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ON REPRESENTATIONS OF QUIVERS AND KAC'S THEOREM

Tesi di Laurea in Teoria delle Rappresentazioni

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Introduction

A quiver is an oriented graph. A representation of a quiver assigns a vector space V_i to each vertex i of the quiver and a linear map from V_i to V_j to each edge oriented from i to j. Representations of quivers are powerful mathematical objects. After the work of [Gab72] it has become clear that many problems in linear algebra can be interpreted within the theory of representations of quivers. A simple example of this fact is given by a quiver with one vertex and one loop. The representations of this quiver encode the problem of classifying the endomorphisms of a given vector space.

In [Gab72], Gabriel characterized quivers with a finite number of isomorphism classes of indecomposable representations. He showed that these correspond to the quivers whose underlying graph is an ADE Dynkin diagram. Moreover, he discovered a remarkable connection between their representations and the positive roots associated with the Dynkin diagram. Notably, this correspondence is independent of the orientation of the quiver. Subsequently, in [BGP73], Bernstein, Gelfand and Ponomarev introduced the reflection functors. These allow to constructively find the indecomposable representations associated with positive roots, starting from the irreducible representations, which are associated with simple roots. In addition they act on the dimension vector of a representation via an element of the Weyl group associated with the Dynkin diagram.

In view of Gabriel's Theorem 2.2.1 it is natural to wonder what happens to quivers which are not of ADE type. The first case to investigate is the so-called tame case, corresponding to affine Dynkin diagrams: various techniques have been developed to study the tame case, but two main problems appeared in the theory. First, to apply the reflection functors, there must exist an admissible vertex, which is a condition on the orientation. The second problem is that not all the roots of the Dynkin diagram can be obtained from simple roots through the action of the Weyl group.

In [Kac80], and later in [Kac82] Kac addressed these problems, as well as the more general (wild) one to loop-free arbitrary quivers, giving a far-reaching generalization of Gabriel's theorem. In this work we investigate the techniques used by Kac in the proof of Kac's Theorem 4.3.2.

In the first chapter we study the structure of the root system associated to a Dynkin diagram. The root system is a combinatorial object that has a great impact on the representations of a quiver, as shown in Gabriel's Theorem 2.2.1. For the proof of Kac's theorem we are interested in the so-called simply-laced Dynkin diagrams, but in the first chapter we consider the general theory of Dynkin diagrams, because they are intrinsically interesting mathematical objects.

First, we study the structure of Dynkin diagrams using the correspondence with the generalized Cartan matrices. Following [Kac90] we give the classification of the Dynkin diagrams of positive and zero type in Theorem 1.1.19.

Then, we define the root system of a Dynkin diagram. To do so, we associate Lie algebras, the so-called Kac-Moody Lie algebras, to Dynkin diagrams. These Lie algebras come together with their root space decomposition and then we will define the root space of the Dynkin diagram as the root space of the associated Kac-Moody Lie algebra. Using techniques of Lie theory, we study some properties that characterize the root systems. We decompose the root system into real roots, obtained by the simple roots acting with the Weyl group, and imaginary roots. We prove that these are generated, by acting with the Weyl group by vectors in the fundamental chamber M. We observe that the Dynkin diagrams of positive type admit only real roots, unlike Dynkin diagrams of zero and negative type, which admit both real and imaginary roots. Then one can define a bilinear form $(\cdot,\cdot)$ associated with the Dynkin diagram.

In the second chapter we introduce the fundamental concepts of the theory of quiver representations. We define the space of representations with fixed dimension vector and we characterize the indecomposable representations in terms of their endomorphism algebra. We recall Gabriel's Theorem 2.2.1 and we presented the example of the 2-Kronecker quiver, a motivating example to generalize Gabriel's Theorem 2.2.1.

In the third chapter we recall the definition of the reflection functors along with their main properties. We then focus on solving the problem of the orientation described above. This is done over finite fields, where it is possible to count the orbits for the action of a group. In fact, we go further. In Lemma 3.2.7 we show that the orbits of an algebraic group G on a vector space $V_1 \oplus V_2$, where the maximal $\mathbb{F}$-split torus of the stabilizer is conjugate to a given $\mathbb{F}$-split torus T, can be related to the orbits satisfying the same property under the action of G on $V_1 \oplus V_2^*$. In the latter case, G acts on V_2^* via the dual (or contragradient) representation.

This observation, combined with the characterization of indecomposable representations Lemma 2.1.19, and the structure of the space $M^\alpha(Q,\mathbb{F})$ of the representations of Q with fixed dimension vector α, allows us, assuming we are working over a finite field, to change the orientation of the quiver without changing the number of indecomposable representations. This result is stated in Lemma 3.2.9.

In the fourth and last chapter, relying on some classical results in algebraic geometry, we prove that the dimension of the space of isomorphism classes of indecomposable representations with fixed dimension vector $\alpha \in M$ is $1 - (\alpha, \alpha)$, where $(\cdot, \cdot)$ is the bilinear form associated to the Dynkin diagram introduced in the first chapter ([Cra92], [Naz73], [Kac80], [Kac82]).

Finally, at the end of the chapter, we prove Kac's theorem in the case of finite fields. Then we apply an argument of reduction modulo p to obtain the statement of the theorem in the case of algebraically closed fields.

Contents

Introduction **i**

1 Dynkin Diagrams, Kac-Moody Algebras and Root Systems **1**

1.1 Cartan matrices and Dynkin diagrams 1

1.2 Kac-Moody algebras and roots system 14

1.3 Real and imaginary roots 23

2 Representations of Quivers **31**

2.1 Quivers and Indecomposable Representations 31

2.2 Gabriel's Theorem 35

3 Orientation and Representations **39**

3.1 Reflection Functors 39

3.2 Independence from the Orientation 42

4 The Space of Representations and Kac's Theorem **47**

4.1 Preliminaries of Algebraic Geometry 47

4.2 Representation With Fixed Dimension Vector 49

4.3 Kac's Theorem 54

4.3.1 Counting $\mathbb{F}_q$-rational points 56

Bibliography **59**

Chapter 1

Dynkin Diagrams, Kac-Moody Algebras and Root Systems

In this chapter, we study Dynkin diagrams and their associated root systems. We begin by dividing Dynkin diagrams into three categories: positive, zero, and negative type. We then classify those of positive and zero type. Subsequently, we associate a root system to each Dynkin diagram. This association is constructed via the Lie algebras corresponding to the diagrams, namely, the Kac–Moody Lie algebras. Using techniques from Lie theory, we analyze the structure of the root systems associated with Dynkin diagrams.

1.1 Cartan matrices and Dynkin diagrams

Definition 1.1.1. Let A and B be matrices in $M_n(\mathbb{R})$. We say that A is equivalent to B if there exists a permutation matrix $\tau \in M_n(\mathbb{R})$ such that $\tau A \tau^{-1} = B$.

We say that A is decomposable if A is equivalent to a matrix of the form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

We say that A is equivalent to the direct sum A_1 and A_2. We say A is indecomposable if it is not decomposable.

Example 1.1.2. Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 5 & 0 & -1 \end{pmatrix}.$$

We observe that A is decomposable, indeed

$$\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}3&0&2&0\\0&1&0&0\\0&0&3&0\\0&5&0&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}3&2&0&0\\0&3&0&0\\0&0&1&0\\0&0&5&-1\end{pmatrix}.$$

It follows that A is equivalent to a block diagonal matrix.

Remark 1.1.3. We observe that the notion of equivalence is strictly stronger than that of similarity: indeed, equivalence implies similarity, but the converse does not hold. A counterexample is provided by the matrices

$$A=\begin{pmatrix}0&2\\0&0\end{pmatrix};\ B=\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

We have that A and B are similar, since

$$\begin{pmatrix}1/2&0\\0&1\end{pmatrix}\begin{pmatrix}0&2\\0&0\end{pmatrix}\begin{pmatrix}2&0\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

However, A and B are not equivalent, since their sets of entries are different.

Definition 1.1.4. Let $A=(a_{ij})\in M_n(\mathbb{R})$. Consider the following properties:

$$a_{ij}\le 0 \text{ for } i\ne j \text{ and } a_{ij}=0 \text{ implies } a_{ji}=0 \tag{C1}$$

$$a_{i_1i_2}\cdots a_{i_{s-1}i_s}a_{i_si_1}=a_{i_1i_s}a_{i_si_{s-1}}\cdots a_{i_2i_1},\ \text{for any set of indices } i_1,\dots,i_s. \tag{S1}$$

We say that A is symmetrizable if Equation (S1) holds and $a_{ij}=0$ implies $a_{ji}=0$ for every $i,j=1,\dots,n$.

We say that A is a generalized Cartan matrix if

- $a_{ij}\in\mathbb{Z}$;
- Equation (C1) holds;
- $a_{ii}=2$.

In the following, for a vector $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, we will write $x\ge 0$ if $x_i\ge 0$ for every $i=1,\dots,n$. Similarly, we will write $x>0$, $x<0$ and $x\le 0$.

Lemma 1.1.5. *Let $A=(a_{ij})$ be an arbitrary real matrix of size $m\times s$ such that there does not exist a vector $u\ge 0$, $u\ne 0$, such that $A^Tu\ge 0$, then there exists $v>0$ such that $Av<0$.*

Lemma 1.1.6. *Let $A = (a_{ij})$ be an indecomposable real matrix that satisfies Equation (C1). Then if $x \in \mathbb{R}^n$, $x \geq 0$, $x \neq 0$ and $Ax \geq 0$ it follows that $x > 0$.*

Proof. We can suppose, up to permutation, that $x_i = 0$ for $i \leq k$ and $x_j \neq 0$ for $j > k$. From $Ax \geq 0$ and Equation (C1) it follows that $a_{ij} = a_{ji} = 0$ for every $i \leq k$ and $j > k$, but this contradicts the indecomposability. □

Theorem 1.1.7. *Suppose that A is an indecomposable matrix that satisfies Equation (C1); then exactly one of the following holds:*

P) A is non-singular and the following holds:

$$Ax \geq 0 \implies x > 0 \text{ or } x = 0$$

Z) A has rank $= n - 1$ *and there exists a vector $x > 0$ such that $Ax = 0$, moreover, we have:*

$$Ax \geq 0 \implies Ax = 0$$

N) There exists a vector $x \geq 0$ such that $Ax < 0$, moreover we have:

$$x \geq 0 \text{ and } Ax \geq 0 \implies x = 0$$

Proof. Let us show that the three options are mutually exclusive:

- P) and Z) are mutually exclusive because the matrices that satisfy P) have full rank, whereas matrices of type Z have rank $n - 1$.
- Replacing x with $-x$ in the cases P) and Z) we obtain the following

$$\nexists\, x \geq 0 \text{ such that } Ax \leq 0 \text{ and } Ax \neq 0.$$

  This condition is incompatible with N).

Let us now suppose that the following holds:

$$\exists \tilde{x} \neq 0 \text{ such that } A\tilde{x} \geq 0. \tag{1.1.1}$$

We aim to show that only P) and Z) can hold.

Let $K_A = \{x \in \mathbb{R} \mid Ax \geq 0\}$. We observe that Lemma 1.1.6 implies that

$$K_A \cap \{x \in \mathbb{R} \mid x \geq 0\} \subseteq \{x \in \mathbb{R} \mid x > 0\} \cup \{0\} \tag{1.1.2}$$

By Equation (1.1.1) it follows that $K_A \neq 0$.

We want to show that exactly one of the following holds:

1. $K_A \subseteq \{x \in \mathbb{R} \mid x > 0\} \cup \{0\}$

2. $K_A = \operatorname{Span} \tilde{x}$, in particular, that $K_A = \ker A$.

If 1. holds, then obviously Equation (1.1.2) holds. Otherwise, if 1. does not hold, then it means that $J := K_A \setminus (\{x \in \mathbb{R} \mid x > 0\} \cup \{0\}) \neq \{0\}$. By Equation (1.1.1) and Equation (1.1.2) it follows that $\tilde{x} > 0$.

Let us suppose, by contradiction, that there exists $y \in J$ such that $y \neq \alpha\tilde{x}$ for every $\alpha \in \mathbb{R}$. Let $t \in [0,1]$, we define $w = (1-t)\tilde{x} + ty$. We observe that $Aw \geq 0$ and moreover, there exists $t \in [0,1]$ such that $w \geq 0$ and $w \notin \{x \in \mathbb{R} \mid x > 0\} \cup \{0\}$, i.e., w contradicts Equation (1.1.2). Such a t exists because, if j is an index such that $\frac{y_j}{\tilde{x}_j} = \min_k \frac{y_k}{\tilde{x}_k}$, we can define $t = -\frac{1}{\frac{y_j}{\tilde{x}_j}-1}$. In this way we observe that

- $y_j < 0$ and we may assume, without loss of generality (by replacing y with a multiple if necessary), that $\frac{y_j}{\tilde{x}_j} > 1$;
- $w_j = (1-t)\tilde{x}_j + ty_j = 0$;
- $w_i \geq 0$ if and only if

$$\begin{aligned} (1-t) + t\frac{y_i}{\tilde{x}_i} \geq 0 &\iff \\ 1 + \frac{1}{\frac{y_j}{\tilde{x}_j} - 1} - \frac{1}{\frac{y_j}{\tilde{x}_j} - 1}\frac{y_i}{\tilde{x}_i} \geq 0 &\iff \\ \frac{y_i}{\tilde{x}_i} \leq \frac{y_j}{\tilde{x}_j}. \end{aligned}$$

  The last inequality is satisfied by definition of j;
- $Aw \geq 0$ because w is a convex combination of elements of K_A.

We have shown that $J \subseteq \operatorname{Span} \tilde{x}$. This also implies that $A\tilde{x} = 0$. Now we have two possibilities:

- if $K_A \subseteq \operatorname{Span} \tilde{x}$ then 2 holds;
- if there exists $z \in K_A \setminus \operatorname{Span} \tilde{x}$, i.e. $z \neq \alpha\tilde{x}$, then $z \geq 0$ because $J \subseteq K_A$. We observe that there exists a sufficiently large $\lambda \geq 0$ such that $z - \lambda\tilde{x} \notin \{x \in \mathbb{R} \mid x \geq 0\}$, moreover $z - \lambda\tilde{x} \in K_A$ because $A\tilde{x} = 0$. From $J \subseteq \operatorname{Span} \tilde{x}$ it follows that $z - \lambda\tilde{x} \in \operatorname{Span} \tilde{x}$, then $z - \lambda\tilde{x} = \alpha\tilde{x}$, so $z = (\lambda + \alpha)\tilde{x}$, that leads to a contradiction.

So, the consequence is that one between 1 and 2 holds. It is obvious that 1 corresponds to P), because $kerA \subseteq K_A \subseteq \{x \in \mathbb{R} \mid x > 0\} \cup \{0\}$ which does not contain vector subspaces. Moreover, if 2 holds, then Z holds, because $\ker A = K_A$.

If Equation (1.1.1) holds, either P) or Z) holds, in particular there is no vector $x \geq 0$ such that $Ax \leq 0$ and $Ax \neq 0$. We observe that we stated the contradiction of Lemma 1.1.5, so its hypothesis cannot hold. It follows that there exists a vector $u \geq 0$, $u \neq 0$, such that $A^T u \geq 0$. This implies that Equation (1.1.1) holds for A^T, then also A^T verifies P) or Z). On the other hand, if Equation (1.1.1) does not hold for both A and A^T, then it follows immediately from Lemma 1.1.5 that N holds for both A and A^T. □

Definition 1.1.8. Let $A \in M_n(\mathbb{R})$ be a real indecomposable matrix and suppose that Equation (C1) holds. Then we define the matrix to be of positive, zero or negative type if it satisfies condition P), Z), or N) in Theorem 1.1.7 respectively.

Generalized Cartan matrices of positive type are also known as finite type, whereas those of zero and negative type are known as affine and indefinite type, respectively.

Corollary 1.1.9. *Let A be a real indecomposable matrix and let Equation (C1) hold, then A is of positive, zero or negative type if and only if there exists $v > 0$ such that $Av > 0$, $Av = 0$ or $Av < 0$ respectively.*

Lemma 1.1.10. *Let A be a matrix of positive or zero type. Then every principal submatrix of A decomposes as a direct sum of matrices of type P.*

Proof. Let A_S be the submatrix associated to the set of indices $S \subseteq \{1, \dots, n\}$. Since A is of positive or zero type, then there exists a vector $v > 0$ such that $Av \geq 0$. Let v_S be the vector associated with S, then $A_S v_S \geq 0$, moreover if $A_S v_S = 0$, then $a_{ij} = 0$ for every $i \in S$, $j \notin S$, indeed for every $i \in S$ we have:

$$\sum_{j=1}^{n} a_{ij} v_j \geq 0 \Rightarrow 0 = A_S v_S = \sum_{j \in S} a_{ij} v_j \geq - \sum_{j \notin S} a_{ij} v_j \geq 0.$$

This means that A is decomposable, against our hypothesis, therefore $A_S v_S > 0$. □

Lemma 1.1.11. *Let A be a real symmetric matrix and let Equation (C1) hold, then A is of positive or zero type if and only if A is positive definite or semipositive definite respectively.*

Proof. By contradiction, if A is of type N, then there exists a vector $v > 0$ such that $Av < 0$, then $v^T Av < 0$, against the hypothesis of A being positive or semipositive definite.

Let A be of positive or zero type, then for every $\lambda > 0$ and for every vector $v > 0$ such that $Av \geq 0$, then $(A + \lambda I)v > 0$. This implies that $\det(A + \lambda I) \neq 0$ and that A has only non-negative eigenvalues. It follows that A is positive definite if $\det A \neq 0$ and positive semidefinite if $\det A = 0$. □

Definition 1.1.12. For a matrix $A \in M_n(\mathbb{R})$ of size n that satisfies Equation (C1), the graph $G(A)$ is defined as the datum of n vertices $\{1, \dots, n\}$ linked by an edge if and only if $a_{ij} \neq 0$.

Lemma 1.1.13. *Let $A = (a_{ij})$, then*

1. *A is symmetrizable if and only if there exist a non-degenerate diagonal matrix D and a symmetric matrix B such that $A = DB$.*

2. *if A is an indecomposable symmetrizable matrix, then there exists a unique decomposition $A = DB$ such that $D = \operatorname{diag}(d_1, \dots, d_n)$ and $B = (b_{ij})$, where $b_{ij} = b_{ji} \in \mathbb{Z}[\frac{1}{2}]$ for $i \neq j$, and $b_{ii} = 2d_i^{-1}$ are relatively prime integers.*

Proof. If $A = DB$ with D a non-degenerate diagonal matrix and B a symmetric matrix, then $a_{ij} = 0$ if and only if $0 = \frac{a_{ij}}{d_i} = b_{ij} = b_{ji} = \frac{a_{ji}}{d_j}$ if and only if $a_{ji} = 0$. Moreover, for every set of indices $i_1, \dots, i_n$, we have:

$$\begin{aligned} a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1} &= d_{i_1} b_{i_1 i_2} \cdots d_{i_{s-1}} b_{i_{s-1} i_s} d_{i_s} b_{i_s i_1} \\ &= d_{i_1} b_{i_1 i_s} d_{i_s} b_{i_s i_{s-1}} \cdots d_{i_2} b_{i_2 i_1} = a_{i_1 i_s} a_{i_s i_{s-1}} \cdots a_{i_2 i_1}. \end{aligned}$$

Let us suppose that A is symmetrizable and suppose, without loss of generality, that A is indecomposable. First, we consider the graph $G(A)$. It is connected because A is indecomposable, so we can consider $T \subseteq G(A)$ a spanning tree of $G(A)$, i.e. a simply connected subgraph with the same vertices as $G(A)$. Let us choose an ordering $(i_1, \dots, i_n)$ of the vertices such that

- i_1 is a leaf for T;
- for every $j > 1$, there exists a unique edge between i_j and $\{i_1, \dots, i_{j-1}\}$.

Let M be a matrix, in the following we will denote $M_{(i)}$ the i-th line of the matrix M. Let us now construct the matrices B and D:

- we choose $d_{i_1} = 1$ and $B_{(i_1)} = A_{(i_1)}$;
- we observe that $a_{i_1 i_2} \neq 0$, and then $b_{i_1 i_2} \neq 0$, so we can choose $d_{i_2} = \frac{a_{i_2 i_1}}{a_{i_1 i_2}} = \frac{a_{i_2 i_1}}{b_{i_1 i_2}}$. We define $B_{(i_2)} = d_{i_2}^{-1} A_{(i_2)}$;
- we proceed inductively: let us suppose to have previously defined lines $B_{(i_1)}, \dots, B_{(i_k)}$ and $d_{i_1}, \dots d_{i_k}$. We want to define $B_{(i_{k+1})}$ and $d_{i_{k+1}}$. Let $j \in \{i_1, \dots, i_k\}$ be the unique index such that $\{j, i_{k+1}\}$ is an edge of T. It holds that $a_{j, i_{k+1}} \neq 0$ and then $b_{j, i_{k+1}} \neq 0$. We define $d_{i_{k+1}} = \frac{a_{i_{k+1} i_j}}{b_{i_j i_{k+1}}}$ and $B_{(i_{k+1})} = d_{i_{k+1}}^{-1} A_{(i_{k+1})}$.

