES03], 2.3.9. First, we need the following definition:

Definition 2.17. Let V be a vector space of finite dimension over a field κ. An element $\sigma \in \mathrm{Aut}_\kappa(V)$ is a *pseudo-reflection* if $\mathrm{rk}(1-\sigma) \leq 1$.

Theorem 2.18 (Serre-Chevalley–Shephard–Todd)**.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian regular local ring. Let G be a finite subgroup of $\mathrm{Aut}_\kappa(R)$ and let R^G be the ring of invariants under the action of G. Suppose that G acts trivially on κ and that R is finite over R^G. Then R^G is regular only if the image of G in $\mathrm{Aut}_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ is generated by pseudo-reflections. If, moreover, the order of G is coprime to the characteristic exponent of κ, the previous necessary condition is also sufficient.*

Example 2.19. Let F be a field and consider the polynomial ring $R = F[x, y, z]_{(x)}$, arising as the local ring at $P = (x)$ of the affine space $\mathbb{A}^3_F$, with residue field $F(y, z)$. Consider the action of $G = \mathbb{Z}/2\mathbb{Z}$ on R via F-automorphisms, where the nontrivial element of G maps $x \longmapsto -x$ and fixes y and z. Then $R^G = F[x^2, y, z]$ and $(x)/(x^2)$ is the $F(y, z)$-vector space with basis $\{x\}$. Clearly R^G is regular and indeed the image of G in $\mathrm{Aut}_{F(y,z)}\left((x)/(x^2)\right)$ is generated by pseudo-reflections.

We will now give a precise definition of what exactly we mean by "tame quotient singularity". The definition is motivated by the following setup: consider a regular S-curve X and a finite group H acting on $X_{/S}$, such that no non-identity element of H acts trivially on a connected component of X. Consider the quotient $X' = X/H$: then, X' has regular generic fiber over S, and the singular locus of X' is composed of finitely many closed points in the closed fibers. Fix a singular point $x' \in X'$ and let $s \in S$ be its image in S, with $\mathrm{char}\, \kappa(x') = \mathrm{char}\, \kappa(s) = p \geq 0$. Let $x \in X$ be any point over x' and suppose that X is *nil-semistable* at x, as per the following definition:

Definition 2.20. Given a fixed closed point $s \in S$ and a point $x \in X$ over s, we say that X is *nil-semistable* at x if the following conditions are satisfied:

i) the reduced fiber X_s^{red} is geometrically connected and geometrically smooth over $\kappa(s)$, except for a finite number of nodes;

ii) X_s^{red} has genus greater or equal to 1;

iii) $\left(X_s^{\mathrm{red}}\right)_{\overline{\kappa(s)}}$ has no irreducible component isomorphic to $\mathbb{P}^1_{\kappa(s)}$ meeting the rest of $\left(X_s^{\mathrm{red}}\right)_{\overline{\kappa(s)}}$ in only one point;

iv) none of the analytic branch multiplicities thorugh x are divisible by the characteristic of $\kappa(s)$.

The first three conditions amount to X_s^{red} being *semistable* over $\kappa(s)$, as per [Stacks], tag 0E6X.

Under these hypotheses, by [CES03], 2.3.2, x has either one or two distinct analytic branches passing through it. If $p > 0$, assume also that $\kappa(x)$ is separable over $\kappa(s)$, that H has order not divisible by p, and that if x is at the intersection of two distinct analytic branches, then H does not interchange them. It follows from Theorem 2.18 that a nil-semistable point $x' \in X'$ is non-regular if there is no line in the tangent space of X' at x' on which H acts trivially.
By [CES03], 2.3.4, in the above hypotheses we have an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \simeq \frac{\widehat{\mathcal{O}_{S,s}}[[x,y]]}{(x^{m_1}y^{m_2} - t_s)},$$

where t_s is the uniformizer of the complete discrete valuation ring $\widehat{\mathcal{O}_{S,s}}$.

Motivated by the above situation, we now give the following definition:

Definition 2.21 (Tame cyclic quotient singularity)**.** Let $X'_{/S}$ be a normal curve over a connected Dedekind scheme S, where either S is excellent or $X'_{/S}$ has smooth generic fiber over S. Consider a closed point $s \in S$ with algebraically closed residue field $\kappa(s)$ of characteristic exponent $p \geq 1$ and a point $x' \in X'_s$ such that X'_s has two analytic branches at x'. We say that x' is a *tame cyclic quotient singularity* if there exists an integer m not divisible by p, a unit $k \in (\mathbb{Z}/m\mathbb{Z})^\times$ and integers $n_1 > 0$ and $n_2 \geq 0$ satisfying

$$n_1 \equiv -kn_2 \mod m,$$

such that we have an isomorphism

$$\widehat{\mathcal{O}_{X',x'}} \simeq \left(\frac{\widehat{\mathcal{O}_{S,s}}[[x,y]]}{(x^{n_1}y^{n_2} - t_s)}\right)^{\mu_m(\kappa(s))},$$

where the action of $\mu_m(\kappa(s))$ is defined by

$$\zeta \cdot x = \zeta x, \quad \zeta \cdot y = \zeta^k y$$

for a primitive m-th root of unity ζ.

Recall the example studied the previous section of this chapter: it is the resolution of a tame cyclic quotient singularity as per the above definition. We will use a general version of that example in the next subsection to prove the following important theorem on the resolution of tame cyclic quotient singularities.

2.2.2 Hirzebruch-Jung algorithm

In the presence of a tame quotient singularity as in Definition 2.21, it is easy to see that the numbers n_1 and n_2 correspond to the analytic branch multiplicities through x'. One might additionally wonder if the data m, k in the definition depend on a choice of coordinates for $\mathcal{O}_{X',x'}$ or if it is intrinsic to x'. The next theorem shows that this data is intrinsic and can be recovered from the regular resolution at x'.

Theorem 2.22. *Let $S = \operatorname{Spec} W$ with W a complete discrete valuation ring with algebraically closed residue field κ, fraction field K, closed point s and generic point η, and let $X' \to S$ be a normal S-curve. Assume that X' has a tame cyclic quotient singularity, as per the previous definition, at a closed point $x' \in X'_s$, which we suppose to have separably closed residue field $\kappa(x')$, with parameters m and k, where we identify $k \in (\mathbb{Z}/m\mathbb{Z})$ with its unique representative $0 \leq k < m$. Consider the Hirzebruch-Jung continued fraction expansion*

$$\frac{m}{k} = b_1 - \frac{1}{b_2 - \frac{1}{\cdots - \frac{1}{b_r}}},$$

with $b_i \geq 2$ for all i. Then, the fiber over $\kappa(x')$ of the minimal regular resolution $X'\left(\{x'\}\right)$ of X' at x' (using the notation of Definition 2.15), equipped with the reduced subscheme structure, is the intersection of divisors E_i intersecting as represented in Figure 2.2, where:

i) the intersections are all transverse with $E_i \simeq \mathbb{P}^1_{\kappa(x')}$;

ii) $E_i.E_i = -b_i < -1$ for all i;

iii) if X'_1 and X'_2 are the analytic branches through x', with multiplicities, respectively, n_2 and n_1 (in the case where $n_2 = 0$, there is only one analytic branch X'_2), then E_1 is transverse to the strict transform $\widetilde{X}'_1$ of X'_1 and E_r is transverse to the strict transform $\widetilde{X}'_2$ of X'_2.

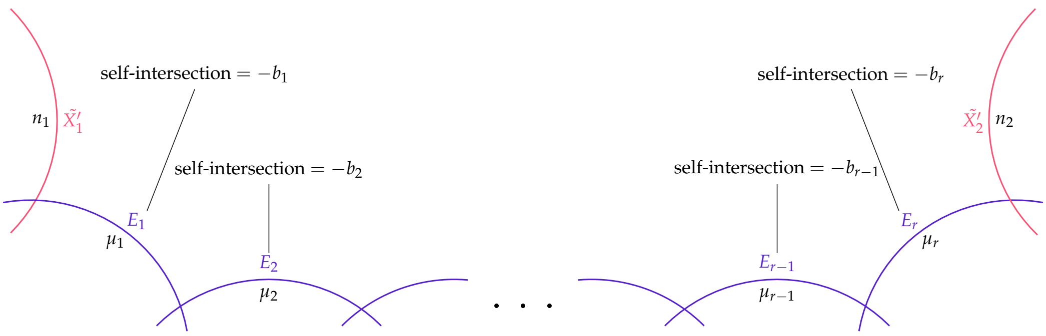


Figure 2.2: The fiber of the minimal regular resolution of X' at x' over x'.

