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Exploring the Relational Approach to Quantum Mechanics: A Comprehensive Analysis of Recent Debates on GHZ correlations

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To my grandparents
Ida, Marion, Agostino, Remo

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Introduction

Quantum mechanics is, at the moment, the best theory available to us to study the microscopic world in all its facets. In the hundred years since its conception, it has resisted all attempts to invalidate it and still surprises us by opening doors that perhaps in the past we didn't even know existed. Even if it will probably be overtaken in the future by some new theory that will include it, its revolutionary significance is undeniable and still allows us to explore worlds that are well beyond our imagination. It is a theory that may have created and may create a certain amount of discomfort in those who face and apply it. This characteristic can be an impediment in some ways, but at the same time it can be a resource, since it forces us to keep an open mind and a vigilant critical spirit, showing us, sometimes clearly and sometimes in a more subtle way, the limits of our knowledge. It is important to underline that this dynamic can occur not only in the context of its most advanced applications, at the limits of our current possibilities - be they intellectual or experimental -, but also when the very foundations of the theory are analyzed. In fact, even the basics of the theory require us to question our way of reasoning, the knowledge we took for granted and much more. They consist of a series of complex postulates and laws, for which advanced mathematical tools are required and which undermine the world we knew before studying it.

We could argue that, since its conception, the theory has, on the one hand, been expanded from a more "technical" point of view to try to extend our knowledge of the universe and, on the other, many scholars have also been devoting themselves to trying to understand what it was telling us about the universe we live in, in particular proposing various interpretations of the theory.

The numerous discussions that have arisen over the decades are very fascinating and have allowed us to explore various paths in the search for an answer. Although a clear and unequivocal answer has not yet been found - who knows if it even exists! -, analyzing with an open mind and a critical spirit the basis of the theory or its more advanced applications, reflects the human need to ask questions and seek answers. Therefore I believe that a scientific theory of this caliber is a wonderful fertile ground also for addressing questions that perhaps could go beyond specific application contexts. We should study it with the passion it deserves, in all its facets.

For this reason, in this thesis a possible reconstruction of the theory, proposed by Rovelli, will be discussed, starting from assumptions different from the original ones. This reconstruction consists of an approach, called "relational", which aims to provide new meaning to the theory, while maintaining its predictive power. As we said, quantum mechanics is made up of a series of concepts and tools that can often create difficulties in their deep understanding. The intent is to keep the predictive and descriptive power of the theory intact while trying to provide it with new foundations that can give us a clearer image of the world - especially the microscopic one.

Therefore, after a review of the concepts underlying the theory in the first chapter, Rovelli's point of view will be discussed. In particular, the first chapter has been divided into four sections, each of which concerns different aspects of the foundations of the theory. The four sections of the first chapter are related to quantum states, measurement in quantum mechanics, the uncertainty relation and quantum dynamics, respectively. In each section you will find the postulates, laws and theorems underlying the theory in its standard formulation. We are especially interested in the concept of quantum state, given that it is taken up precisely by Rovelli. In fact, his idea, presented in the second chapter, is based on the replacement of the concept of state meant in an absolute sense - independent of the observer - with a new concept, namely, of state *relative* to the observer. This is the main revolution proposed by Rovelli, in which the state is meant as the set of information that an observer has about a system. From here, the author conceives three postulates based on the notion of information and reconstructs the formalism of the theory. From this, we thus understand that the relational approach is meant as completely equivalent to standard quantum mechanics in terms of predictions. What changes are the foundations of the theory and in particular how we intend the physical states of systems. This, as we said, was done with the intention of giving a more precise meaning to the theory and the image of the world that it proposes to us.

However, the main part of the thesis consists of the third and final chapter which will take up a recent discussion on Rovelli's approach. In this chapter the points of view of Lawrence, Markiewicz and Żukowski (LMZ from now on), who analyze the relational approach in a specific case, will be presented. Their intent is to refute the relational approach by highlighting contradictions with the predictions of quantum mechanics. In particular, the three authors apply the concepts underlying the relational approach to the case of three qubits in a Greenberger–Horne–Zeilinger (GHZ) state. As we will see, a system of this kind is characterized by a series of constraints on measurement outcomes. In other words, the measurements made on this system are not independent of each other. Such constraints consist of a prediction of quantum mechanics, so we expect that an interpretation or reconstruction of the theory will always be able to respect the predictions of the theory it is trying to reinterpret or reconstruct starting from different hypotheses. LMZ seem to show that contradictions emerge when the relational approach is considered in such case and thus, according to them, cannot be considered a valid approach.

Their article has been resumed several times to try to defend Rovelli's point of view. These responses will be analyzed, as will the counter-responses from LMZ. From all these exchanges an interesting discussion was born which allows us to get an idea of the status of the approach proposed by Rovelli.

The first response we will see will be that of Drezet, who re-discusses the results presented by LMZ from a different point of view. In fact, Drezet puts the concept of quantum state and not of event at the center of his argument, as Rovelli does. Starting from this assumption, the author tries to dismantle the criticism of LMZ by accusing them of assuming non-contextuality, namely, of comparing what according to Drezet are different contexts; something that is indeed not possible to do in the relational approach. This, according to the author, together with mixing the results of different observers, would allow LMZ to reach its own conclusions. We will also see the counter-response to Drezet where LMZ defend themselves by arguing that such accusation has no foundation and that the approach adopted by Drezet is problematic.

Secondly, we will see another response to LMZ from Cavalcanti, Di Biagio and Rovelli himself. The latter also object to LMZ comparing measurements from different observers. In particular, according to Cavalcanti et al. it is necessary to introduce a new observer to deal with the constraints predicted by the theory. In this way, however, the contradiction cited by LMZ does not seem to emerge. Here too, LMZ's counter-response will be analyzed and, as we will see, the question still remains somewhat open.

Finally, we will present the work of Adlam and Rovelli, who introduce into the theory a new postulate regarding the consistency of results from different observers. We will mention the main consequences of this postulate, especially from the ontological point of view of the theory. However, we will focus above all on its connection with what was said before. For this reason, we will also analyze the response of Markiewicz and Żukowski, who apply the relational approach with the addition of the postulate to some situations - including, again, the case of a three-qubit system in a GHZ state - bringing out some interesting facts. For example, the authors argue that the postulate makes the results of some observers hidden variables for others and that in the GHZ case we still have contradictions, even by adding the postulate to the relational approach.

In any case, this thesis does not claim to find a definitive answer to the issue, but rather has the intention of providing a picture of the situation regarding the aforementioned debate. The topics addressed are so complex and vast that in order to arrive at something more substantial a much longer and more in-depth study would be necessary. So, we are content to put the debate in order and understand what the possible paths might be in light of it.

1 | Fundamentals of Quantum Mechanics

> If you want to understand the universe, study physics.
> If you want to understand physics, study philosophy.
>
> Lee Smolin

As we said, the first chapter will present the fundamental concepts of quantum mechanics (QM). The choices made and the style adopted presuppose that the reader already has a basic knowledge of the theory, it is a review rather than a rigorous treatment. In fact, many demonstrations are omitted and we will try to focus only on the reasons that lead to defining certain objects and on the consequences of the results presented. Above all, we want to create a common basis of topics covered, as well as adopt an unambiguous notation, to better address the subsequent chapters.

I felt it was necessary to dedicate a chapter to the fundamental notions of QM since I share the idea that before understanding what the theory *means*, it is good to understand what the theory *does*. And to do this it is necessary to first study its most "technical" part. In fact, the theory, in addition to being conceptually rich, is also complicated from a mathematical point of view, many advanced tools are needed to do QM.

The chapter is divided into four parts which correspond to four cornerstones of the theory, in which we will find the postulates that form its basis. Obviously, it is possible to present the theory differently; in fact, the postulates of a theory are not rigid. What matters is the picture that is formed from the set of postulates and laws, as well as the predictions that we can derive from all this. A historical example is that of Schrödinger's wave mechanics and Heisenberg's matrix mechanics. The two mechanics were developed starting from different assumptions, but they lead to the same results and, if adequately reformulated, they converge into the modern QM.[1] This reminds us that the hierarchical order of principles and laws depends on the formulation we choose, the important thing is that the predictions of measurable quantities are kept unchanged, otherwise we are talking about another theory.

[1] Their equivalence was demonstrated only in 1932 thanks to von Neumann [19]

1.1 Quantum state, superposition principle, observables

So, let us start by trying to understand the mathematical context we are in and therefore the tools we need. In the second section we will find the link between this formalism and the physical world, thus giving physical meaning to the mathematical objects introduced.

Let us consider a generic physical system, meant as a portion of the universe that we want to study. We can imagine carrying out measurements on it to determine the values assumed by the system for each of the observables we measure. To do this we would need a measuring instrument, which is always a macroscopic object, which interacts with the system and, as a result of the instrument-system interaction, we will obtain a real number that corresponds to the outcome of the measurement and the value of the observable. The values obtained correspond to the information we have about the system. We call "system state" the set of information we can have about that system. We can imagine identifying states of the system such that, by measuring the quantity in each of these, we will obtain a defined value.

If we indicate one of these values with α, we can use it to label the corresponding state:

$$|\alpha\rangle \,. \tag{1.1.1}$$

In quantum mechanics this state is to be considered a vector ("state vector", or "ket") of a complex vector space (more precisely a Hilbert space). The dimension of the complex vector space depends on the number of possible outcomes of the measurement of the corresponding quantity. Depending on the observable, the dimension will correspond to a certain finite N or countable infinity with continuous and real variable α.

Within the vector space associated with a quantity, it is possible to have linear combinations of state vectors.

$$|\gamma\rangle = a\,|\alpha\rangle + b\,|\beta\rangle \,, \tag{1.1.2}$$

with complex coefficients a and b. Such combination will still be a state vector belonging to the same vector space. The vector spaces in question are therefore *linear*. Furthermore, two vectors that differ only by a non-zero complex multiplicative constant represent the same state. This means that, for the purposes of the physical description of a certain system, what matters is the "direction" of the vector and not its "module". Because of this, two different vectors can correspond to the same physical state.

The physical quantities, in QM, are identified by linear operators that act on complex vector spaces:

$$A\,|\alpha\rangle \,. \tag{1.1.3}$$

Here the operator A acts on the state vector $|\alpha\rangle$.

We know that in a vector space, when applied to a generic vector, operators generally produce another vector. In the particular case in which the result is the same vector times a multiplicative constant, then the vector in question is called the "eigenvector" or "eigenstate" of the operator and the multiplicative constant is the corresponding "eigenvalue" (generally it corresponds to the label of the eigenstate):

$$A\,|\alpha\rangle = \alpha\,|\alpha\rangle\,. \tag{1.1.4}$$

The eigenstates of an observable are those particular state vectors that are not modified by the measurement process, i.e. by the application of the operator: the measurement simply produces the same eigenstate multiplied by the corresponding eigenvalue, which coincides with the outcome of the measurement itself. As such, the eigenvalue of an observable must be real. In fact, any measuring instrument produces real numbers.

There is also the *dual space*, i.e. a vector space that is "symmetrical" to the vector space of kets. The vectors of the dual space are called "bras" and each bra corresponds to a ket ("dual correspondence"),

$$|\alpha\rangle \longleftrightarrow \langle\alpha|\,. \tag{1.1.5}$$

The new vector space always refers to the same observable and enables us to introduce an internal scalar product, which corresponds to a complex number given by the product between a ket vector and a bra vector,

$$\langle\beta|\alpha\rangle = (\langle\beta|)(|\alpha\rangle). \tag{1.1.6}$$

The internal product, in turn, enables us to define a norm for the vectors which will therefore be real. In QM it is assumed that the product of a vector by itself is non-negative. This postulate is called the "*positive definite metric postulate*" and is the basis of the probabilistic treatment of the measurement of observables. In this way the product of a vector by itself can be square-rooted and we have the norm of the vector:

$$\sqrt{\langle\alpha|\alpha\rangle}. \tag{1.1.7}$$

Additionally, you can also define an external product, which is an operator:

$$|\beta\rangle\,\langle\alpha|\,. \tag{1.1.8}$$

Returning to physical observables, we want them to be represented by linear operators acting on vectors in a Hilbert space. We also want the eigenvalues of the operators representing the observables to be real, since the goal is to identify them with the outcomes of our measurements. For this purpose, these operators must be *Hermitian*, i.e. they must correspond to their conjugated Hermitians,

$$A = A^\dagger. \tag{1.1.9}$$

It is possible to prove that the eigenvalues of a Hermitian operator A are real and the eigenstates $\{|a_1\rangle, |a_2\rangle, |a_3\rangle, ...\}$ corresponding to distinct eigenvalues are orthogonal to each other. This allows us to say that the set of eigenvectors of a Hermitian operator, appropriately normalized, constitute an orthonormal and complete basis for the space of state vectors. This implies that it is possible to write every vector $|\alpha\rangle$ of the space as a linear combination of the eigenstates:

$$|\alpha\rangle = \sum_j c_j |a_j\rangle, \tag{1.1.10}$$

where j covers all eigenstates (finite or infinite). In other words, each state vector can be uniquely represented as a set of complex numbers c_j, which correspond to the coefficients of the linear combination of eigenstates. Having complete information on the state of a system is therefore equivalent to knowing the coefficients of the decomposition of the vector representing that state in the chosen basis.