It is clear that D is non-degenerate and it is also clear that $A = DB$. We also observe that $a_{ij} = 0$ if and only if $b_{ij} = 0$, or equivalently $G(A) = G(B)$. We must show that B is symmetric. Let $\{h, k\}$ be an edge of $G(B)$, i.e. $b_{hk} \neq 0$. If $\{h, k\}$ is an edge of T, then $b_{hk} = b_{kh}$ by definition. Otherwise we observe that T together with the edge $\{h, k\}$ is a graph with a cycle; indeed, there exist $j_1, \ldots, j_r$ such that $j_1 = h$ and $j_r = k$ and $\{j_1, j_2\}, \ldots, \{j_{r-1}, j_r\}$. Then by Equation (S1) we have:

$$d_{j_1} b_{j_1 j_2} \cdots d_{j_{r-1}} b_{j_{r-1} j_r} d_{j_r} b_{j_r j_1} = a_{j_1 j_2} \cdots a_{j_{r-1} j_r} a_{j_r j_1}$$
$$= a_{j_1 j_r} a_{j_r j_{r-1}} \cdots a_{j_2 j_1} = d_{j_1} b_{j_1 j_r} d_{j_r} b_{j_r j_{r-1}} \cdots d_{j_2} b_{j_2 j_1}$$

It follows $b_{hk} = b_{kh}$, because we have already shown that the other parts of the equation are pairwise equal. This concludes the proof of 1) and 2) immediately follows from the construction above. □

Lemma 1.1.14. *Let $A = (a_{ij})$ be a matrix of positive or zero type, such that $a_{ii} = 2$ for every $i = 1, \ldots n$, and $a_{ij} a_{ji} = 0$ or $a_{ij} a_{ji} \geq 1$, then A is symmetrizable. Moreover if there exist $i_1, \ldots i_s$ such that $a_{i_1 i_2} \ldots a_{i_s i_1} \neq 0$, $s \geq 3$, then A is of the form*

$$A = \begin{pmatrix} 2 & -u_1 & 0 & \cdots & -u_n^{-1} \\ -u_1^{-1} & 2 & -u_2 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -u_{n-2}^{-1} & 2 & -u_{n-1} \\ -u_n & 0 & \cdots & -u_{n-1}^{-1} & 2 \end{pmatrix}$$

where $u_1, \ldots, u_n$ are positive integers such that $u_1 \cdots u_n = 1$.

Proof. By hypothesis we have that $a_{ij} = 0$ implies $a_{ji} = 0$ for every $i, j = 1, \ldots n$, so A is symmetrizable if and only if Equation (S1) holds. It is clear that is sufficient to show the statement when there exist $s \geq 3$ and $i_1, \ldots i_s$ such that $a_{i_1 i_2} \ldots a_{i_s i_1} \neq 0$. Let B the principal submatrix of A associated to the set of indices $\{i_1, \ldots, i_s\}$, then B is of the form:

$$B = \begin{pmatrix} 2 & -b_1 & 0 & \cdots & -b'_s \\ -b'_1 & 2 & -b_2 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -b'_{s-2} & 2 & -b_{s-1} \\ -b_s & 0 & \cdots & -b'_{s-1} & 2 \end{pmatrix}.$$

We observe that B is irreducible and that from Lemma 1.1.10 it follows that B is of positive or zero type, in particular, it is of zero type if and only if $B = A$ and A of zero type. Therefore, there exists a vector $v > 0$ such that $Bv \geq 0$, in particular, we can replace B with $(\operatorname{diag} v)^{-1} B \operatorname{diag} v$. We can now suppose $v = (1, \ldots, 1)$. Since $Bv \geq 0$, also the sum of its coefficients is greater than zero, i.e., $2s - \sum_{i=1}^{s} (b_i + b'_i) \geq 0$, moreover by hypothesis $b_i b'_i \geq 1$, hence $b_i + b'_i \geq 2$ and then $2s - \sum_{i=1}^{s} (b_i + b'_i) = 0$ and $b_i = b'_i = 1$. Moreover $\det B = 0$ implies $A = (diag\, v) B (diag\, v)^{-1}$. □

Corollary 1.1.15. *Let A be an indecomposable generalized Cartan matrix of positive or zero type, then A is symmetrizable.*

Definition 1.1.16. Let $A = (a_{ij})$ be a generalized Cartan matrix. We define the graph $S(A)$ associated to A, called the Dynkin diagram of A, in the following way:

- $S(A)$ has n vertices enumerated from 1 to n;
- if $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, then the vertex i is connected to vertex j by $|a_{ij}|$ lines, equipped with an arrow pointing towards i if $|a_{ji}| > 1$;
- if $a_{ij}a_{ji} > 4$, vertices i and j are connected by a bold-faced line equipped with an ordered couple of indices $(|a_{ij}|, |a_{ji}|)$.

We observe that A is completely determined by its Dynkin diagram $S(A)$ and by a numbering of its vertices. We say that a connected Dynkin diagram $S(A)$ is of positive, zero or negative type if A is of that type.

Proposition 1.1.17. *Let A be an indecomposable generalized Cartan matrix, then the following holds:*

1. *A is of positive type if and only if all its principal minors are positive;*
2. *A is of zero type if and only if all its principal minors are positive and* $\det A = 0$*;*
3. *if A is of positive or zero type, then every proper subdiagram of $S(A)$ is a disjoint union of connected Dynkin diagrams of type P;*
4. *if A is of positive type, then $S(A)$ has no cycles;*
5. *if A is of zero type and has a cycle, then $S(A) = A_n^{(1)}$;*
6. *A is of zero type if and only if there exists a vector $\delta > 0$ such that $A\delta = 0$, and such a δ is unique up to scalar multiplication.*

Proof. First of all, we observe that if A is an indecomposable Cartan matrix of positive or zero type, then it satisfies the hypothesis of Lemma 1.1.14 hence it is symmetrizable. By Lemma 1.1.13 it also follows that $A = DB$ where $D = \mathrm{diag}(d_1, \dots, d_n)$ is a diagonal non-degenerate matrix and B is a symmetric matrix. Moreover, Equation (C1) still holds for B and the d_i's are all positive.

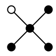
By the above observation and Lemma 1.1.11 1) and 2) follow.

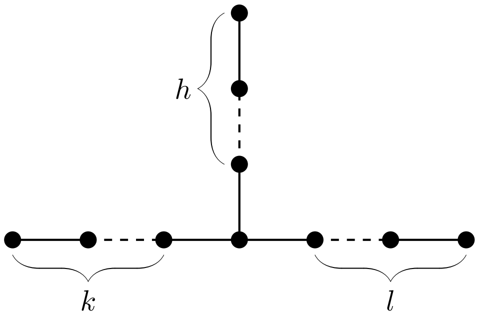
By the above observation and Lemma 1.1.10 3) follows.

By the above observation and Lemma 1.1.14 4) and 5) follow.

Statement 6) follows by Theorem 1.1.7. □

Proposition 1.1.18. *Let G be a connected graph without loops, then either it has a subdiagram of type $A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}$ or $E_8^{(1)}$, or G is a diagram of type A_n, D_n, E_6, E_7 or E_8.*

Proof. Suppose G does not contain a subdiagram of type $A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}$ or $E_8^{(1)}$. Then G has no oriented cycles, because it does not contain $A_n^{(1)}$. Every vertex is in the boundary of at most three edges because G does not contain $D_4^{(1)} =$ . There is at most one vertex which is boundary of three distinct edges because G does not contain $D_n^{(1)}$. If this point does not exist, then $G = A_n$ for some n. Otherwise, G must have a point with three connected edges and three "harms" of length $l \geq k \geq h \geq 1$, i.e. G has the following shape:



Since G does not contain $E_6^{(1)}$, it must be $h = 1$. Moreover, $k \leq 2$ because G does not contain $E_7^{(1)}$. If $k = 1$ then $G = D_n$ for some n. If $k = 2$, then $l \leq 4$ because G does not contain $E_8^{(1)}$. Then $G = E_6, E_7$ or E_8 if $l = 2, 3$ or 4 respectively. □

Theorem 1.1.19. *The Dynkin diagrams of positive and zero type are listed in Table 1.1 and Table 1.2. Moreover, the labels in Table 1.2 are the coordinates of the unique vector $\delta \in \mathbb{Z}^n$ with coprime coordinates and such that $A\delta = 0$.*

Proof. We already know that $A_n^{(1)}$ is a Dynkin diagram of zero type, indeed it can be readily verified that $\delta = (1, \dots, 1)$. We observe that $D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ are of zero type, indeed, it is immediate to check that the the vector of labels in Table 1.2 lies in the kernel of the corresponding generalized Cartan matrices:

$$D_n^{(1)} = \begin{pmatrix} 2 & 0 & -1 & & & & \\ 0 & 2 & -1 & & & & \\ -1 & -1 & 2 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & 2 & -1 & -1 \\ & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix} \qquad E_6^{(1)} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_7^{(1)} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_8^{(1)} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Similarly, a straightforward calculation shows that the Dynkin diagrams in Table 1.2 are of affine type because that the vector $\delta = (\delta_1, \ldots, \delta_n)$ lies in the kernel of the generalized Cartan matrix. Recall that $A\delta = 0$ if and only if $2\delta_i = \sum_{j=1}^n a_{ij}\delta_j$ for every $i = 1, \ldots, n$. Moreover, note that that $a_{ij} \neq 0$ if and only if there exists an edge between vertices i and j, and $a_{ij} = 1$ unless the edge is multiple and with an arrow pointing towards i.

We observe that every Dynkin diagram in Table 1.1 is a subdiagram of a Dynkin diagram in Table 1.2. Therefore, by Proposition 1.1.17 it follows that every Dynkin diagram in Table 1.1 is a Dynkin diagram of positive type.

It remains to show that every connected Dynkin diagram of positive type is listed in Table 1.1 and that every connected Dynkin diagram of zero type is listed in Table 1.2. We proceed by induction on the number n of vertices.

If $n = 2$, the classification of Dynkin diagrams of zero and positive type correspond to classify the pairs of positive integers a_{12}, a_{21} such that $\det A \geq 0$, where

$$A = \begin{pmatrix} 2 & -a_{12} \\ -a_{21} & 2 \end{pmatrix}.$$

Up to equivalence, we can assume $a_{21} \geq a_{12}$. The condition is $a_{12}a_{21} \leq 4$. The only cases are:

- $a_{12} = a_{21} = 1$, corresponding to A_2;

- $a_{12} = 1$ and $a_{21} = 2$, corresponding to C_2;
- $a_{12} = 1$ and $a_{21} = 3$, corresponding to G_2;
- $a_{12} = 1$ and $a_{21} = 4$, corresponding to $A_2^{(2)}$;
- $a_{12} = a_{21} = 2$, corresponding to $A_1^{(1)}$.

Now suppose $n = 3$. Consider the matrix:

$$A = \begin{pmatrix} 2 & -a_{12} & -a_{13} \\ -a_{21} & 2 & -a_{23} \\ -a_{31} & -a_{32} & 2 \end{pmatrix}.$$

the classification of Dynkin diagrams of zero and positive type corresponds to classify the sextuples positive integers $a_{12}, a_{21}, a_{13}, a_{31}, a_{23}, a_{32}$ such that $\det A \geq 0$, and every principal submatrix of A is the generalized Cartan matrix of a Dynkin diagram of positive type of rank 2, listed above.

It follows from Proposition 1.1.17 that the only case where $a_{ij} \neq 0$ for every $i, j = 1, 2, 3$ is when $a_{ij} = 1$ for every $i, j = 1, 2, 3$, that corresponds to $A_2^{(1)}$. Therefore may assume, up to equivalence, that $a_{13} = a_{31} = 0$, and that $a_{12}, a_{21}, a_{23}, a_{32} \neq 0$, otherwise the Dynkin diagram would be disconnected. Moreover, we can suppose up to equivalence, that $a_{12}a_{21} \leq a_{23}a_{32}$. The condition is the following:

$$\begin{cases} a_{12}a_{21} + a_{23}a_{32} \leq 4 \\ a_{12}a_{21} \leq 3 \\ a_{23}a_{32} \leq 3 \end{cases} \tag{1.1.3}$$

The only solutions different from $A_2^{(1)}$ are:

- $a_{12} = a_{21} = a_{23} = a_{32} = 1$ that corresponds to A_3;
- $a_{12} = a_{21} = a_{23} = 1$ and $a_{32} = 2$ corresponding to B_3;
- $a_{12} = a_{21} = a_{32} = 1$ and $a_{23} = 2$ corresponding to C_3;
- $a_{12} = a_{32} = 1$ and $a_{21} = a_{23} = 2$ corresponding to $C_2^{(1)}$;
- $a_{21} = a_{23} = 1$ and $a_{12} = a_{32} = 2$ corresponding to $D_3^{(2)}$;
- $a_{21} = a_{32} = 1$ and $a_{12} = a_{23} = 2$ corresponding to $A_4^{(2)}$;
- $a_{12} = a_{21} = a_{32} = 1$ and $a_{23} = 3$ corresponding to $D_4^{(3)}$;
- $a_{12} = a_{21} = a_{23} = 1$ and $a_{32} = 3$ corresponding to $G_2^{(1)}$.

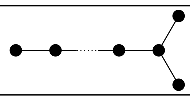
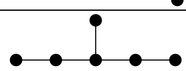
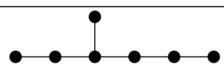
Let $n > 3$ and assume the inductive hypothesis, i.e., that every subdiagram of $S(A)$ appears in *Table* 1.1. Now, suppose that $S(A)$ is a positive Dynkin diagram. If $S(A)$ is simply-laced, i.e., it does not have any multiple edge, then it follows from Proposition 1.1.18 that the only cases are the A_n, D_n, E_6, E_7 and E_8. Suppose that $S(A)$ is not simply laced. The possible configurations are as follows:

- $S(A)$ cannot have a quadruple edge, otherwise it would contain $A_2^{(2)}$.
- If $S(A)$ has one double edge, then every vertex is adjacent with at most two vertices, otherwise $S(A)$ would contain $A_{2n+1}^{(2)}$. The double edge can have a single arrow, because $S(A)$ does not contain $A_1^{(1)}$. Since $S(A)$ cannot contain $F_4^{(1)}$ and $E_6^{(2)}$, if both the vertices of the double edge are adjacent to two vertices, then $S(A) = F_4$. Therefore we can suppose that at least one of the two vertices of the double edge is adjacent to only one vertex. As a result, we obtain the Dynkin diagrams B_n and C_n.
- Suppose that $S(A)$ has a triple edge. Since $S(A)$ does not contain $G_2^{(1)}$ and $D_4^{(3)}$, then the unique case is G_2.

This proves that the Dynkin diagrams of positive type are listed in Table 1.1.

We want to prove that the Dynkin diagrams of zero type are listed in Table 1.2. It is immediate to verify that the Dynkin diagrams obtained by adding a vertex to a Dynkin diagram of positive type in such a way that the new Dynkin diagram is not listed in Table 1.1 and admits only subdiagrams of positive type, are exactly those listed in Table 1.2. This concludes the proof, as it follows from Proposition 1.1.17 that Dynkin diagram of zero type admit only subdiagrams of positive type, therefore they can all be obtained by adding a vertex to a Dynkin diagram of positive type. □

Table 1.1: Positive Dynkin diagrams

Type	Diagram
A_n	
B_n	
C_n	
D_n	
E_6	
E_7	

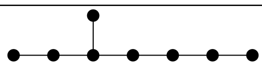
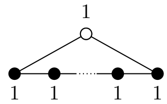
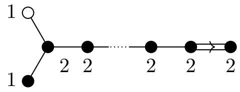
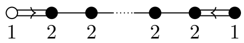
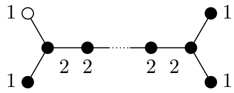
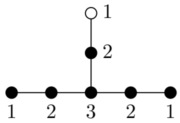
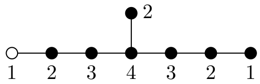
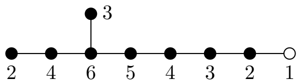
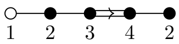
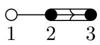
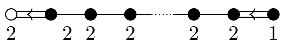
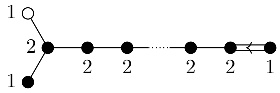
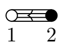
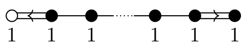
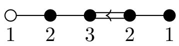
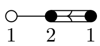
Type	Diagram
E_8	
F_4	
G_2	

Table 1.2: Zero Dynkin diagrams

Type	Diagram
$A_n^{(1)}$	
$B_n^{(1)}$	
$C_n^{(1)}$	
$D_n^{(1)}$	
$E_6^{(1)}$	
$E_7^{(1)}$	
$E_8^{(1)}$	
$F_4^{(1)}$	
$G_2^{(1)}$	
$A_{2n}^{(2)}$	
$A_{2n+1}^{(2)}$	
$A_2^{(2)}$	
$D_n^{(2)}$	
$E_6^{(2)}$	
$D_4^{(3)}$	

1.2 Kac-Moody algebras and roots system

Definition 1.2.1. Given a matrix $A = (a_{ij}) \in M_n(\mathbb{C})$, we say that the triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of A if the following holds:

1. $\mathfrak{h}$ is a $\mathbb{C}$-vector space, $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subseteq \mathfrak{h}$ and $\Pi = \{\alpha_1, \dots, \alpha_n\} \subseteq \mathfrak{h}^*$;
2. both Π and $\Pi^\vee$ are linearly independent sets;
3. $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$;
4. $n - l = \dim \mathfrak{h} - n$, where $l = \operatorname{rank} A$.

We say that two realizations of a matrix are isomorphic if there exists an isomorphism $\phi : \mathfrak{h}_1 \longrightarrow \mathfrak{h}_2$ such that $\phi(\Pi_1^\vee) = \Pi_2^\vee$ and $\phi^*(\Pi_1) = \Pi_2$.

Proposition 1.2.2. *Given a matrix $A = (a_{ij}) \in M_n(\mathbb{C})$, there exists a unique realization of A up to isomorphism.*

Given two matrices $A, B \in M_n(\mathbb{C})$, two realizations of A and B are isomorphic if and only if A and B are equivalent.

Proof. We can assume, up to reordering the indices, that there exists a $l \times n$ matrix A_1 of rank l and a matrix A_2, such that:

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

We define the matrix

$$C = \begin{pmatrix} A_1 & 0 \\ A_2 & I_{n-l} \end{pmatrix}.$$

The realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ is given by $\mathfrak{h} = \mathbb{C}^{2n-l}$, $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ the first n coordinate functions and $\alpha_1^\vee, \dots, \alpha_n^\vee \in \mathfrak{h}$ the rows of the matrix C.

Viceversa, given a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of the matrix A, we complete Π^* to a basis of $\mathfrak{h}$ with vectors $\alpha_{n+1}, \dots, \alpha_{2n-l}$ and we consider the matrix

$$C = \left(\langle \alpha_i^\vee, \alpha_j \rangle\right)_{\substack{i=1,\dots,n \\ j=1,\dots,2n-l}} = \begin{pmatrix} A_1 & B \\ A_2 & D \end{pmatrix}.$$

By adding suitable linear combinations of $\alpha_1^\vee, \dots, \alpha_n^\vee$ to $\alpha_{n+1}^\vee, \dots, \alpha_{2n-l}^\vee$, we can assume that B is the zero matrix. Moreover, properly combining $\alpha_{n+1}^\vee, \dots, \alpha_{2n-l}^\vee$, we may assume $D = I_{2n-l}$. It follows that, for a proper choice of $\alpha_{n+1}^\vee, \dots, \alpha_{2n-l}^\vee$, we can suppose

$$C = \begin{pmatrix} A_1 & 0 \\ A_2 & I_{n-l} \end{pmatrix}.$$

This proves the uniqueness.

The second part of the statement follows immediately by the above observation. Indeed, if A and B admit isomorphic realizations, they are both equal to the matrix given by the first n columns of the matrix:

$$C = \begin{pmatrix} A_1 & 0 \\ A_2 & I_{n-l} \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ B_2 & I_{n-l} \end{pmatrix}.$$

Conversely, if the two matrices are equivalent, then the matrices

$$C_A = \begin{pmatrix} A_1 & 0 \\ A_2 & I_{n-l} \end{pmatrix}; \quad C_B = \begin{pmatrix} B_1 & 0 \\ B_2 & I_{n-l} \end{pmatrix}$$

used to define the realizations of A and B coincide. Therefore, the realizations are isomorphic. □

Definition 1.2.3. Let A be a matrix and $(\mathfrak{h}, \Pi, \Pi^\vee)$ its realization, we call the elements of Π simple roots. We define the root lattice as $\Gamma = \sum_{i=1}^n \mathbb{Z}\alpha_i$ and let $\Gamma_+ = \sum_{i=1}^n \mathbb{Z}_+\alpha_i$.

Definition 1.2.4. Let $A = (a_{ij}) \in M_n(\mathbb{C})$ and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be its realization. We define the Lie algebra $\mathfrak{g}(A)$ associated to A in the following way. Let us consider the auxiliary algebra $\tilde{\mathfrak{g}}(A)$ with generators $\{e_i, f_i \mid i = 1, \dots, n\} \cup \mathfrak{h}$ and the relations:

$$\begin{cases} [e_i, f_j] = \delta_{ij}\alpha_i^\vee & \forall i, j = 1, \dots, n \\ [h, \tilde{h}] = 0 & \forall h, \tilde{h} \in \mathfrak{h} \\ [h, e_i] = \langle h, \alpha_i \rangle e_i & \forall i = 1, \dots, n, \ \forall h \in \mathfrak{h} \\ [h, f_i] = -\langle h, \alpha_i \rangle f_i & \forall i = 1, \dots, n, \ \forall h \in \mathfrak{h} \end{cases}$$

As we will show in Theorem 1.2.7, $\tilde{\mathfrak{g}}(A)$ has a unique maximal ideal τ such that $\tau \cap \mathfrak{h} = 0$. We define $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\tau$. We say that $\mathfrak{g}(A)$ is the Lie algebra of the matrix A. If A is a generalized Cartan matrix, then $\mathfrak{g}(A)$ is called the Kac-Moody algebra associated with the matrix A.

Definition 1.2.5. Let $\mathfrak{h}$ be a commutative Lie algebra and V an $\mathfrak{h}$-module. We say that V is $\mathfrak{h}$-diagonalizable if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V \mid h.v = \lambda(h)v, \forall h \in \mathfrak{h}\}$. We say that V_λ is the weight space and $\lambda \in \mathfrak{h}^*$ a weight if $V_\lambda \neq 0$.

