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Corso di Laurea in Matematica

The Heron-Rota-Welsh Conjecture for matroids representable over $\mathbb{C}$

Tesi di Laurea in Matematica

Relatore:
Chiar.mo Prof.
Roberto Pagaria

Presentata da:
Bianca Biagi

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Introduction

Matroid theory originated in 1935, with the article written by Hassler Whitney *On the abstract properties of linear independence* [Whi35], and with the first of three papers dealing with similar ideas by Takeo Nakasawa [Nak35]. The objective of Whitney and Nakasawa, that introduced this theory independently, was to construct a combinatorial object that captured the abstract properties of dependence that are common to linear algebra and graph theory.
The concept of the characteristic polynomial of a matroid originates directly from graph theory. Specifically, the chromatic polynomial of a graph is a polynomial that encodes the number of proper q-colorings for every integer q. The characteristic polynomial of a matroid constitutes a natural generalization of the chromatic polynomial of a graph. Furthermore, the characteristic polynomial of a matroid shares a profound connection with the Poincaré polynomial associated to a hyperplane arrangement - a polynomial which captures essential cohomological information about the complement space. The characterization of polynomials that occur as characteristic polynomials of matroids remains an open problem, motivating numerous conjectures about their properties. Namely, in the 1970s Heron and Rota conjectured that the absolute values of the coefficients of the characteristic polynomial of a matroid (the so-called Whitney's numbers of the first kind) are unimodal [Rot71], [Her72], and Welsh later conjectured that they are log-concave [Wel76]. The conjecture asserting the log-concavity and unimodality of the Whitney's numbers of the first kind for any matroid is known as the Heron-Rota-Welsh conjecture. This conjecture was proved for matroids realizable over $\mathbb{C}$ and then for matroids realizable over any field by June Huh in 2012 [Huh12]. After that article, it was realized that the key for the proof of the conjecture was the validity of the combinatorial analogues of the hard Lefschetz theorem and of the Hodge-Riemann bilinear relations for the so-called Chow ring associated to a matroid.
The notion of the Chow ring for matroids originates in the work of De Concini and Procesi [CP96] in 1996. In their article, they introduced the *wonderful model* associated to a projective arrangement $\mathcal{A}$ of complex linear subspaces. This model consists of a

smooth projective variety $Y_{\mathcal{A}}$ containing the arrangement complement $\mathcal{M} = \mathbb{P}(V) \setminus \mathcal{A}$, with the property that $Y_{\mathcal{A}} \setminus \mathcal{M}$ forms a divisor with simple normal crossings. Furthermore, De Concini and Procesi provided a complete description of the cohomology ring of these wonderful models. In their 2004 works [FK04], [FY04], Feichtner, Koslov, and Yuzvinsky discussed a generalization of the wonderful model for an atomic lattice, using the concepts of building sets, nested sets and combinatorial blowups. Moreover, they introduced the definition of the Chow ring of a lattice with respect to a building set $\mathcal{G}$, which serves as the combinatorial analogue of the cohomology ring of the wonderful model. In particular, for a simple matroid, its Chow ring is defined as the Chow ring of the associated geometric lattice of flats endowed with the maximal building set.
The Heron-Rota-Welsh conjecture was fully solved in 2018 by Adiprasito, Huh and Katz [AHK18]. In that article, the proof of the hard Lefschetz theorem and the Hodge-Riemann relations for general matroids was inspired by an inductive proof of analogous facts for simple polytopes given by McMullen [McM93].

The aim of this thesis is to prove the Heron-Rota-Welsh conjecture in the case of matroids representable over $\mathbb{C}$. The structure of the thesis is the following.
In the first chapter we introduce some preliminary definitions concerning log-concave and unimodal sequences, posets, and lattices. The central focus of this chapter is the introduction of matroids. Namely, we state some of the equivalent axioms that characterize this construction, and develop several aspects of matroid theory, including the notions of rank, closure, flats, and realizability. Moreover, we define the characteristic polynomial and the reduced polynomial of a matroid, and state the Heron-Rota-Welsh conjecture.

The aim of the second chapter is to describe the De Concini-Procesi wonderful model, its generalization by Feichtner, and to define the Chow ring of a lattice and of a matroid. In particular, we first introduce the concept of blowing up a variety along a subvariety, define the wonderful model and state some of its properties. We then study the combinatorial data of the model, giving the combinatorial definitions of building sets, nested sets and combinatorial blowups, and stating some theorems that link these combinatorial definitions to the wonderful model. Furthermore, we define the Chow ring for both finite atomic lattices and simple matroids. In the case of a matroid representable over $\mathbb{C}$, we claim that this construction coincides with the integral cohomology algebra of the associated wonderful model.

In the third chapter, we present the two fundamental theorems required to prove the Heron-Rota-Welsh conjecture: the Hard Lefschetz theorem (HL) and the Hodge-Riemann bilinear relations (HR). We first formulate these theorems in the setting of a Euclidean vector space endowed with a compatible almost complex structure, and subsequently in the context of Kähler manifolds. This chapter focuses on the essential aspects of Hodge theory necessary to prove HL and HR for the cohomology ring of a smooth projective variety. As a consequence, we deduce the validity of these theorems for the cohomology ring of the wonderful model, which is isomorphic to the Chow ring of a matroid representable over $\mathbb{C}$.

Finally, the purpose of the last chapter is to prove the Heron-Rota-Welsh conjecture for a matroid M representable over $\mathbb{C}$, following the proof in [AHK18]. In particular, we link the coefficients of the reduced polynomial of M to the degree of certain elements in the Chow ring of M. The validity of the Hard Lefschetz theorem (HL) and Hodge-Riemann relations (HR) for this ring serves as the crucial ingredient in our argument.

Contents

Chapter 1

Preliminary notions

The aim of this chapter is to introduce fundamental concepts and definitions that will serve as the foundation for our subsequent discussion. Specifically, we present log-concave and unimodal sequences, discuss key definitions related to partially ordered sets (posets) and lattices, and, most importantly, develop the fundamental concepts of matroid theory.

1.1 Log-concave sequences

In this section, we present the definition of log-concave sequences, a class of sequences that arise naturally in combinatorics, algebra, probability, and statistics.

Definition 1.1. A sequence of real numbers $a_0, \ldots, a_n$ is *log-concave* if

$$a_{i-1}a_{i+1} \leq a_i^2 \text{ , for any } 0 < i < n.$$

Definition 1.2. A sequence of real numbers $a_0, \ldots, a_n$ is *unimodal* if there exists $0 \leq k \leq n$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \geq a_{k+1} \geq \cdots \geq a_n.$$

Remark 1.3. It is simple to see that a log-concave sequence of positive real numbers is also unimodal if it does not have any internal zeros, that is, if $a_i \neq 0$ for $0 < i < n$.

Remark 1.4. The sequence $a_0, \ldots a_n$, with $a_i > 0$ for all $i = 1 \ldots n$, is log-concave if and only if for all $k = 2 \ldots n-1$, the matrix $A_k = \begin{pmatrix} a_{k-1} & a_k \\ a_k & a_{k+1} \end{pmatrix}$ has signature (1,1,0) or is degenerate.

Example 1.5 ([HLP11], p.104)**.** A classical result states that any polynomial with positive real coefficients and only real zeros must have log-concave coefficients. This property directly implies the log-concavity of both binomial coefficients and Stirling numbers.

We now want to prove that the property of log-concavity is preserved under the operation of convolution of sequences. Given two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, their *convolution* is the sequence $(c_n)_{n\geq 0}$ defined by $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, for all $n \geq 0$.

Proposition 1.6 (Proposition 2, [Sta06]). *If $A = a_0, \ldots, a_n$ and $B = b_0, \ldots, b_n$ are log-concave sequences with non-negative elements and no internal zeros, then the convolution $A * B$ is also log-concave.*

Proof. Let $X = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ & a_0 & \cdots & a_{n-1} \\ & & \ddots & \vdots \\ & & & a_0 \end{pmatrix}$ $Y = \begin{pmatrix} b_0 & b_1 & \cdots & b_n \\ & b_0 & \cdots & b_{n-1} \\ & & \ddots & \vdots \\ & & & b_0 \end{pmatrix}$.

We note that $XY = \begin{pmatrix} a_0b_0 & a_0b_1 + a_1b_0 & \cdots & \sum_{i+j=n} a_ib_j \\ & a_0b_0 & \cdots & \sum_{i+j=n-1} a_ib_j \\ & & \ddots & \vdots \\ & & & a_0b_0 \end{pmatrix}$.

In particular, for every $a = 0, \ldots, n$, $(XY)_{k,k+a} = \sum_{i+j=a} a_ib_j = (A*B)_a$, for $k = 1, \ldots, n+1-a$. Since A and B are log-concave sequences, for every $k = 2, \ldots, n-1$ we have $a_k^2 \geq a_{k-1}a_{k+1}$ and $b_k^2 \geq b_{k-1}b_{k+1}$. Equivalently, the 2×2 minors

$$\begin{pmatrix} X_{i-1,j} & X_{i-1,j+1} \\ X_{i,j} & X_{i,j+1} \end{pmatrix} \text{ and } \begin{pmatrix} Y_{i-1,j} & Y_{i-1,j+1} \\ Y_{i,j} & Y_{i,j+1} \end{pmatrix},$$

with $i = \frac{n+2-k}{2}$ and $j = \frac{n+2+k}{2}$, are non-negative for $k = \left[\frac{n}{2}\right], \left[\frac{n}{2}\right] + 2 \ldots, n-4, n-2$. The Cauchy-Binet theorem establishes that the same is true for the matrix XY, that is, $(A * B)_k^2 \geq (A * B)_{k-1}(A * B)_{k+1}$ for $k = 2, \ldots, n-1$. □

1.2 Posets and lattices

This section presents the fundamental definitions and terminology concerning partially ordered sets (posets) and lattices.

Definition 1.7. A *poset* (partially ordered set) is a set endowed with a partial order, a relation that is reflexive, antisymmetric and transitive. If every pair of elements is comparable, the order is said to be *total*.

To any partially ordered set P, one may associate an *order complex* $\Delta(P)$, which is an abstract simplicial complex whose vertices correspond to the elements of P, and whose faces are precisely the finite totally ordered subsets, the *chains*, of P. Furthermore, starting from a simplicial complex Δ, we recover a poset structure through its *face poset* $P(\Delta)$, consisting of the nonempty faces ordered by inclusion.

Definition 1.8. A *lattice* $\mathcal{L}$ is a poset in which every pair of elements has a unique *supremum*, the least upper bound $x \vee y$, and a unique *infimum*, the greatest lower bound $x \wedge y$. A *meet semilattice* is a poset in which every pair of elements has a unique infimum, also called *meet*.

Furthermore, in a poset with least element $\hat{0}$, an element a is said to be an *atom* if $a > \hat{0}$ and there is no x such that $\hat{0} < x < a$. A poset is then called *atomistic* if for every $b > \hat{0}$ there is an atom a such that $b \geq a > \hat{0}$.
We also say that an element y of a poset *covers* another element x, if $y > x$ and there is no z such that $y > z > x$.

To every partially ordered set P, one may associate the so-called *rank function* r, a function that maps to every element x of P an integer $r(x)$, such that

- r is compatible with the ordering, namely, if $x < y$, then $r(x) < r(y)$.
- If y covers x, then $r(y) = r(x) + 1$.

A poset that can be equipped with a rank function r is called *graded*. A graded poset is said to be *semimodular* if this function obeys the identity $r(x)+r(y) \geq r(x\vee y)+r(x\wedge y)$.

Definition 1.9. A geometric lattice is a graded finite atomistic and semimodular lattice.

Example 1.10 (Partition lattice)**.** The partition lattice of rank n, denoted by Π_n, is defined as the set of set partitions of $\{1, \dots, n\}$ ordered by reversed refinement.

We note that the partition lattice Π_n is a geometric lattice.

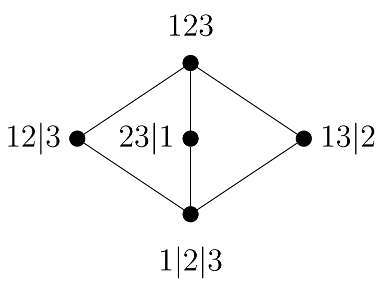


Figure 1.1: *The Hasse diagram of* Π_3.

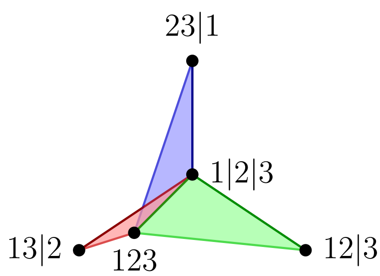


Figure 1.2: *The order complex of* Π_3.

1.3 Matroids

In this section, we present some definitions and results of matroid theory, mostly following [Oxl11].

We first provide the definition of a matroid in terms of independent sets.

Definition 1.11. A matroid M is a pair $(E, \mathcal{I})$, with E a nonempty finite set, and $\mathcal{I} \subseteq \mathcal{P}(E)$ such that:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $J \in \mathcal{I}$ and $I \subset J$, then $I \in \mathcal{I}$.

(I3) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists an element j in $J \backslash I$ such that $I \cup \{j\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called the *independent sets* of M, and E is said to be the *ground set* of M. Moreover, (I3) is called the *exchange property.*

Definition 1.12. A base of a matroid $M = (E, \mathcal{I})$ is a maximal independent set. We note that the third axiom of Definition 1.11 guarantees that any two bases of a matroid must contain the same number of elements. The cardinality of a base is called the *rank* *r(M)* of M.

Proposition 1.13 (Proposition 1.1.1 in [Oxl11])**.** *Let E be the set of column labels of a $m \times n$ matrix over a field $\mathbb{K}$, and let $\mathcal{I}$ be the collection of subsets X of E for which the multiset of columns labeled by X is a linearly independent set over $\mathbb{K}$. Then $\mathcal{I}$ is the collection of independent sets of a matroid on E.*

The matroid $(E, \mathcal{I})$ defined in the last proposition is called *vector matroid.*

Definition 1.14. We say that a matroid $M = (E, \mathcal{I})$ is *representable* (or *realizable*) over a field $\mathbb{K}$ if M is isomorphic to a vector matroid on $\mathbb{K}$, that is, if there exists a $\mathbb{K}$-vector space V and a map $\phi : E \to V$ such that I belongs to $\mathcal{I}$ if and only if $\phi(I)$ is linearly independent in V. We call $\phi(E)$ a *realization* of M .

Some estimates by Knuth in 1974 [Knu74] and then by Nelson in 2018 [Nel18] concerning the cardinality of representable matroids, demonstrate that these constitute a vanishingly small fraction of all matroids. In particular, in [Knu74] Knuth established that the number of non-isomorphic matroid with ground set of cardinality n is at least $\frac{1}{n!}2^{\frac{1}{n}\binom{n}{n/2}-n\log n} \sim 2^{2^n}$, while Nelson in [Nel18] proved that there are at most $2^{n^3/4}$ representable matroids with ground set of cardinality n, for $n \geq 12$. It follows that, as n grows, the proportion of representable matroids among all matroids on a ground set E of n elements tends asymptotically to zero.
We now give an example of a matroid that is not representable over any field.

Example 1.15 (Non-Pappus matroid)**.** Let $M=(E,\mathcal{I})$ be a matroid, where $E = \{1, \dots 9\}$, and the bases are the subsets of 3 elements that do not lie on the same line in Figure 1.3. If M were representable over a field, Pappus's theorem states that the points {7,8,9} have to lie on the same line (Theorem 4.41, pg.38 in [Cox87]). The points then have to be linearly dependent, but this is not true for M. Therefore, the matroid M is not representable over any field.

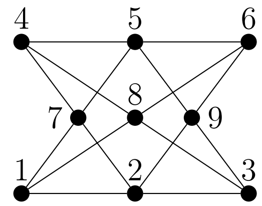


Figure 1.3: *A non-Pappus Matroid.*

We now provide an example of a representable matroid over $\mathbb{C}$: the rank 4 wheel matroid $M(\mathcal{W}_4)$. We first define the class of matroids to which $M(\mathcal{W}_4)$ belongs: the class of graphic matroids.

Proposition 1.16. *Let G be a graph and E its the set of edges. If we denote by $\mathcal{I}$ the collection of edge sets that do not contain any cycles (called the forests of the graph G), then $M = (E, \mathcal{I})$ is a matroid, called the cycle matroid of G, and it is denoted by $M(G)$.*

Proof. To prove that $M = (E, \mathcal{I})$ is a matroid, we show that it satisfies Definition 1.11, that is, $\mathcal{I}$ satisfies the axioms (I1), (I2) and (I3). The axioms (I1) and (I2) are simple to verify: the empty set does not contain any cycle, and the subsets of an acyclic subgraph are acyclic. We now have to verify the exchange property (I3). Let G be a graph, V its set of vertices and E its set of edges. Moreover, let I, J be two forests such that $|I| < |J|$, and let $V(I), V(J)$ be their sets of vertices. We denote by $I_1, \dots, I_{k_1}$ and $J_1, \dots, J_{k_2}$ the connected components respectively of I and J. Since these components are acyclic and connected, they are trees of the graph G, and so the number of edges of each component is equal to the number of vertices of the component minus one. We then have $|I| = |V(I)| - k_1$ and $|J| = |V(J)| - k_2$. Since $|I| < |J|$, then $k_2 - k_1 < V(J) - V(I)$. If $V(J) \subseteq V(I)$, then $V(J) \leq V(I)$ and $k_2 < k_1$. Since J has fewer components than I but more edges, some component of J, that we denote by J_k, intersects with at least two components of I, that we denote by I_1 and I_2. Then, J_k contains an edge $e = (u, v) \notin I$ between I_1 and I_2. Adding e to I does not create a cycle, since u and v were disconnected in I. If $V(J) \nsubseteq V(I)$, there exists a vertex $u \in V(J) \backslash V(I)$. If we pick an edge $e = (u, v)$, adding it to I does not create a cycle: before adding e, u was not connected to any vertex in I. □

A matroid that is isomorphic to the cycle matroid of a graph is called *graphic.*

Definition 1.17. A *wheel graph* of rank n (denoted by $\mathcal{W}_n$) is a graph formed by a cycle of n vertices (whose edges are called *rims*), and a single vertex connecting to each vertex of the cycle through edges called *spikes*.

