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Rationality of cubic hypersurfaces in low dimension

Tesi di laurea in geometria algebrica

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Introduction

The study of cubic hypersurfaces has been a central topic in algebraic geometry for over a century. These hypersurfaces, defined as the zero loci of homogeneous polynomials of degree three in projective space, exhibit rich geometric properties. The investigation into their structure dates back to the 19th century with the work of mathematicians such as Cayley and Salmon, who studied cubic surfaces and their 27 distinguished lines. In the 20th century, major advances were made in understanding cubic threefolds, particularly through the work of Clemens and Griffiths [CG72], who proved their irrationality using intermediate Jacobians. More recently, cubic fourfolds have been extensively studied due to their deep connections with K3 surfaces, Hodge theory, and derived categories.

A fundamental and long-standing question in the study of cubic hypersurfaces is their rationality. The rationality problem asks whether a given algebraic variety is birationally equivalent to a projective space. While cubic surfaces are always rational, the situation becomes much more intricate in higher dimensions. Clemens and Griffiths famously demonstrated that smooth cubic threefolds are irrational by showing that their intermediate Jacobians are non-isomorphic to the Jacobian of a curve as principally polarized abelian varieties. This result provided one of the first concrete examples of a unirational but non-rational variety, exhibiting a counter-example to the long-standing Lüroth problem which asks wether unirationality implies rationality.

In contrast, the rationality of cubic fourfolds remains an open question, with only partial progress made through lattice-theoretic and derived category approaches. Hassett's work on special cubic fourfolds has provided a framework for understanding which of these hypersurfaces might be rational, linking the problem to the existence of associated K3 surfaces. The possibility that rational cubic fourfolds correspond to special divisors in their moduli space remains one of the most compelling open problems in algebraic geometry.

In this thesis, cubic hypersurfaces are examined with an emphasis on their geometric and cohomological properties, particularly in the context of the rationality problem. The approach taken includes the study of abelian and Prym varieties in connection with cubic threefolds.

Chapter 1 provides foundational material necessary for the study, including key results from complex algebraic geometry, such as Serre duality, the Lefschetz hyperplane theorem, and basic notions of Hodge theory. These concepts set the stage for the constructions appearing in later chapters. Chapter 2 introduces abelian and Prym varieties, which play an essential role in understanding the geometry of cubic hypersurfaces, particularly in relation to their Jacobian and theta divisors. The structure of Prym varieties is exploited to analyze cubic threefolds through their associated quadric fibrations.

Chapters 3 and 4 form the core of this thesis. Chapter 3 focuses on cubic threefolds, particularly their irrationality and Torelli-type theorem. The fundamental result of Clemens and Griffiths on the irrationality of a smooth cubic threefold is studied by analyzing the Prym variety associated with a quadric fibration.

Chapter 4 turns to cubic fourfolds, a class of hypersurfaces that have attracted significant attention due to their connection with K3 surfaces and conjectures on their rationality. The relation between cubic fourfolds and K3 categories is examined, particularly through the lattice-theoretic framework developed by Hassett. A key focus is on special cubic fourfolds and their defining properties, including examples of specific surfaces that play a role in rationality questions. Additionally, the Hodge-theoretic perspective is employed to examine conjectures linking cubic fourfolds to K3 surfaces.

Contents

Chapter 1

Preliminaries

1.1 Complex geometry and cohomology tools

In this section, we present some foundational results and constructions that will serve as essential tools throughout the thesis. This includes key short exact sequences and fundamental theorems that form the backbone of the arguments and computations in later sections. By collecting these results here, we aim to provide a convenient reference for applications in subsequent chapters. These tools are standard yet indispensable in modern algebraic geometry and related fields, providing the foundations for many of the results discussed in this work.

The main reference for this section will be [Huy05].

Theorem 1.1.1 (Exponential sequence)**.** *The exponential sequence of sheaves on a complex manifold X*

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0 \tag{1.1.1}$$

is exact.

Theorem 1.1.2 (Euler sequence)**.** *The Euler sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\bigoplus n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0 \tag{1.1.2}$$

is exact.

Theorem 1.1.3. *Let D be an effective Cartier divisor on a complex manifold X. Then*

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \tag{1.1.3}$$

is exact.

Definition 1.1.4 (Normal bundle sequence)**.** Let X be a complex manifold and $Y \subset X$ a complex submanifold we define the *normal bundle* as the cokernel of the injection $\mathcal{T}_Y \hookrightarrow \mathcal{T}_X|_Y$

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{N}_{Y/X} \to 0 \tag{1.1.4}$$

Next we want to highlight three fundamental results in complex algebraic geometry: Serre's duality, the Akizuki–Kodaira–Nakano vanishing theorem, and the weak Lefschetz theorem. Together, these theorems provide essential tools for the study of the cohomological, topological and geometric properties of projective varieties.

Theorem 1.1.5 (Serre duality)**.** *Let X be a compact complex manifold of dimension n. Then, for every vector bundle E, there are natural isomorphisms:*

$$H^q(X, \Omega_X^p \otimes E) \cong H^{n-q}(X, \Omega_X^{n-p} \otimes E^\vee)^\vee.$$

The next definition gives the notion of polarization which will be crucial throughout the whole thesis.

Definition 1.1.6. Let X be a compact Kähler manifold of dimension n and L a line bundle on X. L is called *ample* if and only if L^k for some $k > 0$ defines a closed embedding $\varphi_{L^k} : X \to \mathbb{P}^N$. The datum of an ample line bundle is called polarization.

Theorem 1.1.7 (Akizuki-Kodaira-Nakano vanishing)**.** *Let X be a compact Kähler manifold of dimensio n. Then*

$$H^q(X, \Omega_X^p \otimes L) = 0 \text{ whenever } p + q > n \text{ and } L \text{ ample}$$
$$H^q(X, \Omega_X^p \otimes L) = 0 \text{ whenever } p + q < n \text{ and } L^\vee \text{ ample}$$

Remark 1.1.8. In the following, we state and prove the Lefschetz's Hyperplane section theorem in the context of cohomology with complex coefficients. It is worth noting that the theorem also holds for integral cohomology, but for simplicity, we will give a proof only of the complex case. Here is the more general statement:

Let X be an n-dimensional complex projective algebraic variety in $\mathbb{P}^N$, and let Y be a hyperplane section of X such that

$$U = X \setminus Y$$

is smooth. The Lefschetz theorem refers to any of the following statements:

1. The natural map in singular homology,

$$H_k(Y,\mathbb{Z}) \to H_k(X,\mathbb{Z}),$$

is an isomorphism for $k < n-1$ and is surjective for $k = n-1$.

2. The natural map in singular cohomology,

$$H^k(X,\mathbb{Z}) \to H^k(Y,\mathbb{Z}),$$

is an isomorphism for $k < n-1$ and is injective for $k = n-1$.

3. The natural map in homotopy groups,

$$\pi_k(Y) \to \pi_k(X),$$

is an isomorphism for $k < n-1$ and is surjective for $k = n-1$.

Remark 1.1.9. We need to prove beforehand an excercise of [Huy05] which states that any short exact sequence of holomorphic vector bundles

$$0 \to L \xrightarrow{f} E \xrightarrow{g} F \to 0,$$

where L is a line bundle, induces short exact sequences of the form

$$0 \to L \otimes \wedge^{i-1} F \to \wedge^i E \to \wedge^i F \to 0.$$

Clearly, g induces surgective maps $g_i : \wedge^i E \to \wedge^i F$. Now, consider local trivializations (U,ϕ_L), (U,ϕ_E), (U,ϕ_F) of L, E, and F, respectively. For a point $x \in U$, we define a map $L \otimes \wedge^{i-1} F(x) \to \wedge^i E(x)$ as $l \otimes v \in L \otimes \wedge^{i-1} F(x)$ to $f(l) \wedge g_{i-1}^{-1}(v)$, and extends by linearity. Since L is a line bundle, this map is independent of the choice of the preimage of v and independent of the local trivialization. Hence, it can be glued to a bundle map $h : L \otimes \wedge^{i-1} F \to \wedge^i E$. Now, we claim that h is injective at every fiber to deduce that h is injective. If $h(l \otimes v) = 0$, then there exists a component v_j of v such that $f(l) = g^{-1}(v_j)$ or $f(l) = 0$ or $g^{-1}(v_j) = 0$. By the exactness of the original sequence, this implies that $v_j = 0$ or $l = 0$, which in turn implies that $l \otimes v = 0$.

Actually, the decomposable case above implies the injectivity of h since we can choose a basis of $E(x)$. The exactness of the new sequence at $\wedge^i E$ follows directly from the exactness of the original sequence at E.

Theorem 1.1.10 (Lefschetz Hyperplane Section). *Let X be a compact Kähler manifold of dimension n and let $Y \subset X$ be a smooth hypersurface such that the induced line bundle $\mathcal{O}_X(Y)$ is ample. Then the canonical restriction map*

$$H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C}) \tag{1.1.5}$$

is bijective for $k < n-1$ and injective for $k \leq n-1$.

Remark 1.1.11. Notice that the restriction map in (1.1.5) is compatible with the bidegree decomposition $H^k = \bigoplus H^{p,q}$. Hence it suffices to prove that the map $H^q(X, \Omega_X^p) \to H^q(X, \Omega_Y^p)$ is bijective for $p+q < n-1$ and injective for $p+q \leq n-1$.

Proof. twisting 1.1.3 by Ω_X^p we get a short exact sequence:

$$0 \to \Omega_X^p(-Y) \to \Omega_X^p \to \Omega_X^p|_Y \to 0 \tag{1.1.6}$$

Consider now the dual of 1.1.4 (where $\mathcal{O}_Y(Y) \cong \mathcal{N}_{Y/X}$):

$$0 \to \mathcal{O}_Y(-Y) \to \Omega_X|_Y \to \Omega_Y \to 0$$

since $Y \subset X$ is a divisor and hence the normal bundle (and so its dual) has rank one this yields (by the above remark):

$$0 \to \Omega_Y^{p-1}(-Y) \to \Omega_X^p|_Y \to \Omega_Y^p \to 0 \tag{1.1.7}$$

Now,

$$\begin{aligned} H^q(X, \Omega_X^p(-Y)) &= H^q(X, \Omega_X^p \otimes \mathcal{O}(Y)^\vee) \\ &= 0 \quad \text{for } p+q < n \end{aligned}$$

thanks to AKN vanishing 1.1.7 since $\mathcal{O}_Y(-Y)^\vee$ is ample.
The latter combined with the long exact cohomology sequence induced by 1.1.6:

$$\cdots \to H^q(X, \Omega_X^p(-Y)) \to H^q(X, \Omega_X^p) \to H^q(X, \Omega_X^p) \to H^{q+1}(X, \Omega_X^p(-Y)) \to \ldots$$

implies that the natural restriction map $H^q(X, \Omega_X^p) \to H^q(X, \Omega_X^p|_Y)$ is bijective for $p+q < n-1$ and at least injective for $p+q < n$.

To get the statement we compose this map with the natural map $H^q(X, \Omega_X^p|_Y) \to H^q(X, \Omega_Y^p)$ whose kernel and cokernel are contained in the cohomology groups of the bundle $\Omega_Y^p(-Y)$ thanks to the long exact cohomology sequence induced by 1.1.7:

$$\cdots \to H^q(X, \Omega_Y^{p-1}(-Y)) \to H^q(X, \Omega_X^p|_Y) \to H^q(X, \Omega_Y^p) \to H^{q+1}(X, \Omega_Y^{p-1}(-Y)) \to \ldots$$

Since the restriction of $\mathcal{O}(Y)$ to Y is again positive we can apply the AKN vanishing 1.1.7 to coclude. □

Remark 1.1.12. The Lefschetz hyperplane section theorem extends naturally to the setting of complete intersections of ample divisors just by iterating its application.

Example 1.1.13. Let X a smooth complete intersection of three quadrics in $\mathbb{P}^5$. X is cut out by three general sections of $\mathcal{O}_{\mathbb{P}^5}(2)$ which is very ample on $\mathbb{P}^5$ so at least ample when restricted to the subvariety, so we can iterate the application of the above theorem to get:

$$\begin{array}{ccccc} ? & & ? & & ? \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Example 1.1.14. Despite being a very powerful tool there are cases when the applicability of the hyperplane section theorem fails. For example, consider X a product of projective spaces $X := \mathbb{P}^1 \times \mathbb{P}^N$ with $N > 1$. Thanks to the algebraic version of the Künneth formula:

$$h^{u,v}(M \times N) = \sum_{p+r=u} \sum_{q+s=v} h^{p,q}(M) h^{r,s}(N). \tag{1.1.8}$$

(see for reference [GH14, p. 105]]) we know the whole Hodge diamond of X:

$$\begin{array}{ccccccccc} & \cdots & & \cdots & & \cdots & & \cdots & \\ 0 & & 0 & & 2 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 2 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Consider now a hypersurface $Y \subset X$ cut out by a section of $\mathcal{O}(1,0)$. Then $Y \cong \mathbb{P}^N$, and its Hodge numbers should correspond to those described in 1.2.1. For $N = 2$, the injection stated in the theorem fails. Moreover, for $N > 2$, the isomorphism on the lower-degree cohomology groups also fails. This failure arises because the bundle $\mathcal{O}(1,0)$, which is the normal bundle to Y, is not ample.

Theorem 1.1.15 (Adjunction). *Let Y be a submanifold of a complex manifold X, both compact. Then the canonical bundle K_Y of Y is naturally isomorphic to the line bundle $K_X|_Y \otimes \det(\mathcal{N}_{Y/X})$.*

Theorem 1.1.16 (Hard Lefschetz). *Let X be a compact Kähler then for every $k \leq n = \dim X$, the map*

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism.

Here the operator L is defined as the cup product with the Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R})$.

1.2 Jacobian ring

Consider $X \hookrightarrow \mathbb{P}^{n+1}$ where X is a smooth hypersurface defined by a degree d polynomial f. In the previous section we saw that thanks to 1.1.10 we are able to know the whole Hodge diamond of such X but the middle row. We want to introduce now a widely used tool to compute the remaining Hodge numbers. To start we need the definitions of primitive and vanishing cohomology.

The primitive cohomology is:

$$H^k(X, \mathbb{Z})_{\text{prim}} := \ker\left(L^{n-k+1} : H^k(X, \mathbb{Z}) \to H^{2n-k+2}(X, \mathbb{Z})\right) \quad \text{for} \quad k \leq n$$

$$H^{p,q}(X)_{\text{prim}} = H^{p,q}(X) \cap H^k(X, \mathbb{Z})_{\text{prim}}$$

Remark 1.2.1. Notice that thanks to 1.1.16 the definition can be extended to $k > n$ too.