Lemma 1.2.6. *Let $\mathfrak{h}$ be a commutative Lie algebra and let V be a diagonalizable $\mathfrak{h}$-module. Then the decomposition is induced on every $\mathfrak{h}$-submodule $U \subseteq V$, i.e., $U = \bigoplus_{\lambda \in \mathfrak{h}^*} U_\lambda$ where $U_\lambda = V_\lambda \cap U$ for every $\lambda \in \mathfrak{h}^*$.*

Proof. Let $u = \sum_{i=1}^n v_i \in U$ with $v_i \in V_{\lambda_i}$. We want to prove that $v_i \in U$ for every $i = 1, \ldots, m$. We proceed by induction on m. If $m = 1$ then $u = v_1 \in U$. If $m > 1$, $\lambda_m \neq \lambda_{m-1}$ implies that there exists $x \in \mathfrak{h}^*$ such that $\lambda_m(x) \neq \lambda_{m-1}(x)$. Let $y = x.u - \lambda_m(x)u = \sum_{j\in I} \lambda_j v_j \in U$, where

$$\varnothing \neq I = \{j \in \{1, \ldots, m\} \mid \lambda_j(x) \neq 0\} \subsetneq \{1, \ldots, m\}.$$

We can apply the inductive hypothesis to y, then we have that $v_j \in U$ for every $j \in I$. Moreover, we can apply the inductive hypothesis also to $z = x - \sum_{j\in I} v_j = \sum_{i\in\{1,\ldots,m\}\setminus I} v_i \in U$, and this concludes the proof. □

Theorem 1.2.7. *Let $A = (a_{ij}) \in M_n(\mathbb{C})$ and $(\mathfrak{h}, \Pi, \Pi^\vee)$ its realization, then the following holds:*

1. *$\tilde{\mathfrak{g}}(A) = \tilde{n}_- \oplus \mathfrak{h} \oplus \tilde{n}_+$, where $\tilde{\mathfrak{n}}_-$ and $\tilde{\mathfrak{n}}_+$ are the subalgebras generated respectively by $\{f_i\}$ and $\{e_i\}$;*

2. *$\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$ are freely generated respectively by $\{f_i\}$ and $\{e_i\}$;*

3. *the map $\tilde{\omega} : \tilde{\mathfrak{g}}(A) \longrightarrow \tilde{\mathfrak{g}}(A)$ defined by $e_i \mapsto -f_i$, $f_i \mapsto e_i$, $h \mapsto h$ for every $i = 1, \ldots, n$ and $h \in \mathfrak{h}$, is an involution of $\tilde{\mathfrak{g}}(A)$;*

4. *we have the following decomposition in root spaces with respect to $\mathfrak{h}$:*

$$\tilde{\mathfrak{g}}(A) = (\bigoplus_{0\neq\alpha\in\Gamma_+} \tilde{\mathfrak{g}}_{-\alpha}) \oplus \mathfrak{h} \oplus (\bigoplus_{0\neq\alpha\in\Gamma_+} \tilde{\mathfrak{g}}_{\alpha}).$$

   Here $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}}_(A) \mid [h, x] = \alpha(h)x\}$. Moreover, $\dim \tilde{\mathfrak{g}}_\alpha < \infty$ and $\tilde{\mathfrak{g}}_{\pm\alpha} \subseteq \tilde{\mathfrak{n}}_\pm$ for every $\alpha \in \Gamma_+$;

5. *there exists a unique maximal ideal τ among the ones which intersect trivially $\mathfrak{h}$. Moreover, $\tau = (\tau \cap \tilde{\mathfrak{n}}_-) \oplus (\tau \cap \tilde{\mathfrak{n}}_+)$.*

Proof. Let V be a complex vector space of dimension n. Fix $\lambda \in V^*$ and a basis $\{v_1, \ldots, v_n\}$ of V. We define an action of $\tilde{\mathfrak{g}}(A)$ on $T(V)$ as follows:

- $f_i(a) = v_i \otimes a$ for every $a \in T(V), i = 1, \ldots, n$;
- $h(1) = \langle\lambda, h\rangle 1$ and inductively on s:

$$h(v_j \otimes a) = -\langle\alpha_j, h\rangle v_j \otimes a + v_j \otimes h(a)$$

  for every $a \in T^{s-1}(V), j = 1, \ldots, n, h \in \mathfrak{h}$;

- $e_i(1) = 0$ and inductively on s:

$$e_i(v_j \otimes a) = \delta_{ij}\alpha_i^\vee(a) + v_j \otimes e_i(a)$$

for every $a \in T(V), i, j = 1, \dots n$.

We now need to prove that these maps define a structure of $\tilde{\mathfrak{g}}(A)$-module on $T(V)$. We will prove the relations inductively on s, the base of the induction is trivial, so we will compute only the inductive step.

- We prove that $(h_1h_2 - h_2h_1)(a) = [h_1, h_2](a) = 0$ for every $h_1, h_2 \in \mathfrak{h}, a \in T(V)$:

$$\begin{aligned}&(h_1h_2 - h_2h_1)(v_j \otimes a)\\ &= -h_1(\langle\alpha_j, h_2\rangle v_j \otimes a) + h_1(v_j \otimes a) + h_2(\langle\alpha_j, h_1\rangle v_j \otimes a) - h_2(v_j \otimes h_1(a))\\ &= v_j \otimes h_1h_2(a) - v_j \otimes h_2h_1(a) = v_j \otimes [h_1, h_2](a) = 0 = [h_1, h_2](v_j \otimes a);\end{aligned}$$

- We prove that $(e_if_j - f_je_i)(a) = [e_i, f_j](a) = \delta_{ij}\alpha_i^\vee(a)$ for every $i, j = 1, \dots, n$ and $a \in T(V)$:

$$(e_if_j - f_je_i)(a) = e_i(v_j \otimes a) - v_j \otimes e_i(a) = \delta_{ij}\alpha_i^\vee(a) = [e_i, f_j](a);$$

- We prove that $(hf_i - f_ih)(a) = [h, f_i](a) = -\langle h, \alpha_i\rangle f_i(a)$ for every $h \in \mathfrak{h}, i = 1, \dots, n$ and $a \in T(V)$:

$$(hf_i - f_ih)(a) = h(v_i \otimes a) - v_i \otimes h(a) = -\langle h, \alpha_i\rangle v_i \otimes a = [h, f_i](a);$$

- We prove that $(he_i - e_ih)(a) = [h, e_i](a) = \langle h, \alpha_i\rangle e_i(a)$ for every $h \in \mathfrak{h}, i = 1, \dots, n$ and $a \in T(V)$:

$$\begin{aligned}&(he_i - e_ih)(v_j \otimes a) = \delta_{ij}h(\alpha_i^\vee(a)) + h(v_j \otimes e_i(a)) + \langle h, \alpha_j\rangle e_i(v_j \otimes a) - e_i(v_j \otimes h(a))\\ &= v_j \otimes h(e(a)) + \delta ij\langle h, \alpha_j\rangle\alpha_i^\vee(a) - v_j \otimes e(h(a)) = \delta ij\langle h, \alpha_j\rangle\alpha_i^\vee(a) + v_j \otimes [h, e_i](a)\\ &\delta ij\langle h, \alpha_j\rangle\alpha_i^\vee(a) + \langle h, \alpha_j\rangle v_j \otimes e_i(a) = \langle h, \alpha_j\rangle e_i(v_j \otimes a) = [h, e_i](v_j \otimes e_i).\end{aligned}$$

We will prove by induction that the product of s elements in $\{e_i, f_i \mid i = 1, \dots, n\} \cup \mathfrak{h}$ belongs to $\tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$. In particular, we have to show that:

1. $[f_i, x] \in \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$ for every $x \in \tilde{\mathfrak{n}}_- \cup \mathfrak{h} \cup \tilde{\mathfrak{n}}_+$ and $i = 1, \dots, n$;

2. $[e_i, x] \in \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$ for every $x \in \tilde{\mathfrak{n}}_- \cup \mathfrak{h} \cup \tilde{\mathfrak{n}}_+$ and $i = 1, \dots, n$;

3. $[h, x] \in \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$ for every $x \in \tilde{\mathfrak{n}}_- \cup \mathfrak{h} \cup \tilde{\mathfrak{n}}_+$ and $h \in \mathfrak{h}$.

Property 3. is trivial and the only non-trivial things to prove in properties 1. and 2. are $[f_i, x] \in \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$ for every $x \in \tilde{\mathfrak{n}}_+$ and $[e_i, x] \in \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$ for every $x \in \tilde{\mathfrak{n}}_-$. Let us prove the first one, the second is analogous. We proceed by induction on the length s of the expression $x = [e_{i_1}[\cdots[e_{i_{s-2}}[e_{i_{s-1}}, e_{i_s}]]]]$:

$$[f_i[e_j, \tilde{x}]] = [[f_i, e_j], \tilde{x}] + [e_j[f_i, \tilde{x}]] = \delta_{ij}[h, \tilde{x}] + [e_j[f_i, \tilde{x}]]$$

and we conclude using 3 and the inductive hypothesis.

Suppose now that $u = x + h + y = 0$, with $h \in \mathfrak{h}, x \in \tilde{\mathfrak{n}}_+$ and $y \in \tilde{\mathfrak{n}}_-$. Then u acts on $T(V)$ by: $0 = u(1) = y(1) + \langle \lambda, h \rangle$, hence $\langle \lambda, h \rangle = 0$ for every $\lambda \in V^*$, then $h = 0$.

We observe that the map $\tilde{\mathfrak{n}}_- \longrightarrow V$, $f_i \mapsto v_i$ uniquely defines a map $\tilde{\mathfrak{n}}_- \longrightarrow T(V)$. Since $T(V)$ is freely generated by the images of the generators of $\tilde{\mathfrak{n}}_-$, then $T(V)$ is the universal enveloping algebra $U(\tilde{\mathfrak{n}}_-)$ of $\tilde{\mathfrak{n}}_-$, and the map $y \mapsto y(1)$ is the canonical embedding $\tilde{\mathfrak{n}}_- \hookrightarrow U(\tilde{\mathfrak{n}}_-)$. So y must be zero. This proves 1.

Now, by the Poincaré-Birkhoff-Witt theorem, $\tilde{\mathfrak{n}}_-$ is freely generated by $f_1, \ldots, f_n$. Applying $\tilde{\omega}$, we see that $\tilde{\mathfrak{n}}_+$ is generated by $e_1, \ldots, e_n$. This proves 2.

To prove 4. we prove by induction on s that $[e_{i_1}, [\cdots[e_{i_{s-2}}, [e_{i_{s-1}}, e_{i_s}]]\cdots]] \in \tilde{\mathfrak{g}}_\alpha$ for some $\alpha \in \Gamma_+$. Let $x = [e_{i_1}, [\cdots[e_{i_{s-2}}, [e_{i_{s-2}}, e_{i_{s-1}}]]\cdots]] \in \tilde{\mathfrak{g}}_\alpha$, $\alpha \in \Gamma_+$, then

$$[h, [e_j, x]] = [[h, e_j], x] + [e_j[h, x]] = \alpha_j(h)[e_j, x] + \alpha(h)[e_j, x] = (\alpha + \alpha_j)(h)[e_j, x].$$

We can prove analogously that $[f_{i_1}, [\cdots[f_{i_{s-2}}, [f_{i_{s-1}}, f_{i_s}]]\cdots]] \in \tilde{\mathfrak{g}}_{-\alpha}$ for some $\alpha \in \Gamma_+$. This prove that

$$\tilde{\mathfrak{n}}_\pm = \bigoplus_{0 \neq \alpha \in \Gamma_+} \tilde{\mathfrak{g}}_{\pm\alpha}.$$

We observe that $\dim \tilde{\mathfrak{g}}_\alpha \leq n^{\operatorname{ht} \alpha}$, where, if $\alpha = \sum_{i=1}^n k_i \alpha_i$, $\operatorname{ht} \alpha = \sum_{i=1}^n k_i$. This concludes the proof of 4.

Let $\Omega = \{\eta \subseteq \tilde{\mathfrak{g}}(A) \mid \eta \text{ is an ideal and } \eta \cap \mathfrak{h} = 0\}$. By Lemma 1.2.6 follows that $\eta = \bigoplus_{\alpha \in \Gamma} (\eta \cap \tilde{\mathfrak{g}}_\alpha)$ for every $\eta \in \Omega$. Then for every $\eta_1, \eta_2 \in \Omega$, also $\eta_1 + \eta_2 \in \Omega$. It follows that there is a unique maximal element of Ω:

$$\tau = \sum_{\eta \in \Omega} \eta.$$

In particular, $\tau = (\tau \cap \tilde{\mathfrak{n}}_-) \oplus (\tau \cap \tilde{\mathfrak{n}}_+)$ as vector spaces. We observe that $[f_i, \tau \cap \tilde{\mathfrak{n}}_+] \subseteq \tilde{\mathfrak{n}}_+$ and $[e_i, \tau \cap \tilde{\mathfrak{n}}_-] \subseteq \tilde{\mathfrak{n}}_-$, and this concludes the proof of 5. □

By now on we will consider only generalized Cartan matrices, so that $\mathfrak{g}(A)$ is the Kac-Moody algebra with generalized Cartan matrix A.

Remark 1.2.8. It is clear from Theorem 1.2.7 that the decomposition with respect to the action of $\mathfrak{h} \subseteq \mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\tau$ is

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{0 \neq \alpha \in \Gamma_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

Moreover, it is also clear that, if $\mathfrak{n}_-$ and $\mathfrak{n}_+$ denote the images of $\tilde{\mathfrak{n}}_-$ and $\tilde{\mathfrak{n}}_+$ respectively, there is the diagonal decomposition $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

This remark justifies the following definition.

Definition 1.2.9. Let $\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Gamma} \mathfrak{g}_\alpha$, then we say that $\alpha \in \Gamma \setminus 0$ is a root if $\operatorname{mult}\alpha := \dim \mathfrak{g}_\alpha \geq 1$.

A root α is called positive if $\alpha \in \Gamma_+$ and negative if $-\alpha \in \Gamma_+$. We call Δ, Δ_+ and Δ_- the set of roots, positive roots and negative roots respectively.

Remark 1.2.10. From the proof of Theorem 1.2.7 it follows that $\Delta = \Delta_+ \sqcup \Delta_-$. Moreover, if $\alpha \in \Delta_+$ (respectively in Δ_-), then $\mathfrak{g}_\alpha$ is generated by the elements of the form $[e_{i_1}, [\cdots[e_{i_{s-2}}, [e_{i_{s-2}}, e_{i_{s-1}}]]\cdots]]$ with $i_1 + \ldots + i_s = \operatorname{ht}\alpha$. Analogously, if $\alpha \in \Delta_-$.

Remark 1.2.11. The involution $\tilde{\omega}$ over $\tilde{\mathfrak{g}}(A)$ described in Theorem 1.2.7 is well defined on the quotient by τ, so it induces an involution ω on $\mathfrak{g}(A)$, called the Chevalley involution.

We observe that $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, hence $\Delta_- = -\Delta_+$.

It is also easy to observe that, since every root is either positive or negative, for $\beta \in \Delta_+ \setminus \{\alpha_i\}$, $(\beta + \mathbb{Z}\alpha_i) \cap \Delta \subseteq \Delta_+$.

Lemma 1.2.12. *Let $a \in n_+$ be such that $[a, f_i] = 0$ for every i, then $a = 0$. Similarly if $a \in n_-$ and $[a, e_i] = 0$ for every i, then $a = 0$.*

Proof. We define a $\mathbb{Z}$-grading on $\mathfrak{g}(A)$ setting $\deg e_i = -\deg f_i = 1$ for every $i = 1, \ldots, n$ and $\deg \mathfrak{h} = 0$. We write:

$$\mathfrak{g}(A) = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$$

where $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_1 = \oplus_{i=1}^n \mathbb{C}e_i$ and $\mathfrak{g}_{-1} = \oplus_{i=1}^n \mathbb{C}f_i$.

Let $a \in \mathfrak{n}_+$ such that $[\mathfrak{g}_{-1}, a] = 0$. We define

$$\tau = \sum_{i,j\geq 0} (\operatorname{ad}^i \mathfrak{g}_1)(\operatorname{ad}^j \mathfrak{h})(a) \subseteq \mathfrak{n}_+.$$

We observe that τ is both $\mathfrak{h}$-invariant and $\mathfrak{n}_+$-invariant. Moreover, by the hypothesis $[\mathfrak{g}_{-1}, a] = 0$ it follows that τ is also $\mathfrak{n}_-$-invariant. Then τ is an ideal of $\mathfrak{g}(A)$ such that $\tau \cap \mathfrak{h} = 0$, therefore by Theorem 1.2.7 $\tau = 0$, and then $a = 0$. The proof is analogous when $a \in \mathfrak{n}_-$. □

Definition 1.2.13. Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module, we say that $x \in \mathfrak{g}$ is a locally nilpotent element for V if for every $v \in V$, there exists $N \in \mathbb{N}$ such that $x^N.v = 0$.

Lemma 1.2.14. *Let $y_1, y_2, \ldots$ be a system of generators for a Lie algebra $\mathfrak{g}$ and let $x \in \mathfrak{g}$ be such that for every i, there exists N_i such that $(\operatorname{ad} x)^{N_i}(y_i) = 0$, then $\operatorname{ad} x$ is locally nilpotent on $\mathfrak{g}$.*

Proof. Let $z = [y_{i_1}[\dots[y_{i_{m-2}}[y_{i_{m-1}}, y_{i_m}]]]] \in \mathfrak{g}$, we show by induction on m that there exists $N_z \in \mathbb{N}$ such that $(\operatorname{ad} x)^{N_z} = 0$. If $m = 1$, the statement follows by hypothesis. If $m > 1$, we use the fact that $\operatorname{ad} x$ is a derivation. Say $z = [y_{i_1}, \tilde{z}]$, by inductive hypothesis there exists $N_{\tilde{z}} \in \mathbb{N}$ such that $(\operatorname{ad} x)^{N_{\tilde{z}}} = 0$. Then $N_z = N_{\tilde{z}} + N_i$ is the desired number, indeed:

$$(\operatorname{ad} x)^{N_z}([y_{i_1}, \tilde{z}]) = \sum_{j=0}^{N_z} \binom{N_z}{j} [(\operatorname{ad} x)^j (y_{i_1}), (\operatorname{ad} x)^{N_z - j}(\tilde{z})] = 0$$

□

Definition 1.2.15. We say that a $\mathfrak{h}$-diagonalizable $\mathfrak{g}(A)$-module V is integrable if e_i and f_i are locally nilpotent for every i.

Lemma 1.2.16. *$\operatorname{ad} e_i$ and $\operatorname{ad} f_i$ are both locally nilpotent over $\mathfrak{g}(A)$.*

Proof. By Lemma 1.2.14, it is sufficient to prove that:

1. $(\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0$ for every $i \neq j$;
2. $(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$ for every $i \neq j$.

We will prove relation 1. Relation 2. can be proved similarly.

We let $v = f_j, \theta_{ij} = (\operatorname{ad} f_i)^{1-a_{ij}} f_j$ and $\mathfrak{g}_{(i)} = \mathbb{C} f_i \oplus \mathbb{C}\alpha_i^\vee \oplus \mathbb{C} e_i \cong \mathfrak{sl}_2$. Consider $\mathfrak{g}(A)$ as a $\mathfrak{g}_{(i)}$-module with the restriction of the adjoint representation. We observe that

- $\alpha_i^\vee(v) = \langle \alpha_i^\vee, \alpha_j \rangle v = -a_{ij}$;
- $e_i(v) = 0$ because $i \neq j$.

Then it follows that v generates an irreducible $\mathfrak{g}_{(i)}$-module of dimension $1 - a_{ij}$.

From $i \neq j$ and the relations of $\mathfrak{g}(A)$ it follows:

$$[e_i, \theta_{ij}] = (-a_{ij} + 1 - (1 - a_{ij}))(1 - a_{ij})(\operatorname{ad} f_i)^{-a_{ij}}(f_j) = 0.$$

Moreover, if $k \neq i$, it follows directly that $[e_k, \theta_{ij}] = 0$. By these observations and by Lemma 1.2.12 we conclude that $\theta_{ij} = 0$. □

Definition 1.2.17. An $\mathfrak{h}$-diagonalizable $\mathfrak{g}(A)$-module V is called integrable if e_i and f_i are locally nilpotent on V for every $i = 1, \dots, n$.

Theorem 1.2.18. *$\mathfrak{g}(A)$ is an integrable $\mathfrak{g}(A)$-module.*

Proof. The proof follows by Lemma 1.2.14 and Lemma 1.2.16. □

In the following lemma we will recall some remarkable property of the representations of $\mathfrak{sl}_2(\mathbb{C})$.

Lemma 1.2.19. *Let V be an $\mathfrak{sl}_2(\mathbb{C})$-module and let $v \in V$ be such that $h(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. If we set $v_j := \frac{1}{j!} f^j(v)$, then $h(v_j) = (\lambda - 2j)v_j$. If , in addition, $e(v) = 0$, then $e(v_j) = (\lambda - j + 1)v_{j-1}$.*

Moreover, for each integer $k \geq 0$, there exists a unique, up to isomorphism, irreducible $(k+1)$-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module $V(k)$. There exists a basis $\{v_0, v_1, \dots, v_n\}$ of $V(k)$ such that the action of $\mathfrak{sl}_2(\mathbb{C})$ is given by:

- $h(v_j) = (k - 2j)v_j$;
- $f(v_j) = (j + 1)v_{j+1}$;
- $e(v_j) = (k + 1 - j)v_{j-1}$;

for every $j = 0, 1, \dots, n$, with the convention $v_{-1} = v_{n+1} = 0$.