The above theorem can be stated with more relaxed hypotheses, namely we can take S to be a connected Dedekind scheme (not necessarily local or affine), with either S excellent or $X_{/S}$ with smooth generic fiber, as we assumed in the previous subsection; it is also not necessary that the residue field be algebraically closed, so long as it is separably closed. The proof can ultimately be reduced down to the simpler situation in our statement of the theorem, as seen in [CES03], pp. 348-349.

In the proof of the theorem, we will study an even more special case, that will look a lot like the example in the first section of this chapter. Namely, we will solve the singularity on the affine scheme $Z = \operatorname{Spec} B$ at the origin z, where

$$A = \frac{W[x,y]}{(x^{n_1}y^{n_2} - t)}, \quad B = A^{\mu_m(\kappa)},$$

with t being the uniformizer of W and the action of $\mu_m(\kappa) = \langle \zeta \rangle$ defined by

$$\zeta \cdot x = \zeta x, \quad \zeta \cdot y = \zeta^k y.$$

The reason why we can boil the proof down to this very special case is because by definition, if X' has a tame cyclic quotient singularity at x', then we have an isomorphism $\widehat{\mathcal{O}_{X',x'}} \simeq \widehat{\mathcal{O}_{Z,z}}$. Recalling that the minimal regular resolution is étale-local (Corollary 2.16), we can use the following theorem of Artin to find a common residually trivial connected étale neighborhood (U,u) of (X',x') and (Z,z); then, the resolution of U at $\{u\}$ is the pullback of the resolutions of X' at $\{x'\}$ and Z at $\{z\}$. Since the fibers over u, x' and z are all the same, we conclude that we can compute the minimal regular resolution of Z at $\{z\}$ to prove the statement for (X',x').

Theorem 2.23 (Artin approximation theorem)**.** *Let X', Z be S-schemes of finite type, and let $x' \in X'$, $z \in Z$ be points. If there is an isomorphism of complete local rings $\widehat{\mathcal{O}_{X',x'}} \simeq \widehat{\mathcal{O}_{Z,z}}$, then there is a common étale neighborhood (U,u) of x' in X' and z in Z, i.e. a diagram*

$$\begin{array}{ccccccccc} & U & & & & & u & & \\ \scriptstyle\text{ét.}\swarrow & & \searrow\scriptstyle\text{ét.} & & \textit{mapping} & \swarrow & & \searrow & \\ X' & & Z & & & x' & & z & \end{array}$$

inducing an isomorphism of residue fields $\kappa(x') \simeq \kappa(z) \simeq \kappa(u)$.

Proof. [Art69], 2.6. □

Proof of theorem 2.22. By the above discussion, we may work on the affine scheme $Z = \operatorname{Spec} B$ for B the ring of invariants of $A = W[x,y]/(x^{n_1}y^{n_2} - t)$ by the action of $\mu_m(\kappa)$ defined earlier; we will follow very similar steps to those in Section 2.1, with a little more attention paid to the numerical data of the singularity. The action of G

is free away from $x = y = t = 0$, so Z is normal and regular away from its image point z. Since $n_1 \equiv -kn_2 \mod m$, we have that $n_1 + kn_2 = \mu m$ for some μ. By arguments analogous to those in Section 2.1, we see that

$$B = \frac{\bigoplus_{0 \le j \le \frac{m}{k} i} W u^i v^j}{(u^\mu v^{n_2} - t)},$$

for $u = x^m$ and $v = y/x^k$; in this notatoin, the point z corresponds to the maximal ideal (u, uv). It is clear that Z is a W-curve: it is separated, flat and finitely presented over W; moreover, its generic fiber

$$Z_\eta = (\operatorname{Spec} A)_\eta / G = \operatorname{Spec}\left(\frac{\bigoplus_{0 \le j \le \frac{m}{k} i} K u^i v^j}{(u^\mu v^{n_2} - t)} \right)$$

has dimension 1, so by flatness we see that all the fibers of Z over S have pure dimension 1.

Let us compute the closed fiber Z_ξ of Z:

$$Z_\xi = \operatorname{Spec}(B \otimes_W W/(t)) = \frac{\bigoplus_{0 \le j \le \frac{m}{k} i} \kappa u^i v^j}{(u^\mu v^{n_2})}.$$

Since the action of G does not interchange the x-axis and the y-axis in Spec A, we see that if $n_2 > 0$, Z_ξ is made up of two distinct irreducible components corresponding to the images of the x-axis and the y-axis, intersecting transversally at z; if $n_2 = 0$, there is only one irreducible component, the image of the x-axis.

We now compute the blow-up Z' of Z along the ideal (u, uv): it is the union of two affine open sets $D_+(u)$ and $D_+(uv)$, where

$$D_+(u) = \operatorname{Spec} B\left[\frac{uv}{u}\right] = \operatorname{Spec} B\,[v] = \operatorname{Spec} \frac{W[u,v]}{(u^\mu v^{n_2} - t)};$$

$$D_+(uv) = \operatorname{Spec} B\left[\frac{u}{uv}\right] = \operatorname{Spec} B\left[\frac{1}{v}\right] = \operatorname{Spec} \frac{\bigoplus_{i \ge 0, j \le \frac{m}{k} i} W u^i v^j}{(u^\mu v^{n_2} - t)}.$$

Clearly, $D_+(u)$ is regular; let us write a simpler expression of $D_+(uv)$ by changing variables using the modified Euler algorithm: write $m = b_1 k - k_1$, for $b_1 \ge 2$ and $0 \le k_1 < k$.

Suppose that $k_1 = 0$; then $k = 1$, $b_1 = m$ and $b_1\mu - n_2 = n_1$. Set $j' = i$ and $i' = b_1 i - j$, $u' = 1/v$ and $v' = uv^{b_1}$: then

$$D_+(uv) = \operatorname{Spec} \frac{\bigoplus_{i', j' \ge 0} W u'^{i} v'^{j}}{(u'^{n_1} v'^{\mu} - t)} = \operatorname{Spec} \frac{W[u,v]}{(u'^{n_1} v'^{\mu} - t)},$$

which is regular. In the blow-up $Z' = \operatorname{Bl}_{(u,uv)}(Z)$, let D_1 and D_2 (if $n_2 \ge 0$) be the strict transforms of the irreducible divisors corresponding respectively to the "v-axis" and the "u-axis" in Z. Then, the closed fiber of Z' is the union of the affine open sets

$$Z'_\xi \cap D_+(u) = \operatorname{Spec} \frac{\kappa[u,v]}{(u^\mu v^{n_2})};$$
$$Z'_\xi \cap D_+(uv) = \operatorname{Spec} \frac{\kappa[u',v']}{(u'^{n_1} v'^{\mu})}.$$

Thus, we see that the closed fiber of Z' has three irreducible components: D_2 (if $n_2 \neq 0$), the exceptional divisor $E \simeq \mathbb{P}^1_\kappa$, and D_1, where E meets D_1 and D_2 transversally. So the uniformizer $t \in W$ has Weil divisor

$$\operatorname{div}_{Z'}(t) = n_2 D_2 + \mu E + n_1 D_1.$$

Since $E.\operatorname{div}_{Z'}(t) = 0$ and $n_1 = b_1\mu - n_2$, we see that

$$0 = n_2 D_2.E + \mu E.E + n_1 D_1.E \Longrightarrow E.E = -b_1.$$

Assume now that $k_1 > 0$. As in the case of toric varieties, k will now take the place of m and k_1 the place of k; observe, however, that in the new setting there is no reason why k and k_1 should not be divisible by p, so we cannot infer that $D_+(uv)$ has a cyclic tame quotient singularity. Nevertheless, we can write an expression of $D_+(uv)$ that resembles our original expression of B: since $\frac{m}{k} = b_1 - \frac{k_1}{k}$, the condition $0 \leq j \leq \frac{m}{k} i$ can be rewritten as $0 \leq i \leq \frac{k}{k_1}(b_1 i - j)$. Thus, setting $j' = i$, $i' = b_1 i - j$, $u' = 1/v$, $v' = uv^{b_1}$, we get $u^i v^j = u'^{i'} v' j'$, whence

$$B\left[\frac{1}{v}\right] = \frac{\bigoplus_{0 \leq j' \leq \frac{k}{k_1} i'} W u'^{i'} v'^{j'}}{(u'^{b_1\mu - n_2} v'^{\mu} - t)}.$$

We see again that the closed fiber of Z' has three (or two if $n_2 = 0$) irreducible components: the v'-axis $D_1 \subseteq D_+(uv)$, with multiplicity $b_1\mu - n_2$; if $n_2 \neq 0$, the u-axis $D_2 \subseteq D_+(u)$; the exceptional divisor $E \simeq \mathbb{P}^1_\kappa$ with multiplicity μ. The latter component meets D_1 transversally at a κ-rational point corresponding to the origin of $D_+(uv)$.