The eigenstates of an operator also enable us to express the operators in the form of a square matrix in which the dimension of the matrix depends on the dimension of the space and thus on the number of eigenstates. For example, we can represent the generic operator X with the eigenstates $\{|a_1\rangle, |a_2\rangle, |a_3\rangle, ...\}$ of the operator A:

$$X = \begin{pmatrix} \langle a_1|X|a_1\rangle & \langle a_1|X|a_2\rangle & \langle a_1|X|a_3\rangle & \dots \\ \langle a_2|X|a_1\rangle & \langle a_2|X|a_2\rangle & \langle a_2|X|a_3\rangle & \dots \\ \langle a_3|X|a_1\rangle & \langle a_3|X|a_2\rangle & \langle a_3|X|a_3\rangle & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{1.1.11}$$

In this way we represent the operators and therefore the physical quantities with matrices. Let us keep in mind that we can use different bases and thus different matrices to express the same operator and, if the operator is Hermitian, then the elements of the diagonal are real. In the particular case in which we use the eigenstates of the operator to represent that same operator in the form of a matrix then we obtain a diagonal matrix in which the elements of the diagonal correspond to the eigenvalues of the operator.

The representation of vectors with n-tuple of numbers and operators with square matrices allows us to use all the rules of linear algebra: the products between operators become products between matrices, the application of an operator on a state vector becomes a matrix-vector product, the inner product a row-by-column product of two vectors, etc.

The discussion made so far is valid whether the observable is discrete or continuous. We know that in physics there are observables that admit only discrete values. For example, the spin of a particle can take on only two values. In cases like this, we say that the observable is characterized by a *discrete spectrum* (where "spectrum" means the spectrum of the observable's eigenvalue) and the dimension of the corresponding vector space will be a finite number, equal to the number of possible outcomes.

Obviously, however, there are observables that can take on an infinite number of real values, i.e. the measurable values form a continuum. This is for example the case of position and momentum. The measurement of these observables gives us a real number in a continuous set. In this case we talk about a *continuous spectrum*. We can therefore represent the physical state of a system as a linear combination of the position or momentum eigenstates. The difference compared to a discrete observable is that the summation in the combination will be replaced by an integral since the eigenstates of these observables are infinite. As an example, we can consider the decomposition of $|\alpha\rangle$ using the position basis:

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx' \, |x'\rangle \langle x'|\alpha\rangle \, . \tag{1.1.12}$$

Since the dimension of the Hilbert space representing a continuous observable is infinite, the integral goes from $-\infty$ to $+\infty$. Furthermore, the complex coefficient $\langle x'|\alpha\rangle$ is a complex function of the position variable and is called a *wave function*. We will see in the next section on measurement what its physical meaning is.

So far we have seen that in QM the physical state of a system is represented by a vector in a vector space and contains the information we can have about the system. On the other hand, observables are operators that act on this space and we can associate with them a set of state vectors, called eigenstates, which can constitute an orthonormal and complete basis for the space. In this way, knowing that a superposition principle holds, any state vector can be written as a linear combination of the eigenstates of an observable. Eigenstates also allow us to represent operators in the form of matrices, so we can resort to the rules of linear algebra when dealing with these objects. Now we look for a connection with the physical and experimental world by focusing on the central theme of measurement in QM.

1.2 Measurement in Quantum Mechanics

What do we mean first of all by *measurement*? In physics, measurement typically means an interaction between a physical system that we want to study and a measurement device. This interaction provides us with a real number - in the form of digits on a display, bits stored on a computer, etc. - which corresponds to the outcome of the measurement. The standard formulation of the MQ, which essentially corresponds to the *standard interpretation*, is based on a postulate that predicts a collapse of the state of the system during the measurement process.

Suppose we want to measure the observable A in a given system prepared in a pure state $|\alpha\rangle$. Let $\{|a_1\rangle, |a_2\rangle, |a_3\rangle, ...\}$ be the eigenstates normalized to 1 of the Hermitian operator A, corresponding to the real eigenvalues $a_1, a_2, a_3, ...$. The eigenstates form a basis on which $|\alpha\rangle$ can be decomposed in this way:

$$|\alpha\rangle = \sum_i c_i \, |a_i\rangle = \sum_i |a_i\rangle \langle a_i|\alpha\rangle \, . \tag{1.2.1}$$

We postulate that the act of measuring consists in making the state of the system collapse (or precipitate) in one of the eigenstates of A and that the value measured coincides with the corresponding eigenvalue of A. For example,

$$|\alpha\rangle \longrightarrow |a_j\rangle\,, \tag{1.2.2}$$

when the measured value of the observable A is a_j. We further postulate that the probability P of obtaining the value a_j in a series of measurements carried out on the same system prepared in same conditions, is equal to

$$P(a_j) = |c_j|^2 = |\,\langle a_j|\alpha\rangle\,|^2. \tag{1.2.3}$$

This is the *measurement postulate*. The probability relation becomes empirically effective when we carry out many measurements on the system - or if we have multiple equivalent systems prepared in the same way - in order to extract a probability from a frequency of events. In the particular case in which the state corresponds to one of the eigenstates of operator A, then the measurement does not change the state. After the measurement we will find the system in the state in which we prepared it and the measurement will correspond to the eigenvalue of that eigenstate. In all other cases, the state changes and it is not possible to know which value of the observable we will obtain, we can only calculate the probability of obtaining it. This postulate is fundamental for the theory so that it is anchored to reality through the comparison between its predictions and the results of the experiments. Given the intrinsically probabilistic nature of the predictions, the comparison must be carried out on replicas of the same system prepared in the same state.

The measurement process presented here is therefore an *indeterministic* process. It must be underlined that there are various schools of thought regarding the topic of measurement in QM. Many physicists and philosophers reject the collapse of the state of the system as a consequence of the act of measurement. A particularly interesting position is that of realists - among whom we find Einstein to name one - who support the idea that measurement does not have the power to change the state of a system, but the system was already in the final state and the measurement simply revealed this fact. Realists argue that since QM is unable to provide support for this hypothesis, then it is incomplete. The position of the Orthodox - which essentially corresponds to the Copenhagen interpretation - supports the cited postulate. For them it is therefore not possible to talk about a "defined" state for a system if it is in a superposition of states and the measurement changes the state causing it to collapse into one of the eigenstates of the observable. We could spend a long time debating what the state of the system was before carrying out the measurement and it would also be very interesting, but it is not the purpose of this thesis. What matters is that we know that the postulate we have seen works since the probabilities predicted by the theory on measurement outcomes have excellent agreement with the experiments.

It is also possible to calculate an *expectation value* for an observable out of many measurements carried out on the same system prepared in the same state:

$$\langle\alpha|A|\alpha\rangle = ... = \sum_i |c_i|^2 a_i = \sum_i P(a_i)a_i. \qquad (1.2.4)$$

It is a weighted mean where the weights are the probabilities of obtaining a certain eigenstate as outcome of the collapse. To further underline the agreement with experiments, we point out that, if we carry out repeated measurements of this kind then the mean value of these measurements will tend to the expectation value, if the number of measurements is large.

As we said, the formalism of the theory applies to both discrete and continuous observables. However, it is necessary to make changes since, with continuous observables, the eigenvectors are associated with a continuous set of possible real outcomes of the measurements (for example the summations become integral and for some relations it is necessary to replace some discrete mathematical objects with others that work in the continuos) , but the definitions and relations remain basically the same. The principles underlying the theory apply in both cases.

An important conceptual difference, however, concerns the probabilities of measurement outcomes. To discuss about this, we can take position as "representative" of continuous observables. In the case of discrete spectra we said that the probability of obtaining a certain eigenvalue a_j in a measurement is given by $|\langle a_j|\alpha\rangle|^2$. Can we say something equivalent for the position? The answer is no: the probability of finding the particle somewhere makes sense only for spatial intervals and not for points. The question should be changed by considering the probability of finding the particle in a certain spatial interval, centered around a point. When this interval tends to zero, we can exploit the formalism of differentials, writing:

$$P(x') = dx' |\langle x'|\alpha\rangle|^2, \qquad (1.2.5)$$

where we used the fact that we expect the probability to be proportional to the interval when dx' is small. Therefore, the quantity $|\langle x'|\alpha\rangle|^2$ has the meaning of probability density per unit length, or more simply probability density, and is a function of the position x and depends on how we have prepared the system. We can consider this discussion as an integration to the measurement postulate, including the case of continuous spectrum observables.
For finite intervals the probability is calculated thanks to the integral:

$$P = \int_{x_1}^{x_2} dx' |\langle x'|\alpha\rangle|^2. \qquad (1.2.6)$$

Of course we can also consider the position of the particle in three-dimensional space. For this, it is simply a matter of defining a three-component observable $\mathbf{r} = (x, y, z)$ where x, y, z are observables themselves. In order to assign a position $\mathbf{r}$ to the system it is necessary that these three observables are compatible with each other since we make three measurements and we do not want the state to change between one measurement and another.

The mathematical treatment for other continuous observables other than position is completely similar. The probability density, for example, has the same definition, simply the probability will refer to the eigenvalue space of the new observable and no longer to the x coordinate space.

To conclude this section, let us now discuss about the wave function of a system. We noted that the quantity $\langle x'|\alpha\rangle$ in (1.2.5) is a complex function of the position variable. We called this quantity wave function and its square modulus corresponds to a probability density. It is possible to express inner products, decomposition of state vectors and other quantities in terms of the wave function. For example, the linear combination of the state vectors becomes:

$$\psi_\alpha(x') = \sum_i c_i \phi_i(x'), \tag{1.2.7}$$

where the functions $\phi_i(x')$ are the wave functions of the eigenstates of the observable A and for this reason are called *eigenfunctions* of A. Obviously, all this can be generalized to three spatial dimensions by considering a $\psi(\mathbf{r})$ function.

It is worth underlining that a wave function can also be defined in the momentum space, thus taking the observable momentum to study the state of a system. The treatment is completely similar to that of the position. The observable momentum is in fact characterized by its complete basis of orthonormal eigenstates $|p'\rangle$ and therefore an $|\alpha\rangle$ state can be written as a linear combination of these eigenstates:

$$|\alpha\rangle = \int dp' \, |p'\rangle \langle p'|\alpha\rangle \, . \tag{1.2.8}$$

And we can calculate the probability that the measured momentum lies in a continuous range of values centered around p thanks to $\tilde{\psi}_\alpha(p) = \langle p|\alpha\rangle$.

As with discrete spectra, even in the case of continuous spectra it is possible to move from one basis to another. For discrete observables, it is necessary to use unitary matrices whose matrix elements are the internal products between the elements of the respective basis. In the case of position and momentum, it is possible to show that plane waves are used to pass from one basis to another, i.e. the wave function representing an eigenstate of momentum in coordinate space is a plane wave, and vice versa:

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} exp\left(\frac{ip'x'}{\hbar}\right), \tag{1.2.9}$$

and

$$\langle p'|x'\rangle = \frac{1}{\sqrt{2\pi\hbar}} exp\left(-\frac{ip'x'}{\hbar}\right). \tag{1.2.10}$$

The plane wave formalism is consistent with the *position-momentum uncertainty relation*, which will be discussed shortly. Such relation tells us that, if the indeterminacy of one of the two observables tends to zero, as when we have an eigenstate of that observable, then the indeterminacy of the other goes to infinity since the wave function, in the corresponding space, has constant modulus everywhere.

If, however, we consider a generic state $|\alpha\rangle$, then in order to move from one wave function to another we have to use the *Fourier transforms* which enable us to move from one representation space to another:

$$\psi_\alpha(x) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \tilde{\psi}_\alpha(p)\, exp\left(\frac{ip'x'}{\hbar}\right) \tag{1.2.11}$$

and

$$\tilde{\psi}_\alpha(x) = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, \psi_\alpha(x)\, exp\left(-\frac{ip'x'}{\hbar}\right). \tag{1.2.12}$$

And, again, all this is generalizable to the three-dimensional case. The theory of Fourier transforms is also consistent with the uncertainty relation. In fact, if Δx corresponds to the interval in which the wave function is significantly different to zero, the amplitude of the values of k, Δk, is not independent of Δx. For a Gaussian function, for example, we know that the product $\Delta x \Delta k$ is a constant of order 1. So, again, the more determined one of the two variables is, the more indeterminate the other will be.

1.3 Uncertainty Relation

Suppose two observables A and B defined for the same system are characterized by a complete common set of eigenstates. When this happens, we will say that the observables are *compatible*, otherwise they are *incompatible*. What has to do the coincidence of the eigenstates of two observables with theirs compatibility? If an eigenstate $|ab\rangle$ of A is also an eigenstate of B, with eigenvalue a and b respectively, then we can apply the product operator AB or BA on $|ab\rangle$ always obtaining the same vector:

$$\begin{aligned} AB\,|ab\rangle &= ab\,|ab\rangle \\ BA\,|ab\rangle &= ba\,|ab\rangle\,. \end{aligned} \tag{1.3.1}$$

This means that the order of application of A and B on their common eigenstates is irrilevant. It can be shown that two observables are compatible if and only if the commutator of the operators associated with them is zero:

$$[A, B] = AB - BA = 0. \tag{1.3.2}$$

From a physical point of view, if two observables commute, then, as we have defined measurement, we can measure A, obtaining its eigenstate, and then measure B without this second measurement changing the state of the system. In this way, the final state corresponds to well-defined values of both observable A and B and the order of the measurements is irrelevant. If the observables are incompatible then it is not possible to state all this and the order of measurement becomes crucial.

