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Minimal Length and Coalescing Binaries: Effects on Tidal Heating

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Infinite Thanks to the Great Artist;
Who Gave us Infinite colors

A Sandrone

Abstract

It is known that different proposals which try to study a quantum theory of gravity suggest the presence of a minimal spacetime length, of the order of the Planck scale $L_P \sim 10^{-35}$ m. One of the possible and effective ways to introduce such a bound is through the qmetric, a bitensor that substitutes the classical metric, which can account for the property cited above. The introduction of this kind of metric actually enriches the spacetime with different features; for example, the concept of an irreducible area around a point, which turns out to give, as a consequence, a minimal step in area increment for a black hole horizon.

In this thesis we explore the impact of this feature on the dynamics of an astrophysical system of great interest, namely the coalescence of a black hole binary. After introducing the qmetric, we examine some of its properties, explaining how the limit length leads to the existence of a quantum of area of black hole horizon. Some effects derived by the qmetric are then explored; in particular, as a training step, a possible modification of a well known quantum effect in curved spacetime is shown: the emission of the Hawking radiation.

In the core of the work, the system of a coalescing binary is then presented, starting from the approximate Newtonian point of view and going to the relativistic picture. We describe how to implement the presence of a quantum of the horizon area and its consequences on the evolution of the system, more precisely on the phenomenon of *tidal heating*. As well known, this is essentially the absorption, by the horizons of the two black holes, of a part of the gravitational waves generated by the system in coalescence.

We show that the introduction of a step in area for a black hole horizon can suppress, partially or even completely, tidal heating. We find under what kind of conditions this can happen, computing the consequent phase shift in the emitted gravitational waves.

All this is studied for the (simplistic) case of two Schwarzschild black holes, and then we study the situation of two Kerr black holes, which is more realistic for astrophysical scenarios.

Contents

Notation and conventions

We often work in natural units, with $G = c = 1$, but in several parts, we will restore the physical value of the constants, in order to have numerical values for our computations. We use mostly-plus signature.

Chapter 1

Introduction

Nowadays, different quantum theories suggest that, at a very small scale, spacetime has a non-trivial structure, due to quantum effects. One possible way to have a description of spacetime in such case is based on the implementation of a minimal length, called L_0, of quantum origins. The order of magnitude of this length is supposed to be the Planck scale $L_P \sim 10^{-35}$ m, but the correct value is unknown, being probably derivable from a complete quantum theory of gravity. In the absence of such a quantum theory, or in other words, of a complete quantum description of the spacetime geometry, one can try to introduce via an effective approach this minimal length. Conceptually, there are many issues related to the introduction of such a bound, first of all the possible breaking of Lorentz invariance at small scales. However, a possible way of taking into account the presence of this bound in spacetime is given by the so called qmetric. As suggested by the name, this is essentially a modification of the standard metric: its fundamental property is that it allows to have a distance constant and different from zero between two points even if they are taken to coincide. This paradoxical feature is possible because, instead of a standard tensor, the qmetric is a bitensor, meaning that it depends not only on a point, but on two. Thanks to this structure, it could be possible to study, at least from an effective point of view, some initial effects of quantum spacetime, or possible quantum gravity properties which emerge precisely from the presence of this minimal length. In our work, we follow this approach, focusing on a phenomenon which is of great interest in the recent years: the coalescence of a binary. Essentially, we have a system made by two bodies, or companions, that physically can be stars or black holes, which are orbiting around each other; in this work we will study the case of two black holes. Classically, we could expect that this system is stable; however, from General Relativity we discover that gravitational waves are emitted during the combined motion of the two, resulting in a energy loss, and in a reduction of the distance of the companions (inspiral phase), until they arrive at a limit value of reciprocal distance and start to fuse together (plunge phase); after the fusion, a new black hole is formed, which, to reach stability, starts to emit other gravitational waves based on its characteristic modes, called quasi-normal modes (ringdown phase). The ringdown phase has attracted much interest concerning the study of possible quantum features of spacetime. A very recent example we want to highlight here is in [1], where black holes are regarded as coherent quantum objects whose geometries give possibly detectable quantum features in the ringdown phase. Our interest in this thesis will be, however, in the inspiral phase, specifically in the gravitational waves emitted during it. After being emitted, they go out to the system and arrive at infinity, or they are absorbed by one or both of the black holes of the binary, with the part absorbed being actually very small compared to the one that goes out of the system. This phenomenon of absorption is called *tidal heating*. With the addition of a minimal length L_0, it was proved that we have also a minimal step for the horizon area variation in a black hole, and, due to the thermodynamical laws of black holes, this impacts strongly on its classical absorption of energy, included the one carried by a gravitational wave. In particular, a black hole is not able anymore to absorb each gravitational wave frequency, but only the ones with a frequency higher than a particular threshold. To study this kind of quantum effect in the particular case of a coalescing binary is the main subject of this work. In doing this, we specify that our interest, when some quantity is computed, will be mainly in the order of magnitude of it, instead of the precise numerical value. We follow this line:

In **Chapter one**, we introduce the concept of qmetric, explaining what are the fundamental requirements and how to find its form. In particular, being its form dependent on the distance between the two points considered, it turns out that three different physical cases should be considered: spacelike, timelike and lightlike distances. It is explained also one of the main results of the qmetric description, regarding a modification of the standard Ricci scalar R. From the presence of a minimal length, indeed, we obtain an effective Ricci biscalar $\tilde{R}$, that in the limit for $L_0 \to 0$ does not tend to the classical Ricci scalar R. This suggests important connection with different lines of research, especially the one of the emergent gravity paradigm. It is then shown the fundamental aspect for our work: the presence of a minimal area around a point, which has an important consequence for black holes. Indeed, its horizon area cannot change continuously, by infinitesimal increments, but has a finite step of variation. This point leads to the effect cited above, regarding the limits of absorption of a quantum black hole, that will be explained in the following sections.

In **Chapter two**, we have a collection of different aspects related to the qmetric: we describe initially an approximated version of it, which holds for small distances, and so could be useful to study dynamics which happen at very short scales. Indeed, from the point of view of spacetime distances, qmetric has to be considered precisely when we are at scales of the order of the Planck one, and in this case should emerge its completely non-classical behaviour, that we tried to capture and study. After this, we propose another possible interpretation of the minimal length L_0, namely as a limit radius of curvature; we will show that this way of thinking allows us to classify spacetime regions using standard tensors, and so local objects, despite we are considering a minimal distance which is treated with the language of non-local quantities. We will then link

this discussion to our case of study, explaining what kind of limits, in principle, we could have. The last part of the chapter is about an important quantum effect regarding black holes, the emission of Hawking radiation. As an example of what we will do, and also of what kind of effects can derive from the presence of a minimal step in horizon area, we try to explore the modification of this effect due to our quantum corrections. Remarkably, we find that the greater part of the emitted energy, which is already small, can be suppressed, dependently on the value of the ratio between L_0 and L_P, which is encoded indirectly in a parameter called α. The order of magnitude of the emitted power, or energy, for the standard Hawking emission process will be important for a comparison that we will do in the last chapter, between the radiated power by a black hole via thermal radiation and the power related to a (quantum modification of) tidal heating. This will be important, because both are quantum effects: the first regarding quantum fields in a classical curved spacetime, the second regarding an effective, quantum modification of the spacetime itself. However, we will see that the difference in order of magnitude, and also in the analytical dependence, is very huge, suggesting that to observe some quantum effect combined with a possibly curved spacetime does not mandatory mean that we have to deal with undetectable energies scales, or that a quantum property cannot impact on macroscopical gravitational systems.
In **Chapter three**, we enter the core of the thesis, and we begin by talking about the most simple version of a coalescing binary: the Newtonian one. Here we list some features of the system, from a very classical point of view; nevertheless, it is introduced a first, approximated form of the power emitted by gravitational wave, which signals the difference between the Newtonian and the relativistic case. It is also explained when the first phase of a coalescence, the inspiral one, is expected to end, due to the combined effect of gravitational wave emission and the presence of an horizon for the companions. After this test bench, we briefly cite how the Post-Newtonian (PN) approximation works, which is the one we follow here. In practice, starting from the quantities that derive from the Newtonian version of the problem, an expansion in a small parameter (v/c) is implemented, from which we obtain, at a fixed order in v/c, the analytical forms (templates) of the various functions (like energy, emitted power) we need to study the problem of a quantum-corrected tidal heating. In order to simplify conceptually the discussion, we decided to deal with the problem considering, initially, a binary made of two Schwarzschild black holes, which is the simplest type of black hole. We explain precisely what happens for each Schwarzschild black hole if we add the presence of a minimal step in area, discussing the various cases for α present in the literature, and we explain what is the consequence of that on tidal heating, giving also an explicit computation of some important quantities.

However, in real astrophysical situations, black hole binaries are made by Kerr black holes, which have more complicated geometry and properties. After the Schwarzschild case, so, we proceed with the Kerr one, explaining what are the complications due to the presence of an additional parameter for the black holes, the spin. In doing that, we follow the same line as in the Schwarzschild

case, in order to compare also what is, conceptually and numerically, the difference between the two types of binary. We try in any case to consider a simplified version of the systems; for example, we will consider often equal masses and (for Kerr) equal and aligned spins, each aligned with the orbital angular momentum. We explain what happens in the general case just at a qualitative level, giving numerical results just for the cases cited above. The last part of the chapter involves the comparison, at the level of the emitted power, between Hawking radiation and our q-version of tidal heating. We will see, as anticipated, that the difference is really non-trivial.
In **Chapter four**, we recollect the main results we found, focusing on the ones about the quantum-corrected tidal heating, briefly commenting on the significance of them. We underline also possible continuation of this work, suggesting what lines could be followed.
In the **Appendices**, we find some passages or properties that, for convenience, are not listed of the main text: the interested reader can look at them to understand some details of the principal discussion.

Chapter 2

The qmetric

In this section, we present the qmetric, the effective metric which, at small scales, encodes the presence of a minimal length L_0. We first show what are the requirements for such an object and how to find the general form of the qmetric, then we proceed with some important results which derive from it, and we list some properties. The main result we are interested in, for our discussion, will be the presence of an *irreducible area* around a point, which will be fundamental for the implications regarding the change of area for black holes horizons.

2.1 Form of the qmetric

In order to find the form of the qmetric, we begin by mentioning two important objects: the *Synge world function* $\Omega(p, p_0)$ and the *van Vleck determinant* $\Delta(p, p_0)$. As explicitly indicated in the brackets, each of these two objects depends on two different points: they are indeed non-local objects. Being scalar quantities and functions of two coordinate points, we call them *biscalars*.

2.1.1 The Synge world function and the van Vleck determinant

The definition of $\Omega(p, p_0)$ is simply [2, 3]:

$$\Omega(p, p_0) = \frac{1}{2}(\lambda(p) - \lambda(p_0)) \int_{\lambda(p_0)}^{\lambda(p)} g_{ab} q^a q^b x(\lambda) d\lambda = \frac{1}{2}\sigma(p, p_0)^2; \tag{2.1}$$

where:

- g_{ab} is the standard metric of the spacetime;
- $\sigma(p, p_0)^2$ is the square of *geodesic* distance between the two points;

- q^a is the tangent to the geodesic, with $q^a q_a = \pm 1 = \epsilon$. The explicit form is:
$$q_a = \frac{\nabla_a \sigma^2}{2\sqrt{\epsilon\sigma^2}}. \tag{2.2}$$

We will often consider $\lambda = d(p, p_0) = \sqrt{\epsilon\sigma(p, p_0)^2}$. In practice, Synge world function is simply half the geodesic distance between two points.
The definition of $\Delta(p, p_0)$ is [2, 3]:

$$\Delta(p, p_0) = -\frac{1}{\sqrt{g(p)g(p_0)}} \det\left\{-\nabla_a^{(p)} \nabla_b^{(p_0)} \frac{1}{2}\sigma(p, p_0)^2\right\}. \tag{2.3}$$

From the geometric point of view, this biscalar controls the properties of geodesic congruences which are emanating from the point p_0, as a function of another arbitrary point p.
In flat spacetimes, $\Delta(p, p_0) = 1$ exactly, while in arbitrary curved spacetimes $\Delta(p, p_0) \to 1$ in the coincidence limit $p \to p_0$.

2.1.2 Requirements and complete form

Now, let us introduce the concept of qmetric [2, 4, 5, 6]. Our starting point is the classical metric, g_{ab}, which gives the information on how to compute the distance between two points in a general spacetime, and that in General Relativity is the solution of the well known equation:

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab}; \tag{2.4}$$

with R_{ab} the Ricci tensor, R the Ricci scalar, and T_{ab} the energy-momentum tensor. As we know, if we compute the distance between two different points p_0 and p with any classical metric, and then take the coincidence limit $p \to p_0$, the computation gives zero. We want to work, instead, with an object that gives, in such a limit, a result different from zero, namely the minimal distance L_0. Since our discussion is not based on a complete theory of the spacetime geometry, the idea is to start from the classical metric g_{ab} to construct a new object, called here $\tilde{g}_{ab}$, which satisfies the property above. However, there are other important requirements: the modification which leads to a minimal distance should conserve Lorentz invariance, and of course the new metric should tend to the classical metric for big distances compared to the minimal one L_0, which is expected to be of the order of the Planck length.
It turns out that this new object, instead of a simple tensor, should be a bitensor, depending on two points: a *base point* p_0, and a so called $field\ point$ p. The correct ansatz for searching the qmetric turns out to be [7]:

$$\tilde{g}_{ab} = A g_{ab} - \epsilon B q_a q_b; \tag{2.5}$$

or, in a contravariant form:

$$\tilde{g}^{ab} = A^{-1} g^{ab} + \epsilon Q q^a q^b; \tag{2.6}$$

where A, B and Q are related via:

$$B \equiv \frac{QA}{A^{-1} + Q}. \tag{2.7}$$

The two metrics g_{ab} and $\tilde{g}_{ab}$, related by such a formula, are said to be *disformally coupled*. Some properties of this kind of metrics are in the Appendix, based on [7].
Looking at the previous relations, and remembering that we will find some quantity which depends on two different points, we understand that the non-local behaviour of $\tilde{g}_{ab}$ is probably encoded also in the two functions A and B. Since we demand that, for great distances, the two metrics have to coincide (meaning also that the bitensorial behaviour should tend to a standard tensorial one), we can immediately guess that, for such distances:

$$A \to 1, \quad B \to 0. \tag{2.8}$$

Having the form of the ansatz for the qmetric, now we have to decide how to specify the unknown functions. We follow the procedure of [2], and we demand the two properties:

- The geodesic distance is bounded from below by a Lorentz invariant length L_0. This is, of course, the starting point of our work.
- The modified d'Alembertian $\tilde{\Box}_{p_0,p}$, associated to the qmetric, is such that the two point function $G(p_0, p)$ in *all maximally symmetric spacetimes* satisfies: $G[\sigma^2] \to \tilde{G}[\sigma^2] = G[S_{L_0}(\sigma^2)]$.

In maximally symmetric spacetimes, the leading singular structure of the two point function is just a function of σ^2, and has an expansion in the Hadamard form:

$$G(p_0, p) \equiv \frac{\sqrt{\Delta}}{(\sigma^2)^{\frac{D-2}{2}}} \times (1 + \text{smooth terms}) = G(\sigma^2); \tag{2.9}$$

moreover, S_{L_0} is introduced as the corrected squared distance associated to $\tilde{g}_{ab}$, which can be seen, in our effective discussion, as a function of the *classical* squared distance σ^2. Its specific form derives from a complete quantum theory, and we leave it for now completely free. We require only these properties:

1. $S_{L_0}(0) = L_0^2$;
2. $S_0(\sigma^2) = \sigma^2$ identically;
3. $[|S_{L_0}|/S'^2_{L_0}](0) < \infty$.

Prime indices denote derivation with respect to σ^2.
Before proceeding, we underline an important point about our discussion. If we look at the first requirement, the one of the minimal length, we should remember that we work in spacetime, that means we have a Lorentzian signature of the metric, and we know this gives three cases for the squared geodesic distance: positive, negative, null, which correspond to spacelike, timelike and lightlike intervals. In finding the qmetric, we implicitly suppose that the nature of the geodesics structure is not changed: spacelike/timelike/lightlike intervals remain spacelike/timelike/lightlike. In the study of the first two cases, the strategy is in practice the same, and intuitively we can understand why: when we are dealing with a spacelike geodesic distance between two points p and p_0, and we take the coincidence limit $p \to p_0$, we have $\sigma^2 \to 0^+$; in this case we have to require that, instead of zero, we reach a *positive* constant L_0^2 for the modified squared distance, in order to have still a spacelike interval but bounded from below. For the case of a timelike geodesic distance between p and p_0, we have $\sigma^2 \to 0^-$, so we have to demand that we reach a *negative* constant value, but of the same magnitude: $-L_0^2$. In practice, in this two cases the only difference will be a minus sign, which is encoded in ϵ. The case of lightlike intervals is very different: suppose that we take now p and p_0 to be lightlike separated. In this case, σ^2 is *exactly* 0, because p is on the lightcone of p_0, no matter where. It is not easy to understand how to require that, in the coincidence limit $p \to p_0$, we reach a constant value. The discussion relative to this case will be given in the following; now we start from the spacelike/timelike cases.

Inspired by the Hamilton-Jacobi equation [2], which is satisfied by σ^2 and which yields:

$$g^{ab}\partial_a\sigma^2\partial_b\sigma^2 = 4\sigma^2; \tag{2.10}$$

we demand, to impose the first requirement on the qmetric, that the same equation is still valid also for $\tilde{g}_{ab}$ and S_{L_0}, namely:

$$\tilde{g}^{ab}\partial_a S_{L_0}\partial_b S_{L_0} = 4S_{L_0}. \tag{2.11}$$

This fixes the combination of functions $A^{-1} + Q$:

$$\alpha \equiv A^{-1} + Q = \frac{1}{\sigma^2}\frac{S_{L_0}(\sigma^2)}{S'^2_{L_0}(\sigma^2)}. \tag{2.12}$$

Now we work on the second requirement, and to do so, we start with the form of the d'Alembertian operator associated to $\tilde{g}_{ab}$ in *arbitrary* backgrounds, and not necessarily maximally symmetric. We have [2]:

$$\begin{aligned}\tilde{\Box} = A^{-1}\left\{\Box_g + \frac{1}{2}(D-3)g^{ij}\partial_i(\ln A)\partial_j \ + \epsilon\,\partial\!\!\!/(\ln A)\,\partial\!\!\!/\right\} \\ + \epsilon Q\left\{\left[\nabla_i q^i + \frac{1}{2}(D-1)\ \partial\!\!\!/(\ln A)\right]\partial\!\!\!/ + \ \partial\!\!\!/^2\right\} \\ + \sqrt{\epsilon\sigma^2}\alpha'\ \partial\!\!\!/,\end{aligned} \tag{2.13}$$

where $\not{\partial} \equiv q^i \partial_i$.
In maximally symmetric spacetimes, the functions A and Q depend just on σ^2, and the form of the d'Alembertian operator simplifies a lot, giving:

$$\tilde{\Box} = \alpha\Box \; + \; 2\alpha\sigma^2[\ln(\alpha A^{D-1})]' \frac{\partial}{\partial(\sigma^2)}. \tag{2.14}$$

The standard d'Alembertian operator, instead, has the form:

$$\Box = \frac{\partial^2}{\partial\sigma^2} \; + \; \left(\frac{\partial}{\partial\sigma} \ln \Delta^{-1} \; + \; \frac{D-1}{\sigma}\right)\frac{\partial}{\partial\sigma}. \tag{2.15}$$

In maximally symmetric spacetimes, the van Vleck determinant (VVD) has a relative simple form:

$$\Delta^{-1/(D-1)} = \left\{\frac{\sin(|\sigma|/a)}{|\sigma|/a}, 1, \frac{\sinh(|\sigma|/a)}{|\sigma|/a}\right\}. \tag{2.16}$$

Here, in order, we have the case of positive, zero, negative curvature, with a radius of curvature. For the second requirement, now, we have a differential equation:

$$\frac{d}{d\sigma^2} \ln\left(\frac{A\sigma^2}{S_{L_0}}\left(\frac{\Delta_S}{\Delta}\right)^{2/(D-1)}\right) = 0. \tag{2.17}$$

The quantity Δ_S is simply the VVD with σ^2 replaced by S_{L_0}.
Solving this equation, and fixing the constant of integration requiring that $A = 1$ when $S_{L_0} = \sigma^2$, gives:

$$A = \frac{S_{L_0}}{\sigma^2}\left(\frac{\Delta}{\Delta_S}\right)^{2/(D-1)}. \tag{2.18}$$

Having found A, at this point we have all the unknown functions at hand, and we write them explicitly:

$$A = \frac{S_{L_0}}{\sigma^2}\left(\frac{\Delta}{\Delta_S}\right)^{2/(D-1)}; \tag{2.19}$$

$$B = \frac{S_{L_0}}{\sigma^2}\left(\frac{\Delta}{\Delta_S}\right)^{2/(D-1)} - \frac{\sigma^2 S_{L_0}'^2}{S_{L_0}}; \tag{2.20}$$

$$Q = \frac{1}{\sigma^2}\frac{S_{L_0}}{S_{L_0}'^2} \; - \; \frac{\sigma^2}{S_{L_0}}\left(\frac{\Delta_S}{\Delta}\right)^{2/(D-1)}. \tag{2.21}$$

At this point, of course, we write the form of the qmetric, both in covariant and contravariant version:

$$\tilde{g}_{ab} = \frac{S_{L_0}}{\sigma^2}\left(\frac{\Delta}{\Delta_S}\right)^{2/(D-1)} g_{ab} \; + \; \epsilon\left\{\frac{\sigma^2 S_{L_0}'^2}{S_{L_0}} \; - \; \frac{S_{L_0}}{\sigma^2}\left(\frac{\Delta}{\Delta_S}\right)^{2/(D-1)}\right\} q_a q_b; \tag{2.22}$$

$$\tilde{g}^{ab} = \frac{\sigma^2}{S_{L_0}}\Big(\frac{\Delta}{\Delta_S}\Big)^{-2/(D-1)} g_{ab} \,+\, \epsilon\bigg\{\frac{S_{L_0}}{\sigma^2 S_{L_0}'^2} - \frac{\sigma^2}{S_{L_0}}\Big(\frac{\Delta}{\Delta_S}\Big)^{-2/(D-1)}\bigg\} q^a q^b. \quad (2.23)$$

We can make two observations: the form of the qmetric is fixed but depends on the specific formula of the squared distance S_{L_0}, which derives from a complete quantum theory. Despite this, we see that actually only S_{L_0} itself and its first derivative with respect to σ^2 appears. However, the important point is the following: requiring some conditions about maximally symmetric spacetimes only, we actually found a completely fixed form for the functions A, B, Q, which holds also for *general* spacetimes. This is a remarkable result, since it means that it is enough to require some very basic properties for the qmetric in the simplest case to find the behaviour in the very general cases.

We can compare, also, the line elements ds^2 and $\tilde{ds}^2$, corresponding respectively to the metric g_{ab} and $\tilde{g}_{ab}$. In doing so, we assume maximally symmetric spaces, $\sigma^2 > 0$ and a constant positive curvature. We have:

$$ds^2 = g_{ab}dx^a dx^b = d\sigma^2 + \sigma^2 \Delta^{-2/(D-1)} d\Omega^2_{D-1}, \quad (2.24)$$

$$\tilde{ds}^2 = \tilde{g}_{ab}dx^a dx^b = \Big(d\sqrt{S_{L_0}}\Big)^2 + S_{L_0}\Delta_S^{-2/(D-1)} d\Omega^2_{D-1}; \quad (2.25)$$

another time, we obtain the corrected line element from the standard one with the replacements: $\sigma \to \sqrt{S_{L_0}}$, $\sigma^2 \to S_{L_0}$, $\Delta \to \Delta_S$.

2.2 The Ricci biscalar and the emergent gravity paradigm

In this section, we want to present one of the main and important result of the qmetric approach, namely a highly non trivial limit of one of the objects we can compute from this new metric, the Ricci biscalar $\tilde{R}(p, p_0)$. Classically, once we have the metric of the spacetime g_{ab}, we can compute different quantities, in particular the Riemann tensor R_{manb}, the Ricci tensor $R_{ab} \equiv R^m_{\ amb}$ and the Ricci scalar, $R \equiv g^{ab}R_{ab}$. The last of them is of great importance for Physics, because is the simplest scalar associated to the curvature of the spacetime and also the one that enters in the Lagrangian of GR. Of course, having the qmetric, there will be new quantities associated to it, and these will be, in general, bitensors: the Riemann bitensor, the Ricci bitensor and the Ricci biscalar. However, given the complicated form of the qmetric, to compute such objects is not an easy task. It turns out, by the way, that due to some useful identities, it is possible to find the form of the Ricci biscalar. Before starting, we need to introduce some other important geometrical quantities: the *extrinsic* curvature K_{ab}, the *intrinsic* Ricci scalar $\mathcal{R}$ and the *tidal tensor* $\mathcal{E}_{ab}$.

2.2.1 The tensors $K_{ab}, \mathcal{R}, \mathcal{E}_{ab}$

A central role in working with our kind of bitensors is played by the concept of geodesic, and in particular by the *congruence* of geodesics emanating from a fixed spacetime point. Take some point p_0, and consider some fixed value of geodesic distance, say, G. From all the geodesics emanated from p_0, we take the ones which have a value of the geodesic distance equal to G. Each of these will then connect the point p_0 to another point p, which is at a geodesic distance G from p_0. Now consider the surface made by all these point for a fixed G: this is called the $equi - geodesic$ surface of the point p_0 of distance G, and is denoted by Σ_{G,p_0}.
Consider now one of the points on Σ_{G,p_0}, which we call another time p, and consider the geodesic connecting p_0 with p. Recalling that the affinely parametrized tangent vector q^a to this geodesic is given by (2.2), and noting that it is normal to Σ_{G,p_0}, the extrinsic curvature of the surface Σ_{G,p_0} is defined as:

$$K_{ab} = \nabla_a q_b = \frac{\nabla_a \nabla_b (\sigma^2/2) - \epsilon q_a q_b}{\sqrt{\epsilon \sigma^2}}. \tag{2.26}$$

If we introduce the acceleration vector associated to q^i, defined by $a_i = q^k \nabla_k q_i$, we can also write the extrinsic curvature in another form:

$$K_{ab} = \nabla_a q_b - \epsilon a_b q_a. \tag{2.27}$$

The equi-geodesic foliation of a spacetime characterizes its local geodesic structure, and has interesting properties related to the way we can expand geometrical objects such that, for example, the bitensor $\nabla_a \nabla_b (\sigma^2/2)$ (this object is truly a bitensor, since depends on σ^2. Even if it is not explicit at a first sight, all the quantities which depend on the squared distance are bitensor, so their role is really fundamental in computing physical observables).
Now let us talk about the intrinsic Ricci scalar $\mathcal{R}$. Given a metric g_{ab} in an arbitrary spacetime U_D, where D is the dimension, we identify with $\{x^1, ..., x^D\}$ some set of coordinates, which holds in a certain sufficiently extended region. Take now some subspace u_{D-k}, where $D-k$ underlines that it is a low-dimensional subspace, with coordinates now given by $\{y^1, ..., y^{D-k}\}$. We can restrict the metric of the complete spacetime to the subspace, meaning that if we move just in that subspace, we feel a *restricted* metric which derives from the complete metric g_{ab}. Intuitively, the line element on that subspace will depend on $(D-k)$ variables and will involve $(D-k)$ differential forms, instead of D. The metric on this subspace is called *induced* metric, and is indicated by h_{ab}. We have then:

$$ds^2 = g_{ab} dx^a dx^b, \quad ds^2|_u = h_{ab} dx^a dx^b. \tag{2.28}$$

We can then compute the induced geometrical quantity in the subspace u, starting from the induced metric h_{ab}.
Of all the subspaces we can define out of a spacetime, an important role is played by Σ_{G,p_0}. The induced metric on this surface is simply:

$$h_{ab} = g_{ab} - \epsilon q_a q_b, \tag{2.29}$$

and from it, we can compute the so called intrinsic Ricci scalar $\mathcal{R}_\Sigma$, where now we underline the dependence from the surface Σ_{G,p_0} with the subscript. In the following we will usually call h_{ab} precisely the metric induced on Σ_{G,p_0}, if not differently specified.
We can now give the definition of the tidal tensor $\mathcal{E}_{ab}$, which is simply the contraction of the Riemann tensor with two affinely parametrized tangent vectors to the geodesic, explicitly: $\mathcal{E}_{ab} = R_{ambn}q^m q^n$.