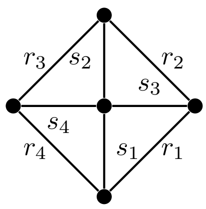


Figure 1.4: *The rank 4 wheel graph* $\mathcal{W}_4$.

We call *wheel matroid* the cycle matroid associated to a wheel graph. We note that this matroid is representable over every field. In fact, all cycle matroids associated to a graph G are representable over any field (Proposition 5.1.2 in [Oxl11]).

We now define the concept of *flats* of a matroid, which serve as the analogue of the linear subspaces generated by sets of vectors in a vector space. Let $M = (E,\mathcal{I})$ be a matroid, we first define the closure operator $cl : \mathcal{P}(E) \to \mathcal{P}(E)$ as

$$cl(X) = \{x \in E : r(X \cup x) = r(X)\}.$$

It is simple to show that $r(X) = r(cl(X))$ (Lemma 1.4.2. pg.26 [Oxl11]).

Lemma 1.18 (Lemma 1.4.3., 1.4.5 in [Oxl11])**.** *The closure operator cl of a matroid on a set E has the following properties:*

(CL1) *If $X \subseteq E$, then $X \subseteq cl(X)$.*
(CL2) *If $X \subseteq Y \subseteq E$, then $cl(X) \subseteq cl(Y)$.*
(CL3) *If $X \subseteq E$, then $cl(cl(X)) = cl(X)$.*
(CL4) *If $X \subseteq E$, $x \in E$, and $y \in cl(X \cup x)\backslash cl(X)$, then $x \in cl(X \cup y)$. This property is called the closure exchange property.*

Moreover, we can consider (CL1)-(CL4) *to be an equivalent set of axioms as* (I1)-(I3) *defined in Definition 1.11. In fact, let E be a set, cl a function from 2^E to 2^E satisfying* (CL1)-(CL4)*, and let*

$$\mathcal{I} = \{X \subseteq E : x \notin cl(X\backslash x) \text{ for all } x \text{ in } X\}.$$

Then, $(E,\mathcal{I})$ is a matroid with closure operator cl.

Definition 1.19. Let $M = (E, \mathcal{I})$ be a matroid. A subset $F \in \mathcal{P}(E)$ is a *flat* if $cl(F) = F$, or equivalently, for any $x \in E \backslash F$

$$r(F \cup x) = r(F) + 1.$$

If $F \subsetneq G$, with F and G flats, we say that G *covers* F if there is no flat X such that $F \subsetneq X \subsetneq G$.

Remark 1.20. It is straightforward to verify that the flats of a matroid M constitute a lattice $\mathcal{L}(M)$, called the *lattice of flats*, ordered by inclusion. We have $X \wedge Y = X \cap Y$ and $X \vee Y = cl(X \cup Y)$.

We can now provide an equivalent definition of a matroid M using flats instead of independent sets.

Definition 1.21. A matroid M is a pair M=$(E, \mathcal{F})$, with E a finite set and $\mathcal{F} \subseteq \mathcal{P}(E)$ the family of flats, such that:

(F1) $E \in \mathcal{F}$.

(F2) If $F, G \in \mathcal{F}$, then $F \cap G \in \mathcal{F}$.

(F3) If $F \in \mathcal{F}$, every element in $E \backslash F$ belongs to one and only one $G \in \mathcal{F}$ that covers F.

In the case of graphic matroids, for example, the flats are all sets of edges such that, adding to the set a generic edge in the graph, the rank of the set grows.

Example 1.22. We describe the lattice of flats of the cycle matroid M associated to the following graph.

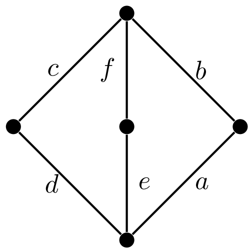


Figure 1.5: *The graph $\mathcal{W}_4$ without the edges s_4 and s_3.*

We note that the rank of M is four, since an independent set (a set that does not contain any cycle) has cardinality at most four. Trivially, the empty set, the six sets consisting of an edge alone, and the ground set E are flats. Moreover, all the sets of two edges are flats: having removed the edges s_3 and s_4, if we add to any pair of edges another generic edge, we do not obtain to a cycle, and so the rank of the set rises. The flats of rank 2 are then $\binom{6}{2}$= 15. Additionally, the flats of rank 3 are of two types: those composed of three edges that do not belong to the same cycle (they are $\binom{6}{3} - 4 \cdot 3 = 8$), and the three cycles $abfe$, $efcd$ and $abcd$. There are then eleven flats of rank three, and so in total the flats are 34 (of which 32 are proper).

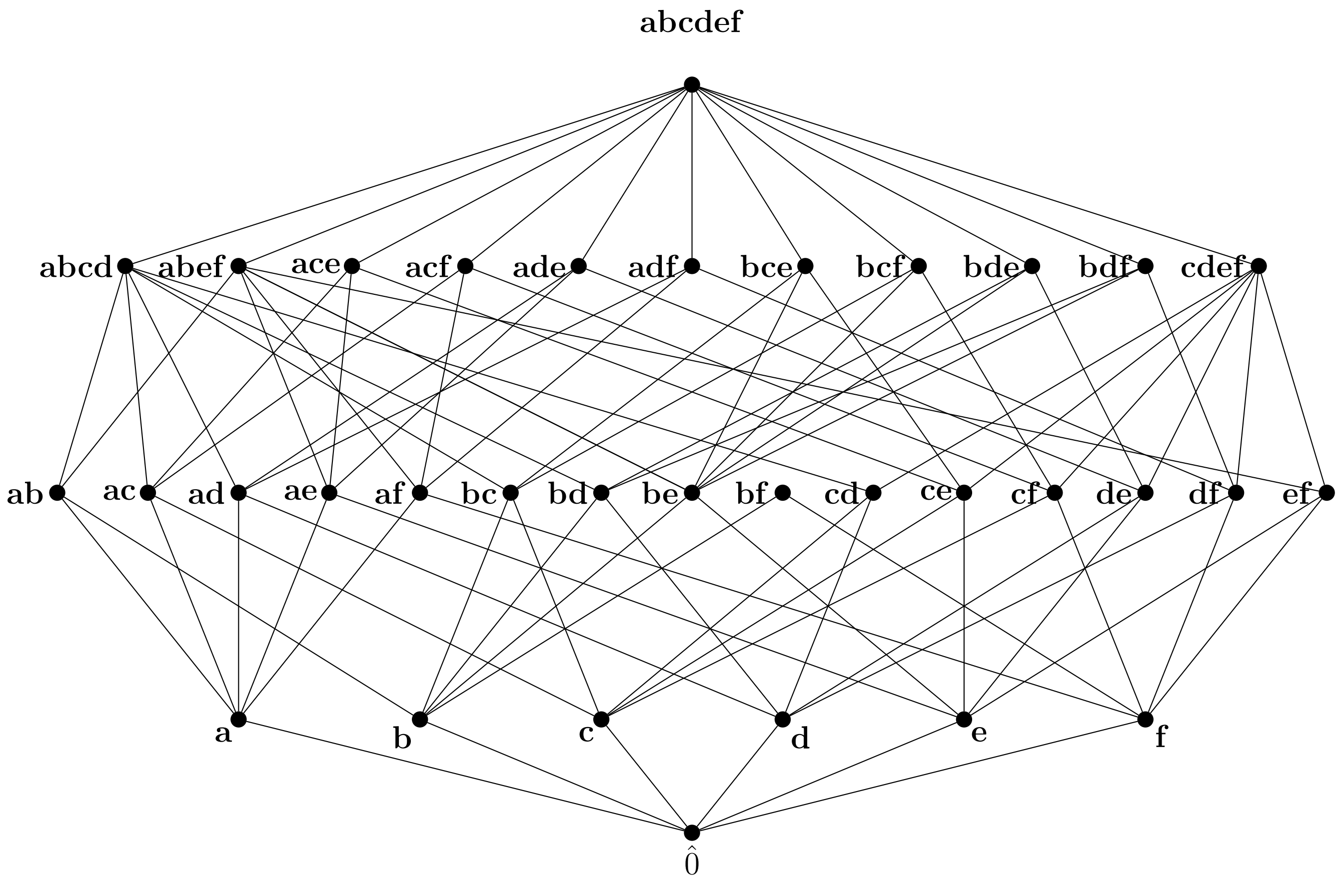


Figure 1.6: *The lattice of flats of M.*

We now prove that Definition 1.11 and Definition 1.21 are, in fact, equivalent. From Lemma 1.18, it follows that the closure axioms (CL1)-(CL4) given in Definition 1.21 are equivalent to the independent sets axioms (I1)-(I4) from Definition 1.11. Hence, establishing the equivalence between (F1)-(F3) and (CL1)-(CL4) completes the proof.

We first define the flats of a matroid through the axioms (F1)-(F3) and fix the closure operator of $X \subseteq E$ as $cl(X) := \bigcap\{F \text{ flat} \mid X \subseteq F\}$, which is the smallest flat containing X. We then prove that the operator we just defined satisfies (CL1)-(CL4):

(CL1) $A \subseteq cl(A)$: the intersection of flats containing A also contains A.

(CL2) If $X \subseteq Y \subseteq E$, then $cl(X) \subseteq cl(Y)$: $cl(Y)$ is a flat for (F2) and contains X since $X \subseteq Y \overset{(CL1)}{\subseteq} cl(Y)$. We then have $cl(X) \subseteq cl(Y)$ for minimality of $cl(X)$.

(CL3) $cl(cl(A)) = cl(A)$: the smallest flat containing $cl(A)$ is $cl(A)$ itself.

(CL4) To prove that if $X \subseteq E$, $x \in E$, and $y \in cl(X \cup x) \backslash cl(X)$, then $x \in cl(X \cup y)$, we first state an equivalent way to express (F3). Let A be a flat and $z \notin A$, then the smallest flat containing $A \cup \{z\}$ covers A (this flat is exactly $cl(A \cup \{z\})$).

We now fix $F = cl(X \cup x)$ and $G = cl(X \cup y)$. We note that G is the minimal flat containing $cl(X)$ and y since $G \overset{(CL1)}{\subseteq} cl(cl(X) \cup \{y\})$. However, F also contains $cl(X)$ and y, so $G \subseteq F$. We then have $cl(X) \subsetneq G \subseteq F$ since $y \in G$ and $y \notin cl(X)$. We also note that $x \notin cl(X)$: if $x \in cl(X)$, then $F \subseteq cl(X)$, that is absurd because $y \in F$ but $y \notin cl(X)$. By (F3), F covers $cl(X)$ ($x \notin cl(X)$ and $F = cl(X \cup \{x\}) = cl(cl(X) \cup \{x\})$), then $G = F$ and $x \in G$.

Assuming conditions (CL1)-(CL4) hold and given the definition of flats in Definition 1.19, we now prove the validity of properties (F1)-(F3):

(F1) $E \in \mathcal{F} : E \overset{(CL1)}{\subseteq} cl(E) \subseteq E$, then $E = cl(E)$.

(F2) If F, G flats, then $F \cap G \in \mathcal{F}$: we have that $F \cap G \overset{(CL1)}{\subseteq} cl(F \cap G) \overset{(CL2)}{\subseteq} cl(F) \cap cl(G) = F \cap G$. Then $F \cap G = cl(F \cap G)$.

(F3) We first prove the existence of a flat G that covers F flat and contains $x \notin F$. We consider $G = cl(F \cup \{x\})$ and show that G covers F. If there exists a flat H such that $F \subsetneq H \subsetneq G$, we find that there exists an element $y \in H \backslash F$. By the exchange property, since $y \in cl(F \cup \{x\}) \backslash cl(F)$, then $x \in cl(F \cup \{y\}) \subseteq cl(H) = H$. Then $H \subsetneq G$ is a flat that contains both F and x, contradicting the minimality of $G = cl(X \cup \{x\})$.
We then prove the uniqueness of such G. Let G and H be two flats that both cover F flat and both contain $x \notin F$. By (F2) $G \cap H$ is a flat, and we also know that it contains both F and x. Then, since $x \notin F$ and $x \in G \cap H$, we get $F \subsetneq G \cap H \subseteq G$. Since G covers F, this implies $G \cap H = G = H$.

Definition 1.23. An element x in a matroid M is a *loop* if $\{x\}$ is a dependent set in M, namely, if x is not contained in any independent set. A matroid M is said to be *loopless* if it does not contain any loops.
Two elements x, y in a matroid M are called *parallel elements* if x and y are not loops but $\{x, y\}$ is a dependent set in M. Moreover, a matroid M is said to be *simple* if it does not contain any loops or parallel elements, that is, if every subset with at most two elements is independent. For example, the matroid M we introduced as the cycle matroid of the graph in Figure 1.5 is simple, since there does not exist any cycle made of one or two edges in this graph.

Theorem 1.24 (Theorem 1.7.5 in[Oxl11])**.** *A lattice $\mathcal{L}$ is geometric if and only if it is the lattice of flats of a matroid. This correspondence is a bijection between the family of finite geometric lattices and the family of simple matroids.*

1.3.1 The characteristic polynomial

In this section, we introduce a key invariant of a matroid M: its *characteristic polynomial* $\mathcal{X}_M$. Throughout this section, $M = (E, \mathcal{I})$ denotes a loopless matroid of rank $r + 1$.

Definition 1.25. The *characteristic polynomial* of M is

$$\mathcal{X}_M(t) = \sum_{A \subseteq E} (-1)^{|A|} \; t^{\; r(M)-r(A)}.$$

The characteristic polynomial $\mathcal{X}_M$ admits an equivalent formulation through the lattice of flats $\mathcal{L}(M)$. Let P be a finite poset, we define recursively the Möbius function associated to P as the map $\mu_P : \; P \times P \to \mathbb{Z}$ such that

$$\mu_P(F, G) = \begin{cases} 0 & \text{if } F \not\subseteq G \\ 1 & \text{if } F = G \\ -\sum_{F \subseteq A \subsetneq G} \; \mu(F, A) & \text{otherwise.} \end{cases} \tag{1.1}$$

We can now reformulate Definition 1.25 in terms of the Möbius function $\mu := \mu_{\mathcal{L}(M)}$ associated to the lattice of flats $\mathcal{L}(M)$ ([Kat16] Theorem 7.12), resulting in

$$\mathcal{X}_M(t) = \sum_{F \in \mathcal{L}(M)} \mu(\emptyset, F) \; t^{r(M)-r(F)}. \tag{1.2}$$

Moreover, from Lemma 7.11 in [Kat16], it follows that, for any $i \in F$,

$$\mu(\emptyset, F) = -\sum_{i \notin G \lessdot F} \mu(\emptyset, G), \tag{1.3}$$

where $G \lessdot F$ means that $G \subsetneq F$ and $r(G) = r(F) - 1$. Using (1.3) and induction on the rank of F, we have that

$$(-1)^{r(F)} \mu(\emptyset, F) > 0. \tag{1.4}$$

In fact

$$(-1)^{r(F)} \mu(\emptyset, F) = (-1)^{r(F)+1} \sum_{i \notin G \lessdot F} \mu(\emptyset, G) = \sum_{i \notin G \lessdot F} (-1)^{r(G)} \mu(\emptyset, G) > 0.$$

We now evaluate (1.2) in $-t$, and get

$$\mathcal{X}_M(-t) = \sum_{F \in \mathcal{L}(M)} \mu(\emptyset, F)(-t)^{r(M)-r(F)} = \sum_{F \in \mathcal{L}(M)} \mu(\emptyset, F)(-1)^{r(F)}(-1)^{r(M)} \; t^{r(M)-r(F)}$$

$$(-1)^{r(M)} \mathcal{X}_M(-t) = \sum_{F \in \mathcal{L}(M)} (-1)^{r(F)} \mu(\emptyset, F) \; t^{r(M)-r(F)}.$$

We fix

$$w_k := \sum_{r(F)=k} (-1)^{r(F)} \mu(\emptyset, F) > 0 \tag{1.5}$$

for Equation (1.4). The absolute values of the coefficients of the characteristic polynomial $w_0, \dots, w_{r+1}$ are known as the *Whitney's numbers of the first kind.* We then get

$$(-1)^{r(M)} \mathcal{X}_M(-t) = w_0 t^{r+1} + w_1 t^r + \cdots + w_r,$$
$$\mathcal{X}_M(t) = w_0 t^{r+1} - w_1 t^r + \dots (-1)^{r+1} w_{r+1}. \tag{1.6}$$

Example 1.26. We now compute the characteristic polynomial of the matroid M which we defined as the cycle matroid of the graph in Figure 1.5. We recall that, by Equation (1.2), $\mathcal{X}_M(t) = \sum_{F \in \mathcal{L}(M)} \mu(\emptyset, F)\ t^{r(M)-r(F)}$. We then have to compute $\mu(\emptyset, F)$ for each flat F, using Equation (1.1). For each flat of rank one we have $\mu(\emptyset, F) = -\mu(\emptyset, \emptyset) = -1$. Moreover, each flat F of rank two contains exactly two flats of rank one, so $\mu(\emptyset, F) = -(1-2) = 1$. We now compute $\mu(\emptyset, F)$ when the rank of F is three. If F is made of three edges, it contains three flats of rank one and three flats of rank two, thus $\mu(\emptyset, F) = -(1-3+3) = -1$. If F is instead one of the three cycles, it contains four flats of rank one and six flats of rank two, so $\mu(\emptyset, F) = -(1-4+6) = -3$. The coefficients of the characteristic polynomial are given by the expression $a_r = \sum_{r(F)=r} \mu(\emptyset, F)$, with $r = 0, \dots, 4$. We then get $a_0 = 1$, $a_1 = -6$, $a_2 = 15$, $a_3 = -8 - 3 \cdot 3 = -17$, and $a_4 = -(1 - 6 + 15 - 17) = 7$. Then, $\mathcal{X}_M(t) = t^4 - 6t^3 + 15t^2 - 17t + 7$.
To obtain the sequence of Whitney's numbers of the first kind, we take the absolute values of $a_0, \dots, a_4$ and get $\{1, 6, 15, 17, 7\}$. We notice that this sequence is log-concave and unimodal.

