Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

CHROMATIC QUASISYMMETRIC FUNCTIONS OF DIGRAPHS

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Sommario

La funzione quasisimmetrica cromatica di un digrafo è una funzione quasisimmetrica associata al digrafo che ne descrive alcune proprietà. Essa generalizza la funzione simmetrica cromatica del grafo, la quale generalizza il polinomio cromatico del grafo. In questa tesi definiamo le funzioni quasisimmetriche cromatiche e ne vediamo le principali proprietà e le principali congetture che le riguardano.

Abstract

The chromatic quasisymmetric function of a digraph is a quasisymmetric function associated with the digraph that describes some of its properties. It generalizes the chromatic symmetric function of the graph, which generalizes the chromatic polynomial of the graph. In this thesis we define the chromatic quasisymmetric functions and see their main properties and the main conjectures that concern them.

Contents

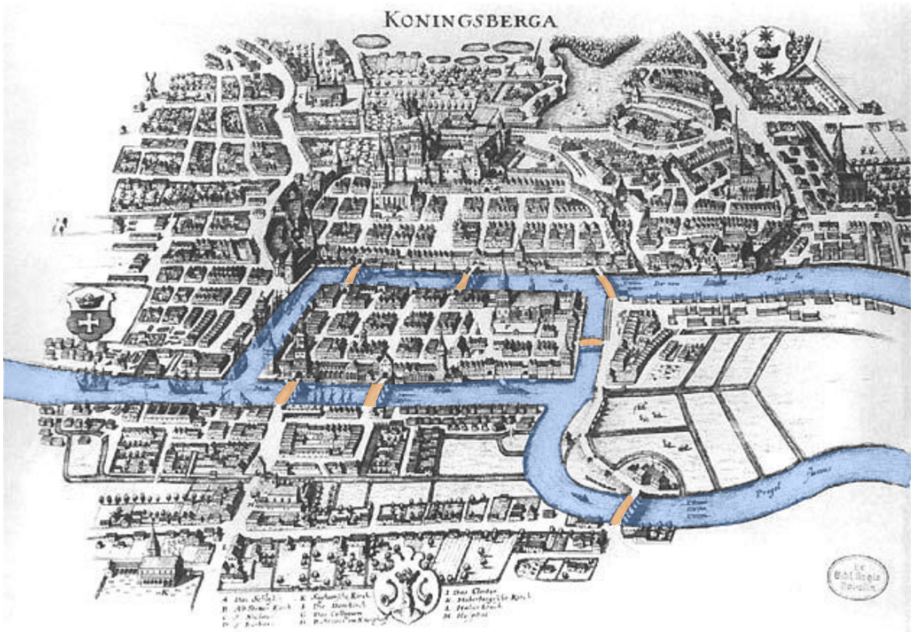
Introduction

In 1995, Richard Stanley introduced the chromatic symmetric function of a graph [Sta95], which generalizes the chromatic polynomial. In 2012, the definition was refined by John Shareshian and Michelle Wachs [SW12] for labeled graphs with the introduction of a parameter. In 2017, Brittney Ellzey [Ell17] presented a generalization in which labeled graphs are replaced by digraphs.
The main conjecture in this area concerns the supposed positive expansion of the chromatic symmetric functions of the incomparability graph of $(\mathbf{3}+\mathbf{1})$-free posets in the basis of elementary symmetric functions [Sta95, Conjecture 5.1] and the related refined conjectures of Shareshian and Wachs for natural unit interval graphs [SW12, Conjecture 4.9] and Ellzey for circular indifference digraphs [Ell17, Conjecture 1.4].
In Chapter 1 we introduce the fundamental concepts of graph theory. In Chapter 2 we introduce the fundamental concepts of combinatorics. In Chapter 3 we introduce the main tool of algebraic combinatorics, the symmetric functions, and their main properties. In Chapter 4 we introduce a generalization of the symmetric functions, the quasisymmetric functions, and their main properties. In Chapter 5 we define the main topic of the thesis: chromatic quasisymmetric functions. In Chapter 6 we see all the main conjectures related to the chromatic quasisymmetric functions.

Chapter 1

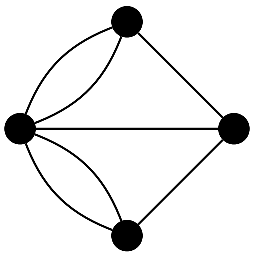
Graph theory

A long time ago, the mathematician Leonhard Euler was asked a bizarre question regarding the bridges in the city of Königsberg [Ale06].
The Prussian city of Königsberg was crossed by the river Pregel, which split into two parts forming an island.



Königsberg, map by Merian-Erben (1652), colorized with blue color represent the river and brown color represent the bridges. (Source: Wikimedia[*].)

There were seven bridges spanning the various sections of the river and the problem was this: *can a person cross all the bridges only once and return home?*
Euler reformulated the problem, replacing each area with a point and each bridge with a line between vertices, and showed that if there are more than two areas with an odd number of bridges, then such a journey is impossible [Eul41].



This solution is considered to be the first theorem of graph theory.

[*] URL: https://commons.wikimedia.org/wiki/File:Konigsberg_Bridge.png.

1.1 Graphs

In this section we will introduce the main concept of graph and its properties, using mainly the notation of [Wil10].

1.1.1 Basic definitions

Definition 1.1.1. A **graph** is a pair

$$G = (V, E)$$

where:

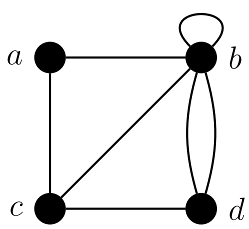
- V is a non-empty finite set, whose elements are called **vertices** (or **nodes**).
- E is a finite multiset of unordered pairs of elements of V called **edges**.

Definition 1.1.2. Let $G = (V, E)$ be a graph.

- An edge $(u, v) \in E$ is said to **join** the vertices u and v, and is denoted by uv.
- Two vertices u and v are called **adjacent** if $uv \in E$, and the vertices u and v are called **incident** with this edge.
- An edge from a vertex to itself is called **loop**.
- Two or more edges are called **multiple edges** (or **parallel edges**) if they are incident to the same two vertices.

A graph $G = (V, E)$ can be represented as a set of points and lines, where each vertex in V is associated with a point and a line exists between two points if there exists an edge in E between the associated vertices.

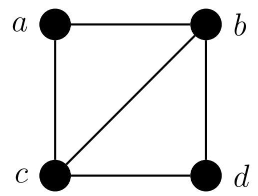
Example 1.1.3. Let $V = \{a, b, c, d\}$ and $E = \{ab, ac, bb, bc, bd, bd, cd\}$. Then $G = (V, E)$ is a graph and can be represented graphically as follows.



Definition 1.1.4. A graph G is called **simple** if the edges are all distinct, and if there are no loops.

Remark 1.1.5. In a simple graph $G = (V, E)$, the multiset of edges E is a set.

Example 1.1.6. Let $V = \{a, b, c, d\}$ and $E = \{ab, ac, bc, bd, cd\}$. Then $G = (V, E)$ is a simple graph.



Definition 1.1.7. Let $G = (V, E)$ be a graph. The **degree** of a vertex $v \in V$, denoted by $\deg(v)$, is the number of edges incident with v, with the convention that a loop at v contributes 2 (rather than 1).

Example 1.1.8. In Example 1.1.3, we have that:

- $\deg(a) = 2$.
- $\deg(b) = 6$.
- $\deg(c) = 3$.
- $\deg(d) = 3$.

Lemma 1.1.9 (Handshaking)**.** Let $G = (V, E)$ be a graph. The sum of the degrees of the vertices is equal to twice the number of edges:

$$\sum_{v \in V} \deg(v) = 2|E|.$$

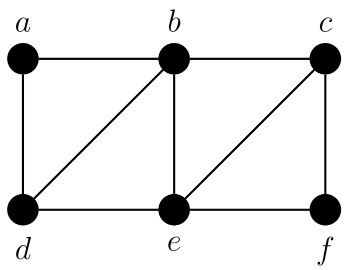
Proof. Each edge contributes exactly 2 to the sum, so the sum of all the vertex-degrees is equal to twice the number of edges.

□

Definition 1.1.10. Let $G = (V, E)$ be a graph.

- A **walk** is a finite sequence of vertices $v_1, \dots, v_m$ in which any two consecutive vertices are adjacent: $v_i v_{i+1} \in E$ for all $i \in \{1, \dots, m-1\}$.
- A **path** is a walk $v_1, \dots, v_m$ in which the vertices are distinct (except, possibly, $v_1 = v_m$).
- A **cycle** of n vertices (or a n-**cycle**) is a path $v_1, \dots, v_n$ in which $v_1 v_n \in E$.

Example 1.1.11. Let G be the following graph



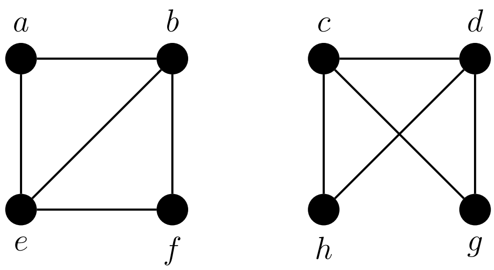
Then:

- The sequence (a, b, d, e, b, c) is a walk but not a path.
- The sequence (a, b, d, e, c, f) is a path.
- The sequence (a, b, e, d) is a cycle.

Definition 1.1.12. A graph G is called **connected** if there exists a path between each pair of distinct vertices. Otherwise, G is called **disconnected**.

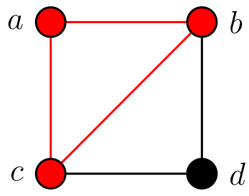
Remark 1.1.13. Any disconnected graph G can be expressed as the union of connected graphs, each of which is called a **connected component** of G.

Example 1.1.14. The following graph is disconnected with two connected components.



Definition 1.1.15. Let $G = (V, E)$ be a graph. A graph $H = (V', E')$ is called **sub-graph** of G if $V' \subseteq V$ and $E' \subseteq E$.

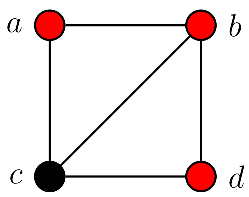
Example 1.1.16. Let $G = (V, E)$ with $V = \{a, b, c, d\}$ and $E = \{ab, ac, bc, bd, cd\}$. Let $H = (V', E')$ with $V' = \{a, b, c\}$ and $E = \{ab, ac, bc\}$. Then H is a sub-graph of G.



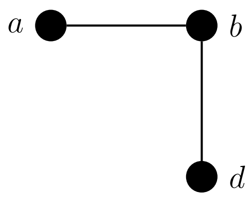
Definition 1.1.17. Let $G = (V, E)$ be a graph and let $S \subseteq V$ be a subset of vertices of G. The **induced sub-graph** of S is the graph $G[S] = (S, E')$ such that, for all $u, v \in S$,

$$uv \in E' \iff uv \in E.$$

Example 1.1.18. Let $G = (\{a, b, c, d\}, E)$ be the following graph and $S = \{a, b, d\}$. Then $G[\{a, b, d\}]$ is a induced sub-graph of G.



G



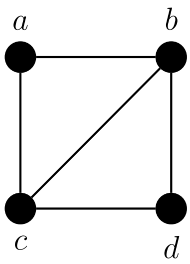
$G[\{a, b, d\}]$

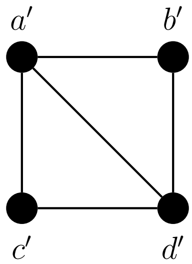
Definition 1.1.19. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. An **isomorphism** $f : G \longrightarrow G'$ is a bijective function $f : V \longrightarrow V'$ for which

$$ab \in E \iff f(a)f(b) \in E'.$$

If there exists an isomorphism, G and G' are called **isomorphic**.

Example 1.1.20. Let $G = (V, E)$ with $V = \{a, b, c, d\}$ and $E = \{ab, ac, bc, bd, cd\}$. Let $G' = (V', E')$ with $V' = \{a', b', c', d'\}$ and $E' = \{a'b', a'c', a'd', b'd', c'd'\}$.



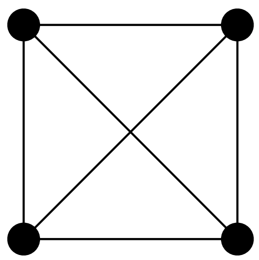


Then G and G' are isomorphic with $f : V \longrightarrow V'$ defined as

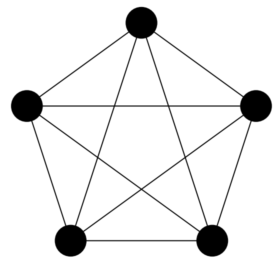
- $f(a) = b'$.
- $f(b) = a'$.
- $f(c) = d'$.
- $f(d) = c'$.

1.1.2 Examples

Definition 1.1.21. The **complete graph** with n vertices, denoted by K_n, is the simple graph in which each pair of distinct vertices are adjacent.



K_4



K_5

Proposition 1.1.22. The complete graph K_n has $\frac{n(n-1)}{2}$ edges.

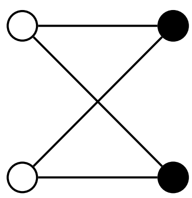
Proof. Since edges are subsets of two elements of the n vertices, the number of edges in K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$.

□

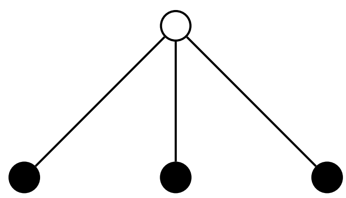
Definition 1.1.23. A **bipartite graph** is a graph $G = (V, E)$ in which V can be partitioned as a disjoint union $A \uplus B$ such that every edge in E joins a vertex of A and a vertex of B.

Definition 1.1.24. A **complete bipartite graph** is a bipartite graph in which each vertex in A is joined to each vertex in B by exactly one edge.

Notation 1.1.25. We denote the complete bipartite graph with disjoint subsets A and B, with $|A| = n$ and $|B| = m$, by $K_{n,m}$.



$K_{2,2}$



$K_{1,3}$

1.1.3 Chromatic polynomial

In 1912, George Birkhoff defined the chromatic polynomial for planar graphs in order to prove what is now the Four color Theorem [Bir12]. Although he failed in his attempt, the chromatic polynomial has become a fundamental object in graph theory.

Notation 1.1.26. We denote the set of positive integers as

$$\mathbb{P} := \mathbb{N} \setminus \{0\} = \{1, 2, 3, \dots\}.$$

Definition 1.1.27. Let G be a graph e let $n \in \mathbb{P}$. An **n-coloring** (or a **coloring** with n colors) of G is a function $k : V \longrightarrow \{1, \dots, n\}$.

Definition 1.1.28. Let G be a graph and let $n \in \mathbb{P}$. A **proper coloring** of G is a coloring k in which adjacent vertices have different colors:

$$k(u) \neq k(v), \quad \text{for all } uv \in E.$$

If there exists a proper coloring of G with n colors, then G is called **n-colorable** (or **colorable** with n colors).

Remark 1.1.29. If a graph G has a loop, then there is no proper coloring of G.

Remark 1.1.30. Let G be a graph and G' the graph obtained by replacing multiple edges with a single edge. Then each proper coloring of G is a proper coloring of G'.

Remark 1.1.31. A graph is 2-colorable if and only if it is bipartite.

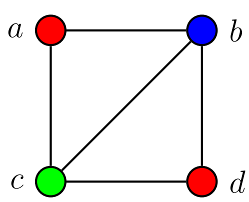
Notation 1.1.32. We denote by $\kappa(G)$ the set of proper colorings of $G = (V, E)$:

$$\kappa(G) := \{k : V \longrightarrow \mathbb{P} \mid k \text{ proper coloring of } G\}.$$

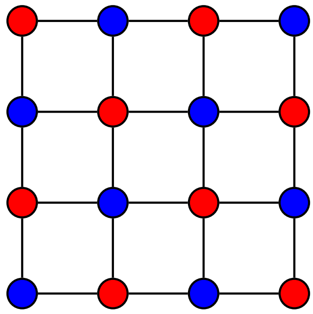
Example 1.1.33. Let $G = (V, E)$ be the simple graph of Example 1.1.6. Let $k : V \longrightarrow \{1, 2, 3\}$ be the function defined as:

- $k(a) = k(d) = 1$.
- $k(b) = 2$.
- $k(c) = 3$.

Then k is a proper 3-coloring of G and G is a 3-colorable graph.



Example 1.1.34. The following graph is 2-colorable.



Proposition 1.1.35. A 2-colorable graph G cannot have any cycles of odd length.

Proof. Since G is 2-colorable, there is a proper coloring k with 2 colors. Suppose that there is a cycle $v_1, \ldots, v_m$, with m odd, so

$$k(v_1) = 1, \; k(v_2) = 2, \; k(v_3) = 1, \; \ldots, \; \text{and } k(v_m) = 1.$$

Then two adjacent vertices (v_1 and v_m) have the same color, which contradicts the fact that k is a proper coloring.

□

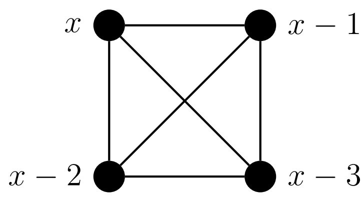
Definition 1.1.36. If the graph G is n-colorable and is not $(n-1)$-colorable, we say that the **chromatic number** of G is n, and write $\chi(G) = n$.

Definition 1.1.37. The **chromatic polynomial** of a graph G is a function

$$P_G : \mathbb{R} \longrightarrow \mathbb{R}$$

such that $P_G(n)$ is the number of proper colorings of G with n colors, for all $n \in \mathbb{P}$.

Example 1.1.38. Let us consider the complete graph with 4 vertices. Suppose we have x colors, so there are x ways to color any vertex of K_4, for a different vertex there are $x-1$ available colors, and so on.



The chromatic polynomial of K_4 is

$$P_{K_4}(x) = x(x-1)(x-2)(x-3).$$

The procedure in this example can be generalized for any K_n.

Remark 1.1.39. The chromatic polynomial of the complete graph with n vertices is

$$P_{K_n}(x) = x(x-1)(x-2)\ldots(x-n+1).$$

The following properties directly follow from the definition of chromatic polynomial.

Remark 1.1.40. Let G be a graph with chromatic number $\chi(G)$ and chromatic polynomial P_G. Let $n \in \mathbb{P}$, if $n < \chi(G)$ then $P_G(n) = 0$.

Remark 1.1.41. Since Remark 1.1.29, if a graph has a loop, then its chromatic polynomial is the zero polynomial.

Remark 1.1.42. Since Remark 1.1.30, multiple edges do not influence the chromatic polynomial.

Proposition 1.1.43. The chromatic polynomial of a graph G is a polynomial with integer coefficients: $P_G(x) \in \mathbb{Z}[x]$.

Proof. To build a proper coloring of $G = (V, E)$ with x colors we follow the steps below:

1. Fix the number of colors to use: $r \in \mathbb{P}$ with $r \leq |V|$.
2. Choose r disjoint subsets of V with vertices not adjacent to other vertices of the same subset.
3. Choose an injective function f from the set of blocks to the set of colors $\{1, \ldots, r\}$.

Let $B_r(V)$ be the number of possible partitions of V in r blocks. The possible injective functions f from blocks to colors are $x(x-1)(x-2)\ldots(x-r)$. So we have that

$$P_G(x) = \sum_{r=0}^{|V|} B_r(V) \cdot x(x-1)(x-2)\ldots(x-r) \in \mathbb{Z}[x].$$

□

As seen in the previous proof, we have that:

Remark 1.1.44. The chromatic polynomial of $G = (V, E)$ is a monic polynomial of degree $|V|$.

Proposition 1.1.45 (Deletion-contraction formula)**.** Let $G = (V, E)$ be a graph and $e = uv \in E$. Then

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$

where

- $G - e$ is the graph obtained from G by deleting the edge e: $G - e := (V, E \setminus e)$.
- G/e is the graph obtained from G by contracting the edge e, that is, the graph obtained from G by
  1. deleting the edge e,
  2. deleting the incident vertices, u and v, and replaced by a new vertex w,

3. for all $x \in V$, replacing each edge of the form either xu or xv by an edge of the form xw.

Proof. Let f be a proper k-coloring of $G - e$. There are two possible cases:

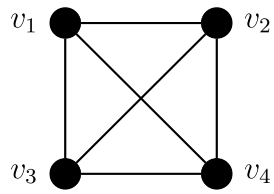
- If u and v have different colors, then f is a proper k-coloring of G.
- If u and v have the same color, then f is a proper k-coloring of G/e.

Therefore, the number of proper k-colorings of $G - e$ is equal to the sum of proper k-colorings of G and proper k-colorings of G/e:

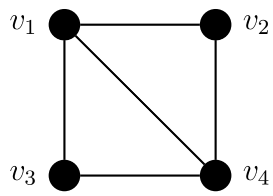
$$P_{G-e}(k) = P_G(k) + P_{G/e}(k).$$

□

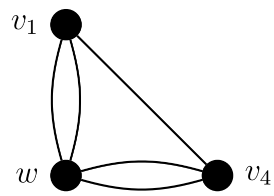
Example 1.1.46. Let G be the following graph.



Let $e = v_2v_3$. The corresponding graphs $G - e$ and G/e are the following.



$G - e$



G/e

The theorem states that

$$x(x-1)(x-2)(x-3) = x(x-1)(x-2)^2 - x(x-1)(x-2).$$

1.1.4 Tutte polynomial

In 1954, William Thomas Tutte construct a polynomial in two variables which generalizes the chromatic polynomial [Tut54].

Definition 1.1.47. The **Tutte polynomial** of a graph $G = (V, E)$ is a function

$$T_G : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

defined as

$$T_G(x, y) := \sum_{S \subseteq E} (x-1)^{c(S)-c(G)} (y-1)^{c(S)+|S|-|V|}$$

where

- $c(G)$ is the number of connected components of the graph G.
- $c(S)$ is the number of connected components of the sub-graph (V, S).