While the vanishing cohomology $H^k(X, \mathbb{Q})_{van}$ is the orthogonal to the image of the cohomology of the ambient space under the inclusion given in theorem 1.1.10.

Recall the Hodge diamond of $\mathbb{P}^{n+1}$:

$$H^q(\mathbb{P}^{n+1}, \Omega^p_{\mathbb{P}^{n+1}}) \cong \begin{cases} \mathbb{C} & \text{if } p = q \leq n+1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.1}$$

$$\begin{array}{ccccccccc} \dots & & \dots & & 1 & & \dots & & \dots \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

(an explicit computation could be made by dualizing 1.1.2 and taking the induced short exact sequence the way we took 1.1.7)

Remark 1.2.2. The vanishing cohomology $H^k(X,\mathbb{Q})_{van} = 0$ for $k \neq n = \dim_{\mathbb{C}}(X)$ for dimension reasons thanks again to 1.1.10.

Take then $k = n = \dim_{\mathbb{C}} X$. While in the general case $H^k(X,\mathbb{Q})_{\text{van}} \subseteq H^k(X,\mathbb{Q})_{\text{prim}}$ in our setting this two definitions coincide; indeed if $n \equiv 1 \quad (2)$ the target space of the Lefschetz operator is 0-dimensional so the whole $H^n(X,\mathbb{Q})$ is primitive, and vanishing too because $H^k(\mathbb{P}^{n+1},\mathbb{Q}) = 0$ for odd k. Otherwie if $k \equiv 0 \quad (2)$ the target space of the Lefschetz operator is 1-dimensional so by linear algebra

$$\dim\Big(\ker(L : H^n(X,\mathbb{Q}) \to H^{n+2}(X,\mathbb{Q}))\Big) = \dim(H^n(X,\mathbb{Q})) - 1$$

and it coincides with the orthogonal of the image of the inclusion of theorem 1.1.10. (For the remainder of the section we will speak in terms of vanishing cohomology). From 1.2.1 we also deduce that $h^{p,q}(X) = h^{p,q}(X)_{\text{van}}$ whenever $p \neq q$ and that $h^{p,p}(X) = h^{p,p}(X)_{\text{van}} + 1$.

Denote

$$R_f := \frac{\mathbb{C}[x_0,\dots,x_{n+1}]}{J_f}$$

the so called Jacobian ring, where $J_f := (\frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_{n+1}})$ is the ideal generated by all partial derivatives of f. [Voi03, Proposition 6.2] states that the vanishing cohomology groups of X are isomorphic to some homogeous components of the Jacobian ring:

$$[R_f]^{(p+1)d-n-2} \cong H^{p,n-p}(X)_{\text{van}}$$

Remark 1.2.3. It is a deep result that Hodge numbers are invariant under deformation, see [Voi02, p. 235].

Let us work through some examples:

Example 1.2.4. Let X be a smooth cubic threefold in $\mathbb{P}^4$. Sicne Hodge numbers are invariat under deformation we can take, without loss of generality, as nonsingular smooth polynomial: $f = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3$. Then we have:

$$h^{3,0}(X) = h^{0,3}(X) = 0 \quad \text{by adjunction} \quad 1.1.15$$

and,

$$h^{2,1}(X) = h^{1,2}(X) = h^{2,1}(X)_{\mathrm{van}} = \dim\left(\left[\frac{\mathbb{C}[x_0, x_1, x_2, x_3, x_4]}{(x_0^2, x_1^2, x_2^2, x_3^2, x_4^2)}\right]^{(2+1)3-3-2}\right) = \binom{5}{4} = 5$$

so the Hodge diamond of a smooth cubic threefold is:

$$\begin{array}{ccccccc} 0 & & 5 & & 5 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Example 1.2.5. Let X be a smooth cubic fourfold in $\mathbb{P}^5$, take again without loss of generality the Fermat cubic: $f = x_0^3 + \cdots + x_5^3$.
We have, again by adjunction 1.1.15:

$$h^{4,0}(X) = h^{0,4}(X) = 0$$

and the Jacobian ring computation yieds:

$$h^{3,1}(X) = h^{1,3}(X) = h^{3,1}(X)_{\mathrm{van}} = \dim\left(\left[\frac{\mathbb{C}[x_0, \dots, x_5]}{(x_0^2 \dots, x_5^2)}\right]^{(3+1)3-4-2}\right) = 1$$

$$h^{2,2}(X) = h^{2,2}(X)_{\mathrm{van}} + 1 = \dim\left(\left[\frac{\mathbb{C}[x_0, \dots, x_5]}{(x_0^2 \dots, x_5^2)}\right]^{(2+1)3-4-2}\right) + 1 = \binom{6}{3} + 1 = 21$$

hence the Hodge diamond of a smooth cubic fourfold is:

$$\begin{array}{ccccccccc} 0 & & 1 & & 21 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

1.3 Chern classes and the 27 lines

Classifying all non-isomorphic vector bundles over a fixed base space is often a challenging problem. Characteristic classes are introduced as topological invariants that help distinguish different classes of vector bundles. These classes live in the cohomology groups of the base space and encode important geometric information. Among characteristic classes, Chern classes are specifically associated with complex vector bundles. They consist of a sequence of invariants $c_1, c_2, \ldots$, where each Chern class $c_i(E) \in H^{2i}(X;\mathbb{Z})$ depends solely on the isomorphism type of the vector bundle $E \to X$. This sequence is uniquely determined by a set of defining properties, which will be outlined in this section.

The same approach could be applied more generally, but for our purposes, let us work on the following framework: let $X \subset \mathbb{P}^N$ be smooth and projective of dimension n and let $E \to X$ be a rank r vector bundle.

Definition 1.3.1. We define $c_i(E) \in H^{2i}(X,\mathbb{Z})$ and the total Chern class $c(E) = \sum c_i(E)$ as classes in cohomology satisfying the following properties:

1. $c_0(E) = 1$
2. (*Functoriality*) If $\phi : Y \to X$ is a morphism of smooth varieties, then
$$\phi^*(c(E)) = c(\phi^* E).$$
3. (*Whitney's formula*) If
$$0 \to E \to F \to G \to 0$$
is a short exact sequence of vector bundles on X, then
$$c(F) = c(E)c(G).$$
4. $c(\mathcal{O}_{\mathbb{P}^n}(1)) = 1 + H$, where H is the hyperplane class of $\mathbb{P}^n$

Remark 1.3.2. When we write $c(X)$ without specifing the vector bundle we mean $c(T_X)$.

Here we collect some properties of the Chern classes that make computations easier and can be proven from the above axioms.

(i) This result can be found in [BT13, p. 275].

Theorem 1.3.3 (Splitting principle)**.** *To prove a polynomial identity in the Chern classes of complex vector bundles, it suffices to prove it under the assumption that the vector bundles are direct sums of line bundles.*

(ii) (*Trivial line bundle*) $c(\mathcal{O}_X) = 1$

(iii) (*Vanishing*) if E is a rank r vector bundle, then $c_i(E) = 0 \ \forall \ i > r$ and $i > n$.

(iv) (*Direct sum*) Let $E = \bigoplus L_i$, then

$$c(E) = \prod c(L_i) = \prod (1 + c_1(L_i))$$

(v) (*Dual*) $c_i(E^\vee) = (-1)^i c_i(E)$

(vi) (*Gauss-Bonnet-Chern*) $e(X) = \deg(c_n(X))$.

(vii) (*Tensor product*) If E has rank r and F has rank s, then

$$c_1(E \otimes F) = sc_1(E) + rc_1(F).$$

If E has rank r and L has rank 1, then

$$c_i(E \otimes L) = \sum_{j=0}^{i} \binom{r-j}{i-j} c_j(E) c_1(L)^{i-j}.$$

(viii) (*Exterior power*) $c_1(\det(E)) = c_1(E)$

(ix) (*Symmetric power*) If E has rank r, then

$$c_1(\operatorname{Sym}^2 E) = (r+1)c_1(E),$$

A classical result in algebraic geometry is that a smooth cubic surface in projective three-space contains exactly 27 lines, a discovery first attributed to mathematicians such as Cayley and Salmon. The computation of these lines has traditionally relied on explicit parametrizations or geometric constructions. However, a more modern approach involves Chern classes which encode essential topological and geometric information about vector bundles. Before getting deep into this example, we need a preliminary computation.
Let E be a rank 2 vector bundle, (thanks to 1.3.3)

$$\begin{aligned} c(\operatorname{Sym}^3 E) = c(\operatorname{Sym}^3(L_1 \oplus L_2)) &= c(L_1^{\otimes 3} \oplus (L_1^{\otimes 2} \otimes L_2) \oplus (L_1 \otimes L_2^{\otimes 2}) \oplus L_2^{\otimes 3}) \\ &= c(L_1^{\otimes 3}) c(L_1^{\otimes 2} \otimes L_2) c(L_1 \otimes L_2^{\otimes 2}) c(L_2^{\otimes 3}) \end{aligned}$$

Using properties above we get: (omitting higher degree terms)

$$c(L_1^{\otimes 3}) = (1+\alpha_1)^3 = 1 + 3\alpha_1$$
$$c(L_1^{\otimes 2} \otimes L_2) = (1+\alpha_1)^2(1+\alpha_2) = 1 + \alpha_2 + 2\alpha_1$$
$$c(L_1 \otimes L_2^{\otimes 2}) = (1+\alpha_1)(1+\alpha_2)^2 = 1 + \alpha_1 + 2\alpha_2$$
$$c(L_2^{\otimes 3}) = (1+\alpha_2)^3 = 1 + 3\alpha_2$$

now, since $\operatorname{rank}(\operatorname{Sym}^3(E)) = 4$ (indeed the Sym^k of a rank r vector bundle has rank $\binom{r+k-1}{k}$) we get:

$$c_1(\operatorname{Sym}^3 E) = 6\alpha_1 + 6\alpha_2 = 6c_1(E)$$
$$c_2(\operatorname{Sym}^3 E) = 11\alpha_1^2 + 32\alpha_1\alpha_2 + 11\alpha_2^2 = 11c_1(E)^2 + 10c_2(E)$$
$$c_3(\operatorname{Sym}^3 E) = 6\alpha_1^3 + 48\alpha_1^2\alpha_2 + 48\alpha_1\alpha_2^2 + 6\alpha_2^3 = 6c_1(E)^3 + 30c_2(E)c_1(E)$$
$$c_4(\operatorname{Sym}^3 E) = 18\alpha_1^3\alpha_2 + 45\alpha_1^2\alpha_2^2 + 18\alpha_1\alpha_2^3 = 9c_2(E)(2c_1(E)^2 + c_2(E))$$

Recall the setting: let $X \subset \mathbb{P}^3$ be a smooth cubic surface cut out by a homogeneous polynomial $f \in \operatorname{Sym}^3(V_4^\vee)$. Consider (the Fano variety of lines)

$$F_1(X) = \{\ell \subset \mathbb{P}^3 \mid \ell \subset X\}$$

this is a subvariety of the variety of lines in $\mathbb{P}^3$ i.e. the planes $\Pi \subset V_4$, that is, $\mathbb{G}r(1,\mathbb{P}^3) \cong \operatorname{Gr}(2,4)$. To get a better understanding of $F_1(X) \subset \operatorname{Gr}(2,4)$ consider:

$$0 \to \mathcal{U} \to \mathcal{O}_G^{\oplus 4} \to \mathcal{Q} \to 0$$

where $\mathcal{U}$ is the natural rank-2 vector bundle whose fiber over a point $[\Pi] \in \operatorname{Gr}(2,4)$ is the 2-dimensional vector space Π and $\mathcal{Q}$, the quotient bundle is another rank-2 vector bundle whose fiber over a point $[\Pi]$ is the quotient of the 4-dimensional vector space over the plane Π.
Since Sym^i preserves surjections, taking the dual and symmetric power this yields a surjection:

$$\varphi : \operatorname{Sym}^3(V_4^\vee \otimes \mathcal{O}_G) \to \operatorname{Sym}^3(\mathcal{U}^\vee)$$

then the composition $\varphi \circ f$ makes f into a global section of $\operatorname{Sym}^3(\mathcal{U}^\vee)$. Thanks to this observation $F_1(X)$ can be seen in $\operatorname{Gr}(2,4)$ cut out by a general section of $\operatorname{Sym}^3(\mathcal{U}^\vee)$. As a result of this argument, we are now able to compute the dimension of $(F_1(X))$ as

$$\dim(F_1(X)) = \dim(\operatorname{Gr}(2,4)) - \operatorname{rk}(\operatorname{Sym}^3(\mathcal{U}^\vee)) = 4 - \binom{3+2-1}{3} = 4 - 4 = 0.$$

$F_1(X)$ is then a finite set of points and we want to know exactly how many of them. In order to do this, we need to compute the top Chern class of the normal bundle, which in the case is the restriction of $\mathrm{Sym}^3(\mathcal{U}^\vee)$ (the restriction symbol will be omitted now on). From of the latter computation we know that:

$$\begin{aligned} c_4(\mathrm{Sym}^3\mathcal{U}^\vee) &= 9c_2(\mathcal{U}^\vee)(2c_1(\mathcal{U}^\vee)^2 + c_2(\mathcal{U}^\vee)) \\ &= 18c_2(\mathcal{U}^\vee)c_1(\mathcal{U}^\vee)^2 + 9c_2(\mathcal{U}^\vee)^2 \end{aligned}$$

Hence we are interested in the study of the Chern classes of $\mathcal{U}^\vee$.

