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Cosmological Wave Asymptotics in the direction of the Initial Singularity

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Abstract

This thesis looks into systems of linear wave equations on cosmological backgrounds. The main aim is to investigate the asymptotics of solutions to such equations, particularly in the direction of the initial singularity of these background models. As an application, the past asymptotics of two linear wave equations on a flat Friedmann-Lemaître solution, coupled to a perfect fluid, are analyzed. The first equation describes linear scalar perturbations of such a background in the presence of a cosmological constant Λ. Of interest is the effect Λ could have on the asymptotics of these perturbations. The second equation, on the other hand, represents the equation of motion of a massive scalar field on the same background. In this respect, the asymptotic expansion, and hence the blow-up profile, of the relevant physical quantities is derived and compared with previous results.

"Progress in physics can proceed both from tolerance and intolerance"

——— C. W. Misner

"Here is a problem with which we must some day come to grips — at least if we are ever to understand this phenomenon called gravitation"

——— R. P. Geroch

Acknowledgments

What a Journey! Thinking about this master degree, and all of the things that contributed to its conclusion, is just mind-binding. Following this degree has been really enriching, both on the scientific and personal levels. And, just like any journey in life, the completion of this thesis, and the degree in general, was only possible in the company of wonderful persons, who I would like to thank infinitely.

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Contents

Chapter 1

Introduction: Asymptotics in relativistic cosmology

Since its birth in 1917, Cosmology, which currently deals with studying the universe on its largest possible scales, was about modeling the universe using a solution to the Einstein's field equations. Based on assumptions made regarding symmetry properties of the spacetime geometry of the universe, and also regarding its matter content, one gets a particular solution, and hence a corresponding model. In this respect, many cosmological solutions had been developed in the early period from 1917 to 1960 [16], most notably are the Friedmann-Lemaître (FL) models, which represent non-stationary cosmological solutions to the field equations, and which later became the mathematical basis of the current standard model of cosmology [18].

In spite of the big success of the FL models, and their perturbations, in accounting for astrophysical observations [44], it is important to remark that they are very special in the space of all cosmological solutions due to the fact that they are maximally symmetric. In other words, they involve spatial sections that are homogeneous and isotropic. Indeed, many possible phenomena are suppressed by assuming this spacetime geometry. Moreover, many models, which are very different from a FL one in the past, evolve to become similar to FL models for some time, before deviating again from such behaviour [56]. Hence, these models could also be in agreement with observations, similar to the standard one. Another important issue is that the universe is not perfectly spatially homogeneous and isotropic on any scale, so models "close" to FL ones, in some appropriate dynamical sense, could be relevant in understanding cosmic structure. In this respect, it is important to remark that using linear perturbation theory to study deviations from a FL background could be insufficient, as it excludes a priori important non-linear effects. Consequently, it becomes important to go beyond the FL models, and analyze the range of solutions that could give behaviour consistent with observations. This means studying cosmological solutions that are spatially inhomogeneous and anisotropic. One important aim of such analysis is to have a classification of all possible asymptotic states which are permitted by the field equations near the cosmological singularity, as this would shed light on how the real universe could have evolved [19]. Moreover, this is also relevant in assessing the non-linear stability of the FL models [51].

Hence, instead of focusing on a given solution, namely the FL model, it became an important task to study qualitative features of the evolution of general classes of cosmological models, beyond the FL ones. This includes investigating their behaviour both to the asymptotic past, where a spacetime singularity is encountered, and asymptotic future, where models either expand forever or recollapse. This paradigm shift started in the 1960s by the work of the Russian school of physicists, as represented by Belinskii, Khalatnikov, and Lifshitz (BKL), which tried to determine the structure of the singularity in a general, inhomogeneous and anisotropic cosmological model, and the associated dynamics in approaching that singularity [35] [9]. Using piece-wise approximation methods, they approximated the evolution of a cosmological model by a sequence of time periods, during each of which certain terms in the Einstein's equations can be neglected. In particular, they argued that the ODEs obtained by dropping spatial derivatives in the original equations should yield a good model of the asymptotic behaviour. Specifically, they emphasized that the relevant ODEs in approaching the singularity are the ones of the spatially homogeneous vacuum solutions of Bianchi type VIII or IX. This indicated that asymptotics in the direction of the singularity are oscillatory. This also implied that matter becomes dynamically insignificant near the singularity, except for particular cases [8]. This body of work came to be known as the **BKL conjecture**. Even though this conjecture gave important insights into the nature of a generic cosmological singularity, it also received a lot of criticism. One reason, for example, was for its use of local methods, whereas it was global techniques, which were introduced by Penrose during the same period, that should have been taken into consideration in analyzing the structure of the singularity [7].

Shortly after, C. Misner and collaborators used the Hamiltonian formalism of general relativity to attack the same problem [40]. In this framework, the field equations are reduced to a time-dependent Hamiltonian system for a particle, which represents the universe point, in two dimensions. Then, the analysis proceeds by replacing the time-dependent potentials by moving potential walls, which reflect the particle instantaneously. Interestingly enough, Misner was able to verify the oscillatory nature of the Bianchi IX model in the direction of the singularity, independently of Belinskii and Khalatnikov [10]. Moreover, MacCallum [37] used the same techniques to study the asymptotic behaviour of a class of spatially homogeneous cosmological models.

It is noteworthy that these two approaches resorted to heuristic arguments when trying to analyze the asymptotic states of general cosmological models, namely ones that are spatially inhomogeneous and anisotropic, and hence without any symmetry assumptions. This is due to the fact that, in this setting, one has to deal with the full Einstein's equations, which represent a system of non-linear PDEs, something which requires an elaborate and careful treatment. In relation to this, a third approach to analyze cosmological asymptotics came with the work of Collins [15], who formulated the field equations for spatially homogeneous models as an autonomous system of first-order differential equations. Then, asymptotic states for special classes of spatially homogeneous models were obtained employing methods of the theory of dynamical systems. In this case, solution curves partition $\mathbb{R}^n$ into orbits, and hence asymptotic states as $t \to \infty$, for example, can be described in terms of sinks, asymptotically stable periodic orbits or more general attractors [45]. This framework was carried forward by the work of Bogoyavlensky and collaboratos [12], and Wainwright and collaborators. Particularly, Wainwright and Hsu [55], based on suitably normalized variables, set a framework to analyze asymptotics of orthogonal spatially homogeneous (OSH) models of class A. These are spatially homogeneous solutions such that the 4-velocity of the perfect fluid, which is the assumed matter model, is orthogonal to the spatial hypersurfaces. Later, the framework was extended to more general situations [30].

Turning again to the issue of analyzing spatially inhomogeneous and anisotropic cosmological models, one of the first classes of solutions to be studied was the so called Gowdy spacetimes [26]. These are vacuum solutions to the field equations with compact spatial sections and a 2-parameter group of isometries. Hence, they admit one spatial degree of freedom. Being vacuum solutions, they could represent idealized cosmological models for the early universe, when matter was not important dynamically. Progress started with the work of Liang [34], which analyzed the dynamics of $G2$ cosmologies, coupled to a perfect fluid, using approximation techniques, where $G2$ refers to the 2-parameter Abelian group of isometries of the model. In this regard, numerical methods were employed as well [13]. However, an important result came with the work of Isenberg and Moncrief [32], which used methods from the theory of partial differential equations to give qualitative results regarding the asymptotic behaviour near the singularity of a class of Gowdy spacetimes without any approximation. The same year, Wainwright and Hewitt [31] extended previous methods of dynamical systems to analyze the asymptotic states of a class of $G2$ cosmologies with a perfect fluid. In this case, the field equations get expressed as an autonomous system of first-order PDEs in two independent variables. Regarding all of these results, it is worth mentioning that such work served to verify or reformulate some of the statements of the BKL conjecture.

Since the year 2000, much progress has been made regarding cosmological asymptotics. For example, Wainwright and collaborators [54] studied asymptotics of completely spatially inhomogeneous and anisotropic models, the so called $G0$ cosmologies, coupled to a perfect fluid and a positive cosmological constant Λ. Also, the authors [36] analyzed the isotropization of these same spatially inhomogeneous models, both to the past and future, in the presence of a positive Λ. They were able to show that there exists an open set of solutions that approaches the de Sitter state at late times, a result which confirms the so called **cosmic no-hair conjecture**. Another example is the work of Ringström [47], which refined previous results on past asymptotics (in the direction of the singularity) of a class of Gowdy spacetimes.

Another direction to study asymptotics for the linearized Einstein's equations is by looking into linear wave equations on a given cosmological background. This is related to the fact that the field equations in a certain gauge, the so-called **harmonic gauge**, take the form of non-linear wave equations for the components of the metric [14]. Hence, such a work should represent a first step in understanding the asymptotic behaviour of the full, non-linear equations. The first result came with the work of Allen and Rendall [3], which studied both past and future asymptotics of linear scalar perturbations of a flat FL model, coupled to a perfect fluid. Then, there is the work of Fournodavlos and others [2], which analyzed asymptotics of the wave operator near the singularity of a flat FL model and Kasner solution. Another closely-related result is that of Bachelot [5], which investigated asymptotics of the Klein-Gordon equation on several FL backgrounds. It is worth remarking that all of these results are relevant for the work of this thesis.

During the same period, Ringström [48] developed a method to study asymptotics of linear systems of wave equations on general cosmological backgrounds, which satisfy certain conditions regarding the spacetime geometry. The important point regarding this technique is that it provides a general framework for studying wave asymptotics, without restricting to a specific cosmological model. To date, this method, along with some improvements, and specifying to a certain class of equations, has only been applied to study asymptotics of the source-free Maxwell's equations on Kasner spacetimes [27]. Hence, in continuing such a work, this thesis deals with studying past asymptotics of the following two, physically interesting systems:

1. Scalar perturbations of a flat FL model coupled to a perfect fluid and a cosmological constant Λ. This shall represent an extension to the results of [3].

2. A massive scalar field on a flat FL model coupled to a perfect fluid. In this respect, the results shall be compared to [5].

It is noteworthy that these two systems are linear with different flavours. The first one comes from the Einstein's equations, in which the given matter model is coupled to the background geometry, but the equations are linearized using perturbation theory. The second system, instead, comes from looking into a scalar field on a fixed background, hence without any coupling between the spacetime geometry and the field.

So, the plan of this thesis is as follows: In chapter 2, some physical background on the systems of interest is given, namely on cosmological perturbation theory and scalar fields in cosmology. In chapter 3, important notions related to the wave equation are discussed, in particular the notions of energy and Sobolev norms. In chapter 4, the relevant method to study asymptotics is demonstrated, along with the logic behind it. In chapter 5, results of the asymptotic analysis of the systems of interest are presented. Finally, some concluding remarks are given.

Chapter 2

Physical background on the systems of interest

In this chapter, the equations of interest, namely those for which asymptotics are to be analyzed, are derived from physical principles. In particular, for the first system, linear perturbation theory, as applied to a FL model with a cosmological constant, is discussed. This also involves looking into the important issue of gauge fixing. For the second system, scalar fields are studied, along with their properties and equations of motion, on a FL background.

2.1 Scalar cosmological perturbations of a FL background

2.1.1 The effect of the cosmological constant on the initial singularity

The background model of interest is that of a flat FL model, hence spatially homogeneous and isotropic, coupled to a perfect fluid and a cosmological constant Λ. As mentioned before, the choice of this particular model is motivated from astrophysical observations. Regarding the cosmological constant, it is interesting to remark that it always had a controversial role in cosmology since the birth of the field [17]. Relevant to our asymptotic analysis is the effect, if any, of Λ on the initial singularity of the background model. This can be seen from looking into the Raychaudhuri equation, which describes the evolution of the expansion of a congruence of timelike curves in the direction of the unit tangent vector field. In fact, it represents the fundamental equation of gravitational attraction.