This is a general fact, as stated in the following conjecture.

Conjecture 1.27 ([Rot71],[Her72],[Wel76])**.** *The Whitney's numbers of the first kind of a matroid M are log-concave and unimodal.*

We now divide (1.2) by $t-1$, since $\mathcal{X}_M(1) = \sum_{A \subseteq E} (-1)^{|A|} = 0$, as the subsets of E are exactly 2^E. We have

$$\tilde{\mathcal{X}_M}(t) = \frac{\mathcal{X}_M(t)}{t-1} = \sum_{k=0}^{r} (-1)^k \mu^k(M)\ t^{r-k}, \tag{1.7}$$

which is called the *reduced polynomial* of a matroid.
We want to study the sequence of integers $\mu^0(M), \mu^1(M), \dots \mu^r(M)$.

Remark 1.28. The first number in this sequence is 1, the last number is $(-1)^{r+1}\mu(\emptyset, E)$, and in general $\mu^k(M)$ is the alternating number of the Whitney's numbers of the first kind:

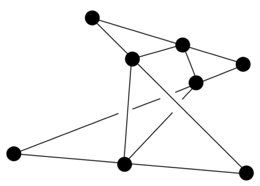
$$\mu^k(M) = w_k - w_{k-1} + \cdots + (-1)^k w_0.$$

If we prove that the sequence of the coefficients $\mu^0(M), \dots \mu^r(M)$ of the reduced polynomial is log-concave, we get Conjecture 1.27, since the convolution of two log-concave sequences is log-concave by Proposition 1.6.

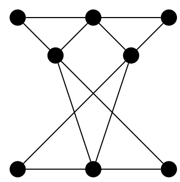
Example 1.29. The reduced polynomial of the matroid M, which we defined as the cycle matroid associated to the graph in Figure 1.5, is $\tilde{\mathcal{X}}_M(t) = t^3 - 5t^2 + 10t - 7$. Then, $\mu^0(M) = 1$, $\mu^1(M) = 5$, $\mu^2(M) = 10$, and $\mu^4(M) = 7$. We note that this sequence is again log-concave and unimodal.

We now define the truncation of a matroid M, a construction particularly useful for inductive arguments.

Definition 1.30. Let $M = (E, \mathcal{I})$ be a matroid of rank $r + 1$. The *truncation* of M, denoted by $tr(M)$, is the matroid on E whose lattice of flats is obtained from the lattice of flats of M by removing all the flats of rank r. The truncation $tr(M)$ has rank function given by $\text{rk}_{tr(M)}(X) := \min(\text{rk}_M(X), r)$, for all $X \subseteq E$. Consequently, $tr(M)$ has rank r.



(a) *Geometric representation of $\mathcal{W}_4$ over $\mathbb{R}$, pg. 651 in [Oxl11].*



(b) *Geometric representation of $tr(\mathcal{W}_4)$ over $\mathbb{R}$.*

Remark 1.31. The coefficients of the reduced characteristic polynomials of M and of $tr(M)$ satisfy the equality:

$$\mu^k(M) = \mu^k(tr(M)),$$

for $0 \leq k < r$.

Indeed, if $w_1, \dots, w_k$ are the Whitney's numbers of the first kind of M and $w'_1, \dots w'_n$ the Whitney's numbers of the first kind of $tr(M)$, we get that $w_k = w'_k$ for $k < r$. In fact, by Equation (1.5),

$$w_k = \sum_{r(F)=k} (-1)^{r(F)} \mu(\emptyset, F) = w'_k$$

for $k < r$, since the flats of M and of $tr(M)$ coincide if their rank is not r. The assertion then follows from Remark 1.28.

Example 1.32. We consider again the matroid M which we defined as the cycle matroid associated to the graph in Figure 1.5. We get that the characteristic polynomial of $tr(M)$ is $t^3 - 6t^2 + 15t - 10$, and its reduced polynomial is $t^2 - 5t + 10$. We observe that the Whitney's numbers of the first kind of M indeed coincide with the Whitney's numbers of the first kind of $tr(M)$ for $k = 0, 1, 2$, and also that $\mu^k(M) = \mu^k(tr(M))$ for $k = 0, 1, 2$. Moreover, we see that these sequences are again log-concave and unimodal.

Chapter 2

De Concini-Procesi wonderful model

In this chapter, we present an important construction in the study of arrangements of linear subspaces: the De Concini-Procesi wonderful model, introduced in 1995 in [CP96]. We also examine the combinatorial structure of the model, present a combinatorial generalization, and introduce the Chow ring associated with a lattice.

2.1 Arrangements

Definition 2.1. An *arrangement* $\mathcal{A} = \{U_1, \dots U_n\}$ is a finite family of linear subspaces in a vector space V. Given an arrangement $\mathcal{A}$, we want to study the complement of $\mathcal{A}$ in the ambient space: $\mathcal{M}(\mathcal{A}) = V \backslash \mathcal{A}$. When V is a real vector space, the study of $\mathcal{M}(\mathcal{A})$ is less intricate, as the complement simply decomposes into a collection of open polyhedral cones. However, when V is endowed with a complex structure, the topology of $\mathcal{M}(\mathcal{A})$ becomes significantly more challenging to characterize.

Definition 2.2. The intersection lattice $\mathcal{L}(\mathcal{A})$ associated to an arrangement $\mathcal{A}$ is the set of intersections of subspaces in $\mathcal{A}$ ordered by reverse inclusion. It is straightforward to see that this is a lattice, with least element V (denoted by $\hat{0}$) and maximum element $\emptyset$ (denoted by $\hat{1}$). Moreover, the elements of $\mathcal{L}(\mathcal{A})$ are often labeled by the codimension of the corresponding intersection.

Example 2.3 (Braid arrangements)**.** *The braid arrangement of rank* $n-1$, denoted by $\mathcal{A}_{n-1}$, is given by the hyperplanes

$$H_{ij} : x_i = x_j \text{ , for } 1 \leq i < j \leq n.$$

The intersection lattice of $\mathcal{A}_{n-1}$ is the *partition lattice* Π_n, as defined in Example 1.10. Indeed, there is a bijection between the elements of Π_n and the elements of $\mathcal{L}(\mathcal{A}_{n-1})$: the blocks of a partition in Π_n correspond to the sets of coordinates with identical entries in $\mathcal{L}(\mathcal{A}_{n-1})$.

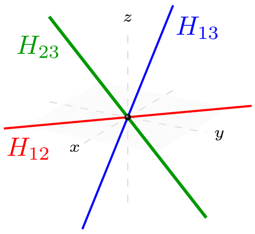


Figure 2.1: *The rank 2 braid arrangement $\mathcal{A}_2$.*

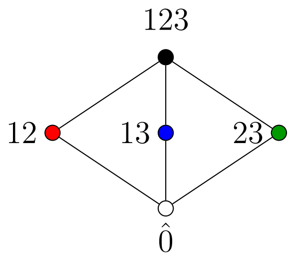


Figure 2.2: *The intersection lattice of $\mathcal{A}_2$.*

Definition 2.4. A *(Cartier) divisor with normal crossings* is a collection of varieties that can locally be defined by one equation, and locally intersect like hyperplanes.

A divisor with *simple* normal crossings is a divisor with normal crossings in which the varieties intersect like coordinate hyperplanes (in codimension k, at most k hyperplanes can intersect).

To better understand arrangements of hyperplanes and their complements, we introduce *arrangement models*, that is, we alter the ambient space preserving the complement and replacing the arrangement by a divisor with simple normal crossings.

2.2 Blowups

To construct the wonderful model, we employ a fundamental operation: the *blowup*. This technique allows us to transform an arrangement of linear subspaces into a divisor with simple normal crossings. In this section, we follow [Smi+00].
We begin by defining the blowup of the affine space at a point. The idea of blowing up the affine space $\mathbb{A}^n$ at a point p is to leave $\mathbb{A}^n$ unaltered except at the point p, which is replaced by the set of all lines through p.

Definition 2.5. Let us choose a coordinate system for $\mathbb{A}^n$ such that p can be assumed to be the origin. Let $B = \{(x,l) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid x \in l\} \in \mathbb{A}^n \times \mathbb{P}^{n-1}$. The *blowing up morphism* of $\mathbb{A}^n$ at p is the natural projection

$$\begin{aligned} B &\xrightarrow{\pi} \mathbb{A}^n \\ (x,l) &\longmapsto x. \end{aligned}$$

We observe that the fiber of π over any point other than the origin is the single point (x,l), where l is the only line through x and the origin; however, the fiber over the origin is a copy of $\mathbb{P}^{n-1}$. We define the *blowup* of $\mathbb{A}^n$ over p the variety B together with the blowing up morphism, and we denote B by $Bl_p\mathbb{A}^n$.

There is an equivalent way to define the blowup up along a point p. We introduce the map

$$\mathbb{A}^n\backslash\{0\} \xrightarrow{\ell} \mathbb{P}^{n-1}$$
$$x = (x_1, \dots, x_n) \longmapsto \ell(x) = [x_1 : \dots : x_n],$$

that attaches to each point $x \in \mathbb{A}^n\backslash\{0\}$ the line through 0 and x. The blowup of $\mathbb{A}^n$ along p is the Zariski closure of the graph of the function ℓ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$:

$$Bl_p\mathbb{A}^n = \overline{\{(x, \ell(x)) \in \mathbb{A}^n \times \mathbb{P}^{n-1}\}}.$$

We can now define the blowup of an arbitrary affine algebraic variety.

Definition 2.6. Let $V \subset \mathbb{A}^n$ be an affine algebraic variety and p a point of V. The blowup of V at p is the Zariski closure of the preimage $\pi^{-1}(V\backslash\{p\})$ in the variety B obtained by blowing up p in $\mathbb{A}^n$, together with the natural projection π to V. We denote the blowup of V at p by $Bl_p(V)$.

Since $Bl_p(\mathbb{A}^n) \xrightarrow{\pi} \mathbb{A}^n$ is an isomorphism when restricted to $Bl_p(\mathbb{A}^n)\backslash\pi^{-1}(p)$, the restriction of π to $Bl_p(V)\backslash\pi^{-1}(p)$ is an isomorphism onto $V\backslash\{p\}$.
We now define the blowup of an algebraic affine variety X along an irreducible subvariety Y.

Definition 2.7. Let $F_1, \dots, F_r$ be functions on the coordinate ring $\mathbb{C}[X]$ of an irreducible affine algebraic variety X, and let I be the ideal they generate. We assume that I is a proper non-zero ideal of $\mathbb{C}[X]$. The blowup of the variety X along the ideal I is the graph B of the rational map

$$X \overset{F}{\dashrightarrow} \mathbb{P}^{r-1}$$
$$x \mapsto [F_1(x) : \dots : F_r(x)],$$

together with the natural projection map $B \subset X \times \mathbb{P}^{r-1} \xrightarrow{\pi} X$. The blowing up of X along I is denoted by $Bl_I(X)$. We see that the projection π defines an isomorphism of quasi-projective varieties between the open sets

$$Bl_I(X)\backslash\pi^{-1}(Y) \to X\backslash Y,$$

with Y the closed set in X defined by the vanishing of $F_1, \dots F_r$. Indeed, the rational map

$$X \dashrightarrow B_I(X)$$
$$x \mapsto (x, [F(x)]),$$

is an inverse of the blowing up map, proving that X and $Bl_I(X)$ are birationally equivalent varieties. The isomorphism class of the blowup does not depend on the choice of the generators but only on the ideal I ([Gat02], Chapter 9, Lemma 9.16).

Definition 2.8. Let Y be an irreducible algebraic subvariety of an affine algebraic variety X. The blowup of X along the subvariety Y is the blowup along the radical ideal $\mathbb{I}(Y)$. We denote this by $Bl_Y(X)$.

The restriction to affine varieties made in the above definitions is not necessary. Indeed, these definitions extend naturally to the case of quasi-projective variety $X \subseteq \mathbb{P}^n$. Moreover, it is possible to blowup along any subvariety Y in any variety X ([Har77], Chapter II, Section 7). The fundamental idea involves replacing X by all directions normal to it.
We now consider $B = Bl_Y X$ with the blowing up morphism π. We define the *exceptional divisor* as $\pi^{-1}(Y)$, the *total transform* of a subspace Z as $\pi^{-1}(Z)$, and the *proper transform* of Z as $\overline{\pi^{-1}(Z \backslash Y)}$.

2.3 The construction of the model

In this and in the following section we will mostly follow [Fei04].
The De Concini-Procesi wonderful model admits two equivalent definitions.

Definition 2.9. Let $\mathcal{A}$ be an arrangement of linear subspaces in a real or complex vector space V. We define the *De Concini-Procesi wonderful model* for $\mathcal{A}$, and we denote it by $Y_{\mathcal{A}}$, the closure of the image of the open embedding

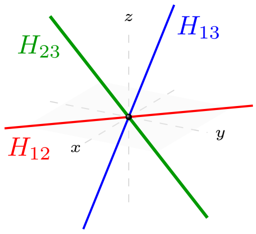
$$\begin{aligned} \Psi :\ \mathcal{M}(\mathcal{A}) &\longrightarrow V \times \prod_{X \in \mathcal{L}(\mathcal{A})_{>\hat{0}}} \mathbb{P}(V \backslash X) \\ x &\longmapsto (x, (\langle x, X\rangle / X)_{X \in \mathcal{L}(\mathcal{A})_{>\hat{0}}}). \end{aligned} \tag{2.1}$$

Definition 2.10. Let $\mathcal{A}$ be an arrangement of linear subspaces in a real or complex vector space V. We consider $X_1, \ldots X_t$ a *linear extension* of the opposite order on $\mathcal{L}(\mathcal{A})_{>\hat{0}}$, that is, a total order that extends the partial order preserving the already existing order relations. The *De Concini-Procesi wonderful model* for $\mathcal{A}$ is obtained successively by blowing up the subspaces $X_1, \ldots X_t$, respectively their proper transforms.

Theorem 2.11 (Thm. 3.1, Thm. 3.2. in [CP96])**.**

1. *The arrangement model $Y_{\mathcal{A}}$ is a smooth variety with a natural projection map $\pi :\ Y_{\mathcal{A}} \longrightarrow V$, which is bijective on $\mathcal{M}(\mathcal{A})$.*
2. *The complement of $\pi^{-1}(\mathcal{M}(\mathcal{A}))$ in $Y_{\mathcal{A}}$ is a divisor with simple normal crossings, and its irreducible components D_X are the proper transforms of elements X in $\mathcal{L}(\mathcal{A})$.*
3. *The irreducible components D_X for $X \in \mathcal{S} \subset \mathcal{L}_{>\hat{0}}$ intersect if and only if $\mathcal{S}$ is a linearly ordered subset in $\mathcal{L}_{>\hat{0}}$.*

Example 2.12 (The wonderful model $Y_{\mathcal{A}_2}$)**.**



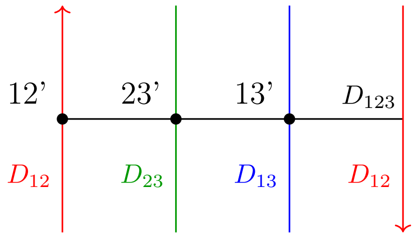


Figure 2.3: *Wonderful model of $\mathcal{A}_2$.*

We observe that in this example, to obtain the wonderful model it is sufficient to blowup in $\{0\}$.

Example 2.13 (Wonderful model of a simple hyperplane arrangement)**.** Let ℓ and r be two distinct lines in $\mathbb{C}^3$ that intersect in 0. While two lines that intersect in a point in $\mathbb{C}^2$ give rise to a simple normal crossing divisor, two lines in $\mathbb{C}^3$ have codimension two, hence they do not form a divisor. We then have to blowup in order to obtain the wonderful model of this arrangement. We first blowup $\mathbb{C}^3$ along the line ℓ and get

$$Y_1 := Bl_\ell \mathbb{C}^3 = \{\ (x,P) \mid P \supseteq \langle x, \ell \rangle \} \subseteq \mathbb{C}^3 \times \mathbb{P}^1,$$

together with the blowing up morphism $\pi_1 : Bl_\ell\mathbb{C}^3 \longrightarrow \mathbb{C}^3$, $\pi_1(x,P) = x$. In the equation above, we have identified $\mathbb{P}^1$ with all the planes of $\mathbb{C}^3$ containing the line ℓ. The exceptional divisor is $E_1 \simeq \pi_1^{-1}(\ell) = \ell \times \mathbb{P}^1$, the proper transform of r is

$$\widetilde{r} = \overline{\pi_1^{-1}(r \backslash \ell)} = \{(x,P) \mid x \in r\backslash\{0\};\ P \supseteq \langle \ell, x\rangle\} \ \cup\ \{(0,P) \mid P = \langle \ell, r\rangle\},$$

and the total transform of r is $\pi^{-1}(r) \simeq r \times \mathbb{P}^1$. We now blowup Y_1 along $\tilde{r}$ and get

$$Y_2 := Bl_{\tilde{r}}(Bl_\ell\mathbb{C}^3) = \left\{ (x,P,Q) \;\middle|\; \begin{array}{ll} l \subset P, & x \in P, \\ r \subset Q, & x \in Q \end{array} \right\},$$

with the blowing up morphism $\pi_2 : Bl_{\tilde{r}}Y_1 \to Y_1$, $\pi_2(x,P,Q) = (x,P)$. The exceptional divisor is

$$E_2 = \pi_2^{-1}(\tilde{r}) = \{\ (x,P,Q) \in Y_2 \mid x \in r\} \simeq \tilde{r} \times \mathbb{P}^1,$$

and the proper transform of E_1 is

$$\tilde{E}_1 := \overline{\pi_2^{-1}(E_1/\tilde{r})} = \{\ (x,P,Q) \in Y_2 \mid x \in \ell,\ \ell \subset Q\}.$$

We then get

$$\tilde{E}_1 \cap E_2 = \{\ (0,P,Q) \mid \ell \subset P;\ \ r,\ell \subset Q\} \simeq \mathbb{P}^1.$$

We obtain that $\tilde{E}_1$ and E_2 form a divisor in Y_2 and that they locally intersect as two hyperplanes: Y_2 is the wonderful model of this arrangement.