Proposition 1.1.48. Let $G = (V, E)$ be a graph and T_G its Tutte polynomial. Then the chromatic polynomial of G is

$$P_G(x) = (-1)^{|V|-c(G)}\, x^{c(G)}\, T_G(1-x, 0)$$

where $c(G)$ is the number of connected components of the graph G.

Proof. [Bol98, Chapter X, Theorem 6]

□

1.2 Digraphs

In this section we will introduce a generalization of the concept of graph where the edges are oriented.

1.2.1 Basic definitions

Definition 1.2.1. A **digraph** (or **directed graph**) is a pair

$$\vec{G} = (V, A)$$

where:

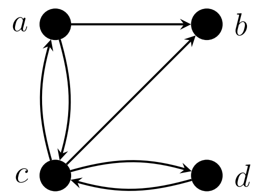
- V is a non-empty finite set, whose elements are called **vertices** (or **nodes**).
- A is a finite family of ordered pairs of elements of V called **arcs** (or **directed edges**).

Definition 1.2.2. Let $\vec{G} = (V, A)$ be a digraph.

- An arc $(u, v) \in A$ is said to be **from** u **to** v.
- An arc from a vertex to itself is called **loop**.
- Two or more arcs are called **multiple arcs** (or **parallel arcs**) if they are from the same vertex u to the same vertex v.

Definition 1.2.3. A digraph $\vec{G}$ is called **simple** if the arcs are all distinct, and if there are no loops.

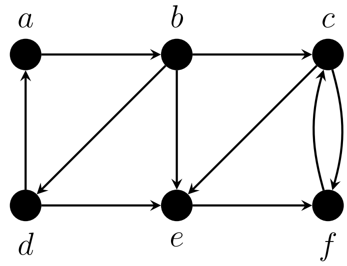
Example 1.2.4. Let $V = \{a, b, c, d\}$ and $A = \{ab, ac, ca, cb, cd, dc\}$. Then $\vec{G} = (V, A)$ is a simple digraph and can be represented graphically as follows.



Definition 1.2.5. Let $\vec{G} = (V, A)$ be a digraph.

- A **directed walk** is a finite sequence of vertices $v_1, \ldots, v_m$ in which $v_i v_{i+1} \in A$ for all $i \in \{1, \ldots, m-1\}$.
- A **directed path** is a directed walk $v_1, \ldots, v_m$ in which the vertices are distinct (except, possibly, $v_1 = v_m$).
- A **directed cycle** of n vertices (or a **directed n-cycle**) is a directed path $v_1, \ldots, v_n$ in which $v_n v_1 \in A$.

Example 1.2.6. Let $\vec{G}$ be the following digraph

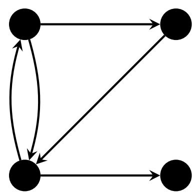


Then:

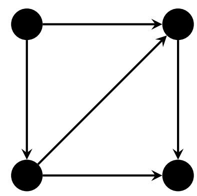
- The sequence (a, b, d, e, f, c, f) is a directed walk but not a directed path.
- The sequence (a, b, d, e, f) is a directed path.
- The sequence (a, b, d) is a directed cycle.

Definition 1.2.7. A digraph is **acyclic** if it does not contain directed cycles.

Example 1.2.8. Let $\vec{G}$ and $\vec{H}$ be the following digraphs.



$\vec{G}$



$\vec{H}$

The digraph $\vec{G}$ has a directed 3-cycle and the digraph $\vec{H}$ is acyclic

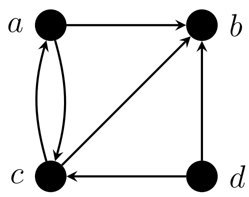
Definition 1.2.9. Let $\vec{G} = (V, A)$ be a digraph.

- The **indegree** of a vertex $v \in V$, denoted by $\deg^-(v)$, is the number of arcs to v.
- The **outdegree** of a vertex $v \in V$, denoted by $\deg^+(v)$, is the number of arcs from v.
- A **source** is a vertex $v \in V$ with $\deg^-(v) = 0$.
- A **sink** is a vertex $v \in V$ with $\deg^+(v) = 0$.

Remark 1.2.10. Let $\vec{G} = (V, A)$ be a digraph. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |A|.$$

Example 1.2.11. Let $\vec{G}$ be the following digraph.



Then

- $\deg^-(a) = 1$.
- $\deg^-(b) = 3$.
- $\deg^-(c) = 2$.
- $\deg^-(d) = 0$ (d is a source).
- $\deg^+(a) = 2$.
- $\deg^+(b) = 0$ (b is a sink).
- $\deg^+(c) = 2$.
- $\deg^+(d) = 2$.

Proposition 1.2.12. Every acyclic digraph has a source vertex and a sink vertex.

Proof. Suppose, by contradiction, that $\vec{G} = (V, A)$ is an acyclic digraph without sink vertices. Let $n = |V|$. Since $v_1 \in V$ is not a sink vertex, there exists a vertex $v_2 \in V$ such that $v_1 v_2 \in A$. Since $v_2 \in V$ is not a sink vertex, there exists a vertex $v_3 \in V$ such that $v_2 v_3 \in A$. By induction, there exists a vertex $v_{n+1} \in V$ such that $v_n v_{n+1} \in A$. Since $n = |V|$, we must have $v_i = v_j$ for some $i \neq j$ and this means that $\vec{G}$ has a directed cycle, so there is a contradiction.
Similarly, is obtained that $\vec{G}$ has a source vertex.

$\square$

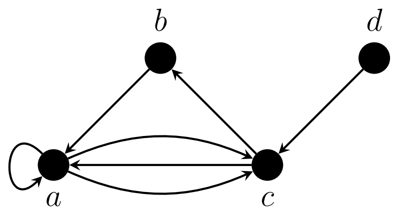
Definition 1.2.13. Let $\vec{G} = (V, A)$ and $\vec{G'} = (V', A')$ be two digraphs. An **isomorphism** $f : \vec{G} \longrightarrow \vec{G'}$ is a bijective function $f : V \longrightarrow V'$ for which

$$ab \in A \iff f(a)f(b) \in A'.$$

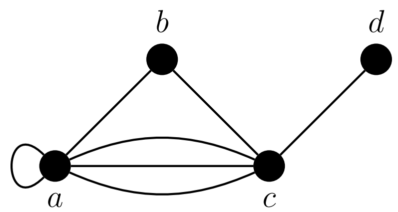
If there exists an isomorphism, then $\vec{G}$ and $\vec{G'}$ are called **isomorphic**.

Definition 1.2.14. The **underlying graph** of the digraph $\vec{G} = (V, A)$ is the graph $G = (V, E)$ obtained by replacing each arc $vw \in A$ by a corresponding edge $vw \in E$.

Example 1.2.15. Let $\vec{G}$ be the following digraph and G the following graph. The underlying graph of $\vec{G}$ is G.



$\vec{G}$



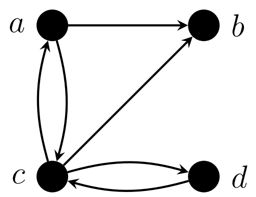
G

Definition 1.2.16. A digraph $\vec{G}$ is called **connected** if the underlying graph G is connected. Otherwise, $\vec{G}$ is called **disconnected**.

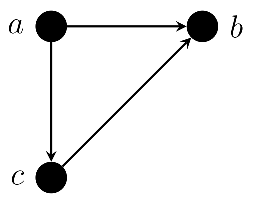
Definition 1.2.17. Let $\vec{G} = (V, A)$ be a digraph. A digraph $\vec{H} = (V', A')$ is called **sub-digraph** of $\vec{G}$ if $V' \subseteq V$ and $A' \subseteq A$.

Remark 1.2.18. Let $\vec{H}$ be an sub-digraph of $\vec{G}$. Then the underlying graph H is a sub-graph of the underlying graph G. The converse is not true.

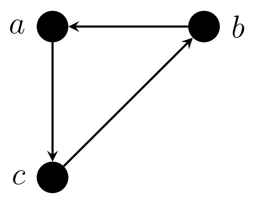
Example 1.2.19. Let $\vec{G}$ be the digraph of Example 1.2.4.



Let $\vec{H}_1$ and $\vec{H}_2$ be the following digraphs.



$\vec{H}_1$



$\vec{H}_2$

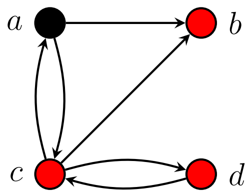
Then:

- $\vec{H}_1$ is an sub-digraph of $\vec{G}$ and H_1 is a sub-graph of G.
- $\vec{H}_2$ is not an sub-digraph of $\vec{G}$, although H_2 is a sub-graph of G.

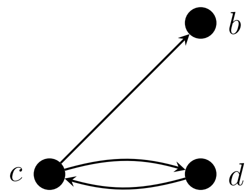
Definition 1.2.20. Let $\vec{G} = (V, A)$ be a digraph and let $S \subseteq V$ be a subset of vertices of $\vec{G}$. The **induced sub-digraph** of S is the digraph $\vec{G}[S] = (S, A')$ such that, for all $u, v \in S$,

$$uv \in A' \iff uv \in A.$$

Example 1.2.21. Let $\vec{G} = (\{a, b, c, d\}, A)$ be the following digraph and $S = \{b, c, d\}$. Then $\vec{G}[\{b, c, d\}]$ is a induced sub-digraph of $\vec{G}$.



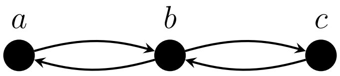
$\vec{G}$



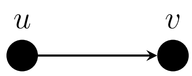
$\vec{G}[\{b, c, d\}]$

Remark 1.2.22. All induced sub-digraphs of $\vec{G}$ are sub-digraphs of $\vec{G}$. The converse is not true.

Example 1.2.23. Let $\vec{G}$ and $\vec{H}$ be the following digraphs. Then $\vec{H}$ is a sub-digraph of $\vec{G}$, but it is not a induced sub-digraph of $\vec{G}$.



$\vec{G}$



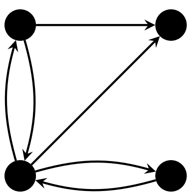
$\vec{H}$

1.2.2 Chromatic polynomial

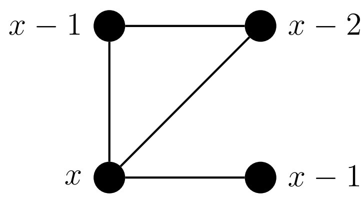
Definition 1.2.24. Let $\vec{G}$ be a digraph. The **chromatic polynomial** of $\vec{G}$ is the chromatic polynomial of his underlying graph G.

Remark 1.2.25. In the definition of coloring, the number of edges between two adjacent vertices is not relevant, so we can ignore multiple edges in the underlying graph for the calculation of the chromatic polynomial.

Example 1.2.26. Let G be the underlying graph of the digraph $\vec{G}$ of Example 1.2.4.



$\vec{G}$



G

The chromatic polynomial of the digraph $\vec{G}$ is

$$P_G(x) = x(x-1)^2(x-2).$$

1.2.3 Orientations

Definition 1.2.27. An **orientation** of the graph G is a digraph $\vec{G}$ whose underlying graph is G.

Using the chromatic polynomial of a graph we can find the number of its orientations.

Proposition 1.2.28. Let G be a graph with chromatic polynomial P_G. Then G has exactly $|P_G(-1)|$ different acyclic orientations.

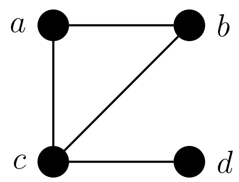
Proof. [Sta73, Corollary 1.3]

□

Definition 1.2.29. An orientation $\vec{G} = (V, A)$ of the graph $G = (V, E)$ is said to be **compatible** with the coloring k if

$$k(u) < k(v), \quad \text{for all } uv \in A.$$

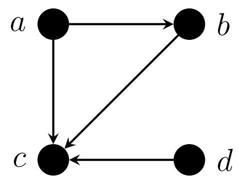
Example 1.2.30. Let G be the following graph



Let k be the proper coloring of G defined as

- $k(a) = k(d) = 1$.
- $k(b) = 2$.
- $k(c) = 3$.

The following digraph is an orientation of G compatible with k.



Proposition 1.2.31. Let k be a proper coloring of the graph $G = (V, E)$. There exists a unique acyclic orientation of G compatible with k.

Proof. There exists a unique orientation $\vec{G} = (V, A)$ of G compatible with k defined as

$$uv \in A \iff k(u) < k(v), \quad \text{for all } uv \in E.$$

The orientation $\vec{G}$ is acyclic because if, by contradiction, there exists an n-cycle $v_1 \dots v_n$ we would have

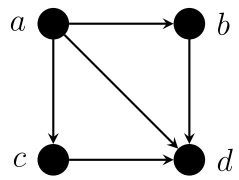
$$k(v_1) < \cdots < k(v_n)$$

but also $k(v_n) < k(v_1)$, so there is a contradiction.

□

The correspondence between colorings and acyclic orientations is not a bijection.

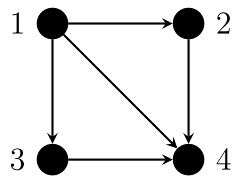
Example 1.2.32. Let $\vec{G}$ be the following digraph

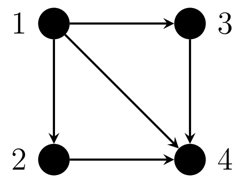


Let k and k' be the following colorings:

- $k(a) = k'(a) = 1$.
- $k(b) = k'(c) = 2$.
- $k(c) = k'(b) = 3$.
- $k(d) = k'(d) = 4$.

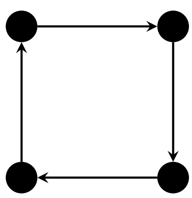
They both have $\vec{G}$ as compatible acyclic orientation.



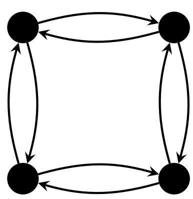


1.2.4 Examples

Notation 1.2.33. We denote by $\vec{C}_n$ the digraph consisting only of a directed cycle of n vertices. We denote by $\overleftrightarrow{C}_n$ the digraph consisting only of a directed cycle of n vertices in which there are also the arc in the opposite direction.

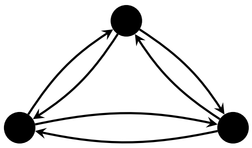


$\vec{C}_4$

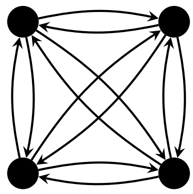


$\overleftrightarrow{C}_4$

Definition 1.2.34. The **complete symmetric digraph** with n vertices, denoted by $\overleftrightarrow{K}_n$, is the simple digraph in which there is exactly one arc from every vertex to every other vertex.



$\overleftrightarrow{K}_3$

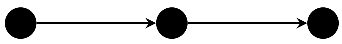


$\overleftrightarrow{K}_4$

Proposition 1.2.35. The complete symmetric digraph $\overleftrightarrow{K}_n$ has $n(n-1)$ arcs.

Proof. Since Proposition 1.1.22, the number of edges of K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$. The number of arcs of $\overleftrightarrow{K}_n$ is twice the number of edges of K_n, so it is $n(n-1)$. □

Notation 1.2.36. We denote by $\vec{P}_n$ the digraph consisting only of a directed path of n vertices. We denote by $\overleftrightarrow{P}_n$ the digraph consisting only of a directed path of n vertices in which there are also the arc in the opposite direction.

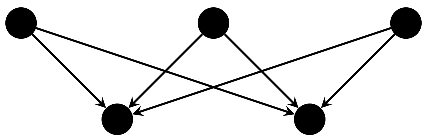


$\vec{P}_3$

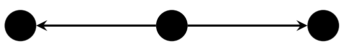
$\overleftrightarrow{P}_3$

Notation 1.2.37. For all $n, m \in \mathbb{P}$, we define

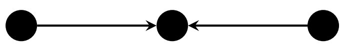
$$\vec{K}_{n,m} := (\{u_1, \dots, u_n, v_1, \dots, v_m\}, \{u_i v_j \mid i = 1, \dots, n,\ j = 1, \dots, m\}).$$



$\vec{K}_{3,2}$

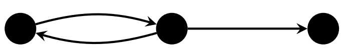


$\vec{K}_{1,2}$

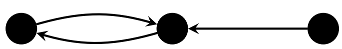


$\vec{K}_{2,1}$

Let us also define the following digraphs that we will use later.



$\overleftrightarrow{K}_{1,2}$



$\overleftrightarrow{K}_{2,1}$

Chapter 2

Combinatorics

In this Chapter we will introduce the main tools of combinatorics that we will use in the following chapters.

2.1 Posets

First of all, we introduce the concept of Symmetric group as in [Sta12, Section 3.1].

Definition 2.1.1. A **poset** (or **partially ordered set**) is a pair $(P, \preceq)$, where:

- P is a non-empty set.
- $\preceq$ is a **partial order** of P: a binary relation on P such that:
  - For all $a \in P$, $a \preceq a$.
  - For all $a, b \in P$, if $a \preceq b$ and $b \preceq a$, then $a = b$.
  - For all $a, b, c \in P$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

Definition 2.1.2. A **strict poset** (or **strict partially ordered set**) is a pair $(S, \prec)$, where:

- S is a non-empty set.
- $\prec$ is a **strict partial order** of S: a binary relation on S such that:
  - For all $a \in S$, $a \not\prec a$.
  - For all $a, b \in S$, if $a \prec b$, then $b \not\prec a$.
  - For all $a, b, c \in S$, if $a \prec b$ and $b \prec c$, then $a \prec c$.

The definitions of poset and strict poset are equivalent in the sense of the following remark:

Remark 2.1.3. If $(P, \preceq)$ is a poset, then $(P, \prec)$ is a strict poset with

$$a \prec b \iff (a \preceq b \text{ and } a \neq b).$$

Conversely, if $(P, \prec)$ is a strict poset, then $(P, \preceq)$ is a poset with

$$a \preceq b \iff (a \prec b \text{ or } a = b).$$

All the definitions that we will give below will be either for poset or strict poset, but they are also valid for the others: just add/remove the equality.

Definition 2.1.4. Let $(P, \preceq)$ be a poset. Two elements $a, b \in P$ are called **comparable** if $s \preceq t$ or $t \preceq s$. Otherwise s and t are called **incomparable**.

Definition 2.1.5. Let $(P, \preceq)$ be a poset. A **chain** in P is a set $\{p_1, \ldots, p_n\} \subseteq P$ such that

$$p_1 \preceq \cdots \preceq p_n.$$

Example 2.1.6. The set of positive integers $\mathbb{P}$ is a poset with the divisibility relation, that is, the relation $\mid$ defined as:

$$a \mid b \iff \exists c \in \mathbb{P} \text{ such that } ac = b.$$

The integers 5 and 7 are incomparable because $5 \nmid 7$ and $7 \nmid 5$. The set of all powers of 2, $\{1, 2, 4, 8, \ldots\}$, is a chain of $(\mathbb{P}, \mid)$.

Definition 2.1.7. Let $(P, \preceq)$ be a poset.

- A **maximal element** of $(P, \preceq)$ is an element $M \in P$ for which there are no $s \in P \setminus \{M\}$ such that $M \preceq s$.
- A **minimal element** of $(P, \preceq)$ is an element $m \in P$ for which there are no $s \in P \setminus \{m\}$ such that $s \preceq m$.

Definition 2.1.8. Two posets $(P, \leq_P)$ and $(Q, \leq_Q)$ are **isomorphic** if there exists a bijection $\phi : P \longrightarrow Q$ such that

$$s \leq_P t \iff \phi(s) \leq_Q \phi(t).$$

Definition 2.1.9. Let $(P, \leq_P)$ be a poset. An **induced sub-poset** of $(P, \leq_P)$ is a poset $(Q, \leq_Q)$, where:

- Q is a subset of P.
- $\leq_Q$ is a partial ordering of Q such that: for $s, t \in Q$, we have

$$s \leq_Q t \iff s \leq_P t.$$

Definition 2.1.10. Let K be a family of posets. A poset $(P, \preceq)$ is called K**-free** if it does not contain any induced sub-poset isomorphic to a poset in K.

Notation 2.1.11. Let $(\mathbf{a}+\mathbf{b})$ denote the poset which is a disjoint union of a chain with a elements and a chain with b elements.

Remark 2.1.12. A poset $(P, \preceq)$ is $(\mathbf{a}+\mathbf{b})$-free if P does not contain a chain of a elements $x_1 \preceq \cdots \preceq x_a$ and a chain of b elements $y_1 \preceq \cdots \preceq y_b$, where $x_1, \ldots, x_a$ are all incomparable with $y_1, \ldots, y_b$.

Example 2.1.13. The set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is a poset with the divisibility relation is not $(\mathbf{3}+\mathbf{1})$-free. Indeed, for example, the induced sub-poset $(\{2, 3, 4, 8\}, \mid)$ is the disjoint union of a chain of three elements $(2, 4, 8)$ and an element (3) which is incomparable with the elements of the chain.

2.2 Compositions and partitions

From now on, to simplify the notation, we will write:

Notation 2.2.1. For all $n \in \mathbb{P}$,

$$[n] := \{1, 2, \ldots, n\}.$$

2.2.1 Partitions

Definition 2.2.2. Let $n \in \mathbb{P}$ and $k \in \mathbb{P}$. A **partition** of n in k parts (or a k-**partition**) is a sequence of integers

$$\lambda = (\lambda_1, \ldots, \lambda_k)$$

such that

- $\lambda_1 \geq \cdots \geq \lambda_k > 0$.
- $\sum\limits_{i=1}^{k} \lambda_i = n$.

The number of parts of λ, denoted by $\ell(\lambda)$, is called the **length** of λ.
By convention, the sequence (0) is a partition of 0.