As we said, there are important properties related to the equi-geodesic foliation. We start from the cited above expansion of the tensor $\nabla_a\nabla_b(\sigma^2/2)$ around the point p_0, which is taken as the base point. Recall that $\lambda = \sqrt{\epsilon\sigma^2}$ represents the distance, here infinitesimal, from the point p_0, and so is the parameter of our expansion. We have:

$$\nabla_a\nabla_b\Big(\frac{\sigma^2}{2}\Big) = g_{ab} - \frac{\lambda^2}{3}\mathcal{E}_{ab} + \frac{\lambda^3}{12}\nabla_{\mathbf{q}}\mathcal{E}_{ab} - \frac{\lambda^4}{60}\Big(\nabla^2_{\mathbf{q}}\mathcal{E}_{ab} + \frac{4}{3}\mathcal{E}_{ia}\mathcal{E}^i_{\ b}\Big) + O(\lambda^5), \quad (2.30)$$

where $\nabla_{\mathbf{q}} \equiv q^i\nabla_i$. We note an important fact: this expansion is completely characterized by the tidal tensor. This holds also for the expansion of the extrinsic and intrinsic curvature (the intrinsic Ricci scalar) and, so, for the one of the trace of the extrinsic curvature, $K \equiv g^{ab}K_{ab}$. They read:

$$K_{ab} = \frac{1}{\lambda}h_{ab} - \frac{1}{3}\lambda\mathcal{E}_{ab} + \frac{1}{12}\lambda^2\nabla_{\mathbf{q}}\mathcal{E}_{ab} - \frac{1}{60}\lambda^3 F_{ab} + O(\lambda^4), \quad (2.31)$$

$$K = \frac{D-1}{\lambda} - \frac{1}{3}\lambda\mathcal{E} + \frac{1}{12}\lambda^2\nabla_{\mathbf{q}}\mathcal{E} - \frac{1}{60}\lambda^3 F + O(\lambda^4), \quad (2.32)$$

$$\mathcal{R}_\Sigma = \frac{\epsilon(D-1)(D-2)}{\lambda^2} + R - \frac{2\epsilon(D+1)}{3}\mathcal{E} + O(\lambda); \quad (2.33)$$

with $\mathcal{E} = g^{ab}\mathcal{E}_{ab} = R_{ab}q^a q^b$, $F_{ab} = \nabla^2_{\mathbf{q}}\mathcal{E}_{ab} + (4/3)\mathcal{E}_{ai}\mathcal{E}^i_{\ b}$, $F = F_{ab}g^{ab}$. Of course, as we can see also from these expansions, being in a flat spacetime does not mean that K_{ab} and $\mathcal{R}$ are null, because they are related to a (hyper)surface embedded in that flat spacetime. For example, for a Minkowskian D-dimensional space, for an equi-geodesic surface Σ_{G,p_0} we have:

$$\mathcal{R}^{flat}_\Sigma = (D-1)(D-2)/\sigma^2, \ K^{flat}_{ab} = \frac{1}{\sqrt{\epsilon\sigma^2}}h_{ab}, \ K^{flat} = (D-1)/\sqrt{\epsilon\sigma^2}. \quad (2.34)$$

Having defined the objects we need, now we go through the study of the relations between geometrical quantities associated to, respectively, a standard metric g_{ab} and the corresponding qmetric $\tilde{g}_{ab}$.

2.2.2 Geometrical quantities for disformally coupled metrics

As we said above, two metric related by the formula (2.5) are said to be disformally coupled, and there are different properties connected to this particular

type of transformation, regarding also the quantities listed in the previous formulas. A complete discussion on that can be found in [7], here we simply recall the results which are fundamental for our task.
We start from the operator ∇_a, which becomes $\tilde{\nabla}_a$ and where we have to correct simply the part of the Christoffel symbol, since the partial derivative is the same:

$$\tilde{\nabla}_a V_b = \partial_a V_b - \tilde{\Gamma}^c{}_{ab} V_c. \tag{2.35}$$

We have:

$$\tilde{\Gamma}^c{}_{ab} = \Gamma^c_{ab} + \frac{1}{2}\tilde{g}^{cm}(-\nabla_m \tilde{g}_{ab} + 2\nabla_{(a}\tilde{g}_{b)m}). \tag{2.36}$$

Remarkably, even if the relation between spacetime metrics g_{ab} and $\tilde{g}_{ab}$ is not conformal (which is a special case of disformal transformations), as in our case, it turns out that the induced metrics h_{ab} and $\tilde{h}_{ab}$ are related to a *conformal* transformation via the function A only. We can se explicitly this property.
When we have defined the vector q_a, it was related to the metric g_{ab} via the condition of normalization $g_{ab}q^a q^b = q^a q_a = \epsilon$. We can define an analogous of this vector also for the qmetric, which we call $\tilde{q}_a$ and that satisfies $\tilde{g}_{ab}\tilde{q}^a\tilde{q}^b = \epsilon$. It turns out that q_a and $\tilde{q}_a$ are related very simply:

$$\tilde{q}_a = \sqrt{A-B}\, q_a; \tag{2.37}$$

and:

$$\tilde{q}^a = \tilde{g}^{ab}\tilde{q}_b = \frac{1}{\sqrt{A-B}} q^a. \tag{2.38}$$

After this definition, we can write that the induced qmetric on Σ_{G,p_0} is:

$$\tilde{h}_{ab} = \tilde{g}_{ab} - \epsilon\tilde{q}_a\tilde{q}_b = (Ag_{ab} - \epsilon B q_a q_b) - \epsilon(A-B)q_a q_b = Ah_{ab}. \tag{2.39}$$

This is a great simplification and has a strong impact on the geometries of equi-geodesic surfaces related by a disformal transformation: it means that the intrinsic geometries of such surfaces are conformal to each other. From this, and from the fact that on Σ_{G,p_0} the function A is constant (because it depends just on σ^2 which on Σ_{G,p_0} is fixed), we have:

$$\tilde{\mathcal{R}}_\Sigma = A^{-1}\mathcal{R}_\Sigma, \tag{2.40}$$

so also the intrinsic Ricci biscalar is simply rescaled by a conformal factor, dependent just on A, and in which of course is encoded the bitensorial nature of this object. We write now the relation between the extrinsic curvature tensor K_{ab} and the corresponding bitensor $\tilde{K}_{ab}$, which are not so simple as in the previous cases:

$$\tilde{K}_{ab} = \frac{1}{\sqrt{A-B}}\left[AK_{ab} + \frac{1}{2}(\nabla_{\mathbf{q}}A)h_{ab}\right]; \tag{2.41}$$

and also the relation between the traces, which derives immediately:

$$\tilde{K} = \tilde{g}^{ab}\tilde{K}_{ab} = \frac{A^{-1}}{\sqrt{A-B}}\left[AK + \frac{D-1}{2}\nabla_{\mathbf{q}}A\right]. \tag{2.42}$$

At this point, we have a set of relations which connects standard objects of the spacetime, associated to g_{ab}, to the new biscalar objects, associated to $\tilde{g}_{ab}$. The next part of the discussion is about the explicit computation of the Ricci biscalar $\tilde{R}$, as a function of these quantities.

2.2.3 The Ricci biscalar $\tilde{R}(p, p_0)$

We recall the Gauss-Codazzi equation, which allows us to compute the Ricci scalar directly in terms of geometrical quantities like the ones we have defined above, and we apply that equation to the case of the Ricci biscalar, getting:

$$\tilde{R} = \tilde{\mathcal{R}}_\Sigma - \epsilon(\tilde{K}^2 + \tilde{K}^2_{ab}) - 2\epsilon\tilde{\nabla}_{\tilde{\mathbf{q}}}\tilde{K} + 2\epsilon\tilde{\nabla}_i\tilde{a}^i; \tag{2.43}$$

where $\tilde{\nabla}_{\tilde{\mathbf{q}}} \equiv \tilde{q}^i\tilde{\nabla}_i$, $\tilde{K}^2_{ab} \equiv \tilde{g}^{ia}\tilde{g}^{jb}\tilde{K}_{ab}\tilde{K}_{ij}$ and of course $\tilde{a}^i$ is the acceleration associated to the vector $\tilde{q}^i$ (remarkably, we have $\tilde{a}_i = a_i$). Substituting all the pieces we need, we obtain that the Ricci biscalar $\tilde{R}$ reads [7]:

$$\tilde{R} = A^{-1}R + \epsilon(\alpha - A^{-1})\mathcal{J}_d - \epsilon\alpha\mathcal{J}_c, \tag{2.44}$$

where:

$$\alpha = \frac{1}{A - B}, \tag{2.45}$$

$$\begin{aligned}\mathcal{J}_c = \epsilon\Big[2(D-1)\frac{\Box\sqrt{A}}{\sqrt{A}} + (D-1)(D-4)A^{-1}(\nabla\sqrt{A})^2\Big] \\ + \Big(K + (D-1)\nabla_{\mathbf{q}}\ln\sqrt{A}\Big)\nabla_{\mathbf{q}}\ln(\alpha A),\end{aligned} \tag{2.46}$$

$$\mathcal{J}_d = 2R_{ij}q^iq^j + K^2_{ij} - K^2 - 2\nabla_i a^i = \epsilon(R - \mathcal{R}_\Sigma - 2\nabla_i a^i). \tag{2.47}$$

Now we have simply to substitute the explicit form of all the functions involved, and this requires a bit long computation. The result is:

$$\begin{aligned}\tilde{R}(p,p_0) = \Bigg[\frac{\sigma^2}{S_{L_0}}\Big(\frac{\Delta}{\Delta_S}\Big)^{-2/(D-1)}\mathcal{R}_\Sigma - \frac{(D-1)(D-2)}{S_{l_0}} + 4(D+1)(\ln\Delta_S)^\bullet\Bigg] \\ - \frac{S_{L_0}}{\lambda^2 S'^2_{L_0}}\Bigg\{K_{ij}K^{ij} - \frac{1}{D-1}K^2\Bigg\} \\ + 4S_{L_0}\Bigg\{-\frac{D}{D-1}[(\ln\Delta_S)^\bullet]^2 + 2(\ln\Delta_S)^{\bullet\bullet}\Bigg\} \\ = Q_0 + Q_K + Q_\Delta;\end{aligned} \tag{2.48}$$

here the dot ($^\bullet$) means derivation with respect to S_{L_0}, and we have called the pieces in the complete expression with three different names, for future utility.

This object is of extreme importance, and as we can see, is an *exact* expression: no Taylor expansion has been used, so it could account also for possible non-perturbative effects due to a zero point length. Since we are not considering covariant Taylor expansion, we have no requirement about the smoothness of the region of spacetime considered: this expression assumes so a strong meaning also for studying regions/point in which the curvature goes to infinity, and so singularities. However, in our discussion we consider just the case of smooth regions of spacetime, so quantities can be expanded via the known methods. Another important point to note is that this object is completely determined by the geodesic structure of the spacetime; in fact, we see that it contains all the functions we studied previously connected to the concept of geodesic.
Now we want to underline another important point. The object computed depends, of course, on two different points, p and p_0, and on the zero point length L_0. We will study in the next the limit of coincidence of this object, $p \to p_0$, which for a generic bitensorial quantity is indicated by square brackets [.], so:

$$\lim_{p \to p_0} \tilde{R}(p, p_0) \equiv [\tilde{R}](p_0). \tag{2.49}$$

It is also of central importance to study the limit $L_0 \to 0$, that means that the zero point length is absent but that, physically, can be associated to the condition $\sigma^2 >> L_0$. However, if we look at the previous expression, we see that performing initially the limit $L_0 \to 0$, we get $S_{L_0} = \sigma^2$ because of the condition we imposed on S_{L_0}, and so $\tilde{R}(p, p_0)$ reduces to $R(p_0)$. If we the perform the coincidence limit, of course the object is still the standard Ricci scalar, because the non-local behaviour is given up by the first limit. Let us try to think about applying this two limits in the opposite order. First we perform the coincidence limit, getting a local quantity, the scalar $[\tilde{R}](p_0)$, which is defined at a general point p_0 in the spacetime but still depends, in principle, on L_0. For L_0 approaching 0, we could expect a behaviour like:

$$[\tilde{R}](p_0) \sim R(p_0) + C(L_0), \tag{2.50}$$

where the term $C(L_0)$ is supposed to go to 0 in the limit $L_0 \to 0$. Actually, this is not case, because if we follow this order in taking the limits, the leading term is not simply $R(p_0)$. This last computation will be the one of our interest, which means that first we extract a scalar from the complete biscalar object, and then we study what happens in the limit in which L_0 is a negligible contribution from the point of view of generic distances.
Consider, so, the scalar $[\tilde{R}](p_0)$ obtained from the coincidence limit of the Ricci biscalar, and in particular each of the pieces Q_0, Q_K and Q_Δ. Start from Q_K. Having the condition $[|S_{L_0}|/S'^2_{L_0}](0) < \infty$, it turns out that this term is convergent in the coincidence limit. This derives from the particular combination:

$$K_{ab}K^{ab} - \frac{1}{D-1}K^2. \tag{2.51}$$

Indeed, we can see from the covariant expansion given above that this combination, precisely for the factor $1/(D-1)$ in front of K^2, gives a behaviour

like λ^2 in the coincidence limit, that vanishes. Any other prefactor would give divergent contribution, because of the terms $O(1/\lambda^2)$ which do not cancel each other. So this term does not give any residue.
Take the case of Q_Δ. This piece is even simpler, because it is smooth and yields other $O(L_0^2)$ contributions, coupled to the background curvature. Also this piece does not give residue. Any possible non-trivial L_0 contributions, so, comes from the term Q_0: let us focus on that term.
We now use the expansions listed in (2.31), and we also write down the following expansion for the VVD:

$$\Delta^{1/2}(p, p_0) = 1 + \frac{1}{12}\lambda^2 R_{ab} q^a q^b + O(\lambda^3). \tag{2.52}$$

Making the substitution $\sigma^2 \to S_{l_0}$, we know that we obtain Δ_S, where now $\lambda = \sqrt{\epsilon\sigma^2} \to \sqrt{\epsilon S_{L_0}}$, so we have, for our two limits:

$$\lim_{L_0\to 0}\lim_{\sigma^2\to 0}(\ln\Delta_S)^\bullet = \frac{1}{6}\epsilon[R_{ab}q^a q^b](p_0). \tag{2.53}$$

Then, using the expansion for $\mathcal{R}_\Sigma$ and the fact that $\Delta(0) = 1$, we have:

$$\lim_{\sigma^2\to 0}\left\{\frac{\sigma^2}{S_{L_0}}\Big(\frac{\Delta}{\Delta_S}\Big)^{-2/(D-1)}\mathcal{R}_\Sigma - \frac{(D-1)(D-2)}{S_{L_0}}\right\} = \\ = \frac{(D-1)(D-2)}{L_0^2}\Big(\Delta_{L_0}^{2/(D-1)} - 1\Big); \tag{2.54}$$

with $\Delta_{L_0}^{1/2} = 1 + \frac{1}{12}\epsilon L_0^2[R_{ab}q^a q^b](p_0) + O(L_0^3)$. Then, performing the limit $L_0 \to 0$ of the RHS, with a bit of algebra we get:

$$\lim_{L_0\to 0}\frac{(D-1)(D-2)}{L_0^2}\Big(\Delta_{L_0}^{2/(D-1)} - 1\Big) = \frac{1}{3}(D-2)\epsilon[R_{ab}q^a q^b](p_0). \tag{2.55}$$

Collecting all pieces, we have now for Q_0:

$$\lim_{L_0\to 0}\lim_{\sigma^2\to 0} Q_0 = \epsilon\left[\frac{(D-2)}{3} + \frac{4(D+1)}{6}\right][R_{ab}q^a q^b](p_0) = \epsilon D[R_{ab}q^a q^b](p_0) = \\ = \epsilon D\mathcal{E}(p_0). \tag{2.56}$$

In other words, we obtained:

$$\lim_{L_0\to 0}[\tilde{R}](p_0) = \epsilon D[R_{ab}q^a q^b](p_0). \tag{2.57}$$

This result is of great importance, and we should comment briefly its meaning. We obtained an object which is independent on L_0 and has a local behaviour, but derives from a strongly non-local structure, which means that it is, in practice, the signature of zero point length, still present at a macroscopic level. This

can be comparable to the various quantum anomalies one encounters in QFT in curved spacetimes. We remark that we obtained this feature without studying a precise quantum theory of gravity, and also without specific assumptions about the form of the corrected distance S_{L_0}: we made just very few assumptions for our structures, apart the condition on zero point length.
In the case of qmetric for for lightlike intervals, which we are going to consider in the next section, the expression of $\lim_{L_0 \to 0}[\tilde{R}](p_0)$ is even more suggestive, since we have [8]: $\lim_{L_0 \to 0}[\tilde{R}](p_0) = (D-1)R_{ab}l^a l^b$; with l^a the null tangent vector to the connecting geodesic, i.e., the heat flow of the horizon having l^a as generator.

2.3 qmetric for lightlike intervals

As we said, in the case of lightlike intervals to find the form of the qmetric is not so trivial. Indeed, our strategy, based on [9, 10], will be different with respect to the previous case. First, we focus on the concept of affine parameter which parametrize a geodesic. For a timelike/spacelike geodesic, we can always find a certain parameter λ which for that geodesic plays the role of proper time/proper distance, respectively. This is not the case for lightlike geodesics. What we can do is, in this case, to take a null affine parameter that has a physical meaning. We decide to take the parameter λ such that it gives a measure of the distance along the geodesic measured by an observer at a certain point x on the geodesic, and parallel transported along it. Suppose we have two points x and x' on a null geodesic, in a classical spacetime with metric g_{ab}, parametrized by λ. Introducing the metric $\tilde{g}_{ab}$, we expect that the parameter λ is mapped into $\tilde{\lambda}(\lambda)$, and that, when $x \to x'$, is satisfied the condition:

$$\lambda(x) - \lambda(x') = 0 \to \tilde{\lambda}(x) - \tilde{\lambda}(x') = L_0. \tag{2.58}$$

In practice, having a constant null measure of the squared distance σ^2, we decide to shift our requirements on the parameter taken to parametrize the geodesic. We require, so, (2.58), and also that in the case of $L_0 = 0$ we get back the classical case:

$$\tilde{\lambda}(x) - \tilde{\lambda}(x') = 0. \tag{2.59}$$

Moreover, following the physical line of the previous cases, we want to require also the same condition on the d'Alembertian operator, so that $\tilde{G}(\sigma^2) = G(S_{L_0})$ in all maximally symmetric spaces. Here, however, we have an obstacle: the form of the Green function $G(\sigma^2)$ on the lightcone is singular. We will see how to fix this problem, but essentially, the strategy will be to take one of the two points *close* but out of the null geodesic, and then to take the limit in which it goes on the geodesic. There is another important modification in the case of lightlike geodesic: the ansatz to construct the metric is not precisely the one we have seen in (2.5). Now we show the new ansatz.
Consider a null geodesic, parametrized by λ, and a point x' on it, and define

the null tangent vector to the geodesic:

$$l^a = \frac{dx^a}{d\lambda}. \tag{2.60}$$

We consider a canonical observer in x' with a four velocity V^a normalized such that $V^a l_a = -1$. In this way, λ selects a particular frame in which λ itself plays the role of distance on the null geodesic. We have actually the relations:

$$\frac{1}{2}\partial_a \sigma^2 = \lambda l_a, \quad \lambda = -\frac{1}{2} V^a \partial_a \sigma^2. \tag{2.61}$$

Taking l^a and V^a, we define another null vector m^a such that:

$$m^a = V^a - \frac{1}{2} l^a, \quad m^a V_a = -\frac{1}{2}, \quad m^a l_a = -1 \tag{2.62}$$

on the geodesic. Our ansatz is:

$$\tilde{g}_{ab} = A g_{ab} + (A - \alpha^{-1})(l_a m_b + m_a l_b), \tag{2.63}$$

$$\tilde{g}^{ab} = A^{-1} g^{ab} + (A^{-1} - \alpha)(l^a m^b + m^a l^b). \tag{2.64}$$

We expressed the metric already with A and α instead of A, B and Q. The quantities we introduced depend on the observer we are taking, since they depend on m^a and so on V^a. The precise form of the metric will be therefore observer-dependent. Let us now show how to derive the functions A and α.

2.3.1 Finding A and α

In the case of spacelike/timelike geodesics, we found the funtion α from a differential equation which follows from the Hamilton-Jacobi identity. In the case of lightlike geodesic, we still find α from a differential equation, but now from another structure, namely the modified geodesic equation.
Let us define:

$$\tilde{l}^a \equiv \frac{dx^a}{d\tilde{\lambda}} = \frac{d\lambda}{d\tilde{\lambda}} l^a \tag{2.65}$$

as the null tangent vector of the geodesic associated to $\tilde{g}_{ab}$. Since this is the tangent vector to the q-geodesic, it makes sense that it satisfies a geodesic equation with the same structure of the classical one, but of course with $\tilde{g}_{ab}$ instead of g_{ab}:

$$\tilde{l}^a \tilde{\nabla}_a \tilde{l}_b = 0. \tag{2.66}$$

Here $\tilde{l}_b = \tilde{g}_{ab} \tilde{l}^a$ is obtained, like the other q-corrected quantities, acting on the indices with $\tilde{g}_{ab}$ and $\tilde{g}^{ab}$, and in this case we have:

$$\tilde{l}_b = \frac{d\lambda}{d\tilde{\lambda}} \alpha^{-1} l_b. \tag{2.67}$$

We already saw some relations between the classical quantities and the q-corrected ones. Recalling these relations, in particular the one regarding $\tilde{\nabla}$

and the connection $\tilde{\Gamma}$, after some algebraic steps we arrive at the following form for the q-geodesic equation:

$$\Big(\frac{d\lambda}{d\tilde{\lambda}}\Big)^2\alpha^{-1}l^a\nabla_a l_c+\Big(\frac{d\lambda}{d\tilde{\lambda}}\Big)l^a\partial_a\Big(\frac{d\lambda}{d\tilde{\lambda}}\alpha^{-1}\Big)-\frac{1}{2}\Big(\frac{d\lambda}{d\tilde{\lambda}}\Big)^2 l^a l^d(\nabla_a\tilde{g}_{dc}+\nabla_c\tilde{g}_{ad}-\nabla_d\tilde{g}_{ac})=0. \tag{2.68}$$

The first term vanishes because $l^a\nabla_a l_c = 0$, since the classical geodesic equation holds for the classical tangent vector l^a with affine parameter λ. We can also substitute $l^a\partial_a = d/d\lambda$ because is the directional derivative. We can also note that, by symmetry:

$$l^a l^d(\nabla_a\tilde{g}_{dc}-\nabla_d\tilde{g}_{ac})=0. \tag{2.69}$$

Now if we compute explicitly the only piece which is left proportional to $l^a l^d$, we get:

$$l^a l^d\nabla_c\tilde{g}_{ad}=-2(A-\alpha^{-1})l^a\nabla_c l_a. \tag{2.70}$$

Inserting all:

$$\Big(\frac{d\lambda}{d\tilde{\lambda}}\Big)\frac{d}{d\lambda}\Big(\frac{d\lambda}{d\tilde{\lambda}}\alpha^{-1}\Big)+\Big(\frac{d\lambda}{d\tilde{\lambda}}\Big)^2(A-\alpha^{-1})l^a\nabla_c l_a=0. \tag{2.71}$$

But the term $l^a\nabla_c l_a$ gives 0, because l^a is the normalized tangent vector to the geodesic, and we arrive at:

$$\Big(\frac{d\lambda}{d\tilde{\lambda}}\Big)\frac{d}{d\lambda}\Big(\frac{d\lambda}{d\tilde{\lambda}}\alpha^{-1}\Big)=0. \tag{2.72}$$

We can suppose that the rate $d\lambda/d\tilde{\lambda}$ is non vanishing, so:

$$\alpha=C\Big(\frac{d\tilde{\lambda}}{d\lambda}\Big)^{-1}. \tag{2.73}$$

In the limit $L_0\to 0$, we have $\lambda\to\infty$ but also $\tilde{\lambda}\to\lambda$, and in this situation we should recover, as usual, the classical metric g_{ab}. This happens if $A\to 1$ and $(A-\alpha^{-1})\to 0$, and we see that we need $\alpha\to 1$, so $C=1$.
Now, we study the case of A. We remember that in the case of spacelike/timelike geodesics we fixed the function A directly via the d'Alembertian equation, introducing the modified Green function $\tilde{G}(\sigma^2)=G(S_{L_0}(\sigma^2))$ and requiring that, having $\Box G(S_{L_0})=0$, we get also $\tilde{\Box}\tilde{G}=0$. As we said, when $\sigma^2=0$ we have a singular behaviour of the Green function, which can be seen from the expansion (2.9). We proceed as follow: having two points x' and x on a null geodesic, we compute $\Box G$ keeping x' but taking as the second point y, which is close to the geodesic but not on it, so such that $\sigma^2(x',y)\neq 0$. Then we will consider the limit $y\to x$. Let us start computing the form of $\Box G(\sigma^2)$ in maximally symmetric spacetimes. We have:

$$\Box G(\sigma^2)=\nabla_a(\nabla^a G)=\nabla_a\Big(\partial^a\sigma^2\frac{dG}{d\sigma^2}\Big)=\nabla_a(\partial^a\sigma^2)\frac{dG}{d\sigma^2}+\partial^a\sigma^2\partial_a\sigma^2\frac{d^2G}{(d\sigma^2)^2}. \tag{2.74}$$

For null geodesic, the second piece gives 0. The piece $\nabla_a \partial^a \sigma^2$, for null separations, gives:

$$\nabla_a \partial^a \sigma^2(x', x) = 2(\lambda \nabla_a l^a + 2). \tag{2.75}$$

Our form of the d'Alembertian of G for lightlike geodesics, in maximally symmetric spaces, is then:

$$\Box G(\sigma^2) = (4 + 2\lambda \nabla_a l^a) \frac{dG}{d\sigma^2}. \tag{2.76}$$

We know the modifications of the various geometrical objects passing to the qmetric description; in particular, we recall the fact that a null geodesic is mapped to a null geodesic, so we can write, for $\tilde{g}_{ab}$:

$$\tilde{\Box}\tilde{G}(\sigma^2) = (4 + 2\tilde{\lambda}\tilde{\nabla}_a \tilde{l}^a) \frac{d\tilde{G}}{dS_{L_0}}, \tag{2.77}$$

in which, recalling the relations between Γ and $\tilde{\Gamma}$, we have:

$$\tilde{\nabla}_a \tilde{l}^a = \nabla_a \Big(\frac{d\lambda}{d\tilde{\lambda}} l^a \Big) + \frac{1}{2} \Big(\frac{d\lambda}{d\tilde{\lambda}} \Big) \tilde{g}^{ac} l^b \nabla_b \tilde{g}_{ac}. \tag{2.78}$$

Now we can insert the ansatz (2.63),(2.64) for the qmetric, obtaining:

$$\tilde{g}^{ac} l^b \nabla_b \tilde{g}_{ac} = (D-2) \frac{d \ln A}{d\lambda} - 2 \frac{d \ln \alpha}{d\lambda}. \tag{2.79}$$

We already fixed the form of α; substituting and rewriting the d'Alembertian equation we have:

$$\tilde{\Box}\tilde{G}(\sigma^2) = \left[4 + 2\tilde{\lambda} \frac{d\lambda}{d\tilde{\lambda}} \nabla_a l^a + \tilde{\lambda}(D-2) \frac{d\lambda}{d\tilde{\lambda}} \frac{d(\ln A)}{d\lambda} \right] \left(\frac{dG(\sigma^2)}{d\sigma^2} \right)_{\sigma^2 = S_{L_0}}. \tag{2.80}$$

Now, we explain the practical meaning we can give to the mapping $\lambda \to \tilde{\lambda}$. In the classical case, we have three points: the base point x', the field point x, both on the null geodesic, and the auxiliary point y, which is at a finite squared distance from x'. The points x and x' are separated by a null squared distance, but by an affine distance λ. After the mapping $\lambda \to \tilde{\lambda}$, we can make the following interpretation: the point x in mapped to a new point $\tilde{x}$, which is at an affine distance $\tilde{\lambda}$ from x' and the auxiliary point y is mapped to $\tilde{y}$, which is at a finite modified squared distance $S(\sigma^2(y, x'))$ from the base point x'. We require that, when the classical Green function $G(\sigma^2)$ satisfies $\Box G = 0$ at the *modified* point $\tilde{x}$, the *modified* Green function $\tilde{G}(\sigma^2) = G(S_{L_0})$ satisfies $\tilde{\Box}\tilde{G} = 0$ at the point x. We have now to evaluate $\Box G$ in x'. Since null geodesic are mapped to null geodesic, we are still dealing with a null distance, and $\Box G$ is singular on $\tilde{x}$. We proceed as in the classical case, starting from $\tilde{y}$ and taking the limit $\tilde{y} \to \tilde{x}$. We have:

$$\Box G(\sigma^2)_{\tilde{x}} = [4 + 2(\lambda \nabla_a l^a)_{\tilde{x}}] \frac{dG(\sigma^2)_{\tilde{x}}}{d\sigma^2} = [4 + 2\tilde{\lambda}(\nabla_a l^a)_{\tilde{x}}] \frac{dG(\sigma^2)_{\tilde{x}}}{d\sigma^2}. \tag{2.81}$$

$G(\sigma^2)$ is solution of $\Box G = 0$ in $\tilde{x}$, so:

$$4 + 2\tilde{\lambda}(\nabla_a l^a)_{\tilde{x}} = 0. \tag{2.82}$$

On the other hand, $\tilde{G}$ solution of $\tilde{\Box}\tilde{G}$ in the point x means:

$$4 + 2\tilde{\lambda}\frac{d\lambda}{d\tilde{\lambda}}\nabla_a l^a + \tilde{\lambda}(D-2)\frac{d\lambda}{d\tilde{\lambda}}\frac{d(\ln A)}{d\lambda} = 0. \tag{2.83}$$

Substituting (2.82), we get:

$$2\tilde{\lambda}\frac{d\lambda}{d\tilde{\lambda}}\nabla_a l^a - 2\tilde{\lambda}(\nabla_a l^a)_{\tilde{x}} + \tilde{\lambda}(D-2)\frac{d\lambda}{d\tilde{\lambda}}\frac{d(\ln A)}{d\lambda} = 0. \tag{2.84}$$

We can write $\nabla_a l^a$ in terms of the VVD:

$$\nabla_a l^a = \frac{D-2}{\lambda} + \frac{d(\ln \Delta^{-1})}{d\lambda}; \tag{2.85}$$

and in the point $\tilde{x}$:

$$(\nabla_a l^a)_{\tilde{x}} = \frac{D-2}{\tilde{\lambda}} + \frac{d\ln(\tilde{\Delta}^{-1})}{d\tilde{\lambda}}, \tag{2.86}$$

Where here we are calling $\Delta = \Delta(x, x')$ and $\tilde{\Delta} = \Delta(\tilde{x}, x')$. We obtain, substituting:

$$\frac{2}{\lambda} + \frac{2}{D-2}\frac{d(\ln \Delta^{-1})}{d\lambda} - \frac{d\tilde{\lambda}}{d\lambda}\frac{2}{\tilde{\lambda}} + \frac{2}{D-2}\frac{d\ln(\tilde{\Delta})}{d\lambda} + \frac{d(\ln A)}{d\lambda} = 0; \tag{2.87}$$

which can be written as:

$$\frac{d}{d\lambda}\left[\frac{\lambda^2}{\tilde{\lambda}^2}\left(\frac{\tilde{\Delta}}{\Delta}\right)^{\frac{2}{D-2}} A\right] = 0. \tag{2.88}$$

The solution is:

$$A = C\frac{\tilde{\lambda}^2}{\lambda^2}\left(\frac{\tilde{\Delta}}{\Delta}\right)^{-\frac{2}{D-2}}; \tag{2.89}$$

requiring that $A \to 1$ when $L_0 \to 0$, we see that C has to be 1.
To summarize, we have that the qmetric for null separation is:

$$\tilde{g}_{ab} = \frac{\tilde{\lambda}^2}{\lambda^2}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{2}{D-2}} g_{ab} - \left[\frac{d\tilde{\lambda}}{d\lambda} - \frac{\tilde{\lambda}^2}{\lambda^2}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{2}{D-2}}\right](l_a m_b + m_a l_b). \tag{2.90}$$

In order to compare the forms of the qmetric in the various cases, we recall the form of the functions found for the spacelike/timelike geodesics. We know that the affine parameter for classical geodesics of this type is $s = \sqrt{\epsilon\sigma^2}$; we can define an analogous affine parameter for the case of modified geodesics, $\tilde{s} = \sqrt{\epsilon S_{L_0}}$. With this definition, α can be written as:

$$\alpha = \left(\frac{ds}{d\tilde{s}}\right)^2. \tag{2.91}$$

We can also write the function A as:

$$A = \frac{\tilde{s}^2}{s^2}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{2}{D-1}}. \tag{2.92}$$

So, we can write the qmetric for spacelike/timelike geodesics as:

$$\tilde{g}_{ab} = \frac{\tilde{s}^2}{s^2}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{2}{D-1}} g_{ab} + \epsilon\left[\left(\frac{d\tilde{s}}{ds}\right)^2 - \frac{\tilde{s}^2}{s^2}\left(\frac{\Delta}{\tilde{\Delta}}\right)^{\frac{2}{D-1}}\right] q_a q_b. \tag{2.93}$$

Comparing the forms (2.90) and (2.93), we can see the main differences: the different dependence on the dimension of the spacetime (of course, in the case of lightlike geodesics, we are dealing with a lower dimensional space), the different power of the function α, and the presence, in the case of null separations, of an additional arbitrary vector m_a, which depends on the choice of an observer in the base point x'. Since in the lightlike case we have a dependence on an observer, we could ask in what sense the Lorentz invariance is preserved. The interpretation is that, once we have specified an observer, so a local frame, the structure of the qmetric will be the same, with a Lorentz invariant minimal length L_0 of the affine parameter, which can have the role of time or distance for the local observer.