If we mean the physical state as the set of information we have about a system relative to observables, we could argue that - if observables A and B commute - a measurement of B after measuring A does not "destroy" the information on the state of the system relatively to A.
An important consequence of the compatibility and incompatibility of observables is a theorem that concerns the distribution of measurement results on the same physical state of a system.

Given a generic observable A, it is possible to define the quantity

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}, \tag{1.3.3}$$

called the standard deviation. Under the square root there is the difference between the expectation value of the observable A squared and the square of the expectation value of A. The standard deviation provides an estimate of how much the outcomes of the measurements performed on the same state are dispersed compared to the average value of that observable. In the case in which the system is prepared in an eigenstate of A, then the dispersion is zero and all the measurements will result in the average value of A. In the general case, we have a more or less dispersed distribution and the uncertainty of the single measurement will be larger the larger ΔA is. As usual, these quantities take on an empirical sense when we carry out many repeated measurements on the same system, or copies of a system, always prepared in the same way.

Having said that, the theorem states that if A and B are two any observables and ΔA and ΔB are the standard deviations of measurement outcomes of A and B on a generic state represented by vector $|\alpha\rangle$, then

$$\Delta A \Delta B \geq |\langle \alpha | [A, B] \alpha \rangle|. \tag{1.3.4}$$

We will not prove the theorem, for those interested it is possible to find a proof on any good book on the basics of QM. Here we focus only on some considerations.

The theorem states that, if we take two incompatible observables, there is a minimum value for the product of the indeterminacies set by the commutator of the two operators representing the observables. This means that it is not possible to obtain two values with zero uncertainty for both. In fact, if the system is prepared in such a way as to reduce the uncertainty on one of the two observables as much as possible, then the uncertainty on the other grows. In other words, having all the information on the state of a system relative to an observable does not imply knowing exactly the value of all the observables that we can define for that system, but only of those that are compatible with each other. The theorem places intrinsic limits - in the sense that they do not depend on our ignorance or on the measurement instruments - to the information we can have on the state of the system.

We have already mentioned the case of position and momentum. In this case, the relation becomes the well-known *Heisenberg uncertainty principle*. In particular, the position along a generic x direction and the momentum component along the same direction, p_x, obey the canonical commutation rule

$$[x, p_x] = i\hbar. \tag{1.3.5}$$

The inequality $\Delta x \Delta p_x \geq \frac{\hbar}{2}$ follows. We underline the fact that a consequence of the position-momentum incompatibility is that, unlike classical physics, in QM it is not possible to define a trajectory for an object, for example a particle, since this would require contextual knowledge of both position and velocity instant by instant, but this is not admitted by the Heisenberg principle.

In this formulation of the theory, the uncertainty relation is not a principle, but a theorem that derives from the postulates of the theory; in particular, it comes from the fact that observables in quantum mechanics in general do not commute.

1.4 Time evolution and the Schrödinger equation

So far we have not considered yet the problem of introducing *time* into the physical description of systems. We considered the state of a prepared system at a certain instant, but we did not care *what* instant it was. Now let us try to introduce time and thus time evolution of physical states into the theory. To do this, first of all it is necessary to ask ourselves whether the dynamics we want to develop should be invariant under Galilean transformations, as in Newtonian mechanics, or under Lorentz transformations, as in special relativity. Here we will focus only on *non-relativistic* dynamics, since we are discussing only the basics of the theory, but it is good to remember that it is possible to reformulate everything so that it adapts to the study of systems traveling at speeds comparable to the speed of light , thus arriving to *quantum field theory*.

Time and space, therefore, are independent quantities. We have already talked about space and we have seen that it is a continuous spectrum observable represented by a Hermitian operator (or three operators in three-dimensional space). We also know what it means to make a measurement of such observable on a system and that we can ask ourselves what the probability of observing the system in a certain spatial interval is. As for time, however, we need to be a little careful. In fact, what was said for position or other observables does not apply to time. It is in fact a parameter, something that passes independently of the systems' configurations. It is not possible to associate a Hermitian operator with it and it makes no sense to ask what the probability of obtaining a certain value as measurement outcome of time is. What we need to focus on is rather what the state of a system is *at* a certain moment.

Therefore we add the time parameter to the physical state of the systems:

$$|\alpha, t_0\rangle\,. \tag{1.4.1}$$

Basically the label t_0 indicates that the system is in the $|\alpha\rangle$ state precisely at the instant t_0. In general, the state of the system at a subsequent instant $t > t_0$ may be different and we indicate the evolution in this way:

$$|\alpha, t_0\rangle \longrightarrow |\alpha, t_0; t\rangle \, . \tag{1.4.2}$$

This transformation can be represented by an operator U ("*time evolution operator*") that acts on the states. In our case we can rewrite the evolution in (1.4.2) as the action of U on the state at time t_0:

$$|\alpha, t_0; t\rangle = U(t, t_0) \, |\alpha, t_0\rangle \, . \tag{1.4.3}$$

However, the new operator must comply with some requests. First of all, it must reduce to the identity if the time interval goes to zero. Furthermore, the norm of the vectors must be constant over time, so that the probabilistic interpretation of measurement results is always valid, and this means that the operator U must be unitary ($U^\dagger U = UU^\dagger = \mathbb{1}$). Finally, it must be composable for subsequent time intervals, i.e. $U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0)$ when $t_0 < t_1 < t_2$. It can be shown that the three properties are respected, for infinitesimal time intervals, if we define U as

$$U(t + dt) = \mathbb{1} - i\Omega dt, \tag{1.4.4}$$

where Ω is a Hermitian operator with the dimensions of $time^{-1}$. Therefore we are representing U as the generator of time translations.

At this point we can make a *conjecture*: given that Ω has the dimensions of $time^{-1}$ and we know from old quantum theory that $E = h\nu$, then we write

$$H = \hbar\Omega, \tag{1.4.5}$$

where H is a new operator having the dimensions of energy which we call the *Hamilton operator*, or *Hamiltonian* for simplicity. We also assume that the Hamilton operator can be obtained from the Hamilton function $H(q, p)$ which in classical physics represents the energy of the system as a function of positions and momenta. In particular, we are conjecturing that the quantum treatment of a system admitting a classical limit is obtained by replacing the spatial coordinates and momenta appearing in the classical Hamilton function with the respective quantum operators. This relation between operator H and function $H(q, p)$ is just a conjecture that can be seen as a tentative to keep in QM the same classical symmetries. In fact, temporal translations are associated with energy in classical physics; for example, if the system is invariant for temporal translations then energy conserves. Let us now look at what its consequences may be.

It is easy to demonstrate that, given a state evolving from t_0 to t and from t to $t+dt$, we can write a differential equation for the time evolution operator, knowing that it is possible to write the differential of an operator just like one does with functions $dU = \frac{dU}{dt}dt$:

$$i\hbar\frac{\partial}{\partial t}U(t, t_0) = HU(t, t_0). \tag{1.4.6}$$

All quantum dynamics is inside this equation. Let us see its application to a state vector $|\alpha, t_0\rangle$:

$$i\hbar\frac{\partial}{\partial t}\left|\alpha, t_0; t\right\rangle = H\left|\alpha, t_0; t\right\rangle . \tag{1.4.7}$$

This equation gives us the time evolution of state vectors.

The two fundamental equations (1.4.6) and (1.4.7) are general but abstract. To translate them into numbers we can consider an observable, for example position. We know that we can associate with it a complete basis of eigenstates that does not depend on time, by definition. We can also insert the time parameter into the wave functions:

$$\langle x|\alpha, t_0; t\rangle = \psi_\alpha(x,t). \tag{1.4.8}$$

Let us replace H with the classical expression $H = \frac{p^2}{2m} + V(x)$, where every quantity corresponds to a Hermitian operator, including the Hamilton operator. It is possible to demonstrate that we can arrive to the equation:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x,t), \tag{1.4.9}$$

which can be extended to the three-dimensional case.

This is the *Schrödinger equation*. He obtained it starting from analogies with geometric optics, thus with different conjectures from those seen here, but the result is the same.

We underline that the Schrödinger equation (1.4.9) is a special case of the equation (1.4.7), which also applies to observables that do not have a classical correspondent.

At this point, we can take the following postulate as valid: *the time evolution of the physical states of a system is given by the Schrödinger equation*.

We note that the equation (1.4.7) is linear in time and therefore deterministic with respect to physical states. This means that, given a state vector at a generic instant and the Hamiltonian (namely, the interaction between the system and the universe), it is possible to uniquely determine the state vector at any other instant. However, this only applies to physical states and not to measurement outcomes for which a probabilistic rule applies. In this sense, we can say that QM is a deterministic theory in a very particular sense, namely with regard to states' evolution within Hilbert spaces.

Thus determinism concerns only the physical states' evolution. There have been attempts to reformulate the theory so as to make it deterministic also with regard to measurement. The most famous is probably the *Bohmian mechanics*, developed by David Bohm in 1952. From considerations on the measurement process in QM, Bohm rejected the completeness of the theory, adding to it variables, called "*hidden*", which could complete it and make it deterministic. The universe thus constructed is hidden (in the sense of not accessible to us in its entirety) and deterministic. Unfortunately, however, Bohm's attempt was not very successful; the theory has very strong hypotheses and is not Lorentz invariant thus does not allow a field theory. Einstein did not consider it adequate to the need to complete the theory.

Let us get back where we were. The Hamiltonian operator H may or may not depend on time. In the first case the solution of equation (1.4.6) is simple:

$$U(t,t_0) = exp\left(-\frac{iH(t-t_0)}{\hbar}\right). \tag{1.4.10}$$

If H changes over time, things get a bit more complicated. H might be different for two different moments t_1 and t_2 and, furthermore, $H(t_1)$ and $H(t_2)$ might not commute. If the operators commute, then the method is similar to the case of time-independent H and we arrive to the result:

$$U(t,t_0) = exp\left(-\left(\frac{i}{\hbar}\right)\int_{t_0}^{t} dt' H(t')\right). \tag{1.4.11}$$

If, however, they do not commute, then the order of the operators becomes crucial and the solution is given by the so-called *Dyson series*. Nevertheless, for our purposes, we can focus on the case of time-independent Hamiltonians.

It is important to see what happens for observables that commute with the operator H (we are assuming that H does not change over time). We know that if two observables commute then they admit a complete set of common eigenstates. If we take the time evolution operator U and apply it to one of the eigenstates of the observable A that commutes with H, then its time evolution corresponds to a simple multiplication by a phase factor, where the frequency is fixed by the energy value for that eigenstate:

$$|\alpha, 0; t\rangle = U(t,0)\,|a_i\rangle = exp\left(-\frac{iE_i t}{\hbar}\right)|a_i\rangle\,. \tag{1.4.12}$$

Eigenstates of this kind are called *stationary states*. The state of the system does not change over time and, if a measurement of A is performed, the same value will always be obtained. For this reason, an observable that commutes with H is called a *constant of motion*.

However, when the initial state is a generic vector, written as a linear combination of the eigenstates of A, it is possible to see that the time evolution corresponds to a multiplication by a phase factor of each coefficient of the decomposition, with frequency fixed by the energy of that stationary state:

$$|\alpha, t_0; t\rangle = \sum_j c_j(t)\,|a_j\rangle\,, \tag{1.4.13}$$

with

$$c_j(t) = c_j(0)\ \exp\left(-\frac{iE_j t}{\hbar}\right). \tag{1.4.14}$$

As a consequence, the probability of obtaining a certain eigenvalue as measurement outcome of A does not depend on time. The probabilities do not depend on when I measure the constant of motion.

In terms of the wave function, it means that if the system is initially in a stationary state of energy E, then its temporal evolution can be represented by the function:

$$\psi(\mathbf{r},t) = \psi(\mathbf{r},0)\ \exp\left(-\frac{iEt}{\hbar}\right). \tag{1.4.15}$$

If we insert this expression into the Schrödinger equation (1.4.9), we obtain the *stationary*, or time-independent, *Schrödinger equation*:

$$\left(-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r})\right)\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{1.4.16}$$

The search for one or more solutions boils down to solving a second-order differential equation in spatial coordinates.

Regarding observables, it is also possible to show that, if the system is initially prepared in a stationary state, then the average value of any observable (even one not compatible with H) does not depend on time. Otherwise, the expectation value may change over time.

At this point we can stop our presentation of the theory. We have seen the fundamental concepts. At the basis of the theory is the notion of quantum state, treated as a vector in a vector space. In these spaces a superposition principle applies and the observables are represented by operators that act on the state vectors. Moreover, we saw the measurement postulate and thus the collapse of state vectors, or of wave functions. It is an indeterministic process and we can only calculate the probability of obtaining a certain value as measurement outcome. The uncertainty relation is a consequence of the fact that observables in QM do not commute, in general. Finally, we talked about the time evolution of systems, governed by the Schrödinger equation.

We have not seen any application of the theory to any system. A typical example in courses on the foundations of QM is the calculation of the energy spectrum of a harmonic oscillator, a useful prototype for many systems in physics. However, our aim was to lay, or rather review, the foundations of the theory in order to be able to tackle the next chapters more easily, so we can stop here. In the next chapter we will discuss an alternative approach to the theory, which starts from different hypotheses in an attempt to give it a more precise meaning.

