Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
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ON THE CONSTRUCTION OF SOME STACKS OF CURVES AND SURFACES

Tesi di Laurea in Geometria Algebrica

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A mamma e papà

Anima mia, fa' in fretta.
Ti presto la bicicletta,
ma corri. E con la gente
(ti prego, sii prudente)
non ti fermare a parlare
smettendo di pedalare.

Giorgio Caproni

Introduction

In algebraic geometry, given a certain type of algebro-geometric objects, one would like to construct a space whose points correspond to isomorphism classes of those objects. These spaces are known as *moduli spaces*. In this thesis, we construct several examples of moduli spaces. In particular, we are interested in the construction of moduli spaces for a certain type of curves and surfaces. Even if curves (and surfaces) are schemes, the moduli spaces that we construct are not schemes. Indeed, in order to find a space which parametrizes isomorphism classes of our objects of interests, we have to enlarge the category of schemes to the category of stacks, which allow to keep track of the automorphisms of those objects.

The first algebro-geometric rigorous construction of a space M_g whose points correspond to isomorphism classes of genus g curves (for $g \geq 2$) is due to Mumford in [MFK94]. In this thesis, we will recover M_g as the *coarse moduli space* (Remark 1.16) of a contravariant functor from the category of schemes to the category of sets. More precisely, let us fix an integer $g \geq 2$ and consider the functor

$$F_{\mathcal{M}_g} : (\mathrm{Sch})^{\mathrm{op}} \to (\mathrm{Set})$$

which sends every scheme T to the set of isomorphism classes of *smooth genus g curves over T* (Definition 1.4), i.e. proper, smooth morphisms of schemes $C \to T$, whose geometric fibres are connected curves of genus g. This functor is not representable in the category of schemes, and the reason is that curves can have non-trivial automorphism group (Remark 1.8).
However, it defines a category fibred in groupoids $\mathcal{M}_g$ over the category of schemes (Remark 1.7). The objects of $\mathcal{M}_g$ are exactly smooth genus g curves over any scheme T; observe in particular that if k is a field, then objects of $\mathcal{M}_g(\operatorname{Spec} k)$ are exactly smooth genus g curve over the field k. It turns out that $\mathcal{M}_g$ is a stack and moreover that $\mathcal{M}_g$ is isomorphic to a quotient stack $[H/\mathrm{PGL}_{5g-5}]$ (Theorem 1.12), where H is a locally closed subscheme of a projective Hilbert scheme. A very important geometric tool that we use is that if C is a smooth genus g curve for $g \geq 2$ over a field k, then the canonical line bundle $\omega_{C/k}$ is ample and $\omega_{C/k}^{\otimes 3}$ is very ample. It is known that the existence of a "natural" ample line bundle allows to descend objects. Moreover, deformation theory for smooth curves gives important tools in understanding geometric properties of the stack $\mathcal{M}_g$.
In particular in this thesis we study why $\mathcal{M}_g$ is smooth and of finite type over $\operatorname{Spec} \mathbb{Z}$ (Proposition 1.14) and why it has relative dimension $3g-3$ (Proposition 1.17). Moreover, the automorphism group scheme of a genus g curve is finite and reduced for $g \geq 2$, and in the language of stacks this is telling us that $\mathcal{M}_g$ is a Deligne-Mumford stack (Theorem 1.12).

Then we study the stack of smooth genus g curves for $g = 0, 1$. We define $\mathcal{M}_0$ as the category whose objects are proper and smooth morphisms of schemes $C \to T$, whose geometric fibres are connected curves of genus 0. The study of $\mathcal{M}_0$ is very similar to the case $g \geq 2$. The reason is that if C is a smooth curve of genus 0 over a field k, then there exists a natural choice of an ample line bundle, namely the anti-canonical one. This allows us to prove that $\mathcal{M}_0$ is indeed an algebraic stack.

However, this argument do not work for the genus 1 case, indeed if C is a smooth curve of genus 1 over a field k, then the canonical line bundle $\omega_{C/k}$ is trivial. If we define $\tilde{\mathcal{M}}_1$ as before for $g = 1$ (Definition 1.21), it turns out that $\tilde{\mathcal{M}}_1$ is not a stack, as there exist examples of ineffective descent data for smooth genus 1 curves. Thus, we define $\mathcal{M}_1$ (Definition 1.28) as the category whose objects are proper and smooth morphisms of algebraic spaces $C \to T$ from an algebraic space C to a scheme T, such that each geometric fibre is a smooth genus 1 curve. Allowing the total space to be an algebraic space, enables us to prove that $\mathcal{M}_1$ is a stack (Proposition 1.33).

Then the thesis develops on the theory of surfaces. One of the goals of this thesis is to construct the stack $\mathcal{M}^{\min}$ (Proposition 3.80) parametrizing minimal surfaces of general type and the stack $\mathcal{M}^{\mathrm{can}}$ (Definition 3.42) parametrizing canonical models of minimal surfaces of general type, in all characteristics. Moduli spaces of surfaces have been of great interest to mathematicians in the twentieth century. Minimal surfaces of general type and their canonical models have been studied by Bombieri [Bom73] in characteristic zero, and Ekedahl [Eke88] generalized Bombieri's result in positive characteristic. Gieseker [Gie77], using techniques of Mumford's geometric invariant theory [MFK94], was the first to construct, over the field of complex numbers, the coarse moduli space $M^{\mathrm{can},\mathbb{C}}_{\chi,K^2}$ of canonical models of minimal surfaces of general type with fixed Euler characteristic χ and self-intersection of the canonical bundle K^2. The geometry of these spaces has been studied, for example, by Catanese [Cat84] and Horikawa [Hor76]. The reader can find an exposition of these topics in [Cat13]. More generally Kollár in [Kol23] studied varieties of general type and their moduli spaces. We will use the language of fibred categories and algebraic stacks: a great work in this field is due to Artin in [Art74b], [Art73], [Art69].

If $S \to \mathrm{Spec}\ k$ is a *minimal surface of general type* over an algebraically closed field k, i.e. a smooth integral projective surface over k with Kodaira dimension 2 (which corresponds to a big canonical bundle) and minimal (which corresponds to a nef canonical bundle), then there exists a *canonical model* X of S, obtained by contracting the (-2)-curves on S. The canonical model X is a normal surface, with at worst Du Val singularities, and with an ample canonical line bundle (Theorem 2.54). An important fact is that X can be obtained by considering the canonical ring of S, namely

$$X = \mathrm{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(S, \omega_{S/k}^{\otimes m}).$$

We aim to study how this process generalizes to families of surfaces.

To do this, the first step is to construct the stack of canonical models of minimal surfaces of general type. We define a category $\mathcal{M}^{\mathrm{can}}$ (Definition 3.42) whose objects are *families of canonical models*, i.e. proper, flat and finitely presented morphisms of schemes $X \to T$, such that each geometric fibre is the canonical model of a minimal surface of general type. The important property is that if $X \to \mathrm{Spec}\ k$ is the canonical model of a minimal surface of general type, the canonical line bundle $\omega_{X/k}$ is ample (Corollary 2.56), and moreover that $\omega_{X/k}^{\otimes 5}$ is very ample by a result of Bombieri and Ekedahl (Theorem 2.60). We use this fact to prove that $\mathcal{M}^{\mathrm{can}}$ is indeed a stack (Proposition 3.41), as we did for $\mathcal{M}_0$ and $\mathcal{M}_g$, $g \geq 2$. In particular, once two integers χ and K^2 are fixed, we consider the stack $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ which parametrizes canonical models X of minimal surfaces of general type with fixed Euler characteristic $\chi = \chi(\mathcal{O}_X)$ and self-intersection of the canonical bundle $K^2 = (\omega_X^2)$ (Definition 3.36). Fixing such invariants allows us to fix the Hilbert polynomial of $\omega_{X/k}^{\otimes 5}$

$$P(m) = \chi + \frac{5m(5m-1)}{2}K^2$$

and find an isomorphism

$$\mathcal{M}^{\mathrm{can}}_{\chi,K^2} \simeq [W/\mathrm{PGL}_N],$$

where W is a locally closed subscheme of the Hilbert scheme that parametrizes closed subschemes of $\mathbb{P}^{N-1}_{\mathbb{Z}}$ with Hilbert polynomial $P(m)$ and $N = P(1) = \chi + 10K^2$ (Theorem 3.48). This isomorphism also allows us to prove that the stack

$$\mathcal{M}^{\mathrm{can}} = \coprod_{\chi,K^2} \mathcal{M}^{\mathrm{can}}_{\chi,K^2}$$

is an algebraic stack (Theorem 3.50).

Subsequently, we construct the stack of minimal surfaces of general type. The construction of this stack requires the use of algebraic spaces because we do not have a natural choice for an ample line bundle on those surfaces, just as with $\mathcal{M}_1$. Thus we first study the stack $\mathcal{S}paces'$ which parametrizes proper, flat and finitely presented morphisms of algebraic spaces (Definition 3.55 and Lemma 3.58). Then we define $\mathcal{M}^{\mathrm{min}}$ as the full subcategory of $\mathcal{S}paces'$ whose objects are proper, smooth and finitely presented morphisms of algebraic spaces $S \to T$ from an algebraic space S to a scheme T, whose geometric fibres are minimal surfaces of general type (Definition 3.72). We prove that $\mathcal{M}^{\mathrm{min}}$ is a stack (Proposition 3.80) and furthermore that it is an algebraic stack using Artin's axioms (Theorem 3.88).

An important property is that in characteristic zero, both $\mathcal{M}^{\mathrm{min}}$ and $\mathcal{M}^{\mathrm{can}}$ are Deligne-Mumford stacks (Proposition 3.90, Proposition 3.54), because the automorphisms groups of minimal surfaces of general type (and of their canonical models), in characteristic zero, are finite and reduced (Proposition 2.72). However, this is no longer true in positive characteristic, as there exist examples of smooth minimal surfaces of general type (and also of their canonical models) with non reduced automorphism group (Remark 2.73).

In this thesis we also construct the stack $\mathcal{M}^{\mathrm{K3}}$ (Proposition 3.106) parametrizing K3 surfaces, i.e. smooth surfaces with trivial canonical line bundle and irregularity zero. The construction of this stack requires the use of algebraic spaces, and it is very similar to the construction of $\mathcal{M}^{\mathrm{min}}$. Finally, we construct the stack $\mathcal{M}^{\mathrm{dP}}$ (Proposition 3.114) parametrizing del Pezzo surfaces, i.e. smooth surfaces with ample anti-canonical line bundle. In this case, we use again the existence of an ample line bundle to descend objects. We observe that the stack $\mathcal{M}^{\mathrm{K3}}$ is not algebraic (Remark 3.107), as there exist examples of formal objects of K3 surfaces which are not effective. On the other hand, the stack $\mathcal{M}^{\mathrm{dP}}$ is algebraic (Theorem 3.115), because we have a "natural" choice of an ample line bundle, namely the anti-canonical one.

In the final chapter of the thesis we study how the process of taking the canonical model of a minimal surfaces of general type generalizes to families. In order to do this, we first prove that if $S \to \operatorname{Spec} k$ is a minimal surface of general type, for a sufficiently large c the c-th canonical ring

$$\bigoplus_{m\geq 0} \mathrm{H}^0(S, \omega_{S/k}^{\otimes cm})$$

is generated in degree 1 (Proposition 2.83). Moreover we can take c depending on some plurigenera of S. This is a key tool in proving the following fact. If $p : S \to T$ is a family of minimal surfaces of general type over T, i.e. an object of $\mathcal{M}^{\min}(T)$, then it is possible to consider the coherent sheaf of $\mathcal{O}_T$-algebras

$$\mathcal{A} = \bigoplus_{m \geq 0} p_* \omega_{S/T}^{\otimes 5m},$$

and we prove that

$$\pi : \underline{\mathrm{Proj}}_T(\mathcal{A}) \to T,$$

is proper, flat and finitely presented (Corollary 4.19, Proposition 4.12 and Proposition 4.20). In particular, we observe that the geometric fibres of π are exactly the canonical models for the geometric fibres of p. Thus, we have proved that π is a family of canonical surfaces, i.e. an object of $\mathcal{M}^{\mathrm{can}}(T)$ (Corollary 4.22).

It follows that we have a morphism of stacks $\alpha : \mathcal{M}^{\min} \to \mathcal{M}^{\mathrm{can}}$ (Proposition 4.23). An interesting property is that α is bijective on geometric points, i.e. for all algebraically closed fields k, the morphism

$$\alpha(k) : \mathcal{M}^{\min}(k) \to \mathcal{M}^{\mathrm{can}}(k)$$

induces a bijection on the sets of isomorphism classes of the objects of these two groupoids (Proposition 2.62). Indeed, if $S \to \operatorname{Spec} k$ is a minimal surface of general type over k, then the canonical model $X \to \operatorname{Spec} k$ of S is an object of $\mathcal{M}^{\mathrm{can}}(k)$. On the other hand, if $X \to \operatorname{Spec} k$ is an object of $\mathcal{M}^{\mathrm{can}}(k)$, taking the minimal resolution

$$S \to X \to \operatorname{Spec} k$$

of singularities of X gives an object of $\mathcal{M}^{\mathrm{can}}(k)$. However α is not an isomorphism (Remark 4.24), because there exist families of canonical surfaces $p' : X \to T$ which do not admit a *simultaneous resolution*, i.e. an object $p : S \to T$ of $\mathcal{M}^{\min}$ with the property that geometric fibres of p are minimal desingularization of geometric fibres of p'.

Contents

CHAPTER 1

Quick review of the stack of curves

In this chapter we study moduli of curves. More precisely, in §1.1 we recall the construction of the stack of smooth genus g curves for $g \geq 2$. We also give a brief presentation of the stack of smooth genus 0 curves, which is very similar to the case $g \geq 2$. In §1.2 we construct the stack of smooth curves of genus 1.

We now fix some notation.

Definition 1.1. Let T be a scheme. A *geometric point* of T is a morphism of schemes

$$\sigma : \operatorname{Spec} k \to T$$

where k is an algebraically closed field.

If $X \to T$ is a morphism of schemes, or if $X \to T$ is a morphism of algebraic spaces from an algebraic space X to a scheme T, we denote by X_k the base change of X to $\operatorname{Spec} k$ given by the following cartesian diagram

$$\begin{tikzcd} X_k \arrow[r, "\varphi"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r, "\sigma"'] & T \end{tikzcd}$$

and call it a *geometric fibre*.
If $\mathcal{L}$ is a line bundle on X, we denote $\mathcal{L}_{X_k} = \varphi^*\mathcal{L}$ the pullback of $\mathcal{L}$ to the fibre X_k.

Suppose now that $t \in T$ is a point. Then we also denote by t the induced morphism of schemes

$$t : \operatorname{Spec} \kappa(t) \to T$$

where $\kappa(t) = \mathcal{O}_{T,t}/\mathfrak{m}_{\mathcal{O}_{T,t}}$ is the residue field of t. Moreover we denote by

$$\bar{t} : \operatorname{Spec} \kappa(\bar{t}) \to T$$

the induced morphism on an algebraic closure $\kappa(\bar{t})$ of $\kappa(t)$.
Finally, we denote by

$$X_t = X \times_T \operatorname{Spec} \kappa(t)$$

the fibre over $\kappa(t)$ and by

$$X_{\bar{t}} = X \times_T \operatorname{Spec} \kappa(\bar{t})$$

the fibre over $\kappa(\bar{t})$, which is in particular a geometric fibre of $X \to T$.

1.1. The stack of curves of genus greater than one

Definition 1.2. Let k be a field. A *curve* over k is a scheme of finite type over k such that all irreducible components have dimension 1.

Definition 1.3. Let $X \to \operatorname{Spec} k$ be a projective, smooth and geometrically connected curve over a field k and let $\omega_X = \omega_{X/k}$ be the canonical sheaf of X. Then we define the (*geometric*) *genus* of X to be

$$g = \dim_k \mathrm{H}^0(X, \omega_X) = \dim_k \mathrm{H}^1(X, \mathcal{O}_X).$$

DEFINITION 1.4. Let T be a scheme and let $g \geq 0$ be an integer such that $g \neq 1$. A *family of smooth curves of genus g over T* is a proper smooth morphism of schemes $C \to T$ such that for every point $t \in T$ the geometric fibre $C_{\bar{t}}$ is a connected curve of genus $g = h^1(C_{\bar{t}}, \mathcal{O}_{C_{\bar{t}}})$.

PROPOSITION 1.5. *Let $C \to T$ be a morphism of schemes. Let $g \geq 0$ be an integer with $g \neq 1$. Then the following are equivalent:*

(1) *$C \to T$ is a family of smooth curves of genus g over T;*
(2) *$C \to T$ is a proper, flat and finitely presented morphism of schemes such that for every point $t \in T$ the fibre C_t is a smooth, projective and geometrically connected curve over $\kappa(t)$ of genus g.*

PROOF. [(1) $\Rightarrow$ (2)]. Since $C \to T$ is smooth, then it is flat by [Stacks, Lemma 01VF] and locally of finite presentation by [Stacks, Lemma 01VE]. Moreover, since $C \to T$ is proper, it is also quasi-compact by [Stacks, Lemma 04XU] and separated. It follows that $C \to T$ is a finitely presented morphism of schemes. Fibres are smooth by [Gro67, Théorème IV.17.5.1] and the genus of a curve over a field is stable under base change by [Liu02, Corollary 5.2.27].
[(2) $\Rightarrow$ (1)]. A flat and finitely presented morphism of schemes with smooth fibres is smooth by [Stacks, Lemma 01V8]. $\square$

DEFINITION 1.6. Let $g \geq 0$ be an integer different from 1. We define the category $\mathcal{M}_g$ as follows.

- Objects are families of smooth curves of genus g.
- An arrow $(C' \to T') \to (C \to T)$ between two objects is a pair (f, g) where $f : C' \to C$, $g : T' \to T$ are morphisms of schemes such that the diagram

$$\begin{tikzcd} C' \arrow[r, "f"] \arrow[d] & C \arrow[d] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

is cartesian.

REMARK 1.7. Observe that $\mathcal{M}_g$ is in a natural way a category fibred in groupoids over Sch, by sending an object $(C \to T)$ of $\mathcal{M}_g$ to the scheme T. The moduli functor associated to the category $\mathcal{M}_g$ is

$$F_{\mathcal{M}_g} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set},$$

where $F_{\mathcal{M}_g}(T)$ is the set of isomorphism classes of families of smooth curves of genus g over T.

REMARK 1.8. We claim that the functor $F_{\mathcal{M}_g}$ is not representable by a scheme, so that it does not exist a fine moduli space for the functor $F_{\mathcal{M}_g}$. Indeed, suppose by contradiction that such a scheme exists, and call it S_g. If $C \to T$ is an *isotrivial family* of genus g curves, i.e. a non-trivial family of smooth curves of genus g whose all fibres are isomorphic to a fixed curve $\tilde{C}$, then this family corresponds to a map $T \to S_g$ which maps every point of T to the same point of S_g. This contradicts the fact that this map should correspond to the trivial family $\tilde{C} \times T$ which is not isomorphic to C by hypothesis. Indeed if a fine moduli space S_g exists, by Yoneda's lemma (A.6), morphisms of schemes from T to S_g correspond to unique families of curves of genus g over T. The reason why isotrivial families exist is that curves can have non-trivial automorphism group. See [Alp24, Example 0.3.32] for an explicit example of an isotrivial family of curves. To keep track of these data, we have to enlarge the category of schemes to the category of stacks.

LEMMA 1.9. *Let k be a field and let A be a k-algebra of finite type such that* $\dim A = 0$. *Then* $\operatorname{Spec} A$ *is finite.*

PROOF. We have that A is isomorphic as a k-algebra to a quotient of $k[x_1, \dots, x_n]$ by an ideal I, for some finite number of indeterminates $x_1, \dots, x_n$. By the Hilbert basis theorem, A is noetherian. Since $\dim A = 0$, A is also an artinian ring. It follows that $\operatorname{Spec} A$ is finite. □

LEMMA 1.10. *Let G be a group scheme of finite type over a field k. Let $e \in G(k)$ be the neutral element of G and let $\mathrm{T}_e G = \mathfrak{g}$ be the tangent space at G at e. Then the following are equivalent:*

(1) $\mathfrak{g} = 0$;
(2) *G is isomorphic as a k-scheme to a finite disjoint union of copies of* $\operatorname{Spec} k$.

PROOF. $[(1) \Rightarrow (2)]$. Consider the local ring $A = \mathcal{O}_{G,e}$ and let $\mathfrak{m}$ be its maximal ideal. We have

$$0 = \mathrm{T}_e G = (\mathfrak{m}/\mathfrak{m}^2)^\vee.$$

By Nakayama's lemma, $\mathfrak{m} = 0$, so that A is a field. Since $e \in G(k)$ it follows that $A = k$. This means that there is a neighbourhood of e which is just a point $A = \operatorname{Spec} k$. The group scheme G acts transitively on itself by translation, so that for all points $g \in G$ there exists an open neighbourhood which is just $\operatorname{Spec} k$. Since G is a scheme of finite type over k, there exists a finite affine covering $\{U_i\}_{i \in I}$ of G such that each U_i is of finite type over k. By Lemma 1.9 each U_i is finite. $[(2) \Rightarrow (1)]$. The local ring $\mathcal{O}_{G,e}$ is just $\operatorname{Spec} k$ by hypothesis, so that the tangent space is zero. □

REMARK 1.11. If G is a group scheme locally of finite type over a field k, using the same notation of Lemma 1.10, we see that if $\mathfrak{g} = 0$ then G is isomorphic to a disjoint union of (not necessarily finite) copies of $\operatorname{Spec} k$.

THEOREM 1.12. *For $g \geq 2$, $\mathcal{M}_g$ is a Deligne-Mumford stack in the étale topology over* $\operatorname{Spec} \mathbb{Z}$. *Moreover, $\mathcal{M}_g \simeq [H/\mathrm{PGL}_{5g-5}]$ where H is a locally closed subscheme of a projective Hilbert scheme.*

PROOF. The fact that $\mathcal{M}_g$ is a stack is proved in [Alp24, Proposition 2.5.14]. See also Remark 1.13 below. The isomorphism with the quotient stack is proved in [Alp24, Theorem 3.1.17]. Finally, to prove that $\mathcal{M}_g$ is a Deligne-Mumford stack, by Theorem A.101 it is sufficient to show that every smooth, projective and geometrically connected curve C over a field k has discrete and reduced automorphism group scheme Aut_C, which is the group scheme representing the functor $\underline{\mathrm{Aut}}_C$ of Example A.64. This is a group scheme locally of finite type over k by [MO67, Theorem 3.7]. Moreover the tangent space at the identity $\mathrm{id}_C \in \mathrm{Aut}_C(k)$ to Aut_C is

$$\mathrm{T}_{\mathrm{id}_C} \mathrm{Aut}_C \simeq \mathrm{Hom}_{\mathcal{O}_C}(\Omega^1_{C/k}, \mathcal{O}_C) = \mathrm{H}^0(C, \mathcal{T}_C)$$

as proved in [MO67, Lemma 3.4], and $\mathrm{H}^0(C, \mathcal{T}_C) = 0$ since the degree of the tangent bundle $\mathcal{T}_C$ is $2 - 2g$, which is negative. But then by Remark 1.11 we have that Aut_C is discrete (i.e. the underlying topological space is discrete) and reduced (i.e. is a reduced scheme), so that $\mathcal{M}_g$ is a Deligne-Mumford stack in the étale topology. □

REMARK 1.13. The key point in proving that $\mathcal{M}_g$ is a stack in the étale topology for $g \geq 2$ is that for a family of smooth curves $C \to T$ of genus g, the canonical bundle $\omega_{C/T}$ is ample relative to the morphism $C \to T$. Indeed if $C \to \operatorname{Spec} k$ is a smooth projective curve of genus $g \geq 2$, then ω_C is a line bundle on C whose degree is

$$\deg(\omega_C) = 2g - 2$$

which is positive for $g \geq 2$. We can use this fact to descent immersions in the projective space. See [Alp24, Proposition 2.5.14] for details. Observe that the existence of a natural ample line bundle can be used to prove that $\mathcal{M}_g$ is a stack in the étale topology also by [Vis08, Theorem 4.38].

PROPOSITION 1.14. *For $g \geq 2$, the stack $\mathcal{M}_g$ is of finite type over* $\operatorname{Spec}\mathbb{Z}$.

PROOF. We have to show that $\mathcal{M}_g$ is locally of finite type over $\operatorname{Spec}\mathbb{Z}$ and that $\mathcal{M}_g$ is quasi-compact (see Definition A.91 and [Stacks, Definition 04YB]). To prove that $\mathcal{M}_g$ is locally of finite type, by Theorem 1.12 we have to show that

$$[H/\operatorname{PGL}_{5g-5}] \to \operatorname{Spec}\mathbb{Z}$$

is locally of finite type. By Remark A.92 it is sufficient to see that $H \to \operatorname{Spec}\mathbb{Z}$ is locally of finite type. In order to prove this, observe that H is a locally closed subscheme of a projective scheme, which is in particular locally of finite type over $\operatorname{Spec}\mathbb{Z}$. Then we use the fact that a locally closed immersion is locally of finite type ([Stacks, Lemma 01T5]) and that the composition of morphisms which are locally of finite type is again locally of finite type ([Stacks, Lemma 01T3]). To prove that $\mathcal{M}_g$ is quasi-compact we use [Stacks, Lemma 04YC], the smooth presentation $H \to [H/\operatorname{PGL}_{5g-5}]$ and the fact that H is quasi-compact in the Zariski topology. □

REMARK 1.15. By Proposition 1.14 it follows that $\mathcal{M}_g$ is locally noetherian, because H is locally of finite type over $\operatorname{Spec}\mathbb{Z}$ and we use [Stacks, Lemma 01T6].

REMARK 1.16. By the Keel-Mori theorem A.103 we know that $\mathcal{M}_g$ admits a coarse moduli space (Definition A.102), which we denote by M_g.

PROPOSITION 1.17. *For $g \geq 2$, the stack $\mathcal{M}_g$ is a smooth Deligne-Mumford stack over* $\operatorname{Spec}\mathbb{Z}$ *of relative dimension $3g-3$, i.e. for every algebraically closed field k, the dimension of $\mathcal{M}_g \times_{\mathbb{Z}} k$ is $3g-3$.*

PROOF. We use the infinitesimal lifting criterion for smoothness ([Alp24, Theorem 3.7.1]). Namely, we have to prove that if we have a 2-commutative diagram ([Alp24, Definition 2.4.17])

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & \mathcal{M}_g \arrow[d] \\
& \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\mathbb{Z}
\end{tikzcd}$$

where $A \to A_0$ is a surjection of artinian local rings with residue field k such that $k = \ker(A \to A_0)$, then there exists a dotted arrow making the diagram 2-commutative. The map $\operatorname{Spec} k \to \mathcal{M}_g$ corresponds to a smooth genus g curve over k by 2-Yoneda's lemma (A.58), while the map $\operatorname{Spec} A_0 \to \mathcal{M}_g$ corresponds to a family of curves $\mathcal{C}_0 \to \operatorname{Spec} A_0$ over A_0. It follows that the dotted arrow of the diagram corresponds to a lifting

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d] & \mathcal{C}_0 \arrow[r, hook, dashed] \arrow[d] & \mathcal{C} \arrow[d, dashed] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} A_0 \arrow[r, hook] & \operatorname{Spec} A
\end{tikzcd}$$

of $\mathcal{C}_0$ over A. Since C is a smooth curve over k, there exists an obstruction theory with obstruction space $\mathrm{H}^2(C, \mathcal{T}_C)$ (see [TV13, Theorem 5.16]). Thus, the existence of a lifting of $\mathcal{C}_0$ over A follows by the fact that $\mathrm{H}^2(C, \mathcal{T}_C) = 0$ because C is a curve.
Let now k be an algebraically closed field and let $C \to \operatorname{Spec} k$ a smooth connected genus g curve over k, which corresponds to a morphism $\varphi_C : \operatorname{Spec} k \to \mathcal{M}_g$. To prove that the relative dimension of $\mathcal{M}_g$ is $3g-3$, by [Alp24, Proposition 3.7.6] it is sufficient to show that the dimension of the tangent space $\mathcal{T}_{\mathcal{M}_g, C}$ at $\mathcal{M}_g$ at $C \in \mathcal{M}_g(\operatorname{Spec} k)$ ([Alp24, Definition 3.5.7]) is

$3g-3$, because by Theorem 1.12 we see that the dimension of Aut_C is 0. By [TV13, Remark 3.24] we have

$$\mathcal{T}_{\mathcal{M}_g,C} \simeq \mathrm{H}^1(C, \mathcal{T}_C).$$

Since $\deg(\mathcal{T}_C) = 2-2g < 0$ it follows that $h^0(C, \mathcal{T}_C) = 0$ and that the dimension of $\mathrm{H}^1(C, \mathcal{T}_C)$ is

$$h^1(C, \mathcal{T}_C) = h^1(C, \mathcal{T}_C) - h^0(C, \mathcal{T}_C) = -(\deg(\mathcal{T}_C) + 1 - g) = 3g - 3$$

where we used the Riemann-Roch theorem for curves. □

However, the stack $\mathcal{M}_g$ for $g \geq 2$ is not proper (see [Edi00, Theorem 1.2]). There exists a *compactification* of $\mathcal{M}_g$ given by adding *nodal stable curves* (see [DM69, Definition 1.1]). For $g \geq 2$, define $\overline{\mathcal{M}}_g$ to be the stack of stable curves of genus g.

THEOREM 1.18. *For $g \geq 2$, the moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus g is a smooth, proper and irreducible Deligne-Mumford stack of relative dimension $3g-3$ over $\operatorname{Spec} \mathbb{Z}$ which admits a projective coarse moduli space $\overline{M}_g$.*

PROOF. The reader can find a complete proof in [Alp24, Theorem A]. See also the original paper [DM69]. □

The construction of the stack of genus 0 curves is very similar to the case $g \geq 2$.
Define the category $\mathcal{M}_0$ as in Definition 1.6 for $g = 0$. Let k be an algebraically closed field. If $C \to k$ is a complete (i.e. proper over k) smooth curve of genus 0, then $C \simeq \mathbb{P}^1_k$, see [Har77, Example IV.1.3.5]. The automorphism functor $\underline{\mathrm{Aut}}_{\mathbb{P}^1_k}$ (Example A.64) is representable by an algebraic group scheme, denoted by $\mathrm{PGL}_{2,k}$. This is an affine algebraic group scheme, as it is identified with the open subset in $\mathbb{P}^3_k$ with coordinates x_0, x_1, x_2, x_3 given by $D(x_0x_3 - x_1x_2)$. In particular, on k-points we have $\mathrm{Aut}(\mathbb{P}^1_k) = \mathrm{PGL}_{2,k}(k)$, where

$$\mathrm{PGL}_{2,k}(k) = \left\{ \begin{bmatrix} x_0 & x_1 \\ x_2 & x_3 \end{bmatrix} \in \mathbb{P}^3_k(k) \;\middle|\; x_0x_3 - x_1x_2 \neq 0 \right\}.$$

REMARK 1.19. Even in the case of genus 0 curves we have a natural choice of an ample line bundle, which can be used to prove that $\mathcal{M}_0$ is a stack in the étale topology by [Vis08, Theorem 4.38]. Indeed if $C \to \operatorname{Spec} k$ is a smooth projective genus 0 curve, then the anticanonical bundle is ample having positive degree

$$\deg \omega_C^\vee = 2 - 2g = 2 > 0.$$

Let now PGL_2 be the affine algebraic group scheme representing the functor $\underline{\mathrm{Aut}}_{\mathbb{P}^1_\mathbb{Z}}$.

PROPOSITION 1.20. *The category $\mathcal{M}_0$ of smooth curves of genus 0 is a stack in the étale topology. Moreover, it is isomorphic to $B\,\mathrm{PGL}_2$, the classifying stack of the affine group scheme PGL_2.*

PROOF. One way to prove this fact is to find a bijective correspondence between smooth genus 0 curves and *Brauer-Severi schemes of relative dimension* 1 and use the correspondence between families of genus 0 curves and PGL_2-torsors. We refer to [Alp24, Exercise 2.5.15] and [Alp24, Exercise B.1.67] for this approach. However, we can also prove the isomorphism of $\mathcal{M}_0$ with $B\,\mathrm{PGL}_2$ using the same method that we will use in the proof of Theorem 3.48, thanks to Remark 1.19. See also [Ols16, Remark 8.4.15]. □

1.2. Curves of genus one

DEFINITION 1.21. We define the category $\tilde{\mathcal{M}}_1$ as follows.

- Objects are families of smooth curves of genus 1.
- An arrow $(C' \to T') \to (C \to T)$ between two objects is a pair (f, g) where $f : C' \to C$, $g : T' \to T$ are morphisms of schemes such that the diagram
$$\begin{tikzcd} C' \arrow[r, "f"] \arrow[d] & C \arrow[d] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$
is cartesian.

The category $\tilde{\mathcal{M}}_1$ is fibred in groupoids over Sch, by sending an object $(C \to T)$ of $\tilde{\mathcal{M}}_1$ to the scheme T. However, $\tilde{\mathcal{M}}_1$ is not a stack over $\mathrm{Sch}_{\text{ét}}$. The reason is that there exist examples of *ineffective descent data* for families of smooth curves of genus 1, i.e. on can find a covering $\{T_i \to T\}_{i\in I}$ in the étale topology and families $C_i \to T_i$ of smooth curves of genus 1 over T_i for all $i \in I$ with isomorphisms
$$C_{i_{|T_{ij}}} \simeq C_{j_{|T_{ij}}}$$
satisfying the cocycle condition, and which do not glue to a family of smooth curves of genus 1 over T. The reader can find an example of this phenomena in [Ray70, Remarques III.3.1.b] or in [Zom18, Theorem 1.2]. Moreover, observe that for $\tilde{\mathcal{M}}_1$ we can not apply [Vis08, Theorem 4.38] using the (anti)canonical bundle, because in this case it is a line bundle of degree 0.

In order to obtain a stack, we will allow the total space to be an algebraic space, see Definition 1.24.

LEMMA 1.22. *Let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be a finite morphism. Then f is projective.*

PROOF. Let $b_1, \ldots, b_n \in B$ such that generate B as an A-module, i.e.
$$B = Ab_1 + \ldots + Ab_n.$$
Define the $\mathbb{N}$-graded A-algebra given by
$$S = A \oplus Bz \oplus Bz^2 \oplus Bz^3 \oplus \ldots$$
with the grading such that $S_0 = A$ and $S_i = Bz^i$ for all $i > 0$. Consider the surjective homomorphism of $\mathbb{N}$-graded A-algebras given by
$$\begin{array}{ccll} A[x_0, \ldots, x_n] & \to & S & \\ x_0 & \mapsto & z & \\ x_i & \mapsto & b_i z & \text{for} \quad 1 \le i \le n. \end{array}$$
It follows that there is a closed immersion
$$\operatorname{Proj} S \hookrightarrow \mathbb{P}^n_A$$
of A-schemes. We claim that $\operatorname{Spec} B \simeq \operatorname{Proj} S$. Let $m \geq 1$ and $b \in B$. Then $bz^m \in S_+$ and
$$(bz^m)^2 = b^2 z^{2m} = b^2 z^{2m-1} z \in Sz.$$
This shows that $S_+ \subseteq \sqrt{Sz}$. Moreover $Sz \subseteq S_+$. It follows that $\sqrt{Sz} = \sqrt{S_+}$. Observe that the localization of S in z gives
$$S_z = \bigoplus_{m\in\mathbb{Z}} Bz^m$$
and the degree 0 part is
$$S_{(z)} \simeq B.$$

Then we have

$$\operatorname{Spec} B \simeq \operatorname{Spec} S_{(z)} \simeq D_+(z) = \operatorname{Proj} S. \qquad \square$$

THEOREM 1.23. *Let k be a field, and let X be a proper algebraic space over k such that $\dim X \leq 1$. Then X is a projective scheme over k.*

PROOF. The fact that X is a scheme follows by [Stacks, Lemma 0ADD]. In order to show that X is projective, it is not restrictive to assume that X is connected. Indeed, suppose that X is the disjoint union of a finite number of projective schemes. Then X is projective, as we now explain. If $X = X_1 \coprod X_2$ with

$$\begin{tikzcd} X_1 \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_k \arrow[d] \\ & k \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_2 \arrow[r, hook] \arrow[dr] & \mathbb{P}^m_k \arrow[d] \\ & k, \end{tikzcd}$$

then

$$X = X_1 \coprod X_2 \hookrightarrow \mathbb{P}^n_k \coprod \mathbb{P}^m_k \hookrightarrow \mathbb{P}^{n+m+1},$$

where the last embedding is given by the inclusion in the first (resp. last) $n+1$ (resp. $m+1$) coordinates. The same argument works for the disjoint union of a finite number of projective schemes, so we can assume that X is connected.

If $\dim X = 0$, we have $X = \operatorname{Spec} A$ with A artinian local k algebra. Since $\operatorname{Spec} A \to \operatorname{Spec} k$ is proper, it is of finite type, so that A is a finitely generated k-algebra. By [AM69, Exercise 8.3] it follows that A is a finite k-algebra. Finally, a finite morphism between affine varieties is projective, by Lemma 1.22. If $\dim X = 1$, we conclude by [Stacks, Lemma 0A26]. $\square$

DEFINITION 1.24. Let T be a scheme. A *family of smooth curves of genus* 1 is a proper, smooth and finitely presented morphism of algebraic spaces $C \to T$, such that for every point $t \in T$ the geometric fibre $C_{\bar{t}}$ is a connected curve genus 1.

PROPOSITION 1.25. *Let $C \to T$ be a morphism of algebraic spaces where T is a scheme. Then the following are equivalent:*

(1) *$C \to T$ is a family of smooth curves of genus 1 over T;*
(2) *$C \to T$ is a proper, flat and finitely presented morphism of algebraic spaces such that for every point $t \in T$ the fibre C_t is a smooth, projective and geometrically connected curve over $\kappa(t)$ of genus 1.*

PROOF. The proof is the same as in Proposition 1.5. $\square$

REMARK 1.26. As we mentioned above, in the definition of a family of smooth curves of genus 1, we allow the total space to be an algebraic space. However, every fibre of such a family is actually a projective scheme by Theorem 1.23.

PROPOSITION 1.27.

DEFINITION 1.28. We define the category $\mathcal{M}_1$ as follows.

- Objects are families of smooth curves of genus 1.
- An arrow $(C' \to T') \to (C \to T)$ between two objects is a pair (f, g) where $f : C' \to C$ is a morphism of algebraic spaces, $g : T' \to T$ is a morphism of schemes and the diagram
$$\begin{tikzcd} C' \arrow[r, "f"] \arrow[d] & C \arrow[d] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$
is cartesian.

REMARK 1.29. The category $\mathcal{M}_1$ is fibred in groupoids over Sch, by sending an object $C \to T$ of $\mathcal{M}_1$ to the scheme T.

DEFINITION 1.30. Let T be a scheme and $Z \to B$, $X \to B$ morphisms of algebraic spaces over T. We define the functor

$$\underline{\mathrm{Hom}}_B(Z, X) : (\mathrm{Sch}\,/B)^{\mathrm{op}} \to \mathrm{Set}$$

sending $(S \to B)$ to the set of morphisms $Z_S \to X_S$ between pullbacks of Z, X to S. A morphism $S' \to S$ of schemes over B is sent to the morphism of sets which maps $f : Z_S \to X_S$ to the unique dotted arrow filling in the diagram

$$\begin{tikzcd}
Z_{S'} \arrow[r] \arrow[dr, dashed] \arrow[ddr, bend right] & Z_S \arrow[dr, bend left, "f"] & \\
& X_{S'} \arrow[r] \arrow[d] & X_S \arrow[d] \\
& S' \arrow[r] & S.
\end{tikzcd}$$

LEMMA 1.31. *Let T be a scheme and $Z \to B$, $X \to B$ morphisms of algebraic spaces over T. If $X \to B$ is separated and $Z \to B$ is proper, flat and of finite presentation, then we have a natural transformation of functors*

$$\Gamma : \underline{\mathrm{Hom}}_B(Z, X) \to \mathcal{H}ilb_{Z \times_B X/B}$$

which is injective and is representable by open immersions.

PROOF. First observe that here $\mathcal{H}ilb_{Z \times_B X/B}$ is the Hilbert functor associated to a morphism of algebraic spaces as in [Stacks, Situation 0CZY].
Let S be a scheme over B and let $f_S : Z_S \to X_S$ be an object in the set $\underline{\mathrm{Hom}}_B(Z, X)(S)$. Define the *graph* of f_S to be

$$\Gamma_{f_S} = (\mathrm{id}, f_S) : Z_S \to Z_S \times_S X_S = (Z \times_B X)_S.$$

By [Stacks, Lemma 03KL] we know that being separated is stable under base change. It follows that $X_S \to S$ is separated and that Γ_{f_S} is a closed immersion by [Stacks, Lemma 03KO]. By [Stacks, Lemma 03MO], [Stacks, Lemma 04WP] and [Stacks, Lemma 03XR], the base change $\Gamma_{f_S}(Z_S) \simeq Z_S \to S$ is proper, flat and of finite presentation. It follows that $\Gamma_{f_S}(Z_S)$ is an element of $\mathcal{H}ilb_{Z \times_B X/B}(S)$. The map $f_S \mapsto \Gamma_{f_S}(Z_S)$ is injective because a graph

$$Y \subseteq (Z \times_B X)_S$$

uniquely determines the morphism $f_S : Z_S \to X_S$ by composing the inverse of $\mathrm{pr}_{1_{|Y}} : Y \to \mathcal{Z}_S$ with $\mathrm{pr}_{2_{|Y}}$.
Finally, Γ is representable by open immersions by [Stacks, Lemma 0D1B]. $\square$

LEMMA 1.32. *The diagonal*

$$\Delta : \mathcal{M}_1 \to \mathcal{M}_1 \times_{\mathrm{Spec}\,\mathbb{Z}} \mathcal{M}_1$$

is representable by algebraic spaces.

PROOF. We use the characterization of Proposition A.80. Let $x = (C_1 \to T)$ and $y = (C_2 \to T)$ be two families of smooth curves of genus 1. We have to show that the sheaf

$$\underline{\mathrm{Isom}}_T(x, y) : (\mathrm{Sch}\,/T)^{\mathrm{op}} \to \mathrm{Set}$$

sending $(S \to T)$ to the set of isomorphisms $f : (C_1)_S \to (C_2)_S$ between the pullbacks to S of the two families of smooth curves is an algebraic space (Definition A.61 and Remark A.63). Consider the natural transformation of functors

$$\mathrm{id}_1 : T \to \underline{\mathrm{Isom}}_T(x, x)$$

which sends every T-scheme $S \to T$ to the identity morphism $\mathrm{id}_{(C_1)_S}$ of the pullback of C_1 to S. Let id_2 be the analogous natural transformation of functors for C_2. Then we have a cartesian diagram

$$\begin{tikzcd}
\underline{\mathrm{Isom}}_T(x,y) \arrow[r] \arrow[d] & T \arrow[d, "(\mathrm{id}_1, \mathrm{id}_2)"] \\
\underline{\mathrm{Isom}}_T(x,y) \times \underline{\mathrm{Isom}}_T(y,x) \arrow[r] & \underline{\mathrm{Isom}}_T(x,x) \times \underline{\mathrm{Isom}}_T(y,y),
\end{tikzcd}$$

where the bottom row is given by $(\phi, \psi) \mapsto (\psi \circ \phi, \phi \circ \psi)$. Moreover, the sheaves

$$\underline{\mathrm{Isom}}_T(x,y) \times \underline{\mathrm{Isom}}_T(y,x) \quad \text{and} \quad \underline{\mathrm{Isom}}_T(x,x) \times \underline{\mathrm{Isom}}_T(y,y)$$

are algebraic spaces because

$$\mathcal{H}ilb_{C_i \times_T C_j / T} \quad \text{for} \quad i, j = 1, 2$$

is an algebraic space (see [Stacks, Proposition 0D01]) and we use Lemma 1.31. The conclusion follows by the fact that fibre products exist in the category of algebraic spaces ([Stacks, Lemma 02X2]) □

Proposition 1.33. *The category $\mathcal{M}_1$ is a stack in the étale topology.*

Proof. We have already proved that morphisms glue in Lemma 1.32. To see that objects glue, let $\{T_i \to T\}_{i \in I}$ be an étale covering of T, and let $C_i \to T_i$ be a family of smooth genus 1 curves over T_i for all $i \in I$. Suppose that we are given isomorphisms

$$\alpha_{ij} : (C_i \to T_i)_{ij} \to (C_j \to T_j)_{ij}$$

for all $i, j \in I$ satisfying the cocycle condition on triple intersections $T_i \times_T T_j \times_T T_k$. In other words, we are given isomorphisms (which we also denote by α_{ij}) of algebraic spaces

$$\alpha_{ij} : C_i \times_T T_j \to C_j \times_T T_i$$

over $T_i \times_T T_j$ satisfying the cocycle condition over $T_i \times_T T_j \times_T T_k$ for all $i, j, k \in I$. Since every morphism $C_i \to T_i$ is of finite presentation, in particular it is also of finite type. Then by [Stacks, Lemma 0ADV.(2)] there exists an algebraic space C over T such that $C \times_T T_i \simeq C_i$ for all $i \in I$. Moreover, being proper and smooth is an étale (even fppf) local property on the base, see [Stacks, Lemma 0429] and [Stacks, Lemma 0422]. It follows that $C \to T$ is a proper and smooth morphism of algebraic spaces.
Then we only have to prove that for every point $t : \operatorname{Spec} \kappa(t) \to T$ of T, the geometric fibre $C_{\overline{t}} \to \operatorname{Spec} \kappa(\overline{t})$ is a connected curve of genus 1.
Denote also by $t \in T$ the unique point in the image of the morphism t. By surjectivity of $\coprod_i T_i \to T$ there exists a point $t' \in T_i$ for some $i \in I$ which is sent to t. Thus we obtain a morphism of schemes

$$\operatorname{Spec} \kappa(t') \to T_i \to T$$

whose image is $t \in T$. Pre-composing with an algebraic closure $\kappa(\overline{t'})$ of $\kappa(t')$ we obtain a geometric point of T

$$\sigma : \operatorname{Spec} \kappa(\overline{t'}) \to T$$

which factorizes as

$$\operatorname{Spec} \kappa(\overline{t'}) \to \operatorname{Spec} \kappa(\overline{t}) \to T. \tag{1}$$

Observe that we have a commutative diagram

$$\begin{array}{ccccc} (C_i)_{\kappa(\overline{t'})} & \longrightarrow & C_i & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\kappa(\overline{t'}) & \longrightarrow & T_i & \longrightarrow & T \end{array}$$

in which the squares on the right and on the left are cartesian. Thus, also the external square is cartesian (Lemma A.3), $C_{\kappa(\overline{t'})} \simeq (C_i)_{\kappa(\overline{t'})}$ and $C_{\kappa(\overline{t'})} \to \operatorname{Spec}\kappa(\overline{t'})$ is a geometric fibre of $C \to T$. Since $C_i \to T_i$ is an object of $\mathcal{M}_1$, then this geometric fibre is a connected curve of genus 1.

By the factorization in Equation (1) it follows that we have a cartesian diagram

$$\begin{array}{ccccc} C_{\kappa(\overline{t'})} & \longrightarrow & C_{\overline{t}} & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ \kappa(\overline{t'}) & \longrightarrow & \operatorname{Spec}\kappa(\overline{t}) & \longrightarrow & T. \end{array}$$

But then also $C_{\overline{t}}$ is a connected curve of genus 1 because being a connected curve of genus 1 is stable under a base change which is a field extension by [Stacks, Lemma 054N], [Gro67, Corollaire 4.1.4] and Corollary B.3. It follows that $C \to T$ is a family of smooth curves of genus 1 over T. □

LEMMA 1.34. *Let X be a locally noetherian scheme and $\mathcal{F} \in \operatorname{Coh}(X)$. Let $p \in X$ be a point. Then the following are equivalent:*

(1) *there exists an open neighbourhood $U \subseteq X$ of p such that $\mathcal{F}_{|U} = 0$;*
(2) *the stalk $\mathcal{F}_p$ is zero;*
(3) *the fibre $\mathcal{F}(p) = \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} \kappa(p)$ is zero.*

PROOF. The statement is local, so that up to restrict $\mathcal{F}$ to an open affine covering, we can assume that $X = \operatorname{Spec} A$ is affine, and $\mathcal{F} = \tilde{M}$ is the sheafification of a finitely generated A-module

$$M = Am_1 + \ldots + Am_k.$$

Let $\mathfrak{p} \in \operatorname{Spec} A$ be the chosen point. Then [(1) $\Rightarrow$ (2) $\Rightarrow$ (3)] are clear. We now prove [(2) $\Rightarrow$ (1)]. If $M_{\mathfrak{p}} = 0$ we have that there exist $s_1, \ldots, s_k \in A \setminus \mathfrak{p}$ such that $s_i m_i = 0$ for all $i = 1, \ldots, k$. Define $f = s_1 \cdot \ldots \cdot s_n \in A$. It follows that $M_f = 0$. Finally we prove the implication [(3) $\Rightarrow$ (2)]. If

$$M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = 0$$

we have $M_{\mathfrak{p}} = (\mathfrak{p}A_{\mathfrak{p}})M_{\mathfrak{p}}$ and by Nakayama's lemma $M_{\mathfrak{p}} = 0$ follows. □

COROLLARY 1.35. *Let $i : X \hookrightarrow X'$ be a closed immersion of locally noetherian schemes and assume that it is a homeomorphism. Let $\mathcal{F}' \in \operatorname{Coh}(X')$ be a coherent sheaf on X'. Let $\mathcal{F} = i^*\mathcal{F}'$ be the pullback of $\mathcal{F}'$ to X. Then $\mathcal{F} = 0$ if and only if $\mathcal{F}' = 0$.*

PROOF. If $\mathcal{F}' = 0$ clearly also $\mathcal{F} = 0$. Suppose now that $\mathcal{F} = 0$ and let $p \in X'$ be a point. By hypothesis, we can view p as a point of X. It follows that

$$\mathcal{F}'(p) = \mathcal{F}(p) = 0$$

and by Lemma 1.34 we have $\mathcal{F}' = 0$. □

PROPOSITION 1.36. *The stack $\mathcal{M}_1$ is algebraic.*

PROOF. To show that $\mathcal{M}_1$ is an algebraic stack, it is sufficient to show that for every smooth and connected genus 1 curve C over a field k, there exist a scheme U and a smooth representable morphism $\phi_U : U \to \mathcal{M}_1$ of fibred categories over Sch (here U stands for the fibred category Sch/U over Sch) such that $(C \to \operatorname{Spec} k) \in \mathcal{M}_1(k)$ is in the image of

$$\phi_U(k) : U(k) \to \mathcal{M}_1(k).$$

Indeed, if that is the case, we claim that the morphism of fibred categories over Sch given by

$$\phi = \coprod_U \phi_U : \coprod_U U \to \mathcal{M}_1.$$

is a smooth presentation from a scheme, where the disjoint union runs all over the morphisms constructed above. In order to prove the claim, we have to show that ϕ is representable, smooth and surjective. For every scheme T and morphism $T \to \mathcal{M}_1$, the fibre product in the category of stacks

$$\begin{tikzcd} X \arrow[r] \arrow[d] & T \arrow[d] \\ \coprod_U U \arrow[r] & \mathcal{M}_1 \end{tikzcd}$$

is

$$X = \coprod_U U \times_{\mathcal{M}_1} T \simeq \coprod_U (U \times_{\mathcal{M}_1} T),$$

hence it is an algebraic space, being the disjoint union of algebraic spaces by Lemma 1.32. Let

$$V = \coprod_U V_U \to X$$

be an étale presentation, where each $V_U \to U \times_{\mathcal{M}_1} T$ is an étale presentation. We have to show that the composition $\alpha : V \to X \to T$ is smooth and surjective. First observe that α is smooth, being the disjoint union of the morphisms

$$\alpha_U : V_U \to U \times_{\mathcal{M}_1} T \to T$$

and each of this morphism is smooth by the definition of smoothness for ϕ_U. To prove the surjectivity of α, consider $t : \operatorname{Spec} k \to T$ a k-point of T, where k is a field. The composition

$$\operatorname{Spec} k \to T \to \mathcal{M}_1$$

corresponds to an object $x = (C \to \operatorname{Spec} k) \in \mathcal{M}_1(k)$ by 2-Yoneda's lemma (A.58). But then there exists a scheme U and a smooth representable morphism $\phi_U : U \to \mathcal{M}_1$ of fibred categories over Sch such that x is in the image as above. It follows that there is a k-point p in the fibre product $U \times_{\mathcal{M}_1} T$ which is sent to t through the projection to T. Since $V_U \to U \times_{\mathcal{M}_1} T$ is an étale presentation, it is surjective, so that there exists a k-point of V_U which is sent to p. Taking the disjoint union $V = \coprod_U V_U$, this shows that $V \to T$ is surjective.