In this respect, the family of spacetimes (M, g) of interest is that of globally hyperbolic ones. These are solutions with the fundamental property that they can be foliated by a 1-parameter family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$. This indicates that there exists a smooth, regular scalar field $\hat{t}$ on M, such that each hypersurface is a level surface of $\hat{t}$ [25]. In other words

$$\forall t \in \mathbb{R}, \qquad \Sigma_t := \{p \in M, \hat{t}(p) = t\}.$$

It follows that the topology of such spacetimes is $M = \mathbb{R} \times \Sigma$. Hence, these solutions can be understood by looking into physical quantities that are tangent (spatial) or normal (temporal) to Σ, which is the $(3+1)$-splitting of these spacetimes. Based on this, coordinates on M can be fixed by first choosing spatial coordinates $x^i = (x^1, x^2, x^3)$ on each Σ_t, such that they vary smoothly between neighboring hypersurfaces, then adding the fourth coordinate t. This yields

$$x^\alpha = (t, x^1, x^2, x^3). \tag{2.1}$$

One important geometrical quantity related to the spatial hypersurfaces is the unit timelike normal, defined as

$$n_\mu \propto \frac{\partial t}{\partial x^\mu}, \tag{2.2}$$

such that $n_\mu n^\mu = -1$. As a consequence of normalization, it follows that [25]

$$\underline{\mathbf{n}} := -N\nabla t, \tag{2.3}$$

or in components

$$n_\mu = (-N, 0, 0, 0), \tag{2.4}$$

where $N = \left(-\vec{\nabla} t \cdot \vec{\nabla} t\right)^{-1/2}$ is called the lapse function, and $\underline{\mathbf{n}}$ indicates the 1-form associated with the vector field $\mathbf{n}$. It is noteworthy that the minus sign is to guarantee that $\mathbf{n}$ is future-directed when t increases to the future. Similarly, the components of the corresponding vector field are given by

$$n^\mu = N^{-1}(1, -\beta^1, -\beta^2, -\beta^3), \tag{2.5}$$

where β^i give the components of the shift vector field $\boldsymbol{\beta}$, which represents the spatial displacement of the curve of constant x^i (to which the time vector $\boldsymbol{\partial}_t$ is tangent) from the unit normal $\mathbf{n}$. This indicates that

$$\boldsymbol{\partial}_t = N\mathbf{n} + \boldsymbol{\beta}.$$

Using coordinates (2.1), the decomposition of the spacetime metric g can be expressed as

$$g_{\mu\nu} dx^\mu \otimes dx^\nu = -N^2 dt \otimes dt + \gamma_{ij}(dx^i + \beta^i dt) \otimes (dx^j + \beta^j dt), \tag{2.6}$$

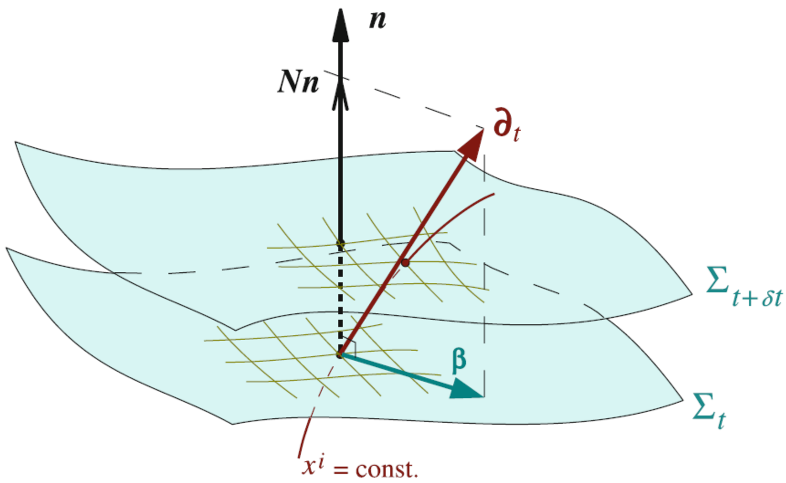

Figure 2.1: Lines of constant x^i cutting through a foliation Σ_t, defining both $\boldsymbol{\partial}_t$ and $\boldsymbol{\beta}$ [25].

where γ_{ij} represents the induced metric on the spatial hypersurfaces Σ.

To be able to look into the Raychaudhuri equation, a smooth congruence of timelike curves is considered. Such a congruence is parametrized by the proper time τ, and admits $\mathbf{n}$ as the unit tangent vector field. In this setting, a quantity of interest is the covariant derivative of $\underline{\mathbf{n}}$, which can be decomposed uniquely as [39]

$$n_{\mu;\nu} = B_{\mu\nu} - a_\mu n_\nu, \tag{2.7}$$

where $B_{\mu\nu}$ represents the spatial part, namely $B_{\mu\nu}n^\nu = B_{\mu\nu}n^\mu = 0$, and $a_\mu n_\nu$ represents the mixed spatio-temporal component, where $a_\mu = n^\alpha n_{\mu;\alpha}$ is the 4-acceleration of the congruence. For a smooth one-parameter family of curves $\gamma_s(\tau)$ in the congruence, an orthogonal deviation vector ξ^α from a reference curve γ_0 can be defined. It represents an infinitesimal spatial displacement from γ_0 to a nearby curve. Then, it can be shown that [57]

$$n^\alpha \xi^\beta{}_{;\alpha} = B^\beta{}_\alpha \xi^\alpha.$$

Hence, $B_{\mu\nu}$ represents the failure of ξ^α to be parallelly propagated in the direction of the unit normal. This can be understood as the curves near γ_0 are being stretched or rotated due to $B_{\mu\nu}$. Consequently, this spatial component can be decomposed as [39]

$$B_{\mu\nu} = \frac{1}{3}\theta\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \tag{2.8}$$

where

- $\theta := n^{\mu}{}_{;\mu}$ represents the overall expansion, or contraction, of the congruence.
- $\mathcal{P}_{\mu\nu} := g_{\mu\nu} + n_{\mu}n_{\nu}$ represents the induced spatial metric.
- $\sigma_{\mu\nu} := \frac{1}{2}\mathcal{P}_{\mu}{}^{\alpha}\mathcal{P}_{\nu}{}^{\beta}(n_{\alpha;\beta}+n_{\beta;\alpha}) - \frac{1}{3}\theta\mathcal{P}_{\mu\nu}$, which is the symmetric traceless part of $B_{\mu\nu}$, gives the shear of the congruence.
- $\omega_{\mu\nu} := \frac{1}{2}\mathcal{P}_{\mu}{}^{\alpha}\mathcal{P}_{\nu}{}^{\beta}(n_{\alpha;\beta} - n_{\beta;\alpha})$, which is the anti-symmetric part, gives the rotation (twist) of the congruence.

and $\mathcal{P}_{\alpha}{}^{\beta} : T(M) \to T(\Sigma)$ represents the projection operator on the tangent space of the spatial hypersurfaces. Another quantity that is relevant for the geometry of Σ is the extrinsic curvature $K_{\mu\nu}$, also referred to as the second fundamental form, defined as

$$K_{\mu\nu} := \mathcal{P}_{\nu}{}^{\lambda}n_{\mu;\lambda} = \frac{1}{3}\theta\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu}, \tag{2.9}$$

where $\omega_{\mu\nu}$ did not appear because it vanishes automatically for any congruence with unit tangent defined by equation (2.3), according to Forbenius' theorem [46].

Based on this, the Raychaudhuri equation is given by the taking the trace of the covariant derivative of $n_{\mu;\nu}$ in the direction of $\mathbf{n}$ to have an evolution equation for the expansion θ. This yields [18]

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} + a^{\mu}{}_{;\mu} - R_{\mu\nu}n^{\mu}n^{\nu}. \tag{2.10}$$

The last term on the right-hand side can be related to the matter content of a given spacetime and a possible cosmological constant through the Einstein's equations, which are expressed as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

Taking the trace, it follows that

$$R = 4\Lambda - 8\pi G T.$$

So, contracting the field equations two times with n^{μ} yields

$$R_{\mu\nu}n^{\mu}n^{\nu} = 8\pi G T_{\mu\nu}n^{\mu}n^{\nu} - \frac{1}{2}(4\Lambda - 8\pi G T) + \Lambda = 8\pi G(T_{\mu\nu}n^{\mu}n^{\nu} + \frac{1}{2}T) - \Lambda.$$

Hence, the Raychaudhuri equation becomes

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} + a^{\mu}{}_{;\mu} - 8\pi G(T_{\mu\nu}n^{\mu}n^{\nu} + \frac{1}{2}T) + \Lambda. \tag{2.11}$$

Employing equation (2.11), a singularity theorem can be given as follows [18].

Theorem 2.1 (Irrotational geodesic singularities) *If for a congruence of timelike curves $\Lambda \leq 0$; $(T_{\mu\nu}n^\mu n^\nu + \frac{1}{2}T) \geq 0$ for a unit timelike vector field n^μ; $a^\mu = 0$; $\omega_{\mu\nu} = 0$; and $\theta_0/3 > 0$ at some time s_0, then a spacetime singularity, where either $\theta \to \infty$ or $\sigma_{\mu\nu} \to \infty$, occurs at a finite proper time $\tau_0 \leq 3/\theta_0$ before s_0.*

It is worth mentioning that this theorem represents a fundamental result, as all subsequent singularity theorems always used it as an essential ingredient, along with other conditions regarding the global structure of spacetime [29]. To see the relevance of this theorem for the chosen background of interest, the spacetime metric needs first to be specified. This is given by [39]

$$g_{FL} := -a^2 d\eta \otimes d\eta + a^2 \delta_{ij} dx^i \otimes dx^j, \tag{2.12}$$

where $\eta \in (0, \infty)$ is the so called conformal time, $a = a(\eta)$ is the scale factor, and δ_{ij} represents the flat metric on $\mathbb{T}^3$. Two important remarks are in order. First, the cosmic time, as measured by observers at fixed comoving spatial coordinates x^i, is given by

$$t = \int a(\eta) d\eta.$$

Second, the choice of spatial topology, namely the 3-torus, is made to have compact spatial sections without a boundary, a setting which is desirable for simplifying mathematical calculations, as it implies spatial hypersurfaces which are bounded in extent and contain finite matter [50]. This is also related to the application of the relevant method of analyzing asymptotics, as it will be demonstrated in chapter 4.

The FL models are globally hyperbolic, with a foliation given by constant η-hypersurfaces Σ_η in the case of (2.12). Hence, coordinates can be fixed as

$$x^\alpha = (\eta, x^1, x^2, x^3).$$

Comparing (2.12) with the decomposition (2.6) yields the following

$$N = a, \qquad \beta^i = 0, \qquad \gamma_{ij} = a^2 \delta_{ij}.$$

Consequently, components of the unit normal $\mathbf{n}$ are expressed as

$$n^\mu = (a^{-1}, 0, 0, 0), \tag{2.13}$$

using (2.5), and similarly

$$n_\mu = (-a, 0, 0, 0). \tag{2.14}$$

From (2.14), the following can be calculated, utilizing the previous definitions

$$
\begin{aligned}
\theta &= g^{\mu\nu} n_{\mu;\nu} = g^{00} n_{0;0} + g^{ij} n_{i;j} \\
&= -a^{-2} \left[(-a)_{,0} - \Gamma^0_{00}(-a) \right] + a^{-2} \delta^{ij} \left[-\Gamma^0_{ij}(-a) \right] \\
&= 3 \frac{a'}{a^2} \equiv 3H,
\end{aligned}
\tag{2.15}
$$

where $' \equiv \frac{\partial}{\partial \eta}$, and H is the so-called Hubble parameter. Alternatively, the conformal Hubble parameter $\mathcal{H}$ is defined as

$$
\mathcal{H} \equiv aH.
$$

Similarly, and as expected from symmetry of the background, it follows that

$$
\sigma_{\mu\nu} = 0. \tag{2.16}
$$

Moreover, for the components of the acceleration, it can be shown that

$$
\begin{aligned}
a_0 &= n_{0;\nu} n^\nu = n_{0;0} n^0 + n_{0;i} n^i \\
&= \left(n_{0,0} - \Gamma^0_{00} n_0 \right) n^0 + \left(-\Gamma^0_{0i} n_0 \right) n^i \\
&= -a^{'} \left(\frac{1}{a} \right) + \frac{a^{'}}{a} = 0,
\end{aligned}
\tag{2.17}
$$

and same thing for a_i.