2 | Relational Approach to Quantum Mechanics

> I stand upon my desk to remind myself that we must constantly look at things in a different way. The world looks very different from up here.
>
> Professor Keating

Let us now discuss the relational approach coined by Rovelli in [21]. Typically when we talk about reinterpretations of QM, we mean theories that try to make sense of the quantum results. Therefore, we *add* something to a theory that already exists and that we know works, with the - often unsuccessful - aim of providing meaning to the quantum concepts which are often counterintuitive, far from our way of reasoning. Having said this premise, the relational approach ("RQM" for Relational Quantum Mechanics) is not to be intended as a reinterpretation of QM in this sense, but as an attempt to "reconstruct" the theory, its formalism and its predictions, starting from different assumptions.

So, what we will do in this chapter is try to arrive at the same notions that we saw in chapter 1 on the basis of specific hypotheses, supported by experimental evidence. In other words, the issue is not to replace or fix QM, but rather to understand what it tells us about the world thanks to new postulates. Rovelli explains all this well by drawing a parallel with special relativity. The author reminds us how Einstein rediscovered the Lorentz formalism, which already existed, starting from his two postulates, giving a new and convincing meaning to the formulas and predictions that were already at our disposal. The author's hope is to do the same in the field of quantum physics, which often still remains obscure to a profound understanding. Furthermore, as the author claims, this approach is not intended to be antagonistic to other visions of QM, but rather an attempt to combine them and complete some of their aspects.

RQM is based on a critique of a notion that constitutes one of the foundations of QM, and is often assumed uncritically, namely the notion of absolute, or observer-independent, state of a system. We know that the state of a physical system can be meant as the set of values of the physical quantities that we have measured on that system, thus, in other words, the criticism refers to the hypothesis that these values are absolute, equal for every measuring apparatus. We will use the terms "measuring apparatus" and "observer" as synonyms, even if the second, although more common, is more easily subject to ambiguity.

Anyway, Rovelli points out that the experimental evidence leads us to question all this, since in some situations, as we will see, different observers provide different descriptions of the same events. The idea is to replace the absoluteness of the quantum state with a new notion of state *relative* to something. According to Rovelli, as in the limit $v \to c$ the concept of absolute time is no longer suitable, in the limit $\hbar \to 0$ the concept of observer-independent quantum state becomes inappropriate.

Hence, RQM was conceived starting from two ideas. First of all, that of deriving the formalism of the theory starting from some reasonable principles from the experimental point of view in order to make more sense of the theory and, secondly, the idea of rejecting the concept of absoluteness of physical quantities when we do QM.

2.1 A theory about information

To begin, let us consider a simple experiment involving the measurement of an observable on a system. We want to study the same sequence of events from the point of view of two different observers, one observer who carries out the measurement and the second who does not. Let us therefore consider a system S and an observable A. Suppose that A is characterized by only two possible outcomes. This means, as we saw in section 1.1, that the Hilbert space associated with the observable A will be two-dimensional and we call its eigenstates $|1\rangle$ and $|2\rangle$, which constitute a complete basis.

Suppose that, at time t_1, the system S is in a generic state, given by the linear superposition of the eigenstates $|1\rangle$ and $|2\rangle$:

$$\alpha\,|1\rangle + \beta\,|2\rangle\,, \tag{2.1.1}$$

with α and β complex numbers, and that at time $t_2 > t_1$ the observer O carries out a measurement on S obtaining the eigenvalue "1" as outcome of the measurement. We can schematically describe the process as:

$$\begin{gathered} t_1 \longrightarrow t_2 \\ \alpha\,|1\rangle + \beta\,|2\rangle \longrightarrow |1\rangle\,, \end{gathered} \tag{2.1.2}$$

where the collapse of the S's state - in O's view - is represented in the second line. We call "ϵ" the sequence of events in (2.1.2).

Let us now consider a second observer P, which describes the sequence of events ϵ. To do this, P must consider the system $S - O$ consisting of the system S and the observer O interacting. Suppose that P does not make any measurements on $S - O$, but that it knows the initial states of S and O, so that it is able to provide a description of ϵ.

If S is described by a vector in a Hilbert space H_S, O will similarly be described by a vector in a H_O space. Therefore the $S-O$ system will be described by the tensor product $H_{S-O} = H_S \times H_O$.

We know that S is described by the eigenstates of A, but what about O? Let us suppose that $|init\rangle$ is the initial state of O in the H_O space at $t = t_1$. At $t = t_2$ the state of O will change since the measurement is a physical interaction. Given that A can provide only two outcomes, O, similarly to S, can "collapse" into two possible states, that we call $|O_1\rangle$ and $|O_2\rangle$, corresponding to the scenario S collapses - according to O - into $|1\rangle$ or $|2\rangle$ respectively. We can think of the states of O as states in which the observer's measurement instrument is providing the observer - in the form of bits, digits or perhaps it has an indicator on a graduated scale - with the values "1" or "2" after having carried out the measurement on S. In P's view, the S's states and O's states are correlated; this means that P knows O has information about S, as far as the observable A is concerned.

After these premises, we can schematically represent the experiment ϵ also from the point of view of P:

$$\begin{gathered} t_1 \longrightarrow t_2 \\ (\alpha\,|1\rangle + \beta\,|2\rangle) \otimes |init\rangle \longrightarrow \alpha\,|1\rangle \otimes |O_1\rangle + \beta\,|2\rangle \otimes |O_2\rangle\,. \end{gathered} \tag{2.1.3}$$

Let us note that, in P's view, the $S-O$ system is in a superposition at time t_2. Such superposition reflects the fact that P knows that O performed a measurement on S but does not know *what* value O obtained, since P has not made any measurement yet. In other words, according to P, S is not necessarily in the state $|1\rangle$ and O is not necessarily in the state $|O_1\rangle$.

Since a consistency condition must hold, Rovelli claims that if P carried out an experiment on $S-O$, it would obtain consistent outcomes; hence, P would observe O in the state $|O_1\rangle$ if O measured "1" on S, and would obtain the $|O_2\rangle$ state if O measured "2" on S, and vice versa. In all scenarios, the observable A and the state of O are correlated according to P. Such correlation is a property of the system $S-O$ and is measurable by any third observer P. We will come back again to this delicate topic in section 3.4.

However, the description in (2.1.3) states that, for P at time $t = t_2$, S has no determined value relative to A. On the other hand, (2.1.2) tells us that, for O, A has value "1". As a result, we conclude that two different observers provide different descriptions of the same sequence of events.

Let us now try to understand what this simple experiment allows us to conclude, supporting the hypotheses underlying the new approach. First of all, we have seen that in the context of QM different observers, in general, can provide a different description of the same event or sequence of events. In particular, when we talk about "description" by an observer, we mean the quantum state associated with a certain system, according to the observer itself, as a result of a measurement. Therefore, we can conclude that the quantum state - or, equivalently, the values of the measured physical quantities of a system - is not absolute, but depends on the observer. Secondly, we have seen that there is no clear distinction between quantum systems and classical systems since even an observer - which we would typically consider a classical macroscopic system - can be described by another observer as is done with any quantum system. In other words, all systems appear to be equivalent. Another hypothesis concerns the completeness of the theory: we postulate that QM is a complete theory, in the sense that it is able to provide a complete and consistent description of physical reality.

All these considerations can also be summarized in a single hypothesis: *quantum mechanics is a theory about the physical description of physical systems relative to other systems, and this is a complete description of the world.* Therefore physics is completely relational and any description of the same system is correct, even if it refers to different observers.

Now that we have talked about the hypotheses underlying the new approach, supporting these with the help of an experiment, let us look at how Rovelli wishes to change the way we should conceive the quantum state and other key concepts of QM.

We have understood that relations are established between observed systems and observers when they interact. In particular we talked about a correlation between the observer O and the system S, which is reflected in the synthetic expression "$A = 1$ *for observer* O" and which concerns the state of the $S-O$ system, which has sense in reference to another observer P. In other words, the state of the $S-O$ system (as well as the state of any other system) does not have an absolute meaning, but we can talk about it and study it only from the point of view of another observer.

Now let us express this correlation in terms of "*information*" in the sense of information theory and replace the sentence above with "O *has the information that* $A = 1$" or, if we want to highlight the fact that such expression has meaning from the point of view of another observer P only, we can say "*with respect to* P*, there is some correlation between the states of* S *and* O". We acknowledge that the two statements just seen are equivalent, they are referring to the same ϵ experiment. In other terms, the information O has about S is explained only in terms of information possessed by P about the $S-O$ system. From this, it is clear how the notion of information reflects the relational features of physics. What is more, the information possessed by P about the $S-O$ system is *probabilistic*. That means P is able to analyze the relation between S and O only quantum mechanically.

We saw that information can be exchanged via physical interaction between any kind of system. Information expresses the fact that a system is in a certain state, and its state is correlated to the state of another system. In this approach information is meant as discrete and thus any acquisition of information can be decomposed into acquisition of elementary bits of information. Because of this, it is possible to characterize any observed system by a family of *yes* or *no* questions that an observer may ask, where any information is meant as the ascription of values to observables. In other words, the act of measuring observables is meant as the creation of a *question-answer* set that specifies the systems' relative states.

2.2 On the reconstruction of QM

Let us now focus on reconstructing the theory starting from some new postulates. The postulates are three in total.

We already discussed that information possessed by systems about other systems play a crucial role in this formulation. How can we express the information characterizing systems? Let us imagine it is possible to "ask" a certain amount of questions to a system. The questions coincide to the measurable observables and can be represented by a string $(Q_1, Q_2, ...)$ which will correspond to a string of possible answers $(e_1, e_2, ...)$, where each e_i can be 0 or 1 and represents the answer to the Q_i question about the observed system. Knowing a portion of $(e_1, e_2, ...)$ provides indications about subsequent answers $(e_{n+1}, e_{n+2}, ...)$. The relevant information an observer has about a system is the part of $(e_1, e_2, ...)$ significant for making predictions about future questions.

However, the first postulate states that:

Postulate 1 There is a maximum amount of relevant information that can be extracted from a system.

This means that any observed system S is characterized by a finite subset of answers $s = (e_1, ..., e_N)$ that represents the maximal *relevant* knowledge any observer O can have about S. The amount of relevant information N can be different for each system. Theoretically, the observer could ask a number of questions bigger than N, but, once it gathered all relevant information about S, it has nothing more to say about it: the N bits constitute the "information capacity" of S.

Nevertheless, if O has gathered the N relevant information about S and asks a further question, the second postulate tells us that:

Postulate 2 It is always possible to acquire new information about a system.

Postulate 2 is motivated by experimental facts: it is always possible to "learn" something new about a system by carrying out a measurement of an observable such that the state of the system is not an eigenstate of that observable.

To prevent postulate 1 and 2 from contradicting each other, when new information is acquired part of the "old" relevant information has to become irrelevant. In other words, at least one bit of the previous information is lost so that new "room" in the $s = (e_1, ..., e_N)$ string is available to gather new information, but the *total* amount of relevant information does not exceed N.

Notice that the number of questions one can ask to a system can be much larger than N since some of these questions may be not independent. They could be related by implication, union or intersection. We call $W(S)$ such set of possible questions.

From Postulate 1, we may assume that one can select in $W(S)$ an ensemble c of N questions Q_i, with $c = \{Q_i, i = 1, ..., N\}$, that are independent from each other. There may be many distinct families of N independent questions in $W(S)$ which correspond to different questions, or, equivalently, to different kind of description of S by observer O.

The answers can be represented as a string $s_c = (e_1, ..., e_N)_c$ which corresponds to the information O has about S as a result of experiments that allowed O to ask the questions in c. The string s_c can take up to $2^N = k$ values $s_c^{(1)}, s_c^{(2)}, ..., s_c^{(k)}$ such that:

$$s_c^{(1)} = (0, 0, ..., 0), s_c^{(2)} = (0, 0, ..., 1), ..., s_c^{(k)} = (1, 1, ..., 1), \qquad (2.2.1)$$

and they are mutually exclusive by construction.

We could also define 2^N new questions such that a *yes* answer to $Q_c^{(i)}$ corresponds to the string of answers $s_c^{(i)}$. Therefore, a *yes* answer to any $Q_c^{(i)}$ of the new set enables us to know the answers of all questions in c; in other words, any $Q_c^{(i)}$ potentially corresponds to N bits of information. The new questions considered $\left\{Q_c^{(1)}, ..., Q_c^{(k)}\right\}$ are referred to as "*complete questions*" and constitute a Boolean algebra. If we consider a different family of independent questions b, a different set of complete questions is obtained. Therefore, the set of questions $W(S)$ has a natural structure of an orthomodular lattice.

It is possible to study the relation between a string of information, like s_c, and another question Q considering the probability that a *yes* answer to Q will follow the string s_c:

$$p(Q, Q_c^{(i)}), \qquad (2.2.2)$$

where we used $Q_c^{(i)}$ for simplicity. Basically, the probability in (2.2.2) is a conditional probability.

If we have two complete families of information b and c, it is possible to consider the probabilities:

$$p^{ij} = p(Q_b^{(i)}, Q_c^{(j)}), \qquad (2.2.3)$$

which constitute a $2^N \times 2^N$ matrix.

The p^{ij} probabilities must comply with some conditions, such as $0 \leq p^{ij} \leq 1$ and $\sum_i p^{ij} = \sum_j p^{ij} = 1$. Such features are satisfied if

$$p^{ij} = \left|U^{ij}\right|^2, \qquad (2.2.4)$$

where U^{ij} is a unitary matrix (not fully determined by p^{ij} though).