Let $0 \neq \sigma \in V_4^\vee$ and by the same argument as above consider it as a section of $\mathcal{U}^\vee$. $V(\sigma)$ (thought as a section of $\mathcal{U}^\vee$) is the set of planes in V_4 contained in $\ker(\sigma)$ which is a 3 dimensional vector subspace of V_4 (thought as a map $V_4 \to \mathbb{C}$):

$$V(\sigma) = \mathrm{Gr}(2, \ker(\sigma)) \cong (\mathbb{P}^2)^\vee$$

Therefore, the zero set of a general section of $\mathcal{U}^\vee \oplus \mathcal{U}^\vee$ is one point, hence:

$$\deg(c_4(\mathcal{U}^\vee \oplus \mathcal{U}^\vee)) = \deg(c_2(\mathcal{U}^\vee)^2) = 1.$$

We proceed by applying Whitney's formula to the dual of the tautological sequence, yielding:

$$c(\mathcal{U}^\vee)c(\mathcal{Q}^\vee) = c(\mathcal{O}_G^{\oplus 4}) = 1.$$

In particular:

$$\begin{aligned} c(\mathcal{Q}^\vee) =& \frac{1}{c(\mathcal{U}^\vee)} = \frac{1}{1 + c_1(\mathcal{U}^\vee) + c_2(\mathcal{U}^\vee)} \\ =& 1 - (c_1(\mathcal{U}^\vee) + c_2(\mathcal{U}^\vee)) + (c_1(\mathcal{U}^\vee)^2 + 2c_1(\mathcal{U}^\vee)c_2(\mathcal{U}^\vee) + c_2(\mathcal{U}^\vee)^2) \\ & - (c_1(\mathcal{U}^\vee)^3 + 3c_1(\mathcal{U}^\vee)^2c_2(\mathcal{U}^\vee) + 3c_1(\mathcal{U}^\vee)c_2(\mathcal{U}^\vee)^2 + c_2(\mathcal{U}^\vee)^3) \\ & + (c_1(\mathcal{U}^\vee)^4 + 4c_1(\mathcal{U}^\vee)^3c_2(\mathcal{U}^\vee) + 6c_1(\mathcal{U}^\vee)^2c_2(\mathcal{U}^\vee)^2 + 4c_1(\mathcal{U}^\vee)c_2(\mathcal{U}^\vee)^3 \\ & + c_2(\mathcal{U}^\vee)^4) \pm \cdots. \end{aligned}$$

By matching terms of the same degree, and noting that $\mathrm{rk}(\mathcal{Q}) = 2$ and $c_2(\mathcal{U}^\vee)^2 = 1$, we find:

$$0 = c_3(\mathcal{Q}^\vee) = 2c_1(\mathcal{U}^\vee)c_2(\mathcal{U}^\vee) - c_1(\mathcal{U}^\vee)^3,$$

$$0 = c_4(\mathcal{Q}^\vee) = c_2(\mathcal{U}^\vee)^2 - 3c_1(\mathcal{U}^\vee)^2c_2(\mathcal{U}^\vee) + c_1(\mathcal{U}^\vee)^4.$$

In particular, we have:

$$0 = c_1(\mathcal{U}^\vee)c_3(\mathcal{Q}^\vee) + c_4(\mathcal{Q}^\vee) = c_2(\mathcal{U}^\vee)^2 - c_1(\mathcal{U}^\vee)^2 c_2(\mathcal{U}^\vee).$$

Therefore the fourth Chern class of the third symmetric power of $\mathcal{U}^\vee$ is given by:

$$c_4(\mathrm{Sym}^3\mathcal{U}^\vee) = 9c_2(\mathcal{U}^\vee)(2c_1(\mathcal{U}^\vee)^2 + c_2(\mathcal{U}_G^\vee)) = 18c_2(\mathcal{U}^\vee)c_1(\mathcal{U}^\vee)^2 + 9c_2(\mathcal{U}^\vee)^2 = 27c_2(\mathcal{U}^\vee)^2.$$

In particular:

$$\deg(c_4(\mathrm{Sym}^3\mathcal{U}^\vee)) = 27\deg(c_2(\mathcal{U}^\vee)^2) = 27.$$

Chapter 2

Abelian varieties and Prym varieties: from complex tori to double covers

2.1 Abelian varieties: polarizations and intermediate Jacobians

In this chapter, we introduce the theory of abelian varieties and their close relatives, Prym varieties. Abelian varieties are fundamental objects in algebraic geometry, serving as higher-dimensional generalizations of elliptic curves. They are complex tori that admit a polarization, which endows them with a projective embedding. Prym varieties, on the other hand, arise naturally in the study of double covers of curves.
The chapter is divided into two sections. The first section focuses on abelian varieties and their intermediate Jacobians, which are crucial for understanding the Hodge structure of complex manifolds. The second section introduces Prym varieties, which play a significant role in the study of theta divisors.

Definition 2.1.1. A *lattice* in a complex vector space $\mathbb{C}^g$ is defined as a discrete subgroup of maximal rank in $\mathbb{C}^g$. Such a lattice is a free abelian group of rank $2g$. A *complex torus* is the quotient $X = \mathbb{C}^g/\Lambda$, where Λ is a lattice in $\mathbb{C}^g$. The complex torus X is a complex manifold of dimension g, and it inherits the structure of a complex Lie group from the vector space $\mathbb{C}^g$. An *abelian variety* is a complex torus that admits a polarization.

The definition of an abelian variety as a polarized complex torus is central to the study of these objects. The polarization ensures that the torus can be embedded into projective space, making it an algebraic variety.

Our interest is to understand which complex tori are abelian varieties. From now on let $X := V/\Lambda$ where V is a finite-dimensional complex vector space and Λ a lattice. Thanks to [BL04, Corollary 1.3.2] we know that $H^2(X,\mathbb{Z}) \cong \Lambda^2(\mathrm{Hom}(\Lambda,\mathbb{Z}))$. So, given a line bundle L on X we can think of $c_1(L)$, which by definition is an element of $H^2(X,\mathbb{Z})$ as an alternating $\mathbb{Z}$-valued form on Λ. This is summed up in:

Proposition 2.1.2. *For an alternating form $E : V \times V \to \mathbb{R}$ the following are equivalent:*

1. *there is a holomorphic line bundle L on X such that E represents the first Chern class $c_1(L)$;*
2. *$E(\Lambda,\Lambda) \subset \mathbb{Z}$ and $E(v,w) = E(iv,iw)$ $\forall$ $v, W \in V$.*

and

Lemma 2.1.3. *There is a 1-1-correspondence between the set of hermitian forms H on V and the set of real valued alternating forms E on V satisfying $E(iv,iw) = E(v,w)$, given by*

$$E(v,w) = \mathrm{Im}H(v,w) \quad \text{and} \quad H(v,w) = E(iv,w) + iE(v,w) \quad \forall\ v,w \in V$$

Proposition 2.1.2 establishes a crucial link between the topology of the torus (via Chern classes) and its geometry (via alternating forms).

Definition 2.1.4. A Hermitian form H on V is said to be a Riemann form with respect to a given lattice Λ in V if the imaginary part of H, $\mathrm{Im}\,(H)$, takes integer values on $\Lambda \times \Lambda$.

Therefore, given a line bundle on X, not only we can associate to it an element in $\Lambda^2(\mathrm{Hom}(\Lambda,\mathbb{Z}))$, but thanks to the latter we can associate to it a Riemann form. Conversely, given a Riemann form H on V with respect to Λ consider

$$\alpha : \Lambda \to \{z \in \mathbb{C}^* \,||z| = 1\}$$

a map such that

$$\alpha(\lambda_1 + \lambda_2) = e^{i\pi E(\lambda_1,\lambda_2)}\alpha(\lambda_1)\alpha(\lambda_2) \quad \lambda_1, \lambda_2 \in \Lambda.$$

Then we can define a line bundle on X given by the quotient of $\mathbb{C} \times V$ by the action of Λ given by:

$$\lambda.(z,w) = (\alpha(\lambda)e^{\pi H(v,\lambda)+\frac{1}{2}\pi H(\lambda,\lambda)}z, w + \lambda)$$

Then Appel Humbert theorem completes the converse of the correspondence between Riemann forms and line bundles:

Theorem 2.1.5 (Appel-Humbert). *Let L be a line bundle on X. Then L is isomorphic to a line bundle constructed as above for a unique choice of H, α.*

Next we need to characterize which Riemann forms correspond to ample line bundles in order to have projective embeddings. This is taken care of by a theorem of Lefschetz:

Theorem 2.1.6 (Lefschetz). *Let X be a complex torus as above $X = V/\Lambda$ with $L = L(H, \alpha)$ the associated line bundle as above.*

$$\text{The Riemann form } H \text{ is positive definite} \iff L \text{ is ample.}$$

The proof can be found in [MRM74] and essentially shows that holomorphic sections of $L(H, \alpha)$ are in 1-1 correspondence with holomorphic functions θ on V which satisfy

$$\theta(z + \lambda) = \alpha(\lambda)e^{\pi H(z,\lambda)+\frac{1}{2}\pi H(\lambda,\lambda)}\theta(z)$$

and proofs that such sections make $L^{\otimes 3}$ very ample.

Given a polarization is then equivalent to the datum of an Hermitian form

$$H : V \times V \to \mathbb{C}$$

that is positive-definite and such that $\operatorname{Im} H|_{\Lambda\times\Lambda}$ is integer-valued.
It can be shown that in a suitable $\mathbb{Z}$-basis of Λ, the matrix representation of $H|_{\Lambda\times\Lambda}$ takes the form:

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where D is a diagonal matrix:

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_g \end{pmatrix}.$$

Here, $d_1, d_2, \ldots, d_g$ are positive integers such that $d_1 \mid d_2 \mid \cdots \mid d_g$ (i.e., each divides the next). The vector $(d_1, \ldots, d_g)$ is called the type of the polarization. If $\prod d_i = 1$ the polarization is said to be principal.

Take the Jacobian of a curve as an example:

Example 2.1.0.1. Let C be a smooth projective complex curve of genus g. Recall the Hodge decomposition:

$$H^1(C,\mathbb{Z}) \subset H^1(C,\mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C)$$

where $H^{1,0}(C) \cong \overline{H^{0,1}(C)}$ and $h^{1,0}(C) = h^{0,1}(C) = g$. From the latter isomorphism it follows that the projection: $H^1(C,\mathbb{R}) \to H^{0,1}(C)$ is a $\mathbb{R}$-linear isomorphism. Consequently, the image Λ of $H^1(C,\mathbb{Z})$ in $H^{0,1}(C)$ forms a lattice. The quotient

$$J(C) := H^{0,1}(C)/\Lambda$$

is a complex torus of dimension g. However, this torus has additional structure. The pairing

$$(\alpha,\beta) \mapsto 2i\int_C \overline{\alpha}\wedge\beta$$

defines a positive definite Hermitian form H on $H^{0,1}(C)$. Furthermore, the imaginary part of H, when restricted to $\Lambda \cong H^1(C;\mathbb{Z})$, coincides with the cup product:

$$H^1(C,\mathbb{Z}) \times H^1(C,\mathbb{Z}) \to H^2(C;\mathbb{Z}) \cong \mathbb{Z}.$$

Thus this induces on Λ a skew-symmetric and integer-valued form. In other words, H is a polarization on $J(C)$. We can say more; in section 2, chapter 2 of [GH14, §2.2] it is proven that it is possible to choose basis $\{\lambda_1,\dots,\lambda_{2g}\}$ of $H_1(C,\mathbb{Z}) \cong H^1(C,\mathbb{Z})$ and $\{\omega_1,\dots,\omega_g\}$ of $H^{0,1}(C)$ such that $\int_{\lambda_i}\omega_j = \delta_{ij}$ for $i,j \leq g$. As consequence, by the so called Riemann conditions discussed in section 6 of the same, the matrix associated to the cup product will be

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

hence the cup product is a unimodular form so the polarization is principal on $J(C)$.

The Jacobian of a curve is one of the most important examples of an abelian variety, and it plays a central role in the study of curves and their moduli.

Definition 2.1.0.2. Let X be a compact Kähler manifold with a Hodge decomposition:

$$H^{2m+1}(X,\mathbb{C}) = H^{m,m+1}(X) \oplus H^{m+1,m}(X)$$

of level one. We define the $(2m+1)$-st *intermediate Jacobian* of X as:

$$J_{2m+1}(X) := H^{m,m+1}(X)/H^{2m+1}(X,\mathbb{Z})$$

(to be more accurate: here by $H^{2m+1}(X,\mathbb{Z})$ we denote the image of the morphism $H^{2m+1}(X,\mathbb{Z}) \to H^{2m+1}(X,\mathbb{C}) \twoheadrightarrow H^{m,m+1}(X)$)

Remark 2.1.7. The most important case is when $2m+1$ is the dimension of X, (in this case we omit the index $2m+1$ in $J(X)$). In this particuar case the cup product induces a skew-symmetric, nondegenerate bilinear map on $H^{2m+1}(X,\mathbb{Z})$ by Poincaré duality wich defines a principal polarization on $J(X)$.

Definition 2.1.8. Given (X, L_X) and (Y, L_Y) two principally polarized abelian varieties, (where L_X, L_Y are the classes of principal polarizations on X and Y as line bundles). Denote $(X, L_X) \times (Y, L_Y)$ the principally polarized abelian variety $(X \times Y, L_X \boxtimes L_Y)$. A principally polarized abelian variety is said to be *irreducible* if it is non zero and non isomorphic to a product of nonzero principally polarized abelian varieties.

In [Deb96] Debarre proves that such a decomposition is unique for every principally polarized abelian variety, i.e. any p.p.a.v. admits a unique decomposition as a product of irreducible p.p.a.v; here is a sketch of proof.

Lemma 2.1.9. *1. A p.p.a.v. (A,Θ) is irreducible if and only if the divisor Θ is irreducible.*

2. Any p.p.a.v. admits a unique decomposition as a product of indecomposable p.p.a.v.

Sketch of proof. Let D be a divisor on an abelian variety A; for $a \in A$ we denote by D_a the translated divisor $D + a$. The map

$$\varphi_D : a \mapsto \mathcal{O}_A(D_a - D)$$

is a homomorphism from A into its dual variety $\hat{A}$, which parametrizes topologically trivial line bundles on A. If D defines a principal polarization, this map is an isomorphism.

Now suppose our p.p.a.v. (A,Θ) is a product $(A_1,\Theta_1) \times \cdots \times (A_p,\Theta_p)$. Then

$$\Theta = \Theta^{(1)} + \cdots + \Theta^{(p)},$$

with

$$\Theta^{(i)} := A_1 \times \cdots \times \Theta_i \times \cdots \times A_p.$$

We recover the summand $A_i \subset A$ as $\varphi_\Theta^{-1}(\varphi_{\Theta^{(i)}}(A))$.

Conversely, let (A, Θ) be a p.p.a.v., and let $\Theta^{(1)}, \dots, \Theta^{(p)}$ be the irreducible components of Θ (each of them occurs with multiplicity one, since otherwise one would have $h^0(A; \mathcal{O}_A(\Theta)) > 1$). Putting

$$A_i := \varphi_\Theta^{-1}(\varphi_{\Theta^{(i)}}(A))$$

and

$$\Theta_i := \Theta^{(i)}|_{A_i},$$

now we should check that (A, Θ) is the product of the (A_i, Θ_i); see [CG72, Lemma 3.20], for the details. □

2.2 Double covers and their Prym varieties: constructions and properties

It is due to recall that an étale morphism is a flat and unramified holomorphic map.

Let C be a smooth curve of genus g. Recall that we have the non-canonical isomorphisms:

$$\mathrm{Pic}^d(C) \cong \mathrm{Pic}^0(C) \cong J(C)$$

Consider a connected étale double cover $\pi : \tilde{C} \to C$. By the Riemann-Hurwitz formula, the genus of $\tilde{C}$ is given by $\tilde{g} = 2g - 1$. Let σ be the involution on $\tilde{C}$ that switches the sheets of the covering π; we will also denote by σ its induced morphism on $\mathrm{Pic}(\tilde{C})$.