We now prove the sufficient condition stated before. Let $C \to \operatorname{Spec} k$ be a smooth and connected genus 1 curve over a field k. First of all, choose an embedding $C \hookrightarrow \mathbb{P}^N_k$ such that $\mathrm{H}^1(C, \mathcal{O}_C(1)) = 0$. For example, if $k = \overline{k}$, we can fix a rational point $P \in C$ and consider the embedding given by the linear system $|3P|$ which is very ample because it has degree 3, see [Har77, Corollary IV.3.2]. In this case,

$$\mathrm{H}^1(C, \mathcal{O}_C(3P)) \simeq (\mathrm{H}^0(C, \mathcal{O}_C(K_C - 3P)))^\vee = 0$$

because $\deg_C(\mathcal{O}_C(K_C - 3P)) = -3 < 0$. Let $P = P(m)$ be the Hilbert polynomial of the embedding, and $H = \mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}})$ the corresponding Hilbert scheme. Let $\pi : \mathcal{C} \to H$ be the universal family, so that there exists a k-point $h \in H(k)$ such that the fibre $\mathcal{C}_h$ is isomorphic to C as closed subschemes of $\mathbb{P}^N_k$. By the definition of the Hilbert scheme (Example A.11), the morphism $\pi : \mathcal{C} \to H$ is proper (because it is projective), flat and finitely presented. Moreover $\mathcal{O}_{\mathcal{C}}(1)$ is a quasi-coherent sheaf on $\mathcal{C}$ which is flat over H, because it is flat as an $\mathcal{O}_{\mathcal{C}}$-module and π is flat. By Corollary B.6 there exists an open neighbourhood $V \subseteq H$ of h such that $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{C}}(1)_{|V} = 0$. By cohomology and base change (Theorem B.5) for all points $t \in V$ it holds $\mathrm{H}^1(\mathcal{C}_t, \mathcal{O}_{\mathcal{C}_t}(1)) = 0$. We restrict further V to another open neighbourhood, which we denote by H', where each fibre is smooth over the base field, geometrically connected and has genus 1. Indeed these are all open conditions, by [Gro67, Théorème 12.2.4.iii], [Gro67, Théorème 12.2.4.viii] and [Har77, Corollary III.9.10]. We get a morphism $H' \to \mathcal{M}_1$ which is given by forgetting the embedding in the projective space, i.e. is the morphism corresponding to the family $\mathcal{C} \times_H H' \to H'$ by 2-Yoneda's lemma (A.58). The morphism $H' \to \mathcal{M}_1$ of stacks is representable by algebraic spaces, since the diagonal of $\mathcal{M}_1$ is representable by algebraic spaces (Lemma 1.32, see also [Alp24, Corollary 3.2.3]). It only remains to show that $H' \to \mathcal{M}_1$ is smooth. In order to do that, we will use the infinitesimal lifting criterion for smoothness ([Alp24, Theorem 3.7.1]). Take a small surjection, i.e. a surjection $\varphi : A' \to A$ of local artinian rings such that $\ker(\varphi) = k$. For every diagram

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, hook] \arrow[rr, bend left] & \operatorname{Spec} A \arrow[r] \arrow[d, hook] & H' \arrow[d] \\
& \operatorname{Spec} A' \arrow[r] \arrow[ur, dashed] & \mathcal{M}_1
\end{tikzcd}$$

we have to show that there exists a dotted arrow completing the diagram. Observe that an arrow from $\operatorname{Spec} A$ to H' corresponds to an embedded family of genus 1 curves curve $\mathcal{C} \subseteq \mathbb{P}^N_A$ by Yoneda's lemma (A.6) and 2-Yoneda's lemma (A.58), while the map from $\operatorname{Spec} k$ to H' corresponds to an embedded curve $C \subseteq \mathbb{P}^N_k$. Moreover, the morphism $\operatorname{Spec} A' \to \mathcal{M}_1$ corresponds to a family of smooth genus 1 curves $\mathcal{C}' \to \operatorname{Spec} A'$. Thus, in order to complete the diagram with the dotted arrow we have to find an embedding $\mathcal{C}' \subseteq \mathbb{P}^N_{A'}$ compatible with the restriction to A. In other words, we have to find a dotted arrow completing the following diagram:

$$\begin{tikzcd}
& \mathbb{P}^N_k & \mathbb{P}^N_A & \mathbb{P}^N_{A'} \arrow[ddl] \\
C \arrow[r, hook] \arrow[ur, hook] \arrow[d] & \mathcal{C} \arrow[r, hook] \arrow[ur, hook] \arrow[d] & \mathcal{C}' \arrow[ur, hook, dashed] \arrow[d] & \\
\operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'. &
\end{tikzcd}$$

We know that morphisms to projective space are determined by line bundles and sections. In particular $\mathcal{C} \hookrightarrow \mathbb{P}^N_A$ is determined by a line bundle $\mathcal{L}$ and by global generating sections $s_0, \dots, s_N \in \mathrm{H}^0(\mathcal{C}, \mathcal{L})$. It is possible to find a line bundle $\mathcal{L}'$ on $\mathcal{C}'$ such that the restriction on $\mathcal{C}$ is isomorphic to $\mathcal{L}$, since an obstruction theory to deforming $\mathcal{L}$ over A' is given by $\mathrm{H}^2(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$, which is zero, see [TV13, Theorem 5.24]. We claim that sections s_i deform to sections $s_i' \in \mathrm{H}^0(\mathcal{C}', \mathcal{L}')$. In order to see this, observe that since $\ker(A' \to A) = k$, we have a short exact sequence

$$0 \to k \to A' \to A \to 0$$

of A'-modules. Tensoring it with A'-module $\mathcal{O}_{\mathcal{C}'}$, which is a flat A'-module (since $\mathcal{C}' \to \operatorname{Spec} A'$ is flat), we get

$$0 \to \mathcal{O}_C \to \mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_{\mathcal{C}} \to 0,$$

and tensoring with $\mathcal{L}'$

$$0 \to \mathcal{L}_{|C} \to \mathcal{L}' \to \mathcal{L} \to 0.$$

Note that $\mathrm{H}^1(C, \mathcal{L}_{|C}) = 0$, because $C \to \operatorname{Spec} k$ corresponds to a point in H'. Passing in cohomology, we get the surjectivity of the map

$$\mathrm{H}^0(\mathcal{C}', \mathcal{L}') \to \mathrm{H}^0(\mathcal{C}, \mathcal{L})$$

so that we can lift the sections s_i as claimed above. The sections s_i are base point free (by the correspondence between maps to the projective space and tuples of line bundles and sections). Moreover $\mathcal{C} \hookrightarrow \mathcal{C}'$ is a closed immersion which is a homeomorphism. It follows that the residue fields at the points of $\mathcal{C}$ and $\mathcal{C}'$ are the same. By Nakayama's lemma, it follows that the sections s_i' are also base point free. This implies that there exists a morphism $i' : \mathcal{C}' \to \mathbb{P}^N_{A'}$ that restricts to $i : \mathcal{C} \hookrightarrow \mathbb{P}^N_A$ over A. The map i' is finite, as it is both proper and quasi-finite. Moreover, i' is a closed immersion. To see this, consider the map

$$\beta : \mathcal{O}_{\mathbb{P}^N_{A'}} \to i'_* \mathcal{O}_{\mathcal{C}'}.$$

The cokernel of β is a coherent sheaf, and it vanishes when restricted to $\mathcal{C}$, since i is a closed immersion. By Corollary 1.35 it follows that $\operatorname{coker}(\beta) = 0$, so that β is surjective. $\square$

CHAPTER 2

Minimal surfaces of general type and their canonical models

In this chapter we study surfaces. More precisely, we are interested in minimal surfaces of general type and in their canonical models. In §2.1 we define the objects of study. In §2.2 we redefine the classical intersection number for proper schemes over an algebraically closed field. Canonical models of minimal surfaces of general type are studied in §2.5, and their singularities, known as *Du Val* singularities are studied in §2.4. In §2.7 we study the automorphism group scheme of minimal surfaces of general type and of their canonical models. Finally in §2.8 we study the finite generation of the pluricanonical rings of minimal surfaces of general type.

2.1. Basics about surfaces

Let k be an algebraically closed field of arbitrary characteristic.

Definition 2.1. A *surface* over k is an integral scheme of dimension 2 which is proper over k. We simply say that S is a surface if the base field is clear from the context.

Observe that if $f : S \to k$ is a surface over k, then f is in particular a finite type morphism of locally noetherian schemes. We will say that S is a *smooth* surface if $f : S \to k$ is smooth, or equivalently if S is a regular scheme ([GW20, Theorem 6.28 and Corollary 6.32]).

Remark 2.2. It is true that every smooth surface over an algebraically closed field k is projective, by a theorem of Zariski and Goodman, see [Băd01, Theorem 1.28].

If S is a Cohen-Macaulay surface over k, then we can define the *dualizing sheaf* $\omega_S = \omega_{S/k}$, which is a coherent sheaf on S (see Section §B.4; we use the fact that a ring of finite type over a noetherian ring is of finite presentation). Recall that ω_S is invertible in a neighbourhood of a point $s \in S$ if and only if the local ring $\mathcal{O}_{S,s}$ is Gorenstein (Proposition B.13). In particular, if S is Gorenstein (e.g. if S is regular or has at most Du Val singularities, see Definition 2.48), then ω_S is an invertible sheaf on S, and it is also called the *canonical sheaf* of S. If S is a smooth surface, then the canonical sheaf is $\omega_S = \det \Omega^1_{S/k}$ by Proposition B.13.

Remark 2.3. If S is a surface, by [Har77, Proposition II.6.15] we have an isomorphism between the group of Cartier divisor modulo linear equivalence on S and $\mathrm{Pic}(S)$. In particular, if ω_S is an invertible sheaf, it corresponds to a class K_S of Cartier divisors modulo linear equivalence. Every Cartier divisor in this class is called a *canonical divisor*.

Definition 2.4. If S is a smooth surface over k, we define the the *m-plurigenus* for $m \geq 1$ as

$$p_m(S) = \dim_k \mathrm{H}^0(S, \omega_S^{\otimes m}).$$

If $m = 1$ we write $p_g(S) = p_m(S)$ and call it the *geometric genus of S*.
We define the *irregularity* as

$$q(S) = \dim_k \mathrm{H}^1(S, \mathcal{O}_S),$$

and the *Euler characteristic* of S as

$$\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S).$$

In particular, these numbers are birational invariants for smooth surfaces, see [Har77, Theorem II.8.19, Exercise II.8.8] for the invariance of p_m and [Har77, Remark III.7.12.3, Corollary V.5.6] for the invariance of the irregularity q. Another birational invariant for smooth surfaces is the Kodaira dimension (see [Har77, Introduction §V.6]).

DEFINITION 2.5. Let S be a smooth surface over k. The *Kodaira dimension of S* is $\kappa(S) = -\infty$ if $p_m(S) = 0$ for all $m \geq 1$, otherwise is defined as

$$\kappa(S) = \min\left\{ d \;\middle|\; \frac{p_m(S)}{m^d} \text{ is bounded from above} \right\}.$$

DEFINITION 2.6. A smooth surface S over k is of *general type* if $\kappa(S) = 2$.

DEFINITION 2.7. If X is a surface over k, we define the *Euler characteristic of X* as

$$\chi(\mathcal{O}_X) = \sum_{i=0}^{2} (-1)^i \dim_k \mathrm{H}^i(X, \mathcal{O}_X).$$

If S is a smooth surface, $\chi(\mathcal{O}_S)$ is the same as in Definition 2.4.

REMARK 2.8. Let $S \to \operatorname{Spec} k$ be a smooth surface over k. Since S is proper over k, and $\operatorname{Spec} k$ is noetherian and separated, it follows that S is also noetherian and separated. Thus, it makes sense to talk about Weil divisors on S. Moreover, since S is smooth, then S is a regular scheme and it is in particular locally factorial, see [Har77, Remark II.6.11.1A]. It follows by [Har77, Corollary II.6.16] that there is an isomorphism between the group of Weil divisors modulo linear equivalence and $\operatorname{Pic}(S)$. By Remark 2.3 these groups are also isomorphic to the group of Cartier divisors modulo linear equivalence.

DEFINITION 2.9. Let S be a smooth surface over k. A *curve over S* is an effective divisor on S.

REMARK 2.10. In the above definition, it makes sense to talk about an effective divisor without specifying if we are considering a Weil divisor or a Cartier divisor. Indeed, by Remark 2.8, the group of Weil divisors is isomorphic to the group of Cartier divisors. Moreover by [Har77, Remark II.6.17.1] effective Cartier divisors correspond exactly to the effective Weil divisor.
Observe further that by [Har77, Remark II.6.17.1] there is a bijective correspondence between curves on S and *locally principal* closed subschemes of S, i.e. subschemes whose sheaf of ideals is locally generated by a single element. This way, we will also see a curve as a locally principal closed subscheme of S. When we say that a curve over S is *connected* we mean that the corresponding closed subscheme is connected.

Let S be a surface over k. Let $Z_1 S$ be the free abelian group generated by closed integral subschemes of dimension 1. Observe that if S is a smooth surface, then $Z_1 S$ is exactly the group of Weil divisors; and if that is the case, the class group $\operatorname{Cl} S$ is the quotient group of $Z_1 S$ by the relation given by the linear equivalence.
Let $C \subset S$ be a closed integral subscheme of dimension 1 and let $\nu : \tilde{C} \to C$ be the normalization of C. If $\mathcal{L} \in \operatorname{Pic}(S)$ is a line bundle, we define

$$\mathcal{L} \cdot C = \deg_{\tilde{C}}(\nu^*(\mathcal{L}_{|C})).$$

We define a pairing between $\operatorname{Pic}(S)$ and $Z_1 S$ given by

$$\begin{array}{ccc} \operatorname{Pic}(S) \times Z_1 S & \dot{\to} & \mathbb{Z} \\ (\mathcal{L}, C) & \mapsto & \mathcal{L} \cdot C. \end{array}$$

DEFINITION 2.11. Let S be a surface over k. We say that two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(S)$ are *numerically equivalent* if $\mathcal{L}_1 \cdot D = \mathcal{L}_2 \cdot D$ for all $D \in Z_1S$. We say that $D_1, D_2 \in Z_1S$ are *numerically equivalent* if $\mathcal{L} \cdot D_1 = \mathcal{L} \cdot D_2$ for all $\mathcal{L} \in \mathrm{Pic}(S)$. We define N^1S as the quotient group of $\mathrm{Pic}(S)$ by the relation given by the numerical equivalence and N_1S as the quotient group of Z_1S by the relation given by the numerical equivalence.

It follows that we have a non degenerate bilinear pairing

$$N^1S \times N_1S \to \mathbb{Z},$$

see [Laz04, Section §1.4] for details. Recall that if S is a smooth surface over k and C is a curve over S, then C is in particular an element of Z_1S.

If S is a smooth surface, by Remarks 2.3 and 2.8, for each curve C there is an associated line bundle of S denoted by $\mathcal{O}_S(C)$.

DEFINITION 2.12. Let S be a smooth surface over k and let $C \subset S$ be a curve. We define the *self-intersection* of C on S as

$$C^2 = \mathcal{O}_S(C) \cdot C.$$

More generally, we define the *intersection number* of two divisors C, D on a smooth surface S as

$$C \cdot D = \mathcal{O}_S(C) \cdot D.$$

REMARK 2.13. If S is a smooth surface, we have that $N^1S = N_1S$ and we call it the *Néron-Severi group*, [Har77, Remark V.1.9.1]. The Néron-Severi group is denoted by Num S. It follows that we have a non degenerate bilinear pairing

$$\mathrm{Num}\ S \times \mathrm{Num}\ S \to \mathbb{Z}.$$

REMARK 2.14. If C, D are two distinct integral closed subschemes of dimension 1 of a smooth surface S, then $C \cdot D \geq 0$ by [Har77, Proposition V.1.4].

DEFINITION 2.15. A curve E on a smooth surface S over k is said to be a (-1)-*curve* or an *exceptional curve of the first kind* if $E \simeq \mathbb{P}^1_k$ and $E^2 = -1$.
A curve E on a smooth surface S over k is said to be a (-2)-*curve* if $E \simeq \mathbb{P}^1_k$ and $E^2 = -2$.

DEFINITION 2.16. Let C be an integral scheme of dimension 1 proper over k. We define the *arithmetic genus of* C as

$$p_a(C) = h^1(C, \mathcal{O}_C).$$

PROPOSITION 2.17. *Let C be an integral scheme of dimension* 1 *projective over k. Then $p_a(C) = 0$ if and only if $C \simeq \mathbb{P}^1_k$.*

PROOF. If $C \simeq \mathbb{P}^1_k$, then the arithmetic genus and the geometric genus coincide by [Har77, Proposition IV.1.1]. The geometric genus of $\mathbb{P}^1_k$ is

$$p_g(\mathbb{P}^1_k) = \dim_k \mathrm{H}^0(\mathbb{P}^1_k, \omega_{\mathbb{P}^1_k}) = 0$$

because $\deg(\omega_{\mathbb{P}^1_k}) = -2$. It follows that $p_a(\mathbb{P}^1_k) = 0$.
Suppose now that $p_a(C) = 0$. Consider the normalization

$$\nu : \tilde{C} \to C$$

of C and the induced morphism of sheaves

$$\nu^{\#} : \mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}}.$$

For every closed point $P \in C$ we have

$$\nu^{\#}_P : \mathcal{O}_{C,P} \hookrightarrow (\nu_* \mathcal{O}_{\tilde{C}})_P \simeq \tilde{\mathcal{O}}_{C,P}.$$

by definition of normalization, see also [Stacks, Lemma 0C3B]. It follows that $\nu^\#$ is injective. The coherent sheaf $F = \mathrm{coker}(\nu^\#)$ is supported on the singular locus of C, and we have an exact sequence of sheaves on C given by

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}} \to F \to 0.$$

Passing in cohomology we get a long exact sequence

$$0 \to \mathrm{H}^0(C, \mathcal{O}_C) \to \mathrm{H}^0(C, \nu_* \mathcal{O}_{\tilde{C}}) \to \mathrm{H}^0(C, F) \to 0$$

because $\mathrm{H}^1(C, \mathcal{O}_C) = 0$ by hypothesis. Since $C, \tilde{C}$ are integral schemes over an algebraically closed field, then $\mathrm{H}^0(C, \mathcal{O}_C) \simeq \mathrm{H}^0(C, \nu_* \mathcal{O}_{\tilde{C}}) \simeq k$. It follows that $\mathrm{H}^0(C, F) = 0$ and in particular $F = 0$. This means that for all $P \in C$ closed, $\mathcal{O}_{C,P}$ is normal, hence C is normal. But since C has dimension 1 and k is perfect, C is smooth over k and ν is an isomorphism away from singular points. Since $F = 0$, ν is an isomorphism of schemes. Thus, C is a smooth curve, and $p_a(C) = p_g(C) = 0$. In particular, $C \simeq \mathbb{P}^1_k$ ([Har77, Example IV.1.3.5]). □

PROPOSITION 2.18. *Let S be a smooth surface and let $E \subseteq S$ be a closed integral subscheme of dimension* 1.

(1) *E is a (-1)-curve if and only if $p_a(E) = 0$ and $E^2 = -1$;*
(2) *E is a (-2)-curve if and only if $p_a(E) = 0$ and $E^2 = -2$.*

PROOF. We know that S is projective over k by Remark 2.2. Since E is a closed subscheme of S, then E is also projective over k. Then $E \simeq \mathbb{P}^1$ if and only if $p_a(E) = 0$ by Proposition 2.17. □

THEOREM 2.19 (Castelnuovo). *If E is a (-1)-curve on a smooth surface S, then there exists a morphism $f : S \to S_0$ to a smooth surface S_0, and a point $P \in S_0$, such that S is isomorphic via f to the blow-up of S_0 with centre P, and E is the exceptional curve.*

PROOF. See [Har77, Theorem V.5.7]. □

On the other hand, if S_0 is a smooth surface and $f : S \to S_0$ is the blow-up of S_0 at a point P, the inverse image of P through f is a curve $E = f^{-1}(P)$. In particular, S is again a smooth surface, and E is a (-1)-curve, see [Har77, Proposition V.3.1].

DEFINITION 2.20. We say that a smooth surface S is *minimal* if S does not contain (-1)-curves.

By Castelnuovo's theorem (Theorem 2.19), a surface is minimal if and only if there does not exist another smooth surface S' such that S is isomorphic to the blow-up of S' in a point. Again by Castelnuovo's theorem, we see that every smooth surface is birational to a minimal smooth surface. Thus, in order to classify smooth surfaces up to birationality, it is sufficient to classify the birational classes of smooth minimal surfaces.
In characteristic zero, the classification is due to Enriques and is based on the Kodaira dimension. The classification has been generalized in positive characteristic by the work of Mumford and Bombieri in [Mum69],[BM77],[BM76].
The classification of minimal surfaces of general type has not be established yet.

In this chapter, we will mainly consider minimal surfaces of general type.

NOTATION 2.21. When we say that a surface is minimal of general type we are assuming that S is smooth.

If S is a minimal surface of general type, we have already seen that there exists a canonical divisor K_S (Remark 2.3), which by Remark 2.8 is both a Cartier divisor and Weil divisor. Thus it makes sense to define

$$K_S^2 := \omega_S \cdot K_S.$$

In the next section we generalize the self-intersection of the canonical bundle for Gorenstein surfaces. More generally, we define the intersection between two line bundles on a surface.

2.2. Intersection theory

We fix an algebraically closed field k of arbitrary characteristic. We will define intersection theory for a scheme X which is proper over k. We follow [Kle66, § I.1, II.2]. See also [GW23, §23.15].

Definition 2.22. Let $\mathbb{Q}[n_1, \ldots, n_t]$ be the polynomial ring in variables $n_1, \ldots, n_t$ over the rational numbers. If $f \in \mathbb{Q}[n_1, \ldots, n_t]$, then f is called a *numerical polynomial* if for every $(n_1, \ldots, n_t) \in \mathbb{Z}^t$ the value of f at $(n_1, \ldots, n_t)$ is an integer.

Theorem 2.23 (Snapper). *Let X be a scheme proper over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_t$ be t invertible $\mathcal{O}_X$-modules, $t \geq 0$. Then the function*

$$f_{\mathcal{F}}(n_1, \ldots, n_t) = \chi(\mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_t^{\otimes n_t})$$

is a numerical polynomial in $n_1, \ldots, n_t$ of degree equal to $\dim(\mathrm{Supp}(\mathcal{F}))$.

Proof. [Kle66, § I.1, Theorem], [Băd01, Theorem 1.1]. □

We now redefine the classical intersection number.

Definition 2.24. Let $t \geq 0$ be an integer and let $\mathcal{L}_1, \ldots, \mathcal{L}_t$ be t invertible sheaves on a scheme X proper over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module such that $\dim(\mathrm{Supp}(\mathcal{F})) \leq t$. The *intersection number* of $\mathcal{L}_1, \ldots, \mathcal{L}_t$ with $\mathcal{F}$ is by definition the coefficient of the monomial $n_1 n_2 \ldots n_t$ in the numerical polynomial $\chi(\mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_t^{\otimes n_t})$. We denote this integer by $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_t \cdot \mathcal{F})$.

If $\mathcal{L}_1 = \mathcal{L}_2 = \ldots = \mathcal{L}_t = \mathcal{L}$ we write $(\mathcal{L}^t \cdot \mathcal{F})$ instead of $(\mathcal{L} \cdot \ldots \cdot \mathcal{L} \cdot \mathcal{F})$. If $\mathcal{F} = \mathcal{O}_Z$ with Z a closed subscheme of X, we write $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_t \cdot Z)$ instead of $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_t \cdot i_*\mathcal{O}_Z)$, where $i : Z \hookrightarrow X$ is the corresponding closed immersion. If $\mathcal{F} = \mathcal{O}_X$ we simply write $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_t)$ instead of $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_t \cdot \mathcal{O}_X)$. Finally, if $\mathcal{L}_i = \mathcal{O}_X(D_i)$ for Cartier divisors D_i we also write $(D_1 \cdot \ldots \cdot D_t \cdot \mathcal{F})$ instead of $(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_t \cdot \mathcal{F})$.
Suppose now that S is a surface over k and that $\mathcal{L}_1, \mathcal{L}_2$ are two invertible sheaves on S. Considering $\mathcal{F} = \mathcal{O}_S$ in Definition 2.24 we obtain the definition of the intersection number $(\mathcal{L}_1 \cdot \mathcal{L}_2)$.

Definition 2.25. If S is a Gorenstein surface, then ω_S is a line bundle on S, and we define the self-intersection of the canonical bundle as

$$K_S^2 := (\omega_S \cdot \omega_S).$$

If S is smooth, then by Remark 2.3 we have that K_S^2 is also equal to

$$(K_S \cdot K_S).$$

Proposition 2.26. *If S is a smooth surface and $\mathcal{L}_1, \mathcal{L}_2$ are two line bundles on S, then $(\mathcal{L}_1 \cdot \mathcal{L}_1)$ coincides with the classical intersection number of Remark 2.13. In this case, we will denote $(\mathcal{L}_1 \cdot \mathcal{L}_2)$ simply by $\mathcal{L}_1 \cdot \mathcal{L}_2$.*

Proof. See [Băd01, Corollary 1.20]. □

Definition 2.27. Let X be a scheme of dimension $r \geq 1$ which is proper over k and let $\mathcal{L}$ be a line bundle on X. We say that $\mathcal{L}$ is *nef* or *numerically effective* if $(\mathcal{L} \cdot C) \geq 0$ for all integral closed subscheme of X of dimension 1.
In particular, if S is a surface and $\mathcal{L}$ is a line bundle on S, then $\mathcal{L}$ is nef if $(\mathcal{L} \cdot C) \geq 0$ for all integral curves on S.

By Lemma [GW23, Lemma 23.64] we have that formation of the numerical polynomial of Definition 2.23 is invariant under changing the base field. In particular we obtain the following lemma.

LEMMA 2.28. *Let k be a field, and let X be a scheme proper over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and $\mathcal{L}_1, \dots, \mathcal{L}_t$ be t invertible $\mathcal{O}_X$-modules. Suppose that $k \subseteq K$ is a field extension and let $\mathcal{L}_{1,K}, \dots, \mathcal{L}_{t,K}, \mathcal{F}_K$ be the pullbacks of $\mathcal{L}_1, \dots, \mathcal{L}_t, \mathcal{F}$ to $X \otimes_k K$. Then*

$$(\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_t \cdot \mathcal{F}) = (\mathcal{L}_{1,K} \cdot \ldots \cdot \mathcal{L}_{t,K} \cdot \mathcal{F}_K).$$

PROOF. See [GW23, Remark 23.71]. □

If X is a quasi-compact and quasi-separated scheme, in Definition B.7 we defined what an ample invertible $\mathcal{O}_X$-module is.
If X is a scheme of finite type over a noetherian ring A, then an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is ample if and only if some tensor power $\mathcal{L}^{\otimes n}$, $n > 0$, is very ample over $\operatorname{Spec} A$, see [Har77, Theorem II.7.6].

THEOREM 2.29 (Nakai-Moishezon Criterion). *Let X be a scheme proper over k and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is ample if and only if for every integral closed subscheme $Z \subseteq X$ of dimension $t > 0$ we have $(\mathcal{L}^t \cdot Z) > 0$.*

PROOF. See [Băd01, Theorem 1.22]. □

COROLLARY 2.30. *If S is a surface over k and $\mathcal{L}$ is an invertible $\mathcal{O}_S$-module, then $\mathcal{L}$ is ample if and only if $(\mathcal{L}^2) > 0$ and $(\mathcal{L} \cdot C) > 0$ for every integral closed subscheme $C \subset S$ of dimension 1. If in addition $\mathrm{H}^0(S, \mathcal{L}) \neq 0$, then the condition $(\mathcal{L}^2) > 0$ is not needed.*

PROOF. See [Băd01, Corollary 1.24]. □

2.3. Characterization of minimal surfaces of general type

We start this section by proving that for a surface of general type (Definition 2.6), requiring the canonical bundle ω_S to be nef (Definition 2.27) is equivalent to requiring that S is minimal (Definition 2.20).

LEMMA 2.31. *Let S be a smooth surface.*

- *If ω_S is nef, then S is minimal.*
- *If $\kappa(S) \geq 0$, then S is minimal if and only if ω_S is nef.*

PROOF. Suppose that ω_S is nef, and suppose by contradiction that $E \subset S$ is a (-1)-curve on S. By adjunction formula we have

$$-2 = 2p_a(E) - 2 = (\mathcal{O}_S(E) \otimes_{\mathcal{O}_S} \omega_S) \cdot E$$

where $p_a(E) = 0$ is the arithmetic genus of $E \simeq \mathbb{P}^1_k$. Since $E^2 = -1$ it follows that $\omega_S \cdot E = -1$ which is an absurd because ω_S is nef.
Suppose now that $\kappa(S) \geq 0$. It follows that there exists an integer $m > 0$ such that $\dim_k \mathrm{H}^0(S, \omega_S^{\otimes m}) \neq 0$. Let

$$D = \sum_{a_i > 0} a_i C_i \in |\omega_S^{\otimes m}|$$

be an effective divisor on S, where each C_i is a closed integral subscheme of S of dimension 1. Suppose by contradiction that ω_S is not nef; thus there exists an integral closed subscheme $C \subset S$ of dimension 1 such that $\omega_S \cdot C < 0$. Since $D \in |\omega_S^{\otimes m}|$ we also have

$$\sum_i a_i C_i \cdot C = D \cdot C < 0.$$

If $C \neq C_i$ for all i, then $C \cdot C_i \geq 0$ for all i by Remark 2.14, and then $D \cdot C \geq 0$. This means that there exists an index i such that $C = C_i$. From $D \cdot C < 0$ we deduce that $C^2 < 0$. Again by adjunction formula we have

$$-2 \leq 2p_a(C) - 2 = (\mathcal{O}(C) \otimes \omega_S) \cdot C = C^2 + \omega_S \cdot C \leq -2,$$

so that the only possibility is $p_a(C) = 0$ and $C^2 = \omega_S \cdot C = -1$. It follows by Proposition 2.18 that C would be a (-1)-curve, which is absurd. □

Definition 2.32. Let X be an integral scheme proper over a field k and let $\mathcal{L}$ be a line bundle on X. We say that $\mathcal{L}$ is *big* if there exists $m \in \mathbb{N}^+$ such that

$$\dim \varphi_{|\mathcal{L}^{\otimes m}|}(X \setminus \mathrm{Bs}|\mathcal{L}^{\otimes m}|) = \dim X$$

where $\varphi_{|\mathcal{L}^{\otimes m}|}$ is the map associated to the complete linear system $|\mathcal{L}^{\otimes m}|$. Equivalently, $\mathcal{L}$ is big if the dimension of the space of global sections of $\mathcal{L}^{\otimes m}$ grows as $m^{\dim X}$ when m tends to infinity, see [Laz04, Definition 2.1.3 and Corollary 2.1.37].

Example 2.33. If S is a surface and $\mathcal{L} \in \mathrm{Pic}(S)$ is a line bundle, then $\mathcal{L}$ is big if there exists a positive integer m such that

$$\dim \varphi_{|\mathcal{L}^{\otimes m}|}(S \setminus \mathrm{Bs}|\mathcal{L}^{\otimes m}|) = \dim S = 2.$$

Equivalently, $\mathcal{L}$ is big if the dimension of the space of global sections of $\mathcal{L}^{\otimes m}$ grows quadratically with m when m tends to infinity.

Corollary 2.34. *Let S be a smooth surface. Then S is a minimal surface of general type if and only if ω_S is big and nef.*

Proof. The bigness of the canonical bundle ω_S means that the Kodaira dimension of S is 2, so by definition S is of general type. Finally, S is minimal if and only if ω_S is nef by Lemma 2.31. □

Remark 2.35. Let S be a minimal surface of general type. It holds $K_S^2 = \omega_S^2 > 0$ because otherwise $h^0(S, \omega_S^{\otimes m})$ would not increase quadratically with m as m tends to infinity by asymptotic Riemann-Roch theorem [Laz04, Corollary 1.4.41]. Moreover $\omega_S \cdot C \geq 0$ for every integral closed subscheme $C \subset S$ of dimension 1 because ω_S is nef.
Thus, by the Nakai-Moishezon criterion of ampleness (Theorem 2.29 and Corollary 2.30), ω_S is ample if and only if for every integral closed subscheme $C \subset S$ of dimension 1, we have $\omega_S \cdot C \neq 0$.

Proposition 2.36 (Hodge index theorem). *Let S be a smooth surface. Let D be a divisor on S such that $(D^2) = (D \cdot D) > 0$. Then for every divisor E such that $(D \cdot E) = 0$ we have $(E^2) \leq 0$. Moreover $(E^2) = 0$ if and only if E is numerically equivalent to 0.*

Proof. See [Băd01, Corollary 2.4]. □

Corollary 2.37. *Let S be a minimal surface of general type. The following are equivalent:*

(i) *ω_S is ample;*
(ii) *S does not contain (-2)-curves.*

Proof. Suppose first that ω_S is ample. Suppose by contradiction that there exists a (-2)-curve $C \subset S$. In particular, $p_a(C) = 0$ and $(C^2) = -2$. By adjunction formula we have

$$-2 = 2p_a(C) - 2 = (C^2) + (\omega_S \cdot C) = -2 + (\omega_S \cdot C).$$

Thus, $(\omega_S \cdot C) = 0$, which is an absurd because ω_S is ample.
Conversely, suppose that S does not contain (-2)-curves. Suppose by contradiction that ω_S is not ample. By Remark 2.35 there exists an integral closed subscheme $C \subset S$ of dimension 1

such that $(\omega_S \cdot C) = 0$. By Hodge index theorem (Proposition 2.36) we have $(C^2) < 0$ because $\omega_S^2 > 0$ and C can not be numerically equivalent to zero [Bǎd01, Proof of Theorem 9.1]. Thus by adjunction formula we have

$$2p_a(C) - 2 = C^2 + (\omega_S \cdot C) = (C^2) < 0.$$

It follows that $p_a(C) = 0$ and $(C^2) = -2$. In other words, C is a (-2)-curve on S. □

2.4. Du Val singularities

Let k be an algebraically closed field and let X be a normal scheme of dimension 2 which is proper over k (i.e. a normal surface).

Definition 2.38. A *resolution* or *desingularization* of X is a proper morphism $\pi : S \to X$ from a surface S which is smooth over k, such that $\pi_* \mathcal{O}_S = \mathcal{O}_X$ and

$$\pi_{|\pi^{-1}(X \setminus \mathrm{Sing}(X))} : \pi^{-1}(X \setminus \mathrm{Sing}(X)) \to X \setminus \mathrm{Sing}(X)$$

is an isomorphism.

Remark 2.39. If $\pi : S \to X$ is a resolution of X, then by Zariski's main theorem ([Har77, Corollary III.11.4]) we have that the fibres of π are connected.

Definition 2.40. A resolution is said to be *minimal* if for every other resolution $\pi' : S' \to X$, there exists a unique morphism $u : S' \to S$ such that $\pi' = \pi \circ u$ as in the following diagram:

$$\begin{tikzcd} S' \arrow[rr, "u"] \arrow[dr, "\pi'"'] & & S \arrow[dl, "\pi"] \\ & X. & \end{tikzcd}$$

Proposition 2.41. *Let $\pi : S \to X$ be a resolution. Then π is minimal if and only if for every $x \in \mathrm{Sing}(X)$, there are no exceptional curves of the first kind among the components of the reduced fibre $E = \pi^{-1}(x)_{\mathrm{red}}$.*

Proof. Suppose first that $\pi : S \to X$ is a minimal resolution. If X is smooth over k, then $\mathrm{Sing}(X)$ is empty, thus the condition on the preimage of singular points is trivially satisfied. On the other hand, if $x \in \mathrm{Sing}(X)$, suppose by contradiction that $C \subset S$ is a (-1)-curve which is an irreducible component of the reduced fibre $\pi^{-1}(x)_{\mathrm{red}}$. By Castelnuovo's theorem (Theorem 2.19) the curve C can be contracted to a point over another smooth surface S'. Then π factors through S' and π would not be minimal.
Suppose now that $\pi : S \to X$ is a resolution such that there are no exceptional curves of the first kind among the components of the reduced fibres of singular points. Let $\pi' : S' \to X$ another desingularization. Let $u = \pi^{-1} \circ \pi'$. Then u is a birational map. By the structure theorem of birational maps between two smooth projective surfaces S' and S (S' and S are projective because they are smooth and proper over k and the morphisms π and π' are also projective), there exists a morphism $v : S'' \to S$ that is composite of finitely many blow-ups, such that $u \circ v$ is a morphism

$$\begin{tikzcd} & S'' \arrow[dl, "v"'] \arrow[dr] & \\ S' \arrow[rr, dashed, "u"'] & & S. \end{tikzcd}$$

Choose v such that it is a composite of a minimal number of blow-ups. We must show that this number is zero, i.e. that u is a morphism.
Assume by contradiction that the number of blow-ups involved is strictly positive. Then S'' contains an exceptional curve of the first kind, E'', contained in the fibre $(\pi' \circ v)^{-1}(x)$. If

$(u \circ v)(E'')$ is a point, then we can contract E'' to a surface $\tilde{S}$ which dominates both S' and S, and this would contradict the minimality of v. Thus $(u \circ v)(E'')$ is a component, say E_1, of the fibre $\pi^{-1}(x)$. Since $u \circ v$ is a birational morphism, $u \circ v$ is a composite of a finite number of blow-ups (and isomorphisms). Since $\pi^{-1}(x)_{\mathrm{red}}$ does not contain (-1)-curves, we have $(E_1^2) \leq -2$. However, the proper transform of a curve C by a blow-up has self-intersection less than or equal to (C^2). Hence we would get $(E'')^2 \leq -2$, which contradicts the hypotheses that E'' is an exceptional curve of the first kind. It follows that u is a morphism. $\square$

REMARK 2.42. Let X be a normal surface. Then X has a finite number of singularities, i.e. non regular points. Indeed, since X is normal, then X is regular in codimension one by [Har77, Theorem I.6.2A]. It follows that $\mathrm{Sing}(X)$ is just a set of points. Since X is noetherian, there is just a finite number of such points.

PROPOSITION 2.43. *Let X be a normal surface. Then there exists a minimal resolution resolution of X.*

PROOF. First, a resolution of X exists by a theorem of Zariski-Abhyankar [Zar39], [Zar42], [Abh57].
It is not restrictive to assume that X is singular in a unique point $x \in X$. If not, for every singular point $x' \in X$ we can choose an affine open $x' \subset U \subseteq X$ in which x' is the unique singular point. At this point we work locally, and we eventually glue all the minimal resolutions of the singular points.
Suppose now that $x \in X$ is the unique singular point and let $\pi : S \to X$ be an arbitrary desingularization. If π is minimal we are done. Otherwise, by Proposition 2.41, in the reduced fibre $E = \pi^{-1}(x)_{\mathrm{red}}$ there exists a component E_1 of E which is a (-1)-curve. Thus we can contract E_1 to another smooth surface S_1 and we get a desingularization $\pi_1 : S_1 \to X$. If π_1 is minimal we are done. Otherwise, notice that the fibre $\pi_1^{-1}(x)_{\mathrm{red}}$ has $n-1$ components, where n is the number of components of E. Repeating this process, we necessarily arrive in a finite number of steps to a desingularization in which the reduced fibre over x does not contains exceptional curves of the first kind. In other words, we necessarily arrive to a minimal desingularization. $\square$

REMARK 2.44. It is clear that the minimal resolution of Proposition 2.43 is unique up to isomorphism, by Definition 2.40.

DEFINITION 2.45. Let X be a normal surface. Let $x \in X$ be a singular point. We say that

(1) $x \in X$ is a *rational* singularity if there exist a desingularization $\pi : S \to X$ and an affine open neighbourhood U of x in X such that $U \setminus \{x\}$ is smooth over k and $(R^1\pi_*\mathcal{O}_S)_{|U} = 0$;
(2) $x \in X$ is a *Du Val* singularity or a *rational double point* if x is a rational singularity and x is a Gorenstein point (i.e. $\mathcal{O}_{X,x}$ is Gorenstein).

REMARK 2.46. Definition 2.45.(1) is independent on the choice of the desingularization, in the sense that if U is a neighbourhood of x in X such that $U \setminus \{x\}$ is smooth over k, and if $\pi : S \to X$ is a desingularization such that $(R^1\pi_*\mathcal{O}_S)_{|U} = 0$, then for every other desingularization $\pi' : S' \to X$ it holds that $(R^1\pi'_*\mathcal{O}_{S'})_{|U} = 0$, see [Băd01, Definition 3.17].

PROPOSITION 2.47. *Let X be a normal surface over k. Then each singular point of X is a rational singularity if and only if there exists a desingularization $\pi : S \to X$ such that $R^1\pi_*\mathcal{O}_S = 0$.*

PROOF. Suppose first that there exists a desingularization $\pi : S \to X$ such that $R^1\pi_*\mathcal{O}_S = 0$. Then each singular point of $x \in X$ is rational because it is sufficient to consider an open

neighbourhood U of x in X in which x is the unique singular point of U.
Suppose now that X has only rational singularities. Let $x_1, \ldots, x_n$ be the singular points of X. For all $i = 1, \ldots, n$, let U_i be an affine open neighbourhood of x_i, such that $U_i \setminus \{x_i\}$ is smooth over k and let $\pi_i : S_i \to U_i$ be a desingularization. Since x_i is a rational singularity, we have $R^1(\pi_i)_*\mathcal{O}_{S_i} = 0$. Denote $U_i' = U_i \setminus \{x_i\}$. For all i, the restriction

$$\pi_{i|\pi_i^{-1}(U_i')} : \pi_i^{-1}(U_i') \to U_i'$$

is an isomorphism. Let $S_{n+1} = X \setminus \left(\bigcup_{i=1}^n U_i\right)$ and consider also the identity morphism

$$\mathrm{id}_{S_{n+1}} : S_{n+1} \to S_{n+1}.$$

It follows that we can glue the π_i's and $\mathrm{id}_{S_{n+1}}$ together to a morphism

$$\pi : S \to X$$

where

$$S = \left(\coprod_{i=1}^{n+1} S_i\right) \Big/ \sim$$

with the relation $\sim$ given by $s \sim s'$ in S if $s = s'$ or if $s \in S_i$, $s' \in S_j$ and $\pi_i(s) = \pi_j(s')$. The morphism π is proper because being proper is a local property on the target [GW20, Proposition 12.58], and each π_i is proper. Moreover, S is smooth over k because each S_i is smooth over k. For all $i = 1, \ldots, n$ we have that

$$\pi_{|S_i} : S_i \to U_i$$

induces an isomorphism of sheaves

$$\varphi_i : \mathcal{O}_{U_i} \simeq (\pi_{|S_i})_*\mathcal{O}_{S_i}.$$

The same holds also for $\mathrm{id}_{S_{n+1}}$. Then π induces an isomorphism $\mathcal{O}_X \simeq \pi_*\mathcal{O}_S$ of sheaves on X, by glueing the isomorphisms φ_i.
For all $x \in X$, we have

$$(R^1\pi_*\mathcal{O}_S)_x \simeq \mathrm{H}^1(\pi^{-1}(x), \mathcal{O}_{|\pi^{-1}(x)})$$

by [Har77, Proposition III.8.1]. Since π is an isomorphism outside the singular locus of X, it follows that $R^1\pi_*\mathcal{O}_S$ is supported on the singular locus of X, because if $x \in X$ is a regular point, then $\pi^{-1}(x)$ is just a point. Working locally, if $x \in U_i$ is a singular point, we observe that

$$(R^1\pi_*\mathcal{O}_S)_x = (R^1(\pi_i)_*\mathcal{O}_{S_i})_x = 0.$$

It follows that $R^1\pi_*\mathcal{O}_S = 0$. □

Definition 2.48. If X is a normal Gorenstein surface whose singular points are rational singularities, we will say that X *has at most Du Val singularities.*

Proposition 2.49. *Let $\pi : S \to X$ be the minimal desingularization of a normal surface with at most Du Val singularities, then*

$$\pi_*\omega_S = \omega_X.$$

Proof. See [Băd01, Corollary 4.19]. □

Proposition 2.50. *Let X be a normal surface and let $\pi : S \to X$ be a resolution. The following are equivalent:*

(1) $R^1\pi_*\mathcal{O}_S = 0$;
(2) $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_X)$.

PROOF. The Leray spectral sequence of Theorem B.15 gives an exact sequence

$$\begin{aligned}0 \to \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^1(S, \mathcal{O}_S) \to \mathrm{H}^0(X, R^1\pi_*\mathcal{O}_S) \to \\ \to \mathrm{H}^2(X, \mathcal{O}_X) \to \mathrm{H}^2(S, \mathcal{O}_S) \to 0.\end{aligned}$$

By a standard dimension argument it follows that

$$\chi(\mathcal{O}_X) - \chi(\mathcal{O}_S) = \dim_k \mathrm{H}^0(X, R^1 f_*\mathcal{O}_S).$$

Since $R^1 f_*\mathcal{O}_S$ is supported on the singular locus of X, the equivalence of conditions (1) and (2) follows. □

LEMMA 2.51. *Let X be a scheme of finite type over a field k and let K/k be a field extension. Let $X_K = X \otimes_k \operatorname{Spec} K$ be the base change of X to $\operatorname{Spec} K$. Then X is Gorenstein if and only if X_K is Gorenstein.*

PROOF. See [Stacks, Lemma 0C03]. □

LEMMA 2.52. *Let k be an algebraically closed field and let $X \to \operatorname{Spec} k$ be a normal surface over k. Let K be another algebraically closed field and let K/k be a field extension. Denote by $X' = X \otimes_k K$ the base change of X to K. Then X has at most Du Val singularities if and only if X' has at most Du Val singularities.*

PROOF. First observe that X is Gorenstein if and only if X' is Gorenstein by Lemma 2.51. Therefore we can assume that X and X' are Gorenstein. Let $f : S \to X$ be a desingularization of X and $f' : S' \to X'$ the base change of f through $X' \to X$. Thus, we have the following diagram

$$\begin{tikzcd} S' \arrow[r, "g'"] \arrow[d, "f'"'] & S \arrow[d, "f"] \\ X' \arrow[r, "g"'] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

We have that S' is again smooth over $\operatorname{Spec} K$ by [Stacks, Lemma 01VB]. Moreover, f' is again proper by [Stacks, Lemma 01W4]. Observe further that

$$f'_*\mathcal{O}_{S'} = f'_*(g')^*\mathcal{O}_S \simeq g^* f_*\mathcal{O}_S = g^*\mathcal{O}_X = \mathcal{O}_{X'}$$

where the second isomorphism comes from flat base change (Lemma B.1). Using the characterization of Proposition 2.50 we have to show that $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_X)$ if and only if $\chi(\mathcal{O}_{S'}) = \chi(\mathcal{O}_{X'})$. This is clear by Corollary B.3. □

2.5. Canonical models of surfaces

Let S be a minimal surface of general type over an algebraically closed field k. We define the *canonical ring of S* as

$$R(S) = \bigoplus_{m \geq 0} \mathrm{H}^0(S, \omega_S^{\otimes m}).$$

In an appendix to Zariski [Zar62], Mumford proved in [Mum62, Theorem] that $R(S)$ is a finitely generated ring over $k = \mathrm{H}^0(S, \mathcal{O}_S)$. In particular $R(S)$ is noetherian. The ring $R(S)$ has a natural $\mathbb{N}$-graduation given by $R(S)_m = \mathrm{H}^0(S, \omega_S^{\otimes m})$ for all integers $m \geq 0$.

DEFINITION 2.53. With notation as above, we define the *canonical model of S* as

$$X = \operatorname{Proj} R(S).$$

THEOREM 2.54. *Let S be a minimal surface of general type. Then for sufficiently large n the complete linear system $|nK_S|$ is base point free and defines a morphism*

$$\phi_n = \phi_{|nK_S|} : S \to \mathbb{P}^N_k$$

where $N = \dim_k \mathrm{H}^0(S, \omega_S^{\otimes n}) - 1$ with the following properties: the image $X_n = \phi_n(S)$ is a normal surface having at most Du Val singularities, and ϕ_n is an isomorphism of $S \backslash \phi_n^{-1}(\mathrm{Sing}(X_n))$ onto $X_n \setminus \mathrm{Sing}(X_n)$, where $\mathrm{Sing}(X_n)$ denotes the singular locus of X_n. Hence $\phi_n : S \to \phi_n(S)$ is a desingularization.

PROOF. See [Băd01, Theorem 9.1]. □

REMARK 2.55. The morphism ϕ_n of Theorem 2.54 is given by contracting (-2)-curves on S to points, see [Băd01, Proof of Theorem 9.1].
If there are no such curves on S, then by Corollary 2.37, ω_S is ample. Then there exists an integer $n > 0$ such that $\omega_S^{\otimes n}$ is very ample. In this case, ϕ_n is a closed embedding and in particular an isomorphism on the image.
Therefore there only remains to consider the case when such curves do exist on S. It turns out that there is just a finite number of these curves, see [Băd01, Proof of Theorem 9.1], say $E_1, \dots, E_r$ and the morphism $\pi_n : S \to X_n$ is given by contracting (-2)-curves $E_1, \dots, E_r$ to points. Moreover, the intersection matrix $||(E_i \cdot E_j)||_{i,j}$ is negative definite ([Băd01, Proof of Theorem 9.1]) and thus

$$\phi_n^* \omega_{X_n} = \omega_S$$

by [Art62, Theorem 2.7], see also [Băd01, Theorem 3.15]. We also note here that since ω_{X_n} is a line bundle, we have

$$\phi_n^*(\omega_{X_n}^{\otimes m}) = \omega_S^{\otimes m}$$

for all integers $m \in \mathbb{Z}$.
It is clear that S is the minimal desingularization of X_n, and thus by Proposition 2.49 we also have

$$\omega_{X_n} = (\phi_n)_* \omega_S.$$

Finally, by projection formula we also find

$$\omega_{X_n}^{\otimes m} \simeq (\phi_n)_* \phi_n^* \omega_{X_n}^m \simeq (\phi_n)_* \omega_S^{\otimes m}$$

for all $m \in \mathbb{Z}$.

COROLLARY 2.56. *With notation as in Theorem 2.54, the canonical line bundle ω_{X_n} is ample.*

PROOF. Let F be a closed integral subscheme of X_n of dimension 1 and let E be the proper transform of F by ϕ_n. It is clear that E is not among the curves E_i with $(\omega_S \cdot E_i) = 0$ because otherwise it would be contracted to a point by Corollary 2.37. Then

$$(\omega_{X_n} \cdot F) = (\phi_n^*(\omega_{X_n}) \cdot E) = (\omega_S \cdot E) > 0$$

where the positivity is because ω_S is nef and $(\omega_S \cdot E) \neq 0$. Moreover

$$(\omega_{X_n} \cdot \omega_{X_n}) = (\omega_S \cdot \omega_S) > 0.$$

By the Nakai-Moishezon criterion (Corollary 2.30) we conclude. □

DEFINITION 2.57. Let S be a minimal surface of general type. With notation as in Theorem 2.54, the surface $X_n = \phi_n(S)$ is called the *n-canonical model* of S. We denote by $\pi_n : S \to X_n$ the map from S to its n-canonical model, which is given by restricting ϕ_n to its image.

Remark 2.58. By Remark 2.55 it follows that the canonical ring $R(S)$ is identified with

$$R(S)=\bigoplus_{m\geq 0}\mathrm{H}^0(S,\omega_S^{\otimes m})=\bigoplus_{m\geq 0}\mathrm{H}^0(X_n,\omega_{X_n}^{\otimes m}),$$

and that we have an isomorphism

$$X_n\simeq \operatorname{Proj}\big(\bigoplus_{m\geq 0}\mathrm{H}^0(X_n,\omega_{X_n}^{\otimes m})\big)=\operatorname{Proj}R(S)=X,$$

where the first isomorphism is given by [GW20, Corollary 13.75], because ω_{X_n} is ample by Corollary 2.56. Thus, for sufficiently large n, the n-canonical model of S is isomorphic to the canonical model of S. In particular, post-composing π_n with this isomorphism we get a morphism

$$\pi: S\to X.$$

Proposition 2.59. *Let S be a minimal surface of general type and let $\pi: S\to X$ be the map from S to its canonical model. Then for all $m\in\mathbb{Z}$ we have*

(1) $\pi_*(\omega_S^{\otimes m})\simeq\omega_X^{\otimes m}$ *and*
(2) $\omega_S^{\otimes m}\simeq\pi^*(\omega_X^{\otimes m})$.

Moreover, ω_X is ample.

Proof. If $m=0$, then $\pi_*\mathcal{O}_S\simeq\mathcal{O}_X$ and $\pi^*\mathcal{O}_X\simeq\mathcal{O}_S$. If $m\neq 0$, everything is clear by Remark 2.55 and by Corollary 2.56. □

Theorem 2.60 (Bombieri, Ekedahl). *Let k be an algebraically closed field of arbitrary characteristic. Let S be a smooth surface such that $\omega_S:=\det\Omega^1_{S/k}$ is big and nef, i.e. S is a minimal surface of general type. Then*

(1) *for each integer $i\geq 4$, $\omega_S^{\otimes i}$ is globally generated;*
(2) *for each integer $i\leq -1$, $\mathrm{H}^0(S,\omega_S^{\otimes i})=0$;*
(3) *for each $i\in\mathbb{Z}\setminus\{-1,0,1,2\}$, $\mathrm{H}^1(S,\omega_S^{\otimes i})=0$;*
(4) *assuming* char $k\neq 2$, *for each $i\in\mathbb{Z}\setminus\{0,1\}$, $\mathrm{H}^1(S,\omega_S^{\otimes i})=0$;*
(5) *for each integer $i\geq 2$, $\mathrm{H}^2(S,\omega_S^{\otimes i})=0$;*
(6) *if X is the canonical model of S, for each integer $m\geq 5$ the line bundle $\omega_X^{\otimes m}$ is very ample on X.*

Proof. (1): in characteristic zero Bombieri states the result in [Bom73, Theorem 2.(i)]. In positive characteristic, Ekedahl states the result in [Eke88, Main Theorem.(ii)].

(2): for $i\leq -4$, by (1) we obtain that $\mathrm{H}^0(S,\omega_S^{\otimes(-i)})\neq 0$ hence $\mathrm{H}^0(S,\omega_S^{\otimes i})=0$ because $\omega_S^{\otimes(-i)}$ is not trivial, as it is big, being a power of ω_S which is big.

The cases $i=-1,-2,-3$ are obtained by taking suitable powers of sections and using what we already know for $i\leq -4$.

(3,4): if $m\geq 1$, then $\omega_S^{\otimes m}$ is again big and nef. Thus, in characteristic zero, if $i\geq 2$ we have that $\mathrm{H}^1(S,\omega_S^{\otimes i})=0$ by the Kawamata–Viehweg vanishing [Kaw82, Theorem 1], [Vie82, Theorem I]. On the other hand, if $i\leq -1$, by Serre duality it holds

$$\mathrm{H}^1(S,\omega_S^{\otimes i})\simeq\left(\mathrm{H}^1\left(S,\omega_S^{\otimes(1-i)}\right)\right)^\vee=0$$

which vanishes as we said above.
In positive characteristic, Ekedahl states the result for $i\leq -2$ [Eke88, Main theorem.(i)]. If $i\geq 3$, by Serre duality we obtain

$$\mathrm{H}^1(S,\omega_S^{\otimes i})\simeq\left(\mathrm{H}^1\left(S,\omega_S^{\otimes(1-i)}\right)\right)^\vee=0$$

(5): follows from (2) by Serre duality.

(6): Bombieri proved this in characteristic zero [Bom73, Main Theorem.(i)] and Ekedahl generalized the result in positive characteristic [Eke88, Main therem.(iii)]. See also [Bea10, Remark X.2] or [Cat+99, Theorem 1.2]. □

COROLLARY 2.61. *In Theorem 2.54 we can take $n \geq 5$.*

PROOF. By looking at the proof of [Băd01, Theorem 9.1], we see that it is sufficient to consider n sufficiently large such that the n-th power of the canonical line bundle $\omega_{X_n}^{\otimes n}$ on X_n is very ample. Thus it is sufficient to take $n \geq 5$ by Theorem 2.60.(6). □

PROPOSITION 2.62. *Let k be an algebraically closed field of arbitrary characteristic. There is a bijective correspondence between*

$$A = \left\{ S \to \operatorname{Spec} k \;\middle|\; \begin{array}{c} S \text{ is an integral scheme, proper and smooth} \\ \text{over } k \text{ of dimension 2 with canonical} \\ \text{line bundle } \omega_S \text{ big and nef} \end{array} \right\}$$

and

$$B = \left\{ X \to \operatorname{Spec} k \;\middle|\; \begin{array}{c} S \text{ is an integral and normal scheme, proper} \\ \text{over } k \text{ of dimension 2 with at most} \\ \text{Du Val singularities and ample canonical} \\ \text{line bundle } \omega_X \end{array} \right\}$$

which is given by sending $(S \to \operatorname{Spec} k) \in A$ to $(X \to \operatorname{Spec} k)$ where X is the canonical model of S.