Proposition 1.2.20. *Let V be an integrable $\mathfrak{g}(A)$-module, then the following holds:*

1. *as $\mathfrak{g}_{(i)}$-module, V decomposes in direct sum of $\mathfrak{h}$-invariant irreducible modules;*
2. *let $\lambda \in \mathfrak{h}^*$ a weight for V and α_i a simple root of $\mathfrak{g}(A)$. Let $M = \{t \in \mathbb{Z} \,|\, \lambda + t\alpha_i$ is a weight for $V\}$ and let $m_t = \mathrm{mult}_V(\lambda + t\alpha_i)$, then*
   - *(a) M is a closed interval of integers $[-p, q]$ with p and q non negative integers such that $p - q = \langle \lambda, \alpha_i^\vee \rangle$ when $p, q < \infty$. If $\mathrm{mult}_V \lambda < \infty$, then $p, q < \infty$;*
   - *(b) the map $e_i : V_{\lambda + t\alpha_i} \longrightarrow V_{\lambda + (t+1)\alpha_i}$ is injective and the function $t \mapsto m_t$ is increasing for $t \in [-p, -\frac{1}{2}\langle \lambda, \alpha_i^\vee \rangle]$;*
   - *(c) the map $t \mapsto m_t$ is symmetric with respect to $t = -\frac{1}{2}\langle \lambda, \alpha_i^\vee \rangle$;*
   - *(d) if λ and $\lambda + \alpha_i$ are both weights of V, then $e_i(V_\lambda) \neq 0$.*

Proof. By the proof of Lemma 1.2.16 follows that

$$e_i f_i^k.v = k(1 - k + \langle \lambda, \alpha_i^\vee \rangle) f_i^{k-1}.v + f_i^k e_i.v$$

for every $v \in V_\lambda$. Let us fix $v \in V_\lambda$, we define the $\mathfrak{g}_{(i)}$-submodule $U = \sum_{k,m \geq 0} f_i^k e_i^m.v$. This space is clearly $\mathfrak{h}$-invariant. Moreover, e_i and f_i are both locally nilpotent, then it follows that $\dim U < +\infty$. By Weyl's completely reducibility theorem, it follows that U decomposes as a direct sum of finite dimensional irreducible $\mathfrak{g}_{(i)}$ modules. These modules are also $\mathfrak{h}$-invariant, because the eigenvalues of $\mathfrak{h}$ are the same as $\alpha_i^\vee$. It follows that we can decompose V as a direct sum of finite dimensional irreducible $\mathfrak{h}$-invariant $\mathfrak{g}_{(i)}$-modules, that implies 1.

Let now define $U = \sum_{k \in \mathbb{Z}} V_{\lambda + k\alpha_i}$. We observe that U is $(\mathfrak{g}_{(i)} + \mathfrak{h})$-invariant, so it decomposes as a direct sum of finite dimensional irreducible $(\mathfrak{g}_{(i)} + \mathfrak{h})$-modules. Set $p = -\inf M$ and

$q = \sup M$. Both p and q are non-negative because $0 \in M$. By definition, $M = [-p, q] \cap \mathbb{Z}$. We observe that $\langle \lambda + t\alpha_i, \alpha_i^\vee \rangle = 0$ if and only if $t = -\frac{1}{2}\langle \lambda, \alpha_i^\vee \rangle$. There follow by the symmetry of $\mathfrak{sl}_2$-representations properties 2b and 2c. Moreover, we have $p - \frac{1}{2}\langle \lambda, \alpha_i^\vee \rangle = q + -\frac{1}{2}\langle \lambda, \alpha_i^\vee \rangle$, which implies $p - q = \langle \lambda, \alpha_i^\vee \rangle$. Property 2d follows by the structure of $\mathfrak{sl}_2$-representations. □

Corollary 1.2.21. *Let λ be a weight for an integrable $\mathfrak{g}(A)$-module V. Then we have:*

1. *if $\lambda + \alpha_i$ is not a weight, then $\langle \lambda, \alpha_i^\vee \rangle \geq 0$;*
2. *$\lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ is a weight of the same multiplicity as λ.*

Proof. If $\lambda + \alpha_i$ is not a weight, then $1 \notin [-p, q]$, as defined in Proposition 1.2.20. It implies that $q = 0$, and then $\langle \lambda, \alpha_i^\vee \rangle = p \geq 0$.

We observe that $-\langle \lambda, \alpha_i^\vee \rangle = q - p$ and $-p \leq q - p \leq q$, then $-\langle \lambda, \alpha_i^\vee \rangle \in [-p, q]$. Statement 2 follows by Lemma 1.2.23. □

Definition 1.2.22. For $i = 1, \ldots, n$ we define the fundamental reflections $r_i \in GL(\mathfrak{h}^*)$: for $\lambda \in h^*$,

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i.$$

We define $W \subseteq \mathrm{GL}(h^*)$ as the group generated by the fundamental reflections defined above. We say W is the Weyl group of $\mathfrak{g}(A)$.

Lemma 1.2.23. *Let $\mathfrak{g}(A)$ be a Kac-Moody Lie algebra, let Δ be its root system and W the associated Weyl group. The following hold:*

1. *Δ is a W-invariant set and $\operatorname{mult} \alpha = \operatorname{mult} w(\alpha)$ for every $\alpha \in \Delta$ and $w \in W$. Moreover $\Delta_+ \setminus \{\alpha_i\}$ is an r_i-invariant set;*
2. *the set Δ_+ is uniquely defined by the following properties:*
   - *(a) $\Pi \subseteq \Delta_+ \subseteq \Gamma_+$ and $2\alpha \notin \Delta_+$ for every $\alpha \in \Pi$;*
   - *(b) if $\alpha \in \Delta_+ \setminus \{\alpha_i\}$, then $\alpha + k\alpha_i \in \Delta_+$ if and only if $-p \leq k \leq q$ for $p, q \in \mathbb{Z}{\geq 0}$ such that $p - q = \langle \alpha, \alpha_i^\vee \rangle$.*
3. *if A is an indecomposable generalized Cartan matrix of zero or negative type, then for every $\beta \in \Delta_+$, there exists $\alpha \in \Pi$ such that $\beta + \alpha \in \Delta_+$.*

Proof. We have shown in Theorem 1.2.18 that $g(A)$ is an integrable $g(A)$-module, so we can apply Corollary 1.2.21. It follows that, for every $\alpha \in \Delta$, $r_i(\alpha) = \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i$ is a root of the same multiplicity as α. It follows that $w(\alpha) \in \Delta$ and $\operatorname{mult} w(\alpha) = \operatorname{mult} \alpha$ for every $\alpha \in \Delta$ and every $w \in W$. By Remark 1.2.11 follows that for every $\alpha \in \Delta_+ \setminus \{\alpha_i\}$, holds

$$r_i(\alpha) = \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i \in \Delta_+ \setminus \{\alpha_i\}.$$

Let us prove that a) and b) hold for a root system. We have that $\Pi \subseteq \Delta_+ \subseteq \Gamma_+$ by definition. By the construction of $\mathfrak{g}(A)$, follows that $2\alpha_i \notin \Delta$ for every $\alpha_i \in \Pi$, indeed $\mathfrak{g}_{\alpha_i}$ is generated by $[e_i, e_i] = 0$. Moreover, b) holds because of Proposition 1.2.20.

On the other hand, if there exists $\Pi \subseteq \tilde{\Delta}_+ \subseteq \Gamma_+$, then one can show by induction on $\operatorname{ht}\alpha$ that $\alpha \in \Delta_+$ if and only if $\alpha \in \tilde{\Delta}_+$.

To prove 3, we observe that if A is of negative type, then for every $\beta \in \Delta_+$, then $A\beta \ngeq 0$, in particular, there exists $i \in S(A)$ such that $(A\beta)_i = r_i(\beta) < 0$. It follows that $\beta + \alpha_i \in \Delta_+$. If A is of zero type, then we have two cases:

- if $A\beta \ngeq 0$, then, as in the negative case, there exists $\alpha_i \in \Pi$ such that $\alpha_i + \beta \in \Delta_+$;
- if $A\beta = 0$, then $\beta \notin \Pi$. By 2 follows that there exist $\alpha_i \in \Pi$, $k \in \mathbb{N}_{>0}$ and $\tilde{\beta} \in \Delta_+$ such that $\beta = \tilde{\beta} + k\alpha_i$, then $\tilde{\beta} = \beta - k\alpha_i \in \Delta_+$. Then $q \geq 1$, since $0 = \langle \beta, \alpha_i^\vee \rangle = p - q$, with $p \geq 1$. It follows that $\beta + \alpha_i \in \Delta_+$.

□

1.3 Real and imaginary roots

Definition 1.3.1. Let $\alpha \in \Delta$, we call α a real root if there exists a $w \in W$ such that $w(\alpha) \in \Pi$, otherwise we say that α is an imaginary root. We denote by Δ^{re}, Δ^{re}_+, Δ^{im}, Δ^{im}_+ the sets of real, positive real, imaginary and positive imaginary roots respectively.

Remark 1.3.2. We observe that, since our aim is to study the root system associated with a Cartan matrix A, we may assume that A is an indecomposable Cartan matrix. Indeed, if $A = A_1 \oplus \cdots \oplus A_k$, then also $\mathfrak{g}(A) \cong \mathfrak{g}(A_1) \oplus \cdots \oplus \mathfrak{g}(A_k)$ and consequently $\Delta(A) = \coprod_{i=1}^k \Delta(A_i)$ and $W(A) = \prod_{i=1}^k W(A_i)$.

Definition 1.3.3. Let $\alpha = \sum_{i=1}^n k_i\alpha_i \in \Gamma$, we define the support of α as the subgraph of $\operatorname{Supp}\alpha \subseteq S(A)$ consisting of the vertices i such that $k_i \neq 0$ together with every edge between these vertices.

Definition 1.3.4. We define the fundamental chamber $K \subseteq \Gamma \otimes \mathbb{R}$ the set of vectors $\alpha = \sum_{i=1}^n k_i\alpha_i \in \Gamma$ such that:

1. there exists at least one i such that $k_i \geq 0$, and this is required especially when $S(A) \setminus \{p_i\}$ contains a connected component of negative type;
2. $\varphi_i(\alpha) = \sum_{j=1}^n k_j a_{ij} \leq 0$ for every $i = 1, \ldots, n$.

We denote by M the set of $\alpha \in K \cap \Gamma$ such that $\operatorname{Supp}\alpha$ is connected.

Lemma 1.3.5. *Let A be an indecomposable Cartan matrix, then the following hold:*

1. *if $\alpha = \sum_{i=1}^n k_i\alpha_i \in K$, then $k_i \geq 0$ for every $i = 1, \dots, n$;*
2. *the set Δ_+^{im} is W-invariant and if $\alpha \in \Delta_+^{im}$, then $w(\alpha) \in M$ for some $w \in W$;*
3. *if A is of positive type, then $K = 0$;*
4. *if A is of zero type, then $K = \mathbb{R}_+\delta$, where δ is the vector found in Theorem 1.1.19 and reported in Table 1.2;*
5. *if A is of negative type, then K is a solid cone;*
6. *if $\alpha \in \Delta_+^{re}$, then there exists a sequence $r_{i_1}, \dots, r_{i_k}$ of minimal length such that*

$$r_{i_k} \cdots r_{i_1}(\alpha) \in \Pi.$$

Proof. Let us prove the first statement by contradiction: suppose that $\alpha = \sum_{i=1}^n k_i\alpha_i = \beta - \gamma \in K$. Let $\beta = \sum_{i\in S_1}^n c_i\alpha_i$ and $\gamma = \sum_{i\in S_2}^n c_i\alpha_i$, with $S_1 \sqcup S_2 = S(A)$ such that $c_i = k_i \geq 0$ for every $i \in S_1$ and $c_j = -k_j > 0$ for every $j \in S_2$. From the definition of K it follows that S_1 contains only subdiagrams of positive or zero type, otherwise $S_2 = \varnothing$, and moreover $\varphi(\beta) \leq 0$ for every $i \in S_1$. This together with Theorem 1.1.7 proves that S_1 has all the connected components of zero type, which implies $\varphi(\beta) = 0$ for every $i \in S_1$. Because $S(A)$ is connected, there exist $i \in S_1$, $j \in S_2$ such that $a_{ij} \neq 0$. We conclude the proof of 1 because the following leads to a contradiction together with $\alpha \in K$:

$$\varphi_i(\alpha) = \varphi_i(\beta) - \varphi_i(\gamma) = -\varphi_i(\gamma) = -\sum_{j\in S_2} a_{ij}c_j > 0.$$

To prove 2, let $\alpha \in \Delta_+^{im}$, then $\alpha \in \Delta_+ \setminus \Pi$, then follows by Lemma 1.2.23 and the definition of Δ_+^{im} that $r_i(\alpha) \in \Delta_+ \setminus \Pi$. It follows that $W(\Delta_+^{im}) \subset \Delta_+^{im}$. Let $\tilde{w} \in W$ such that $\operatorname{ht}\tilde{w}(\alpha) = \min_{w\in W} w(\alpha)$, then $\tilde{w}(\alpha) - \alpha_i \notin \Delta_+$ for every $i = 1, \dots, n$. By Lemma 1.2.23 follows that $\varphi_i(\tilde{w}(\alpha)) = -q \leq 0$ for every $i = 1, \dots, n$, which proves 2.

Statements 3,4 and 5 follow by 1 and Theorem 1.1.7. Statement 6 follows immediately by definition of real root. □

Lemma 1.3.6. *Let $\alpha \in M$, then $\alpha \in \Delta_+^{im}$.*

Proof. Let $\alpha = \sum_{i=1}^n k_i\alpha_i \in M$ and $\Omega = \{\gamma \in \Delta_+ \,|\, \alpha - \gamma \in \Gamma_+\}$. At least one element of Π belongs to Ω, so it is not empty. Let $\beta = \sum_{i=1}^n m_i\alpha_i$ be an element of maximal height of Ω. Suppose by contradiction that $\alpha \notin \Delta_+$, then the following holds:

1. $P = \{i \in S(A) \,|\, k_i = m_i\} \neq \varnothing$;

2. $\beta + \alpha_i \notin \Delta_+$ if $i \notin P$, i.e. if $k_i > m_i$.

Indeed, by Lemma 1.3.5 and the hypothesis $M \neq \{0\}$ follows that $S(A)$ is of negative or zero type, then by Lemma 1.2.23 we can add simple roots to β until at least $m_i = k_i$ for some $i \in S(A)$. This proves 1. To prove 2 it suffices to observe that if $\beta + \alpha_i \in \Delta_+$ for $i \notin P$, then $\beta + \alpha_i \in \Omega$ with $\operatorname{ht}(\beta + \alpha_i) > \operatorname{ht}\beta$, against the assumption that β is of maximal height.

Let R be a connected component of the subdiagram $\operatorname{Supp}(\beta - \alpha) = S(A) \setminus P = \{i \in S(A) \mid k_i > m_i\}$. Define $\beta' = \sum_{i\in R} m_i\alpha_i$ and $\beta'' = \beta - \beta'$. We observe that

a) $\varphi_i(\beta) \geq 0$ for every $i \in R$;

b) $\varphi_i(\beta'') \leq 0$ for every $i \in R$;

c) there exists $i \in R$ such that $\varphi_i(\beta'') > 0$.

In fact, Lemma 1.2.23 says that $\beta + k\alpha_i \in \Delta_+$ if and only if $-p \leq k \leq q$ with $\varphi_i(\beta) = p - q$, $p, q \in \mathbb{N}$, and 1 says $\beta + \alpha_i \notin \Delta_+$, so it implies $q = 0$ and then $\varphi_i(\beta) = p \geq 0$, which proves a). For every $i \in R$ holds

$$\varphi_i(\beta) = \sum_{j\in S(A)\setminus R} m_j a_{ji} < 0$$

that implies b). Moreover, if $\varphi_i(\beta) = 0$ for every $i \in R$, then $a_{ij} = 0$ for every $i \in R$, $j \in S(A) \setminus R$, i.e. R is a connected component of $S(A)$ that is connected. It follows from a), b), and c) that $\varphi_i(\beta') \geq 0$ for every $i \in R$ and there exists $j \in R$ such that $\varphi_i(\beta') > 0$. This implies that R is a Dynkin diagram of positive type. Consider now the element $\alpha' = \sum_{i\in R}(k_i - m_i)\alpha_i$ and we observe that $\operatorname{Supp}\alpha' = R$ and that $\varphi_i(\alpha') = \varphi_i(\alpha - \beta)$ for every $i \in R$. Resuming, we know that $\alpha \in M \subseteq K$, then $\varphi_i(\alpha) \leq 0$ and $\varphi_i(\beta) \geq 0$ for every $i \in R$. It follows that $\varphi_i(\alpha') = \varphi_i(\alpha) - \varphi_i(\beta) \leq 0$ for every $i \in R$. This leads to a contradiction because we have already shown that R is a Dynkin diagram of positive type. It follows that $\alpha \in \Delta_+$.

We observe that for the element 2α we have what we have shown for α, particularly $2\alpha \in \Delta_+$. By Lemma 1.2.23 follows that $\alpha \notin \Delta_+^{re}$, then $\alpha \in \Delta_+^{im}$. □

By Lemma 1.1.13 we can give the following definition.

Definition 1.3.7. Let A be a symmetrizable Cartan matrix, then $A = DB$, with $B = (b_{i,j})$ a symmetric matrix and $b_{ij} \in \mathbb{Z}[\frac{1}{2}]$. Then we define a bilinear form $(\cdot,\cdot)$ on Γ by $(\alpha_i, \alpha_j) = b_{ij}$.

Remark 1.3.8. If A is a symmetrizable Cartan matrix and $\alpha = \sum_{i=1}^n c_i\alpha_i \in \Gamma$, then $(\alpha, \alpha) \in \mathbb{Z}$. Indeed:

$$(\alpha, \alpha) = (\sum_{i=1}^n c_i\alpha_i, \sum_{j=1}^n c_j\alpha_j) = 2\sum_{i<j} c_ic_jb_{ij} + \sum_{i=1}^n c_{ii}b_ii.$$

Both summands belong to $\mathbb{Z}$.

Lemma 1.3.9. *Let A be a symmetrizable Cartan matrix and $(\cdot,\cdot)$ the associated bilinear form, then the following hold:*

1. *the bilinear form $(\cdot,\cdot)$ is W-invariant, i.e. $(w(\alpha), w(\beta)) = (\alpha, \beta)$ for every $\alpha, \beta \in \Gamma$ and $w \in W$;*
2. *$\alpha \in \Gamma$ is a real root if and only if $(\alpha,\alpha) > 0$;*
3. *$\alpha \in \Gamma$ is an imaginary root if and only if $(\alpha,\alpha) \leq 0$;*
4. *$\alpha \in \Gamma$ is isotropic if and only if there exist a connected subdiagram $\tilde{S}$ of $S(A)$ of type Z and $w \in W$ such that $w(\alpha)$ is an imaginary root of $\tilde{S}$.*

Proof. First of all, we observe that $(\alpha, \alpha_i) = 0$ for every $\alpha \in U_i = \{\alpha \in \Gamma \mid \langle \alpha, \alpha_i^\vee \rangle = 0\}$. Indeed, if $\alpha = \sum_{j=1}^n c_j \alpha_j$, then:

$$(\alpha, \alpha_i) = \sum_{j=1}^n c_j(\alpha_j, \alpha_i) = d_i^{-1} \sum_{j=1}^n c_j d_i b_{ij} = d_i^{-1}\langle \alpha, \alpha_i^\vee \rangle = 0.$$

In particular we have that $(r_i(\alpha), \alpha_i) = 0$ for every $\alpha \in \Gamma$ and every $\alpha_i \in \Pi$.

We will prove 1) on the generators $\{r_1, \dots, r_n\} \subseteq W$. Let $\beta, \gamma \in \Gamma$, then:

$$(r_i(\beta), r_i(\gamma)) = (r_i(\beta), \gamma - \langle \gamma, \alpha_i^\vee \rangle \alpha_i) = (r_i(\beta), \gamma) - \langle \gamma, \alpha_i^\vee \rangle (r_i(\beta), \alpha_i).$$

In particular it follows that, for every $\beta, \gamma \in \Gamma$, holds $(r_i(\beta), r_i(\gamma)) = (r_i(\beta), \gamma) = (\beta, r_i(\gamma))$. Then using this identity and the fact that $r_i^2 = id$, we have

$$(r_i(\beta), r_i(\gamma)) = (r_i(\beta), \gamma) = (r_i(\beta), r_i^2(\gamma)) = (\beta, r_i^2(\gamma)) = (\beta, \gamma).$$

Let $\alpha \in \Delta^{re}$, then there exists $w \in W$ such that $w(\alpha) = \alpha_i \in \Pi$. By 1) it follows that

$$(\alpha, \alpha) = (w(\alpha), w(\alpha)) = (\alpha_i, \alpha_i) = b_{ii} \geq 0.$$

Let $\alpha = \sum_{i=1}^n c_i \alpha_i \in \Delta^{im}$. By 1) it follows that, without loss of generality, we can assume $\alpha \in K$. By the definition of K we have:

$$(\alpha, \alpha) = \sum_{i=1}^n c_i(\alpha_i, \alpha) = \sum_{i=1}^n d_i^{-1} c_i \langle \alpha_i, \alpha \rangle \leq 0.$$

Since $\Delta = \Delta^{re} \sqcup \Delta^{im}$, 2) and 3) are proved.

Let α be an isotropic root. Then, from 3), it follows that it is an imaginary root. As above, we can assume $\alpha = \sum_{i=1}^n c_i \alpha_i \in M$. Then $0 = (\alpha, \alpha) = \sum_{i=1}^n c_i(\alpha_i, \alpha)$. It follows that $\langle \alpha_i, \alpha \rangle = 0$ for every $i \in \operatorname{Supp} \alpha$. Then it follows that α is a root for a subdiagram of $S(A)$ of type Z and that $\alpha = m\delta$ with $m \in \mathbb{N}^+$ and δ as defined in Theorem 1.1.19. □

Definition 1.3.10. Let A be a Cartan matrix, we say that A is of hyperbolic type if it is of negative type and every proper connected subdiagram of $S(A)$ is of type P or Z.

From now on, we will focus on symmetrizable indecomposable Cartan matrices of positive, zero, or hyperbolic type. For simplicity, we will refer to them as Cartan matrices.

Lemma 1.3.11. *Suppose that, for any two disjoint subdiagrams of $S(A)$, one of them is of positive type. Then any $\alpha \in K \cap \Gamma$ has a connected support. Equivalently, $M = K \cap \Gamma$.*

Proof. Suppose that $\alpha = \beta + \gamma$ and that $P = \operatorname{Supp} \beta$ and $P' = \operatorname{Supp} \gamma$ are disjoint. Without loss of generality, we may assume that P is a diagram of positive type, then $\varphi_i(\beta) > 0$ for some i such that $p_i \in P$. It follows that $\varphi_i(\alpha) = \varphi_i(\beta) > 0$, which contradicts the hypothesis $\alpha \in K$. □

Remark 1.3.12. By Lemma 1.1.10 follows that Cartan matrices of positive and zero type verify the hypotheses of Lemma 1.3.11. If A is a Cartan matrix of hyperbolic type, it also verifies the hypotheses of Lemma 1.3.11. Indeed, if S_1 and S_2 are disjoint subdiagrams of $S(A)$ of zero type, then there must be a connected subdiagram $\tilde{S} \subseteq S(A)$ such that $S_1 \subseteq \tilde{S}$ and $\varnothing \neq \tilde{S} \cap S_2 \subsetneq S_2$. This is possible because S_2 is of zero type and then it has more than one vertex. Then $\tilde{S}$ is a proper subdiagram of $S(A)$ and is of negative type, because it contains S_1: a proper subdiagram of zero type. This leads to a contradiction because A is of hyperbolic type.