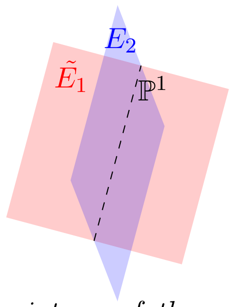


Figure 2.4: *A local picture of the wonderful model of two lines in* $\mathbb{C}^3$.

2.4 The combinatorial data

We now present the combinatorial definitions of *building sets*, *nested sets* and *combinatorial blowups* for a finite lattice $\mathcal{L}$. Using these constructions, we establish a combinatorial generalization of the De Concini-Procesi wonderful model and define the graded algebra $D(\mathcal{L}, \mathcal{G})$.

Definition 2.14. Let $\mathcal{L}$ be a finite meet-semilattice. A *combinatorial building set* $\mathcal{G}$ is a subset $\mathcal{G} \subseteq \mathcal{L}_{>\hat{0}}$ such that, for any $X \in \mathcal{L}_{>\hat{0}}$ and $\max \mathcal{G}_{\leq X} = \{G_1, \ldots G_n\}$, there is an isomorphism of posets

$$\varphi_X : \prod_{j=1}^{n} [\hat{0}, G_j] \longrightarrow [\hat{0}, X], \qquad (2.2)$$

with $\varphi_X(\hat{0}, \ldots, G_j, \ldots, \hat{0}) = G_j$ for $j = 1, \ldots, n$, and $\mathcal{G}_{\leq X}$ the *set of factors* of X in $\mathcal{G}$: the elements in $\mathcal{G}$ there are less than or equal to X.
The building sets represent the subsets to be blown up in the construction of the wonderful model.

We now relate this definition to the context of the De Concini-Procesi wonderful model, introducing the notion of *geometric building set.*

Definition 2.15. Let $\mathcal{L}$ be the intersection lattice of an arrangement of subspaces in a vector space V, and cd: $\mathcal{L} \to \mathbb{N}$ the function assigning to every element of $\mathcal{L}$ the codimension of the corresponding subspace. A subset $\mathcal{G}$ in $\mathcal{L}$ is a *geometric building set* if it is a combinatorial building set such that, for any $X \in \mathcal{L}$, the codimension of X is equal to the sum of the codimensions of its factors: $\mathrm{cd}(X) = \sum_{j=1}^{n} \mathrm{cd}(G_j)$.

Example 2.16 (Boolean lattice)**.** We define the boolean lattice (or boolean algebra) on three elements as the following lattice.

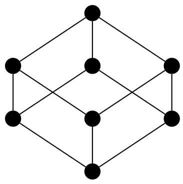


Figure 2.5: *The boolean lattice on 3 elements.*

We note that the atoms form the minimal combinatorial building set $\mathcal{G}_{\min}$. We see that, if we consider the arrangement of hyperplanes $A : x_1 = 0,\ B : x_2 = 0$ and $C : x_3 = 0$ in $\mathbb{R}^3$, the boolean lattice is the intersection lattice of this arrangement. Moreover, if we consider the arrangement of hyperplanes in $\mathbb{R}^4$ $A : x_1 = 0,\ B : x_2 = x_3 = 0$ and $C : x_3 = x_4 = 0$, the boolean lattice is also the intersection lattice of this arrangement. We now label each vertex of the lattice with the codimension of the corresponding intersection of hyperplanes.

We see that in the first case, $\mathcal{G}_{\min}$ is also a geometric building set, while in the second arrangement of hyperplanes, $\mathcal{G}_{\min}$ is not a geometric building set, since the codimension of $B \cap C$ is not the sum of the codimensions of B and C. In fact, the minimal geometric building set in this case is $\mathcal{G}_{\min} \cup (B \cap C) \cup \hat{1}$.

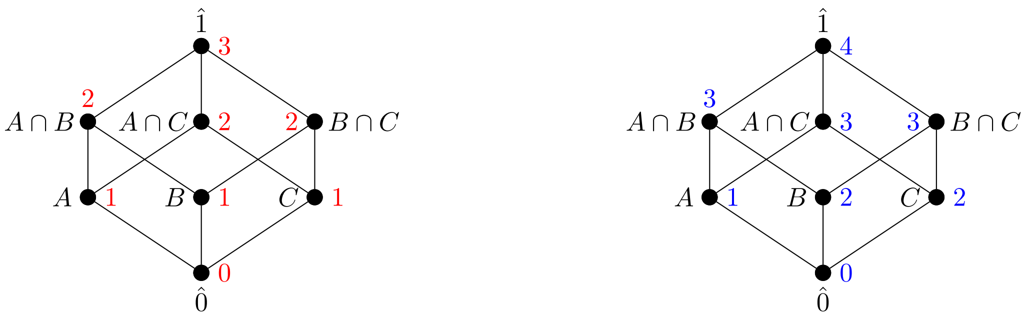


Figure 2.6: *Intersection lattices of two arrangements.*

Moreover, every building set $\mathcal{G}$ gives rise to a family of *nested sets*, that in the context of the construction of the wonderful model represent the non-empty intersections of irreducible divisors components.

Definition 2.17. Let $\mathcal{L}$ be a finite meet-semilattice and $\mathcal{G}$ a combinatorial building set. A subset $\mathcal{S}$ in $\mathcal{G}$ is *nested* if, for any set $\{X_1, \dots X_k\}$ of incomparable elements in $\mathcal{S}$ with cardinality at least two, $X_1 \vee \cdots \vee X_k$ exists and does not belong to $\mathcal{G}$. The nested sets in $\mathcal{G}$ form an abstract simplicial complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ on the vertex set $\mathcal{G}$, called *the nested set complex.*

Example 2.18 (Partition lattice)**.** It is easy to see that the minimal building set $\mathcal{G}_{\min}$ of the partition lattice Π_n is given by the partitions with one block consisting of more than one element: they are all the elements X of Π_n that do not allow for a product decomposition of $[\hat{0}, X]$. A subset of such partitions is nested if, for any pair of elements, either they are disjoint (incomparable and their join does not belong to $\mathcal{G}$) or one is contained in the other (they are comparable). In fact, if two elements intersect non-trivially, they do not belong to any $\mathcal{G}_{\min}$-nested set (they are incomparable but their join belongs to $\mathcal{G}_{\min}$).

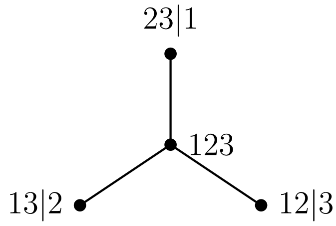
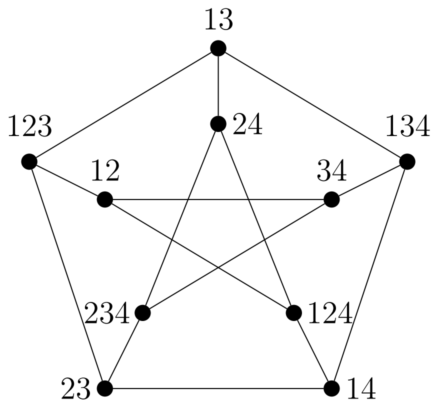


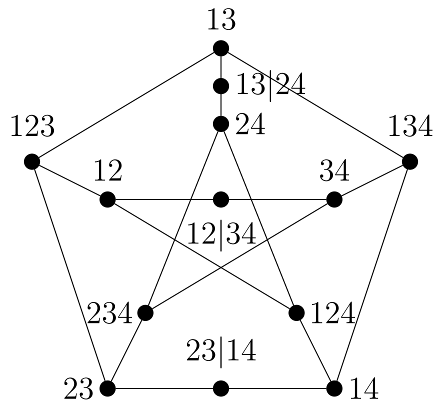
Figure 2.7: *The nested set complex* $\mathcal{N}(\mathcal{L}(\mathcal{A}_2), \mathcal{G})$.

We observe that the nested set complex $\mathcal{N}(\mathcal{L}(\mathcal{A}_2), \mathcal{G})$ corresponds exactly to the order complex of $\mathcal{L}(\mathcal{A}_2)_{>\hat{0}}$.

Let us now consider a less trivial example: the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$, with $\mathcal{L}$ the intersection lattice of $\mathcal{A}_3$ and $\mathcal{G} = \mathcal{G}_{\min}$.



(a) *The base of the complex* $\mathcal{N}(\mathcal{L}(\mathcal{A}_3), \mathcal{G}_{\min})$.



(b) *The base of the complex* $\Delta(\mathcal{L}(\mathcal{A}_3)_{>\hat{0}})$.

Figure 2.8: *Comparison between the bases of* $\mathcal{N}(\mathcal{L}(\mathcal{A}_3), \mathcal{G}_{\min})$ *and of* $\Delta(\mathcal{L}(\mathcal{A}_3)_{>\hat{0}})$.

We note that the nested set complex $\mathcal{N}(\mathcal{L}(\mathcal{A}_3), \mathcal{G}_{\min})$ and the order complex of $\mathcal{L}(\mathcal{A}_3)_{>\hat{0}}$ are two cones with $\hat{1}$ as the vertex and with the bases depicted in Figure 2.8. We observe that also in this example, the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ is homotopy equivalent to the order complex of $\mathcal{L}_{>\hat{0}}$. This is a general fact, as stated in the following theorem.

Theorem 2.19 (Prop 3.3 in [FM05])**.** *Let $\mathcal{L}$ be a finite meet-semilattice and $\mathcal{G}$ a building set. The nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ is homotopy equivalent to the order complex of $\mathcal{L}_{>\hat{0}}$*

$$\mathcal{N}(\mathcal{L}, \mathcal{G}) \simeq \Delta(\mathcal{L}_{>\hat{0}}).$$

We can also give a combinatorial definition of the blowup of a semilattice $\mathcal{L}$ along an element X.

Definition 2.20. Let $\mathcal{L}$ be a semilattice and X an element in $\mathcal{L}_{>\hat{0}}$. The *combinatorial blowup* $(Bl_X\mathcal{L}, \prec)$ is a poset (in fact again a semilattice) such that

$$Bl_X\mathcal{L} = \{Y | Y \in \mathcal{L}, Y \ngeq X\} \cup \{Y' | Y' \in \mathcal{L}, Y' \ngeq X, \text{ and } Y' \vee X \text{ exists in } \mathcal{L}\}.$$

The order relation $\prec$ is defined as

$$Y \prec Z, \text{ for } Y < Z \text{ in } \mathcal{L},$$
$$Y' \prec Z', \text{ for } Y < Z \text{ in } \mathcal{L},$$
$$Y \prec Z', \text{ for } Y < Z \text{ in } \mathcal{L}.$$

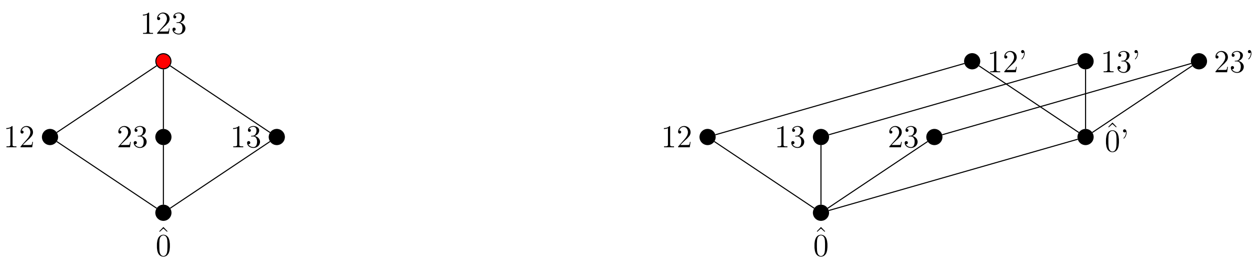
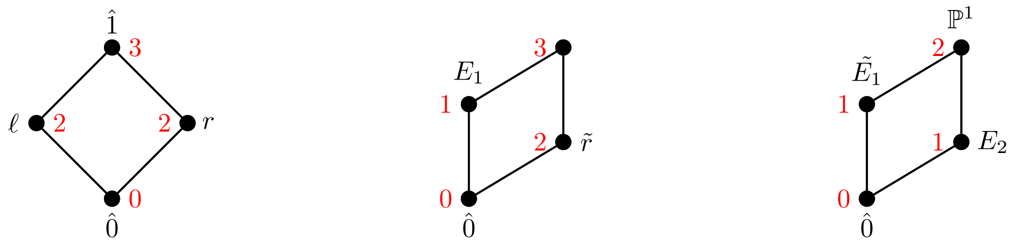


Figure 2.9: *The combinatorial blowup of* Π_3 *in 123.*

Comparing this result with Example 2.12, we observe that $Bl_{123}\Pi_3$ is the face poset of $Y_{\mathcal{A}_2} = Bl_{\{0\}}V$. Moreover, we note that also the combinatorial blowup of the arrangement introduced in Example 2.13 coincides with the face poset of the geometric blowup (Figure 2.4).



(a) *Intersection lattice.* **(b)** *Blowup along* ℓ. **(c)** *Blowup along* $\tilde{r}$.

Figure 2.10: *Combinatorial blowup of two lines in* $\mathbb{C}^3$.

We note that ℓ and r form a combinatorial (but not geometric) building set in the intersection lattice in Figure 2.10a. The fact that the combinatorial blowup coincides with the face poset of the geometric blowup is a general result, as stated in the following theorem.

Theorem 2.21 (Prop. 4.7 (1) in [FK04]). *Let $\mathcal{A}$ be a complex subspace arrangement, $\mathcal{G}$ a geometric building set in $\mathcal{L}(\mathcal{A})$, and $G_1, \dots G_i$ a non-increasing linear order on $\mathcal{G}$. We call $\mathcal{L}_i$ the face poset of $Bl_i(\mathcal{A})$, obtained by blowing up $G_1, \dots G_i$. We have that $\mathcal{L}_i$ coincides with the combinatorial blowups of $\mathcal{L}$ in $G_1, \dots G_i$.*

$$\mathcal{L}_i \;=\; Bl_{G_i}(\dots(Bl_{G_2}(Bl_{G_1}\mathcal{L}))\dots).$$

The definitions introduced above (building sets, nested sets and combinatorial blowups) naturally induce a combinatorial generalization of the De Concini-Procesi wonderful model, as established by the following theorem.

Theorem 2.22 (Thm. 3.4 in [FK04]). *Let $\mathcal{L}$ be a semilattice, $\mathcal{G}$ a combinatorial building set and $G_1 \dots G_t$ a non-increasing linear order on $\mathcal{G}$. The face poset of the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ coincides with blowing up the semilattice in $G_1 \dots G_t$.*

$$Bl_{G_i}(\dots(Bl_{G_2}(Bl_{G_1}\mathcal{L}))\dots) = \mathcal{F}(\mathcal{N}(\mathcal{L}, \mathcal{G})).$$

Since the combinatorial blowup coincides with the face poset of the geometric blowup by Theorem 2.21, the face poset of the wonderful model coincides with the face poset of $\mathcal{N}(\mathcal{L}, \mathcal{G})$. Indeed, we note that the face poset of Figure 2.7 corresponds exactly to Figure 2.9.

2.5 Chow ring of a lattice

Using the notions introduced above, we can now define a graded commutative algebra for any finite atomic lattice $\mathcal{L}$.

Definition 2.23. Let $\mathcal{L}$ be a finite atomic lattice, $\mathfrak{U}(\mathcal{L})$ its set of atoms, and $\mathcal{G}$ a building set in $\mathcal{L}$. The *Chow ring* $D(\mathcal{L}, \mathcal{G})$ of $\mathcal{L}$ with respect to $\mathcal{G}$ is defined as

$$D(\mathcal{L}, \mathcal{G}) \;:=\; \mathbb{Z}[\{x_G\}_{G \in \mathcal{G}}]/\mathcal{I},$$

where $\mathcal{I}$ is generated by

$$\begin{aligned} &\prod_{i=1}^{t} x_{G_i} \quad \text{for } \{G_1, \dots, G_t\} \notin \mathcal{N}(\mathcal{L}, \mathcal{G}), \\ &\sum_{G \geq H} x_G \quad \text{for } H \in \mathfrak{U}(\mathcal{L}). \end{aligned} \tag{2.3}$$

For the projective wonderful model of a complex hyperplane arrangement, this algebra admits a surprising geometric interpretation: it is isomorphic to the integral cohomology algebra of the projective arrangement.

Let $\mathcal{A}$ be an arrangement of linear subspaces. If we replace the ambient space V by its projectivization $\mathbb{P}V$ in the constructions Definition 2.9 and Definition 2.10, we get a wonderful model for $\mathbb{P}\mathcal{A}$. We denote by $Y^{\mathbb{P}}_{\mathcal{A},\mathcal{G}}$ the wonderful model for $\mathbb{P}\mathcal{A}$ with respect to a geometric building set $\mathcal{G}$ in $\mathcal{L}(\mathcal{A})$.

Theorem 2.24 ([CP96][FY04])**.** *Let $\mathcal{L} = \mathcal{L}(\mathcal{A})$ be the intersection lattice of an essential arrangement of complex hyperplanes $\mathcal{A}$. We recall that an arrangement is called essential if the overall intersection is $\{0\}$. Let $\mathcal{G}$ be a building set in $\mathcal{L}$ that contains the total intersection of $\mathcal{A}$. Then, the algebra $D(\mathcal{L}, \mathcal{G})$ is isomorphic to the integral cohomology algebra of the projective arrangement model $Y^{\mathbb{P}}_{\mathcal{A},\mathcal{G}}$.*

$$D(\mathcal{L}, \mathcal{G}) \;\cong\; H^*(Y^{\mathbb{P}}_{\mathcal{A},\mathcal{G}}, \mathbb{Z}).$$

For instance, we can see the interpretation of the first relations in (2.3). We have that $Y^{\mathbb{P}}_{\mathcal{A},\mathcal{G}} \setminus \mathcal{M}(\mathcal{A}) = \bigcup_{G\in\mathcal{G}} D_G$, and we map $[D_G] \in H^2(Y^{\mathbb{P}}_{\mathcal{A},\mathcal{G}})$ to $x_G \in D(\mathcal{L}, \mathcal{G})$. If D and D' intersect transversely, the cup product is $[D] \cdot [D'] = [D \cap D']$. A subset $S \in \mathcal{G}$ is $\mathcal{G}$-nested if and only if $\bigcap_{G\in S} D_G \neq \emptyset \;\subseteq Y^{\mathbb{P}}_{\mathcal{A},\mathcal{G}}$. We then get, if S is not $\mathcal{G}$-nested, that $\prod_{G\in S}[D_G] = [\,\bigcap_{G\in S} D_G] = 0$.