Notation 2.2.3. If λ is a partition of n then we write either $\lambda \vdash n$ or $|\lambda| = n$.

Notation 2.2.4. The set of all partitions of n is

$$\mathrm{Par}(n) := \{\lambda \vdash n\}.$$

The set of all partitions is

$$\mathrm{Par} := \bigcup_{n \in \mathbb{N}} \mathrm{Par}(n).$$

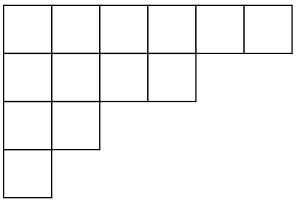
Example 2.2.5. For instance,

$$\begin{aligned}
&\mathrm{Par}(1) = \{(1)\}.\\
&\mathrm{Par}(2) = \{(2), (1,1)\}.\\
&\mathrm{Par}(3) = \{(3), (2,1), (1,1,1)\}.\\
&\mathrm{Par}(4) = \{(4), (3,1), (2,2), (2,1,1), (1,1,1,1)\}.\\
&\mathrm{Par}(5) = \{(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)\}.
\end{aligned}$$

Notation 2.2.6. The partition $\lambda = (1, \ldots, 1) \vdash n$ can be denote by (1^n).

Definition 2.2.7. The **Young diagram** of $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \vdash n$ is a finite collection of cells, arranged in left-justified rows, with λ_1 cells in the first row, λ_2 cells in the second row, etc.

Example 2.2.8. The Young diagram of $(6, 4, 2, 1)$ is

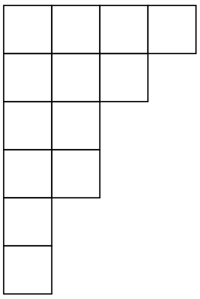


Definition 2.2.9. The **conjugate partition** (or **transpose partition**) of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_p) \vdash n$ such that

$$\lambda'_i := \max\{m \in \{1, \ldots, k\} \mid \lambda_m \geq i\}, \quad \text{for all } i \in \{1, \ldots, \lambda_1\}.$$

Remark 2.2.10. The Young diagram of the conjugate partition of λ is the Young diagram of λ with rows and columns exchanged.

Example 2.2.11. The conjugate partition of $\lambda = (6, 4, 2, 1)$ is $\lambda' = (4, 3, 2, 2, 1, 1)$.



2.2.2 Orders

Definition 2.2.12. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \vdash n$. Then λ **dominates** μ if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$$

for all $i \in \mathbb{N}$. If $i > k$ (respectively, $i > m$), then we take $\lambda_i = 0$ (respectively, $\mu_i = 0$).

Notation 2.2.13. Let $\lambda, \mu \vdash n$.

- If λ dominates μ, we write $\lambda \trianglerighteq \mu$.
- If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$, we write $\lambda \triangleright \mu$.

Remark 2.2.14. The relation $\trianglerighteq$ is a partial order on $\mathrm{Par}(n)$, called **dominance order**.

Example 2.2.15. For instance,

$$(3, 3) \trianglerighteq (2, 2, 1, 1).$$

Instead, $(3, 3)$ and $(4, 1, 1)$ are incomparable since $3 < 4$ and $3 + 3 > 4 + 1$.

Definition 2.2.16. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \vdash n$. Then λ is less than μ in **lexicographic order** if, for some index $i \in \mathbb{P}$,

$$\lambda_j = \mu_j \quad \text{for all } j < i \text{ and } \lambda_i < \mu_i\,.$$

Notation 2.2.17. Let $\lambda, \mu \vdash n$.

- If λ is less than μ in lexicographic order, we write $\lambda \geq_{\text{lex}} \mu$.
- If $\lambda \geq_{\text{lex}} \mu$ and $\lambda \neq \mu$, we write $\lambda >_{\text{lex}} \mu$.

Remark 2.2.18. The lexicographic order is a total ordering on $\text{Par}(n)$.

Example 2.2.19. For partitions of 5, we have

$$(1,1,1,1,1) <_{\text{lex}} (2,1,1,1) <_{\text{lex}} (2,2,1) <_{\text{lex}} (3,1,1) <_{\text{lex}} (3,2) <_{\text{lex}} (4,1) <_{\text{lex}} (5).$$

The lexicographic order is a *refinement* of the dominance order in the sense of the following Proposition.

Proposition 2.2.20. Let $\lambda, \mu \vdash n$. If $\lambda \trianglerighteq \mu$, then $\lambda \geq_{\text{lex}} \mu$.

Proof. If $\lambda = \mu$, then $\lambda \geq_{\text{lex}} \mu$. If $\lambda \neq \mu$, then there exists an index k such that $\lambda_i = \mu_i$ for all $i < k$ and $\lambda_k \neq \mu_k$. Therefore

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i = \mu_1 + \mu_2 + \cdots + \mu_i$$

for all $i < k$ and, since $\lambda \triangleright \mu$,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k > \mu_1 + \mu_2 + \cdots + \mu_k.$$

$\square$

2.2.3 Compositions

Definition 2.2.21. Let $n \in \mathbb{P}$ and $k \in \mathbb{P}$. A **composition** of n in k parts (or a k**-composition** of n) is a sequence of integers

$$a = (a_1, \ldots, a_k)$$

such that

- $a_1 > 0, \ldots, a_k > 0$.
- $\sum\limits_{i=1}^{k} a_i = n$.

The number of parts of a, denoted by $\ell(a)$, is called the **length** of a.
By convention, the sequence (0) is a composition of 0.

Example 2.2.22. There are eight compositions of 4:

- A 1-composition: (4).
- Three 2-compositions: $(3,1)$, $(2,2)$ and $(1,3)$.
- Three 3-compositions: $(2,1,1)$, $(1,2,1)$ and $(1,1,2)$.
- A 4-composition: $(1,1,1,1)$.

Notation 2.2.23. The set of all compositions of n is denoted by $\mathrm{Comp}(n)$. The set of all compositions is

$$\mathrm{Comp} := \bigcup_{n\in\mathbb{N}} \mathrm{Comp}(n).$$

Lemma 2.2.24. There exists a bijection between the compositions of n in k parts and the subsets of $[n-1]$ with $k-1$ elements through

$$a = (a_1, \dots, a_k) \longrightarrow S_a := \{a_1, a_1 + a_2, \dots, a_1 + \dots + a_{k-1}\}$$

and

$$S = \{s_1, \dots, s_{k-1}\} \longrightarrow \mathrm{co}(S) := (s_1, s_2 - s_1, s_3 - s_2, \dots, n - s_{k-1}).$$

Proof. The two functions are the inverse of each other:

- For all $a = (a_1, \dots, a_k) \models n$,

$$\begin{aligned}\mathrm{co}(S_\alpha) &= \mathrm{co}(\{a_1, a_1 + a_2, \dots, a_1 + \dots + a_{k-1}\}) = \\ &= (a_1, (a_1 + a_2) - a_1, (a_1 + a_2 + a_3) - (a_1 + a_2), \dots) = \\ &= (a_1, a_2, a_3, \dots) = a.\end{aligned}$$

- For all $S = \{s_1, \dots, s_{k-1}\} \subseteq [n-1]$,

$$\begin{aligned}S_{\mathrm{co}(S)} &= S_{(s_1, s_2 - s_1, s_3 - s_2, \dots, n - s_{k-1})} = \\ &= \{s_1, (s_2 - s_1) + s_1, (s_3 - s_2) + (s_2 - s_1) + s_1, \dots\} = \\ &= \{s_1, s_2, s_3, \dots\} = S.\end{aligned}$$

□

Proposition 2.2.25. The number of compositions of n in k parts is $\binom{n-1}{k-1}$. The number of compositions of n is 2^{n-1}.

Proof. The number of subsets of $[n-1]$ with $k-1$ elements is $\binom{n-1}{k-1}$ and so, by previous lemma, is the number of compositions of n in k parts. The number of subsets of $[n-1]$ is 2^{n-1} and so is the number of compositions of n.

□

2.2.4 Weak compositions

Definition 2.2.26. Let $n \in \mathbb{P}$ and $k \in \mathbb{P}$. A **weak composition** of n in k parts (or a **weak k-composition** of n) is a sequence of integers

$$\alpha = (\alpha_1, \dots, \alpha_k)$$

such that

- $\alpha_1 \geq 0, \dots, \alpha_k \geq 0$.
- $\sum\limits_{i=1}^{k} \alpha_i = n$.

Example 2.2.27. The sequence $(2, 0, 1)$ is not a composition, but it is a weak composition.

Notation 2.2.28. If α is a weak composition of n, then we write $\alpha \models n$.

Proposition 2.2.29. The number of weak compositions of n in k parts is $\binom{n+k-1}{k-1}$.

Proof. Given a weak composition $\alpha = (\alpha_1, \dots, \alpha_k) \models n$, by adding 1 in each entry we have a composition $(\alpha_1 + 1, \dots, \alpha_k + 1)$ of $(n + k)$. So, the number of weak compositions of n in k parts is equal to the number of compositions of $(n + k)$ in k parts. By Proposition 2.2.25, the number of compositions of $(n + k)$ in k parts is $\binom{n+k-1}{k-1}$ and so is the number of weak compositions of n in k parts. □

Definition 2.2.30. The **associated partition** to the composition $\alpha \in \mathrm{Comp}(n)$ is the partition $\lambda(\alpha) \vdash n$ obtained by sorting the parts of α in descending order.

Definition 2.2.31. The **associated partition** to the weak composition $\alpha \models n$ is the partition $\lambda(\alpha) \vdash n$ obtained by:

- Sorting the parts of α in descending order.
- Removing the zeros.

Example 2.2.32. The partition associated to $\alpha = (1, 2, 2, 0, 1) \models 6$ is

$$\lambda(\alpha) = (2, 2, 1, 1) \vdash 6.$$

Chapter 3

Symmetric functions

In this Chapter we will define an important tool of Algebraic combinatorics, the symmetric functions, and its main properties.

3.1 Symmetric group

First of all, we introduce the concept of symmetric group as in [Sag01, Section 1.1].

Definition 3.1.1. The **symmetric group** of degree $n \in \mathbb{P}$ is the group of all bijections from $[n]$ to itself with composition as the operation.
An element of the symmetric group of degree n is called **permutation** of n.

Notation 3.1.2. We denote the symmetric group of degree n by S_n.

Notation 3.1.3. We multiply permutations from right to left: $\pi\sigma$ is the bijection obtained by first applying σ, followed by π.

Notation 3.1.4. We can represent a permutation using $\pi \in S_n$ using **one-line notation** defined as follows:

$$\pi = \pi_1, \dots, \pi_n$$

where $\pi_i := \pi(i)$ for all $i \in [n]$.

Notation 3.1.5. We can represent $\pi \in S_n$ using **cycle notation** defined as follows:

- The elements of the sequence $1, \pi(1), \pi^2(1), \dots$ cannot be all distinct. Let p be the first power such that $\pi^p(1) = 1$, we write the cycle

$$\left(1, \pi(1), \pi^2(1), \dots, \pi^{p-1}(1)\right).$$

- Now pick the smallest $j \in [n]$ not in the cycle containing 1 and repeat the operations of the previous point.
- Iterate this process until all elements of $[n]$ have been used.

We write π in cycle notation as

$$\pi = \left(1, \pi(1), \pi^2(1), \dots, \pi^{p-1}(1)\right) \left(j, \pi(j), \pi^2(j), \dots\right) \dots.$$

If this does not cause confusion, we will omit the commas in the notations.

Example 3.1.6. Let $\pi \in S_6$ be the permutation given by

$$\pi(1) = 2, \quad \pi(2) = 3, \quad \pi(3) = 1, \quad \pi(4) = 6, \quad \pi(5) = 5, \quad \pi(6) = 4.$$

In one-line notation,

$$\pi = 231654.$$

In cycle notation,

$$\pi = (123)(46)(5).$$

We will use one-line notation from now on.

Definition 3.1.7. Let $\sigma \in S_n$. The **reverse permutation** of σ is the permutation $\sigma^{\mathrm{rev}} \in S_n$ defined as

$$\sigma^{\mathrm{rev}}(i) := \sigma(n+1-i), \quad \text{for all } i \in [n].$$

In one-line notation, the reverse permutation of $\pi = \pi_1 \dots \pi_n$ is $\pi^{\mathrm{rev}} = \pi_n \dots \pi_1$.

Notation 3.1.8. Let $\mathbf{x} = \{x_1, x_2, \dots\}$ be an infinite set of variables. Let $n \in \mathbb{P}$. We denote the action of $\pi \in S_n$ on the formal power series $f(\mathbf{x})$ as

$$\pi \cdot f(x_1, x_2, \dots) := f(x_{\pi(1)}, x_{\pi(2)}, \dots)$$

with $\pi(i) := i$ for all $i > n$.

Proposition 3.1.9. Let $\sigma \in S_n$. Then

$$\sigma^{-1}(i) < \sigma^{-1}(j) \iff (\sigma^{\mathrm{rev}})^{-1}(i) > (\sigma^{\mathrm{rev}})^{-1}(j).$$

Proof. Let $i, j \in [n]$, then

$$\sigma^{-1}(i) < \sigma^{-1}(j) \iff i \text{ precedes } j \text{ in } \sigma \text{ in one-line notation}$$

and

$$(\sigma^{\mathrm{rev}})^{-1}(i) > (\sigma^{\mathrm{rev}})^{-1}(j) \iff j \text{ precedes } i \text{ in } \sigma^{\mathrm{rev}} \text{ in one-line notation.}$$

Furthermore,

$$i \text{ precedes } j \text{ in } \sigma \iff j \text{ precedes } i \text{ in } \sigma^{\mathrm{rev}}.$$

□

3.2 Symmetric functions

Now we introduce the concept of symmetric functions as in [Sta99, Section 7.1].

Definition 3.2.1. Let $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ be a set of variables. Let R be a non-zero unitary commutative ring. A **homogeneous symmetric function** of degree $n \in \mathbb{P}$ over R in the variables $\mathbf{x}$ is a formal power series

$$f(\mathbf{x}) = \sum_{\alpha \models n} c_\alpha \mathbf{x}^\alpha \in R[[\mathbf{x}]]$$

such that

$$\pi \cdot f(\mathbf{x}) = f(\mathbf{x}) \quad \text{for all permutations } \pi,$$

where

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_k^{\alpha_k}, \quad \text{for all } \alpha = (\alpha_1, \dots, \alpha_k) \models n.$$

Notation 3.2.2. The set of all homogeneous symmetric functions of degree n over R is denoted by Sym_R^n (or Λ_R^n).

Remark 3.2.3. For every non-zero unitary commutative ring R,

$$\mathrm{Sym}_R^0 = R.$$

Notation 3.2.4. The direct sum of all sets of homogeneous symmetric functions over R is

$$\mathrm{Sym}_R := \bigoplus_{n\in\mathbb{N}} \mathrm{Sym}_R^n = \mathrm{Sym}_R^0 \oplus \mathrm{Sym}_R^1 \oplus \mathrm{Sym}_R^2 \oplus \dots .$$

From now on, we will always take $R = \mathbb{Q}$. Therefore:

Notation 3.2.5.

$$\mathrm{Sym}^n := \mathrm{Sym}_{\mathbb{Q}}^n .$$

Notation 3.2.6.

$$\mathrm{Sym} := \bigoplus_{n\in\mathbb{N}} \mathrm{Sym}^n = \mathrm{Sym}^0 \oplus \mathrm{Sym}^1 \oplus \mathrm{Sym}^2 \oplus \dots .$$

Definition 3.2.7. A **symmetric function** is a formal power series in Sym.

Remark 3.2.8. The set Sym is a $\mathbb{Q}$-algebra.

So we can define some bases for Sym.

3.3 Bases of Sym

In this section, we will define some of the main bases of Sym^n.

3.3.1 Monomial symmetric functions

Definition 3.3.1. The **monomial symmetric function** of $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ is

$$m_\lambda := \sum_{\substack{\alpha \models n \\ \lambda(\alpha)=\lambda}} \mathbf{x}^\alpha$$

where

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_k^{\alpha_k}, \quad \text{for all } \alpha = (\alpha_1, \ldots, \alpha_k) \models n,$$

and $m_{(0)} := 1$.

Example 3.3.2. For instance,

$$\begin{aligned} m_{(2)} &= \sum_{i \in \mathbb{P}} x_i^2 \,. \\ m_{(1,1)} &= \sum_{\substack{i,j \in \mathbb{P} \\ i<j}} x_i x_j \,. \\ m_{(2,1)} &= \sum_{\substack{i,j \in \mathbb{P} \\ i \neq j}} x_i^2 x_j \,. \end{aligned}$$

Proposition 3.3.3. Let

$$f(\mathbf{x}) = \sum_{\alpha \models n} c_\alpha \mathbf{x}^\alpha \in \mathrm{Sym}^n \,.$$

Then

$$f(\mathbf{x}) = \sum_{\lambda \vdash n} c_\lambda m_\lambda \,.$$

Proof. Since $f(\mathbf{x})$ is a symmetric function of degree n, then $f(\mathbf{x}) = \pi \cdot f(\mathbf{x})$ for all $\pi \in S_n$, therefore

$$\sum_{\alpha \models n} c_\alpha \mathbf{x}^\alpha = \sum_{\alpha \models n} c_\alpha \mathbf{x}^{\sigma \cdot \alpha}.$$

So, the coefficients c_α are equal for all monomials $\mathbf{x}^\alpha$ with same associated partition $\lambda(\alpha)$ of α.

$\square$

Theorem 3.3.4. The set $\{m_\lambda \mid \lambda \vdash n\}$ is a basis for Sym^n.

Proof. By definition, the m_λ are linearly independent. By the previous proposition, the m_λ are generators of Sym^n.

$\square$

Corollary 3.3.5. The set $\{m_\lambda \mid \lambda \in \mathrm{Par}\}$ is a basis for Sym.

Remark 3.3.6. The dimension of Sym^n is equal to the number of partitions of n: $\dim(\mathrm{Sym}^n) = |\,\mathrm{Par}(n)|$, for all $n \in \mathbb{N}$.

3.3.2 Elementary symmetric functions

Definition 3.3.7. The **elementary symmetric function** of $\lambda = (\lambda_1, \dots, \lambda_k) \vdash N$ is

$$e_\lambda := e_{\lambda_1} \cdot \cdots \cdot e_{\lambda_k}$$

where

$$e_n := m_{(1^n)} = \sum_{\substack{i_1,\dots,i_n \in \mathbb{P} \\ i_1 < \cdots < i_n}} x_{i_1} \dots x_{i_n}\,, \quad \text{for all } n \in \mathbb{P},$$

and $e_0 := 1$.

Example 3.3.8. For instance,

$$\begin{aligned} e_3 &= m_{(1,1,1)} = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \dots, \\ e_{(1,1)} &= m_1 \cdot m_1 = x_1^2 + x_1x_2 + x_1x_3 \cdots + x_2x_1 + x_2^2 + x_2x_3 + \cdots = m_2 + 2m_{(1,1)}. \end{aligned}$$

Definition 3.3.9. Let $A = (a_{ij})_{i,j \in \mathbb{P}}$ be an integer matrix with finitely many non-zero entries and with row and column sum

$$\begin{aligned} r_i &:= \sum_{j \in \mathbb{P}} a_{ij}\,, \quad \text{for all } i \in \mathbb{P}, \\ c_j &:= \sum_{j \in \mathbb{P}} a_{ij}\,, \quad \text{for all } j \in \mathbb{P}. \end{aligned}$$

The **row-sum vector** of A is

$$\mathrm{row}(A) := (r_1, r_2, \dots).$$

The **column-sum vector** of A is

$$\mathrm{col}(A) := (c_1, c_2, \dots).$$

Definition 3.3.10. A **(0,1)-matrix** is a matrix whose entries are all 0 or 1.

Proposition 3.3.11. Let $\lambda \vdash n$ and $\alpha = (\alpha_1, \alpha_2, \dots) \models n$. The coefficient $\varepsilon_{\lambda,\alpha}$ of $\mathbf{x}^\alpha$ in e_λ, i.e.

$$e_\lambda = \sum_{\alpha \models n} \varepsilon_{\lambda,\alpha}\, \mathbf{x}^\alpha\,,$$

is equal to the number of $(0,1)$-matrices $A = (a_{ij})_{i,j \in \mathbb{P}}$ satisfying $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \alpha$. Hence

$$e_\lambda = \sum_{\mu \vdash n} \varepsilon_{\lambda,\mu}\, m_\mu\,.$$

Proof. Let X be the matrix defined as

$$X := \begin{pmatrix} x_1 & x_2 & x_3 & \dots \\ x_1 & x_2 & x_3 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

To obtain $\mathbf{x}^\alpha$ in $e_\lambda = e_{\lambda_1} \cdot e_{\lambda_2} \cdot \dots$, we choose λ_1 different entries from the first line, λ_2 from the second line, etc., such that the product of the chosen entries is $\mathbf{x}^\alpha$. Let A be the matrix obtained from X by replacing the chosen variables with 1 and the non-chosen variables with 0. Then A is a $(0,1)$-matrix with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \alpha$.