When we make a change of variables

$$(m, k, n_1, n_2, \mu) \rightsquigarrow (k, k_1, n_1, \mu, b_1\mu - n_2),$$

we see that $D_+(uv)$ is like our original situation, with a different set of parameters. Observe that since n_2 is replaced by $\mu > 0$, we no longer have to worry about the second axis being absent. Indeed, when we focus our attention to $D_+(uv)$, we see that the role of the D_2 axis is taken by the affine chart $E \cap D_+(uv)$ and the role of z is taken by the intersection of E and D_1. Thus, when we blow up Z' at the origin of $D_+(uv)$, everything remains the same outside of $D_+(uv)$, and we get three irreducible components in the closed fiber over $D_+(uv)$: the strict transform E_1 of

E, the new exceptional divisor E', the axis D_1. When we blow up again, everything around E_1 remains the same, so we can identify it with its strict transform; the strict transform of E', which we will call E_2, will now play the role of the D_2 axis, and the strict transform of D_1 (which we will call again D_1 by abuse of notation) the role of D_1. Since at every step m is replaced by a strictly smaller k, we infer that this iterative process of blowing up at the singular point must end at a regular scheme Z^{reg} in a finite number of steps.
By repeating the same argument as the case $k_1 = 0$, we see easily that $E_1.E_1 = -b_1$. Now, since

$$\frac{m}{k} = b_1 - \frac{1}{k/k_1},$$

we can conclude the proof inductively and see that the closed fiber in the final resolution is indeed as stated in the theorem, with $E_i.E_i = -b_i$. Moreover, since $b_i \geq 2$ for all i, we conclude that the final regular resolution is indeed minimal, as at no point in the process did we ever produce a (-1)-curve. □

Corollary 2.24. *In the hypotheses and notations of Theorem 2.22, the components E_i have multiplicities μ_i in the closed fiber of the minimal regular resolution $(X')^{\mathrm{reg}}$ at $\{x'\}$, where:*

i) if $k = 1$, then $r = 1$ and $\mu_1 = (n_1 + n_2)/m$;

ii) if $k > 1$, the μ_i are the unique solution to the linear system

$$\begin{pmatrix} b_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & b_2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & b_3 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & b_{r-1} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & b_r \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_{r-1} \\ \mu_r \end{pmatrix} = \begin{pmatrix} n_2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ n_1 \end{pmatrix} \tag{2.25}$$

Proof. The case $k = 1$ was shown in the proof of the previous theorem. Suppose then that $k > 1$. The closed fiber of $(X')^{\mathrm{reg}}$ has Weil divisor

$$\mathrm{div}(t) = n_1 \widetilde{X_1'} + \sum_{i=1}^{r} \mu_i E_i + n_2 \widetilde{X_2'} + [\ldots \text{components that do not meet the } E_i\text{'s} \ldots].$$

Recall that:

- E_1 meets $\widetilde{X_1'}$ and E_2 transversally;
- E_r meets E_{r-1} and $\widetilde{X_2'}$ (if $n_2 > 0$) transversally;
- for $1 < i < r$, E_i meets E_{i-1} and E_{i+1} transversally;
- $E_i.E_i = -b_i$.

Then, the conditions $E_i.\operatorname{div}(t) = 0$ yield precisely the system of equations in 2.25. □

Example 2.26. In the example in Section 2.1, we had $r = 3$, $b_1 = 3$, $b_2 = b_3 = 2$, $n_1 = 1$, $n_2 = 2$. Indeed:

$$\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

We conclude this chapter with the following definition:

Definition 2.27 (Dual graph)**.** Let C be a vertical divisor contained in a closed fiber X_s of a regular S-curve $X_{/S}$. Let $\Gamma_1, \dots, \Gamma_r$ be its irreducible components. The *dual graph* of C is the graph G whose vertices are the irreducible components of C, and there are $\Gamma_i.\Gamma_j$ edges between Γ_i and Γ_j if $i \neq j$.

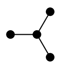
For a quotient singularity X with minimal regular resolution $X^{\text{reg}} \to X$, the dual graph of the singularity is the dual graph of the exceptional divisor of the resolution. Thus, we see that in the hypotheses and notations of Theorem 2.22, the dual graph of a tame cyclic quotient singularity is a Dynkin graph of type A_r.

Example 2.28 (D_4 singularity)**.** Let K be an algebraically closed field of characteristic zero and consider the surface

$$X = \operatorname{Spec} \frac{K[s,t,u]}{(s^2 + t^3 + u^3)},$$

which is a singular surface whose only singular point is the origin; it can be proven ([Reid]) that X arises as a quotient of $\mathbb{A}^2_K$ by the action of the group of order 16 generated by

$$\alpha : p(x,y) \longmapsto p(ix, -iy); \quad \beta : p(x,y) \longmapsto p(y, -x).$$

The blow-up $X_1 \to X$ of X at the origin is covered by three affine charts U_s, U_t, U_u. An immediate computation shows that U_s is regular, while U_t and U_u each have three singular points that are identified by the glueing. The minimal regular resolution is obtained by blowing up these three points; the resulting exceptional divisor is the union of four -2-curves $\Gamma_0, \dots, \Gamma_4$, where $\Gamma_1, \dots, \Gamma_3$ are disjoint and intersect Γ_0 transversally. The dual graph of the singularity is thus a Dynkin graph of type D_4:  . This kind of singularity is called a *du Val singularity* of type D_4.

Chapter 3

Wild quotient singularities

In this chapter we will study some properties of wild quotient singularities. Since the nature of singularities and their resolution is local, we will work in the following setting: let A be a regular local Noetherian ring of dimension 2 and G a finite group acting faithfully on A; then $R = A^G$ is a normal local Noetherian ring with residue field κ and fraction field K. Recall that if R is non-regular and the order of G is divisible by the characteristic of κ, we say that R is a *wild quotient singularity*; it is *tame* if R is non-regular and the order of G is prime to the characteristic exponent of κ. We will look at dual graphs of wild quotient singularities and see that they always contain a *node*, i.e. a vertex of valency ≥ 3. We will then see an example of wild quotient singularity whose dual graph has two nodes.

Compared to tame quotient singularities, wild quotient singularities can be much more difficult to describe, even in the cyclic case. For example, the quotient itself can be difficult to compute even in the simplest cases, and it is unknown if a resolution exists in higher dimensions.

3.1 Étale fundamental group

In this section, we will briefly introduce an important geometrical tool that we will use to prove that dual graphs arising from wild quotient singularities always have at least two nodes: the *étale fundamental group*, or *algebraic fundamental group*. It is an algebraic analogue to the topological fundamental group, defined by using finite étale morphisms as a replacement for covering spaces in general topology.

3.1.1 Topological fundamental group

Recall that in general topology, the fundamental group of a topological space (X, τ) at a point x is defined as the group $\pi_1(X, x)$ of homotopy classes of loops at x, with the operation of concatenation. Since a loop (and in general a path) is a continuous map $([0, 1], \mathcal{E}) \to (X, \tau)$, with $\mathcal{E}$ denoting the Euclidean topology, it is apparent that this tool is of little use when working with the Zariski topology, as it is much coarser than the Euclidean topology. In this section, we will generalize the notion of fundamental group in a way that makes sense for schemes, using the étale topology, which in a way serves as a "refinement" of the Zariski topology.