We are now ready to give the definition of the Chow ring of a matroid M. We begin by defining this ring for a representable matroid M over $\mathbb{C}$, and then extend this definition by introducing the Chow ring of a generic matroid.
Let $M{=}(E, \mathcal{I})$ be a representable matroid over $\mathbb{C}$. E can be then considered to be a finite set of non-null vectors, that span a complex vector space U, and $\mathcal{I}$ the collection of linearly independent subsets of E. We can now define an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{P}(U^*)$: every vector can be seen as a linear form on U^*, and so defines a hyperplane $\mathbb{P}(e^{\perp}) \subset \mathbb{P}(U^*)$, with

$$e^{\perp} = \{\phi \in U^* | \langle e, \phi\rangle = 0\}.$$

The flat F corresponds to the element

$$F^{\perp} = \{\phi \in U^* | \langle e, \phi\rangle = 0 \;\forall e \in F\}$$

of the partition lattice $\mathcal{L}(\mathcal{A})$. We consider the wonderful model $Y^{\mathbb{P}}_{\mathcal{A},\mathcal{G}_{\max}}$ of this arrangement. The Chow ring of the matroid M is then defined as the Chow ring of the intersection lattice $\mathcal{L}(\mathcal{A})$. By Theorem 2.24, this ring is isomorphic to the integral cohomology ring of $Y^{\mathbb{P}}_{\mathcal{A},\mathcal{G}_{\max}}$, since the arrangement is essential.
Moreover, in Definition 2.23 we introduced the Chow ring of an arbitrary finite atomic lattice. Using the correspondence between simple matroids and geometric lattices stated in Theorem 1.24, we define the Chow ring of a generic matroid M precisely as in Definition 2.23.

In the following chapter we will use an equivalent definition:

Definition 2.25. Let $M = (E, \mathcal{I})$ be a simple matroid. The *Chow ring* of M is

$$A^*(M) := S_M/(I_M + J_M),$$

where $S_M := \mathbb{Z}[x_F |\ F$ is a nonempty proper flat of $M]$. Furthermore, I_M is the ideal generated by $x_F x_G$, with F and G two incomparable nonempty proper flats of M, and J_M is the ideal generated by $\sum\limits_{i_1 \in F} x_F - \sum\limits_{i_2 \in F} x_F$, with i_1 and i_2 distinct elements of E.

We now prove that this definition is equivalent to Definition 2.23.

If we consider $\mathcal{G}$ as the maximum building set $\mathcal{G}_{\max}$, the $\mathcal{G}_{\max}-$nested sets are exactly the flags of flats: the join of any set of elements always exists and belongs to $\mathcal{G}_{\max}$, so a $\mathcal{G}_{\max}$-nested set cannot contain incomparable elements. Moreover, the relations $\prod_{i=1}^t x_{G_i}$, for $\{G_1, \dots, G_t\} \notin \mathcal{N}(\mathcal{L}, \mathcal{G})$, are overabundant: it is sufficient to consider the relations $x_F x_G$, with F and G incomparable flats. Furthermore, we have to prove that the relations $\sum_{G \geq H} x_G$ for $H \in \mathfrak{U}(\mathcal{L})$, are equivalent to the relations $\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F$, with i_1 and i_2 distinct elements of E. We find that if the matroid M is simple, then the atoms of $\mathcal{L}(M)$ are exactly the elements of the ground set of E. We then define a homomorphism φ from $D(\mathcal{L}(M), \mathcal{G}_{\max})$ (Definition 2.23), to $A^*(M)$ (Definition 2.25). We note that in Definition 2.23 we consider x_F with $F \in \mathcal{L}_{>\hat{0}}$, while in Definition 2.25 F has to be a proper flat, so $F \in \mathcal{L} \backslash \{\hat{0}, \hat{1}\}$. We then define the map

$$\varphi(x_F) := \begin{cases} x_F & \text{if } F \text{ is a proper flat.} \\ -\sum_{i \in F} x_F & \text{if } F = \hat{1}. \end{cases}$$

We see that the map φ is well-defined, since $\varphi(x_{\hat{1}})$ does not depend on $i \in E$, and since

$$\varphi\left(\sum_{F \geq i} x_F\right) = \varphi\left(\sum_{i \in F} x_F\right) + \varphi(x_{\hat{1}}) = \sum_{i \in F} x_F - \sum_{i \in F} x_F = 0.$$

Moreover, we can also define an homomorphism ψ from $A^*(M)$ to $D(\mathcal{L}(M), \mathcal{G}_{\max})$, and prove that it is well-defined and the inverse of φ. We fix $\psi : x_F \longmapsto x_F$, and note that

$$\psi\left(\sum_{i \in F} x_F - \sum_{j \in F} x_F\right) = \sum_{i \in F} x_F + x_{\hat{1}} - x_{\hat{1}} - \sum_{j \in F} x_F = \sum_{F \geq i} x_F - \sum_{F \geq j} x_F = 0.$$

We now show that ψ is the inverse of φ on the generators. If we consider x_F with F a proper flat, the maps are both the identity, and so the assertion is trivial. If $F = \hat{1}$, we have that

$$\psi(\varphi(x_{\hat{1}})) = -\sum_{i \in F} x_F - x_{\hat{1}} + x_{\hat{1}} = -\sum_{F \geq i} x_F + x_{\hat{1}} = x_{\hat{1}}.$$

Example 2.26. We now describe the Chow ring $A^*(M)$ of the matroid M which we defined as the cycle matroid associated with the graph in Figure 1.5.
We recall that an element $x_{F_1} \cdots x_{F_k}$ in $A^*(M)$ is zero if the flats $F_1, \cdots, F_k$ do not form a $\mathcal{G}_{\max}$-nested set, that is, if two of them are not comparable. Thus, if $x_{F_1} \cdots x_{F_k}$ is non-zero, then the flats $F_1, \cdots, F_k$ form a flag. Moreover, for each i, j in E, we have that $\sum_{i \in F} x_F = \sum_{j \in F} x_F$. We also recall that $A^*(M)$ is a graded ring of dimension $r(M) - 1$, and so, $A^*(M) = \sum_{k=0}^{3} A^k(M)$, since we already showed that the rank of M is four.
First of all, we give an ordering to the ground set E of M.

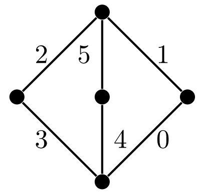


Figure 2.11: *M with an ordering on the ground set E.*

We now compute $A^k(M)$ for each $k = 0, \ldots, 3$. We have that $A^0(M)$ is trivially $\mathbb{Z}$, while $A^1(M)$ is generated by elements of the form x_F, with F a nonempty proper flat of M. We showed that the non empty proper flats of this matroid are exactly 32, but between these elements there are exactly five relations given by $\sum_{0 \in F} x_F = \sum_{i \in F} x_F$ for each i in $\{1, \cdots, 5\}$. A base of $A^1(M)$ has then cardinality $32 - 5 = 27$ and so $A^1(M) \cong \mathbb{Z}^{27}$.
We now compute $A^2(M)$, whose elements are of the form $x_{F_1} x_{F_2}$, with $F_1 \subsetneq F_2$. The more efficient way to compute this, is to use Corollary 1 of Theorem 1 in [FY04]. This corollary states that the monomials $x_{F_1}^{a_1} \cdots x_{F_k}^{a_k}$, where $F_1 \subsetneq \cdots \subsetneq F_k$, $F_i \in \mathcal{L}_{>\hat{0}}$, and $a_i < rk(F_i) - rk(F_{i-1})$ for $i = 1, \ldots, k$, form a basis of $A^*(M)$. We see that, using this basis, we get the same dimension of $A^1(M)$ we computed before. In fact, a basis of $A^1(M)$, according to this theorem, is made of all the elements of the form x_F, with $r(F) > 2$: they are $15 + 8 + 3 + 1 = 27$. Moreover, we can now compute more easily the dimension of $A^2(M)$. A basis for $A^2(M)$ is given by elements of the form x_F^2, with $r(F) > 2$, and elements of the form $x_F x_G$, with $F \subsetneq G$, $rk(F) > 1$, and $rk(G) > rk(F) + 1$. The elements of the first form are $8 + 3 + 1 = 12$, while the elements of the second form are 15 (since $r(F) = 2$ and $r(G) = 4$). In total they are 27, and so $A^2(M) \cong \mathbb{Z}^{27}$. Lastly, a basis of $A^3(M)$ is made of just elements of the form x_F^3, with $r(F) > 3$, since the existence of elements of the form $x_F x_G^2, x_F^2 x_G$ or $x_F x_G x_H$ would imply that M contains flats of rank greater than four. There is just one element in this basis, $x_{\hat{1}}$, and so $A^3(M) \cong \mathbb{Z}$. In conclusion, we get $A^0(M) \cong \mathbb{Z}$, $A^1(M) \cong \mathbb{Z}^{27}$, $A^2(M) \cong \mathbb{Z}^{27}$ and $A^3(M) \cong \mathbb{Z}$.

Chapter 3

The Kähler package on Kähler manifolds

The aim of this chapter is to prove the hard Lefschetz theorem and the Hodge-Riemann bilinear relations first for an euclidean vector space of dimension $2n$ with a compatible almost complex structure, and then for a compact Kähler manifold. We will see how this result implies the validity of the two theorems for the De Concini-Procesi wonderful model. In this chapter, we will mostly follow [Huy05].

3.1 Complex and hermitian structures

We begin by studying complex and hermitian structures on a real vector space V. In this section, V will be a finite-dimensional real vector space.

Definition 3.1. An endomorphism $I : V \to V$ with $I^2 = -\mathrm{id}$ is called an *almost complex structure* on V.

Lemma 3.2 (Lemma 1.2.2 in [Huy05])**.** *If I is an almost complex structure on a real vector space V, then V admits in a natural way the structure of a complex vector space.*

Corollary 3.3 (Corollary 1.2.3 in [Huy05])**.** *Any almost complex structure on V induces a natural orientation on V.*

For a real vector space V, the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ is denoted by $V_{\mathbb{C}}$.

Definition 3.4. Let I be an almost complex structure on a real vector space V and let $I : V_{\mathbb{C}} \to V_{\mathbb{C}}$ be its linear extension. Then, the $\pm i$ eigenspaces are respectively denoted by $V^{1,0}$ and $V^{0,1}$.

Lemma 3.5 (Lemma 1.2.5 in [Huy05]). *Let V be a real vector space. Then, $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$.*

If V is a real vector space of dimension d, the natural decomposition of its exterior algebra is

$$\bigwedge^* V = \bigoplus_{k=0}^{d} \bigwedge^k V.$$

The complex vector space $\bigwedge^* V_{\mathbb{C}}$ decomposes analogously.

Definition 3.6. We now define $\bigwedge^{p,q} V$ as

$$\bigwedge^{p,q} V := \bigwedge^p V^{1,0} \otimes_{\mathbb{C}} \bigwedge^q V^{0,1}.$$

Proposition 3.7 (Proposition 1.2.8 in [Huy05]). *We have that*

$$\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V_{\mathbb{C}}.$$

With respect to the above decomposition, we consider the natural projection $\Pi^{p,q} : \bigwedge^* V_{\mathbb{C}} \to \bigwedge^{p,q} V$. Furthermore, we define $\mathbf{I}$: $\bigwedge^* V_{\mathbb{C}} \to \bigwedge^* V_{\mathbb{C}}$ as the linear operator that acts on $\bigwedge^{p,q} V$ as

$$\mathbf{I} = \sum_{p,q} i^{p-q} \cdot \Pi^{p,q}.$$

We note that $\mathbf{I}$ is the multiplicative extension of I on $V_{\mathbb{C}}$. We also denote by $\mathbf{I}$ the corresponding operator on $\bigwedge^* V_{\mathbb{C}}^*$.

Let now $\langle , \rangle$ be a scalar product on V.

Definition 3.8. An almost complex structure I on V is *compatible* with the scalar product $\langle , \rangle$ if $\langle I(v), I(w) \rangle = \langle v, w \rangle$ for all $v, w \in V$.

From now on, $(V, \langle , \rangle, I)$ will be a real vector space endowed with a scalar product $\langle , \rangle$ and a compatible almost complex structure I.

Definition 3.9. The *fundamental form* associated to $(V, \langle , \rangle, I)$ is the form

$$\omega := -\langle (), I() \rangle = \langle I(), () \rangle.$$

Lemma 3.10. *The fundamental form ω associated to $(V, \langle , \rangle, I)$ is real and of type* (1,1).

Proof. We have that for all $v, w \in V$

$$\langle v, I(w) \rangle = \langle I(v), I(I(w)) \rangle = -\langle I(v), w \rangle = -\langle w, I(v) \rangle.$$

Then, the form ω is alternating: $\omega \in \bigwedge^2 V^*$. Moreover $\omega \in \bigwedge^{1,1} V_{\mathbb{C}}^*$, since

$$(\mathbf{I}\omega)(v, w) = \omega(\mathbf{I}(v), \mathbf{I}(w)) = \langle I(I(v)), I(w) \rangle = \omega(v, w).$$

□

We can also consider the extension of the scalar product $\langle , \rangle$ to a positive definite hermitian form on $V_{\mathbb{C}}$ defined by $\langle v \otimes \lambda, w \otimes \mu \rangle_{\mathbb{C}} := (\lambda \overline{\mu}) \cdot \langle v, w \rangle$.

Definition 3.11. Let ω be the fundamental form associated to $(V, \langle , \rangle, I)$. The *Lefschetz operator* $L : \bigwedge^* V_{\mathbb{C}} \to \bigwedge^* V_{\mathbb{C}}$ is given by $a \mapsto \omega \wedge a$.

Remark 3.12. It is easy to verify that the Lefschetz operator is of bidegree (1,1):

$$L\left(\bigwedge^{p,q} V^*\right) \subset \bigwedge^{p+1,q+1} V^*.$$

Moreover, let now $(V, \langle , \rangle)$ be an oriented euclidean vector space of dimension d, such as an euclidean vector space endowed with an almost complex structure. Then, $\langle , \rangle$ defines scalar products on all exterior powers $\bigwedge^k V$.

Definition 3.13. Let $e_1, \ldots, e_d \in V$ be an orthonormal basis of V, and vol $\in V$ the orientation of V given by $\mathrm{vol} = e_1 \wedge \cdots \wedge e_d$. Then, the *Hodge operator* $* : \bigwedge^k V \to \bigwedge^{d-k} V$ is defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \cdot \mathrm{vol},$$

for every $0 \leq k \leq d$ and for every $\alpha, \beta \in \bigwedge^* V$.

Proposition 3.14 (Proposition 1.2.20 in [Huy05]). *Let $(V, \langle , \rangle)$ be an oriented euclidean vector space of dimension d. Let $e_1, \ldots, e_d \in V$ be an orthonormal basis of V and vol $\in V$ the orientation of V given by vol $= e_1 \wedge \cdots \wedge e_d$. Then, the Hodge $*$-operator has the following properties:*

1. *For every $0 \leq k \leq d$, if $\{i_1, \ldots, i_k, j_1, \ldots, j_{d-k}\} = \{1, \ldots, n\}$, then*

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \varepsilon \cdot e_{j_1} \wedge \cdots \wedge e_{j_k},$$

   where $\varepsilon = sgn(i_1, \ldots, i_k, j_1, \ldots, j_{d-k})$. In particular, $*1 = vol$.

2. *The operator is self-adjoint up to sign: for $\alpha, \beta \in \bigwedge^k V$, for every $0 \leq k \leq d$*

$$\langle \alpha, *\beta \rangle = (-1)^{k(d-k)} \langle *\alpha, \beta \rangle.$$

3. *The operator is involutive up to sign: for every $0 \leq k \leq d$*

$$(*|_{\bigwedge^k V})^2 = (-1)^{k(d-k)}.$$

4. *The operator is an isometry on $(\bigwedge^* V, \langle , \rangle)$: for $\alpha, \beta \in \bigwedge^k V$, for every $0 \leq k \leq d$*

$$\langle \alpha, \beta \rangle = \langle *\alpha, *\beta \rangle.$$

Remark 3.15. We can extend $\langle , \rangle_{\mathbb{C}}$ to a positive definite hermitian form on $\bigwedge^* V_{\mathbb{C}}^*$ and extend $\mathbb{C}-$linearly the Hodge operator.

From now on, we will consider $d = 2n$ and $\langle , \rangle$ in $\bigwedge^* V^*$.

Definition 3.16. The *dual Lefschetz operator* Λ is the operator $\Lambda : \bigwedge^* V^* \to \bigwedge^* V^*$ that is adjoint to L with respect to $\langle , \rangle$:

$$\langle \Lambda\alpha, \beta \rangle = \langle \alpha, L\beta \rangle \quad \text{for all } \alpha, \beta \in \bigwedge^* V^*.$$

We denote by Λ also the linear extension $\bigwedge^* V^*_{\mathbb{C}} \to \bigwedge^* V^*_{\mathbb{C}}$ of the dual Lefschetz operator.

Remark 3.17. We have that $\Lambda = *^{-1} \circ L \circ *$, and that Λ is of bidegree $(-1, -1)$ (Lemma 1.2.23, Lemma 1.2.24 in [Huy05]).

Definition 3.18. The *counting operator* $H := \bigwedge^* V^* \to \bigwedge^* V^*$ is defined by $H|_{\bigwedge^k V^*} = (k - n) \cdot id$, where $\dim_{\mathbb{R}} V^* = 2n$.