PROOF. Clearly the set A is the set of minimal surfaces of general type over k. By Theorem 2.54 the canonical model X of S is normal a surface with at most Du Val singularities. Moreover by Proposition 2.59, the canonical line bundle ω_X is ample. It follows that $(X \to \operatorname{Spec} k)$ is an element of B.
On the other hand, let $(X \to \operatorname{Spec} k)$ be an element of B, and let $S \to X$ be the minimal resolution of X, which exists by Proposition 2.43. Then by [Kle66, Proposition I.4.1] we have that ω_S is nef, because $\omega_S = \pi^*\omega_X$ and ω_X is nef being ample. Moreover, since by Proposition 2.59 it holds $\pi_*\omega_S^{\otimes m} = \omega_X^{\otimes m}$ for all $m \geq 0$, we have

$$\mathrm{H}^0(S, \omega_S^{\otimes m}) = \mathrm{H}^0(X, \pi_*\omega_S^{\otimes m}) = \mathrm{H}^0(X, \omega_S^{\otimes m}).$$

Then it is clear that ω_S is big, as ω_S is big, being ample. It follows that $(S \to \operatorname{Spec} k)$ is an element of A. □

DEFINITION 2.63. We say that a surface X over k is a *canonical model of a minimal surface of general type* if $X \to \operatorname{Spec} k$ is an element of the set B in Proposition 2.62.

2.6. Examples

EXAMPLE 2.64. Let $S = C_1 \times C_2$ be the product of two nonsingular projective curves over k of genus g_1 and g_2 respectively. If ω_{C_1} and ω_{C_2} are the canonical bundles of the two curves, then the canonical bundle of S is $\omega_S \simeq p_1^*\omega_{C_1} \otimes p_2^*\omega_{C_2}$ where p_1, p_2 denote the projections of S to C_1 and C_2 respectively (see for example [Har77, Exeercise II.8.3]). It follows that for $n > 0$ we have

$$\mathrm{H}^0(S, \omega_S^{\otimes n}) = \mathrm{H}^0(C_1, \omega_{C_1}^{\otimes n}) \otimes \mathrm{H}^0(C_2, \omega_{C_2}^{\otimes n}).$$

In particular, $h^0(S, \omega_S^{\otimes n}) = h^0(C_1, \omega_{C_1}^{\otimes n}) \cdot h^0(C_2, \omega_{C_2}^{\otimes n})$. Thus, if $g_1 \geq 2$ and $g_2 \geq 2$ we have $\kappa(S) = 2$.

Example 2.65. Let S be a smooth projective surface embedded in $\mathbb{P}^n_k$ such that S is complete intersection in $\mathbb{P}^n_k$, i.e. the homogeneous ideal $\mathcal{I}(S) \subseteq k[x_0, \dots, x_n]$ defining S is generated by $n-2$ homogeneous polynomial $f_1, \dots, f_{n-2}$ of positive degree $d_1, \dots, d_{n-2}$. If $H_i \subseteq \mathbb{P}^n_k$ is the hypersurface with equation $f_i = 0$ for $i = 1, \dots, n-2$ we also say that S is the *complete intersection* of the hypersurfaces $H_1, \dots, H_{n-2}$. Define the restriction of $\mathcal{O}_{\mathbb{P}^n_k}$ to S as $\mathcal{O}_S(m) = \mathcal{O}_{\mathbb{P}^n_k}(m) \otimes \mathcal{O}_S$. Then by adjunction formula we have $\omega_S = \mathcal{O}_S((\sum_{i=1}^{n-2} d_i) - n - 1)$. We see that ω_S is an ample invertible $\mathcal{O}_S$-module if $\sum_{i=1}^{n-2} d_i > n+1$. Moreover, if this is the case, if E is a (-1)-curve on S, then by

$$-2 = 2p_a(E) - 2 = E^2 + \omega_S \cdot E = -1 + \omega_S \cdot E$$

we have that $\omega_S \cdot E = -1$ which is an absurd. Thus S is a minimal surface of general type. One can also prove that the irregularity of S is $q(S) = 0$ (see, for example, [Băd01, Example 9.6.1]).
As a particular case, if $n = 3$ we see that smooth hypersurfaces of degree $d \geq 5$ in $\mathbb{P}^3_k$ are minimal surfaces of general type with $q = 0$.

Example 2.66. Let $S' \subset \mathbb{P}^3_k$ be the Fermat hypersurface of degree 5 given by

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0,$$

where x_i are the coordinates of $\mathbb{P}^3_k(k)$. By the previous example, we know that if char $k \neq 5$ then S' is a smooth minimal surface of general type with $q(S') = 0$. If ξ is a primitive root of unity of order 5, we have an automorphism of S'

$$u : S' \to S'$$

given by $u(t_0, t_1, t_2, t_3) = (t_0, \xi t_1, \xi^2 t_2, \xi^3 t_3)$, and it is such that $u^5 = \mathrm{id}_{S'}$. One can see that u has no fixed points on X', and if we define G to be the subgroup of the automorphisms of S' generated by u, then we know that the quotient surface $S = S'/G$ is a smooth projective surface. One can prove that the homomorphism of $\mathcal{O}_{S'}$-module $\alpha : \mathcal{O}_S \to f_*\mathcal{O}_{S'}$ defined by $f : S' \to S$, induces an injective map in cohomology

$$0 \to \mathrm{H}^1(S, \mathcal{O}_S) \to \mathrm{H}^1(S', \mathcal{O}_{S'}),$$

see, for example, [Băd01, Example 9.6.2]. It follows that $\mathrm{H}^1(S, \mathcal{O}_S) = \mathrm{H}^1(S', \mathcal{O}_{S'}) = 0$.
Observe that $p_g(S') = h^0(S', \mathcal{O}_{S'}(1)) = 4$. Thus $\chi(\mathcal{O}_{S'}) = 1 - q(S') + p_g(S') = 5$. Since $f : S' \to S$ is an étale morphism of degree 5, then $\chi(\mathcal{O}_{S'}) = 5\chi(\mathcal{O}_S)$ ([Băd01, Proposition 9.7]) and $f^*\omega_S \simeq \omega_{S'}$, which implies that ω_S is ample on S. It follows that $\chi(\mathcal{O}_S) = 1$ and by $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ we also have $p_g(S) = 0$. Thus S is an example of a minimal surface of general type with $q(S) = p_g(S) = 0$. Surfaces with these properties are called *Godeaux surfaces.*

Example 2.67. Togliatti in [Tog40] constructed an example of a degree 5 surface with 31 isolated rational double points of type A_1, i.e. in the minimal desingularization, over any singular point there is exactly one (-2)-curve. Moreover, Beauville proved that 31 is the maximum number of *ordinary double points* (i.e. rational double points of type A_1) on a degree 5 surface.
Barth in [Bar96] constructed an example of a degree 6 hypersurface S in $\mathbb{P}^3_{\mathbb{C}}$ with 65 isolated rational double points of type A_1. An explicit equation defining S in $\mathbb{P}^3_{\mathbb{C}}$ with coordinates x, y, z is

$$4(\Phi^2 x^2 - y^2)(\Phi^2 y^2 - z^2)(\Phi^2 z^2 - x^2) - (1 + 2\Phi)(x^2 + y^2 + z^2 - 1)^2 = 0$$

where $\Phi = \frac{\sqrt{5}+1}{2}$. Moreover, 65 is the maximum number of ordinary double points on a sextic ([JR97]).

2.7. Automorphism group

Let S be a minimal surface of general type over an algebraically closed field k of arbitrary characteristic and let $\pi : S \to X$ be the map to its canonical model. We want to study the automorphism groups of S and X. We will work in a more general context.

If V is a scheme over another scheme T, we consider the functor

$$\underline{\mathrm{Aut}}_{V/T} : (\mathrm{Sch}\,/T)^{\mathrm{op}} \to \mathrm{Grp}$$

given by sending $(T' \to T)$ to the group $\mathrm{Aut}_{T'}(T' \times_T V)$. If $T = \operatorname{Spec} k$ we simply write $\underline{\mathrm{Aut}}_V = \underline{\mathrm{Aut}}_{V/k}$. We denote by $\mathrm{Aut}(V)$ the group of k-automorphisms of V, i.e. it is the group $\underline{\mathrm{Aut}}_V(k)$ of k-points of $\underline{\mathrm{Aut}}_V$. See also Example A.64.

LEMMA 2.68. *Let k be an algebraically closed field and let V be a projective scheme over k. Then $\underline{\mathrm{Aut}}_V$ is represented by a group scheme Aut_V which is locally of finite type over k. If V is Gorenstein and moreover V is such that either the canonical bundle ω_V or the anti-canonical bundle $\omega_V^\vee$ is ample, then $\underline{\mathrm{Aut}}_V$ is represented by a group scheme Aut_V which is of finite type over k.*

PROOF. The first assertion follows by [MO67, Theorem 3.7]. Suppose now that V is Gorenstein and that ω_V is ample, the other case being analogous. We only have to prove that Aut_V is quasi-compact. First, define $\underline{\mathrm{Hom}}_k(V, V)$ as in Example A.64. If $f \in \underline{\mathrm{Hom}}_k(V, V)(k)$ we consider the *graph* of f to be

$$\Gamma_f : V \to V \times_k V$$

and we denote by $W = \Gamma_f(V) \subset V \times_k V$ the associated closed subscheme. Consider now the scheme $V \times_k V$. The line bundle

$$\mathcal{L} = \omega_V \boxtimes \omega_V = \mathrm{pr}_1^*\omega_V \otimes \mathrm{pr}_2^*\omega_V$$

is ample on $V \times_k V$. Observe that $W = \Gamma_f(V) \simeq V$ and under this isomorphism we have that $\mathcal{L}_{|W}$ is

$$(\omega_V \boxtimes \omega_V)_{|W} = \mathrm{id}^*\omega_V \otimes_{\mathcal{O}_V} f^*\omega_V \simeq \omega_V^{\otimes 2}.$$

The Hilbert polynomial of W with respect to the line bundle $\mathcal{L}_{|W}$ is then

$$P = P(m) = \chi(W, \mathcal{L}_{|W}^{\otimes m}) = \chi(V, \omega_V^{\otimes 2m}).$$

We consider now the Hilbert scheme $\mathrm{Hilb}_{V\times_k V/k}^{\mathcal{L},P}$, which parametrizes closed subschemes $Z \subset V \times_k V$ with Hilbert polynomial P with respect to the line bundle $\mathcal{L}_{|Z}$. This is a projective scheme over k, see [Fan+05, Theorem 5.16].
By [Fan+05, Theorem 5.23], the functor $\underline{\mathrm{Hom}}_k(V, V)$ is representable by an open subscheme $\mathrm{Hom}_k(V, V)$ of $\mathrm{Hilb}_{V\times_k V/k}^{\mathcal{L},P}$, and moreover $\underline{\mathrm{Aut}}_V$ is representable by an open subscheme Aut_V of $\mathrm{Hom}_k(V, V)$. Since $\mathrm{Hilb}_{V\times_k V/k}^{\mathcal{L},P}$ is quasi-compact and noetherian, it follows that Aut_V is quasi-compact. □

COROLLARY 2.69. *Let k be an algebraically closed field. Let X be the canonical model of a minimal surface of general type over k. Then Aut_X is a group scheme of finite type over k.*

PROOF. This follows by Lemma 2.68 because ω_X is ample. □

LEMMA 2.70. *Let S be a minimal surface of general type over k and let X be its canonical model. The morphism $\pi : S \to X$ induces a group homomorphism $\mathrm{Aut}(S) \to \mathrm{Aut}(X)$ which is an isomorphism.*

Proof. Let $\varphi \in \mathrm{Aut}(S)$ be an automorphism of S over k. Since the canonical bundle ω_S is preserved by φ, it follows that φ induces a unique automorphism of X over k, because by definition $X = \operatorname{Proj} R(S)$. Let now $\psi \in \mathrm{Aut}(X)$. In particular, ψ is a birational map from X to itself. Since S is birational to X, it follows that ψ induces a birational map ϕ from S to itself. By [Bǎd01, Theorem 10.21] (see also [Bea10, Remarks II.13.(1)]) we have that ϕ is necessarily an isomorphism. These are group homomorphisms which are one inverse of the other. $\square$

Proposition 2.71. *Let k be a field of characteristic 0. Let $X \to \operatorname{Spec} k$ be a normal integral scheme of dimension 2, proper over k, with at most Du Val singularities such that the canonical bundle ω_X is ample. Then $\mathrm{H}^0(X, \mathcal{T}_X) = 0$.*

Proof. Suppose first that X is smooth. By Grothendieck duality (Theorem B.14) we have

$$\mathrm{H}^0(X, \mathcal{T}_X) \simeq (\mathrm{H}^2(X, \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \omega_X))^\vee = 0$$

where the vanishing follows by the Kodaira-Nakano vanishing theorem (recall that char $k = 0$). If X is not smooth, then the vanishing of $\mathrm{H}^0(X, \mathcal{T}_X)$ follows by [Bha+13, Lemma 2.5]. $\square$

Proposition 2.72. *Let k be a field of characteristic 0 and let S be a minimal surface of general type over k. Let $\pi : S \to X$ be the projection to the canonical model. Then Aut_S and Aut_X are isomorphic to the disjoint union of finitely many copies of $\operatorname{Spec} k$. Moreover,*

$$\alpha : \mathrm{Aut}_S \to \mathrm{Aut}_X$$

is an isomorphism of group schemes.

Proof. The tangent space at the identity at Aut_X is

$$\mathrm{T}_{\mathrm{id}_X} \mathrm{Aut}_X \simeq \mathrm{H}^0(X, \mathcal{T}_X)$$

by [MO67, Lemma 3.4]. Thus $\mathrm{T}_{\mathrm{id}_X} \mathrm{Aut}_X = 0$ by Proposition 2.71. Hence Aut_X is isomorphic to a finite disjoint union of copies of $\operatorname{Spec} k$ by Lemma 1.10.
By [BW74, Proposition 1.2] it holds that $\pi_* \mathcal{T}_S \simeq \mathcal{T}_X$. It follows that

$$\mathrm{H}^0(S, \mathcal{T}_S) \simeq \mathrm{H}^0(X, \pi_* \mathcal{T}_S) \simeq \mathrm{H}^0(X, \mathcal{T}_X) = 0$$

by Proposition 2.71. Then Aut_S is isomorphic to a disjoint union of copies of $\operatorname{Spec} k$ by Remark 1.11. Finally, there is just a finite numbers of copies of $\operatorname{Spec} k$ in Aut_S because $\mathrm{Aut}_S(k) = \mathrm{Aut}(S)$ is finite, being in bijection with $\mathrm{Aut}(X) = \mathrm{Aut}_X(k)$ by Lemma 2.70.
It follows that $\alpha : \mathrm{Aut}_S \to \mathrm{Aut}_X$ is a morphism of group schemes which are the disjoint union of finitely many copies of $\operatorname{Spec} k$ and that it induces an isomorphism of groups

$$\mathrm{Aut}(S) \to \mathrm{Aut}(X)$$

on k-points by Lemma 2.70. Thus, α is necessarily an isomorphism. $\square$

Remark 2.73. However, if k is an algebraically closed field such that char $k = p > 0$, an analogous result of Proposition 2.72 do not hold. Suppose that X is the canonical model of a minimal surface of general type over such a field k. By Corollary 2.69 we know that Aut_X is a group scheme of finite type over k. Thus by Lemma 1.10 we know that Aut_X is isomorphic to a finite disjoint union of copies of $\operatorname{Spec} k$ if and only if the tangent space in the identity of Aut_X is zero. This do not always happen in positive characteristic; indeed there are examples in which

$$\mathrm{T}_{\mathrm{id}_X} \mathrm{Aut}_X \simeq \mathrm{H}^0(X, \mathcal{T}_X) \neq 0,$$

see [Tzi22, Proposition 3.1]. Moreover, there exists also examples of smooth surfaces S with ample canonical line bundle and non-trivial tangent space at the identity $\mathrm{T}_{\mathrm{id}_X} \mathrm{Aut}_S$. See, for example, [Lan83] or [She96].

2.8. Finite generation of pluricanonical rings

Throughout this section, we prove that if S is a minimal surface of general type over an algebraically closed field k, then the graded k-algebra

$$\bigoplus_{m\geq 0} \mathrm{H}^0(S, \omega_S^{\otimes cm})$$

is generated in degree 1 for sufficiently large c (Proposition 2.83).

Lemma 2.74. *Let $r \geq 1$ be an integer, let X be a topological space and let*

$$0 \to F_r \to \cdots \to F_2 \to F_1 \to F_0 \to 0$$

be an exact sequence of sheaves of abelian groups on X. Assume $\mathrm{H}^{j-1}(X, F_j) = 0$ whenever $2 \leq j \leq r$.

Then $\mathrm{H}^0(X, F_1) \to \mathrm{H}^0(X, F_0)$ is surjective.

Proof. We proceed by induction on r. If $r = 1$ then we have an isomorphism $F_1 \simeq F_0$. If $r = 2$ there is a short exact sequence

$$0 \to F_2 \to F_1 \to F_0 \to 0$$

and we use the long exact sequence in cohomology and $\mathrm{H}^1(X, F_2) = 0$.

The inductive step is proved as follows. Assume $r \geq 3$. Suppose that the statement of the lemma holds for $r-1$ and we want to prove for r. Set $G = \ker(F_{r-2} \to F_{r-3})$. From the long exact sequence in cohomology induced by the short exact sequence

$$0 \to F_r \to F_{r-1} \to G \to 0$$

we deduce $\mathrm{H}^{r-2}(X, G) = 0$. So we can apply the inductive hypothesis to the exact sequence

$$0 \to G \to F_{r-2} \to \cdots \to F_2 \to F_1 \to F_0 \to 0.$$

and conclude. □

Definition/Proposition 2.75. Let A be a ring and let $f_1, \ldots, f_n \in A$. We define a complex of A-modules as follows. Set $K_0 = A$ and $K_p = 0$ if p is not in the range $1 \leq p \leq n$. For $1 \leq p \leq n$, let

$$K_p = \bigoplus A e_{i_1 \ldots i_p}$$

be the free A module of rank $\binom{n}{p}$ with basis

$$\left\{ e_{i_1 \ldots i_p} \;\middle|\; 1 \leq i_1 < \ldots < i_p \leq n \right\}.$$

The differential $d : K_p \to K_{p-1}$ is defined by

$$\mathrm{d}(e_{i_1 \ldots i_p}) = \sum_{r=1}^{p} (-1)^{r-1} f_{i_r} e_{i_1 \ldots \hat{i}_r \ldots i_p};$$

and for $p = 1$ set $\mathrm{d}(e_i) = f_i$. One check easily that $\mathrm{d}^2 = 0$. This complex is called the *Koszul complex* of $f_1, \ldots, f_n$.

Proof. See [Mat89, §16]. □

We refer to [Mat89, §16] for properties of the Koszul complex.

Lemma 2.76. *Let A be a ring and let $f_1, \ldots, f_r \in A$ be generators of the unit ideal A. Then the Koszul complex of $f_1, \ldots, f_r$ is exact.*

Proof. It is well known that the cohomology of the Koszul complex of $f_1, \ldots, f_r$ is annihilated by the ideal generated by $f_1, \ldots, f_r$. □

LEMMA 2.77. *Let X be a proper scheme over the field k and let L be a globally generated invertible sheaf on X. Set $V = \mathrm{H}^0(X, L)$ and $r = \dim_k V$. Then the Koszul complex*

$$0 \to \wedge^r V \otimes_k L^{\otimes(-r)} \to \cdots \to \wedge^2 V \otimes_k L^{\otimes(-2)} \to V \otimes_k L^{\otimes(-1)} \to \mathcal{O}_X \to 0 \tag{2}$$

is an exact sequence of locally free sheaves on X.

PROOF. Since L is globally generated, $V \otimes_k \mathcal{O}_X \to L$ is an epimorphism of sheaves. Then $V \otimes_k L^\vee \to \mathcal{O}_X$ is an epimorphism.

Let $s_1, \ldots, s_r$ be a k-basis of $V = \mathrm{H}^0(X, L)$. Let $U \subseteq X$ be an open affine subset of X such that $L|_U$ is trivial. Fix an isomorphism $L|_U \simeq \mathcal{O}_U$. Then $s_i|_U$ corresponds to $f_i \in \Gamma(U, \mathcal{O}_X)$. It is clear that $f_1, \ldots, f_r$ generate the unit ideal in $\Gamma(U, \mathcal{O}_X)$. By taking sections on U of the complex (2) we obtain the Koszul complex of $f_1, \ldots, f_r$ with respect to the ring $\Gamma(U, \mathcal{O}_X)$. This is exact by Lemma 2.76. □

PROPOSITION 2.78. *Let X be a proper scheme over the field k. Let L be a globally generated invertible sheaf on X. Let $b \geq 1$ be an integer. Assume $\mathrm{H}^q(X, L^{\otimes(b-q)}) = 0$ whenever $1 \leq q \leq h^0(X, L) - 1$. Then the multiplication map*

$$\mathrm{H}^0(X, L) \otimes_k \mathrm{H}^0(X, L^{\otimes b}) \to \mathrm{H}^0(X, L^{\otimes(b+1)})$$

is surjective.

PROOF. Set $V = \mathrm{H}^0(X, L)$ and $r = \dim_k V = h^0(X, L)$. We tensor the Koszul complex (2), which is exact by Lemma 2.77, by $L^{\otimes(b+1)}$ and get the exact sequence

$$0 \to \wedge^r V \otimes_k L^{\otimes(b+1-r)} \to \cdots \to \wedge^2 V \otimes_k L^{\otimes(b-1)} \to V \otimes_k L^{\otimes b} \to L^{\otimes(b+1)} \to 0.$$

We now verify that hypothesis of Lemma 2.74 are satisfied. For all $2 \leq j \leq r$ we have

$$\mathrm{H}^{j-1}(X, \wedge^j V \otimes_k L^{\otimes(b+1-j)}) = \wedge^j V \otimes_k \mathrm{H}^{j-1}(X, L^{\otimes(b-(j-1))}) = 0$$

where the last vanishing holds by hypothesis. Thus, Lemma 2.74 implies that $V \otimes_k \mathrm{H}^0(X, L^{\otimes b}) \to \mathrm{H}^0(X, L^{\otimes(b+1)})$ is surjective. □

LEMMA 2.79. *Let A be a commutative $\mathbb{N}$-graded ring which is generated by A_1 as an A_0-algebra, namely $A = A_0[A_1]$. Let M be an $\mathbb{N}$-graded A-module. Suppose that $\{y_\lambda\}_{\lambda \in L}$ is a system of homogeneous generators for M such that $\deg(y_\lambda) \leq n_0$ for all $\lambda \in L$. Then for all $n \geq n_0$ and for all $k \geq 0$ we have $M_{n+k} = A_k \cdot M_n$.*

PROOF. We follow the proof of [Bou61, §III.1.3 Lemme 1].

Let $n \geq n_0$. Let $x \in M_{n+1}$. Write $x = \sum a_\lambda y_\lambda$ as a finite linear combination of generators of M with coefficients $a_\lambda \in A$. Up to considering the homogeneous part of each coefficient α_λ, we can suppose without loss of generality that the a_λ's are homogeneous in A, so that $\deg(a_\lambda) = n + 1 - \deg(y_\lambda) \geq 1$. Since $A = A_0[A_1]$ and $\deg(a_\lambda) > 0$, we can write a_λ as a linear combination of elements of the form $a' \cdot a$ with $a' \in A_1$ and $a \in A$. It follows that $x \in A_1 \cdot M_n$ and in particular $M_{n+1} = A_1 \cdot M_n$. Let now $k \geq 2$. Then

$$M_{n+k} = A_1 \cdot M_{n+k-1} = \underbrace{A_1 \cdot A_1 \cdot \ldots \cdot A_1}_{k \text{ times}} \cdot M_n$$

and since $A = A_0[A_1]$ we have $M_{n+k} = A_k \cdot M_n$. □

LEMMA 2.80. *Let A be an $\mathbb{N}$-graded commutative ring such that $A = A_0[A_1]$. Let $S = \bigoplus_{i \geq 0} S_i$ be an $\mathbb{N}$-graded A-algebra, which is a finite A-module. Then there exists a finite set of homogeneous generators $\{s_\lambda\}_{\lambda \in L}$ for S as an A-module. Let n_0 be a positive integer such that $\deg(s_\lambda) \leq n_0$ for all $\lambda \in L$. Then*

(1) *for all $n \geq n_0$ and $k \geq 0$, $S_{n+k} = S_k \cdot S_n$;*

(2) *for all* $d \geq n_0$, $S^{(d)} = S_0[S_d]$.

Proof. We follow the proof of [Bou61, §III.1.3 Lemme 2]. Since S is a finite A-module, there exists a finite set of homogeneous generators $\{s_\lambda\}_{\lambda \in L}$ for S as an A-module. Since L is finite, there exists a positive integer n_0 such that $\deg(s_\lambda) \leq n_0$ for all $\lambda \in L$. Then (1) follows by Lemma 2.79. If $d \geq n_0$ and $m > 0$, we immediately obtain by (1) that $S_{md} = (S_d)^m$. It follows that $S^{(d)} = S_0[S_d]$. □

Lemma 2.81. *Let k be a field, let $R = \bigoplus_{i \geq 0} R_i$ be an $\mathbb{N}$-graded k-algebra with $R_0 = k$. Assume R is generated as a k-algebra by R_1, R_2, R_3, which are k-vector spaces of finite dimension.*

Then for every integer $e \geq 5 \dim_k R_1 + 4 \dim_k R_2 + 3 \dim_k R_3$, the $(6e)$-th Veronese subring $R^{(6e)}$ is generated in degree 1.

Proof. We follow the proof of [Bou61, §III.1.3 Proposition 3]. Set $d_i = \dim_k R_i$. Pick a k-basis $\{x_1, \dots, x_{d_1}\}$ of R_1, a k-basis $\{y_1, \dots, y_{d_2}\}$ of R_2 and a k-basis $\{z_1, \dots, z_{d_3}\}$ of R_3. These are homogeneous generators of the k-algebra R. Let B be the sub-k-algebra of R generated by

$$x_1^6, \dots, x_{d_1}^6, y_1^3, \dots, y_{d_2}^3, z_1^2, \dots, z_{d_3}^2.$$

Observe that the degree of the elements of B is divisible by 6. Let $\mathcal{F}$ be the set consisting of

$$x_1^{\alpha_1} \cdots x_{d_1}^{\alpha_{d_1}} y_1^{\beta_1} \cdots y_{d_2}^{\beta_{d_2}} z_1^{\gamma_1} \cdots z_{d_3}^{\gamma_{d_3}} \tag{3}$$

as $0 \leq \alpha_i < 6$, $0 \leq \beta_i < 3$, $0 \leq \gamma_i < 2$ are such that

$$6 \mid \alpha_1 + \cdots \alpha_{d_1} + 2\beta_1 + \cdots + 2\beta_{d_2} + 3\gamma_1 + \cdots + 3\gamma_{d_3}.$$

The degree of the element in (3) is at most $5d_1 + 4d_2 + 3d_3$.
Let A be the $\mathbb{N}$-graded ring whose underlying ring is B with the grading given by $A_t = B_{6t}$. By definition of B, we have that A is generated as a k-algebra by A_1. In other words $A = A_0[A_1]$. Let S be the $\mathbb{N}$-graded ring whose underlying ring is the 6-th Veronese subring $R^{(6)}$ and the grade is given by $S_t = R_{6t}$. It is obvious that S is an A-algebra. Moreover elements of $\mathcal{F}$ are in particular elements of the ring S. We want to show that that S is generated by $\mathcal{F}$ as a B-module. In order to do that, it is sufficient to show that every element of the form

$$p = x_1^{\alpha_1} \cdots x_{d_1}^{\alpha_{d_1}} y_1^{\beta_1} \cdots y_{d_2}^{\beta_{d_2}} z_1^{\gamma_1} \cdots z_{d_3}^{\gamma_{d_3}}$$

with $\alpha_i \geq 0$, $\beta_i \geq 0$, $\gamma_i \geq 0$ such that

$$6 \mid \alpha_1 + \cdots \alpha_{d_1} + 2\beta_1 + \cdots + 2\beta_{d_2} + 3\gamma_1 + \cdots + 3\gamma_{d_3}$$

is a linear combination of elements of $\mathcal{F}$ with coefficients in B. Write

$$\begin{aligned} \alpha_i &= 6k_i + r_i & &1 \leq i \leq d_1,\ 0 \leq r_i < 6; \\ \beta_i &= 3k'_i + r'_i & &1 \leq i \leq d_2,\ 0 \leq r'_i < 3; \\ \gamma_i &= 2k''_i + r''_i & &1 \leq i \leq d_3,\ 0 \leq r''_i < 2. \end{aligned}$$

Define

$$p_1 = (x_1^6)^{k_1} \cdot \ldots \cdot (x_{d_1}^6)^{k_{d_1}} \cdot (y_1^3)^{k'_1} \cdot \ldots \cdot (y_{d_2}^3)^{k'_{d_2}} \cdot (z_1^2)^{k''_1} \cdot \ldots \cdot (z_{d_3}^2)^{k''_{d_3}}$$
$$p_2 = x_1^{r_1} \cdot \ldots \cdot x_{d_1}^{r_{d_1}} \cdot y_1^{r'_1} \cdot \ldots \cdot y_{d_2}^{r'_{d_2}} \cdot z_1^{r''_1} \cdot \ldots \cdot z_{d_3}^{r''_{d_3}}.$$

We see that $p_1 \in B$ because is a multiplication of generators of B. Moreover $p_2 \in \mathcal{F}$ because if

$$6 \mid \alpha_1 + \cdots \alpha_{d_1} + 2\beta_1 + \cdots + 2\beta_{d_2} + 3\gamma_1 + \cdots + 3\gamma_{d_3}$$

then also

$$6 \mid r_1 + \ldots + r_{d_1} + 2r'_1 + \ldots + 2r'_{d_2} + 3r''_1 + \ldots + 3r''_{d_3}.$$

In particular we see that $p = p_1 \cdot p_2$ so that S is generated by $\mathcal{F}$ as a B-module, and so also as an A-module. By the description of the elements of $\mathcal{F}$, we see that the cardinality of $\mathcal{F}$ is finite. In other words, S is a finite A-module. By Lemma 2.80, there exists an integer $n_0 \geq 0$ such that for all $d \geq n_0$ we have $S^{(d)} = S_0[S_d]$. By Lemma 2.79, we see that it is sufficient to take n_0 as the maximum degree of a set of generators for S as a finite A-module. Thus we can take $n_0 = 5d_1 + 4d_2 + 3d_3$ because this is the maximum degree of the elements of $\mathcal{F}$.
Since $S = R^{(6)}$ we have $S^{(d)} = R^{(6d)}$. Hence for all $d \geq 5d_1 + 4d_2 + 3d_3$ we have that $R^{(6d)}$ is generated in degree 1. □

Proposition 2.82. *Let k be an algebraically closed field of arbitrary characteristic. Let S be a minimal smooth surface of general type over k. Let $a \geq 4$ be an integer. Then:*

(1) *for every integer $b \geq 3$, then the multiplication map*
$$\mathrm{H}^0(S, \omega_S^{\otimes a}) \otimes_k \mathrm{H}^0(S, \omega_S^{\otimes ab}) \to \mathrm{H}^0(S, \omega_S^{\otimes a(b+1)})$$
is surjective;

(2) *the $\mathbb{N}$-graded k-algebra*
$$\bigoplus_{m\geq 0} \mathrm{H}^0(S, \omega_S^{\otimes am})$$
is generated by degrees ≤ 3.

Proof. (1): we have to show that the hypotheses of Proposition 2.78 are satisfied. Fix an integer $a \geq 4$ and denote $L = \omega_S^{\otimes 4}$. Observe first that L is globally generated by Theorem 2.60. We now show that for all $b \geq 3$ and for all $1 \leq q \leq h^0(S, L) - 1$ we have
$$\mathrm{H}^q(S, L^{\otimes(b-q)}) = 0.$$
Since $\dim S = 2$, we only have to check that it holds for $q = 1, 2$. But $\mathrm{H}^1(S, L^{\otimes(b-1)}) = 0$ because $a \cdot (b-1) \geq 8$ and we apply Theorem 2.60.(3). Finally $\mathrm{H}^2(S, L^{\otimes(b-2)}) = 0$ because $a \cdot (b-2) \geq 4$ and we apply Theorem 2.60.(5). Thus (1) follows by Proposition 2.78. To conclude, (2) follows immediately from (1). □

Proposition 2.83. *Let k be an algebraically closed field of arbitrary characteristic. Let S be a minimal smooth surface of general type over k. For every integer $i \geq 1$, let $p_i = h^0(S, \omega_S^{\otimes i})$ be the ith plurigenus of X. If $e \geq 5p_4 + 4p_8 + 3p_{12}$, then the $(24e)$th-canonical ring*
$$\bigoplus_{m\geq 0} \mathrm{H}^0(X, \omega_X^{\otimes 24em})$$
is generated in degree 1.

Proof. By Proposition 2.82 we know that the $\mathbb{N}$-graded k-algebra
$$R = \bigoplus_{m\geq 0} \mathrm{H}^0(S, \omega_S^{\otimes 4m})$$
is generated by degrees ≤ 3. Denote by $R_m = \mathrm{H}^0(S, \omega_S^{\otimes 4m})$ the graded part of degree m. Let
$$d_1 = \dim_k R_1 = \dim_k \mathrm{H}^0(S, \omega_S^{\otimes 4}) = p_4,$$
$$d_2 = \dim_k R_2 = \dim_k \mathrm{H}^0(S, \omega_S^{\otimes 8}) = p_8,$$
$$d_3 = \dim_k R_3 = \dim_k \mathrm{H}^0(S, \omega_S^{\otimes 12}) = p_{12}.$$
By Lemma 2.81 we have that if $e \geq 5p_4 + 4p_8 + p_{12}$, then
$$R^{(6e)} = \bigoplus_{m\geq 0} R_{6em} = \bigoplus_{m\geq 0} \mathrm{H}^0(S, \omega_S^{\otimes 24em})$$
is generated in degree 1. □

CHAPTER 3

Stacks of surfaces

In this chapter we study the construction of some stacks of surfaces. In order to do that, we first recall properties of schemes over fields which are stable under field extension in §3.1 and open or ind-constructible subsets of the target of a morphism of schemes in §3.2. Subsequently we construct the stack of canonical models of minimal surfaces of general type in §3.4, the stack of minimal surfaces of general type in §3.7, the stack of del Pezzo surfaces in §3.10 and the stack of K3 surfaces in §3.9. In particular, we prove that the stack of canonical models is algebraic (Theorem 3.50 and Theorem 3.97) and that the stack of minimal surfaces of general type is algebraic (Theorem 3.88).

3.1. Stable properties under field extension

We will use notation introduced in Definition 1.1.

Definition 3.1. Let k be a field and let $\overline{k}$ be an algebraic closure of k. Let X be a scheme of finite type over k. Let $\mathcal{P}$ be one of the following property of schemes

(1) integral;
(2) connected;
(3) regular;
(4) reduced;
(5) irreducible.

We say that the morphism $X \to \operatorname{Spec} k$ has property $\mathcal{P}$ *geometrically* if the scheme $X_{\overline{k}} = X \otimes_k \operatorname{Spec} \overline{k}$ has property $\mathcal{P}$.

The reader should note that, with notation as above, being geometrically $\mathcal{P}$ is a property of the morphism $X \to \operatorname{Spec} k$, and not of the scheme X. If $X \to \operatorname{Spec} k$ has property $\mathcal{P}$ geometrically, by abuse of notation we will also say that X has property $\mathcal{P}$ geometrically. We have the following key remark.

Remark 3.2. If $k = \overline{k}$, then $X \to \operatorname{Spec} k$ is geometrically $\mathcal{P}$ if and only if X has property $\mathcal{P}$.

Lemma 3.3. *Let X be a scheme of finite type over a field k. Let K/k be a field extension and let $X_K = X \otimes_k \operatorname{Spec} K$ be the base change. Then*

(1) X is geometrically integral if and only if X_K is geometrically integral;
(2) X is geometrically normal if and only if X_K is geometrically normal.

Proof. A scheme is geometrically integral if and only if it is both geometrically reduced and geometrically irreducible. Then (1) follows by [Stacks, Lemma 0384] and [Stacks, Lemma 054P]. Finally, (2) follows by [Stacks, Lemma 038P]. □

Proposition 3.4. *Let X be an integral scheme of finite type over a field k and let K/k be a field extension. Then every irreducible component of $X_K = X \otimes_k \operatorname{Spec} K$ has dimension equal to the dimension of X.*

Proof. See [Gro67, Corollaire IV.4.1.4] or [Har77, Exercise II.3.20]. □

PROPOSITION 3.5. *Let X be a proper scheme over a field k. Let $\mathcal{L}$ be a line bundle on X. Let K/k be a field extension and let*

$$\begin{tikzcd} X_K = X \otimes_k \operatorname{Spec} K \arrow[r, "\varphi"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

be the base change. Denote $\mathcal{L}_{X_K} = \varphi^*\mathcal{L}$. Then

(1) *$\mathcal{L}$ is ample on X if and only if $\mathcal{L}_{X_K}$ is ample on X_K;*
(2) *$\mathcal{L}$ is nef for X if and only if $\mathcal{L}_{X_K}$ is nef for X_K;*
(3) *the pullback map*
$$\varphi^* : \operatorname{Pic}(X) \to \operatorname{Pic}(X_K)$$
is injective;
(4) *if X, X_K are both integral schemes, then $\mathcal{L}$ is big for X if and only if $\mathcal{L}_{X_K}$ is big for X_K;*
(5) *if X is integral and Gorenstein then $\omega_{X_K} = \varphi^*\omega_X$ and*
$$(\omega_{X_K}^n) = (\omega_X^n)$$
where $n = \dim(X) = \dim(X_K)$.

PROOF. (1): follows by [Alp24, Proposition B.2.9], [GW20, Proposition 13.64], [GW20, Proposition 14.58].

(2): follows by [Alp24, Proposition B.2.15]. See also [Kle66, Proposition I.4.1].

(3): the scheme X is quasi-compact and quasi-separated, because it is proper over $\operatorname{Spec} k$. Moreover, $\mathrm{H}^0(X, \mathcal{O}_X)$ is a finite-dimensional k-vector space because X is proper over k, see [GW20, Theorem 12.65] or [GW23, Corollary 23.18]. Then by [Stacks, Lemma 0CC5] we conclude.

(4): follows by [Alp24, Proposition B.2.24].

(5): observe that X_K is Gorenstein by Lemma 2.51 and that X_K is again proper over K, because $X_K \to \operatorname{Spec} K$ is the pullback of a proper and flat morphism of schemes. Then ω_X (resp. ω_{X_K}) is a line bundle on X (resp. X_K) by Proposition B.13. Then by §B.4 we know that $\omega_{X_K} \simeq \varphi^*\omega_X$. Hence the statement follows by Lemma 2.28. □

PROPOSITION 3.6. *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of schemes. Let*

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

be the base change of f to Y. Let $\mathcal{P}$ be one of the following properties of morphism of schemes

(1) *proper;*
(2) *flat;*
(3) *of finite presentation;*
(4) *smooth.*

If f has property $\mathcal{P}$ then f' has property $\mathcal{P}$.
Moreover, the same fact holds if X, X', Y, Y' are algebraic spaces.

PROOF. We use [Stacks, Lemma 01W4], [Stacks, Lemma 01U9], [Stacks, Lemma 01TS], [Stacks, Lemma 01VB] for schemes.
The statement for algebraic spaces follows by [Stacks, Lemma 03XR], [Stacks, Lemma 03MO], [Stacks, Lemma 04WP], [Stacks, Lemma 03ZE]. □

3.2. Open and ind-constructible properties on the target

Let $f : X \to Y$ be a morphism of schemes. Suppose that $\mathcal{P}$ is property of a schemes over fields. In this section we want to study the subsets of Y of the form

$$Y_{\mathcal{P}} = \{y \in Y \mid \mathcal{P} \text{ holds for } X_y \text{ over } \kappa(y)\}\,.$$

Under certain hypotheses, we will show that $Y_{\mathcal{P}}$ is open in Y for some $\mathcal{P}$, while in other cases it will be an *ind-constructible* subset of Y (Definition 3.7).

We follow conventions of [Gro67], which are the same as [Stacks], but they differ from the second version of [GD71].

DEFINITION 3.7. Let X be a topological space and let E be a subset of X.

- We say that E is *retro-compact* if for every quasi-compact open $V \subseteq X$, the intersection $E \cap V$ is quasi-compact.
- We say that E is *constructible* in X if E is a finite union of subsets of the form $U \cap V^{\mathrm{c}}$, where U and V are open and retro-compact in X.
- We say that E is *locally constructible* in X if there exists an open covering $X = \bigcup_i V_i$ such that each $E \cap V_i$ is constructible in V_i.
- We say that E is *ind-constructible* in X if for every point $x \in X$ there exists an open neighbourhood $x \in U$ in X such that $E \cap U$ is the union of locally constructible subsets of U.

REMARK 3.8. It is clear that if E is locally constructible in X, then it is also ind-constructible in X.

PROPOSITION 3.9. *Let X be a noetherian topological space. The constructible sets in X are precisely the finite unions of locally closed subsets of X. In particular, open subsets and closed subsets of X are constructible.*

PROOF. See [Stacks, Lemma 005L]. □

PROPOSITION 3.10. *Let X be a noetherian scheme and let $E \subseteq X$ be a subset.*

(1) *E is closed if and only if E is locally constructible and stable under specialization.*
(2) *E is open if and only if E is locally constructible and stable under generization.*

PROOF. See [GW20, Lemma 10.17]. □

PROPOSITION 3.11. *Let $f : X \to Y$ be a proper, flat and finitely presented morphism of schemes. Then the following subset of Y are open in Y:*

(1) *the set of $y \in Y$ such that the fibre X_y is geometrically integral;*
(2) *the set of $y \in Y$ such that the fibre X_y is geometrically normal.*

PROOF. See [Gro67, Théorème IV.12.2.4 (iv) and (viii)]. □

PROPOSITION 3.12. *Let $f : X \to Y$ be a proper, flat and finitely presented morphism of schemes. Let*

$$\eta_{X/Y} : Y \to \{0, 1, 2, 3, \dots, \infty\}$$

be the function which associate to $y \in Y$ the dimension of the fibre X_y. Then $\eta_{X/Y}$ is locally constant.

Proof. See [Stacks, Lemma 0D4J]. □

Proposition 3.13. *Let $f : X \to Y$ be a proper, flat and finitely presented morphism of schemes in which every fibre is a Gorenstein integral normal scheme scheme of dimension 2. Then the function*

$$Y \to \mathbb{Z}, \qquad y \mapsto (K^2_{X_y}) = (\omega_{X_y} \cdot \omega_{X_y})$$

is locally constant on Y.

Proof. We know that $\omega_{X/Y}$ is a line bundle by Proposition B.13. Then $\omega_{X/Y}^{\otimes(n_1+n_2)}$ is again a line bundle for all integers $n_1, n_2 \geq 0$. By Theorem B.4, the function

$$y \mapsto \sum_{i\geq 0}(-1)^i \dim_{\kappa(y)} \mathrm{H}^i\left(X_y, \omega_{X_y}^{\otimes(n_1+n_2)}\right) = \chi\left(X_y, \omega_{X_y}^{\otimes(n_1+n_2)}\right)$$

is locally constant. By Definition 2.24, the self-intersection of the canonical bundle of X_y is the coefficient of the monomial n_1n_2 in the numerical polynomial $\chi\left(X_y, \omega_{X_y}^{\otimes(n_1+n_2)}\right)$; in particular, it is locally constant. □

Proposition 3.14. *Let $f : X \to Y$ be a morphism of schemes of finite presentation and let $\mathcal{L}$ be a line bundle on X. Then the set*

$$\left\{y \in Y \;\middle|\; \mathcal{L}_{X_y} \text{ is ample over } \kappa(y)\right\}$$

is ind-constructible in Y, and it is open if f is also proper.

Proof. See [Gro67, Proposition IV.9.6.2(II)] and [Gro67, Proposition IV.9.6.4]. □

Proposition 3.15. *Let $f : X \to Y$ be a proper, smooth and finitely presented morphism of schemes such that for all points $y \in Y$ the geometric fibre $X_{\overline{y}}$ is a minimal surface of non-negative Kodaira dimension. Then the set*

$$Y_{\mathrm{big}} = \left\{y \in Y \;\middle|\; \omega_{X_y} \text{ is big}\right\}$$

is open in Y.

Proof. Recall that the dualizing sheaf $\omega_{X/Y}$ exists and it is a line bundle by Proposition B.13.
Observe that for all $y \in Y$ the line bundle $\omega_{X_{\overline{y}}}$ is nef by Lemma 2.31. It follows that for all $y \in Y$ the line bundle ω_{X_y} is nef by Proposition 3.5. By the characterization of bigness for nef divisor ([Laz04, Theorem 2.2.16] and [Alp24, Corollary B.2.23]) we have that

$$\begin{aligned} Y_{\mathrm{big}} &= \left\{y \in Y \;\middle|\; \omega_{X_y} \text{ is big}\right\} = \\ &= \left\{y \in Y \;\middle|\; (\omega_{X_y} \cdot \omega_{X_y}) > 0\right\}. \end{aligned}$$

Then the statement follows by Proposition 3.13. □

Definition 3.16. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally noetherian schemes.

(1) Let $x \in X$ and let $y = f(x)$. We say that f is *Gorenstein at x* if f is flat at x and the local ring $\mathcal{O}_{X_y,x}$ is Gorenstein.
(2) We say that f is a Gorenstein morphism if f is Gorenstein at every point of x.

Lemma 3.17. *Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally noetherian schemes. The following are equivalent:*

(1) *f is Gorenstein;*
(2) *f is flat and its fibres are Gorenstein.*

PROOF. This is clear from Definition 3.16. ☐

LEMMA 3.18. *Let $f : X \to Y$ be a morphism of schemes which is flat and locally of finite presentation. Define*

$$X_{\mathrm{Gor}} = \{x \in X \mid f \text{ is Gorenstein at } x\}.$$

Then formation of X_{Gor} commutes with arbitrary base change and is open in X.

PROOF. See [Stacks, Lemma 0E0Q], [Stacks, Lemma 0C09]. ☐

LEMMA 3.19. *Let $f : X \to Y$ be a morphism of schemes which is proper, flat and of finite presentation. Define*

$$Y_{\mathrm{Gor}} = \{y \in Y \mid X_y \text{ is a Gorenstein scheme}\}.$$

Then Y_{Gor} is open in Y.

PROOF. We claim that

$$Y_{\mathrm{Gor}} = Y \setminus f(X \setminus X_{\mathrm{Gor}}).$$

Suppose first that $y \in Y_{\mathrm{Gor}}$. Suppose by contradiction that $y = f(x)$ for $x \in X \setminus X_{\mathrm{Gor}}$. Then f would not be Gorenstein at x, and by Definition 3.16 we would have that $\mathcal{O}_{X_y,x}$ is not Gorenstein. This is absurd because X_y is a Gorenstein scheme.
Suppose now that $y \in Y \setminus f(X \setminus X_{\mathrm{Gor}})$. Suppose by contradiction that X_y is not a Gorenstein scheme. Then there exists $x \in X_y$ such that $\mathcal{O}_{X_y,x}$ is not Gorenstein. Then $x \notin X_{\mathrm{Gor}}$, so that $x \in X \setminus X_{\mathrm{Gor}}$. Since $f(x) = y$ we have that $y \in f(X \setminus X_{\mathrm{Gor}})$ which is absurd.
We already know that X_{Gor} is open in X by Lemma 3.18. Then $X \setminus X_{\mathrm{Gor}}$ is closed. Since f is proper, then $f(X \setminus X_{\mathrm{Gor}})$ is closed in Y. It follows that Y_{Gor} is open in Y. ☐

Consider now the following situation. Let $f : X \to Y$ be a proper, flat and finitely presented morphism of schemes whose geometric fibres are integral normal Gorenstein schemes of dimension 2. We define

$$Y_{\mathrm{DV}} = \{y \in Y \mid X_{\overline{y}} \text{ has at most Du Val singularities}\}.$$

DEFINITION 3.20. Let $f : X \to Y$ be a proper, flat and finitely presented morphism of schemes whose geometric fibres are integral normal Gorenstein schemes of dimension 2. We define a *simultaneous resolution of f* to be a proper and finitely presented map $r : X' \to X$ such that the composition $f' = f \circ r$ is smooth and its geometric fibres are resolutions of the geometric fibres of f (Definition 2.38).

LEMMA 3.21. *Let $f : X \to Y$ be a proper, flat and finitely presented morphism of schemes whose geometric fibres are integral normal Gorenstein schemes of dimension 2. If there exists a simultaneous resolution of f, then Y_{DV} is open and closed in Y.*

PROOF. Let $r : X' \to X$ be a simultaneous resolution of f. Since f is proper, flat and finitely presented morphism, by Theorem B.4 we have that the function

$$a : y \mapsto \chi(X_y, \mathcal{O}_{X_y})$$

is locally constant on Y. Since $f' : X' \to Y$ is smooth and proper by definition, then again by Theorem B.4 we have that also

$$b : y \mapsto \chi(X'_y, \mathcal{O}_{X'_y})$$

is locally constant on Y.
Since $r_*\mathcal{O}_{X'_{\overline{y}}} = \mathcal{O}_{X_{\overline{y}}}$, applying Leray spectral sequence (Theorem B.15) we obtain a long exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^1(X_{\overline{y}}, \mathcal{O}_{X_{\overline{y}}}) \to \mathrm{H}^1(X'_{\overline{y}}, \mathcal{O}_{X'_{\overline{y}}}) \to \mathrm{H}^0(X_{\overline{y}}, R^1 r_* \mathcal{O}_{X'_{\overline{y}}}) \to \\ \to \mathrm{H}^2(X_{\overline{y}}, \mathcal{O}_{X_{\overline{y}}}) \to \mathrm{H}^2(X'_{\overline{y}}, \mathcal{O}_{X'_{\overline{y}}}) \to 0. \end{aligned}$$

A standard dimensional argument shows that

$$\chi(X_{\overline{y}}, \mathcal{O}_{X_{\overline{y}}}) - \chi(X'_{\overline{y}}, \mathcal{O}_{X'_{\overline{y}}}) = h^0(X_{\overline{y}}, R^1(r_{\overline{y}})_*\mathcal{O}_{X'_{\overline{y}}}).$$

But Y_{DV} is exactly the locus where $R^1(r_{\overline{y}})_*\mathcal{O}_{X'_{\overline{y}}} = 0$. Thus it is the locus where $a(y)-b(y) = 0$, because if $y \in Y$ is any point, then

$$\chi(X_{\overline{y}}, \mathcal{O}_{X_{\overline{y}}}) = \chi(X_y, \mathcal{O}_{X_y})$$

by Corollary B.3. Since a and b are locally constant on Y, it follows that Y_{DV} is open and closed in Y. $\square$

LEMMA 3.22. *Let $f : X \to Y$ be a proper, flat and finitely presented morphism of schemes whose geometric fibres are integral normal Gorenstein schemes of dimension 2. Assume further that $Y = \operatorname{Spec} R$ is affine. Then there exists a noetherian affine scheme Y_0 and a cartesian diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X_0 \arrow[d, "f_0"] \\ Y \arrow[r] & Y_0 \end{tikzcd}$$

where f_0 is a proper, flat and finitely presented morphism of schemes whose geometric fibres are integral normal Gorenstein schemes of dimension 2.

PROOF. First observe that every fibre of f is a geometrically integral, geometrically normal Gorenstein scheme of dimension 2 by Lemma 3.3, Proposition 3.4 and Lemma 2.51. Consider the set

$$\{R_\lambda \mid R_\lambda \subseteq R \text{ such that } R_\lambda \text{ is of finite type over } \mathbb{Z}\}.$$

We denote $Y_\lambda = \operatorname{Spec} R_\lambda$. Then by [Stacks, Lemma 01ZM] there exists an index α and a morphism of finite presentation $f_\alpha : X_\alpha \to Y_\alpha$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X_\alpha \arrow[d, "f_\alpha"] \\ Y \arrow[r] & Y_\alpha \end{tikzcd}$$

is cartesian. Consider now the projective system $\{Y_\lambda\}_{\lambda \geq \alpha}$ of affine schemes such that R_λ are subrings of R which are finitely generated extension of R_α. Denote $f_\lambda : X_\lambda \to Y_\lambda$ the base change of f_α to Y_λ. Then f is the projective limit of the f_λ's. Denote $u_\lambda : Y \to Y_\lambda$ and $u_{\lambda\mu} : Y_\mu \to Y_\lambda$, for $\mu \geq \lambda$ the morphisms induced by the inclusions. Since f is proper and flat, there exists an index $\beta \geq \alpha$ such that f_β is proper and flat by [Stacks, Lemma 04AI] and [Stacks, Lemma 081F]. Moreover, f_β is of finite presentation because it comes from f_α by base change.

It remains to verify that the geometric fibres of some f_λ, $\lambda \geq \beta$ are geometrically integral and geometrically normal Gorenstein schemes of dimension 2. For every $\lambda \geq \beta$ consider $E_\lambda \subseteq Y_\lambda$ made out of points $y \in Y_\lambda$ such that the fibre $(X_\lambda)_y$ satisfies the following properties:

(1) $(X_\lambda)_y$ is a geometrically integral and geometrically normal scheme;
(2) $(X_\lambda)_y$ has dimension 2;
(3) $(X_\lambda)_y$ is a Gorenstein scheme.

These conditions do not depend on the base field; i.e. they hold over $\kappa(y)$ if and only if they hold over some (every) field extension K of $\kappa(y)$ by Lemma 3.3, Lemma 2.51 and Proposition 3.4; therefore by transitivity of the fibres ([Gro67, Proposition I.3.6.4]) we have the equalities $u_{\lambda\mu}^{-1}(E_\lambda) = E_\mu$ and $u_\lambda^{-1}(Y_\lambda) = Y$ by hypothesis.