We may also consider probabilities of the form:

$$p^{i(jk)i} = p(Q_b^{(i)}, Q_c^{(jk)} Q_b^{(i)}), \qquad (2.2.5)$$

where $Q_c^{(jk)} = Q_c^{(j)} \vee Q_c^{(jk)}$ (union). It is possible to prove that $p^{i(jk)i}$ is given by

$$p^{i(jk)i} = \left|U^{ij}U^{ji} + U^{ik}U^{ki}\right|^2. \qquad (2.2.6)$$

Having said that, the third postulate states that

Postulate 3 If c and b are two complete families of questions, then the unitary matrix U_{cb} in

$$p(Q_c^{(i)}, Q_b^{(j)}) = \left|U_{cb}^{ij}\right|^2 \tag{2.2.7}$$

can be chosen so that for every c, b and d, we have $U_{cd} = U_{cb}U_{bd}$ and the effect of composite questions is given by eq. (2.2.6).

Postulate 3 reflects the fact that a superposition principle holds for probabilities and limits the structure of the set of questions.

As a consequence, any question can be considered as a vector of a Hilbert space, a basis $|Q_c^{(i)}\rangle$ can be fixed and any other question $|Q_b^{(j)}\rangle$ can be represented as a linear combination of it:

$$|Q_b^{(j)}\rangle = \sum_i U_{bc}^{ji} |Q_c^{(i)}\rangle . \tag{2.2.8}$$

After all, we saw that any complete question $Q_c^{(i)}$ corresponds to a string of possible answers $s_c^{(i)}$ that contains N bits of information about the system. Because of this, representing a question with a vector is not so strange at this point. Any complete question $|Q_c^{(i)}\rangle$ related to the questions c represents the system state since it contains the maximal amount N of relevant information extractable from that system but it is always possible to obtain a different set of relevant information (Postulate 2). The two sets are related as in (2.2.8) and the matrices U_{bc}^{ji} emerge as unitary change of basis from the $Q_c^{(i)}$ to the $Q_b^{(j)}$ basis.

To sum up, the Hilbert spaces constituted by the vectors $|Q_c^{(i)}\rangle$ are no longer associated with a specific observable since the vectors $|Q_c^{(i)}\rangle$ may contain information about more than one observable. Nevertheless, as the basis of an observable gave us all the possible system's configurations - relatively to that observable -, similarly a set of $k = 2^N$ complete independent questions $\left\{|Q_c^{(1)}\rangle, ..., |Q_c^{(k)}\rangle\right\}$ constitute a complete basis for all possible system's descriptions - relatively to N specific questions.

Moreover, if the system is initially in a state $|Q_c^{(i)}\rangle$, then the probability of measuring the state $|Q_b^{(j)}\rangle$ is given by:

$$\left|\langle Q_c^{(i)}|Q_b^{(j)}\rangle\right|^2 , \tag{2.2.9}$$

which corresponds to $p^{ij} = |U^{ij}|^2$. Therefore the three postulates lead to the conventional formalism of the theory and to the probability rules.

So far we did not discuss about dynamics. Any question is characterized by a set of operations, which are performed by the observer at a certain time. If these same operations are performed at different times, we might consider the respective questions as different. Therefore, the time parameter t at which the question is asked becomes relevant. We can add a t label to any question Q:

$$|Q(t)\rangle \,. \tag{2.2.10}$$

If we consider two subsequent moments t_1 and t_2, with $t_1 < t_2$, then, like in section 1.4, it is possible to represent the evolution of Q as:

$$|Q(t_1)\rangle \longrightarrow |Q(t_2)\rangle \,, \tag{2.2.11}$$

and, in general, the family of questions Q defined by the same procedure performed at different times can be represented as $t \longrightarrow Q(t)$.

Since the set of questions at time t_2 has to be isomorphic to the set of questions at time t_1 if we assume time evolution is a symmetry of the theory, there should be a unitary operator $U(t_2 - t_1)$ such that

$$|Q(t_2)\rangle = U(t_2 - t_1)\,|Q(t_1)\rangle\, U^{-1}(t_2 - t_1), \tag{2.2.12}$$

for any question $|Q\rangle$. Such relations reflects the fact we used the Heisenberg picture instead of the Schödinger's, but the unitary operator U is the same of section 1.4. Indeed, from this follows equation (1.4.6) and $U(t_2 - t_1) = \exp(-i(t_2 - t_1)H)$ if H does not depend on time.

In conclusion, we can re-discuss our considerations of section 2.1 in light of the new notions introduced. We saw that any system S is characterized by a set of N relevant information that can be known by a observer O in the form of bits of information. If O extracts all the relevant information about S, it can associate with S a specific state $|Q\rangle$ - *relative to O* - which is a vector in a Hilbert space and contains N *yes* or *no* answers. Such process is possible only if an interaction between the observer and the observed system occurs.

So, we can say that "O has information about S". However, such statement refers to the state of O and, as such, it has to be relative to another observer, say P, observing O. P can get information about the information possessed by O about S only by interacting with the $S - O$ system. In P's view, the information possessed by O is expressed by the fact that O and S's states are correlated and P would obtain consistent results if it carried out measurements on S and O. Relative to P, the interaction between S and O is a fully unitary evolution. On the other hand, O is incapable of giving a full description of the interaction between S and itself, since there is no meaning in being correlated with oneself. Because of this, its description of the unitary evolution of S breaks down at the time of the interaction.

To sum up, we saw how Rovelli, starting from some postulates, reconstructs the formalism of QM. As we said at the beginning of the chapter, his intent is to understand what the theory is telling us about the world. We could state that, in some way, Rovelli wanted to concretize what many physicists and philosophers have argued throughout history, namely that properties become relational in the microscopic. In fact, many situations suggest this idea.

Having said that, each case has its specificity, it is necessary to see if the theory actually works in individual contexts leading to intuitively acceptable consequences. There are many situations where RQM could be applied; in this thesis we will look at the case of three entangled qubits in a GHZ state recently presented in [16] and the related discussion that has arisen.

3 Relative facts

> To know, is to know that you know nothing. That is the meaning of true knowledge.
>
> Plato, *Apology of Socrates*

As we said, our analysis continues with an interesting discussion that has arisen in recent years regarding an application of RQM. Our presentation begins with the work [16] of Lawrence, Markiewicz and Żukowski (LMZ from now on), where the case of a three-qubit GHZ state is presented. The authors of [16] argue that in such situation the RQM's predictions give rise to inconsistencies with the QM's ones based on GHZ correlations. Because of this, RQM seems to be incompatible with QM. The context of a three-qubit system has been considered several times, even by Rovelli himself, since the publication of [16]. We will focus on the relevant aspects of this discussion, trying to put it sufficiently in order.

In addition, at the end of the chapter, we will delve deeper into what has been said, focusing on an important postulate introduced in [1] by Adlam and Rovelli and the related response by Markiewicz and Żukowski in [17] who resume the GHZ case and make other interesting considerations.

3.1 A three-qubit GHZ state and quantum correlations

Our analysis begins considering an experiment with two observers carrying out measurements on a system of three qubit. So, let us consider a system S consisting of three entangled qubits S_m, with $m = 1, 2, 3$, in a GHZ state.

A qubit is typically meant as a simple system that can take up two values relatively to three non-commutative observables. An example is the spin of a particle measured in the three axial directions x, y and z. However, in order to be as general as possible, we will not use such notation; instead, the indices $n = 1, 2, 3$ will be considered (z, x, y respectively for the spin case) when referring to the three observables.

The relative bi-dimensional basis consists of $|+1^{(n)}\rangle$ and $|-1^{(n)}\rangle$ (for $n = 1, 2, 3$) and we add the subscript S_m in order to indicate the considered qubit, $|\pm1^{(n)}\rangle_{S_m}$, with $m = 1, 2, 3$. The basis are related as following:

$$|\pm1^{(2)}\rangle_{S_m} = \frac{1}{\sqrt{2}}(|+1^{(1)}\rangle_{S_m} \pm |-1^{(1)}\rangle_{S_m}), \tag{3.1.1}$$

and

$$|\pm1^{(3)}\rangle_{S_m} = \frac{1}{\sqrt{2}}(|+1^{(1)}\rangle_{S_m} \pm i\,|-1^{(1)}\rangle_{S_m}), \tag{3.1.2}$$

for a given qubit S_m.

The GHZ state for S, $|GHZ\rangle_S$, in terms of the state of each S_m, in the $n = 1$ basis is:

$$|GHZ\rangle_S = \frac{1}{\sqrt{2}}(|+1^{(1)}, +1^{(1)}, +1^{(1)}\rangle_{S_1S_2S_3} + |-1^{(1)}, -1^{(1)}, -1^{(1)}\rangle_{S_1S_2S_3}), \tag{3.1.3}$$

where $S_1 \otimes S_2 \otimes S_3 = S_1S_2S_3$ for the sake of brevity and, as in the next formulas, the labels in the kets refer respectively to S_1, S_2 and S_3.

Hence, relative to the $n = 1$ basis, there are only two scenarios: all qubits S_m are concurrently either in the state $|+1^{(1)}\rangle$ or in the state $|-1^{(1)}\rangle$. In the spin case, the superposition (3.1.3) can be written as $\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle_{zzz} + |\downarrow\downarrow\downarrow\rangle_{zzz})$, where the subscripts specify the basis. If an observer carries out measurements on S, the outcomes of the qubits are not independent from each other.

We can schematically re-express the $|GHZ\rangle_S$ state as:

$$|GHZ\rangle_S = \frac{1}{\sqrt{2}}(\otimes_{m=1}^{3}|+1^{(1)}\rangle_{S_m} + \otimes_{m=1}^{3}|-1^{(1)}\rangle_{S_m}). \tag{3.1.4}$$

Let us consider now an observer A that carries out measurements on S, thus on every qubit S_m, in the $n = 3$ basis. In order to study such experiment, we would need to re-write the $|GHZ\rangle_S$ state in terms of the $n = 3$ basis. This is not a big deal, the $|GHZ\rangle_S$ state will be of the form:

$$|GHZ\rangle_S = \sum_{p,q,r=\pm1} c^{333}_{pqr}\,|p^{(3)}\rangle_{S_1}|q^{(3)}\rangle_{S_2}|r^{(3)}\rangle_{S_3}\,, \tag{3.1.5}$$

where the coefficients c^{333}_{pqr} are given by inner products of vectors of the $n = 1$ and $n = 3$ basis. Note that there are no correlations between measurement outcomes in the $n = 3$ basis, since all possible scenarios - eight in total - are characterized by the same probability given by $|c^{333}_{pqr}|^2 = \frac{1}{8}$.

As we saw in chapter 2, any RQM-measurement provides an outcome *relative* to the observer. Because of this, we call such outcomes *"relative facts"* obtained by an observer. As the relative approach argues, an outcome which is a relative fact for an observer is not a relative fact for a different observer. Hence, A obtains three relative facts $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$, where every $\mathbf{A}_i$ can be either $+1$ or -1 and refers to the measurement on the qubit S_i.

Furthermore, we saw that any RQM-measurement is a quantum interaction between the observer A and the system S resulting in a entangled state of the the system $S - A$:

$$|init\rangle_A \, |GHZ\rangle_S \longrightarrow |GHZ\rangle_{SA} \, , \tag{3.1.6}$$

where SA is for $S \otimes A$.

In this case, S is composed of three subsystems so, if the entangling processes commute, we can see the entangling mechanism as regarding each qubit:

$$|init\rangle_A \sum_{p,q,r=\pm 1} c^{333}_{pqr} \, |p^{(3)}\rangle_{S_1} \, |q^{(3)}\rangle_{S_2} \, |r^{(3)}\rangle_{S_3} \longrightarrow \sum_{p,q,r=\pm 1} c^{333}_{pqr} \, |p^{(3)}\rangle_{SA_1} \, |q^{(3)}\rangle_{SA_2} \, |r^{(3)}\rangle_{SA_3} \, , \tag{3.1.7}$$

and such process (although [16] does not specify it) is to be meant as "seen" from the point of view of another observer.

Let us now consider a second observer B that carries out measurements on the $S - A$ system in the $n = 2$ basis - namely, three measurements for each subsystem SA_m. Similarly to A, B will obtain three relative facts $\{\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$, relative to the $n = 2$ basis.

Again, the $S - A$ system is to be studied in the $n = 2$ basis and thus its state will be of the form:

$$|GHZ\rangle_{SA} = \sum_{p,q,r=\pm 1} c^{222}_{pqr} \, |p^{(2)}\rangle_{SA_1} \, |q^{(2)}\rangle_{SA_2} \, |r^{(2)}\rangle_{SA_3} \, , \tag{3.1.8}$$

and the quantum interaction between B and the $S - A$ system results in an entangled $|GHZ\rangle_{SAB}$ state, where $SAB = S \otimes A \otimes B$, relative to an external observer. Explicitly, the final state of the system SAB will be:

$$|GHZ\rangle_{SAB} = \sum_{p,q,r=\pm 1} c^{222}_{pqr} \, |p^{(2)}\rangle_{SAB_1} \, |q^{(2)}\rangle_{SAB_2} \, |r^{(2)}\rangle_{SAB_3} \, . \tag{3.1.9}$$

Now we are going to see the correlations between measurement outcomes.