2.4 Minimal area

Now we introduce one of the most important concept for our work, which is the one of a minimal area. Indeed, having a minimal length in a space or spacetime gives some non-trivial consequences, one of these regarding the concepts of area and volume around a point [9, 10, 11]. This is actually the effect which leads to the presence of a quantum of area for black holes horizons, and so is a key property for the discussion of dynamics which involves variation of area. In order to explain the basic concept, we start from the simplest case, namely an Euclidean space modified with the qmetric. We strictly follow the discussion of [9].

2.4.1 qEuclidean space

In a D-dimensional Euclidean space, $\mathbb{R}^{\mathrm{D}}$, we know that in Cartesian coordinates $\{x^1, ..., x^D\}$ the metric is simply $g_{ab} = \delta^a_b$, so the line elements is:

$$ds^2 = \sum_{i=1}^{D} (dx^i)^2. \tag{2.94}$$

Taking as our base point the origin, $x' \equiv \{0, ..., 0\}$, we have that, selected a field point x, there is one and only one geodesic connecting x' with x, and the geodesic distance is simply:

$$\sigma^2(x, 0) = \sum_{i=1}^{D} (x^i)^2. \tag{2.95}$$

Since we deal with radial geodesics, it is common to change coordinates from Cartesian to polar:

$$\{x^1, ..., x^D\} \to \{\rho, y^1, ..., y^{D-1}\}, \tag{2.96}$$

where ρ is the radial coordinate and $\{y^1, ..., y^{D-1}\}$ are angular coordinates in the $(D-1)$-dimensional space orthogonal to ρ. Taking the explicit laws of transformation, we have that for the line element we can write:

$$ds^2 = d\rho^2 + \rho^2 d\Omega^2_{D-1}, \tag{2.97}$$

where $\sigma^2 d\Omega^2_{D-1}$ is the line element on the surface of an hypersphere of radius ρ, centered in the origin of $\mathbb{R}^{\mathrm{D}}$. We can see that in this system of coordinate the metric is still diagonal, and for the determinant holds:

$$\sqrt{\delta} = \rho^{D-1} \times [\text{Angular terms}]; \tag{2.98}$$

here [Angular terms] encodes all contributions from angular variables.
The equigeodesic surface $\Sigma_0(l^2)$ is simply the set of points x which have a fixed squared distance l^2 from the origin, $\sigma^2(x) = l^2$. On this (hyper)surface, we have the induced metric h_{ab} which gives a line element:

$$ds^2|_\Sigma = l^2 d\Omega^2_{D-1}. \tag{2.99}$$

For the square root of the determinant of h_{ab} holds:

$$\sqrt{h} = l^{D-1} \times [\text{Angular terms}]. \tag{2.100}$$

Now we can introduce two main concepts: the area around a point and the volume around a point. We start from the area. Take a point x on $\Sigma_0(l^2)$. This point is at a fixed distance from the origin, and its position is identified by the set of angular coordinates. Without loss of generality, we can take $y^i = 0$ for each $i = 1, ..., D-1$. The infinitesimal area element around x is:

$$dA(x) = \sqrt{h} d^{D-1} y = l^{D-1} d\Omega_{D-1}. \tag{2.101}$$

If we integrate over all the angular variables, we can compute actually the total area around the base point associated to a given equigeodesic surface. We would have simply:

$$A_0(l) = l^{D-1}\Omega_{D-1}, \tag{2.102}$$

where Ω_{D-1} is the geometric factor arising from the integration over all the angular range. We note that, if we take the coincidence limit $x \to 0$, so in this case $\sigma^2 \to 0$, both $dA(x)$ and A_0 goes to 0. We can define also the volume around a point x on the equigeodesic surface, via:

$$dV = \sqrt{\delta} d^D x = l^{D-1} dl d\Omega_{D-1}. \tag{2.103}$$

We are moving also the radial coordinate in this definition, but we still write l, because we underline that we are referring to a point x on the surface $\Sigma_0(l^2)$. To identify the notion of infinitesimal volume around x we can integrate l in a small arbitrary range, for example $[l-\epsilon, l+\epsilon]$.
Also for this case, we can define the volume around the base point, integrating over all the angular variables and from 0 to l the radial coordinate, which for clarity now we call ρ. We have:

$$V_0(l) = \int_0^l A_0(\rho) d\rho = \frac{l^D}{D}\Omega_{D-1} \tag{2.104}$$

Here also, both the infinitesimal volume around x and the total volume around the base point go to zero in the coincidence limit. Indeed, for $l \to 0$, we have $V_0(l) \to 0$, and if x tends to the origin, the same happens to all the points on the geodesic connecting the origin to x, and so $l-\epsilon$ and $l+\epsilon$ both tend to 0. These considerations hold for the classical flat space $\mathbb{R}^{\mathrm{D}}$. Now let us explore the case of the same space with the introduction of the qmetric.
Recall that, in the case of flat spaces, we have $\Delta = \tilde{\Delta} = 1$, moreover since we are considering just space coordinates, we have $\epsilon = 1$. We have then:

$$\tilde{\delta}_{ab} = \frac{S_{L_0}}{\sigma^2}\delta_{ab} + \left(\frac{\sigma^2}{S_{L_0}} S'^2_{L_0} - \frac{S_{L_0}}{\sigma^2}\right) q_a q_b, \tag{2.105}$$

and the associated line element squared, after some computations, is:

$$d\tilde{s}^2 = \tilde{g}_{ab} dx^a dx^b = (d\sqrt{S_{L_0}})^2 + S_{L_0} d\Omega^2_{D-1}. \tag{2.106}$$

As we have seen before, $d\tilde{s}^2$ has the same structure of ds^2 in the classical case, with the substitution $\sigma \to \sqrt{S_{L_0}}$. The main difference is that now we are dealing with a strange topology: all the points at a distance less than L_0 from the origin are removed, leaving a hole of radius L_0. Now we focus on the modified area and volume around a point. The definitions we gave in the case of a standard Euclidean space are still valid; the difference now is that, instead of the quantities $\sqrt{h}$ and $\sqrt{\delta}$, we have $\sqrt{\tilde{h}}$ and $\sqrt{\tilde{\delta}}$. On the hypersurface made by all the points at a fixed modified squared distance from the origin $S_{L_0} = S_{L_0}(l^2)$, we define the infinitesimal q-area and the infinitesimal q-volume via:

$$d\tilde{\Sigma} = \sqrt{\tilde{h}} d^{D-1}y, \quad d\tilde{V} = \sqrt{\tilde{\delta}} d^D x; \tag{2.107}$$

Changing the name of the infinitesimal area to avoid confusion with the function A. We already saw some relations between the determinant of the classical metrics and the qmetrics, and they yield:

$$\sqrt{\tilde{\delta}} = \frac{A^{\frac{D-1}{2}}}{\sqrt{\alpha}} \sqrt{\delta}, \quad \sqrt{\tilde{h}} = A^{\frac{D-1}{2}} \sqrt{h}. \tag{2.108}$$

From these, we obtain:

$$d\tilde{\Sigma} = \left(\sqrt{S_{L_0}}\right)^{D-1} d\Omega_{D-1}, \tag{2.109}$$

$$d\tilde{V} = \frac{1}{2}\left(\sqrt{S_{L_0}}\right)^{D-2} dS_{L_0} d\Omega_{D-1}. \tag{2.110}$$

Now, we examine the coincidence limit. For the case of the volume of the entire equigeodesic ball centered on the origin, we have:

$$\tilde{V}_0(l) = \frac{1}{2}\int_{S_{L_0}(0)}^{S_{L_0}(l^2)} (S_{L_0})^{\frac{D-2}{2}} dS_{L_0} \int d\Omega_{D-1} = \frac{\Omega_{D-1}}{D}[S_{L_0}(l^2)^{D/2} - L_0^{D/2}]; \tag{2.111}$$

which goes to zero in the coincidence limit. About the volume around a point on the equigeodesic surface, we can for example integrate the radial coordinate S_{L_0} between two arbitrary values $S_{L_0}(l_-^2)$ and $S_{L_0}(l_+^2)$ and the angular variables over all the range, obtaining:

$$\tilde{V}_0(l) = \frac{\Omega_{D-1}}{D}[S_{L_0}(l_+^2)^{D/2} - S_{L_0}(l_-^2)^{D/2}]. \tag{2.112}$$

In the coincidence limit, another time we have $l_+^2, l_-^2 \to 0$, so we have that also this volume goes to zero. Now we study the case of the area. Taking the infinitesimal q-area, in the limit $l^2 \to 0$ we see clearly that:

$$\lim_{l^2 \to 0} d\tilde{\Sigma} = L_0^{D-1} d\Omega_{D-1}, \tag{2.113}$$

which is non vanishing. Integrating over all the angles, we have:

$$\tilde{\Sigma}_0 = L_0^{D-1}\Omega_{D-1}. \tag{2.114}$$

This is a remarkable result: introducing a minimal length in our space leads to a minimal area around the base point, but still with a null volume, in the coincidence limit. This non-trivial behaviour can be interpreted as the fact that, having a minimal length, we have also a minimal value for defining a q-equigeodesic surface around a point, and this gives a minimal value for the area,

which can be interpreted as the area of a point. We studied this concept in the case of an Euclidean space, so with zero curvature and without time coordinate, but the same feature appears also in a Minkowskian spacetime, and, in general, in a spacetime with non zero components of the Riemann tensor (and actually with a curvature which is not too high).
We conclude this paragraph briefly talking about the case of a four-dimensional spacetime of Lorentzian signature, which is the physical case we are interested in. In particular, we want to focus on the case of area of a null surface. Suppose we have a base point x', and consider the lightcone of x'. The qmetric induced on the lightcone gives a squared line element that can be written as:

$$d\tilde{s}^2 = \tilde{\lambda}^2 d\Omega^2. \tag{2.115}$$

Taking now a point x on the lightcone, the area element around x reads:

$$d\tilde{\Sigma}(x) = \tilde{\lambda}_x^2 d\Omega, \tag{2.116}$$

where we specified that the parameter $\tilde{\lambda}_x$ is considered in the point x. Integrating over the solid angle, we get:

$$\tilde{\Sigma}(x) = 4\pi\tilde{\lambda}_x^2. \tag{2.117}$$

Now, we know that in the coincidence limit we have $\tilde{\lambda}_x - \tilde{\lambda}_0 \to L_0$; setting $\tilde{\lambda}_0 = 0$ we obtain $\tilde{\lambda}_x \to L_0$ and so:

$$\lim_{x\to 0} \tilde{\Sigma}(x) = 4\pi L_0^2. \tag{2.118}$$

We have then a two dimensional surface around a point, which makes sense, since considering the lightcone of a point we are actually restricting the dimension of the spacetime, and then also the dimension of the area and volume around a point.

2.4.2 Quantization of black hole horizon's area

Having discussed the presence of a minimal area around a point given by the qmetric description, now we show the consequence of this property on black holes; in particular, we show that qmetric implements a minimal step in the increment of the horizon's area of a black hole. The discussion we follow is essentially from an operational point of view.
Suppose to have a Schwarzschild black hole with a classical metric in the usual coordinates $\{t, r, \theta, \phi\}$:

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right)dt^2 + \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 + r^2 d\Omega^2; \tag{2.119}$$

where $2Gm \equiv R_s$ is the radius at which is located the horizon. First, let us explain the approximations we are going to consider. Of course, we could from

the very beginning go to the qmetric description, computing the squared line element $d\tilde{s}^2$ associated to this kind of spacetime, and study the modification of the area and other characteristics of the black hole with the new metric $\tilde{g}_{ab}$. However, we already saw that, from the computational point of view, qmetric is a very difficult structure to deal with, due also to its non-local behaviour. For this reason, we decide to do not simply a modification of the metric, but instead to focus on the effects generated by the qmetric itself.
Another fact is that, as we know, despite the high symmetry of the Schwarzschild solution, to study the complete geometry of the spacetime around a black hole is not a trivial task, especially near the horizon, which is the region of our interest. For this reason, we make a reasonable approximation: considering a small piece of the horizon, we actually work in a flat null surface of spacetime, making easier calculations and underlining the central point of the effect. We then consider no curvature effects, so using simply the Minkowski metric for a small region of spacetime around the horizon.
Consider now a photon, or a quanta of radiation, which is arriving at the horizon, ingoing to the center of the black hole at the speed of light. If we go to the rest frame of the photon, now the photon itself is at rest, and the horizon is moving towards it at the speed of light. We can imagine, from a practical point of view, that the horizon is made by radially outgoing photons, which are moving on null geodesics. To study the system corrected with effective features deriving from the qmetric description, we need to fix a base point in the spacetime. This base point, called x', is taken to be the spacetime point at which the horizon reaches the ingoing photon. Now, without loss of generality, it is possible to take this point as the origin of our coordinate system, setting $x' = (0, 0, 0, 0)$. We also rotate our frame so that the radial coordinate is completely along one of the Cartesian axes, say x, with increasing value. It means that we arrive at the point $(0, 0, 0, 0)$ increasing the value of x, so we start from negative value of it. Having motion on a null geodesic, this holds also for the time coordinate, which starts then from a negative value and arrives at 0 when the horizon meets the photon. Since we consider flat geometry, we have, at a generic point x before x', one geodesic which links x and x'. We can parametrize this geodesic via the affine parameter λ_x. In our frame, the photon frame, this parameter has the meaning of space or time distance from the point x'. The generic point x is then:

$$x = (\lambda_x, \lambda_x, 0, 0), \tag{2.120}$$

with $\lambda_x < 0$; and the tangent vector to the geodesic is:

$$l^a = \frac{dx^a}{d\lambda} = (1, 1, 0, 0). \tag{2.121}$$

Going to the point x', the parameter λ_x tends to $\lambda_{x'}$, which is 0. We have, in the coincidence limit:

$$\lim_{x \to x'} |\lambda_x - \lambda_{x'}| = \lim_{x \to x'} |\lambda_x| = 0. \tag{2.122}$$

Considering the area of the surface made of all the points at an affine geodesic distance from the base point x', which is λ_x, we have a spherical surface such that:

$$A(\lambda_x) = 4\pi\lambda_x^2. \quad (2.123)$$

Classically, approaching the base point, we have that λ_x approaches $\lambda_{x'} = 0$, so the area goes to 0. Introducing the qmetric, we know that in practice we map the parameter λ_x to $\tilde{\lambda_x}$ and $\lambda_{x'}$ to $\tilde{\lambda}_{x'}$. In this case, in the coincidence limit, we have:

$$\lim_{x\to x'} |\tilde{\lambda}_x - \tilde{\lambda}_{x'}| = \lim_{x\to x'} |\tilde{\lambda_x}| = L_0. \quad (2.124)$$

This means that, around the base point x', we have a non-vanishing value of the area given by: $A_{x'} = 4\pi L_0^2$. In the radial motion of the photon into the center of the black hole, when the horizon and the photon meet, we have so two kinds of area: the area of the horizon, or of a region of the horizon around the base point, and the area arising from the coincidence limit. Both of them are irreducible: the first from thermodynamical properties of the black hole, the second from the properties of the qmetric. When the photon crosses the horizon, or in other words, when the radiation is absorbed, this two areas should be added together, leading to the concept of minimal area variation, dependent on the minimal length in the spacetime:

$$\Delta A_{min} = 4\pi L_0^2. \quad (2.125)$$

After the absorption event, the black hole can be seen in a situation of perturbed geometry, with a point on the horizon which is gaining an amount of area ΔA_{min}. For the no-hair theorem, the black hole should reach stability keeping spherical geometry of the horizon, but increasing its value to A', where:

$$A' \geq A + \Delta A_{min}. \quad (2.126)$$

Now we know that, from the thermodynamics of black holes, when matter or energy crosses an horizon and is absorbed by the black hole, the area of the horizon increases of an amount which is proportional to the amount of energy which is absorbed. If the amount of energy which would be absorbed is not enough in order to satisfy (2.126), so to give an area increment at least of ΔA_{min}, the matter or energy can not be absorbed, according to our qmetric description.

Chapter 3

Effects of qmetric: properties and examples

In this chapter, we present some examples of applications of the qmetric, principally from an effective point of view, explaining some features and properties of this structure. First, we want to present a form of the qmetric which is based on an expansion in small lengths. This is motivated by the fact that, since the qmetric differs significantly from the standard metric only when we reach scales of the order of the Planck length, we could think that when we want to study some physical system (dynamics of a particle, trajectories, etc.) at such scales through the implementation of the qmetric, it can be useful to have a simplified form which captures the essential difference from the classical case.
In the second section, we present a method to characterize a spacetime based on its curvature invariants, and in particular the Kretschmann scalar $R_{abcd}R^{abcd}$. We will show what kind of role plays the minimal length in such a characterization, and how it is interpreted.
The third part is dedicated to one of the most famous phenomenon which combine quantum effects with a curved spacetime, which is the emission of Hawking radiation. After a discussion about the changing in this phenomenon due to the presence of a quantum of horizon area for a black hole, we propose a simple modification to the law which gives the density of emitted particles in function of the particle frequency ω, still from an effective point of view.

3.1 qmetric for small distances

As said before, we begin the chapter proposing a form of the qmetric which holds for small distances. Our strategy will be simple, based on suitable expansions of quantities. For simplicity, we focus on the case of spacelike geodesics. Start from the complete form of the qmetric, in a maximally symmetric space, which we recall here:

$$\tilde{g}_{ab} = \frac{S_{L_0}}{\sigma^2}\Big(\frac{\Delta}{\Delta_S}\Big)^{2/(D-1)} g_{ab} \;+\; \epsilon\Bigg\{\; \frac{\sigma^2 S'^2_{L_0}}{S_{L_0}} \;-\; \frac{S_{L_0}}{\sigma^2}\Big(\frac{\Delta}{\Delta_S}\Big)^{2/(D-1)}\Bigg\} q_a q_b. \quad (3.1)$$

Now, we know the qualitative behaviour of the qmetric, which is reflected also by the qualitative behaviour of the modified distance $S_{L_0}(\sigma^2)$: for great values of the squared distance σ^2, compared to the squared minimal length L_0^2, $\tilde{g}_{ab}$ tends to g_{ab}, and S_{L_0} tends to σ^2. When we go to the limit $\sigma^2 \to 0$, however, we get a real modified squared distance which tends to L_0^2. At the same time, $\tilde{g}_{ab}$ tends to a form which diverges for vanishing squared distances σ^2, precisely to balance the 0 coming out from it, giving a finite result. In approaching this limit, in which emerges strongly the behaviour of the *non classical part* of the qmetric, we could think that are predominant some of the properties of the quantum nature of the spacetime, and that they can be underlined specializing the general form $\tilde{g}_{ab}$ for $\sigma^2 \to 0$. In doing this, we simply apply basic notions of Taylor expansions, starting precisely from the unknown function S_{L_0}. As usual, to simplify the discussion, we study the simplest case of spacetime: we take a completely flat spacetime in four dimensions, so our classical metric g_{ab} is simply η_{ab}, and (2.1) becomes:

$$\tilde{\eta}_{ab} = \frac{S_{L_0}}{\sigma^2}\eta_{ab} + \epsilon\Bigg\{\frac{\sigma^2 S'^2_{L_0}}{S_{L_0}} - \frac{S_{L_0}}{\sigma^2}\Bigg\} q_a q_b; \quad (3.2)$$

because $\Delta/\Delta_S = 1$ in this case. Writing explicitly the form of q_a and q_b, we get:

$$\tilde{\eta}_{ab} = \frac{S_{L_0}}{\sigma^2}\eta_{ab} + \Bigg\{\frac{\sigma^2 S'^2_{L_0}}{S_{L_0}} - \frac{S_{L_0}}{\sigma^2}\Bigg\}\frac{\partial_a\sigma^2\partial_b\sigma^2}{4\sigma^2}. \quad (3.3)$$

Now we start to work in the case of small distances, which is analogous to the limit $\sigma^2 \to 0$. Of course, the function S_{L_0}, which represents the corrected distance, is unknown. In our treatment, we only know that it is a function of σ^2. Supposing that S_{L_0} is sufficiently regular, meaning that, being a physical function, it has not exotic features, at least around $\sigma^2 = 0$, we can make the hypothesis that for squared distances which tend to 0 it admits such an expansion:

$$S_{L_0} \simeq L_0^2 + k\sigma^2, \quad \text{for } \sigma^2 \to 0, \quad (3.4)$$

where k is an unknown parameter, that is the first coefficient in its expansion for small value of the squared distance. We can see that, in going through this limit, since we are considering just the first order of S_{L_0} we deal with only one unknown real parameter instead of an entire function. This can be useful for the study of different systems, in which we analyze dynamics around the minimal length scale: in such cases, we have just one free parameter in the description, that maybe can also be fixed imposing some physical requirement. An important fact to underline is that k should be positive: it must be different from zero to satisfy the third fundamental requirement of the qmetric ($[|S_{L_0}|/S'^2_{L_0}](0) < \infty$), and, having $k < 0$, for $\sigma^2 \to 0$ we obtain $S_{L_0} < L_0^2$, so $k > 0$. Now, plugging the condition (3.4) in the form of $\tilde{\eta}_{ab}$, we get:

$$\tilde{\eta}_{ab} = \frac{L_0 + k\sigma^2}{\sigma^2}\eta_{ab} + \left\{ \frac{\sigma^2 k^2}{L_0^2 + k\sigma^2} - \frac{L_0^2 + k\sigma^2}{\sigma^2} \right\} \frac{\partial_a \sigma^2 \partial_b \sigma^2}{4\sigma^2}, \tag{3.5}$$

having clearly $S'_{L_0} = k$. In what follows, we will perform some calculation, keeping just the relevant terms in σ^2, and to do that could be important to underline this: while σ^2 is tending *exactly* to 0, the value L_0^2, despite is expected to be very small, is fixed. So, for example, in ratios between σ^2 and L_0^2 the higher term is L_0^2:

$$\frac{\sigma^2}{L_0^2} \to 0, \quad \text{for } \sigma^2 \to 0. \tag{3.6}$$

The construction is of course independent on the precise numerical value (and order) of L_0^2; we know that physically it should be $L_0 \simeq 10^{-35}$, but in any case, in our treatment this is a constant value, so we have the condition indicated in the last equation. We get:

$$\tilde{\eta}_{ab} = \left(k + \frac{L_0^2}{\sigma^2} \right) \eta_{ab} - \left(\frac{L_0^2 (L_0^2 + 2k\sigma^2)}{4\sigma^4 (L_0^2 + k\sigma^2)} \right) \partial_a \sigma^2 \partial_b \sigma^2. \tag{3.7}$$

We note that, in doing the explicit computation, a piece quadratic in k already simplifies. Now, let us collect all terms at the numerator:

$$\tilde{\eta}_{ab} = \frac{4\sigma^2 (L_0^2 + k\sigma^2)^2 \eta_{ab} - (L_0^4 + 2kL_0^2\sigma^2) \partial_a \sigma^2 \partial_b \sigma^2}{4\sigma^4 (L_0^2 + k\sigma^2)}. \tag{3.8}$$

Until now, we did not approximate anything in the form, up to the function S_{L_0} for which we wrote a hypothetical expansion. Now, we look at the denominator. Factorizing out L_0^2, we have the term in brackets which becomes $(1 + k\sigma^2/L_0^2)$. We apply the expansion:

$$\frac{1}{4\sigma^4 L_0^2 (1 + \frac{k\sigma^2}{L_0^2})} \simeq \frac{1}{4\sigma^4 L_0^2} \Big(1 - \frac{k\sigma^2}{L_0^2} \Big). \tag{3.9}$$

Inserting this expansion in the form above, and doing computations, after collecting terms in front of η_{ab} and $\partial_a \sigma^2 \partial_b \sigma^2$, we can arrive at:

$$\tilde{\eta}_{ab} \simeq \left(\frac{L_0^2}{\sigma^2} + k - \frac{k^2 \sigma^2}{L_0^2} - \frac{k^3 \sigma^4}{L_0^4} \right) \eta_{ab} + \left(-\frac{L_0^2}{4\sigma^4} - \frac{k}{4\sigma^2} + \frac{k^2}{2L_0^2} \right) \partial_a \sigma^2 \partial_b \sigma^2. \tag{3.10}$$

This is the approximated form we found, in which we ordered the terms in growing powers of σ^2. As we see, there are three terms, one in front of η_{ab} and two in front of $\partial_a \sigma^2 \partial_b \sigma^2$, that are divergent in $\sigma^2 \to 0$. This makes sense, because without diverging terms we cannot balance the behaviour of the standard squared distance in the studied limit, as anticipated. Now, we want to contract with $dx^a dx^b$, to find the approximated form of the line element in this limit. Before doing that, however, we underline an important point: the quantity we find has the meaning of infinitesimal distance *around* L_0^2, in the sense that we are considering increments of square distance that start already from the minimal one L_0^2. One possible way to formalize this feature is to consider the difference between the metric we are considering and another formal one, which has $k = 0$ and so gives only the constant contribution equal to L_0^2 in the limit $\sigma^2 \to 0$ (as we said, the case $k = 0$ is actually forbidden). Let us denote it with $\delta_k \sqrt{S_{L_0}}$, not to be confused with $d\sqrt{S_{L_0}}$, which denotes instead the differential of the effective geodesic distance, that diverges in the $\sigma^2 \to 0$ limit.

We have, from the relations of the standard metric:

$$\begin{aligned} &\eta_{ab} dx^a dx^b = (d\sigma)^2, \\ &\partial_a \sigma^2 \partial_b \sigma^2 dx^a dx^b = 4\sigma^2 (d\sigma)^2; \end{aligned} \tag{3.11}$$

from these we then see what terms cancel out in taking the contraction. Remarkably, terms linear in k cancel out, giving a contribution which is quadratic and cubic in k:

$$\left(\delta_k \sqrt{S_{L_0}}\right)^2 \simeq \frac{k^2 \sigma^2}{L_0^2} (d\sigma)^2 - \frac{k^3 \sigma^4}{L_0^4} (d\sigma)^2. \tag{3.12}$$

This result is consistent with the same line element computed starting only from our hypothesis about the modified squared distance S_{L_0}, namely $S_{L_0} \simeq L_0^2 + k\sigma^2$. Let us see this:

$$\delta \sqrt{S_{L_0}} \simeq \frac{d\sqrt{S_{L_0}}}{d\sigma^2} d\sigma^2 = \frac{k d\sigma^2}{2\sqrt{L_0^2 + k\sigma^2}} \simeq \frac{k}{2L_0} \left(1 - \frac{k\sigma^2}{2L_0^2}\right) d\sigma^2; \tag{3.13}$$

So (recalling that $d\sigma^2 = 2\sigma d\sigma$):

$$\left(\delta \sqrt{S_{L_0}}\right)^2 = \left[\frac{k}{2L_0} \left(1 - \frac{k\sigma^2}{2L_0^2}\right) 2\sigma d\sigma\right]^2 \simeq \frac{k^2\sigma^2}{L_0^2} (d\sigma)^2 - \frac{k^3\sigma^4}{L_0^4} (d\sigma)^2. \tag{3.14}$$

In the last, we considered the term in σ^6 negligible, but it would give:

$$\left(\delta \sqrt{S_{L_0}}\right)^2 = \frac{k^2\sigma^2}{L_0^2} (d\sigma)^2 - \frac{k^3\sigma^4}{L_0^4} (d\sigma)^2 + \frac{k^4\sigma^6}{4L_0^6} (d\sigma)^2. \tag{3.15}$$

So, from these we have an hint about the presence of an alternative sign; this could have a deep meaning, suggesting that, in the field of all the possible forms of $S_{L_0}(\sigma^2)$, we have to look for some function which, if admitting an expansion around $\sigma^2 = 0$, it occurs with alternative signs.
There are more considerations that we can say about this discussion; however, since we have to focus on another point, we just wanted to show some basic features. This topic could be explored further in the future.