Property (1) gives an open condition by Proposition 3.11. By Proposition 3.12 the dimension

of the fibres is locally constant, hence condition (2) is open. Finally, property (3) gives an open condition by Lemma 3.19. Since an open set in a noetherian topological space is in particular ind-constructible by Proposition 3.10 and Remark 3.8, then properties (1),(2) and (3) define an ind-constructible subset of E_λ.
Since the limit of ind-constructible subset E_λ, $\lambda \geq \beta$, coincides with the limit of Y_λ's according to [Gro67, Corollaire IV.8.3.5], there exists an index $\gamma \geq \beta$ such that $E_\gamma = Y_\gamma$. $\square$

LEMMA 3.23. *Let $X \to Y$ be a proper, flat and finitely presented morphism of schemes whose fibres are geometrically integral, geometrically normal, of dimension 2 and Gorenstein. Then Y_{DV} is locally constructible in Y.*

PROOF. Up to considering an affine covering of Y, we assume that Y is affine. Moreover, by an argument of noetherian approximation, it is not restrictive to assume that Y is noetherian, as we now explain.

By Lemma 3.22 there exists a morphism of schemes $X' \to Y'$ with the same properties as $X \to Y$ and a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "u"'] & Y' \end{tikzcd}$$

such that Y' is noetherian. If Y'_{DV} is locally constructible, then $Y_{\mathrm{DV}} = u^{-1}(Y'_{\mathrm{DV}})$ is again locally constructible by [GW20, Proposition 10.43]. Thus, it is not restrictive to assume Y noetherian.
By [Art74a, Theorem 1] there exists an algebraic space R which is locally of finite type over Y ([Art69, Theorem 5.3]) representing the functor of simultaneous resolutions (see [Art74a, Introduction]). It follows that for an étale presentation $U \to R$, there exists a simultaneous resolution of $X \times_Y U \to U$, where U is a scheme, corresponding to the étale presentation $U \to R$. By Lemma 3.21 we have that U_{DV} is open and closed in U. In particular, U_{DV} is locally constructible by Proposition 3.9. Denote by $f : U \to Y$ the composition, which is a morphism of schemes.
Since Y is noetherian, then R is also locally of finite presentation over Y. It follows in particular that U is locally of finite presentation over Y. Since $U_{DV} = f^{-1}(Y_{DV})$ it follows that Y_{DV} is locally constructible by [GD71, Corollaire 7.2.10]. $\square$

LEMMA 3.24. *Let $X \to Y$ be a proper, flat and finitely presented morphism of schemes whose fibres are geometrically integral, geometrically normal, of dimension 2 and Gorenstein. Assume further that $Y = \operatorname{Spec} R$ is the spectrum of a discrete valuation ring. Let y_0 be the special point of Y and let ξ be the generic point of Y. If $y_0 \in Y_{\mathrm{DV}}$ then $\xi \in Y_{\mathrm{DV}}$.*

PROOF. See [Lie08, Proposition 6.1]. $\square$

LEMMA 3.25. *Let $X \to Y$ be a proper, flat and finitely presented morphism of schemes whose fibres are geometrically integral, geometrically normal, of dimension 2 and Gorenstein. Let $y_0, y_1 \in Y$ such that $y_0 \in \overline{\{y_1\}}$. If $y_0 \in Y_{\mathrm{DV}}$ then $y_1 \in Y_{\mathrm{DV}}$.*

PROOF. Suppose by contradiction that $y_1 \notin Y_{\mathrm{DV}}$. We have that $y_0 \in Y$ is in the closure of the image of the morphism

$$f : U \setminus U_{\mathrm{DV}} \to Y$$

which is the restriction to $U \setminus U_{\mathrm{DV}}$ of the morphism that we constructed in the proof of Lemma 3.23. Then by [Stacks, Lemma 0CM2] there exists a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & U \setminus U_{\mathrm{DV}} \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & Y \end{tikzcd}$$

where R is a discrete valuation ring, $K = \operatorname{frac}(R)$, and the closed point of $\operatorname{Spec} R$ maps to y_0. We can assume moreover that the generic point of $\operatorname{Spec} R$ maps to y_1 (see proof of [Stacks, Lemma 0CM2]).

If we base change the morphism $(X \to Y)$ to $T = \operatorname{Spec} R$ we obtain a cartesian diagram

$$\begin{tikzcd} X_T \arrow[r] \arrow[d, "g"'] & X \arrow[d] \\ T \arrow[r] & Y \end{tikzcd}$$

where g is again a proper, flat and finitely presented morphism of schemes (Proposition 3.6) whose fibres are geometrically integral, geometrically normal, of dimension 2 and Gorenstein, because they are in particular fibres of the starting family.

The fibre over the special point $t_0 \in T = \operatorname{Spec} R$ is given by a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_T \arrow[r] \arrow[d, "g"'] & X \arrow[d] \\ \operatorname{Spec} \kappa(t_0) \arrow[r] & T = \operatorname{Spec} R \arrow[r] & Y \end{tikzcd}$$

and $t_0 \in T_{\mathrm{DV}}$ because t_0 maps to y_0 and $y_0 \in Y_{\mathrm{DV}}$ by hypothesis. However, the generic point $\xi \in T$ is not in T_{DV} because ξ maps to y_1 and $y_1 \notin Y_{\mathrm{DV}}$ by assumption. This is absurd by Lemma 3.24. $\square$

Corollary 3.26. *Let $X \to Y$ be a proper, flat and finitely presented morphism of schemes whose fibres are geometrically integral, geometrically normal, of dimension 2 and Gorenstein. Assume that Y is noetherian. Then Y_{DV} is open in Y.*

Proof. This follows immediately by Lemma 3.25, Lemma 3.23 and Proposition 3.10. $\square$

Proposition 3.27. *Let $f : X \to Y$ be a proper, smooth and finitely presented morphism of schemes whose fibres are integral schemes. Then the set*

$$Y_{\geq 0} = \left\{ y \in Y \;\middle|\; \exists\, m \in \mathbb{N}_{\geq 1} \text{ such that } \dim_{\kappa(y)} \mathrm{H}^0(X_y, \omega_{X_y}^{\otimes m}) \neq 0 \right\}$$

is ind-constructible in Y.

Proof. Define the following subset of Y:

$$E = \left\{ y \in Y \;\middle|\; \dim_{\kappa(y)} \mathrm{H}^0(X_y, \omega_{X_y}^{\otimes m}) = 0 \ \ \forall m \geq 1 \right\}.$$

It is clear that

$$Y_{\geq 0} = Y \setminus E = \bigcup_{m \geq 1} \left\{ y \in Y \;\middle|\; \dim_{\kappa(y)} \mathrm{H}^0(X_y, \omega_{X_y}^{\otimes m}) > 0 \right\}.$$

Denote $A_m = \left\{ y \in Y \;\middle|\; \dim_{\kappa(y)} \mathrm{H}^0(X_y, \omega_{X_y}^{\otimes m}) > 0 \right\}$. Each A_m is a locally constructible set by Theorem B.4. In particular, each A_m is ind-constructible by Remark 3.8. It follows that $Y_{\geq 0}$ is union of ind-constructible sets, and thus it is ind-constructible by [Gro67, Proposition IV.1.9.5]. $\square$

COROLLARY 3.28. *Let $f : X \to Y$ be a proper, smooth and finitely presented morphism of schemes whose fibres are integral schemes of dimension 2. Then the set*

$$\{y \in Y \mid \kappa(X_{\overline{y}}) \geq 0\}$$

is ind-constructible in Y, where $\kappa(X_{\overline{y}})$ is the Kodaira dimension of the geometric fibre over $\kappa(\overline{y})$.

PROOF. Observe that the set in the statement is

$$\begin{aligned} &\{y \in Y \mid \exists\ m \in \mathbb{N}_{\geq 1} \text{ such that } p_m(X_{\overline{y}}) \neq 0\} = \\ =&\left\{y \in Y \;\middle|\; \exists\ m \in \mathbb{N}_{\geq 1} \text{ such that } \dim_{\kappa(\overline{y})} \mathrm{H}^0(X_{\overline{y}}, \omega_{X_{\overline{y}}}^{\otimes m}) \neq 0\right\} = \\ =&\left\{y \in Y \;\middle|\; \exists\ m \in \mathbb{N}_{\geq 1} \text{ such that } \dim_{\kappa(y)} \mathrm{H}^0(X_y, \omega_{X_y}^{\otimes m}) \neq 0\right\}, \end{aligned}$$

where in the last equality we used Corollary B.3. Then the statement follows by Proposition 3.27. □

PROPOSITION 3.29. *Let $f : X \to Y$ be a proper, smooth and finitely presented morphism of schemes whose fibres are geometrically integral schemes of dimension 2. Consider the set*

$$Y_{\min} = \{y \in Y \mid X_{\overline{y}} \text{ is a minimal surface}\}.$$

Then

(1) *$Y_{\min}$ is closed in Y.*
(2) *If the geometric fibres of f have non-negative Kodaira dimension, then $Y_{\min}$ is also open in Y.*

PROOF. (1): Let $H = \mathrm{Hilb}_{X/Y}$ be the relative Hilbert scheme ([Stacks, Section 0CZX]) which is locally of finite presentation over Y by [Stacks, Proposition 0D01]. For every point $y \in Y$ we have a cartesian diagram

$$\begin{tikzcd} \mathrm{Hilb}_{X_{\overline{y}}} \arrow[r] \arrow[d] & \mathrm{Hilb}_{X_y} \arrow[r] \arrow[d] & H \arrow[d, "\pi"] \\ \kappa(\overline{y}) \arrow[r] & \kappa(y) \arrow[r] & Y \end{tikzcd}$$

where $\mathrm{Hilb}_{X_{\overline{y}}}$ (resp. Hilb_{X_y}) is the Hilbert scheme parametrizing closed subschemes of $X_{\overline{y}}$ (resp. X_y), proper over $\kappa(\overline{y})$ (resp. $\kappa(y)$). Suppose that $C \subset X_{\overline{y}}$ is a (-1)-curve on $X_{\overline{y}}$ (Definition 2.15), and denote by $i : C \hookrightarrow X_{\overline{y}}$ the closed immersion. Then the normal bundle of C in $X_{\overline{y}}$ is $\mathcal{N}_{C/X_{\overline{y}}} \simeq \mathcal{O}_{\mathbb{P}^1_{\kappa(\overline{y})}}(-1)$. It follows that

$$\mathrm{H}^0(C, \mathcal{N}_{C/X_{\overline{y}}}) = \mathrm{H}^1(C, \mathcal{N}_{C/X_{\overline{y}}}) = 0. \tag{4}$$

By deformation theory, there exists an isomorphism $\mathrm{T}_{[C]}\mathcal{H}ilb_{X_{\overline{y}}} \simeq \mathrm{H}^0(C, \mathcal{N}_{C/X_{\overline{y}}})$, where $\mathcal{H}ilb_{X_{\overline{y}}}$ is the functor defined in [TV13, §2.4.2], see [TV13, Theorem 3.26]. Moreover, the space $\mathrm{H}^1(C, \mathcal{N}_{C/X_{\overline{y}}})$ is an obstruction space for the functor $\mathcal{H}ilb_{X_{\overline{y}}}$ and the object $C \in \mathcal{H}ilb_{X_{\overline{y}}}(\kappa(\overline{y}))$, because C is a local complete intersection in $X_{\overline{y}}$, see [TV13, Theorem 5.21]; note that we may also consider the deformation category $\mathcal{H}ilb_{C \hookrightarrow X_{\overline{y}}}$ associated to the category $\mathcal{H}ilb_{X_{\overline{y}}}$ and the object $C \in \mathcal{H}ilb_{X_{\overline{y}}}(\kappa(\overline{y}))$, as in [TV13, §2]. The category $\mathcal{H}ilb_{C \hookrightarrow X_{\overline{y}}}$ contains only objects of $\mathcal{H}ilb_{X_{\overline{y}}}$ that restrict to C over $\operatorname{Spec} \kappa(\overline{y})$.
By above arguments and Equation (4) we have that $\mathrm{T}_{[C]}\mathrm{Hilb}_{X_{\overline{y}}} = 0$ and thus $[C]$ is a reduced isolated point of the scheme $\mathrm{Hilb}_{X_{\overline{y}}}$. Again by above arguments and Equation (4) it follows that the *deformation problem* $\mathcal{H}ilb_{C \hookrightarrow X_{\overline{y}}}$ is unobstructed (Definition [TV13, Definition 5.1]). It follows that the scheme $\mathrm{Hilb}_{X_{\overline{y}}}$ is smooth in a neighbourhood of $[C]$, and moreover that $\pi : H \to Y$ is smooth in a neighbourhood of $[C]$ by [Kol96, Theorem 2.10].

In particular, one can show that there exists an open neighbourhood V of $[C]$ in H such that $\pi_{|V} : V \to \pi(V)$ is étale. We choose V sufficiently small such that points of V corresponds to (-1)-curves of some fibres near to $X_{\overline{y}}$. The fact that C deforms to (-1) curves follows from the fact that $C \simeq \mathbb{P}^1_{\kappa(\overline{y})}$ and $\mathbb{P}^1_{\kappa(\overline{y})}$ is rigid, having $\mathrm{H}^1(\mathbb{P}^1_{\kappa(\overline{y})}, \mathcal{T}_{\mathbb{P}^1_{\kappa(\overline{y})}}) = 0$ [Ser06, Corollary 1.2.15 and Example 1.2.16]; while the self-intersection of a deformation of C remains -1 because of the invariance of the self-intersection under deformations, see also [Ser06, Example 3.4.24]. Since étale maps are open, then $\pi(V)$ is open in Y. Equivalently, $Y_{\min}$ is closed in Y.

(2): Suppose now that the geometric fibres of f have non-negative Kodaira dimension. Working locally, we may assume that Y is affine. By an argument of noetherian approximation, we may also assume that Y is noetherian, see also the proof of Lemma 3.22. To show that Y' is open in Y, it is sufficient to show that Y' is stable under generization by Proposition 3.10. Moreover, by [Stacks, Lemma 0CM2] it is sufficient to consider the case in which $Y = \operatorname{Spec} R$ is the spectrum of a discrete valuation ring, see also the proof of Lemma 3.25. Then the statement follows by [Art74a, Lemma 2.1] or by [Alp24, Proposition B.2.17] using Lemma 2.31. □

3.3. Families

In this chapter, we will define categories whose objects are morphisms of schemes, or more generally, morphisms of algebraic spaces from an algebraic space X to a scheme T. We study here some properties that will hold in all categories that we consider.
Consider the following list of properties of morphisms of schemes or algebraic spaces.

P_1: proper;
P_2: flat;
P_3: of finite presentation;
P_4: smooth.

We now define an *example category* which is representative of all the categories that we will consider in this chapter.

Definition 3.30. Let $\mathcal{P}_{\mathrm{mor}} \subseteq \{P_1, P_2, P_3, P_4\}$ and let $\mathcal{P}_{\mathrm{fib}}$ be a finite set of properties of schemes over a field. We define a category C as follows:

- objects of C are morphisms $(X \to T)$ of schemes with properties in $\mathcal{P}_{\mathrm{mor}}$ and such that each geometric fibre has properties in $\mathcal{P}_{\mathrm{fib}}$ over the base field;
- an arrow $(X' \to T') \to (X \to T)$ between two objects is a pair (f, g) where $f : X' \to X$, $g : T' \to T$ are morphisms of schemes such that the diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

is cartesian, i.e. a pullback.

Proposition 3.31. *With notation as above, there exists a forgetful functor*

$$\begin{array}{rccc} F: & C & \to & \mathrm{Sch} \\ & (X \to T) & \mapsto & T \\ & (f,g) & \mapsto & g \end{array}$$

which makes C fibred in groupoids over Sch.

Proof. First, we prove that C is fibred over Sch, Definition A.45. Let $T \in \mathrm{Sch}$ and $(a : X \to T) \in C$. Let $g : T' \to T$ be a morphism of schemes, i.e. an arrow in Sch. We have to show that there exists a cartesian arrow $(f, g) : (X' \to T') \to (X \to T)$ in C such that $F((f, g)) = g$. Consider the cartesian diagram of schemes

$$\begin{tikzcd} X' = X \times_T T' \arrow[r, "f"] \arrow[d, "a'"'] & X \arrow[d, "a"] \\ T' \arrow[r, "g"'] & T. \end{tikzcd}$$

Observe that a' is a morphism of schemes with properties in $\mathcal{P}_{\mathrm{mor}}$ because these properties are stable under base change by Proposition 3.6. Moreover, each geometric fibre of a' is in particular a geometric fibre of a as we have a cartesian diagram as follows

$$\begin{tikzcd} (X')_k = X_k \arrow[r] \arrow[d] & X' \arrow[r, "f"] \arrow[d, "a'"'] & X \arrow[d, "a"] \\ \operatorname{Spec} k \arrow[r] & T' \arrow[r, "g"'] & T \end{tikzcd}$$

for all geometric points $\operatorname{Spec} k \to T'$ of T'. It follows that geometric fibres of $(a' : X' \to T')$ have properties $\mathcal{P}_{\mathrm{fib}}$ and thus a' is an object of C. It is clear that (f, g) is a cartesian arrow, as it is given by a cartesian diagram.
To prove that C is fibred in groupoids over Sch (Definition A.50) we have to show that for all schemes T the fibre category $C(T)$ is a groupoid. This follows from the fact that an arrow of $C(T)$ is (f, id), i.e. is a morphism of schemes $f : X' \to X$ such that the diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ T \arrow[r, "\mathrm{id}"] & T \end{tikzcd}$$

is cartesian. Then f is necessarily an isomorphism. □

The same fact holds if we consider categories in which objects are morphism of algebraic spaces from an algebraic space X to a scheme T.

Definition 3.32. Let $\mathcal{P}_{\mathrm{mor}} \subseteq \{P_1, P_2, P_3, P_4\}$ and let $\mathcal{P}_{\mathrm{fib}}$ be a finite set of properties of schemes over a field. We define a category C' as follows:

- objects of C' are morphisms $(X \to T)$ of algebraic spaces from an algebraic space X to a scheme T with properties in $\mathcal{P}_{\mathrm{mor}}$ and such that each geometric fibre is a scheme and has properties in $\mathcal{P}_{\mathrm{fib}}$ over the base field;
- an arrow $(X' \to T') \to (X \to T)$ between two objects is a pair (f, g) where $f : X' \to X$ is a morphism of algebraic spaces, and $g : T' \to T$ is a morphism of schemes such that the diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

is cartesian.

Proposition 3.33. *With notation as above, there exists a forgetful functor*

$$\begin{array}{rccc} F' : & C' & \to & \mathrm{Sch} \\ & (X \to T) & \mapsto & T \\ & (f, g) & \mapsto & g \end{array}$$

which makes C' fibred in groupoids over Sch.

Proof. The proof is the same as in Proposition 3.31. □

Proposition 3.34. *With above notation, suppose that C (resp. C') is a category in which $\mathcal{P}_{\text{fib}}$ are properties which are stable under a base change which is a field extension. Let $(X \to T)$ be a morphism of schemes (resp. of algebraic spaces from an algebraic space X to a scheme T). Then the following are equivalent:*

(1) *$(X \to T)$ is an object of C;*
(2) *$(X \to T)$ has properties $\mathcal{P}_{\text{mor}}$ and for all points $t \in T$ the geometric fibre $X_{\bar{t}}$ over the geometric point $\operatorname{Spec} \kappa(\bar{t}) \to T$ has properties $\mathcal{P}_{\text{fib}}$.*

Proof. The fact that (2) implies (1) is clear. To prove that (2) implies (1), consider a geometric point

$$\sigma : \operatorname{Spec} k \to T$$

and let $t \in T$ be the unique point in the image of σ. Then σ induces an homomorphism of rings $\alpha : \kappa(t) \to k$ between fields. Thus, the morphism σ factors as

$$\operatorname{Spec} k \to \operatorname{Spec} \kappa(t) \to T.$$

Since k is algebraically closed, the homomorphism of rings α factors as

$$\kappa(t) \to \kappa(\bar{t}) \to k$$

and we obtain another factorization of σ as

$$\operatorname{Spec} k \to \operatorname{Spec} \kappa(\bar{t}) \to T.$$

It follows that we have a commutative diagram

$$\begin{array}{ccccc} X_k & \longrightarrow & X_{\bar{t}} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} \kappa(\bar{t}) & \longrightarrow & T \end{array}$$

in which the external square and the right square are cartesian. Thus also the square on the left is cartesian by Lemma A.3. By hypotheses, properties $\mathcal{P}_{\text{fib}}$ are stable under a base change which is a field extension. Thus, X_k has properties in $\mathcal{P}_{\text{fib}}$. It follows that $(X \to T)$ is an object of C. $\square$

Definition 3.35. With notation as above, we will say that an object $(X \to T)$ of C or C' is a family, and we will say that X is the *total space* of the family, while T is the *base scheme* of the family. Indeed an object $(a : X \to T)$ of C or C' must be interpreted as a family of schemes, namely the fibres of a, parametrized by the base scheme T.

3.4. The stack of canonical models of minimal surfaces of general type

In what follows, χ, K^2 are two fixed integers.

Definition 3.36. Let T be a scheme. A *family of canonical surfaces with invariants χ, K^2 over T* is a proper, flat and finitely presented morphism of schemes $X \to T$ such that for all geometric points $\sigma : \operatorname{Spec} k \to T$, the fibre X_k is an integral (which implies connected) normal scheme of dimension 2, with at most Du Val singularities and ample dualizing sheaf ω_{X_k} such that $\chi(\mathcal{O}_{X_k}) = \chi$ and $K^2_{X_k} = K^2$.

In other words, a family of canonical surfaces is a (proper, flat and finitely presented) morphism of schemes such that every geometric fibre is the canonical model of a minimal surface of general type by Proposition 2.62.

Proposition 3.37. *Let $X \to T$ be a morphism of schemes. Then the following are equivalent:*

(1) *$X \to T$ is a family of canonical surfaces with invariants χ, K^2;*
(2) *$X \to T$ is a proper, flat and finitely presented morphism of schemes such that for all points $t \in T$ the geometric fibre $X_{\bar{t}}$ is an integral normal scheme of dimension 2, with at most Du Val singularities and ample dualizing sheaf $\omega_{X_{\bar{t}}}$ such that $\chi(\mathcal{O}_{X_{\bar{t}}}) = \chi$ and $K^2_{X_{\bar{t}}} = K^2$.*

Proof. This is clear by Proposition 3.34 because the properties of being an integral normal scheme of dimension 2 with at most Du Val singularities, ample dualizing sheaf, Euler characteristic χ and self-intersection of the canonical bundle K^2 are stable under a base change which is a field extension by Lemma 3.3, Lemma 2.51, Proposition 3.5, Proposition 3.4 and Corollary B.3. □

Remark 3.38. Let $p : X \to T$ be a family of canonical surfaces and let $t \in T$ be a point. Even if $\kappa(t)$ is not algebraically closed, we have a cartesian diagram

$$\begin{tikzcd} X_{\bar{t}} \arrow[r] \arrow[d] & X_t \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}\kappa(\bar{t}) \arrow[r] & \operatorname{Spec}\kappa(t) \arrow[r] & T. \end{tikzcd}$$

Then X_t is a Gorenstein geometrically integral scheme of dimension 2 with ample canonical bundle ω_{X_t} by Lemma 3.3, Lemma 2.52 Proposition 3.4 and Proposition 3.5. Then the relative canonical sheaf $\omega_{X/T}$ exists, is a line bundle by Proposition B.13, and is p-ample (Definition B.8) by Corollary B.11.

Definition 3.39. We define the category $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ as follows.

- Objects are families of canonical surfaces with invariants χ, K^2.
- An arrow $(X' \to T') \to (X \to T)$ between two objects is a pair (f, g) where $f : X' \to X$, $g : T' \to T$ are morphisms of schemes such that the diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

is cartesian.

Proposition 3.40. *The forgetful functor*

$$\begin{array}{rccc} F: & \mathcal{M}^{\mathrm{can}}_{\chi,K^2} & \to & \mathrm{Sch} \\ & (X \to T) & \mapsto & T \end{array}$$

makes $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ a category fibred in groupoids over Sch.

Proof. This follows immediately by Proposition 3.31. □

Proposition 3.41. *The category $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ is a stack over* $\mathrm{Sch}_{\text{ét}}$.

Proof. We will prove that $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ is a stack in the fpqc topology, which implies the statement by Remark A.30. We prove that the hypotheses of [Vis08, Theorem 4.38] are satisfied. First we show that families of canonical surfaces with fixed invariants χ and K^2 form a class of morphisms $\mathcal{P}$ which is stable (Definition A.59) in the fpqc site $\mathrm{Sch}_{\mathrm{fpqc}}$. Given an object

$$(p : X \to T) \in \mathcal{M}^{\mathrm{can}}_{\chi,K^2}(T)$$

and isomorphisms $X' \simeq X$, $T \simeq T'$, then the morphism $p' : X' \to T'$ given by the compositions is again a family of canonical surfaces with invariants χ, K^2. Indeed p' is a proper, flat and finitely presented morphism of schemes, because it is the composition of p with two isomorphisms. Moreover, geometric fibres of p' are isomorphic to geometric fibres of p, so that the

conditions on the fibres are satisfied.
The condition (ii) on the base change in the definition of a stable class of arrows (Definition A.59) is satisfied, because we have already proved that $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ is fibred over Sch. Hence being a family of canonical surfaces with fixed invariants χ, K^2 is a stable condition.
In order to prove that $\mathcal{P}$ is local (Definition A.60), let $X \to T$ be any morphism of schemes. Let $\{T_i \to T\}_{\{i\in I\}}$ be an fpqc covering, and suppose that $X_i = T_i \times_T X \to T_i$ is a family of canonical surfaces with fixed invariants χ, K^2 for all $i \in I$. We have to show that also $X \to T$ is a family of canonical surfaces with the same invariants χ, K^2. The properties of being proper, flat and finitely presented for a morphism of schemes are fpqc local properties on the base, see [Stacks, Lemma 02L1], [Stacks, Lemma 02L2] and [Stacks, Lemma 02L0]. It follows that $X \to T$ is a proper, flat and finitely presented morphism of schemes. By Proposition 3.37, we have to show that for all points $t : \operatorname{Spec}\kappa(t) \to T$, if $\kappa(\bar{t})$ is an algebraic closure of $\kappa(t)$, then the geometric fibre over $\kappa(\bar{t})$ is a canonical surface with invariants χ, K^2. By surjectivity of $\coprod_i T_i \to T$ there exists a point $t' \in T_i$ for some $i \in I$ which is sent to t. Thus we obtain a morphism of schemes

$$\operatorname{Spec}\kappa(t') \to T_i \to T$$

whose image is $t \in T$. Pre-composing with an algebraic closure $\kappa(\overline{t'})$ of $\kappa(t')$ we obtain a geometric point of T

$$\sigma : \operatorname{Spec}\kappa(\overline{t'}) \to T$$

which factorizes as

$$\operatorname{Spec}\kappa(\overline{t'}) \to \operatorname{Spec}\kappa(\bar{t}) \to T \tag{5}$$

as proved in Proposition 3.34.
Observe that we have a commutative diagram

$$\begin{tikzcd}
(X_i)_{\kappa(\overline{t'})} \arrow[r] \arrow[d] & X_i \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}\kappa(\overline{t'}) \arrow[r] & T_i \arrow[r] & T
\end{tikzcd}$$

in which the squares on the right and on the left are cartesian. Thus, also the external square is cartesian, $X_{\kappa(\overline{t'})} \simeq (X_i)_{\kappa(\overline{t'})}$ and $X_{\kappa(\overline{t'})} \to \operatorname{Spec}\kappa(\overline{t'})$ is a geometric fibre of $X \to T$. Since $X_i \to T_i$ is a family of canonical surfaces with fixed invariants χ, K^2, then this geometric fibre is a canonical surface with invariants χ, K^2.

By Equation (5) it follows that we have a cartesian diagram

$$\begin{tikzcd}
X_{\kappa(\overline{t'})} \arrow[r] \arrow[d] & X_{\bar{t}} \arrow[r] \arrow[d] & X \arrow[d] \\
\kappa(\overline{t'}) \arrow[r] & \operatorname{Spec}\kappa(\bar{t}) \arrow[r] & T.
\end{tikzcd}$$

But then also $X_{\bar{t}}$ is a canonical surface with invariants χ, K^2 by Lemma 3.3, Lemma 2.52, Proposition 3.5 and Proposition 3.4. Then $X \to T$ is a family of canonical surfaces with invariants χ, K^2 by Proposition 3.37. This shows that being a family of canonical surfaces is a local condition in the fpqc topology.
For each object $p : X \to T$ of $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}(T)$ we have an invertible sheaf $\omega_{X/T}$ which is ample relative to the morphism p, see Remark 3.38. Moreover, formation on $\omega_{X/T}$ is compatible with base change, i.e. if we have a cartesian diagram of schemes

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d] & R \arrow[r, "g"] \arrow[d] & X \arrow[d] \\ V \arrow[r] & U \arrow[r] & T \end{tikzcd}$$

whose columns are objects of $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$, then the diagram

$$\begin{tikzcd} f^*g^*\omega_{X/T} \arrow[r] \arrow[d] & (g\circ f)^*\omega_{X/T} \arrow[d] \\ f^*\omega_{R/U} \arrow[r] & \omega_{Q/V} \end{tikzcd}$$

of invertible sheaves on Q commutes, see [Con00, Theorem 3.6.1]. Then $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ is a stack over $\mathrm{Sch}_{\mathrm{fpqc}}$ by [Vis08, Theorem 4.38] and then also over $\mathrm{Sch}_{\text{ét}}$. □

DEFINITION 3.42. We define the *stack of canonical models* as

$$\mathcal{M}^{\mathrm{can}} = \coprod_{\chi,K^2} \mathcal{M}^{\mathrm{can}}_{\chi,K^2}.$$

as χ, K^2 runs in $\mathbb{Z}$ and define *families of canonical surfaces* to be the objects of $\mathcal{M}^{\mathrm{can}}$. An object $(X \to \operatorname{Spec} k) \in \mathcal{M}^{\mathrm{can}}(k)$ over an algebraically closed field k is the canonical model of a minimal surface of general type over k (Definition 2.63).

Let $f : S \to k$ be a minimal surface of general type over an algebraically closed field k of arbitrary characteristic (Definitions 2.6 and 2.20). By Theorem 2.60(3) we know that $\mathrm{H}^1(S,\omega_S^{\otimes 5}) = \mathrm{H}^2(S,\omega_S^{\otimes 5}) = 0$. We now prove the same vanishing for canonical surfaces.

LEMMA 3.43. *Let $X \to \operatorname{Spec} k$ be a canonical surface over an algebraically closed field k. Let $m \geq 1$ be an integer. Then*

$$\mathrm{H}^1(X,\omega_X^{\otimes 5m}) = \mathrm{H}^2(X,\omega_X^{\otimes 5m}) = 0.$$

PROOF. This is just an application of the Leray spectral sequence (Theorem B.15) and Lemma B.16. We will spell out the result in this particular case. Let $\pi : S \to X$ be the minimal desingularization of X (Definition 2.40, so that S is a minimal surface of general type and X is its canonical model, see also Proposition 2.62.

We know that $\pi^*\omega_X^{\otimes n} \simeq \omega_S^{\otimes n}$ for all $n \in \mathbb{Z}$ by Proposition 2.59. It follows that for all integers $i \geq 0$ we have

$$R^i\pi_*\omega_S^{\otimes(-5m+1)} = R^i\pi_*\pi^*\omega_X^{\otimes(-5m+1)} \simeq \omega_X^{\otimes(-5m+1)} \otimes_{\mathcal{O}_X} R^i\pi_*\mathcal{O}_S, \tag{6}$$

where the last isomorphism is given by projection formula [Stacks, Lemma 01E8]. Recall that $R^1\pi_*\mathcal{O}_S = 0$ by Proposition 2.47. Moreover, the fibres of π have dimension ≤ 1, being only points or curves, so that the higher direct images $R^i\pi_*\mathcal{O}_S$ are zero for all $i \geq 2$ by [GW23, Corollary 23.146] (observe that π is proper and hence also of finite presentation because X and S are noetherian schemes). We conclude by Equation (6) that

$$R^i\pi_*\omega_S^{\otimes(-5m+1)} = 0 \text{ for all } i \geq 1 \text{ and}$$
$$\pi_*\omega_S^{\otimes(-5m+1)} \simeq \omega_X^{\otimes(-5m+1)}.$$

It follows that the second page of the Leray spectral sequence for $\omega_S^{\otimes(-5m+1)}$ is all 0 except from the first column. This spectral sequence converges to $\mathrm{H}^{p+q}(S,\omega_S^{\otimes(-5m+1)})$, so that

$$\mathrm{H}^p(X,\omega_X^{\otimes(-5m+1)}) = \mathrm{H}^p(X,\pi_*\omega_S^{\otimes(-5m+1)}) \simeq \mathrm{H}^p(S,\omega_S^{\otimes(-5m+1)}). \tag{7}$$

By Theorem 2.60 we have $\mathrm{H}^1(S,\omega_S^{\otimes(-5m+1)})=0$ for all $m\geq 1$ and thus by Equation (7) for $p=1$ we immediately obtain

$$\mathrm{H}^1(X,\omega_X^{\otimes 5m})\simeq(\mathrm{H}^1(X,\omega_X^{\otimes(-5m+1)}))^\vee=0.$$

Again by Theorem 2.60 we have $\mathrm{H}^0(S,\omega_S^{\otimes(-5m+1)})=0$ for all $m\geq 1$ and thus by Equation (7) with $p=0$ it follows that $\mathrm{H}^0(X,\omega_X^{\otimes(-5m+1)})=0$. Then

$$\mathrm{H}^2(X,\omega_X^{\otimes 5m})\simeq(\mathrm{H}^0(X,\omega_X^{\otimes(-5m+1)}))^\vee=0. \qquad \square$$

COROLLARY 3.44. *Let $X\to T$ be a family of canonical surfaces and let $m\geq 1$ be an integer. Then for every point $t:\operatorname{Spec}\kappa(t)\to T$ we have*

$$\mathrm{H}^1(X_t,\omega_{X_t}^{\otimes 5m})=\mathrm{H}^2(X_t,\omega_{X_t}^{\otimes 5m})=0.$$

PROOF. If $\kappa(t)$ is an algebraically closed field, then everything is clear by Lemma 3.43. Otherwise, consider $\kappa(\bar{t})$ to be an algebraic closure of $\kappa(t)$. Then the statement follows by Lemma 3.43, because by Corollary B.3, we have that $h^i(X_t,\omega_{X_t}^{\otimes 5m})=h^i(X_{\bar{t}},\omega_{X_{\bar{t}}}^{\otimes 5m})$ for $i=1,2$. $\square$

LEMMA 3.45. *Let $p:X\to T$ be an object of $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$. Then*

(i) *$p_*\mathcal{O}_X\simeq\mathcal{O}_T$;*
(ii) *$p_*(\omega_{X/T}^{\otimes 5})$ is a locally free sheaf on T of rank $\chi+10K^2$;*
(iii) *for any morphism $g:T'\to T$, the natural map*

$$g^*p_*\omega_{X/T}^{\otimes 5}\to p'_*\omega_{X'/T'}^{\otimes 5}$$

is an isomorphism, where

$$p':X'=X\times_T T'\to T'$$

denotes the base change of p to T'.

PROOF. For every point $t:\operatorname{Spec}\kappa(t)\to T$ of T we have that the fibre X_t over $\kappa(t)$ is geometrically connected and geometrically reduced by Remark 3.38.

It follows that $\mathrm{H}^0(X_t,\mathcal{O}_{X_t})=\kappa(t)$ by [Stacks, Lemma 0FD2]. Then the natural map

$$\beta^0(\kappa(t)):p_*\mathcal{O}_X\otimes_{\mathcal{O}_T}\kappa(t)\to\mathrm{H}^0(X_t,\mathcal{O}_{X_t})=\kappa(t)$$

of cohomology and base change (Theorem B.5) is surjective, and by cohomology and base change (Theorem B.5(2)) we have that $p_*\mathcal{O}_X$ is locally free sheaf of finite rank. We can compute the rank of $p_*\mathcal{O}_X$ on geometric fibres of $X\to T$ over geometric points of the form $\operatorname{Spec}\kappa(\bar{t})\to T$. In particular, we see that $p_*\mathcal{O}_X$ is a line bundle because

$$h^0(X_{\bar{t}},\mathcal{O}_{X_{\bar{t}}})=\dim_{\kappa(\bar{t})}\kappa(\bar{t})=1.$$

If $t:\operatorname{Spec}\kappa(t)\to T$ is a point of T, then the natural map $\mathcal{O}_T\to p_*\mathcal{O}_X$ induces a map of $\kappa(t)$-modules

$$\kappa(t)\to p_*\mathcal{O}_X\otimes_{\mathcal{O}_T}\kappa(t),$$

which is surjective because post-composing with $\beta^0(\kappa(t))$ is the identity. It follows that $\mathcal{O}_T\to p_*\mathcal{O}_X$ is a surjective homomorphism of line bundles, hence it is an isomorphism and (i) is proved.
Let $t:\operatorname{Spec}\kappa(t)\to T$ be a point of T and consider the first map of cohomology and base change

$$\beta^1(\kappa(t)):R^1p_*(\omega_{X/T}^{\otimes 5})\otimes_{\mathcal{O}_T}\kappa(t)\to\mathrm{H}^1(X_t,\omega_{X_t}^{\otimes 5}),$$

which is surjective because $\mathrm{H}^1(X_t,\omega_{X_t}^{\otimes 5})=0$ by Corollary 3.44. By cohomology and base change (Corollary B.6) it follows that $R^1p_*(\omega_{X/T}^{\otimes 5})=0$, because this holds over a neighbourhood of the point in the image of t, and t was any point of T.

Again by cohomology and base change (Corollary B.6) we have that $p_*\omega_{X/T}^{\otimes 5}$ is a locally free sheaf of finite rank. We can compute the rank of this locally free sheaf on fibres. By Corollary 3.44 we have in particular that $h^1(X_t, \omega_{X_t}^{\otimes 5}) = h^2(X_t, \omega_{X_t}^{\otimes 5}) = 0$. Then

$$\begin{aligned} h^0(X_t, \omega_{X_t}^{\otimes 5}) = \chi(\omega_{X_t}^{\otimes 5}) = & \\ &= \chi(\mathcal{O}_{X_t}) + \frac{(5K_{X_t})(5K_{X_t} - K_{X_t})}{2} = \\ &= \chi + \frac{5K_{X_t} \cdot 4K_{X_t}}{2} = \chi + 10K^2. \end{aligned}$$

Finally, (iii) follows by cohomology and base change because again by Corollary B.6, formation of $p_*\omega_{X/T}^{\otimes 5}$ commutes with base change [GW23, Definition 23.138]. □

LEMMA 3.46. *Let $p : X \to T$ be an object of $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$. Then the map*

$$p^* p_* \omega_{X/T}^{\otimes 5} \to \omega_{X/T}^{\otimes 5}$$

is surjective, and the resulting T-map

$$X \to \mathbb{P}(p_*\omega_{X/T}^{\otimes 5})$$

is a closed embedding.

PROOF. By Lemma 3.45.(iii) and cohomology and base change (Theorem B.5), up to restricting to geometric fibres over geometric points $\operatorname{Spec} \kappa(\bar{t}) \to T$, it is sufficient to verify the statement when $T = \operatorname{Spec} k$ is the spectrum of an algebraically closed field k, see [Ols16, Lemma 8.4.6]. Then the statement follows by the fact that $\omega_{X/k}^{\otimes 5}$ is very ample in this case by Theorem 2.60(6). □

LEMMA 3.47. *Let $A \to B$ a morphism of rings such that B is flat as an A-module. Let M be a B-module which is flat over B. Then M is flat over A.*

PROOF. See [Mat89, Transitivity, p.46, §3.7]. □

We will now to prove the algebraicity of the stack $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$. Recall that a stack $\mathcal{X}$ is algebraic if there exists a smooth surjective morphism $U \to \mathcal{X}$ where U is a scheme. We will prove the algebraicity of $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ by exhibiting an isomorphism of $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ with a quotient stack.

THEOREM 3.48. *The stack $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ is an algebraic stack over* $\mathrm{Sch}_{\text{ét}}$.

PROOF. Define $N = \chi + 10K^2$ and let

$$\tilde{M}^{\mathrm{can}}_{\chi,K^2} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$$

be the functor which sends any scheme T to the set of isomorphism classes of pairs

$$(p : X \to T, \sigma : \mathcal{O}_T^N \simeq p_*\omega_{X/T}^{\otimes 5}),$$

where $(p : X \to T)$ is an object of $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}(T)$, and an isomorphism of two pairs

$$(p' : X' \to T, \sigma' : \mathcal{O}_T^N \simeq p'_*\omega_{X'/T}^{\otimes 5}) \to (p : X \to T, \sigma : \mathcal{O}_T^N \simeq p_*\omega_{X/T}^{\otimes 5})$$

is given by an isomorphism of T-schemes $\alpha : X' \to X$ such that the diagram

$$\begin{tikzcd} & \mathcal{O}_T^N \arrow[dl, "\sigma'"'] \arrow[dr, "\sigma"] & \\ p'_*\omega_{X'/T}^{\otimes 5} \arrow[rr] & & p_*\omega_{X/T}^{\otimes 5} \end{tikzcd}$$

commutes, where the horizontal arrow is the one induced by α.

Let $(p : X \to T, \sigma) \in \tilde{M}^{\mathrm{can}}_{\chi,K^2}(T)$ be an isomorphism class of a pairs as above. Thus by Lemma 3.46 we have a closed embedding

$$\begin{tikzcd} X \arrow[rr, hook, "\iota"] \arrow[dr] & & \mathbb{P}(p_*\omega_{X/T}^{\otimes 5}) \simeq \mathbb{P}_T^{N-1} \arrow[dl] \\ & T & \end{tikzcd}$$

where the isomorphism in the top right corner is the one induced σ. By the construction of the closed embedding ι, we also have an isomorphism

$$\tau : \iota^*\mathcal{O}_{\mathbb{P}_T^{N-1}}(1) \to \omega_{X/T}^{\otimes 5}. \tag{8}$$

If $X_{\bar{t}} \to \operatorname{Spec} \kappa(\bar{t})$ is a geometric fibre of the family $p : X \to T$, then the very ample line bundle $\omega^{\otimes 5}_{X_{\bar{t}}/\kappa(\bar{t})}$ (Theorem 2.60(6)) induces a closed immersion $X_{\bar{t}} \hookrightarrow \mathbb{P}^{N-1}_{\kappa(\bar{t})}$ with Hilbert polynomial

$$\begin{aligned} P(m) &= \chi(\omega_{X_{\bar{t}}}^{\otimes 5m}) \\ &= \chi(\mathcal{O}_{X_{\bar{t}}}) + \frac{(5mK_{X_{\bar{t}}})(5mK_{X_{\bar{t}}} - K_{X_{\bar{t}}})}{2} \\ &= \chi + \frac{5m(5m-1)}{2}K^2. \end{aligned}$$

We fix the polynomial $P = P(m)$ and consider the Hilbert scheme

$$\mathcal{H} = \mathrm{Hilb}^P(\mathbb{P}_{\mathbb{Z}}^{N-1})$$

parametrizing closed subschemes of $\mathbb{P}_{\mathbb{Z}}^{N-1}$ with Hilbert polynomial P (Example A.11), with respect to $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^{N-1}}(1)$. The scheme $\mathrm{Hilb}^P(\mathbb{P}_{\mathbb{Z}}^{N-1})$ represents the functor $\mathcal{H}ilb^P(\mathbb{P}_{\mathbb{Z}}^{N-1})$ of Example A.11 and we can consider the universal object in $\mathcal{H}ilb^P(\mathbb{P}_{\mathbb{Z}}^{N-1})(\mathcal{H})$ given by

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[d, "\pi"'] & \mathbb{P}_{\mathcal{H}}^{N-1} = \mathbb{P}_{\mathbb{Z}}^{N-1} \times_{\mathbb{Z}} \mathcal{H} \arrow[dl] \\ \mathcal{H}. & \end{tikzcd}$$

Observe that by definition of the Hilbert functor, π is flat and finitely presented. Moreover π is projective, hence proper.
We claim that there exists a maximal open subscheme $H_1 \subseteq \mathcal{H}$ such that the restriction X_{H_1} of $\mathcal{Z}$ to H_1 has the following property: for all geometric points $\operatorname{Spec} \kappa(\bar{h}) \to H_1$, the fibre $X_{\bar{h}} \to \operatorname{Spec} \kappa(\bar{h})$

(a) is geometrically integral;
(b) is geometrically normal;
(c) has at most Du Val singularities;
(d) has invariants χ, K^2.

In order to prove the claim, it is sufficient to show that conditions (a) – (d) are open conditions. This follows for (a) and (b) by Proposition 3.11. Once we assume (a) and (b), having Gorenstein singularities is an open condition by Lemma 3.19 and then condition (c) is open by Corollary 3.26. The Euler characteristic is locally constant by Theorem B.4 and once we assume conditions (a), (b), (c), also the self-intersection of the canonical bundle is locally constant by Proposition 3.13 and the claim is proved. Observe further that every fibre has dimension 2, because the Hilbert polynomial has degree 2. In other words, $X_{H_1} \to H_1$ is a family of

canonical surfaces with invariants χ, K^2.
Moreover, there exists a maximal open subscheme $H \subseteq H_1$ where the embedding

$$\begin{tikzcd} X_H \arrow[r, "i"] \arrow[d] & \mathbb{P}^{N-1}_H \arrow[dl] \\ H & \end{tikzcd}$$

satisfies the following condition. The sheaf $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^{N-1}_H}(1)$ is *strongly ample*, i.e. for all points $\operatorname{Spec}\kappa(h) \to H$, the pullback $\mathcal{L}_{X_h/\kappa(h)}$ of $\mathcal{L}$ to the fibre is very ample and

$$\mathrm{H}^i(X_h, \mathcal{L}^{\otimes m}_{X_h/\kappa(h)}) = 0 \quad \forall\, i, m > 0.$$

The fact that this is an open condition follows by [Kol23, §8.4]. It follows that we have a diagram

$$\begin{tikzcd} X_H \arrow[r] \arrow[d, "f"'] & \mathcal{Z} \arrow[r] \arrow[d] & \mathbb{P}^{N-1}_{\mathcal{H}} \arrow[dl] \\ H \arrow[r] & \mathcal{H} & \end{tikzcd}$$

where $(f : S_H \to H)$ is an object of $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}(H)$ and $\mathcal{L}$ is strongly ample. Moreover, by cohomology and base change (Corollary B.6), we have that $f_*\mathcal{L}$ is a locally free sheaf whose formation commutes with base change and it has rank N because on each fibre $X_h \to \operatorname{Spec}\kappa(h)$ defines the embedding in $\mathbb{P}^{N-1}_{\kappa(h)}$.
If we have a pair $(X \to T, \sigma) \in \tilde{M}^{\mathrm{can}}_{\chi,K^2}(T)$, it follows that $X \to T$ is given by the pullback of the universal family $X_H \to H$ through a unique morphism $X \to H$ as in the following diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_H \arrow[d, "f"] \\ T \arrow[r] & H \end{tikzcd}$$

because the isomorphism of equation (8) shows that $\iota^*\mathcal{O}_{\mathbb{P}^{N-1}_T}(1)$ is strongly ample by Lemma 3.43. It follows that we have a natural transformation of functors

$$\overline{F} : \tilde{M}^{\mathrm{can}}_{\chi,K^2} \to H$$

sending $(X \to T, \sigma) \in \tilde{M}^{\mathrm{can}}_{\chi,K^2}(T)$ to the unique morphism of schemes $T \to H$.

Consider now the following functor

$$\begin{array}{rccc} \tilde{H}: & (\mathrm{Sch}/H)^{\mathrm{op}} & \longrightarrow & \mathrm{Set} \\ & (g : T \to H) & \longmapsto & \left\{ \begin{array}{c} \text{isomorphisms of sheaves} \\ \lambda : g^*f_*(\omega^{\otimes 5}_{X_H/H}) \simeq g^*f_*(\mathcal{L}) \end{array} \right\}. \end{array}$$

We claim that $\tilde{H}$ is representable by an affine smooth H-scheme. It is sufficient to verify the claim locally around any point $h \in H$. Let

$$h \in \operatorname{Spec}(A) \subseteq H$$

be an affine neighbourhood of h such that both $f_*(\omega^{\otimes 5}_{X_H/H})$ and $f_*(\mathcal{L})$ are trivial, and choose a trivialization for these locally free sheaves of rank N. It follows that the functor $\tilde{H}$ restricted to the category of schemes over $\operatorname{Spec}(A)$ is identified with the functor which to any A-scheme $T \to \operatorname{Spec}(A)$ associate the set

$$\mathrm{GL}_N(\Gamma(T, \mathcal{O}_T)),$$

and this functor is representable by an affine scheme by Example A.14. We also denote by $\tilde{H}$ the H-scheme representing the functor $\tilde{H}$.

We claim now that the natural transformation of functors $\overline{F}$ lifts to a natural transformation of functors

$$F : \tilde{M}^{\mathrm{can}}_{\chi,K^2} \to \tilde{H}.$$

To prove this, recall that $\overline{F}$ maps $(f' : X \to T, \sigma)$ to the unique morphism of schemes $g : T \to H$ as above. We have to show that the morphism $g : T \to H$ factors through $\tilde{H}$. In other words, by the definition of $\tilde{H}$, we have to exhibit an isomorphism of sheaves $g^* f_*(\omega^{\otimes 5}_{X_H/H}) \simeq g^* f_*(\mathcal{L})$. In order to do this, consider the following diagram

$$\begin{tikzcd}
\mathbb{P}^{N-1}_T \arrow[r] & \mathbb{P}^{N-1}_H \\
X \arrow[u, hook, "\iota"] \arrow[r, "g'"] \arrow[d, "f'"'] & X_H \arrow[u, hook, "i"'] \arrow[d, "f"] \\
T \arrow[r, "g"'] & H.
\end{tikzcd}$$

We have

$$g^* f_*(\omega^{\otimes 5}_{X_H/H}) \simeq f'_* \omega^{\otimes 5}_{X/T} \simeq f'_* \iota^* \mathcal{O}^{N-1}_T(1) \simeq g^* f_* \mathcal{L},$$

where the second isomorphism is given by Equation (8).

Next we show that the morphism of functors F identifies $\tilde{M}^{\mathrm{can}}_{\chi,K^2}$ with a closed subscheme of $\tilde{H}$. Let

$$\begin{tikzcd}
X_{\tilde{H}} \arrow[r, hook, "e"] \arrow[d, "\tilde{f}"'] & \mathbb{P}^{N-1}_{\tilde{H}} \arrow[dl] \\
\tilde{H} &
\end{tikzcd}$$

be the universal object over $\tilde{H}$. The identity morphism on the scheme $\tilde{H}$ corresponds to an isomorphism of sheaves

$$\lambda^u : \tilde{f}_* \omega^{\otimes 5}_{X_{\tilde{H}}/\tilde{H}} \to \tilde{f}_* e^* \mathcal{O}_{\mathbb{P}^{N-1}_{\tilde{H}}}(1)$$

by the definition of $\tilde{H}$. Let $T \to \tilde{H}$ be a morphism of schemes, and let $p : X_T \to T$ be the pullback of $X_{\tilde{H}}$ to T. Let $j : X_T \hookrightarrow \mathbb{P}^{N-1}_T$ be the induced closed immersion, and define $\mathcal{L}' = j^* \mathcal{O}_{\mathbb{P}^{N-1}_T}(1)$. The isomorphism λ^u induces a diagram of sheaves on X_T

$$\begin{tikzcd}
p^* p_* \omega^{\otimes 5}_{X_T/T} \arrow[r] \arrow[d, two heads] & p^* p_* \mathcal{L}' \arrow[d, two heads] \\
\omega^{\otimes 5}_{X_T/T} \arrow[r, dashed] & \mathcal{L}'
\end{tikzcd}$$

where the surjectivity of the left vertical arrow follows by Lemma 3.46 and the same proof of Lemma 3.46 works also for the surjectivity of the right vertical arrow. This way, the functor $\tilde{M}^{\mathrm{can}}_{\chi,K^2}$ is identified with the subfunctor of $\tilde{H}$ which to any morphism of schemes $T \to \tilde{H}$ associates the unital set if there exists a dotted arrow filling in this diagram, and the empty set otherwise.

This condition is represented by a closed subscheme of $\tilde{H}$. Indeed, denote by K the kernel of the map

$$\tilde{f}^* \tilde{f}_* \omega^{\otimes 5}_{X_{\tilde{H}}/\tilde{H}} \to \omega^{\otimes 5}_{X_{\tilde{H}}/\tilde{H}},$$

and for every $T \to \tilde{H}$ denote by K_T the kernel of the map

$$\tilde{p}^* \tilde{p}_* \omega_{X_T/T}^{\otimes 5} \to \omega_{X_T/T}^{\otimes 5}.$$

Then a dotted arrow filling in the diagram exists over $T \to \tilde{H}$ if and only if the composite map

$$K_T \hookrightarrow p^* p_* \omega_{X_T/T}^{\otimes 5} \to p^* p_* \mathcal{L}' \to \mathcal{L}' \tag{9}$$

is zero. Let r be an integer such that the map

$$f^* f_*(K \otimes \mathcal{L}^{\otimes r}) \to K \otimes \mathcal{L}^{\otimes r}$$

is surjective, and such that $f_*(K \otimes \mathcal{L}^{\otimes r})$ is a locally free sheaves on $\tilde{H}$ whose formation commutes with arbitrary base change on $\tilde{H}$. Then the condition that the map in Equation (9) is zero over some $T \to \tilde{H}$ is equivalent to the condition that the map of locally free shaves on $\tilde{H}$

$$f_*(K \otimes \mathcal{L}^{\otimes r}) \to f_* \mathcal{L}^{\otimes r+1}$$

pulls back to the zero bundle on T. This condition is represented by a closed subscheme of $\tilde{H}$ by Example A.15. This shows that $\tilde{M}^{\text{can}}_{\chi,K^2}$ is representable by a quasi-projective scheme.