The pushforward map $\pi_* : \mathrm{Div}(\tilde{C}) \to \mathrm{Div}(C)$ induces a norm map

$$N : \mathrm{Pic}(\tilde{C}) \to \mathrm{Pic}(C)$$
$$[\sum n_i p_i] \mapsto [\sum n_i \pi(p_i)]$$

Definition 2.2.1. Define the Prym variety associated with $(\tilde{C}, C)$ as

$$P := (1 - \sigma)(J(\tilde{C})) = \ker(N)^0.$$

This is an abelian variety of dimension $g-1$, isomorphic to the quotient $J(\tilde{C})/\pi^*(J(C))$. [ref: PippoInzaghi culomerdo]

Remark 2.2.2. The fiber $N^{-1}([D])$ consists of two disjoint translates of P, denoted by P_0 and P_1, where

$$P_i \cong (1 - \sigma)(\mathrm{Pic}^i(\tilde{C})) \quad \text{for } i = 0, 1.$$

For a divisor D of degree d on C, the fiber $N^{-1}([D])$ is the union of two subvarieties which we will denote P_0^D and P_1^D. If $[\tilde{D}] \in P_0^D$ and A is another divisor on $\tilde{C}$, then the class of $\tilde{D} + A - \sigma(A)$ belongs to P_i^D, where $i \equiv \deg(A) \mod 2$.

Setting $D = K_C$, we can label the components P_i^K such that for any $[D] \in N^{-1}(K_C)$, we have:

$$[D] \in P_i^K \iff h^0(D) \equiv i \mod 2.$$

It is convenient to consider $P^* := P_0^K$ instead of P. Set-theoretically, this is given by:

$$P^* = \{[D] \in \mathrm{Pic}^{\tilde{g}-1}(\tilde{C}) \mid \pi_*(D) = K_C \text{ and } h^0(D) \text{ is even}\}.$$

Now, define the divisor $\theta^* \subset P^*$ as:

$$\theta^* = \{[D] \in \mathrm{Pic}^{\tilde{g}-1}(\tilde{C}) \mid \pi_*(D) = K_C \text{ and } h^0(D) \geq 2 \text{ is even}\}.$$

Under the identification $P^* = P$, the divisor θ^* defines a principal polarization on P.

Here we have a useful characterization of the singuarities of the Prym variety.

According to the Riemann's singularity theorem [Nar92], a point $[D] \in \theta^*$ is singular if one of the following conditions holds:

1. $D = \pi^*(M) + E$, where $|M|$ is a mobile linear system on C and E is an effective divisor on $\tilde{C}$ such that $\pi_* E \in |K_C - 2M|$.

2. $h^0(D) \geq 4$.

Chapter 3

The rationality problem for cubic threefolds

3.1 Lüroth's problem

Definition 3.1.1. A projective variety X is said to be rational if it is birationally equivalent to some $\mathbb{P}^n$, i.e., there exist mutually inverse dominant rational maps

$$X \dashrightarrow \mathbb{P}^n \quad \text{and} \quad \mathbb{P}^n \dashrightarrow X.$$

In this case such rational maps are said to be birational.

Remark 3.1.2. We can understand rational varieties in terms of algebra: dominant rational maps define inclusions of function fields, and this yields an arrow-reversing equivalence between the category of varieties with dominant rational maps and the category of finitely generated field extensions of $\mathbb{C}$. Thus, X being rational is equivalent to

$$k(X) \cong k(\mathbb{P}^n) = \mathbb{C}(x_1, \dots, x_n) \quad \text{for some } n.$$

Rationality strongly depends on the base field, in the next example we will make this statement more clear.

Definition 3.1.3. A projective variety X is said to be unirational if there exist a dominant rational map

$$\mathbb{P}^n \dashrightarrow X \quad \text{for some } n$$

Remark 3.1.4. In algebraic terms: X is unirational if $k(X)$ is a subfield of $k(\mathbb{P}^n) = \mathbb{C}(x_1, \dots, x_n)$

Example 3.1.5 (Smooth Quadrics)**.** Every quadric $Q = V(q) \subset \mathbb{P}^{n+1}$ is rational, indeed a birational correspondence can be given as follows: fix a point $p \in Q$ and consider the set of lines L in $\mathbb{P}^{n+1}$ through p, $L \cong \mathbb{P}^n$. For the generic line $l \in L$ the intersection $l \cap Q = \{p, P_l\}$ hence we can give a birational morphism mapping each suitable line l to the residual point of intersection P_l.
Just for this one example consider a quadric defined over $\mathbb{Q}$. Let $C = V(x^2 + y^2 - pz^2) \subset \mathbb{P}^2(\mathbb{Q})_{[x:y:z]}$ where p is a prime number or 1.

- When $p = 1$, we can write down an explicit rational parametrization $f : \mathbb{P}^1 \dashrightarrow C$

$$f(t) = [t^2 - 1 : 2t : t^2 + 1]$$

  This parametrization is given by the inverse of stereographic projection from the point $[0 : 1 : 0]$.

- When $p \equiv 1 \mod 4$, we can write $p = a^2 + b^2$ and so we have a $\mathbb{Q}$-point given by $[a : b : 1]$ and again using projection from this point gives a birational map $C \dashrightarrow \mathbb{P}^1$.

- When $p \equiv 1 \mod 4$, then in fact C has no $\mathbb{Q}$-points, $C(\mathbb{Q}) = \varnothing$ so we cannot project from any point as we did before. However C becomes rational over the field extension $k = \mathbb{Q}(\sqrt{p})$, just note that there is a k-point $[0 : \sqrt{p} : 1]$ and projection from this point gives us a birational map $C \dashrightarrow \mathbb{P}^1$.

Remark 3.1.6. It is worth noting here that if we let $Q \subset \mathbb{P}^{n+1}$ be a smooth quadric then Q is rational over the field k if and ony if $Q(k) \neq \varnothing$. (In particular, unirational quadrics are rational). cf. [Bej22, Proposition 3.1]

Example 3.1.7 (Smooth cubic surfaces)**.** We want to prove that every cubic surface is isomorphic to the bowup of the plane $\mathbb{P}^2$ along 6 points in general position. Let $X = (\mathbb{P}^2_x \times \mathbb{P}^3_y, \mathcal{O}(1,1)^{\oplus 3})$ where we assigned the coordinates $\underline{x}$ to $\mathbb{P}^2$ and $\underline{y}$ to $\mathbb{P}^3$. Up to rearranging we can write the equations for X either as:

$$X = V\left(\sum_{i=1}^{3} x_i f_i^1(y), \sum_{i=1}^{3} x_i f_i^2(y), \sum_{i=1}^{3} x_i f_i^3(y)\right)$$

or as,

$$X = V\left(\sum_{i=1}^{4} y_i l_i^1(x), \sum_{i=1}^{4} y_i l_i^2(x), \sum_{i=1}^{4} y_i l_i^3(x)\right).$$

where f_i^j, l_i^j are linear forms in the coordinates, respectively, $\underline{y}$ and $\underline{x}$). To these equations we can associate matrices (respectively):

$$M(\underline{y}) = \begin{pmatrix} f_1^1(\underline{y}) & f_2^1(\underline{y}) & f_3^1(\underline{y}) \\ f_1^2(\underline{y}) & f_2^2(\underline{y}) & f_3^2(\underline{y}) \\ f_1^3(\underline{y}) & f_2^3(\underline{y}) & f_3^3(\underline{y}) \end{pmatrix}$$

the second matrix:

$$N(\underline{x}) = \begin{pmatrix} l_1^1(\underline{x}) & l_2^1(\underline{x}) & l_3^1(\underline{x}) & l_4^1(x) \\ l_1^2(\underline{x}) & l_2^2(\underline{x}) & l_3^2(\underline{x}) & l_4^2(\underline{x}) \\ l_1^3(\underline{x}) & l_2^3(\underline{x}) & l_3^3(\underline{x}) & l_4^3(\underline{x}) \end{pmatrix}$$

via these matricese it is easier to study the fibers of the projections. Start with the projection π_y (restricted to X) onto $\mathbb{P}^3_y$, the generic fiber of this projection is empty as it is cut out by 3 linear equation in $\mathbb{P}^2$. The fiber is non-empty exactly where the matrix M drops rank i.e. $V(\det(M(\underline{y})))$ which is a cubic surface $V(F_3(\underline{y})) \subset \mathbb{P}^3_y$. (The 2×2 minors would vanish in a codimension 4 locus in $\mathbb{P}^3$ hence it is empty [FP06]). Consider now the projection to $\mathbb{P}^2$, now the generic fiber is a point. However if we take $p \in \mathbb{P}^2$ such that $\mathrm{rk}(N(p)) \leq 2$ then $\pi_x^{-1}(p)$ is defined by two linear equations in $\mathbb{P}^3$ hence it is a $\mathbb{P}^1$. The locus where N drops rank is of codimension 2 and degree 6 [FP06], hence 6 points in $\mathbb{P}^2$. To conclude the proof we need to know that every cubic surface is determinantal, luckily this is proven in [Bea00].

We want to prove now the unirationality of cubic hypersurfaces of dimension at least two by induction: Fix a smooth cubic $X \subset \mathbb{P}^{n+1}$ with $n > 2$ and consider two generic hyperplane sections $H_1, H_2 \subset X$ and the associated rational map:

$$\begin{aligned} H_1 \times H_2 &\dashrightarrow X \\ (y_1, y_2) &\mapsto x \end{aligned}$$

where x is the residual point of intersection of $\{y_1, y_2\} \subset \overline{y_1 y_2} \cap X$. The map is dominant, in fact for a generic point x and any point $y_1 \in H_1$ the intersection of line $\overline{x y_1}$ with H_2 is a unique point y_2. Thus we have a rational dominant map from $H_1 \times H_2$ to X and by the induction assumption H_1 and H_2 are unirational, hence X is unirational. It is left to prove the base case i.e. when X is a surface. We could just notice that we already proved that a cubic surface is rational, hence it is unirational. Another way of proving this is by almost following the previous argument, it is true in fact that any smooth cubic surface contains two skew lines (which are obviously rational varieties) so the map

$$l_1 \times l_2 \dashrightarrow X$$

concludes the proof.

It is an obvious observation that a rational variety X is in particular unirational. It is also true that any variety birationally equivalent to a unirational variety is unirational. A natural question to ask is "are all unirational varieties rational?". Again, in algebraic terms, are all non-trivial subfields of the field of rational functions in fact isomorphic to the field of rational functions? This question, posed by Lüroth in 1861, is now known as the Lüroth problem. The answer is positive in dimension 1, thanks to the so called Lüroth theorem. Here is a sketch of the proof:
let C be a curve (which we may assume smooth and projective) and φ a dominant rational map $\varphi : \mathbb{P}^1 \dashrightarrow C$. By Riemann-Hurwitz [Mir95] we have that:

$$\chi(\mathbb{P}^1) = \deg(\varphi)\chi(C) - \mathrm{ram}(\varphi)$$

thus,

$$2 - 2g_{\mathbb{P}^1} = \deg(\varphi)(2 - 2g_C) - \mathrm{ram}(\varphi)$$

i.e.

$$g_C = 1 - \frac{1}{\deg(\varphi)}(1 + \mathrm{ram}(\varphi)) < 1$$

In conclusion the genus g_C is zero and hence C is rational. The answer is positive again in dimension 2 (over an algebraically closed field of characteristic zero) thanks to a criterion by Castelnuovo. The theorem states that a smooth, projective surface S is rational if and only if $p_2(S) = q(S) = 0$, where $p_m(S) := h^0(S, \mathcal{O}(mK_S))$ and $q(S) := h^0(S, \mathcal{O}(K_S))$ and a on unirational surface these invariants vanish. Indeed, a dominant rational map $\varphi : \mathbb{P}^2 \dashrightarrow S$ defines a pullback map on forms. Thus, if $p_2(S) = 0$ is non-zero, then the pull-back of $0 \neq \omega \in H^0(S, \mathcal{O}(2K_S))$ has to vanish on $\mathbb{P}^2$. Hence the Jacobian of φ must be zero everywhere; but this contradicts the fact that $\varphi : U \to V$ (U, V open subsets in $\mathbb{P}^2$ and S respectively) is surjective, by the Implicit Function theorem. Analogousy for $q(S)$.

The first counterexamples to the Lüroth problem were found in dimension 3 by three different groups with different techniques:

Authors	Example	Method
Clemens–Griffiths	all smooth cubic threefolds	intermediate Jacobian
Iskovskikh–Manin	all smooth quartic threefolds	birational automorphisms
Artin–Mumford	some quartic double solids	torsion of $H^3(\bullet, \mathbb{Z})$

Table 3.1: table from [Deb24]

More precisely:

- Clemens and Griffiths' method will be described later on.
- Iskovskikh–Manin proved in [**iskovskih1971three**] that all smooth quartic threefold are irrational. They showed that the group of birational automorphism of X is finite, while the corresponding group for $\mathbb{P}^3$ is huge.
- Artin–Mumford proved in [AM72] that a particular double covering X of $\mathbb{P}^3$, branched along a quartic surface with ten nodes, is unirational but not rational. They showed that the torsion subgroup of $H^3(X, \mathbb{Z})$ is nontrivial, and is a birational invariant which is trivial for $\mathbb{P}^3$.

3.2 Quadric fibration and Prym variety associated to a cubic threefold

Having introduced Prym varieties and their theta divisors in the previous chapter, we now shift our focus to a specific and particularly significant example: the Prym variety associated with a cubic threefold. This will ultimately lead us to prove the main theorem from which will follow the irrationality of the cubic threefold a Torelli theorem.