3.2 A spacetime characterization based on L_0

Here we want to explore a possible way of interpreting the minimal length L_0, namely, the identification of this scale as a *radius of curvature* for a given classical spacetime. In particular, L_0 will give a limit about the maximal value of curvature we can have in a classical spacetime, or in other words, that the spacetime support. There are different references in which, to study such a limit, are taken generic components of the curvature tensor in the theory, or some general combination of them [12]; in what follows, we propose another method, which instead of general components works with scalars, and more precisely, the Kretschmann one. The advantage of doing this is that we work with coordinates independent objects; moreover, even if we are dealing with non-local concepts, like L_0 and the qmetric, in this discussion we look just at classical scalars, so associated only to a point in the spacetime, being computed from standard tensors.

3.2.1 Maximal Kretschmann scalar for Schwarzschild spacetime

Our interpretation is very simple, and to start with, we take the important case of a Schwarzschild spacetime. It is precisely in this case that, due to the particular form of the metric and the properties of Einstein equation, it particularly makes sense to consider the Kretschmann invariant.
We know the Schwarzschild metric, which we recall in dimensional units:

$$ds^2 = -\left(1 - \frac{2Gm}{rc^2}\right)dt^2 + \left(1 - \frac{2Gm}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2; \tag{3.16}$$

where m is the mass of the star, or the black hole, concentrated in a point which is the origin of coordinates, and r is the distance from it. Now, we know that such a spacetime is surely not flat, being characterized by a non-vanishing value of the Riemann tensor R_{abcd}. It is usual to define different types of scalar quantities, computed contracting the various tensors we have in the theory. This scalars can give a *summary* of the spacetime curvature, depending on the spacetime point itself. For example, is well known that for a two-dimensional sphere of radius r, embedded in a three-dimensional space, we have a constant curvature, which can be found by the Ricci scalar R, that gives $R = 2/r^2$. We see that this quantity is inversely proportional to the square of the radius of the sphere, which is consistent with an intuitive picture one can have: the more a sphere is big, the less you feel you are curving if you move along the surface. In analogy, it is common to talk about the concept of *radius of curvature* also in a general manifold, and this also makes sense from a dimensional analysis, since from the definition in terms of the derivative of the metric tensor, we have:

$$R \sim l^{-2}, \tag{3.17}$$

where for l we mean length. As we said, the are many scalars we can write down with different combinations of tensors; for Physics, of course the simplest and most important is the Ricci one, because it enters in the gravitational action giving the usual Einstein equation. Nevertheless, to study some properties of the spacetime itself, it could be relevant also to look at other quantities.
Take now the previous situation: a Schwarzschild spacetime. Suppose that we want to consider the spacetime property of a minimal length, knowing that we can associate a length also to the curvature scalars. Specifically, we want to ask ourselves: having a minimal length, does it happen that we have a *limit* about the magnitude of curvature in the spacetime considered? In our simple example of a Schwarzschild spacetime, to answer this question we start to list the main tensors we have: the metric tensor g_{ab}, the Ricci tensor R_{ab} and the Riemann tensor R_{abcd}. Combining them, we can define the principal scalars in the theory; the simplest are: the Ricci scalar $R = g^{ab}R_{ab}$, the scalar $R_{ab}R^{ab}$ and the Kretschmann scalar $R_{abcd}R^{abcd}$. However, if we are in vacuum, it is easy to take Einstein equation and, contracting with the metric and substituting, to obtain the following results:

$$R = 0, \quad R_{ab} = 0; \tag{3.18}$$

which means that, of the three scalars above, two are identically zero in vacuum: $R = R_{ab}R^{ab} = 0$. The only scalar which for sure is different from zero, being in practice the square of the most general tensor that encodes information about curvature, is the Kretschmann scalar, which we call now K, and which for Schwarzschild gives (in dimensional units):

$$K = \frac{48G^2m^2}{c^4r^6}. \tag{3.19}$$

We see immediately two consistent facts: the scalar is radially symmetric, since it depends just on the distance r from the center of the black hole, which is defined from the area of the surface around the center via $r = \sqrt{A/4\pi}$; then, we see that the only divergence we have is for $r = 0$, meaning that the only singular point in this spacetime is the center of the black hole. Looking at this scalar, how can we effectively implement a minimal length L_0 to study some possible bounds of the curvature? One of the possibilities is to plug directly $r = L_0$, interpreting the minimal length as a minimal distance from the point in which the mass is located, and this would give a value of the Kretschmann scalar which can be interpreted as the limiting value of curvature for Schwarzschild spacetime. This method, however, could have some issues: of course we cannot go closer than L_0 to the center, for the limits imposed by the qmetric, but this could give a value of the Kretshmann invariant which depends a lot on the particular spacetime we are in. We could think, instead, that each spacetime we have should be compared with the same value of the curvature, in other words, with the same radius of curvature. Our choice, so, is to interpret L_0 as a curvature radius, and since K has dimensions of l^{-4}, with l being a length, we take the following condition for the limit curvature:

$$|K| \simeq \frac{1}{L_0^4}. \tag{3.20}$$

The absolute value is taken because we consider the magnitude of the scalar K. So, for a Schwarzschild spacetime in four dimensions, we can define the region of spacetime of allowed curvature by:

$$\frac{48G^2m^2}{c^4r^6} < \frac{1}{L_0^4} \rightarrow r > \left(\frac{48G^2m^2L_0^4}{c^4}\right)^{1/6}. \tag{3.21}$$

This has of course the same radial symmetry as the Kretschmann scalar, meaning that we are defining a three-dimensional sphere centered in the point mass: in this sphere, the curvature is too high for the limits imposed by the presence of a minimal length L_0. The radius of this sphere can be called r_0. We see an interesting dependence in the mass m:

$$r_0 \sim m^{1/3}, \tag{3.22}$$

so, the more massive is our point-particle located at the origin, the bigger is the distance from this point at which curvature starts to be too high. We are actually removing a sphere from our spacetime, with radius r_0, and if we think at the volume V_0 of this sphere, we see that is linear in the mass:

$$V_0 \sim m. \tag{3.23}$$

Now, let us see some numerical values of this radius, to have an idea of the physical order of magnitude that r_0 can have.
For L_0, we take the Planck length $L_P = \sqrt{(\hbar G/c^3)}$. Of course, as we know, L_0 could be not precisely this values, but the order of magnitude is expected to be the same. For astrophysical objects like black holes, we have different orders for the mass: from a few times the solar mass $M_\odot$ for standard black holes to incredibly large values like $10^6 M_\odot$ for supermassive black holes. Take the solar mass as an example. We have:

$$r_0 = \left(\frac{48L_P^4G^2M_\odot^2}{c^4}\right)^{1/6} \simeq 1.4 \times 10^{-22} \text{ m}. \tag{3.24}$$

While for a mass of $10^6 M_\odot$, since the mass enters with a power of $1/3$, we get:

$$r_0 \sim 10^{-20} \text{ m}. \tag{3.25}$$

These orders could have an unexpected meaning: introducing a minimal scale of the order of the Planck one, so 10^{-35} m, actually for Schwarzschild we get a limit distance from the center of the body of order of 10^{-20} m, which is very big compared to the Planck length. It means that we lose the concept of usual

spacetime at a distance which is more and more large with respect to L_P, at least for a Schwarzschild spacetime. Now we try to deal with the relation for r_0 from a different point of view: requiring that the limiting distance r_0 is truly the same as L_0, we ask: what kind of mass can give such a result?
Setting as before $L_0 = L_P$, we obtain:

$$m_0 = \frac{L_0 c^2}{4\sqrt{3}G} = \frac{1}{4}\sqrt{\frac{\hbar c}{3G}} \simeq 3.1 \times 10^{-9} \text{ Kg}. \tag{3.26}$$

This makes sense, since the order found is precisely comparable to the one of the Planck mass, and so we have consistency about the procedure applied, even if it is a naive method to study bounds on the spacetime curvature.

3.2.2 Limit curvature and gravitational waves

In this part, we treat the concept introduced above in the case of a spacetime with nearly constant metric; more precisely, one through which are passing gravitational waves. We want to study a possible limit on curvature in the case of spacetime perturbations given by gravitational wave, eventually emitted by a coalescing binary. Take the case of a flat spacetime. We can understand from the very beginning that, probably, to reach this value of limiting curvature we need a GW frequency which is incredibly high, and so the study of this problem could seem to have no sense; however, different constants enter in the computation, giving an interesting contribution. Moreover, it could be conceptually interesting to see what kind of curvature can give a gravitational wave, even if it is very weak. We start recalling that, in the approximation of nearly flat spacetime, we have for the metric:

$$g_{ab} = \eta_{ab} + h_{ab}; \tag{3.27}$$

where η_{ab} is the Minkowski metric and h_{ab} is the dynamical metric, for which we suppose $|h_{ab}| << 1$. We know from the standard theory of General Relativity that, in vacuum, setting a suitable gauge condition we obtain that h_{ab} has to satisfy a wave equation, which is precisely the Einstein equation for small deviation from the Minkowski metric [13]. It turns out that the general solution h_{ab}, which can be seen as a 4×4 matrix in function of the spacetime coordinates, can be written in a very useful form, in which only two components, called $h_\times$ and h_+, are independent and different from zero. In this case, we have:

$$h_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.28}$$

in which all the spacetime dependence is in the functions $h_\times$ and h_+, which are called *cross* and *plus* polarizations. Of course the specific form of these functions depends on the characteristics of the waves, and in particular on the source. A binary system of black holes in coalescence, that is the main subject of our work, actually produces gravitational waves, a point that will be explained in the next chapter in more details; for now, we cite just the form of $h_\times$ and h_+ in this case, describing the various factor that enter:

$$\begin{aligned} h_+(t) &= \frac{4}{r}\Big(\frac{GM_c}{c^2}\Big)^{5/3}\Big(\frac{\pi f_{gw}}{c}\Big)^{2/3}\frac{1+\cos^2\theta}{2}\cos\left(2\pi f_{gw}t+2\phi\right), \\ h_\times(t) &= \frac{4}{r}\Big(\frac{GM_c}{c^2}\Big)^{5/3}\Big(\frac{\pi f_{gw}}{c}\Big)^{2/3}\cos\theta\sin\left(2\pi f_{gw}t+2\phi\right). \end{aligned} \tag{3.29}$$

In the last formulas, M_c is the *chirp mass*, a function of the two masses of the black holes m_1 and m_2:

$$M_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}; \tag{3.30}$$

f_{gw} is the frequency of the gravitational wave; t is actually the retarded time, which takes into account the fact that gravitational waves travel at a finite speed (the speed of light) through the space; r is the distance from the center of the coalescing binary; θ is geometrical quantity which depends on the orientation of the point of detection with respect to the position of the binary, ϕ is a constant shift. As explicitly indicated, the dependence in which we focus for physical application (like the real detection of gravitational waves on Earth or in the space) is the time dependence, an important point for what follows. It is clear that the particular kind of gravitational waves we are considering depends strongly on the characteristic of the source; this can be seen easily for M_c, and it holds also for f_{gw}, since it turns out to be proportional to the frequency at which the two components of the binary are orbiting around each other, called orbital frequency or source frequency f_s.
Now, return to the concept of curvature, and specifically to the one of scalars quantities. As we said, we consider a situation in which the spacetime is almost flat, and is crossed by gravitational waves produced in some region, probably far. It means that we are in vacuum, and the simplest scalars we can define from the tensors we have at hand are again identically zero. Another time, we look at the Kretschmann scalar $K = R_{abcd}R^{abcd}$. Luckily, in the case of nearly flat spacetime, due to the particular form of the metric and the fact that in the computations h_{ab} is treated as a small parameter, the Riemann tensor assumes a very simplified form, that can be written as [13]:

$$R_{abcd} = \frac{1}{2}(\partial_b\partial_c h_{ad} + \partial_a\partial_d h_{bc} - \partial_a\partial_c h_{bd} - \partial_b\partial_d h_{ac}). \tag{3.31}$$

We can now compute the Kretschmann scalar K; after some passages, relabelling some dummy variables, we have:

$$\begin{aligned} &K = R_{abcd}R^{abcd} = (\partial_b\partial_c h_{ad})(\partial^b\partial^c h^{ad}) + (\partial_b\partial_c h_{ad})(\partial^a\partial^d h^{bc}) \\ &+ (\partial_b\partial_c h_{ad})(\partial^a\partial^c h^{bd}) + (\partial_b\partial_c h_{ad})(\partial^b\partial^d h^{ac}). \end{aligned} \tag{3.32}$$

Now, consider the gravitational waves indicated above. Since we have just two spatial components different from zero, keeping only time derivatives terms, we have:

$$K \sim (\ddot{h}_\times)^2 + (\ddot{h}_+)^2. \tag{3.33}$$

This is a very simple result, symmetric in the two polarizations: a property which makes sense, looking at the form of h_{ab}. Plugging the explicit form of $h_\times(t)$ and $h_+(t)$, we obtain:

$$\begin{aligned}(\ddot{h}_\times)^2 + (\ddot{h}_+)^2 = \frac{16}{r^2c^4}\left(\frac{GM_c}{c^2}\right)^{10/3}\left(\frac{\pi f_{gw}}{c}\right)^{4/3}(4\pi^2 f_{gw}^2)^2 \\ \times\left[\left(\frac{1+\cos^2(\theta)}{2}\right)^2\cos^2(\Psi(t)) + \cos^2(\theta)\sin^2(\Psi(t))\right],\end{aligned} \tag{3.34}$$

where we added the correct power of c to have a dimension of m^{-4}, and $\Psi(t)$ is the total phase dependent on t. The double derivative had the effect to bring in front of the function another term $\sim f_{gw}^2$, then the square produced a total additional factor $\sim f_{gw}^4$. Now, we have a scalar dependent on time, in particular a periodic function: it is common in these situations to take a mean value of the quantity considered on several periods, indicated by $\langle\rangle$. In our case, since we have $\langle\cos^2(2\pi f_{gw}t + 2\phi)\rangle = \langle\sin^2(2\pi f_{gw}t + 2\phi)\rangle = 1/2$, we get an additional factor of $1/2$ in front of all. Then, we can look at the factor in square brackets, which depends just on θ. One can integrate over the complete solid angle and divide by 4π, obtaining the average value of the term which is $4/5$. So we obtain:

$$\langle K\rangle \sim \langle(\ddot{h}_\times)^2 + (\ddot{h}_+)^2\rangle = \frac{32}{5r^2c^4}\left(\frac{GM_c}{c^2}\right)^{10/3}\left(\frac{\pi f_{gw}}{c}\right)^{4/3}(4\pi^2 f_{gw}^2)^2. \tag{3.35}$$

The quantity in the last equation represents the mean curvature of spacetime, in the angle, and in time, due to gravitational waves generated by a coalescing binary. Here we suppose to be at a fixed, great distance r from the source, in a region of almost null curvature, as happens in the real cases. Suppose that we are at a generic point in space, fixing the angle θ. While time is passing, that space point feels a value of curvature (in our discussion identified with K) which is periodically varying in time, of which we took the mean value. We underline that the forms of $h_\times$ and h_+ considered here hold in the case of great distances compared to the size of the source, so, for example, the case of gravitational waves emitted by a coalescing binary and detected on Earth. Of course, seeing the constants that appear in (3.35), we can understand that we need a huge frequency value f_{gw} to have a mean value of curvature comparable with the maximal one, if we have $L_0 \sim L_P$. As anticipated, from gravitational waves moving in a flat spacetime endowed with a minimal length of the order of the Planck one, the issue of reaching a critical value of K is for now irrelevant.
The discussion above, however, is based on the assumption of nearly flat spacetime: we are expanding the metric around the Minkowskian one, and so the

predominant part of the metric is flat, giving intuitively that the curvature is very weak. We know, in any case, that gravitational waves propagate also in a curved background, for example, a region close to a black hole. In this case, we have for the metric:

$$g_{ab} = g_{ab,0} + h_{ab}; \tag{3.36}$$

where $g_{ab,0}$ is the constant, leading part of the metric, but different from a flat one, and h_{ab} is, as usual, the dynamical part. We explain briefly what happens for such a metric. In this case, we still have a wave equation for h_{ab}, but the differential operator associated with its evolution takes into account the presence of non-vanishing Christoffel symbols [13]. In particular, we will have that the form of the Riemann tensor is different with respect to (3.31), having part related to the covariant derivative ∇_a. This, in principle, gives a different scalar K, which depends of course on the particular geometry connected to $g_{ab,0}$. In practice, now the curvature value associated with K is not near to zero, but instead is near to the one given by $g_{ab,0}$, and it oscillates periodically around its mean value, as before. Computing this mean value, now we can truly find a value higher than the maximal one, since we start from a background curvature different from zero, which could be also high. The gravitational waves add a small amount of curvature, depending on their frequency, but this amount can be enough to reach the maximal curvature.

Instead of searching particular numerical values, we want to finish this part talking more about the case of a Kretschmann scalar dependent on time. We take the particular example of a scalar periodic in time, but the central point of the discussion holds also for a general dependence in time. Let us call this scalar $K(t)$, which in the case seen above is the Kretschmann scalar without considering a mean value over time. Now we consider cases of positive values for $K(t)$ at each instant of time, otherwise we have to consider absolute values. Suppose for a moment that we have a value of the frequency so high that we reach a mean value of the curvature which is less than the maximal one, but close, in the sense that it is almost of the same order, just a little smaller: $\langle K \rangle \lesssim K_{max}$. For the case of gravitational waves propagating in flat spacetime, we see that it is not easy to have a sufficiently high frequency; however, working in an already curved spacetime, as explained before, it could happen. Evolving in time, $K(t)$ will oscillate around its mean value, between its minimum K_- and its maximum value K_+. So, at a generic instant of time, we have:

$$K_- \leq K(t) \leq K_+. \tag{3.37}$$

Now, suppose that $K_+ > K_{max}$; in other words, suppose that, while having anyway a mean value less than the limiting one, our $K(t)$ reaches values which are greater than this limiting one, periodically in time. So, while time is passing, restricting to a period of the wave we actually have that the spacetime curvature is in an allowed range of values for a certain interval of time, and here we can still apply the description via the concept of metric, which at most should be the qmetric, because we are at an order of the radius of curvature which is in any case comparable with L_0. In the other interval, we are above the limit, and spacetime has not the standard meaning of a coherent structure describable by a metric tensor. After an interval of time passed above the limiting level of curvature, the scalar $K(t)$ will return to values which are less then K_{max}, then it will go to K_-, and restart to grow, and this continues periodically in time. Of course, is not easy to understand completely what it means such a behaviour; for example, when $K(t)$ is above K_{max}, we should think that the concept of spacetime actually loses its meaning, and consequently also the concept of time t we are considering for the evolution of $K(t)$. If something like this phenomenon can happen, even if for now it is probably not common to deal with such situations, should we think that spacetime returns to its previous structure after having passed a limit value of curvature, or that it goes to another configuration, different with respect to the standard one? According to the effective qmetric description, such circumstances in which K_{max} is passed should not occur.

In any case, in this section we wanted to give a simple method with which one can characterize a spacetime under study, looking at a limit imposed by a minimal length. In regions with a curvature much smaller than the maximal, one can describe spacetime curvature locally with usual tensors, like R_{abcd}. When we are in a region with an allowed curvature, but close to the maximal one, spacetime is describable, but we should look at bitensors instead of tensors, corrected with the effective qmetric, giving $\tilde{R}_{abcd}$. Over the maximal value, actually we cannot say anything, since we lose consistency with the quantities associated with the standard spacetime properties. The qmetric, in any case, suggests that the curvature, after reached the maximal value, cannot increase further, giving a sort of frozen behaviour for the spacetime. Schematically, we can summarize with the following classification:

$$\begin{array}{ccc} K << K_{max}, & K \lesssim K_{max}, & K > K_{max} \\ R_{abcd}, & \tilde{R}_{abcd}, & ? \end{array} \tag{3.38}$$

with, according to the qmetric description, the case of $K > K_{max}$ being *collapsed* to $K = K_{max}$. In this way, instead of bitensors, we consider standard tensors to classify spacetime regions of allowed curvature, indicating how we actually need to look at a bitensorial description.

3.3 Hawking radiation with quantum of area

To proceed in our discussion of effective properties given by the introduction of a minimal length, and consequently, a step in area in a black hole horizon, we now study the case of a well known phenomenon: the emission of Hawking radiation [14],[15]. This works as follow: combining a QFT with a curved spacetime background, we discover the presence of a non-trivial value of temperature for the black hole, connected with the value of its surface gravity on the horizon. Due to the presence of this temperature, we observe the emission of particles,

here considered as scalar fields, with a spectral distribution which is a function of the particle frequency ω. We recall some basic concepts.
Taking the case of a Schwarzschild black hole of mass m, area $A = 16\pi m^2 G^2/c^4$, we have, on the horizon, a temperature T_{H}:

$$T_{\mathrm{H}} = \frac{\hbar c^3}{8\pi m G k_{\mathrm{B}}}, \tag{3.39}$$

inversely proportional to the mass of the black hole. The spectral distribution, in function of the frequency ω of the field, is found to be:

$$N(\omega) = \frac{1}{e^{\frac{\hbar\omega}{k_{\mathrm{B}} T_{\mathrm{H}}}} - 1}. \tag{3.40}$$

This form is found considering, in a first approximation, the effective potential of the black hole $V_{eff}(r)$ completely negligible. However, this function gives problems for $\omega \to 0$; indeed, we get a divergence. To solve this issue, and obtain a more physical distribution, one can consider that actually the function above is obtained considering that all the radiation emitted arrives at infinity; in other words, we are ignoring possible reflections given by the potential. If we define, as usual, the two coefficients $R(\omega)$ and $T(\omega)$ related respectively to the reflection and transmission of signals of frequency ω, we have of course:

$$|R(\omega)|^2 + |T(\omega)|^2 = 1. \tag{3.41}$$

In (3.40), since we have not the reflection term, we have $|T(\omega)|^2 = 1$, which is actually at the numerator of the function. In the general case, we would have:

$$N(\omega) = \frac{|T(\omega)|^2}{e^{\frac{\hbar\omega}{k_{\mathrm{B}} T_{\mathrm{H}}}} - 1}, \tag{3.42}$$

where the term $|T(\omega)|^2$ is also called *grey body factor*. Through the technique of the asymptotic match, it is possible to fix the coefficient, obtaining ($G = c = 1$):

$$N_{gb}(\omega) = \frac{16 m^2 \omega^2}{e^{\frac{\hbar\omega}{k_{\mathrm{B}} T_{\mathrm{H}}}} - 1}, \tag{3.43}$$

where now with N_{gb} we indicate the distribution corrected with the grey body factor. Now we see that in the limit for $\omega \to 0$, the function is well defined, giving a behaviour linear in ω and then convergent to 0. Now, let us study what can happen if we add the presence of a quantum in horizon area.
In the standard case, having the emission of radiation by a black body, we actually have a changing in the mass, and, consequently, of the horizon area. But since we have a finite step in area, it is not possible to increase continuously m and A, because we need an amount of energy high enough to produce the minimal step in area ΔA. Suppose that we consider each particle singularly, and so having an amount of energy proportional to its own frequency ω. If the frequency is not high enough, the energy of that particle cannot give a sufficient

variation of mass, or area, to the black hole. It means that we have, actually, the effective emission of radiation only starting from a specific frequency which we call ω_{crit}. Since our step in area is $\Delta A = 4\pi L_0^2$, in units $G = c = 1$ we can factorize out the coefficient which relates the Planck length L_P and the minimal one L_0. We have:

$$\Delta A = 4\pi L_0^2 = \alpha L_P^2 = \alpha\hbar. \tag{3.44}$$

Take now the mass-area relation for Schwarzschild black hole: $A = 16\pi m^2$. In a variational form, we can say that a small variation of mass δm and a small variation of area δA are related by:

$$\delta m = \frac{\delta A}{32\pi m}. \tag{3.45}$$

It means that we also have a step in mass connected to the one in the area, leading:

$$\delta m = \frac{\alpha\hbar}{32\pi m}. \tag{3.46}$$

Suppose we have a quantum particle with a specific frequency ω, that gives an amount of energy $E = \hbar\omega$. This particle can be emitted only if, in absolute value, it gives a variation of mass δm equal or greater than the one in the last equation. We obtain:

$$\hbar\omega_{crit} = \delta m \rightarrow \omega_{crit} = \frac{\alpha}{32\pi m}. \tag{3.47}$$

This is the form of the critical frequency, at which the black hole is able to emit or absorb energy. Return now to the spectral distribution $N(\omega)$, in its simple form (3.40).
We can write in a more useful form the exponent at the denominator of (3.40), defining:

$$\omega_0 = \frac{k_{\mathrm{B}} T_{\mathrm{H}}}{\hbar}. \tag{3.48}$$

In this way, we get:

$$N(\omega) = \frac{1}{e^{\frac{\omega}{\omega_0}} - 1}. \tag{3.49}$$

It is interesting to note that the ratio ω_{crit}/ω_0 is independent on the mass of the black hole, depending only on the parameter α:

$$\frac{\omega_{crit}}{\omega_0} = \frac{\alpha}{4}. \tag{3.50}$$

Consequently, we have that, changing the mass of the black hole, we can change the value of T_{H} and also the value of the frequency at which we start to have emission, but the value of the density of particles emitted at that frequency remains the same:

$$N(\omega_{crit}) = \frac{1}{e^{\frac{\alpha}{4}} - 1}. \tag{3.51}$$

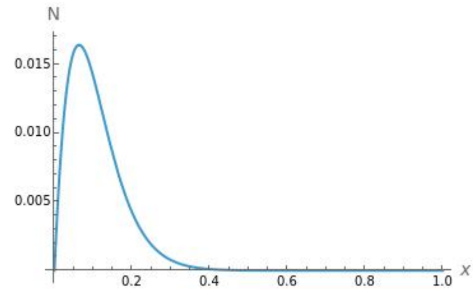

Figure 3.1: Graph of the function $N_{gb}(x)$.

If we consider the addition of the transmission coefficient, we have another time that $N(\omega_{crit})$ depends just on α. Indeed:

$$|T(\omega_{crit})|^2 = 16m^2\omega_{crit}^2 = \frac{\alpha^2}{64\pi^2}, \tag{3.52}$$

and we arrive at:

$$N_{gb}(\omega_{crit}) = \frac{\alpha^2}{64\pi^2}\frac{1}{e^{\frac{\alpha}{4}} - 1}. \tag{3.53}$$

It could be useful to have a graph of $N_{gb}(\omega)$. Setting $x \equiv m\omega$, we can see the behaviour in Figure 3.1.

Consider the addition of the critical frequency ω_{crit}. We can understand that, for the discussion above, the emission of radiation for frequencies in the range $[0; \omega_{crit})$ is suppressed, and the function should be modified in the form. We call the corrected distribution function, including the grey body factor, $N_{gb,\alpha}(\omega)$. Following a naive modification, we can initially think that the effective distribution now takes the form:

$$N_{gb,\alpha}(\omega) = \begin{cases} 0 & \text{if } \omega < \omega_{crit}, \\ N_{gb}(\omega) & \text{if } \omega \geq \omega_{crit}. \end{cases} \tag{3.54}$$

Here, we have explicitly wrote the label α to remember that the effective modification is controlled by the value of this parameter. Since we are *cutting* an entire piece of the function, setting it to zero, also an entire piece of the graph and of the subtended area in the figure is now forced to be identically zero. It is clear, now, that depending on the value of ω_{crit}, we can have a large range of frequencies in which the emission is suppressed. Since we see clearly that we deal with only a maximum point, to understand more quantitatively when it happens that we have the bigger part of the emission suppressed, we can take as a reference point the position of the maximum value, which we call ω_{max}. In practice, we want to know when is satisfied the condition:

$$\omega_{crit} > \omega_{max}. \tag{3.55}$$

The first point is to understand what is the value ω_{max}. Let us compute the derivative of $N_{gb}(\omega)$:

$$N'_{gb}(\omega) = \frac{32m^2\omega}{e^{\frac{\hbar\omega}{k_B T_H}} - 1} - \frac{16m^2\omega^2}{\left(e^{\frac{\hbar\omega}{k_B T_H}} - 1\right)^2}. \tag{3.56}$$

We should impose $N'_{gb}(\omega) = 0$, and solve for ω. However, we see easily that this is not an equation solvable analytically. To find at least an approximated value, we should proceed numerically. Before that, nevertheless, we try to understand by hand what is the value of ω_{max}, then we can compare with the more precise value found numerically. Our procedure is simple: recall that we can write the exponent with the help of ω_0, as in (3.49). From a certain point of view, ω_0 is a sort of characteristic frequency of the distribution. Try to consider values of ω which are multiples of ω_0. Start with ω_0 itself, and plug into the derivative $N'_{gb}(\omega)$. Easily we can see:

$$N'_{gb}(\omega_0) > 0, \tag{3.57}$$

so the value ω_{max} is after ω_0. Now we try with $2\omega_0$. Plugging in the derivative, we see:

$$N'_{gb}(2\omega_0) < 0, \tag{3.58}$$

meaning that ω_{max} is before $2\omega_0$. Naively, we can suppose that the position of ω_{max} is *almost* in the middle of this range, and we take the indicative value:

$$\omega' = \frac{3}{2}\frac{k_B T_H}{\hbar} = 1.5\omega_0. \tag{3.59}$$

This is obtained from a very naive procedure, looking just at the form of the graph and the sign of $N'_{gb}(\omega)$. Now, if we solve numerically the equation $N'_{gb}(\omega) = 0$, we actually find for the value of ω_{max}:

$$\omega_{max} = 1.5833\omega_0 \simeq 1.6\omega_0. \tag{3.60}$$

We see, then, that this value is not so far from the one found before. Now we can solve the inequality $\omega_{crit} > \omega_{max}$; substituting the form of T_H, we have:

$$\frac{\alpha}{32\pi m} > \frac{1.5833}{8\pi m} \to \alpha \gtrsim 6.3. \tag{3.61}$$

For such a value of α, then, we have a big suppression of Hawking radiation. In literature, there are different proposals for the constant α [9]; in this work, we will focus essentially on four possibilities: $\alpha = 4\ln 2$, $\alpha = 4\ln 3$, $\alpha = 4\pi$, $\alpha = 8\pi$. The last two values are bigger a lot compared to the one in the last inequality; in these cases, the position of the value of ω_{crit} is probably far from ω_{max}, and the great part of the Hawking radiation is cut. Consequently, if we deal with such values for α, to detect Hawking radiation will be even more difficult than the standard case, because, in addition to the probably small value of the horizon temperature (for macroscopic black holes), we are left, due to

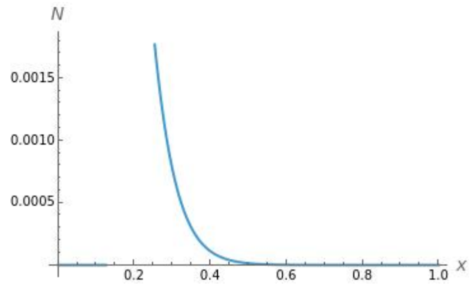

Figure 3.2: The graph of $N_{gb,\alpha}(x)$ in a first approximation.

effective quantum corrections, just with a small part of the complete radiation. Now, we can think that the form of $N_{gb,\alpha}(\omega)$ we are considering takes into account drastically the effects of quantization: our function now has two pieces, one identically equal to zero, the other following the standard distribution. We can see this in the graph of Figure 3.2.