To complete the picture, it is necessary to look into the matter model of the background solution. As stated before, the relevant model is that of a perfect fluid with a 4-velocity given by [39]

$$
u^\mu = \frac{dx^\mu}{d\tau}, \tag{2.18}
$$

such that $u^\mu u_\mu = -1$, and τ represents the proper time along the flow lines of the fluid. From the definition, it follows that

$$
u^\mu = (a^{-1}, 0, 0, 0), \tag{2.19}
$$

where the fact that $N = d\tau / d\eta$ has been used, along with the vanishing of the spatial part of the fluid 4-velocity as a consequence of the background symmetry [25]. This implies that

$$
\mathbf{u} = \mathbf{n}. \tag{2.20}
$$

The other important quantity for specifying the matter model is the corresponding energy-momentum tensor, expressed as [39]

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}, \tag{2.21}$$

where ρ and P are the matter energy density and isotropic pressure as measured by an observer comoving with the fluid, respectively. Based on (2.21), it follows that

$$8\pi G(T_{\mu\nu}n^\mu n^\nu + \frac{1}{2}T) = 8\pi G(\rho + \frac{1}{2}(-\rho + 3P)) = 4\pi G(\rho + 3P). \tag{2.22}$$

To be able to go forward, an equation of state for the perfect fluid has to be specified. The standard choice in cosmology is that of a linear, barotropic equation of state [41]

$$P = \omega\rho, \tag{2.23}$$

where ω is a constant belonging to the interval $[0, 1]$. This implies a constant adiabatic speed of sound propagation in the fluid. Values of ω that are most relevant in cosmology are $\omega = 0$, which represents non-relativistic gas with zero pressure, or the so called dust, and $\omega = \frac{1}{3}$, which represents ultra-relativistic gas, or radiation. Consequently, (2.22) can be re-expressed as

$$8\pi G(T_{\mu\nu}n^\mu n^\nu + \frac{1}{2}T) = 4\pi G\rho(1 + 3\omega), \tag{2.24}$$

which is non-negative for the indicated range of ω. Such a condition for a given matter model is referred to as the **strong energy condition**.

Hence, taking the previous results into consideration, and neglecting the cosmological constant Λ for a moment, it follows that theorem 2.1 indeed applies to an expanding FL model coupled to a perfect fluid, where in this case the singularity $\theta \to \infty$ corresponds to $a \to 0$. However, a positive cosmological constant that is big enough in magnitude with respect to $\rho+3P$ can have an effect on preventing the singularity [1]. For this reason, Geroch [23] argued that for closed universes, namely spacetimes that contain compact, spacelike 3-dimensional submanifolds, in the case of a negative Λ, there shall be no non-singular closed-universe solution in the set of all solutions. This is closely related to the fact that for a closed universe, without any cosmological constant, there are theorems guarantying that it must be singular if it satisfies generic conditions, and if causality is not violated. Based on all of this, it would be interesting to see if a similar pattern holds for linear perturbations of this background, namely if Λ has any effect on past asymptotics of linear perturbations of a FL model.

2.1.2 Linear cosmological perturbation theory

Perturbation theory was introduced in cosmology with the aim of trying to account for complex structure in the universe, ranging from stars and galaxies, and up to clusters

[1] It is worth remarking that such a possibility is excluded for the physical universe based on observations [18].

and super clusters of galaxies [39]. This is due to the fact that, being spatially homogeneous and isotropic, the FL models cannot account for any type of cosmic structure, even though being successful in describing the average expansion and evolution of the universe on large scales. So, instead of considering directly spatially inhomogeneous and anisotropic solutions to the non-linear field equations, a simpler strategy would be adding spatial inhomogeneities and anisotropies perturbatively, order by order, to the rather spatially homogeneous and isotropic background.

A relevant issue in carrying out such a procedure is that of choosing coordinates or a gauge. In particular, this amounts to choosing a mapping between events in the spatially homogeneous background and those in the inhomogeneous, perturbed universe. The problem is that such a choice is not unique, something that allows for different descriptions of the same physical phenomenon. Moreover, this leads to introducing extra, or unnecessary degrees of freedom that could be mistaken for physical perturbations [24]. All of this follows from splitting quantities of interest into a background and perturbation [39]. This issue of choosing a gauge, the so called **gauge fixing**, will be discussed in more detail in the next section.

In particular, and as discussed in the previous section, the background model is chosen to be a flat FL model coupled to a perfect fluid and a cosmological constant Λ. Hence, it can be assumed that physical quantities can be decomposed into a background, which depends only on the conformal time η, and perturbation, which depends on both η and x^i. In other words, for a general tensorial quantity $\mathbf{T}(\eta, x^i)$, the following split is assumed [39]

$$\mathbf{T}(\eta, x^i) = \mathbf{T}_0(\eta) + \delta\mathbf{T}(\eta, x^i). \tag{2.25}$$

By virtue of perturbation theory, it follows that

$$\delta\mathbf{T}(\eta, x^i) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \delta\mathbf{T}_n(\eta, x^i), \tag{2.26}$$

where n indicates the order of perturbation, and ϵ is a small parameter to keep track of the expansion. In linear perturbation theory, only expansions up to the order of ϵ are considered. In what follows, the parameter ϵ is going to be omitted, so that the equations do not become too complicated.

As indicated earlier, the background model can be foliated by a one-parameter family of spacelike Cauchy hypersurfaces Σ_η. As a result, this motivates splitting quantities of interest into a temporal part and a spatial part. So, for a general 4-vector $\mathcal{U}^\mu$, this is expressed as

$$\mathcal{U}^\mu = [\mathcal{U}^0, \mathcal{U}^i],$$

where $\mathcal{U}^0$ represents a scalar on the spatial hypersurfaces. In addition, based on the Helmholtz theorem [1], $\mathcal{U}^i$ can be decomposed as

$$\mathcal{U}^i = \delta^{ij}\mathcal{U}_{,j} + \mathcal{U}^i_{vec},$$

where $\mathcal{U}$ is a scalar, curl-free part, and $\mathcal{U}^i_{vec}$ is a vector, divergence-free part. Again, the designation as a scalar or vector refers to behaviour from the point of view of transformations on the spatial hypersurfaces [39].

In a similar fashion, a general rank-2 tensor can be split into a temporal part, spatial part, and mixed part. In this case, the mixed part would represent a vector. One of the most important tensor quantities is the metric tensor $g_{\mu\nu}$, for which perturbations are defined as [39]

$$\delta g_{00} = -2a^2\phi, \tag{2.27}$$

$$\delta g_{0i} = a^2 B_i, \tag{2.28}$$

$$\delta g_{ij} = 2a^2 C_{ij}. \tag{2.29}$$

The Hodge theorem [1], along with the Helmholtz one, allow to decompose the $0-i$ and $i-j$ components as follows

$$B_i = B_{,i} - S_i, \tag{2.30}$$

$$C_{ij} = -\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2}h_{ij}, \tag{2.31}$$

where

- ϕ, B, ψ and E represent scalar metric perturbations. As indicated before, these are curl-free by construction.
- S_i and F_i represent vector metric perturbations. As before, these are divergence-free.
- h_{ij} give tensor metric perturbations, which are both transverse (divergence-free) ${h_{ij,}}^j = 0$ and trace-free $h^i_i = 0$.

Given that S_i and F_i are divergence-free, they are subject to 2 constraints. Also, h_{ij}, being divergence-free and trace-free, is subject to 4 constraints. Hence, the total number of degrees of freedom of the system is $16 - 2 - 4 = 10$, as expected.

The reason for considering these three types of perturbations is that the corresponding governing equations decouple at a linear order. Hence, each type can be investigated separably [33], a property which does not hold at second and higher order expansions.

This is due to the fact that the evolution equations of each type of perturbation still decouple at higher orders ($n > 1$), but they are sourced by terms composed of perturbations of lower order [39].

Now, employing equations (2.27)-(2.29), expansions for the components of the metric tensor $g_{\mu\nu}$ at a linear order, based on (2.12) and (2.25), can be given as

$$g_{00} = -a^2(1 + 2\phi_1), \tag{2.32}$$

$$g_{0i} = a^2 B_{1i}, \tag{2.33}$$

$$g_{ij} = a^2(\delta_{ij} + 2C_{1ij}). \tag{2.34}$$

From the fact that

$$g_{\mu\nu}g^{\nu\lambda} = \delta_\mu{}^\lambda,$$

similar expressions for the components of the inverse metric $g^{\mu\nu}$ can be derived. Another way of looking into this is by remarking that ϕ_1 represents perturbation of the lapse function N, and that B_{1i} represents perturbation of the shift vector field β_i. This can be seen from expanding both N and β_i, according to (2.25), and employing the decomposition (2.6) of $g_{\mu\nu}$. For the lapse function, it follows that

$$N = N_0 + \delta N_1.$$

So, at a linear order

$$N^2 = N_0{}^2 + 2N_0\delta N_1.$$

Given that $g_{00} = -N^2$, where the term $\beta_k\beta^k$ is neglected because it is second order, it follows that

$$-N_0{}^2 - 2N_0\delta N_1 = -a^2 - 2a^2\phi_1,$$

which yields

$$N_0 = a, \qquad \delta N_1 = a\phi_1.$$

Similarly for β_i

$$\delta\beta_{1i} = a^2 B_{1i}.$$

Now, recalling the (3+1)-decomposition of $g^{\mu\nu}$ [25]

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{\beta^j}{N^2} \\ \frac{\beta^i}{N^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{N^2} \end{pmatrix},$$

the expression for g^{00}, for example, can be given as

$$g^{00} = -{N_0}^{-2}\left[1 + 2\frac{\delta N_1}{N_0}\right]^{-1} = -a^{-2}[1 - 2\phi_1]. \tag{2.35}$$

Similarly, the other components are given by [39]

$$g^{0i} = a^{-2}{B_1}^i, \tag{2.36}$$

$$g^{ij} = a^{-2}[\delta^{ij} - 2{C_1}^{ij}]. \tag{2.37}$$

Just as the components of the metric define the unit timelike normal to constant η-hypersurfaces $\mathbf{n}$, same thing also holds regarding its perturbations. So, based on equation (2.4), it follows that

$$n_\mu = (-N, \mathbf{0}) = -a(1 + \phi_1, \mathbf{0}). \tag{2.38}$$

In a similar fashion, and based on (2.5)

$$n^\mu = \left(\frac{1}{N}, -\frac{\beta^i}{N}\right) = a^{-1}\left(1 - \phi_1, -{B_1}^i\right). \tag{2.39}$$