Proposition 3.19 (Proposition 1.2.26 in [Huy05])**.** *If we consider on $\bigwedge^* V^*$ the Lefschetz operator L, its dual Λ and the counting operator H, they satisfy:*

1. $[H, L] = 2L$, *2.* $[H, \Lambda] = -2\Lambda$, *3.* $[L, \Lambda] = H$.

Corollary 3.20. *The action of $L, \Lambda,$ and H defines a natural $\mathfrak{sl}(2)-$representation on $\bigwedge^* V^*$.*

Proof. We recall that $\mathfrak{sl}(2)$ is the Lie algebra of all 2×2-matrices of trace zero. A base of $\mathfrak{sl}(2)$ is given by $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It is easy to see that they satisfy $[B, X] = 2X$, $[B, Y] = 2Y$, and $[X, Y] = B$. Now, mapping $X \mapsto L$, $Y \mapsto \Lambda$, and $B \mapsto H$, we get a Lie algebra homomorphism $\mathfrak{sl}(2) \longrightarrow \mathrm{End}(\bigwedge^* V^*)$. Tensorizing with $\mathbb{C}$, we also obtain the $\mathfrak{sl}(2, \mathbb{C})$-representation. □

Corollary 3.21. $[L^i, \Lambda](\alpha) = i(k - n + i - 1)L^{i-1}(\alpha)$ *for all* $\alpha \in \bigwedge^k V^*$.

Proof. We use induction on i. If $i = 1$, we have 3) of Proposition 3.19. If $i > 1$, we use 3) of Proposition 3.19 and the induction hypotesis, and get

$$\begin{aligned}
[L^i, \Lambda] &= L^i \Lambda\alpha - \Lambda L^i \alpha \\
&= L(L^{i-1}\Lambda\alpha - \Lambda L^{i-1}\alpha) + L\Lambda L^{i-1}\alpha - \Lambda L L^{i-1}\alpha \\
&= L[L^{i-1}, \Lambda](\alpha) + [L, \Lambda](L^{i-1}\alpha) \\
&= (i - 1)(k - n + (i - 1) - 1)L^{i-1}(\alpha) + (2i - 2 + k - n)L^{i-1}(\alpha) \\
&= i(k - n + i - 1)L^{i-1}(\alpha).
\end{aligned}$$

□

Definition 3.22. An element $\alpha \in \bigwedge^k V^*$ is called *primitive* if $\Lambda\alpha = 0$. The linear subspace of all primitive elements $\alpha \in \bigwedge^k V^*$ is denoted by $P_k \subset \bigwedge^k V^*$. Accordingly, an element $\alpha \in \bigwedge^k V^*_{\mathbb{C}}$ is called primitive if $\Lambda\alpha = 0$.

Theorem 3.23 (Hard Lefschetz)**.** *Let $(V, \langle , \rangle, I)$ be an euclidean vector space of dimension $2n$ with a compatible almost complex structure, and let L and Λ be the Lefschetz operators.*

1. *There exists a direct sum decomposition, called the* Lefschetz decomposition*:*
$$\bigwedge^k V^* = \bigoplus_{i \geq 0} L^i(P^{k-2i}). \tag{3.1}$$
Moreover, 3.1 is orthogonal with respect to $\langle , \rangle$.
2. *If $k > n$, then $P^k = 0$.*
3. *The map $L^{n-k} : P^k \to \bigwedge^{2n-k} V^*$ is injective for $k \leq n$.*
4. *The map $L^{n-k} : \bigwedge^k V^* \to \bigwedge^{2n-k} V^*$ is bijective for $k \leq n$.*
5. *If $k \leq n$, then $P^k = \{\alpha \in \bigwedge^k V^* \mid L^{n-k+1}\alpha = 0\}$.*

Proof. 1) From representation theory, we know that the $\mathfrak{sl}(2)$-representations are completely reducible, and so they can be decomposed as the direct sum of their irreducibile subrepresentations. Moreover, we know that each heighest weight vector (a vector v such that $\Lambda v = 0$) determines an irreducible representation of $\mathfrak{sl}(2)$, with basis $\{v, Lv, \ldots, L^r v\}$, where $L^{r+1}v = 0$. We also know that every irreducible representation is of this form. In our setting, the heighest weight vectors correspond to the elements of the primitive subspaces. Then, $\bigwedge^k V^*$ is an $\mathfrak{sl}(2)$-representation by Corollary 3.20, and its irreducible representation are of the form $L^i(P^{k-2i})$. Moreover, this decomposition is orthogonal with respect to $\langle , \rangle$: let $v \in P^{k-2r}(X)$ and $w \in P^{k-2s}(X)$, then

$$\langle L^r v, L^s w\rangle = \langle v, \Lambda^r L^s V\rangle = c\langle v, L^{s-r} w\rangle.$$

Indeed, applying repeatedly 3.21, we get $\Lambda^r L^s w = c \cdot L^{s-r} w$, with $c = \prod_{j=0}^{r-1}(s-j)(k-n+s-1-j)$. If $r > s$ the product is 0, while if $r < s$ we obtain that c is a nonzero constant. Then, if $r > s$ it follows that $\langle L^r v, L^s w\rangle = 0$, while if $r < s$ we get

$$\langle L^r v, L^s w\rangle = c\langle v, L^{s-r} w\rangle = c\langle \Lambda^{s-r} v, w\rangle = 0,$$

since v is primitive.

2) If $\alpha \in P^k$, with $k > n$, we consider $i \geq 0$ minimal such that $L^i\alpha = 0$. By Corollary 3.21, we get $0 = [L^i, \Lambda](\alpha) = i(k-n+i-1)L^{i-1}\alpha$. Then, $i = 0$ and $\alpha = 0$.

3) Let $0 \neq \alpha \in P^k$, $k \leq n$, and $i > 0$ minimal such that $L^i\alpha = 0$. Again by Corollary 3.21, we have that $0 = [L^i, \Lambda](\alpha) = i(k-n+i-1)L^{i-1}\alpha$ and, therefore, $k-n+i-1 = 0$. In particular, $L^{n-k}\alpha \neq 0$ and $L^{n-k+1}\alpha = 0$.

We note that, from the injectivity of P^{n-k} on L^k, it follows that $L^r : P^i \to L^r P^i$ is an isomorphism for all $i \leq n-r$. This implies $\dim(P^i) = \dim(L^r P^i)$ for all $i \leq n-r$.

4) We now prove that $L^{k-n} : \bigwedge^k V^* \longrightarrow \bigwedge^{2n-k} V^*$ is bijective. We already know from 3) that the map is injective, thus if we show that $\bigwedge^k V^*$ and $\bigwedge^{2n-k} V^*$ have the same dimension, then L^{n-k} is also bijective. Thanks to 1) and 2), if we fix $s = r + (n-k)$, we have that

$$\bigwedge^k V^* = \bigoplus_{r \geq 0} L^r(P^{k-2r}), \text{ and } \bigwedge^{2n-k} V^* = \bigoplus_{s \geq \frac{n-k}{2}} L^s(P^{2n-k-2s}) = \bigoplus_{r \geq 0} L^{n-k+r} P^{k-2r}.$$

We then get, again for 3), that $\dim \bigwedge^k V^* = \sum_r \dim P^{k-2r}$ and $\dim \bigwedge^{2n-k} V^* = \sum_r \dim P^{k-2r}$.

5) We have seen in 3) that $P^k \subseteq \mathrm{Ker}(L^{n-k+1})$. Conversely, if $\alpha \in \bigwedge^k V^*$, with $L^{n-k+1}\alpha = 0$, then by Corollary 3.21 we have $[L^{n-k+2}, \Lambda](\alpha) = L^{n-k+2}\Lambda\alpha = (n-k+2)L^{n-k-1}\alpha = 0$. Since L^{n-k+2} is injective on $\bigwedge^{k-2} V^*$, then $\Lambda\alpha = 0$. □

Remark 3.24. Since L, Λ and H are of pure type $(1,1)$, $(-1,-1)$, and $(0,0)$ respectively, the Lefschetz decomposition is compatible with the bidegree decomposition. Indeed, $P^k_{\mathbb{C}} = \bigoplus_{p+q=k} P^{p,q}$, with $P^{p,q} := P^k_{\mathbb{C}} \cap \bigwedge^{p,q} V^*$.

Proposition 3.25. *For all $\alpha \in P^k$,*

$$*L^j\alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} \cdot L^{n-k-j} \boldsymbol{I}(\alpha).$$

Proof. Proposition 1.2.31 on [Huy05]. □

Definition 3.26. Let $(V, \langle , \rangle, I)$ be as before, we introduce

$$\int_V : \bigwedge^{2n} V^* \longrightarrow \mathbb{R}, \quad c \cdot \mathrm{vol} \overset{\int_V}{\longmapsto} c.$$

Definition 3.27. Let ω be the fundamental form associated to $(V, \langle , \rangle, I)$. The *Hodge-Riemann pairing* is the bilinear form

$$Q : \bigwedge^k V^* \times \bigwedge^k V^* \longrightarrow \mathbb{R}, \quad (\alpha, \beta) \longmapsto (-1)^{\frac{k(k-1)}{2}} \int_V \alpha \wedge \beta \wedge \omega^{n-k}.$$

We will also denote by Q the $\mathbb{C}$-linear extension of the Hodge-Riemann pairing to $\bigwedge^* V^*_{\mathbb{C}}$.

Theorem 3.28 (Hodge-Riemann bilinear relations)**.** *We have that*

$$i^{p-q} Q(\alpha, \overline{\alpha}) = (n - (p+q))! \cdot \langle \alpha, \alpha \rangle_{\mathbb{C}},$$

for $0 \neq \alpha \in P^{p,q}$, with $p + q \leq n$. In particular, $i^{p-q}Q(\alpha, \overline{\alpha})$ is definite positive on $P^{p,q}$.

Proof. We fix $k = p + q$ and $\beta \in \bigwedge^k V^*$ such that $*\overline{\beta} = L^{n-k}\overline{\alpha}$. By definition we get

$$(-1)^{\frac{k(k-1)}{2}} \alpha \wedge \overline{\alpha} \wedge \omega^{n-k} = (-1)^{\frac{k(k-1)}{2}} \alpha \wedge L^{n-k}\overline{\alpha} = (-1)^{\frac{k(k-1)}{2}} \langle \alpha, \beta \rangle_{\mathbb{C}} \cdot \mathrm{vol}.$$

We then have $(-1)^{\frac{k(k-1)}{2}} \int_V \alpha \wedge \beta \wedge \omega^{n-k} = (-1)^{\frac{k(k-1)}{2}} \langle \alpha, \beta \rangle_{\mathbb{C}}$. We now recall that, by Proposition 3.14, $*^2\overline{\beta} = (-1)^k\overline{\beta}$. We also have

$$*^2\overline{\beta} = *L^{n-k}\overline{\alpha} = (-1)^{\frac{k(k+1)}{2}} (n-k)! \cdot i^{p-q} \cdot \overline{\alpha}$$

by Proposition 3.25. Thus, $\beta = (-1)^{k+\frac{k(k+1)}{2}} (n-k)! \cdot i^{q-p} \cdot \alpha$ and

$$Q(\alpha, \overline{\alpha}) = (-1)^{k+\frac{k(k+1)}{2}+\frac{k(k-1)}{2}} (n-k)! \cdot i^{q-p} \cdot \langle \alpha, \alpha \rangle_{\mathbb{C}}.$$

This implies

$$i^{p-q} Q(\alpha, \overline{\alpha}) = (n-k)! \cdot \langle \alpha, \alpha \rangle_{\mathbb{C}} > 0.$$

□

Remark 3.29. Since the hard Lefschetz decomposition is orthogonal with respect to $\langle , \rangle$, then it is also orthogonal with respect to Q.

Remark 3.30. If $k = 2j$, then $i^{p-q} Q(\alpha, \overline{\alpha}) = (-1)^j Q(\alpha, \overline{\alpha})$.

3.2 Kähler manifolds

We begin by stating the Poincaré duality, a fundamental theorem applicable to all compact oriented manifolds..

Theorem 3.31 (Poincaré duality, Theorem 3.30 in [Hat02])**.** *If M is a compact oriented manifold of dimension n, there is an isomorphism between the cohomology groups*

$$H^k(M) \cong H^{n-k}(M)^*,$$

for $0 \leq k \leq n$.

Now, let M be a differentiable manifold, and let TM denote its tangent bundle.

Definition 3.32. A *holomorphic atlas* on a differentiable manifold is an atlas $\{(U_i, \varphi_i)\}$ of the form $\varphi_i : U_i \simeq \varphi_i(U_i) \subseteq \mathbb{C}^n$, such that the transition functions $\varphi := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are holomorphic. Two holomorphic atlases $\{(U_i, \varphi_i)\}$ and $\{(U_i', \varphi_i')\}$ are called equivalent if all maps $\varphi_i \circ \varphi_j'^{-1} : \varphi_j'(U_i \cap U_j') \to \varphi_i(U_i \cap U_j')$ are holomorphic.
A *complex manifold* X of dimension n is a (real) differentiable manifold of dimension $2n$ endowed with an equivalence class of holomorphic atlases.

Definition 3.33. An *almost complex manifold* is a differentiable manifold X together with a vector bundle endomorphism

$$I : TX \longrightarrow TX, \quad \text{with } I^2 = -\mathrm{id}.$$

Proposition 3.34 (Proposition 2.6.2 in [Huy05]). *Any complex manifold X admits a natural almost complex structure.*

Definition 3.35. Let X be an almost complex manifold. We denote by $T_{\mathbb{C}}X$ the complexification of TX, that is $T_{\mathbb{C}}X = TX \otimes \mathbb{C}$, and we introduce the complex vector bundles

$$\bigwedge^k (T_{\mathbb{C}}X)^* \quad \text{and} \quad \bigwedge^p (T^{1,0}X)^* \otimes_{\mathbb{C}} \bigwedge^q (T^{0,1}X)^*.$$

Their sheaves of sections are denoted by $\mathcal{A}^k(X)$ and $\mathcal{A}^{p,q}(X)$.

Let now X be a complex manifold of dimension n and I the induced almost complex structure.

Definition 3.36. A Riemannian metric g on X is an *hermitian structure* on X if for any point $x \in X$ the scalar product g_x on T_xX is compatible with the almost complex structure I_x. The induced real $(1,1)$-form $\omega := g(I(),())$ is called the *fundamental form.* The complex manifold X endowed with an hermitian structure g is called an *hermitian manifold.*

Moreover, we define on $\bigwedge^k (T_{\mathbb{C}}X)^*$ the Lefschetz operator, the Hodge $*$-operator and the dual Lefschetz operator as we did in the previous section.

Definition 3.37. Let (X, g) be an hermitian manifold. We define the *adjoint operator* d^* as

$$d^* = * \circ d \circ * : \mathcal{A}^k(X) \longrightarrow \mathcal{A}^{k-1}(X),$$

and the *Laplace operator* as $\Delta = d^*d + dd^*$.

Definition 3.38. A *Kähler structure* is an hermitian structure g for which the fundamental form ω is closed, that is, $d\omega = 0$. In this case, ω is called the *Kähler form.* A complex manifold endowed with a Kähler structure is called a *Kähler manifold.*

Proposition 3.39. *Let X be a Kähler manifold. Then Δ commutes with L and Λ.*

Proof. Proposition 3.1.12 in [Huy05]. □

Definition 3.40. Let (X, g) be a Riemannian manifold. A form $\alpha \in \mathcal{A}^k(X)$ or $\alpha \in \mathcal{A}^{p,q}(X)$ is called d-harmonic if $\Delta(\alpha) = 0$. Moreover, we denote by $\mathcal{H}^k(X, g)$ the space of d-harmonic k-forms and by $\mathcal{H}^{p,q}(X, g)$ the space of d-harmonic (p,q)-forms.

Proposition 3.41. *Let (X, g) be a Riemannian manifold. Then, every de Rham cohomology class $[\alpha] \in H^k(X)$ contains a unique harmonic representative α. Thus, the natural map $\mathcal{H}^k(M, g) \to H^k(M, g)$ that associates to each harmonic form its cohomology class is bijective.*

Proof. Theorem 6.11 in [War83]. □

Definition 3.42. Let X be a compact Kähler manifold. The *Kähler class* associated to a Kähler structure on X is the cohomology class $[\omega] \in H^{1,1}(X)$ of its Kähler form. The *Kähler cone*

$$\mathcal{K}_X \subseteq H^{1,1}(X) \cap H^2(X, \mathbb{R})$$

is the set of all Kähler classes associated to any Kähler structure on X.

Definition 3.43. Let (X, g) be a compact Kähler manifold. Then the *primitive cohomology* is defined as

$$H^k(X, \mathbb{R})_p := \operatorname{Ker}\left(\Lambda : H^k(X, \mathbb{R}) \longrightarrow H^{k-2}(X, \mathbb{R})\right).$$

Moreover, we define on X the *Hodge-Riemann pairing*

$$Q : H^k(X) \times H^k(X) \longrightarrow \mathbb{R} \qquad (\alpha, \beta) \longmapsto (-1)^{\frac{k(k-1)}{2}} \int_X \alpha \wedge \beta \wedge [\omega]^{n-k},$$

with $[\omega]$ the Kähler class.

Theorem 3.44 (Hard Lefschetz). *Let (X, g) be a compact Kähler manifold of dimension n. Then, for $k \leq n$*

$$L^{n-k} : H^k(X, \mathbb{R}) \cong H^{2n-k}(X, \mathbb{R}),$$

and

$$H^k(X, \mathbb{R}) = \bigoplus_{i \geq 0} L^i H^{k-2i}(X, \mathbb{R})_p.$$

Proof. From Theorem 3.23, it follows that $L^{n-k} : \mathcal{A}^k(X) \to \mathcal{A}^{2n-k}(X)$ is bijective. Moreover, by Proposition 3.39 we know that $[L, \Delta] = 0$ and $[\Lambda, \Delta] = 0$, so L and Λ map harmonic forms to harmonic forms. We then find that $L^{n-k} : \mathcal{H}^k(X) \to \mathcal{H}^{2n-k}(X)$ is bijective. Then, the first assertion follows from Proposition 3.41. The second assertion is a consequence of the decomposition proved in Theorem 3.23, the fact that L and Λ respect harmonicity, and Proposition 3.41. □

Remark 3.45. We note that the primitive cohomology does not depend on the chosen Kähler structure, but only on the Kähler class. Once a Kähler structure is chosen, any class in $H^k(X)_p$ can be realized by an harmonic form that is primitive at every point.