□

Example 3.3.12. For instance,

$$\begin{aligned}
e_{(1)} &= m_{(1)} \\
\\
e_{(1,1)} &= m_{(2)} + 2\, m_{(1,1)} \\
e_{(2)} &= \phantom{m_{(2)} + 2\,} m_{(1,1)} \\
\\
e_{(1,1,1)} &= m_{(3)} + 3\, m_{(2,1)} + 6\, m_{(1,1,1)} \\
e_{(2,1)} &= \phantom{m_{(3)} + 3\,} m_{(2,1)} + 3\, m_{(1,1,1)} \\
e_{(3)} &= \phantom{m_{(3)} + 3\, m_{(2,1)} + 3\,} m_{(1,1,1)} \\
\\
e_{(1,1,1,1)} &= m_{(4)} + 4\, m_{(3,1)} + 6\, m_{(2,2)} + 12\, m_{(2,1,1)} + 24\, m_{(1,1,1,1)} \\
e_{(2,1,1)} &= m_{(3,1)} + 2\, m_{(2,2)} + 5\, m_{(2,1,1)} + 12\, m_{(1,1,1,1)} \\
e_{(2,2)} &= m_{(2,2)} + 2\, m_{(2,1,1)} + 6\, m_{(1,1,1,1)} \\
e_{(3,1)} &= m_{(2,1,1)} + 4\, m_{(1,1,1,1)} \\
e_{(4)} &= m_{(1,1,1,1)}
\end{aligned}$$

Lemma 3.3.13. Let $\lambda, \mu \vdash n$. Then $\varepsilon_{\lambda,\mu} = \varepsilon_{\mu,\lambda}$.

Proof. The $(0,1)$-matrix A satisfies $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$ if and only if the transpose A^T satisfies $\mathrm{row}(A^T) = \mu$ and $\mathrm{col}(A^T) = \lambda$.

□

Lemma 3.3.14. Let $\lambda, \mu \vdash n$ and let λ' be the conjugate partition of λ. Then

- $\varepsilon_{\lambda,\mu} = 0$, unless $\mu \trianglelefteq \lambda'$.
- $\varepsilon_{\lambda,\lambda'} = 1$.

Proof. Suppose $\varepsilon_{\lambda,\mu} \neq 0$, by Proposition 3.3.11, there exists a $(0,1)$-matrix with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$.

Let A' be the matrix with $\operatorname{row}(A') = \lambda$ and with its 1's left-justified, i.e., $A'_{ij} = 1$ for $1 \le j \le \lambda_i$ and $A'_{ij} = 0$ for $j > \lambda_i$, for all $i \in \mathbb{P}$.
For instance,

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \longrightarrow A' = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

For any $i \in \mathbb{P}$, the number of 1's in the first i columns of A' clearly is not less than the number of 1's in the first i columns of A, so by definition of dominance order we have $\operatorname{col}(A') \trianglerighteq \operatorname{col}(A) = \mu$. By Definition 2.2.9, $\operatorname{col}(A') = \lambda'$, so $\lambda' \trianglerighteq \mu$.
We observe that A' is the only $(0,1)$-matrix such that $\operatorname{row}(A) = \lambda$ and $\operatorname{col}(A') = \lambda'$, so $\varepsilon_{\lambda,\lambda'} = 1$.

□

Theorem 3.3.15. The set $\{e_\lambda \mid \lambda \vdash n\}$ is a basis for Sym^n.

Proof. Let $\preceq$ be a total order on $\operatorname{Par}(n)$ that extends the dominance order $\trianglelefteq$ (for example, the lexicographic order $\le_{\text{lex}}$). Let $E = (\varepsilon_{\lambda,\mu})_{\lambda,\mu \in \operatorname{Par}(n)}$.
Let λ^1, λ^2, ..., $\lambda^{|\operatorname{Par}(n)|}$ be the partitions of n in the $\preceq$ order:

$$\lambda^1 \preceq \lambda^2 \preceq \cdots \preceq \lambda^{|\operatorname{Par}(n)|}.$$

By the previous lemma, the matrix E, with the row order λ^1, λ^2, ... and column order $(\lambda^1)'$, $(\lambda^2)'$, ..., is triangular with 1's on the main diagonal. Hence E is invertible. Since $\{m_\lambda \mid \lambda \vdash n\}$ is a basis for Sym^n, by Proposition 3.3.11, then $\{e_\lambda \mid \lambda \vdash n\}$ is a basis for Sym^n.

□

Corollary 3.3.16. The set $e = \{e_\lambda \mid \lambda \in \operatorname{Par}\}$ is a basis for Sym.

3.3.3 Complete homogeneous symmetric functions

Definition 3.3.17. The **complete homogeneous symmetric function** of $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N$ is

$$h_\lambda := h_{\lambda_1} \cdot \cdots \cdot h_{\lambda_k}$$

where

$$h_n := \sum_{\lambda \vdash n} m_\lambda = \sum_{\substack{i_1,\ldots,i_n \in \mathbb{P} \\ i_1 \le \cdots \le i_n}} x_{i_1} \ldots x_{i_n}, \quad \text{for all } n \in \mathbb{P},$$

and $h_0 := 1$.

Example 3.3.18. For instance,

$$\begin{aligned} h_3 &= m_{(1,1,1)} + m_{(2,1)} + m_{(3)} = \\ &= x_1^3 + x_2^3 + \cdots + x_1^2 x_2 + x_1 x_2^2 + \cdots + x_1 x_2 x_3 + x_1 x_2 x_4 + \ldots . \end{aligned}$$

Definition 3.3.19. A $\mathbb{N}$**-matrix** is a matrix whose entries are all elements of $\mathbb{N}$.

Proposition 3.3.20. Let $\lambda \vdash n$ and $\alpha = (\alpha_1, \alpha_2, \dots) \models n$. The coefficient $N_{\lambda,\alpha}$ of $\mathbf{x}^\alpha$ in h_λ, i.e.

$$h_\lambda = \sum_{\alpha \models n} N_{\lambda,\alpha}\, \mathbf{x}^\alpha\,,$$

is equal to the number of $\mathbb{N}$-matrices $A = (a_{ij})_{i,j\in\mathbb{P}}$ satisfying $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \alpha$. Hence

$$h_\lambda = \sum_{\mu \vdash n} N_{\lambda,\mu}\, m_\mu\,.$$

Proof. Let X be the matrix in the proof of Proposition 3.3.11:

$$X := \begin{pmatrix} x_1 & x_2 & x_3 & \dots \\ x_1 & x_2 & x_3 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

To obtain $\mathbf{x}^\alpha$ in $h_\lambda = h_{\lambda_1} \cdot h_{\lambda_2} \cdot \dots$, we choose λ_1 entries from the first line (a_{11} times x_1, a_{12} times x_2, etc.), λ_2 from the second line (a_{21} times x_1, a_{22} times x_2, etc.), etc., such that the product of the chosen entries is $\mathbf{x}^\alpha$.
Then $A = (a_{ij})_{i,j\in\mathbb{P}}$ is a $\mathbb{N}$-matrix with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \alpha$.

$\square$

Example 3.3.21. For instance,

$$\begin{aligned}
h_{(1)} &= m_{(1)} \\[1ex]
h_{(1,1)} &= 2m_{(1,1)} + m_{(2)} \\
h_{(2)} &= m_{(1,1)} + m_{(2)} \\[1ex]
h_{(1,1,1)} &= 6m_{(1,1,1)} + 3m_{(2,1)} + m_{(3)} \\
h_{(2,1)} &= 3m_{(1,1,1)} + 2m_{(2,1)} + m_{(3)} \\
h_{(3)} &= m_{(1,1,1)} + m_{(2,1)} + m_{(3)} \\[1ex]
h_{(1,1,1,1)} &= 24m_{(1,1,1,1)} + 12m_{(2,1,1)} + 6m_{(2,2)} + 4m_{(3,1)} + m_{(4)} \\
h_{(2,1,1)} &= 12m_{(1,1,1,1)} + 7m_{(2,1,1)} + 4m_{(2,2)} + 3m_{(3,1)} + m_{(4)} \\
h_{(2,2)} &= 6m_{(1,1,1,1)} + 4m_{(2,1,1)} + 3m_{(2,2)} + 2m_{(3,1)} + m_{(4)} \\
h_{(3,1)} &= 4m_{(1,1,1,1)} + 3m_{(2,1,1)} + 2m_{(2,2)} + 2m_{(3,1)} + m_{(4)} \\
h_{(4)} &= m_{(1,1,1,1)} + m_{(2,1,1)} + m_{(2,2)} + m_{(3,1)} + m_{(4)}
\end{aligned}$$

Theorem 3.3.22. The set $\{h_\lambda \mid \lambda \vdash n\}$ is a basis for Sym^n.

Proof. [Sta99, Corollary 7.6.2]

$\square$

Corollary 3.3.23. The set $h = \{h_\lambda \mid \lambda \in \mathrm{Par}\}$ is a basis for Sym.

3.3.4 Power sum symmetric functions

Definition 3.3.24. The **power sum symmetric function** of $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N$ is

$$p_\lambda := p_{\lambda_1} \cdot \cdots \cdot p_{\lambda_k}$$

where

$$p_n := m_{(n)} = \sum_{i \in \mathbb{P}} x_i^n \,, \quad \text{for all } n \in \mathbb{P},$$

and $p_0 := 1$.

Example 3.3.25. For instance,

$$\begin{aligned} p_3 &= m_3 = x_1^3 + x_2^3 + x_3^3 + \ldots, \\ p_{(1,1)} &= m_1 \cdot m_1 = e_{(1,1)} = m_2 + 2m_{(1,1)}. \end{aligned}$$

Theorem 3.3.26. The set $\{p_\lambda \mid \lambda \vdash n\}$ is a basis for Sym^n.

Proof. [Sta99, Corollary 7.7.2] □

Corollary 3.3.27. The set $p = \{p_\lambda \mid \lambda \in \mathrm{Par}\}$ is a basis for Sym.

3.4 Chromatic symmetric functions

In 1995, Richard Stanley introduced a symmetric function generalization of the chromatic polynomial [Sta95].

Definition 3.4.1. Let $G = (V, E)$ be a graph with $V = \{v_1, \ldots, v_n\}$. The **chromatic symmetric function** of G is

$$X_G(\mathbf{x}) \;:=\; \sum_{k \in \kappa(G)} \mathbf{x}_k$$

where

$$\mathbf{x}_k := x_{k(v_1)} \cdots x_{k(v_n)} \,.$$

Remark 3.4.2. For all graphs G,

$$X_G(\mathbf{x}) \;=\; \sum_{k \in \kappa(G)} x_1^{|k^{-1}(1)|} \, x_2^{|k^{-1}(2)|} \ldots .$$

Remark 3.4.3. By Remark 1.1.29, if a graph has a loop, then its chromatic symmetric function is the zero polynomial.

Remark 3.4.4. By Remark 1.1.30, multiple edges do not influence the chromatic symmetric function.

Example 3.4.5. The chromatic symmetric function of K_n is

$$X_{K_n}(\mathbf{x}) = n!\, x_1x_2 \dots x_{n-1}x_n + n!\, x_1x_2 \dots x_{n-1}x_{n+1} + \cdots = n!\, m_{(1^n)}$$

because:

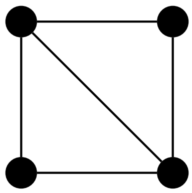
- There are $n!$ different ways to color the vertices of K_n with n distinct colors.
- There is no proper coloring with less than n colors.

Example 3.4.6. The chromatic symmetric function of $G = ([n], \varnothing)$ is

$$X_G(\mathbf{x}) = x_1^n + \cdots + x_1^{n-1}x_2 + \cdots + x_1 \dots x_n + \cdots = h_n$$

because all functions $k : [n] \longrightarrow \mathbb{P}$ are colorings of G.

Example 3.4.7. Let $G = (V, E)$ be the following graph



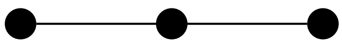
Then:

- There are $|V|! = 4! = 24$ different ways to assign 4 distinct colors to vertices.
- There are 6 different ways to assign 3 different colors (one color must be used twice since $|V| = 4$).
- There are no proper colorings with less than 3 colors.

Then the chromatic symmetric function of G is

$$\begin{aligned} X_G(\mathbf{x}) &= 24x_1x_2x_3x_4 + 24x_1x_2x_3x_5 + \cdots + 6x_1^2x_2x_3 + 6x_1^2x_2x_4 + \cdots = \\ &= 24m_{(1,1,1,1)} + 6m_{(2,1,1)}. \end{aligned}$$

Example 3.4.8. Let $G = (V, E)$ be the following graph



Then:

- There are $|V|! = 6$ different ways to assign 3 distinct colors to vertices.
- There are 2 different ways to assign 2 different colors.
- There are no proper colorings with less than 2 colors.

Then the chromatic symmetric function of G is

$$X_G(\mathbf{x}) = 6m_{(1,1,1)} + 2m_{(2,1)}.$$

Let us now see the main properties of X_G, including its relation to the chromatic polynomial.

Proposition 3.4.9. Let $G = (V, E)$ be a graph. Then:

1. $X_G(\mathbf{x})$ is a homogeneous polynomial of degree $|V|$.

2. $X_G(\mathbf{x})$ is a symmetric function.

3. If $x_1 = \cdots = x_n = 1$ and $x_i = 0$ for all $i > n$, then X_G is the chromatic polynomial of G:
$$X_G(1^n) = P_G(n).$$
Hence $X_G(1^n)$ is the number of proper colorings of G with n colors.

Proof.

1. By definition, every monomial in X_G has a factor x_i for each vertex, so every monomial has the same degree: $|V|$.

2. If $k : V \longrightarrow \{1, 2, 3, \dots\}$ is a proper coloring, then the function k' defined as
$$k' : V \longrightarrow \sigma \cdot \{1, 2, 3, \dots\} = \{\sigma(1), \sigma(2), \sigma(3), \dots\}$$
is also a proper coloring. So, permuting the indices of X_G leaves the function unchanged.

3. By the hypotheses, every monomial of X_G will be 1 or 0. The surviving monomials are exactly those that use only the first n colors. So their sum is $P_G(n)$ by definition of a chromatic polynomial.

$\square$

For symmetric chromatic functions there is no formula analogous to 1.1.45, because they are always homogeneous, so trying to formulate such a relation encounters difficulties when considering edge contraction. In 2019, Logan Crew and Sophie Spirkl introduce a vertex-weighted version of the chromatic symmetric functions for which there is a deletion-contraction relation [CS20].

Chapter 4

Quasisymmetric functions

In this Chapter we will generalize the concept of symmetric function, mainly following [Sta99, Sezione 7.19] as a reference.

4.1 Quasisymmetric functions

Notation 4.1.1. Let $f(\mathbf{x})$ be a formal power series. We denote with

$$[x_1^{\alpha_1} \dots x_n^{\alpha_n}]f$$

the coefficient of the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ in $f(\mathbf{x})$.

Example 4.1.2. Let

$$f(x_1, x_2) = 3x_1 + 4x_2 + x_2^2 \,.$$

Then

- $[x_1]f = 3$.
- $[x_2]f = 4$.
- $[x_1^2]f = 0$.
- $[x_2^2]f = 1$.

Definition 4.1.3. Let $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ be a set of variables. A **quasisymmetric function** in the variables $\mathbf{x}$ is a formal power series $f(\mathbf{x}) \in \mathbb{Q}[[\mathbf{x}]]$ such that

$$[x_1^{\alpha_1} \dots x_n^{\alpha_n}]f = [x_{i_1}^{\alpha_1} \dots x_{i_n}^{\alpha_n}]f$$

for each $\alpha_1, \dots, \alpha_n \in \mathbb{P}$ and for every sequence of positive integers $i_1 < \dots < i_n$.

If in the definition we had every sequence of positive integers $i_1, \dots, i_n$ distinct (and not $i_1 < \dots < i_n$), then we would have the symmetric functions.

Notation 4.1.4. The set of all homogeneous quasisymmetric functions of degree n is denoted by QSym^n (or $\mathcal{Q}^n$). The set of all quasisymmetric functions is denoted by QSym (or $\mathcal{Q}$).

Example 4.1.5.

$$f(\mathbf{x}) = \sum_{\substack{i,j\in\mathbb{P}\\ i<j}} x_i^2 x_j = x_1^2x_2 + x_1^2x_3 + \dots + x_2^2x_3 + x_2^2x_4 + \dots \in \mathrm{QSym}^3 \,.$$

Proposition 4.1.6. Symmetric functions are quasisymmetric functions:

$$\mathrm{Sym} \subseteq \mathrm{QSym}\,.$$

Proof. If $f(\mathbf{x}) \in \mathrm{Sym}$, then

$$(1, i_1)\ldots(n, i_n)f = f$$

for every sequence of positive integers $i_1 < \cdots < i_n$. So

$$[x_1^{\alpha_1}\ldots x_n^{\alpha_n}]f = [x_{i_1}^{\alpha_1}\ldots x_{i_n}^{\alpha_n}]f$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbb{P}$.

$\square$

Remark 4.1.7. The converse does not hold: not all quasisymmetric functions are symmetric functions.

Example 4.1.8. The quasisymmetric function of Example 4.1.5 is not symmetric, for instance $f \neq (1,2)f$, because $x_1^2x_2 \in f$ and $x_1^2x_2 \notin (1,2)f$.

4.2 Bases of QSym

In this section, we will see the main bases for QSym^n.

4.2.1 Monomial quasisymmetric functions

Definition 4.2.1. The **monomial quasisymmetric function** of $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$ is

$$M_\alpha := \sum_{\substack{i_1,\ldots,i_k\in\mathbb{P}\\ i_1<\cdots<i_k}} x_{i_1}^{\alpha_1}\ldots x_{i_k}^{\alpha_k}$$

and $M_{(0)} := 1$.

Example 4.2.2. For instance,

$$M_{(2)} = \sum_{i\in\mathbb{N}} x_i^2\,.$$

$$M_{(1,1)} = \sum_{\substack{i,j\in\mathbb{P}\\ i<j}} x_ix_j\,.$$

$$M_{(2,1)} = \sum_{\substack{i,j\in\mathbb{P}\\ i<j}} x_i^2x_j\,.$$

Remark 4.2.3. For all $n \in \mathbb{P}$,

$$M_{(1^n)} := \sum_{\substack{i_1,\ldots,i_n\in\mathbb{P}\\ i_1<\cdots<i_n}} x_{i_1}\ldots x_{i_n} = m_{(1^n)} = e_n \in \mathrm{Sym}^n\,.$$

More generally, the following result holds.

Proposition 4.2.4. For all $\lambda \vdash n$,

$$m_\lambda = \sum_{\substack{\alpha \in \mathrm{Comp}(n) \\ \lambda(\alpha)=\lambda}} M_\alpha \,.$$

Proof. For all $\lambda \vdash n$,

$$m_\lambda := \sum_{\substack{(\alpha_1,\dots,\alpha_k)\models n \\ \lambda(\alpha)=\lambda}} x_1^{\alpha_1} \dots x_k^{\alpha_k} = \sum_{\substack{(\alpha_1,\dots,\alpha_k)\in \mathrm{Comp}(n) \\ \lambda(\alpha)=\lambda}} \underbrace{\left(\sum_{\substack{i_1,\dots,i_k \in \mathbb{P} \\ i_1<\dots<i_k}} x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k} \right)}_{M_\alpha} .$$

$\square$

Example 4.2.5. For instance,

$$\begin{aligned}
m_{(2,2)} &= M_{(2,2)} \,. \\
m_{(3,3)} &= M_{(3,3)} \,. \\
m_{(2,1)} &= M_{(2,1)} + M_{(1,2)} \,. \\
m_{(2,1,1)} &= M_{(2,1,1)} + M_{(1,2,1)} + M_{(1,1,2)} \,. \\
m_{(2,1,1,1)} &= M_{(2,1,1,1)} + M_{(1,2,1,1)} + M_{(1,1,2,1)} + M_{(1,1,1,2)} \,.
\end{aligned}$$

Remark 4.2.6. If $f(\mathbf{x}) \in \mathrm{QSym}^n$, then

$$f(\mathbf{x}) = \sum_{\alpha \models n} \Big([x_1^{\alpha_1} \dots x_k^{\alpha_k}] f \Big) M_\alpha \,.$$

Proposition 4.2.7. The set $\{M_\alpha \mid \alpha \models n\}$ is a basis for QSym^n.

Proof. By definition, the M_α are linearly independent. By the previous remark, the M_α are generators of QSym^n.

$\square$

Remark 4.2.8. The dimension of QSym^n is equal to the number of compositions of n: $\dim(\mathrm{QSym}^n) = |\,\mathrm{Comp}(n)| = 2^{n-1}$, for all $n \in \mathbb{N}$.

Remark 4.2.9. The set $\{M_\alpha \mid \alpha \in \mathrm{Comp}\}$ is a basis for QSym.

4.2.2 Gessel's fundamental quasisymmetric functions

Notation 4.2.10. Let $n \in \mathbb{P}$ and $S \subseteq [n-1]$. We denote with $D_n(S)$ the set of all functions $f : [n] \longrightarrow \mathbb{P}$ such that:

1. $f(i) \geq f(i+1)$ for all $i \in [n-1]$.
2. $f(i) > f(i+1)$ for all $i \in S$.

Definition 4.2.11. Let $S \subseteq [n-1]$. **Gessel's fundamental quasisymmetric function** of S is

$$F_{n,S} := \sum_{f \in D_n(S)} \mathbf{x}_f$$

where

$$\mathbf{x}_f := x_{f(1)}\, x_{f(2)}\, \ldots\, x_{f(n)}.$$

Example 4.2.12. Let $n = 3$ and $S = \{1\}$. Then $D_3(\{1\})$ is the set of all functions $f : \{1,2,3\} \longrightarrow \mathbb{P}$ such that $f(1) > f(2) \geq f(3)$. Therefore,

$$\begin{aligned} F_{3,\{1\}} &= \sum_{\substack{i_1,i_2,i_3 \in \mathbb{P} \\ i_1 \leq i_2 < i_3}} x_{i_1} x_{i_2} x_{i_3} = \\ &= \sum_{\substack{i_1,i_2,i_3 \in \mathbb{P} \\ i_1 < i_2 < i_3}} x_{i_1} x_{i_2} x_{i_3} + \sum_{\substack{i_1,i_2 \in \mathbb{P} \\ i_1 < i_2}} x_{i_1}^2 x_{i_2} = \\ &= M_{(1,1,1)} + M_{(2,1)}\,. \end{aligned}$$

Example 4.2.13. Let $n = 3$ and $S = \{2\}$. Then $D_3(\{2\})$ is the set of all functions $f : \{1,2,3\} \longrightarrow \mathbb{P}$ such that $f(1) \geq f(2) > f(3)$. Therefore,

$$\begin{aligned} F_{3,\{2\}} &= \sum_{\substack{i_1,i_2,i_3 \in \mathbb{P} \\ i_1 < i_2 \leq i_3}} x_{i_1} x_{i_2} x_{i_3} = \\ &= \sum_{\substack{i_1,i_2,i_3 \in \mathbb{P} \\ i_1 < i_2 < i_3}} x_{i_1} x_{i_2} x_{i_3} + \sum_{\substack{i_1,i_2 \in \mathbb{P} \\ i_1 < i_2}} x_{i_1} x_{i_2}^2 = \\ &= M_{(1,1,1)} + M_{(1,2)}\,. \end{aligned}$$

Remark 4.2.14. $F_{n,S} \in \mathrm{QSym}^n$, for all $n \in \mathbb{P}$ and $S \subseteq [n-1]$.