If we explicit a generic GHZ state in the $n = 2$ basis (like (3.1.8)) the result will be:

$$\begin{aligned} |GHZ\rangle_S = \frac{1}{\sqrt{2}} (& |{+1^{(2)}}, {+1^{(2)}}, {+1^{(2)}}\rangle_{S_1S_2S_3} + |{+1^{(2)}}, {-1^{(2)}}, {-1^{(2)}}\rangle_{S_1S_2S_3} + \\ & |{-1^{(2)}}, {+1^{(2)}}, {-1^{(2)}}\rangle_{S_1S_2S_3} + |{-1^{(2)}}, {-1^{(2)}}, {+1^{(2)}}\rangle_{S_1S_2S_3} \, . \end{aligned} \tag{3.1.10}$$

It is clear how the products of all possible sets of measurement outcomes are always $+1$. Hence, the first correlation between outcomes is:

$$p^{(2)} q^{(2)} r^{(2)} = +1, \tag{3.1.11}$$

for $p, q, r = \pm 1$.

Note that the correlation (3.1.11) is a quantum prediction for a system of this kind, thus is independent of the approach chosen. In other words, if a theory claims to be a re-interpretation of QM (we use the term "re-interpretation" for simplicity), it has to respect the quantum predictions, otherwise it is incompatible. We will see four constraints in total and all of these correspond to quantum predictions, so we expect all four to be respected by re-interpretations of the theory.

The remaining three correlations are similar, so we can focus on just one of those. For example, we can imagine an experiment in which we want to make a measurement in the $n = 2$ basis on the S_1 qubit and a measurement in the $n = 3$ basis for each of the remaining qubits S_2 and S_3. The global GHZ state of S would be:

$$\begin{aligned}|GHZ\rangle_S = \frac{1}{\sqrt{2}}(&|-1^{(2)}, -1^{(3)}, -1^{(3)}\rangle_{S_1S_2S_3} + |-1^{(2)}, +1^{(3)}, +1^{(3)}\rangle_{S_1S_2S_3} + \\ &|+1^{(2)}, -1^{(3)}, +1^{(3)}\rangle_{S_1S_2S_3} + |+1^{(2)}, +1^{(3)}, -1^{(3)}\rangle_{S_1S_2S_3}.\end{aligned} \tag{3.1.12}$$

It is not so confusing if we keep in mind that the first label in the kets refers to S_1 in the $n = 2$ basis and the remaining ones refer respectively to S_2 and S_3 in the $n = 3$ basis.

In this case, the correlation arises from the fact that the products of measurement outcomes give -1 in all four scenarios. Hence:

$$p^{(2)}q^{(3)}r^{(3)} = -1, \tag{3.1.13}$$

for $p, q, r = \pm 1$.

If we permute the qubits, similar correlations emerge:

$$p^{(3)}q^{(2)}r^{(3)} = -1, \tag{3.1.14}$$

and

$$p^{(3)}q^{(3)}r^{(2)} = -1. \tag{3.1.15}$$

The four correlations (3.1.11), (3.1.13), (3.1.14) and (3.1.15) can be put together:

$$\begin{aligned} p^{(2)}q^{(2)}r^{(2)} &= +1 \\ p^{(2)}q^{(3)}r^{(3)} &= -1 \\ p^{(3)}q^{(2)}r^{(3)} &= -1 \\ p^{(3)}q^{(3)}r^{(2)} &= -1, \end{aligned} \tag{3.1.16}$$

and those are individually true, although it is not possible to verify all of them in a single experiment, since there are six possible variables in total. Furthermore, it is not even possible to solve them simultaneously, given that there are not six numbers that solve all the four constraints. In the context of standard QM, such contradiction leads us to only a few options. Either QM is to be considered non-local and that means signals can travel faster than speed of light or QM's values are indefinite before measurement - that is similar to an agnostic approach to QM. Actually, there is also a little known third option, namely "superdeterminism" [13], which preserves locality and value definiteness.

Now, in the RQM context, we said that observer A gets three relative facts $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$ after measuring the S system in the $n = 3$ basis and, similarly, observer B obtains three relative facts $\{\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$ after carrying out measurements on the $S-A$ system in the $n = 2$ basis. We can put together all the outcomes in a unique set $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$ and re-express the four constraints in terms of relative facts of A and B:

$$\begin{aligned} \mathbf{B}_1\mathbf{B}_2\mathbf{B}_3 &= +1 \\ \mathbf{B}_1\mathbf{A}_2\mathbf{A}_3 &= -1 \\ \mathbf{A}_1\mathbf{B}_2\mathbf{A}_3 &= -1 \\ \mathbf{A}_1\mathbf{A}_2\mathbf{B}_3 &= -1, \end{aligned} \tag{3.1.17}$$

and, if we multiply all equations, we get:

$$\mathbf{A}_1^2\mathbf{A}_2^2\mathbf{A}_3^2\mathbf{B}_1^2\mathbf{B}_2^2\mathbf{B}_3^2 = -1. \tag{3.1.18}$$

Now, only two scenarios are possible at this point. We can have that observer A obtained three relative facts $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$, but B can not produce relative facts $\{\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$ that satisfy the constraints in (3.1.17). On the other hand, B could obtained three relative facts $\{\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$, but A could not obtain three relative facts satisfying (3.1.17).

In light of this, the authors of [16] argue that RQM's relative facts arise contradictions with quantum predictions and, thus, RQM is incompatible with QM.

To sum up, we saw that a system S composed of three entangled qubits S_m ($m = 1, 2, 3$) is characterized by the four constraints (3.1.16). If the numbers p, q, r in the formulas are interpreted as relative facts of two different observers A and B, we obtain eq. (3.1.18) that apparently shows the impossibility of contextually having two sets of real numbers $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$ and $\{\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$.

Remember that the quantum predictions (3.1.16) are independent of RQM or any other re-interpretation. As we said, any prediction of QM is to be satisfied by RQM. In the next sections, we are going to discuss whether this line of reasoning is somehow justified or, in other words, whether LMZ *really* proved the incorrectness of RQM.

3.2 Density matrix - a first attempt to save RQM

Let us discuss the results presented in [9] by Aurelien Drezet, in which the author re-express the experiment in terms of the density matrix.

As we saw, a system's state can be described by a vector $|\alpha\rangle$ in the form of a linear combination of eigenstates relative to an observable. $|\alpha\rangle$ is a *pure state*. We could have a mechanism that casually produces several copies of the same system in different pure states $|\alpha_1\rangle$, $|\alpha_2\rangle$, $|\alpha_3\rangle$, ..., and the probability of obtaining the $|\alpha_p\rangle$ pure state out of the mixture is w_p (with $\sum_p w_p = 1$). Such set of pure states corresponds to a *mixed state*.

The mixed state can be described by the operator:

$$\rho = \sum_p w_p \left|\alpha\right\rangle \left\langle\alpha\right| , \tag{3.2.1}$$

and, as any other operator, is expressible in terms of a matrix by using the eigenstates of any operator. ρ is called the *density matrix*.

In the case of a system composed of two subsystems M and N, each of the subsystem is described by a *reduced density matrix*. For example, if we want to describe the subsystem M, its reduced density matrix will be:

$$\rho_M^{(red.)} = Tr[\rho_{MN}], \tag{3.2.2}$$

where ρ_{MN} is the density matrix of the global system.

In the context of RQM, any quantity describing a system, including the density matrix, is to be meant as relative to a certain observer. Because of this, in our experiment involving system S and observers A and B, the density matrix of S, in A's view, can be written as:

$$\rho_{S_A}^{(red.)} = \frac{1}{8}\sum_{p,q,r} |p^{(3)}\rangle_{S_1\ S_1}\langle p^{(3)}| \otimes |q^{(3)}\rangle_{S_2\ S_2}\langle q^{(3)}| \otimes |r^{(3)}\rangle_{S_3\ S_3}\langle r^{(3)}|, \tag{3.2.3}$$

in which we sum the entangled states of the subsystems S_m relative to A in the $n = 3$ basis.

Observer A can use ρ_{S_A} in order to evaluate the expectation value of any observables associated with the qubits. In the spin notation the observables are of the form σ_{nS_m} (with $n = x, y, z$) and, due to decoherence [9], the expectation values after the measurement are:

$$\begin{aligned}
Tr_S(\sigma_{xS_1}\sigma_{xS_2}\sigma_{xS_3}\rho_{S_A}^{(red.)}) &= 0\\
Tr_S(\sigma_{xS_1}\sigma_{yS_2}\sigma_{yS_3}\rho_{S_A}^{(red.)}) &= 0\\
Tr_S(\sigma_{yS_1}\sigma_{xS_2}\sigma_{yS_3}\rho_{S_A}^{(red.)}) &= 0\\
Tr_S(\sigma_{yS_1}\sigma_{yS_2}\sigma_{xS_3}\rho_{S_A}^{(red.)}) &= 0,
\end{aligned} \tag{3.2.4}$$

while before the measurement the expectation values were:

$$\begin{aligned}
Tr_S(\sigma_{xS_1}\sigma_{xS_2}\sigma_{xS_3}\rho_{S}^{(red.)}) &= +1\\
Tr_S(\sigma_{xS_1}\sigma_{yS_2}\sigma_{yS_3}\rho_{S}^{(red.)}) &= -1\\
Tr_S(\sigma_{yS_1}\sigma_{xS_2}\sigma_{yS_3}\rho_{S}^{(red.)}) &= -1\\
Tr_S(\sigma_{yS_1}\sigma_{yS_2}\sigma_{xS_3}\rho_{S}^{(red.)}) &= -1.
\end{aligned} \tag{3.2.5}$$

It is not so clear what $\rho_S^{(red.)}$ means since it is a quantity given by the possible states of S but it is not specified respect to whom such states are considered. Anyway, we may consider the expectation values in (3.2.5) as the outcomes that A should expect before measuring the spins since the constraints are to be satisfied.

In any case, Drezet argues that the expectation values become zero since the measurement from A causes loss of coherence and correlations between the qubits. This is part of the central reasoning of Drezet. The author claims that the measurement process is more "invasive" than what LMZ think.

The measurement performed by B on the entangled system $S \otimes A$ in the $n = 2$ basis gives us the state (3.1.9) and the density matrix used by B to describe the $S \otimes A$ system will be:

$$\rho_{SA_B}^{(red.)} = \frac{1}{4} \sum_{p,q,r} |p^{(2)}\rangle_{SA_1\ SA_1}\langle p^{(2)}| \otimes |q^{(2)}\rangle_{SA_2\ SA_2}\langle q^{(2)}| \otimes |r^{(2)}\rangle_{SA_3\ SA_3}\langle r^{(2)}|, \quad (3.2.6)$$

where the states of the SA_m qubits are explicit.

Again, B uses $\rho_{SA_B}^{(red.)}$ to evaluate the expectation values after its measurement:

$$\begin{aligned} Tr_{SA}(\sigma_{xSA_1}\sigma_{xSA_2}\sigma_{xSA_3}\rho_{SA_B}^{(red.)}) &= +1 \\ Tr_{SA}(\sigma_{xSA_1}\sigma_{ySA_2}\sigma_{ySA_3}\rho_{SA_B}^{(red.)}) &= 0 \\ Tr_{SA}(\sigma_{ySA_1}\sigma_{xSA_2}\sigma_{ySA_3}\rho_{SA_B}^{(red.)}) &= 0 \\ Tr_{SA}(\sigma_{ySA_1}\sigma_{ySA_2}\sigma_{xSA_3}\rho_{SA_B}^{(red.)}) &= 0, \end{aligned} \quad (3.2.7)$$

where a partial coherence is preserved.

Furthermore, associated with the state (3.1.9), Drezet explicits the constraint

$$p_{SAB_1}^{(2)} q_{SAB_2}^{(2)} r_{SAB_3}^{(2)} = +1. \quad (3.2.8)$$

Drezet agrees with LMZ about the fact that the three variables p, q, r in the constraint (3.2.8) are to be meant as relative facts of B. In contrast, he argues that the remaining three constraints do not enable us to mix the relative facts of A and the relative facts of B and that we cannot compare the constraints with one another. In other words, those cannot be used as a proof supporting the incorrectness of RQM. His reasoning goes as following.

Drezet argues that the remaining constraints should be associated respectively with a single interaction between B and a qubit SA_m. Such interactions can be meant as interactions between a qubit of B, B_m, and the respective qubit SA_m. For example, the constraint (3.1.13) is associated with a process leading to the entangled state:

$$|GHZ'\rangle_{SAB} = \frac{1}{2} \sum_{p,q,r} |p^{(2)}\rangle_{SAB_1} |q^{(2)}\rangle_{SA_2} |init\rangle_{B_2} |r^{(2)}\rangle_{SA_3} |init\rangle_{B_3}, \quad (3.2.9)$$

where B_1 interacted with SA_1 while B_2 and B_3 remained in their initial state.

Basically, while the authors of [16] argue that B's measurements can be considered simultaneous since the three entangling processes commute, Drezet focuses on each interaction $B_m - SA_m$ thinking that LMZ considered those non-contextual and absolute.

In the context of RQM, the correlation (3.1.13) becomes:

$$p^{(2)}_{SAB_1} q^{(3)}_{SA_2} r^{(3)}_{SA_3} = -1, \tag{3.2.10}$$

associated with state $|GHZ'\rangle_{SAB}$ and we could associate similar relations with analogous processes for $m = 2$ and $m = 3$ leading to states $|GHZ''\rangle_{SAB}$ and $|GHZ'''\rangle_{SAB}$. In total we would have three constraints.

Now, LMZ interpreted the variables in the three constraints as relative facts: in (3.2.10), $p^{(2)}_{SAB_1}$ is a relative fact for B and $q^{(2)}_{SA_2}$ and $r^{(2)}_{SA_3}$ are relative facts for A. In constrast, Drezet argues that, while $p^{(2)}_{SAB_1}$ is a relative fact for B, there is no reason to consider $q^{(2)}_{SA_2}$ and $r^{(2)}_{SA_3}$ relative facts for A.