In conclusion, if $S(A)$ is of positive, zero or negative type, for every two proper disjoint subdiagrams of $S(A)$, at most one of them is of zero type.

Lemma 1.3.13. *Let A be a Cartan matrix of positive, zero or hyperbolic type and $\alpha \in \Gamma$, then the following hold:*

1. *if $(\alpha, \alpha) \leq 1$, then either $\alpha \in \Gamma_+$, $-\alpha \in \Gamma_+$ or $\alpha = 0$;*
2. *if $(\alpha, \alpha) = 1$, then $\alpha \in \Delta^{re}$.*

Proof. Suppose by contradiction that $\pm\alpha \notin \Gamma_+$ and $\alpha \neq 0$. Then we can write $\alpha = \beta - \gamma$ with $\beta, \gamma \in \Gamma_+$. Let $P = \operatorname{Supp} \beta$ and $P' = \operatorname{Supp} \gamma$. We observe that P and P' are disjoint subdiagrams of $S(A)$. Moreover, both P and P' are disjoint unions of connected subdiagrams, at most one of which is of zero type, as follows by Remark 1.3.12. We conclude by computing

$$(\alpha, \alpha) = (\beta - \gamma, \beta - \gamma) = (\beta, \beta) + (\gamma, \gamma) + 2(-\beta, \gamma). \tag{1.3.1}$$

Both (β, β) and (γ, γ) are non-negative and at least one of them is positive. Moreover, we observe that:

$$2(-\beta, \gamma) = -2(\sum_{i \in P} \sum_{j \in P'} c_i c_j b_{ij}) \geq 1$$

because B has coefficients in $\mathbb{Z}[\frac{1}{2}]$ and $b_{ij} \leq 0$ for every $i \in P, j \in P'$ and at least one of them is strictly positive, because $S(A)$ is connected. Then in Equation (1.3.1) there is a summand that is at least one, and two non-negative summands, at least one of which is positive. It follows that $(\alpha, \alpha) \geq 1$, that is a contradiction.

By 1) it follows that we can suppose $\alpha \in \Gamma_+$. Now consider the orbit of α under the action of W. By Lemma 1.3.9 we have that $(w(\alpha), w(\alpha)) = (\alpha, \alpha) = 1$, and then by 1) it follows that $w.\alpha \subseteq \Gamma_+ \cup \Gamma_- \cup \{0\}$. Let β be an element of minimal height of $W.\alpha \cap \Gamma_+$, then $\beta \in \Pi$, otherwise there exists $p_i \in \operatorname{Supp} \beta$ such that $r_i(\beta) \in W.\alpha \cap \Gamma_+$ and $\operatorname{ht} \beta > \operatorname{ht} r_i(\beta)$. □

Lemma 1.3.14. *Let A be a Cartan matrix and $\alpha \in \Gamma_+$, then the following hold:*

1. *α is an imaginary root if and only if $(\alpha, \alpha) \leq 0$;*
2. *if A is a symmetric matrix, α is a real root if and only if $(\alpha, \alpha) = 1$.*

Proof. By Lemma 1.3.13, we may assume without loss of generality that $\alpha \in \Gamma_+$. Let $\beta \in \Gamma_+ \cup W.\alpha$ be an element of minimal hight. It follows that $\langle \beta, \alpha_i^\vee \rangle \leq 0$ for every i, otherwise $\operatorname{ht} r_i(\beta) = \operatorname{ht}(\beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i) < \operatorname{ht} \beta$. It follows that $\beta \in K \cup \Gamma_+$. By Lemma 1.3.11 it follows that $\operatorname{Supp} \beta$ is connected and then $\beta \in M$. By Lemma 1.3.5 it follows that $\alpha \in \Delta^{im}$, which proves 1).

By Lemma 1.3.13 it follows that if $(\alpha, \alpha) = 1$, then α is a real root. Vice versa, we observe that if A is symmetric, then $B = \frac{1}{2} A$ and then $(\alpha_i, \alpha_i) = 1$ for every i. The thesis follows by the definition of real root and Lemma 1.3.9. □

Let us make some useful example. We will consider only symmetric Dynkin diagrams, indeed in the second part of the work we will work with these diagrams.

Example 1.3.15. Let us consider the Dynkin diagram of finite type A_2 : ●—●, and the corresponding Cartan matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. We want to compute its root system: we have the elements $\alpha_1, \alpha_2 \in \Pi$, and we need to apply the simple reflections to find the real roots:

- $r_1(\alpha_2) = \alpha_2 - \langle \alpha_1, \alpha_2 \rangle \alpha_1 = \alpha_2 + \alpha_1$;
- $r_2(\alpha_1) = \alpha_1 - \langle \alpha_2, \alpha_1 \rangle \alpha_2 = \alpha_1 + \alpha_2$;
- $r_1(\alpha_1 + \alpha_2) = r_1(\alpha_1) + r_1(\alpha_2) = -\alpha_1 + \alpha_1 + \alpha_2 = \alpha_2$;
- $r_2(\alpha_1 + \alpha_2) = r_2(\alpha_1) + r_2(\alpha_2) = \alpha_1 + \alpha_2 - \alpha_2 = \alpha_1$.

It follows that the set of positive real roots is $\Delta_+^{re} = \alpha_1, \alpha_2, \alpha_1 + \alpha_2$. Since A_2 is a Dynkin diagram of finite type, it does not admit imaginary roots, hence $\Delta_+ = \Delta_+^{re}$. Generalizing to the case of A_n : ●—●⋯●—●, we observe that for every $i = 1, \ldots, n$, the following properties hold:

- $r_i(\alpha_i) = -\alpha_i$;
- if $i \neq n$, then $r_{i+1}(\alpha_i) = \alpha_i + \alpha_{i+1}$;
- if $i \neq 1$, then $r_{i-1}(\alpha_i) = \alpha_i + \alpha_{i-1}$;
- $r_j(\alpha_i) = \alpha_i$ for every $j \notin \{i-1, i, i+1\}$.

It is straightforward to deduce that the positive root system of A_n is:

$$\Delta_+ = \{ \sum_{i \leq k \leq j} \alpha_k \mid 1 \leq i \leq j \leq n \}.$$

Example 1.3.16. Let us consider the Dynkin diagram of affine type $A_1^{(1)}$: ○⟺●, and the corresponding Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. By Lemma 1.3.5 together with Lemma 1.3.6, it follows that $\Delta_+^{im}(A) = \mathbb{N}\delta$, where δ is the vector $(1,1)$ found in Theorem 1.1.7. Let us compute the real roots of $A_1^{(1)}$ using two different methods. First, we apply the criterion found in Lemma 1.3.14, namely we impose:

$$1 = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a^2 - 2ab + b^2 = (a-b)^2$$

This implies that the real roots are the vectors $(a, a+1)$ and $(a, a-1)$, with $a \in \mathbb{N}$. Alternatively, we can proceed more directly by observing that:

- $r_1(\alpha_2) = 2\alpha_1 + \alpha_2$;
- $r_2(\alpha_1) = \alpha_1 + 2\alpha_2$.

Using these reflections, we can inductively construct the real roots:

- $r_1(k\alpha_1 + (k+1)\alpha_2) = -k\alpha_1 + (k+1)(2\alpha_1 + \alpha_2) = (k+2)\alpha_1 + (k+1)\alpha_2$;
- $r_2((k+1)\alpha_1 + k\alpha_2) = (k+1)(\alpha_1 + 2\alpha_2) - k\alpha_2 = (k+1)\alpha_1 + (k+2)\alpha_2$.

It follows that $\Delta_+^{re} = \{(k, k+1) \in \Gamma_+, |, k \in \mathbb{N}\} \cup \{(k+1, k) \in \Gamma_+, |, k \in \mathbb{N}\}$.

Chapter 2

Representations of Quivers

In this chapter we recall the fundamental definitions and standard results from the theory of quiver representations. We present Gabriel's Theorem 2.2.1 and then analyze the case of the 2-Kronecker quiver, which motivates the search for a generalization of Gabriel's Theorem 2.2.1.

2.1 Quivers and Indecomposable Representations

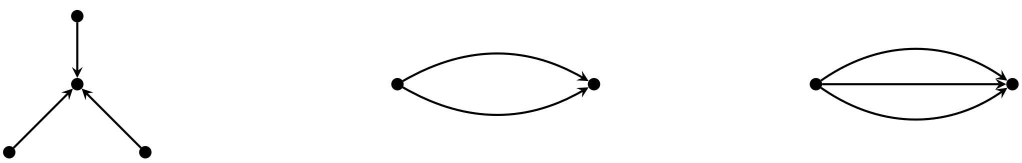
Definition 2.1.1. A quiver $Q = (S, \Omega)$ is an oriented graph, i.e. it is defined by its underlying graph S:

- a set of vertices S_0;
- a set of edges S_1;

and by an orientation Ω of the graph, given by two functions $s, t : S_1 \longrightarrow S_0$, called the source function and the target function, respectively.

We will consider only quivers without loops, that is, quivers for which for every $l \in S_1$, $s(l) \neq t(l)$.

Example 2.1.2. Some examples of quivers are:



Notice that the quivers

are two different quivers, with same underlying graph S but different orientations Ω and Ω'.

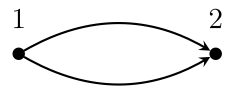
Definition 2.1.3. A representation of a quiver Q is defined by

- a collection of vector spaces $(V_i)_{i \in S_0}$, one for every vertex of the quiver Q;
- a collection of linear maps $(\varphi_l : V_{s(l)} \longrightarrow V_{t(l)})_{l \in S_1}$, one for every oriented edge of the quiver Q.

We will denote the representation $(V_i, \varphi_l)_{i \in S_0, l \in S_1}$ simply by V.

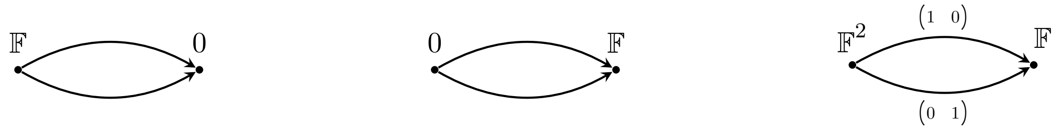
Definition 2.1.4. To every representation V of a quiver Q, we can associate a vector, called the dimension vector, defined as $\alpha = (\dim V_i)_{i \in S_0}$.

Example 2.1.5. Let Q be the quiver



also known as the 2-Kronecker quiver.

The following are three different representations of the quiver Q:



The dimension vectors of the above representations are $\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\alpha_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively.

Definition 2.1.6. Let $U = (U_i, \psi_l)_{i \in S_0, l \in S_1}$ and $V = (V_i, \varphi_l)_{i \in S_0, l \in S_1}$ be two representations of the quiver Q. We say that U is a subrepresentation of V if

- $U_i \subseteq V_i$ is a vector subspace;
- ψ_l is the restriction of φ_l to $U_{s(l)}$.

Definition 2.1.7. Given two representations U and W of the quiver Q, we define the representation $V = U \oplus W$ as

- $V_i = U_i \oplus W_i$ for every $i \in S_0$;
- $\theta_l = \varphi_l \oplus \psi_l : U_{s(l)} \oplus W_{s(l)} \longrightarrow U_{t(l)} \oplus W_{t(l)}$ for every $l \in S_1$.

A representation of a quiver Q is called irreducible if it does not admit any non-trivial subrepresentation and is called indecomposable if it cannot be written as the direct sum of two representations of Q.

Example 2.1.8. The simplest examples of indecomposable objects are the irreducible representations $U^{(i)}$, defined, for every $i \in S_0$, as follows:

- $(U^{(i)})_j = 0$ for every $i \neq j$ and $(U^{(i)})_i = \mathbb{F}$;
- $\varphi_l = 0$ for every $l \in S_1$.

More precisely, the $U^{(i)}$'s are the only irreducible representations of a quiver Q.

Definition 2.1.9. A morphism of representations of the quiver Q between V and W is defined by a collection of linear maps $(f_i : V_i \longrightarrow W_i)_{i \in S_0}$, one for every vertex of the quiver Q, such that for every $l \in S_1$, $\psi_l \circ f_{s(l)} = f_{t(l)} \circ \varphi_l : V_{s(l)} \longrightarrow V_{t(l)}$. In particular, we require that the following diagram commutes for every $l \in S_1$:

$$\begin{array}{ccc} V_{s(l)} & \xrightarrow{\varphi_l} & V_{t(l)} \\ {\scriptstyle f_{s(l)}}\downarrow & & \downarrow{\scriptstyle f_{t(l)}} \\ W_{s(l)} & \xrightarrow[\psi_l]{} & W_{t(l)} \end{array}$$

Remark 2.1.10. To every quiver $Q = (S, \Omega)$, we can associate a Dynkin diagram by considering the underlying graph S. It is associated with its Cartan matrix, which is a symmetric Cartan matrix.

Definition 2.1.11. Let Q be a quiver with numbered vertices $S_0 = \{1, \dots, n\}$ and consider $\alpha = \sum_{i=1}^{n} k_i \alpha_i \in \Gamma_+$. We define $M^\alpha(Q, \mathbb{F})$ as the space of all representations of Q with fixed dimension vector α.

We observe that a representation V with dimension vector α can be described by a tuple of linear maps $(\varphi_l)_{l \in S_1}$, where $\varphi_l \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^{k_{s(l)}}, \mathbb{F}^{k_{t(l)}})$. Equivalently

$$M^\alpha(Q, \mathbb{F}) = \bigoplus_{l \in S_1} \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^{k_{s(l)}}, \mathbb{F}^{k_{t(l)}}) \simeq \bigoplus_{l \in S_1} (\mathbb{F}^{k_{t(l)}} \otimes (\mathbb{F}^{k_{s(l)}})^*).$$

We will set $GL(\alpha) = GL_{k_1} \times \cdots \times GL_{k_n}$ and $\mathrm{End}(\alpha) = \bigoplus_{i \in S_0} \mathrm{End}_{\mathbb{F}}(\mathbb{F}^{k_i})$. We also set $g = \dim GL(\alpha)$. Moreover, we will denote $q(\beta) = (\beta, \beta)$ for $\beta \in \Gamma_+$.

An isomorphism between two representations $V_1, V_2 \in M^\alpha(Q, \mathbb{F})$ is given by a tuple $(f_i)_{i \in S_0}$ of invertible endomorphisms, i.e., $(f_i)_{i \in S_0} \in GL(\alpha)$. This means that the isomorphism class of $V \in M^\alpha(Q, \mathbb{F})$ is the orbit of V in $M^\alpha(Q, \mathbb{F})$ for the action of $GL(\alpha)$. The action is given by:

$$(f_i)_{i \in S_0}.(\varphi_l)_{l \in S_1} = (f_{s(l)}^{-1} \circ \varphi_l \circ f_{t(l)})$$

We observe that the normal subgroup C generated by $(I_{k_i})_{i\in S_0}$ acts trivially on $M^{\alpha}(Q,\mathbb{F})$, thus we consider the quotient action of $G^{\alpha} := GL_{k_1} \times \ldots \times GL_{k_n}/C$. G^{α} is a linear algebraic group and $M^{\alpha}(Q,\mathbb{F})$ has a natural structure of algebraic variety. We have also the following equalities:

- $\dim M^{\alpha}(Q,\mathbb{F}) = \sum_{l\in S_1} k_{s(l)}k_{t(l)} = -\frac{1}{2}\sum_{i\neq j} a_{ij}k_ik_j$;
- $\dim G^{\alpha} + 1 = \sum_{i=1}^{n} k_i^2$.

It follows that

$$\dim M^{\alpha}(Q,\mathbb{F}) - \dim G^{\alpha} = 1 - (\alpha,\alpha) = 1 - q(\alpha). \tag{2.1.1}$$

Definition 2.1.12. Let Q be a quiver and let $\alpha \in \Gamma_+$. We define $M^{\alpha}_{ind}(Q,\mathbb{F}) \subseteq M^{\alpha}(Q,\mathbb{F})$ as the set of indecomposable representations of Q of dimension vector α.

Definition 2.1.13. Let G be a linear algebraic group over $\mathbb{F}$. A subgroup $T \subseteq G$ is called a torus if $T \otimes \overline{\mathbb{F}} \simeq \overline{\mathbb{F}}^{\times} \times \cdots \times \overline{\mathbb{F}}^{\times}$. We say that T is an $\mathbb{F}$-split torus if $T \simeq \mathbb{F}^{\times} \times \cdots \times \mathbb{F}^{\times}$.

Definition 2.1.14. Let G be a linear algebraic group acting on an algebraic variety X. For $x \in X$, we denote by G_x the stabilizer of x, i.e.,

$$G_x = \{g \in G \mid g.x = x\}.$$

We denote by $\mathfrak{g}_x$ the Lie algebra of G_x.

We say that $x \in X$ is:

- a free point if $G_x = 0$;
- an infinitesimally free point if $\mathfrak{g}_x = 0$;
- a quasi-free point if G_x does not contain any non-trivial $\mathbb{F}$-split torus.

Definition 2.1.15. Let A be an associative unital algebra over $\mathbb{F}$, and let V be an A-module. We say that V is a brick if $\mathrm{End}_A(V) \simeq \mathbb{F}$. Similarly, a representation U of a quiver Q is a brick if $\mathrm{End}_Q(U) \simeq \mathbb{F}$.

Definition 2.1.16. Let A be an associative unital algebra. We define the radical of A as $\operatorname{rad} A = \{a \in A \mid aM = 0$ for every irreducible module M of $A\}$.

The following is a classical result of representation theory known as the Fitting Lemma. We formulate it within the framework of quiver representation theory.

Theorem 2.1.17. *Let Q be a quiver and let V be a representation of Q. Then V is indecomposable if and only if every element of $\mathrm{End}_Q(V)$ is either an isomorphism or nilpotent.*

Remark 2.1.18. If V is an indecomposable representation of Q, then it follows from Theorem 2.1.17 that, for every $\varphi \in \mathrm{End}_Q(V)$, we can write $\varphi = \lambda \,\mathrm{Id} + N$, where $N \in \mathrm{End}_Q(V)$ is a nilpotent endomorphism.

Lemma 2.1.19. *Let $U \in M^\alpha(Q, \mathbb{F})$. Then the following are equivalent:*

1. *U is an indecomposable representation;*
2. *$\mathrm{End}_Q U$ does not contain non-scalar $\mathbb{F}$-split semisimple elements;*
3. *U is a quasi-free point of $M^\alpha(Q, \mathbb{F})$ with respect to the action of G^α;*
4. *$\mathfrak{g}_U$ does not contain non-zero $\mathbb{F}$-split tori.*

The following properties are equivalent:

a. *$\mathrm{End}_Q(U) = \mathbb{F}$;*

b. *U is a free point;*

c. *U is an infinitesimally free point.*

Proof. First, we observe that for $U \in M^\alpha(Q, \mathbb{F})$, $G_U = \mathrm{Aut}_Q(U)/\langle \mathrm{Id}_U \rangle$ and $\mathfrak{g}_U = \mathrm{End}_Q(U)/\langle \mathrm{Id}_U \rangle$.

The equivalence between 2), 3) and 4) follows directly from the previous observation, and similarly the equivalence between a),b) and c).

We will prove the equivalence between 1) and 2).

If U is decomposable, then there exist $\pi : U \longrightarrow U_1$ projection and $\iota : U_1 \longrightarrow U$ such that $\pi \circ \iota = \mathrm{Id}_{U_1}$, moreover $\iota \circ \pi \in \mathrm{End}_Q(U)$ and it is a scalar split semisimple element.

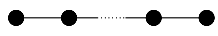
Now, we suppose that U is indecomposable. Let f be a split semisimple element and suppose that $\lambda \in \mathbb{F}$ is one of its eigenvalues. It follows from Theorem 2.1.17 that $f - \lambda \,\mathrm{Id}_U$ is a nilpotent endomorphism of U. Moreover, $f - \lambda \cdot \mathrm{Id}_U$ is also split semisimple, since both f and $\lambda \,\mathrm{Id}_U$ are split semisimple and they commute. Hence, their difference is split semisimple as well. A nilpotent split semisimple endomorphism is the zero map, therefore $f = \lambda \,\mathrm{Id}_U$. □

2.2 Gabriel's Theorem

The following theorem due to Gabriel gives us a fundamental result on the representations of quivers. We omit the proof here, since we will later present a more general result. The proof can be found in [Gab72].

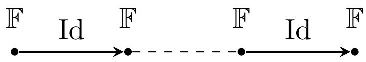
Gabriel's Theorem 2.2.1. *A quiver Q admits a finite number of isomorphism classes of indecomposable representations if and only if its underlying graph is a Dynkin diagram of finite type. Moreover, in this case, the indecomposable representations of the graph are in bijection with the positive roots of the Dynkin diagram.*

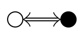
We now illustrate some applications of Gabriel's theorem with explicit examples.

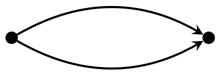
Example 2.2.2. Consider the Dynkin diagram A_n: . We have seen in Example 1.3.15 that the positive root system of A_n is $\Delta_+ = \{\sum_{i\le k\le j}\alpha_k \mid 1 \le i \le j \le n\}$. We enumerate the indecomposable representations of A_n inductively on n. If $n = 1$, then there exists a unique indecomposable representation, which is the irreducible one.

For $n > 1$, we restrict our attention to indecomposable representations whose dimension vector has support on all vertices of A_n, since otherwise the representation would correspond to a proper subquiver isomorphic to A_m with $m < n$. Thus, we consider the indecomposable representation corresponding to the positive root $\alpha = \sum_{i=1}^n \alpha_i$.