Remark 4.2.15. For all $n \in \mathbb{P}$,

$$\begin{aligned} F_{n,\varnothing} &= \sum_{\substack{i_1,\ldots,i_n \in \mathbb{P} \\ i_1 \leq \cdots \leq i_n}} x_{i_1} \ldots x_{i_n} = h_n \in \mathrm{Sym}^n\,. \\ F_{n,[n-1]} &= \sum_{\substack{i_1,\ldots,i_n \in \mathbb{P} \\ i_1 < \cdots < i_n}} x_{i_1} \ldots x_{i_n} = e_n \in \mathrm{Sym}^n\,. \end{aligned}$$

Using the definitions of S_α and $\mathrm{co}(S)$ seen in Lemma 2.2.24, we can define a different form of Gessel's fundamental quasisymmetric functions.

Proposition 4.2.16. Let

$$L_\alpha := \sum_{\substack{i_1,\dots,i_n\in\mathbb{P}\\ i_1\le\cdots\le i_n\\ i_j<i_{j+1} \text{ if } j\in S_\alpha}} x_{i_1}x_{i_2}\dots x_{i_n}\,, \quad \text{for all } \alpha=(\alpha_1,\alpha_2,\dots,\alpha_k)\models n.$$

Then

$$F_{n,S} = L_{\mathrm{co}(S)}\,, \quad \text{for all } S=\{s_1,s_2,\dots,s_{k-1}\}\subseteq[n-1].$$
$$F_{n,S_\alpha} = L_\alpha\,, \quad \text{for all } \alpha=(\alpha_1,\alpha_2,\dots,\alpha_k)\models n.$$

Proof. By Lemma 2.2.24, we have $S_{\mathrm{co}(S)}=S$ for all sets S. Therefore

$$L_{\mathrm{co}(S)} = \sum_{\substack{i_1,\dots,i_n\in\mathbb{P}\\ i_1\le\cdots\le i_n\\ i_j<i_{j+1} \text{ if } j\in S}} x_{i_1}x_{i_2}\dots x_{i_n} = \sum_{f\in D_n(S)} x_{f(1)}\,x_{f(2)}\,\dots\,x_{f(n)} = F_{n,S}.$$

The second equation is proved similarly.

□

Actually, L_α is the standard definition of Gessel's fundamental function.

Proposition 4.2.17. For all $\alpha\models n$,

$$L_\alpha = \sum_{\substack{T\subseteq[n-1]\\ S_\alpha\subseteq T}} M_{\mathrm{co}(T)}\,.$$
$$M_\alpha = \sum_{\substack{T\subseteq[n-1]\\ S_\alpha\subseteq T}} (-1)^{|T\setminus S_\alpha|} L_{\mathrm{co}(T)}\,.$$

Hence the set $\{L_\alpha \mid \alpha\models n\}$ is a basis for QSym^n.

Proof. [Sta99, Proposition 7.19.1]

□

Proposition 4.2.18. The set $\{F_{n,S}\mid S\subseteq[n-1]\}$ is a basis for QSym^n. Hence the set

$$F := \bigcup_{n\in\mathbb{P}} \{F_{n,S}\mid S\subseteq[n-1]\}$$

is a basis for QSym.

Proof. Since $\{L_\alpha \mid \alpha\models n\}$ is a basis for QSym^n, by Proposition 4.2.16, also $\{F_{n,S}\mid S\subseteq[n-1]\}$ is a basis for QSym^n.

□

4.3 QSym-automorphism

Definition 4.3.1. The QSym-**automorphism** is the homomorphism

$$\omega : \mathrm{QSym} \longrightarrow \mathrm{QSym}$$

defined as

$$\omega(F_{n,S}) := F_{n,[n-1]\setminus S} \text{ for each } n \in \mathbb{N} \text{ and } S \subseteq [n-1].$$

Example 4.3.2. For instance, $\omega(F_{4,\{1,3\}} + F_{4,\{1\}}) = F_{4,\{2\}} + F_{4,\{2,3\}}$.

Proposition 4.3.3. The QSym-automorphism ω is an involution.

Proof. Let $F_{n,S}$ be a generic Gessel's fundamental quasisymmetric function. Then

$$\omega^2(F_{n,S}) = \omega(F_{n,[n-1]\setminus S}) = F_{n,[n-1]\setminus([n-1]\setminus S)} = F_{n,S}\,.$$

So, $\omega = \omega^{-1}$.

□

Remark 4.3.4. By Remark 4.2.15, for all $n \in \mathbb{P}$, we have that

$$\omega(h_n) = \omega(F_{n,\varnothing}) = F_{n,[n-1]} = e_n\,.$$
$$\omega(e_n) = \omega(F_{n,[n-1]}) = F_{n,\varnothing} = h_n\,.$$

Therefore, we have $\omega(h_\lambda) = e_\lambda$ and $\omega(e_\lambda) = h_\lambda$, for all $\lambda \in \mathrm{Par}$.

Proposition 4.3.5. Let $f \in \mathrm{QSym}^n$. Then

$$\omega(f) \in \mathrm{Sym}^n \iff f \in \mathrm{Sym}^n\,.$$

Proof. Since $\{h_\lambda \mid \lambda \vdash n\}$ and $\{e_\lambda \mid \lambda \vdash n\}$ are bases of Sym^n, if f is in Sym^n, then f can be write as sum of h_λ. So $\omega(f)$ is sum of e_λ. Therefore $\omega(f)$ is in Sym^n. The converse is obtained analogously.

□

Chapter 5

Chromatic quasisymmetric functions

In this Chapter we introduce a generalization of the chromatic symmetric functions, mainly following [Ell17] as a reference.
In this Chapter we consider only loop-free graphs and digraphs.

5.1 Chromatic quasisymmetric functions

In this section, we will define a generalization of the chromatic symmetric function.

5.1.1 Chromatic quasisymmetric functions of digraphs

Definition 5.1.1. Let $\vec{G} = (V, A)$ be a digraph and let $k : V \longrightarrow \mathbb{P}$ be a proper coloring of $\vec{G}$. An **ascent** of k is an arc $uv \in A$ such that $k(u) < k(v)$. A **descent** of k is an arc $uv \in A$ such that $k(u) > k(v)$.

Notation 5.1.2. We denote by $\mathrm{asc}(k)$ and $\mathrm{des}(k)$, the number of ascents and descents of the proper coloring k:

$$\mathrm{asc}(k) := \Big|\{uv \in A \mid k(u) < k(v)\}\Big|.$$

$$\mathrm{des}(k) := \Big|\{uv \in A \mid k(u) > k(v)\}\Big|.$$

Remark 5.1.3. Let $\vec{G} = (V, A)$ be a digraph. If $k : V \longrightarrow \mathbb{P}$ is a proper coloring of $\vec{G}$, then

$$|A| = \mathrm{asc}(k) + \mathrm{des}(k).$$

Definition 5.1.4. Let $\vec{G} = (V, A)$ be a digraph with $V = \{v_1, \dots, v_n\}$. The **chromatic quasisymmetric function** of $\vec{G}$ is

$$X_{\vec{G}}(\mathbf{x}, t) := \sum_{k \in \kappa(G)} \mathbf{x}_k \, t^{\mathrm{asc}(k)} \in \mathrm{QSym}^n[t]$$

where

$$\mathbf{x}_k := x_{k(v_1)} \cdots x_{k(v_n)} \,.$$

Remark 5.1.5. If we set the parameter t equal to 1 we obtain the chromatic symmetric function:

$$X_{\vec{G}}(\mathbf{x}, 1) = X_G(\mathbf{x}).$$

This remark leads us to two further observations:

Remark 5.1.6. Since Proposition 3.4.9, if we set the parameter t equal to 1 and $x_1 = \cdots = x_n = 1$ and $x_i = 0$ for all $i > n$, we obtain the chromatic polynomial of G:

$$X_{\vec{G}}(1^n, 1) = P_G(n).$$

Hence $X_{\vec{G}}(1^n, 1)$ is the number of proper colorings of $\vec{G}$ with n colors.

Remark 5.1.7. Since Remark 1.1.29, if a digraph has a loop, then its chromatic quasisymmetric function is the zero polynomial.

Example 5.1.8. Let $G = (V, \varnothing)$, so every map $k : V \longrightarrow \mathbb{P}$ is a proper coloring. Since there are no arcs, $\operatorname{asc}(k) = 0$ for all map k. If $n = |V|$, then

$$X_G(\mathbf{x}, t) = \sum_{i_1,\ldots,i_n \in \mathbb{P}} x_{i_1} \ldots x_{i_n} = \Big(\sum_{i_1 \in \mathbb{P}} x_{i_1} \Big)^n = m_1^n \in \operatorname{Sym}^n .$$

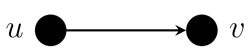
Example 5.1.9. The chromatic quasisymmetric function of $\overleftrightarrow{K}_n$ is

$$\begin{aligned} X_{\overleftrightarrow{K}_n}(\mathbf{x}, t) &= \sum_{\substack{i_1,\ldots,i_n \in \mathbb{P} \\ i_1 \neq \ldots \neq i_n}} x_{i_1} \ldots x_{i_n}\, t^{\binom{n}{2}} = \\ &= n! \sum_{\substack{i_1,\ldots,i_n \in \mathbb{P} \\ i_1 < \cdots < i_n}} x_{i_1} \ldots x_{i_n}\, t^{\binom{n}{2}} = n!\, e_n\, t^{\binom{n}{2}} \in \operatorname{Sym}^n[t]. \end{aligned}$$

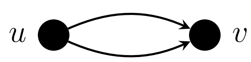
Because, by construction, every map $k : [n] \longrightarrow \mathbb{P}$, with $k(1) \neq \ldots \neq k(n)$, is a proper coloring and half of the arcs are ascents. Since the number of arcs of $\overleftrightarrow{K}_n$ is $n(n-1)$, by Proposition 1.2.35, then the number of ascents is $\binom{n}{2} = \frac{n(n-1)}{2}$.

Remark 5.1.10. Unlike coloring and chromatic symmetric functions, multiple arcs influence the chromatic quasisymmetric function.

Example 5.1.11. Let $\vec{G} = (\{u, v\}, \{uv\})$ and $\vec{H} = (\{u, v\}, \{uv, uv\})$ be the following digraphs.



$\vec{G}$



$\vec{H}$

In $\vec{G}$ and $\vec{H}$, every map $k : \{u, v\} \longrightarrow \mathbb{P}$, with $k(u) \neq k(v)$, is a proper coloring. Then

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\substack{i_1,i_2 \in \mathbb{P} \\ i_1 > i_2}} x_{i_1} x_{i_2} + \sum_{\substack{i_1,i_2 \in \mathbb{P} \\ i_1 < i_2}} x_{i_1} x_{i_2}\, t = M_{(1,1)} + M_{(1,1)}\, t\, .$$

$$X_{\vec{H}}(\mathbf{x}, t) = \sum_{\substack{i_1,i_2 \in \mathbb{P} \\ i_1 > i_2}} x_{i_1} x_{i_2} + \sum_{\substack{i_1,i_2 \in \mathbb{P} \\ i_1 < i_2}} x_{i_1} x_{i_2}\, t^2 = M_{(1,1)} + M_{(1,1)}\, t^2\, .$$

Proposition 5.1.12. Let $\vec{G} = (V, A)$ and $\vec{H} = (V', A')$ be digraphs on disjoint vertex sets ($V \cap V' = \varnothing$). Let $\overrightarrow{G+H} = (V \cup V', A \cup A')$ be the union of $\vec{G}$ and $\vec{H}$. Then

$$X_{\overrightarrow{G+H}}(\mathbf{x}, t) = X_{\vec{G}}(\mathbf{x}, t)\, X_{\vec{H}}(\mathbf{x}, t).$$

Proof. Since $V \cap V' = \varnothing$, then $A \cap A' = \varnothing$. Therefore, each proper coloring $k : V \cupdot V' \longrightarrow \mathbb{P}$ of $G + H$ can be restricted to the proper coloring $g : V \longrightarrow \mathbb{P}$ of G and the proper coloring $h : V' \longrightarrow \mathbb{P}$ of H.

$$\begin{aligned} X_{\overrightarrow{G+H}}(\mathbf{x}, t) &= \sum_{k \in \kappa(G+H)} \mathbf{x}_k\, t^{\operatorname{asc}(k)} = \sum_{g \in \kappa(G)} \sum_{h \in \kappa(H)} \mathbf{x}_g\, \mathbf{x}_h\, t^{\operatorname{asc}(g)+\operatorname{asc}(h)} = \\ &= \sum_{g \in \kappa(G)} \mathbf{x}_g\, t^{\operatorname{asc}(g)} \sum_{h \in \kappa(H)} \mathbf{x}_h\, t^{\operatorname{asc}(h)} = X_{\vec{G}}(\mathbf{x}, t)\, X_{\vec{H}}(\mathbf{x}, t). \end{aligned}$$

$\square$

In 2016, Jordan Awan and Olivier Bernardi introduce a quasisymmetric generalization of the Tutte polynomial to directed graphs [AB20].

5.1.2 Chromatic quasisymmetric functions of labeled graphs

The chromatic quasisymmetric function can also be defined in a similar way on labeled graphs.

Definition 5.1.13. A **labeled graph** is a graph whose vertex set is a subset of $\mathbb{P}$.

Without loss of generality, we will assume that labeled graphs will have $[n]$ as sets of vertices, with $n = |V|$.

Definition 5.1.14. Let $G = ([n], E)$ be a labeled graph. The chromatic quasisymmetric function of G is

$$X_G(\mathbf{x}, t) := \sum_{k \in \kappa(G)} \mathbf{x}_k\, t^{\operatorname{asc}(k)} \in \mathrm{QSym}^n[t]$$

where

$$\mathbf{x}_k := x_{k(v_1)} \cdots x_{k(v_n)}$$

and

$$\operatorname{asc}(k) := \Big|\{uv \in E \mid u < v \text{ and } k(u) < k(v)\}\Big|.$$

Remark 5.1.15. Let $G = ([n], E)$ be a labeled graph. Let $\vec{G}$ be the digraph constructed by orienting each edge from the smaller vertex to the larger vertex. Then

$$X_G(\mathbf{x}, t) = X_{\vec{G}}(\mathbf{x}, t).$$

Conversely, every acyclic digraph can be obtained from a labeled graph constructed as follows:

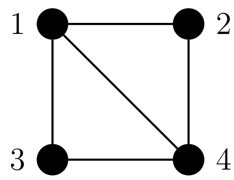
- By Proposition 1.2.12, $\vec{G}$ has a source vertex w and we label that vertex 1.

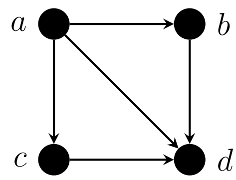
- The digraph $\vec{G}'$ obtained from $\vec{G}$ by removing the vertex w and the arcs from w is still an acyclic digraph. So, by Proposition 1.2.12, $\vec{G}'$ has a source vertex and we label that vertex 2.

- We continue this process until all vertices are labeled.

In other words, the definition of chromatic quasisymmetric function for labeled graphs is equivalent to the definition of chromatic quasisymmetric function for acyclic digraphs.

Example 5.1.16. Let G be the following labeled graph and $\vec{G}$ the following digraph.





Then $\vec{G}$ is obtained from G by orienting each edge from the smaller vertex to the larger vertex and G can be obtained from $\vec{G}$ following the steps of the previous remark.

5.2 Palindromicity

In this section we prove an important property of chromatic quasisymmetric functions.

Definition 5.2.1. Let $\alpha = (\alpha_1, \dots, \alpha_k) \models n$. The **reverse composition** of α is the composition

$$\alpha^{\text{rev}} := (\alpha_k, \dots, \alpha_1).$$

Definition 5.2.2. Let $\rho : \text{QSym} \longrightarrow \text{QSym}$ be the involution defined on the monomial quasisymmetric function basis, by

$$\rho(M_\alpha) := M_{\alpha^{\text{rev}}}, \quad \text{for all } \alpha \in \text{Comp}.$$

We can extend ρ to $\text{QSym}[t]$ by linearity.

Remark 5.2.3. Every symmetric function is fixed by ρ.

Theorem 5.2.4. Let $\vec{G} = (V, A)$ be a digraph on n vertices. Then

$$\rho(X_{\vec{G}}(\mathbf{x}, t)) = X_{\vec{G}}(\mathbf{x}, t^{-1})\, t^{|A|}.$$

Proof. We define the involution $\gamma : \kappa(G) \longrightarrow \kappa(G)$ as

$$\gamma(k) := \gamma_k\,, \quad \text{for all } k \in \kappa(G),$$

where the function $\gamma_k : V \longrightarrow \mathbb{P}$ is the proper coloring defined as

$$\gamma_k(v) := \max_{u\in V}\{k(u)\} + \min_{u\in V}\{k(u)\} - k(v) \quad \text{for all } v \in V.$$

By definition, we have $\operatorname{asc}(\gamma_k) = |A| - \operatorname{asc}(k)$ for all $k \in \kappa(G)$, therefore

$$X_{\vec{G}}(\mathbf{x}, t) := \sum_{k\in\kappa(G)} \mathbf{x}_k\, t^{\operatorname{asc}(k)} = \sum_{\gamma_k\in\kappa(G)} \mathbf{x}_{\gamma_k}\, t^{|A|-\operatorname{asc}(k)}.$$

This implies that the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m} t^j$ in $X_{\vec{G}}(\mathbf{x}, t)$ is equal to the coefficient of $x_m^{\alpha_1} x_{m-1}^{\alpha_2} \dots x_1^{\alpha_m} t^{|A|-j}$ in $X_{\vec{G}}(\mathbf{x}, t)$ for all $j \in \{0, 1, \dots, |A|\}$ and for all compositions $(\alpha_1, \alpha_2, \dots, \alpha_m)$ of $|V|$.
Therefore, the coefficient of $M_\alpha t^j$ in the expansion of $X_{\vec{G}}(\mathbf{x}, t)$ in the basis $\{M_\alpha t^j\}_{\alpha,j}$ is equal to the coefficient of $M_{\alpha^{\mathrm{rev}}} t^{|A|-j}$ in the expansion of $X_{\vec{G}}(\mathbf{x}, t)$, which is equal to the coefficient of $M_\alpha t^{|A|-j}$ in the expansion of $\rho(X_{\vec{G}}(\mathbf{x}, t))$:

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\alpha,j} c_{\alpha,j} M_\alpha t^j \implies \rho(X_{\vec{G}}(\mathbf{x}, t)) = \sum_{\alpha,j} c_{\alpha,j} M_\alpha t^{|A|-j} = \underbrace{\sum_{\alpha,j} c_{\alpha,j} M_\alpha t^{-j}}_{X_{\vec{G}}(\mathbf{x},t^{-1})} t^{|A|}.$$

Hence the desired result holds.

□

Corollary 5.2.5. Let $\vec{G}$ be a digraph on n vertices. Then

$$\rho(X_{\vec{G}}(\mathbf{x}, t)) = \sum_{k\in\kappa(\vec{G})} \mathbf{x}_k\, t^{\operatorname{des}(k)}.$$

Proof. By the previous theorem and Remark 5.1.3, we have

$$\rho(X_{\vec{G}}(\mathbf{x}, t)) = X_{\vec{G}}(\mathbf{x}, t^{-1})\, t^{|A|} = \sum_{k\in\kappa(\vec{G})} \mathbf{x}_k\, t^{-\operatorname{asc}(k)}\, t^{|A|} = \sum_{k\in\kappa(\vec{G})} \mathbf{x}_k\, t^{\operatorname{des}(k)}.$$

□

Proposition 5.2.6. Let $\vec{G} = (V, A)$ be a digraph. If $X_{\vec{G}}(\mathbf{x}, t) \in \mathrm{Sym}[t]$, then $X_{\vec{G}}(\mathbf{x}, t)$ is palindromic in t with center of symmetry $\frac{|A|}{2}$, in the sense that

$$X_{\vec{G}}(\mathbf{x}, t) = t^{|A|}\, X_{\vec{G}}(\mathbf{x}, t^{-1}).$$

Proof. Since $X_{\vec{G}}(\mathbf{x}, t)$ is symmetric, by the previous proposition, we have

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{k\in\kappa(\vec{G})} \mathbf{x}_k\, t^{\operatorname{des}(k)}.$$

Therefore, by Definition 5.1.4, we have

$$\sum_{k\in\kappa(\vec{G})} \mathbf{x}_k\, t^{\operatorname{asc}(k)} = X_{\vec{G}}(\mathbf{x}, t) = \sum_{k\in\kappa(\vec{G})} \mathbf{x}_k\, t^{\operatorname{des}(k)}.$$

So, by the Remark 5.1.3, we have palindromicity.

□

5.3 Expansion in Gessel's fundamental basis

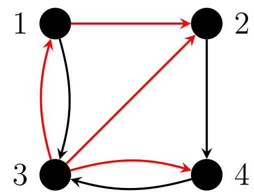
In this section, without loss of generality, we assume that the vertex set of a digraph $\vec{G}$ is $[n]$. The chosen labeling does not influence the chromatic quasisymmetric function.