Theorem 3.2.1. *Let $X \subset \mathbb{P}^4$ be a cubic hypersurface, and let $(J(X), \theta)$ denote its intermediate Jacobian. Then, the theta divisor $\theta \subset J(X)$ has a unique singular point at $0 \in \theta$, which has multiplicity three. In particular the projective tangent cone to the unique singular point is isomorphic to the cubic threefold $TC_0(\theta) \cong X$.*

Now we develop a construction that will play a key role in the remainder of this work. Let $\mathbb{P}(W_k) \subset \mathbb{P}(V_{n+2}) =: \mathbb{P}$ be a linear subspace and pick $\mathbb{P}(U) \subset \mathbb{P}$ a generic linear subspace of codimension k. Hence, $U \oplus W = V$ and $\mathbb{P}(W) \cap \mathbb{P}(U) = \emptyset$. The linear projection

$$\pi : \mathbb{P}(V) \dashrightarrow \mathbb{P}(U) \cong \mathbb{P}(V/W)$$
$$x \longmapsto < x, \mathbb{P}(W) > \cap \mathbb{P}(U)$$

where $x \in \mathbb{P} \setminus \mathbb{P}(W)$ is mapped to the unique point of intersection of the linear subspace generated by $\{x, \mathbb{P}(W)\}$ and $\mathbb{P}(U)$. The easiest way to understand this map is to think of it as the projectified of the canonical surjection of vector spaces:

$$V \longrightarrow V/W$$
$$v \longmapsto [v] + W$$

The rational map π is the one associated to the linear system $|\mathcal{I}_{\mathbb{P}(W)} \otimes \mathcal{O}_{\mathbb{P}}(1)| \subset |\mathcal{O}_{\mathbb{P}}(1)|$. The map is resolved by a blow-up:

$$\begin{array}{ccc} & \mathrm{Bl}_{\mathbb{P}(W)}\mathbb{P} & \\ {}^{\tau}\swarrow & & \searrow^{\phi} \\ \mathbb{P}(V) & \overset{|\mathcal{I}_{\mathbb{P}(W)}\otimes\mathcal{O}_{\mathbb{P}}(1)|}{\dashrightarrow} & \mathbb{P}(V/W) \end{array}$$

The map τ is completely understood, we want now to get a deeper understanding of the map ϕ. The fiber over a point $y \in \mathbb{P}(V/W) \cong \mathbb{P}(U)$ is the strict transform of the linear subspace $< y, \mathbb{P}(W) > \subset \mathrm{Bl}_{\mathbb{P}(W)}\mathbb{P}$ which is a projective space $\mathbb{P}^k$. Geometrically $E \cap \phi^{-1}(y)$ is a section of the blow-up restricted to the exceptional divisor $\tau_{|E} : E \to \mathbb{P}(W)$ which maps a point $x \in \mathbb{P}(W)$ to the normal direction in $\mathbb{P}(\mathcal{N}_{\mathbb{P}(W)/\mathbb{P}})$ corresponding to the line $< x, y > \cong \mathbb{P}^1$. We proved that all fibers of ϕ are projective spaces indeed $\mathrm{Bl}_{\mathbb{P}(W)}\mathbb{P}$ is a projective bundle over $\mathbb{P}(V/W)$, indeed it is the projective bundle $\mathrm{Bl}_{\mathbb{P}(W)}\mathbb{P} \cong \mathbb{P}(\mathcal{F}^\vee)$ with $\mathcal{F} := \phi_* \tau^* \mathcal{O}(1)$, in particular, we have, [Fat24]:

Lemma 3.2.2. *For a vector subspace $W \subset V$, consider the blow-up $X := \mathrm{Bl}_{\mathbb{P}(W)} \mathbb{P}(V)$. Then*

$$X = \mathbb{P}_{\mathbb{P}(V/W)}(W \otimes \mathcal{O} \oplus \mathcal{O}(-1))$$

Let now $X \subset \mathbb{P} = \mathbb{P}(V)$ be a cubic hypersurface containing $\mathbb{P}(W)$ with equation $F \in H^0(\mathbb{P}, \mathcal{O}(3) \otimes \mathcal{I}_{\mathbb{P}(W)})$. Then then the total transform has two components, the exceptional divisor E and the strict transform of $X \subset \mathbb{P}$. The latter is the blow-up $\mathrm{Bl}_{\mathbb{P}(W)}X$ of X along $\mathbb{P}(W)$. Therefore $\tau^*(F) \in H^0(\mathrm{Bl}_{\mathbb{P}(W)}X, \tau^*\mathcal{O}(3) \otimes \mathcal{O}(-E))$, hence

$$\mathrm{Bl}_{\mathbb{P}(W)}X = V(\tau^* F) \in |\tau^*\mathcal{O}(3) \otimes \mathcal{O}(-E)|$$

Consequently, given the above considerations, we can see the natural projection of $\mathrm{Bl}_{\mathbb{P}(W)}X$ to $\mathbb{P}(V/W)$ as a quadric fibration. Indeed, if we compute the direct image of τ^*F under ϕ:

$$\phi_*(\tau^*\mathcal{O}(3) \otimes \mathcal{O}(-E)) \cong \phi_*(\mathcal{O}_\phi(2) \otimes \phi^*\mathcal{O}(1)) \cong \mathrm{Sym}^2(\mathcal{F}) \otimes \mathcal{O}(1)$$

where we used that:

$$\tau^*\mathcal{O}(3) \otimes \mathcal{O}(-E) \cong \tau^*\mathcal{O}(2) \otimes (\tau^*\mathcal{O}(1) \otimes \mathcal{O}(-E)) \cong \mathcal{O}_\phi(2) \otimes \phi^*\mathcal{O}(1)$$

We now have τ^*F as a section of $H^0(\mathbb{P}(V/W), \mathrm{Sym}^2(\mathcal{F}) \otimes \mathcal{O}(1))$. Hence, the fiber of $\mathrm{Bl}_{\mathbb{P}(W)} X \subset \mathrm{Bl}_{\mathbb{P}(W)}(\mathbb{P}) \cong \mathbb{P}(\mathcal{F}^\vee)$ over $y \in \mathbb{P}(V/W)$, i.e. the residual quadric Q_y of the intersection $\mathbb{P}(W) \subset < y, \mathbb{P}(W) > \cap X$, is the quadric defined by $q_y \in S^2(\mathcal{F}(y))$. In particular, the fiber is smooth if and only if this quadric is non-degenerate.

Thus, the discriminant divisor D_P of $\mathrm{Bl}_P(X) \to \mathbb{P}(V/W)$ is

$$D_P = V(\det(q)) \subset \mathbb{P}(V/W).$$

Here, $\det(q) : \det(\mathcal{F})^\vee \to \det(\mathcal{F}) \otimes \mathcal{O}(k+1)$ is viewed as a section of the line bundle $\det(\mathcal{F})^2 \otimes \mathcal{O}(k+1) \cong \mathcal{O}(k+3)$.
To conclude here is a result from [Huy23] that summarizes the above discussion:

Proposition 3.2.3. *Assume that the smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ contains a linear subspace $P = \mathbb{P}^{k-1}$ such that there exists no linear subspace $\mathbb{P}^k \subset X$ containing P. Then the linear projection from P defines a morphism*

$$\pi : \mathrm{Bl}_P(X) \to \mathbb{P}^{n+1-k} \tag{3.2.1}$$

with the following properties:

(i) The fibre over $y \in \mathbb{P}^{n+1-k}$ is the residual quadric Q_y of $P \subset y\mathbb{P} \cap X$, i.e.

$$< y, P > \cap X = P \cup Q_y.$$

(ii) The fibres are singular exactly over the discriminant divisor $D_P \in |\mathcal{O}(k+3)|$.

(iii) The morphism $\pi : \mathrm{Bl}_P(X) \to \mathbb{P}^{n+1-k}$ is flat.

We want now discuss the case $\dim(X) = 3$ and $\dim(W) = 2$ i.e. a line $l = \mathbb{P}(W)$ in a smoooth cubic threefold. The rational map discussed before $\varphi_{|\mathcal{I}_l \otimes \mathcal{O}_X(1)|}$ geometrically

is the projection from l onto a plane disjoint to l.
The fiber over a point $y \in \mathbb{P}^2$ is the residual conic of $l \subset < y, l > \cap X \subset < y, l > \cong \mathbb{P}^2$, the conic is smooth or a union of two lines l_1, l_2, possibly non-reduced, i.e. $l_1 = l_2$, or with $l_i = l$. Thanks to [Bea06] and [Huy23, Corollary 5.1.23] if l is in general position the generic fiber is smooth while the fiber over a point in the discriminant divisor D_l is the union of two lines $l_1 \neq l_2$ with $l_1 \neq l_2$ and $l_i \neq l$; in particular D_l, which is a plane quintic by the previous proposition, si smooth for generic l.

Let $\tilde{C} \subset F(X)$ be the subvariety of lines in X intersecting l, then $\phi_{|\tilde{C}} : \tilde{C} \to C$ is an étale double cover [Bea06]. The involution σ associates to a line r of $\tilde{C}$ the residual line in the intersection $< l, r > \cap X$.

Proposition 3.2.4. *The Prym variety associated to $(\tilde{C}, C)$ is isomorphic to the intermediate Jacobian $J(X)$.*

The proof can be found in [Bea77] or [Tyu72].

Before proving the main theorem we need to establish and remark some properties of the curves $C, \tilde{C}$ (further details can be found in [Bea06]):

1. $C := D_P$ is a smooth degree 5 plane curve
2. $\tilde{C}$ is connected and there exist on $\tilde{C}$ a base point free pencil $|L|$ of degree 5 such that $\pi_* L = H$
3. let D be a degree 5 mobile linear system on C, then $D \cong H - p$ or $D \cong H - p + q$, with $p, q \in C$
4. $\pi^* H \in \mathrm{Pic}^{10}(\tilde{C}) \cap P^*$
5. Let D be a divisor on $\tilde{C}$ such that $h^0(D) \geq 2$ and $\pi_* H \equiv H$, then, either $D \equiv L$ or $D \equiv \sigma L$.

Now we are able to prove the first part of theorem 3.2.1:

Proof. Under the theorem 2.1.7 we can prove the assertion for P or either P^*, so we prove the theorem for $\theta^* \subset P^*$. We need to study two types of singularities:

1. This type of singularities is of the form $\pi^*(M) + E$ where $|M|$ is a mobile linear system and E an effective divisor.

Remark 3.2.5. Such a divisor M needs to be by computation a degree 5 divisor on C; in fact;

$$\begin{aligned}10 = \tilde{g} - 1 &= \deg(\pi^*(M)) + \deg(E)\\ &= 2\deg(M) + \frac{1}{2}(deg(K_C) - 2\deg(M))\\ &= \deg(M) + g - 1\\ &= \deg(M) - 5\end{aligned}$$

thanks to this condition on the degree of M it is proven in [Bea06] that such singularities need to be of the form:

$$\pi^*(H - p) + E \text{ where } p \in C \text{ and } \pi_*(E) = 2p$$

$$h^0(\pi^*(H - p) + E) \text{ even}$$

Let $\pi^{-1}(p) = \{p_1, p_2\}$, the possible divisors are $\pi^*(H)$ and $\pi^*(H) + p_i - \sigma p_i \;\; i = 1, 2$. Now, since $[\pi^* H]$ belongs to P^* then $[\pi^* H + p_i - \sigma p_i] \notin P^*$ by Remark 2.2.2. Hence, the only singularity of this type is the point $[\pi^* H]$.

2. Let $[D] \in P^*$ such that $h^0(D) \geq 4$; it is a matter of proving that $D \equiv \pi^* H$. We already said that on $\tilde{C}$ there exist a base point free pencil $|L|$ such that $\pi_* L = H$ so, let $s, t \in H^0(\tilde{C}, \mathcal{O}(L))$ whose divisors have no common points, we have a short exact sequence:

$$0 \to \mathcal{O}(D - L) \xrightarrow{(t,-s)} \mathcal{O}(D)^{\oplus 2} \xrightarrow{(s,t)} \mathcal{O}(D + L) \to 0$$

from which we deduce:

$$h^0(D - L) + h^0(D + L) \geq 2h^0(D) \geq 8$$

On the other hand, by Riemann-Roch, we obtain

$$h^0(D + L) = h^0(K - D - L) + 5 = h^0(\sigma D - L) + 5 = h^0(D - \sigma L) + 5,$$

hence, finally,

$$h^0(D - L) + h^0(D - \sigma L) \geq 3.$$

Thus, one of the linear systems $|D - L|$ or $|D - \sigma L|$ is a pencil projecting into $|H|$, so it is equal to $|L|$ or $|\sigma L|$ (5. of the properties listed before this proof). Consequently, D is linearly equivalent to $\pi^* H$, $2L$, or $2\sigma L$. But

$$2L \equiv \pi^* H + (L - \sigma L)$$

and

$$2\sigma L \equiv \pi^* H + (\sigma L - L)$$

do not belong to P^*, thus

$$D \equiv \pi^* H.$$

□

Corollary 3.2.6. *Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold, then its intermediate jacobian $(J(X), \theta)$ is irreducible as a principally polarized abelian variety.*

3.3 Irrationality and Torelli theorem

Lemma 3.3.1. *The Jacobian $J(C)$ of a smooth curve is irreducible.*

Now we have everyhing we need to prove the irrationality.

Lemma 3.3.2 (Voisin thm 7.31)**.** *Let X be a complex manifold, $Y \subset X$ a closed submanifold of codimension c. Consider $\mathrm{Bl}_Y X$, the variety obtained by blowing up X along Y. There are natural isomorphisms*

$$H^p(\mathrm{Bl}_Y X, \mathbb{Z}) \cong H^p(X, \mathbb{Z}) \oplus \sum_{k=1}^{c-1} H^{p-2k}(Y, \mathbb{Z})$$

This yields Lemma 3.11 [CG72]:

Proposition 3.3.3. *Let C be a curve on a smooth projective threefold X, then:*

$$J(\mathrm{Bl}_C X) \cong J(X) \times J(C)$$

We state here a preliminary theorem that provides a structured way to understand birational transformations as composition of blowups and blowdowns (cf. [Wlo99]).

Theorem 3.3.4 (Weak Factorization Theorem). *Let $f : X \dashrightarrow Y$ be a birational map of smooth complete varieties over a field of characteristic zero, which is an isomorphism over an open set U. Then f can be factored as*

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \dots \overset{f_{n-1}}{\dashrightarrow} X_n = Y,$$

where each X_i is a smooth complete variety and f_i is a blow-up or blow-down at a smooth center, which is an isomorphism over U.