Let us first recall some facts on fundamental groups and their relationship to covering spaces. A continuous map $p : Y \to X$ is a *covering space* of X if X has an open cover $\{U_i\}_i$ such that for all indices i, $p^{-1}(U_i)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U_i by p. Given two covering spaces (Y, p) and (Y', p') of X, a *morphism of covering spaces* is a map $f : Y \to Y'$ making the following diagram commutative:

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array}$$

Fix a base point $x \in X$. If $p : Y \to X$ is a covering space and $y \in Y$ a point mapping to x, we can consider the subgroup $p_*(\pi_1(Y, y))$ of $\pi_1(X, x)$. If X is path connected and locally path connected, we have that two covering spaces $p : Y \to X$ and $p' : Y' \to X$ are isomorphic via an isomorphism mapping a point $y \in p^{-1}(x)$ to a point $y' \in {p'}^{-1}(x)$ if and only if they induce the same subgroup of $\pi_1(X, x)$ (see [Hat02], 1.37). This yields a *Galois correspondence* between isomorphism classes of covering spaces and subgroups of the fundamental group. If X is also semi-locally simply connected, the function associating to each covering space the corresponding subgroup of $\pi_1(X, x)$ is surjective; in particular, there is a covering space corresponding to the trivial subgroup, i.e. a simply connected covering space $(\widetilde{Y}, \widetilde{p})$. If we fix a point $\widetilde{y} \in \widetilde{Y}$ mapping to x, we have the following universal property of $(\widetilde{Y}, \widetilde{p}, \widetilde{y})$: for all covering spaces $p : Y \to X$ and $y \in Y$ mapping to x, there is a unique morphism of covering spaces $\widetilde{Y} \to Y$ mapping $\widetilde{y}$ to y. In particular, $\widetilde{Y}$ is unique up to unique isomorphism, and it is a covering space for all covering spaces of X; we therefore call it the *universal covering space* of (X, x).

Consider the group $\mathrm{Aut}_X(\widetilde{X})$ of morphisms of covering spaces $\widetilde{X} \to \widetilde{X}$. If $\alpha \in \mathrm{Aut}_X(\widetilde{X})$, then $\alpha\widetilde{x} = x$, so any path connecting $\widetilde{x}$ to $\alpha\widetilde{x}$ will be mapped to a loop in

X. Since $\widetilde{X}$ is simply connected, we then have a well defined isomorphism

$$\mathrm{Aut}_X(\widetilde{X}) \simeq \pi_1(X, x).$$

Now consider the category $\mathbf{Cov}(X)$ of covering spaces of X with a finite number of connected components, with the arrows being the morphisms of covering spaces. Define the functor $F_x : \mathbf{Cov}(X) \to \mathbf{Set}$ mapping a covering space $p : Y \to X$ to $p^{-1}(x)$. By the universal property of the universal covering space, we see that this functor is representable by $\widetilde{X}$, i.e. given a covering space $p : Y \to X$,

$$F_x(Y) = \mathrm{Hom}_X(\widetilde{X}, Y).$$

Since $\mathrm{Aut}_X(\widetilde{X})$ acts on $\mathrm{Hom}_X(\widetilde{X}, Y)$ via $\alpha \cdot f = f \circ \alpha$, we see in particular that F_x is a functor from $\mathbf{Cov}(X)$ to the category of $\mathrm{Aut}_X(\widetilde{X})$ (or $\pi_1(X, x)$)-sets. It can be proven that F_x induces an equivalence of categories between $\mathbf{Cov}(X)$ and the category of $\pi_1(X, x)$-sets with finite orbits, yielding a classification of covering spaces of X with finitely many connected components.

3.1.2 Étale fundamental group

Let us now use the previous properties of the topological fundamental group to define a schematic analogue.

Let X be a connected scheme and let $p : Y \to X$ be a finite étale morphism. Then p has the following properties: it is open ([Stacks], tag 03WT) and closed (because it is finite), thus, if $Y \neq \emptyset$, it is surjective; for all $x \in X$ there is an étale neighborhood $(U, u) \longmapsto (X, x)$ such that $Y \times_X U$ is a disjoint union of affine open subschemes, each of which is mapped isomorphically onto U by $p \times \mathrm{id}$ ([Stacks], tag 04HN). This motivates the following nomenclature:

Definition 3.1 (Étale covering)**.** An *étale covering* of X is a finite étale morphism $p : Y \to X$.

The reader should not confuse an étale covering (in French: *revêtement étale*) with an étale cover, or cover(ing) for the étale topology (in French: *famille couvrante étale*); the former generalizes the notion of a covering space, the latter of an open cover.

Fix a geometric point $\overline{x} : \operatorname{Spec} \overline{k} \to X$ ($\overline{k}$ being an arbitrary algebraically closed field) lying over $x \in X$. Consider the category $\mathbf{FÉt}/X$ of étale coverings of X (the morphisms being the X-morphisms) and the functor $F_{\overline{x}} : \mathbf{FÉt}/X \to \mathbf{Set}$ mapping each covering (Y, p) to the set $\mathrm{Hom}_X(\overline{x}, Y)$ of $\overline{x}$-points of Y lying over x. Contrarily to the previous subsection, in this case the functor $F_{\overline{x}}$ is not, in general, representable, due to the finiteness of the étale coverings; it is however *pro-representable*.

This means that there is a projective family $\widetilde{X} = (X_i)_{i \in I}$ of étale coverings of X indexed by a directed set I, such that for all étale coverings Y (by abuse of notation we omit the étale map p) of X we have

$$F_{\bar{x}}(Y) = \mathrm{Hom}(\widetilde{X}, Y) := \varinjlim_{i \in I} \mathrm{Hom}_X(X_i, Y),$$

functorially in Y. We call such a family $\widetilde{X}$ "the" *universal étale covering* of X.

Example 3.2. Consider the spectrum of a field $X = \operatorname{Spec} K$, and a geometric point $\overline{x} : \operatorname{Spec} \overline{k} \to \operatorname{Spec} K$ where we suppose that $\overline{k}$ is separable over K. An étale covering Y of X can be identified with a finite separable field extension E/K; we have $F_{\bar{x}}(Y) \simeq \mathrm{Hom}_K(E, \overline{k})$ and if $\overline{k}/K$ is finite, then $F_{\bar{x}}$ is representable and we have

$$F_{\bar{x}}(Y) = \mathrm{Hom}_{\mathbf{FÉt}/X}(\overline{k}, Y) = \mathrm{Hom}_X(\overline{k}, Y).$$

In general, however, F will only be pro-representable, with

$$F_{\bar{x}}(Y) = \varinjlim \mathrm{Hom}_X(\operatorname{Spec} L_i, Y)$$

for an indexed family $(L_i)_i$ of finite Galois extensions of K.

In the previous notations, the X_i can always be chosen to be Galois over X, i.e. such that their degree over X is equal to the order of $\mathrm{Aut}_X(X_i)$ (see [Mil13], p. 26). A morphism $X_i \to X_j$ yields a homomorphism $\mathrm{Aut}_X(X_i) \to \mathrm{Aut}_X(X_j)$, so we can define

$$\pi_1(X, \overline{x}) = \mathrm{Aut}_X(\widetilde{X}) := \varprojlim \mathrm{Aut}_X(X_i).$$

This is the *étale fundamental group*, endowed with the canonical topology as a limit of finite discrete groups. In the previous example, the étale fundamental group of X at $\overline{x}$ is isomorphic to the Galois group of the extension $\overline{k}/K$ ([SGA 1], Exposé V, Proposition 8.1).

For each étale covering Y of X, we have a left action of $\pi_1(X, \overline{x})$ on $F(Y)$ induced by the action of $\pi_1(X, \overline{x})$ on $\widetilde{X}$. We have the following theorem:

Theorem 3.3. *In the previous notations:*

i) The functor $F_{\bar{x}}$ defines an equivalence of categories

$$\mathbf{FÉt}/X \longrightarrow (\text{finite } \pi_1(X, \overline{x})\text{-sets}).$$

ii) Given a different geometric point $\overline{x}'$ of X, there is an isomorphism $\eta : F_{\bar{x}} \to F_{\bar{x}'}$, yielding an isomorphism $\pi_1(X, \overline{x}) \to \pi_1(X, \overline{x}')$ compatible with the equivalence in (i). The latter isomorphism is independent of the choice of η, up to inner conjugation.

iii) Given a morphism $f : X \to Y$ of connected schemes, set $\overline{y} = f \circ \overline{x}$. Then f induces a canonical homomorphism

$$f_* : \pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y}),$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \textbf{FÉt}\,/Y & \xrightarrow{\text{base change}} & \textbf{FÉt}\,/X \\ {\scriptstyle F_{\overline{y}}}\downarrow & & \downarrow{\scriptstyle F_{\overline{x}}} \\ \textit{finite } \pi_1(Y, \overline{y})\textit{-sets} & \xrightarrow[f^*]{} & \textit{finite } \pi_1(X, \overline{x})\textit{-sets} \end{array}$$

Proof. [Stacks], tag 0BND. □

In particular, if X is irreducible we can take the étale fundamental group at a geometric point over the generic point of X; for example, if K is the stalk at the generic point, we can take $\overline{x}$ to be the map $\operatorname{Spec} \overline{K} \to X$. In this case, we write $\pi_1(X) := \pi_1(X, \overline{x})$ and call it the *fundamental group* of X. The fundamental group of the regular locus of X is called the *local fundamental group*, denoted by $\pi_1^{\text{loc}}(X)$.