Observe now that there is an action of the group scheme $G = \mathrm{GL}_N$ (Example A.14) on $\tilde{M}^{\text{can}}_{\chi,K^2}$. Namely, if T is a scheme, the action of $g \in G(T) = \mathrm{GL}_N(\Gamma(T, \mathcal{O}_T))$ on $(X \to T, \sigma) \in \tilde{M}^{\text{can}}_{\chi,K^2}(T)$ is

$$g \cdot (X \to T, \sigma) \mapsto (X \to T, \sigma \circ g).$$

Consider the map of stacks

$$\alpha : \tilde{M}^{\text{can}}_{\chi,K^2} \to \mathcal{M}^{\text{can}}_{\chi,K^2}$$

which maps $(X \to T, \sigma)$ to $(X \to T) \in \mathcal{M}^{\text{can}}_{\chi,K^2}(T)$. For any scheme T and morphism $f : T \to \mathcal{M}^{\text{can}}_{\chi,K^2}$ corresponding to a family of canonical surfaces $X \to T$, the diagram

$$\begin{tikzcd} \tilde{M}^{\text{can}}_{\chi,K^2} \times_{\mathcal{M}^{\text{can}}_{\chi,K^2}} T \arrow[r] \arrow[d] & \tilde{M}^{\text{can}}_{\chi,K^2} \\ T & \end{tikzcd}$$

is a principal G-bundle. Indeed the action of an element $g \in G(T)$ on an object $((X \to T, \sigma), (X \to T)) \in \tilde{M}^{\text{can}}_{\chi,K^2} \times_{\mathcal{M}^{\text{can}}_{\chi,K^2}} T$ only changes the isomorphism σ. It follows that we have an isomorphism

$$\mathcal{M}^{\text{can}}_{\chi,K^2} \simeq [\tilde{M}^{\text{can}}_{\chi,K^2}/G].$$

In particular, $\mathcal{M}^{\text{can}}_{\chi,K^2}$ is an algebraic stack by Proposition A.85. $\square$

REMARK 3.49. We will prove the algebraicity of $\mathcal{M}^{\text{can}}_{\chi,K^2}$ also in §3.8 using a different method. Namely, we will verify that hypotheses of Artin's axioms (Theorem A.119) are satisfied.

THEOREM 3.50. *The stack $\mathcal{M}^{\text{can}}$ is algebraic.*

PROOF. We have

$$\mathcal{M}^{\text{can}} = \coprod_{\chi,K^2} \mathcal{M}^{\text{can}}_{\chi,K^2}$$

and each $\mathcal{M}^{\text{can}}_{\chi,K^2}$ is algebraic by Theorem 3.48. In particular, for all χ, K^2 there exists a smooth presentation $\tilde{M}^{\text{can}}_{\chi,K^2} \to \mathcal{M}^{\text{can}}_{\chi,K^2}$. It follows that

$$\coprod_{\chi,K^2} \tilde{M}^{\text{can}}_{\chi,K^2} \to \mathcal{M}^{\text{can}}$$

is a smooth presentation. $\square$

Proposition 3.51. *For every K^2 and χ, the stack $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ is of finite type over* $\operatorname{Spec}\mathbb{Z}$.

Proof. We know by the proof of Theorem 3.48 that $\mathcal{M}^{\mathrm{can}}_{\chi,K^2} \simeq [\tilde{M}^{\mathrm{can}}_{\chi,K^2}/G]$ where $\tilde{M}^{\mathrm{can}}_{\chi,K^2}$ is a quasi-projective scheme over $\operatorname{Spec}\mathbb{Z}$ and G is an affine group scheme. It follows in particular that $\tilde{M}^{\mathrm{can}}_{\chi,K^2}$ is of finite type over $\mathbb{Z}$, and then also $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ is of finite type over $\mathbb{Z}$ (Definition A.91 and Remark A.92). $\square$

Remark 3.52. By Proposition 3.51 it follows that $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ is noetherian, because $\tilde{M}^{\mathrm{can}}_{\chi,K^2}$ is of finite type over $\operatorname{Spec}\mathbb{Z}$ and we use [Stacks, Lemma 01T6].

Remark 3.53. However, $\mathcal{M}^{\mathrm{can}}$ is not a Deligne-Mumford stack. Indeed condition (iii) of Theorem A.101 is not satisfied by Remark 2.73.

If we work in characteristic zero however, the stack $\mathcal{M}^{\mathrm{can}}$ is a Deligne-Mumford stack. We use the following notation:

$$\mathcal{M}^{\mathrm{can},\mathbb{Q}} = \mathcal{M}^{\mathrm{can}} \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{Q}.$$

Proposition 3.54. *The stack $\mathcal{M}^{\mathrm{can},\mathbb{Q}}$ is a Deligne-Mumford stack.*

Proof. Since $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ is an algebraic stack by Theorem 3.50, then also $\mathcal{M}^{\mathrm{can},\mathbb{Q}}$ is an algebraic stack. By Theorem A.101 we have to show that for every canonical surface X over a field k of characteristic 0, the group scheme Aut_X is discrete and reduced. This is what we proved in Proposition 2.72. $\square$

3.5. The stack of algebraic spaces

We now introduce the more general stack of proper, flat and finitely presented morphisms of algebraic spaces. The main reference is [Stacks, Section 0D1D].

Definition 3.55. We define the category $\mathcal{S}paces'$ as follows.

- Objects are proper, flat and finitely presented morphisms of algebraic spaces $X \to T$ from an algebraic space X to a scheme T.
- An arrow $(X' \to T') \to (X \to T)$ between two objects is a pair (f, g) where $f : X' \to X$ is a morphism of algebraic spaces and $g : T' \to T$ is a morphism of schemes such that the diagram
$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$
is cartesian.

Lemma 3.56. *The forgetful functor*

$$\begin{array}{rccc} F: & \mathcal{S}paces' & \to & \mathrm{Sch} \\ & (X \to T) & \mapsto & T \end{array}$$

makes $\mathcal{S}paces'$ a category fibred in groupoids over Sch.

Proof. It follows immediately by Proposition 3.33. Observe that $\mathcal{P}_{\mathrm{fib}}$ is the empty set here. $\square$

Lemma 3.57. *The diagonal*

$$\Delta : \mathcal{S}paces' \to \mathcal{S}paces' \times_{\operatorname{Spec}\mathbb{Z}} \mathcal{S}paces'$$

is representable by algebraic spaces.

Proof. Let $\alpha : T \to \mathcal{S}paces' \times_{\operatorname{Spec}\mathbb{Z}} \mathcal{S}paces'$ be a morphism of algebraic spaces from a scheme T. By 2-Yoneda's lemma A.58, α corresponds to two objects of $\mathcal{S}paces'$ over T, say $(x : X \to T)$ and $y : (Y \to T)$. By Proposition A.80, it is sufficient to show that the sheaf $\underline{\mathrm{Hom}}_T(x, y)$ (Definition A.61) is an algebraic space (see also [Stacks, Lemma 045G]). This follows from the fact that the diagram of functors

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_T(x,y) \arrow[rr] \arrow[d] & & T \arrow[d] \\
\underline{\mathrm{Hom}}_T(x,y) \times \underline{\mathrm{Hom}}_T(y,x) \arrow[rr] & & \underline{\mathrm{Hom}}_T(x,x) \times \underline{\mathrm{Hom}}_T(y,y)
\end{tikzcd}$$

is cartesian, where the bottom arrow is given by $(\phi, \psi) \mapsto (\psi \circ \phi, \phi \circ \psi)$, and the right vertical arrow is just $(\mathrm{id}, \mathrm{id})$. The morphism functors $\underline{\mathrm{Hom}}_T(-,-)$ are all algebraic spaces, by [Stacks, Proposition 0D1C]. Then the statement follows from the fact that fibre products exist in the category of algebraic spaces, see Remark A.90. □

Allowing the total space to be an algebraic space in the definition of an object of $\mathcal{S}paces'$ allow us to prove that $\mathcal{S}paces'$ is a stack over $\mathrm{Sch}_{\mathrm{fppf}}$ and thus also over $\mathrm{Sch}_{\text{ét}}$. The philosophy here is that an fppf descent data for algebraic spaces is effective. This is one of the reason why algebraic spaces have been introduced.

Lemma 3.58. *The category $\mathcal{S}paces'$ is a stack over* $\mathrm{Sch}_{\mathrm{fppf}}$.

Proof. We know that $\mathcal{S}paces'$ is a category fibred in groupoids over $\mathrm{Sch}_{\mathrm{fppf}}$ by Lemma 3.56. We prove that conditions of Definition A.67 of a stack hold. First observe that morphisms glue by Lemma 3.57. Suppose now that $\{T_i \to T\}_{i \in I}$ is an fppf covering in $\mathrm{Sch}_{\mathrm{fppf}}$. Let $X_i \to T_i$ be an object of $\mathcal{S}paces'$ for all $i \in I$, and $\alpha_{ij} : X_i \times_T T_j \to X_j \times_T T_i$ isomorphisms over $T_i \times_T T_j$ for each $i, j \in I$, such that the cocycle condition on $T_i \times_T T_j \times_T T_k$ is satisfied. We have to show that there exists an object $X \to T$ of $\mathcal{S}paces'$ which is compatible with restrictions. The existence of an algebraic space over T compatible with restrictions follows by [Stacks, Lemma 0ADV.(2)], because if $(f : X \to T) \in \mathcal{S}paces'$, then f is in particular of finite type. To show that this is an object of $\mathcal{S}paces'$ we use the fact that for a morphism of algebraic spaces, the properties of being proper, flat and of finite presentation are local properties in the fppf topology by [Stacks, Lemma 041W], [Stacks, Lemma 0422], [Stacks, Lemma 041V]. It follows that $(X \to T)$ is an object of $\mathcal{S}paces'$ and we are done. □

Remark 3.59. One could also define the category $\mathcal{S}paces$ whose objects are morphisms from an algebraic space X to a scheme T and morphisms are defined as for $\mathcal{S}paces'$. This also satisfies conditions (i) and (ii) of a stack, as proved in [Stacks, Lemma 04UA].

Remark 3.60. The category $\mathcal{S}paces$ is not an algebraic stack. The reason is that Artin's axioms (Theorem A.119) are not satisfied. Indeed, consider for example the object $(\xi : \mathbb{A}^1_k \to \operatorname{Spec} k) \in \mathcal{S}paces$. Consider the functor $\underline{\mathrm{Aut}}_{\mathbb{A}^1_k}$ of Example A.64. By deformation theory, the tangent space at the identity to $\underline{\mathrm{Aut}}_{\mathbb{A}^1_k}$ is $\mathrm{H}^0(\mathbb{A}^1_k, \mathcal{T}_{\mathbb{A}^1_k}) \simeq k[x]$, which is an infinite dimensional k-vector space. Thus condition (d) of Artin's axioms (Theorem A.119) is not satisfied.

Remark 3.61. The category $\mathcal{S}paces'$ is not an algebraic stack. Indeed the effectiveness axiom (f) of Artin's axioms (Theorem A.119) is not satisfied, and this is a necessary condition for the algebraicity, see [Stacks, Lemma 07X8]. The reason why this condition is not satisfied is, for example, that there exists a formal object of K3 surfaces (Section §3.9) which is not effective, see [Stacks, Section 0D1Q], [Ser06, Example 2.5.12] and [TV13].

3.6. The stack of smooth surfaces

DEFINITION 3.62. We define the category $\mathit{Surfaces}^{\mathrm{sm}}$ as the full subcategory of $\mathcal{S}paces'$ (i.e. we consider some objects of $\mathcal{S}paces'$, but all possible morphisms between them) whose objects are *families of smooth surfaces*, i.e. proper, smooth and finitely presented morphisms $f : S \to T$ of algebraic spaces, where T is a scheme and for all geometric points $\sigma : \operatorname{Spec} k \to T$ the geometric fibre of f given by a cartesian diagram

$$\begin{tikzcd} S_k \arrow[r] \arrow[d] & S \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r, "\sigma"'] & T \end{tikzcd}$$

is an integral scheme of dimension 2.

In other words, if $S_k \to \operatorname{Spec} k$ is a geometric fibre of a family of smooth surfaces, then S_k is a smooth surface over k.

By the definition of the category $\mathit{Surfaces}^{\mathrm{sm}}$, we have a fully faithful embedding

$$\mathit{Surfaces}^{\mathrm{sm}} \subset \mathcal{S}paces'. \tag{10}$$

REMARK 3.63. If $f : S \to T$ is a family of smooth surfaces, then f is also flat by [Stacks, Lemma 04TA].

REMARK 3.64. If S is an algebraic space of dimension 2, smooth and of finite type over a field k, then S is a scheme, by [Art73, Théorème 4.7]. Thus, in the definition of a family of smooth surfaces it is not restrictive to assume that fibres are schemes. We will use this result also when we discuss fibres of the stack of minimal surfaces of general type.

PROPOSITION 3.65. *Let $S \to T$ be a morphism of algebraic spaces where T is a scheme. Then the following are equivalent:*

(1) *$S \to T$ is a family of smooth surfaces;*
(2) *$S \to T$ is a proper, smooth and finitely presented morphism of algebraic spaces such that for every point $t : \operatorname{Spec} \kappa(t) \to T$, the fibre $S_{\bar{t}}$ over an algebraic closure $\kappa(\bar{t})/\kappa(t)$ of $\kappa(t)$ is an integral scheme of dimension 2;*
(3) *$S \to T$ is a proper, smooth and finitely presented morphism of algebraic spaces such that for every point $t : \operatorname{Spec} \kappa(t) \to T$, the fibre S_t is a geometrically integral scheme of dimension 2.*

PROOF. The equivalence between (1) and (2) is clear by Proposition 3.34 because the property of being an integral scheme of dimension 2 is stable under base change which is a field extension by Lemma 3.3 and Proposition 3.4. The equivalence between (2) and (3) follows again by the fact the properties of being geometrically integral and of dimension 2 are stable under a base change which is a field extension. □

LEMMA 3.66. *Let $f : S \to T$ be a proper and flat morphism of algebraic spaces from an algebraic space S to a scheme T. Suppose that the fibres of f are schemes. Let $t : \operatorname{Spec} \kappa(t) \to T$ be a point of T and let $\kappa(\bar{t})/\kappa(t)$ be an algebraic closure. Then $f_t : S_t \to \operatorname{Spec} \kappa(t)$ is smooth if and only if $f_{\bar{t}} : S_{\bar{t}} \to \operatorname{Spec} \kappa(\bar{t})$ is smooth.*

PROOF. The morphisms f_t and $f_{\bar{t}}$ are both flat morphisms of finite type of locally noetherian schemes, being the pullback of a proper and flat morphism. Then f_t is smooth if and only if $S_{\bar{t}}$ is regular by [GW20, Corollary 6.28] and this happens if and only if $S_{\bar{t}} \to \operatorname{Spec} \kappa(\bar{t})$ is smooth, again by [GW20, Corollary 6.28]. □

LEMMA 3.67. *Let $f : S \to T$ be a flat and locally of finite presentation morphism of algebraic spaces where T is a scheme and such that for all points $t : \operatorname{Spec}\kappa(t) \to T$ the fibre S_t is a scheme which is smooth over $\kappa(t)$. Then f is smooth. Moreover, it is sufficient to require that every geometric fibre is smooth.*

PROOF. By definition of a smooth morphism of algebraic spaces [Stacks, Definition 03ZC] we have to prove that there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "h"'] & S \arrow[d, "f"] \\ V \arrow[r, "b"'] & T \end{tikzcd}$$

where U, V are schemes, a, b are étale and a is surjective, and h is smooth.
Since T is a scheme, we consider b to be $\mathrm{id} : T \to T$ which is obviously an étale morphism. Choose $a : U \to S$ to be an étale presentation, i.e. a surjective étale morphism from a scheme U, which exists because S is an algebraic space. Thus we only have to show that the composition

$$h : U \xrightarrow{a} S \xrightarrow{f} T$$

is smooth. Observe that h is flat and locally of finite presentation because f is flat and locally of finite presentation by hypotheses, see definitions in [Stacks, Definition 03ML] and [Stacks, Definition 03XP]. Finally over any point $t : \operatorname{Spec}\kappa(t) \to T$ of T we have a diagram

$$\begin{tikzcd} U_t \arrow[r] \arrow[d] & U \arrow[d] \\ S_t \arrow[r] \arrow[d] & S \arrow[d] \\ \operatorname{Spec}\kappa(t) \arrow[r] & T \end{tikzcd}$$

where U_t is the pullback of U to S_t. Hence the two squares are cartesian, and then also the external diagram is cartesian (Lemma A.3). Thus, the fibre U_t is smooth over $\kappa(t)$, being the composition of a smooth morphism $S_t \to \kappa(t)$ with an étale morphism $U_t \to S_t$. Observe that by Lemma 3.66, this holds even if we only assume that geometric fibres are smooth. It follows that h is smooth by [Stacks, Lemma 01V8] and we are done. $\square$

LEMMA 3.68. *The forgetful functor*

$$\begin{array}{rccc} F: & Surfaces^{\mathrm{sm}} & \to & \mathrm{Sch} \\ & (S \to T) & \mapsto & T \end{array}$$

makes $Surfaces^{\mathrm{sm}}$ a category fibred in groupoids over Sch.

PROOF. Follows immediately by Proposition 3.33. $\square$

PROPOSITION 3.69. *The cateogory $Surfaces^{\mathrm{sm}}$ is a stack over* $\mathrm{Sch}_{\mathrm{fppf}}$.

PROOF. We know that $\mathcal{S}paces'$ is a stack over $\mathrm{Sch}_{\mathrm{fppf}}$, and that $Surfaces^{\mathrm{sm}} \subset \mathcal{S}paces'$. In particular, morphisms in $Surfaces^{\mathrm{sm}}$ glue because they glue in $\mathcal{S}paces'$. We now prove that objects glue in the fppf topology. Since $\mathcal{S}paces'$ is a stack by Lemma 3.58, we only have to prove that if $S \to T$ is an object of $\mathcal{S}paces'$ and $\{T_i \to T\}_{i \in I}$ is an fppf covering, the following are equivalent:

(1) $S \to T$ is an object of $Surfaces^{\mathrm{sm}}$;
(2) for each i, the base change $S_i \to T_i$ is an object of $Surfaces^{\mathrm{sm}}$.

The fact that (1) implies (2) is clear, since fibres of each pullback are in particular fibres of the family $S \to T$. To see the converse, assume (2) and recall that we have already proved in the previous section that $S \to T$ is proper and finitely presented (Lemma 3.58). Moreover, $S \to T$ is smooth because being smooth is a fpqc local property on the base by [Stacks, Lemma 0429] (see [Stacks, Definition 03YH]). Then by Proposition 3.65 we only have to prove that for each point $t : \operatorname{Spec}\kappa(t) \to T$ of T, the geometric fibre $S_{\bar{t}} \to \operatorname{Spec}\kappa(\bar{t})$ is an integral scheme of dimension 2.

Denote also by $t \in T$ the unique point in the image of the morphism t. By surjectivity of $\coprod_i T_i \to T$ there exists a point $t' \in T_i$ for some $i \in I$ which is sent to t. Thus we obtain a morphism of schemes

$$\operatorname{Spec}\kappa(t') \to T_i \to T$$

whose image is $t \in T$. Pre-composing with an algebraic closure $\kappa(\overline{t'})$ of $\kappa(t')$ we obtain a geometric point of T

$$\sigma : \operatorname{Spec}\kappa(\overline{t'}) \to T$$

which factorizes as

$$\operatorname{Spec}\kappa(\overline{t'}) \to \operatorname{Spec}\kappa(\bar{t}) \to T \tag{11}$$

as proved in Proposition 3.34.

Observe that we have a commutative diagram

$$\begin{tikzcd}
(S_i)_{\kappa(\overline{t'})} \arrow[r] \arrow[d] & S_i \arrow[r] \arrow[d] & S \arrow[d] \\
\operatorname{Spec}\kappa(\overline{t'}) \arrow[r] & T_i \arrow[r] & T
\end{tikzcd}$$

in which the squares on the right and on the left are cartesian. Thus, also the external square is cartesian, $S_{\kappa(\overline{t'})} \simeq (S_i)_{\kappa(\overline{t'})}$ and $S_{\kappa(\overline{t'})} \to \operatorname{Spec}\kappa(\overline{t'})$ is a geometric fibre of $S \to T$. Since $S_i \to T_i$ is an object of $Surfaces^{\mathrm{sm}}$, then this geometric fibre is an integral scheme of dimension 2.

By the factorization in Equation (11) it follows that we have a cartesian diagram

$$\begin{tikzcd}
S_{\kappa(\overline{t'})} \arrow[r] \arrow[d] & S_{\bar{t}} \arrow[r] \arrow[d] & S \arrow[d] \\
\kappa(\overline{t'}) \arrow[r] & \operatorname{Spec}\kappa(\bar{t}) \arrow[r] & T.
\end{tikzcd}$$

But then also $S_{\bar{t}}$ is an integral scheme of dimension 2 by Lemma 3.3 and Proposition 3.4. Then $S \to T$ is a family of smooth surfaces. It follows that $S \to T$ is a family of smooth surfaces over T. $\square$

Proposition 3.70. *The diagonal*

$$\Delta : Surfaces^{\mathrm{sm}} \to Surfaces^{\mathrm{sm}} \times_{\operatorname{Spec}\mathbb{Z}} Surfaces^{\mathrm{sm}}$$

is representable by algebraic spaces.

Proof. We know that the diagonal of $\mathcal{S}paces'$ is representable by algebraic spaces by Lemma 3.57. It follows that also the diagonal of $Surfaces^{\mathrm{sm}}$ is representable by algebraic spaces, because the proof of Lemma 3.57 only uses the properties of morphisms of algebraic spaces, and $Surfaces^{\mathrm{sm}}$ is a full subcategory of $\mathcal{S}paces'$, see Equation (10). $\square$

Remark 3.71. However, $Surfaces^{\mathrm{sm}}$ is not an algebraic stack. See Remark 3.61.

3.7. The stack of minimal surfaces of general type

Definition 3.72. Let T be a scheme. A *family of minimal surfaces of general type over T* is a proper, smooth and finitely presented morphism of algebraic spaces $S \to T$ such that for all geometric points $\sigma : \operatorname{Spec} k \to T$, the geometric fibre S_k is an integral scheme, of dimension 2 such that the canonical line bundle ω_{S_k} is big and nef.

Remark 3.73. Recall that by Remark 3.64 it is not restrictive to assume that each fibre is a scheme, as for $Surfaces^{\mathrm{sm}}$.

Remark 3.74. If S is a smooth surface over an algebraically closed field k, requiring $\omega_S = \omega_{S/k}$ to be big and nef it means that S is a minimal surface of general type by Corollary 2.34. Thus, geometric fibres of a family of minimal surfaces of general type are precisely minimal surfaces of general type.

Proposition 3.75. *Let $S \to T$ be a morphism of algebraic spaces where T is a scheme. Then the following are equivalent:*

(1) *$S \to T$ is a family of minimal surfaces of general type;*
(2) *$S \to T$ is a proper, smooth and finitely presented morphism of algebraic spaces such that for every point $t : \operatorname{Spec} \kappa(t) \to T$, the fibre $S_{\bar{t}}$ over an algebraic closure $\kappa(\bar{t})/\kappa(t)$ of $\kappa(t)$ is an integral scheme of dimension 2 such that the canonical line bundle $\omega_{S_{\bar{t}}}$ is big and nef;*
(3) *$S \to T$ is a proper, smooth and finitely presented morphism of algebraic spaces such that for every point $t : \operatorname{Spec} \kappa(t) \to T$, the fibre S_t is a geometrically integral scheme of dimension 2 such that the canonical line bundle ω_{S_t} is big and nef.*

Proof. The equivalence between (1) and (2) follows immediately by Proposition 3.34 because the properties of being integral, of dimension 2 with canonical line bundle big and nef are stable properties under a base change which is a field extension by Lemma 3.3, Proposition 3.4, and Proposition 3.5.
The equivalence between (2) and (3) is clear because for every point $t \in T$ we have a cartesian diagram

$$\begin{tikzcd} S_{\bar{t}} \arrow[r] \arrow[d] & S_t \arrow[r] \arrow[d] & S \arrow[d] \\ \operatorname{Spec} \kappa(\bar{t}) \arrow[r] & \operatorname{Spec} \kappa(t) \arrow[r] & T \end{tikzcd}$$

and the properties required on the fibres are stable under field extensions as we have already explained before. □

Definition 3.76. We define the category $\mathcal{M}^{\min}$ as follows.

- Objects are families of minimal surfaces of general type.
- An arrow $(S' \to T') \to (S \to T)$ between two objects is a pair (f, g) where $f : S' \to S$ is a morphism of algebraic spaces, $g : T' \to T$ is a morphism of schemes and they are such that the diagram

$$\begin{tikzcd} S' \arrow[r, "f"] \arrow[d] & S \arrow[d] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

  is cartesian.

Remark 3.77. The category $\mathcal{M}^{\min}$ is a subcategory of $Surfaces^{\mathrm{sm}}$. More precisely it is the full subcategory whose objects are families of smooth surfaces $(S \to T)$ whose geometric fibres have big and nef canonical line bundle. It follows that we have a fully faithful embedding

$$\mathcal{M}^{\min} \subset Surfaces^{\mathrm{sm}}, \tag{12}$$

and in particular by Equation (10) a fully faithful embedding

$$\mathcal{M}^{\min} \subset Surfaces^{\mathrm{sm}} \subset \mathcal{S}paces'. \tag{13}$$

Remark 3.78. In the definition of a family of minimal surfaces of general type we allow the total space to be an algebraic space, as for $\mathcal{M}_1$ (Definition 1.28). The reason is that, unlike for $\mathcal{M}^{\mathrm{can}}$, we do not have a natural choice of an ample line bundle. Indeed, on each geometric fibre we require the canonical line bundle only to be big and nef. For this reason, descent data of families of minimal surfaces of general type may fail to glue to a family in which the total space is a scheme. Thus we consider families in which the total space is an algebraic space.

Lemma 3.79. *The forgetful functor*

$$\begin{array}{rccc} F: & \mathcal{M}^{\min} & \to & \mathrm{Sch} \\ & (X \to T) & \mapsto & T \end{array}$$

makes $\mathcal{M}^{\min}$ *a category fibred in groupoids over* Sch.

Proof. Follows immediately by Proposition 3.33. □

The next proposition shows that allowing the total space to be an algebraic space in the definition of a family of minimal surfaces of general type, enables to prove that $\mathcal{M}^{\min}$ is a stack in the étale topology.

Proposition 3.80. *The category* $\mathcal{M}^{\min}$ *is a stack over* $\mathrm{Sch}_{\text{ét}}$.

Proof. The fact that morphisms glue follows from the fully faithful embedding in Equation (13) and the fact that $Surfaces^{\mathrm{sm}}$ is a stack (Proposition 3.69). To see that objects glue, let $\{T_i \to T\}_{i \in I}$ be an étale covering of T, which is in particular an fppf covering, and let $S_i \to T_i$ be a family of minimal surfaces of general type for all $i \in I$.
Since $Surfaces^{\mathrm{sm}}$ is a stack, we only have to prove that if $S \to T$ is an object of $Surfaces^{\mathrm{sm}}$ and $\{T_i \to T\}_{i \in I}$ is an fppf covering, the following are equivalent:

(1) $S \to T$ is an object of $\mathcal{M}^{\min}$;
(2) for each i, the base change $S_i \to T_i$ is an object of $\mathcal{M}^{\min}$.

The fact that (1) implies (2) is clear, since fibres of each pullback are in particular fibres of the family $S \to T$. The proof of the fact that (2) implies (1) is the same as the proof of Proposition 3.69. Observe that we are able to reply the proof of Proposition 3.69 thanks to Proposition 3.75. □

We want now to prove the algebraicity of the stack $\mathcal{M}^{\min}$. In order to do that, we will verify the hypotheses of Artin's axioms (Theorem A.119). The reader can find definitions in §A.6.

Lemma 3.81. *The diagonal*

$$\Delta : \mathcal{M}^{\min} \to \mathcal{M}^{\min} \times_{\operatorname{Spec} \mathbb{Z}} \mathcal{M}^{\min}$$

is representable by algebraic spaces.

Proof. By Proposition 3.70 the diagonal of $Surfaces^{\mathrm{sm}}$ is representable by algebraic spaces and $\mathcal{M}^{\min}$ is a full subcategory of $Surfaces^{\mathrm{sm}}$. Thus the same proof of Proposition 3.70 works here. □

LEMMA 3.82. *Let $W \to W'$ be a thickening in the category of schemes (i.e. $W \to W'$ is a closed immersion of schemes which is a homeomorphism) and let $W \to T$ be an affine morphism of schemes. Consider the pushout in the category of schemes as in [Stacks, Lemma 07RT]*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & W' \arrow[d] \\ T \arrow[r] & T'. \end{tikzcd}$$

Then the functor p of groupoids completing the diagram

$$\begin{tikzcd} \mathcal{M}^{\min}(T') \arrow[dr, dashed, "p"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathcal{M}^{\min}(T) \times_{\mathcal{M}^{\min}(W)} \mathcal{M}^{\min}(W') \arrow[r] \arrow[d] & \mathcal{M}^{\min}(W') \arrow[d] \\ & \mathcal{M}^{\min}(T) \arrow[r] & \mathcal{M}^{\min}(W) \end{tikzcd}$$

is an equivalence of groupoids.

PROOF. The same fact holds for $\mathcal{S}paces'$ by [Stacks, Lemma 0D1J]. Thus we have an equivalence of categories

$$\tilde{p} : \mathcal{S}paces'(T') \to \mathcal{S}paces'(T) \times_{\mathcal{S}paces'(W)} \mathcal{S}paces'(W').$$

Moreover $\mathcal{M}^{\min}$ is a full subcategory of $Surfaces^{\mathrm{sm}}$ which is a full subcategory of $\mathcal{S}paces'$. It follows that we only have to prove the following. Given $S' \to T'$ an object of $\mathcal{S}paces'$, then $S' \to T'$ is in $\mathcal{M}^{\min}$ if and only if $T \times_{T'} S' \to T$ and $W' \times_{T'} S' \to W'$ are in $\mathcal{M}^{\min}$.
Suppose first that $(f : S' \to T')$ is an object of $\mathcal{M}^{\min}(T')$. Then the pullbacks of f over T and W' are objects of $\mathcal{M}^{\min}(T)$ and $\mathcal{M}^{\min}(W')$ respectively, because the geometric fibres of the pullbacks are in particular geometric fibres of f and being smooth is preserved under base change by Proposition 3.6.
Suppose now that $T \times_{T'} S' \to T$ and $W' \times_{T'} S' \to W'$ are in $\mathcal{M}^{\min}(T)$ and $\mathcal{M}^{\min}(W')$ respectively. Observe that $T \to T'$ is also a thickening by [Stacks, Lemma 07RT]. Then the statement follows from the fact that the fibres of $S' \to T'$ are the same as the fibres of $T \times_{T'} S' \to T$ (see also [Stacks, Lemma 0D56]). In particular, $S' \to T'$ is smooth by Lemma 3.67 and we are done. $\square$

COROLLARY 3.83. *The stack $\mathcal{M}^{\min}$ satisfies the Rim-Schlessinger condition.*

PROOF. The (RS) condition (Definition A.105) is just a particular case of Lemma 3.82. $\square$

LEMMA 3.84. *The category $\mathcal{M}^{\min}$ satisfy openness of versality over* $\operatorname{Spec}\mathbb{Z}$. *Similarly, after base change openness of versality holds over any noetherian base scheme T.*

PROOF. The same fact holds for $\mathcal{S}paces'$ by [Stacks, Lemma 0D3X]. Then the statement follows from the fully faithful embedding $\mathcal{M}^{\min} \subset \mathcal{S}paces'$ of Equation (13), because if U is a scheme of finite type over $\mathbb{Z}$ and $x \in \mathcal{M}^{\min}(U)$ is an object, then in particular $x \in \mathcal{S}paces'(U)$ and we use that $\mathcal{S}paces'$ satisfy openness of versality over $\operatorname{Spec}\mathbb{Z}$. See also [Stacks, Lemma 0D59]. $\square$

LEMMA 3.85. *Let k be a field of finite type over $\mathbb{Z}$ and let $x_0 = (S \to \operatorname{Spec} k)$ be an object of $\mathcal{M}^{\min}(\operatorname{Spec} k)$. Then the k-vector spaces*

$$T\mathcal{F}_{\mathcal{M}^{\min},k,x_0} \quad \text{and} \quad \operatorname{Inf}(\mathcal{F}_{\mathcal{M}^{\min},k,x_0})$$

are finite dimensional.

Proof. The fact holds for $\mathcal{S}paces'$ by [Stacks, Lemma 0D1K]. Then the statement follows from the fully faithful embedding $\mathcal{M}^{\min} \subset \mathcal{S}paces'$ of Equation (13). Indeed the spaces $T\mathcal{F}_{\mathcal{M}^{\min},k,x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{M}^{\min},k,x_0})$ only depends on the morphisms of the objects, see Definitions A.106 and A.107. See also [Stacks, Lemma 0D57]. $\square$

Lemma 3.86. *Every formal object of $\mathcal{M}^{\min}$ is effective.*

Proof. Let R be a noetherian complete local $\mathbb{Z}$-algebra such that $k = R/\mathfrak{m}_R$ is a field of finite type over $\mathbb{Z}$. For all $n \geq 1$ let $\xi_n = (S_n \to \mathrm{Spec}(R/\mathfrak{m}_R^n)) \in \mathcal{M}^{\min}(\mathrm{Spec}(R/\mathfrak{m}_R^n))$ be an object and $f_n : \xi_n \to \xi_{n+1}$ be morphisms in $\mathcal{M}^{\min}$ over $\mathrm{Spec}(R/\mathfrak{m}_R^n) \to \mathrm{Spec}(R/\mathfrak{m}_R^{n+1})$. In other words, we are given cartesian diagrams

$$f_n : \quad \begin{tikzcd} S_n \arrow[r] \arrow[d] & S_{n+1} \arrow[d] \\ \mathrm{Spec}(R/\mathfrak{m}_R^n) \arrow[r] & \mathrm{Spec}(R/\mathfrak{m}_R^{n+1}). \end{tikzcd}$$

Denote by $\xi = (R, \xi_n, f_n)$ the given formal object and let $T_n = \mathrm{Spec}(R/\mathfrak{m}_R^n)$. We have to show that ξ is effective.
Consider first the base change of $S_1 \to \mathrm{Spec}\, k$ to an algebraic closure

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S_1 \arrow[d] \\ \mathrm{Spec}\, \overline{k} \arrow[r] & \mathrm{Spec}\, k. \end{tikzcd}$$

It follows that $S' \to \mathrm{Spec}\, \overline{k}$ is a minimal surface of general type, because it is a geometric fibre of ξ_1, which is a family of minimal surfaces of general type. By Theorem 2.54 and Proposition 2.61 we have that the linear system $|\omega_{S'}^{\otimes 5}|$ defines a morphism $S' \to \mathbb{P}^N_{\overline{k}}$ (where $N = h^0(S', \omega_{S'}^{\otimes 5}) - 1 = \chi(\mathcal{O}_{S'}) + 10K_{S'}^2 - 1$), whose image is the canonical model X' of S', i.e. X' is a normal surface birational to S', having at most Du Val singularities. The morphism $S' \to X'$ is given by sections $s_0, \ldots, s_N \in \mathrm{H}^0(S', \omega_{S'}^{\otimes 5})$.
We can extend this morphism to a morphism $S_1 \to X_1$ where X_1 is projective over k, i.e. there exists a morphism filling in the diagram

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S_1 \arrow[d, dashed] \\ X' \arrow[r, dashed] \arrow[d] & X_1 \arrow[d, dashed] \\ \overline{k} \arrow[r] & k \end{tikzcd}$$

with X_1 projective over k. Indeed, by Corollary B.3 we have

$$\mathrm{H}^0(S_1, \omega_{S_1}^{\otimes 5}) \otimes_k \overline{k} \simeq \mathrm{H}^0(S', \omega_{S'}^{\otimes 5}).$$

It follows that sections s_i lift to sections $s_i' \in \mathrm{H}^0(S_1, \omega_{S_1}^{\otimes 5})$. By Nakayama's lemma, the sections s_i' are also base point free. This implies that there exists a morphism $h_1 : S_1 \to \mathbb{P}^N_k$ which restricts to $S' \to X'$. Then we restrict h_1 to its image X_1.
We want now to show that the map $S_1 \to X_1$ also extends to infinitesimal deformations. In other words, we have to show that for all $n \geq 2$ the diagram

$$\begin{tikzcd} S_1 \arrow[r] \arrow[d] & S_n \arrow[d, dashed] \\ X_1 \arrow[r, dashed] \arrow[d] & X_n \arrow[d, dashed] \\ T_1 \arrow[r] & T_n \end{tikzcd}$$

can be filled in by a projective morphism $X_n \to T_n$. Arguing by induction, suppose that we have constructed X_n, so that we have

$$S_n \to X_n \hookrightarrow \mathbb{P}^N_{T_n} \to T_n,$$

where the morphism $S_n \to X_n$ is given by sections $s_{0,n}, \ldots, s_{N,n} \in \mathrm{H}^0(S_n, \omega^{\otimes 5}_{S_n/T_n})$ which are lifting of $s_0, \ldots, s_N$ to S_n. Observe that S_i are infinitesimal thickenings of S_1, hence they are schemes with the same underlying topological space. The kernel of the morphism of rings

$$R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$$

is $V_n = \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1}$, which is a finite dimensional k-vector space, being annihilated by $\mathfrak{m}_R$. Thus we have a short exact sequence

$$0 \to V_n \to R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n \to 0$$

of $R/\mathfrak{m}_R^{n+1}$-modules. Tensoring with the flat $R/\mathfrak{m}_R^{n+1}$-module $\mathcal{O}_{S_{n+1}}$ we get

$$0 \to V_n \otimes_k \mathcal{O}_{S_1} \to \mathcal{O}_{S_{n+1}} \to \mathcal{O}_{S_n} \to 0.$$

Finally, we tensor again by the flat $\mathcal{O}_{S_{n+1}}$-module $\omega^{\otimes 5}_{S_{n+1}/T_{n+1}}$ and we get an exact sequence

$$0 \to (\omega^{\otimes 5}_{S_1/T_1})^{\oplus \dim_k V_n} \to \omega^{\otimes 5}_{S_{n+1}/T_{n+1}} \to \omega^{\otimes 5}_{S_n/T_n} \to 0.$$

Passing in cohomology, we get

$$\ldots \to \mathrm{H}^0(S_{n+1}, \omega^{\otimes 5}_{S_{n+1}/T_{n+1}}) \xrightarrow{\varphi} \mathrm{H}^0(S_n, \omega^{\otimes 5}_{S_n/T_n}) \to (\mathrm{H}^1(S_1, \omega^{\otimes 5}_{S_1/T_1}))^{\oplus \dim_k V_n}.$$

It follows that φ is surjective, because $\mathrm{H}^1(S_1, \omega^{\otimes 5}_{S_1/T_1}) = 0$ by Theorem 2.60. Thus we lift $s_{0,n}, \ldots, s_{N,n}$ to sections $s_{0,n+1}, \ldots, s_{N,n+1} \in \mathrm{H}^0(S_n, \omega^{\otimes 5}_{S_n/T_n})$.
Moreover $S_n \hookrightarrow S_{n+1}$ is a closed immersion which is a homeomorphism. It follows that the residue fields at the points of S_n and S_{n+1} are the same. By Nakayama's lemma, it follows that the sections $s_{i,n+1}$ are also base point free. This implies that there exists a morphism $h_{n+1} : S_{n+1} \to \mathbb{P}^N_{T_{n+1}}$ which restricts to $S_1 \to X_1$. We restrict h_{n+1} to its image X_{n+1}. The morphism $X_{n+1} \to T_{n+1}$ is projective, because it is projective when restricted to X_n and we argue as in the proof of Proposition 1.36 using Corollary 1.35.
Thus, for all $n \geq 2$ we have a projective scheme X_n filling in the commutative diagram

$$\begin{tikzcd} S_1 \arrow[r] \arrow[d] & S_n \arrow[d] \\ X_1 \arrow[r] \arrow[d] & X_n \arrow[d] \\ T_1 \arrow[r] & T_n. \end{tikzcd}$$

For all $n \geq 1$, let $i_n : X_n \hookrightarrow \mathbb{P}^N_{T_n}$ be the closed immersion in the projective space, and let $\mathcal{L}_n = i_n^* \mathcal{O}_{\mathbb{P}^N_{T_n}}(1)$. We have a formal object (R, X_n, g_n) given by

$$\begin{tikzcd}
S_1 \arrow[r] \arrow[d] & S_2 \arrow[r] \arrow[d] & \dots \arrow[r] & S_n \arrow[r, "f_n"] \arrow[d] & S_{n+1} \arrow[d] \\
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \dots \arrow[r] & X_n \arrow[r, "g_n"] \arrow[d] & X_{n+1} \arrow[d] \\
\mathrm{Spec}(R/\mathfrak{m}_R) \arrow[r] & \mathrm{Spec}(R/\mathfrak{m}_R^2) \arrow[r] & \dots \arrow[r] & \mathrm{Spec}(R/\mathfrak{m}_R^n) \arrow[r] & \mathrm{Spec}(R/\mathfrak{m}_R^{n+1}).
\end{tikzcd}$$

By Grothendieck existence theorem (see [Alp24, Theorem C.5.8] for a reference) the formal object (R, X_n, g_n) is effective. It follows that there exists a projective morphism $X \to \mathrm{Spec}(R)$, an ample line bundle $\mathcal{L}$ on X and compatible isomorphisms $X_n \simeq X \times_{\mathrm{Spec}\, R} \mathrm{Spec}(R/\mathfrak{m}_R^n)$ and $\mathcal{L}_n \simeq \mathcal{L}_{|X_n}$. Moreover, the formal object ξ is a formal object for the *resolution functor* (see [Art74a]), i.e. for all $n \geq 1$, S_n is a resolution of X_n. Since every formal object of this form is effective by [Art74a, Lemma 2.2], it follows that there exists a scheme S and a morphism $S \to X \to \mathrm{Spec}(R)$ completing the diagram

$$\begin{tikzcd}
S_1 \arrow[r] \arrow[d] & S_2 \arrow[r] \arrow[d] & S_3 \arrow[r] \arrow[d] & \dots \arrow[r] & S \arrow[d] \\
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & \dots \arrow[r] & X \arrow[d] \\
\mathrm{Spec}(R/\mathfrak{m}_R) \arrow[r] & \mathrm{Spec}(R/\mathfrak{m}_R^2) \arrow[r] & \mathrm{Spec}(R/\mathfrak{m}_R^3) \arrow[r] & \dots \arrow[r] & \mathrm{Spec}\, R.
\end{tikzcd}$$

This shows that the formal object ξ is effective. $\square$

LEMMA 3.87. *The stack $\mathcal{M}^{\min}$ is limit preserving.*

PROOF. Let T be an affine scheme which is the limit $T = \lim T_i$ of a directed inverse system of affine schemes over a directed set I. We have to show that colim $\mathcal{M}^{\min}(T_i) \to \mathcal{M}^{\min}(T)$ is an equivalence of categories. The properties of being proper, flat and finitely presented descend to limits; in other words, the stack $\mathcal{S}paces'$ is limit preserving, see [Stacks, Lemma 0D1I]. Also the property of smoothness descends to limits by [Stacks, Lemma 0CN2]. Thus it is sufficient to prove the following. Let $0 \in I$ and let $f_0 : S_0 \to T_0 \in \mathcal{S}paces'(T_0)$ such that f_0 is smooth. Suppose that $f : S = S_0 \times_{T_0} T \to T \in \mathcal{M}^{\min}(T)$. We have to show that there exists an index $i \in I$, $i \geq 0$, such that $f_i : S_i = T_i \times_{T_0} S_0 \to T_i$ is an object of $\mathcal{M}^{\min}(T_i)$. Observe first that f_i is smooth for all $i \geq 0$ by Proposition 3.6, being the base change of f_0 which is smooth.
It remains to verify that the geometric fibres of some f_i, $i \geq 0$, are minimal surfaces of general type. For every $i \geq 0$ consider $E_i \subseteq T_i$ made out of points $t \in t_i$ such that the fibre $(S_i)_t$ satisfies the following properties:

(1) $(S_i)_t$ is integral;
(2) $(S_i)_t$ has dimension 2;
(3) $(S_i)_t$ has big canonical bundle;
(4) $(S_i)_t$ has nef canonical bundle.

These conditions do not depend on the base field; i.e. they hold over $\kappa(t)$ if and only if they hold over some (every) field extension K of $\kappa(t)$ by Lemma 3.3, Proposition 3.4, and Proposition 3.5.
Property (1) gives an open condition by Proposition 3.11. By Proposition 3.12 the dimension of the fibres is locally constant, hence condition (2) is open. Finally, properties (3) and (4) both give an open condition by Proposition 3.15 and Proposition 3.29 up to restricting to an ind-constructible set where the Kodaira dimension is non-negative, which is possible by Corollary 3.28.

Since the limit of ind-constructible subsets E_i, $i \geq 0$, coincides with the limit of T_i's according to [Gro67, Corollaire IV.8.3.5], there exists an index $i \geq 0$ such that $E_i = T_i$ and we are done. $\square$

THEOREM 3.88. *The stack* $\mathcal{M}^{\min}$ *is an algebraic stack over* $\mathrm{Sch}_{\text{ét}}$.

PROOF. We prove that conditions of Theorem A.119 are satisfied. Conditions (b),(d),(e),(f) and (g) are proved in Lemmas 3.87, 3.85, 3.81, 3.86 and 3.84. Condition (a) is given by Proposition 3.80 and finally condition (c) is proved in Corollary 3.83. $\square$

REMARK 3.89. The stack $\mathcal{M}^{\min}$ is not a Deligne-Mumford stack. Indeed condition (iii) of Theorem A.101 is not satisfied as for $\mathcal{M}^{\mathrm{can}}$ by Remark 2.73.

As for $\mathcal{M}^{\mathrm{can}}$, if we work in characteristic zero we obtain a Deligne-Mumford stack. Define

$$\mathcal{M}^{\min,\mathbb{Q}} = \mathcal{M}^{\min} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Q}.$$

PROPOSITION 3.90. *The stack* $\mathcal{M}^{\min,\mathbb{Q}}$ *is a Deligne-Mumford stack.*

PROOF. Since $\mathcal{M}^{\min}$ is an algebraic stack by Theorem 3.88, then also $\mathcal{M}^{\min,\mathbb{Q}}$ is an algebraic stack. By Theorem A.101 we have to show that for every minimal surface of general type S over an algebraically closed field k of characteristic 0, the group scheme Aut_S is discrete and reduced. This is what we proved in Proposition 2.72. $\square$

3.8. Algebraicity of the stack of canonical models with Artin's axioms

Throughout this section we give an alternative proof of the algebraicity of the stack $\mathcal{M}^{\mathrm{can}}$. Recall that in Theorem 3.50 we have already proved the algebraicity of $\mathcal{M}^{\mathrm{can}}$, using the theory of Hilbert schemes. We will now verify that Artin's axioms (Theorem A.119) are satisfied.

REMARK 3.91. The key point here is that $\mathcal{M}^{\mathrm{can}}$ is a subcategory of $\mathcal{S}paces'$. More precisely, it is the full subcategory whose objects are families of canonical surfaces (Definition 3.36). Hence we have a fully faithful embedding

$$\mathcal{M}^{\mathrm{can}} \subset \mathcal{S}paces'. \tag{14}$$

Indeed in both cases, arrows are given by cartesian diagrams. Observe that we do not fix the Euler characteristic χ and the self-intersection of the canonical bundle K^2, because we consider the stack $\mathcal{M}^{\mathrm{can}}$ as in Definition 3.42. Observe further that the total space of a family of canonical surface is a scheme.

REMARK 3.92. We already know that $\mathcal{M}^{\mathrm{can}}$ is a stack over the étale topology, see Proposition 3.41 and Definition 3.42.
The diagonal

$$\Delta : \mathcal{M}^{\mathrm{can}} \to \mathcal{M}^{\mathrm{can}} \times_{\operatorname{Spec}\mathbb{Z}} \mathcal{M}^{\mathrm{can}}$$

is representable by algebraic spaces because the diagonal of $\mathcal{S}paces'$ is representable by algebraic spaces (Lemma 3.57) and we argue as in Proposition 3.70.
The fact that the category $\mathcal{M}^{\mathrm{can}}$ satisfy openness of versality over $\operatorname{Spec}\mathbb{Z}$ is clear as in Lemma 3.84 by the fully faithful embedding of Equation (14).
Finally, if k is a field of finite type over $\operatorname{Spec}\mathbb{Z}$ and $x_0 = (S \to \operatorname{Spec} k)$ is an object of $\mathcal{M}^{\mathrm{can}}(\operatorname{Spec} k)$, the k-vector spaces $T\mathcal{F}_{\mathcal{M}^{\mathrm{can}},k,x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{M}^{\mathrm{can}},k,x_0})$ are finite dimensional, as in Lemma 3.85, again by the fully faithful embedding of Equation (14).

Thus we only have to prove that $\mathcal{M}^{\mathrm{can}}$ satisfy the Rim-Schlessinger condition, effectiveness of formal objects and the property of being limit preserving.

LEMMA 3.93. *Let $W \to W'$ be a thickening in the category of schemes (i.e. $W \to W'$ is a closed immersion of schemes which is a homeomorphism) and let $W \to T$ be an affine morphism of schemes. Consider the pushout in the category of schemes as in [Stacks, Lemma 07RT]*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & W' \arrow[d] \\ T \arrow[r] & T'. \end{tikzcd}$$

Then the functor p of groupoids completing the diagram

$$\begin{tikzcd} \mathcal{M}^{\mathrm{can}}(T') \arrow[dr, dashed, "p"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathcal{M}^{\mathrm{can}}(T) \times_{\mathcal{M}^{\mathrm{can}}(W)} \mathcal{M}^{\mathrm{can}}(W') \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{can}}(W') \arrow[d] \\ & \mathcal{M}^{\mathrm{can}}(T) \arrow[r] & \mathcal{M}^{\mathrm{can}}(W) \end{tikzcd}$$

is an equivalence of groupoids.

PROOF. The same fact holds for $\mathcal{S}paces'$ by [Stacks, Lemma 0D1J]. Thus we have an equivalence of categories

$$\tilde{p} : \mathcal{S}paces'(T') \to \mathcal{S}paces'(T) \times_{\mathcal{S}paces'(W)} \mathcal{S}paces'(W').$$

Moreover $\mathcal{M}^{\mathrm{can}}$ is a full subcategory of $\mathcal{S}paces'$. It follows that we only have to prove the following. Given $X' \to T'$ an object of $\mathcal{S}paces'$, then $X' \to T'$ is in $\mathcal{M}^{\mathrm{can}}$ if and only if $T \times_{T'} X' \to T$ and $W' \times_{T'} X' \to W'$ are in $\mathcal{M}^{\mathrm{can}}$.
Suppose first that $(f : X' \to T')$ is an object of $\mathcal{M}^{\mathrm{can}}(T')$. Then the pullbacks of f over T and W' are objects of $\mathcal{M}^{\mathrm{can}}(T)$ and $\mathcal{M}^{\mathrm{can}}(W')$ respectively, because the fibres of the pullbacks are in particular fibres of f.
Suppose now that $T \times_{T'} X' \to T$ and $W' \times_{T'} X' \to W'$ are in $\mathcal{M}^{\mathrm{can}}(T)$ and $\mathcal{M}^{\mathrm{can}}(W')$ respectively. Observe that $T \to T'$ is also a thickening by [Stacks, Lemma 07RT]. Thus, $T \times_{T'} X' \to X'$ is also a thickening by [Stacks, Lemma 09ZX]. In particular, X' is a scheme by [Stacks, Lemma 05ZR], because $T \times_{T'} X'$ is a scheme. Then the statement follows from the fact that the fibres of $X' \to T'$ are the same as the fibres of $T \times_{T'} X' \to T$ (see also [Stacks, Lemma 0D56]). ☐

COROLLARY 3.94. *The stack $\mathcal{M}^{\mathrm{can}}$ satisfies the Rim-Schlessinger condition.*

PROOF. The (RS) condition (Definition A.105) is just a particular case of Lemma 3.93. ☐

LEMMA 3.95. *Every formal object of $\mathcal{M}^{\mathrm{can}}$ is effective.*

PROOF. Let R be a noetherian complete local $\mathbb{Z}$-algebra such that $k = R/\mathfrak{m}_R$ is of finite type over $\mathbb{Z}$. For all $n \geq 1$ let $\xi_n = (X_n \to \mathrm{Spec}(R/\mathfrak{m}_R^n)) \in \mathcal{M}^{\mathrm{can}}(\mathrm{Spec}(R/\mathfrak{m}_R^n))$ be an object and $f_n : \xi_n \to \xi_{n+1}$ be morphisms in $\mathcal{M}^{\mathrm{can}}$ over $\mathrm{Spec}(R/\mathfrak{m}_R^n) \to \mathrm{Spec}(R/\mathfrak{m}_R^{n+1})$. In other words, we are given cartesian diagrams

$$f_n : \quad \begin{tikzcd} X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[d] \\ \mathrm{Spec}(R/\mathfrak{m}_R^n) \arrow[r] & \mathrm{Spec}(R/\mathfrak{m}_R^{n+1}). \end{tikzcd}$$

Denote by $\xi = (R, \xi_n, f_n)$ the formal object and let $T_n = \mathrm{Spec}(R/\mathfrak{m}_R^n)$. We have to show that ξ is effective.

We define, for all $n \geq 1$ the dualizing sheaf $\mathcal{L}_n = \omega_{X_n/T_n}$ which is a line bundle because each fibre of $X_n \to T_n$ is a canonical surface (Proposition B.13). If we consider an algebraic closure $\overline{k}/k$ of k, then the pullback of X_1 to $\overline{k}$ given by

$$\begin{tikzcd} \tilde{X} \arrow[r, "\varphi"] \arrow[d] & X_1 \arrow[d] \\ \operatorname{Spec} \overline{k} \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

is a canonical surface over an algebraically closed field $\overline{k}$, because ξ_1 is an object of $\mathcal{M}^{\mathrm{can}}$. It follows that $\omega_{\tilde{X}/\overline{k}} = \varphi^* \omega_{X_1/k} = \varphi^* \mathcal{L}_1$ is ample by Theorem 2.60(6). Thus, also $\mathcal{L}_1$ is ample by [Gro67, Corollaire 2.7.2], because $X_1 \to \operatorname{Spec} k$ is in particular quasi-compact, being proper. By Grothendieck existence theorem [Alp24, Theorem C.5.8], it follows that there exists a projective morphism $p : X \to \operatorname{Spec} R$ and an ample line bundle $\mathcal{L}$ on X and compatible isomorphisms $X_n \simeq X \times_{\operatorname{Spec} R} T_n$ and $\mathcal{L}_n \simeq \mathcal{L}_{|X_n}$. Since p is projective over a noetherian scheme, then p is in particular proper and of finite presentation. Moreover, p is flat by [Stacks, Lemma 0D4G], because every morphism $X_n \to T_n$ is flat. To conclude, it is clear that $X \to \operatorname{Spec} R$ is a family of canonical surfaces, because the fibre over the unique point of R is exactly $X_1 \to \operatorname{Spec}(R/\mathfrak{m}_R)$. □

Lemma 3.96. *The stack $\mathcal{M}^{\mathrm{can}}$ is limit preserving.*

Proof. Let T be an affine scheme which is the limit $T = \lim T_i$ of a directed inverse system of affine schemes over a directed set I. We have to show that colim $\mathcal{M}^{\mathrm{can}}(T_i) \to \mathcal{M}^{\mathrm{can}}(T)$ is an equivalence of categories. The properties of being proper, flat and finitely presented descend to limits; in other words, the stack $\mathcal{S}paces'$ is limit preserving, see [Stacks, Lemma 0D1I]. Moreover, it is not restrictive to assume that total spaces are schemes by [Stacks, Lemma 07SR]. Otherwise, one can also work directly in the category of schemes and show that being proper, flat and finitely presented are properties that descend through limits by [Stacks, Lemma 01ZM] and [Stacks, Lemma 04AI]. Thus it is sufficient to prove the following. Let $0 \in I$ and let $f_0 : X_0 \to T_0 \in \mathcal{S}paces'(T_0)$, where the total space X_0 can be assumed to be a scheme. Suppose that $f : X = X_0 \times_{T_0} T \to T \in \mathcal{M}^{\mathrm{can}}(T)$. We have to show that there exists an index $i \in I$, $i \geq 0$, such that $f_i : X_i = T_i \times_{T_0} X_0 \to T_i$ is an object of $\mathcal{M}^{\mathrm{can}}(T_i)$.
In other words, it remains to verify that the geometric fibres of some f_i, $i \geq 0$ are minimal surfaces of general type. For every $i \geq 0$ consider $E_i \subseteq T_i$ made out of points $t \in t_i$ such that the fibre $(X_i)_t$ satisfies the following properties:

(1) $(X_i)_t$ is geometrically integral;
(2) $(X_i)_t$ is geometrically normal;
(3) $(X_i)_t$ has dimension 2;
(4) $(X_i)_t$ has ample canonical bundle;
(5) $(X_i)_t$ is a Gorenstein scheme;
(6) $(X_i)_{\overline{t}}$ has at most Du Val singularities.