Theorem 3.3.5 (Clemens-Griffiths criterion). *Let X be a rational smooth projective threefold. The intermediate Jacobian $J(X)$ is isomorphic as a pricipally polarized abelian variety to a product of jacobians of curves.*

Proof. Since X is rational, there exist a birational map $\phi : \mathbb{P}^3 \dashrightarrow X$. By the weak factorization theorem 3.3.4 we have a commutative diagram:

$$\begin{array}{ccc} & P & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle f} \\ \mathbb{P}^3 & \overset{\phi}{\dashrightarrow} & X \end{array}$$

where π is a composition of blowups, either of points or of smooth curves, and f is a rational morphism. By this diagram $J(P)$ is a product of Jacobians of curves, indeed by Lemma 3.3.2: $H^3(P,\mathbb{Z}) \cong H^3(\mathbb{P}^3,\mathbb{Z}) \oplus \bigoplus_{i=1}^{p} H^1(C_i,\mathbb{Z})$ and this is compatible with the Hodge decomposition and the cup-products. As a consequence, by the above Proposition:

$$J(P) \cong J(C_1) \times J(C_2) \times \cdots \times J(C_p)$$

Going back to $J(X)$ we have $f : P \to X$ a birational morphism, so we have homomorphisms $f^* : H^3(X,\mathbb{Z}) \to H^3(P,\mathbb{Z})$ and $f_* : H^3(P,\mathbb{Z}) \to H^3(X,\mathbb{Z})$ with $f_*f^* = 1$ again compatible with the Hodge decomposition and the cup-products in an appropriate sense. Thus $H^3(X,\mathbb{Z})$ with its polarized Hodge structure is a direct factor of $H^3(P,\mathbb{Z})$ hence $J(X)$ is a direct factor of $J(P)$ i.e. there exists a p.p.a.v. A such that

$$J(X) \times A \cong J(C_1) \times \cdots \times J(C_p)$$

□

Corollary 3.3.6. *A smooth cubic threefold $X \subset \mathbb{P}^4$ is not rational.*

Proof. Thanks to Corollary 3.2.6 and Lemma 3.3.1 and the above theorem we have (for some curve C):

$$J(X) \cong J(C)$$

Classical Brill–Noether theory shows that the singular set $\theta_{\text{sing}} \subset J(C)$ of the theta divisor

$$\theta = W^0_{g-1} = \{L \in Pic^{g-1}(C) | h^0(L) > 0\} \subset Pic^{g-1}(C) \cong J(C)$$

of the Jaobian of a smooth curve C of genus g is the Brill–Noether locus

$$\theta_{\text{sing}} = W^1_{g-1} = \{L \in Pic^{g-1}(C) | h^0(L) > 1\}$$

the lower bound for the dimension of the Brill-Noether locus is:

$$\dim(W^1_{g-1}) \geq g - (1+1)(g - g + 1 + 1) = g - 4$$

Since in our case $g = 5$ one finds $\dim(\theta_{\text{sing}}) \geq 1$. This contraddicts Theorem 3.2.1. □

Now we prove the second part of Theorem 3.2.1.

Theorem 3.3.7. *The projectified of the tangent cone at the origin of θ is isomorhpic to X.*

The idea behind the proof is that we can realize the theta divisor as a parametrized space and apply the technique in [SV88].
To be more precise, since $(\text{Alb}(F(X)), \Xi) \cong (J(X), \theta)$ as principally polarized abelian varieties by [Huy23, Corollary 5.3.3], we use a parametrization of $(\text{Alb}(F(X)), \Xi)$. The parametrizing map is the composition cf. [Huy23, Corollary 5.3.12]:

$$\alpha : F(X) \times F(X) \xrightarrow{a \times a} \text{Alb}(F(X)) \times \text{Alb}(F(X)) \xrightarrow{(1,-1)} \text{Alb}(F(X)) \cong J(X)$$

the image Im (α) is indeed the theta divisor $\Xi \subset \text{Alb}(F(X))$ by corollary 5.3.12 [Huy23]. In particular tha map α collapses the diagonal Δ to $0 \in \Xi$, the unique singular point.

Sketch of proof. The universal property of the blow-up provides us with a diagram:

$$\begin{array}{ccccccccc}
\mathbb{P}(\mathcal{T}_F) & \hookrightarrow & \text{Bl}_\Delta(F(X) \times F(X)) & \longrightarrow & \text{Bl}_0(\Xi) & \hookrightarrow & \text{Bl}_0(\text{Alb}(F(X))) & \hookleftarrow & \mathbb{P}(T_0\text{Alb}(F(X))) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Delta & \hookrightarrow & F(X) \times F(X) & \xrightarrow{\alpha} & \Xi & \hookrightarrow & \text{Alb}(F(X)) & \hookleftarrow & \{0\}.
\end{array}$$

where we use that the exceptional divisor $\mathcal{N}_{\Delta/F(X)\times F(X)}$ is isomorphic to $\mathcal{T}_{F(X)}$ which is again isomorphic, thanks to [Huy23] Proposition 5.2.2] to the restriction of the universal sub-bundle $\mathcal{U}_{\mathbb{G}(1,\mathbb{P}^4)|F(X)} = \mathbb{L} \cong \{(x,l)|x \in l \subset X\}0$.
The fibre of the blow-up $\mathrm{Bl}_0(\Xi)$ over the origin $0 \in \Xi$ is by definition the projective tangent cone of $0 \in \Xi$, which is regarded as a closed subscheme $\mathrm{TC}_0(\Xi) \subset \mathbb{P}(T_0 Alb(F(X)))$. By the irriducibility of Ξ we have that the blow-up $\mathrm{Bl}_0(\Xi)$ is irreducibe too, ([Har13] II. Prop. 7.16). Therefore, the induced morphism between the exceptional divisors $\mathbb{P}(\mathcal{T}_F) \to \mathrm{TC}_0(\Xi)$ is surjective. Consider then the composition:

$$r : \mathbb{L} = \mathbb{P}(\mathcal{U}_F(X)) \cong \mathbb{P}(\mathcal{T}_F) \to TC_0(\Xi) \subset \mathbb{P}(T_0\mathrm{Alb}(F(X))) \cong \mathbb{P}^4$$

Up to a linear coordinate change r is the projection $\mathbb{L} \to \mathbb{P}(V)$ (for the details [Huy23, p.244]).

In particular, we recover X as the image of r.

□

Corollary 3.3.8 (Clemens-Griffiths, Tyurin). *For two smooth cubic hypersurfaces $Y, Y' \subset \mathbb{P}^4$ over $\mathbb{C}$, the following assertions are equivalent:*

1. *There exists an isomorphism $Y \cong Y'$.*
2. *There exists a Hodge isometry $H^3(Y,\mathbb{Z}) \cong H^3(Y',\mathbb{Z})$.*
3. *There exists an isomorphism of polarized varieties $(J(Y),\Theta) \cong (J(Y'),\Theta')$.*

Remark 3.3.9. Recall the classical Torelli theorem for curves whose statement is the same as the one for cubic threefolds. However there is a strong form of the Torelli theorem which asserts that for two smooth projective, irreducible curves C and C' any isomorphism of principally polarized abelian varieties $(J(C);\theta)$ ' $(J(C');\theta')$ is up to a sign induced by a unique isomorphism $C \cong C'$. This in particular applies to automorphisms and leads for a nonhyperelliptic curve C to the isomorphism

$$\mathrm{Aut}(C) \cong \mathrm{Aut}(J(C),\theta)/\{\pm\mathrm{id}\}$$

If C is hyperelliptc, then $\mathrm{Aut}(C) \cong \mathrm{Aut}(J(C),\theta)$. (an analogous assertion holds for smooth cubic threefolds [Huy23]).

3.3.1 A particular cubic

We now focus on a particular cubic threefold, the so-called Klein cubic threefold, whose irrationality is established by the discussion in the previous section. However, we will prove its irrationality directly using an ad-hoc technique. The Klein cubic threefold is the hypersurface in $\mathbb{P}^4$ defined by:

$$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0 = 0$$

The prove will revolve around the existence of two particular automorphisms of X. Let ζ be a primitive 11-th root of unity.

$$\delta : (X_0, X_1, X_2, X_3, X_4) \mapsto (X_0, \zeta X_1, \zeta^{-1}X_2, \zeta^3X_3, \zeta^6X_4),$$
$$\sigma : (X_0, X_1, X_2, X_3, X_4) \mapsto (X_1, X_2, X_3, X_4, X_0),$$

which satisfy $\delta^{11} = \sigma^5 = 1$ and $\sigma\delta\sigma^{-1} = \delta^{-2}$.

They induce automorphisms δ^*, σ^* of $J(X)$. Suppose that $J(X)$ is isomorphic as a principally polarized abelian variety to the Jacobian $J(C)$ of a curve C. By the above remark we have a short exact sequence

$$1 \to \mathrm{Aut}(C) \to \mathrm{Aut}(J(C)) \to \mathbb{Z}/2;$$

since δ^* and σ^* have odd order, they are induced by automorphisms δ_C, σ_C of C, satisfying

$$\sigma_C\delta_C\sigma_C^{-1} = \delta_C^{-2}.$$

Now we apply the Lefschetz fixed point formula [Hat02]. The automorphism δ of X fixes the 5 points corresponding to the basis vectors of $\mathbb{C}^5$; it acts trivially on $H^{2i}(X, \mathbb{Q})$ for $i = 0, \dots, 3$. Therefore, we find

$$\mathrm{Tr}(\delta^*_{|H^3(V,\mathbb{Q})}) = -5 + 4 = -1.$$

Similarly, σ fixes the 4 points $(1, \alpha, \alpha^2, \alpha^3, \alpha^4)$ of V with $\alpha^5 = 1, \alpha \neq 1$, so

$$\mathrm{Tr}(\sigma^*_{|H^3(V,\mathbb{Q})}) = -4 + 4 = 0.$$

Applying now the Lefschetz formula to C, we find that σ_C has two fixed points on C and δ_C three. But since σ_C normalizes the subgroup generated by δ_C, it preserves the

3-point set $\mathrm{Fix}(\delta_C)$; since it is of order 5, it must fix each of these 3 points, which gives a contradiction.

Finally, suppose $J(X)$ is isomorphic to a product $A_1 \times \cdots \times A_p$ of p.p.a.v. By the uniqueness of the decomposition Lemma 2.1.9, the automorphism δ^* permutes the factors A_i. Since δ has order 11 and $p \leq 5$, this permutation must be trivial, so δ^* induces an automorphism of A_i for each i, hence of $H^1(A_i, \mathbb{Q})$; but the group $\mathbb{Z}/11$ has only one nontrivial irreducible representation defined over $\mathbb{Q}$, given by the cyclotomic field $\mathbb{Q}(\zeta)$, with $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 10$. Since $\dim(A_i) < 5$, we see that the action of δ^* on each A_i, and therefore on $J(X)$, is trivial. But this contradicts the relation $\mathrm{Tr}(\delta^*_{|H^3(V,\mathbb{Q})}) = -1$. □

Remark 3.3.10. It is worth noting that another proof of irrationality comes from [Adl78]. In this article the author proves that the group $\mathrm{PSL}_2(\mathbb{F}_{11})$ admits a faithful action on the Klein cubic threefold and it is in fact isomorphic to its automorphism group. The thing is

$$|\mathrm{PSL}_2(\mathbb{F}_{11})| = 660$$

but by Hurwitz bound [Mir95] on a compact Riemann surface of genus 5 the maximum admissible order for a group acting on the curve is

$$84(5-1) = 336 < 660.$$

3.4 Nodal cubic hypersurfaces

Assume $X \subset \mathbb{P}^{n+1}$ is a nodal hypersurface with exactly one node at the point $[0 : \cdots : 0 : 1] \in \mathbb{P}^{n+1}$. Then, after a change of coordinates, X is defined by an equation of the form

$$f_3(x_0, \ldots, x_n) + x_{n+1} \cdot q_2(x_0, \ldots, x_n) = 0,$$

where f_3 and q_2 define smooth hypersurfaces of degree three and two in $V(x_{n+1}) \cong \mathbb{P}^n$.

An isolated singularity $x_0 \in X$ is an ordinary double point or a node if the exceptional fiber $E_{x_0} \subset \mathrm{Bl}_{x_0}X$ is a non-degenerate quadric in the exceptional divisor $\mathbb{P}^n \cong E \cong \mathrm{Bl}_{x_0}\mathbb{P}^{n+1}$. Then for any line $x_0 \in l \cong \mathbb{P}^1 \subset \mathbb{P}^{n+1}$ the intersection $l \cap X$ has multiplicity at least two at x_0.

$$\begin{array}{ccccccc} E_{x_0} & \hookrightarrow & \mathrm{Bl}_{x_0}(X) & \hookrightarrow & \mathrm{Bl}_{x_0}(\mathbb{P}^{n+1}) & \xrightarrow{\phi} & \mathbb{P}^n \\ \downarrow & & \downarrow & & \downarrow \tau & & \\ \{x_0\} & \hookrightarrow & X & \hookrightarrow & \mathbb{P}^{n+1} & & \end{array}$$

Let F be the equation defining X, then $F \in |\mathcal{O}(3) \otimes \mathcal{I}_{x_0}^{\otimes 2}|$. Hence

$$\tau^* F \in |\tau^* \mathcal{O}(3) \otimes \mathcal{O}(-2E)|$$

describes $\mathrm{Bl}_{x_0} X$ as a subvariety of $\mathrm{Bl}_{x_0} \mathbb{P}^{n+1}$, indeed the equation defining X vanishes along E to order two. From the identity $\phi^* \mathcal{O}(1) \cong \tau^* \mathcal{O}(1) \otimes \mathcal{O}(-E)$ we get that:

$$\tau^* \mathcal{O}(3) \otimes \mathcal{O}(-2E) \cong \tau^* \mathcal{O}(1) \otimes \phi^* \mathcal{O}(2)$$

i.e.

$$\mathrm{Bl}_{x_0} X \in |\mathcal{O}(1,2)|$$

This is equivalent to decomposing $V_{n+2} = V_{n+1} \oplus < v_{n+2} >$ and seeing that the polynomial F lies in the vector space $\mathrm{Sym}^2 V_{n+1}^\vee \otimes V_{n+2}^\vee$ and observing that

$$\mathrm{Sym}^2 V_{n+1} \otimes V_{n+2}^\vee \cong H^0(\mathbb{P}_{\mathbb{P}^n}(\mathcal{O} \oplus \mathcal{O}(-1)), \mathcal{O}(1,2)),$$

since from the discussion in the previous chapter and Lemma 3.2.2 we had that

$$\mathrm{Bl}_{x_0}(\mathbb{P}^{n+1}) \cong \mathbb{P}(\mathcal{F}^\vee) =: P \quad \text{where} \quad \mathcal{F} \cong \mathcal{O}(1) \oplus \mathcal{O}.$$

Let P' be:

$$P' := \mathbb{P}_{\mathbb{P}^n}(\mathcal{O}(-2) \oplus \mathcal{O}(-3))$$

By the isomorphism

$$\mathbb{P}_{\mathbb{P}^n}(\mathcal{O} \oplus \mathcal{O}(-1)) \cong \mathbb{P}_{\mathbb{P}^n}(\mathcal{O}(-2) \oplus \mathcal{O}(-3))$$

we have that:

$$\mathrm{Bl}_{x_0} X \subset \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(-2) \otimes \mathcal{O}(-3)) \quad \text{cut out by a section of } \mathcal{O}_{P'}(1)$$

where $\mathcal{O}_P(1)$ indicates the relative bundle on P.

If we consider now the projection morphism to $\mathbb{P}^n$ we see that the fiber is generically a point, and a whole $\mathbb{P}^1$ when $q_2 = f_3 = 0$.
Hence we conclude that:

$$\mathrm{Bl}_{x_0} X \cong \mathrm{Bl}_Z \mathbb{P}^n$$

where Z is the (generically smooth) complete intersection $V(q_2, f_3)$.