Indeed, if $\alpha \in H^k(X)_p$ is harmonic, then also $\Lambda\alpha$ is harmonic, since $[\Lambda, \Delta] = 0$ by Proposition 3.39. Thus, if $\alpha \in H^k(X)_p$ is harmonic and primitive in cohomology ($[\Lambda\alpha] = 0$), then $\Lambda\alpha = 0$ pointwise, since by Proposition 3.41 there is a unique harmonic form in the zero cohomology class: the 0 form.

Theorem 3.46 (Hodge-Riemann bilinear relations)**.** *Let (X, g) be a compact Kähler manifold of dimension n with Kähler class $[\omega]$. Then, the Hodge-Riemann pairing multiplied by i^{p-q} is definite positive on the primite cohomology, that is, if $\alpha \neq 0 \in H^k(X)_p$, then*

$$(-1)^{\frac{k(k-1)}{2}} \cdot i^{p-q} \int_X \alpha \wedge \beta \wedge [\omega]^{n-k} > 0.$$

Furthermore, the Lefschetz decomposition is orthogonal with respect to Q.

Proof. By Remark 3.45, any $\alpha \in H^k(X)_p$ can be represented by an harmonic form $\alpha \in \mathcal{H}^k(X)_p$ which is primitive at any point $x \in X$. The assertion then follows from Theorem 3.28. □

3.2.1 Projective varieties

We want to establish the validity of the Kähler package for the projective wonderful model (a smooth projective variety).

We recall that a differentiable submanifold Y of real dimension $2k$ of a complex manifold X is a *complex submanifold* if there exists a holomorphic atlas $\{(U_i, \varphi_i)\}$ of X such that $\varphi_i : U_i \cap Y \cong \varphi_i(U_i) \cap \mathbb{C}^k$. Moreover, a complex manifold X is *projective* if it is isomorphic to a closed complex submanifold of some projective space $\mathbb{P}^n$.

Proposition 3.47. *A smooth projective variety is a projective Kähler manifold.*

Proof. We know that $\mathbb{P}^n$ is a complex manifold (pg.56 in [Huy05]) and that a smooth projective variety X is a projective complex manifold. Indeed, we can consider $X \cap U_i$ to be an atlas covering X, where U_i are the standard charts of $\mathbb{P}^n$. By the holomorphic implicit function theorem (Proposition 1.1.11 in [Huy05]), which we can apply since the variety is smooth, $X \cap U_i$ is locally biholomorphic to $\mathbb{C}^{n-k}$, with $k = n - \dim(X)$. Furthermore, on $U_i \cap U_j$ the transition maps are given by rational functions, and the restrictions of these maps to X are holomorphic (where defined).

We also know that $\mathbb{P}^n$ admits a canonical Kähler structure induced by the *Fubini-Study* metric (Example 3.1.9 in [Huy05]). Moreover, the complex submanifolds of a Kähler manifold are also Kähler (Proposition 3.1.10 in [Huy05]). It then follows that any projective complex manifold is Kähler, and so is any smooth projective variety. □

Definition 3.48. Let L be a line bundle on a complex manifold X. The line bundle is called *very ample* if for some linear system in $H^0(X, L)$ (a subspace of $H^0(X, L)$), and for $s_0, \ldots, s_N \in H^0(X, L)$ a base of the linear system, the map

$$\varphi : X \backslash B_S(L) \longrightarrow \mathbb{P}^n, \quad x \longmapsto (s_0(x), \cdots, s_N(x))$$

is an embedding.

We recall that $B_S(L)$ is the set of all base points of L, namely, the points $x \in X$ such that $s(x) = 0$ for all $s \in H^0(X, L)$.
Moreover, a line bundle is called *ample* if for some $k > 0$, L^k is a very ample line bundle.

By definition, a compact complex manifold is projective if and only if it admits an ample line bundle: the tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ is very ample, and if $X \subseteq \mathbb{P}^n$ is a projective manifold, then the restriction $\mathcal{O}_{\mathbb{P}^n}(1)_{|X}$ is very ample on X.
We recall that the first Chern class of a holomorphic line bundle $L \in \mathrm{Pic}(X)$ is the image of L under the boundary map

$$c_1 : \ \mathrm{Pic}(X) \cong H_1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}).$$

The boundary map arises from the long exact cohomology sequence induced by the exponential sequence on the complex manidold X.

Definition 3.49. A line bundle L is called *positive* if its first Chern class $c_1(L) \in H^2(X, \mathbb{R})$ can be represented by a closed positive (1,1)-form.

We note that a compact complex manifold X that admits a positive line bundle L is Kähler: the closed positive real $(1,1)$-form ω representing $c_1(L)$ defines a Kähler structure on X.

Theorem 3.50 (Kodaira embedding theorem)**.** *Let X be a compact Kähler manifold. A line bundle L on X is positive if and only if it is ample. In this case, the manifold is projective.*

Proof. Proposition 5.3.1 in [Huy05]. □

The wonderful model X, being a smooth projective variety over $\mathbb{C}$, naturally carries the structure of a projective Kähler manifold. An ample line bundle L on X, which exists since X is projective, determines a Kähler structure through its first Chern class $c_1(L)$. Thus, the standard results of Hodge theory hold for X. In particular, the cohomology ring of X satisfies the hard Lefschetz theorem and the Hodge-Riemann bilinear relations with respect to $c_1(L)$.

We now characterize ampleness for line bundles in terms of intersection numbers.

Theorem 3.51 (Nakai-Moishezon criterion)**.** *Let X be a compact complex manifold and let L be a holomorphic line bundle on X. Then, L is ample if and only if $\int_Y c_1(L)^{\dim(Y)} > 0$ for every analytic subvariety $Y \subseteq X$. Moreover, $\int_Y c_1(L)^{\dim(Y)}$ are called intersection numbers.*

Proof. Theorem 1.2.23 pg.33 [Laz04]. □

We recall that an *analytic subvariety* Y of a complex manifold X is a closed subset $Y \subseteq X$ such that, for any point $x \in X$, there exists an open neighborhood $x \in U \subseteq X$ such that $Y \cap U$ is the zero set of finitely many holomorphic functions.

Moreover, by Serre's GAGA principle ([Ser56]), if X is a projective algebraic variety, every closed analytic subvariety of X^{an} (the complex analytic space obtained by equipping X with the Euclidean topology) is algebraic, and every smooth algebraic subvariety of X is a smooth analytic subvariety of X^{an}. Thus, Theorem 3.51 naturally extends to smooth projective varieties X, and their algebraic smooth subvarieties Y.

Chapter 4

Proof of the conjecture

The key ingredient of the proof of Conjecture 1.27 for a matroid M is the validity of Hodge theory, specifically of the Hodge-Riemann bilinear relations, for the Chow ring of M. In this chapter, we establish the conjecture when M is a representable matroid over $\mathbb{C}$. The general case was resolved in 2018 [AHK18] by Adiprasito, Huh, and Katz, who demonstrated that the Chow ring of an arbitrary matroid M satisfies the Poincaré duality, the hard Lefschetz theorem and the Hodge-Riemann bilinear relations, therefore proving Conjecture 1.27 in full generality.
In this chapter, M will be a matroid of rank $r+1$, with an ordering on the ground set $E = \{0, 1, \dots, n\}$, and lattice of flats $\mathcal{F}$.

4.1 The Kähler package

In this section, we characterize what it means for a graded algebra to satisfy the Kähler package (Poincaré duality, hard Lefschetz theorem, and Hodge-Riemann bilinear relations). We note that this package holds for the Chow ring of any representable matroid over $\mathbb{C}$, as established in the previous chapter.

Let A^* a graded algebra. We fix an isomorphism called the *degree map*, of the form:

$$\deg : A^r \to \mathbb{R}.$$

Definition 4.1. A graded algebra A^* satisfies the *Poincaré duality of dimension* r if

- There are isomorphism $A^0 \simeq \mathbb{R}$ and $A^r \simeq \mathbb{R}$.
- $A^q = 0$ for every integer $q > r$.
- For $q \leq r$, we have an isomorphism

$$A^{r-q} \to Hom_{\mathbb{R}}(A^q, A^r), \quad x \mapsto (y \mapsto \deg(x \cdot y)).$$

In this case we say that A^* is a *Poincaré duality algebra of dimension r.*

From now on, A^* will be a Poincaré duality algebra of dimension r.

Definition 4.2. Let ℓ be an element of A^1, and $0 \leq q \leq \frac{r}{2}$.

1. The *Lefschetz operator* on A^q associated to ℓ is the linear map
$$L_\ell^q : A^q \to A^{r-q}, \qquad a \mapsto \ell^{r-2q}a.$$
2. The *Hodge-Riemann form* on A^q associated to ℓ is the symmetric bilinear form
$$Q_\ell^q : A^q \times A^q \to \mathbb{R}, \qquad (a_1, a_2) \mapsto (-1)^q \deg(a_1 \cdot L_\ell^q(a_2)).$$
3. The *primitive subspace* of A^q associated to ℓ is
$$P_\ell^q := \{a \in A^q \mid \ell \cdot L_\ell^q(a) = 0\} \subseteq A^q.$$

The definitions introduced here are consistent with those of the previous chapter, where X is a smooth projective variety, $A^q = H^{2q}(X)$, and $\deg = \int_X$.

Definition 4.3. We say that

1. A^* satisfies the *hard Lefschetz property* (HL(ℓ)) if the Lefschetz operator L_ℓ^q is an isomorphism on A^q for $0 \leq q \leq \frac{r}{2}$.
2. A^* satisfies the *Hodge-Riemann bilinear relations* (HR(ℓ)) if the Hodge-Riemann form Q_ℓ^q is positive definite on P_ℓ^q for $0 \leq q \leq \frac{r}{2}$.

Remark 4.4. If the Lefschetz operator L_ℓ^q is an isomorphism, there is a decomposition
$$A^{q+1} = P_\ell^{q+1} \oplus \ell A^q.$$
We then have, if A^* satisfies HL(ℓ), the *Lefschetz decomposition* of A^q for $q \leq \frac{r}{2}$:
$$A^q = P_\ell^q \oplus \ell P_\ell^{q-1} \oplus \cdots \oplus \ell^q P_\ell^0.$$
We also know that the Lefschetz decomposition of A^q is orthogonal with respect to the Hodge-Riemann bilinear form Q_ℓ^q.

We now consider A^* to be the Chow ring of a matroid M. In Theorem 2.24, we stated that if M is a representable matroid over $\mathbb{C}$, then the Chow ring of M is isomorphic to the integral cohomology ring of a smooth projective variety $Y_{\mathcal{A}}^{\mathbb{P}}$.
Moreover, in the previous chapter we showed that if Y is a smooth projective variety, then the cohomology ring $H^*(Y)$ is a Poincaré duality algebra that satisfies HL(ℓ) and HR(ℓ), with ℓ an ample class. Then, it follows that the Chow ring of any matroid M representable over $\mathbb{C}$ is a Poincaré duality algebra of dimension $r(M) - 1$, that satisfies the hard Lefschetz theorem and the Hodge-Riemann bilinear relations. By Theorem 8.8 of [AHK18], this result extends fully to the Chow ring of an arbitrary matroid M.

Example 4.5. We recall that the Chow ring of the matroid M which we defined as the cycle matroid associated with the graph in Figure 1.5, is of the form $\bigoplus_{k=0}^{3} A^k(M)$, with $A^0(M) \cong \mathbb{Z}$, $A^1(M) \cong \mathbb{Z}^{27}$, $A^2(M) \cong \mathbb{Z}^{27}$, and $A^3(M) \cong \mathbb{Z}$. We then note that in this example the Chow ring of M satisfies the Poincaré duality.

4.2 The proof

In this section, we present one of the main results of [AHK18], following a mini-course held by Corrado de Concini "*Hodge theory for matroids*" at the fourth edition of the winter school "Geometry, Algebra and Combinatorics of Moduli Spaces and Configurations" in 2020 in Dobbiaco (IT).

From now on we will denote by $A^*(M)$ the Chow ring of a matroid M with rank $r+1$.

Definition 4.6. A class $\ell \in A^1(M)$ is *ample* if for any flag of flats $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_s$, there exists a representative $\sum\limits_{G \in \mathcal{F}} a_G x_G$ of ℓ such that

- $a_{F_i} = 0$ for $i = 1, \ldots, s$.
- If G is comparable to every F_i, but $G \neq F_i$, then $a_G > 0$.

If in the second part of the definition we have $a_G \geq 0$ instead of $a_G > 0$, the ample class is said to be *nef*.

We see that this definition of an ample class is analogous to the characterization of ample line bundles stated in Theorem 3.51. Indeed, in the context of the wonderful model, each stratum (smooth subvariety) is an intersection of $\{D_{F_i}\}_i$, which corresponds to a flag of flats. Then, if ℓ is ample according to Definition 4.6, it intersects positively with all strata, due to the condition $a_{G_i} > 0$. This condition is analogous to the ampleness criterion stated in Theorem 3.51.

Definition 4.7. We define two linear forms

$$\alpha_{M,i} := \sum_{i \in F} x_F, \qquad \beta_{M,i} := \sum_{i \notin F} x_F,$$

for every i in E. It is simple to see that their classes in $A^*(M)$ are independent from i, and we denote them by α_M and β_M. In particular, the independence of $\alpha_{M,i}$ from i follows directly from Definition 2.25, while the independence of $\beta_{M,i}$ from i follows from:

$$\beta_{M,i} - \beta_{M,j} = \sum_{i \notin F} x_F - \sum_{j \notin F} x_F = \sum_{i \notin F, j \in F} x_F - \sum_{j \notin F, i \in F} x_F = \sum_{j \in F} x_F - \sum_{i \in F} x_F = 0.$$

Lemma 4.8. *The set of all ample classes has the structure of a convex cone, called the ample cone. Moreover, the ample cone is non-empty. In fact, there exists $\varepsilon > 0$ such that, if $0 < \tau_F < \varepsilon$, then $\ell = \alpha_M - \sum_{F \in \mathcal{F}} \tau_F x_F$ is ample.*

Lemma 4.9. *The element $\beta_M \in A^1(M)$ is nef.*

Proof. Let $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_s$ be a flag of flats, and $j \in F_1$. If we write $\beta_M = \sum\limits_{j \notin G} x_G$, the definition of nef is easily satisfied. □

Proposition 4.10. *Let $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ be a flag of non-empty proper flats of M.*

1. *If the rank of F_i is not i for some $i \leq k$, then*
$$x_{F_1} x_{F_2} \cdots x_{F_k} \alpha_M^{r-k} = 0 \in A^r(M).$$
2. *If the rank of F_i is i for every $i \leq k$, then*
$$x_{F_1} x_{F_2} \cdots x_{F_k} \alpha_M^{r-k} = \alpha_M^r \in A^r(M).$$

Before proving the theorem, we state a useful general observation.

Remark 4.11. For any i not in F (a non-empty proper flat),

$$x_F \alpha_M = x_F \left(\sum_{i \in G} x_G \right) = x_F \left(\sum_{i \in G, F \subset G} x_G \right) \in A^*(M).$$

The flats G that contain $\{i\}$ but not F are not comparable with F, hence $x_F x_G = 0$. It follows that the sum on the right-hand side is over all proper flats that contain F and $\{i\}$. We note that if the rank of F is r, there do not exist proper flats that contain F and $\{i\}$: in this case $x_F \alpha_M$ is zero.

Proof. We prove the first assertion by descending induction on $k < r$. If $k = r - 1$, the rank or F_k has to be r, hence the product $x_{F_k} \alpha_M$ is zero by Remark 4.11. We now consider $k < r - 1$ and i not in F_k. We have

$$x_{F_1} x_{F_2} \cdots x_{F_k} \alpha_M^{r-k} = x_{F_1} x_{F_2} \cdots x_{F_k} \left(\sum_{i \in G} x_G \right) \alpha_M^{r-k-1} = \sum_{i \in G} x_{F_1} x_{F_2} \cdots x_{F_k} x_G \alpha_M^{r-(k+1)},$$

where the sum is all over the proper flats G containing F_k and $\{i\}$ (again by Remark 4.11). We apply the induction hypothesis for $k+1$ to every term of the sum on the right-hand side of the equation, which then is zero.

We prove the second part of the proposition by ascending induction on k. When $k = 1$, considering i in F_1, we get $\alpha_M^r = \left(\sum_{i \in G} x_G \right) \alpha_M^{r-1}$. By the first assertion for $k = 1$, $x_G \alpha_M^{r-1}$ is nonzero if and only if the rank of G is one, so $\alpha_M^r = x_{F_1} \alpha_M^{r-1}$, with $F_1 = \{i\}$.

We now consider $k > 1$ and i in $F_k \backslash F_{k-1}$. Using first the induction hypothesis and then Remark 4.11, we get

$$\alpha_M^r = x_{F_1} \cdots x_{F_{k-1}} \alpha_M^{r-(k-1)} = x_{F_1} \cdots x_{F_{k-1}} \left(\sum_{i \in G} x_G \right) \alpha_M^{r-k} = \sum_{i \in G} x_{F_1} \cdots x_{F_{k-1}} x_G \alpha_M^{r-k},$$

where the sum is over all proper flats G containing F_{k-1} and $\{i\}$. It follows from the first assertion that the only non zero term of the sum on the right-hand side is the one in which the rank of G is k,. We then obtain

$$\alpha_M^r = x_{F_1} x_{F_2} \cdots x_{F_{k-1}} x_{F_k} \alpha_M^{r-k}.$$

□

Definition 4.12. Let $\mathscr{F} = \{ F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\}$ be a k-step flag of non-empty proper flats of M. The flag $\mathscr{F}$ is

1. *Initial* if $r(F_i) = i$ for all $i = 1, \dots, k$.
2. *Descending* if $\min(F_1) > \min(F_2) > \cdots > \min(F_k) > 0$.