3.2 Dual graphs of wild quotient singularities

Let A be a regular local Noetherian ring of dimension 2, G a finite group acting faithfully on A, $R = A^G$. Assume that the action of G on $\operatorname{Spec} A$ is free away from its closed point x, so that its image in $X = \operatorname{Spec} R$ is the only singular point of X; assume also for simplicity that R is complete and its residue field κ is algebraically closed of characteristic $p \geq 0$. By 2.9, a minimal regular resolution exists; moreover, its dual graph is a tree:

Theorem 3.4. *In the above assumptions, let $\pi : X^{\text{reg}} \to X$ be the minimal regular resolution of X at x and suppose that the exceptional divisor $\pi^{-1}(x)$ has normal crossings and its irreducible components are smooth. Then the associated graph is a tree, and each component is a rational curve.*

Proof. [Lor13], 2.8. □

More specifically, dual graphs of tame quotient singularities are always star-shaped, i.e. they have at most one node, and no node if the group is abelian ([IS15], p. 5). Recall that we saw in the previous chapter that the dual graph of a tame cyclic quotient singularity is a Dynkin graph of type A_r. In general, we say that R is a *Hirzebruch-Jung singularity* if each irreducible component of the exceptional divisor is isomorphic to a projective line, and the corresponding intersection matrix looks as follows:

$$(E_i.E_j)_{i,j} = \begin{pmatrix} -b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -b_2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -b_3 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -b_{r-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -b_r \end{pmatrix},$$

for integers $b_i \geq 2$. In other words, R is a Hirzebruch-Jung singularity if and only if its dual graph has no nodes. We can associate to the singularity the continued fraction

$$\frac{m}{k} = b_1 - \frac{1}{b_2 - \frac{1}{\cdots - \frac{1}{b_r}}},$$

and therefore talk about Hirzebruch-Jung singularity *of type m/k*. We will show that such singularities can only arise in the tame case, whence the dual graph of a wild quotient singularity always has at least one node. As we mentioned earlier, the algebraic fundamental group will play an important role in the proof; therefore, we state the following proposition, generalizing [SGA 1], Exposé V, Proposition 8.1.

Proposition 3.5. *Let $B' \subseteq B \subseteq B''$ be a chain of complete local Noetherian normal domains, with respective fields of fractions $K' \subseteq K \subseteq K''$. Assume the following:*

i) The extensions $B' \subseteq B$ and $B \subseteq B''$ are finite.

ii) The local ring B' has separably closed residue field.

iii) The local ring B'' is regular.

iv) The extension $K' \subseteq K$ is Galois.

v) The ring extension $B' \subseteq B$ is étale in codimension 1.

vi) The ring extension $B \subseteq B''$ is totally ramified at some prime $\mathfrak{q} \subseteq B$ of height 1.

Then there is a natural isomorphism $\pi_1^{\text{loc}}(\operatorname{Spec} B') = \operatorname{Gal}(K/K')$, where $\pi_1^{\text{loc}}(\operatorname{Spec} B')$ is taken with respect to some separable closure of K.

Proof. We will only give an idea of the proof, with omitted details; for a complete version, see [IS15], 2.3. Consider the regular locus $Y' \subseteq \operatorname{Spec} B'$, an open subscheme by [EGA IV], 6.12.7, and let Y be its preimage in $\operatorname{Spec} B$ through the morphism of schemes induced by the inclusion $B' \subseteq B$. We know that there is an equivalence of categories between the category $\mathcal{C}$ of finite étale morphisms to

Y' and the category of finite $\pi_1(Y')$-sets; thus, it can be shown that $\pi_1(Y')$ actually coincides with the automorphism group of the forgetful functor $\mathcal{C} \to \mathbf{Set}$ that forgets about the $\pi_1(Y')$-action. The idea is to show that there is also an equivalence of categories between $\mathcal{C}$ and the category $\mathcal{C}'$ of finite $\mathrm{Gal}(K/K')$-sets. Then, since the Galois group $\mathrm{Gal}(K/K')$ coincides with the automorphism group of the forgetful functor $\mathcal{C}' \to \mathbf{Set}$ forgetting about the Galois action, it will follow that the two groups are equal. We thus consider the functor

$$\begin{aligned} \Psi : \mathcal{C} &\to \mathcal{C}' \\ \widetilde{Y} &\longmapsto \widetilde{Y}(K), \end{aligned}$$

where $\widetilde{Y}(K)$ is the set of K-points of the base change $\widetilde{Y}_{K'} = \widetilde{Y} \times_{Y'} \operatorname{Spec} K'$. This functor is faithful because it is the composition of the faithful functors $\widetilde{Y} \longmapsto \widetilde{Y}_{K'}$ and $\widetilde{Y}_{K'} \longmapsto \widetilde{Y}(K)$. The proofs that Ψ is full and essentially surjective are omitted. □

The remaning part of this section will be dedicated to sketching out a proof of the following theorem:

Theorem 3.6. *Let R be a Hirzebruch-Jung singularity of type m/k, and let $p \geq 1$ be the characteristic exponent of the residue field $\kappa = R/\mathfrak{m}_R$. Then the local fundamental group $\pi_1^{\mathrm{loc}}(R) := \pi_1^{\mathrm{loc}}(\operatorname{Spec} R)$ is isomorphic to the prime-to-p part of the cyclic group $\mathbb{Z}/m\mathbb{Z}$. In particular, it has no element of order p.*

Recall that the prime-to-p part of a group G is the quotient of G by the subgroup generated by all its p-Sylow subgroups.

To motivate why we should want to prove the previous statement on local fundamental groups, consider our initial setting where A is a regular local Noetherian ring of dimension 2 and G is finite a group acting faithfully on A, such that the action is free away from the closed point of $\operatorname{Spec} A$. Then the hypotheses of the previous proposition are satisfied for $B' = R = A^G$, $B = B'' = A$. The action of G on A extends trivially to its fraction field K and fixes the fraction field K' of R. Thus, G acts on K as a subgroup of the Galois group $\mathrm{Gal}(K/K')$; since the action is faithful, it is clear that if $\mathrm{Gal}(K/K')$ has no element of order p, then G must have no element of order p either. It follows then by the previous theorem and the previous proposition that the dual graph of a wild quotient must have at least one node:

Corollary 3.7. *The dual graph of a wild quotient singularity in dimension 2 contains at least one node.*

Proof of Theorem 3.6 (Sketch). We will sketch out only the fundamental steps of the proof of this theorem, which uses ideas from logarithmic algebraic geometry; for a complete proof, see Section 3 of [IS15]. The idea is to find suitable rings B and B'' such that the triplet $R = B' \subseteq B \subseteq B''$ satisfies the hypotheses of Proposition 3.5. Consider then the minimal regular resolution $f : X^{\mathrm{reg}} \to \operatorname{Spec} R$ of the Hirzebruch-Jung singularity and let $E = E_1 + \ldots + E_r$ be its reduced exceptional divisor. By loc. cit., 3.1, E coincides with the closed fiber of the resolution. Pick κ-points (recall that κ is the separably closed residue field of the complete ring R) $x \in E_1 \smallsetminus E_2$ and $x' \in E_r \smallsetminus E_{r-1}$, and extend them to effective Cartier divisors D, D' in X^{reg} such that $D \cap E_1 = \{x\}$ and $D' \cap E_r = \{x'\}$. Then, their images $C = f(D)$ and $C' = f(D')$ are Weil divisors on $\operatorname{Spec} R$. Regard R as a commutative multiplicative monoid and consider its submonoid

$$M = \{g \in R \mid V(g) \subseteq C \cup C'\}.$$

Define a homomorphism of monoids

$$\begin{aligned} \nu : M &\to \mathbb{N} \oplus \mathbb{N} \\ g &\longmapsto \left(\mathrm{val}_C(g), \mathrm{val}_{C'}(g)\right), \end{aligned}$$

with valuations taken at the generic points of C and C', and let $P = \operatorname{Im} \nu \subseteq \mathbb{N} \oplus \mathbb{N}$ be its image. Then we have an "exact" sequence

$$1 \to R^\times \to M \to P \to 0,$$

where exact in this instance means that P is isomorphic to the quotient $M/\!\sim$ by the equivalence relation $g \sim g' \iff g = ug'$ for some $u \in R^\times$.
Recall that a commutative monoid P is said to be:

- *fine* if it is finitely generated and integral;
- *saturated* if any element a in its groupification P^{gp} that has a multiple in P is itself an element of P (similar to a saturated semigroup);
- *sharp* if the subset $P^\times$ of invertible elements of P is trivial.