Remark 3.4.1. The sections (q_2, f_3) belong to the linear system $|\mathcal{O}(2) \oplus \mathcal{O}(3)|$ where the zero locus of f_3 is the intersection of X with $\mathbb{P}^n$ while the zero locus of q_2 is the intersection of the non-degenerate quadric E with $\mathbb{P}^n$.

Chapter 4

Cubic fourfolds

4.1 Generalities on cubic fourfolds

Consider a smooth cubic fourfold in $\mathbb{P}^5$. In general, smooth cubic fourfolds are parametrized by a smooth Zariski open subset

$$\mathcal{U} \subset \mathbb{P}(\mathrm{Sym}^3(V_6^\vee)) \cong \mathbb{P}^{55}.$$

If we want to consider smooth cubic fourfolds up to isomorphism we have to quotient $\mathcal{U}$ with PGL_6, yielding a 20-dimensional moduli space of cubic fourfolds denoted $\mathcal{C}$. Understanding this moduli space is crucial for studying the geometry of these varieties, as well as their connections to K3 surfaces.

Let X be a smooth cubic fourfold, as we discussed in the first chapter, its Hodge diamond is:

$$\begin{array}{ccccccccc} 0 & & 1 & & 21 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

where the middle row contains all the nontrivial Hodge-theoretic informations. The middle cohomoogy $H^4(X,\mathbb{Z})$ of a smooth cubic fourfold has rank 23. There is a special class $h^2 \in H^{2,2}(X)$, which is the square of the hyperplane class, whose orthogonal consists in the primitive cohomology $< h^2 >^{\perp} = H^4(X,\mathbb{Z})_{\mathrm{prim}}$. Hassett proves [Has00, Prop. 2.1.2] that $H^4(X,\mathbb{Z})$ is a unimodular lattice under the intersection form $<,>$ of signature $(21,2)$ i.e.

$$H^4(X,\mathbb{R}) \cong (+1)^{\oplus 21} \oplus (-1)^{\oplus 2}$$

and the primitive cohomology is:

$$H^4(X,\mathbb{Z})_{\text{prim}} \cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus H \oplus H \oplus E_8 \oplus E_8$$

where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the hyperbolic plane, and E_8 is the positive definite quadratic form associated with the corresponding Dynkin diagram,

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

A key role in Hassett's work is played by the Fano variety of lines $F(X) \subset \mathbb{G}(1,5)$. Its importance follows from the isomorphism

$$\alpha : H^4(X,\mathbb{Z})_{\text{prim}} \to H^2(F(X),\mathbb{Z})_{\text{prim}}(-1)$$

(the -1 means that the weight is shifted by two; this reverses the sign of the intersection form). Where α is the Abel-Jacobi map, i.e. the composition

$$\alpha = p_* q^* : H^4(X,\mathbb{Z}) \to H^2(F(X),\mathbb{Z})$$

of the maps:

$$\begin{array}{ccc} \mathbb{L} & \xrightarrow{p} & F(X) \\ {\scriptstyle q}\downarrow & & \\ X & & \end{array}$$

here $\mathbb{L}$ is the universal line $\mathbb{L} := \{(l,x) \mid x \in l \subset X\}$ and p,q the natural projections.

The Hodge structure may be given by specifying the one-dimensional subspace

$$H^{3,1}(X) \subset H^4(X,\mathbb{C}).$$

As we computed in the first chapter via the Jacobian ring method the space $H^{3,1}(X)$ is one-dimensional geneated by $\sigma := x_0x_1x_2x_3x_4x_5$, with σ non-degenerate. Since $h^{3,1}(X) = 1$ this subspace corresponds to a line in a 23-dimensional space. Note that $H^{3,1}(X)$ needs to be orthogonal to the square of the hyperplane class h^2, thus we see it as a line in a 22-dimensional space $H^4(X,\mathbb{Z})_{\text{prim}}$ satisfying the following properties:

(i) $H^{3,1}(X)$ is isotropic with respect to the intersection form $<,>$

(ii) The Hermitian form $H(u,v) = -<u,\overline{v}>$ on $H^{3,1}(X)$ is positive.

This conditions combined give us the Hodge structure as a point on a euclidean open subset of a quadric hypersurface $Q \subset \mathbb{P}^{21}$. Therefore the set of all possible Hodge structures is 20-dimensional. Exactly the dimension of the moduli space. This leads to a theorem due to Voisin [Voi86]: which asserts thst the period map for cubic fourfolds $\tau : X \mapsto H^{3,1}(X)$ is an open immersion of analytic spaces. In particular, if X_1 and X_2 are cubic fourfolds and there exists an isomorphism of Hodge structures

$$\psi : H^4(X_1,\mathbb{Z}) \to H^4(X_2,\mathbb{Z}),$$

(preserving the class h^2) then X_1 and X_2 are isomorphic.
Here I have merely provided an argument that should suggest the validity of Torelli's theorem for cubic fourfolds. However, Voisin's proof relies on cubics containing a plane, which we will discuss later.

4.2 Rationality questions and relation to K3 surfaces

As we already saw for smooth cubic curves, rationality can be ruled out by a simple invariant: their genus, which is equal to one, whereas any rational curve has genus zero. In contrast, all smooth cubic surfaces are rational, as they are isomorphic to the blow-up of the projective plane at six general points, providing an explicit birational parametrization. As shown in the previous chapter, all smooth cubic threefolds are irrational, a fact first established by Clemens and Griffiths using intermediate Jacobians. For smooth cubic fourfolds, the general case is expected to be irrational, though the question remains open in full generality. However, special examples of rational cubic fourfolds are known, and we will get to such examples later. Before doing so, I first want to present an argument, inspired by ideas from [CG72], that explains why the general

cubic fourfold is expected to be irrational.
As we already observed in theorem 3.3.4, every birational morphism is a composition of blow-ups and blow-downs. When we blow up a surface on a fourfold we add its Hodge structure to the middle Hodge stucture of the four dimensional ambient variety as a summand. As a consequence of the Voisin's Torelli theorem, the primitive Hodge structure $H^4(X)_{\mathrm{prim}}$ of a cubic fourfold X is irreducible. Then, if there existed a birational isomorphism between X and $\mathbb{P}^4$, it would necessarily factor as a sequence of blow-ups and blow-downs. At each step, a new piece of Hodge structure would either be introduced as a direct summand or removed. However, due to the irreducibility of $H^4(X)_{\mathrm{prim}}$, at some point in this sequence, it would have to appear as a direct summand the entire primitive Hodge structure of the cubic fourfold. This would require the existence of a surface whose Hodge structure contains at least $H^4_{\mathrm{prim}}(X)$ as a direct summand. Yet, no such surface is known to exist, suggesting that the generic cubic fourfold is not birational to $\mathbb{P}^4$. To conclude, while the argument above suggests that the general cubic fourfold should be irrational, we do not yet have a single explicit example of a proven irrational cubic fourfold. Establishing such an example remains an open problem and a major challenge in the field.
However we have conjectures on the subject due to Hassett and others such as Kuznetsov, Russo and Staglianò which relate K3 surfaces to the study of the rationality of cubic fourfolds.

4.2.1 An overview of K3 surfaces

Definition 4.2.1. Let X be a compact connected complex manifold of complex dimension two, we say that X is a K3 surface if:

- the irregularity is zero i.e. $H^1(X, \mathcal{O}_X) = 0$,
- the canonical bundle is trivial: $\omega_X = \mathcal{O}_X$.

Let X be a projective K3 surface, a pair (X, L) is a polarized K3 surface of degree d if L is an ample primitive line bundle over X such that $(L)^2 = d > 0$. Because the intersection pairing is even, the degree of a polarized K3 surface can be written as $d = 2g - 2$, where g is called the genus of X and, in fact, it corresponds to the genus of any smooth curve in the linear system of the polarization. Hence we know almost all the Hodge numbers of a K3 surface S; having a trivial canonical bundle implies that $h^{2,0}(S) = h^{0,2}(S) = 1$ and irregularity zero means $h^{1,0}(S) = h^{0,1}(S) = 0$, of course we want it to be connected and hence $h^{0,0}(S) = 0$. The only number left to be computed is then $h^{1,1}(S)$. To compute it we employ the Noether's formula for surfaces:

$$\chi(\mathcal{O}_S) = \frac{K_S.K_S + e(S)}{12}$$

The left hand side equals

$$\chi(\mathcal{O}_S) = \sum_{i=0}^{2} (-1)^i h^i(O_S) = 1 + 0 + 1 = 2.$$

On the right hand side we have

$$K_S.K_S = \mathcal{O}_S.\mathcal{O}_S = 0$$

and by putting together the Hodge numbers we already know

$$e(S) = 2 + \dim H^2(S, \mathbb{C}) = 4 + h^{1,1}(S).$$

Thus we have $h^{1,1}(S) = 20$. So the Hodge diamond is:

$$\begin{matrix} 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

We just state here a very deep result on K3 surfaces. The global Torelli theorem for K3 surfaces seems to have been conjectured first in the late fifties by Andreotti and Weil. Together with Nirenberg, Kodaira, Grauert, and Tyurina, they showed that the periods of a K3 surface give local coordinates in moduli, work which was clarified and greatly generalized by Griffiths in his study of the period map. The global Torelli theorem was proved for algebraic K3 surfaces by Piatetskii-Shapiro and Shafarevitch in 1971 [PŠ71].

Theorem 4.2.2 (Global Torelli)**.** *Two complex K3 surfaces X and Y are isomorphic if and only if there exists an isomorphism of integral Hodge structures between $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ respecting the intersection pairing. Moreover, two polarised K3 surfaces (X, L) and (X', L') are isomorphic if and only if there exists an isomorphism of integral Hodge structures between $H^2(X, \mathbb{Z})$ and $H^2(X', \mathbb{Z})$ mapping $[L]$ to $[L']$.*

Proof. See section 6.3 of [Huy17]. □

The most relevant aspect of the Global Torelli theorem for this work is the surjectivity of the global period map. This ensures that whenever we construct an admissible lattice, we can associate to it a corresponding K3 surface.

Here, we introduce a few definitions that will be useful in the following sections.

Definition 4.2.3. A Hodge structure of weight m is defined by a finitely generated group $H_{\mathbb{Z}}$ and a decomposition of $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$ into a direct sum $H_{\mathbb{C}} = \bigoplus_{p+q=m} H^{p,q}$, where $H^{p,q}$ is a complex subspace satisfying the symmetry condition $H^{p,q} = \overline{H^{q,p}}$.

Definition 4.2.4. Let $H = \bigoplus_{p+q=k} H^{p,q}$ be a non-zero Hodge structure of weight k. The level $\lambda(H)$ of H is defined as

$$\lambda(H) := \max\{q - p \mid H^{p,q} \neq 0\}.$$

Definition 4.2.5. Let H be a Hodge structure of weight k. We define H to be of K3 type if

$$\lambda(H) = 2 \quad \text{and} \quad h^{\frac{k+2}{2},\frac{k-2}{2}} = 1.$$

4.2.2 Conjectures

Based on the previous definitions, the Hodge structure of a cubic fourfold is said to be of K3 type. Indeed this is just the starting point of the deep relation between K3 surfaces and cubic fourfolds. As a lattice $H^4(X,\mathbb{Z})_{\text{prim}}$ can be embedded primitively and isometrically (up to a global sign) into the full cohomology lattice of a K3 surface:

$$H^4(X,\mathbb{Z})_{\text{prim}} \hookrightarrow H^*(S,\mathbb{Z})$$

Note that the complement $H^4(X,\mathbb{Z})_{\text{prim}}^{\perp} \subset H^*(X,\mathbb{Z})$, which is the rank five lattice $H^0_{\mathbb{Z}} \oplus H^2_{\mathbb{Z}} \oplus \mathbb{Z} \cdot h^2 \oplus H^6_{\mathbb{Z}} \oplus H^8_{\mathbb{Z}}$, carries a trivial Hodge structure.

This deep relation with K3 surfaces allowed Hassett to formulate a conjecture on the rationality of cubic fourfolds.

This is Hassett's conjecture as stated in [Huy17]; in the next section, we will be more faithful to the author's original phrasing.

Conjecture 4.2.6 (Hassett)**.** A smooth cubic fourfold $X \subset \mathbb{P}^5$ is rational if and only if there exists a primitive isometric embedding of Hodge structures $H^2(S,\mathbb{Z})_{\text{prim}} \hookrightarrow H^4(X,\mathbb{Z})_{\text{prim}}$ for some polarised K3 surface (S,H):

$$X \text{ rational} \quad \Leftrightarrow \quad \exists\, H^2(S,\mathbb{Z})_{\text{prim}} \hookrightarrow H^4(X,\mathbb{Z})_{\text{prim}}. \tag{4.2.1}$$

More recently Kuznetsov [Kuz06] proposed another conjecture which surprisingly turned out to be equivalent to Hassett conjecture by a work of Addington and Thomas [AT12].

4.3 Special cubic fourfolds

Trying to relate K3 surfaces and cubic fourfolds one gets the prolem that the middle Hodge structure of a K3 has dimension one less that the cubic fourfold one. Hence we could ask if there are any particular cubic fourfolds whose primitive Hodge structure contains a rank one algebraic summand coming from $H^{2,2}$ whose complement is the Hodge structure of a K3.

Definition 4.3.1. A cubic fourfold X is special if it contains an algebraic surface T which is not homologous to a complete intersection of ample divisors from $\mathbb{P}^5$.

Let X be a cubic fourfold and let $A(X)$ denote the lattice $H^{2,2}(X) \cap H^4(X,\mathbb{Z})$. This lattice is positive definite by the Riemann bilinear relations.

The Hodge conjecture is true for cubic fourfolds [Zuc77], so $A(X)$ is generated (over $\mathbb{Q}$) by the classes of algebraic cycles.

X is special if and only if the rank of $A(X)$ is at least two. Equivalently, X is special if and only if

$$A(X)^0 := A(X) \cap H^4(X,\mathbb{Z})^0 \neq 0.$$

Definition 4.3.2. Let $(K, <,>)$ be a positive definite rank-two lattice containing a distinguished element h^2 with $< h^2, h^2 >= 3$. A marked special cubic fourfold is a special cubic fourfold X with the data of a primitive imbedding of lattices $K \hookrightarrow A(X)$ preserving the class h^2. A labelled special cubic fourfold is a special cubic fourfold with the data of the image of a marking, i.e. a saturated rank-two lattice of algebraic classes containing h^2. A special cubic fourfold is typical if it has a unique labelling.

Proposition 4.3.3. *The special cubic fourfolds form a countably infinite union of irreducible divisors in $\mathcal{C}$.*

Proof. see [Has00, §3] □

Definition 4.3.4. Let (X, K) be a labelled special cubic fourfold. The discriminant of the pair (X, K) is the determinant of the intersection matrix of K.