These conditions do not depend on the base field; i.e. they hold over $\kappa(t)$ if and only if they hold over some (every) field extension K of $\kappa(t)$ by Lemma 3.3, Proposition 3.4, Proposition 3.5 and Lemma 2.52.
Properties (1),(2) (3) give an open condition by Proposition 3.11, Proposition 3.12. Property (4) is ind-constructible by Proposition 3.14. Property (5) gives an open condition by Lemma 3.19 and finally property (6) is ind-constructible by Lemma 3.23 (it is also open by Corollary 3.26).
Since the limit of ind-constructible subset E_i, $i \geq 0$, coincides with the limit of T_i's according

to [Gro67, Corollaire IV.8.3.5], there exists an index $i \geq 0$ such that $E_i = T_i$ and we are done. $\square$

THEOREM 3.97. *The stack $\mathcal{M}^{\mathrm{can}}$ is an algebraic stack.*

PROOF. In Remark 3.92, Corollary 3.94 and Lemmas 3.95, 3.96 we proved that conditions of Theorem A.119 are satisfied. Thus $\mathcal{M}^{\mathrm{can}}$ is algebraic. $\square$

DEFINITION 3.98. Let $\mathcal{M}^{\mathrm{Smcan}}$ be the full subcategory of both $\mathcal{M}^{\mathrm{min}}$ and $\mathcal{M}^{\mathrm{can}}$ whose objects are proper, smooth and finitely presented morphisms of schemes $S \to T$ such that for all $t \in T$, the geometric fibre $S_{\bar{t}}$ over $\kappa(t)$, is an integral scheme of dimension 2 with ample dualizing sheaf $\omega_{S_{\bar{t}}}$.

REMARK 3.99. Combining Theorem 3.88 and Theorem 3.97 we have that $\mathcal{M}^{\mathrm{Smcan}}$ is an algebraic stack over $\mathrm{Sch}_{\text{ét}}$. Thus, both $\mathcal{M}^{\mathrm{min}}$ and $\mathcal{M}^{\mathrm{can}}$ contains $\mathcal{M}^{\mathrm{Smcan}}$ as an algebraic open substack. Observe that $\mathcal{M}^{\mathrm{Smcan}}$ is not a Deligne-Mumford stack by Remark 2.73.
However, we have that

$$\mathcal{M}^{\mathrm{Smcan},\mathbb{Q}} = \mathcal{M}^{\mathrm{Smcan}} \otimes_{\mathbb{Z}} \operatorname{Spec} \mathbb{Q}$$

is a Deligne-Mumford stack, because both $\mathcal{M}^{\mathrm{can},\mathbb{Q}}$ and $\mathcal{M}^{\mathrm{can},\mathbb{Q}}$ are Deligne-Mumford stacks by Proposition 3.54 and Proposition 3.90.

3.9. The stack of K3 surfaces

DEFINITION 3.100. Let k be an algebraically closed field. A K3 *surface over* k is a smooth surface S over k such that the canonical line bundle $\omega_S = \omega_{S/k}$ is trivial and the irregularity $q(S) = \dim_k \mathrm{H}^1(S, \mathcal{O}_S)$ is zero.

Recall that every smooth surface is projective, see Remark 2.2.

DEFINITION 3.101. Let T be a scheme. A *family of* K3 *surfaces over* T is a proper, smooth and finitely presented morphism of algebraic spaces $S \to T$ such that for all geometric points $\sigma : \operatorname{Spec} k \to T$, the geometric fibre S_k is an integral scheme, of dimension 2, with trivial canonical bundle $\omega_{S_k} \simeq \mathcal{O}_{S_k}$ and irregularity $q(S_k)$ zero.

PROPOSITION 3.102. *Let $S \to T$ be a morphism of algebraic spaces where T is a scheme. Then the following are equivalent:*

(1) *$S \to T$ is a family of* K3 *surfaces;*
(2) *$S \to T$ is a proper, smooth and finitely presented morphism of algebraic spaces such that for all points $t : \operatorname{Spec} \kappa(t) \to T$, the fibre $S_{\bar{t}}$ over an algebraic closure $\kappa(\bar{t})/\kappa(t)$ of $\kappa(t)$ is an integral scheme of dimension 2 with trivial canonical line bundle $\omega_{S_{\bar{t}}} \simeq \mathcal{O}_{S_{\bar{t}}}$ and irregularity $q(S_{\bar{t}}) = 0$.*

PROOF. It follows immediately by Proposition 3.34 because the properties of being integral, of dimension 2 with trivial canonical line bundle and irregularity zero are stable properties under a base change which is a field extension by Lemma 3.3, Proposition 3.4, Proposition 3.5 and Corollary B.3. $\square$

DEFINITION 3.103. We define the category $\mathcal{M}^{\mathrm{K3}}$ as follows.

- Objects are families of K3 surfaces.
- An arrow $(S' \to T') \to (S \to T)$ between two objects is a pair (f, g) where $f : S' \to S$ is a morphism of algebraic spaces, $g : T' \to T$ is a morphism of schemes and they are such that the diagram

$$\begin{tikzcd} S' \arrow[r, "f"] \arrow[d] & S \arrow[d] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

is cartesian.

REMARK 3.104. The category $\mathcal{M}^{\mathrm{K3}}$ is the full subcategory of $Surfaces^{\mathrm{sm}}$ whose objects are families of smooth surfaces $(S \to T)$ whose geometric fibres have trivial canonical bundle and irregularity zero.

LEMMA 3.105. *The forgetful functor*

$$\begin{array}{rccc} F: & \mathcal{M}^{\mathrm{K3}} & \to & \mathrm{Sch} \\ & (S \to T) & \mapsto & T \end{array}$$

makes $\mathcal{M}^{\mathrm{K3}}$ a category fibred in groupoids over Sch*.*

PROOF. This is clear by Proposition 3.33. $\square$

PROPOSITION 3.106. *The category $\mathcal{M}^{\mathrm{K3}}$ is a stack over* $\mathrm{Sch}_{\text{ét}}$*.*

PROOF. We use the same argument of the proof of Proposition 3.80. Indeed, we can reply the same proof thanks to Proposition 3.102. $\square$

REMARK 3.107. The stack $\mathcal{M}^{\mathrm{K3}}$ is not algebraic, as we already discussed in Remark 3.61.

3.10. The stack of del Pezzo surfaces

DEFINITION 3.108. Let k be an algebraically field. A *del Pezzo surface* over k is a smooth surface S over k with ample anti-canonical bundle $\omega_S^\vee = \omega_{S/k}^\vee$.

As we have seen in the previous sections, the existence of a natural ample line bundle is a very important geometric tool which allow us to obtain the stack parametrizing families of del Pezzo surfaces (Proposition 3.114) whose total space is a scheme (Definition 3.109).

DEFINITION 3.109. Let T be a scheme. A *family of del Pezzo surfaces over T* is a proper, smooth and finitely presented morphism of schemes $S \to T$ such that for all geometric points $\sigma : \operatorname{Spec} k \to T$, the fibre S_k is an integral scheme, of dimension 2 with ample anti-canonical line bundle $\omega_{S_k}^\vee$.

PROPOSITION 3.110. *Let $S \to T$ be a morphism of schemes. Then the following are equivalent:*

1. *$S \to T$ is a family of del Pezzo surfaces;*
2. *$S \to T$ is a proper, smooth and finitely presented morphism of schemes such that for all points $t \in T$ the geometric fibre $S_{\bar{t}}$ is an integral scheme of dimension 2 with ample anti-canonical line bundle $\omega_{S_{\bar{t}}}^\vee$;*
3. *$S \to T$ is a proper, smooth and finitely presented morphism of schemes such that for all points $t \in T$ the fibre S_t is a geometrically integral scheme of dimension 2 with ample anti-canonical line bundle $\omega_{S_t}^\vee$.*

PROOF. The equivalence between (1) and (2) is clear by Proposition 3.34 because the properties of being an integral scheme of dimension 2 with ample anti-canonical line bundle are stable under a base change which is a field extension by Lemma 3.3, Proposition 3.5 and Proposition 3.4.
The equivalence between (2) and (3) follows again by the fact that these properties are stable under a base change which is a field extension. $\square$

REMARK 3.111. Observe in particular that if $S \to T$ is a family of del Pezzo surfaces, by Proposition 3.110 it follows that for all points $t : \operatorname{Spec} \kappa(t) \to T$ the anti-canonical bundle $\mathcal{L} = \omega_{S_t}^\vee$ is ample. Then there exists a line bundle $\omega_{S/T}^\vee$ by Proposition B.13 which is p-ample (Definition B.8 by Corollary B.11).

Definition 3.112. We define the category $\mathcal{M}^{\mathrm{dP}}$ as follows.

- Objects are families of del Pezzo surfaces.
- An arrow $(S' \to T') \to (S \to T)$ between two objects is a pair (f, g) where $f : S' \to S$ and $g : T' \to T$ are morphisms of schemes such that the diagram

$$\begin{tikzcd} S' \arrow[r, "f"] \arrow[d] & S \arrow[d] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

is cartesian.

Proposition 3.113. *The forgetful functor*

$$\begin{array}{rccc} F: & \mathcal{M}^{\mathrm{dP}} & \to & \mathrm{Sch} \\ & (S \to T) & \mapsto & T \end{array}$$

makes $\mathcal{M}^{\mathrm{dP}}$ *a category fibred in groupoids over* Sch.

Proof. This follows immediately by Proposition 3.31. $\square$

Proposition 3.114. *The category* $\mathcal{M}^{\mathrm{dP}}$ *is a stack over* $\mathrm{Sch}_{\text{ét}}$.

Proof. We will prove that $\mathcal{M}^{\mathrm{dP}}$ is a stack in the fpqc topology, which implies the statement by Remark A.30. We prove that hypotheses of [Vis08, Theorem 4.38] are satisfied. First we show that families of del Pezzo surfaces form a class of morphisms $\mathcal{P}$ which is stable (Definition A.59) in the fpqc site $\mathrm{Sch}_{\mathrm{fpqc}}$. Given an object

$$(p : S \to T) \in \mathcal{M}^{\mathrm{dP}}(T)$$

and isomorphisms $S' \simeq S$, $T \simeq T'$, then the morphism $p' : S' \to T'$ given by the compositions is again a family of del Pezzo surfaces. Indeed p' is a proper, flat and finitely presented morphism of schemes, because it is the composition of p with two isomorphisms. Moreover, geometric fibres of p' are isomorphic to geometric fibres of p, so that the conditions on the fibres are satisfied.
The condition (ii) on the base change in the definition of a stable class of arrows (Definition A.59) is satisfied, because we have already proved that $\mathcal{M}^{\mathrm{dP}}$ is fibred over Sch. Hence being a family of del Pezzo surfaces is a stable condition.
In order to prove that $\mathcal{P}$ is local (Definition A.60), let $S \to T$ be any morphism of schemes. Let $\{T_i \to T\}_{\{i \in I\}}$ be an fpqc covering, and suppose that $S_i = T_i \times_T S \to T_i$ is a family of del Pezzo surfaces for all $i \in I$. We have to show that also $S \to T$ is a family of del Pezzo surfaces. The properties of being proper, smooth and finitely presented for a morphism of schemes are fpqc local properties on the base, see [Stacks, Lemma 02L1], [Stacks, Lemma 02VL] and [Stacks, Lemma 02L0]. It follows that $S \to T$ is a proper, smooth and finitely presented morphism of schemes. By Proposition 3.110, we have to show that for all points $t : \operatorname{Spec} \kappa(t) \to T$, the fibre over $\kappa(t)$ is a geometrically integral scheme of dimension 2 with ample anti-canonical line bundle $\omega_{S_t}^{\vee}$. By surjectivity of $\coprod_i T_i \to T$ there exists a point $t' \in T_i$ for some $i \in I$ which is sent to t. Thus we obtain a morphism of schemes

$$\alpha : \operatorname{Spec} \kappa(t') \to T_i \to T$$

whose image is $t \in T$. The morphism α factorizes as

$$\operatorname{Spec} \kappa(t') \to \operatorname{Spec} \kappa(t) \to T. \tag{15}$$

Observe that we have a commutative diagram

$$\begin{tikzcd}
(S_i)_{t'} \arrow[r] \arrow[d] & S_i \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}\kappa(t') \arrow[r] & T_i \arrow[r] & T
\end{tikzcd}$$

in which the squares on the right and on the left are cartesian. Thus, also the external square is cartesian, so that $S_{\kappa(t')} \simeq (S_i)_{t'}$. Since $S_i \to T_i$ is a family of del Pezzo surfaces, then $(S_i)_{t'}$ is a geometrically integral scheme of dimension 2 with ample anti-canonical line bundle $\omega^{\vee}_{(S_i)_{t'}}$ by Proposition 3.110.

By Equation (15) we have a cartesian diagram

$$\begin{tikzcd}
S_{\kappa(t')} \arrow[r] \arrow[d] & S_t \arrow[r] \arrow[d] & X \arrow[d] \\
\kappa(t') \arrow[r] & \operatorname{Spec}\kappa(t) \arrow[r] & T.
\end{tikzcd}$$

But then also S_t is a geometrically integral scheme of dimension 2 with ample anti-canonical line bundle $\omega^{\vee}_{S_t}$ by Lemma 3.3, Proposition 3.5, Proposition 3.4. Then $S \to T$ is a family of del Pezzo surfaces by Proposition 3.110. This shows that being a family of del Pezzo surfaces is a local condition in the fpqc topology.
For each object $p : S \to T$ of $\mathcal{M}^{\mathrm{dP}}(T)$ we have an invertible sheaf $\omega^{\vee}_{S/T}$ which is ample relative to the morphism p, see Remark 3.111. Moreover, formation on $\omega^{\vee}_{S/T}$ is compatible with base change, i.e. if we have a cartesian diagram of schemes

$$\begin{tikzcd}
Q \arrow[r, "f"] \arrow[d] & R \arrow[r, "g"] \arrow[d] & X \arrow[d] \\
V \arrow[r] & U \arrow[r] & T
\end{tikzcd}$$

whose columns are objects of $\mathcal{M}^{\mathrm{dP}}$, then the diagram

$$\begin{tikzcd}
f^*g^*\omega_{S/T} \arrow[r] \arrow[d] & (g\circ f)^*\omega_{S/T} \arrow[d] \\
f^*\omega_{R/U} \arrow[r] & \omega_{Q/V}
\end{tikzcd}$$

of invertible sheaves on Q commutes, see [Con00, Theorem 3.6.1]. Then $\mathcal{M}^{\mathrm{dP}}$ is a stack over $\mathrm{Sch}_{\mathrm{fpqc}}$ by [Vis08, Theorem 4.38] and then also over $\mathrm{Sch}_{\text{ét}}$. $\square$

THEOREM 3.115. *The stack $\mathcal{M}^{\mathrm{dP}}$ is an algebraic stack over* $\mathrm{Sch}_{\text{ét}}$.

PROOF. One can prove this theorem using the theory of the Hilbert scheme, as we have done for $\mathcal{M}^{\mathrm{can}}_{\chi,K^2}$ in Theorem 3.48. The key point here is the existence of a natural ample line bundle, namely the anti-canonical one.
However, it is also clear that Artin's axioms (Theorem A.119) are satisfied. Indeed the verification of Artin's axioms is the same as for $\mathcal{M}^{\mathrm{can}}$ in §3.8 and $\mathcal{M}^{\mathrm{min}}$ in §3.7. $\square$

REMARK 3.116. The stack $\mathcal{M}^{\mathrm{dP}}$ is not a Deligne-Mumford stack. Indeed $\mathbb{P}^2_{\mathbb{C}}$ is a del Pezzo surface over $\mathbb{C}$ and the automorphism group scheme of $\mathbb{P}^2_{\mathbb{C}}$ is $\mathrm{PGL}_{3,\mathbb{C}}$, and thus it is not discrete.

CHAPTER 4

Canonical models of minimal surfaces of general type in families

The aim of this chapter is to construct a morphism of stacks

$$\alpha : \mathcal{M}^{\min} \to \mathcal{M}^{\mathrm{can}}$$

from the stack of minimal surfaces of general type to the stack of canonical models, see Proposition 4.23.

4.1. The relative dualizing sheaf for Deligne-Mumford stacks

First of all, we want to observe that we have a notion of a *structure sheaf* $\mathcal{O}_{\mathcal{X}}$ for a Deligne-Mumford stack $\mathcal{X}$ and the notion of *$\mathcal{O}_{\mathcal{X}}$-modules.* Recall that every algebraic space is in particular a Deligne-Mumford stack, see Remark A.88.

Definition 4.1. Let $\mathcal{X}$ be a Deligne-Mumford stack. The *small étale site of* $\mathcal{X}$ is the category $\mathcal{X}_{\text{ét}}$ whose objects are schemes étale over $\mathcal{X}$ provided with the following Grothendieck topology. A covering of $U \to \mathcal{X}$ is a collection of étale morphism of schemes $\{\varphi_i : U_i \to U\}_{i\in I}$ over $\mathcal{X}$ such that $\bigcup_i \varphi_i(U_i) = U$.

Definition 4.2. Let $\mathcal{X}$ be a Deligne-Mumford stack. The *structure sheaf* $\mathcal{O}_{\mathcal{X}}$ of $\mathcal{X}$ is the sheaf on $\mathcal{X}_{\text{ét}}$ defined by

$$\begin{array}{rccc} \mathcal{O}_{\mathcal{X}} : & (\mathcal{X}_{\text{ét}})^{\mathrm{op}} & \longrightarrow & \mathrm{Ring} \\ & (U \to \mathcal{X}) & \longmapsto & \Gamma(U, \mathcal{O}_U) \end{array}$$

for every étale $\mathcal{X}$-scheme U.

Definition 4.3. Let $\mathcal{X}$ be a Deligne-Mumford stack over a scheme T. The *relative sheaf of differentials* $\Omega^1_{\mathcal{X}/T}$ is the sheaf on $\mathcal{X}_{\text{ét}}$ defined by $\Omega^1_{\mathcal{X}/T}(U \to \mathcal{X}) = \Gamma(U, \Omega^1_{U/T})$ for every étale $\mathcal{X}$-scheme U.

We define $\mathcal{O}_{\mathcal{X}}$-modules for a Deligne-Mumford stack as module objects over $\mathcal{O}_{\mathcal{X}}$.

Definition 4.4. Let $\mathcal{X}$ be a Deligne-Mumford stack. A *sheaf of $\mathcal{O}_{\mathcal{X}}$-modules* (or simply a *$\mathcal{O}_{\mathcal{X}}$-module*) is a sheaf F on $\mathcal{X}_{\text{ét}}$ which is a module object for $\mathcal{O}_{\mathcal{X}}$ in the category of sheaves, i.e. for every étale $\mathcal{X}$-scheme U, $F(U \to \mathcal{X})$ is a module over $\mathcal{O}_{\mathcal{X}}(U \to \mathcal{X}) = \Gamma(U, \mathcal{O}_U)$ and the module structure is compatible with respect to restrictions along morphisms $V \to U$ of étale $\mathcal{X}$-schemes.

We denote by $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ the category of $\mathcal{O}_{\mathcal{X}}$-modules. Let F be an $\mathcal{O}_{\mathcal{X}}$-module on a Deligne-Mumford stack $\mathcal{X}$. Let U be an étale $\mathcal{X}$-scheme. We can restrict $F_{|U_{\text{ét}}}$ on the small étale site of U (Example A.24). Namely, if $V \to U$ is an object of $U_{\text{ét}}$, we have

$$F_{|U_{\text{ét}}}(V \to U) = F(V \to \mathcal{X}).$$

Furthermore, we can also consider the restriction of $F_{|U_{\mathrm{Zar}}}$ to the small Zariski site of U (Example A.22).

DEFINITION 4.5. Let $\mathcal{X}$ be a Deligne-Mumford stack. An $\mathcal{O}_{\mathcal{X}}$-module F is *quasi-coherent* if

(i) for every étale $\mathcal{X}$-scheme U, the restriction $F_{|U_{\mathrm{Zar}}}$ is a quasi-coherent $\mathcal{O}_{U_{\mathrm{Zar}}}$-module;
(ii) for every étale morphism $f : U \to V$ of étale $\mathcal{X}$-schemes, the natural morphism

$$f^* : (F_{|V_{\mathrm{Zar}}}) \to F_{|U_{\mathrm{Zar}}}$$

is an isomorphism.

A quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module F is a *locally free sheaf of rank* r (resp. *line bundle*) if $F_{|U_{\mathrm{Zar}}}$ is a locally free sheaf of rank r (resp. line bundle) for every étale morphism $U \to \mathcal{X}$ from a scheme.

EXAMPLE 4.6. If $\mathcal{X}$ is a Deligne-Mumford stack over a scheme T, the relative sheaf of differential $\Omega^1_{\mathcal{X}/T}$ is quasi-coherent, see [Alp24, Example 4.1.19]. If moreover $\mathcal{X} \to T$ is smooth, then $\Omega^1_{\mathcal{X}/T}$ is a locally free sheaf.

Let $(f : S \to T)$ be an object of $\mathcal{M}^{\min}(T)$. We know that S is an algebraic space, and in particular is a Deligne-Mumford stack over T. By Definition 3.72, the morphism f is proper, smooth and of finite presentation. Moreover, f has relative dimension 2 (i.e. fibres of f are smooth of dimension 2). It follows that $\Omega^1_{S/T}$ is a locally free sheaf of rank 2, see [Stacks, Lemma 0CK5] and [Stacks, Lemma 02G1].

DEFINITION 4.7. Let $(f : S \to T) \in \mathcal{M}^{\min}(T)$. The $\mathcal{O}_S$-module

$$\omega_{S/T} = \det \Omega^1_{S/T}$$

is defined by $\omega_{S/T}(U \to \mathcal{X}) = \det \Omega^1_{U/T}$ for every étale S-scheme U. We will call $\omega_{S/T}$ the *relative dualizing sheaf of* f.

REMARK 4.8. Let $(f : S \to T) \in \mathcal{M}^{\min}(T)$. Observe that the $\mathcal{O}_S$-module $\omega_{S/T}$ is a line bundle because for every étale S-scheme U, we have that $\omega_{U/T}$ is a line bundle. Moreover, formation of $\omega_{S/T}$ commutes with arbitrary base change by [Stacks, Lemma 05ZC]. In particular, for every point $t \in T$, the pullback of $\omega_{S/T}$ to S_t is isomorphic to the canonical line bundle ω_{S_t} of S_t.

4.2. Construction of the morphism

In what follows, we fix an object of $\mathcal{M}^{\min}$. More precisely, we consider a base scheme T, and we fix $p : S \to T$ to be a family of minimal surfaces of general type parametrized by T. We recall that the morphism $p : S \to T$ is by definition proper, smooth (in particular flat) and of finite presentation. The dualizing sheaf $\omega_{S/T}$ exists and it is a line bundle by §4.1.

Consider now the sheaf

$$\mathcal{A} = \bigoplus_{m \geq 0} p_* \omega_{S/T}^{\otimes 5m}$$

on T.

LEMMA 4.9. *The sheaf* $\mathcal{A}$ *is a quasi-coherent sheaf of* $\mathbb{N}$*-graded* $\mathcal{O}_T$*-algebras.*

PROOF. To show that $\mathcal{A}$ is a quasi-coherent sheaf, it is sufficient to show that for a fixed $m \geq 0$ the sheaf $p_* \omega_{S/T}^{\otimes 5m}$ is a quasi-coherent sheaf, since any direct sum of quasi-coherent sheaf is quasi-coherent. The morphism $p : S \to T$ is finitely presented, and in particular is quasi-compact and quasi-separated. Moreover $\omega_{S/T}^{\otimes 5m}$ is a line bundle, hence it is a quasi-coherent $\mathcal{O}_S$-module ([Stacks, Definition 03G9]). By [Stacks, Lemma 03M9] it follows that $p_* \omega_{S/T}^{\otimes 5m}$ is a quasi-coherent $\mathcal{O}_T$-module. We want now to show that $\mathcal{A}$ is a sheaf of $\mathcal{O}_T$-algebras. Consider first the quasi-coherent sheaf

$$\mathcal{B} = \bigoplus_{m \geq 0} \omega_{S/T}^{\otimes 5m}$$

on S. This is a sheaf of $\mathcal{O}_S$-algebras, where for all $m', m \geq 0$ the product is given by the natural map

$$\alpha : \omega_{S/T}^{\otimes 5m} \otimes_{\mathcal{O}_S} \omega_{S/T}^{\otimes 5m'} \to \omega_{S/T}^{\otimes 5(m+m')}.$$

We now apply the covariant functor

$$p_* : \mathrm{Mod}_{\mathcal{O}_S} \to \mathrm{Mod}_{\mathcal{O}_T}$$

to α and we get a morphism of $\mathcal{O}_T$-modules

$$p_*(\alpha) : p_*(\omega_{S/T}^{\otimes 5m} \otimes_{\mathcal{O}_S} \omega_{S/T}^{\otimes 5m'}) \to p_*\omega_{S/T}^{\otimes 5(m+m')}.$$

By adjunction formula it holds that

$$\mathrm{Hom}_{\mathcal{O}_T}(p_*\omega_{S/T}^{\otimes 5m}, p_*\omega_{S/T}^{\otimes 5m}) \simeq \mathrm{Hom}_{\mathcal{O}_S}(p^*p_*\omega_{S/T}^{\otimes 5m}, \omega_{S/T}^{\otimes 5m})$$

so that we get a natural morphism of sheaves

$$\beta : p^*p_*\omega_{S/T}^{\otimes 5m} \to \omega_{S/T}^{\otimes 5m}$$

on S. Tensoring β with the identity morphism on $\omega_{S/T}^{\otimes 5n}$ we get a morphism of $\mathcal{O}_S$-modules

$$p^*p_*\omega_{S/T}^{\otimes 5m} \otimes_{\mathcal{O}_S} \omega_{S/T}^{\otimes 5m'} \to \omega_{S/T}^{\otimes 5m} \otimes_{\mathcal{O}_S} \omega_{S/T}^{\otimes 5m'}$$

and applying the functor p_* to this morphism we obtain

$$\gamma : p_*(p^*p_*\omega_{S/T}^{\otimes 5m} \otimes_{\mathcal{O}_S} \omega_{S/T}^{\otimes 5m'}) \to p_*(\omega_{S/T}^{\otimes 5m} \otimes_{\mathcal{O}_S} \omega_{S/T}^{\otimes 5m'}).$$

Finally observe that by projection formula [Stacks, Lemma 01E8], we have an isomorphism

$$\delta : p_*\omega_{S/T}^{\otimes 5m} \otimes_{\mathcal{O}_S} p_*\omega_{S/T}^{\otimes 5m'} \simeq p_*(p^*p_*\omega_{S/T}^{\otimes 5m} \otimes_{\mathcal{O}_S} \omega_{S/T}^{\otimes 5m'}).$$

The composition

$$p_*\alpha \circ \gamma \circ \delta : p_*\omega_{S/T}^{\otimes 5m} \otimes_{\mathcal{O}_S} p_*\omega_{S/T}^{\otimes 5m'} \to p_*\omega_{S/T}^{\otimes 5(m+m')}$$

gives the structure of $\mathcal{O}_T$-algebra to the sheaf $\mathcal{A}$. □

Observe that there exists a natural $\mathbb{N}$-graduation on the sheaf $\mathcal{A}$, where the part of degree m, for $m \geq 0$, is $\mathcal{A}_m = p_*\omega_{S/T}^{\otimes 5m}$.

For every $U \subseteq T$ affine open subset, we have that

$$\Gamma(U, \mathcal{A}) = \bigoplus_{m \geq 0} \Gamma(U, \mathcal{A}_m)$$

is a graded $\Gamma(U, \mathcal{O}_T)$-algebra. In particular, for every $U \subseteq T$ as above, there exists a morphism of schemes

$$\pi_U : \operatorname{Proj} \Gamma(U, \mathcal{A}) \to U.$$

By [GW20, §13.7] there exists a T-scheme

$$\pi : \underline{\mathrm{Proj}}(\mathcal{A}) \to T$$

with U-isomorphisms $\eta_U : \pi^{-1}(U) \xrightarrow{\sim} \operatorname{Proj} \Gamma(U, \mathcal{A})$ for all $U \subseteq T$ affine open subset. Moreover, $\underline{\mathrm{Proj}}(\mathcal{A})$ and η_U are unique up to a unique isomorphism. We will call the scheme $\underline{\mathrm{Proj}}(\mathcal{A})$ the *projective spectrum of* $\mathcal{A}$.

REMARK 4.10. Formation of $\underline{\mathrm{Proj}}(\mathcal{A})$ is compatible with base change, i.e. if $f : T' \to T$ is a morphism of schemes, then

$$\underline{\mathrm{Proj}}(f^*\mathcal{A}) \simeq \underline{\mathrm{Proj}}(\mathcal{A}) \times_T T'.$$

See [GW20, Remark 13.27].

LEMMA 4.11. *For all $m \geq 0$, the quasi-coherent $\mathcal{O}_T$-module $\mathcal{A}_m = p_*\omega_{S/T}^{\otimes 5m}$ is a locally free sheaf of finite rank whose formation commutes with arbitrary base change [GW23, Definition 23.138].*

PROOF. We are in hypotheses of cohomology and base change (Theorem B.5), which holds also for morphisms of algebraic spaces, see [Stacks, Section 073I]. We claim that for every point $t \in T$ the $\kappa(t)$-vector space $\mathrm{H}^1(S_t, \omega_{S_t/\kappa(t)}^{\otimes 5m})$ is zero. If $\kappa(t)$ is an algebraically closed field, then $\mathrm{H}^1(S_t, \omega_{S_t/\kappa(t)}^{\otimes 5m}) = 0$ by Theorem 2.60.(3). On the other hand, if $t \in T$ is any point of T, then we consider an algebraic closure of the residue field $\kappa(\bar{t})/\kappa(t)$. If $S_{\bar{t}}$ and S_t are the fibres of p over $\kappa(\bar{t})$ and $\kappa(t)$ respectively, then $\mathrm{H}^1(S_t, \omega_{S_t}^{\otimes 5m}) = \mathrm{H}^1(S_{\bar{t}}, \omega_{S_{\bar{t}}}^{\otimes 5m}) = 0$ by Corollary B.3, and the claim follows. Finally, by Corollary B.6, it follows that $\mathcal{A}_m$ is a locally free sheaf of finite rank whose formation commutes with arbitrary base change. □

In particular it follows that $\mathcal{A}$ is a quasi-coherent flat $\mathcal{O}_T$-module, because any direct sum of flat $\mathcal{O}_T$-modules is flat, see [Stacks, Lemma 05NG].

PROPOSITION 4.12. *Let $(p : S \to T) \in \mathcal{M}^{\min}(T)$ and let $\mathcal{A} = \bigoplus_{m\geq 0} p_*\omega_{S/T}^{\otimes 5m}$. Then the map $\pi : \underline{\mathrm{Proj}}(\mathcal{A}) \to T$ is flat.*

PROOF. We know by Lemma 4.11 that $\mathcal{A}_m$ is in particular a flat $\mathcal{O}_T$-module for all $m \geq 0$. Then the statement follows by [Stacks, Lemma 0D4C]. □

LEMMA 4.13. *Let $p : S \to T$ be an object of $\mathcal{M}^{\min}(T)$. Let $m \geq 3$ be an integer. Then the function*

$$\tilde{P}_m : T \to \mathbb{Z}, \qquad t \mapsto \tilde{P}_m(t) = \dim_{\kappa(t)} \mathrm{H}^0(X_t, \omega_{S_t}^{\otimes m})$$

is locally constant on T.

PROOF. Observe that we are in hypotheses of cohomology and base change (Theorem B.5). Let $t \in T$ be a point. For a fixed integer $m \geq 3$ we denote $\mathcal{L} = \omega_{S/T}^{\otimes m}$. The natural map

$$\beta^1(\kappa(t)) : R^1 f_*(\omega_{S/T}^{\otimes m}) \otimes_{\mathcal{O}_T} \kappa(t) \to \mathrm{H}^1(S_t, \omega_{S_t}^{\otimes m})$$

of cohomology and base change is surjective as we now explain. If $\kappa(t)$ is an algebraically closed field, then $\mathrm{H}^1(S_t, \omega_{S_t/\kappa(t)}^{\otimes m}) = 0$ by Theorem 2.60; otherwise we can consider an algebraic closure $\kappa(\bar{t})/\kappa(t)$ and we argue as in Lemma 4.11.

Then by Corollary B.6 there exists an open neighbourhood V of t such that $R^1 p_*\mathcal{L}_{|V} = 0$. Since $t \in T$ was any point of T, it follows by cohomology and base change part (2) that $\tilde{P}_m(t)$ is locally constant on T. □

We will need the following algebraic lemma.

LEMMA 4.14. *Let A be a ring and let B be an $\mathbb{N}$-graded A-algebra such that B_m is a finite A-module for all $m \geq 0$. Suppose that for all algebrically closed field k and every homomorphism of rings $A \to k$, $B \otimes_A k$ is generated by $B_1 \otimes_A k$ as a k-algebra. Then B is generated by B_1 as an A-algebra.*

PROOF. Let $\{x_1 \dots, x_s\}$ with $x_i \in B$ for $i = 1, \dots s$ be a generating set for B_1 as an A-module. We claim that the 0-degree homomorphism of A-algebras

$$\phi : A[x_1, \dots, x_s] \to B$$

is surjective. In order to prove that, it is sufficient to show that for all $m \geq 0$ the homomorphism of A-algebras

$$\phi_m : A[x_1, \dots x_s]_m \to B_m$$

is surjective. Let $L = \operatorname{coker}(\phi_m)$, which is a finite A-module since B_m is a finite A-module. Fix an algebraically closed field k and a ring homomorphism $A \to k$. Then $B_1 \otimes_A k$ is a finite dimensional k-vector space with generators $\{x_1 \otimes 1, \ldots, x_s \otimes 1\}$. By hypotheses, for every $m \geq 1$ the k-algebra $B_m \otimes_A k$ is generated as a k-vector space by elements of the form

$$x_1^{\alpha_1} \cdots x_s^{\alpha_s} \otimes 1 \quad \text{with} \quad \alpha_1 + \ldots + \alpha_s = m.$$

In other words, the k-linear map $\phi_m \otimes \mathrm{id}_k$ is surjective and $L \otimes_A k = 0$. It follows that for every $\mathfrak{p} \in \operatorname{Spec} A$

$$0 = L \otimes_A \kappa(\overline{\mathfrak{p}}) = (L \otimes_A \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} \kappa(\overline{\mathfrak{p}}),$$

and in particular

$$0 = L \otimes_A \kappa(\mathfrak{p}) = L_{\mathfrak{p}}/\mathfrak{p}L_{\mathfrak{p}}$$

so that $L_{\mathfrak{p}} = \mathfrak{p}L_{\mathfrak{p}}$. The maximal ideal of the local ring $A_{\mathfrak{p}}$ is $\mathfrak{p}A_{\mathfrak{p}}$ and $L_{\mathfrak{p}}$ is a finite $A_{\mathfrak{p}}$-module. By Nakayama's Lemma, it follow that $L_{\mathfrak{p}} = 0$, and this fact holds for all $\mathfrak{p} \in \operatorname{Spec} A$. Since being 0 is a local property for modules, it follows that $L = 0$ and that ϕ_m is surjective. □

Definition 4.15. Let T be a scheme. We say that a quasi-coherent $\mathbb{N}$-graded $\mathcal{O}_T$-algebra $\mathcal{A}$ is *(locally) generated by* $\mathcal{A}_1$ or *(locally) generated in degree* 1 if there exists an open affine covering $\{U_i\}_{i\in I}$ such that the $\Gamma(U_i, \mathcal{O}_T)$-algebra $\Gamma(U_i, \mathcal{A})$ is generated by $\Gamma(U_i, \mathcal{A}_1)$ (equivalently, the $\Gamma(U_i, \mathcal{O}_T)$-algebra $\Gamma(U_i, \mathcal{A})$ is generated by $\Gamma(U_i, \mathcal{A}_1)$ for all open affine subschemes $U \subseteq T$).

Notation 4.16. Let T be a scheme. If $\mathcal{A} = \bigoplus_{m\geq 0} \mathcal{A}_m$ is a quasi-coherent graded $\mathcal{O}_T$-algebra, we denote by $\mathcal{A}^{(d)}$ the quasi-coherent graded $\mathcal{O}_T$-algebra

$$\mathcal{A}^{(d)} = \bigoplus_{m\geq 0} \mathcal{A}_{md}.$$

It is clear that $\mathcal{A}^{(d)}$ is a sub-$\mathcal{O}_T$-algebra of $\mathcal{A}$.

Notation 4.17. In what follows, we will use the definition of a *projective morphism* according to [Stacks, Section 01W7] and [Gro67, Definition II.5.5]. This definition is slightly different from the one in Hartshorne's book [Har77, Definition, p.103]. If a morphism is projective according to Hartshorne, we say that it is H-projective.

Proposition 4.18. *Let $(p : S \to T) \in \mathcal{M}^{\min}(T)$ and let $\mathcal{A} = \bigoplus_{m\geq 0} p_*\omega_{S/T}^{\otimes 5m}$. Then for every point $t \in T$ there exist an open affine neighbourhood U of t in T and an integer $d \geq 1$ (which depends on U) such that $\mathcal{A}_{|U}^{(d)}$ is generated in degree 1 as an $\mathcal{O}_U$-algebra. Moreover the associated U-scheme*

$$\pi_U : \underline{\operatorname{Proj}}(\mathcal{A}_{|U}) = \underline{\operatorname{Proj}}(\mathcal{A}_{|U}^{(d)}) \to U$$

is H-projective.

Proof. The function $\tilde{P}_m$ defined in Lemma 4.13 is locally constant on T for $m \geq 3$. Thus we can choose an open covering $\{U_i\}_{i\in I}$ of T such that over any open U_i the functions $\tilde{P}_4, \tilde{P}_8$ and $\tilde{P}_{12}$ are constant. Up to restricting the covering, we may assume that $\{U_i\}_{i\in I}$ is an affine open covering of T.

Fix $U = U_i = \operatorname{Spec} R$ to be one of the affine opens. Denote by P_4, P_8 and P_{12} the values of the constant functions $\tilde{P}_4, \tilde{P}_8$ and $\tilde{P}_{12}$ respectively on U. Then there exists an integer $n \geq 1$ such that $5n \geq 5P_4 + 4P_8 + 3P_{12}$. For such an integer n, consider the $\mathbb{N}$-graded R-algebra given by

$$\mathcal{A}^{(24n)}(U) = \bigoplus_{m\geq 0} \mathcal{A}_{120nm}(U).$$

For each geometric point t of U we have

$$\begin{aligned}\mathcal{A}^{(24n)}(U) \otimes_R \kappa(t) &= \left(\bigoplus_{m\geq 0} p_*\omega_{S/T}^{\otimes 120nm}(U)\right) \otimes_R \kappa(t) \simeq \\ &\simeq \bigoplus_{m\geq 0}(p_*\omega_{S/T}^{\otimes 120nm}(U) \otimes_R \kappa(t)) \simeq \\ &\simeq \bigoplus_{m\geq 0} \mathrm{H}^0(S_t, \omega_{S_t/\kappa(t)}^{\otimes 120nm}),\end{aligned}$$

where the last isomorphism is given by Corollary B.6. Since S_t is a minimal surface of general type and $5n \geq 5P_4+4P_8+3P_{12}$, by Proposition 2.83 we have that $\mathcal{A}^{(24n)}(U)\otimes_R\kappa(t)$ is generated in degree 1 as an R-algebra. By Lemma 4.14 we conclude that $\mathcal{A}^{(24n)}(U)$ is generated in degree 1 as an R-algebra. In particular, for $d = 24n$, we have that $\mathcal{A}^{(d)}_{|U}$ is generated in degree 1 as an $\mathcal{O}_U$-algebra.

Then we use Lemma 4.11 to conclude that the associated U scheme

$$\underline{\mathrm{Proj}}(\mathcal{A}^{(d)}_{|_U}) \to U$$

is projective by [Stacks, Lemma 0B3U], and hence it is H-projective because U is affine, see [Har77, §II, Definition p.103]. ☐

Corollary 4.19. *Let* $(p : S \to T) \in \mathcal{M}^{\min}(T)$ *and let* $\mathcal{A} = \bigoplus_{m\geq 0} p_*\omega_{S/T}^{\otimes 5m}$. *The morphism of schemes* $\pi : \underline{\mathrm{Proj}}(\mathcal{A}) \to T$ *is locally projective, and in particular proper.*

Proof. By Proposition 4.18, there exists an affine open cover $\{U_i\}_{i\in I}$ and integers $d_i \geq 1$ for $i \in I$ such that the morphism

$$\underline{\mathrm{Proj}}(\mathcal{A}^{(d_i)}_{|_{U_i}}) \to U_i$$

is projective. As U_i is affine, we have

$$\underline{\mathrm{Proj}}(\mathcal{A}_{|_{U_i}}) \simeq \mathrm{Proj}\,\mathcal{A}(U_i)$$

and thus the inclusion of sheaves $\mathcal{A}^{(d_i)}_{|U_i} \subseteq \mathcal{A}_{|_{U_i}}$ induces an isomorphism

$$\underline{\mathrm{Proj}}(\mathcal{A}^{(d_i)}_{|_{U_i}}) \simeq \underline{\mathrm{Proj}}(\mathcal{A}_{|_{U_i}}).$$

for all $i \in I$. It follows that the morphism

$$\pi_{|U_i} : \underline{\mathrm{Proj}}(\mathcal{A}_{|U_i}) \to U_i$$

is projective for all $i \in I$ by Proposition 4.18. Then $\pi : \underline{\mathrm{Proj}}(\mathcal{A}) \to T$ is locally projective ([Stacks, Definition 01W8]). Finally, π is proper by [Stacks, Lemma 01WC]. ☐

Proposition 4.20. *Let* $(p : S \to T) \in \mathcal{M}^{\min}(T)$ *and let* $\mathcal{A} = \bigoplus_{m\geq 0} p_*\omega_{S/T}^{\otimes 5m}$. *The map* $\pi : \underline{\mathrm{Proj}}(\mathcal{A}) \to T$ *is finitely presented.*

Proof. Since by Corollary 4.19 the map π is proper, then it is also quasi-compact and quasi-separated. Thus to prove that π is of finite presentation it is sufficient to show that it is locally of finite presentation.

We already know that there exist an affine open covering $\{U_i\}_{i\in I}$ of T and integers $d_i \geq 1$ for all $i \in I$ such that such that $\mathcal{A}^{(d_i)}(U_i)$ is a $\Gamma(U_i, \mathcal{O}_{U_i})$-algebra generated in degree 1. It follows in particular that $\mathcal{A}^{(d_i)}(U_i)$ is an $\Gamma(U_i, \mathcal{O}_{U_i})$-algebra of finite type for all $i \in I$. Fix $U = U_i = \mathrm{Spec}\, R$ to be one of the affine opens, and let $d = d_i$. If R is a noetherian ring, we conclude that $\mathcal{A}^{(d)}(U)$ is a finitely presented R-algebra. If that is the case, by [Stacks, Lemma 0D4D] we conclude that

$$\pi_{|U} : \underline{\mathrm{Proj}}(\mathcal{A}_{|_U}) \simeq \underline{\mathrm{Proj}}(\mathcal{A}^{(d)}_{|_U}) \to U$$

is of finite presentation.

Suppose now that R is not noetherian. We will use an argument of noetherian approximation. In order to do that, up to restricting p to an affine open U_i, we suppose that $p : S \to T$ is a family of minimal surfaces of general type with $T = \operatorname{Spec} R$ affine and not noetherian (the noetherian case being already done). Recall that the morphism p is smooth and of finite presentation by the definition of a family of minimal surface of general type. Consider now the set

$$\{R_\lambda \mid R_\lambda \subseteq R \text{ such that } R_\lambda \text{ is of finite type over } \mathbb{Z}\}\,.$$

We denote $T_\lambda = \operatorname{Spec} R_\lambda$. Then by [Stacks, Lemma 01ZM] there exists an index α and a morphism of finite presentation $p_\alpha : S_\alpha \to T_\alpha$ such that the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, "p"'] & S_\alpha \arrow[d, "p_\alpha"] \\ T \arrow[r] & T_\alpha \end{tikzcd}$$

is cartesian. Consider now the projective system $\{T_\lambda\}_{\lambda \geq \alpha}$ of affine schemes such that R_λ are subrings of R which are finitely generated extension of R_α. Denote $p_\lambda : X_\lambda \to Y_\lambda$ the base change of p_α to T_λ. Then p is the projective limit of the p_λ's. Denote $u_\lambda : T \to T_\lambda$ and $u_{\lambda\mu} : T_\mu \to T_\lambda$, for $\mu \geq \lambda$ the morphisms induced by the inclusions. Since p is proper and smooth, there exists an index $\beta \geq \alpha$ such that p_β is proper and smooth by [Stacks, Lemma 08K1] and [Stacks, Lemma 0CN2]. Moreover, p_β is of finite presentation by Proposition 3.6 because it comes from p_α by base change.
It remains to verify that the geometric fibres of some p_λ, $\lambda \geq \beta$, are minimal surfaces of general type. This has already been done in the proof of Lemma 3.87. It follows that there exists an index $\lambda \geq \beta$ such that $p_\lambda : S_\lambda \to T_\lambda$ is a family of minimal surfaces of general type. In other words, we have an arrow between two objects of $\mathcal{M}^{\min}$, i.e. a cartesian diagram

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "p"'] & S_\lambda \arrow[d, "p_\lambda"] \\ T \arrow[r, "f"'] & T_\lambda \end{tikzcd}$$

where T_λ is noetherian. Define

$$\mathcal{A}' = \bigoplus_{m \geq 0} (p_\lambda)_* \omega_{S_\lambda/T_\lambda}^{\otimes 5m}.$$

By above arguments it follows that

$$\pi' : \underline{\operatorname{Proj}}(\mathcal{A}') \to T_\lambda$$

is a finitely presented morphism. The functor

$$f^* : \operatorname{Mod}(\mathcal{O}_{T_\lambda}) \to \operatorname{Mod}(\mathcal{O}_{T_\lambda})$$

commutes with direct sums by [Stacks, Lemma 01AJ] and formation of $p_*\omega_{S/T}^{\otimes 5m}$ commutes with arbitrary base change by Lemma 4.11. It follows that

$$f^*\big(\bigoplus_{m \geq 0} p_*\omega_{S/T}^{\otimes 5m}\big) = \bigoplus_{m \geq 0} f^* p_*\omega_{S/T}^{\otimes 5m} \simeq \bigoplus_{m \geq 0} p'_*\omega_{S'/T'}^{\otimes 5m}. \tag{16}$$

It follows by Remark 4.10 and by Equation (16) that we have a cartesian diagram

$$\begin{tikzcd} \underline{\operatorname{Proj}}(\mathcal{A}) \arrow[r] \arrow[d, "\pi"'] & \underline{\operatorname{Proj}}(\mathcal{A}') \arrow[d, "\pi'"] \\ T \arrow[r, "f"'] & T_\lambda. \end{tikzcd} \tag{17}$$

Then π is finitely presented, being the pullback of a finitely presented morphism, see Proposition 3.6. $\square$

REMARK 4.21. For all $m \geq 0$ we know that $p_*\omega_{S/T}^{\otimes 5m}$ is a locally free sheaf of finite rank whose formation commutes with arbitrary base change by Lemma 4.11. It follows in particular that for every geometric point $\sigma : \operatorname{Spec} k \to T$ the cartesian diagram

$$\begin{tikzcd} S_k \arrow[r] \arrow[d, "p_k"'] & S \arrow[d, "p"] \\ \operatorname{Spec} k \arrow[r, "\sigma"'] & T \end{tikzcd}$$

yields an isomorphism

$$\bigoplus_{m\geq 0} (p_k)_*\omega_{S_k}^{\otimes 5m} \simeq \bigoplus_{m\geq 0} \sigma^* p_*\omega_{S/T}^{\otimes 5m} \simeq \bigoplus \mathrm{H}^0(S_k, \omega_{S_k}^{\otimes 5m})$$

as in Equation (16). Thus, for every geometric point σ of T, we have a cartesian diagram

$$\begin{tikzcd} \tilde{X}_k = \operatorname{Proj}\left(\bigoplus_{m\geq 0} \mathrm{H}^0(S_k, \omega_{S_k}^{\otimes 5m})\right) \arrow[r] \arrow[d] & \underline{\operatorname{Proj}}\left(\bigoplus_{m\geq 0} p_*\omega_{S/T}^{\otimes 5m}\right) \arrow[d] \\ \operatorname{Spec} k \arrow[r, "\sigma"'] & T. \end{tikzcd}$$

as in Equation (17). The geometric fibre $\tilde{X}_k$ is isomorphic to the canonical model

$$X_k = \operatorname{Proj}\left(\bigoplus_{m\geq 0} \mathrm{H}^0(S_k, \omega_{S_k}^{\otimes m})\right),$$

of S_k, the isomorphism being induced by the inclusion

$$\bigoplus_{m\geq 0} \mathrm{H}^0(S_k, \omega_{S_k}^{\otimes 5m}) \subseteq \bigoplus_{m\geq 0} \mathrm{H}^0(S_k, \omega_{S_k}^{\otimes m}).$$

In other words, the geometric fibres of π are canonical models of minimal surfaces of general type.

COROLLARY 4.22. *Let $(p : S \to T) \in \mathcal{M}^{\min}(T)$ and let $\mathcal{A} = \bigoplus_{m\geq 0} p_*\omega_{S/T}^{\otimes 5m}$. Then $(\pi : \underline{\operatorname{Proj}}(\mathcal{A}) \to T)$ is an object of $\mathcal{M}^{\mathrm{can}}$.*

PROOF. The morphism π is proper, flat and finitely presented by Corollary 4.19, Proposition 4.12 and Proposition 4.20. Moreover, every geometric fibre of π is the canonical model of a minimal surface of general type by Remark 4.21. $\square$

PROPOSITION 4.23. *There exists a morphism of stacks $\alpha : \mathcal{M}^{\min} \to \mathcal{M}^{\mathrm{can}}$.*

PROOF. Recall that a morphism of stacks is a morphism of fibred categories (Definition A.71). First, we define the action of α on objects. Given $(p : S \to T) \in \mathcal{M}^{\min}$, we define $\alpha(p)$ as the object $(\pi : \underline{\operatorname{Proj}}_T(\mathcal{A}) \to T)$ of $\mathcal{M}^{\mathrm{can}}$, where $\mathcal{A} = \bigoplus_{m\geq 0} p_*\omega_{S/T}^{\otimes 5m}$. This is indeed an object of $\mathcal{M}^{\mathrm{can}}$ by Corollary 4.22.

Suppose now that we are given an arrow in $\mathcal{M}^{\min}$. In other words, we are given a cartesian diagram of algebraic spaces

$$\xi : \quad \begin{tikzcd} S' \arrow[r] \arrow[d, "p'"'] & S \arrow[d, "p"] \\ T' \arrow[r] & T \end{tikzcd}$$

where T, T' are schemes and $(p : S \to T)$ and $(p' : S' \to T')$ are objects of $\mathcal{M}^{\min}$. Since formation of $\mathcal{A}$ commutes with arbitrary base change by Lemma 4.11, we obtain a cartesian diagram of schemes

$$\eta : \quad \begin{tikzcd} \underline{\mathrm{Proj}}(\mathcal{A}') \arrow[r] \arrow[d, "\pi"'] & \underline{\mathrm{Proj}}(\mathcal{A}) \arrow[d, "\pi"] \\ T' \arrow[r] & T \end{tikzcd}$$

as in Equation (17), where $\mathcal{A}' = \bigoplus_{m\geq 0} p'_*\omega_{S'/T'}^{\otimes 5m}$. In particular, the diagram η is an arrow of $\mathcal{M}^{\mathrm{can}}$. We define the action of α on arrows by $\alpha(\xi) = \eta$. Since every arrow is cartesian, it is clear that α sends cartesian arrows in cartesian arrows. Finally, the map α obviously commutes with the projection to Sch:

$$\begin{tikzcd} \mathcal{M}^{\min} \arrow[rr, "\alpha"] \arrow[dr] & & \mathcal{M}^{\mathrm{can}} \arrow[dl] \\ & \mathrm{Sch}\,. & \end{tikzcd}$$

□

Remark 4.24. The morphism $\alpha : \mathcal{M}^{\min} \to \mathcal{M}^{\mathrm{can}}$ is clearly a bijection on geometric points, by Proposition 2.62. However, α is not an isomorphism, because if $(p : S \to T)$ is an object of $\mathcal{M}^{\mathrm{can}}(T)$, then a simultaneous resolution of p could not exist. This is due to the work of Artin in [Art74a].

4.3. Related topics

In this section, we provide a concise exposition on topics related to moduli spaces of surfaces.