We could want to define a function which, taking into account quantum effects, is modified also in the shape and in the behaviour, giving in any case a distribution equal to zero before the frequency ω_{crit}. However, such a distribution derives probably from a complete quantum theory of spacetime applied to the case of Hawking radiation. Nevertheless we can try to write down at hand some modified version of the function, and see if could make sense to study a modified spectral distribution. In practice, still considering (3.54), we change the function $N_{gb}(\omega)$ in the second line, looking for a less drastic alternative. One possibility is to do the replacement $\omega \to (\omega - \omega_{crit})$, keeping in any case the complete function equal to zero until we arrive at ω_{crit}. In this way, we conserve precisely the original behaviour, but shifted in ω by an amount of ω_{crit}. However, in this case the integral of the function from zero to infinity would be unchanged, meaning that the total power emitted remains the same. This seems to be unphysical, since we are actually cutting a part of the emission, and not redistributing it.
Before considering reflection effects, the transmission coefficients was equal to 1, with all the radiation going to infinity. After the introduction of not negligible reflections, we got a value $|T(\omega)|^2 = 16m^2\omega^2$. It is clear that our considerations impacts precisely on this coefficient, and in a first approximation the modification was:

$$|T(\omega)|^2 \to \theta(\omega - \omega_{crit})|T(\omega)|^2. \tag{3.62}$$

So, to have a lighter changing, instead of a Heaviside theta we can define a general function $f(\omega)$, which modifies the transmission coefficients with some suitable conditions:

$$|T(\omega)|^2 \to f(\omega)|T(\omega)|^2, \quad f(\omega) = 0 \text{ for } \omega \in [0;\omega_{crit}). \tag{3.63}$$

Now, in finding the form of $|T(\omega)|$, being interested in a correction that avoids divergencies around $\omega = 0$, an expansion for small frequencies was applied.

Since we are in any case considering a $|T(\omega)|$ derived for small ω, we can follow the same strategy, considering the function $f(\omega)$ expandable for small values of ω, getting:

$$f(\omega) \simeq a + b\omega; \tag{3.64}$$

with the parameters a and b to be determined imposing some conditions.
We can, for example, require that in approaching ω_{crit} both the function $N_{gb,\alpha}$ and its derivative go to zero. This would give as a consequence a very smooth behaviour; in addition, we also get probably another maximum value different from the one found before. Since in this case we are forcing the distribution function to be zero in the critical frequency, we are also losing the feature seen in the discussion above, for which we got that $N_{gb}(\omega_{crit})$ is dependent on α only. In order to keep this characteristic information, we want to propose another kind of modification, based on different conditions. We demand that at ω_{crit} we still have the value computed above, namely:

$$N_{gb,\alpha}(\omega_{crit}) = N_{gb}(\omega_{crit}). \tag{3.65}$$

Then, demanding that the emission starts with the same density of particles at that frequency, we also try to demand that the emission starts smoothly, and we require:

$$N'_{gb,\alpha}(\omega_{crit}) = 0. \tag{3.66}$$

These two conditions are enough to fix the values of a and b. The first condition translates into:

$$(a + b\omega)|_{\omega_{crit}} = 1 \rightarrow a = 1 - b\omega_{crit}. \tag{3.67}$$

So we have:

$$(a + b\omega) = 1 + b(\omega - \omega_{crit}). \tag{3.68}$$

For the second condition, having found a in function of b, we substitute and compute the derivative. Imposing it equal to zero for $\omega = \omega_{crit}$, we get b:

$$b = \frac{1}{\omega_0} \frac{e^{\frac{\omega_{crit}}{\omega_0}}}{e^{\frac{\omega_{crit}}{\omega_0}} - 1} - \frac{2}{\omega_{crit}} = 8\pi m \left(\frac{e^{\frac{\alpha}{4}}}{e^{\frac{\alpha}{4}} - 1} - \frac{8}{\alpha} \right). \tag{3.69}$$

Consequently, for a:

$$a = 1 - b\omega_{crit} = 3 - \frac{\alpha}{4} \frac{e^{\frac{\alpha}{4}}}{e^{\frac{\alpha}{4}} - 1}, \tag{3.70}$$

completing the first order form of $f(\omega)$.

Chapter 4

Minimal length and quantum of area: tidal heating

In this section we present the main subject of our work, the coalescence of two black holes, and we discuss the effects which take place in this dynamics due to the presence of a quantum of area for a black hole: in particular, we will focus on an important phenomenon, the tidal heating. Despite the fact that the quantization of the area, derived from a minimal length $L_0 \sim L_P$, scales as L_0^2, and so is expected to have an incredibly small magnitude, it turns out that in some physical processes it can leave a signature also at a macroscopical level, which, connected to basic parameters of the system, can also be observed. The coalescence of a binary of black holes is one of these cases.
Classically, the dynamics of a coalescing binary is divided in three parts: the first, called the inspiral part, is the one in which the two compact bodies orbit around each other, at a distance which is big compared to a characteristic size of both the two objects (for example, this characteristic size can be the Schwarzschild radius in the case of non rotating black holes); the second, called merging (plunge), is the one in which, naively speaking, starts the fusion between the two black holes, and it takes place when the separation between the two bodies approaches the characteristic size of them and strong field effects can not be neglected; the third is called ringdown and is the last part of the dynamics, in which a final Kerr black hole is formed by the initial two, and emits a specific spectrum of gravitational waves before reaching stability. A complete study of the coalescence of a binary is an extremely hard task, which requires to take into account the full (non-linear) structure of General Relativity and the application of the right formalism for each part of the dynamics. However, our interest is specifically in one of the effect that takes place during the inspiral phase: the tidal heating, which is the absorption of gravitational waves, emitted by the binary, by the companions of the binary itself ([16], [17], [18]). For this

reason, we will focus just on this part, eventually mentioning some properties of the other phases.
In order to have, at least, a classical and simplified picture of the problem, we begin by describing the dynamics of the two black holes from a completely Newtonian point of view, meaning that we consider simply two pointlike particles orbiting around each other. This extremely simplified study will actually be the basis for all the corrections. We give then a brief explanation about the formalism we need, which is the Post-Newtonian formalism, in which corrections to the zero-order Newtonian dynamics arises as v/c series in the analytical expressions of the quantities which characterize the system, like the orbital energy and the emitted power. Then, we look at the laws of black hole mechanics, briefly mentioning the change on the parameters of each of the black holes during the inspiral phase. We proceed by explaining how the presence of a step in area changes strongly the classical picture, and we underline the signature of this modification present in the gravitational waves signal. We will give particular emphasis to the modification in observable quantities, for example the gravitational wave phase. Subsequently, we compare the magnitude of these effects to the one of another well known physical effect, the Hawking radiation, finding an interesting result regarding the comparison of two quantum effects in curved spacetime. Also some numerical values of the quantities involved are given, taking specific cases of the parameters involved, for example, suitable initial masses ratio.

4.1 Newtonian inspiral of compact binaries

We follow essentially the discussion of Chapter four of [13]. For now, we also restore the constants G and c.
Consider the case of two pointlike bodies in the empty space, with masses m_1 and m_2 and positions $\mathbf{r}_1$ and $\mathbf{r}_2$. Going to the center-of-mass frame (CM), the dynamics reduces to a one-body problem:

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}; \tag{4.1}$$

where:

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}, \quad M \equiv m_1 + m_2, \quad \mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1, \quad r \equiv |\mathbf{r}| \tag{4.2}$$

are, respectively, the reduced mass, the total mass, the relative coordinate and the distance between the two bodies. For simplicity, we consider the case of circular orbits; also in the Post-Newtonian treatment, actually, is considered often the case of quasi-circular orbit, due to the fact that in the typical dynamics of the system, effects of circularization of the orbit are present.
Considering the orbital frequency ω_s and the orbital radius R, we can write the relation $v^2/R = GM/R^2$, with $v = \omega_s R$, and via the Kepler law:

$$\omega_s^2 = \frac{GM}{R^3}. \tag{4.3}$$

This system has been studied from the point of view of the production of gravitational waves, as it is one of the simplest physical system from which we have such a production. We simply recall the main results. First, we recall the *chirp mass*:

$$M_c \equiv \mu^{3/5} M^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}, \tag{4.4}$$

which is useful to write compactly some results. Recalling the basic features about gravitational waves present in Chapter two, we know that the gravitational wave signal is encoded in h_{ab}. The form of h_{ab} is largely simplified in the DeDonder gauge, and in the form of plus and cross polarizations, as we have seen. We have, in time dependence:

$$h_+(t) = \frac{4}{r}\Big(\frac{GM_c}{c^2}\Big)^{5/3}\Big(\frac{\pi f_{gw}}{c}\Big)^{2/3}\frac{1+\cos^2\theta}{2}\cos\left(2\pi f_{gw} t + 2\phi\right), \tag{4.5}$$

$$h_\times(t) = \frac{4}{r}\Big(\frac{GM_c}{c^2}\Big)^{5/3}\Big(\frac{\pi f_{gw}}{c}\Big)^{2/3}\cos\theta\sin\left(2\pi f_{gw} t + 2\phi\right). \tag{4.6}$$

As before, t is the retarded time, and the gravitational wave frequency is f_{gw}. We recall that, in the quadrupole approximation, for the relevant mode holds $f_{gw} = 2f_s$, with $f_s = (\omega_s/2\pi)$ the orbital frequency. In the following we will work essentially in that approximation.
We recall also the power radiated in gravitational waves, per unit solid angle:

$$\frac{dP}{d\Omega} = \frac{2c^5}{\pi G}\Big(\frac{GM_c\omega_{gw}}{2c^3}\Big)^{10/3} g(\theta); \tag{4.7}$$

where

$$g(\theta) = \Big(\frac{1+\cos^2\theta}{2}\Big)^2 + \cos^2\theta. \tag{4.8}$$

Another time, we want to look at quantities like (4.5) and (4.6) after taking the mean value over a long time scale compared to the characteristic period of the wave. We know that $\langle\cos^2(2\omega t + 2\phi)\rangle = \langle\sin^2(2\omega t + 2\phi)\rangle = 1/2$, independent of ϕ. The angular distribution of the radiated power, which is proportional to $\left\langle \dot h_+^2 + \dot h_\times^2 \right\rangle$, is then independent of ϕ. Recalling also the angular average of the factor $g(\theta)$, which gives:

$$\int \frac{g(\theta)}{4\pi} d\Omega = \frac{4}{5}, \tag{4.9}$$

we have, for the total radiated power:

$$P = \frac{32c^5}{5G}\Big(\frac{GM_c\omega_{gw}}{2c^3}\Big)^{10/3}. \tag{4.10}$$

This is, actually, the Newtonian picture with the additional emitted power due to the presence of gravitational waves: indeed, without explicitly mentioning this, we have already corrected the complete Newtonian version of the problem, simply introducing P.

If we have an energy emitted to infinity, however, we have a loss of energy by the binary system. The source of the radiated energy should be the total energy of the system, which is the sum of the kinetic and potential energies of the orbit, namely:

$$E_{orbit} = E_{kin} + E_{pot} = -\frac{Gm_1 m_2}{2R}. \tag{4.11}$$

Looking at the previous, since we must have the conservation of the total energy, it is clear that emitting (positive) energy via gravitational waves, the orbital energy must decrease, in the sense that it becomes more and more negative, but this means that R has to decrease in time, since we are considering fixed the two masses. Looking at (4.3), if R decreases, ω_s increases. If ω_s increases, from (4.10) we see that it increases the radiated power. We have, so, a sort of ripple effect, in which the emitted energy continuously increases, and R continuously decreases. This, on a sufficiently large scale of time, will lead the system to the plunge, a sort of a fusion between the two black holes, and in general to the phenomenon of coalescence. It is remarkable that we can see this behaviour just considering the Newtonian version with the addition of an emitted power due to gravitational waves.
Let us now look at (4.3). We see that the radial velocity $\dot{R}$ can be written also:

$$\dot{R} = -\frac{2}{3} R \frac{\dot{\omega}_s}{\omega_s} = -\frac{2}{3} (\omega_s R) \frac{\dot{\omega}_s}{\omega_s^2}. \tag{4.12}$$

As long as the condition $\dot{\omega_s} << \omega_s^2$ is satisfied, $|\dot{R}|$ is much smaller than the tangential velocity $\omega_s R$, and so we can treat the orbit as a circular one with a slowly varying radius. Working in this condition, we study a first approximation of the back-reaction of GWs, searching explicit laws of the quantities that characterize the system, for example the frequency. We can write the orbital energy introducing ω_{gw} instead of R, obtaining:

$$E_{orbit} = -\Big(\frac{G^2 M_c^5 \omega_{gw}^2}{32}\Big)^{1/3}. \tag{4.13}$$

Now, balancing the energy, we write:

$$P = -\frac{dE_{orbit}}{dt}; \tag{4.14}$$

and we have:

$$\dot{\omega}_{gw} = \frac{12}{5} 2^{1/3} \Big(\frac{GM_c}{c^3}\Big)^{5/3} \omega_{gw}^{11/3}. \tag{4.15}$$

Substituting $f_{gw} = \omega_{gw}/2\pi$ and integrating, we get a function $f_{gw}(t)$ which diverges at a certain finite value of t, which is identified as the time of coalescence and which we call t_{coal}. We introduce, then, a new time coordinate $\tau \equiv t_{coal} - t$, called *time to coalescence*; in terms of this, we have:

$$f_{gw}(\tau) = \frac{1}{\pi} \Big(\frac{5}{256\tau}\Big)^{3/8} \Big(\frac{GM_c}{c^3}\Big)^{-5/8}. \tag{4.16}$$

It is useful to work with τ because we can read easily the divergence of the frequency value at $\tau = 0$. We can also insert numerical values and a mass of reference, which is often the solar mass $M_\odot \approx 1.98 \times 10^{30}$ kg, to have an idea of the order of magnitude. We have then:

$$f_{gw}(\tau) \simeq 151 \text{ Hz}\Big(\frac{M_\odot}{M_c}\Big)^{5/8}\Big(\frac{1\ s}{\tau}\Big)^{3/8}. \tag{4.17}$$

Inverting this equation for τ, we can see at which instant of time before the coalescence corresponds a certain frequency. Taking the case of $M_c = 1.21M_\odot$ (which corresponds for example to a binary of two stars each one of $1.4M_\odot$), we have: a radiation of about 10 Hz, which is the order of the lowest frequencies accessible to ground-based interferometers, is emitted at $\tau = 17$ min to coalescence; radiation of about 100 Hz is emitted at 2 seconds to coalescence; a 1000Hz radiation is emitted around the last few milliseconds. Looking also at (4.3), we can see that the separation between the two bodies in the case of $m_1 = m_2 = 1.4M_\odot$ when the emitted frequency is 1kHz is $R \approx 33$ km. This order of distances can be reached, for example, by neutron stars or black holes, but at such distances we are actually close to the phase at which the two bodies, in our case black holes, start to feel the reciprocal strong field; in other words, at these scales they are approaching the plunge phase.

We can compute also the evolution of the orbital radius. From (4.15),(4.12) and (4.3), we have (the dot means derivation with respect to t):

$$\frac{\dot{R}}{R} = -\frac{2}{3}\frac{\dot{\omega}_{gw}}{\omega_{gw}} = -\frac{1}{4\tau}, \tag{4.18}$$

and integrating in τ:

$$R(\tau) = R_0\Big(\frac{\tau}{\tau_0}\Big)^{1/4} = R_0\Big(\frac{t_{coal} - t}{t_{coal} - t_0}\Big)^{1/4}; \tag{4.19}$$

where R_0 is the value of R at the initial time t_0, and $\tau_0 = t_{coal} - t_0$. This function decreases very smoothly for a long time range, which is consistent with our approximation of quasi-circular orbits. Then, it starts to decrease rapidly, close to the value t_{coal}, which signals the beginning of the plunge phase. Looking at (4.3) and (4.16), we can also find the relation between the initial value of the radius R_0 and the time to coalescence τ_0, which leads:

$$\tau_0 = \frac{5}{256}\frac{c^5 R_0^4}{G^3 M^2 \mu}. \tag{4.20}$$

An important parameter to take trace of the gravitational wave signal is the number of cycles spent by the gravitational wave in the detector bandwidth. Usually, we can take trace of the signal between two frequencies f_{min} and f_{max}:

f_{min} is the lowest accessible frequency, which in our cases is around 20 Hz; f_{max} is the frequency at which we stop to count cycles, that it could be taken as the frequency at which the phase under study ends and the next is starting. Supposing that the period $T(t)$ of the wave is a slowly varying function of time, we can write that the number of cycles in a small time interval dt is:

$$d\mathcal{N} = \frac{dt}{T(t)} = f_{gw}(t)dt, \tag{4.21}$$

and for the total number we have so:

$$\mathcal{N} = \int_{t_{min}}^{t_{max}} f_{gw}(t)dt = \int_{f_{min}}^{f_{max}} \frac{f_{gw}}{\dot{f}_{gw}} df_{gw}, \tag{4.22}$$

where t_{min} and t_{max} are the instants of time corresponding to the two frequencies we are considering. For practical measurements, this is a very useful quantity, and to have a precise theoretical predictions we need an accurate study of the gravitational waveform, in particular of the accumulated phase Φ. In fact, this will be one of the principal tasks of our work. Let us give a first description of the waveform. Consider a particle moving in the plane (x, y) on a quasi-circular orbit, with radius $R(t)$ and angular velocity $\omega_s(t)$; this particle has Cartesian coordinates given by:

$$x(t) = R(t) \cos\Big(\frac{\Phi(t)}{2}\Big), \tag{4.23}$$

$$y(t) = R(t) \sin\Big(\frac{\Phi(t)}{2}\Big); \tag{4.24}$$

where $\Phi(t)$, since the frequency is varying, is given by an integral:

$$\Phi(t) = 2\int_{t_0}^{t} \omega_s(t')dt' = \int_{t_0}^{t} \omega_{gw}(t')dt'. \tag{4.25}$$

In principle, in computing GW production from the relations deriving from the quadrupole approximation, we should consider that: ω_{gw} is not constant but is a function of time; R also is not constant and so there are contributions from its derivatives, and, in the arguments of the trigonometric functions, instead of $\omega_{gw}t$ we have to put $\Phi(t)$. However, we work in the condition $\dot{\omega}_s << \omega_s^2$, and in practice, in a considerable range of frequencies, we can neglect terms proportional to $\dot{R}(t)$ or $\dot{\omega}_s(t)$. Moreover, the frequencies at which we should start to consider such contributions turn out to be of the order of 10 kHz, which are after the order of frequencies at which plunge phase begins. So, in practice, we just substitute $\omega_{gw}t$ with $\Phi(t)$ in the trigonometric functions and ω_{gw} with $\omega_{gw}(t)$ in the prefactor. All these quantities will be actually evaluated at the retarded time. Having the frequency as a function of τ, we can find that $\Phi(\tau)$ is:

$$\Phi(\tau) = -2\Big(\frac{5GM_c}{c^3}\Big)^{-5/8} \tau^{5/8} + \Phi_0. \tag{4.26}$$

Here $\Phi_0 = \Phi(\tau = 0)$ is the value of Φ at coalescence. Since we are interested in the radiation emitted in the direction from the bodies toward us, we introduce the angle γ as the angle between the normal to the orbit and the line-of-sight. Expressing the two polarizations of the wave in terms of γ and τ, we get:

$$h_+(\tau) = \frac{1}{r}\Big(\frac{GM_c}{c^2}\Big)^{5/4}\Big(\frac{5}{c\tau}\Big)^{1/4}\Big(\frac{1+\cos^2\gamma}{2}\Big)\cos(\Phi(\tau)), \tag{4.27}$$

$$h_\times(\tau) = \frac{1}{r}\Big(\frac{GM_c}{c^2}\Big)^{5/4}\Big(\frac{5}{c\tau}\Big)^{1/4}\cos\gamma\sin(\Phi(\tau)). \tag{4.28}$$

We already said the the frequency of the wave tends to a divergence approaching the time of coalescence; now we can see that this feature holds also for the amplitude. This behaviour is called *chirping*.
To compare experimental data with the theoretical waveform, however, requires the Fourier transform of the wave amplitude. To compute such a Fourier transform could be not so easy, because our functions $h_+,\ h_\times$ are defined only in the region $-\infty < t < t_{coal}$. The result turns out to be (here with a tilde we indicate the transform):

$$\tilde{h}_+(f) = \Big(\frac{5}{24}\Big)^{1/2}\frac{1}{\pi^{2/3}}e^{i\Psi_+(f)}\frac{c}{r}\Big(\frac{GM_c}{c^3}\Big)^{5/6}\frac{1}{f^{7/6}}\Big(\frac{1+\cos^2\gamma}{2}\Big), \tag{4.29}$$

$$\tilde{h}_\times(f) = \Big(\frac{5}{24}\Big)^{1/2}\frac{1}{\pi^{2/3}}e^{i\Psi_\times(f)}\frac{c}{r}\Big(\frac{GM_c}{c^3}\Big)^{5/6}\frac{1}{f^{7/6}}\cos\gamma. \tag{4.30}$$

The two phases in the exponents are related by $\Psi_\times = \Psi_+ + \pi/2$ and:

$$\Psi_+(f) = 2\pi f\Big(t_c + \frac{r}{c}\Big) - \Phi_0 - \frac{\pi}{4} + \frac{3}{4}\Big(\frac{GM_c}{c^3}8\pi f\Big)^{-5/3}. \tag{4.31}$$

The phase of the wave is one of the most important objects of our study, because the effect due to quantization of the area of a black hole imprints the form of this phase, leaving an observable effect of this quantum correction. We will give, in the following sections, the Post-Newtonian correction to this phase function, with the other PN-corrected quantities related to the binary inspiral.
As we said, our discussion makes sense until we approach a specific orbital frequency, corresponding to a value of the distance between the two bodies at which our strong field effects are no longer negligible. Indeed, in our treatment we did not consider any modification of the dynamics due to a modified background geometry in which the bodies are moving. For our discussion, is crucial to understand how can be identified this characteristic scale. In order to give just an idea, we now consider the case of two Schwarzschild black holes orbiting in a quasi-circular orbit, with a decreasing separation $R(t)$. In the Schwarzschild geometry, in the limit of a test mass (so one of the two bodies is much lighter than the other), there is a minimum value of the radial distance beyond which stable circular orbits are not allowed; this limit orbit is called Innermost Stable Circular Orbit (ISCO). Taking usual coordinates (t, r, θ, ϕ), the location of this $r_{\rm ISCO}$ is:

$$r_{\rm ISCO} = \frac{6GM}{c^2}, \tag{4.32}$$

with as usual M the total mass. So, our discussion takes place in a regime in which $r \geq r_{\rm ISCO}$. From the Kepler law (4.3), we see that the corresponding source frequency f_s is:

$$f_{s,\rm ISCO} = \frac{1}{12\pi\sqrt{6}}\frac{c^3}{GM}. \tag{4.33}$$

We can insert numerical values and get:

$$f_{s,\rm ISCO} \approx 2.2\ \text{kHz}\Big(\frac{M_\odot}{M}\Big). \tag{4.34}$$

This means that, if we take for example the case of a BH binary with total mass $M = 20M_\odot$, we obtain $f_{s,\rm ISCO} \approx 110$ Hz. Actually, we are precisely interested in the case of two black holes with similar mass, with almost $m_1 \approx m_2 \approx 10M_\odot$, so this order of magnitude could be relevant, even if this is the case of Schwarzschild black holes, and in a real coalescence, each of the two black holes is spinning, so we need to consider Kerr geometries. We present also the energy spectrum in the inspiral phase, in frequency domain:

$$\frac{dE}{df} = \frac{\pi^{2/3}}{3G}(GM_c)^{5/3}f^{-1/3}, \tag{4.35}$$

which, after integration up to a maximum value f_{max}, gives the total energy radiated:

$$\Delta E_{\rm rad} = \frac{\pi^{2/3}}{2G}(GM_c)^{5/3}f_{\rm max}^{2/3}. \tag{4.36}$$

In numerical value, setting $f_{\rm max} = 2f_{s,\rm ISCO}$ for the quadrupole approximation, we have that the total radiated energy during inspiral depends just on the reduced mass of the system:

$$\Delta E_{\rm rad} \simeq 8 \times 10^{-2}\mu c^2, \tag{4.37}$$

that is a huge amount of energy. Actually we can also make the computation from another point of view: in the Schwarzschild metric, the binding energy of the ISCO is given by:

$$E_{\rm binding,ISCO} = \Big(1 - \frac{2\sqrt{2}}{3}\Big)\mu c^2 \simeq 5.7 \times 10^{-2}\mu c^2, \tag{4.38}$$

and this is the energy emitted by the inspiraling binary from a large distance of separation to the ISCO one. The PN correction to this binding energy are of the order of a few per cent. So, this first computation gives in any case correct orders of magnitude.

4.2 PN expansion and templates for inspiral binaries

We now present a brief explanation about the concept of Post-Newtonian expansion (PN expansion), which is the mathematical structure implemented to

study systems like coalescing binaries of black holes or neutron stars. In such systems, the dynamics is governed by gravitational force, and for this reason they are called self-gravitating. In our work, we will refer just to the analytical results which derive from this approach applied to the case of a coalescing black holes binary; for a complete treatment of the PN expansion we remand to [13], [16].
For a self-gravitating system with total mass M, we have:

$$\left(\frac{v}{c}\right)^2 \sim \frac{2GM}{c^2 d} = \frac{R_S}{d}, \tag{4.39}$$

with v the orbital velocity and d a typical size of the system; in the case of a binary, is the orbital distance. Actually, R_S/d gives a measure of the strength of the gravitational field near the source, so an expansion in powers of v/c should be associated to an expansion in R_S/d, and so a deviation from the case of non-relativistic speeds should be associated to a deviation from the case of flat spacetime background. This means that we deal with cases in which $(v/c)^2$ and R_S/d are comparable; in this approach, are considered slowly-moving and weakly self-gravitating sources of gravitational waves, where the two previous parameters are small, comparable but not negligible. We will also consider that the matter energy-momentum tensor of the source, T^{ab}, has a compact support. As in the case of electromagnetism, it is usual to distinguish between two regions: the *near zone* and the $far\ zone$. They are defined as follow. The reduced wavelength of the radiation emitted $\bar{\lambda}$ is larger than d by a factor of (c/v), so supposing that (v/c) is small we have $d \ll \bar{\lambda}$. The near region is defined as the region in which $r \ll \bar{\lambda}$, and it is possible to identify also a subregion, called *exterior* near zone, in which $d < r \ll \bar{\lambda}$. In the near region retardation effects are negligible, and almost we have static potential. Here, a PN expansion in powers of (v/c) is consistent. The far zone is the defined by the condition $r \gg \bar{\lambda}$. Here, retardation effects are important, and this is the region in which the observed waves take place. A modified approach is required, basically implemented via considering Minkowski spacetime. The near and far regions meet in an intermediate region of superposition, in which $r \sim \bar{\lambda}$. From an initial analysis of the problem, we could think that we can correct the equation of motion of the sources at the decided order of (v/c), and then compute the corrected GW production by these sources. However, due to the complex structure of the theory (in particular its non-linearity), it is not consistent to split in two steps this procedure: the emission of gravitational waves costs energy, and at a certain order the back-reaction of GWs on the sources is not negligible and affects their equation of motion. Moreover, the gravitational field can be itself a source of gravitational waves, and the GWs computed at a certain order in (v/c) will be the source of other GWs at higher orders (this is a typical effect in a non-linear theory).