Theorem 4.3.5 (prop 3.2.2, th 3.2.3)**.** *Let (X, K) be a labelled special cubic fourfold of discriminant d then*

$$d > 0 \quad \text{and } d \equiv 0, 2 \ (mod\ 6) \quad (*)$$

In particular, the special cubic fourfolds possessing a labelling of discriminant d form an irreducible (possibly empty) algebraic divisor $\mathcal{C}_d \subset C$.

Theorem 4.3.6. *Let $d > 6$ be an integer with $d \equiv 0, 2 \ (mod\ 6)$. Then the divisor $\mathcal{C}_d$ is nonempty.*

Moreover, there exists a polarized K3 surface S such that $K^{\perp} \subset H^4(X, \mathbb{Z})$ is Hodge-isometric to $H^2(S, \mathbb{Z})_{\text{prim}}(-1)$ if and only if d satisfies a further condition

$$d \text{ is not divisible by } 4, 9, \text{ or any odd prime } p \equiv 2 \ (\text{mod } 3) \quad (**)$$

Here is a list of the first values fo d satisfying the conditions $(*)$ and $(**)$.

$(*)$ $d = 8, 12, 14, 18, 20, 24, 26, 30, 32, 36, 38, 42, 44, 48, 50, 54, 56, 60, 62, 66, 68, 72, 74, 78, \ldots$
$(**)$ $d = 14, 26, 38, 42, 62, 74, 78, 86, 98, 114, 122, 134, 146, \ldots$

Remark 4.3.7. Spelling out the numerical condition $(**)$, one finds that Conjecture 0.2.6 predicts rationality of all cubics contained in the Hassett divisors $\mathcal{C}_{14}$, $\mathcal{C}_{26}$, $\mathcal{C}_{38}$, $\mathcal{C}_{42}$, $\mathcal{C}_{62}$, $\mathcal{C}_{74}$, ... For $d = 14$ this is classical, and for the next two values $d = 26$ and $d = 38$ it was confirmed recently by Russo and Staglianò [RS18]. The case $d = 42$ was settled again by Russo and Staglianò [RS22], so that the first Hassett divisor in this series for which rationality is not yet known is $\mathcal{C}_{62}$.

4.4 Examples

Here we collect some examples of special cubic fourfolds for small values of d. First, we make some preliminary computations with Chern classes. Let X be a cubic fourfold containing a surface T and assume

$$K_d = \mathbb{Z}h^2 + \mathbb{Z}T$$

To compute the discriminant d in practice, we have to compute the determinant of the matrix:

$$\begin{pmatrix} < h^2, h^2 > & < h^2, T > \\ < T, h^2 > & < T, T > \end{pmatrix}$$

hence we need to determine the self-intersection $< T, T >$ on X. We interpret $< T, T >$ as $c_2(\mathcal{N}_{T/X})$, the highest Chern class of the normal bundle to T in X. First

$$c(\mathcal{T}_X|_T) = 1 + 3ht + 6h^2t^2,$$

where we are omitting the restriction to T, $h = h|_T$, and

$$c(\mathcal{T}_T) = 1 - K_T t + e_T t^2,$$

where e_T denotes the topological Euler characteristic if T. Using the exact sequence

$$0 \to \mathcal{T}_T \to \mathcal{T}_X|_T \to \mathcal{N}_{T/X} \to 0$$

we get:

$$1 + 3ht + 6h^2t^2 = (1 - K_+ e_T t^2)(1 + c_1(\mathcal{N}_{T/X})t + c_2(\mathcal{N}_{T/X})t^2)$$
$$1 + 3ht + 6h^2t^2 = 1 + (c_1(\mathcal{N}_{T/X} - K_T))t + (c_2(\mathcal{N}_{T/X} - K_T) - K_T c_1(\mathcal{N}_{T/X} + e_T))t^2 + ...$$

thus, we have

$$c_1(\mathcal{N}_{T/X}) = 3h + K_T$$

and finally,

$$6h^2 = c_2(\mathcal{N}_{T/X} - K_T) - 3hK_T - K_T^2 + e_T$$
$$c_2(\mathcal{N}_{T/X} - K_T) = 6h^2 + 3hK_T + K_T^2 - e_T$$

4.4.1 The case of discriminant 8

Observe that the value $d = 8$ satisfies condition $(*)$ and hence $\mathcal{C}_8 \subset \mathcal{C}$ is irreducible and nonempty, but doesn't satisfy condition $(**)$. Let's look at cubic fourfolds containg a plane. A dimension count reveals that the set of smooth cubic fourfolds containing a plane forms a divisor in the moduli space of all smooth cubic fourfolds. Indeed, fix a plane $\mathbb{P}^2 \cong P \subset \mathbb{P}^5$ and compute the linear system $|\mathcal{I}_P \otimes \mathcal{O}_{\mathbb{P}^5}(3)|$ of all cubic containing P. Its dimension is

$$h^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)) - h^0(P, \mathcal{O}_P(3)) - 1 = 56 - 10 - 1 = 45.$$

The subgroup of PGL_6 preserving P as a subvariety (but not necessarily pointwise) is of dimension $3^2 + 6 \cdot 3 - 1 = 26$. This proves that within the 20-dimensional moduli space $\mathcal{C}$ of all smooth cubic fourfolds the set of cubics containing a plane is an irreducible divisor. It is a result of Voisin [Voi86] that every cubic in $\mathcal{C}_8$ contains a plane. We now want to verify that the discriminant of the lattice generated by h^2 and P is equal to eight. Indeed by the Chern classes computation above the matrix of this lattice takes the form

$$K_8 = \begin{array}{c|cc} & h^2 & P \\ \hline h^2 & 3 & 1 \\ P & 1 & 3 \end{array}$$

whose discriminant is eight.

Example 4.4.1. Let $[x_0 : \ldots : x_5]$ be the coordinates of $\mathbb{P}^5$ and fix $P = V(x_0, x_1, x_2)$ a plane. The cubic polynomial $F = x_0q_1 + x_1q_2 + x_2q_3$, where q_1, q_2, q_3 are quadrics, define a cubic fourfold $X = V(F)$ containing the plane P. As we saw in the previous chapter we have a quadric fibration

$$\phi : \mathrm{Bl}_P X \to \mathbb{P}^2 \supset \Delta_6$$

with discriminant divisor a sextic $\Delta_6 \subset \mathbb{P}^2$.The fiber of the projection is generically a smooth quadric, thus it defines a maximum rank bilinear form on a 4-dimensional vector space, the set of the totally isotropic subspaces has 2 one-dimensionl components, it is the locus cut out by a section of $\mathrm{Sym}^2(\mathcal{U}^\vee)$ on $\mathrm{Gr}(2, \mathcal{E})$, where $\mathcal{E} = 3\mathcal{O} \oplus \mathcal{O}(1)$. While over the discriminant locus Δ_6 the fiber is connected. By Stein factorization we obtain a surface with a 2:1 map over $\mathbb{P}^2$ ramified along a sextic curve and hence a degree 2 K3 surface.
This is the K3 associated to X in the sense of Hassett but this isn't the only K3 surface associated to such cubic hypersurface. In fact X contains $V(q_1, q_2, q_3)$ which is a complete intersection of type $(2, 2, 2)$ in $\mathbb{P}^5$, so a degree 2 K3 surface.

The rationality of a cubic containing a plane is not known in general but there are examples of cubic fourfolds containing a fixed plane and another surface yielding rationality. If the existence of a second, complementary linear space contained in the cubic hypersurface X is assumed rationality of X can be deduced.

Example 4.4.2 (Cubic fourfold containing two skew planes)**.** Here is a result from [Huy23] that holds not only in dimension 4 but for every even-dimensional cubic.

Proposition 4.4.3. *Assume that a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ of even dimension $n = 2m$ contains two complementary linear subspaces $\mathbb{P}^m \cong \mathbb{P}(W) \subset X$ and $\mathbb{P}^m \cong \mathbb{P}(W') \subset X$, i.e., such that $W \oplus W' = V$ and, in particular, $\mathbb{P}(W) \cap \mathbb{P}(W') = \varnothing$. Then the quadric fibration 3.2.3 admits a section and X is rational.*

Proof. The section is given by the inclusion $\mathbb{P}^m \cong \mathbb{P}(W') \subset X$, for which the linear projection induces an isomorphism $\mathbb{P}(W') \cong \mathbb{P}(U)$. As any quadric admitting a rational point is rational, the scheme-theoretic generic fibre $q^{-1}(\xi)$ is a rational quadric over $K(\mathbb{P}^m)$. Hence, $\mathrm{Bl}_{\mathbb{P}}(X)$ is rational and, therefore, X itself is. □

Let's work through an explicit example, fix disjoint planes:

$$\Pi_1 = \{x_0 = x_1 = x_2 = 0\},\ \Pi_2 = \{x_3 = x_4 = x_5 = 0\} \subset \mathbb{P}^5$$

and take as a cubic fourfold X containing both planes:

$$X = V(x_0x_3^2 + x_1x_4^2 + x_2x_5^2 - x_0^2x_3 + x_1^2x_4 - x_2^2x_5) \subset \mathbb{P}^5$$

which is smooth. More generally, fix forms

$$c_1, c_2 \in \mathbb{C}[x_0, \ldots, x_2; x_3, \ldots, x_5]$$

of bidegree $(1,2)$ and $(2,1)$ in the variables $\{x_0, \ldots, x_2\}$ and $\{x_3, \ldots, x_5\}$. Then the cubic hypersurface defined by:

$$X = V(\tilde{c}_1 + \tilde{c}_2) \subset \mathbb{P}^5$$

contains Π_1, Π_2. The birational parametrization of such X can be given geometrically. For generic points $p_1 \in \Pi_1$ and $p_2 \in \Pi_2$, the line $< p_1, p_2 >$ intersects X in a unique other point, say P. The case when the intersection $< p_1, p_2 > \cap X$ is more than three points is when the line $< p_1, p_2 >$ is contained in X, that is:

$$S := V(c_1, c_2) \subset \Pi_1 \times \Pi_2 \cong \mathbb{P}^2 \times \mathbb{P}^2$$

which is a complete intersection of hypersurfaces of bidegree $(1,2)$ and $(2,1)$ and hence (by adjunction and Lefschetz Hyperplane section theorem) S is a K3 surface. Hence we have a birational morphism

$$\Pi_1 \times \Pi_2 \smallsetminus S \longrightarrow X$$
$$(p_1, p_2) \mapsto X \cap < p_1, p_2 >$$

Remark 4.4.1.1. The linear series inducing this birational parametrization is given by the forms of bidegree $(2,2)$ containing S and the inverse by the quadrics in $\mathbb{P}^5$ containing Π_1 and Π_2.

4.4.2 The case of discriminant 14

$\mathcal{C}_{14}$ is the locus of cubic fourfolds containing a quartic rational normal scroll Σ. Observe first that 14 satisfies both $(*)$ and $(**)$.
A quartic scroll is a smooth rational ruled surface $\Sigma \subset \mathbb{P}^5$ with degree 4, with the rulings embedded as lines. There are two possibilities:

- $\mathbb{P}^1 \times \mathbb{P}^1$ embedded via the linear series $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)|$
- an embedding of the Hirzebruch surface $\mathbb{F}_2$

we will analize the case where the quartic scroll is realized as the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark 4.4.4. it is worth noting that all scrolls of the first class are projectively equivalent and have equations given by the 2×2 minors of:

$$\begin{pmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{pmatrix}$$

where $[x_0 : x_1 : x_2 : x_3 : x_4 : x_5]$ are the coordinates of $\mathbb{P}^5$.

Let X be a cubic fourfold containing a rational normal quartic scroll Σ. We have $<\Sigma, \Sigma>= 10$ so our marking is

$$K_{14} = \begin{array}{c|cc} & h^2 & \Sigma \\ \hline h^2 & 3 & 4 \\ \Sigma & 4 & 10 \end{array}$$

Lemma 4.4.5. *Let Σ be a quartic scroll, realized as the image of $\mathbb{P}^1\times\mathbb{P}^1$ under the linear series $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)|$. Then a generic point $p \in \mathbb{P}^5$ lies on a unique secant to Σ.*

Sketch of proof. The embedding:

$$\varphi_{|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)|} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$$

can be seen as:

$$\begin{aligned} \mathbb{P}(V_2) \times \mathbb{P}(V_2) &\hookrightarrow \mathbb{P}(V_2 \otimes \mathrm{Sym}^2 V_2) \\ (v, w) &\longmapsto v \otimes w^2 \end{aligned}$$

A generic point in $p \in \mathbb{P}^5 \setminus \Sigma$ is a rank-2 tensor and hence can be written as $v_1 \otimes t_1 + v_2 \otimes t_2$. The line $< t_1, t_2 > \subset \mathbb{P}(\mathrm{Sym}^2 V_2)$ is determined by p. The intersection of $< t_1, t_2 >$ with the conic of rank 1 tensors is $\{w_1^2, w_2^2\}$. Thus p can be written as $v_1' \otimes w_2^2 + v_2' \otimes w_2^2$ where the points (v_1', w_1) and (v_2', w_2) of $\mathbb{P}(V_2) \times \mathbb{P}(V_2)$ are uniquely determined, and span the unique secant line passing through p. □

Remark 4.4.6. The locus of points on more than one secant equals the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2$ associated with the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ of the second factor

Sketch of proof If two secants to Σ, $l(s_1; s_2)$ and $l(s_3; s_4)$ intersect, then $s_1, \ldots, s_4$ are coplanar. But points $s_1, \ldots, s_4 \in \Sigma$ that fail to impose independent conditions on $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2)|$ necessarily have at least three points on a line or all the points on a conic contained in Σ.
see for reference [Has00].

In a similar fashion to the argument used to prove tha rationality of the cubic fourfolds containing two skew planes we want to establish rationality of a smooth cubic fourfold containing a rational normal scroll. Consider the map:

$$\begin{aligned} \Sigma^{(2)} &\dashrightarrow X \\ (p,q) &\mapsto x \end{aligned}$$

from the symmetric product of Σ to X mapping a couple of points on Σ to the unique residual point of the intersection of the secant line to Σ with X, $x = l(p,q) \cap X \smallsetminus \{p,q\}$. Thanks to the previous lemma one can prove that this map defines a birational isomorphism:

$$X \overset{\sim}{\dashrightarrow} \Sigma^{(2)} \cong (\mathbb{P}^1 \times \mathbb{P}^1)^{(2)}.$$

Thus we can conclude thanks to [Mat68] in which the author proves that the n-fold symmetric product of a rational variety is again rational.

Remark 4.4.7. Tregub in [Tre84] proves the rationality using via the map induced by the linear systems $|\mathcal{O}(2) \otimes \mathcal{I}_\Sigma|$ on X i.e. the projectified of the $\binom{4}{2} = 6$-dimensional vector space generated by the 2×2-minors of the matrix in Remark 4.4.4.

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