Connected components and number of moduli. If S is a minimal surface of general type over $\mathbb{C}$, then the topological space underlying S is an oriented compact real manifold of dimension 4.
By results of Bombieri [Bom73], it is known that surfaces of general type with fixed numerical invariants $\chi = \chi(\mathcal{O}_S)$ and K^2, where K^2 is the self-intersection of a canonical divisor of the minimal model S, belong to a finite number of families.
Moreover, K^2 and χ, are invariants under orientation-preserving homeomorphisms of the underlying topological space of S.
As in §3.4 we can consider the Deligne-Mumford stack $\mathcal{M}^{\min,\mathbb{C}}_{\chi,K^2}$ of complex minimal surfaces of general type with invariants χ, K^2.
Then the isomorphism classes of surfaces S with these invariants χ, K^2 are parametrized by a quasi-projective variety $M^{\min,\mathbb{C}}_{\chi,K^2}$. According to [Cat84], this moduli space consists of a finite number of irreducible components $M_1, \dots, M_k$, and any two points belonging to the same connected component of $M^{\min,\mathbb{C}}_{\chi,K^2}$ correspond to isomorphism classes $[S]$ and $[S']$ of minimal models S and S' that are diffeomorphic to each other, by Ehresmann theorem (see, for example, [Voi07, Theorem 9.3])
For example, Horikawa in [Hor75] studied the component $M^{\min,\mathbb{C}}_{5,5}$ (equivalently $K^2 = 5, p_g = 4, q = 0$, by Definition 2.4) and he showed that this moduli space is connected but consists of two irreducible components, each of dimension 40.
We may also define $M^{\min,\mathbb{C}} = \coprod_{\chi,K^2} M^{\min,\mathbb{C}}_{\chi,K^2}$ as χ and K^2 runs over integers and we may ask whether the converse of what we said above holds. In other words, let S be an oriented compact real manifold of dimension 4. Define

$$M_S = \left\{ [X] \in M^{\min,\mathbb{C}} \;\middle|\; X \text{ is diffeomorphic to } S \right\}.$$

The space M_S may be empty. If not, by Ehresmann theorem, we know that M_S is union of connected components of $M^{\min,\mathbb{C}}$.

$\mathcal{Q}$: how many connected components does M_S have?

Manetti gave an answer to this problem in [Man01, Theorem A]: for all $n > 0$ there exists a smooth oriented compact differentiable 4-manifold S, such that M_S has at least n connected components.

Another interesting question concerns the dimension of moduli spaces of surfaces.

Definition 4.25. Let S be a minimal surfaces of general type over $\mathbb{C}$ with invariants χ and K^2. The *number of moduli of* S, denoted by $M(S)$ is the dimension of $M^{\min,\mathbb{C}}_{\chi,K^2}$ at $[S]$, the isomorphism class of S.

As we said above, for fixed χ and K^2, we have that $M^{\min,\mathbb{C}}_{\chi,K^2}$ has a finite number of irreducible components. It follows in particular that if S is a minimal surface of general type with fixed χ and K^2, the number of moduli $M(S)$ can assume only a finite number of values.

Catanese also gives bounds for $M(S)$ in terms of χ and K^2. First of all, deformation theory for smooth surfaces provides two inequalities as follows: if $\mathcal{T}_S$ is the tangent bundle of S, then

$$h^1(S,\mathcal{T}_S) \geq M(S) \geq h^1(S,\mathcal{T}_S) - h^2(S,\mathcal{T}_S). \tag{18}$$

See, for example, [Ser06, Corollary 2.4.7]. The statement is true in a more general context: it holds for smooth projective *algebraic varieties*, i.e. proper schemes over a field k. Since for curves the second cohomology groups vanish, this is the reason why for smooth genus g curves, the dimension of M_g (Remark 1.16) at a point $[C]$ is exactly $h^1(C,\mathcal{T}_C) = 3g-3$, as we proved in Proposition 1.17.

However, $h^2(S,\mathcal{T}_S)$ may be not zero for minimal surfaces of general type, as we observed in Remark 2.73. In fact, surfaces may be obstructed, as we explain later, because $H^2(S,\mathcal{T}_S)$ provides an obstruction space for S.

We also have the following bound due to Catanese.

Proposition 4.26. *Let S be a minimal surface of general type over $\mathbb{C}$ with invariants χ and K^2. Then*

$$M(S) \leq 10\chi + 3K^2 + 108.$$

Proof. See [Cat84, Theorem B]. ☐

An example of obstructed surface. We want now to briefly present an example of an obstructed minimal surface of general type. We will work over the field of complex numbers. Consider the first Hirzebruch surface

$$\mathbb{F}_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)),$$

which is a del Pezzo surface (Definition 3.108), i.e. $-K_{\mathbb{F}_1}$ is ample. It holds that $-K_{\mathbb{F}_1}$ is very ample, and it realizes a closed immersion of $\mathbb{F}_1$ in $\mathbb{P}^8_{\mathbb{C}}$, because $\dim_{\mathbb{C}} \mathrm{H}^0(\mathbb{F}_1, -K_{\mathbb{F}_1}) = 9$. Moreover the anticanonical ring

$$R = R(\mathbb{F}_1, -K_{\mathbb{F}_1}) = \bigoplus_{m\geq 0} \mathrm{H}^0(\mathbb{F}_1, \omega_{\mathbb{F}_1}^{\otimes -m})$$

is generated in degree 1. Define

$$U = \operatorname{Spec} R \subseteq \mathbb{A}^9_{\mathbb{C}}$$
$$X = \operatorname{Proj} R[t] \subseteq \mathbb{P}^9_{\mathbb{C}}$$

to be the *affine cone* and the *projective cone*, respectively, of $\mathbb{F}_1$ with respect to the anti-canonical line bundle ([GW20, §13.9]). A result of Altmann [Alt97] states that the base of the miniversal deformation of U is $\operatorname{Spf}\mathbb{C}[t]/(t^2)$, i.e. there exists a *formal versal couple* (R,ξ) ([Ser06, Definition 2.2.6]) such that $R=\mathbb{C}[\epsilon]=\mathbb{C}[t]/(t^2)$ and ξ is a formal element over R of the deformation functor Def_U of U ([Ser06, §2.4.1]).

PROPOSITION 4.27. *Every formal deformation of U over $\mathbb{C}[[x]]$ is trivial.*

PROOF. Let $\xi\to\operatorname{Spf}\mathbb{C}[t]/(t^2)$ be the versal formal deformation of U. Let $\eta\to\operatorname{Spf}\mathbb{C}[[x]]$ be a formal deformation of U over $\mathbb{C}[[x]]$. Since ξ is versal, η is induced by a local homomorphism of local $\mathbb{C}$-algebras

$$\phi:\mathbb{C}[t]/(t^2)\to\mathbb{C}[[x]]$$

so that the diagram

$$\begin{tikzcd}
\eta \arrow[r] \arrow[d] & \xi \arrow[d] \\
\operatorname{Spf}\mathbb{C}[[x]] \arrow[r, "f"'] & \operatorname{Spf}\mathbb{C}[t]/(t^2)
\end{tikzcd}$$

is cartesian. Since $t^2=0$ in $\mathbb{C}[t]/(t^2)$ and $\mathbb{C}[[x]]$ is a domain, we have that $\phi(t)=0$. Thus, ϕ factorizes as

$$\begin{tikzcd}
\mathbb{C}[t]/(t^2) \arrow[rr, "\phi"] \arrow[dr, "a"'] & & \mathbb{C}[[x]] \\
& \mathbb{C} \arrow[ur] &
\end{tikzcd}$$

where $a(t)=0$. It follows that f factorize through $\operatorname{Spec}\mathbb{C}$ and the pullback of ξ to $\operatorname{Spec}\mathbb{C}$ is simply U. Then the pullback of U to $\operatorname{Spf}\mathbb{C}[[x]]$ is the trivial deformation, and thus η is trivial. $\square$

REMARK 4.28. Using spectral sequences, it is possible to show that the restriction

$$\mathrm{Def}_X\to\mathrm{Def}_U$$

is an isomorphism. In particular, the base of the miniversal deformation of X is $\operatorname{Spf}\mathbb{C}[t]/(t^2)$.

Consider now a smooth projective surface $S\subseteq X$ obtained by intersecting X with a general hypersurface in $\mathbb{P}^9_{\mathbb{C}}$ of sufficiently positive degree m. One can show that the natural transformation of functors

$$\begin{tikzcd}
& \mathrm{Def}_{S\hookrightarrow X} \arrow[dl] \arrow[dr] & \\
\mathrm{Def}_S & & \mathrm{Def}_X
\end{tikzcd}$$

are smooth. It follows that the base of the miniversal deformation of S is

$$\operatorname{Spf}\frac{\mathbb{C}[[t,u_1,\dots,u_d]]}{(t^2)}$$

for some $d\geq 0$. In particular, S is an obstructed surface.

Moreover, the canonical line bundle of S is ample. Thus, S is a minimal surface of general type over $\mathbb{C}$ and S coincides with its canonical model. In other words, S provides a point that belongs to both $\mathcal{M}^{\min}$ and $\mathcal{M}^{\mathrm{can}}$. By deformation theory, this implies that neither $\mathcal{M}^{\min}$ nor $\mathcal{M}^{\mathrm{can}}$ is smooth, and that this property does not hold even in characteristic zero.

The geography of surfaces of general type. Another fundamental question regards the *geography* of surfaces of general type over the field of complex numbers. We can formulate it as follows.

$$\mathcal{Q}\text{: for which values of } \chi \text{ and } K^2 \text{ is } M^{\min,\mathbb{C}}_{\chi,K^2} \text{ non empty?}$$

In other words, we are asking for which values of χ and K^2 there exists a minimal surface of general type over $\mathbb{C}$ with Euler characteristic χ and self-intersection of the canonical bundle K^2. This problem is constrained by well-known inequalities.

PROPOSITION 4.29. *Let S be a minimal surface of general type over the field of complex numbers. Then the following inequalities holds.*

(1) $K_S^2 \geq 1$;
(2) $\chi(\mathcal{O}_S) \geq 1$;
(3) $K_S^2 \geq 2p_g(S) - 4$;
(4) $K_S^2 \geq 2\chi - 6$ *(Noether's inequality);*
(5) $K_S^2 \leq 9\chi$ *(Bogomolov-Miyaoka-Yau inequality).*

PROOF. The first inequality is due to [Kod68, §3]. For (2) see [Bea10, Theorem X.4]. For (3) see [Bea10, Exercise X.13(1)]. Then (4) follows immediately by (3) since $\chi(S) = 1 - q(S) + p_g(S)$. Finally (5) was proved independently by Miyaoka [Miy77] and Yau [Yau77] after Bogomolov [Bog78] proved a weaker version of it. $\square$

Therefore, the geography of complex minimal surfaces of general type can be represented as in Figure 1.

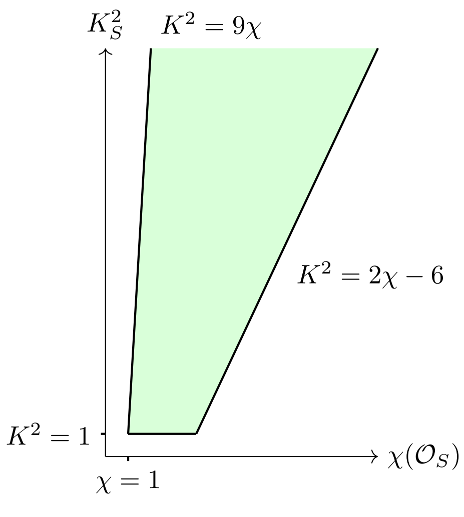

FIGURE 1. The geography of minimal surfaces of general type

Surfaces on the line $K^2 = 2\chi - 6$ have been classified by Horikawa [Hor76]. They exist for every value $\chi \geq 4$. Persson [Per81, Theorem 2] found a complex minimal surface of general type for every values of χ and K^2 such that $2\chi - 6 \leq K^2 \leq 8\chi$. Moreover, Sommese in [Som84] proved that the possible ratios of χ/K^2 for which a minimal surface of general type exists with those invariants, form a dense subset of the interval $[2, 9]$.

Compactification. Beyond the moduli space of smooth surfaces, one can construct a compactification of $\overline{M}^{\min}_{\chi,K^2}$ by considering the projective KSBA (Kollár-Shepherd-Barron-Alexeev) moduli space $M^{\mathrm{KSBA}}_{\chi,K^2}$, which parametrizes stable surfaces, including possibly reducible ones. The moduli space $M^{\min,\mathbb{C}}_{\chi,K^2}$ is an open subset of $M^{\mathrm{KSBA}}_{\chi,K^2}$.

This construction generalizes the Deligne-Mumford compactification of M_g using stable curves. However, the theory of stable surfaces is significantly more involved, due to the fact that stable surfaces need not be Gorenstein, which complicates the definition of the canonical sheaf and stable families. We refer the reader to [Kol23].

Singularity. A theorem of Vakil shows that the stack of smooth surfaces with ample canonical divisor can have arbitrarily bad singularities, highlighting the complexity of these moduli spaces.

THEOREM 4.30 (Vakil's "Murphy's law"). *For every singularity type of finite type over $\mathbb{Z}$, there exists a Gieseker moduli space $M^{\mathrm{can}}_{\chi,K^2}$ and a minimal surface of general type S with ample canonical divisor K_S (hence S coincides with its canonical model X) such that $(M^{\mathrm{can}}_{\chi,K^2}, X)$ realizes the given singularity germ, up to a smooth factor.*

PROOF. See [Vak06]. $\square$

APPENDIX A

Stack theory

We recall basics of stack theory. We will follow [Vis08].

A.1. Representable functors

Let $\mathcal{C}$ be a category. We write $U \in \mathcal{C}$ to denote an object of $\mathcal{C}$. We denote by

$$\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

the category whose objects are contravariant functors $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and arrows are the natural transformations.

NOTATION A.1. More generally, if $\mathcal{C}$ and $\mathcal{D}$ are two categories, we denote by $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ the set of functors from $\mathcal{C}$ to $\mathcal{D}$.
If F, G are two functors between the same categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathrm{Hom}(F, G)$ the set of natural transformations of functors from F to G.

NOTATION A.2. Let $\mathcal{C}$ be a category and let $(F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}) \in \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$. If $f : V \to U$ is an arrow in $\mathcal{C}$, we will denote by

$$f^* : F(U) \to F(V)$$

the corresponding map of sets $f^* = F(f)$. If $u \in F(U)$, we call $f^*(u) \in F(V)$ the *pullback* of u on V.

LEMMA A.3. *Let $\mathcal{C}$ be a category which admits fibre products. Suppose that we have a commutative diagram of morphisms between objects of $\mathcal{C}$*

$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "u"] & P \arrow[r, "r"] \arrow[d, "s"] & A \arrow[d, "f"] \\ D \arrow[r, "h"'] & B \arrow[r, "g"'] & C. \end{tikzcd}$$

Then both squares are cartesian if and only if the right hand square and the square

$$\begin{tikzcd} Q \arrow[r, "rt"] \arrow[d, "u"'] & A \arrow[d, "f"] \\ D \arrow[r, "gh"'] & C \end{tikzcd}$$

obtained by composing the rows, are cartesian.

PROOF. If both squares are cartesian, then it is clear that also the square obtained by composing the rows is cartesian.
Suppose now that the right hand square and the external square are cartesian. Let $D \times_B P$ be the pullback of D and P along h and s. Since by hypothesis the diagram

$$\begin{tikzcd} Q \arrow[r, "rt"] \arrow[d, "u"'] & A \arrow[d, "f"] \\ D \arrow[r, "gh"'] & C \end{tikzcd}$$

is cartesian, by the universal property of the pullback there exists a unique dotted arrow making the diagram

$$\begin{tikzcd} D \times_B P \arrow[rrd, bend left, "r\circ \mathrm{pr}_P"] \arrow[rd, dotted, "\alpha"] \arrow[rdd, bend right, "\mathrm{pr}_D"'] & & \\ & Q \arrow[r, "rt"] \arrow[d, "u"'] & A \arrow[d, "f"] \\ & D \arrow[r, "gh"'] & C \end{tikzcd}$$

commutative. It follows that also

$$\begin{tikzcd} D \times_B P \arrow[rrd, bend left, "\mathrm{pr}_P"] \arrow[rd, dotted, "\alpha"] \arrow[rdd, bend right, "\mathrm{pr}_D"'] & & \\ & Q \arrow[r, "t"] \arrow[d, "u"'] & P \arrow[d, "s"] \\ & D \arrow[r, "h"'] & B \end{tikzcd}$$

is commutative, so that the universal property of the pullback is satisfied for Q along the maps t and u. □

EXAMPLE A.4. Let $\mathcal{C}$ be a category and let $U \in C$ be an object. Then there is a functor

$$h_U : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

which sends an object $V \in \mathcal{C}$ to the set $h_U(V) = \mathrm{Hom}_{\mathcal{C}}(V, U)$. If $\alpha : V' \to V$ is an arrow in $\mathcal{C}$ then $h_U(\alpha) : h_U(V) \to h_U(V')$ is defined to be composition with α. Moreover, an arrow $f : U \to V$ defines a morphism $h_U \to h_V$ in a natural way, see [Vis08, §2.1]. We will also denote by $U(V)$ the set $h_U(V)$.
Sending an object U of $\mathcal{C}$ to h_U defines a functor

$$h_- : \mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}),$$

see [Vis08, §2.1.1] for details.

DEFINITION A.5. A *representable functor* on the category $\mathcal{C}$ is a functor

$$F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

which is isomorphic to a functor of the form h_U for some object U of $\mathcal{C}$. If this happens, we say that F is *represented* by U.

LEMMA A.6 (Yoneda's lemma). *Let $\mathcal{C}$ be a category, $U \in \mathcal{C}$ be an object, and $F : \mathcal{C}^{op} \to \mathrm{Set}$ be a contravariant functor.*

(1) *The function*

$$\begin{array}{rccc} \phi : & \mathrm{Hom}(h_U, F) & \longrightarrow & F(U) \\ & \alpha & \longmapsto & \alpha_U(\mathrm{id}_U) \end{array}$$

is a bijection.

(2) *The functor*

$$h_- : \mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

is fully faithful.

PROOF. Consider the map of sets

$$\begin{array}{rccc} \psi : & F(U) & \longrightarrow & \mathrm{Hom}(h_U, F) \\ & p & \longmapsto & \psi_p \end{array}$$

where ψ_p is the natural transformation from h_U to F such that for every object $V \in \mathcal{C}$ we have

$$\begin{array}{rccc} \psi_{p,V}: & h_U(V) & \longrightarrow & F(V) \\ & f & \longmapsto & F(f)(p). \end{array}$$

It is easy to see that ϕ and ψ are one the inverse of the other. Then, (2) follows immediately by (1) by considering $F = h_{U'}$ for another object $U' \in \mathcal{C}$. $\square$

DEFINITION A.7. Let $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor. A *universal object* for F is a pair (U, ξ), where $U \in \mathcal{C}$ and $\xi \in F(U)$ are such that for each object V of $\mathcal{C}$ and each $\sigma \in F(V)$, there exists a unique arrow $f : V \to U$ such that $F(f)(\xi) = \sigma \in F(V)$.

In other words, (U, ξ) is a universal object if the morphism $h_U \to F$ defined by ξ as in Lemma A.6 is an isomorphism.

We will now give some examples in the category of schemes. If S is a scheme, then the functor

$$h_S : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{Set}$$

is called the *functor of points of* S. Moreover, if a functor

$$F : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{Set}$$

is represented by a scheme M, we say that M is a *fine moduli space* for F.

The first point of the theory of stacks is to view a scheme as a functor. Namely, thanks to Yoneda's lemma (A.6) we can confuse a scheme S with its *functor of points* h_S.

For a contravariant functor F as above, being representable by some scheme T is in some sense the best condition that we can ask for. The reason is that there exists a universal object which induces all of the others via unique morphisms, as in Definition A.7.

EXAMPLE A.8. Let $S = \operatorname{Spec} R$ be an affine scheme. Consider the functor

$$\mathcal{O} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$$

that sends an S-scheme U to $\mathcal{O}(U)$. The functor $\mathcal{O}$ is represented by $\operatorname{Spec} R[x]$ and $(\operatorname{Spec} R[x], x)$ is a universal object, see [Vis08, Example 2.4].

EXAMPLE A.9. Let $S = \operatorname{Spec} R$ be an affine scheme. Consider the functor

$$\mathcal{O}^* : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$$

that sends an S-scheme U to $\mathcal{O}^*(U)$ (invertible sections of the structure sheaf). The functor $\mathcal{O}^*$ is represented by $\mathbb{G}_{m,S} = \operatorname{Spec} R[x, x^{-1}]$ and $(\mathbb{G}_{m,S}, x)$ is a universal object, see [Vis08, Example 2.5].

EXAMPLE A.10. Let $S = \operatorname{Spec} R$ be an affine scheme. It is known that morphisms from an S-scheme T to $\mathbb{P}^N_S$ are in one-to-one correspondence with isomorphism classes of tuples $(\mathcal{L}, s_0, \ldots, s_N)$ where $\mathcal{L}$ is a line bundle on T and $s_i \in \mathrm{H}^0(T, \mathcal{L})$ are global sections generating $\mathcal{L}$. It follows that the functor

$$F : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$$

sending an S-scheme T to the set of isomorphism classes of tuples defined above, is represented by $\mathbb{P}^N_S$. The universal object over $\mathbb{P}^N_S$ is $(\mathcal{O}_{\mathbb{P}^N_S}(1), x_0, \ldots, x_N)$, see [Vis08, Example 2.6] for details.

EXAMPLE A.11. For every noetherian scheme T and every polynomial $P \in \mathbb{Q}[m]$, we can consider *Hilbert functor*

$$\mathcal{H}ilb^P(\mathbb{P}^n_T) : (\mathrm{Sch}\,/T)^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

sending a morphism $(f : S \to T)$ to the set of closed subschemes $Z \subseteq \mathbb{P}^N_S$, flat and finitely presented over S, such that for all $s \in S$ the fibre $Z_s = f^{-1}(s)$ is a closed subscheme of $\mathbb{P}^n_{\kappa(s)}$ with Hilbert polynomial P.
The Hilbert functor is represented by a projective scheme over T ([Fan+05, Theorem 5.20] or [Alp24, Theorem 1.1.2]).
We define the *Hilbert scheme of* T *and* P to be the scheme representing the functor $\mathcal{H}ilb^P(\mathbb{P}^n_T)$, and we denote it by $\mathrm{Hilb}^P(\mathbb{P}^n_T)$.

DEFINITION A.12. Let $\mathcal{C}$ be a category. A *group object* of $\mathcal{C}$ is an object G of $\mathcal{C}$, together with a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Grp}$ in the category of groups, whose composite with the forgetful functor $\mathrm{Grp} \to \mathrm{Set}$ equals h_G. Equivalently, a group object is an object G, together with a group structure on $G(U)$ for each object U of $\mathcal{C}$, so that the function $f^* : G(V) \to G(U)$ associated with an arrow $f : U \to V$ in $\mathcal{C}$ is always a homeomorphism of groups. See [Vis08, Definition 2.11].

EXAMPLE A.13. Let $S = \mathrm{Spec}\, R$. For each scheme U, the set $\mathcal{O}(U)$ has an additive group structure. Then $\mathbb{G}_{a,S} = \mathrm{Spec}\, R[x]$ is a group scheme by Example A.8.
Analogously, $\mathbb{G}_{m,S}$ of Example A.9 has an obvious structure of group scheme.

EXAMPLE A.14. Let $S = \mathrm{Spec}\, R$, and consider the contravariant functor from $\mathrm{Sch}\,/S$ to Set which sends each S-scheme U to the set of matrices in $\mathrm{M}_n(\mathcal{O}(U))$ with invertible determinants. This functor is represented by an open subscheme of $\mathbb{A}^{n^2}_S$, which we denote by $\mathrm{GL}_{n,S}$. Matrices with invertible determinants form a group, and this gives to $\mathrm{GL}_{n,S}$ a group scheme structure. If $S = \mathrm{Spec}\,\mathbb{Z}$ we simply write $\mathrm{GL}_n = \mathrm{GL}_{n,\mathrm{Spec}\,\mathbb{Z}}$, and this scheme is

$$\mathrm{Spec}\,\mathbb{Z}[(x_{ij}) \mid 1 \le i \le n,\, 1 \le j \le n,\, \det(x_{ij})^{-1}].$$

EXAMPLE A.15. Let H be a scheme and let $f : E \to E'$ be a morphism of locally free sheaves on H of finite rank r and s respectively. Let

$$F : (\mathrm{Sch}\,/H)^{\mathrm{op}} \to \mathrm{Set}$$

be the functor sending $g : T \to H$ to the unital set if $g^*(f) : g^*E \to g^*E'$ is the zero map, and the empty set otherwise. We claim that F is represented by a closed subscheme of H. To see this, consider $\{U_i = \mathrm{Spec}\, A_i\}_{i \in I}$ an open affine cover of E on which both E, E' are trivialized. Then, for each $i \in I$, the restriction $f_{|U_i}$ is represented by an $s \times r$ matrix

$$\left(f^{(i)}_{j,k}\right) : \mathcal{O}^{\oplus r}_{U_i} \to \mathcal{O}^{\oplus s}_{U_i}$$

where $1 \le j \le s$, $1 \le k \le r$ and $f^{(i)}_{j,k} \in \mathcal{O}_H(U_i)$ for all i, j, k. Consider the ideal $\mathcal{I}_i$ of A_i generated by the elements $f^{(i)}_{j,k}$ for $1 \le j \le s$ and $1 \le k \le r$. Then $V(\mathcal{I}_i)$ is a closed subscheme of U_i and a map $g : T \to U_i$ is such that the pullback $g^* f_{|U_i}$ is zero if and only if g factors through $V(\mathcal{I}_i)$. It follows that glueing the ideals $\mathcal{I}_i$ for all $i \in I$ we get an ideal sheaf $\mathcal{I} \subset \mathcal{O}_H$, and the corresponding closed subscheme represents the functor F.

A.2. Sites, sheaves and algebraic spaces

DEFINITION A.16. Let $\mathcal{C}$ be a category, and let $C \in \mathcal{C}$ be an object. We define the *comma category* $\mathcal{C}/C$ whose objects are arrows $C' \to C$ of $\mathcal{C}$ with target C, and morphisms from $(f : C' \to C)$ to $(g : C'' \to C)$ are given by morphisms $h : C' \to C''$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} C' \arrow[rr, "h"] \arrow[dr, "f"'] & & C'' \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

commutes.

Definition A.17. Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is the data, for each object C of $\mathcal{C}$, of a collection of sets of arrows $\{C_i \to C\}_{i\in I}$, called *coverings of* C so that the following conditions are satisfied.

(i) If $C' \to C$ is an isomorphism, then $\{C' \to C\}$ is a covering.
(ii) If $\{C_i \to C\}_{iıI}$ is a covering and $C' \to C$ is any arrow, then the fibre products $C_i \times_C C'$ exist, and the collection of projections $\{C_i \times_C C' \to C'\}_{i\in I}$ is a covering.
(iii) If $\{C_i \to C\}_{i\in I}$ is a covering, and for each index i we have a covering $\{C_{ij} \to C_i\}_j$ (here j varies on a set depending on i), the collection of composites $\{C_{ij} \to C_i \to C\}_{ij}$ is a covering of C.

Notation A.18. We will simply denote by $\{C_i \to C\}$ a covering (or a set of arrows), namely we will not write the subscript $i \in I$ to avoid making the notation too heavy.

Definition A.19. A *site* is a pair $(\mathcal{C}, \tau)$ where $\mathcal{C}$ is a category and τ is a Grothendieck topology. We will also say that $\mathcal{C}$ is a site, implicitly assuming that it is equipped with a Grothendieck topology.

Definition A.20. A set $\{U_i \to U\}$ of functions, or morphisms of schemes, is called *jointly surjective* when the set-theoretic union of their images equals U.

Example A.21. If $(\mathcal{C}, \tau)$ is a site, and $C \in \mathcal{C}$ is an object, then τ induces an obvious Grothendieck topology on the comma category $\mathcal{C}/C$.

Example A.22. Let X be a topological space. We denote by X_{op} the category in which objects are the open subsets of X, and the arrows are given by inclusions. We define a Grothendieck topology on X_{op} by associating with each open subset $U \subseteq X$ the set of open coverings of U. Note that if $U_1 \subseteq U$ and $U_1 \subseteq U$ are arrows in X_{op}, the fibred product $U_1 \times_U U_2$ is the intersection $U_1 \cap U_2$.
In particular, if X is a scheme, the Zariski topology on X defines the *small Zariski site on* X.

Example A.23. Let (Top) be the category of topological spaces. We define the *global étale topology for topological spaces* as follows. If X is a topological space, then a covering of X is a jointly surjective collection of local homeomorphisms $X_i \to X$.

Example A.24. Let X be a scheme. Consider the full subcategory $X_{\text{ét}}$ of (Sch/X), consisting of morphisms $U \to X$ locally of finite presentation, that are étale. If $U \to X$ and $V \to X$ are objects of $X_{\text{ét}}$, then an arrow $U \to V$ over X is necessarily étale. The *small étale site of* X on $X_{\text{ét}}$ is defined as follows. A covering of $U \to X$ is a jointly surjective collection of morphisms $U_i \to U$.

Let S be a scheme. We now give examples of Grothendieck topologies on the comma category (Sch/S). If $S = \operatorname{Spec}\mathbb{Z}$, this is just (Sch).

Example A.25. The *global Zariski topology* is defined as follows. A covering $\{U_i \to U\}$ is a collection of open embeddings covering U. Recall that an open embedding is a morphism $V \to U$ that gives an isomorphism of V with an open subscheme of U, and not simply an embedding of an open subscheme.

Example A.26. The *global étale topology* is defined as follows. A covering $\{U_i \to U\}$ is a jointly surjective collection of étale maps locally of finite presentation.

EXAMPLE A.27. The *global fppf topology* is defined as follows. A covering $\{U_i \to U\}$ is a jointly surjective collection of flat maps locally of finite presentation. Here fppf stands for *fidèltment plat et de présentation finie.*

DEFINITION/PROPOSITION A.28. An *fpqc morphism of schemes* is a faithfully flat morphism $f : X \to Y$ that satisfies the following equivalent conditions.

(i) Every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.
(ii) There exists a covering $\{V_i\}$ of Y by open affine subschemes, such that each V_i is the image of a quasi-compact open subset of X.
(iii) Given a point $x \in X$, there exists an open neighbourhood U of x in X, such that the image $f(U)$ is open in Y, and the restriction $U \to f(U)$ of f is quasi-compact.
(iv) Given a point $x \in X$, there exists a quasi-compact open neighbourhood U of x in X, such that the image $f(U)$ is open and affine in Y.

PROOF. See [Vis08, Proposition 2.33]. □

Here fpqc stands for *fidèltment plat et quasi-compact.*

EXAMPLE A.29. The *fpqc topology* is defined as follows. A covering $\{U_i \to U\}$ is a collection of morphisms such that the induced map $\coprod U_i \to U$ is fpqc.

REMARK A.30. The fpqc topology is finer than the fppf topology, which is finer than the étale topology, which is in turn finer than the Zariski topology. Indeed, an open immersion is étale, an étale map is flat, and an fppf covering $\{U_i \to U\}$ induces an fpqc map $\coprod U_i \to U$. See also [Vis08, §2].

DEFINITION A.31. Let $\mathcal{C}$ be a category. A *presheaf* on $\mathcal{C}$ is a contravariant functor

$$F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}\,.$$

from the category $\mathcal{C}$ to the category of sets.

DEFINITION A.32. Let $\mathcal{C}$ be a site and let $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor.

(i) F is *separated* if, given a covering $\{U_i \to U\}$ and two sections a and b in $F(U)$ whose pullbacks to each $F(U_i)$ coincide, it follows $a = b$.
(ii) F is a *sheaf* if the following condition is satisfied. Suppose that we are given a covering $\{U_i \to U\}$ in $\mathcal{C}$, and a set of elements $a_i \in F(U_i)$. Denote by $\mathrm{pr}_1 : U_i \times_U U_j \to U_i$ and $\mathrm{pr}_2 : U_i \times_U U_j \to U_j$ the first and the second projection respectively, and assume that $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j$ in $F(U_i \times_U U_j)$ for all i and j. Then there is a unique section $a \in F(U)$ whose pullback to $F(U_i)$ is a_i for all i.

If F and G are sheaves on a site $\mathcal{C}$, a *morphism of sheaves* $F \to G$ is simply a natural transformation of functors.

REMARK A.33. Let S be a scheme and let $F : (\mathrm{Sch}\,/S)^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf. If F is a sheaf in the fpqc topology, then it is clear by Remark A.30 that F is also a sheaf in the fppf topology, in the étale topology and in the Zariski topology.

THEOREM A.34 (Grothendieck). *Let S be a scheme. A representable functor on* $(\mathrm{Sch}\,/S)$ *is a sheaf in the fpqc topology.*

PROOF. See [Vis08, Theorem 2.55]. □

REMARK A.35. In particular, every representable functor on $(\mathrm{Sch}\,/S)$ is a sheaf also in the étale and fppf topology by Remark A.33.

Definition A.36. Let $\alpha : F \to G$ and $\beta : G' \to G$ be morphisms of presheaves on a category $\mathcal{C}$. The *fibre product of F and G' along α and β* is the presheaf $F \times_G G' : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ such that

$$F \times_G G'(C) = F(C) \times_{G(C)} G'(C) = \{(a,b) \in F(C) \times G'(C) \mid \alpha_C(a) = \beta_C(b)\}\,.$$

Definition A.37. A morphism $F \to G$ of (pre)sheaves over $\mathrm{Sch}_{\text{ét}}$ is *representable by schemes* if for every morphism $T \to G$ from a scheme T (i.e. from the functor of points of T), the fibre product $F \times_G T$ is a scheme (i.e. is a representable functor).

Definition A.38. If $\mathcal{P}$ is a property of morphism of schemes stable under base change, a morphism $F \to G$ of sheaves representable by schemes *has property* $\mathcal{P}$ if for every morphism $T \to G$ from a scheme T, the morphism of schemes $F \times_G T \to T$ has property $\mathcal{P}$.

Example A.39. The properties of schemes of being surjective or étale are stable under base change.

Definition A.40. An *algebraic space* is a sheaf $F : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ in the étale topology such that there exist a scheme U and a morphism $U \to F$ representable by schemes which is surjective and étale. The morphism $U \to F$ is called an *étale presentation*.

Example A.41. Every scheme is obviously an algebraic space.

A.3. Fibred categories

Notation A.42. Let $\mathrm{p}_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$ be a functor between two categories. We will say that $\mathcal{F}$ is a category *over* $\mathcal{C}$. We write $\xi \mapsto U$ if $\mathrm{p}_{\mathcal{F}}(\xi) = U$ and by a commutative diagram

$$\begin{tikzcd} \xi \arrow[r, "\alpha"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "f"'] & V \end{tikzcd}$$

we mean $\mathrm{p}_{\mathcal{F}}(\alpha) = f$.

Definition A.43. Let $\mathcal{F}$ be a category over $\mathcal{C}$. An arrow $\phi : \xi \to \eta$ of $\mathcal{F}$ is *cartesian* if for any arrow $\psi : \zeta \to \eta$ in $\mathcal{F}$ and any arrow $h : \mathrm{p}_{\mathcal{F}}(\zeta) \to \mathrm{p}_{\mathcal{F}}(\xi)$ in $\mathcal{C}$ with $\mathrm{p}_{\mathcal{F}}(\phi) \circ h = \mathrm{p}_{\mathcal{F}}(\psi)$, there exists a unique arrow $\theta : \zeta \to \xi$ with $\mathrm{p}_{\mathcal{F}}(\theta) = h$ and $\phi \circ \theta = \psi$.
If $\xi \to \eta$ is a cartesian arrow of $\mathcal{F}$ mapping to an arrow $U \to V$, we also say that ξ is a *pullback of η to U*.

Remark A.44. The pullback is not unique. However, it is unique up to a unique isomorphism, in the sense of [Vis08, Remark 3.3].

Definition A.45. Let $\mathcal{C}$ be a category. A *fibred category over* $\mathcal{C}$ is a category $\mathcal{F}$ over $\mathcal{C}$, such that given an arrow $f : U \to V$ in $\mathcal{C}$ and an object η of $\mathcal{F}$ mapping to V, there is a cartesian arrow $\phi : \xi \to \eta$ with $\mathrm{p}_{\mathcal{F}}(\phi) = f$.

Definition A.46. If $\mathcal{F}$ and $\mathcal{G}$ are fibred categories over $\mathcal{C}$, then a *morphism of fibred categories* $F : \mathcal{F} \to \mathcal{G}$ is a functor such that

- F is *base-preserving*, i.e. $\mathrm{p}_{\mathcal{G}} \circ F = \mathrm{p}_{\mathcal{F}}$;
- F sends cartesian arrows to cartesian arrows.

Definition A.47. Let $\mathcal{F}$ be a fibred category over $\mathcal{C}$. Given an object U of $\mathcal{C}$ the *fibre* $\mathcal{F}(U)$ of $\mathcal{F}$ over U is the subcategory of $\mathcal{F}$ whose objects are the objects ξ of $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}(\xi) = U$, and whose arrows are arrows ϕ in $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}(\phi) = \mathrm{id}_U$.

Definition A.48. A *cleavage* of a fibred category $\mathcal{F}$ over $\mathcal{C}$ is a class K of cartesian arrows in $\mathcal{F}$ such that for each arrow $f : U \to V$ in $\mathcal{C}$ and each object $\eta \in \mathcal{F}(V)$ there exists a unique arrow in K with target η mapping to f in $\mathcal{C}$. In other words, if we choose a cleavage, pullbacks are uniquely determined.

By the axiom of choice, every fibred category has a cleavage. Given a fibred category $\mathcal{F} \to \mathcal{C}$ with a cleavage K, we can associate to each object U of $\mathcal{C}$ a category $\mathcal{F}(U)$ and to each arrow $f : U \to V$ a functor $f^* : \mathcal{F}(V) \to \mathcal{F}(U)$ which sends each object η of $\mathcal{F}(V)$ to $f^*\eta = \xi$, if $\phi : \xi \to \eta$ is the unique arrow in K with target η mapping to f.

Remark A.49. Suppose that $\mathcal{F}$ is a fibred category over $\mathcal{C}$. Sending an object U of $\mathcal{C}$ to the category $\mathcal{F}(U)$ only defines a *pseudo-functor* (or *lax* 2*-functor* [Vis08, Definition 3.10]. The reason is that pullbacks are only unique up to a unique isomorphism by Remark A.44. See [Vis08, Proposition 3.11].

Definition A.50. Let $\mathcal{C}$ be a category. A *category fibred in groupoids over* $\mathcal{C}$ is a category $\mathcal{F}$ fibred over $\mathcal{C}$, such that the category $\mathcal{F}(U)$ is a groupoid for any object U of $\mathcal{C}$.

Proposition A.51. *Let $\mathcal{F}$ be a category over $\mathcal{C}$. Then $\mathcal{F}$ is fibred in groupoids over $\mathcal{C}$ if and only if the followings hold.*

(i) *Every arrow in $\mathcal{F}$ is cartesian.*
(ii) *Given an object η of $\mathcal{F}$ and an arrow $f : U \to \mathrm{p}_{\mathcal{F}}(\eta)$ of $\mathcal{C}$, there exists an arrow $\phi : \xi \to \eta$ of $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}(\phi) = f$.*

Proof. See [Vis08, Proposition 3.22]. □

Definition A.52. Let $\mathcal{C}$ be a category. A *category fibred in sets over* $\mathcal{C}$ is a category $\mathcal{F}$ fibred over $\mathcal{C}$, such that the category $\mathcal{F}(U)$ is a set for any object U of $\mathcal{C}$.

Proposition A.53. *Let $\mathcal{C}$ be a category, and let $\mathcal{F}$ be a category over $\mathcal{C}$. Then $\mathcal{F}$ is fibred in sets if and only if for any object η of $\mathcal{F}$ and any arrow $f : U \to \mathrm{p}_{\mathcal{F}}(\eta)$ of $\mathcal{C}$, there is a unique arrow $\phi : \xi \to \eta$ of $\mathcal{F}$ with $\mathrm{p}_{\mathcal{F}}(\phi) = f$.*

Proof. See [Vis08, Proposition 3.25]. □

Remark A.54. If $\mathcal{F}$ is a category fibred in sets over $\mathcal{C}$, the pullbacks are uniquely determined. It follows that there is a well-defined functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ by sending U to the set $\mathcal{F}(U)$, see [Vis08, §3.4] for details. Moreover, this is an equivalence of the category of categories fibred in sets over $\mathcal{C}$ and the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, see [Vis08, Proposition 3.26].

Example A.55. Given an object U of a category $\mathcal{C}$, the representable functor h_U is associated to the comma category $(\mathcal{C}/U)$, which is a category fibred in sets over $\mathcal{C}$ through the forgetful functor $(\mathcal{C}/U) \to \mathcal{C}$ that forgets the arrow.

Remark A.56. Let $\mathcal{C}$ be a category. By Yoneda's lemma A.6 we have that $\mathcal{C}$ is embedded in the category $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, and by Remark A.54 we have that $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is embedded in the 2-category of fibred categories over $\mathcal{C}$. Combining these two embeddings, we have that $\mathcal{C}$ is embedded in the 2-category of fibred categories over $\mathcal{C}$. This embedding sends an object U of $\mathcal{C}$ into the fibred category (in sets) $(\mathcal{C}/U) \to \mathcal{C}$ of Example A.55.

Remark A.57. Suppose now that $\mathcal{F}$ is a category fibred over $\mathcal{C}$ and that U is an object of $\mathcal{C}$. Let $F : (\mathcal{C}/U) \to \mathcal{F}$ be a morphism of fibred categories over $\mathcal{C}$. We can associate an object $F(\mathrm{id}_U) \in \mathcal{F}(U)$. Moreover, to each base-preserving natural transformation $\alpha : F \to G$ of functors $F, G : (\mathcal{C}/U) \to \mathcal{F}$ we associate the arrow $\alpha_{\mathrm{id}_U} : F(\mathrm{id}_U) \to G(\mathrm{id}_U)$. This defines a functor

$$\mathrm{Hom}_{\mathcal{C}}((\mathcal{C}/U), \mathcal{F}) \to \mathcal{F}(U).$$

LEMMA A.58 (2-Yoneda's lemma). *Let $\mathcal{F}$ be a category fibred over $\mathcal{C}$ and let $U \in \mathcal{C}$ be an object. The functor*

$$\mathrm{Hom}_{\mathcal{C}}((\mathcal{C}/U), \mathcal{F}) \to \mathcal{F}(U)$$

defined in Remark A.57 defines an equivalence of categories.

PROOF. It is sufficient to define the inverse functor. See [Vis08, §3.6.2]. □

DEFINITION A.59. Let $\mathcal{C}$ be a category. Let $\mathcal{P}$ be a class of arrows of $\mathcal{C}$. We say that $\mathcal{P}$ is *stable* if the following conditions hold:

(i) If $f : C \to D$ is in $\mathcal{P}$, and $\phi : C' \simeq C$, $\psi : D \simeq D'$ are isomorphisms, then the composition
$$\psi \circ f \circ \phi : C' \to D'$$
is in $\mathcal{P}$.

(ii) Given $C \to D$ an arrow in $\mathcal{P}$ and $C' \to D$ any other arrow, then a fibred product $C \times_D C'$ exists and the projection $C \times_D C' \to C'$ is in $\mathcal{P}$.

DEFINITION A.60. Let $\mathcal{C}$ be a site, where $\mathcal{C}$ is a category with fibred products. Let $\mathcal{P}$ be a stable class of arrows. We say that $\mathcal{P}$ is *local* if the following condition holds. Given a covering $\{U_i \to U\}$ and an arrow $C \to U$ such that the projections $U_i \times_U C \to U_i$ are in $\mathcal{P}$ for all i, then also $C \to U$ is in $\mathcal{P}$.

A.4. Stacks

DEFINITION A.61. Let $\mathcal{F} \to \mathcal{C}$ be a fibred category and let K be a fixed cleavage. Let S be an object of $\mathcal{C}$ and let ξ, η be two objects of $\mathcal{F}(S)$. We can define a functor

$$\underline{\mathrm{Hom}}_S(\xi, \eta) : (\mathcal{C}/S)^{\mathrm{op}} \to \mathrm{Set}$$

by sending each object $u : U \to S$ to the set $\mathrm{Hom}_U(u^*\xi, u^*\eta)$ of arrows in the category $\mathcal{F}(U)$. An arrow $f : U_1 \to U_2$ from $u_1 : U_1 \to S$ to $u_2 : U_2 \to S$ is sent to

$$f^* : \mathrm{Hom}_{U_2}(u_2^*\xi, u_2^*\eta) \to \mathrm{Hom}_{U_1}(u_1^*\xi, u_1^*\eta).$$

We also define

$$\underline{\mathrm{Aut}}_S(\xi) = \underline{\mathrm{Hom}}_S(\xi, \xi)$$

as the *automorphism functor of* ξ.

REMARK A.62. The functor $\underline{\mathrm{Hom}}_S(\xi, \eta)$ of Definition A.61 is independent on the choice of the cleavage K, i.e. different cleavage gives canonically isomorphic functors, see [Vis08, §3.7].

REMARK A.63. The functor $\underline{\mathrm{Hom}}_S(\xi, \eta)$ of Definition A.61 in [Stacks] is denoted by $\mathrm{Mor}(\xi, \eta)$, see [Stacks, Definition 02ZB]. We may also define

$$\underline{\mathrm{Isom}}_S(\xi, \eta) : (\mathcal{C}/S)^{\mathrm{op}} \to \mathrm{Set}$$

as the subfunctor of $\underline{\mathrm{Hom}}_S(\xi, \eta)$ which sends each object $u : U \to S$ to the set $\mathrm{Isom}_U(u^*\xi, u^*\eta) \subseteq \mathrm{Hom}_U(u^*\xi, u^*\eta)$ of isomorphisms between $u^*\xi$ and $u^*\eta$ in the category $\mathcal{F}(U)$. It is clear that if $\mathcal{F}$ is fibred in groupoids over $\mathcal{C}$, then there is no difference between the functors $\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Isom}}$, because each fibre $\mathcal{F}(S)$ is a groupoid.

EXAMPLE A.64. Let X be a scheme over a base S. Then we have the *automorphism functor of* X defined as

$$\underline{\mathrm{Aut}}_X = \underline{\mathrm{Aut}}_S(X \to S) : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$$

which maps an object $(T \to S)$ to the set of automorphisms over T of the pullback $X \times_S T$. More generally, if X and Y are two schemes over a base S, we define the functor

$$\underline{\mathrm{Hom}}_S(X, Y) : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$$

which maps an object $(T \to S)$ to the set morphisms of schemes $\mathrm{Hom}(X \times_S T, Y \times_S T)$.

Definition A.65. Let $\mathcal{C}$ be a site and let $\mathcal{U} = \{U_i \to U\}$ be a covering in $\mathcal{C}$. An *object with descent data* $(\{\xi_i\}, \{\phi_{ij}\})$ on $\mathcal{U}$, is a collection of objects $\xi_i \in \mathcal{F}(U_i)$ together with isomorphisms $\phi_{ij} : \mathrm{pr}_2^*\xi_j \simeq \mathrm{pr}_1^*\xi_i$ in $\mathcal{F}(U_i \times_U U_j)$, such that the following cocycle condition is satisfied: for any triple of indices i, j and k, we have the equality

$$\mathrm{pr}_{13}^*\phi_{ik} = \mathrm{pr}_{12}^*\phi_{ij} \circ \mathrm{pr}_{23}^*\phi_{jk} : \mathrm{pr}_3^*\xi_k \to \mathrm{pr}_1^*\xi_i$$

where pr_{ab} and pr_a are projections on the a^{th} and b^{th} factor, or the a^{th} factor respectively.

Definition A.66. An object with descent data $(\{\xi_i\}, \{\phi_{ij}\})$ on a covering $\mathcal{U} = \{\sigma_i : U_i \to U\}$ is *effective* if there exists an object $\xi \in \mathcal{F}(U)$, together with cartesian arrows $\xi_i \to \xi$ over $\sigma_i : U_i \to U$, such that the diagram

$$\begin{tikzcd}
\mathrm{pr}_2^*\xi_j \arrow[rr, "\phi_{ij}"] \arrow[d] & & \mathrm{pr}_1^*\xi_i \arrow[d] \\
\xi_j \arrow[dr] & & \xi_i \arrow[dl] \\
& \xi &
\end{tikzcd}$$

commutes for all i and j.

Definition A.67. Let $\mathcal{C}$ be a category and let $\mathcal{F}$ be a fibred category over a site $\mathcal{C}$. We say that $\mathcal{F}$ is a *stack* if the following conditions are satisfied.

(i) (morphisms glue) For any object S of $\mathcal{C}$ and any two objects ξ and η in $\mathcal{F}(S)$, the functor $\underline{\mathrm{Hom}}_S(\xi, \eta)$ is a sheaf in the comma topology.
(ii) (objects glue) All objects with descent data in $\mathcal{F}$ are effective.

Proposition A.68. *Let $\mathcal{C}$ be a site and let $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor. We can also consider it as a category fibred in set $F \to \mathcal{C}$. Then F is a stack if and only if it is a sheaf.*

Proof. See [Vis08, Proposition 4.9]. □

Example A.69. In the category of schemes, every representable functor in is a stack in the étale (even fppf) topology by Theorem A.34 and Proposition A.68. This is how we view a scheme as a stack.

Remark A.70. Every algebraic space is a stack, since every algebraic space is in particular a sheaf.

Definition A.71. A *morphism of stacks* is a morphism of fibred categories, between two stacks.

Definition A.72. Let $\mathcal{F}$ be a stack over the category of schemes Sch with the étale topology. Let k be a field. A morphism $x : \mathrm{Spec}\, k \to \mathcal{F}$ of stacks is a *field-valued point* of the stack $\mathcal{F}$. This corresponds to an object, which we also denote by x, of $\mathcal{F}(\mathrm{Spec}\, k)$ by 2-Yoneda's lemma (A.58). The *automorphism group* or *stabilizer* of x is defined as the sheaf

$$G_x = \underline{\mathrm{Aut}}_{\mathrm{Spec}\, k}(x) = \underline{\mathrm{Hom}}_{\mathrm{Spec}\, k}(x, x).$$

Definition A.73. Let $G \to S$ be a group scheme with multiplication $m : G \times_S G$ and identity $e : S \to G$. An *action* of G on a scheme $p : X \to S$ over S is a morphism $a : G \times_S X \to X$ of schemes over S such that the following diagrams commute:

$$\begin{tikzcd} G \times_S G \times_S X \arrow[r, "\mathrm{id}\times a"] \arrow[d, "m\times\mathrm{id}"'] & G \times_S X \arrow[d, "a"] \\ G \times_S X \arrow[r, "a"'] & G \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "(e\circ p,\mathrm{id})"] \arrow[dr, "\mathrm{id}"'] & G \times_S X \arrow[d, "a"] \\ & X. \end{tikzcd}$$

If X, Y are S-schemes with actions a_X, a_Y of G, a morphism $f : X \to Y$ of S-schemes is *G-equivariant* if $a_Y \circ (\mathrm{id} \times f) = f \circ a_X$, and is *$G$-invariant* if is G-equivariant and Y has the trivial G action.

Definition A.74. Let $G \to S$ be a smooth affine group scheme over a scheme S. A *principal G-bundle over an S-scheme T* is a morphism $P \to T$ of schemes with an action $\sigma : G \times_S P \to P$ of G on P such that $P \to T$ is a G-invariant smooth morphism and

$$\begin{array}{rccc} (\sigma, \mathrm{pr}_2) : & G \times_S P & \longrightarrow & P \times_T P \\ & (g, p) & \longmapsto & (g \cdot p, p) \end{array}$$

is an isomorphism, where $g \cdot p$ denotes the action of g on p, i.e. $g \cdot p = \sigma(g, p)$.

Definition A.75. Let $G \to S$ be a smooth affine group scheme over a scheme S. The *classifying stack* $\mathrm{B}G$ is the category over Sch/S whose objects are principal G-bundles $P \to T$ and a morphism $(P \to T) \to (P' \to T')$ is the data of a G-equivariant morphism $P \to P'$ such that the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd}$$

is cartesian.

Definition A.76. Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. We define the *quotient stack* $[U/G]$ as the category over Sch/S whose objects over an S-scheme T are diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T \end{tikzcd}$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of S-schemes. A morphism between two objects $(T \leftarrow P \to U) \to (T \leftarrow P \to U)$ is given by a morphism $T \to T'$ of schemes and a G-equivariant morphism $P \to P'$ of schemes such that the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[rr, bend left] & P' \arrow[r] \arrow[d] & U \\ T \arrow[r] & T' \end{tikzcd}$$

is commutative, and such that the square on the left is cartesian.

Remark A.77. Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. Consider the trivial principal G-bundle given by

$$\begin{tikzcd} U \times G \arrow[r, "\sigma"] \arrow[d, "\mathrm{pr}_1"'] & U \\ U \end{tikzcd}$$

where σ is the multiplication given by the group action. By 2-Yoneda's lemma (A.58) we have a corresponding morphism $U \to [U/G]$ of stacks.

PROPOSITION A.78. *Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. Then the quotient stack $[U/G]$ is a stack over the étale topology. In particular, the classifying stack* $\mathrm{B}G = [S/G]$ *is a stack over the étale topology.*

PROOF. See [Alp24, Proposition 2.5.13]. $\square$

A.5. Algebraic stacks and Deligne-Mumford stacks

We now work in the category of schemes.

DEFINITION A.79. Let $\mathcal{X}, \mathcal{Y}$ be two fibred categories over Sch. A morphism $\mathcal{X} \to \mathcal{Y}$ of fibred categories is *representable* if for every morphism $T \to \mathcal{Y}$ from a scheme T, the fibre product $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic space.

PROPOSITION A.80. *Let $\mathcal{X}$ be a category fibred in groupoids over* Sch. *The following are equivalent:*

(1) *the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable (by algebraic spaces);*
(2) *for every scheme T and any $x, y \in \mathrm{ob}(\mathcal{X}(T))$, the sheaf* $\underline{\mathrm{Isom}}_T(x, y)$ *is an algebraic space.*

PROOF. Let $x, y \in \mathrm{ob}(\mathcal{X}(T))$, and denote also by x, y the two morphisms of fibred categories over Sch

$$x : \mathrm{Sch}/T \to \mathcal{X} \quad \text{and} \quad y : \mathrm{Sch}/T \to \mathcal{X}$$

which corresponds to these objects by 2-Yoneda's lemma A.58. The 2-fibre product

$$\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}, (x,y)} \mathrm{Sch}/T$$

is a category fibred in setoids over Sch/T which corresponds to the presheaf

$$\underline{\mathrm{Isom}}_T(x, y)$$

by [Stacks, Lemma 04SI]. $\square$

DEFINITION A.81. Let $\mathcal{P}$ be a property of morphisms of schemes. We say that $\mathcal{P}$ is *étale* (resp. *smooth*) *local on the source* if for every étale (resp. smooth) surjection $X' \to X$ of schemes, a morphism $X \to Y$ satisfies $\mathcal{P}$ if and only if $X' \to X \to Y$ does. We say that $\mathcal{P}$ is *étale* (resp. *smooth*) *local on the target* if for every étale (resp. smooth) surjection $Y' \to Y$ of schemes, a morphism $X \to Y$ satisfies $\mathcal{P}$ if and only if $X \times_Y Y' \to Y$ does. We say that $\mathcal{P}$ is *étale* (resp. *smooth*) *local* if for every étale (resp. smooth) surjection of schemes $X \to Y$, then X has $\mathcal{P}$ if and only if Y has $\mathcal{P}$.