The Post-Newtonian approximation starts with the following expansion of the metric:

$$
\begin{aligned}
g_{00} &= -1 + g_{00}^{(2)} + g_{00}^{(4)} + g_{00}^{(6)} + ..., \\
g_{0i} &= \qquad\; g_{0i}^{(3)} + g_{0i}^{(5)} + ..., \\
g_{ij} &= \delta_{ij} + g_{ij}^{(2)} + g_{ij}^{(4)} + ...;
\end{aligned}
\tag{4.40}
$$

where $g_{ab}^{(n)}$ means that the term is of order $(v/c)^n \equiv \epsilon^n$ in the expansion. Similarly, the energy-momentum tensor is expanded as:

$$
\begin{aligned}
T^{00} &= T^{00,(0)} + T^{00,(2)} + ... \\
T^{0i} &= T^{0i,(1)} + T^{0i,(3)} + ... \\
T^{ij} &= T^{ij,(2)} + T^{ij,(4)} + ...
\end{aligned}
\tag{4.41}
$$

These expansions are motivated by symmetries considerations. Now, in the previous section we saw what is, from a Newtonian point of view, the binary evolution. We can say that this is the case of the Newtonian limit of the complete metric, which can be identified with the first order of the expansion above. We can think that, if we try to expand at higher orders the metric and the energy momentum tensor, we get corrections to the evolution of the main parameters of the binary in power of (v/c), dependent on the order at which we stop our expansions. This is the main concept of the PN expansion, and in our work we will look at forms of the various parameters of the binary at a certain order in (v/c) (or, adimensionally, v), which are called *templates*. In particular, our focus will be on the flux of energy in function of the dimensionless velocity v, indicated with $F(v)$. Now, there is an important point to underline. In the evolution of the motion, we saw that are emitted gravitational waves, which carry energy from the binary to infinity. This is precisely the flux mentioned above, which can be called $F_\infty(v)$. However, going through higher orders in the expansions, so, to higher powers of v, we have another contribution to the energy flux, which does not go to infinity but instead it is directed to the horizons and absorbed by the two black holes. This is called $F_{\rm H}(v)$. This is connected to the presence of a back-reaction in the system: the emission of gravitational waves impacts on the horizons, modifying the parameters of the black holes and also the evolution of the system. This is called *tidal heating*. Being an effect which appears at a (relative) high order in v, it is completely absent in the first terms of the expansion. The total flux of energy out of the orbital energy is therefore $F(v) = F_\infty(v) + F_{\rm H}(v)$. In the following, we give the templates of both these fluxes of energy, which are taken from [19] (here, we set $G = c = 1$):

$$
\begin{aligned}
F_\infty(v) &= \frac{32}{5}\eta^2 v^{10}\left[1 - \left(\frac{1247}{336} + \frac{35}{12}\eta\right)v^2 + (4\pi + F_{\rm SO})v^3\right], \\
F_{\rm H}(v) &= \frac{32}{5}\eta^2 v^{10}\left[-\frac{\Psi_5}{4}v^5 + \frac{\Psi_8}{2}v^8\right],
\end{aligned}
\tag{4.42}
$$

where:

$$\eta \equiv \frac{m_1 m_2}{M^2}, \quad v = (\pi M f)^{1/3}. \tag{4.43}$$

In η, m_1 and m_2 are as usual the masses of the black holes, and $M = m_1 + m_2$; in the definition of v, f is the instantaneous frequency of the gravitational waves emitted. Ψ_5 and Ψ_8 are factors related to the changing in the masses of the black holes, with their complete form given in the appendix, and F_{SO} is a spin-orbit term which is taken from [20]. They are dependent, in general, on a dimensionless ratios for each black hole, $\chi_i \equiv J_i/m_i^2$. Near to these expansions we have, of course, the one of the orbital energy, which for consistency is another time in function of the (dimensionless) relative velocity [19]:

$$E(v) = -\frac{\eta}{2} v^2 \left[1 - \frac{9+\eta}{12} v^2\right]. \tag{4.44}$$

An interesting fact is that, having the expansions above at a certain order in v, we can compute the accumulated phase $\psi(v)$ of the gravitational waves emitted [21]. It works as follow. Take the definition of v given above, and consider the relation between the instantaneous frequency f and the variation of the orbital phase:

$$\frac{d\phi}{dt} = \pi f. \tag{4.45}$$

Consider also the energy balance equation:

$$\frac{dE(v)}{dt} = -F(v). \tag{4.46}$$

Combining these, we can write:

$$\phi(v) = \phi_c - \int_{v_i}^{v} d\bar{v} \bar{v}^3 \frac{E'(\bar{v})}{F(\bar{v})}, \tag{4.47}$$

where ϕ_c and v_i are constants. Recalling that the only multipole mode of the gravitational waves considered is the one with $m = 2$, we know that:

$$h(t) = A(t) \cos\left(\phi_{gw}(t)\right), \tag{4.48}$$

with $A(t)$ is a slowly varying in time amplitude. The Fourier transform of the waveform is then:

$$\tilde{h}(f) = B(f) e^{i\psi(f)}. \tag{4.49}$$

Here $B(f)$ is a prefactor dependent by the frequency, and $\psi(f)$ is the phase, for which, in the stationary phase approximation, holds:

$$\psi(f) = 2\pi f t(v) - 2\phi(v) - \frac{\pi}{4}. \tag{4.50}$$

We can also find an useful parametrization for the time $t(v)$ in function of v, another time using (4.45),(4.47):

$$t(v) = t_c - \int_{v_i}^{v} d\bar{v} \frac{E'(\bar{v})}{F(\bar{v})}. \tag{4.51}$$

Combining the last equations, we get the form of the phase in the frequency domain, in which we write explicitly the dependence on v:

$$\psi(v) = \frac{2t_c v^3}{M} - 2\phi_c - \frac{\pi}{4} - 2\int_{v_i}^{v} d\bar{v}(v^3 - \bar{v}^3)\frac{E'(\bar{v})}{F(\bar{v})}. \tag{4.52}$$

This is an important object in our discussion: the first three terms are constant, so we will focus on the term dependent on the integral. This term can be called $\delta\psi(v)$ and for which we can write:

$$\delta\psi(v) = -2\int^{v} d\bar{v}(v^3 - \bar{v}^3)\frac{dE(\bar{v})}{d\bar{v}}\frac{1}{F_\infty(\bar{v}) + F_{\rm H}(\bar{v})}. \tag{4.53}$$

Now, we can see, looking at the templates given above, that the two fluxes F_∞ and $F_{\rm H}$ have very different orders in v:

$$\frac{F_\infty(v)}{F_{\rm H}(v)} \sim v^{-5}, \tag{4.54}$$

so we could expect that the contribution to the phase from tidal heating is small compared to the one from the flux at infinity. However, in general is not negligible. We can identify the various contribution inside $\delta\psi(v)$. We write:

$$\delta\psi(v) = \delta\psi_{\rm PP}(v) + \delta\psi_{\rm TH}(v), \tag{4.55}$$

where we distinguish the *point particle* contribution, so without effects of absorption by the presence of an horizon, and the *tidal heating* contribution. There are different references which give the form of both of these phase contributions [16, 22]; in our discussion, we compute $\delta\psi_{\rm PP}(v)$ neglecting completely the flux to the horizon in the denominator in (4.53), so our form could differ a bit with respect to some other found in references. In any case, this is done just to give an example of the orders of magnitude, since our focus is, instead, $\delta\psi_{\rm TH}(v)$, which is computed expanding the denominator in the integral of $\delta\psi(v)$ and considering just first order contributions coming from $F_{\rm H}(v)$. This is done because, since we are working with expansion in v, could be not consistent to neglect completely a term with lower powers of v keeping terms which are of higher order. In doing this we follow essentially [19]. The details of the computation will be shown in the appendix.
We now want to specify that, in the entire work, if it is not specified differently, we look at the case in which $m_1 = m_2$, and so $M = 2m_1 = 2m_2$. Changing a bit the notation, we will call the single mass of each black hole simply $M/2$. Moreover, when we study the case of Kerr black holes, which actually is the case closer to the physical system in the known universe, we will always assume, if not specified differently, that the two spins of the black holes are aligned each other, and aligned with the orbital angular momentum; moreover, for the ratios χ_i, we assume that $\chi_1 = \chi_2$. Combined with the condition of equal masses, we get that $J_1 = J_2$. For simplicity, however, let us begin from the case of Schwarzschild black holes, and compute both $\delta\psi_{\rm PP}(v)$ and $\delta\psi_{\rm TH}(v)$.

4.2.1 Templates of $\delta\psi(v)$ for Schwarzschild black holes with equal masses

Let us start computing the term $\delta\psi_{\rm PP}(v)$.
Starting from (4.53), in the term containing the fluxes functions we set:

$$\eta = \frac{m_2 m_2}{M^2} = \frac{1}{4}, \quad F_{\rm H}(v) = 0. \tag{4.56}$$

Since this is the case of two black holes with spins equal to zero, we have also, as we can see in the appendix:

$$F_{\rm SO} = 0, \quad \Psi_5 = 0, \quad \Psi_8 = \frac{1}{4}. \tag{4.57}$$

We obtain:

$$\frac{1}{F_\infty(\bar{v})} = \frac{5}{2\bar{v}^{10}\left[1 - \left(\frac{1247}{336} + \frac{35}{48}\right)\bar{v}^2 + 4\pi\bar{v}^3\right]} \simeq \frac{5}{2\bar{v}^{10}}\left[1 + \frac{373}{84}\bar{v}^2 - 4\pi\bar{v}^3\right], \tag{4.58}$$

in which we have approximated the fraction, considering the term dependent in v small compared to 1. On the other hand, we have for the orbital energy:

$$\frac{dE(\bar{v})}{d\bar{v}} = -\frac{\bar{v}}{4}\left(1 - \frac{37}{24}\bar{v}^2\right). \tag{4.59}$$

Now we plug in this expansions in the definition of $\delta\psi_{\rm PP}(v)$ given above, getting:

$$\delta\psi_{\rm PP}(v) = -2\int^v d\bar{v}(v^3 - \bar{v}^3)\left(-\frac{\bar{v}}{4}\left(1 - \frac{37}{24}\bar{v}^2\right)\right)\left(\frac{5}{2\bar{v}^{10}}\left[1 + \frac{373}{84}\bar{v}^2 - 4\pi\bar{v}^3\right]\right). \tag{4.60}$$

Making the integration for a generic v, we obtain the general form of the shift of the phase $\delta\psi_{\rm PP}(v)$:

$$\delta\psi_{\rm PP}(v) = -\frac{185\pi}{24}\ln(v) - \frac{185\pi}{72} - \frac{69005}{10752}\frac{1}{v} - \frac{3\pi}{2}\frac{1}{v^2} + \frac{2435}{4032}\frac{1}{v^3} + \frac{3}{32}\frac{1}{v^5}. \tag{4.61}$$

Beside this contribution, we have the one coming from the total flux, and to compute that, as we said, we consider the horizon flux term $F_{\rm H}(v)$ small compared to $F_\infty(v)$. Then, we isolate the contribution of the tidal heating alone, identified in the expansion. Starting from the general form (4.53), we have, for the fraction in the integral, the substitution:

$$\frac{1}{F_\infty(\bar{v}) + F_{\rm H}(\bar{v})} \to -\frac{5}{16}\frac{1}{\bar{v}^2}; \tag{4.62}$$

then for $\delta\psi_{\rm TH}(v)$ we have:

$$\delta\psi_{\rm TH}(v) \simeq -\frac{5v^3}{96}\Big(3\ln(v) - 1\Big) + \frac{37}{512}v^5. \tag{4.63}$$

Of course, since the quantities computed derive from expansions stopped at a certain PN order, they have still the meaning of expansions of a precise order in v, but we kept all the orders deriving from the integration, without neglecting pieces in the middle steps. Often in our discussion, instead of the precise form of the functions we work with, we are interested in order of magnitude of the parameters of the binary, and in comparing them.

We recall that, since our study is focused on the inspiral phase, we know that the discussions we make are consistent only up to a certain frequency of the source (binary), which in the previous section, for Schwarzschild black holes, was identified with the ISCO frequency (setting now $G = c = 1$):

$$f_{s,\text{ISCO}} = \frac{1}{12\pi\sqrt{6}}\frac{1}{M}. \tag{4.64}$$

Since the only relevant multipole mode for the emitted gravitational waves is $m = 2$, we set always, otherwise specified differently, $f_{gw} = 2f_s$, with f_{gw} the frequency of the emitted gravitational waves and f_s the source frequency. The ISCO frequency is then associated to a limit gravitational wave frequency, which we call $f_{gw,\text{ISCO}}$. In the case of Schwarzschild, is simply:

$$f_{gw,\text{ISCO}} = \frac{1}{6\pi\sqrt{6}}\frac{1}{M}. \tag{4.65}$$

Since the parameter v is related to the instantaneous gravitational wave frequency by (4.43), we can compute immediately the critical value of v at which our study, and concretely our templates, stop to be consistent. That is:

$$v_{\text{ISCO}} = (\pi M f_{gw,\text{ISCO}})^{1/3} = \left(\frac{1}{6\sqrt{6}}\right)^{1/3} \simeq 0.41. \tag{4.66}$$

We also have to consider the frequency at which a typical detector is able to follow the evolution of a gravitational wave, and the corresponding value of v, that we call $v_{gw,0}$. We can take for the minimum gravitational wave frequency $f_{gw,0} = 20$ Hz ($f_{s,0} = 10$ Hz), that is realistic for modern GW interferometers, that gives:

$$v_{gw,0} \simeq (\pi M f_{gw,0})^{1/3} \simeq 0.054; \tag{4.67}$$

where here the frequency is constant, depending just on the characteristics of the detector; consequently, $v_{gw,0}$ *depends* on the total mass of the binary, for which we took an indicative real value of $20M_\odot$. We see that the order of magnitude is clearly small compared to v_{ISCO}, so it is surely possible to follow the evolution of the inspiral phase via the emitted gravitational waves, through a typical GW detector.

Plugging these values in the form of $\delta\psi_{\text{PP}}(v)$ and $\delta\psi_{\text{TH}}(v)$, we can compute the total shift in the phase during the inspiral, due to the presence of both

$F_\infty(v)$ and $F_{\mathrm{TH}}(v)$. We have:

$$\begin{aligned} &\delta\psi_{\mathrm{PP}}(v_{gw,0}) \simeq 2.0 \times 10^{5}, \\ &\delta\psi_{\mathrm{TH}}(v_{gw,0}) \simeq 8.0 \times 10^{-5}, \\ &\delta\psi_{\mathrm{PP}}(v_{\mathrm{ISCO}}) \simeq -13; \\ &\delta\psi_{\mathrm{TH}}(v_{\mathrm{ISCO}}) \simeq 0.014. \end{aligned} \tag{4.68}$$

We see immediately that the quantities have a very different order of magnitude. Moreover, we see that the two contributions at v_{ISCO} have different sign; this is related to the fact that, during the inspiral, tidal heating acts as a sort of *friction* in the dynamics of the two black holes, retarding the standard evolution made up by the emission of energy to infinity via gravitational waves.
After this example regarding the case of two Schwarzschild black holes, we want to follow the same steps for the case of two Kerr black holes. As we said, in order to simplify the discussion, we study the case of equal masses and equal spins of the bodies.

4.2.2 Templates of $\delta\psi(v)$ for Kerr black holes with equal spins and masses

As before, we start from the term $\delta\psi_{\mathrm{PP}}(v)$ (to avoid heavy notation, we keep the same symbols used before, despite the fact that now we refer to another type of black holes). Of course, since now we are considering the case of a non-null value for the spins, it turns out that our templates depend on an additional parameter, which should be fixed to an initial value in order to get some numerical results. In order to have, at least qualitatively, an idea of the order of magnitude in this case, we decide to fix $\chi_1 = \chi_2 = 0.5$. We have again $\eta = 1/4$, but now we get:

$$F_{\mathrm{SO}} = \frac{59}{6}, \quad \Psi_5 = \frac{7}{32}, \quad \Psi_8 = \frac{7}{32}\left[1 + \frac{\sqrt{3}}{2}\right]. \tag{4.69}$$

Starting from the case without the term F_{H}, we have:

$$\frac{1}{F_\infty(\bar{v})} = \frac{5}{2}\frac{1}{\bar{v}^{10}}\left[1 + \frac{373}{84}\bar{v}^2 - (4\pi + F_{\mathrm{SO}})\bar{v}^3\right]. \tag{4.70}$$

For the orbital energy, we still have (4.44).
Plugging these terms into the definition of $\delta\psi_{\mathrm{PP}}(v)$ and integrating, we obtain:

$$\begin{aligned} \delta\psi_{\mathrm{PP}}(v) \simeq \frac{5}{4}\Bigg[&-\frac{37}{24}\left(4\pi + \frac{59}{6}\right)\ln(v) - \frac{37}{72}\left(4\pi + \frac{59}{6}\right) \\ &- \frac{41403}{8064}\frac{1}{v} - \frac{3}{10}\left(4\pi + \frac{59}{6}\right)\frac{1}{v^2} + \left(\frac{487}{1008}\right)\frac{1}{v^3} + \frac{3}{40}\frac{1}{v^5}\Bigg]. \end{aligned} \tag{4.71}$$

To calculate the contribution from tidal heating alone, we follow again the steps in [19], expanding as usual the denominator containing both the terms F_∞ and

$F_{\rm H}$ and keeping only the terms which contain the first order contributions from $F_{\rm H}$. We get, for the fraction:

$$\frac{1}{F_\infty + F_{\rm H}} = \frac{5}{2}\frac{1}{v^{10}}\left[\frac{7}{128}v^5 + \frac{373}{768}v^7 - \left(\frac{7}{64}\left(1+\frac{\sqrt{3}}{2}\right) + \frac{7}{64}\left(4\pi + \frac{59}{6}\right)\right)v^8\right]. \tag{4.72}$$

Inserted in the definition of $\psi(v)$, this gives a contribution which is identified with the one of tidal heating alone, which is:

$$\begin{aligned}\delta\psi_{\rm TH}(v) = \frac{5}{4}\Bigg[&- \frac{7}{384}(1+3\ln(v)) - \frac{1233}{2048}v^2 - \\ &\left(\frac{7}{64}\left(1+\frac{\sqrt{3}}{2}\right) + \frac{7}{64}\left(4\pi+\frac{59}{6}\right)\right)\frac{v^3}{3}(3\ln(v)-1)\Bigg].\end{aligned} \tag{4.73}$$

We can also write this phase underlining the various PN terms and their order, indeed we have:

$$\delta\psi_{\rm TH}(v) = \frac{3}{32}\left[\psi_{2.5} + \psi_{3.5}v^2 + \psi_4 v^3\right], \tag{4.74}$$

where $\psi_{2.5}$, $\psi_{3.5}$ and ψ_4 are respectively the terms of 2.5, 3.5, 4 PN order; and it holds:

$$\begin{aligned}\psi_{2.5} &= -\frac{35}{144}(3\ln(v)+1), \\ \psi_{3.5} &= -\frac{2055}{256}, \\ \psi_4 &= -\frac{20}{9}(3\ln(v)-1)\left[\frac{7}{32}\left(4\pi+\frac{59}{6}\right) + \frac{7}{32}\left(1+\frac{\sqrt{3}}{2}\right)\right].\end{aligned} \tag{4.75}$$

We also list $\delta\psi_{\rm TH}(v)$ corresponding to the extreme ratio $\chi = 1$. In this case, for the coefficients we have:

$$\Psi_5 = 1, \quad \Psi_8 = \frac{1}{2}, \quad F_{\rm SO} = \frac{59}{3}. \tag{4.76}$$

From these, we obtain:

$$\delta\psi_{\rm TH}(v) = \frac{3}{32}\left[-\frac{10}{9}(3\ln(v)+1) - \frac{2055}{56}v^2 - \frac{20}{9}\left(\frac{121}{6}+4\pi\right)(3\ln(v)-1)v^3\right]. \tag{4.77}$$

These are the structures we need to make our discussion. In the next section, we begin the explanation of the effect of a minimal increment of area on the dynamics of a black holes binary, specifically on tidal heating and on the phase shift due to the energy flux related to it. We begin by explaining what the *critical frequency* is, a value of the orbital frequency (or, equivalently, of the parameter v) which plays an important role in our analysis and should be compared to the ISCO frequency (equivalently, $v_{\rm ISCO}$). In order to have a simplified picture of the situation, we will start another time from the Schwarzschild case.

4.3 Tidal heating with qmetric effects

Here starts the core of our discussion, namely the modification of the phenomenon of tidal heating due to quantization of area effects. Firstly, we will explain in details what is the consequence of a quantum of area in the black hole horizon, with particular emphasis on the orders of magnitude which enter. Indeed, we will see remarkably that, despite the fact that quantization of area is (probably) of the order of L_P^2, so really hard to detect, its effect on the behaviour of a black hole impacts in an accessible range of frequencies. In our discussion, we initially explore the case of Schwarzschild black holes, and then we describe the more physical situation of Kerr black holes.
We will explain in detail what is expected in the tidal heating of each companion of the binary, underlining what happens in observable parameters like the phase of the emitted gravitational waves, ψ. In this we focus on specific configurations, and we describe also, at a qualitative level, what is expected for a more general situation.
In the last part, we compare the order of magnitude obtained for different physical parameters in the case of binaries under study with the one from another interesting phenomenon studied in the section above, the emission of the Hawking radiation. We do that principally to make a comparison between two quantum effects in a curved spacetime, and to make clear that quantum corrections can take place at very different orders of magnitude. The important fact to underline, in this case, is also another: the two effects taken into account are both of quantum nature, but the emission of Hawking radiation is related to the presence of quantum fields in a *classical* curved spacetime, while the modification of tidal heating regards the modification of the spacetime itself by quantum effects, being related so to the behaviour of *quantum* curved spacetime. It is interesting to compare, at least from the point of view of the order of magnitude, this two kinds of effects, which both involve quantum effects in a curved spacetime.

4.3.1 Quantum of area and tidal heating: Schwarzschild black holes

Let us start from the case of a Schwarzschild black hole, in which we have just one parameter: the mass. For completeness, in what follows we take up again what said in chapter two, in the discussion about the critical frequency for a Schwarzschild black hole. From the area-mass relation and from thermodynamics of black holes, we have, in units with $G = c = 1$ [23]:

$$A = 16\pi m^2, \quad \delta A = 32\pi m \delta m; \tag{4.78}$$

where A is the horizon area of the black hole and m is its mass. If the black hole gains a small amount of mass δm, the area must change consequently of a small amount δA given by the formula above. Since the horizon of a black hole is a null surface, implementing the structure of the qmetric we can apply what we have discussed in the first chapter, finding that there is a minimal step in area $\Delta A = 4\pi L_0^2$. In other words, the mass and the horizon area of the black hole can't change continuously by infinitesimal increments, because the area A is forced to be incremented, at least initially, by a finite amount ΔA. From the formula above, we have immediately that also the mass m must initially change by a finite amount Δm given by:

$$\Delta m = \frac{L_0^2}{8m}. \tag{4.79}$$

We can understand that, since the mechanism which allows the black hole to gain mass is principally the absorption of energy, adding minimal length effects has as a result that black holes are not able to absorb all the energy which impacts the horizon: if an amount of energy arrives at the surface, it can be absorbed only if it would cause an increment in the area equal or greater than ΔA.
Suppose that this amount of energy is carried by a physical wave, with specific values of frequency f and wavelength λ. Going to the quantum description, we can say that such a wave carries an energy $E = \hbar\omega$, where ω is the pulsation. We have so that the condition for the absorption becomes:

$$\omega = \frac{\Delta m}{\hbar} = \frac{L_0^2}{8m\hbar}; \tag{4.80}$$

giving a condition on the frequency that can be absorbed. This is precisely what we called *critical frequency*. With a little redundancy in the terminology, we will call critical frequency equivalently ω, f or the templates parameter v, and they will be indicated respectively with ω_{crit}, f_{crit} and v_{crit}. Now, we know that it is usual to suppose that the minimal length squared L_0^2 is, if not precisely equal to, of the order of the Planck length L_P^2, with the constant between them which is an unknown parameter of the theory. We define this ratio parameter from the point of view of quantum of area, writing:

$$\Delta A = 4\pi L_0^2 = \alpha L_P^2 = \alpha\hbar. \tag{4.81}$$

From the last, it follows that if the minimal length is exactly equal to the Planck length, $\alpha = 4\pi$. With this definition, we can write ω_{crit} as:

$$\omega_{crit} = \frac{\alpha}{32\pi m}. \tag{4.82}$$

It is expected that only a complete theory of quantum gravity can give a precise value of this constant. In the literature, however, there are different proposals which come from trials of giving a complete theory of quantum gravity or simply by heuristic arguments [24]. Here we cite four of them; in growing order, their numerical values are:

$$\begin{aligned} \alpha_1 &= 4\ln 2, \\ \alpha_2 &= 4\ln 3, \\ \alpha_3 &= 4\pi, \\ \alpha_4 &= 8\pi. \end{aligned} \tag{4.83}$$

The first two values derive from a proposal by Bekenstein and Mukhanov, based on statistical considerations, for which:

$$\Delta A = 4\ln(k)L_P^2, \quad k > 1. \tag{4.84}$$

They also proposed the value $k = 2$, while Hod proposed $k = 3$ [9]. About the other two values, which have a more trigonometric form, we can say: the value of 4π is simply from the qmetric description, identifying $L_0 = L_P$, while 8π derives still from Bekenstein, in an earlier analysis. Let us see what happens by substituting into the formula of ω_{crit} these values of α. We have:

$$\begin{aligned} \omega_{crit,1} &= \frac{\ln 2}{8\pi m}, \\ \omega_{crit,2} &= \frac{\ln 3}{8\pi m}, \\ \omega_{crit,3} &= \frac{1}{8m}, \\ \omega_{crit,4} &= \frac{1}{4m}. \end{aligned} \tag{4.85}$$

In order to have an idea of the numerical value of the pulsations (in Hertz) associated to these values of α, let us restore the constants G and c. We have to add simply the factor c^3/G. Plugging the values in all the four frequencies, and considering an indicative value for the mass, for example $m = 10M_\odot$ we see that we move almost in a range $\omega_{crit} \in [564\text{Hz}; 5112\text{Hz}]$. This is surprisingly, because it means that, for a macroscopic object like a black hole with a mass of ten solar masses, with an horizon area which is also macroscopic, the implementation of a minimal length of the order of the Planck length has implications at frequencies of the order of kHz, which is, in principle, an observable range.

Now, return to our discussion about tidal heating. As we said above, in the classical description, in the evolution of the binary motion a part of the gravitational waves generated are captured by the horizons of the two black holes involved, causing a changing of their parameters. This is controlled by the function $F_{\mathrm{H}}(v)$. Take still the case of Schwarzschild black holes. The gravitational waves generated by the binary have a frequency which, in our quadrupole approximation, is two times the frequency of the source: $f_{gw} = 2f_s$. So, at the horizons of both the black holes arrive gravitational waves which, instant per instant, have a frequency directly proportional to the one of the source. The orbital frequency evolves in this way: it starts from an initial value $f_{s,0}$, which is the minimum value at which a detector is sensitive; grows up continuously, and it would diverge analytically, but we have seen that it is consistent to stop the study of the inspiral phase at the value of the ISCO frequency $f_{s,\mathrm{ISCO}}$. We already saw the values of v_{ISCO} and $v_{gw,0}$. Let us now introduce the critical frequency in the system. Adding the presence of a quantum in the horizon area, each of the two black holes has now an associated critical frequency $f_{crit,i}$, which depends on its mass m_i. Since we are considering the case of equal masses, we see that this critical frequency is the same for both the black holes, and so we drop the label i, writing:

$$f_{crit} = \frac{\omega_{crit}}{2\pi} = \frac{\alpha}{64\pi^2 m} = \frac{\alpha}{32\pi^2 M}, \tag{4.86}$$

where $M = m_1 + m_2 = 2m$. Before the gravitational wave emitted reaches this value, both the black holes cannot absorb the flux going through the horizon, because the energy carried is not enough to give a gain of area at least of the minimal amount. So, it is crucial to understand where this frequency is in the range of interest. Looking at the numerical value computed above, we can consistently assume that f_{crit} is above $2f_{s,0}$, so, each of the black holes starts to absorb *after* the frequency at which a detector is able to detect the emitted gravitational wave, and this is important for the observations. Let us now compare f_{crit} with the limit value f_{ISCO}. Leaving α as a free parameter, we set the following inequality:

$$f_{crit} > 2f_{\text{ISCO}} \rightarrow \frac{\alpha}{32\pi^2 M} > \frac{1}{6\pi\sqrt{6}M} \rightarrow \alpha > \frac{16\pi}{3\sqrt{6}} \simeq 6.8. \tag{4.87}$$

This means that, for α greater than 6.8, the frequency at which each of the two black holes starts to absorb is higher than the maximum frequency reached by the emitted gravitational waves in the inspiral phase, and this result *does not depend* on the masses involved. Suppose that we have such a value of α. It means that, during the entire phase under study, we have that the phenomenon of tidal heating is completely neglected by quantum effects. This is remarkable, because quantum corrections of the spacetime are supposed to take place at very short scales, but, in this case, their consequences impact on a macroscopic phenomenon like tidal heating.
A graph of the situation is indicated in Figure 4.1, where on the x-axis we have α, and on the y-axis we have the values of $v_{gw,0}$, v_{ISCO}, v_{crit}.