Then, it can be proven (loc. cit., 3.4) that P is fine, saturated and sharp, and there exists a section $P \to M$. Once we choose such a section, we can write $M = R^\times \oplus P$ and identify P with a multiplicative submonoid of R. Recall that any (commutative) monoid has a unique maximal ideal consisting of its non invertible elements; since P is sharp, this coincides with $P \smallsetminus \{1\}$. By loc. cit., 3.6, for any choice of a section $M \to P$, the maximal ideal $P \smallsetminus \{1\}$ of P generates the maximal ideal $\mathfrak{m}_R \subseteq R$ of R.
Now we have two possible cases: either R contains a field, or it does not. In the former case, let $W \subseteq R$ be a subfield of R such that $W \to \kappa$ is bijective; in the latter case, let $W \subseteq R$ be a subring such that W is a complete discrete valuation ring

with uniformizer $p \in W$ (recall that $p \geq 1$ is the characteristic exponent of κ), and $W/pW \to \kappa$ is bijective. In either case, the above assertion on the maximal ideal of P implies that there is a surjection $W[[P]] \twoheadrightarrow R$, where, if $I \subseteq W[P]$ is the ideal generated by $P \smallsetminus \{1\}$, $W[[P]]$ is the I-adic completion of the monoid ring $W[P]$:

$$W[[P]] = \varprojlim \frac{W[P]}{I^n W[P]}.$$

An element of $W[[P]]$ is a formal series of the form $\sum_{g \in P} w_g g$ with $w_g \in W$; in particular, it can be shown (loc. cit., 3.7) that R is a quotient of $W[[P]]$ by the principal ideal generated by an element of the form

$$\psi = \begin{cases} p + \sum_{g \in P \smallsetminus \{1\}} w_g g & \text{if } R \text{ does not contain a field;} \\ 0 & \text{otherwise.} \end{cases}$$

Now let P^{gp} denote the groupification of P and let Q^{gp} be the unique intermediate subgroup $P^{\mathrm{gp}} \subseteq Q^{\mathrm{gp}} \subseteq \mathbb{Z} \oplus \mathbb{Z}$ such that $[\mathbb{Z} \oplus \mathbb{Z} : Q^{\mathrm{gp}}]$ is a power of p and $[Q^{\mathrm{gp}} : P]$ is prime to p; let Q be a monoid making the square on the right in the following diagram cartesian:

$$\begin{array}{ccccc} P & \longrightarrow & Q & \longrightarrow & \mathbb{N} \oplus \mathbb{N} \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ P^{\mathrm{gp}} & \longrightarrow & Q^{\mathrm{gp}} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

Like P, Q is also sharp, fine and saturated.
We have now finally found our "suitable" rings: define

$$\begin{aligned} B &:= R \otimes_{W[[P]]} W[[Q]] = W[[Q]]/(\psi); \\ B'' &:= R \otimes_{W[[P]]} W[[\mathbb{N} \oplus \mathbb{N}]] = W[[\mathbb{N} \oplus \mathbb{N}]]/(\psi). \end{aligned}$$

The triplet $R = B' \subseteq B \subseteq B''$ satisfies the hypotheses of Proposition 3.5 by loc. cit., 3.8; thus, if K is the fraction field of B and K' the fraction field of R, we have $\pi_1^{\mathrm{loc}}(R) = \mathrm{Gal}(K/K')$.
We are now ready to complete the proof. Let $\mathbb{Z}/m'\mathbb{Z}$ be the prime-to-p part of $\mathbb{Z}/m\mathbb{Z}$; by definition of Q^{gp}, we have an isomorphism $\mathbb{Z}/m'\mathbb{Z} \simeq Q^{\mathrm{gp}}/P^{\mathrm{gp}} =: H$. Let G be the group scheme representing the **Set**-valued functor on R-algebras mapping $\mathcal{A} \longmapsto \mathrm{Hom}_{\mathbf{Grp}}(H, \mathcal{A}^\times)$; since m' is invertible in W and the residue field κ_W is separably closed, G actually coincides with the finite group scheme $\mathbb{Z}/m'\mathbb{Z}$. It can be shown that the morphism $\operatorname{Spec} K \to \operatorname{Spec} K'$ is a G-torsor for the étale topology on $\operatorname{Spec} R$, so we see that K is fixed by the G-action and that $[K : K'] = |G| = m'$. It follows then that $K' \subseteq K$ is Galois with $\mathrm{Gal}(K/K') \simeq \mathbb{Z}/m'\mathbb{Z}$; by Proposition 3.5, we conclude that $\pi_1^{\mathrm{loc}}(R) \simeq \mathbb{Z}/m'\mathbb{Z}$. □

3.3 Dual graphs with two nodes

In this section, we will present an example of a wild quotient singularity whose dual graph is not star-shaped, and in particular has (at least) two nodes, following [IS15].

Let $p \geq 0$ be a prime number, κ an algebraically closed field of characteristic p and fix a p-power $q = p^m$. Consider the following smooth projective curve of affine equation $C : y^q - y = x^{q+1}$, so

$$C = \operatorname{Proj} \frac{\kappa[x_0, x_1, x_2]}{\left(x_1^q x_2 - x_1 x_2^q - x_0^{q+1}\right)}.$$

By [Har77], Chapter IV, 3.11.1, C has genus $g = \frac{q(q-1)}{2}$. Curves of this type are called *Hermitian curves* and are known to have extremely large automorphism groups, thus behaving pathologically with respect to rational points; in particular, by [Sti73], Theorem 7, in this case Aut(C) has order $q^3(q^3+1)(q^2-1)$ if $q \neq 2$. We wish to find a suitable subgroup of Aut(C) to act on C. For this purpose, let us perform a change of variables; we can work with the affine equation $y^q - y = x^{q+1}$ to define automorphisms on an affine chart, which will naturally extend to automorphisms on C that fix the point at infinity. Set

$$x = x' + r; \quad y = y' + sx' + t$$

for scalars r, s and t, whence our equation becomes

$$\begin{aligned} &(y' + sx' + t)^q - (y' + sx' + t) = (x' + r)^{q+1} \\ \iff &y'^q + s^q x'^q + t^q - y' - sx' - t = x'^{q+1} + r^{q+1} + \sum_{i=1}^{q} \tbinom{q+1}{i} x'^i r^{q+1-i}. \end{aligned}$$

For $2 \leq i \leq q-1$, the binomial coefficient $\binom{q+1}{i}$ is zero in κ, so we get the equation

$$y'^q - y' = (s + r^q)x' + (r - s^q)x'^q + x'^{q+1} + r^{q+1} - t^q + t.$$

Since we want our new variables to satisfy the original affine equation for C, we require that

$$\begin{cases} s + r^q = 0 \\ r - s^q = 0 \\ r^{q+1} - t^q + t = 0 \end{cases} \implies \begin{cases} s = -r^q \\ r^{q^2} + r = 0 \\ r^{q+1} - t^q + t = 0. \end{cases}$$

Thus, we define the group $G = \{(t, r) \in \kappa^2 \,|\, r^{q^2} + r = 0, r^{q+1} = t^q - t\}$ with composition law $(t, r) \cdot (t', r') = (t + t' - r^q r', r + r')$, acting on C via

$$(t, r) : x \longmapsto x + r; \quad (t, r) : y \longmapsto y - r^q x + t.$$

Proposition 3.8. *The group G has order $|G| = q^3$; if $q \neq 2$, it is a p-Sylow subgroup of* Aut(C).

Proof. Since κ is algebraically closed and the polynomials $T^{q^2} + T$ and $T^q - T - \lambda$ are separable, it follows that $|G| = q^2 \cdot q = q^3$. Since we have observed earlier that when $q \neq 2$, Aut(C) has order $q^3(q^3+1)(q^2-1)$, in this case G is a p-Sylow of Aut(C). □

Let Z denote the center of G and G' its derived subgroup. Consider also the following subgroup of G:

Definition 3.9 (Frattini subgroup)**.** Let G be a group. The *Frattini subgroup* is the intersection Φ of all maximal proper subgroups of G.

The Frattini subgroup can be thought of as the subgroup of all "small elements" of G, i.e. all the non-generators of the group.

Definition 3.10 (Special group)**.** A *special group* is a p-group that is either elementary abelian, or it is of class 2 and its derived subgroup, Frattini subgroup and center all coincide and are elementary abelian.