Definition 5.3.1. Let $\vec{G} = ([n], A)$ be a digraph and let $\sigma \in S_n$. A **$\vec{G}$-inversion** of σ is an arc $uv \in A$ such that $\sigma^{-1}(u) > \sigma^{-1}(v)$.

Notation 5.3.2. Let $\mathrm{inv}_{\vec{G}}(\sigma)$ be the number of $\vec{G}$-inversions of σ:

$$\mathrm{inv}_{\vec{G}}(\sigma) := \Big|\{uv \in A \mid \sigma^{-1}(u) > \sigma^{-1}(v)\}\Big|.$$

Example 5.3.3. Let $\sigma = 2143 \in S_4$ and $\vec{G}$ be the following digraph. We color in red the arcs of $\vec{G}$ that are $\vec{G}$-inversions, that is, the arcs from a vertex i to a vertex j for which j precedes i in the permutation σ written in one-line notation.

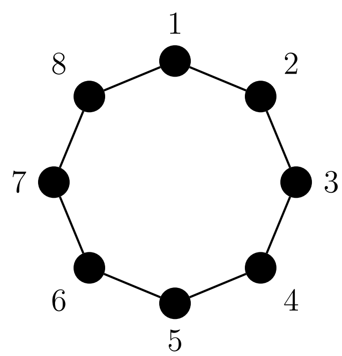


Then $\mathrm{inv}_{\vec{G}}(\sigma) = 4$.

Definition 5.3.4. Let $G = ([n], E)$ be a graph and let $\sigma = \sigma_1 \dots \sigma_n \in S_n$. The **$(G, \sigma)$-rank** of $v \in [n]$, denoted by $\mathrm{rank}_{(G,\sigma)}(v)$, is the length of the longest sequence $(\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k})$ of σ such that:

- $\sigma_{i_k} = v$.
- $\sigma_{i_j}\sigma_{i_{j+1}} \in E$, for each $1 \leq j < k$.

Example 5.3.5. Let G be the cycle C_8 on 8 vertices labeled with $[8]$ in cyclic order.



Let $\sigma = 25413786 \in S_8$. Then

- $\mathrm{rank}_{(G,\sigma)}(1) = 2 \quad (\sigma_1\sigma_4 = 21)$.
- $\mathrm{rank}_{(G,\sigma)}(2) = 1$.
- $\mathrm{rank}_{(G,\sigma)}(3) = 3 \quad (\sigma_2\sigma_3\sigma_5 = 543)$.
- $\mathrm{rank}_{(G,\sigma)}(4) = 2 \quad (\sigma_2\sigma_3 = 54)$.
- $\mathrm{rank}_{(G,\sigma)}(5) = 1$.
- $\mathrm{rank}_{(G,\sigma)}(6) = 2 \quad (\sigma_2\sigma_8 = 56 \text{ or } \sigma_6\sigma_8 = 76)$.
- $\mathrm{rank}_{(G,\sigma)}(7) = 1$.
- $\mathrm{rank}_{(G,\sigma)}(8) = 3 \quad (\sigma_1\sigma_4\sigma_7 = 218)$.

Definition 5.3.6. Let $G = ([n], E)$ be a graph, let $\sigma = \sigma_1 \dots \sigma_n \in S_n$ and $i \in \mathbb{P}$ such that $i < n$. We say that i is a ***G*-descent** of σ if either of the following conditions holds:

- $\mathrm{rank}_{(G,\sigma)}(\sigma_i) > \mathrm{rank}_{(G,\sigma)}(\sigma_{i+1})$.
- $\mathrm{rank}_{(G,\sigma)}(\sigma_i) = \mathrm{rank}_{(G,\sigma)}(\sigma_{i+1})$ and $\sigma_i > \sigma_{i+1}$.

Notation 5.3.7. The set of G-descents of σ is denoted by $\mathrm{DES}_G(\sigma)$.

Example 5.3.8. The set of G-descents of $\sigma = 25413786$ in Example 5.3.5 is

$$\mathrm{DES}_G(\sigma) = \{3, 5, 7\}$$

because

- $\mathrm{rank}_{(G,\sigma)}(\sigma_1) = \mathrm{rank}_{(G,\sigma)}(2) = 1 = 1 = \mathrm{rank}_{(G,\sigma)}(5) = \mathrm{rank}_{(G,\sigma)}(\sigma_2)$ but
$$\sigma_1 = 2 \not> 5 = \sigma_2.$$
- $\mathrm{rank}_{(G,\sigma)}(\sigma_2) = \mathrm{rank}_{(G,\sigma)}(5) = 1 < 2 = \mathrm{rank}_{(G,\sigma)}(4) = \mathrm{rank}_{(G,\sigma)}(\sigma_3)$.
- $\mathrm{rank}_{(G,\sigma)}(\sigma_3) = \mathrm{rank}_{(G,\sigma)}(4) = 2 = 2 = \mathrm{rank}_{(G,\sigma)}(1) = \mathrm{rank}_{(G,\sigma)}(\sigma_4)$ and
$$\sigma_3 = 4 > 1 = \sigma_4.$$
- $\mathrm{rank}_{(G,\sigma)}(\sigma_4) = \mathrm{rank}_{(G,\sigma)}(1) = 2 < 3 = \mathrm{rank}_{(G,\sigma)}(3) = \mathrm{rank}_{(G,\sigma)}(\sigma_5)$.
- $\mathrm{rank}_{(G,\sigma)}(\sigma_5) = \mathrm{rank}_{(G,\sigma)}(3) = 3 > 1 = \mathrm{rank}_{(G,\sigma)}(7) = \mathrm{rank}_{(G,\sigma)}(\sigma_6)$.
- $\mathrm{rank}_{(G,\sigma)}(\sigma_6) = \mathrm{rank}_{(G,\sigma)}(7) = 1 < 3 = \mathrm{rank}_{(G,\sigma)}(8) = \mathrm{rank}_{(G,\sigma)}(\sigma_7)$.
- $\mathrm{rank}_{(G,\sigma)}(\sigma_7) = \mathrm{rank}_{(G,\sigma)}(8) = 3 > 2 = \mathrm{rank}_{(G,\sigma)}(6) = \mathrm{rank}_{(G,\sigma)}(\sigma_8)$.

Theorem 5.3.9. Let $\vec{G} = (V, A)$ be a digraph. Then

$$\omega(X_{\vec{G}}(\mathbf{x}, t)) = \sum_{\sigma \in S_n} F_{n,\mathrm{DES}_G(\sigma)}\, t^{\mathrm{inv}_G(\sigma)}.$$

Hence $\omega(X_{\vec{G}}(\mathbf{x}, t))$ is F-positive.

Proof. Let us assume, without loss of generality, that the vertex set V of $\vec{G}$ is $[n]$. Let $\mathrm{AO}(G)$ be the set of all acyclic orientations of the underlying graph G. Let $\vec{G}_{\bar{a}} = ([n], A_{\bar{a}})$ be an acyclic orientation of the underlying graph G. Let $\mathrm{asc}(\vec{G}_{\bar{a}})$ be the number of arcs of $\vec{G}_{\bar{a}}$ that are also arcs of $\vec{G}$:

$$\mathrm{asc}(\vec{G}_{\bar{a}}) := \Big|A \cap A_{\bar{a}}\Big| = \Big|\{uv \in A \mid uv \in A_{\bar{a}}\}\Big|.$$

For each acyclic orientation $\vec{G}_{\bar{a}}$, we define $\mathcal{C}(\vec{G}_{\bar{a}})$ as the set of proper colorings k of G such that if uv is a directed edge of $\vec{G}_{\bar{a}}$ then $k(u) < k(v)$:

$$\mathcal{C}(\vec{G}_{\bar{a}}) := \{k \in \kappa(G) \mid uv \in A_{\bar{a}} \implies k(u) < k(v)\}.$$

Since $\vec{G} = ([n], A)$ and $\vec{G}_{\bar{a}} = ([n], A_{\bar{a}})$ have the same underlying graph G, then

$$uv \in A \implies uv \in A_{\bar{a}} \ \text{ or } \ vu \in A_{\bar{a}}\,.$$

Let $uv \in A$ and let $k \in \mathcal{C}(\vec{G}_{\bar{a}})$ so:

- If $k(u) < k(v)$, then $uv \in A_{\bar{a}}$, otherwise we would have $vu \in A_{\bar{a}}$ and hence $k(u) > k(v)$.
- If $k(u) > k(v)$, then $uv \notin A_{\bar{a}}$, otherwise we would have $uv \in A_{\bar{a}}$ and hence $k(u) < k(v)$.

Therefore, the number of ascents of a proper coloring k of $\vec{G}$ is equal to the number of arcs of $\vec{G}_{\bar{a}}$ that are also arcs of $\vec{G}$:

$$\mathrm{asc}(k) = \mathrm{asc}(\vec{G}_{\bar{a}}), \quad \text{for all } k \in \mathcal{C}(\vec{G}_{\bar{a}}).$$

Furthermore, by Proposition 1.2.31, for each proper coloring k there exists a unique compatible acyclic orientation $\vec{G}_{\bar{a}}$. In other words, for all $k \in \kappa(G)$, there exists a unique set $\mathcal{C}(\vec{G}_{\bar{a}})$ such that $k \in \mathcal{C}(\vec{G}_{\bar{a}})$. So, the set $\kappa(G)$ of all proper colorings of G is equal to the disjoint union of all $\mathcal{C}(\vec{G}_{\bar{a}})$:

$$\kappa(G) = \bigcupdot_{\vec{G}_{\bar{a}} \in \mathrm{AO}(G)} \mathcal{C}(\vec{G}_{\bar{a}}).$$

Therefore

$$X_{\vec{G}}(\mathbf{x}, t) := \sum_{k \in \kappa(G)} \mathbf{x}_k\, t^{\mathrm{asc}(k)} = \sum_{\vec{G}_{\bar{a}} \in \mathrm{AO}(G)} \left(t^{\mathrm{asc}(\vec{G}_{\bar{a}})} \sum_{k \in \mathcal{C}(\vec{G}_{\bar{a}})} \mathbf{x}_k \right). \tag{5.1}$$

From each acyclic orientation $\vec{G}_{\bar{a}} = ([n], A_{\bar{a}})$, we can define a strict poset $P_{\bar{a}} = ([n], <_{P_{\bar{a}}})$ by

$$ij \in A_{\bar{a}} \implies i <_{P_{\bar{a}}} j\,, \quad \text{for all } i, j \in [n],$$

and extending transitively.

For each strict poset $P = ([n], <_P)$ and for each bijection $w : [n] \longrightarrow [n]$, let

$$L(P, w) := \{\pi = \pi_1 \dots \pi_n \in S_n \mid x <_P y \implies \pi^{-1}(w(x)) < \pi^{-1}(w(y))\}.$$

For any subset $S \subseteq [n-1]$, let

$$n - S := \{i \in [n-1] \mid n - i \in S\}.$$

Let

$$\mathrm{DES}(\sigma) := \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}, \quad \text{for all } \sigma \in S_n.$$

Then, by [Sta99, Corollary 7.19.5], we have

$$\sum_{k \in \mathcal{C}(\vec{G}_{\bar{a}})} \mathbf{x}_k = \sum_{\sigma \in L(P_{\bar{a}}, w_{\bar{a}})} F_{n, n-\mathrm{DES}(\sigma)}\,, \tag{5.2}$$

where $w_{\bar{a}} : [n] \longrightarrow [n]$ is a bijection such that

$$x <_{P_{\bar{a}}} y \implies w_{\bar{a}}(x) > w_{\bar{a}}(y).$$

Let $e : [n] \longrightarrow [n]$ be the identity map on $[n]$. Then

$$L(P_{\bar{a}}, e) = \{\pi = \pi_1 \dots \pi_n \in S_n \mid x <_{P_{\bar{a}}} y \implies \pi^{-1}(x) < \pi^{-1}(y)\}.$$

Let $w : [n] \longrightarrow [n]$ be a bijection and $\sigma = \sigma_1 \dots \sigma_n \in S_n$, then we can define

$$w\,\sigma := w(\sigma_1) \dots w(\sigma_n) = w(\sigma(1)) \dots w(\sigma(n)) \in S_n$$

then

$$(w\,\sigma)^{-1} = \sigma^{-1}(w^{-1}(1)) \,\dots\, \sigma^{-1}(w^{-1}(n)) \in S_n\,.$$

Therefore, for $\sigma \in S_n$, we have

$$w_{\bar{a}}\,\sigma \in L(P_{\bar{a}}, w_{\bar{a}}) \iff \sigma \in L(P_{\bar{a}}, e).$$

So, equation (5.2) becomes

$$\sum_{k \in \mathcal{C}(\vec{G}_{\bar{a}})} \mathbf{x}_k = \sum_{\sigma \in L(P_{\bar{a}}, e)} F_{n, n-\mathrm{DES}(w_{\bar{a}}\sigma)}\,.$$

Therefore, equation (5.1) becomes

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\vec{G}_{\bar{a}} \in \mathrm{AO}(G)} \left(t^{\mathrm{asc}(\vec{G}_{\bar{a}})} \sum_{\sigma \in L(P_{\bar{a}}, e)} F_{n, n-\mathrm{DES}(w_{\bar{a}}\sigma)} \right). \tag{5.3}$$

Since each $\sigma \in S_n$ can be seen as a proper coloring, by assigning to each vertex $i \in [n]$ the color $\sigma^{-1}(i) \in [n]$, by Proposition 1.2.31, there exists a unique acyclic orientation $\vec{G}_{\bar{a}(\sigma)}$ such that

$$uv \in A_{\bar{a}(\sigma)} \iff \sigma^{-1}(u) < \sigma^{-1}(v).$$

Therefore, there exists a unique poset $P_{\bar{a}(\sigma)}$ such that $\sigma \in L(P_{\bar{a}(\sigma)}, e)$. So,

$$S_n = \bigcupdot_{\vec{G}_{\bar{a}} \in \mathrm{AO}(G)} L(P_{\bar{a}}, e).$$

Furthermore, for all $\sigma \in S_n$,

$$\mathrm{asc}(\vec{G}_{\bar{a}(\sigma)}) := \left|\{uv \in A \mid uv \in A_{\bar{a}(\sigma)}\}\right| = \left|\{uv \in A \mid \sigma^{-1}(v) > \sigma^{-1}(u)\}\right|,$$

so, by Proposition 3.1.9,

$$\mathrm{asc}(\vec{G}_{\bar{a}(\sigma)}) = \left|\{uv \in A \mid (\sigma^{\mathrm{rev}})^{-1}(u) > (\sigma^{\mathrm{rev}})^{-1}(v)\}\right| = \mathrm{inv}_{\vec{G}}(\sigma^{\mathrm{rev}}).$$

Therefore, equation (5.3) becomes

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\sigma \in S_n} t^{\mathrm{asc}(\vec{G}_{\bar{a}(\sigma)})} F_{n, n-\mathrm{DES}(w_{\bar{a}(\sigma)}\sigma)}, \tag{5.4}$$

where $w_{\bar{a}(\sigma)}$ is a bijection of $[n]$ such that

$$x <_{P_{\bar{a}(\sigma)}} y \implies w_{\bar{a}(\sigma)}(x) > w_{\bar{a}(\sigma)}(y).$$

Let

$$\mathrm{ASC}(\sigma) := \{i \in [n-1] \mid \sigma(i) < \sigma(i+1)\}, \quad \text{for all } \sigma \in S_n.$$

For all $\sigma \in S_n$, we have that

$$\begin{aligned} n - \mathrm{DES}(\sigma) &= \{i \in [n-1] \mid n - i \in \mathrm{DES}(\sigma)\} = \\ &= \{i \in [n-1] \mid \sigma(n-i) > \sigma(n-i+1)\} = \\ &= \{i \in [n-1] \mid \sigma^{\mathrm{rev}}(i+1) > \sigma^{\mathrm{rev}}(i)\} = \\ &= \mathrm{ASC}(\sigma^{\mathrm{rev}}). \end{aligned}$$

So, for all $\sigma \in S_n$,

$$n - \mathrm{DES}(w_{\bar{a}(\sigma)}\sigma) = \mathrm{ASC}((w_{\bar{a}(\sigma)}\sigma)^{\mathrm{rev}}).$$

So, equation (5.4) becomes

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\sigma \in S_n} F_{n, \mathrm{ASC}((w_{\bar{a}(\sigma)}\sigma)^{\mathrm{rev}})}\, t^{\mathrm{inv}_{\vec{G}}(\sigma^{\mathrm{rev}})}.$$

Then, by reversing σ, we have that

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\sigma \in S_n} F_{n, \mathrm{ASC}(\tilde{w}_{\bar{a}(\sigma)}\sigma)}\, t^{\mathrm{inv}_{\vec{G}}(\sigma)} \tag{5.5}$$

where $\tilde{w}_{\bar{a}(\sigma)}$ is a bijection of $[n]$ such that

$$x <_{P_{\bar{a}(\sigma)}} y \implies \tilde{w}_{\bar{a}}(x) < \tilde{w}_{\bar{a}}(y).$$

Applying the QSym-automorphism ω to both sides of equation (5.5) gives us

$$\omega(X_{\vec{G}}(\mathbf{x}, t)) = \sum_{\sigma \in S_n} F_{n, \mathrm{DES}(\tilde{w}_{\bar{a}(\sigma)}\sigma)}\, t^{\mathrm{inv}_{\vec{G}}(\sigma)}. \tag{5.6}$$

For each acyclic orientation $\vec{G}_{\bar{a}}$ and for each vertex $v \in [n]$, define $\mathrm{rank}_{\bar{a}}(v)$ as the length of the longest chain of $P_{\bar{a}}$ from a minimal element of $P_{\bar{a}}$ to v.
We want choose a specific bijection $\tilde{w}_{\bar{a}(\sigma)}$ To do this we follow the following steps:

- We label the smallest vertex $v \in [n]$, with $\mathrm{rank}_{\bar{a}(\sigma)}(v) = 0$, as 1.
- We label the next smallest vertex $v \in [n]$, with $\mathrm{rank}_{\bar{a}(\sigma)}(v) = 0$, as 2.
- We continue this process until all vertices with $\mathrm{rank}_{\bar{a}(\sigma)}$ equal to 0 are labeled.
- We repeat this process with the vertices with $\mathrm{rank}_{\bar{a}(\sigma)}$ equal to 1.
- We continue until all vertices are labeled.

For all $v \in [n]$, by definition of $P_{\bar{a}}$, we have

$$\mathrm{rank}_{(G,\sigma)}(v) = \mathrm{rank}_{\bar{a}(\sigma)}(v) + 1.$$

If $i \in \mathrm{DES}(\tilde{w}_{\bar{a}(\sigma)}\sigma)$, then

$$\tilde{w}_{\bar{a}(\sigma)}(\sigma(i)) > \tilde{w}_{\bar{a}(\sigma)}(\sigma(i+1)).$$

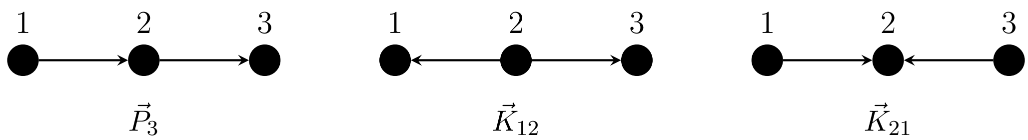
Since $\sigma(i+1)$ was labeled before $\sigma(i)$ in the labeling $\tilde{w}_{\bar{a}(\sigma)}$, by construction of $\tilde{w}_{\bar{a}(\sigma)}$, we have two possibilities:

- $\mathrm{rank}_{\bar{a}(\sigma)}(\sigma(i+1)) = \mathrm{rank}_{\bar{a}(\sigma)}(\sigma(i))$.
- They have the same $\mathrm{rank}_{\bar{a}(\sigma)}$ and $\sigma(i) > \sigma(i+1)$.

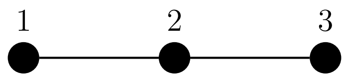
So, i is a G-descent of σ.
Similarly, it can be proved that if i is a G-descent of σ, then $i \in \mathrm{DES}(\tilde{w}_{\bar{a}(\sigma)}\sigma)$.
Hence $\mathrm{DES}(\tilde{w}_{\bar{a}(\sigma)}\sigma) = \mathrm{DES}_G(\sigma)$. Using this equivalence in equation (5.6), the theorem is proven.

□

Example 5.3.10. Let us give labelings to the digraphs $\vec{P}_3$, $\vec{K}_{12}$ and $\vec{K}_{21}$ as follows:



So, the labeled underlying graph is the same:



Then $\mathrm{DES}(\sigma)$ will be the same for each of $\vec{P}_3$, $\vec{K}_{12}$ and $\vec{K}_{21}$.
Let us calculate $\mathrm{DES}_G(\sigma)$ for each $\sigma \in S_3$.

σ	$\mathrm{rank}_{(G,\sigma)}(1)$	$\mathrm{rank}_{(G,\sigma)}(2)$	$\mathrm{rank}_{(G,\sigma)}(3)$
123	1	2	3
132	1	2	1
213	2	1	2
231	2	1	2
312	1	2	1
321	3	2	1

Therefore, we can calculate the sets of G-descents:

σ	$\mathrm{rank}_{(G,\sigma)}(\sigma_1)$	$\mathrm{rank}_{(G,\sigma)}(\sigma_2)$	$\mathrm{rank}_{(G,\sigma)}(\sigma_3)$	$\mathrm{DES}_G(\sigma)$
123	1	2	3	$\varnothing$
132	1	1	2	$\varnothing$
213	1	2	2	$\varnothing$
231	1	2	2	$\{2\}$
312	1	1	2	$\{1\}$
321	1	2	3	$\varnothing$

Let us calculate $\mathrm{inv}_{\vec{G}}(\sigma)$ for each digraphs and for each $\sigma \in S_3$.