For the different parts of the covariant derivative of $\mathbf{n}$, similar perturbative expressions can be given. For example, the expansion θ can be calculated as

$$\theta = g^{00}n_{0;0} + g^{ij}n_{i;j} = -a^{-2}[1 - 2\phi_1]\left[-a^{'}(1 + \phi_1) - a{\phi_1}' - \left(\frac{a^{'}}{a} + {\phi_1}'\right)(-a(1 + \phi_1))\right]$$
$$+ a^{-2}[\delta^{ij} - 2{C_1}^{ij}]\left[a(1 + \phi_1)\left(\left(\frac{a^{'}}{a} - 2\frac{a^{'}}{a}\phi_1\right)\delta_{ij} + {C_{1ij}}' + 2\frac{a^{'}}{a}C_{1ij} - \frac{1}{2}(B_{i,j} + B_{j,i})\right)\right], \tag{2.40}$$

which eventually gives

$$\frac{1}{a}\left[3\frac{a^{'}}{a} - 3\frac{a^{'}}{a}\phi_1 + C_{1i}{}^{i'} - B_{1i,}{}^{i}\right]. \tag{2.41}$$

Recalling that

$$C_{1i}{}^{i} = -3\psi_1 + \nabla^2 E_1, \qquad B_{1i}{}^{i} = \nabla^2 B_1,$$

the result becomes

$$\theta = \frac{3}{a}\left[\mathcal{H} - \mathcal{H}\phi_1 - \psi_1 + \frac{1}{3}\nabla^2\sigma_1\right], \tag{2.42}$$

where ∇^2 denotes the Laplace-Beltrami operator on the spatial sections, and $\sigma_1 \equiv E_1{}^{'} - B_1$ defines the shear potential. This quantity can be better understood from calculating the scalar part of the shear, which gives [39]

$$\sigma_{1ij} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)a\sigma_1. \tag{2.43}$$

It is also worth remarking that perturbations of the scalar curvature of the spatial hypersurfaces Σ are related to the scalar metric perturbations ψ_1 through [39]

$$\delta^3 R_1 = \frac{4}{a^2}\nabla^2\psi_1. \tag{2.44}$$

Regarding perturbations of the matter model, it is worth reviewing first a couple of important elements from the (3+1)-decomposition of perfect fluids. Based on the previous definition of $\mathbf{u}$, the fluid velocity relative to the unit normal $\mathbf{n}$, denoted $\mathbf{U}$, can be defined as [25]

$$\mathbf{U} := \frac{d\boldsymbol{\ell}}{d\tau}, \tag{2.45}$$

where $d\boldsymbol{\ell}$ represents the spatial displacement of the fluid flow lines from the congruence of $\mathbf{n}$, and τ is the proper time along that congruence. Based on the definition, it follows that $\mathbf{U}$ is tangent to Σ, namely

$$U^\mu n_\mu = 0.$$

Hence, it follows that $\mathbf{u}$ can be decomposed as

$$\mathbf{u} = \Gamma(\mathbf{n} + \mathbf{U}), \tag{2.46}$$

where the Lorentz factor $\Gamma = (1 - U^k U_k)^{1/2}$ represents the proportionality between proper time along the congruence of $\mathbf{n}$ and proper time along the fluid flow. $\mathbf{U}$ can be further decomposed by introducing the fluid coordinate velocity

$$\mathbf{v} := \frac{d\mathbf{x}}{d\eta}, \tag{2.47}$$

where $d\mathbf{x}$ represents the spatial displacement of the fluid flow from the curve of constant comoving spatial coordinates x^i. Given that this curve drifts from the unit timelike normal a distance of $\boldsymbol{\beta} d\eta$ between η and $\eta + d\eta$, then the following holds

$$d\boldsymbol{\ell} = \boldsymbol{\beta} d\eta + d\mathbf{x}.$$

Dividing by $d\tau$, it follows

$$\mathbf{U} = \frac{1}{N}(\boldsymbol{\beta} + \mathbf{v}). \tag{2.48}$$

Inserting this expression back into (2.46), the components of $\mathbf{u}$ get expressed as

$$u^\mu = \left(\frac{\Gamma}{N}, \frac{\Gamma v^i}{N}\right). \tag{2.49}$$

Invoking the previous expansion of N, and noticing that the Lorentz factor is of second order, the following expansions of u^μ follow

$$u^0 = a^{-1}(1 - \phi_1), \tag{2.50}$$

$$u^i = a^{-1} {v_1}^i. \tag{2.51}$$

Using equations (2.32)-(2.34), the components of $\underline{\mathbf{u}}$ can be calculated. Similar to what has been done before, the Helmholtz theorem can be employed to give

$$v^i = \delta^{ij} v_{,j} + v^i_{vec},$$

where v^i_{vec}, being divergence-free, represents vorticity of the perturbed fluid. Based on (2.48), it is worth remarking that for the choice of coordinates called the orthogonal coordinate system, where $B^i = 0$, $\mathbf{U}$ and $\mathbf{v}$ are proportional [39].

The other important ingredient to fully understand the perturbed fluid is perturbations of the energy-momentum tensor $T_{\mu\nu}$. For example, the T_{00} component is calculated as

$$\begin{aligned} T_{00} = (\rho_0 + P_0 + \delta\rho_1 + \delta P_1) a^2 (1 + \phi_1)^2 + (P_0 + \delta P_1)[-a^2(1 + 2\phi_1)], \\ = a^2(\rho_0 + \delta\rho_1)(1 + 2\phi_1), \\ = a^2[\rho_0 + \delta\rho_1 + 2\rho_0\phi_1]. \end{aligned} \tag{2.52}$$

Similarly, the other components are evaluated as

$$T_{0i} = a^2[-v_{1i}(\rho_0 + P_0) - \rho_0 B_{1i}], \tag{2.53}$$

$$T_{ij} = a^2[P_0\delta_{ij} + 2P_0 C_{1ij} + \delta P_1 \delta_{ij}]. \tag{2.54}$$

Finally, it is noteworthy that for a general fluid, there is an extra piece in the energy-momentum tensor, namely the anisotropic stress tensor $\pi_{\mu\nu}$. Such a quantity is defined subject to the following constraints [39]

$$\pi_{\mu\nu}u^\nu = 0, \qquad \pi^\mu{}_\mu = 0.$$

2.1.3 Gauge transformations and gauge fixing

As indicated earlier, splitting variables into background and perturbation introduces an ambiguity in identifying spacetime points of the perturbed universe based on spacetime points of the background. This is due to the fact that there is freedom in choosing a specific gauge that relates the two spacetimes, and hence this procedure is not covariant [39]. So, for this whole perturbative analysis to work out, it is necessary to eliminate the gauge dependence from expressions of perturbations, something that will be shown shortly after.

In this respect, it is important to recall some background on coordinate or gauge transformations. In particular, it is more convenient to do this in a geometric, coordinate-independent way, and then fix coordinates later to do calculations [38]. Starting with a one-parameter family of 4-manifolds $\mathcal{M}_\epsilon$ embedded in a 5-manifold $\mathcal{N}$, it follows that each one of these manifolds can be interpreted as a perturbed spacetime with respect to an unperturbed one given by $\mathcal{M}_0$. Then, an identification map

$$P_\epsilon : \mathcal{M}_0 \to \mathcal{M}_\epsilon,$$

can be introduced, which relates points in the two manifolds. This identifies a vector field X on $\mathcal{N}$, such that points lying on the same integral curve γ of X are regarded to be equivalent. Fixing coordinates x^μ on $\mathcal{M}_0$ and parameterizing γ by ϵ, then coordinates on $\mathcal{N}$ can be given as $\{x^A = (x^\mu, \epsilon)\}$, where $A = 0, 1, .., 4$ and $\mu = 0, 1, .., 3$. This vector field X induces a local one-parameter group of transformations of $\mathcal{N}$, denoted ϕ_ϵ, which can be expressed as [4]

$$\phi_\epsilon : \mathcal{M}_0 \to \mathcal{M}_\epsilon.$$

Based on this, a choice of gauge, or gauge fixing, is just a choice of a particular vector field X, which relates $\mathcal{M}_0$ and $\mathcal{M}_\epsilon$. Indeed, X is referred to as the gauge generator. In this respect, there are two approaches to demonstrate the issue of gauge dependence, the active and passive approaches [38]. For the active approach, a vector field X, which maps a point $p \in \mathcal{M}_0$ to a point $u \in \mathcal{M}_\epsilon$, and a vector field Y, which maps a point $q \in \mathcal{M}_0$ to the same point u, are identified. Then, a gauge transformation in this case is defined on $\mathcal{M}_0$, and hence it is evaluated at the same coordinate point. In the passive approach, instead, a point $p \in \mathcal{M}_0$ is fixed, and a gauge choice X identifies it with a point $q \in \mathcal{M}_\epsilon$, whereas another choice Y identifies it with a different point $u \in \mathcal{M}_\epsilon$. So, the gauge transformation in this case is defined on $\mathcal{M}_\epsilon$, and it is carried out at the same physical point. This is explained diagrammatically in figure 2.2.

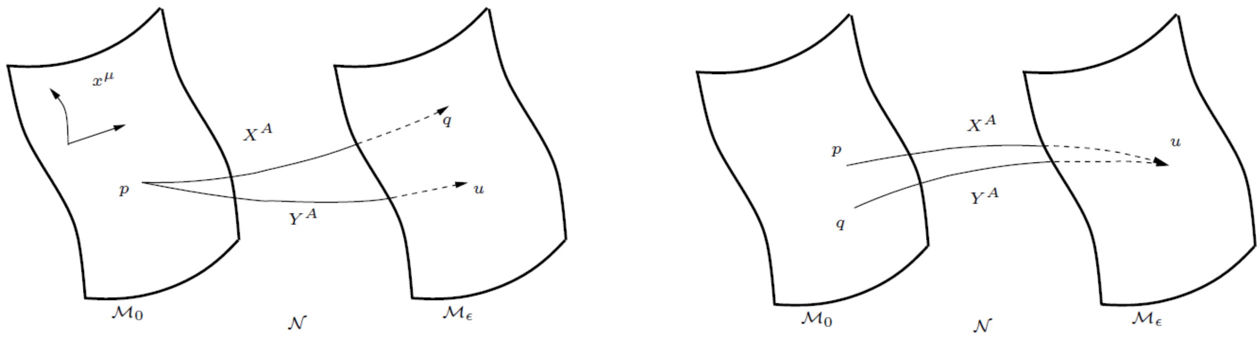

Figure 2.2: Active (right) v.s. passive (left) approach to gauge transformations [38].

Having discussed the two approaches, it is the active approach that is going to be utilized in discussing the behaviour of different quantities under gauge transformations. The basic relation relating transformed (tilted) and original variables is the exponential map, defined as [39]

$$\widetilde{\mathbf{T}} = e^{\mathcal{L}_\xi}\mathbf{T}, \tag{2.55}$$

where ξ is the vector field generating the transformation, and $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ. Component-wise, it can be expressed that

$$\xi^\mu \equiv \epsilon\xi_1^\mu, \tag{2.56}$$

and hence the exponential map gives

$$\exp(\mathcal{L}_\xi) = 1 + \epsilon\mathcal{L}_{\xi_1} \tag{2.57}$$

Now, $\mathbf{T}$ can be expanded according to equation (2.25), which gives the following expressions upon comparing with (2.55)

$$\widetilde{\mathbf{T}}_0 = \mathbf{T}_0, \tag{2.58}$$

$$\epsilon\widetilde{\delta\mathbf{T}_1} = \epsilon(\delta\mathbf{T}_1 + \mathcal{L}_{\xi_1}\mathbf{T}_0). \tag{2.59}$$

These equations show clearly that background quantities remain unaffected, whereas perturbations receive gauge dependence.