The fixed ordering of the ground set E ensures that $\min(F)$ is well-defined for any flat F of the matroid. We denote by $D_k(M)$ the set of initial descending k-step flags of non-empty proper flags of M.

Example 4.13. We show that Proposition 4.10 holds for the matroid M, the cycle matroid associated with the graph in Figure 1.5. We fix $\alpha := \alpha_M,\ \beta := \beta_M,\ \alpha_i := \alpha_{M,i}$, and $\beta_i := \beta_{M,i}$. We want to obtain, for each $k = 0, \dots, 3$, that $x_{\mathscr{F}}\alpha^{3-k} = \alpha^3$ if $\mathscr{F}$ is an initial k-step flag, and that $x_{\mathscr{G}}\alpha^{3-k} = 0$ if $\mathscr{G}$ is a non-initial k-step flag. We first consider the case $k = 1$. If $\mathscr{F} = 0123$, then $x_{\mathscr{F}}\alpha^2 = (x_{0123}\alpha_4)\alpha = 0$, since there does not exist any proper flat that contains 4 and is comparable with 0123. If $\mathscr{F} = 12$, then $x_{\mathscr{F}}\alpha^2 = (x_{12}\alpha_3)\alpha = (x_{12}x_{0123})\alpha_4 = 0$. If $\mathscr{F} = 1$, we get

$$x_{\mathscr{F}}\alpha^2 = (x_1\alpha_2)\alpha = x_1x_{12}\alpha_3 + x_1x_{124}\alpha_3 + x_1x_{125}\alpha_3 + x_1x_{0123}\alpha_4.$$

Since the only non-zero term on the right-hand side of the equation is the first one, we get $x_1\alpha^2 = x_1x_{12}\alpha_3 = x_1x_{12}x_{0123}$. We obtain that $x_1\alpha^2$ coincides with α^3, since

$$\begin{aligned}\alpha^3 &= \alpha_1\alpha^2 = \\ &= x_1\alpha^2 + x_{12}\alpha_0\alpha_4 + x_{13}\alpha_0\alpha_4 + x_{14}\alpha_0\alpha_2 + x_{15}\alpha_0\alpha_2 + x_{01}\alpha_2\alpha_4 + \\ &+ (x_{0123}\alpha_4 + x_{0145}\alpha_2 + x_{124}\alpha_0 + x_{125}\alpha_0 + x_{143}\alpha_0 + x_{153}\alpha_0)\alpha = \\ &= x_1\alpha^2 = x_1x_{12}x_{0123}.\end{aligned}$$

We now consider $k = 2$. We get that if $\mathscr{F} = 1 \subsetneq 0123$, then $x_{\mathscr{F}}\alpha = x_1x_{0123}\alpha_4 = 0$; and if $\mathscr{F} = 12 \subsetneq 0123$, then $x_{\mathscr{F}}\alpha = x_{12}x_{0123}\alpha_4 = 0$. However, if $\mathscr{F} = 1 \subsetneq 12$, then $x_{\mathscr{F}}\alpha = x_1x_{12}\alpha_3 = x_1x_{12}x_{0123} = \alpha^3$.

We now define the map deg analogously to the construction in the preceding section, namely as an isomorphism deg: $A^r(M) \to \mathbb{Z}$. We fix that the isomorphism deg maps α_M^r to 1, since by Proposition 4.10 α_M^r generates $A^r(M)$.

Our goal now is to demonstrate that $\mu^k(M)$, the k-th coefficient of the reduced polynomial of M, coincides with the degree of $\alpha_M^{r-k}\beta_M^k$, where α_M and β_M are the elements defined in Definition 4.7. The proof requires two preliminary lemmas.

Let F be a flat of a matroid $M = (E, \mathcal{I})$. We introduce the matroid M^F as the matroid with F as ground set, and $\mathcal{I}_F = \{I \in \mathcal{I} | \ I \subset F\}$ as the collection of independent sets.

Lemma 4.14. *For every $k \le r$, we have*

$$\mu^k(M) = |D_k(M)|.$$

Proof. We consider the truncation of the matroid M as defined in Definition 1.30, and we use descending induction on the rank of the matroid. As we noted in Remark 1.31, $\mu^k(M) = \mu^k(tr(M))$ for $k < r$; so if we use the induction hypothesis on $tr(M)$ (that by definition has rank r), we get $|D_k(M)| = |D_k(tr(M))| = \mu^k(tr(M)) = \mu^k(M)$ for $k < r$.

We now have to prove the assert for $k = r$. From Remark 1.28, it follows that $\mu^r(M) = (-1)^{r+1}\mu(\emptyset, E)$, and from Equation (1.3)

$$\mu(\emptyset, E) = -\sum_{0 \notin F \lessdot E} \mu(\emptyset, F) = -\sum_{0 \notin F, r(F)=r} \mu(\emptyset, F). \tag{4.1}$$

We now use the induction hypothesis on the flats F of rank r and get $(-1)^r\mu(\emptyset, F) = |D_{r-1}(M^F)|$. We call $D_r(M, F)$ the set of initial descending r-step flags of non-empty proper flats, with $F_r = F$. It is easy to see that there is a bijection between $D_{r-1}(M^F)$ and $D_r(M, F)$, given by the map

$$(F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{r-1}) \longmapsto (F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq F).$$

Another simple remark is that $D_r(M) = \dot{\bigcup}_{0 \in F, r(F)=r} D_r(M, F)$. Using Equation (4.1), then the induction hypothesis and finally these remarks, we get

$$\begin{aligned} \mu^r(M) = (-1)^{r+1}\mu(\emptyset, E) \overset{4.1}{=} \sum_{0 \notin F, r(F)=r} (-1)^r \mu(\emptyset, F) &= \sum_{0 \notin F, r(F)=r} |D_{r-1}(M^F)| \\ &= \sum_{0 \notin F, r(F)=r} |D_r(M, F)| = |D_r(M)|. \end{aligned}$$

□

Example 4.15. We prove the validity of Lemma 4.14 for the matroid M which we defined as the cycle matroid associated with the graph in Figure 1.5. We recall that $\mu^0(M) = 1,\ \mu^1(M) = 5,\ \mu^2(M) = 10,$ and $\mu^3(M) = 7$. We have:

$D_1(\mathcal{W}_4) = \{1,\ 2,\ 3,\ 4,\ 5\}$.

$D_2(\mathcal{W}_4) = \{2 \subsetneq 21,\ 3 \subsetneq 31,\ 3 \subsetneq 32,\ 4 \subsetneq 41,\ 4 \subsetneq 42,\ 4 \subsetneq 43,\ 5 \subsetneq 51,\ 5 \subsetneq 52,\ 5 \subsetneq 53,\ 5 \subsetneq 54\}$.

$D_3(\mathcal{W}_4) = \{4 \subsetneq 43 \subsetneq 431,\ 4 \subsetneq 42 \subsetneq 421,\ 5 \subsetneq 54 \subsetneq 5432,\ 5 \subsetneq 54 \subsetneq 541,\ 5 \subsetneq 53 \subsetneq 5432,$
$5 \subsetneq 53 \subsetneq 5231, 5 \subsetneq 52 \subsetneq 521\}$.

We then get $|D_0(M)| = 1$ trivially, $|D_1(M)| = 5,\ |D_2(M)| = 10,$ and $|D_3(M)| = 7$.

Lemma 4.16. *For every $0 < k \leq r$, we have*

$$\beta_M^k = \sum_{\mathscr{F}} x_{\mathscr{F}} \in A^*(M),$$

where the sum is over all descending k-step flags of non empty proper flats of M.

Proof. We use ascending induction on k. When $k = 1$, the only initial descending 1-step flags of proper flats are the single flats F, with $0 \notin F$. We then get

$$\beta_M = \beta_{M,0} = \sum_{0 \notin F} x_F = \sum_{\mathscr{F}} x_{\mathscr{F}} \in A^*(M).$$

For $k > 1$, we use the induction hypothesis and obtain $\beta_M^{k+1} = \sum_{\mathscr{F}} \beta_M x_{\mathscr{F}}$, where the sum is over all descending k-step flags of non empty proper flats of M. For each of these flags $\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\}$, we take $i_{\mathscr{F}} := \min(F_1)$ and $\sum_{i_{\mathscr{F}} \notin F} x_F$ as representative of β_M. We then have

$$\beta_M^{k+1} = \sum_{\mathscr{F}} \beta_M x_{\mathscr{F}} = \left(\sum_{i_{\mathscr{F}} \notin F} x_F \right) x_{\mathscr{F}} = \sum_{\mathscr{G}} x_{\mathscr{G}}.$$

The sum on the right-hand side is over all descending flags of non-empty proper flats of M of the form $\mathscr{G} = \{F \subsetneq F_1 \subsetneq \cdots \subsetneq F_k\}$, since we chose $i_{\mathscr{F}}$ to be $\min(F_1)$. □

Example 4.17. We prove that matroid M which we defined as the cycle matroid of the graph in Figure 1.5 satisfies this lemma. We have $\beta = \sum_{0 \notin F} x_F$ for the definition of β. Moreover,

$$\beta^2 = \beta_1 \left(\sum_{0 \notin F,\ 1 \in F} x_F \right) + \beta_2 \left(\sum_{0,1 \notin F,\ 2 \in F} x_F \right) + \cdots + \beta_4 \left(\sum_{0,1,2,3 \notin F,\ 4 \in F} x_F \right).$$

The first term of the sum represents the descending flags $x_F x_G$ with $\min(G) = 1$, the second term the descending flags $x_F x_G$ with $\min(G) = 2$, and so on. We now compute β^3 as $\beta^3 = \beta\beta^2$, with β^2 the sum of all descending 2-step flags $x_F x_G$, with $F \subsetneq G$.

$$\beta^3 = \beta_2 \left(\sum_{0,1 \notin F,\ 2 \in F} x_F x_G \right) + \cdots + \beta_4 \left(\sum_{0,1,2,3 \notin F,\ 4 \in F} x_F x_G \right).$$

The first term represents the 3-step descending flags $x_F x_G x_H$ with $\min(H) = 1$, the second term the 3-step descending flags $x_F x_G x_H$ with $\min(H) = 2$, and so on.

We now have all the elements to prove the following proposition.

Proposition 4.18. *For every $0 \leq k \leq r$, we have*

$$\mu^k(M) = deg(\alpha_M^{r-k}\beta_M^k).$$

Proof. By Lemma 4.16 we know that $\beta_M^k = \sum_{\mathscr{F}} x_{\mathscr{F}} \in A^*(M)$, where the sum is over all descending k-steps flags of non empty proper flats of M.
We then have

$$\alpha_M^{r-k}\beta_M^k = \sum_{\mathscr{F}} \alpha_M^{r-k} x_{\mathscr{F}}.$$

By Proposition 4.10, if $\mathscr{F}$ is an initial descending flag, then $x_{\mathscr{F}}\alpha_M^{r-k} = \alpha_M^r$; if $\mathcal{F}$ is not initial, then $x_{\mathscr{F}}\alpha_M^{r-k} = 0$. Then,

$$\deg(\alpha_M^{r-k}\beta_M^k) = \deg\left(\sum_{\mathscr{F}\in D_k(M)} \alpha_M^r\right) = \sum_{\mathscr{F}\in D_k(M)} \deg(\alpha_M^r) = |D_k(M)|.$$

Since $\deg(\alpha_M^r) = 1$, the assertion then follows from Lemma 4.14. □

Example 4.19. We prove that the cycle matroid M of the graph in Figure 1.5 satisfies this proposition. For each $k = 0, \dots, 3$, we want to show that $\deg(\alpha^{3-k}\beta^k) = |D_k(M)|$. We just saw that β^3 is the sum of 3-step (so initial) descending flags: $\deg(\beta^3) = |D_3(M)|$. Moreover, we also recall that β^2 is the sum of all descending 2-step flags, it then follows that $\alpha\beta^2 = \alpha_0 \sum x_F x_G = \sum x_i x_{cl(ij)} x_{cl(ij0)}$. The terms of this sum are in bijection with the 2-step initial descending flags ($x_i x_{cl(ij)} x_{cl(ij0)} \mapsto x_i x_{cl(ij)}$), hence $\deg(\alpha\beta^2) = |D_2(M)|$. Furthermore, we write $\alpha^2\beta$ as $\alpha_0(\alpha_0\beta_0)$. We have that $\alpha_0\beta_0 = \sum x_i x_{cl(i0)} + \sum x_i x_{cl(ij0)} + \sum x_{cl(ij)} x_{cl(ij0)}$. We multiply again by α_0 the last two sums, and each element of the first sum by α_k (with $k \notin cl(i0)$). We get that just $\sum x_i x_{cl(i0)} \alpha_k = \sum x_i x_{cl(i0)} x_{cl(i0k)}$ is not zero, and that the terms of this sum are in bijection with the 1-step descending initial flags: $x_i x_{cl(i0)} x_{cl(i0k)} \mapsto x_i$. We then get $\deg(\alpha^2\beta) = |D_1(M)|$. Lastly, $\deg(\alpha^3) = 1$ for the definition of deg.

We now show how the validity of the hard Lefschetz theorem and of the Hodge-Riemann bilinear relations for $A^*(M)$ is the key of the proof of Conjecture 1.27.

Proposition 4.20. *Let M be a matroid realizable over $\mathbb{C}$, and ℓ_1 and ℓ_2 be two elements of $A^1(M)$. If ℓ_2 is nef, we get*

$$deg(\ell_1\ell_1\ell_2^{r-2})deg(\ell_2\ell_2\ell_2^{r-2}) \leq deg(\ell_1\ell_2\ell_2^{r-2})^2.$$

Proof. We first consider ℓ_2 ample. We recall the definition of the Hodge-Riemann bilinear form associated to ℓ_2:

$$Q_{\ell_2}^1 : A^1(M) \times A^1(M) \longrightarrow \mathbb{R} \qquad (a_1, a_2) \longmapsto -\deg(a_1\ell_2^{r-2}a_2).$$

Since M is a representable matroid over $\mathbb{C}$, $A^*(M)$ satisfies HL(ℓ_2) and HR(ℓ_2). It then follows that the Lefschetz decomposition of $A^1(M)$ is valid :

$$A^1(M) = \langle \ell_2 \rangle \oplus P^1_{\ell_2}(M).$$

From HR(ℓ_2), it follows that $Q^1_{\ell_2}$ positive definite on $P^1_{\ell_2}(M)$ and negative definite on $\langle \ell_2 \rangle$. Indeed, $Q^1_{\ell_2}(c\ell_2, c\ell_2) = -c^2 \deg(\ell_2) < 0$, since $\deg(\ell_2) > 0$ because ℓ_2 is ample. We now want to compute the signature of $Q^1_{\ell_2}$ restricted to the subspace $\langle \ell_1, \ell_2 \rangle$.
Since $Q^1_{\ell_2}$ restricted to $\langle \ell_2 \rangle$ is negative definite, the signature is $(1, 1, 0)$ or $(0, 1, 1)$. Hence, the determinant of the matrix $\begin{pmatrix} Q^1_{\ell_2}(\ell_1, \ell_1) & Q^1_{\ell_2}(\ell_1, \ell_2) \\ Q^1_{\ell_2}(\ell_2, \ell_1) & Q^1_{\ell_2}(\ell_2, \ell_2) \end{pmatrix}$ is minor or equal to zero. Then,

$$Q^1_{\ell_2}(\ell_1, \ell_1) Q^1_{\ell_2}(\ell_2, \ell_2) \leq Q^1_{\ell_2}(\ell_1, \ell_2)^2.$$

Replacing $Q^1_{\ell_2}$ with $-\deg(a_1 \ell_2^{r-2} a_2)$, we obtain

$$\deg(\ell_1 \ell_1 \ell_2^{r-2}) \deg(\ell_2 \ell_2 \ell_2^{r-2}) \leq \deg(\ell_1 \ell_2 \ell_2^{r-2})^2.$$

We now prove the statement when ℓ_2 is nef. Since the ample cone is non-empty by Lemma 4.8, we can choose ℓ an ample class. It is easy to see that, if ℓ_2 is nef and ℓ is ample, then $\ell_2(t) := l_2 + t\ell$ is ample for any t positive real number. For the first part of the proof we have

$$\deg(\ell_1 \ell_1 \ell_2(t)^{r-2}) \deg(\ell_2(t) \ell_2(t) \ell_2(t)^{r-2}) \leq \deg(\ell_1 \ell_2(t) \ell_2(t)^{r-2})^2.$$

We obtain the statement by taking the limit $t \to 0$. □

Theorem 4.21 (Theorem 8.8 in [AHK18]). *The Chow ring of a generic matroid M satisfies the hard Lefschetz theorem and the Hodge-Riemann bilinear relations.*

Proof. The proof involves a delicate double induction argument, which we omit here as it falls beyond the scope of this thesis. □

From Theorem 4.21, it follows that Proposition 4.20 holds also for a generic matroid M. We are now ready to prove Conjecture 1.27.

Theorem 4.22. *Let M be a matroid of rank $r+1$. For every integer $0 < k < r$, we have*

$$\mu^{k-1}(M) \mu^{k+1}(M) \leq \mu^k(M)^2.$$

Proof. We use descending induction on $r(M)$. If $k < r - 1$, we apply the induction hypothesis on the truncation of M, and conclude using $\mu^k(tr(M)) = \mu^k(M)$ for $k < r-1$ (Remark 1.31). When $k = r - 1$, by Proposition 4.18 the statement is equivalent to

$$\deg(\alpha_M^2 \beta_M^{r-2}) \deg(\beta_M^2 \beta_M^{r-2}) \leq \deg(\alpha_M^1 \beta_M^{r-1})^2.$$

This inequality follows from Proposition 4.20 with $\ell_2 = \beta_M$, which is nef by Lemma 4.9. □

Corollary 4.23 (Conjecture 1.27)**.** *Let M be a matroid of rank $r+1$. We denote by $w_0, \ldots, w_{r+1}$ the Whitney's numbers of the first kind of M. For every positive integer $k \leq r$, we have*

$$w_{k-1}w_{k+1} \leq w_k^2.$$

Proof. The assertion follows from Theorem 4.22 and Remark 1.28. □

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