σ	$\mathrm{inv}_{\vec{P}_3}(\sigma)$	$\mathrm{inv}_{\vec{K}_{12}}(\sigma)$	$\mathrm{inv}_{\vec{K}_{21}}(\sigma)$
123	0	1	1
132	1	2	0
213	1	0	2
231	1	0	2
312	1	2	0
321	2	1	1

In summary, we have

σ	$\mathrm{DES}_G(\sigma)$	$\mathrm{inv}_{\vec{P}_3}(\sigma)$	$\mathrm{inv}_{\vec{K}_{12}}(\sigma)$	$\mathrm{inv}_{\vec{K}_{21}}(\sigma)$
123	$\varnothing$	0	1	1
132	$\varnothing$	1	2	0
213	$\varnothing$	1	0	2
231	$\{2\}$	1	0	2
312	$\{1\}$	1	2	0
321	$\varnothing$	2	1	1

Using Theorem 5.3.9, we have

$$\begin{aligned}
\omega\left(X_{\vec{P}_3}(\mathbf{x},t)\right) &= (F_{3,\varnothing}) + (2F_{3,\varnothing} + F_{3,\{1\}} + F_{3,\{2\}})\, t + (F_{3,\varnothing})\, t^2. \\
\omega\left(X_{\vec{K}_{12}}(\mathbf{x},t)\right) &= (F_{3,\varnothing} + F_{3,\{2\}}) + (2F_{3,\varnothing})\, t + (F_{3,\varnothing} + F_{3,\{1\}})\, t^2. \\
\omega\left(X_{\vec{K}_{21}}(\mathbf{x},t)\right) &= (F_{3,\varnothing} + F_{3,\{1\}}) + (2F_{3,\varnothing})\, t + (F_{3,\varnothing} + F_{3,\{2\}})\, t^2.
\end{aligned}$$

By Proposition 4.3.3, we have

$$\begin{aligned}
X_{\vec{K}_{12}}(\mathbf{x},t) &= (F_{3,\{1,2\}} + F_{3,\{1\}}) + (2F_{3,\{1,2\}})\, t + (F_{3,\{1,2\}} + F_{3,\{2\}})\, t^2. \\
X_{\vec{K}_{21}}(\mathbf{x},t) &= (F_{3,\{1,2\}} + F_{3,\{2\}}) + (2F_{3,\{1,2\}})\, t + (F_{3,\{1,2\}} + F_{3,\{1\}})\, t^2.
\end{aligned}$$

By Proposition 5.2.6, $X_{\vec{K}_{12}}(\mathbf{x},t)$ and $X_{\vec{K}_{12}}(\mathbf{x},t)$ do not have symmetric functions as coefficients. Instead, $X_{\vec{P}_3}(\mathbf{x},t)$ has symmetric functions as coefficients:

$$\begin{aligned}
\omega\left(X_{\vec{P}_3}(\mathbf{x},t)\right) &= (F_{3,\varnothing}) + (2F_{3,\varnothing} + F_{3,\{1\}} + F_{3,\{2\}})\, t + (F_{3,\varnothing})\, t^2 = \\
&= h_3 + (2h_3 + F_{3,\{1\}} + F_{3,\{2\}})\, t + h_3\, t^2
\end{aligned}$$

using what we saw in Examples 4.2.12 and 4.2.13,

$$\begin{aligned}\omega\left(X_{\vec{P}_3}(\mathbf{x},t)\right) &= h_3 + (2h_3 + F_{3,\{1\}} + F_{3,\{2\}})\,t + h_3\,t^2 = \\ &= h_3 + (2h_3 + 2M_{(1,1,1)} + M_{(1,2)} + M_{(2,1)})\,t + h_3\,t^2\end{aligned}$$

by Proposition 3.3.20 and 4.2.4,

$$\begin{aligned}\omega\left(X_{\vec{P}_3}(\mathbf{x},t)\right) &= h_3 + (2h_3 + 2M_{(1,1,1)} + M_{(1,2)} + M_{(2,1)})\,t + h_3\,t^2 = \\ &= h_3 + (2h_3 + 2m_{(1,1,1)} + m_{(2,1)})\,t + h_3\,t^2 = \\ &= h_3 + (h_3 + h_{(2,1)})\,t + h_3\,t^2\end{aligned}$$

So, by Proposition 4.3.5, we have $X_{\vec{P}_3}(\mathbf{x},t) \in \mathrm{Sym}^3[t]$.

Chapter 6

Conjectures

In this Chapter we introduce the main conjectures about chromatic quasisymmetric functions.
In this Chapter we consider only loop-free graphs and digraphs.

6.1 Stanley-Stembridge (1995)

The Stanley-Stembridge is one of the most known conjecture in algebraic combinatorics. The conjecture was stated by Richard Stanley in 1995 [Sta95, Conjecture 5.1], but takes its name from an equivalent conjecture stated by Richard Stanley and John Stembridge in 1993 [SS93, Conjecture 5.5].
In 2024, Tatsuyuki Hikita posted a proof of the conjecture on arXiv [Hik24].

6.1.1 Incomparability graphs

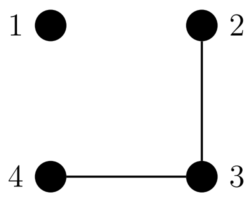
Definition 6.1.1. Let $(P, \prec)$ be a strict poset. The **incomparability graph** of $(P, \prec)$ is the graph $\mathrm{inc}(P) = (V, E)$, where:

- $V = P$.
- $E = \{(v, w) \in P^2 \mid\ v \text{ and } w \text{ are incomparable for } \prec\}$.

Example 6.1.2. Let $(\{1, 2, 3, 4\}, \mid)$ be the strict poset (let us consider $x \nmid x$). Then

- $1 \mid 2$, $1 \mid 3$ and $1 \mid 4$.
- $2 \nmid 3$ and $3 \nmid 2$.
- $2 \mid 4$.
- $3 \nmid 4$ and $4 \nmid 3$.

The incomparability graph of $(\{1, 2, 3, 4\}, \mid)$ is the following:



Using the incomparability graphs, we have a characterization of $(\mathbf{3}+\mathbf{1})$-free posets.

Remark 6.1.3. A poset is $(\mathbf{3}+\mathbf{1})$-free if and only if his incomparability graph does not contain a sub-graph isomorphic to $K_{1,3}$.

6.1.2 Positivity and unimodality

Definition 6.1.4. Let b be a basis of Sym. A symmetric function $f(\mathbf{x}) \in \mathrm{Sym}$ is **b-positive** if the expansion of f in terms of the b basis has non-negative coefficients.

Definition 6.1.5. Let b be a basis of Sym and let

$$f(\mathbf{x},t) = \sum_{j=0}^{n} a_j(\mathbf{x})\, t^j \in \mathrm{Sym}[t].$$

Then

- $f(\mathbf{x},t)$ is **b-positive** if all $a_j(\mathbf{x})$ are b-positive.
- $f(\mathbf{x},t)$ is **b-unimodal** if there exists $k \in \{0,\dots,n\}$ such that:
  1. $a_{j+1}(\mathbf{x}) - a_j(\mathbf{x})$ is b-positive for all $j \in \{0,\dots,k-1\}$.
  2. $a_j(\mathbf{x}) - a_{j+1}(\mathbf{x})$ is b-positive for all $j \in \{k,\dots,n-1\}$.

Remark 6.1.6. Let b be a basis of Sym and let

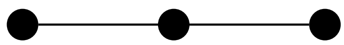
$$f(\mathbf{x},t) = \sum_{j=0}^{n} a_j(\mathbf{x})\, t^j \in \mathrm{Sym}[t]$$

be palindrome, then: $f(\mathbf{x},t)$ is b-unimodal if and only if $a_{j+1}(\mathbf{x}) - a_j(\mathbf{x})$ is b-positive for all $j \in \{1,\dots,\frac{|E|-1}{2}\}$.

6.1.3 Stanley-Stembridge conjecture

Conjecture 6.1.7 (Stanley-Stembridge [Sta95]). Let G be a incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset. Then $X_G(\mathbf{x})$ is e-positive.

Example 6.1.8. Let P be the poset with the following incomparability graph G:



By Remark 6.1.3, the poset P is $(\mathbf{3}+\mathbf{1})$-free. We observe that:

- There are 6 different ways to assign 3 distinct colors to vertices.
- There are 2 different ways to assign 2 distinct colors.
- There are no proper colorings with 1 color.

Then the chromatic symmetric function of G is

$$\begin{aligned} X_G(\mathbf{x}) &= 6x_1x_2x_3 + 6x_1x_2x_4 + \cdots + 2x_1^2x_2 + 2x_1^2x_3 + \cdots = \\ &= 6m_{(1,1,1)} + 2m_{(2,1)}. \end{aligned}$$

By Proposition 3.3.11, we have that

$$e_{(2,1)} = m_{(2,1)} + 3m_{(1,1,1)}.$$

Therefore, the chromatic symmetric function of G is e-positive:

$$X_G(\mathbf{x}) = 6m_{(1,1,1)} + 2m_{(2,1)} = 2e_{(2,1)}.$$

6.2 Shareshian-Wachs (2012)

In 2012, Shareshian-Wachs generalized the Stanley-Stembridge conjecture [SW12].

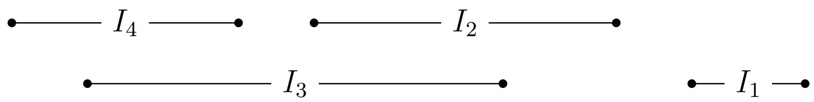
6.2.1 Natural interval unit orders

Definition 6.2.1. An **interval order** is a strict poset $(P, \prec)$ isomorphic to a set $\mathcal{I}$ of closed intervals of $\mathbb{R}$ partially ordered with

$$[a, a'] \prec_{\mathcal{I}} [b, b'] \iff a' < b.$$

Definition 6.2.2. A **proper interval order** is an interval order in which no interval properly contains another. A **proper interval graph** is the incomparability graph of a proper interval order.

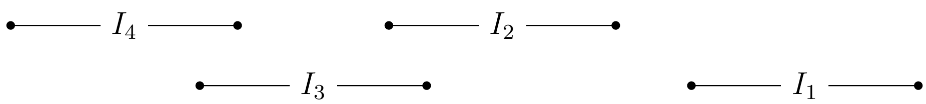
Example 6.2.3. The poset $P = ([4], |\,)$ of Example 6.1.2 is a proper interval order isomorphic to $\mathcal{I} = \{I_1, I_2, I_3, I_4\}$ defined as follows



by the isomorphism $\phi(i) = I_i$ for all $i \in \{1, 2, 3, 4\}$.

Definition 6.2.4. A **unit interval order** (or **semi-order**) is an interval order in which all closed intervals have length one. A **unit interval graph** is the incomparability graph of a unit interval order.

Example 6.2.5. The poset $P = ([4], |\,)$ of Example 6.1.2 is a unit interval order isomorphic to $\mathcal{I} = \{I_1, I_2, I_3, I_4\}$ defined as follow



The fact that the poset of previous examples is both a proper interval order and a unit interval order is no coincidence.

Proposition 6.2.6. Let $P = ([n], \prec)$ be a strict poset. The following statements are equivalent:

1. P is a proper interval order.

2. P is a unit interval order.

3. P is a $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free poset.

Proof. The equivalence of 1. and 2. was shown by Fred S. Roberts [Rob69]. The equivalence of 2. and 3. was shown by Dana Scott and Patrick Suppers [SS58].

□

Proposition 6.2.7. Let $P = ([n], \prec)$ be a finite strict poset. Then, P is a unit interval order if and only if there exist n real numbers $y_1 < \cdots < y_n$ such that

$$y_i + 1 < y_j \iff i \prec j\,, \quad \text{for all } i, j \in [n].$$

Proof. Let P be a unit interval order with n elements. Then P is isomorphic to a set of n unit closed real intervals $\mathcal{I} = \{[y_1, y_1+1], \ldots, [y_n, y_n+1]\}$ partially ordered with

$$[y_i, y_i+1] \prec_{\mathcal{I}} [y_j, y_j+1] \iff y_i + 1 < y_j\,.$$

Let us assume, without loss of generality, that $y_1 < \cdots < y_n$.
Similarly, proceeding backwards, the converse is demonstrated.

□

Definition 6.2.8. A **natural unit interval order** is a strict poset $P = ([n], \prec)$ such that:

1. $x \prec y$ implies $x < y$.
2. For each induced sub-poset $\{x, y, z\}$ of P, with $x \prec z$ and y incomparable to x and z, we have that $x < y < z$.

A **natural unit interval graph** is the incomparability graph of a natural unit interval order.

Proposition 6.2.9. Let $P = ([n], \prec)$ be a strict poset. The following statements are equivalent:

1. P is a unit interval order.
2. P is a natural unit interval order.

Proof. [SW16, Theorem 4.1]

□

Example 6.2.10. The poset $P = ([4], \mid)$ of Example 6.1.2 is a natural unit interval order because:

1. For all $x, y \in [4]$, $x \mid y$ implies $x < y$.
2. The only induced sub-poset $\{x, y, z\}$ of P, with $x \prec z$ and y incomparable to x and z, is $\{2, 3, 4\}$.

Corollary 6.2.11. Let $G = ([n], E)$ be a labeled graph. The following statements are equivalent:

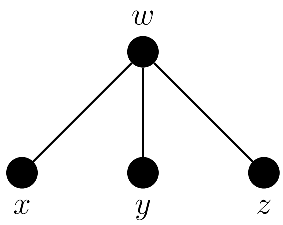
1. G is a proper interval graph.
2. G is a unit interval graph.
3. G is a natural unit interval graph.

Proof. By Propositions 6.2.6 and 6.2.9 and by Definitions 6.2.2, 6.2.4 and 6.2.8, the statements are equivalent.

□

Lemma 6.2.12. Let G be a natural unit interval graph and let k be a proper coloring of G. For each color $a \in \mathbb{P}$, define $G_{k,a}$ as the induced sub-graph of G of all vertices colored by a or $a+1$. Then each connected component of $G_{k,a}$ is a path $(i_1, i_2, \dots i_n)$, with $i_1 < \dots < i_n$.

Proof. Since $G_{k,a}$ is composed of vertices of two colors, $G_{k,a}$ does not contain cycles of odd length, and consequently does not contain cycles of length 3. By definition of natural unit interval graph, if xy and yz are edges in G and xz is not an edge, then either $x < y < z$ or $z < y < x$. Consequentially, if $(i_1, i_2, \dots, i_n)$ is a path of $G_{k,a}$ then either $i_1 < i_2 < \dots < i_n$ or $i_1 > i_2 > \dots > i_n$, which implies that $G_{k,a}$ does not contain cycles. If we prove that each vertex has at most two adjacent vertices we are finished. Suppose, by contradiction, that x, y, z are all adjacent to w in $G_{k,a}$.



Suppose, without loss of generality, that $x < w$. As seen before

- Since (x, w, y) is a path in $G_{k,a}$, then $x < w < y$.
- Since (x, w, z) is a path in $G_{k,a}$, then $x < w < z$.
- Since (y, w, z) is a path in $G_{k,a}$, then $y < w < z$.

Then $y < w < y$, but this is a contradiction. So there are no vertices with more than two adjacent vertices.

□

Theorem 6.2.13. Let G be a natural unit interval graph. Then

$$X_G(\mathbf{x}, t) \in \mathrm{Sym}[t].$$

Proof. For each proper coloring k of G, let $G_{k,a}$ be the induced sub-graph of G containing only the vertices colored by a and $a+1$. By Lemma 6.2.12, each connected component of $G_{k,a}$ is a path $(i_1, i_2, \dots, i_n)$ with $i_1 < \dots < i_n$.
For each $a \in \mathbb{P}$, we define a function $\phi_a : \kappa(G) \longrightarrow \kappa(G)$ such that $\phi_a(k)$ is the coloring of $\vec{G}$ obtained from k by:

- Replacing each color a by $a+1$ and each $a+1$ by a in the connected components $(i_1, i_2, \dots, i_n)$ of $G_{k,a}$ if n is odd.
- Leaving unchanged the connected components $(i_1, i_2, \dots, i_n)$ of $G_{k,a}$ if n is even.

By definition, the function ϕ_a is an involution.
The number of occurrences of the color a in $\phi_a(k)$ is equal to the number of occurrences of $a+1$ in k, and the number of occurrences of the color $a+1$ in $\phi_a(k)$ is equal to the number of occurrences of a in k.
Since $G_{k,a}$ contains only two colors, in a path of $i_1 < \cdots < i_n$, with an odd number of vertices, exactly half of the edges are ascents of k. Hence, all ascents of k are descents of $\phi_a(k)$ and all descents of k are ascents of $\phi_a(k)$. So $\mathrm{asc}(k) = \mathrm{asc}(\phi_a(k))$.
Therefore, for each $a \in \mathbb{P}$, the function ϕ_a is an involution that switches the number of vertices with color a and the number of vertices with color $a+1$, leaves the number of vertices of all other colors the same, and does not change the number of ascents of the coloring. Hence all coefficients of $X_G(\mathbf{x}, t)$ are symmetric functions.

□

6.2.2 Shareshian-Wachs conjecture

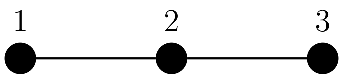
Conjecture 6.2.14 (Shareshian-Wachs [SW12])**.** Let G be a natural unit interval graph. Then $X_G(\mathbf{x}, t)$ is e-positive and e-unimodal.

An overview of e-positivity results is [Ban+24, Table 1].
By Remark 5.1.15, we can equivalently rewrite the conjecture using digraphs.

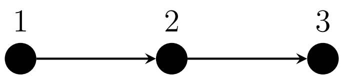
Definition 6.2.15. A **natural unit interval digraph** is an orientation of a natural unit interval graph $G = ([n], E)$, constructed by orienting each edge from the smaller vertex to the larger vertex.

Conjecture 6.2.16 (Shareshian-Wachs, natural unit interval digraph version)**.** Let $\vec{G}$ be a natural unit interval digraph. Then $X_{\vec{G}}(\mathbf{x}, t)$ is e-positive and e-unimodal.

Example 6.2.17. Let $P = ([3], \prec)$ be the poset whose only order relation is $1 \prec 3$. Then P is a natural unit interval order and his natural unit interval graph is



By Remark 5.1.15, the chromatic quasisymmetric function of the natural unit interval graph is the same of $\vec{P}_3$:



So, as seen in Example 5.3.10,

$$\omega\left(X_{\vec{P}_3}(\mathbf{x}, t)\right) = h_3 + (h_3 + h_{(2,1)})\, t + h_3\, t^2.$$

By Remark 4.3.4,

$$X_{\vec{P}_3}(\mathbf{x}, t) = e_3 + (e_3 + e_{(2,1)})\, t + e_3\, t^2.$$

So, the natural unit interval graph of P is e-positive and e-unimodal.

6.2.3 Shareshian-Wachs implies Stanley-Stembridge

We now show that the Shareshian-Wachs conjecture generalizes the Stanley-Stembridge conjecture.

Theorem 6.2.18. If every $(\mathbf{3}+\mathbf{1})$- and $(\mathbf{2}+\mathbf{2})$-free poset is e-positive, then every $(\mathbf{3}+\mathbf{1})$-free poset is e-positive.

Proof. [Gua13, Theorem 5.1] □

Corollary 6.2.19. The Shareshian-Wachs conjecture implies the Stanley-Stembridge conjecture.

Proof. By Proposition 6.2.9, the class of natural unit interval orders is equivalent to the class of unit interval orders. By Proposition 6.2.6, the class of unit interval orders is equivalent to the class of $(\mathbf{3}+\mathbf{1})$- and $(\mathbf{2}+\mathbf{2})$-free posets. Hence, by Theorem 6.2.18, the Shareshian-Wachs conjecture implies the Stanley-Stembridge conjecture. □

6.3 Ellzey (2017)

In 2017, Ellzey generalized the Shareshian-Wachs conjecture [Ell17].

6.3.1 Circular indifference digraph

Definition 6.3.1. Let $a, b, n \in \mathbb{P}$. The **circular interval** $[a, b]$ of $[n]$ is the set defined as

$$[a,b] := \begin{cases} \{a, a+1, a+2, \ldots, b\} & \text{if } a \leq b \\ \{a, a+1, a+2, \ldots, n, 1, 2, \ldots, b\} & \text{if } a > b \end{cases}$$

Example 6.3.2. If $n = 6$, then

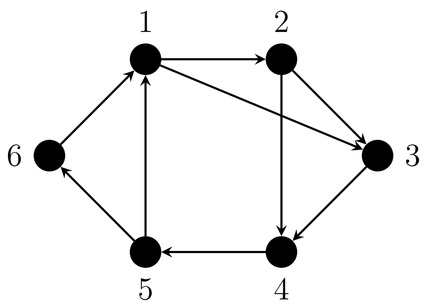
$$[1,4] = \{1,2,3,4\},$$
$$[4,1] = \{1,4,5,6\}.$$

Definition 6.3.3. A **circular indifference digraph** is a digraph $\vec{G} = (V, A)$ in which there exists a set $\mathcal{I}$ of circular intervals of $[n]$ such that

$$A = \{(i,j) \in V^2 \mid i \neq j \text{ and } [i,j] \text{ is contained in a circular interval of } \mathcal{I}\}.$$

A **circular indifference graph** is the underlying graph of a circular indifference digraph.