Using equation (2.59), the transformation behaviour of different tensorial quantities can be investigated under gauge transformations. For this purpose, components of the generating vector field ξ_1^μ are split into a temporal part, and a spatial scalar and vector parts as [39]

$$\xi_1^\mu = (\alpha_1, \beta_1,{}^i + \gamma_1{}^i), \tag{2.60}$$

where as usual $\gamma_1{}^i{}_{,i} = 0$. The simplest case to consider is that of 4-scalars. Taking the energy density of the fluid ρ as an example, it follows that

$$\widetilde{\delta\rho_1} = \delta\rho_1 + \mathcal{L}_{\xi_1}\rho_0 = \delta\rho_1 + \alpha_1 {\rho_0}', \tag{2.61}$$

which is completely specified by fixing the temporal gauge α_1.

For 4-vectors, applying (2.59) to the fluid 4-velocity u^μ gives

$$\widetilde{\delta u_{1\mu}} = \delta u_{1\mu} + \mathcal{L}_{\xi_1} u_{0\mu}, \tag{2.62}$$

Recalling that for a 1-form ω_μ, the Lie derivative is evaluated as [14]

$$\mathcal{L}_{\xi_1}\omega_\mu = \omega_{\mu,\alpha}\xi_1^\alpha + \omega_\alpha \xi_1{}^\alpha{}_{,\mu},$$

it follows that

$$\widetilde{\delta u_{1\mu}} = \delta u_{1\mu} + u'_{0\mu}\alpha_1 + u_{0\alpha}\xi_1{}^\alpha{}_{,\mu} \tag{2.63}$$

For $\mu = i$, this relation gives

$$\widetilde{v_{1i}} + \widetilde{B_{1i}} = v_{1i} + B_{1i} - \alpha_{1,i}. \tag{2.64}$$

As it will be shown later

$$\widetilde{B_{1i}} = B_{1i} + \xi'_{1i} - \alpha_{1,i}.$$

Substituting this into (2.64) yields

$$\widetilde{v_{1i}} = v_{1i} - \xi'_{1i}. \tag{2.65}$$

Considering the corresponding scalar and vector parts, it follows that

$$\widetilde{v_1} = v_1 - \beta'_1, \tag{2.66}$$

$$\widetilde{v^i_{vec(1)}} = v^i_{vec(1)} - {\gamma_1}^{i'}. \tag{2.67}$$

Similarly, equation (2.59) can be employed to calculate the transformation behaviour of general tensors, in particular the metric tensor. An essential ingredient is the following fact for a (0,2)-tensor field $h_{\mu\nu}$ [14]

$$\mathcal{L}_{\xi_1} h_{\mu\nu} = h_{\mu\nu,\alpha}\xi_1^\alpha + h_{\alpha\nu}\xi^\alpha_{1,\mu} + h_{\mu\alpha}\xi^\alpha_{1,\nu}.$$

Hence, for δg_{00}, which is a scalar, it can be calculated that

$$\begin{aligned} \widetilde{\delta g_{(1)00}} &= \delta g_{(1)00} + g'_{(0)00}\alpha_1 + 2g_{(0)00}\alpha_1', \\ &= \delta g_{(1)00} - 2aa'\alpha_1 - 2a^2\alpha_1'. \end{aligned} \tag{2.68}$$

From (2.27), it follows that

$$\widetilde{\phi_1} = \phi_1 + \mathcal{H}\alpha_1 + \alpha_1'. \tag{2.69}$$

It is worth remarking that the same equation follows from (2.63) by setting $\mu = 0$. Following the same procedure for δg_{oi} is a little bit more involved, as this perturbation contains both scalar and vector parts. Applying (2.59), it follows

$$\begin{aligned} \widetilde{\delta g_{(1)0i}} &= \delta g_{(1)0i} + g_{(0)00}\alpha_{1,i} + g_{(0)ij}{\xi_1^{j}}', \\ &= \delta g_{(1)0i} - a^2\alpha_{1,i} + a^2\xi_{1i}'. \end{aligned} \tag{2.70}$$

Based on (2.28), it follows that

$$\widetilde{B_{1i}} = B_{1i} - \alpha_{1,i} + \xi_{1i}', \tag{2.71}$$

which is the equation encountered above. Taking the divergence to eliminate the vector part, the equation yields

$$\widetilde{B_1} = B_1 - \alpha_1 + \beta_1'. \tag{2.72}$$

Subtracting this from (2.71) yields the following

$$\widetilde{S_{1i}} = S_{1i} - \gamma_{1i}'. \tag{2.73}$$

For δg_{ij}, applying (2.59) gives

$$\widetilde{\delta g_{(1)ij}} = \delta g_{(1)ij} + g'_{(0)ij}\alpha_1 + g_{(0)ik}\xi^k_{1,j} + g_{(0)kj}\xi^k_{1,i}. \tag{2.74}$$

From (2.29), it follows that

$$\widetilde{C_{1ij}} = C_{1ij} + \mathcal{H}\alpha_1\delta_{ij} + \xi_{1(i,j)}. \tag{2.75}$$

To analyze the transformation behaviour of the scalar part of C_{ij}, it is convenient to first take the trace

$$-3\widetilde{\psi_1} + \nabla^2\widetilde{E_1} = -3\psi_1 + \nabla^2 E_1 + 3\mathcal{H}\alpha_1 + \nabla^2\beta_1, \tag{2.76}$$

then acting by the operator $\partial^i\partial^j$ on (2.75) to get a second equation as

$$-\nabla^2\widetilde{\psi_1} + \nabla^2\nabla^2\widetilde{E_1} = -\nabla^2\psi_1 + \nabla^2\nabla^2 E_1 + \mathcal{H}\nabla^2\alpha_1 + \nabla^2\nabla^2\beta_1. \tag{2.77}$$

Based on (2.76) and (2.77), it follows that

$$\widetilde{\psi_1} = \psi_1 - \alpha_1\mathcal{H}, \tag{2.78}$$

$$\widetilde{E_1} = E_1 + \beta_1. \tag{2.79}$$

To get an insight into the vector part, the divergence ${C_{ij,}}^j$ of (2.75) is calculated as

$$-\widetilde{\psi_{1,i}} + \nabla^2\widetilde{E_{1,i}} + \frac{1}{2}\nabla^2\widetilde{F_{1i}} = -\psi_{1,i} + \nabla^2 E_{1,i} + \frac{1}{2}\nabla^2 F_{1i} + \mathcal{H}\alpha_{1,i} + \frac{1}{2}\nabla^2\xi_{1i} + \frac{1}{2}\nabla^2\beta_{1,i}. \tag{2.80}$$

Inserting the expressions from (2.78) and (2.79), the equation gives

$$\widetilde{F_{1i}} = F_{1i} + \gamma_{1i}. \tag{2.81}$$

Finally, by substituting (2.78), (2.79) and (2.81) into (2.75), it can be verified that

$$\widetilde{h_{1ij}} = h_{1ij}, \tag{2.82}$$

hence they are gauge invariant.

As it will be useful later, it is worth mentioning the transformation behaviour of the scalar shear potential, $\sigma_1 = E_1' - B_1$, and the combination $v_1 + B_1$ as

$$\widetilde{\sigma_1} = \sigma_1 + \alpha_1, \tag{2.83}$$

$$\widetilde{v_1} + \widetilde{B_1} = v_1 + B_1 - \alpha_1. \tag{2.84}$$

As was shown by equation (2.59), transformations of perturbations have dependence on the chosen gauge. Indeed, if it occurs that a given perturbation transforms as the Lie derivative of a given background quantity, then it does not represent an actual physical perturbation, but it is a fictitious one, as physical observables should not depend on the choice of coordinates. However, combinations of suitable matter and metric variables can give rise to gauge-invariant quantities, which coincide with metric and matter perturbations in a specific gauge [39] [2]. It is important not to confuse this with the notion of gauge-independence, as exemplified by the behaviour of tensor metric perturbations h_{1ij}. For this reason, gauge-invariant expressions are constructed only for scalar and vector perturbations.

One particular choice of gauge which yields gauge-invariant quantities, and which will later prove useful in deriving evolution equations for the perturbations in the next section, is the so called **longitudinal**, **orthogonal zero-shear**, or **conformal Newtonian** gauge. This is implemented by choosing to work with spatial hypersurfaces of vanishing shear σ_{ij} [39]. Looking into scalar perturbations, and recalling equation (2.83) for σ_1, this implies that

$$\alpha_{1\ell} = -\sigma_1 = B_1 - E_1', \tag{2.85}$$

starting from an arbitrary gauge. To completely fix the longitudinal gauge, equation (2.85) is complemented by the choice $\widetilde{E_{1\ell}} = 0$, which, from the previous requirement, implies that $\widetilde{B_{1\ell}} = 0$. Recalling (2.79), this gives

$$\beta_{1\ell} = -E_1. \tag{2.86}$$

Having fixed $\alpha_{1\ell}$ and $\beta_{1\ell}$, expressions of scalar metric perturbations ϕ_1 and ψ_1 in this gauge can be given as

$$\widetilde{\phi_{1\ell}} = \phi_1 + \mathcal{H}(B_1 - E_1') + (B_1 - E_1')', \tag{2.87}$$

$$\widetilde{\psi_{1\ell}} = \psi_1 - \mathcal{H}(B_1 - E_1'), \tag{2.88}$$

which agree with the gauge-invariant Bardeen potentials Φ and Ψ [6]. Considering $\widetilde{\phi_{1\ell}}$, for example, and recalling (2.69), (2.72) and (2.79), it follows that

$$\begin{aligned}\widetilde{\phi}_1 + \mathcal{H}(\widetilde{B}_1 - \widetilde{E}_1{}') + (\widetilde{B}_1 - \widetilde{E}_1{}')' &= \phi_1 + \mathcal{H}\alpha_1 + \alpha_1' + \mathcal{H}(B_1 - \alpha_1 + \beta_1' - E_1' - \beta_1') \\ &\quad + (B_1 - \alpha_1 + \beta_1' - E_1' - \beta_1')' \\ &= \phi_1 + \mathcal{H}(B_1 - E_1') + (B_1 - E_1')',\end{aligned}$$

[2] In mathematical terms, what is important is not the vector space of solutions to the linearized equations, but its quotient by the linearizations of 1-parameter families of diffeomorphisms, or the so-called gauge transformations. So, looking into gauge-invariant perturbations amounts to representing this quotient space as a subspace.

and similarly for $\widetilde{\psi_{1\ell}}$. From (2.61) and (2.66), it can be shown that

$$\widetilde{\delta\rho_{1\ell}} = \delta\rho_1 + (B_1 - E_1')\rho_0', \tag{2.89}$$

$$\widetilde{v_{1\ell}} = v_1 + E_1', \tag{2.90}$$

which again are gauge-invariant. To employ the longitudinal gauge for vector perturbations, it is necessary to impose that $n^i = 0$, which implies that

$$\widetilde{B_{1\ell,}}^{i} = \widetilde{S_{1\ell}}^{i} = 0.$$

So, from (2.73), the vector part of the spatial gauge transformation can be fixed to

$$\gamma^i_{1\ell} = \int S^i_1 d\eta + \hat{C}^i_1(x^j), \tag{2.91}$$

where $\hat{C}^i_1(x^j)$ is an arbitrary constant 3-vector, determined by the choice of spatial coordinates on an initial hypersurface. From (2.81), the remaining vector metric perturbation is expressed as

$$\widetilde{F_{1\ell}}^{i} = F_{1\ell}{}^{i} + \int S^i_1 d\eta + \hat{C}^i_1(x^j). \tag{2.92}$$

It is worth mentioning that there are many other choices of gauge which yield gauge-invariant quantities, for example the spatially-flat gauge, the synchronous gauge, or the comoving-orthogonal gauge. Some of these, such as the synchronous one, have residual gauge degrees of freedom, which results from the coordinate choice not being fixed in an unambiguous manner [39].