Proposition 3.11. *The group G defined above is a special group with center*

$$Z = \Phi = G' = \{(t, 0) \mid t \in \mathbb{F}_q\}.$$

Proof. It is immediate to check that the centralizer of an element $(t, r) \in G$ is the set of all $(t, \lambda r)$ for $\lambda \in \mathbb{F}_q$, whence $Z = \{(t, 0) \mid t \in \mathbb{F}_q\}$ and it is elementary abelian. By [Mac68], 9.26, for finite p-groups

$$\Phi = \langle [x, y], z^p \mid x, y, z \in \Phi \rangle\,;$$

so we have the inclusion $G' \subseteq \Phi$ and $\Phi \subseteq Z$ (because G/Φ is elementary abelian). Finally, observe that G' is a nontrivial subspace of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^2$, whence, since $G' \subseteq Z$ and Z is a one-dimensional subspace of $\mathbb{F}_q^2$, $G' = Z$. □

Let us return to our group action on C. Observe that G acts freely on the affine part of C and fixes the point at infinity ∞. Consider the diagonal action of G on the smooth proper surface $C \times C$ and take the quotient $Y = (C \times C)/G$ by this action, which is a normal proper surface whose singular locus coincides with the image y of the fixed point (∞, ∞). In this section, we shall sketch out a proof for the following theorem:

Theorem 3.12. *The dual graph for the minimal regular resolution of the quotient singularity $y \in Y$ contains at least two nodes.*

Since we are working with a wild quotient singularity, it can be difficult to study the resolution of this singularity explicitly. We shall therefore consider a different, more manageable morphism. Observe first that $C/G \simeq \mathbb{P}^1_\kappa$, which follows from the Riemann-Hurwitz formula for computing the genus of a curve ([Har77], Chapter IV, 2.4):

$$2g(C) - 2 = |G| \left(2g(C/G) - 2\right) + \sum_{P \in C} d_P,$$

In this case:

$$\begin{aligned}
&2\frac{q(q-1)}{2} - 2 = q^3 \left(2g(C/G) - 2\right) + \sum_{P \in C} \underbrace{d_P}_{\substack{=0 \\ \text{for all } P \neq \infty}} \\
\iff & q^2 - q - 2 = 2q^3 g(C/G) - 2q^3 + 2q^3 - 2 + \varepsilon \\
\iff & g(C/G) = \frac{q^2 - q - \varepsilon}{2q^3} = \frac{1}{2}\left(\frac{1}{q} - \frac{1}{q^2} - \frac{\varepsilon}{q^3}\right) \leq \frac{1}{2q} \\
\Longrightarrow & g(C/G) = 0.
\end{aligned}$$

We thus see that C/G is a smooth projective curve of genus 0, and it is therefore isomorphic to $\mathbb{P}^1_\kappa$. Since the composition $C \times C \xrightarrow{\mathrm{pr}_1} C \to C/G$ is G-invariant, we have a unique factorization through $\varphi : (C \times C)/G \to C/G$ making the following diagram commutative:

$$\begin{array}{ccccc}
C \times C & \xrightarrow{\mathrm{pr}_1} & C & \longrightarrow & C/G \\
& \searrow & & \nearrow_{\varphi} & \\
& & {}^{C \times C}\!/_{G} & &
\end{array}$$

Now we consider the following commutative diagram

$$\begin{array}{ccccc}
& & \widetilde{Y} & & \\
& \swarrow & & \searrow & \\
Z & & & & Y = {}^{C \times C}\!/_{G} \longleftarrow C \times C \\
& \searrow^{\psi} & & \swarrow^{\varphi} & \qquad\qquad \Big\downarrow \mathrm{pr}_1 \\
& & \mathbb{P}^1_\kappa = G/C & \longleftarrow & \qquad\qquad C
\end{array}$$

where $\widetilde{Y} \to Y$ is the minimal resolution of singularities and $\psi : Z \to G/C$ is obtained by contracting all the (-1)-curves in the fiber of $\widetilde{Y} \to G/C$ over ∞ (where by a slight abuse of notation we call ∞ the image of the point at infinity in G/C).

As we mentioned above, we shall work on the morphism ψ to deduce the properties of the dual graph of the minimal resolution of singularities $\widetilde{Y} \to Y$. Indeed, Theorem 3.12 is an immediate consequence of the following proposition:

Proposition 3.13. *The reduced singular fiber $\psi^{-1}(\infty)_{\text{red}} \subseteq Z$ is a divisor with strictly normal crossing whose irreducible components are copies of $\mathbb{P}^1_\kappa$, with dual graph as in Figure 3.1, where:*

- *there are q strings on the right side of the graph, each of length $q-1$;*
- *the long string on the left has length $q-1$ as well;*
- *the numbers indicate the multiplicities of the integral irreducible components in the schematic fiber $\psi^{-1}(\infty)$;*
- *the black vertices correspond to integral components of self-intersection -2, while the white vertices correspond to integral components of self-intersection $-q$.*

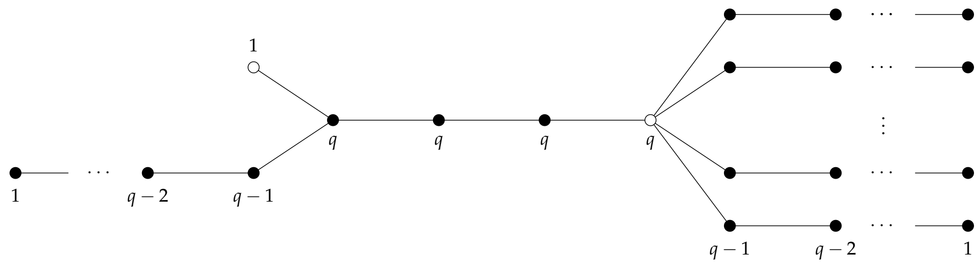


Figure 3.1: The dual graph of $\psi^{-1}(\infty)$.

Proof of Theorem 3.12. The fiber over ∞ through the morphism $\widetilde{Y} \to \mathbb{P}^1_\kappa$ is the union of the strict transform $F \subseteq \widetilde{Y}$ of the fiber over ∞ of $\varphi : Y \to \mathbb{P}^1_\kappa$, and the exceptional divisor $E \subseteq \widetilde{Y}$ of the resolution of singularities $\widetilde{Y} \to Y$. By [IS12], 2.1, the integral divisor F has multiplicity q^3 in the schematic fiber; since by the previous proposition $\psi^{-1}(\infty)$ has no integral component of multiplicity greater than q, it means that F has been contracted by the map $\widetilde{Y} \to Z$. This means that the dual graph of $\psi^{-1}(\infty)$ is obtained by the dual graph of E by a series of vertex contractions. Since the former graph has exactly two nodes, it follows that the dual graph of E has at least two nodes. □

We will now sketch out a proof of Proposition 3.13 to complete the proof of Theorem 3.12. Some details are omitted; for a complete proof, see [IS15].

Proof of Proposition 3.13 (Sketch). By [IS15], 7.4, if $\eta \in \mathbb{P}^1_\kappa$ is the generic point of $\mathbb{P}^1_\kappa$ and z a uniformizer of the local ring $\mathcal{O}_{\mathbb{P}^1_\kappa,\infty}$, the generic fiber $\varphi^{-1}(\eta) = \psi^{-1}(\eta)$ is the smooth projective curve over $\kappa(z)$ obtained by the homogenization of the affine equation

$$y^q - z^{q^2-1}y - x^{q+1} - z^{q-1}x^q = 0. \tag{3.14}$$

An immediate computation shows that the only singular point of this curve lies in its affine part; we will therefore compute the minimal regular resolution of the affine scheme

$$X = \operatorname{Spec} \frac{k[z,x,y]}{(y^q - z^{q^2-1}y - x^{q+1} - z^{q-1}x^q)},$$

which we will see has no (-1)-curve in its singular fiber, thus yielding our desired description of the fiber $\psi^{-1}(\infty)$. Let us first consider the case where $q = 2$: then 3.14 becomes

$$y^2 - z^3y - x^3 - zx^2 = 0,$$

which is an elliptic surface; the morphism $\psi : S \to \mathbb{P}^1_\kappa$ is then an elliptic fibration whose singular fiber has Kodaira symbol I_3^* by [Lan94], page 429, case 13C, corresponding to a Dynkin graph of type D_7 (see [Kod63], 6.2), as desired.
Now let $q \neq 2$. It might be convenient to label some vertices in Figure 3.1 to make things clearer, as in Figure 3.2.