It is straightforward to verify that the representation given by $V_i = \mathbb{F}$ for every $i = 1,\dots,n$ and $\varphi_{(i,j)} = \mathrm{Id} : \mathbb{F} \longrightarrow \mathbb{F}$ for every $(i,j) \in S_1$ is an indecomposable representation of A_n with dimension vector α. Moreover, we need not explicitly consider the orientation since we can choose a numeration of the vertices such that (i,j) is an oriented edge from i to j. By Gabriel's Theorem 2.2.1, this is the unique indecomposable representation with dimension vector equal to α. This representation, in the case of the equioriented quiver with underlying graph A_n, is the following:

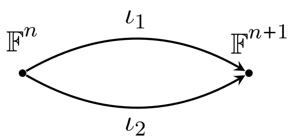


Example 2.2.3. Consider the Dynkin diagram $A_1^{(1)}$: . We have seen in Example 1.3.16 that the real positive root system of $A_1^{(1)}$ is $\Delta_+^{re} = \{(k, k+1) \in \Gamma_+ \mid k \in \mathbb{N}\} \cup \{(k+1, k) \in \Gamma_+ \mid k \in \mathbb{N}\}$, and that the imaginary positive roots are $\Delta_+^{im} = \{(n,n) \mid n \in \mathbb{N} \setminus \{0\}\}$. We now examine the indecomposable representations of the Kronecker quiver



whose underlying graph corresponds to the diagram $A_1^{(1)}$. We expect to find a relation between the dimensions of the indecomposable representations of the Kronecker quiver and the roots of $A_1^{(1)}$. In particular, we will exhibit an indecomposable representation for every root.

- W_n is the representation



where $\iota_1(x_1,\dots,x_n) = (x_1,\dots,x_n,0)$ and $\iota_2(x_1,\dots,x_n) = (0,x_1,\dots,x_n)$. This representation is indecomposable, indeed if $n = 1$ it is an irreducible representation. Otherwise, if

$n > 1$, we will show that $\mathrm{End}_Q(W_n)$ has only invertible and nilpotent element, and then we will conclude that W_n is indecomposable by Theorem 2.1.17. Consider an endomorphism $(A, B) \in \mathrm{End}_{\mathbb{F}}(\mathbb{F}^n) \oplus \mathrm{End}_{\mathbb{F}}(\mathbb{F}^{n+1})$. We will denote $a_1, \dots, a_n \in \mathbb{F}^n$ the rows of A, $b_1, \dots, b_n \in \mathbb{F}^{n+1}$ the rows of B. For every $i = 1, \dots, n+1$, we set $\tilde{b}_i = (b_{i1}, \dots, b_{in}) \in \mathbb{F}^n$ and $\hat{b}_i = (b_{i2}, \dots, b_{i,n+1}) \in \mathbb{F}^n$. Then we have that (A, B) is an endomorphism of W_n if and only if the following conditions hold:

1. $\iota_1 A = B\iota_2$, i.e., if and only if

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \\ \underline{0} \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \\ \tilde{b}_{n+1} \end{pmatrix}.$$

   It follows that $\tilde{b}_{n+1} = 0$ and $a_i = \tilde{b}_i$ for every $i = 1, \dots, n$.

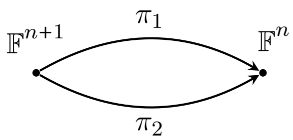
2. $\iota_1 A = B\iota_2$, i.e., if and only if

$$\begin{pmatrix} \underline{0} \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \hat{b}_1 \\ \vdots \\ \hat{b}_n \\ \hat{b}_{n+1} \end{pmatrix}.$$

   It follows that $\hat{b}_1 = 0$ and $a_i = \hat{b}_{i+1}$ for every $i = 1, \dots, n$.

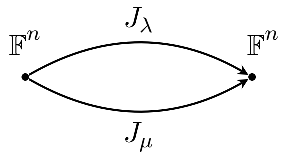
It follows that $A = \lambda I_n$ and $B = \lambda I_{n+1}$ for $\lambda \in \mathbb{F}$.

- Z_n is the representation



where $\pi_1(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$ and $\pi_2(x_1, \dots, x_{n+1}) = (x_2, \dots, x_{n+1})$. The proof that Z_n is indecomposable is similar to the proof that W_n is indecomposable.

- $V_{\lambda\mu}$ is the representation



where J_λ is the matrix of the Jordan block of size n with eigenvalue λ. Also in this case, every endomorphism in $\operatorname{End}_Q(V_{\lambda\mu})$ is either an isomorphism or nilpotent. Indeed, if $n = 1$, we can observe that $\operatorname{End}_Q(V_{\lambda\mu}) = \eta \operatorname{Id}_{V_{\lambda\mu}}$, therefore it is a brick. Otherwise, for $n > 1$, with calculation similar to the case of W_n, we can prove that $\operatorname{End}_Q(V_{\lambda\mu})$ is the set of matrices $A \in M_n(\mathbb{F})$ such that

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & a_1 & a_2 \\ 0 & \cdots & \cdots & 0 & a_1 \end{pmatrix}$$

for some $a_1, \dots, a_n \in \mathbb{F}$. We observe that $V_{\lambda\mu}$ has only indecomposable or nilpotent endomorphism, therefore from Theorem 2.1.17 follows that $V_{\lambda\mu}$ is indecomposable. However, in this case, $V_{\lambda\mu}$ is not a brick.

The following theorem, that we will not prove, is the general result concerning the indecomposable representations of the 2-Kronecker quiver.

Theorem 2.2.4. *Let V be an indecomposable representation of the 2-Kronecker quiver over the algebraically closed field $\mathbb{F}$, then it is isomorphic to exactly one of the following:*

- *W_n for $n \in \mathbb{N}$;*
- *Z_n for $n \in \mathbb{N}$;*
- *$V_{\lambda\mu}$ for $[\lambda : \mu] \in \mathbb{P}^1(\mathbb{F})$;*

where W_n, Z_n and $V_{\lambda\mu}$ are the indecomposable representations defined in Example 2.2.3.

Chapter 3

Orientation and Representations

In this chapter, we analyze the problem of orientation. Gabriel's theorem is independent of the orientation, and thus we expect that any generalization of Gabriel's theorem will also exhibit this invariance. We then introduce reflection functors, which allow us to act on the dimension vector of a representation using elements of the Weyl group, while preserving indecomposability. In addition, we present a method for changing the orientation of a quiver in order to apply reflection functors. This method has the desirable property of preserving the number of indecomposable representations, but unfortunately, it only works over finite fields.

3.1 Reflection Functors

Definition 3.1.1. Let Q be a quiver with vertex set S_0 and arrow set S_1. A vertex $i \in S_0$ is called a sink if no arrow has i as its starting point, i.e., if $s(l) \neq i$ for all $l \in S_1$. Similarly, i is a source if no arrow ends at i, i.e., if $t(l) \neq i$ for all $l \in S_1$. We say that i is admissible if it is a source or a sink.

Definition 3.1.2. Let Q be a quiver, and let $i \in S_0$ be an admissible vertex. We define the quiver $\tilde{r}_i(Q)$ to be the quiver with the same underlying graph as Q, but with the orientation modified by reversing all arrows with i as an endpoint. That is, the new orientation $\tilde{r}_i(\Omega)$ is defined by the following rules:

- For every arrow $l \in S_1$ such that $i \neq s(l)$ and $i \neq t(l)$, set $\tilde{r}_i(s)(l) = s(l)$ and $\tilde{r}_i(t)(l) = t(l)$;
- For every arrow $l \in S_1$ such that either $i = s(l)$ or $i = t(l)$, set $\tilde{r}_i(s)(l) = t(l)$ and $\tilde{r}_i(t)(l) = s(l)$.

Definition 3.1.3. Let Q be a quiver, and let $i \in S_0$ be a sink. We define the functor

$$F_i^+ : \mathrm{Rep}(Q) \longrightarrow \mathrm{Rep}(\tilde{r}_i(Q)).$$

Given a representation $V = (V_k, \varphi_l)_{k\in S_0, l\in S_1}$ the image $F_i^+(V)$ is given by:

- $(F_i^+(V))_k = V_k$, for every $k \neq i$;
- $(F_i^+(V))_l = \varphi_l : V_{s(l)} \longrightarrow V_{t(l)}$, for every $l \in S_1$ such that $t(l) \neq i$;
- $(F_i^+(V))_i = \ker \varphi^+$, where
$$\varphi^+ : \bigoplus_{l\in S_1, t(l)=i} V_{s(l)} \longrightarrow V_i$$
is the unique map such that $\varphi^+_{|V_{s(l)}} = \varphi_{s(l)}$ for every $l \in S_1$ such that $t(l) = i$;
- for every $l \in S_1$ such that $t(l) = i$, we define $(F_i^+(V))_l : \ker\varphi^+ \longrightarrow (F_i^+(V))_{t(l)}$ as the composition:
$$\ker\varphi^+ \hookrightarrow \bigoplus_{l\in S_1, t(l)=i} V_{s(l)} \twoheadrightarrow V_{s(l)}.$$

Now, we have to define the image $F_i^+(f)$ of a morphism $f = (f_k)_{k\in S_0} \in \mathrm{Hom}_Q(V, W)$:

- for every $k \neq i$, we define $(F_i^+(f))_k := f_k : V_k \longrightarrow W_k$;
- let $S_{(i)}$ be the set of vertices near i, i.e. the vertices $k \in S_0$ such that there exists $l \in S_1$, $t(l) = i$ and $s(l) = k$. We define $(F_i^+(f))_i$ as the restriction of the map $f_{(i)} := \bigoplus_{k\in S_{(i)}} f_k : \bigoplus_{k\in S_{(i)}} V_k \longrightarrow \bigoplus_{k\in S_{(i)}} W_k$ to $\ker\varphi^+$. Then $(F_i^+(f))_i : \ker\varphi^+ \longrightarrow \ker\psi^+$ is well defined by the commutativity of the following diagram

$$\begin{array}{ccc} \bigoplus_{k\in S_{(i)}} V_k & \xrightarrow{\varphi^+} & V_i \\ \Big\downarrow{\scriptstyle f_{(i)}} & & \Big\downarrow{\scriptstyle f_i} \\ \bigoplus_{k\in S_{(i)}} W_k & \xrightarrow[\psi^+]{} & W_i \end{array}$$

Similarly, we define the functor $F_i^- : \mathrm{Rep}(Q) \longrightarrow \mathrm{Rep}(\tilde{r}_i(Q))$ when $i \in S_0$ is a source. On the objects we define:

- $(F_i^-(V))_k = V_k$, for every $k \neq i$;
- $(F_i^-(V))_l := \varphi_l : V_{s(l)} \longrightarrow V_{t(l)}$, for every $l \in S_1$ such that $s(l) \neq i$;
- $(F_i^-(V))_i = \mathrm{coker}\,\varphi^-$, where
$$\varphi^- : V_i \longrightarrow \bigoplus_{l\in S_1, s(l)=i} V_{t(l)}$$
is the unique map such that $\pi_{s(l)} \circ \varphi^- = \varphi_{t(l)}$ for every $l \in S_1$ such that $s(l) = i$;

- for every $l \in S_1$ such that $s(l) = i$, we define $(F_i^-(V))_l : (F_i^-(V))_{t(l)} \longrightarrow \operatorname{coker}\varphi^-$ as the composition:

$$V_{t(l)} \hookrightarrow \bigoplus_{l \in S_1, t(l)=i} V_{t(l)} \twoheadrightarrow \operatorname{coker}\varphi^-.$$

On the morphisms, we define:

- for every $k \neq i$, we define $(F_i^-(f))_k := f_k : V_k \longrightarrow W_k$;

- let $S_{(i)}$ be the set of vertices near i, i.e. the vertices $k \in S_0$ such that there exists $l \in S_1$ such that $s(l) = i$ and $t(l) = k$. We define $(F_i^-(f))_i$ as the projection on $\operatorname{coker}\varphi^-$ of the map $f_{(i)} := \bigoplus_{k \in S_{(i)}} f_k : \bigoplus_{k \in S_{(i)}} V_k \longrightarrow \bigoplus_{k \in S_{(i)}} W_k$. Then $(F_i^-(f))_i : \operatorname{coker}\varphi^- \longrightarrow \operatorname{coker}\psi^-$ is well defined because of the commutativity of the following diagram:

$$\begin{array}{ccc} V_i & \xrightarrow{\varphi^-} & \bigoplus_{k \in S_{(i)}} V_k \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{(i)}} \\ W_i & \xrightarrow[\psi^-]{} & \bigoplus_{k \in S_{(i)}} W_k \end{array}$$

The functors F_i^+ and F_i^- are called reflection functors.

Remark 3.1.4. In the definition of the reflection functors we should check that the image of a morphism of Q-representations is a morphism of $\tilde{r}_i(Q)$-representations. We will check it in the case of F_i^+, for F_i^- the proof is analogous.

If $f \in \operatorname{Hom}_Q(V, W)$, we have to prove that $F_i^+(f)_{t(l)} \circ F_i^+(\varphi_l) = F_i^+(\psi_l) \circ F_i^+(f)_{s(l)}$ for every $l \in S_1$, where $F_i^+(\varphi_l) = (F_i^+(V))_l$ and $F_i^+(\psi_l) = (F_i^+(W))_l$. It follows by the definition that the condition holds for every $l \in S_1$ such that $t(l) \neq i$. Consider $l \in S_1$ such that $t(l) = i$ and $v = (v_k)_{k \in S_{(i)}} \in \ker\varphi^+ \subseteq \bigoplus_{l \in S_1, t(l)=i} V_{s(l)}$. The thesis follows by the following computations:

- $(F_i^+(f)_{t(l)} \circ F_i^+(\varphi_l))(v) = F_i^+(f)_{t(l)}(v_{s(l)}) = f_{s(l)}(v_{s(l)})$;

- $(F_i^+(\psi_l) \circ F_i^+(f)_{s(l)})(v) = F_i^+(\psi_l)((f_k(v_k))_{k \in S_{(i)}}) = f_{s(l)}(v_{s(l)})$.

The following theorem lists the properties of reflection functors which we will use in the proof of the main result. A proof of this theorem can be found in [BGP73].

Theorem 3.1.5. *Let Q be a quiver and let $i \in S_0$ be an admissible vertex. The following properties hold for the reflection functors F_i^+ and F_i^- (the statements are given for F_i^+; if i is a source, replace $+$ with $-$ accordingly):*

1. *$F_i^+(U \oplus W) = F_i^+(U) \oplus F_i^+(W)$ for every $U, W \in \mathrm{Rep}(Q)$;*

2. *let $U \in \mathrm{Rep}(Q)$ be an indecomposable representation. Then exactly one of the following holds:*

   (a) *$U \simeq U^{(i)}$ and $F_i^+(U^{(i)}) = 0$. Here, $U^{(i)}$ is the indecomposable object defined in Example 2.1.8;*

   (b) *$U \not\simeq U^{(i)}$. In this case, $F_i^+(U)$ is an indecomposable object and $F_i^- F_i^+(U) = U$. Moreover, $\dim F_i^+(U) = r_i(\dim U)$.*

3.2 Independence from the Orientation

In this section we will consider only finite fields, i.e., $\mathbb{F} = \mathbb{F}_q$, where $q = p^t$, p prime.

Lemma 3.2.1. *Let G be a linear algebraic group acting on an $\mathbb{F}_q$-vector space V of dimension n, and let V^* be the dual representation of V. Then the number of G-orbits in V and V^* coincide.*

Proof. Let $\chi : \mathbb{F}_q \longrightarrow \mathbb{C}^*$ be a nontrivial character. Set $A = \mathrm{Hom}_{Set}(V, \mathbb{C})$ and $A^* = \mathrm{Hom}_{Set}(V^*, \mathbb{C})$. Both A and A^* are finite-dimensional $\mathbb{C}$-vector spaces. We define the discrete Fourier transform $\mathcal{F} : A \longrightarrow A^*$ defined by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = q^{-\frac{n}{2}} \sum_{v \in V} f(v)\chi(\xi(v))$$

for every $\xi \in V^*$.

First, we want to show that $\widehat{\widehat{f}}(v) = f(-v)$. For every $v \in V$, we have:

$$\begin{aligned} \widehat{\widehat{f}}(v) &= q^{-\frac{n}{2}} \sum_{\varphi \in V^*} (q^{-\frac{n}{2}} \sum_{w \in V} f(w)\chi(\varphi(w)))\chi(v(\varphi)) \\ &= q^{-n} \sum_{w \in V} \sum_{\varphi \in V^*} f(w)\chi(\varphi(w + v)) \end{aligned}$$

where in the above, we identified V with $(V^*)^*$ via $v \mapsto (\varphi \mapsto \varphi(v))$. We now aim to study the value of the innermost summation as $w \in W$ varies:

- if $w = -v$, then we obtain

$$\sum_{\varphi \in V^*} f(w)\chi(\varphi(v + w)) = f(-v) \sum_{\varphi \in V^*} \chi(0) = q^n f(-v);$$

- if $w \neq -v$, then let $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ be a basis of V^* such that $\varphi_1(v + w) = 1$ and $\varphi_j(v + w) = 0$ for every $j > 1$. For every $\varphi \in V^*$ we can write $\varphi = \sum_{j=1}^n \lambda_j \varphi_j$. Then we obtain

$$\sum_{\varphi \in V^*} \chi(\varphi(v+w)) = \sum_{\lambda_1,\dots,\lambda_n \in \mathbb{F}_q} \sum_{i=1}^{n} \chi(\lambda_i \varphi_i(v+w)) = q^{n-1} \sum_{\lambda \in \mathbb{F}_q} \chi(\lambda) = 0.$$

The last equality follows by the fact that the image of χ lies in a set of root of unity.

We can therefore conclude that $\widehat{\widehat{f}}(v) = f(-v)$. It follows that $f \in A$ is a G-invariant function if and only if $\widehat{f} \in A^*$ is a G-invariant function. Moreover, $\dim A^G$ is equal to the number of orbits of G in V, because every G-invariant function is uniquely identified by its values on the orbits of G in V. Similarly, $\dim (A^*)^G$ is the number of orbits of G in V^*. The thesis follows directly by $\dim A^G = \dim (A^*)^G$. □

Remark 3.2.2. Let G be a linear algebraic group acting on an $\mathbb{F}_q$-vector space V and let $\mathcal{O}$ be an orbit of this action. For every point $x \in \mathcal{O}$, we can consider $T_x \subseteq G_x$ maximal $\mathbb{F}_q$-split torus of the stabilizer. These tori are all in the same conjugacy class. Indeed, if $x, y \in \mathcal{O}$, then $y = g.x$ for some $g \in G$. It follows that $G_y = gG_xg^{-1}$, and so we conclude that $T_y = gT_xg^{-1}$.

In particular, we can define the conjugacy class of the maximal $\mathbb{F}_q$-split torus of the stabilizer of the orbit $\mathcal{O}$ as the conjugacy class of T_x for any $x \in \mathcal{O}$.

Definition 3.2.3. Let G be a linear algebraic group acting on an $\mathbb{F}_q$-vector space V and let V^* be its dual representation. Let $T \subseteq G$ be an $\mathbb{F}_q$-split torus, we set $d(T, V)$ the number of orbits $\mathcal{O}$ of G over V such that the conjugacy class of T in G is the same of the maximal $\mathbb{F}_q$-split torus of the stabilizer of the orbit $\mathcal{O}$.

Lemma 3.2.4. *Let G be a linear algebraic group acting on an $\mathbb{F}_q$-vector space V and let V^* be its dual representation. Let $T \subseteq G$ be an $\mathbb{F}_q$-split torus, then $d(T, V) = d(T, V^*)$.*

Proof. First, we observe that, if it does not exists $T \subseteq G$ non-trivial $\mathbb{F}_q$-split torus, then the result follows by Lemma 3.2.1, because every orbit is quasi-free.

Suppose that there exists at least one $T \subseteq G$ non-trivial $\mathbb{F}_q$-split torus and let us fix it. We will proceed by induction on $\dim V$.

Let $C \subseteq V$ and let $C^\vee \subseteq V^*$ be the set of points fixed by the action of T. We will prove several properties on C, and all these proofs can be repeated also for $C^\vee$.

Set $W = N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. The action of T on C is trivial, then we can define the action of W on C and $C^\vee$. In fact, if $g \in N_G(T)$, $t \in T$ and $x \in C$, then $g^{-1}tg.x = \tilde{t}.x = x$, therefore $tg.x = g.x$ that implies $g.x \in C$.

We now want to show that $x \in C$ is quasi-free with respect to the action of W if and only if T is a maximal $\mathbb{F}_q$-split torus of G_x. Suppose that $\tau \subseteq W_x$ is a non-trivial $\mathbb{F}_q$-split torus and let $\tilde{T} \subseteq N_G(T)$ be its preimage. Then $T \subseteq \tilde{T}$ and $\tilde{T}$ is an $\mathbb{F}_q$-split torus. It follows that $\tilde{T} = T$, and therefore $\tau = e$. Conversely, if there exists an $\mathbb{F}_q$-split torus $\tilde{T}$ such that $G_x \supseteq \tilde{T} \supseteq T$, then

$\tilde{T}/T \subseteq W_x$ is an $\mathbb{F}_q$-split torus, which must be trivial since x is a quasi-free point. It follows that $\tilde{T} = T$.

Let x, y be quasi-free points of C for the action of W, and suppose that they belong to the same orbits with respect to the action of G, i.e., $y = g.x$. We aim to show that they belong to the same orbit with respect to the action of W. We know that both T and $T_1 = gTg^{-1}$ are maximal $\mathbb{F}_q$-split tori of G_y, then they are conjugate via $g_1 \in G_y$, i.e., $T = g_1 T_1 g_1^{-1} = g_1 g T g^{-1} g_1^{-1}$. It follows that $g_1 g \in N_g(T)$, then its image in W is the element we were searching.

From the above results, it follows that $d(T, V)$ is the number of quasi-free orbits of W in C, and analogously $d(T, V^*)$ is the number of quasi-free orbits of W in $C^\vee$.