Computing the numerical values of the four cases of α we have, we easily see that two of them, α_1 and α_2, are lesser than 6.8, while the other two, α_3 and α_4, are greater. So we arrive at this result: for Schwarzschild black holes with equal masses, if the value α is equal to 4π or 8π, *tidal heating is completely suppressed*.
What happens for the other two cases? The binary starts to evolve, and the gravitational wave frequency grows up with the source frequency. In the first part of the inspiral phase, when the frequency is lesser than the critical value, tidal heating is absent. When the GW frequency reaches the critical value, classical tidal heating is *turned on*, and the evolution starts to follow the classical picture, until we arrive at the ISCO point and the inspiral phase stops. Now it is interesting to see quantitatively the difference between the complete classical description of the system and the one corrected with these quantum effects.
We know that, for a specific value of f_{gw}, we have associated a value of v, which plugged in the templates listed in the section above gives us the accumulated phase of the gravitational wave $\delta\psi(v)$. We have identified also a piece of $\delta\psi(v)$,

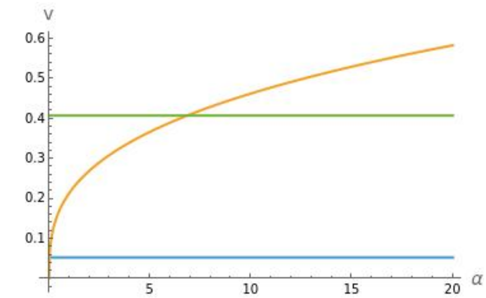

Figure 4.1: The comparison between the relevant values of v in the case of Schwarzschild black holes, as a function of α. In blue, $v_{gw,0}$; in green, v_{ISCO}; in yellow, v_{crit}, which is the only one dependent on α. We took for $v_{gw,0}$ a total mass of $20M_\odot$.

called $\delta\psi_{\text{TH}}(v)$, which is associated precisely to the presence of the tidal heating. In the classical description of the system, this piece gives contribution to the shift in the phase at all frequencies, but in our description, we have to change this feature, knowing that for a range of frequencies (which can also correspond to the entire inspiral phase) this term is absent, being absent tidal heating itself. More concretely, we can in practice substitute $\delta\psi_{\text{TH}}(v)$ with an effective version:

$$\delta\psi_{\text{TH}}(v) \to \theta(v - v_{crit})\delta\psi_{\text{TH}}(v), \tag{4.88}$$

where a Heaviside theta is added to formally take into account the presence or absence of tidal heating, controlled by the parameter v.
Now, it is clear that, with respect to the classical description, we have actually a *lack* of the term $\delta\psi_{\text{TH}}(v)$ in the phase of the observed gravitational wave: this phase shift can be calculated via the templates of the previous section. In order to explore what are the orders of magnitude of this effect, we want to explicitly compute this difference between classical and quantum case, for the four values of α listed in (4.83). After that, we will show the general behaviour of $\delta\psi_{\text{TH}}(v_{crit})$, leaving α free. We go in numerical order, and we divide the cases: *tidal heating starts in inspiral phase* and *tidal heating is absent in inspiral phase*. So we start from α_1 and α_2.
Let's compute the values of v associated to $\omega_{crit,1}$ and $\omega_{crit,2}$. We have:

$$\begin{aligned} v_{crit,1} &= (\pi M f_{crit,1})^{1/3} = \left(\frac{\ln 2}{8\pi}\right)^{1/3} \simeq 0.30 \\ v_{crit,2} &= (\pi M f_{crit,2})^{1/3} = \left(\frac{\ln 3}{8\pi}\right)^{1/3} \simeq 0.35. \end{aligned} \tag{4.89}$$

In order to take into account the fact that the detector starts to follow the evolution of the gravitational waves, and so of their phase, from a $v_{gw,0}$, we should compute $\delta\psi_{\rm TH}(v_{crit})$ and then subtract the contribution of $\delta\psi_{\rm TH}(v_{gw,0})$. Plugging the values in $\delta_{\rm TH}\psi(v)$, we have:

$$\begin{aligned}
&\delta\psi_{\rm TH}(v_{gw,0}) \simeq 7.9\times 10^{-5},\\
&\delta\psi_{\rm TH}(v_{crit,1}) \simeq 6.8\times 10^{-3},\\
&\delta\psi_{\rm TH}(v_{crit,2}) \simeq 9.8\times 10^{-3}.
\end{aligned} \tag{4.90}$$

From these, we obtain:

$$\begin{aligned}
\Delta\psi_{\rm TH1} &= \delta\psi_{\rm TH}(v_{crit,1}) - \delta\psi_{\rm TH}(v_{gw,0}) \simeq 6.7\times 10^{-3},\\
\Delta\psi_{\rm TH2} &= \delta\psi_{\rm TH}(v_{crit,2}) - \delta\psi_{\rm TH}(v_{gw,0}) \simeq 9.7\times 10^{-3}.
\end{aligned} \tag{4.91}$$

The last two quantities represent the phase shift observed in the gravitational wave signals via a typical detector, in the cases of α_1 and α_2, due to the absence of tidal heating derived from the addition of a finite step in area for black hole horizons. It is clear that the orders of magnitude computed are small; as said, tidal heating gives a contribution to the phase which is not easy to detect directly.
Now we can proceed with the other two case, namely, α_3 and α_4. In these cases, we have seen that the numerical value of α is such that we do not have tidal heating at all in inspiral. This means, in practice, that the phase shift with respect to the classical case is computed during the total inspiral phase, and so stopped at the value of $v_{\rm ISCO} \simeq 0.41$. We have already computed $\delta\psi_{\rm TH}(v_{\rm ISCO})$ in (4.68), we have just to subtract $\delta\psi_{\rm TH}(v_{gw,0})$ and we obtain:

$$\Delta\psi_{\rm TH3} = \Delta\psi_{\rm TH4} = \delta\psi_{\rm TH}(v_{\rm ISCO}) - \delta\psi_{\rm TH}(v_{gw,0}) = 1.4\times 10^{-2}. \tag{4.92}$$

This is another time of small order of magnitude. From this we understand that, at least for the case of two Schwarzschild black holes, the maximum value of the phase shift considered has order comparable to (4.92).

Since, in principle, we are also interested in study what happens varying α, it is interesting to see what is the behaviour of $\delta\psi_{\rm TH}(v_{crit,\alpha})$, where now we underline that α is taken as free parameter. For each value of α we have a different v_{crit} and we obtain actually a function of α, $\delta\psi_{\rm TH}(\alpha)$. Taking the analytical form (4.63) and plugging $v_{crit} = (\pi M f_{crit})^{1/3}$, we obtain:

$$\delta\psi_{\rm TH}(\alpha) = -\frac{5\alpha}{3072\pi}\Big(\ln\Big(\frac{\alpha}{32\pi}\Big) - 1\Big) + \frac{37}{512}\Big(\frac{\alpha}{32\pi}\Big)^{5/3}. \tag{4.93}$$

This represents the phase shift, due to the suppression of tidal heating, because of the introduction of a quantum step in horizon area. We recall that we are in the case of two Schwarzschild black holes, with equal masses. We can take a plot of that function, considering, of course, positive values for α (Figure 4.2).

We see from the graph that the function grows for growing values of α, but we know that, after $\alpha = 6.8$, our phase shift is stopped at the value $v_{\rm ISCO}$. A more practical graph, for our purpose, should be given by Figure 4.3, in which the function is constant after $\alpha = 6.8$, equal to a value $\delta\psi_{\rm TH}(v_{\rm ISCO})$.

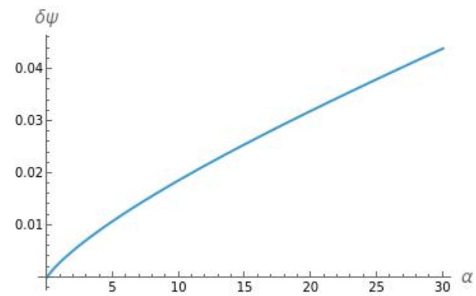

Figure 4.2: The function $\delta\psi_{\mathrm{TH}}(\alpha)$. Since v_{crit} is actually a function of α, we obtain a dependence in that parameter for the phase shift due to reduction of tidal heating.

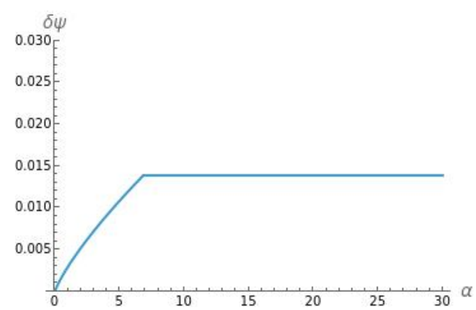

Figure 4.3: A different version of the previous graph, in which is taken into account that after $\alpha = 6.8$ the phase shift we measure is equal to $\delta\psi_{\mathrm{TH}}(v_{\mathrm{ISCO}})$.

In the next subsection, we want to study the more physical case of two Kerr black holes, and see the orders of magnitude in that system. For that situation, it is clear we have to deal with a more complicated structure, since the parameters of each black hole are two: the mass and the angular momentum. Instead of work directly with the angular momentum, we take the ratio $\chi = J/m^2$, and we will see that now the relevant values of the parameter v we have to deal with are not constant, but dependent on this χ. After having shown what are the analytical changes, we go through a numerical evaluation of the phase shift, to see what kind of orders of magnitude comes into play.

4.3.2 Quantum of area and tidal heating: Kerr black holes

Here we follow the methods presented in [23].
For Kerr black holes, since we have the addition of a non vanishing angular momentum, we have to modify the thermodynamical relation seen above. In particular, take the case of a Kerr black hole of mass m, angular momentum J and horizon area A. We have the new mass condition ($G = c = 1$):

$$m = \sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}} \tag{4.94}$$

If we follow Bekenstein's heuristic method of quantization, for the minimal step in area we have as usual $\Delta A = \alpha L_P^2$. Supposing that the step in area is not just initial, but is actually constant and finite for each value of A, we get that the total value of the horizon area can be written as:

$$A = \alpha L_P^2 N = \alpha \hbar N, \tag{4.95}$$

where N plays the role of a quantum number. The same we do with the spin, since from standard quantum mechanics we know that is quantized: $J = \hbar j$, where j is a semi-integer number bounded by $0 \leq j \leq \alpha N/8\pi$. We have consequently that the total mass m of the black hole depends on the two quantum numbers N and j:

$$m_{N,j} = \sqrt{\frac{\alpha \hbar N}{16\pi} + \frac{4\pi \hbar j^2}{N\alpha}}. \tag{4.96}$$

This relation gives the mass (or energy) spectrum of the Kerr black hole under consideration. Since we see that it is not uniform, we could ask ourselves if this kind of spectrum can give similar results to the case of Schwarzschild. We start to study this problem focusing on the mass, and we take $m_{N,j}$ as a set of eigenvalues for some set of quantum states of the black hole. When the black hole undergoes a transition, we can think that for the mass eigenvalue we have simply $m_{N,j} \to m_{N+\Delta N, j+\Delta j}$. We are interested mainly in the transition of the mass given by the interaction between the black hole and a gravitational wave of a given frequency, for which we can write the carried energy, as usual, as $E = \hbar\omega$. For the mass holds then:

$$|m_{N+\Delta N, j+\Delta j} - m_{N,j}| = \hbar\omega. \tag{4.97}$$

Since we want to study the case of the minimal jump in the energy level, we can set $\Delta N = 1$. On the other hand, the interaction of our interest is the one between a gravitational wave emitted by a binary and one of the two components of the binary itself, so we know that for such gravitational waves the relevant mode is the quadrupolar one with $l = 2, m = 2$ [13]. From angular momentum conservation, we have then the selection rule $\Delta j = 2$, giving for (4.97):

$$m_{N+1,j+2} - m_{N,j} = \hbar\omega. \tag{4.98}$$

Now, remember that we are interested in physical system like astrophysical black holes: to get a macroscopic value for the mass $m_{N,j}$, knowing also the bounds for the semi-integer number j, we should assume a large value of N. Take the difference in (4.98) with the specific form of $m_{N,j}$ given by (4.96), and perform the limit for $N \to \infty$. Calling now explicitly the frequency of the absorption ω_{crit}, we obtain:

$$\hbar\omega_{crit} = \frac{\kappa\hbar\alpha}{8\pi} + 2\hbar\Omega_{\mathrm{H}}, \tag{4.99}$$

where with κ and Ω_{H} we indicate, respectively, the surface gravity and the angular velocity of the horizon of the black hole. Their explicit expression are both function of the ratio $\chi = J/m^2$, with $0 \leq \chi \leq 1$, and they lead:

$$\kappa = \frac{\sqrt{1-\chi^2}}{2m(1+\sqrt{1-\chi^2})}, \quad \Omega_{\mathrm{H}} = \frac{\chi}{2m(1+\sqrt{1-\chi^2})}. \tag{4.100}$$

The equation (4.99) is of great importance for us; this is the equivalent version of (4.82) but for Kerr black holes. Indeed, if we set $\chi = 0$, we recover precisely (4.82).

Suppose now we have our binary system, made of two Kerr black holes, of the same mass $m_1 = m_2 = m$ and with the same angular momentum, aligned each other and aligned with the orbital angular momentum. We indicate as usual the total mass with $M = 2m$. For the parameter v_{crit} holds:

$$\begin{gathered} v_{crit} = (\pi M f_{gw})^{1/3} = \left(\pi M \frac{\omega_{crit}}{2\pi}\right)^{1/3} = \\ \left(\frac{\alpha\sqrt{1-\chi^2}}{16\pi(1+\sqrt{1-\chi^2})} + \frac{\chi}{(1+\sqrt{1-\chi^2})}\right)^{1/3}. \end{gathered} \tag{4.101}$$

It is obvious that now we have a more involved structure for the critical frequency, and also for v_{crit}; in order to have a picture of its behaviour, it could be useful to see the graph Figure 4.4. In this, we took the value α_3.

Seeing that the function is monotonous, we can take an intermediate value of χ, for example $\chi = 1/2$, and compute numerically the values of v_{crit} for the various values of α proposed, to compare with the ones from the Schwarzschild case. We get:

$$v_{crit}(\chi = 0.5) = \left(\frac{\alpha\sqrt{3}}{16\pi(2+\sqrt{3})} + \frac{1}{(2+\sqrt{3})}\right)^{1/3}. \tag{4.102}$$

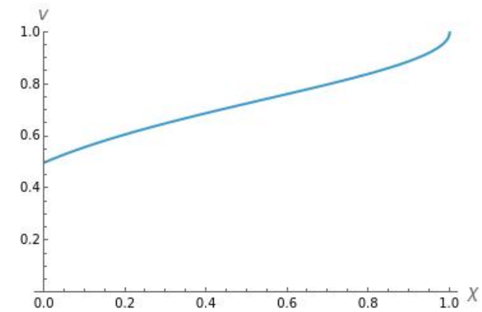

Figure 4.4: Graph of v_{crit} in function of the ratio χ, with $\alpha = \alpha_3$.

Plugging the four values of α in literature, we obtain:

$$\begin{aligned} v_{crit,1}(\chi = 0.5) &\simeq 0.67, \\ v_{crit,2}(\chi = 0.5) &\simeq 0.68, \\ v_{crit,3}(\chi = 0.5) &\simeq 0.73, \\ v_{crit,4}(\chi = 0.5) &\simeq 0.79. \end{aligned} \tag{4.103}$$

Beside, we have the values for the Schwarzschild case, which can be obtained from (4.101) setting $\chi = 0$. We have:

$$\begin{aligned} v_{crit,1}(\chi = 0) &\simeq 0.30, \\ v_{crit,2}(\chi = 0) &\simeq 0.35, \\ v_{crit,3}(\chi = 0) &= 0.50, \\ v_{crit,4}(\chi = 0) &\simeq 0.63. \end{aligned} \tag{4.104}$$

We see that the spin increases the value of v_{crit}, but for higher values of α, the difference between the values for $\chi = 0.5$ and $\chi = 0$ tends to decrease.
Now, we know that in our analysis we are considering the coalescence of two Kerr black holes, and another important value of v is the one corresponding to the ISCO frequency. Another time, in the case of Kerr black holes in coalescence, the form that gives this value of frequency is more involved with respect to the Schwarzschild case, and is dependent on χ. From [16], we have the following dependence:

$$\pi M f_{gw,\mathrm{ISCO}} = \{(3 + Z_2 - [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2})^{3/2} + \chi\}^{-1}, \tag{4.105}$$

where we have already written the gravitational wave frequency corresponding to ISCO, $f_{gw,\mathrm{ISCO}}$, and where Z_1 and Z_2 are defined as:

$$\begin{aligned} Z_1 &= 1 + (1 - \chi^2)^{1/3}[(1 + \chi)^{1/3} + (1 - \chi)^{1/3}], \\ Z_2 &= (3\chi^2 + Z_1^2)^{1/2}. \end{aligned} \tag{4.106}$$

So we have consequently:

$$v_{\mathrm{ISCO}} = \{(3 + Z_2 - [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2})^{3/2} + \chi\}^{-1/3}. \tag{4.107}$$

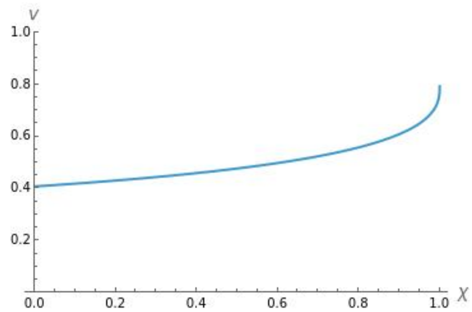

Figure 4.5: A graph of $v_{\rm ISCO}$ for the Kerr case, as a function of χ.

As before we would like to have a picture of this behaviour. The graph of $v_{\rm ISCO}$ in function of χ in the range $[0;1]$ is shown in Figure 4.5.

This relation is again monotonous. The extreme values are:

$$v_{\rm ISCO}(\chi=0) = \left(\frac{1}{6}\right)^{1/2}, \qquad v_{\rm ISCO}(\chi=1) = \left(\frac{1}{2}\right)^{1/3} \tag{4.108}$$

Of course, for $\chi = 0$ we get the Schwarzschild value, while for the extreme ratio $\chi = 1$ we get $v_{\rm ISCO}(\chi=1) \simeq 1.9 v_{\rm ISCO}(\chi=0)$. Following the same strategy as before, we should understand at which value of v starts tidal heating in the system, due to the presence of a step in area. In principle, we have to solve the following inequality, where we leave free both α and χ:

$$\begin{aligned} &v_{crit} > v_{\rm ISCO} \to \\ &\to \left(\frac{\alpha\sqrt{1-\chi^2}}{16\pi(1+\sqrt{1-\chi^2})} + \frac{\chi}{(1+\sqrt{1-\chi^2})}\right)^{1/3} > \\ &\{(3+Z_2-[(3-Z_1)(3+Z_1+2Z_2)]^{1/2})^{3/2}+\chi\}^{-1/3}. \end{aligned} \tag{4.109}$$

To solve analytically for χ is, of course, a mess, but we can manage the inequality in a different way: we solve for α, meaning that we get a condition for α dependent on the ratio χ. For each of the four values of α under consideration, we will have a specific value of χ at which the condition $v_{crit} > v_{\rm ISCO}$ is satisfied. We define, for simplicity, the following two functions:

$$W_1 = \frac{\sqrt{1-\chi^2}}{16\pi(1+\sqrt{1-\chi^2})}, \quad W_2 = \frac{\chi}{(1+\sqrt{1-\chi^2})}. \tag{4.110}$$

In this way, after having reorganized, we have:

$$\alpha > \frac{1}{W_1\{(3+Z_2-[(3-Z_1)(3+Z_1+2Z_2)]^{1/2})^{3/2}+\chi\}} - \frac{W_2}{W_1}. \tag{4.111}$$

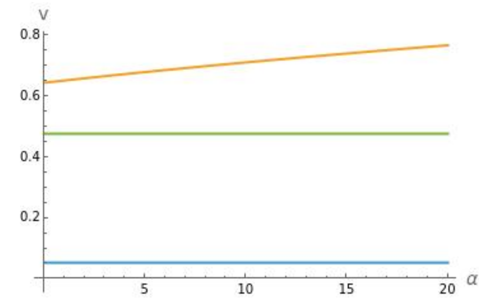

Figure 4.6: Here the graphs of $v_{gw,0}$ (blue line), $v_{\rm ISCO}$ (green line), v_{crit} (yellow line) as functions of α. Having fixed $\chi_1 = \chi_2 = 0.5$, the only one function that depends on α is v_{crit}. The important point of the graph is that $v_{crit} > v_{\rm ISCO}$ always, for α positive.

Let us see the case $\chi = 0.5$. For the ISCO value we have $v_{\rm ISCO} \simeq 0.48$, and for the inequality we have:

$$\alpha > -17,3. \tag{4.112}$$

This means that, for initial value of $\chi = 0.5$ for each black hole, we have $v_{crit} > v_{\rm ISCO}$ for each of the four values of α considered. This is an interesting result, and we show the graph in Figure 4.6.

As we see, the value of v_{crit} (yellow line), at which classical tidal heating can start, is always above the value of $v_{\rm ISCO}$ (green line). So, in this case, the phase shift due to the presence of a step in area is computed considering the integral in the definition of $\delta\psi(v)$ from v_0 to $v_{\rm ISCO}$. In analogy with the Schwarzschild case, we could call this phase shift $\Delta\psi_{\rm TH(1,2,3,4)}$, underlying that it holds for each value of α considered. However, we have a stronger result: this shift holds for *all* values of $\alpha > 0$, so we call it simply $\Delta\psi_{\rm TH}$. We indicate also that we are in the Kerr case, with an explicit dependence on the ratio χ. Recalling the template for Kerr black holes (4.73), we have:

$$\Delta\psi_{\rm TH}(\chi = 0.5) = \\ \delta\psi_{\rm TH}(\chi = 0.5, v_{\rm ISCO}) - \delta\psi_{\rm TH}(\chi = 0.5, v_{gw,0}) \simeq 0.066 - 0.176 \simeq -0.11. \tag{4.113}$$

This is a small order of magnitude, but in any case is of great conceptual importance.

We want to see what we obtain in the case of extreme ratio $\chi = 1$. We have $v_{\rm ISCO} \simeq 0.79$, and since the term containing α goes away, the inequality (4.111) leads:

$$1 > \left(\frac{1}{2}\right)^{1/3}, \tag{4.114}$$

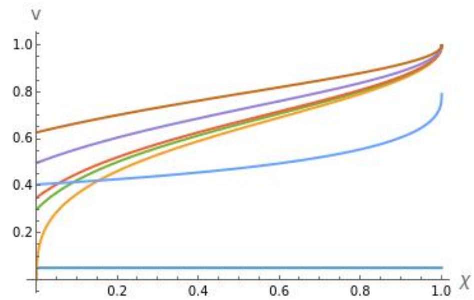

Figure 4.7: A summary of the various relevant values of v, now functions of χ. The constant blue line is as usual $v_{gw,0}$; then, we can distinguish the five functions of v_{crit} for different values of α; the other blue line is v_{ISCO}.

which is always satisfied, and so we get that also in this case the phase shift due to the absence of tidal heating is during the entire inspiral phase. The graph in this case is not necessary, since we deal with three constant functions. Looking at the template for $\delta\psi_{\mathrm{TH}}$ in the case $\chi = 1$ (4.77), we get numerically:

$$\begin{aligned} &\Delta\psi_{\mathrm{TH}}(\chi = 1) = \\ &\delta\psi_{\mathrm{TH}}(\chi = 1, v_{\mathrm{ISCO}}) - \delta\psi_{\mathrm{TH}}(\chi = 1, v_{gw,0}) \simeq 3.6 - 0.81 = 2.8. \end{aligned} \tag{4.115}$$

Compared to the last cases, this is some orders of magnitude bigger; in any case, having two Kerr black holes in a binary with initial values of ratio χ equal to or close to 1 is not a common situation in astrophysical observations for now. It is interesting, another time, to see that also for this value of χ for each black hole we have the complete suppression of tidal heating during the inspiral phase. In order to have a graphical picture of the relation between the relevant values of v in function of χ, it should be interesting to plot them in the same graph, taking now fixed values for α, that are the four considered in this work and also $\alpha = 0$, to see what happens if the quantum of area is actually reduced to an infinitesimal one. This is shown in Figure 4.7.

We see clearly that, after a certain value of χ, almost each function $v_{crit,\alpha}$ is above the one of v_{ISCO}, suggesting that, actually, the real astrophysical case of two Kerr black holes, despite being the most difficult analytically, is also the most interesting from the point of view of quantum-of-area impact. In the case of equal masses and equal spins, it is sufficient to have a certain amount of ratio χ for each black hole to have a suppression of tidal heating at all, for each relevant value of α in literature. Moreover, seeing that, in the region of small values of χ, for higher α the graph of $v_{crit,\alpha}$ seems to rise, we should expect that this feature is enforced if α is greater than the values in the literature. Another important fact should be underlined: we see that, even in the graph of $v_{crit,\alpha=0}$, we get a non-trivial dependence in χ. We should ask why, since we are setting $\alpha = 0$, and we are then neglecting the quantum of area. The point is that, in finding the condition of absorption for Kerr black holes, the quantization of the area was not the only quantum correction we made: we treated the entire black hole as a quantum object, quantizing by hand also its angular momentum, imposing $J = j\hbar$ and introducing *another* quantum number, independent on the one related to the quantization of the area. Searching the minimal step in energy, we then set $j = 2$ for angular momentum, and so it is consistent that we still have a non-constant behaviour for $v_{crit,\alpha=0}$.

4.3.3 General case of different masses: qualitative description

It is important to underline a property of the cases examined in the previous examples: since we considered equal masses, and in the Kerr case also equal spins, in the evolution of the binary the two black holes follow the same changing in their parameters, and the system is really symmetric in the formal transformation BH1 $\rightarrow$ BH2, and viceversa. One of the consequences of this symmetry is that, in the quantum description, when tidal heating is turned on for one of the two black holes, the same happens for the other. Intuitively, this allows us to divide the inspiral phase in two effective subphases: the first completely without tidal heating, the second with the presence of tidal heating felt by each of the two black holes. Of course, this happens in the case of $v_{crit} < v_{\mathrm{ISCO}}$, otherwise the entire inspiral phase takes place without tidal heating. What happens in the case of different masses? Take, for example, the case of two Schwarzschild black holes, with $m_1 < m_2$. Now, for the value of ω_{crit}, we have not just one value which holds for both black holes, but each of the two has its own critical frequency:

$$\begin{aligned} \omega_{crit,BH1} &= \frac{\alpha}{32\pi m_1}, \\ \omega_{crit,BH2} &= \frac{\alpha}{32\pi m_2}, \end{aligned} \tag{4.116}$$

with $\omega_{crit,BH1} > \omega_{crit,BH2}$. Since we deal with macroscopic values of the masses, and the minimal frequency of the detector is around $f_0 \simeq 20\mathrm{Hz}$, we suppose that both the two frequency are higher than the minimum one for detection, so in any case, tidal heating is supposed to start in the observable range. We distinguish three cases:

1. Both critical frequencies are less than the ISCO frequency.
2. One of the two frequencies is less than the ISCO one, the other is greater. Since we have $\omega_{crit,BH1} > \omega_{crit,BH2}$, the frequency inside the inspiral frequency range is $\omega_{crit,BH2}$.
3. Both the critical frequencies are greater than the ISCO one.

Start from the first case. When the system starts to evolve (or, if we want, the system starts to be observed), each black hole does not feel tidal heating. Increasing the value of the orbital frequency, we arrive at a point in which we

reached the value of gravitational wave frequency sufficient to turn on tidal heating for one of the two black holes, BH2. At this point, we start to have a *partial* tidal heating, which, from the point of view of the gravitational wave phase, corresponds to a *partial* phase shift. The system continues to evolve with this partial tidal heating until we reach the value of the other critical frequency $\omega_{crit,BH1}$, at which also BH1 starts to absorb gravitational waves emitted by the system. In this last part of the inspiral phase, we have the complete classical tidal heating, and so the complete accumulated phase associated. We can say that, colloquially, the inspiral phase is made by three subphases: the first without tidal heating, the second with a partial tidal heating, the third with the complete classical tidal heating.
In the second case, one of the two values of the frequency is higher than the ISCO one, namely $\omega_{crit,BH1}$. For the first black hole, we have the complete absence of tidal heating during the inspiral phase. The system starts to evolve, and in the initial subphase there is no tidal heating, until we arrive at the frequency $\omega_{crit,BH2}$. At this point, it starts the tidal heating for BH2. The system evolves with this partial tidal heating until we arrive at ISCO frequency, and the inspiral phase stops. In this case we can then identify two subphases, the first without tidal heating as before, the second with only partial.
The third case is the simplest one: the two critical frequencies are both higher than the ISCO one, meaning that during the entire inspiral phase tidal heating is completely neglected.
At this point, it is important to understand one point: what does it mean precisely *partial*? Let us explain that. When we computed the templates for $\delta\psi_{\mathrm{TH}}(v)$, we had $\eta = 1/4$, and we inserted the coefficients Ψ_5, Ψ_8 already. In the general case, we have clearly another value for η, since the masses are no longer equal. For the coefficients Ψ_5 and Ψ_8, we have to pay attention: they depend again on the mass ratio, but, as explained in the appendix, they are the sum of contributions of tidal heating of both the black holes. More precisely:

$$\Psi_5 = A_5^{(1)} + A_5^{(2)}, \quad \Psi_8 = A_8^{(1)} + A_8^{(2)}; \tag{4.117}$$

where $A_5^{(i)}$ and $A_8^{(i)}$ are terms which derive from the capacity of absorption of BH1 and BH2. In the case of equal masses, they were initially set to zero, and when tidal heating starts, each $A^{(i)}$ becomes, in principle, different from zero at the same time, and in (4.117) we get the complete sum. In the general case, while they start from zero as in the equal-masses case, when we reach the first of the two critical frequencies, only the terms $A^{(i)}$ corresponding to the black hole that started to absorb gravitational waves become different from zero, and so each of the sums in (4.117) has actually just one term; in other words, these sums become *partial*. If also the other critical frequency is less than ISCO frequency, after having reached this, we start to have the complete sum for both Ψ_5 and Ψ_8, as is considered in the previous sections. More formally, remembering the way in which we wrote the effective term of tidal heating modified by the introduction of a quantum in the horizon area, in equation (4.88), we can write some similar form also for the general case. We define the new template $\delta\psi^{p}_{\rm TH}(v)$, which corresponds to the phase shift given by the partial tidal heating: the one in which enter just the partial sums derived from (4.117). Recalling that to $\omega_{crit,BH1}$ and $\omega_{crit,BH2}$ correspond, respectively, two values of v that we call $v_{crit,BH1}$ and $v_{crit,BH2}$, we write:

$$\delta\psi_{\rm TH}(v) \to \theta(v - v_{crit,BH2})\delta\psi^{p}_{\rm TH}(v) + \theta(v - v_{crit,BH2})\theta(v - v_{crit,BH1})\delta\psi_{\rm TH}(v), \tag{4.118}$$

where as usual θ means the Heaviside function.