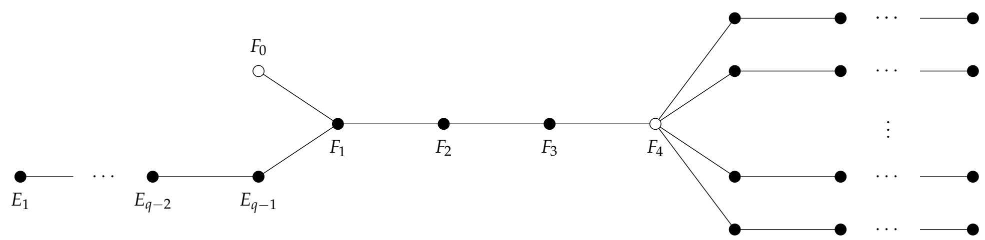


Figure 3.2: The dual graph of $\psi^{-1}(\infty)$, with labeled vertices.

We will start by computing the blow-up of X at the non reduced singular point (z^{q-1}, x, y), and then blow up again any singularities that may arise.
Consider first the z^{q-1}-chart of the blow-up, which has variables z, $Y := y/z^{q-1}$ and $X := x/z^{q-1}$; we can then substitute $x = X \cdot z^{q-1}$, $y = Y \cdot z^{q-1}$ and obtain the equation

$$Y^q - z^{2q-2}Y - X^{q+1}z^{q-1} - X^q z^q = 0. \tag{3.15}$$

The fiber over ∞ is obtained by requiring $z = 0$, yielding an affine line with multiplicity q, which coincides with the singular locus of this chart; we will call this component F_1.
The y-chart of the blow-up has variables z, $Z := z^{q-1}/y$, $\widetilde{X} := x/y$, and y, and it is defined by the equations

$$z^{q-1} = Zy, \quad 1 - Z^{q+1}y^2 - \widetilde{X}^{q+1}y - Zy\widetilde{X}^q = 0,$$

so by the Jacobian criterion we see that it is smooth. The fiber $z = 0$ here yields two irreducible components $qF_1 + F_0$.

Lastly, the x-chart has variables z, $\widetilde{Z} := z^{q-1}/x$, $\widetilde{Y} := y/x$, and y, and it is defined by the equations

$$z^{q-1} = \widetilde{Z}x, \quad \widetilde{Y}^q - \widetilde{Z}^{q+1}\widetilde{Y}x^2 - x - \widetilde{Z}x = 0.$$

The fiber over ∞ in this chart is given by $\widetilde{Z}x = 0$, yielding the equation $\widetilde{Z} \cdot \widetilde{Y}^q = 0$, whence we have again F_1 with multiplicity q and a new component that we will call E_1. It can be proven, by computing the completion of the local ring at the origin, that there is a Hirzebruch-Jung singularity at the origin of parameters $m = q-1$, $k = q-2$. We thus have the following continued fraction, with $q-1$ 2's appearing in the expansion:

$$\frac{q-1}{q-2} = 2 - \frac{1}{2 - \frac{1}{\cdots - \frac{1}{2}}}.$$

It follows that by computing successive blow-ups to obtain a minimal resolution, we end up with $q-2$ new irreducible components $E_2, \ldots, E_{q-1}$ with multiplicities as in Figure 3.2.

Let us now get back to the z^{q-1}-chart and compute the blow-up of 3.15 along the ideal (z, Y). In the Y-chart, we have variables X, Y and $Z := z/Y = z^q/y$, with the law $z = Z \cdot Y$. The strict transform of 3.15, then, is

$$Y - Z^{q-1}Y^q - X^{q+1}Z^{q-1} + X^qZ^{q-1} = 0,$$

which is smooth. The closed fiber on this chart is obtained by requiring $Z \cdot Y = 0$, yielding $Z^{q-1}X^q(X-1) = 0$; therefore, it has three irreducible components $qF_1 + qF_2 + F_0$. When we now look at the z-chart of the blow-up, we have variables z, X and $Y/z = y/z^q$, and the equation

$$\left(\frac{Y}{z}\right)^q z - z^q\frac{Y}{z} - X^{q-1} - X^q = 0, \tag{3.16}$$

which has a single isolated singularity at the origin. The fiber $z = 0$ is given by the ring $\kappa[X, Y/z]$ modulo the equation $X^q(X-1)$, showing that there are no intersections that we have not already considered.

We now blow up 3.16 along the ideal (z, X). The z-chart has variables z, X/z, and Y/z, the strict transform of 3.16 is given by

$$\left(\frac{Y}{z}\right)^q - z^{q-1}\frac{Y}{z} - \left(\frac{X}{z}\right)^{q+1} z^q - z^{q-1}\left(\frac{X}{z}\right)^q = 0, \tag{3.17}$$

and the fiber $z = 0$ is an affine line with multiplicity q, corresponding to the irreducible component qF_3. A similar computation for the X-fiber shows that there are no more singularities and irreducible components to list.

Finally, we will blow up 3.17 along the ideal $(z, Y/z)$. First we look at the z-chart: it has variables z, X/z, and Y/z^2, equation

$$\left(\frac{Y}{z^2}\right)^q z - z\frac{Y}{z} - \left(\frac{X}{z}\right)^{q+1} z - \left(\frac{X}{z}\right)^q = 0,$$

and fiber $z = 0$ isomorphic to a copy of an affine line with multiplicity q: this is the irreducible component qF_4. The Jacobian criterion shows that the singular locus here is given by the q points satisfying $z = X/z = 0$ and $(Y/z)^{q+1} - Y/z = 0$. Like before, computing the formal completion of the local rings at these singular points shows that each point is a rational double point of type A_{q-1}, yielding the q strings on the right in Figure 3.2. Lastly, an immediate computation shows that the Y/z-chart of the blow-up is smooth and no new irreducible components show up in its fiber. This concludes our proof that the dual graph of $\psi^{-1}(\infty)$ looks like the one in Figure 3.1.

Now that we have established the shape of the dual graph of $\psi^{-1}(\infty)$, we can compute the self-intersection numbers of its irreducible components, using that $\psi^{-1}(\infty)$ has self-intersection zero. Indeed, $\psi^{-1}(\infty)$ is equal to the sum

$$\sum_{i=1}^{q-1} iE_i + \sum_{i=0}^{4} F_i + \sum_{i=1}^{q-1}\sum_{j=1}^{q} iD_{i,j},$$

where the $D_{i,j}$ correspond to the vertices on the right side of the graph, with $D_{i,j}$ appearing with multiplicity i. From $\psi^{-1}(\infty).\psi^{-1}(\infty) = 0$ we get:

- $E_1.E_1 + 2E_2.E_1 = 0 \Longrightarrow E_1.E_1 = -2;$
- for all $2 \leq i \leq q-1$,

$$(i-1)E_{i-1}.E_i + iE_i.E_i + (i+1)E_{i+1}.E_i = 0 \Longrightarrow E_i.E_i = -2;$$

- $F_0.F_0 + qF_1.F_0 = 0 \Longrightarrow F_0.F_0 = -q;$
- $(q-1)E_{q-1}.F_1 + qF_1.F_1 + F_1.F_0 + qF_2.F_1 = 0 \Longrightarrow F_1.F_1 = -2;$
- for $i = 2, 3$,

$$qF_{i-1}.F_i + qF_i.F_i + qF_{i+1}.F_i = 0 \Longrightarrow F_i.F_i = -2;$$

- $qF_3.F_4 + qF_4.F_4 + \sum_{j=1}^{q}(q-1)D_{q-1,j}.F_4 = 0 \Longrightarrow F_4.F_4 = -q;$
- for all $1 \leq j \leq q$,

$$qF_4.D_{q-1,j} + (q-1)D_{q-1,j}.D_{q-1,j} + (q-2)D_{q-2,j}.D_{q-1,j} = 0$$
$$\Longrightarrow D_{q-1,j}.D_{q-1,j} = -2;$$

- for all $2 \leq i \leq q-2$,

$$(i-1)D_{i-1,j}.D_{i,j} + iD_{i,j}.D_{i,j} + (i+1)D_{i+1,j}.D_{i,j} = 0 \Longrightarrow D_{i,j}.D_{i,j} = -2;$$

- for all $1 \leq j \leq q$,

$$2D_{2,j}.D_{1,j} + D_{1,j}.D_{1,j} = 0 \Longrightarrow D_{1,j}.D_{1,j} = -2.$$

The previous computations conclude the proof. □

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