We want now to apply inductive hypothesis to the action of W on C and $C^\vee$, but previously we have to verify that $C^\vee$ is isomorphic to the dual representation C^* of C. We have that C^* is naturally isomorphic to $V^*/\operatorname{ann}(C)$. Since T is semisimple, this quotient admits a retraction and so it is canonically a subrepresentation of V^*. Our aim is to show that it coincides with the subrepresentation of V^* on which T acts trivially. Consider $\varphi \in V^*$ and $g \in T$. We observe that $\varphi - g.\varphi \in \operatorname{ann}(C)$, indeed $(\varphi - g.\varphi)(v) = \varphi(v) - \varphi(g^{-1}.v) = \varphi(v) - \varphi(v) = 0$ for every $v \in C$. Since $g.\varphi = \varphi - (\varphi - g.\varphi)$, then we can conclude that T acts trivially on $V^*/\operatorname{ann}(C)$.

By applying the inductive hypothesis on W and its action on C and $C^\vee$, we obtain:

$$d(T, V) = d_W(\{e\}, C) = d_W(\{e\}, C^\vee) = d(T, V^*)$$

where we set $d_W(\{e\}, C)$ and $d_W(\{e\}, C^\vee)$ the number of quasi-free orbits of W in C and $C^\vee$.

We showed that $d(T, V) = d(T, V^*)$ for every $T \subseteq G$ non-trivial $\mathbb{F}_q$-split torus. If $T = \{e\}$, then

$$\begin{aligned} d(\{e\}, V) &= \#\{\mathcal{O} \subseteq V \mid G.\mathcal{O} = \mathcal{O}\} - \sum_{[T]\in\Theta} d(T, V) \\ &= \#\{\mathcal{O} \subseteq V^* \mid G.\mathcal{O} = \mathcal{O}\} - \sum_{[T]\in\Theta} d(T, V^*) = d(\{e\}, V^*) \end{aligned}$$

where Θ is the set of conjugacy classes of tori in G. □

Corollary 3.2.5. *Let G be a linear algebraic group acting on an $\mathbb{F}_q$-vector space V and let V^* be its dual representation. Then the number of quasi-free orbits of G in V and in V^* is the same.*

Proof. The result follows by Lemma 3.2.4 and by the fact that the number of quasi-free orbits of G in V is $d(\{e\}, V)$ and similarly the number of quasi-free orbits of G in V^* is $d(\{e\}, V^*)$. □

Lemma 3.2.6. *Let G be a linear algebraic group acting on two $\mathbb{F}_q$-vector spaces V_1 and V_2. Then, the number of orbits of $V_1 \oplus V_2$ equals the number of orbits of $V_1 \oplus V_2^*$.*

Proof. Consider $x, y \in V_1$ and suppose that they are in the same orbit $\mathcal{O} \subseteq V_1$, i.e., $y = g.x$. Let $\mathcal{O}_x \subseteq V_2$ be an orbit with respect to the action of G_x on V_2, then $g.\mathcal{O}_x \subseteq V_2$ is an orbit with respect to the action of G_y. In fact, consider $v_1, v_2 \in \mathcal{O}_x$, $v_2 = h.v_1$ with $h \in G_x$, since $G_x = g^{-1}G_y g$, it follows that $h = g^{-1}\tilde{h}g$ with $\tilde{h} \in G_y$, then $g.v_2 = \tilde{h}g.v_1$, and so $g.v_1$ and $g.v_2$ belong to the same orbit $\mathcal{O}_y = g.\mathcal{O}_x$. Therefore, for every orbit $\mathcal{O} \subseteq V_1$ with respect to the action of G, the numbers $o(\mathcal{O}) = \#\{\mathcal{O}_x \subseteq V_2 \,|\, G_x.\mathcal{O}_x = \mathcal{O}_x, x \in \mathcal{O}\}$ and $o^*(\mathcal{O}) = \#\{\mathcal{O}_x \subseteq V_2^* \,|\, G_x.\mathcal{O}_x = \mathcal{O}_x, x \in \mathcal{O}\}$ are well defined. We want to show that $\#\{\mathcal{O} \subseteq V_1 \oplus V_2 \,|\, G.\mathcal{O} = \mathcal{O}\} = \sum_{\mathcal{O} \subseteq V_1} o(\mathcal{O})$. This follows from the fact that the orbits of G in $V_1 \oplus V_2$ are $\bigcup_{g \in G}(g.x, g.\mathcal{O}_x)$, for some $x \in V_1$ and some $\mathcal{O}_x \subseteq V_2$ orbit with respect to the action of G_x, and by the observation that, chosen $x \neq y$ in the same orbit of G in V_1, they produce the same set of orbits of $V_1 \oplus V_2$. The fact that these are the orbits follows by the following construction: consider $(x, v) \in V_1 \oplus V_2$, then the orbit generated by this element is $\mathcal{O}_{12} = \{(g.x, g.v) \,|\, g \in G\}$. In particular, $G_x.(x, v) = (x, \mathcal{O}_x) \subseteq \mathcal{O}_{12}$. Moreover, $g.(x, \mathcal{O}_x) \subseteq \mathcal{O}_{12}$ for every $g \in G$, and the union of these set clearly cover the orbit.

By Lemma 3.2.1 follows that $o(\mathcal{O}) = o^*(\mathcal{O})$. We conclude because, using the above identity for V_2 and V_2^*, we obtain:

$$\#\{\mathcal{O} \subseteq V_1 \oplus V_2 \,|\, G.\mathcal{O} = \mathcal{O}\} = \sum_{\mathcal{O} \subseteq V_1} o(\mathcal{O}) = \sum_{\mathcal{O} \subseteq V_1} o^*(\mathcal{O}) = \#\{\mathcal{O} \subseteq V_1 \oplus V_2^* \,|\, G.\mathcal{O} = \mathcal{O}\}.$$

□

Lemma 3.2.7. *Let G be a linear algebraic group acting on two $\mathbb{F}_q$-vector spaces V_1 and V_2. Let $T \subseteq G$ be an $\mathbb{F}_q$-split torus, then $d(V_1 \oplus V_2, T) = d(V_1 \oplus V_2^*, T)$.*

Proof. The proof is the same as the one of Lemma 3.2.4, using Lemma 3.2.6 in place of Lemma 3.2.1. □

Definition 3.2.8. Let Q be a quiver, $\alpha \in \Gamma_+$, provided that $\mathbb{F} = \mathbb{F}_q$ with $q = p^t$ for some prime number p, we define $n_\alpha(Q, q)$ as the number of isomorphism classes of isomorphism of indecomposable representations of Q with dimension vector α defined over $\mathbb{F}_q$.

Lemma 3.2.9. *Let Q be a quiver, $\alpha \in \Gamma_+$ and $\alpha_i \in \Pi$. Assume that $\alpha \neq \alpha_i$ and that $n_\alpha(Q, q) \neq 0$, then $r_i(\alpha) \in \Gamma_+$ and $n_{r_i(\alpha)}(Q, q) \neq 0$. Moreover, $n_\alpha(Q, q) = n_{r_i(\alpha)}(Q, q)$.*

Proof. Recall that $M^\alpha(Q, \mathbb{F}_q) = \bigoplus_{l \in S_1} V_{s(l)}^* \otimes V_{t(l)}$. In this sum, if we substitute a subset of the summands indexed by $\tilde{S}_1 \subseteq S_1$ with their dual, we obtain the space $M^\alpha(\tilde{Q}, \mathbb{F}_q)$, where $\tilde{Q}$ is the quiver with the same underlying graph as Q, and with the reverse orientation on the edges belonging to $\tilde{S}_1$. Moreover, it follows by Lemma 2.1.19 and Lemma 3.2.7 that $n_\alpha(\tilde{Q}, q) = n_\alpha(Q, q)$. Consider $\tilde{S}_1 \subseteq S_1$ such that i is an admissible vertex in $\tilde{Q}$. By Theorem 3.1.5 it follows that $n_\alpha(\tilde{Q},q) = n_{r_i(\alpha)}(r_i(\tilde{Q}),q)$. Using again the result above for $\widehat{S_1} \subseteq S_1$ such that $\widehat{r_i(\tilde{Q})} = Q$, we obtain:

$$n_\alpha(Q,q) = n_\alpha(\tilde{Q},q) = n_{r_i(\alpha)}(r_i(\tilde{Q}),q) = n_{r_i(\alpha)}(Q,q).$$

□

Chapter 4

The Space of Representations and Kac's Theorem

In this chapter, we use algebraic geometry to develop the tools necessary for the proof of the main result, namely Kac's theorem. In particular, we consider the space of representations of a quiver with a fixed dimension vector. Within this algebraic variety, we observe that the set of indecomposable representations forms a constructible subset. This observation allows us to define the dimension of the space of isomorphism classes of indecomposable representations with a fixed dimension vector.

We note that this space corresponds to the space of orbits under the action of an algebraic group. This correspondence enables us to compute the dimensions of the orbits, ultimately leading to a formula for the dimension of the orbit space, which depends only on the Dynkin diagram of the quiver and the dimension vector.

At the end of the chapter, we prove the theorem, first over finite fields, and then, using a standard reduction modulo p argument, we extend the proof to the case of algebraically closed fields.

In this chapter, unless otherwise specified, we work over algebraically closed fields.

4.1 Preliminaries of Algebraic Geometry

In this section all the topological notions are relative to the Zariski topology.

Definition 4.1.1. Let X be an algebraic variety, a subset $Y \subseteq X$ is called locally closed if $Y = U \cap V^c$, where $U, V \subseteq X$ open subsets. We say that Y is constructible if it is a finite union of locally closed set.

Remark 4.1.2. The collection of constructible sets is closed under finite union, finite intersection and taking complements.

We will use two classical results. The references give statements in the more general setup of morphism of schemes of finite type.

The following is a useful result due to Chevalley concerning constructible sets. For more details, see [DG67, pp. IV, 1.8.4].

Theorem 4.1.3. *Let $\Phi : X \longrightarrow Y$ be a morphism of varieties, then $\Phi(X)$ is constructible, and more generally, Φ sends constructible sets to constructible sets.*

Definition 4.1.4. If X is an algebraic variety and $f : X \longrightarrow \mathbb{Z}$. We say that f is upper semicontinuous if the set $\{x \in X \mid f(x) \geq n\}$ is closed for all $n \in \mathbb{N}$.

The following result can be found in [DG67, pp. IV, 13.1.3].

Theorem 4.1.5. *If X and Y are algebraic varieties and $f : X \longrightarrow Y$ is a morphism of varieties, then the function $X \longrightarrow \mathbb{Z}$ defined by $\dim_X f^{-1}(f(x))$ is upper semicontinuous.*

The following lemma is a classical result of algebraic geometry. The proof can be found in [Mum04, corollary to theorem 2, section 8, chapter 7].

Lemma 4.1.6. *Let $\pi : X \longrightarrow Y$ be a dominant morphism of irreducible algebraic varieties. Then every irreducible component of a fiber $\pi^{-1}(y)$, provided it is non empty, has dimension at least $\dim X - \dim Y$. Moreover, there is a non-empty open subset $U \subset Y$ such that $\dim \pi^{-1}(u) = \dim X - \dim Y$, for every $u \in U$.*

Lemma 4.1.7. *Let X be an algebraic variety endowed with the action of a connected algebraic group G. Then the following hold:*

1. *each orbit Gx is locally closed and irreducible;*

2. $\dim Gx = \dim G - \dim \mathrm{Stab}_G(x)$.

Proof. We observe that Gx is the image of the map $G \longrightarrow X$ defined by $g \mapsto g.x$, then $\overline{Gx}$ is irreducible and Gx is a constructible set. There exists $\varnothing \neq U \subseteq Gx$ with U open in Gx. The set $G.U = \bigcup_{g \in G} g.U$ is contained in Gx and is G-invariant, then it is equal to Gx. Each $g.U$ is open in $\overline{Gx}$, then Gx is open in $\overline{Gx}$, therefore Gx is locally closed.

Every fiber of $G \longrightarrow Gx$ is a subgroup conjugate to $\mathrm{Stab}_G(x)$. Therefore they have the same dimension. By Lemma 4.1.6, we get that $\dim Gx = \dim G - \dim \mathrm{Stab}_G(x)$. □

Definition 4.1.8. Let X be an algebraic variety endowed with the action of an algebraic group G. For every $s \in \mathbb{N}$, we define $X_{(s)} = \{x \in X \mid \dim Gx = s\}$ and $X_{(\leq s)} = \{x \in X \mid \dim Gx =\leq s\}$.

If $Y \subseteq X$ is a constructible set, we can define the numbers

- $\mu(Y) = \max_s(\dim Y \cap X_{(s)}) = \max_s(\dim Y \cap X_{(\leq s)})$;

- $t(Y) = \sum_s \#\{Z \subseteq \overline{Y \cap X_{(s)}} \mid Z$ is an irreducible component of dimension $s + \mu_G(Y)\}$.

Remark 4.1.9. If the set Y is G-invariant and if the set of orbits Y/G were an algebraic variety, then the number $\mu(Y)$ would represent its dimension.

4.2 Representation With Fixed Dimension Vector

We recall the definition of some important object defined in Section 2.1. In particular, we have seen in Definition 2.1.11 that $M^\alpha(Q,\mathbb{F})$ is the space of representation of the quiver Q over the field $\mathbb{F}$ with fixed dimension vector $\alpha \in \Gamma_+$. Similarly we defined $M^\alpha_{ind}(Q,\mathbb{F})$ in Definition 2.1.12 as the subspace of $M^\alpha(Q,\mathbb{F})$ of indecomposable representations.

In section Section 2.1 we also defined, for a vector $\alpha = \sum_{i=1}^n k_i\alpha_i$, the following objects:

- $GL(\alpha) = GL_{k_1} \times \cdots \times GL_{k_n}$;
- $G^\alpha = GL(\alpha)/C$ where C is the normal subgroup generated by Id_U;
- $g = \dim GL(\alpha)$;
- $q(\alpha) = (\alpha, \alpha)$;
- $\mathrm{End}(\alpha) = \bigoplus_{i=1}^n \mathrm{End}_{\mathbb{F}}(\mathbb{F}^{k_i})$.

Lemma 4.2.1. *The set $M^\alpha_{ind}(Q,\mathbb{F}) \subseteq M^\alpha(Q,\mathbb{F})$ is a constructible set.*

Proof. We will show that the complement of $M^\alpha_{ind}(Q,\mathbb{F})$ is constructible. If $U \in M^\alpha_{ind}(Q,\mathbb{F})^c$, i.e., U is decomposable, then there exist $U_1 \in M^{\alpha_1}(Q,\mathbb{F})$ and $U_2 \in M^{\alpha_2}(Q,\mathbb{F})$ such that $U = U_1 \oplus U_2$, with $\alpha = \alpha_1 + \alpha_2$. In particular, $U \in \mathrm{Im}(\Theta^\alpha_{\alpha_1,\alpha_2})$, where we define the map

$$\Theta^\alpha_{\alpha_1,\alpha_2} : M^{\alpha_1}(Q,\mathbb{F}) \times M^{\alpha_2}(Q,\mathbb{F}) \longrightarrow M^\alpha(Q,\mathbb{F}), \quad (U_1,U_2) \mapsto U_1 \oplus U_2.$$

It follows that

$$M^\alpha_{ind}(Q,\mathbb{F})^c = \bigcup_{\alpha_1+\alpha_2=\alpha} \mathrm{Im}(\Theta^\alpha_{\alpha_1,\alpha_2}).$$

Since, by Theorem 4.1.3, the image of a regular map between algebraic varieties is constructible, then $M^\alpha_{ind}(Q,\mathbb{F})^c$ is constructible because it is a finite union of constructible sets. □

Lemma 4.2.2. *Consider $\alpha = \sum_{i=1}^n k_i\alpha_i \in \Gamma_+$ and $\beta = \sum_{i=1}^n m_i\alpha_i$ such that $0 \le \beta \le \alpha$ and $\beta \neq 0, \alpha$. If $m_i' = k_i - m_i$ for every i, then the following identity holds:*

$$(\alpha - \beta, \beta) = \sum_{j=1}^n \frac{m_j m_j'}{k_j}\Big(\sum_{i=1}^n a_{ij}k_i\Big) + \frac{1}{2}\sum_{i,j=1}^n a_{ij}\Big(\frac{m_i}{k_i} - \frac{m_j}{k_j}\Big)k_ik_j. \qquad (4.2.1)$$

Proof. The following identities hold:

- the left hand side is $(\alpha-\beta,\beta) = \sum_{i=1}^{n} a_{ij}m_im_j'$;
- the first summand of the right hand side is:

$$\sum_{j=1}^{n}\frac{m_jm_j'}{k_j}(\sum_{i=1}^{n}a_{ij}k_i) = \sum_{i,j=1}^{n}a_{ij}m_jm_j'\frac{k_i}{k_j}$$
$$= \sum_{i\neq j}a_{ij}(m_im_j'\frac{m_j}{k_j}+m_jm_i'\frac{m_j'}{k_j}) + \sum_{i=1}^{n}a_{ii}m_im_i' = \sum_{i\neq j}a_{ij}m_im_j'(\frac{m_j}{k_j}+\frac{m_i'}{k_i}) + \sum_{i=1}^{n}a_{ii}m_im_i';$$

- the second summand of the right hand side is:

$$\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}(\frac{m_i}{k_i}-\frac{m_j}{k_j})k_ik_j = \sum_{1\leq i<j\leq n}a_{ij}(\frac{m_i^2k_j}{k_i}+\frac{m_j^2k_i}{k_j}-2m_im_j)$$
$$= \sum_{1\leq i<j\leq n}a_{ij}(m_im_j'\frac{m_i}{k_i}+m_j'm_i'\frac{m_j}{k_j}+m_im_j(\frac{m_i}{k_i}+\frac{m_j}{k_j}-2))$$
$$= \sum_{i\neq j}a_{ij}(m_im_j'\frac{m_i}{k_i}+m_im_j(\frac{m_j}{k_j}-1)) = \sum_{i\neq j}a_{ij}m_im_j'(\frac{m_i}{k_i}-\frac{m_j}{k_j}).$$

By adding the last two, we find that the right hand side is:

$$\sum_{i=1}^{n}a_{ii}m_im_i' + \sum_{i\neq j}a_{ij}m_im_j'(\frac{m_i}{k_i}-\frac{m_j}{k_j}+\frac{m_j}{k_j}+\frac{m_i'}{k_i}) = \sum_{i=1}^{n}a_{ij}m_im_j'$$

that is equal to the left hand side. □

Lemma 4.2.3. *Let Q be a quiver and let α be a vector in the fundamental chamber M. Then exactly one of the following holds:*

1. Supp α *belongs to* Table 1.2 *and* $q(\alpha) = (\alpha,\alpha) = 0$*;*
2. *for every $\beta_1,\dots,\beta_r$ non-zero vectors $\beta_i \geq 0$ such that $\alpha = \beta_1+\dots+\beta_r$, then $q(\alpha) < \sum_{i=1}^{r} q(\beta_i)$.*

Proof. Suppose without loss of generality that $Q = \operatorname{Supp}\alpha$ and suppose that 2 does not holds. There exist $\beta_1,\dots,\beta_r$ non-zero vectors $\beta_i \geq 0$ such that $\alpha = \beta_1+\dots+\beta_r$ and $q(\alpha) \geq \sum_{i=1}^{r} q(\beta_i)$. It follows that

$$\sum_{i=1}^{r}(\alpha-\beta_i,\beta_i) = (\alpha,\alpha) - \sum_{i=1}^{r}(\beta_i,\beta_i) \geq 0$$

and in particular there exists $\beta = \beta_i = \sum_{i=1}^{n} m_i\alpha_i$ such that $(\alpha-\beta,\beta)\geq 0$. By Equation (4.2.1) follows that:

$$0 \leq (\alpha-\beta,\beta) = \sum_{j=1}^{n}\frac{m_jm_j'}{k_j}(\alpha,\alpha_j) + \frac{1}{2}\sum_{i,j=1}^{n}a_{ij}(\frac{m_i}{k_i}-\frac{m_j}{k_j})k_ik_j$$

where $m_i' = k_i - m_i$, for every $i = 1, \dots, n$. Both summands are less than or equal to 0, so both must be 0. It follows that $\frac{m_i}{k_i} = \frac{m_j}{k_j}$, for every $i, j = 1, \dots, n$ such that $a_{ij} \neq 0$. Because $\operatorname{Supp} \alpha$ is connected, then $\alpha = k\beta$. Moreover, for every $i = 1, \dots, n$, we have $(\alpha, \alpha_i) = 0$, then $q(\alpha) = 0$ and $\operatorname{Supp} \alpha$ appears in Table 1.2. □

Lemma 4.2.4. *Consider a quiver Q and a vector α in the fundamental chamber M, and suppose that Item 2 of Lemma 4.2.3 holds for α, then $M^\alpha_{ind}(Q, \mathbb{F}) \subseteq M^\alpha(Q, \mathbb{F})$ is a dense set, i.e., $M^\alpha_{ind}(Q, \mathbb{F})$ contains a non-empty open set.*

Proof. If $\alpha = \beta + \gamma$, then we define the map:

$$\Theta^\alpha_{\beta,\gamma} : GL(\alpha) \times M^\beta(Q, \mathbb{F}) \times M^\gamma(Q, \mathbb{F}) \longrightarrow M^\alpha(Q, \mathbb{F}), \quad (g, U, W) \mapsto g.(U \oplus W).$$

We observe that this map is constant on the orbits of $H = GL(\beta) \times GL(\gamma) \subseteq GL(\alpha)$, so

$$\begin{aligned} \dim \overline{\operatorname{Im}}(\Theta^\alpha_{\beta,\gamma}) &\leq \dim GL(\alpha) + \dim M^\beta + \dim M^\gamma - \dim H \\ &= q(\alpha) + \dim M^\alpha + \dim M^\beta + \dim M^\gamma - \dim H. \end{aligned}$$

Since there is only a finite number of maps $\Theta^\alpha_{\beta,\gamma}$, and

$$\dim M^\alpha - \dim \overline{\operatorname{Im}}(\Theta^\alpha_{\beta,\gamma}) \geq q(\beta) + q(\gamma) - q(\alpha) > 0$$

the statement follows. □

We now proceed to stratify the algebra $\operatorname{End}(\alpha)$ according to the Jordan type of each component.