2.1.4 Evolution equations for scalar metric perturbations

In this section, the Einstein's equations for linear perturbations of the background metric are derived. In particular, focus is going to be on scalar perturbations, as these are directly related to structure formation [41], and also because of the method of asymptotic analysis to be employed later. The field equations in the presence of a cosmological constant Λ are given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

For globally hyperbolic spacetimes, these equations can be projected two times in the direction of $\mathbf{n}$, and one time along $\mathbf{n}$ and one time tangent to Σ, to give two constraint equations, the so called energy and momentum constraints. Projecting, instead, two times tangent to Σ gives the main evolution equation for the metric [25]. Moreover, from the Bianchi-identities

$$\nabla_\mu (R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \Lambda g^{\mu\nu}) = 0,$$

it follows that

$$\nabla_\mu T^{\mu\nu} = 0,$$

which gives the energy conservation of the specified matter model when projected along $\mathbf{n}$, while momentum conservation follows from projection tangent to Σ.

For the background model, the components of the Ricci tensor and Ricci scalar are evaluated as

$$R_{00} = 3\left(\mathcal{H}^2 - \frac{a''}{a}\right), \tag{2.93}$$

$$R_{ij} = (\mathcal{H}' + 2\mathcal{H}^2)\delta_{ij}, \tag{2.94}$$

$$R = \frac{6}{a^2}\left(\mathcal{H}' + \mathcal{H}^2\right). \tag{2.95}$$

Combining these with (2.21) gives the following Friedmann's constraint and evolution equations

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \rho_0 + \frac{\Lambda}{3} a^2, \tag{2.96}$$

$$\mathcal{H}' = -\frac{4\pi G}{3} a^2 (\rho_0 + 3P_0) + \frac{\Lambda}{3} a^2. \tag{2.97}$$

The covariant conservation of the energy-momentum tensor yields

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \nabla_0 T^{00} + \nabla_i T^{ij} + \nabla_0 T^{0i} + \nabla_0 T^{io}, \\ &= T^{00}{}_0 + 2\Gamma^0_{00} T^{00} + \Gamma^i_{i0} T^{00} + \Gamma^0_{ij} T^{ij}, \\ &= \rho_0' + 3\mathcal{H}(\rho_0 + P_0) = 0, \end{aligned} \tag{2.98}$$

which is the continuity equation for the perfect fluid, expressing energy conservation. Momentum conservation, on the other hand, is satisfied trivially due to background symmetry. It is worth remarking that the evolution equation (2.97) is nothing other than the Raychaudhuri equation for the congruence of flow lines of the perfect fluid.

Carrying out the same procedure for linear scalar perturbations, the following expressions for the components of the perturbed Ricci tensor and Ricci scalar, in an arbitrary gauge, can be given

$$R_{00} = 3\mathcal{H}(\phi_1' + \psi_1') - \nabla^2\sigma_1' - \mathcal{H}\nabla^2\sigma_1 + \nabla^2\phi_1 + 3\psi_1'', \tag{2.99}$$

$$R_{0i} = 2\mathcal{H}\phi_{1,i} + 2\psi_{1,i}' + 2\mathcal{H}^2 B_{1,i} + \mathcal{H}' B_{1,i}, \tag{2.100}$$

$$\begin{aligned} R_{ij} = (&-\mathcal{H}\phi_1' - 2\phi_1\mathcal{H}' - 4\mathcal{H}^2\phi_1 + \mathcal{H}C_1{}^{k\prime}_{k} - \mathcal{H}B_1{}^{k}{}_{,k})\delta_{ij} + C_{1ij}'' + (2\mathcal{H}' + 4\mathcal{H}^2)C_{1ij} \\ &+ 2\mathcal{H}C_{1ij}' - \frac{1}{2}(B_{1i,j}' + B_{1j,i}') - \mathcal{H}(B_{1i,j} + B_{1j,i}) - \mathcal{H}(B_{1i,j} - B_{1j,i}) + C_1{}^{k}{}_{j,ik} \\ &- \nabla^2 C_{1ij} - C_1{}^{k}{}_{k,ij} + C_{1ik,}{}^{k}{}_{j} - \phi_{1,ij}, \end{aligned} \tag{2.101}$$

$$\begin{aligned} R = &-6\mathcal{H}a^{-2}\phi_1' - 3\mathcal{H}a^{-2}\psi_1' + 2a^{-2}\nabla^2\sigma_1' + 6\mathcal{H}a^{-2}\nabla^2\sigma_1 - 2a^{-2}\nabla^2\phi_1 \\ &- 6a^{-2}\psi_1'' - 12a^{-2}\phi_1(\mathcal{H}' + \mathcal{H}^2) - 15a^{-2}\mathcal{H}\psi_1' + 4a^{-2}\nabla^2\psi_1, \end{aligned} \tag{2.102}$$

where B_{1i} and C_{1ij} in the expression for R_{ij} were not decomposed into their corresponding scalar, vector and tensor parts for convenience. Based on (2.52) and (2.53), the energy and momentum constraints for scalar perturbations can be given as [39]

$$3\mathcal{H}(\psi_1' + \mathcal{H}\phi_1) - \nabla^2(\psi_1 + \mathcal{H}\sigma_1) = -4\pi G a^2 \delta\rho_1, \tag{2.103}$$

$$\psi_1' + \mathcal{H}\phi_1 = -4\pi G a^2(\rho_0 + P_0)V_1, \tag{2.104}$$

where $V_1 \equiv v_1 + B_1$ is the covariant velocity perturbation. In the longitudinal gauge, the scalar shear σ_1 vanishes, and the previous equations become

$$3\mathcal{H}(\Psi' + \mathcal{H}\Phi) - \nabla^2\Psi = -4\pi G a^2 \delta\rho_{1\ell}, \tag{2.105}$$

$$\Psi' + \mathcal{H}\Phi = -4\pi G a^2(\rho_0 + P_0)v_{1\ell}. \tag{2.106}$$

In a similar fashion, and with a little bit of work, the evolution equation for scalar perturbations can be derived as

$$\begin{aligned} [2\mathcal{H}\phi_1' + (4\mathcal{H}' + 2\mathcal{H}^2)\phi_1 + 2\psi_1'' + 4\mathcal{H}\psi_1' - \nabla^2\sigma_1' - 2\mathcal{H}\nabla^2\sigma_1 + \nabla^2\phi_1 - \nabla^2\psi_1]\delta_{ij} \\ + \sigma_{1,ij}' + 2\mathcal{H}\sigma_{1,ij} + \psi_{1,ij} - \phi_{1,ij} = 8\pi G a^2 \delta P_1 \delta_{ij}. \end{aligned} \tag{2.107}$$

Taking the trace of this equation yields

$$3[2\mathcal{H}\phi_1'+(4\mathcal{H}'+2\mathcal{H}^2)\phi_1+2\psi_1''+4\mathcal{H}\psi_1']-2\nabla^2[\sigma_1'+2\mathcal{H}\sigma_1+\psi_1-\phi_1] = 8\pi G(3a^2\delta P_1). \quad (2.108)$$

The traceless part, instead, gives

$$\nabla^2[\sigma_1' + 2\mathcal{H}\sigma_1 + \psi_1 - \phi_1]\delta_{ij} = 0. \quad (2.109)$$

Hence, the evolution equation (2.107) can be recast as

$$\psi_1'' + 2\mathcal{H}\psi_1' + \mathcal{H}\phi_1' + (2\mathcal{H}' + \mathcal{H}^2)\phi_1 = 4\pi G a^2 \delta P_1, \quad (2.110)$$

$$\sigma_1' + 2\mathcal{H}\sigma_1 + \psi_1 - \phi_1 = 0. \quad (2.111)$$

Equation (2.111) represents an evolution equation for the scalar shear potential σ_1. However, in the longitudinal gauge, it gives the following important identity

$$\Psi = \Phi. \quad (2.112)$$

So, equation (2.110) becomes an evolution equation for the metric perturbation Ψ as

$$\Psi'' + 3\mathcal{H}\Psi' + (2\mathcal{H}' + \mathcal{H}^2)\Psi = 4\pi G a^2 \delta P_{1\ell}. \quad (2.113)$$

For adiabatic pressure perturbations, the case which is relevant for a single fluid [41], it follows that

$$\delta P = c_s^2 \delta\rho,$$

where c_s^2 is the square of the adiabatic speed of sound in the fluid. Based on this, and invoking the energy constraint (2.105) to substitute for $\delta\rho_{1\ell}$, the following closed equation is obtained

$$\Psi'' - c_s^2\nabla^2\Psi + 3\mathcal{H}(1 + c_s^2)\Psi' + [2\mathcal{H}' + \mathcal{H}^2(1 + 3c_s^2)]\Psi = 0, \quad (2.114)$$

which represents a scalar wave equation for Ψ. Invoking the background equations (2.96) and (2.97), the term $2\mathcal{H}' + \mathcal{H}^2(1 + 3c_s^2)$ can be expressed as

$$2\mathcal{H}'+\mathcal{H}^2(1+3c_s^2) = -\frac{8\pi G}{3}a^2(\rho_0+3P_0)+\frac{2\Lambda}{3}a^2+\frac{8\pi G}{3}a^2\rho_0+\frac{\Lambda}{3}a^2+3c_s^2\left(\frac{8\pi G}{3}a^2\rho_0 + \frac{\Lambda}{3}a^2\right),$$

$$= 8\pi G a^2(c_s^2\rho_0 - P_0) + \Lambda a^2(1 + c_s^2),$$

$$= 8\pi G a^2 \rho_0 (c_s^2 - \omega) + \Lambda a^2 (1 + c_s^2).$$

where $\omega = P_0/\rho_0$. Inserting this back, equation (2.114) becomes

$$\Psi'' - c_s^2 \nabla^2 \Psi + 3\mathcal{H}(1 + c_s^2)\Psi' + [8\pi G a^2 \rho_0 (c_s^2 - \omega) + \Lambda a^2 (1 + c_s^2)]\Psi = 0. \tag{2.115}$$

Choosing a linear, barotropic equation of state (2.23) for the perfect fluid as before, it follows that $c_s^2 = \omega$. In this case, the continuity equation (2.98) can be integrated to yield

$$\rho_0 \propto \frac{1}{a^{3(1+\omega)}}, \tag{2.116}$$

which implies that dust is the relevant matter model towards the asymptotic future, whereas radiation is the relevant model towards the asymptotic past.

Similarly, given (2.23), the Friedmann's equation (2.96) and (2.97) can be solved exactly for the scale factor a to give [41]

$$\frac{a(\eta)}{a(\eta_0)} = \left(\frac{\eta}{\eta_0}\right)^{2/(1+3\omega)}, \tag{2.117}$$

for some arbitrary $\eta_0 \in (0, \infty)$. Consequently, it follows that

$$\mathcal{H} = \frac{2}{(1 + 3\omega)\eta}. \tag{2.118}$$

Taking equations (2.23), (2.117) and (2.118) into account, the evolution equation (2.115) can be expressed as

$$\Psi'' - \omega \nabla^2 \Psi + \frac{6(1+\omega)}{(1+3\omega)} \frac{1}{\eta} \Psi' + \Lambda a_m \eta^{4/(1+3\omega)} (1 + \omega)\Psi = 0, \tag{2.119}$$

where a_m is an arbitrary constant. It is noteworthy that once the behaviour of Ψ is understood from the previous equation, the behaviour of perturbations of the fluid energy density can be determined from the energy constraint as

$$\delta\rho_{1\ell} = \frac{1}{4\pi G a^2} [\nabla^2 \Psi - 3\mathcal{H}\Psi' - 3\mathcal{H}^2 \Psi]. \tag{2.120}$$

Equation (2.119) is the first equation that is going to be analyzed asymptotically, in the direction of the initial singularity, in chapter 5.