4.4 Quantum effects in curved spacetime: a comparison

In this section, we want to make a simple comparison between two quantum effects, which both take place in a curved spacetime and so contain non trivial gravitational effect: the first is the Hawking radiation and the second is the q-version of tidal heating. Our comparison will be from the energy point of view; more specifically, we want to show what kind of order of magnitude enter in both the effects, and to see if there is a consistent difference between them. To compare these two effects makes sense because of the following fact: the Hawking radiation is a quantum effect in a *classical* curved spacetime, meaning that we have quantum fields which are evolving in a classical gravitational background; on the other hand, (classical) tidal heating is a complete gravitational effect, but after the effective correction given by the presence of the step in area, we deal actually with a quantum version of tidal heating, but *quantum* here means that we include effective quantum corrections of the spacetime itself. In what follows, we assume that the total mass of the two system is the same, in order to have the same order of magnitude of the mass. This means that, if we assume a binary system with total with total mass $m = m_1 + m_2$, the black hole which emits thermal radiation is taken with mass m.
Start from the case of a Schwarzschild black hole of mass m, and consider its thermal emission. The law which gives the total power emitted by a black body of temperature T and with surface A is well known, and leads:

$$P = \sigma A T^4, \tag{4.119}$$

where σ is the Stefan-Boltzmann constant, which is dependent on several physical constant:

$$\sigma = \frac{2\pi^5}{15}\frac{k_B^4}{c^2h^3} = 5.670373 \times 10^{-8}\frac{\rm W}{\rm m^2K^4}. \tag{4.120}$$

Here, k_B is the Boltzmann constant and h the Planck constant. We can consider the black hole as a black body with surface equal to the horizon area, and with temperature given by the Hawking temperature $T_{\rm H}$, for which we have:

$$T_{\rm H} = \frac{hc^3}{16\pi^2 mGk_B}. \tag{4.121}$$

We see the characteristic behaviour of T_H: being inversely proportional to the mass m, the more massive the black hole is, the more difficult becomes the observation of the temperature. This is one of the reason for which Hawking radiation is so hard to detect. For the horizon area A, we have as usual $A = 16\pi m^2$. Inserting constant to get dimension of squared meters, we have:

$$A = \frac{16\pi m^2 G^2}{c^4}. \tag{4.122}$$

Plugging the values in (4.119), we obtain:

$$P_\mathrm{T} = \frac{2\pi^5 k_B^4}{15c^2h^3}\frac{16\pi m^2 G^2}{c^4}\left(\frac{hc^3}{16\pi^2 mGk_B}\right)^4 \sim m^{-2}, \tag{4.123}$$

where the label to the subscript stays for thermal, to identify that we are talking about the power emitted due to the presence of a temperature T_H. We are interested specifically in the mass dependence of the power emitted: is inversely proportional to the square of the black hole mass, or, in other words, proportional to the squared black hole temperature.
Now, let us consider a binary system with total mass m, and let's focus on the flux term related to tidal heating, F_H. We recall the form used in the text, already in dimensional form:

$$F_\mathrm{H}(v) = \frac{32}{5}\frac{c^5}{G}\eta^2 v^{10}\left[-\frac{\Psi_5}{4}v^5 + \frac{\Psi_8}{2}v^8\right], \tag{4.124}$$

The factor c^5/G is to have a term with dimensions of J/s. Now, we know that v is a small parameter, such that $0 < v < 1$. In order to study the order of magnitude, we can consider just the leading term of (4.124):

$$F_\mathrm{H}(v) \simeq \frac{32}{5}\frac{c^5}{G}\eta^2 v^{10}\left(-\frac{\Psi_5}{4}v^5\right) = -\frac{8}{5}\frac{c^5}{G}\eta^2\Psi_5 v^{15}. \tag{4.125}$$

We recall that the parameter η is such that $0 < \eta < 1/4$, so, even if we consider generic mass ratios, this factor enters with an order which is almost 1. For the coefficients Ψ_5 and Ψ_8, then, we have a dependence in mass ratios and spins of the black holes (if we consider spinning bodies); in any case, they also enter with an order which is almost 1. For the frequency, having in mind what is the order of magnitude of the ISCO frequency, we can consider a range which is almost between 1Hz and 100Hz. Now, we recall also that $v = (\pi m f)^{1/3}$; inserted in (4.125), gives us a dependence in m to the fifth power:

$$F_\mathrm{H}(v) = -\frac{8\pi^5}{5}\frac{c^5}{G}\eta^2\Psi_5 f^5 m^5; \tag{4.126}$$

so $F_\mathrm{H} \sim m^5$. We have also a dependence on the gravitational wave frequency, or equivalently, a dependence in the source frequency, to the fifth power again.

This is the template associated with tidal heating flux, at the leading order in m: since it is a classical effect, why we are considering it for a comparison between quantum effect? As we explained in the section above, due to the presence of a step in area, tidal heating could be largely suppressed, giving a phase shift with respect to the classical case but also a lack in the power absorbed by the horizons; from this point of view, this lack of energy is a complete quantum effect, despite the fact that the analytical form of the function is the same. Since we have a dependence on a positive power of m, and we consider macroscopic values of the mass, as for real astrophysical systems, we can have a huge amount of energy which is missing in the effective quantum case.

Now we can see clearly that for the ratio $|F_{\rm H}/P_{\rm T}|$ holds:

$$\frac{|F_{\rm H}|}{|P_{\rm T}|} \sim m^7. \tag{4.127}$$

This is a remarkable result: comparing two quantum systems, both with a curved spacetime, we discovered that the dependence on the mass is very different. While the Hawking radiation, from the emitted-power point of view, is for now almost undetectable, the lack of absorbed power by the black holes of a binary is incredibly large, and is still a quantum effect on macroscopic objects, connected this time to quantum properties of the spacetime itself. However, since as we said this flux of energy regards the horizons of the black holes, and so, is confined in an inaccessible region of the spacetime, it is not, in any case, detectable in a direct way. We wanted to show that, even if one is considering a quantum effect in a macroscopic system, with a non-negligible gravitational field, this does not mean that all the physical consequences are almost undetectable: quantum gravitational effects are supposed to be significant when we are at the Planck length, but the consequences of them can be considerable at a much higher scale.
We can also make the discussion more quantitative, introducing the numerical values of all the constants, to control also the order of magnitude of the coefficient in front of the ratio. For the power radiated via Hawking radiation, we get, keeping the mass free:

$$P_{\rm T} \simeq 3.6 \times 10^{34}\ m^{-2}. \tag{4.128}$$

For the tidal heating flux, we can write:

$$|F_{\rm H}| \simeq 1.8 \times 10^{55} f^5 m^5. \tag{4.129}$$

Now, recalling what we said about the coefficients in the form of $F_{\rm H}$, we have just to consider the frequency in a range $[1; 100]$Hz. The ratio leads:

$$\frac{|F_{\rm H}|}{|P_{\rm T}|} \simeq [5.0; 500] \times 10^{20}\ m^7. \tag{4.130}$$

Even if we look at the numerical coefficient, we discover a great difference between the two effects. This underlines another time the important point: despite the fact that we have two quantum effects, which both depend, directly or indirectly, on the Planck scale or $\hbar$, the orders in play are very different. This gives a remarkable result: while the energy, or power, emitted by thermal radiation by a black hole is truly hard to detect (at least, for now and for macroscopic black holes), the one suppressed in tidal heating due to the presence of a minimal length of the order of the Planck scale, being in any case an energy not accessible at infinity but present in the black holes region, is incredibly large.

Chapter 5

Conclusions and outlooks

We have finished our discussion about minimal length and correlated effects, specifically in the case of a coalescing binary. Now we want to collect the main results we found, briefly recalling the path followed.
To introduce the qmetric was necessary to understand the important consequence of a *minimal area* around a point, given by a minimal length in spacetime, which leads to the presence of a minimal step in the horizon area for a black hole $\Delta A_{min} = 4\pi L_0^2 = \alpha L_P^2$. We saw some features derived from the implementation of such properties in the spacetime; for example, the way we proposed to interpret a Lorentz invariant minimal length as a limit radius of curvature. The case of Schwarzschild spacetime suggests that is the Kretschmann scalar that we should consider, like in (3.20) and (3.21), to study possible limits on local curvature. It would be interesting to generalize the method and to study also more involved geometries, like the Kerr one.
As a preamble of the main work of the thesis, we decided to consider an effective modification of the emission of Hawking radiation, due to the presence of a step in area for the black hole horizon, studying the dependence in α. It is remarkable that the great part of the emitted radiation seems to be actually strongly suppressed, if we have $\alpha > 6.3$ (see (3.61), (3.55)).

Now, we talk about the results of the main subject of the work: the effect of area quantization for a coalescing binary. We start from the case of two Schwarzschild black holes with equal masses. In this simple case, the first important function we have to consider is the template $\delta\psi_{\mathrm{TH}}(v)$, that gives the GW phase shift due to the presence of *tidal heating*, which is the absorption, by the two black hole horizons, of the gravitational waves produced by the binary. It is in function of the dimensionless orbital velocity v.

We obtained that the frequency at which each of the two black holes is able to absorb the energy $\hbar\omega$ carried by a gravitational wave (called critical frequency) is linear in α and inversely proportional to the mass m of the black hole (4.82). This frequency has to be compared with the ISCO one, at which the inspiral phase goes into the merge, that is again inversely proportional to the mass. A graphical situation is represented in Figure 4.1, where the dimensionless parameter $v = (\pi M f_{gw})^{1/3}$ is considered. It is clear that, after a value $\alpha \simeq 6.8$, the critical frequency is higher than the ISCO one, meaning that each of the black holes is not able to absorb gravitational waves during the entire inspiral phase. This is a strong result: for certain values of α, tidal heating is completely suppressed. This, in particular, holds for $\alpha_3 = 4\pi$ and $\alpha_4 = 8\pi$, two of four relevant values present in the literature. For these, the total phase shift due to the absence of tidal heating is computed (4.92).

For the other two values, we have different phase shifts, in these cases explicitly dependent on α (see (4.91)). The phase shifts are computed starting from the minimum frequency at which a detector is able to follow the gravitational wave evolution. Since we are, in general, interested to the case of free α, an important function is the template (4.93), which gives the phase shift in function of α, with graph in Figure 4.2. Of course, for $\alpha > 6.8$, we have actually a *condensation* of the phase shift to the value $\delta\psi_{\text{TH}}(v_{\text{ISCO}})$, because at v_{ISCO} stops in any case the inspiral phase. A more practical graph is then given in Figure 4.3.
For the more realistic situation of two Kerr black holes in a binary, the dynamics of the system is more involved, because of the addition of the spins, which are taken to be equal and aligned for the two black holes, and aligned to the orbital angular momentum. As usual, the masses are taken to be equal. This choice allows us to work in a completely symmetric system, as in the case of Schwarzschild. First, it is explained what is the relation between the critical frequency of each black hole and α, which turns out to depend, in addition to α and the mass m, also on the dimensionless ratio $\chi_i = J_i/m_i^2$, for which of course holds $\chi_1 = \chi_2$. The behaviour of v_{crit} is shown in Figure 4.4 for $\alpha = \alpha_3$. The function of the ISCO frequency is also given, as a function of χ, that has to be compared with the critical one. The graph of v_{ISCO} is plotted in Figure 4.5. In order to explore what happens, at least indicatively, for $\chi \in [0; 1]$, we decided to fix χ and to follow the same procedure as the case of Schwarzschild. Having already studied the case $\chi = 0$ (spin equal to zero), we decided to set a middle value $\chi = 0.5$, and then explore the case of extreme ratio $\chi = 1$. The results are really remarkable: both for $\chi = 0.5$ and for $\chi = 1$, it turns out that tidal heating is suppressed for the entire inspiral phase, giving the maximum values for the phase shift, namely $\delta\psi_{\text{TH}}(\chi = 0.5, v_{\text{ISCO}})$ and $\delta\psi_{\text{TH}}(\chi = 1, v_{\text{ISCO}})$. The graphical situation for $\chi = 0.5$ is shown in Figure 4.6, while for $\chi = 1$ a graphical representation is not necessary, since the dependence on α disappears in v_{crit}, leaving just constant functions . The numerical values for the respective cases are given in (4.113), (4.115); as before, these are computed starting from the minimum frequency for the GWs detection. A graph which summarize the qualitative comparison between all the relevant frequencies in function of χ, fixing different values of α, is then shown in Figure 4.7. It is easy to see what is the effect of the spin: it is sufficient to have a reasonably large value of χ to obtain that v_{crit} is always above the value of v_{ISCO}, giving a complete suppression of tidal heating. We think that, conceptually, this is one of the most important results.

It is then explained qualitatively what is expected for the case of generic values of masses. For the case of $m_1 = m_2$, we have that in the inspiral phase tidal heating is *turned on* at the same time for each black hole, because of the symmetry of the system: the critical frequency is the same for both the black holes. This allows us to identify, naively speaking, two *sub*phases: one without tidal heating at all, and the other with a complete, classical tidal heating. In the case of different masses, we have no longer one critical frequency, but two, one for each black hole. We can say that, if both these two frequencies are less than the ISCO one, we have to divide the inspiral phase in three *sub*phases, with an effective correction of $\delta\psi_{\rm TH}$ given by (4.118).
Before we conclude, just a final remark. Prototypical of quantum effects in curved spacetime is the Hawking radiation, which is tied to Planck scale physics as it is evident from the fact that the entropy it foresees corresponds to ~ 1 degree of freedom per L_P^2 unit of area of the horizon; this radiation turns out to be really feeble for solar mass black holes. The quantum effect we have studied in this thesis also comes from Planck scale physics (recall that the quantum of area is $O(1)L_P^2$), then how can we have hope to detect it? To answer this question the emitted power by Hawking radiation has been compared with the (lack of) power associated to a quantum-corrected version of tidal heating. In doing this, the two systems are considered to have the same total mass. We underline again the important conceptual difference between the two: the Hawking radiation involves the presence of *quantum* fields evolving in a *classical* curved spacetime near a black hole horizon, while our correction of tidal heating implements *quantum* properties of the horizon spacetime which impact on dynamics of *classical* gravitational fields. It is surprising to discover that, despite both are quantum effects in curved spacetime, the lack of power we can have for suppression of tidal heating is incredibly big compared to the radiated power for Hawking radiation: we see from (4.130) that for the ratio $|F_{\rm TH}|/|P_{\rm T}|$ we have an order of 10^{20} and a dependence in the mass which is m^7. This could happen, essentially, because in the quantum correction of tidal heating we have a classical amount of energy which is neglected due to quantum effects (the presence of a critical frequency for the exchange of that energy), but the neglected flux of energy $F_{\rm H}$, even if relatively small compared to F_∞, derives in any case from one of the most powerful phenomenon of the universe, that is the coalescence of two black holes. What actually happens is that a Planck scale effect is able to inhibit or turn on a classical one, giving a strong impact on macroscopical systems like a coalescing binary. These arguments should be studied more concretely in the future, implementing a more suitable form also for the modified template associated to the phase shift, and focusing also on the dependence on α, which, via the observations, can help in discriminating between the possible proposals for a quantum theory of spacetime.

Appendix A

Properties of disformally coupled metrics

Here we list some properties of disformally coupled metrics. A more complete discussion can be found in [7]. Take a spacetime described by the metric g_{ab}, and take a certain scalar field $\varphi(x)$, dependent on the spacetime point x. We can define another metric $\tilde{g}_{ab}$ such that:

$$\tilde{g}_{ab} = \Omega^2 g_{ab} - \epsilon \Lambda t_a t_b, \tag{A.1}$$

where $\Omega = \Omega[\varphi]$, $\Lambda = \Lambda[\varphi]$ are functions of the scalar field $\varphi(x)$ and:

$$t_a = \frac{\partial_a \varphi}{\sqrt{\epsilon g^{ij} \nabla_i \varphi \nabla_j \varphi}}, \quad g^{ab} t_a t_b = \epsilon. \tag{A.2}$$

Two metric related by the last equation are said to be $disformally$ coupled. Note that here, instead of A and B, we are using the symbols Ω^2 and Λ; this is done also to underline that the case we studied in the text is actually a bit different: instead of a scalar φ, we have a biscalar σ^2 in our functions Ω and Λ. Although the terminology seems to suggest it, the law relating the two metrics g_{ab} and $\tilde{g}_{ab}$ is not a sort of contrary of the conformal one. Indeed, in the given form, this relation is a more general case which allows, as a subcase, the one of conformal relation between two metrics. To be precise, we have two important special cases: the case with $\Lambda = 0$ and the one with $\Lambda = \Omega^2 - \Omega^{-2}$. The first is precisely a $conformal$ relation, while the second, with an improper convention, is said to be disformal again. In the case studied in the text, we have $\varphi(x) \to \sigma^2(x, x')$, and for the function Ω^2:

$$\Omega^2(x, x') = 1 + \frac{L_0^2}{\sigma^2(x, x')}. \tag{A.3}$$

Now we just list some important results for the two special cases cited. Starting from the conformal one, we have in D dimensions:

- $\tilde{g}_{ab} = \Omega^2 g_{ab}$,
- $\tilde{h}_{ab} = \Omega^2 h_{ab}$,
- $\tilde{K}_{ab} = \Omega K_{ab} + (\nabla_{\mathbf{t}}\Omega) h_{ab}$,
- $\tilde{K} = \Omega^{-1} K + (D-1)\Omega^{-2}\nabla_{\mathbf{t}}\Omega$;

where the quantities written were introduced in the main text. In particular, h_{ab} is the induced metric on a level surface Σ of φ, called also $first\ fundamental\ form$; K_{ab} is the *second* $fundamental\ form$ and $\tilde{K} = \tilde{g}^{ab}\tilde{K}_{ab}$ its trace; $\nabla_{\mathbf{t}} = t^a \nabla_a$, with $t^a = g^{ab} t_b$. In the other case, we have:

- $\tilde{g}_{ab} = \Omega^2 g_{ab} - \epsilon(\Omega^2 - \Omega^{-2}) t_a t_b$,
- $\tilde{h}_{ab} = \Omega^2 h_{ab}$,
- $\tilde{K}_{ab} = \Omega^3 K_{ab} + (\Omega^2 \nabla_{\mathbf{q}}\Omega) h_{ab}$,
- $\tilde{K} = \Omega K + (D-1)\nabla_{\mathbf{q}}\Omega$.

We see that the relation between the induced metrics is the same, while for the second forms, we obtain in practice an additional factor Ω^2. It is interesting to see also the composition of two disformal coupling. Actually, the rule turns out to be very simple, after having introduced the function α, already defined in the text, which read:

$$\alpha = (\Omega^2 - \Lambda)^{-1}. \tag{A.4}$$

Take now a metric g_{ab}, and apply a disformal coupling defined by the function Ω'^2 and α'. Here we call the new metric g'_{ab}, obtaining:

$$g'_{ab} = \Omega'^2 g_{ab} - \epsilon(\Omega'^2 - \alpha'^{-1}) t_a t_b. \tag{A.5}$$

Now, apply a second disformal transformation, characterized by the functions Ω''^2 and α'', obtaining g''_{ab}:

$$g''_{ab} = \Omega''^2 g'_{ab} - \epsilon(\Omega''^2 - \alpha''^{-1}) t'_a t'_b. \tag{A.6}$$

It is easy to see, by explicit calculation, that we can go directly from g_{ab} to g''_{ab} simply multiplying the functions Ω^2 and α:

$$g''_{ab} = (\Omega'\Omega'')^2 g_{ab} - \epsilon\Big[(\Omega'\Omega'')^2 - (\alpha'\alpha'')^{-1}\Big] t_a t_b. \tag{A.7}$$

In other words, if we work with Ω^2 and α, the composition is a simple multiplication at the level of this functions, giving the coefficients of the direct transformation:

$$\Omega''' = \Omega'\Omega'', \quad \alpha''' = \alpha'\alpha''. \tag{A.8}$$

Appendix B

Coefficients Ψ_5, Ψ_8

Here we show how to obtain the form of the coefficients Ψ_5 and Ψ_8 written in the main text, and is explained from where they come from. We follow [19].
Consider a binary made up by two black holes, with masses m_1 and m_2 and total mass $M = m_1 + m_2$. As we said, during the inspiral each of the black hole absorbs a part of the emitted gravitational wave, changing its parameters (mass, angular momentum...) due to the exchange of energy and angular momentum carried by the gravitational waves. This is what is called *tidal heating*. For now, we identify the flux of absorbed energy formally with $F_{\rm H}$, while the flux related to the energy carried to infinity is called F_∞. Now we give an explanation on how to find the form of such flux $F_{\rm H}$, precisely in terms of the rate of change of the masses m_1 and m_2. This turns out to be, compared to the main flux F_∞, a very small effect. Being both the fluxes expansions for small value of parameter v, the smallness of $F_{\rm H}$ means actually that its first order contribution in v has a power very bigger compared to the one of the first order of F_∞. Start from the rate of change, for example, of the mass m_1 with respect to time, which can be computed via the resolution of the Teukolsky equation in the Weyl formalism [19]. We have:

$$\frac{dm_1}{dt} = \left(\frac{dE}{dt}\right)_{\rm N} \left(\frac{m_1}{m}\right)^3 \frac{v^5}{4}\left[-\chi_1(\hat{\rm L}\cdot\hat{\rm J}_1) + 2\frac{v^3 r_{+1}}{m}\right]\left[1 + 3\chi_1^2\right], \tag{B.1}$$

where $(dE/dt)_{\rm N} = (32/5)\eta^2 v^{10}$ is the energy loss due to emission of gravitational waves in the quadrupole approximation, χ_1 is the dimensionless ratio J_1/m_1^2 of BH1, $\hat{\rm L}$ is the orbital angular momentum, $\hat{\rm J}_1$ is the spin of BH1. The form of r_+ is given for rotating black holes by:

$$r_{\pm i} = m_i \pm \sqrt{m_i^2 - \frac{J_i}{m_i}} \tag{B.2}$$

Of course, for BH2 we have the same structure but with the replacement $1 \to 2$. Now, two functions are defined, which are called $A_i^{(5)}$ and $A_i^{(8)}$, where

$i = 1, 2$ label stays for BH1 or BH2. They read:

$$A_i^{(5)} = \left(\frac{m_i}{m}\right)^3 \chi_i(\hat{\mathrm{L}} \cdot \hat{\mathrm{J}}_i)\left[1 + 3\chi_i^2\right],$$
$$A_i^{(8)} = \left(\frac{m_i}{m}\right)^4 \left[1 + \sqrt{1 - \chi_i^2}\right]\left[1 + 3\chi_i^2\right]. \tag{B.3}$$

As we can see, they depend just on ratios, like m_i/m or χ_i. With these definitions, the rate of change of the mass m_i can be written then:

$$\frac{dm_i}{dt} = \left(\frac{dE}{dt}\right)_{\mathrm{N}}\left[-A_i^{(5)}\frac{v^5}{4} + A_i^{(8)}\frac{v^8}{2}\right]. \tag{B.4}$$

Now, since to absorption of energy corresponds surely to a change in the mass of the black hole, and since the above function has the correct dimensions, we identify precisely the power absorbed by each black hole with:

$$F_{\mathrm{H},i} = \frac{dm_i}{dt}. \tag{B.5}$$

In this way, we can define the total absorbed flux absorbed by the two black holes, as:

$$F_{\mathrm{H}} = \sum_i F_{\mathrm{H},i} = \left(\frac{dE}{dt}\right)_{\mathrm{N}}\left[-\Psi_5\frac{v^5}{4} + \Psi_8\frac{v^8}{2}\right], \tag{B.6}$$

where Ψ_5 and Ψ_8 are precisely the coefficients of interest.

Appendix C

Approximation for templates of $\delta\psi_{\rm TH}$

In this appendix, we show better how to identify the term related to tidal heating in the expansion of the fraction containing the two fluxes contributions, namely:

$$\frac{1}{F_\infty + F_{\rm H}}. \tag{C.1}$$

Recall that we have:

$$\begin{aligned} F_\infty(v) &= \frac{32}{5}\eta^2 v^{10}\left[1 - \left(\frac{1247}{336} + \frac{35}{12}\eta\right)v^2 + (4\pi + F_{\rm SO})v^3\right], \\ F_{\rm H}(v) &= \frac{32}{5}\eta^2 v^{10}\left[-\Psi_5\frac{v^5}{4} + \Psi_8\frac{v^8}{2}\right]; \end{aligned} \tag{C.2}$$

where we are considering a generic ratio η and a non-vanishing value of the spin. For the sum, we have:

$$F_\infty(v) + F_{\rm H}(v) = \frac{32}{5}\eta^2 v^{10}\left[1 + C_2 v^2 + C_3 v^3 + C_5 v^5 + C_8 v^8\right], \tag{C.3}$$

without ambiguity about the values of C_i, since each one is connected to the correspondent order in v. We underline that the coefficients coming from tidal heating are C_5 and C_8. Let us define $x \equiv C_2 v^2 + C_3 v^3 + C_5 v^5 + C_8 v^8$, and consider now the fraction:

$$\frac{1}{F_\infty(v) + F_{\rm H}(v)} = \frac{5}{32\eta^2}\frac{1}{v^{10}}\frac{1}{1+x} \simeq \frac{5}{32\eta^2}\frac{1}{v^{10}}(1 - x + x^2 + ...), \tag{C.4}$$

where we expand the denominator at the second order in x. In doing so, since the maximum order of the denominator is 8, we will neglect all the terms which are at an order higher than 8. After having to that, we keep just terms which contain C_5 or C_8 at least one time. We have:

$$\begin{aligned} &1 - x + x^2 \simeq 1 - C_2 v^2 - C_3 v^3 - C_5 v^5 - C_8 v^8 \\ &+ C_2^2 v^4 + C_3^2 v^6 + 2C_2C_3 v^5 + 2C_2C_5 v^7 + 2C_3C_5 v^8 + ..., \end{aligned} \tag{C.5}$$

where we stopped at order 8. Now, let's keep just the terms containing something coming from $F_\mathrm{H}(v)$. They are:

$$-C_5 v^5,\ -C_8 v^8,\ +2C_2C_5 v^7,\ +2C_3C_5 v^8. \tag{C.6}$$

This gives:

$$\frac{1}{F_\infty + F_\mathrm{H}} \to \frac{5}{32\eta^2}\frac{1}{v^{10}}\left[\Psi_5 \frac{v^5}{4} - \Psi_8 \frac{v^8}{2} + \Psi_5 \left(\frac{1247}{336} + \frac{35}{12}\eta\right)\frac{v^7}{2} - \Psi_5(4\pi + F_\mathrm{SO})\frac{v^8}{2}\right], \tag{C.7}$$

this expression is precisely the one considered to find the various templates connected to tidal heating in the text.

Bibliography

[1] T. Antonelli, A. Giusti, R. Casadio, and L. Heisenberg, "Quasinormal modes for coherent quantum black holes," May 2025. arXiv:2505.08415.

[2] D. J. Stargen and D. Kothawala, "Small scale structure of spacetime: van Vleck determinant and equi-geodesic surfaces," *Phys. Rev. D 92, 024046*, Mar. 2015. arXiv:1503.03793.

[3] E. Poisson, A. Pound, and I. Vega, "The motion of point particles in curved spacetime," *Living Reviews in Relativity, Volume 7, Article number 6*, Sept. 2011. arXiv:1102.0529.

[4] D. Kothawala and T. Padmanabhan, "Entropy density of spacetime from the zero point length," *Phys. Lett. B 748: 67-69*, Aug. 2014. arXiv:1408.3963.

[5] D. Kothawala, "Minimal Length and Small Scale Structure of Spacetime," *Phys. Rev. D 88, 104029*, Nov. 2013. arXiv:1307.5618.

[6] D. Kothawala and T. Padmanabhan, "Grin of the Cheshire cat: Entropy density of spacetime as a relic from quantum gravity," *Phys. Rev. D 90, 124060*, Dec. 2014. arXiv:1405.4967.

[7] D. Kothawala, "Intrinsic and Extrinsic curvatures in Finsler-esque spaces," *Gen. Rel. Grav. 46:1836*, Nov. 2014. arXiv:1406.2672.

[8] A. Pesci, "Minimum-length Ricci scalar for null separated events," *Phys. Rev. D 102, 124057*, Jan. 2021. arXiv:1911.04135.

[9] A. Perri, "Minimum length metric and horizon area variation," *AMS Laurea, Institutional Theses Repository*, 2022.

[10] A. Pesci, "Quantum metric for null separated events and spacetime atoms," *Class. Quantum Grav. 36 075009*, Oct. 2019. arXiv:1812.01275.

[11] T. Padmanabhan, "Distribution function of the Atoms of Spacetime and the Nature of Gravity," *Entropy, 17, 7420-7452*, Nov. 2015. arXiv:1508.06286.

[12] T. Padmanabhan, *Gravitation: foundations and frontiers.* Cambridge: Cambridge university press, 2010.

[13] M. Maggiore, *Gravitational waves.* Oxford: Oxford university press, 2017.

[14] S. M. Carroll, *Spacetime and geometry: an introduction to general relativity.* San Francisco Munich: Addison-Wesley, nachdr. ed.

[15] A. Fabbri and J. Navarro-Salas, *Modeling black hole evaporation.* London: Imperial College Press World Scientific Pub, 2005.

[16] S. Isoyama and H. Nakano, "Post-Newtonian templates for binary black-hole inspirals: the effect of the horizon fluxes and the secular change in the black-hole masses and spins," *Class. Quant. Grav. 35 (2018) no.2, 024001*, May 2023. arXiv:1705.03869.

[17] S. Datta, "Tidal heating of Quantum Black Holes and their imprints on gravitational waves," *Phys. Rev. D 102, 064040 (2020)*, May 2021. arXiv:2002.04480.

[18] S. Chakraborty, S. Datta, and S. Sau, "Tidal heating of black holes and exotic compact objects on the brane," *Physical Review D 104, 104001 (2021)*, Nov. 2021. arXiv:2103.12430.

[19] N. V. Krishnendu and S. Chakraborty, "Probing black hole 'charge' from the binary black hole inspiral," *Phys. Rev. D 109, 124047, Published 20 June 2024*, June 2024. arXiv:2402.15336.

[20] S. Mukherjee, S. Datta, S. Tiwari, K. S. Phukon, and S. Bose, "Towards establishing the presence or absence of horizons in coalescing binaries of compact objects by using their gravitational wave signals," *Phys. Rev. D 106 (2022) 10, 104032*, Sept. 2022. arXiv:2202.08661.

[21] W. Tichy, E. E. Flanagan, and E. Poisson, "Can the post-Newtonian gravitational waveform of an inspiraling binary be improved by solving the energy balance equation numerically?," *Phys. Rev. D61 (2000) 104015*, Dec. 1999. arXiv:gr-qc/9912075.

[22] C. K. Mishra, A. Kela, K. G. Arun, and G. Faye, "Ready-to-use post-Newtonian gravitational waveforms for binary black holes with non-precessing spins: An update," *Phys. Rev. D 93, 084054 (2016)*, Jan. 2016. arXiv:1601.05588.

[23] I. Agullo, V. Cardoso, A. d. Rio, M. Maggiore, and J. Pullin, "Potential gravitational-wave signatures of quantum gravity," *Phys. Rev. Lett. 126, 041302*, 2021. arXiv:gr-qc/2007.13761.

[24] J. D. Bekenstein and V. F. Mukhanov, "Spectroscopy of the quantum black hole," *Phys. Lett. B360 (1995) 7-12*, May 1995. arXiv:gr-qc/9505012.