Definition 4.2.5. Consider $\alpha = \sum_{i=1}^n k_i \alpha_i \in \Gamma_+$. Let for $i = 1, \dots, n$, $\lambda_i = (\lambda_i^1, \dots, \lambda_i^r, \dots)$ be a partition of k_i. We say that $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition of α and we denote $\lambda^r_\cdot \in \Gamma_+$ the vector $(\lambda_1^r, \dots, \lambda_n^r)$, for every $r \in \mathbb{N}$.

For $\theta \in \operatorname{End}(\alpha)$, we say θ is of type λ when the maps $\theta_i \in \operatorname{End}(\mathbb{F}^{k_i})$ are nilpotent maps of type λ_i, i.e., λ_i^r is the number of Jordan blocks of size ≥ 0. We denote the space of endomorphism of type λ with $N_\lambda = \{\theta \in \operatorname{End}(\alpha) \mid \theta \text{ is of type } \lambda\}$.

Let $\theta \in \operatorname{End}(\alpha)$ be a fixed endomorphism, we define the space $Mod_\theta = \{U \in M^\alpha \mid \theta \in \operatorname{End}_Q(U)\}$.

Remark 4.2.6. We denote by z the partition associated to the zero map, i.e., $z_i = (k_i, 0, \dots)$, for every $i = 1, \dots, n$.

Lemma 4.2.7. *The following dimension formulas hold:*

1. *if $\theta \in \operatorname{End}(\alpha)$, then $\dim Mod_\theta = \sum_{l \in S_1} \sum_r \lambda^r_{s(l)} \lambda^r_{t(l)}$;*

2. $\dim N_\lambda = \dim GL(\alpha) - \sum_{i=1}^{n} \sum_r (\lambda_i^r)^2$.

Proof. Consider vector spaces V_1, V_2 of dimensions n_1 and n_2 respectively, and let λ_1 be a partition of n_1 and λ_2 be a partition of n_2. Let $f_1 \in \mathrm{End}(V_1)$ and $f_2 \in \mathrm{End}(V_2)$ be nilpotent endomorphisms of type λ_1 and λ_2 respectively. We will show that the dimension of the space $C = \{h : V_1 \longrightarrow V_2 \mid f_2 h = h f_1\}$ is $\sum_r \lambda_1^r \lambda_2^r$.

We observe that a map $h \in C$ is uniquely determined by the choice of images of the vectors generating the Jordan blocks for f_1. In particular, let $v_1, \dots, v_k$ be these vectors. If we choose $h(v_i)$, we only need to determine what $h(f_1^s(v_i))$ is for every s, but we have $h(f_1^s(v_i)) = f_2^s h(v_i)$. Moreover, we observe that $h(v_i)$ must be a vector with order of nilpotency less than or equal to that of v_i.

Since the number of vectors generating the Jordan blocks for f_1 with order of nilpotency r is $\lambda_1^r - \lambda_1^{r+1}$, it follows that:

$$\dim\{h : V_1 \longrightarrow V_2, |, f_2 h = h f_1\} = \sum_r (\lambda_1^r - \lambda_1^{r+1}) \left(\sum_{s \le r} \lambda_2^s \right) = \sum_s \lambda_2^s \sum_{r \ge s} (\lambda_1^r - \lambda_1^{r+1}) = \sum_s \lambda_1^s \lambda_2^s. \tag{4.2.2}$$

In order to compute $\dim Mod_\theta$, we need to compute

$$\dim\{\varphi_l : V_{s(l)} \longrightarrow V_{t(l)} \mid \theta_{t(l)} \varphi_l = \varphi_l \theta_{s(l)}\}$$

for every $l \in S_1$. It follows from equation Equation (4.2.2) that $\dim Mod_\theta = \sum_{l \in S_1} \sum_r \lambda_{s(l)}^r \lambda_{t(l)}^r$.

We observe that N_λ is an orbit for the conjugation action of $GL(\alpha)$ on $\mathrm{End}(\alpha)$. Fixed $\theta \in N_\lambda$, by Lemma 4.1.7 follows

$$\begin{aligned} \dim N_\lambda &= \dim GL(\alpha) - \dim \mathrm{Stab}(\theta) = \dim GL(\alpha) - \dim\{f \in GL(\alpha) \mid f\theta = \theta f\} \\ &= \dim GL(\alpha) - \dim\{f \in \mathrm{End}(\alpha) \mid f\theta = \theta f\} = \dim GL(\alpha) - \sum_{i=1}^{n} \sum_r (\lambda_i^r)^2 \end{aligned}$$

where the last equality follows by Equation (4.2.2) in the same way as 1). □

Definition 4.2.8. Consider a quiver Q and $\alpha = \sum_{i=1}^{n} k_i \alpha_i \in \Gamma_+$. We denote $I = M_{ind}^\alpha(Q, \mathbb{F})$ and $B = \{U \in M^\alpha(Q, \mathbb{F}) \mid U \text{ is a brick}\}$. We set $N = \{\theta \in \mathrm{End}(\alpha) \mid \theta \neq 0, \theta \text{ is nilpotent}\}$. Moreover, we will call $MN = \{(U, \theta) \in M^\alpha \times N \mid \theta \in \mathrm{End}_Q(U)\}$ and $I_{(s)} N = \{(U, \theta) \in I_{(s)} \times N \mid \theta \in \mathrm{End}_Q(U)\}$.

Remark 4.2.9. We observe that:

- $I = \bigcup_{s<g} I_{(s)}$;
- $N = \bigcup_{\lambda \neq z} N_\lambda$;
- $MN = \bigcup_{\lambda \neq z} MN_\lambda$;

- $I_{(s)}N \subseteq MN$.

From Lemma 2.1.19, it follows that $B = I_{(g-1)}$. Moreover, $I_{(s)}$ is locally closed in M^α, indeed, by Theorem 4.1.5 it follows that the map $I \longrightarrow \mathbb{N}$ defined by $U \mapsto \dim \operatorname{End}_Q(U)$ is an upper semicontinuous map, and $I_{(s)} = \{U \in I \mid \dim \operatorname{End}_Q(U) = g-s-1\} = \{U \in I \mid \dim \operatorname{End}_Q(U) \leq g-s-1\} \cap \{U \in I \mid \dim \operatorname{End}_Q(U) \geq g-s-1\}$ and both sets are locally closed.

Lemma 4.2.10. *Let Q be a quiver satisfying Item 2 of Lemma 4.2.3, $\alpha \in \Gamma_+$ and λ a partition of α. The following hold:*

1. *if $\lambda \neq z$, then $\dim MN_\lambda < g - q(\alpha)$, and as a consequence $\dim MN < g - q(\alpha)$;*
2. *for every $s < g-1$, we have $\dim I_{(s)} < s + 1 - q(\alpha)$;*
3. *B is a non-empty open subset of M^α. Moreover, $\mu(B) = 1 - q(\alpha)$ and $t(B) = 1$.*

Proof. Consider the projection $\pi : MN_\lambda \longrightarrow N_\lambda$. We observe that $\pi^{-1}(\theta) = Mod_\theta$, it follows by Lemma 4.2.7 that the dimension of the fibers is constant, then:

$$\dim MN_\lambda \leq \dim N_\lambda + \dim Mod_\theta = g - \sum_r q(\lambda^r_\cdot) < g - q(\alpha)$$

where the last inequality follows by the assumption that Q verifies Item 2 and by the observation that $\alpha = \sum_r \lambda^r_\cdot$. It follows immediately that $\dim MN < g - q(\alpha)$ since $MN = \bigcup_\lambda MN_\lambda$.

If $s < g-1$, then U is indecomposable and is not a brick, therefore there exist non-zero nilpotent endomorphisms of U. It follows that the projection $\pi : I_{(s)}N \longrightarrow I_{(s)}$ is surjective, and then:

$$\dim \pi^{-1}(U) = \dim(\operatorname{End}_Q(U) \cap N) = \dim(\operatorname{rad}(\operatorname{End}_Q(U))) = g - s - 1$$

where the last equality follows by the fact that $\operatorname{End}_Q(U)$ is a local algebra, and then, using Remark 2.1.18 we conclude that the codimension of the radical in $\operatorname{End}_Q(U)$ is 1. It follows that

$$\dim I_{(s)} = \dim I_{(s)}N - \dim \pi^{-1}(U) = \dim I_{(s)}N - (g-s-1) \leq \dim MN - g + s + 1 < s + 1 - q(\alpha)$$

and this conclude the proof of 2.

If $s < g-1$, we have $\dim I_{(s)} < \dim M^\alpha - (g-s-1) < \dim M^\alpha$, where the first inequality follows by Equation (2.1.1). It follows that $\overline{I}_{(s)}$ is a proper closed subset of M^α. Moreover, it follows from Lemma 4.2.4 that the decomposable representation are contained in a proper closed subset of M^α, then B is a non-empty open subset of M^α, and B is irreducible. It follows that $\mu(B) = \dim B - g + 1 = 1 - q(\alpha)$ and $t(B) = 1$ □

Theorem 4.2.11. *Let Q be a quiver and $\alpha \in M$ such that Item 2 of Lemma 4.2.3 holds. Then, $\mu_\alpha := \mu(M^\alpha_{ind}(Q,\mathbb{F})) = 1 - q(\alpha)$ and $t_\alpha := t(M^\alpha_{ind}(Q,\mathbb{F})) = 1$.*

Proof. By Lemma 4.2.10 and because $I = B \cup \bigcup_{s<g-1} I_{(s)}$, it follows that

$$\mu(M^{\alpha}_{ind}) = \max_{(s)} \mu(I_{(s)}) = \mu(B) = 1 - q(\alpha).$$

□

Theorem 4.2.12. *Let Q be a quiver with underlying graph S listed in Table 1.2. If $\alpha = k\delta \in M$, then $\mu_\alpha = \mu(M^{\alpha}_{ind}) = 1 - q(\alpha) = 1$ and $t(M^{\alpha}_{ind}(Q, \mathbb{F})) = 1$.*

The isomorphism classes of indecomposable representations of a quiver whose underlying graph is an affine Dynkin diagram are well-known. In particular, we can find a complete description of such representations in [Naz73].

The classification relies on the fact that it is possible to prove that every indecomposable representation of $A_n^{(1)}$ can be constructed from a representation of $A_1^{(1)}$, or can be seen as an indecomposable representation of a subgraph of $A_n^{(1)}$. The indecomposable representations of $A_1^{(1)}$ are described in Theorem 2.2.4.

A similar argument applies to indecomposable representation with associated Dynkin diagram $D_n^{(1)}$. The idea is to study the indecomposable representation of $D_4^{(1)}, D_5^{(1)}$ and $D_6^{(1)}$ and then to show that every indecomposable representation of $D_n^{(1)}$ can be constructed from a representation of one of these three Dynkin diagrams or it can be seen as an indecomposable representation of a subgraph of $D_n^{(1)}$.

Finally, one has to study the indecomposable representations of the exceptional cases $E_6^{(1)}, E_7^{(1)}$ and $E_8^{(1)}$.

4.3 Kac's Theorem

Theorem 4.3.1. *Let Q be a quiver and $\mathbb{F} = \mathbb{F}_q$ where $q = p^t$ for some prime p. For $\alpha \in \Gamma_+$, the following hold:*

1. *if $\alpha \notin \Delta_+$, then every representation of Q with dimension vector α is decomposable, i.e., $n_\alpha(Q, \mathbb{F}_q) = 0$;*

2. *if $\alpha \in \Delta_+^{re}$, then there exists a unique isomorphism class of indecomposable representations of dimension vector α, i.e., $n_\alpha(Q, \mathbb{F}_q) = 1$;*

3. *if $\alpha \in \Delta_+^{im}$, then $\lim_{t\to+\infty} \frac{n_\alpha(Q,\mathbb{F}_{p^t})}{p^{(1-(\alpha,\alpha))t}} = 1$, in particular*

$$\lim_{t\to+\infty} n_\alpha(Q, \mathbb{F}_{p^t}) = +\infty;$$

4. *$n_\alpha(Q, \mathbb{F}_q)$ does not depend on the orientation of Q.*

Proof. Consider $\alpha \in \Gamma_+ \setminus \Delta_+$ and suppose $m^\alpha(Q, \mathbb{F}_q)$ contains an indecomposable object. By Lemma 3.2.9 it follows that, for every $w \in W$, $w(\alpha) \in \Gamma_+$ and $m^{w(\alpha)}(Q, \mathbb{F}_q)$ contains an indecomposable object. Let $\beta = \sum_{i=1}^n k_i \alpha_i$ be the element of $W.\alpha$ of minimal height. The minimality of the height of β and the definition of $r_i(\beta) = \beta - (\sum_{j=1}^n a_{ij} k_j)\alpha_i$ imply that $\sum_{j=1}^n a_{ij} k_j \leq 0$ for every $i = 1, \dots, n$. Moreover, $\operatorname{Supp}\beta$ is connected, otherwise it could not have any indecomposable representation. It follows that $\beta \in M \subseteq \Delta_+^{im}$, and so $\alpha \in \Delta_+^{im}$.

Consider $\alpha \in \Delta_+^{re}$, by definition there exists sequence of minimal length $i_1, \dots, i_k$ such that $r_{i_k} \cdot \dots \cdot r_{i_1}(\alpha) = \alpha_i \in \Pi$. There exists a unique representation of dimension vector α_i, that is $U^{(i)}$. By Lemma 3.2.9 it follows that $n_\alpha(Q, \mathbb{F}_q) = 1$, which proves 2.

Consider the $\mathbb{F}_q$-rational points of $M_{ind}^\alpha(Q, \overline{\mathbb{F}}_p)$, where $\overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$. If $\alpha \in M$, it follows by Theorem 4.2.11 and Theorem 4.2.12 that the dimension is $\mu_\alpha = 1 - (\alpha, \alpha)$. Using elementary arguments from algebraic geometry and and the results mentioned above, one can conclude that, for $\alpha \in M$, we have:

$$\lim_{t \to +\infty} \frac{n_\alpha(Q, \mathbb{F}_{p^t})}{p^{(1-(\alpha,\alpha))t}} = 1.$$

An argument that partially prove this fact can be found Section 4.3.1. Since $\Delta_+^{im} = \bigcup_{w \in W} w(M)$, it follows from Lemma 3.2.9 that 3 holds for every $\alpha \in \Delta_+$.

Statement 4 follows immediately by Lemma 3.2.9 and by its proof. □

Kac's Theorem 4.3.2. *Let Q be a quiver with $\mathbb{F} = \overline{\mathbb{F}}$. For $\alpha \in \Gamma_+$, the following hold:*

1. *if $\alpha \notin \Delta_+$, then any representation of Q with dimension vector α is decomposable;*
2. *if $\alpha \in \Delta_+$, if α is non-divisible in Γ_+, then there exists an indecomposable representation of Q with dimension vector α. Moreover, $\mu_\alpha = 1 - (\alpha, \alpha)$, which does not depend on the orientation of Q;*
3. *if $\alpha \in \Delta_+^{re}$, there exists a unique isomorphism class of indecomposable representations of dimension vector α.*

Proof. Consider an object $U \in M^\alpha(Q, \mathbb{F})$, let $\mathbb{K}$ be the smallest subfield of $\mathbb{F}$, and let $\bar{\mathbb{K}}$ be its algebraic closure. We observe that U is defined over a subfield $\mathbb{F}_0 = \mathbb{K}(\xi_1, \dots, \xi_s)(\eta_1, \dots, \eta_t) \subseteq \mathbb{F}$, where $\xi_1, \dots, \xi_s$ are transcendental elements over $\mathbb{K}$, and $\eta_1, \dots, \eta_t$ generate the ring of integers of $\mathbb{K}[\xi_1, \dots, \xi_s]$.

The object U is decomposable if and only if there exists at least one projection in $\operatorname{End}_Q U$, meaning there exists a non-trivial solution $(X_i)i \in S_0 \in \bigoplus i \in S_0 \operatorname{End}_{\mathbb{F}}(U_i)$ to the system of equations:

$$\begin{cases} X_{s(l)}\varphi_l = \varphi_l X_{t(l)} & \forall l \in S_1 \\ X_i^2 = X_i & \forall i \in S_0 \end{cases} \tag{4.3.1}$$

where $(\varphi_l)_{l \in S_1}$ are the maps defining U and are defined over $\mathbb{F}_0$.

We analyze this system as follows: first, consider the case where $\operatorname{char} \mathbb{F} = p > 0$. Take an arbitrary specialization $\xi_i \mapsto \beta_i$ for $i = 1, \ldots, s$, $\eta_j \mapsto \gamma_j$ for $j = 1, \ldots, t$, with $\beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_t \in \bar{\mathbb{K}}$. By Theorem 4.3.1, if $\alpha \notin \Delta_+$, then Equation (4.3.1) admits a solution over the finite field $\mathbb{K}(\beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_t)$, and consequently, it has a solution over $\mathbb{F}$.

Now, suppose $\operatorname{char} \mathbb{F} = 0$. Let $R = \mathbb{Z}[\xi_1, \ldots, \xi_s][\eta_1, \ldots, \eta_t]$. Consider the reduction modulo p of R. As previously demonstrated, for $\alpha \notin \Delta_+$, there exists a solution of Equation (4.3.1) over the field of fractions of R_p/I for every prime number p and every prime ideal $I \subseteq R_p$. This implies that Equation (4.3.1) has a solution over $\mathbb{F}$, which completes the proof of part 1

To prove parts 2 and 3, we apply a similar argument. In particular, if U is an indecomposable representation of Q over a field $\mathbb{K}$ with a non-divisible dimension vector $\alpha \in \Gamma_+$, then U remains indecomposable over any finite Galois extension $\tilde{\mathbb{K}} \supseteq \mathbb{K}$. Indeed, let $U = U_1 \oplus \ldots \oplus U_k$ be the decomposition of U into indecomposable representations over $\tilde{\mathbb{K}}$. The Galois group $Gal(\tilde{\mathbb{K}}/\mathbb{K})$ permutes these indecomposable components, so $\dim U_1 = \ldots = \dim U_k = \beta$. Thus, $\alpha = k\beta$, and since α is non-divisible, we conclude that $k = 1$, meaning U is indecomposable over $\tilde{\mathbb{K}}$.

From this observation and Theorem 4.3.1, it follows that if α is a non-divisible root or a real root, there exists a representation admitting no solution to Equation (4.3.1) for any subfield of $\mathbb{F}$, and hence no solution exists in $\mathbb{F}$. Moreover, if $\alpha \in \Delta_+$, there is a unique representation as described above for every subfield, ensuring its uniqueness.

To conclude, note that over a finite subfield of cardinality p^t (or under reduction modulo p when $\operatorname{char} \mathbb{K} = 0$), the number of indecomposable representations is of order $p^{t(1-(\alpha,\alpha))}$, yielding $\mu_\alpha = 1 - (\alpha, \alpha)$. □

4.3.1 Counting $\mathbb{F}_q$-rational points

In order to make this thesis more self-contained we will sketch an elementary argument proving the inequality

$$\liminf_{t \to \infty} \frac{n_\alpha(Q, \mathbb{F}_{p^t})}{p^{t(1-(\alpha,\alpha))}} \geq 1.$$

This is a slighter weaker result than Theorem 4.3.1, but still significant.

The following lemma ensures that the notion of isomorphism between two quiver representations is independent by the field.

Lemma 4.3.3 (Derksen and Weyman [DW17], §5, p. 45). *Let U_1, U_2 be two indecomposable representations defined over $\mathbb{F}_q$. If $U_1 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r} \simeq U_2 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$, then $U_1 \simeq U_2$.*

In particular, $U_1, U_2 \in M^\alpha(Q, \mathbb{F}_{q^t})$ are isomorphic representations if and only if they belong to the same $GL(\alpha, \mathbb{F}_{q^t})$-orbit, where $GL(\alpha, \mathbb{F}_{q^t})$ is the group $GL(\alpha)$ defined over the field $\mathbb{F}_{q^t}$.

The following estimate is of elementary nature:

Lemma 4.3.4 (Borevich and Shafarevich [BS86], §5, p. 45)**.** *If* $f \in \mathbb{F}_q[x_1,\ldots,x_N]$ *is a non-zero polynomial and*

$$X_f(\mathbb{F}_{q^t}) = \{(a_1,\ldots,a_N) \in \mathbb{F}_{q^t}^N \mid f(a_1,\ldots,a_N) = 0\}.$$

Then there exists a positive constant C *such that*

$$\#X_f(F_{q^t}) \leq Cq^{Nt-1}.$$

Let Q be a quiver that satisfies Item 2 of Lemma 4.2.3 and let $\alpha = \sum_{i=1}^n k_i\alpha_i \in M$. We proved in Lemma 4.2.10 that the complement of bricks in $M^\alpha(Q,\mathbb{F})$ is contained in a Zariski closed set Z, which is defined over some finite field $\mathbb{F}_q$. It follows from Lemma 4.3.4 that

$$\#\{U \in M^\alpha(Q,\mathbb{F}_{q^t}) \,|\, U \text{ is a brick}\} \approx q^{t(\dim M^\alpha(Q,\mathbb{F}_{q^t}))}$$

where $f(t) \approx g(t)$ indicates that $\lim \frac{f(t)}{g(t)} = 1$.

Since U is a brick, its stabilizer for the action of $GL(\alpha,\mathbb{F}_{q^t})$ is of dimension one. Hence, the $GL(\alpha,\mathbb{F}_{q^t})$-orbit of U has cardinality

$$\#GL(\alpha,\mathbb{F}_{q^t})/(q^t-1) \approx q^{t(\sum_{i=1}^n k_i^2-1)}.$$

It follows that

$$\#\{\text{isomorphism classes of bricks } U \in M^\alpha(Q,\mathbb{F}_{q^t})\} \approx q^{(1-(\alpha,\alpha))t}.$$

Since every brick is indecomposable, we conclude that

$$\liminf_{t\to\infty} \frac{n_\alpha(Q,\mathbb{F}_{q^t})}{q^{(1-(\alpha,\alpha))t}} \geq 1. \tag{4.3.2}$$

Moreover, if Q has a Dynkin diagram of zero type as underlying graph, then Equation (4.3.2) still holds, as proven in [Pag16].

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