In Drezet's view, the authors of [16] compared the three constraints with the first (3.2.8) because they thought the interactions $B_m - SA_m$ were non-contextual. This, together with the possibility of mixing the relative facts of A and B as in (3.2.10) enable LMZ to reach their conclusion, Drezet thinks.

For the latter, in the process leading to $|GHZ'\rangle_{SAB}$, B should describe the $S \otimes A$ system by using the density matrix:

$$\begin{aligned} \rho^{(red.)}_{SA_B} =& \frac{1}{2} \, |{+1}^{(2)}\rangle_{SA_1\, SA_1}\langle {+1}^{(2)}| \\ & \otimes |\phi\rangle_{SA_2,SA_3\; SA_2,SA_3}\langle\phi| \\ +& \frac{1}{2} \, |{-1}^{(2)}\rangle_{SA_1\, SA_1}\langle {-1}^{(2)}| \\ & \otimes |\psi\rangle_{SA_2,SA_3\; SA_2,SA_3}\langle\psi|, \end{aligned} \tag{3.2.11}$$

where

$$|\phi\rangle_{SA_2,SA_3} = \frac{1}{\sqrt{2}}(|{+1}^{(3)}\rangle_{SA_2} |{-1}^{(3)}\rangle_{SA_3} + |{-1}^{(3)}\rangle_{SA_2} |{+1}^{(3)}\rangle_{SA_3}), \tag{3.2.12}$$

and

$$|\psi\rangle_{SA_2,SA_3} = \frac{1}{\sqrt{2}}(|{+1}^{(3)}\rangle_{SA_2} |{+1}^{(3)}\rangle_{SA_3} + |{-1}^{(3)}\rangle_{SA_2} |{-1}^{(3)}\rangle_{SA_3}). \tag{3.2.13}$$

Again, B uses the density matrix to calculate the expectation values after the interaction $B_1 - SA_1$:

$$\begin{aligned} Tr_{SA}(\sigma_{xSA_1}\sigma_{xSA_2}\sigma_{xSA_3}\rho^{(red.)}_{SA_B}) &= +1 \\ Tr_{SA}(\sigma_{xSA_1}\sigma_{ySA_2}\sigma_{ySA_3}\rho^{(red.)}_{SA_B}) &= -1 \\ Tr_{SA}(\sigma_{ySA_1}\sigma_{xSA_2}\sigma_{ySA_3}\rho^{(red.)}_{SA_B}) &= 0 \\ Tr_{SA}(\sigma_{ySA_1}\sigma_{ySA_2}\sigma_{xSA_3}\rho^{(red.)}_{SA_B}) &= 0, \end{aligned} \tag{3.2.14}$$

where a partial coherence is preserved. The second line is not the same of (3.2.10), since the traces in (3.2.14) concern only B and not A. Therefore, following Drezet's reasoning, B gives a description of $S \otimes A$ that seems to provide a constraint on the outcomes. However, such relation regards B's description of $S \otimes A$ and we are not allowed to mix B's relative facts with the ones of A. Furthermore, Drezet says, the process leading to (3.1.9) and the processes leading to $|GHZ'\rangle_{SAB}$, $|GHZ''\rangle_{SAB}$ and $|GHZ'''\rangle_{SAB}$ belong to different contexts and we can not compare the relative constraints to deduce an incorrectness of RQM, given that different contexts are not comparable in RQM.

Now, is it really true that the authors of [16] assume non-contextuality? And, is it appropriate to use an approach based on quantum states for RQM, as Drezet does?

Regarding the first question, LMZ do not assume non-contextuality, but simply replace the variables in the constraints with the relative facts of A and B. It is not even a matter of comparing relative facts of different observers - this would in fact be against the axioms of the theory [8] - but simply verifying whether the relative facts satisfy the predictions of QM, as they should do, given that they must follow the same constraints as the eigenvalues of observables in the standard formulation. To do this, LMZ study products between relative facts of different observers and deduce a contradiction.

Secondly, an approach like that of Drezet, thus based on the concept of physical state, is not similar to Rovelli's conception of RQM, given that the latter bases the ontology of the theory not on states but on relative facts [8]. Drezet instead gives a central role to the physical states of the systems from an ontological point of view. As a consequence, his equations are less correlated than those of LMZ. For example, in (3.2.14) we do not mix relative facts of A and B since the equations simply regard B's description of $S \otimes A$ and thus do not reveal any contradiction. However, this is precisely due to the fact that we are focusing on states and not on relative facts, and not because the theory - at least as Rovelli intends it - predicts that there are no correlations.

As the authors of [16] remind us, Rovelli conceives RQM as a theory that concerns facts, i.e. events, relative to observers, and not physical states. This is Rovelli's conception. Then obviously it is not a given that we can not deviate from what he says and create different approaches within RQM, thus changing the hierarchical order of the fundamental concepts of the theory. Many philosophers - rather than physicists - tried to give reality to the physical state, but this is a controversial deviation from the more accepted idea that the physical state is simply a useful object for making predictions. In any case, Drezet seems to combine the principles presented by Rovelli with his own view and ultimately creates an approach partly different from the relational one that is unable to answer the issue raised in [16].

3.3 Quantum predictions: directions for use - a second attempt to save RQM

Another interesting response to LMZ is [5] from Cavalcanti, Di Biagio and Rovelli (CDR from now on). In this latter article the issue raised in [16] is criticized by re-discussing the experiment of A and B from the point of view of a third observer W. The main criticism made by CDR concerns the "comparison" that LMZ do between outcomes obtained by different observers. We used the inverted commas to specify that it is necessary to decide whether LMZ *really* compare relative facts of different observers, as the authors of [5] argue.

So, CDR remind us that "it is meaningless to compare events relative to different systems, unless this is done relative to a (possibly third) system" and "comparisons can only be made by a (quantum-mechanical) interaction". Hence, following their reasoning, the constraints do not make sense if the three observers do not interact with each other or with another external observer.

Hence, the authors consider an external observer (W) who interacts with A and B after their measurements. W obtains relative facts that must satisfy the constraints (3.1.16). For example, if W interacts with one of the qubits of B, e.g. B_1, after it made the measurement, W would obtain a relative fact, B_1^{W1}, where the 1 in "$W1$" stands for "first set of measurements of W". Similarly, if W interacts with B_2 and B_3, W would obtain relative facts B_2^{W1} and B_3^{W1}, respectively. This first set of relative facts of W satisfies (3.1.11). Explicitly:

$$B_1^{W1} B_2^{W1} B_3^{W1} = +1, \tag{3.3.1}$$

where we simply substituted the variables in (3.1.11) with the relative facts of W.

Let us now suppose that, in a second experiment, W interacts with B_1 and then with A_2 and A_3. Analogously, W would obtain three relative facts: B_1^{W2}, A_2^{W2} and A_3^{W2}. The second set of relative facts satisfies (3.1.13). Explicitly:

$$B_1^{W2} A_2^{W2} A_3^{W2} = -1. \tag{3.3.2}$$

We could obtain similar equations associated with (3.1.14) and (3.1.15) considering analogous experiments.

The point made by CDR is that, for observer W, all four constraints (3.1.16) are to be individually satisfied by its relative facts and W might use those to predict its outcomes before a specific set of measurements, *but* one can not argue that all four constraints are to be simultaneously satisfied, given that W can only perform one set of measurements and can not simultaneously possess all six corresponding relative facts that we would substitute inside the constraints (3.1.16). As we said, W can at most verify that the constraints hold individually and use them to predict its triplet of outcomes.

Now, are we really sure that Lawrence et al. compared the relative facts of different observers? In other words, is it really necessary to introduce a third observer W to discuss about the constraints (3.1.16) in the context of RQM?

LMZ responded to[5] in a recent paper [14]. They are not sure that CDR actually have proved them wrong.

The authors point out that the situation presented in [5] is very different from the one they consider. CDR consider six different observers A_i, B_j (with $i, j = 1, 2, 3$) and an external observer W. The latter can have access to the measurement results (relative facts) of the other observers by interacting with them. In particular, they consider four scenarios - experiments performed by W - in which several sets of interactions between different systems are considered. We can associate with each set of measurements a constraint for the relative facts obtained W.

The point - as we have seen above - is that each constraint is satisfied if we consider one specific experiment of W, *but* not all of them can be satisfied at the same time since they refer to different scenarios, thus the relative facts that appear in different constraints are not comparable to each other.

However, LMZ consider a different situation. For them the experiment (i.e. the sequence of measurements) is only one and is conducted by two different observers, namely A and B. Thus their criticism - that it makes no sense to claim that the four constraints are to be satisfied simultaneously - is not applicable to their case since CDR arrive at this criticism starting from a very different situation. In other words, the two articles seem to start from two different points that lead to the same constraints. One demonstrates that the relative facts do not satisfy the constraints, the other does not.

To sum up, LMZ are not convinced that it is necessary to split the two observers A and B into three sub-systems or even introduce a new observer W to discuss the constraints. To them, it looks like a forced effort by CDR to solve a problem that does not actually emerge. The central point of the discussion is to understand whether LMZ make an actual comparison between relative facts of different observers or not. In the first case we must give credence to [5] and thus it is not possible to demand that all four constraints are to be satisfied. In such scenario, RQM is free from criticisms regarding inconsistencies in the case of a GHZ state for three spins. On the other hand, if LMZ have not carried out a comparison, then inconsistencies with quantum predictions actually emerge.

The situation we described in section 3.1 can be reformulated in these terms. Systems A and B make a measurement on S. This act can be meant as the attribution of one or more properties to S - i.e. the value of one or more of the observables $n = 1, 2, 3$ - with respect to observer A or B. If we call any of the attributions Z, we can say that Z is absolute. However, if I somehow want to talk about Z, I cannot do so except by considering another system W. This is simply a reformulation of what has already been said. According to CDR, LMZ are wrong since they talk about Z independently of any external system W. On the other hand, for LMZ this is not necessary.

Finally, LMZ do not agree with an operation introduced in [5] regarding the measurement of one of the qubits of B. In their article, CDR explain that the measurement performed by B_1 on the entangled system SA_1, for example, can be meant as an operation that includes a "re-establishment" of the measurement previously made by A_1 and a measurement of B_1 "directly" on S_1 (and not on the entangled system). It is as if the initial states of A_1 and S_1 were re-established, and then the measurement by B_1 on S_1 was considered. However, this does not convince LMZ; according to them, it conflicts with the postulates of RQM and it opens up a question about what it means to realize a relative fact by an observer. For example, if A is human, he or she will witness the measurement process on one of S's qubits and see the process as non-unitary. But, what remains of this non-unitary interaction, as we inverse such process? Is it destroyed? And, if A is a simple qubit, what does it mean for a qubit to realize a relative fact? LMZ believe that such "re-establishment" of the measurement process is meaningless in QM and that it can not be tested by any experiment.

These questions are related to other issues in [5], which the authors of [14], or rather two of them, address in a recent article that we will discuss in the next section.

3.4 Are relative facts hidden variable?

In this last section we will see a postulate introduced in [1] by Adlam and Rovelli with the aim of solving some problems of RQM. The postulate leads to some interesting consequences which we will mention, but what interests us most is the connection with relative facts. For this reason, we will also analyze the response from Markiewicz and Żukowski in [17], which attempt to show other inconsistencies of the theory, even with the new postulate. Markiewicz and Żukowski propose a new scenario with two entangled photons where to apply the postulate and finally consider again the context of a three-qubit system in a GHZ state.

So, in section 2.1 we mentioned that a consistency condition regarding outcomes of different observers must hold. If an observer A makes a measurement on a system S, it will obtain a relative fact. Let us consider also a second observer B who performs a measurement on S and interacts with A to find out what value A obtained in its previous measurement. The consistency condition tells us that the two measurements of B will be coherent. We can consider this a fundamental postulate of RQM.

Anyway, RQM, in this formulation, does not guarantee that the measurement of A on S and the interaction of B with A are coherent [1]. Indeed, asking whether these two results are consistent does not even make sense within the theory because a comparison presupposes an interaction and thus an observer with respect to which this comparison can actually be made - this is typically referred to as *"postulate of relativity of comparisons"*. However, even when considering a third observer W the problem persists. Indeed, the third observer would be able to do nothing but take measurements on the systems - that is interact with them - and obtain relative facts. Nevertheless, what W has obtained is information valid only for W; even if these are consistent with each other, we can not argue that the measurement made by A on S is consistent with the measurement that B made on A. In fact, the third observer is comparing *its* relative facts and not the relative facts of A and B. To be sure one would have to compare *their* relative facts, but with this formulation of the theory this is not possible. Therefore, it seems that each observer can only rely on its own results and can not state anything about the results obtained by other observers. All this might make us think that it is not even possible to make comparisons between measurements and data from different observers to empirically confirm hypotheses.

The authors of [1] present the so-called "*cross-perspective links* (CPL)" postulate to overcome this problem. The CPL postulate is meant to substitute the postulate about relativity of comparisons and it tells us that relative facts obtained by two different observers - under appropriate conditions - will be consistent. Now, thanks to this postulate, the measurement of B on A will be consistent, not only with the measurement of B on S - which was already guaranteed -, but also with the measurement of A on S. Hence, previously, B used to have information only on its representation of A's knowledge, but now B has information *also* on A's knowledge - namely, B is certain that, for A, S has a certain value relative to the observable considered. In the CPL postulate, information on the knowledge of A are physical variables (or rather are contained in variables) accessible to other observers. In a nutshell, by transitive property, the measurement of B on S and the measurement of A on S will be consistent.