2.2 Scalar fields on a FL background

Scalar fields represent another example of a matter model which can be coupled to spacetime geometry through the Einstein's equations. They, however, came to play an important role in cosmology only since the 1980s with the proposal of a rapid, exponential expansion in the early universe, the so called inflation [28]. For a minimally-coupled scalar field ϕ, the energy momentum-tensor on a spacetime (M, g) can be expressed as [39]

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \partial^\lambda \phi \partial_\lambda \phi + U(\phi) \right). \tag{2.121}$$

In the case of a massive scalar field, the potential $U(\phi)$ can be expressed as

$$U(\phi) = \frac{1}{2} m^2 \phi^2, \tag{2.122}$$

for m the mass of the field. Comparing (2.121) with (2.21), it follows that the energy-momentum tensor for a scalar field can be put in a perfect fluid form with the following identifications

$$u_\mu = \frac{\partial_\mu \phi}{\sqrt{-\partial^\lambda \phi \partial_\lambda \phi}}, \tag{2.123}$$

$$\rho_\phi = -\frac{1}{2} \partial^\lambda \phi \partial_\lambda \phi + U, \tag{2.124}$$

$$P_\phi = -\frac{1}{2} \partial^\lambda \phi \partial_\lambda \phi - U. \tag{2.125}$$

In particular, for a homogeneous scalar field, as would be the case for a field coupled to a FL background, the energy density and pressure simplify to

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + U, \qquad P_\phi = \frac{1}{2} \dot{\phi}^2 - U. \tag{2.126}$$

It is worth mentioning that, in contrast to barotropic fluids, the ratio $\omega = P/\rho$ for a scalar field is not in general a constant, but time-dependent. This is related to the fact that the adiabatic speed of sound c_s^2 is not the true speed of propagation of field fluctuations [18]. Another interesting feature of scalar fields can be seen in the case of a constant field $\phi(t) = \phi_0$, hence $\dot{\phi} = 0$. As a result, the field is potential dominated, and, based on (2.126), it follows that

$$\rho_\phi + P_\phi = 0,$$

which implies that the strong energy condition $\rho_\phi + 3P_\phi \geq 0$ is not satisfied anymore. Moreover, the energy-momentum tensor simplifies to

$$T_{\mu\nu} = -U(\phi_0) g_{\mu\nu},$$

which mimics the effect of a cosmological constant given by

$$\Lambda = 8\pi G\, U(\phi_0).$$

Hence, there are two perspectives regarding the identity of the cosmological constant. These are the geometric perspective, which considers Λ as an extra degree of freedom in the Einstein tensor, and the matter perspective, which views it as a contribution from a fluid satisfying a $P = -\rho$ equation of state in the energy-momentum tensor. In this respect, and in relation to equation (2.119), it is the first perspective to be emphasized in this thesis.

The equation of motion of a scalar field, namely the Klein-Gordon equation, follows from the covariant conservation of $T_{\mu\nu}$. From (2.121), it follows that

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= (\nabla_\mu \partial^\mu \phi)\partial^\nu \phi + \partial^\mu \phi(\nabla_\mu \partial^\nu \phi) - g^{\mu\nu}(\nabla_\mu \partial^\lambda \phi)\partial_\lambda \phi - g^{\mu\nu}\nabla_\mu U, \\ &= (\nabla^\mu \partial_\mu \phi)\partial^\nu \phi + (\nabla^\mu \partial^\nu \phi)\partial_\mu \phi - (\nabla^\nu \partial^\lambda \phi)\partial_\lambda \phi - g^{\mu\nu}\nabla_\mu U, \\ &= \left(\nabla^\mu \partial_\mu \phi - \frac{dU}{d\phi}\right)\partial^\nu \phi, \end{aligned} \tag{2.127}$$

where the identity $\nabla^\nu \partial^\lambda = \nabla^\lambda \partial^\nu$ has been employed. This implies, for the case of (2.122), that

$$\nabla^\mu \nabla_\mu \phi - m^2 \phi = 0. \tag{2.128}$$

Moreover, the d'Alembert operator $\Box_g \equiv g^{\mu\nu}\nabla_\mu \nabla_\nu$ for a scalar function u can be expressed as

$$\Box_g u = \frac{1}{\sqrt{-det\, g}}\partial_\mu \left(\sqrt{-det\, g}\, \partial^\mu u\right), \tag{2.129}$$

which follows from the fact that for a scalar u

$$\Box_g u = \nabla_\mu \nabla^\mu u = \nabla_\mu \partial^\mu u = \partial_\mu \partial^\mu u + \Gamma^\mu_{\mu\alpha}\partial^\alpha u = \partial_\mu \partial^\mu u + \frac{1}{\sqrt{-det\, g}}\partial_\alpha \left(\sqrt{-det\, g}\right)\partial^\alpha u.$$

Based on (2.129), the Klein-Gordon equation for a massive scalar field ϕ on a flat FL background, coupled to a perfect fluid, can be obtained as

$$
\begin{aligned}
\Box_{g_{FL}}\phi - m^2\phi &= \frac{1}{\eta^{8/(1+3\omega)}}\left[-\partial_\eta\left(\eta^{\frac{4}{1+3\omega}}\phi'\right) + \eta^{\frac{4}{1+3\omega}}\sum_{i=1}^{3}\phi_{ii}\right] - m^2\phi, \\
&= \frac{1}{\eta^{8/(1+3\omega)}}\left[-\eta^{\frac{4}{1+3\omega}}\phi'' - \frac{4}{1+3\omega}\eta^{\frac{3(1-\omega)}{1+3\omega}}\phi' + \eta^{\frac{4}{1+3\omega}}\sum_{i=1}^{3}\phi_{ii}\right] - m^2\phi, \qquad (2.130) \\
&= -\frac{1}{\eta^{4/(1+3\omega)}}\phi'' - \frac{4}{1+3\omega}\frac{1}{\eta^{(5+3\omega)/(1+3\omega)}}\phi' + \frac{1}{\eta^{4/(1+3\omega)}}\sum_{i=1}^{3}\phi_{ii} - m^2\phi,
\end{aligned}
$$

which yields

$$
\phi'' - \sum_{i=1}^{3}\phi_{ii} + \frac{4}{1+3\omega}\frac{1}{\eta}\phi' + \eta^{4/(1+3\omega)}m^2\phi = 0. \qquad (2.131)
$$

This is the second equation to be analyzed past-asymptotically in chapter 5.

Chapter 3

Mathematical background on the wave equation

In this chapter, some background on the fundamental notions of the wave equation is presented, in particular the concepts of **energy** and **Sobolev norms**. Due to the fact that the equations of interest in this thesis are systems of wave equations, these notions will be of relevance for the upcoming discussion. In this respect, some preliminaries from the theory of ODEs are discussed first. Then, the relevant notions for linear wave equations in (n+1)-dimensions are introduced. Even though it is the case $n = 3$ that is important for cosmology, the discussion is carried out for a general space dimension. It is worth remarking that most of the discussion of this chapter is based on [49], where all the proofs are provided.

3.1 Preliminaries from the theory of ODEs

Before tackling the wave equation directly, it is simpler to start looking into relevant concepts from the theory of ODEs that help to illustrate some of the main notions related to PDEs, above all the wave equation. In particular, one central issue in the study of differential equations, in general, is that of local existence of solutions. In other words, for a function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$, which is at least continuous, this amounts to showing that for the problem

$$\begin{aligned} \frac{dx}{dt}(t) &= f[t, x(t)], \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

where $x_0 \in \mathbb{R}^n$, there exists a continuously differentiable function x defined on an interval $(-\epsilon, \epsilon)$, for some $\epsilon > 0$, such that (3.1) is satisfied. This is done by setting up a sequence of approximations as

$$x_n = x_0 + \int_0^t f[s, x_{n-1}(s)]ds, \quad (3.2)$$

for $n \geq 1$, and proving that this sequence converges. But, to be able to achieve this, the important concepts of metric spaces and completeness are needed. This motivates the following definition.

Definition 3.1 Let X be a set. A function $d : X \times X \to \mathbb{R}$ satisfying the following properties:

- For each pair of points $x, y \in X$, d is non-negative

  $$d(x, y) \geq 0,$$

  for all $x, y \in X$.

- $$d(x, y) = 0 \Leftrightarrow x = y.$$

- d is symmetric

  $$d(x, y) = d(y, x),$$

  for all $x, y \in X$.

- d satisfies the triangle inequality

  $$d(x, y) \leq d(x, z) + d(z, y),$$

  for all $x, y, z \in X$.

is called a metric on X. In this case, (X, d) is referred to as a metric space.

The concept of metric spaces allows to characterize the convergence of a sequence, without knowledge of the element to which it converges. This is related to the important notion of a Cauchy sequence.

Definition 3.2 Let (X, d) be a metric space. A sequence $x_n \in X$, $n \geq 1$, is called a Cauchy sequence if for every $\epsilon > 0$, there is an N such that for $n, m \geq N$

$$d(x_n, x_m) \leq \epsilon.$$

Having a Cauchy sequence, however, does not mean that it always converges. This is instead a property of the underlying metric space.

Definition 3.3 Let (X, d) be a metric space. If for every Cauchy sequence $\{x_n\}$, $n \geq 1$, there is an $x \in X$ such that $x_n \to x$, then the metric space is called complete.

Consequently, complete metric spaces are very important in the analysis of ODEs, and differential equations in general, as they allow to show that a solution exists. One example of a complete metric space is the real numbers, where the metric in this case is $d(x, y) = |x - y|$.

Spaces with more structure than that of metric spaces can also be defined, namely Banach spaces.

Definition 3.4 A normed linear space is a vector space X (over $\mathbb{R}$ or $\mathbb{C}$) on which there is a function $\|\cdot\|$ defined, called a norm, with the following properties

$$\|x\| \geq 0, \quad \|x\| = 0 \Leftrightarrow x = 0,$$

$$\|\lambda x\| = |\lambda| \, \|x\| \, ,$$

$$\|x + y\| \leq \|x\| + \|y\| \, .$$

It is worth remarking from the previous definition that if X is a normed linear space with a norm $\|\cdot\|$, then $d(x, y) = \|x - y\|$ induces a metric on X. Based on this, there is the following definition.

Definition 3.5 Let X be a normed linear space with a norm $\|\cdot\|$ and let $d(x, y) = \|x - y\|$. Then, X is said to be a Banach space if (X, d) is complete.

So, a Banach space is just a complete normed space, which is bigger than a complete metric space. This is highlighted by the fact that not every metric is induced by a norm. In fact, a typical example of a Banach space is again $\mathbb{R}^n$, equipped with the usual norm

$$|x| = \left(\sum_{i=1}^{n} (x^i)^2 \right)^{1/2} ,$$

for $x = (x^1, x^2, ..., x^n)$.