DEFINITION A.82. Let $\mathcal{P}$ be a property of morphisms of schemes stable under base change and étale local on the source. We say that a representable morphism $\mathcal{X} \to \mathcal{Y}$ of fibred categories over Sch has property $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme T and for every étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} T$ by a scheme U, the composition

$$U \to \mathcal{X} \times_{\mathcal{Y}} T \to T$$

has property $\mathcal{P}$.

EXAMPLE A.83. The properties of being surjective, étale or smooth are stable under base change and étale local on the source.

DEFINITION A.84. Let $\mathcal{X}$ be a stack over the étale topology. We say that $\mathcal{X}$ is an *algebraic stack* if there exists a scheme U and a representable morphism $U \to \mathcal{X}$ which is surjective and smooth. The morphism $U \to \mathcal{X}$ is called a *smooth presentation.*

PROPOSITION A.85. *Let G be an affine smooth group scheme acting on an a scheme U. The quotient stack $[U/G]$ is an algebraic stack such that $U \to [U/G]$ is a smooth presentation.*

PROOF. See [Alp24, Theorem 3.1.10]. □

DEFINITION A.86. Let $\mathcal{X}$ be a stack over the étale topology. We say that $\mathcal{X}$ is a *Deligne-Mumford stack* if there exist a scheme U and a representable morphism $U \to \mathcal{X}$ which is surjective and étale. The morphism $U \to \mathcal{X}$ is called an *étale presentation*.

REMARK A.87. Every Deligne-Mumford stack is in particular an algebraic stack.

REMARK A.88. Every algebraic space is a Deligne-Mumford stack.

PROPOSITION A.89. *The following hold:*

(1) *the diagonal of an algebraic space is representable by schemes;*
(2) *the diagonal of an algebraic stack is representable.*

PROOF. See [Alp24, Theorem 3.2.1]. □

REMARK A.90. Fibre products exist for algebraic spaces, Deligne-Mumford stacks, and algebraic stacks, see [Alp24, Exercise 3.1.9].

Remark A.90 allows us to give the following definition.

DEFINITION A.91. Let $\mathcal{P}$ be a property of morphism of schemes. If $\mathcal{P}$ is étale local (resp. smooth local) on the source and target and is stable under composition and base change, a morphism $\mathcal{X} \to \mathcal{Y}$ of Deligne-Mumford stacks (resp. algebraic stacks) *has property* $\mathcal{P}$ if for all étale (resp. smooth) presentations $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ yielding a diagram

$$\begin{array}{ccccc} U & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} V & \longrightarrow & V \\ & & \downarrow & & \downarrow \\ & & \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

the composition $U \to V$ has $\mathcal{P}$. It is enough to check this for specific presentations $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$.

REMARK A.92. If $G \to S$ is a smooth affine group scheme acting on a S-scheme U, then $[U/G] \to S$ is flat (resp. smooth, surjective, locally of finite presentation, locally of finite type) if and only if $U \to S$ is. Indeed we can use the diagram

$$\begin{array}{ccccc} U & \longrightarrow & [U/G] & \longrightarrow & S \\ & & \downarrow & & \downarrow \\ & & [U/G] & \longrightarrow & S \end{array}$$

where $U \to [U/G]$ is the smooth presentation as in Proposition A.85.

DEFINITION A.93. Let $\mathcal{P}$ be a property of schemes which is étale (resp. smooth) local. We say that a Deligne-Mumford stack (resp. algebraic stack) $\mathcal{X}$ *has property* $\mathcal{P}$ if for every étale (resp. smooth) presentation $U \to \mathcal{X}$, the scheme U has $\mathcal{P}$. It is enough to check this for a specific presentation $U \to \mathcal{X}$.

We want now to give a characterization of Deligne-Mumford stacks. To do this, we first recall the definition of an unramified morphism of schemes and the formal criterion for smoothness, unramifiedness and étaleness.

Definition A.94. A *first-order thickening of affine schemes* is a closed immersion of schemes

$$\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A.$$

where A is a ring and $I \subset A$ is an ideal of A such that $I^2 = 0$.

Definition A.95. A ring map $R \to A$ is *unramified* if it is of finite type and $\Omega_{A/R} = 0$.

Definition A.96. A morphism of schemes $f : X \to Y$ is *unramified at $x \in X$* if there exists an affine open neighbourhood $\operatorname{Spec} A = U \subseteq X$ of x and an affine open $\operatorname{Spec} R = V \subseteq Y$ with $f(U) \subseteq V$ such that the induced map $R \to A$ is unramified. A morphism of schemes $f : X \to Y$ is *unramified* if it is unramified at every point of X.

Definition A.97. Let $f : X \to Y$ be a morphism of schemes. Consider a commutative diagram

$$\begin{tikzcd} Z_0 \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ Z \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where $i : Z_0 \hookrightarrow Z$ is a first-order thickening of affine schemes over Y. We say that

(1) f is *formally smooth* if there exists a dotted arrow making the above diagram commute;
(2) f is *formally unramified* if there exists at most one dotted arrow making the above diagram commute;
(3) f is *formally étale* if there exists exactly one dotted arrow making the above diagram commute.

Lemma A.98. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*
(i) *the morphism f is unramified;*
(ii) *the morphism f is locally of finite type and formally unramified.*

Proof. See [Stacks, Lemma 02HE]. □

Lemma A.99. *Let $k \subseteq K$ be a finite field extension. Then $k \subseteq K$ is separable if and only if, for every (some) algebraically closed extension Ω of k, we have an isomorphism*

$$\Omega \otimes_k K \simeq \Omega \times \cdots \times \Omega$$

as Ω-algebras.

Proof. Suppose first that $k \subseteq K$ is separable. By the primitive element theorem, there exists $\alpha \in K$ such that $K = k(\alpha)$. Let $p(t) \in k[t]$ be the minimal polynomial of α over k, so that $K \simeq k[t]/p(t)$. By hypothesis we know that $p(t)$ has distinct roots $\alpha_1, \dots, \alpha_n$ in an algebraic closure of k. From $K \simeq k[t]/p(t)$ we have isomorphisms of Ω-algebras:

$$\begin{aligned} \Omega \otimes_k K &\simeq \Omega \otimes_k k[t]/p(t) \simeq \Omega[t]/p(t) \\ &= \Omega[t]/((t-\alpha_1) \cdot \ldots \cdot (t-\alpha_n)) \simeq \prod_{i=1}^{n} \Omega[t]/(t-\alpha_i) \\ &\simeq \Omega \times \ldots \times \Omega. \end{aligned}$$

Conversely, suppose that there exists an isomorphism $\phi : \Omega \otimes_k K \to \Omega^{\oplus n}$ of Ω-algebras, where $n = [K : k]$. For $i = 1, \dots, n$, consider the ith projection $\mathrm{pr}_i : \Omega^{\oplus n} \to \Omega$ and the homomorphism $\sigma_i : K \to \Omega$ defined by $\sigma_i(a) = (\mathrm{pr}_i \circ \phi)(1 \otimes a)$ for every $a \in K$. Observe that $\sigma_1, \dots, \sigma_n$ are k-linear, being composition of k-linear homomorphism. Moreover, they are immersions of K into Ω, as they are composition of an isomorphism with a projection. We must show that they are distinct. If $\sigma_i = \sigma_j$, then the Ω-linear maps $\mathrm{pr}_i \circ \phi$, $\mathrm{pr}_j \circ \phi$ coincide on the subset $\{1 \otimes a \mid a \in K\}$, which generates $\Omega \otimes_k K$ as Ω-vector space; then $\mathrm{pr}_i \circ \phi = \mathrm{pr}_j \circ \phi$, hence $i = j$, because ϕ is an isomorphism. □

PROPOSITION A.100. *Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Then the following are equivalent:*

(i) *f is formally unramified;*
(ii) *for every point $y \in Y$ the fibre $X_y = X \times_Y \operatorname{Spec} k(y)$ is a disjoint union of spectra of finite separable field extensions of $k(y)$;*
(iii) *for every geometric point $\operatorname{Spec} \Omega \to Y$ the fibre product $X \times_Y \operatorname{Spec} \Omega$ is isomorphic to a disjoint union of copies of $\operatorname{Spec} \Omega$;*
(iv) $\Omega_{X/Y} = 0$.

PROOF. The equivalence between (i) and (iv) follows directly from the definitions A.95 and A.96. The equivalence between (i) and (ii) follows by [Stacks, Lemma 02G7] and does not require f to be of finite type. Finally the equivalence between (ii) and (iii) follows by Lemma A.99. □

THEOREM A.101. *Let $\mathcal{X}$ be an algebraic stack over the étale topology. The following are equivalent:*

(i) *$\mathcal{X}$ is a Deligne-Mumford stack;*
(ii) *the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified;*
(iii) *every field valued point of $\mathcal{X}$ has discrete and reduced stabilizer group.*

PROOF. First, observe that the diagonal of an algebraic stack is locally of finite type ([Stacks, Lemma 04XS]). Then the equivalence between (ii) and (iii) follows by Lemma A.98, Proposition A.100 and by the fact that the stabilizer of a point $x : \operatorname{Spec} k \to \mathcal{X}$ (Definition A.72) is identified with the fibre product

$$\begin{tikzcd} G_x = \underline{\mathrm{Aut}}_{\operatorname{Spec} k}(x) \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "{(x,x)}"] \\ \mathcal{X} \arrow[r, "\Delta"'] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

by [Alp24, Exercise 2.4.39]. The equivalence between (i) and (ii) is [LM00, Théorème 8.1]. □

DEFINITION A.102. Let $\mathcal{X}$ be an algebraic stack. A morphism $\pi : \mathcal{X} \to X$ from $\mathcal{X}$ to an algebraic space X is called a *coarse moduli space* for $\mathcal{X}$ if

(i) for every algebraically closed field k, the induced map
$$\mathcal{X}(k)/\sim \to X(k)$$
from the set of isomorphism classes of objects of $\mathcal{X}$ over k is bijective, and
(ii) π is universal for maps to algebraic spaces, i.e. every map $\mathcal{X} \to Y$ to an algebraic space Y factors uniquely as
$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \arrow[dr] & \\ X \arrow[r, dashed] & Y. \end{tikzcd}$$

THEOREM A.103 (Keel-Mori). *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi : \mathcal{X} \to X$ with $\mathcal{O}_X = \pi_* \mathcal{O}_{\mathcal{X}}$ such that*

(1) *X is separated and of finite type over S,*

(2) *π is a proper universal homeomorphism, and*
(3) *for every flat morphism $X' \to X$ of algebraic spaces, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.*

Proof. See [KM97, Corollary 1.3], [Stacks, Theorem 0DUT], or [Alp24, Theorem 4.4.6]. □

A.6. Artin's axioms of algebraicity

In this section we recall the necessary definitions to define Artin's axioms to prove the algebraicity of a stack. The main result is Theorem A.119. The main reference is [Stacks, Chapter 07SZ].

Let $p : \mathcal{X} \to \mathrm{Sch}$ be a category fibred in groupoids. Let k be a field of finite type over $\operatorname{Spec}\mathbb{Z}$, and let x_0 be an object of $\mathcal{X}(\operatorname{Spec} k)$. By [Stacks, Lemma 01TA] there exists an affine open $\operatorname{Spec}(\Lambda) \subset \operatorname{Spec}\mathbb{Z}$ such that the map $\mathbb{Z} \to k$ factorizes as $\mathbb{Z} \to \Lambda \to k$ and $\Lambda \to k$ is finite.

We define the category $\mathcal{A}_\Lambda$ whose objects are pairs (A, φ) where A is an artinian local Λ-algebra and $\varphi : A/\mathfrak{m}_A \to k$ is a Λ-algebra isomorphism. Morphisms $f : (B, \psi) \to (A, \varphi)$ in $\mathcal{A}_\Lambda$ are local Λ-algebra homomorphisms $f : B \to A$ such that $\varphi \circ \tilde{f} = \psi$, where $\tilde{f} : B/\mathfrak{m}_B \to A/\mathfrak{m}_A$ is the map induced by f. The category $\mathcal{A}_\Lambda$, up to canonical equivalence, does not depend on the choice of the affine open $\operatorname{Spec}\Lambda \subset \operatorname{Spec}\mathbb{Z}$, see [Stacks, Section 07T2].

Definition A.104. We define the category $\mathcal{F}_{\mathcal{X},k,x_0}$ as follows.

- Objects are morphisms $x_0 \to x$ of $\mathcal{X}$ where $p(x) = \operatorname{Spec} A$ with A an artinian local Λ-algebra and $p(x_0) = \operatorname{Spec} k \to p(x) = \operatorname{Spec} A$ corresponds to a ring map $A \to k$ which identifies k with the residue field of A.
- Morphisms $(x_0 \to x) \to (x_0 \to x')$ are commutative diagrams

$$\begin{tikzcd} x & & x' \arrow[ll] \\ & x_0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in $\mathcal{X}$.

Observe that we obtain a functor

$$\mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{A}_\Lambda \tag{19}$$

sending an object $(x_0 \to x)$ to A if $p(x) = \operatorname{Spec} A$ and sending a morphism $(x_0 \to x) \to (x_0 \to x')$ to the corresponding ring map $A \to A'$. In particular, $\mathcal{F}_{\mathcal{X},k,x_0}$ is cofibred in groupoids over $\mathcal{A}_\Lambda$ and the fibre groupoid $\mathcal{F}_{\mathcal{X},k,x_0}(k)$ over k is equivalent to a set with a single object and a single morphism, see [Stacks, Lemma 07T5].

Let $F : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over Sch. Let k be a field of finite type over $\mathbb{Z}$ and let x_0 be an object of $\mathcal{X}$ lying over $\operatorname{Spec} k$. Let $y_0 = F(x_0)$ which is an object of $\mathcal{Y}$ lying over $\operatorname{Spec} k$. Then F induces a functor

$$\alpha : \mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{F}_{\mathcal{X},k,y_0} \tag{20}$$

of cofibred categories over $\mathcal{A}_\Lambda$. An object $(x_0 \to x)$ of $\mathcal{F}_{\mathcal{X},k,x_0}(A)$ is sent to the object $(\alpha(x_0) \to \alpha(x))$ of $\mathcal{F}_{\mathcal{Y},k,y_0}(A)$.

We now define the Rim-Schlessinger condition of classical deformation theory for a fibred category in groupoids.

Definition A.105. Let $\mathcal{X}$ be a category fibred in groupoids over Sch. We say that $\mathcal{X}$ satisfies the *Rim-Schlessinger condition* or simply *(RS)* if for every pushout of schemes

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

such that X, X', Y, Y' are spectra of local artinian rings of finite type over $\operatorname{Spec}\mathbb{Z}$, and $X \to X'$ is a closed immersion, then the induced functor of fibre categories

$$\mathcal{X}(Y') \to \mathcal{X}(Y) \times_{\mathcal{X}(X)} \mathcal{X}(X')$$

is an equivalence of categories.

This definition is compatible with classical Rim-Schlessinger condition. Indeed, if $\mathcal{X} \to \mathrm{Sch}$ is a category fibred in groupoids satisfying (RS), then for any finite type field k over $\mathbb{Z}$ (i.e. $\operatorname{Spec} k \to \operatorname{Spec}\mathbb{Z}$ is of finite type) and any object x_0 of $\mathcal{X}$ over k, the functor $\mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{A}_\Lambda$ (Equation (19)) satisfies classical Rim-Schlessinger condition [Stacks, Definition 06J2]. See [Stacks, Lemma 07WU].

Let $\mathcal{X} \to \mathrm{Sch}$ be a category fibred in groupoids, let k be a field of finite type over $\operatorname{Spec}\mathbb{Z}$ and let $x_0 \in \mathcal{X}(\operatorname{Spec} k)$. Consider the category $\mathcal{F}_{\mathcal{X},k,x_0}$ (Definition A.104).

Definition A.106. We define the *tangent space* of $\mathcal{F}_{\mathcal{X},k,x_0}$ as

$$T\mathcal{F}_{\mathcal{X},k,x_0} = \left\{ \begin{array}{c} \text{isomorphism classes of morphisms} \\ x_0 \to x \text{ of } \mathcal{X} \text{ over } \operatorname{Spec} k \to \operatorname{Spec} k[\epsilon] \end{array} \right\}$$

where $k[\epsilon]$ are the dual numbers (i.e. $k[\epsilon] = k[t]/(t^2)$).

Using the natural morphism $k \to k[\epsilon]$ we define x_0' as the pullback of x_0 to $\operatorname{Spec} k[\epsilon]$. On the other hand, we also have a morphism $k[\epsilon] \to k$ and a map $x_0 \to x_0'$ over $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]$. In particular, we have a morphism of fibre categories $\mathcal{X}(\operatorname{Spec} k[\epsilon]) \to \mathcal{X}(\operatorname{Spec} k)$ and hence a map $\operatorname{Aut}_{\mathcal{X}(\operatorname{Spec} k[\epsilon])}(x_0') \to \operatorname{Aut}_{\mathcal{X}(\operatorname{Spec} k)}(x_0)$ from the set of automorphisms of x_0' in the fibre category $\mathcal{X}(\operatorname{Spec} k[\epsilon])$ to the set of automorphisms of x_0 in the fibre category $\mathcal{X}(\operatorname{Spec} k)$.

Definition A.107. With notations as above, we define

$$\operatorname{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) = \operatorname{Ker}(\operatorname{Aut}_{\mathcal{X}(\operatorname{Spec} k[\epsilon])}(x_0') \to \operatorname{Aut}_{\mathcal{X}(\operatorname{Spec} k)}(x_0)).$$

Remark A.108. If $\mathcal{X}$ satisfies (RS) (Definition A.105), the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\operatorname{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ are equipped with a natural structure of k-vector spaces, see [Stacks, Lemma 06IH], [Stacks, Lemma 06JX].

Definition A.109. Let $p : \mathcal{X} \to \mathrm{Sch}$ be a category fibred in groupoids. A *formal object* ξ of $\mathcal{X}$ is a triple $\xi = (R, \xi_n, f_n)$ where R is a noetherian complete local $\mathbb{Z}$-algebra such that $R/\mathfrak{m}_R$ is a field of finite type over $\mathbb{Z}$, $\xi_n \in \mathcal{X}(\operatorname{Spec}(R/\mathfrak{m}_R^n))$ for all $n \geq 1$, and $f_n : \xi_n \to \xi_{n+1}$ are morphisms of $\mathcal{X}$ over $\operatorname{Spec}(R/\mathfrak{m}_R^n) \to \operatorname{Spec}(R/\mathfrak{m}_R^{n+1})$ for all $n \geq 1$.
A *morphism of formal objects* $a : \xi = (R, \xi_n, f_n) \to \eta = (T, \eta_n, g_n)$ is the data of morphisms $a_n : \xi_n \to \eta_n$ for all $n \geq 1$ such that for every n the diagram

$$\begin{array}{ccc} \xi_n & \xrightarrow{f_n} & \xi_{n+1} \\ {\scriptstyle a_n}\downarrow & & \downarrow{\scriptstyle a_{n+1}} \\ \eta_n & \xrightarrow[g_n]{} & \eta_{n+1} \end{array}$$

is commutative. The *category of formal objects of* $\mathcal{X}$ is the category whose objects are formal objects of $\mathcal{X}$ and arrows are morphisms of formal objects.

Remark A.110. Suppose that $\xi = (R, \xi_n, f_n)$ is a formal object of a category fibred in groupoids $\mathcal{X} \to \mathrm{Sch}$. Set $k = R/\mathfrak{m}_R$ and $x_0 = \xi_1$. By definition of a formal object, the field $R/\mathfrak{m}_R$ is of finite type over $\mathbb{Z}$, so that it makes sense to define the category $\mathcal{F}_{\mathcal{X},k,x_0}$, which is a category cofibred in groupoids over $\mathcal{A}_\Lambda$. Then the formal object ξ defines a formal object ξ for $\mathcal{F}_{\mathcal{X},k,x_0}$ (see [Stacks, Definition 06H3], [Stacks, Remark 0CXH])).

Suppose now that $p : \mathcal{X} \to \mathrm{Sch}$ is a category fibred in groupoids and R is a noetherian complete local $\mathbb{Z}$-algebra such that the residue field $R/\mathfrak{m}_R$ is of finite type over $\mathbb{Z}$. Let $x \in \mathcal{X}(R)$. For all $n \geq 1$ we have a morphism of affine schemes $\alpha_n : \mathrm{Spec}(R/\mathfrak{m}_R^n) \to \mathrm{Spec}\, R$ so that we can define $\xi_n = x_{|_{\mathrm{Spec}(R/\mathfrak{m}_R^n)}}$, where for all $n \geq 1$ we denote by $x_{|_{\mathrm{Spec}(R/\mathfrak{m}_R^n)}}$ the choice of a pullback $\alpha_n^* x$ (Definition A.43). We also have natural compatible morphisms $f_n : \xi_n \to \xi_{n+1}$ coming from transitivity of restrictions

$$\begin{tikzcd} \xi_n \arrow[r, "f_n"] \arrow[d, mapsto] & \xi_{n+1} \arrow[r] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ \mathrm{Spec}(R/\mathfrak{m}_R^n) \arrow[r] & \mathrm{Spec}(R/\mathfrak{m}_R^{n+1}) \arrow[r] & \mathrm{Spec}(R). \end{tikzcd}$$

Thus, $\xi = (R, \xi_n, f_n)$ is a formal object of $\mathcal{X}$. If $\mathrm{Spec}\, R \to \mathrm{Spec}\, T$ is a morphism of affine schemes induced by a local homomorphism of rings $R \to T$ where R, T are noetherian complete local rings, then for all $n \geq 1$ there is a induced morphism of affine schemes $\mathrm{Spec}(R/\mathfrak{m}_R^n) \to \mathrm{Spec}(T/\mathfrak{m}_T^n)$. In particular, one can show that the construction of a formal object $\xi = (R, \xi_n, f_n)$ coming from $x \in \mathcal{X}(\mathrm{Spec}\, R)$ is functorial in x. In other words, we have a functor

$$G : \begin{pmatrix} \text{full subcategory of } \mathcal{X} \text{ whose objects} \\ \text{are } x \in \mathcal{X} \text{ such that } p(x) = \mathrm{Spec}(R) \\ \text{where } R \text{ is noetherian complete local} \\ \text{with } R/\mathfrak{m}_R \text{ of finite type over } \mathbb{Z} \end{pmatrix} \to (\text{category of formal objects of } \mathcal{X})\,.$$

DEFINITION A.111. With notations as above, we say that a formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ is *effective* if it is in the essential image of the functor G.

EXAMPLE A.112. If $\mathcal{X}$ is an algebraic stack over $\mathrm{Sch}_{\mathrm{fppf}}$, then F is an equivalence of categories. See [Stacks, Lemma 07X8].

DEFINITION A.113. Let I be a set and $\leq$ a binary relation on I.

- We say $\leq$ is a *preorder* if it is transitive (i.e. $i \leq j$ and $j \leq k$ imply $i \leq k$) and reflexive (i.e. $i \leq i$ for all $i \in I$).
- We say that $(I, \leq)$ is a *preordered set* if $\leq$ is a preorder.
- We say that $(I, \leq)$ is a *directed set* if it is a preordered set such that $I \neq \emptyset$ and such that for all $i, j \in I$ there exists $k \in I$ such that $i \leq k$, $j \leq k$.

DEFINITION A.114. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. An *inverse system over I in $\mathcal{C}$* is the data of objects M_i of $\mathcal{C}$ for every $i \in I$ and for every $j \leq i$ a morphism $f_{ij} : M_i \to M_j$ such that $f_{ii} = \mathrm{id}$ and such that $f_{ik} = f_{jk} \circ f_{ij}$ whenever $k \leq j \leq i$. An inverse system is called a *directed inverse system* if the preordered set $(I, \leq)$ is directed.

DEFINITION A.115. Let $\mathcal{X}$ be a category fibred in groupoids over Sch. We say that $\mathcal{X}$ is *limit preserving* if for every affine scheme T which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i, we have an equivalence

$$\mathrm{colim}\ \mathcal{X}(T_i) \to \mathcal{X}(T)$$

of fibre categories.

Details about the above notions can be found in [Stacks, Section 07XK].

Recall that in classical deformation theory if $\mathcal{F} \to \mathcal{A}_\Lambda$ is a deformation category and R is a complete noetherian Λ-algebra, then a formal object ξ of $\mathcal{F}$ over R defines a morphism of fibred categories $\tilde{\xi} : \underline{R} \to \mathcal{F}$ over $\mathcal{A}_\Lambda$, where $\underline{R}$ is the category fibred in sets over $\mathcal{A}_\Lambda$ corresponding to the functor $\mathrm{Mor}_{\mathcal{A}_\Lambda}(R, -) : \mathcal{A}_\Lambda \to \mathrm{Set}$ (see [Stacks, Remark 06HC]). The formal object ξ is said to be *versal* if $\tilde{\xi}$ is smooth.

Definition A.116. Let $p : \mathcal{X} \to \mathrm{Sch}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object of $\mathcal{X}$. Set $k = R/\mathfrak{m}_R$ and $x_0 = \xi_1$. We say that ξ is *versal* if ξ is versal as a formal object of $\mathcal{F}_{\mathcal{X},k,x_0}$.

Suppose that A, B are artinian local rings with residue field k and suppose that $\xi_1 \to y \to z$ is a morphism of $\mathcal{X}$ over closed immersions $\operatorname{Spec} k \to \operatorname{Spec} A \to \operatorname{Spec} B$. Suppose moreover that for an integer $n \geq 1$ we have a commutative diagram

$$\begin{tikzcd} & y \arrow[dl] \\ \xi_n & \xi_1 \arrow[l] \arrow[u] \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & \operatorname{Spec} A \arrow[dl] \\ \operatorname{Spec}(R/\mathfrak{m}_R^n) & \operatorname{Spec} k. \arrow[l] \arrow[u] \end{tikzcd}$$

If ξ is versal, there exists an $m \geq n$ and a commutative diagram

$$\begin{tikzcd} & & z \arrow[dll] \\ & & y \arrow[dl] \arrow[u] \\ \xi_m & \xi_n \arrow[l] & \xi_1 \arrow[l] \arrow[u] \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & & \operatorname{Spec} B \arrow[dll] \\ & & \operatorname{Spec} A \arrow[dl] \arrow[u] \\ \operatorname{Spec}(R/\mathfrak{m}_R^m) & \operatorname{Spec}(R/\mathfrak{m}_R^n) \arrow[l] & \operatorname{Spec} k. \arrow[l] \arrow[u] \end{tikzcd}$$

Suppose now that U is a scheme over $\mathbb{Z}$ such that $U \to \operatorname{Spec} \mathbb{Z}$ is locally of finite type. Let $u_0 \in U$ be a *finite type point* of U, i.e. the natural morphism $\operatorname{Spec} k(u_0) \to U$ is of finite type. Let $k = k(u_0)$. Observe first that the composition

$$\operatorname{Spec} k \to U \to \operatorname{Spec} \mathbb{Z}$$

is of finite type, being composition of finite type morphisms. Let $p : \mathcal{X} \to \mathrm{Sch}$ be a category fibred in groupoids, let $x \in \mathcal{X}(U)$ be an object and let x_0 be the pullback of x over u_0

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ \operatorname{Spec} k \arrow[r, "u_0"'] & U. \end{tikzcd}$$

By 2-Yoneda's lemma (A.58), x corresponds to a morphism of categories fibred in groupoids

$$x : U \to \mathcal{X}$$

where the scheme U corresponds to the category fibred in groupoids $(\mathrm{Sch}/U) \to \mathrm{Sch}$. Then by the functoriality as in Equation (20) we have an associated functor

$$\hat{x} : \mathcal{F}_{U,k,u_0} \to \mathcal{F}_{\mathcal{X},k,x_0}.$$

Definition A.117. Let $\mathcal{X} \to \mathrm{Sch}$ be a category fibred in groupoids and let U be a scheme locally of finite type over $\mathbb{Z}$. Let $u_0 \in U$ be a finite type point and $x \in \mathcal{X}(U)$. We say that x is *versal at* u_0 if $\hat{x}$ is smooth.

Definition A.118. Let $\mathcal{X} \to \mathrm{Sch}$ be a category fibred in groupoids. We say that $\mathcal{X}$ satisfies *openness of versality* if given a scheme U locally of finite type over $\mathbb{Z}$, an object $x \in \mathcal{X}(U)$ and a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subseteq U$ such that x is versal at every finite type point of U'.

THEOREM A.119 (Artin's axioms for algebraicity). *Let $p : \mathcal{X} \to \mathrm{Sch}$ be a category fibred in groupoids. Suppose that*

(a) *$\mathcal{X}$ is a stack over $\mathrm{Sch}_{\text{ét}}$;*
(b) *$\mathcal{X}$ is limit preserving;*
(c) *$\mathcal{X}$ satisfies the Rim-Schlessinger condition;*
(d) *the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathrm{Inf}(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional for any finite type field k over $\mathbb{Z}$ and any $x_0 \in \mathcal{X}(\operatorname{Spec} k)$;*
(e) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces;*
(f) *every formal object of $\mathcal{X}$ is effective;*
(g) *$\mathcal{X}$ satisfies openness of versality.*

Then $\mathcal{X}$ is an algebraic stack.

PROOF. See [Stacks, Lemma 07Y4]. See also [Alp24, Section § C.7] and the original paper [Art74b]. □

REMARK A.120. In [Stacks, Lemma 07Y4] everything is done in a more general case with respect to a scheme S, and it is required that for all finite type point $s \in S$, the local ring $\mathcal{O}_{S,s}$ is a G-ring ([Stacks, Definition 07GH]). In our case, $S = \mathbb{Z}$, which is a G-ring by [Stacks, Proposition 07PX], and every localization of a G-ring is again a G-ring by [Stacks, Proposition 07PV].

APPENDIX B

Some results of cohomology

B.1. Flat base change

LEMMA B.1. *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ T' & \xrightarrow[g]{} & T \end{array}$$

where f is affine or f is quasi-compact and quasi-separated and g is flat. For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, there is a natural isomorphism

$$g^* f_* \mathcal{F} \simeq (f')_* (u')^* \mathcal{F}.$$

PROOF. See [GW20, Lemma 22.88]. □

COROLLARY B.2. *Let $f : X \to \operatorname{Spec} R$ be a quasi-compact and quasi-separated morphism of schemes and let $R \to R'$ be a flat map of rings. Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the base change morphism induces for all $i \geq 0$ isomorphisms*

$$\mathrm{H}^i(X, \mathcal{F}) \otimes_R R' \simeq \mathrm{H}^i(X_{R'}, \mathcal{F}_{R'}),$$

where $X_{R'} = X \otimes_{\operatorname{Spec} R} \operatorname{Spec} R'$ and $\mathcal{F}_{R'}$ is the pullback of $\mathcal{F}$ to $X_{R'}$.

PROOF. See [GW20, Corollary 22.91]. □

COROLLARY B.3. *Let k be a field and let X be a proper scheme over k. Let K/k be a field extension. Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the base change morphism induces for all $i \geq 0$ isomorphisms*

$$\mathrm{H}^i(X, \mathcal{F}) \otimes_k K \simeq \mathrm{H}^i(X_K, \mathcal{F}_K).$$

In particular it holds that

$$h^i(X, \mathcal{F}) = h^i(X_K, \mathcal{F}_K)$$

and

$$\chi(X, \mathcal{F}) = \chi(X_K, \mathcal{F}_K).$$

PROOF. This is obvious by Corollary B.2, since a proper morphism is separated, and it is also quasi-compact, being universally closed ([Stacks, Lemma 04XU]). Moreover, any field extension is a flat map. □

B.2. Cohomology and base change

Consider the following situation:

($\triangle$) Let $f : X \to S$ be a proper, flat and finitely presented morphism of schemes. Let $\mathcal{F}$ be a line bundle on X.

In situation ($\triangle$), if $s \in S$ is any point of S we can consider the fibre over s given by the following cartesian diagram of schemes

$$
\begin{tikzcd}
X_s \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} \kappa(s) \arrow[r] & S
\end{tikzcd}
$$

According to [GW23, Section 23.28], for all $i \in \mathbb{Z}$ we have natural maps

$$\beta^i(\kappa(s)) : R^i f_*(\mathcal{F}) \otimes_{\mathcal{O}_S} \kappa(s) \to \mathrm{H}^i(X_s, \mathcal{F}_s)$$

where we denoted by $\mathcal{F}_s$ the pullback of $\mathcal{F}$ to X_s.

THEOREM B.4. *In situation ($\triangle$) one has the following assertions.*

(1) *The Euler characteristic*

$$\chi_{\mathcal{F}} : S \to \mathbb{Z}, \qquad s \mapsto \sum_{i \geq 0} (-1)^i \dim_{\kappa(s)} \mathrm{H}^i(X_s, \mathcal{F}_s)$$

is locally constant on S.

(2) *For each $i \in \mathbb{Z}$ the function*

$$S \to \mathbb{Z}, \qquad s \mapsto \dim_{\kappa(s)} \mathrm{H}^i(X_s, \mathcal{F}_s)$$

is upper semi-continuous and locally constructible, i.e. for all $n \geq 0$ the subset

$$\left\{ s \in S; \dim_{\kappa(s)} \mathrm{H}^i(X_s, \mathcal{F}_s) \geq n \right\}$$

is closed and locally constructible in S.

PROOF. See [GW23, Thoerem 23.139]. □

THEOREM B.5 (Cohomology and base change). *In situation ($\triangle$), fix $i \in \mathbb{Z}$ and a point $s \in S$.*

(1) *The following conditions are equivalent:*
   (i) *The map $\beta^i(\kappa(s)) : R^i f_* \mathcal{F} \otimes \kappa(s) \to \mathrm{H}^i(X_s, \mathcal{F}_s)$ is surjective.*
   (ii) *There exists an open neighbourhood U of s such that the formation of $R^i f_* \mathcal{F}_{|U}$ commutes with base change.*

(2) *Assume that $\beta^i(\kappa(s))$ is surjective. Then the following conditions are equivalent:*
   (i) *The map $\beta^{i-1}(\kappa(s))$ is surjective (and hence formation of $R^{i-1} f_* \mathcal{F}$ commutes with base change in an open neighbourhood of s);*
   (ii) *there exists an open neighbourhood V of s such that the $\mathcal{O}_V$-module $R^i f_* \mathcal{F}_V$ is finite locally free.*

   In this case, the function $s \mapsto \dim_{\kappa(s)} \mathrm{H}^i(X_s, \mathcal{F}_s)$ is locally constant on V.

PROOF. See [GW23, Theorem 23.140]. □

COROLLARY B.6. *Let $f : X \to S$ be a proper morphism of finite presentation, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation that is flat over S (e.g. $\mathcal{F}$ is a line bundle and f is flat). Suppose that there exists an $s \in S$ such that $\mathrm{H}^1(X_s, \mathcal{F}_s) = 0$. Then there exists an open neighbourhood U of s such that $R^1 f_* \mathcal{F}_{|U} = 0$, $f_* \mathcal{F}_{|U}$ is locally free of finite rank and its formation commutes with base change.*

PROOF. See [GW23, Corollary 23.144]. □

B.3. Relatively ample line bundle

DEFINITION B.7. Let X be a quasi-compact and quasi-separated scheme. An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is called *ample* if for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, there exists an integer n_0 such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections for all $n \geq n_0$.

Definition B.8. Let $f : X \to Y$ be a finite type morphism of schemes. An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is called *relatively ample for* f or f*-ample* if $\mathcal{L}_{|f^{-1}(U)}$ is an ample line bundle for all affine open subschemes U of Y.

Proposition B.9. *Let $f : X \to Y$ be a finite type morphism of schemes and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is f-ample if and only if there exists an open covering $\{U_i\}_{i\in I}$ of Y such that $\mathcal{L}_{|f^{-1}(U_i)}$ is ample for all $i \in I$.*

Proof. See [GW20, Proposition 13.63]. □

Proposition B.10. *Let $f : X \to Y$ be a proper morphism of schemes and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $y \in Y$ and assume that $\mathcal{L}_y$ is ample on the fibre X_y. Then there exists an open affine neighbourhood $U \subseteq Y$ of y such that $\mathcal{L}_{|f^{-1}(U)}$ is ample.*

Proof. See [GW23, Theorem 24.46]. □

Corollary B.11. *Let $f : X \to Y$ be a proper morphism of schemes and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that for all $y \in Y$ the line bundle $\mathcal{L}_y$ is ample on the fibre X_y. Then $\mathcal{L}$ is f-ample.*

Proof. This is an immediate consequence of Proposition B.9 and Proposition B.10. □

B.4. Quick review of Grothendieck duality

The reader can find details in [Con00], [Har66], [AK70].

Definition B.12. A morphism of schemes is called *Cohen-Macaulay with pure relative dimension* n if it is flat, locally of finite presentation and the fibres are Cohen-Macaulay schemes of pure dimension n.

For every proper Cohen-Macaulay morphism of schemes $f : X \to Y$, we can define a *dualizing sheaf* $\omega_f \in \mathrm{QCoh}(X)$ and a *trace map*

$$\gamma_f : R^n f_*(\omega_f) \to \mathcal{O}_Y$$

as in [Con00, Chapters 3 and 4]. These constructions are compatible with base change. We will also use the notation $\omega_{X/Y} = \omega_f$.

Proposition B.13. *Let $f : X \to Y$ be a proper Cohen-Macaulay morphism of pure relative dimension n. Then:*

(a) *if f is smooth, then $\omega_f \simeq \det \Omega^1_{X/Y}$;*
(b) *ω_f is flat, of finite presentation;*
(c) *ω_f is invertible if and only if all fibres of f are Gorenstein;*
(d) *if there exists a factorization*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & P \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

where i is a closed immersion and π is proper, smooth with pure relative dimension N, then

$$\omega_f \simeq \mathcal{E}xt_P^{N-n}(i_*\mathcal{O}_X, \omega_\pi);$$

(e) *γ_f is surjective;*
(f) *if f has geometrically reduced and geometrically connected fibres, then γ_f is an isomorphism.*

Proof. See [Con00, Chapters 3 and 4]. □

THEOREM B.14 (Grothendieck duality). *Let $f : X \to Y$ be a proper Cohen-Macaulay morphism with pure relative dimension n, let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank, and let $m \in \mathbb{Z}$ be an integer. Suppose that $R^i f_* \mathcal{F}$ is a locally free sheaf of finite rank on Y for all $i > m$.*
Then, for every $\mathcal{G} \in \mathrm{QCoh}(Y)$ and every $i \geq m$ there is an isomorphism

$$R^{n-i} f_*(\mathcal{F}^\vee \otimes \omega_f \otimes f^*\mathcal{G}) \simeq \mathcal{H}om_{\mathcal{O}_Y}(R^i f_*(\mathcal{F}), \mathcal{G})$$

PROOF. See [Con00, Theorem 5.1.2]. □

B.5. Leray spectral sequence

For a topological space X, let $\mathrm{Sh}(X)$ be the category of sheaves of abelian groups on X. Let $f : X \to Y$ be a morphism of schemes. We have an associated map

$$f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

and a commutative diagram

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[rr, "f_*"] \arrow[dr, "\Gamma_X"'] & & \mathrm{Sh}(Y) \arrow[dl, "\Gamma_Y"] \\ & \mathrm{Ab} & \end{tikzcd}$$

where the Γ's functions are simply taking global sections.

THEOREM B.15 (Leray spectral sequence, [Ler50]). *Let $f : X \to Y$ be a continuous map of topological spaces and let $\mathcal{F} \in \mathrm{Sh}(X)$. There is a spectral sequence whose second page is*

$$E_2^{pq} = H^p(Y, R^q f_* \mathcal{F})$$

and which converges to

$$E^{p+q} = H^{p+q}(X, \mathcal{F}).$$

LEMMA B.16. *Let $f : X \to Y$ a morphism of schemes. Suppose that $R^i f_* \mathcal{O}_X = 0$ for all $i \geq 1$ and that $f_* \mathcal{O}_X = \mathcal{O}_Y$. Let $\mathcal{L} \in \mathrm{Pic}(Y)$. Then*

$$H^p(X, f^*\mathcal{L}) \simeq H^p(Y, \mathcal{L}).$$

PROOF. By the projection formula [Stacks, Lemma 01E8] we have

$$R^q f_* f^* \mathcal{L} \simeq \mathcal{L} \otimes_{\mathcal{O}_Y} R^q f_* \mathcal{O}_X = \begin{cases} 0 & \text{if } q \geq 1 \\ \mathcal{L} & \text{if } q = 0. \end{cases}$$

Applying the Leray spectral sequence to $f^*\mathcal{L}$ we get

$$E_2^{p,q} = H^p(Y, R^q f_* f^* \mathcal{L}) = \begin{cases} 0 & \text{if } q \geq 1 \\ H^p(Y, L) & \text{if } q = 0. \end{cases}$$

The second page of the spectral sequence the only non-zero terms are in the first column, so that it is already stabilized. Since this spectral sequence converges to the cohomology groups $H^i(X, f^*\mathcal{L})$ we obtain the result. □

Bibliography

[Abh57] Shreeram Abhyankar. “On the field of definition of a nonsingular birational transform of an algebraic surface”. In: *Ann. of Math. (2)* 65 (1957), pp. 268–281.

[AK70] Allen Altman and Steven Kleiman. *Introduction to Grothendieck duality theory*. Vol. Vol. 146. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1970, pp. ii+185.

[Alp24] J. Alper. *Stacks and Moduli*. `https://sites.math.washington.edu/~jarod/moduli.pdf`. 2024.

[Alt97] Klaus Altmann. “The versal deformation of an isolated toric Gorenstein singularity”. In: *Invent. Math.* 128.3 (1997), pp. 443–479.

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969, pp. ix+128.

[Art62] Michael Artin. “Some numerical criteria for contractability of curves on algebraic surfaces”. In: *Amer. J. Math.* 84 (1962), pp. 485–496.

[Art69] M. Artin. “Algebraization of formal moduli. I”. In: *Global Analysis (Papers in Honor of K. Kodaira)*. Univ. Tokyo Press, Tokyo, 1969, pp. 21–71.

[Art73] Michael Artin. *Théorèmes de représentabilité pour les espaces algébriques*. Vol. No. 44 (Été, 1970). Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics]. En collaboration avec Alexandru Lascu et Jean-François Boutot. Les Presses de l’Université de Montréal, Montreal, QC, 1973, p. 282.

[Art74a] M. Artin. “Algebraic construction of Brieskorn’s resolutions”. In: *J. Algebra* 29 (1974), pp. 330–348.

[Art74b] M. Artin. “Versal deformations and algebraic stacks”. In: *Invent. Math.* 27 (1974), pp. 165–189.

[Băd01] Lucian Bădescu. *Algebraic surfaces*. Universitext. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author. Springer-Verlag, New York, 2001, pp. xii+258.

[Bar96] W. Barth. “Two projective surfaces with many nodes, admitting the symmetries of the icosahedron”. In: *J. Algebraic Geom.* 5.1 (1996), pp. 173–186.

[Bea10] Arnaud Beauville. “Surfaces algébriques complexes”. In: *Algebraic surfaces*. Vol. 76. C.I.M.E. Summer Sch. Springer, Heidelberg, 2010, pp. 5–56.

[Bha+13] Bhargav Bhatt et al. “Moduli of products of stable varieties”. In: *Compos. Math.* 149.12 (2013), pp. 2036–2070.

[BM76] E. Bombieri and D. Mumford. “Enriques’ classification of surfaces in char. p. III”. In: *Invent. Math.* 35 (1976), pp. 197–232.

[BM77] E. Bombieri and D. Mumford. “Enriques’ classification of surfaces in char. p. II”. In: *Complex analysis and algebraic geometry*. Iwanami Shoten Publishers, Tokyo, 1977, pp. 23–42.

[Bog78] F. A. Bogomolov. “Holomorphic tensors and vector bundles on projective manifolds”. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 42.6 (1978), pp. 1227–1287, 1439.

[Bom73] E. Bombieri. "Canonical models of surfaces of general type". In: *Inst. Hautes Études Sci. Publ. Math.* 42 (1973), pp. 171–219.

[Bou61] N. Bourbaki. *Éléments de mathématique. Fascicule XXVIII. Algèbre commutative. Chapitre 3: Graduations, filtrations et topologies. Chapitre 4: Idéaux premiers associés et décomposition primaire.* Vol. No. 1293. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics]. Hermann, Paris, 1961, p. 183.

[BW74] D. M. Burns Jr. and Jonathan M. Wahl. "Local contributions to global deformations of surfaces". In: *Invent. Math.* 26 (1974), pp. 67–88.

[Cat+99] Fabrizio Catanese et al. "Embeddings of curves and surfaces". In: *Nagoya Math. J.* 154 (1999), pp. 185–220.

[Cat13] F. Catanese. "A superficial working guide to deformations and moduli". In: *Handbook of moduli. Vol. I.* Vol. 24. Adv. Lect. Math. (ALM). Int. Press, Somerville, MA, 2013, pp. 161–215.

[Cat84] F. Catanese. "On the moduli spaces of surfaces of general type". In: *J. Differential Geom.* 19.2 (1984), pp. 483–515.

[Cat89] F. Catanese. "Everywhere nonreduced moduli spaces". In: *Invent. Math.* 98.2 (1989), pp. 293–310.

[Con00] Brian Conrad. *Grothendieck duality and base change.* Vol. 1750. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2000, pp. vi+296.

[DM69] P. Deligne and D. Mumford. "The irreducibility of the space of curves of given genus". In: *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), pp. 75–109.

[Edi00] Dan Edidin. "Notes on the construction of the moduli space of curves". In: *Recent progress in intersection theory (Bologna, 1997).* Trends Math. Birkhäuser Boston, Boston, MA, 2000, pp. 85–113.

[Eke88] Torsten Ekedahl. "Canonical models of surfaces of general type in positive characteristic". In: *Inst. Hautes Études Sci. Publ. Math.* 67 (1988), pp. 97–144.

[Fan+05] Barbara Fantechi et al. *Fundamental algebraic geometry.* Vol. 123. Mathematical Surveys and Monographs. Grothendieck's FGA explained. American Mathematical Society, Providence, RI, 2005, pp. x+339.

[FFP16] Elisabetta Fortuna, Roberto Frigerio, and Rita Pardini. *Projective geometry.* Italian. Vol. 104. Unitext. Solved problems and theory review, La Matematica per il 3+2. Springer, [Cham], 2016, pp. xii+266.

[GD71] A. Grothendieck and J. A. Dieudonné. *Éléments de géométrie algébrique. I.* Vol. 166. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1971, pp. ix+466.

[Gie77] D. Gieseker. "Global moduli for surfaces of general type". In: *Invent. Math.* 43.3 (1977), pp. 233–282.

[Gro67] A. Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV". In: *Inst. Hautes Études Sci. Publ. Math.* 32 (1967), p. 361.

[GW20] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I. Schemes—with examples and exercises.* Second. Springer Studium Mathematik—Master. Springer Spektrum, Wiesbaden, [2020] ©2020, pp. vii+625.

[GW23] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry II: Cohomology of schemes—with examples and exercises.* Springer Studium Mathematik—Master. Springer Spektrum, Wiesbaden, [2023] ©2023, pp. vii+869.

[Har66] Robin Hartshorne. *Residues and duality.* Vol. No. 20. Lecture Notes in Mathematics. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard

1963/64, With an appendix by P. Deligne. Springer-Verlag, Berlin-New York, 1966, pp. vii+423.

[Har77] Robin Hartshorne. *Algebraic geometry.* Vol. No. 52. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496.

[Hor75] Eiji Horikawa. "On deformations of quintic surfaces." In: *Invent. Math.* 31.1 (1975), pp. 43–85.

[Hor76] Eiji Horikawa. "Algebraic surfaces of general type with small C_1^2. I". In: *Ann. of Math. (2)* 104.2 (1976), pp. 357–387.

[JR97] David B. Jaffe and Daniel Ruberman. "A sextic surface cannot have 66 nodes". In: *J. Algebraic Geom.* 6.1 (1997), pp. 151–168.

[Kaw82] Yujiro Kawamata. "A generalization of Kodaira-Ramanujam's vanishing theorem". In: *Math. Ann.* 261.1 (1982), pp. 43–46.

[Kle66] Steven L. Kleiman. "Toward a numerical theory of ampleness". In: *Ann. of Math. (2)* 84 (1966), pp. 293–344.

[KM97] Seán Keel and Shigefumi Mori. "Quotients by groupoids". In: *Ann. of Math. (2)* 145.1 (1997), pp. 193–213.

[Kod68] Kunihiko Kodaira. "Pluricanonical systems on algebraic surfaces of general type". In: *J. Math. Soc. Japan* 20 (1968), pp. 170–192.

[Kol23] János Kollár. *Families of varieties of general type.* Cambridge Tracts in Mathematics. With the collaboration of Klaus Altmann and Sándor J. Kovács. Cambridge University Press, Cambridge, 2023, pp. xviii+471.

[Kol96] János Kollár. *Rational curves on algebraic varieties.* Vol. 32. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1996, pp. viii+320.

[Lan83] William E. Lang. "Examples of Surfaces of General Type with Vector Fields". In: *Arithmetic and Geometry: Papers Dedicated to I.R. Shafarevich on the Occasion of His Sixtieth Birthday. Volume II: Geometry.* Ed. by Michael Artin and John Tate. Birkhäuser Boston, 1983, pp. 167–173.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I.* Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387.

[Ler50] Jean Leray. "L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue". In: *J. Math. Pures Appl. (9)* 29 (1950), pp. 1–80, 81–139.

[Lie08] Christian Liedtke. "Algebraic surfaces of general type with small c_1^2 in positive characteristic". In: *Nagoya Math. J.* 191 (2008), pp. 111–134.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves.* Vol. 6. Oxford Graduate Texts in Mathematics. Translated from the French by Reinie Erné, Oxford Science Publications. Oxford University Press, Oxford, 2002, pp. xvi+576.

[LM00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques.* Vol. 39. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2000, pp. xii+208.

[Man01] Marco Manetti. "On the moduli space of diffeomorphic algebraic surfaces". In: *Invent. Math.* 143.1 (2001), pp. 29–76.

[Mat89] Hideyuki Matsumura. *Commutative ring theory.* Second. Vol. 8. Cambridge Studies in Advanced Mathematics. Translated from the Japanese by M. Reid. Cambridge University Press, Cambridge, 1989, pp. xiv+320.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory.* Third. Vol. 34. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, 1994, pp. xiv+292.

[Miy77] Yoichi Miyaoka. "On the Chern numbers of surfaces of general type". In: *Invent. Math.* 42 (1977), pp. 225–237.

[MO67] Hideyuki Matsumura and Frans Oort. "Representability of group functors, and automorphisms of algebraic schemes". In: *Invent. Math.* 4 (1967), pp. 1–25.

[Mum62] David Mumford. "The canonical ring of an algebraic surface". In: *Ann. of Math. (2)* 76 (1962), pp. 612–615.

[Mum69] David Mumford. "Enriques' classification of surfaces in char p. I". In: *Global Analysis (Papers in Honor of K. Kodaira).* Univ. Tokyo Press, Tokyo, 1969, pp. 325–339.

[Ols16] Martin Olsson. *Algebraic spaces and stacks.* Vol. 62. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2016, pp. xi+298.

[Per81] Ulf Persson. "Chern invariants of surfaces of general type". In: *Compositio Math.* 43.1 (1981), pp. 3–58.

[Ray70] Michel Raynaud. *Faisceaux amples sur les schémas en groupes et les espaces homogènes.* Vol. Vol. 119. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1970, pp. ii+218.

[Ser06] Edoardo Sernesi. *Deformations of algebraic schemes.* Vol. 334. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006, pp. xii+339.

[She96] N. I. Shepherd-Barron. "Some foliations on surfaces in characteristic 2". In: *J. Algebraic Geom.* 5.3 (1996), pp. 521–535.

[Som84] Andrew John Sommese. "On the density of ratios of Chern numbers of algebraic surfaces". In: *Math. Ann.* 268.2 (1984), pp. 207–221.

[Stacks] The Stacks Project Authors. *Stacks Project.* `https://stacks.math.columbia.edu`.

[Tog40] Eugenio G. Togliatti. "Una notevole superficie de 5^o ordine con soli punti doppi isolati". In: *Vierteljschr. Naturforsch. Ges. Zürich* 85 (1940), pp. 127–132.

[TV13] Mattia Talpo and Angelo Vistoli. "Deformation theory from the point of view of fibered categories". In: *Handbook of moduli. Vol. III.* Vol. 26. Adv. Lect. Math. (ALM). Int. Press, Somerville, MA, 2013, pp. 281–397.

[Tzi22] Nikolaos Tziolas. "Vector fields on canonically polarized surfaces". In: *Math. Z.* 300.3 (2022), pp. 2837–2883.

[Vak06] Ravi Vakil. "Murphy's law in algebraic geometry: badly-behaved deformation spaces". In: *Invent. Math.* 164.3 (2006), pp. 569–590.

[Vie82] Eckart Viehweg. "Vanishing theorems". In: *J. Reine Angew. Math.* 335 (1982), pp. 1–8.

[Vis08] A. Vistoli. *Notes on Grothendieck topologies, fibered categories and descent theory.* 2008.

[Voi07] Claire Voisin. *Hodge theory and complex algebraic geometry. I.* English. Vol. 76. Cambridge Studies in Advanced Mathematics. Translated from the French by Leila Schneps. Cambridge University Press, Cambridge, 2007, pp. x+322.

[Yau77] Shing Tung Yau. “Calabi’s conjecture and some new results in algebraic geometry”. In: *Proc. Nat. Acad. Sci. U.S.A.* 74.5 (1977), pp. 1798–1799.

[Zar39] Oscar Zariski. “The reduction of the singularities of an algebraic surface”. In: *Ann. of Math. (2)* 40 (1939), pp. 639–689.

[Zar42] Oscar Zariski. “A simplified proof for the resolution of singularities of an algebraic surface”. In: *Ann. of Math. (2)* 43 (1942), pp. 583–593.

[Zar62] Oscar Zariski. “The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface”. In: *Ann. of Math. (2)* 76 (1962), pp. 560–615.

[Zom18] W. Zomervrucht. “Ineffective descent of genus one curves”. In: *arXiv* (2018).

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