The authors explain that the act of measurement of B on A is conceivable as looking back in time to the moment of A's last interaction - in this case with S - in which the variable took on a defined value for A. Therefore, interacting with a system means this and not looking at the state of A at the moment of the interaction with A. Furthermore, the authors point out that, in order to understand which information is accessible to other observers, it is necessary to understand when information, in general, is no longer accessible. This, they argue, happens when another variable that does not commute with the variable we are interested in takes on a defined value. According to Adlam and Rovelli, this hardly happens if the observer is human, it is more likely if the observer is, for example, another qubit.

Moreover, the authors point out that by adding the CPL postulate, it is no longer necessary to say that the facts obtained by a certain observer are *relative* to that observer, since any other observer could theoretically know the information. In any case, until an interaction between observers occurs, it is fair to say that the value obtained is relative to the observer who made the measurement. This reflects a change in RQM from an ontological point of view.

The state of one system with respect to another consists in the description of the common history of the two systems, thus it concerns the set of interactions between the two systems (and the "indirect" interactions, i.e. interactions of one of the two systems with an external system that is somehow connected to the second system by some continuous chain of interactions) and allows us to predict the results of a possible future interaction between the two. Therefore, the state always concerns the common history, so it can happen that there will be irrelevant information that concerns one system but not the other. The authors also underline the fact that when a new interaction occurs the relative state must update since, when this happens, one or more past interactions become irrelevant, in the sense that they are not useful for predicting new interactions. Indeed, the purpose of a relative state is to describe information regarding the common history of the two systems that is relevant for their future interactions. This is what Rovelli says in his first article [21], when he states that if new information is acquired, part of the old information that was relevant is now no longer relevant (Postulate 2).

Moreover, the authors of [1] claim that, in general, the state of one system with respect to another is an objective physical fact and not simply a set of knowledge possessed by one system about another - in this sense "*information is physical*". In any case, when the observer has a consciousness we can state without fear that the state of a system he/she wants to describe essentially corresponds to the information that the observer has about the system - therefore it corresponds to his knowledge. This is also because the theory, for practical purposes during experiments, must assume that this can be said. Furthermore, an interesting thing that the authors highlight is that, if the information on a certain system is physical, it means that our knowledge of that system is a physical object and all this brings new life to the dichotomy that often exists between epistemic and ontic, since, in this picture, knowledge becomes ontological.

Finally, it is pointed out that, in general, an interaction between two systems does not have the form of a measurement process, since it will not generally result in the definition of a single variable with respect to the observing system. This means that we could have that during the interaction more than one variable, with respect to the observer, takes on a defined value. This is because the interaction Hamiltonian can be written using a different basis than that of the observable that interests us so as to describe the measurement of another quantity. The authors propose two solutions for this issue. The first looks like an ad hoc solution. An idea could be to note that all measurements can be traced back to position measurements, so we could use the position basis as a privileged basis with respect to which a defined value will be assumed. An alternative could be to consider that, in general, there are more variables that take on a defined value with respect to the observer. In the case of two qubits the theory does not say what happens, so it is sufficient to concentrate on the case of a macroscopic observer and show that in this case there is only one variable that takes on a defined value in the interaction. According to the authors, the solution could be decoherence since it seems to be the process that allows us to "select" a privileged basis with respect to which a certain observable of the observed system takes on a defined value, while all the others are no longer accessible.

In conclusion, the main novelty brought by Adlam and Rovelli is the introduction of the CPL postulate which guarantees consistency between relative facts of different observers. This postulate leads us to reconsider what is meant by measurement - in particular what is meant by exchange of information between observers. Indeed, the information possessed by an observer becomes accessible to other observers. Because of this, the postulate revolutionizes our way of understanding events in RQM, given that, although physical states are still relative - they describe the relation between the observer and the system, rather than an absolute feature of the system -, quantum events are to be meant as objective. As a matter of fact, if a system interacts with two different observers, the relative facts obtained by the two observers must match. We can never have the situation in which the values are defined but different for the two observers. At most, it can happen that the value is defined for one of the two observers and still related to the system preparation for the other observer.

Now we continue our discussion regarding the coherence of relative facts of different observers by analyzing a recent article, [17], by Markiewicz and Żukowski, in which some scenarios are discussed that allow to study RQM in detail with the addition of the CPL postulate.

The authors of [17] consider the case of a system composed of two entangled photons S_1 and S_2 which constitute a single system S:

$$|ENT\rangle_S = \sum_{l=0,1} c_l \, |l\rangle_{S_1} \, |l\rangle_{S_2} \,, \tag{3.4.1}$$

and the two photons can be meant as two qubits. Note that the observable considered is either 0 or +1 for both photons.

We also consider two observers in two different laboratories who measure the same observable on one of the two photons. Observer A performs the measurement on S_1 and observer B on S_2. Let us begin by considering A who interacts with S_1 and therefore obtains - in an RQM measurement - a relative fact with respect to itself.

B does not know what the state of the composite system $S = S_1 \otimes S_2$ is, but it knows that the system $S \otimes A$ is entangled, since it knows that A and S interacted. The point is that - if we follow RQM - B can only state that the probability of obtaining a certain value of the observable for S_2 will be given by the square of the modulus of the corresponding coefficient in the decomposition of the state $S \otimes A$:

$$|ENT\rangle_{SA} = \sum_{l=0,1} c_l \, |l\rangle_{S_1} \, |l\rangle_{S_2} \, |l\rangle_A \,. \tag{3.4.2}$$

As a matter of fact, the state (3.4.2) that B attributes to $S \otimes A$ contains all the information at its disposal to make predictions on future results. The most important aspect is that, if we imagined changing A's result on S_1, we would have no change in B's attributions about the results of his measurement. Indeed, the probabilities would not change since A's result is not contained in the state (3.4.2), which is the one and only "source" of information for B to make predictions. The authors claim that the relative facts of A and B are not "EPR correlated", although perhaps it would be more correct to say that there is no causal nexus.

In a standard QM situation, B will always be able to say that - if B knows that A has made the measurement - its result on S_2 will have to match A's result on S_1. And, if A tells B its result, B's attribution of probability on its results will change - it becomes a deterministic attribution. We might wonder what happens in an RQM context in case B interacts with A to "know" its result on S_1. Without the CPL postulate, this operation is an end in itself since B, by interacting with A, obtains a certain relative fact which we cannot argue will be consistent with the relative fact obtained by A after its measurement on S_1. As we said before, a third observer W would be needed, but even this does not solve the problem since the relative facts of W are valid only for W.

However, if we consider A and B entangled as a single system which performs measurements on the system S, then we have correlated results also in RQM - and the order of the measurements does not matter in this case. The reason for this is that, in the case where A and B are entangled, the measurement of A on S_1 and the measurement of B on S_2 are to be considered as a single measurement on the system S by the system $A \otimes B$. Indeed, system S will be described by system $A \otimes B$ by a state of two entangled qubits. Therefore, the measurements of A and B - that is the measurement of the $A \otimes B$ system - will always be correlated. Hence, the predictions of RQM change depending on whether we consider the two observers separately or as a single system. On the other hand, in a standard QM context, the results are always correlated.[2]

If, however, we assume the CPL postulate is valid, the relative fact of A is to be considered a hidden variable for B. The reason lies in the fact that, if B interacts with A to know the result of its measurement on S_1, B will be able to say, in light of the CPL postulate, that A's measurement on S_1 will be consistent - thus correlated - with its measurement on S_2. As a matter of fact, the CPL postulate guarantees coherence between relative facts of different observers obtained by making measurements on the same system. S_1 and S_2 are entangled systems which are part of a single system S and therefore what was said previously about the coherence of the results also applies to them. As a consequence, A's relative facts causally influences the results of B since it is precisely the result of A which implies that the result of B must be 0 or +1, depending on the scenario.

However, RQM does not predict that A's relative fact is available to B in the state (3.4.2) of $S \otimes A$ to make predictions, so it is not included in the formalism. This means that - according to the authors of [17] - relative facts have no predictive power; instead, if we assume the CPL postulate to be true, then they become hidden variables - therefore RQM becomes a theory with hidden variables, and this is against one of the axioms of the theory.

As we said, the authors also revisit the situation studied in the first article [16] in which a system composed of three qubits in a GHZ state and two observers are considered. They show that, even considering the CPL postulate, we arrive at contradictions equivalent to the case of RQM without CPL. Furthermore, they show that CPL is in disagreement with another postulate of RQM. In particular, the one which states that, from the point of view of an external observer, a measurement/interaction between two systems is described

[2] Regarding this, Rovelli's article [23] is interesting.

as a unitary process that entangles the two systems.

Let us look at this last thing first. After A has performed its measurement on system S, for B the system $S \otimes A$ is entangled. In order to know something about the two systems, B must make a measurement. In B's view the entangled state of $S \otimes A$ is of the form:

$$\sum_j c_j \left|S_j\right\rangle_S \left|A_j\right\rangle_A . \tag{3.4.3}$$

Suppose B makes a measurement on A. At this point, B is entangled with the system $S \otimes A$ and it is possible to describe the global system with a state that includes the kets of all three entangled subsystems:

$$\sum_j c_j \left|S_j\right\rangle_S \left|A_j\right\rangle_A \left|B_j\right\rangle_B . \tag{3.4.4}$$

By applying Born's rule to (3.4.4), there is a non-zero probability that B gets a "wrong" result - namely, inconsistent with the result of A. This is what the axiom we mentioned predicts: according to it, the systems S, A and B are entangled after the measurement of B and thus there are coefficients that lead to a non-zero probability of obtaining a wrong outcome.

However, if we assume CPL to be true, then we find a contradiction. In fact, CPL guarantees that the relative facts of A and B match, therefore the probability that B obtains a "wrong" result is zero. All this shows that the cross-perspective axiom can only be considered as something that introduces hidden variables into the theory. Indeed, A's relative fact causally forces B's relative fact to be consistent with A's one. And this is not included in the RQM formalism if we consider the state (3.4.3) as the proper description of the situation in B's view. These hidden variables - unlike what happens for other hidden variable theories - are produced during the measurement process and not before. So the question may arise as to when exactly they are produced during the measurement process.

Finally, the authors show that the same contradictions as in [16] emerge even by including the CPL postulate in the case of a three-qubit system in a GHZ state. Here, again, we have two observers A and B. A makes three measurements and obtains three relative facts. Then it is B's turn. B obtains three relative facts by performing measurements on S. Then, B decides to make measurements on A and obtains three relative facts here as well. However, given that CPL holds, B is able to state that the results it obtained from the measurements on A match the results obtained by A in A's measurements on S.

Furthermore, the same relation (3.1.11) which reflects a constraint on B's results holds. And, if B decides to apply the CPL postulate, then B can infer that A's relative facts must be such that:

$$B_1 A_2 A_3 = -1, \tag{3.4.5}$$

which is simply (3.1.13) applied to the case of relative facts B_1, A_2 and A_3.

Analogously, B could state that, if, again, we substitute the variables with the relative facts of A and B, (3.1.14) and (3.1.15) will hold. It is easy to show that all this leads to something absurd because it would mean that the product between the three relative facts of A should be:

$$A_1 A_2 A_3 = \pm i, \tag{3.4.6}$$

when we know that the relative facts of A must be real numbers.

Conclusion

We have reached the end. We started from the application of the relational approach to the case of three qubits in a GHZ state and returned there. What we have said obviously goes well beyond the single case we have considered, given that, as we have seen, the application considered has allowed us to extend our knowledge of the relational approach in its various facets.

We have presented Rovelli's approach which aims to maintain the descriptive and predictive power of quantum mechanics while giving it a new meaning. We then considered the application of this approach by discussing the attempted confutation by Lawrence, Markiewicz and Żukowski (LMZ). We know that in the case of three qubits in a GHZ state, constraints on the results emerge. These constraints lead to some possible consequences within standard quantum mechanics. In the case of the relational approach, LMZ believe that contradictions emerge which make this approach invalid.

We have seen a first response from Drezet who adopts an approach based on the concept of physical state and not of event. An argument of this kind does not reveal any contradiction with quantum mechanics, but is somewhat unconvincing given that the attempt to base the theory on the concept of state and consider it an element of reality is controversial.

A more convincing answer is that of Cavalcanti, Di Biagio and Rovelli. The three authors argue that LMZ compare facts relative to different observers, thus it is necessary to introduce a third observer W in order to discuss the constraints provided by the theory. Cavalcanti et al. discuss what was said in [16] in W's view and underline that it makes no sense to expect the constraints to be satisfied at the same time since those refer to different contexts - namely, different sets of W's measurements. In this case the question probably remains more open, since it would be necessary to understand whether what is done in [16] is actually a comparison between results relative to different observers or simply a replacement of eigenvalues with relative facts of A and B.

As regards the last section, we saw a brief in-depth analysis in which we discussed the main consequences of the CPL postulate. We analyzed Markiewicz and Żukowski's response according to whom the postulate merely reiterates their point of view on relative facts. In fact, the essence of our discussion does not change: Markiewicz and Żukowski argue that contradictions still arise in the case of a GHZ state, even with the CPL postulate. Here too, however, it would be necessary to understand whether a third observer is necessary to draw their conclusions with a right justification.

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