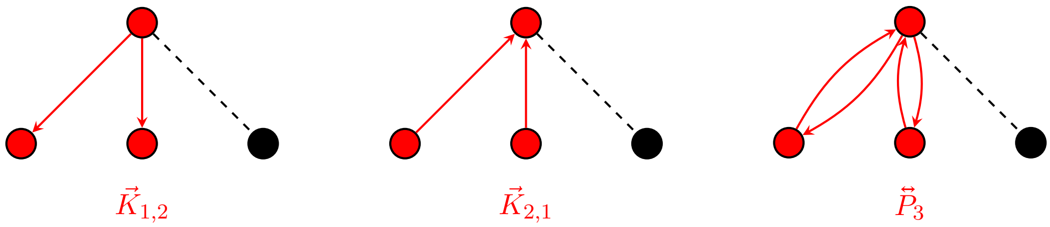
Example 6.3.4. The following digraph $\vec{G} = ([6], A)$ is a circular indifference digraph with $\mathcal{I} = \{[1,3],[2,4],[4,5],[5,1]\}$.



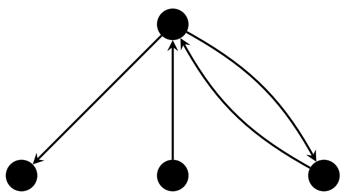
Let us repeat the work done in the previous section, generalized to digraphs. To do this, we will use the digraphs introduced in Subsection 1.2.4.

Lemma 6.3.5. Let $\vec{G}$ be a digraph that has no induced sub-digraphs isomorphic to $\vec{K}_{1,2}$, $\vec{K}_{2,1}$, $\overleftrightarrow{K}_{1,2}$, $\overleftrightarrow{K}_{2,1}$ or $\overleftrightarrow{P}_3$. Then the underlying graph G does not contain a sub-graph isomorphic to $K_{1,3}$.

Proof. Let $\vec{H}$ be an induced sub-digraph of $\vec{G}$ whose underlying graph is $K_{1,3}$. We observe that the degree-three vertex in $K_{1,3}$ cannot be connected to two other vertices by the same type of arcs in $\vec{H}$, otherwise $\vec{G}$ would have an induced sub-digraph isomorphic to $\vec{K}_{1,2}$, $\vec{K}_{2,1}$ or $\overleftrightarrow{P}_3$.



Then $\vec{H}$ must be the following digraph:

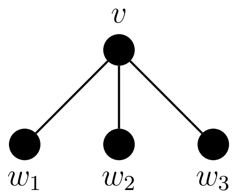


But this digraph contains a sub-digraph isomorphic to $\overleftrightarrow{K}_{1,2}$ and a sub-digraph isomorphic to $\overleftrightarrow{K}_{2,1}$. Therefore, any digraph that contains $K_{1,3}$ as sub-graph must contain a forbidden sub-digraph.

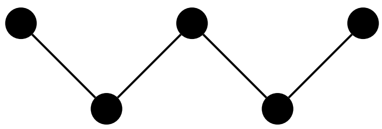
□

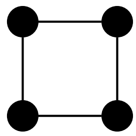
Lemma 6.3.6. Let $\vec{G}$ be a digraph that has no induced sub-digraphs isomorphic to $\vec{K}_{1,2}$, $\vec{K}_{2,1}$, $\overleftrightarrow{K}_{1,2}$, $\overleftrightarrow{K}_{2,1}$ or $\overleftrightarrow{P}_3$. Let k be a proper coloring of $\vec{G}$. For each color $a \in \mathbb{P}$, define $\vec{G}_{k,a}$ as the induced sub-digraph of $\vec{G}$ of all vertices colored by a or $a+1$. Then each connected component of $\vec{G}_{k,a}$ is either a directed cycle with an even number of vertices or a directed path.

Proof. Since the underlying graph $G_{k,a}$ is constructed from only two colors of $\vec{G}$, then $G_{k,a}$ is a 2-colorable graph. Then, by Proposition 1.1.35, the graph $G_{k,a}$ cannot have any cycles of odd length. Furthermore, $G_{k,a}$ cannot have any vertex adjacent to more than two other vertices. Indeed, suppose vertex v were adjacent to vertices w_1, w_2, and w_3 in $G_{k,a}$.

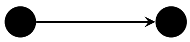


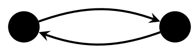
Since $G_{k,a}$ has no 3-cycles, w_1, w_2, and w_3 have no edges between them. Then $G_{k,a}$ contains a sub-graph isomorphic to $K_{1,3}$, but this contradicts Lemma 6.3.5. Since every vertex has degree at most 2, every connected component of $G_{k,a}$ must be either a path or a cycle (of even length, as seen before).



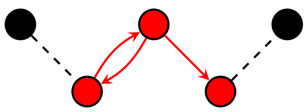


Note that every digraph with 2 vertices is a directed path or cycle of length 2.

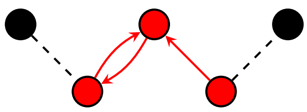




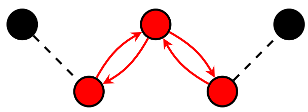
We observe that we cannot have two vertices with both arcs in a path with 3 or more vertices.



$\overleftrightarrow{K}_{1,2}$

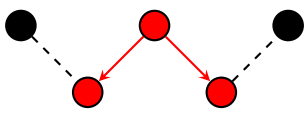


$\overleftrightarrow{K}_{2,1}$

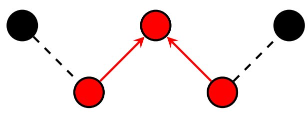


$\overleftrightarrow{P}_3$

And we cannot have two arcs from the same vertex or two arcs to the same vertex.



$\vec{K}_{1,2}$



$\vec{K}_{2,1}$

Then $\vec{G}_{k,a}$ can be a directed path or directed cycle with an even number of vertices. □

Theorem 6.3.7. Let $\vec{G}$ be a digraph that has no induced sub-digraphs isomorphic to $\vec{K}_{1,2}$, $\vec{K}_{2,1}$, $\overleftrightarrow{K}_{1,2}$, $\overleftrightarrow{K}_{2,1}$ or $\overleftrightarrow{P}_3$. Then $X_{\vec{G}}(\mathbf{x}, t) \in \mathrm{Sym}[t]$.

Proof. By Proposition 5.1.12, we can assume without loss of generality that $\vec{G}$ is connected. For each proper coloring k of $\vec{G}$, let $\vec{G}_{k,a}$ be the induced sub-digraph of $\vec{G}$ containing only the vertices colored by a and $a+1$. By Lemma 6.3.6, each connected component of $\vec{G}_{k,a}$ is a directed path or a directed cycle of even length.
For each $a \in \mathbb{P}$, we define a function $\phi_a : \kappa(G) \longrightarrow \kappa(G)$ such that $\phi_a(k)$ is the coloring of $\vec{G}$ obtained from k by:

- Replacing each occurrence of a with $a+1$ and replacing each $a+1$ with a in the connected components of $\vec{G}_{k,a}$ that are directed paths with an odd number of vertices.

- For the other connected components of $\vec{G}_{k,a}$ (directed paths with an even number of vertices and directed cycles with an even number of vertices), the colors of $\phi_a(k)$ are the same of k.

By definition, the function ϕ_a is an involution.
Since $\vec{G}_{k,a}$ contain only two colors, in a directed path with an odd number of vertices in $\vec{G}_{k,a}$, exactly half of the arcs are ascents of k. Hence, if we invert the color a with the color $a+1$ and vice versa, we will change all ascents to descents and vice versa, but the number of ascents is preserved.
Therefore, for each $a \in \mathbb{P}$, the function ϕ_a is an involution that switches the number of vertices with color a and the number of vertices with color $a+1$, leaves the number of vertices of all other colors the same, and does not change the number of ascents of the coloring. Hence all coefficients of $X_{\vec{G}}(\mathbf{x}, t)$ are symmetric functions.

□

Lemma 6.3.8. Circular indifference digraphs do not have any induced sub-digraphs isomorphic to $\vec{K}_{1,2}$, $\vec{K}_{2,1}$, $\overleftrightarrow{K}_{1,2}$, $\overleftrightarrow{K}_{2,1}$ or $\overleftrightarrow{P}_3$.

Proof. Let $\vec{G}$ be a circular indifference digraph with set $\mathcal{I}$ of circular intervals on $[n]$. Suppose $\vec{G}$ contains an induced sub-digraph $\vec{H}$ isomorphic to $\vec{K}_{1,2}$ with vertex set $\{a, b, c\}$ and arc set $\{ba, bc\}$. Then the circular intervals $[b, a]$ and $[b, c]$ are both contained in circular intervals of $\mathcal{I}$. Then there are two options:

- $[b, a] \subset [b, c]$ and hence $[a, c] \subset [b, c]$, therefore $[a, c]$ is contained in a circular interval of $\mathcal{I}$,

- $[b, c] \subset [b, a]$ and hence $[c, a] \subset [b, a]$, therefore $[c, a]$ is contained in a circular interval of $\mathcal{I}$.

Either way there exists an edge between a and c in $\vec{G}$, which is a contradiction.
Similarly, it is shown that $\vec{G}$ does not contain any induced sub-digraphs isomorphic to $\vec{K}_{2,1}$, $\overleftrightarrow{K}_{1,2}$, $\overleftrightarrow{K}_{2,1}$ or $\overleftrightarrow{P}_3$.

□

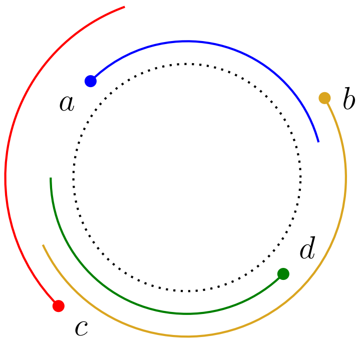
Corollary 6.3.9. Let $\vec{G}$ be a digraph such that all connected components of $\vec{G}$ are circular indifference digraphs. Then $X_{\vec{G}}(\mathbf{x}, t)$ is symmetric.

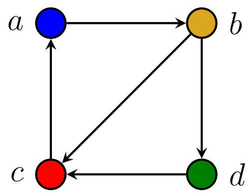
6.3.2 Circular-arc digraphs

Definition 6.3.10. Let $\{A_1, \ldots, A_n\}$ be a finite collection of circular arcs. A **circular-arc digraph** is a digraph constructed by assigning a vertex v_i to each arc A_i and adding an arc from v_i to v_j if the starting point (clockwise) of A_j is contained in A_i. A **circular-arc graph** is the underlying graph of a circular-arc digraph.

Definition 6.3.11. A **proper circular-arc digraph** is a circular-arc digraph in which no circular arc properly contains another.

Example 6.3.12. Let $\mathcal{A}$ be the following collection of proper circular arcs and let $\vec{G}$ be the corresponding proper circular arc digraph.





$\vec{G}$

Theorem 6.3.13. Let $\vec{G}$ be a connected digraph. The following statements are equivalent:

1. $\vec{G}$ is a proper circular-arc digraph.

2. $\vec{G}$ is a circular indifference digraph.

Proof. Let $\vec{G}$ be a proper circular-arc digraph with $\{A_1, \ldots, A_n\}$ as circular arcs set. We label the circular arcs as follows:

- Label a random circular arc as 1.

- Find the first circular arc that begins clockwise after circular arc 1 and label this as 2.

- Find the first circular arc that begins clockwise after circular arc 2 and Label this as 3.

- Continue until all n circular arcs are labeled.

Let $\mathcal{I}$ be the set of all circular interval $[i, j]$ such that circular arc i contains the starting point of circular arc j. If circular arc i contains the starting point of circular arc j, then it must also contain the starting point of all arcs in $[i, j]$. Hence, the proper circular-arc digraph is a circular indifference digraph with $\mathcal{I}$.

Let $\vec{G}$ be a circular indifference digraph with $\mathcal{I} = \{[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]\}$ as set of circular intervals of $[n]$. Let $\mathcal{I}' = \{[1, c_1], [2, c_2], \ldots, [n, c_n]\}$ be the set of circular intervals of $[n]$ such that $[i, c_i]$ is the largest circular interval of the form $[i, x]$ that is contained in an circular interval of $\mathcal{I}$.
By definition, each circular interval $[i, c_i] \in \mathcal{I}'$ is contained in a circular interval of $\mathcal{I}$. Vice versa, each circular interval $[a_i, b_i] \in \mathcal{I}$ is equal or is contained in the circular interval $[a_i, c_{a_i}]$ of $\mathcal{I}'$. So $\mathcal{I}$ and $\mathcal{I}'$ are both associated to $\vec{G}$.
Let us define a set of n circular arcs as follow:

- Draw a circle and place n points around the circle.
- Label these points in clockwise order from 1 to n.
- For each circular interval $[i, c_i] \in \mathcal{I}'$, place a circular arc A_i on the circle from slightly before i to slightly after c_i.
- Continue this process with all the circular intervals of $\mathcal{I}'$.

Since the circular arc A_i contain the starting points of all the arcs corresponding to the vertices in $[i, c_i]$, we have that ij is an arc of the proper circular-arc digraph for each $i < j \leq c_i$.
Hence, $\vec{G}$ is a proper circular-arc digraph with $\{A_1, \ldots, A_n\}$ as set of circular arcs.
□

Corollary 6.3.14. Let G be a simple connected graph. The following statements are equivalent:

1. G is a proper circular-arc graph.
2. G is a circular indifference graph.

Proof. A proper circular-arc graph is the underlying graph of a proper circular-arc digraph. By previous theorem, a proper circular-arc digraph is a circular indifference digraph and a circular indifference graph is the underlying graph of a circular indifference digraph.
□

For more information on the relationships between the various classes of graphs see [Skr82].

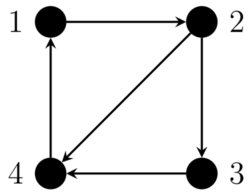
6.3.3 Ellzey conjecture

Conjecture 6.3.15 (Ellzey [Ell17])**.** Let $\vec{G}$ be a circular indifference digraph. Then $X_{\vec{G}}(\mathbf{x}, t)$ is e-positive and e-unimodal.

As seen in the previous section, the conjecture can be expressed equivalently in terms of proper circular-arc digraphs instead of circular indifference digraphs.

Conjecture 6.3.16 (Ellzey, circular-arc digraphs version)**.** Let $\vec{G}$ be a proper circular-arc digraph. Then $X_{\vec{G}}(\mathbf{x}, t)$ is e-positive and e-unimodal.

Example 6.3.17. Let $\vec{G}$ be the following digraph.



Note that it is the circular-arc digraph of Example 6.3.12. It is a circular indifference digraph with $\mathcal{I} = \{[1,2],[2,4],[4,1]\}$.
Let us calculate $\mathrm{DES}_G(\sigma)$ and $\mathrm{inv}_{\vec{G}}(\sigma)$, for each $\sigma \in S_4$.

σ	$\mathrm{DES}_G(\sigma)$	$\mathrm{inv}_{\vec{G}}(\sigma)$
1234	$\varnothing$	2
1243	$\varnothing$	3
1324	$\varnothing$	3
1342	$\varnothing$	2
1423	$\varnothing$	2
1432	$\varnothing$	3
2134	$\varnothing$	3
2143	$\varnothing$	4
2314	$\{2\}$	3
2341	$\varnothing$	2
2413	$\varnothing$	3
2431	$\{3\}$	3

σ	$\mathrm{DES}_G(\sigma)$	$\mathrm{inv}_{\vec{G}}(\sigma)$
3124	$\{1\}$	3
3142	$\{1\}$	2
3214	$\varnothing$	4
3241	$\varnothing$	3
3412	$\varnothing$	1
3421	$\varnothing$	2
4123	$\varnothing$	1
4132	$\varnothing$	2
4213	$\varnothing$	2
4231	$\{3\}$	2
4312	$\{2\}$	2
4321	$\varnothing$	3

By Theorem 5.3.9, the chromatic quasisymmetric function of $\vec{G}$ is

$$\begin{aligned}\omega(X_G(\mathbf{x}, t)) = {} & 2F_{4,\varnothing}t + (7F_{4,\varnothing} + F_{4,\{1\}} + F_{4,\{2\}} + F_{4,\{3\}})t^2 + \\ & + (7F_{4,\varnothing} + F_{4,\{1\}} + F_{4,\{2\}} + F_{4,\{3\}})t^3 + 2F_{4,\varnothing}t^4.\end{aligned}$$

As seen in the Examples 4.2.12 and 4.2.13, we calculate

$$\begin{aligned}F_{4,\{1\}} &= M_{(1,1,1,1)} + M_{(2,1,1)} + M_{(1,2,1)} + M_{(3,1)}.\\ F_{4,\{2\}} &= M_{(1,1,1,1)} + M_{(2,1,1)} + M_{(1,1,2)} + M_{(2,2)}.\\ F_{4,\{3\}} &= M_{(1,1,1,1)} + M_{(1,2,1)} + M_{(1,1,2)} + M_{(1,3)}.\end{aligned}$$

By Proposition 4.2.4, we have that

$$\begin{aligned} m_{(1,1,1,1)} &= M_{(1,1,1,1)}. \\ m_{(2,1,1)} &= M_{(2,1,1)} + M_{(1,2,1)} + M_{(1,1,2)}. \\ m_{(3,1)} &= M_{(3,1)} + M_{(1,3)}. \\ m_{(2,2)} &= M_{(2,2)} \end{aligned}$$

Therefore

$$\begin{aligned} &F_{4,\{1\}} + F_{4,\{2\}} + F_{4,\{3\}} = \\ &\quad = \underbrace{3M_{(1,1,1,1)}}_{3m_{(1,1,1,1)}} + \underbrace{2M_{(2,1,1)} + 2M_{(1,2,1)} + 2M_{(1,1,2)}}_{2m_{(2,1,1)}} + \underbrace{M_{(2,2)}}_{m_{(2,2)}} + \underbrace{M_{(3,1)} + M_{(1,3)}}_{m_{(3,1)}}. \end{aligned}$$

By Remark 4.2.15, we have that

$$F_{4,\varnothing} = h_4 = m_{(1,1,1,1)} + m_{(2,1,1)} + m_{(2,2)} + m_{(3,1)} + m_{(4)}.$$

By Proposition 3.3.20, we have that

$$h_{(3,1)} = 4m_{(1,1,1,1)} + 3m_{(2,1,1)} + 2m_{(2,2)} + 2m_{(3,1)} + m_{(4)}.$$

So, we have that

$$\begin{aligned} 2F_{4,\varnothing} &= 2h_4. \\ 7F_{4,\varnothing} + F_{4,\{1\}} + F_{4,\{2\}} + F_{4,\{3\}} &= 6h_4 + h_{(3,1)}. \end{aligned}$$

Therefore

$$\omega(X_G(\mathbf{x}, t)) = 2h_4 t + (6h_4 + h_{(3,1)})t^2 + (6h_4 + h_{(3,1)})t^3 + 2h_4 t^4.$$

By Remark 4.3.4,

$$X_G(\mathbf{x}, t) = 2e_4 t + (6e_4 + e_{(3,1)})t^2 + (6e_4 + e_{(3,1)})t^3 + 2e_4 t^4.$$

So, the chromatic quasisymmetric function of the proper circular-arc digraph $\vec{G}$ is e-positive and e-unimodal.

6.3.4 Ellzey implies Shareshian-Wachs

We now show that the Ellzey conjecture generalizes the Shareshian-Wachs conjecture.

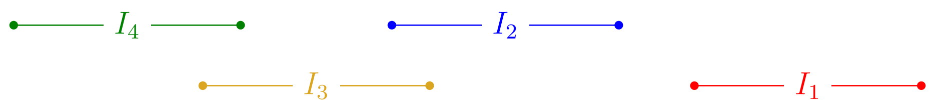
Theorem 6.3.18. Let $\vec{G}$ be a connected digraph. The following statements are equivalent:

1. $\vec{G}$ is a natural unit interval digraph.

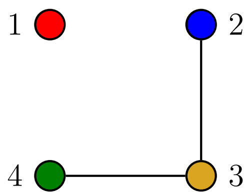
2. $\vec{G}$ is an acyclic circular indifference digraph.

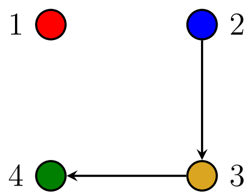
Proof. Let $\vec{G}$ be a natural unit interval digraph. By Definition 6.2.15 and Corollary 6.2.11, $\vec{G}$ is an orientation of a proper interval graph $G = ([n], E)$, constructed by orienting each edge from the smaller vertex to the larger vertex. Let us consider the set of proper intervals to which the corresponding proper interval order is isomorphic. So $\vec{G}$ is a proper circular-arc digraph with circular arcs constructed by placing proper intervals around a circle, positioning the first and last intervals such that one is not above the other. Since, by construction, the circular arcs do not cover the entire circle, then $\vec{G}$ is acyclic. Therefore, by Theorem 6.3.13, $\vec{G}$ is an acyclic circular indifference digraph. Similarly, proceeding backwards, the converse is demonstrated. $\square$

Example 6.3.19. Let P be the natural unit interval order of Example 6.1.2. It is a unit interval order with the following intervals, as seen in Example 6.2.5.

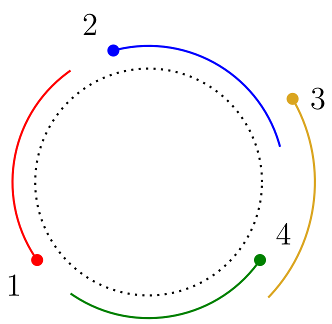


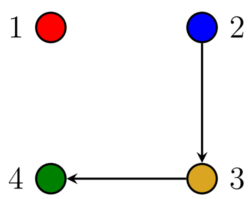
Then the natural unit interval graph and the natural unit interval digraph of P are the following





We place the intervals around a circle, as stated in the previous proof, and we see that the unit interval digraph is the correspondent circular-arc digraph.





Finally, we note that it is a circular indifference digraph with $\mathcal{I} = \{[2, 3], [3, 4]\}$.

Corollary 6.3.20. The Ellzey conjecture implies the Shareshian-Wachs conjecture.

Proof. By previous theorem, if the chromatic quasisymmetric functions of circular indifference digraphs are e-positive and e-unimodal, then the chromatic quasisymmetric functions of natural unit interval digraphs are e-positive and e-unimodal. $\square$

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