Hence, the strategy to show local existence of solutions to differential equations is to first identify a suitable complete metric space, or a Banach space, for the initial data

(x_0 in the case of (3.1)), and then construct a Cauchy sequence of approximations in such a space. This in turn determines the corresponding space of the solutions. For the particular case of (3.1), it is the space $C([-\epsilon, \epsilon], \mathbb{R}^n; \|\cdot\|)$ that does the job. In other words, the space of continuous functions from the interval $[-\epsilon, \epsilon]$ to $\mathbb{R}^n$, equipped with a suitable norm $\|\cdot\|$. Such a space is of the general form

$$[C_b(X, Y), \|\cdot\|_C],$$

for (X, d) a metric space and $(Y, \|\cdot\|)$ a Banach space, where for $f \in C_b(X, Y)$

$$\| f\|_C := \sup_{x \in X} \| f(x)\| \,.$$

The choice of this space is justified by the following theorem.

Theorem 3.1 *Let (X, d) be a metric space and let $(Y, \|\cdot\|)$ be a Banach space. Then, $[C_b(X, Y), \|\cdot\|_C]$ is a Banach space.*

Even though the concepts are similar, the situation is a little bit more elaborate for the linear wave equation in (n+1)-dimensions due to some technical details, as it will be demonstrated in the next section.

3.2 The linear wave equation in (n+1)-dimensional Minkowski spacetime

3.2.1 Energies and norms

Given a Minkowski spacetime (M, η), where η is the $(n+1) \times (n+1)$ matrix given by

$$\eta = \mathrm{diag}(-1, 1, ..., 1),$$

then the linear, homogeneous wave equation on such a background is expressed as

$$\Box_\eta u \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu u = -u_{tt} + \Delta u = 0, \tag{3.3}$$

where

$$u_{tt} = \partial_t^2 u, \qquad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial(x_i)^2}.$$

Similar to (3.1), an important goal is to show local existence of solutions to the following initial-value problem

$$
\begin{aligned}
&\Box u = 0, \\
&u(0,x) = f(x), \\
&u_t(0,x) = g(x),
\end{aligned}
\tag{3.4}
$$

where the initial data in this case are the given functions $f(x)$ and $g(x)$. The question now becomes: What are the suitable function spaces, to which f and g should belong, such that local existence for (3.4) is guaranteed ? A hint to the answer comes from looking into the following quantity

$$
E(t) := \frac{1}{2}\int_{\mathbb{R}^n} [u_t^2 + |\nabla u|^2](t,x)dx, \tag{3.5}
$$

which is referred to as the energy, and $\nabla u = (\partial_1 u, ..., \partial_n u)$. Assuming that f and g have a compact support, $E(t)$ can be shown to be conserved as follows

$$
\frac{dE}{dt} = \int_{\mathbb{R}^n} [u_t u_{tt} + \nabla u \cdot \nabla u_t]dx = \int_{\mathbb{R}^n} [u_{tt} - \Delta u]u_t dx = 0, \tag{3.6}
$$

where for $x, y \in \mathbb{R}^n$

$$
x \cdot y = \sum_{i=1}^{n} x^i y^i.
$$

Based on this fact and equation (3.5), it seems that the norm to be considered should be something of the following form

$$
\|u\|_{H^k(\mathbb{R}^n)} = \left(\sum_{|\alpha| \le k} \int_{\mathbb{R}^n} |\partial^\alpha u|^2 dx \right)^{1/2}, \tag{3.7}
$$

where for a multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$

$$
|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n,
$$

and the meaning of $H^k(\mathbb{R}^n)$ is to become clear later. That (3.7) indeed defines a norm, according to definition (3.4), is something to be verified. Moreover, it is not clear what function space corresponds to (3.7). A good starting point to attack these questions is the Fourier transform. However, before proceeding, two remarks are in order. First, when considering the inhomogeneous wave equation

$$
u_{tt} - \Delta u = F,
$$

the energy $E(t)$ is not conserved anymore. Instead, (3.6) yields

$$\frac{dE}{dt} = \int_{\mathbb{R}^n} u_t F dx.$$

Second, the discussion of this section is carried out for linear wave equations on a Minkowski background, as the main motivation is trying to explain the essential concepts without affronting too many technical details. For a general Lorentzian manifold (M, g), instead, the treatment becomes more involved, as in this situation

$$\Box_g u \equiv g^{\mu\nu}(u, \partial u)\partial_\mu \partial_\nu u,$$

and the dependence of g on u and ∂u results in additional technical complications.

3.2.2 Schwartz functions and the Fourier transform

The function space most relevant for discussing the Fourier transform is the set of Schwartz functions, which is defined as follows.

Definition 3.6 The Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is the subset of $C^\infty(\mathbb{R}^n, \mathbb{C})$ (smooth, complex-valued functions) such that for every pair of multi-indices α and β, there is a real constant $C_{\alpha,\beta}$ such that

$$|x^\alpha \partial^\beta f(x)| \leq C_{\alpha,\beta}, \tag{3.8}$$

for all $x \in \mathbb{R}^n$

In other words, Schwartz functions are smooth functions whose derivatives decay at infinity faster than any inverse power of x. Another function space that is closely related to $\mathcal{S}(\mathbb{R}^n)$ is $C_0^\infty(\mathbb{R}^n, \mathbb{C})$, namely smooth, complex-valued functions with a compact support. In fact,

$$C_0^\infty(\mathbb{R}^n, \mathbb{C}) \subset \mathcal{S}(\mathbb{R}^n).$$

For example, $f(x) = \exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$, but it is not in $C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Given $\mathcal{S}(\mathbb{R}^n)$, the Fourier transform can be defined as follows.

Definition 3.7 Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then, the Fourier transform of f, denoted $\hat{f}$, is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx. \tag{3.9}$$

It is worth remarking that, based on (3.8), definition (3.9) makes sense. Moreover, the Fourier transform can be differentiated as

$$\partial_\xi^\alpha \hat{f}(\xi) = \int_{\mathbb{R}^n} (-ix)^\alpha e^{-ix\cdot\xi} f(x)dx. \tag{3.10}$$

Similarly, and using integration by parts

$$\xi^\alpha \hat{f}(\xi) = \int_{\mathbb{R}^n} i^{|\alpha|}\partial_x^\alpha(e^{-ix\cdot\xi})f(x)dx = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}(-i)^{|\alpha|}\partial_x^\alpha f(x)dx. \tag{3.11}$$

Based on (3.10) and (3.11), it follows that

$$\xi^\alpha \partial_\xi^\beta \hat{f}(\xi) = \int_{\mathbb{R}^n} (-i)^{|\alpha|+|\beta|} e^{-2ix\cdot\xi} x^\beta \partial_x^\alpha f(x)dx, \tag{3.12}$$

which shows that $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$. Consequently, the Fourier transform represents an automorphism on the space $\mathcal{S}(\mathbb{R}^n)$. Another important fact related to the Fourier transform is stated by the following theorem.

Theorem 3.2 *For all $f \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform is invertible, namely*

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)d\xi.$$

Now, defining a function $h(x) = (2\pi)^{-n}\bar{\hat{g}}(x)$, where $g \in \mathcal{S}(\mathbb{R}^n)$, and $\overline{(\quad)}$ denotes the complex conjugate, then the following identity

$$\int_{\mathbb{R}^n} \hat{f}(\xi)h(\xi)d\xi = \int_{\mathbb{R}^n} f(x)\hat{h}(x)dx,$$

can be recast as

$$(2\pi)^{-n}\int_{\mathbb{R}^n} \hat{f}(\xi)\bar{\hat{g}}(\xi)d\xi = \int_{\mathbb{R}^n} f(x)\bar{g}(x)dx,$$

which is known as the Parseval's formula. Applying this to $f = g = \partial^\alpha u$, where $u \in \mathcal{S}(\mathbb{R}^n)$, it follows that

$$(2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{\partial^\alpha u}\overline{\widehat{(\partial^\alpha u)}}d\xi = \int_{\mathbb{R}^n} \partial^\alpha u\overline{(\partial^\alpha u)}dx.$$

Inserting the fact that $\widehat{\partial^\alpha u} = i^{|\alpha|}\xi^\alpha\hat{u}$, based on (3.11), the previous formula becomes

$$(2\pi)^{-n}\int_{\mathbb{R}^n} \xi^{2\alpha}|\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\partial^\alpha u(x)|^2 dx. \tag{3.13}$$

So, using (3.13), the norm (3.7) for $u \in \mathcal{S}(\mathbb{R}^n)$ can be alternatively expressed as

$$\|u\|^2_{H^k(\mathbb{R}^n)} = (2\pi)^{-n} \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi. \tag{3.14}$$

At this point, it may be convenient to express the Fourier side of the previous expression in another form. This is done by appealing to the following lemma.

Lemma 3.1 *For every positive integer k, there are positive real constants $c_{1,k}, c_{2,k}$ such that*

$$c_{1,k}(1+|\xi|^2)^k \leq \sum_{|\alpha|\leq k} \xi^{2\alpha} \leq c_{2,k}(1+|\xi|^2)^k.$$

Employing lemma 3.1, the following alternative, but equivalent norm can be defined.

Definition 3.8 Define for $u \in \mathcal{S}(\mathbb{R}^n)$ and $s \in \mathbb{R}$

$$|u|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{1/2}. \tag{3.15}$$

Thus, for the case $s = k$ a non-negative integer, it follows that there are constants $C_{i,k},\ i = 1, 2$, such that

$$C_{1,k}|u|_{H^k(\mathbb{R}^n)} \leq \|u\|_{H^k(\mathbb{R}^n)} \leq C_{2,k}|u|_{H^k(\mathbb{R}^n)}, \tag{3.16}$$

which shows the equivalence of the norms (3.7) and (3.15). This also shows how s, which is related to the degree of differentiability of the function, makes sense even being real, and not only integer.

Based on the previous observations, the motivation for discussing the Fourier transform becomes clear from the fact that the norm (3.15) can be used to bound the supremum norm of the derivatives of a function, a property which is relevant for proving local existence. This is based on the following theorem.

Theorem 3.2 *Let k be a non-negative integer and assume that $s > k + n/2$. Then, there is a constant C, depending on k, n and s such that for all $f \in \mathcal{S}(\mathbb{R}^n)$*

$$\|f\|_{C^k_b(\mathbb{R}^n,\mathbb{C})} \leq C|f|_{H^s(\mathbb{R}^n)}, \tag{3.17}$$

where

$$\|f\|_{C_b^k(\mathbb{R}^n,\mathbb{C})} = \sum_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} |\partial^\alpha f(x)|. \tag{3.18}$$

It is worth remarking that the norm (3.18) is basically defined for $f \in C_b^k(\mathbb{R}^n, \mathbb{C})$, which denotes the set of functions $f \in C^k(\mathbb{R}^n, \mathbb{C})$ such that for each multi-index α with $|\alpha| \leq k$, there is a real constant $C_\alpha < \infty$ such that

$$|\partial^\alpha f(x)| \leq C_\alpha,$$

for all $x \in \mathbb{R}^n$. However, based on this definition and (3.8), it is clear that

$$\mathcal{S}(\mathbb{R}^n) \subset C_b^k(\mathbb{R}^n, \mathbb{C}).$$

The result of theorem 3.2 can be related to the norm (3.7) through the use of inequality (3.16). This leads to the following corollary.

Corollary 3.1 *Let k and m be non-negative integers such that $m > k + n/2$. Then, there is a constant C, depending on k, n and m such that for all $f \in \mathcal{S}(\mathbb{R}^n)$*

$$\|f\|_{C_b^k(\mathbb{R}^n,\mathbb{C})} \leq C \, \|f\|_{H^m(\mathbb{R}^n)} . \tag{3.19}$$

3.2.3 L^2-spaces and equivalence classes

To get an insight into the function space for which the norm (3.7) is defined, such that the corresponding space is a Banach space, it is worth considering the following simpler norm.

Definition 3.9 For $f \in C(\mathbb{R}^n)$, define

$$\|f\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f|^2(x)dx\right)^{1/2}, \tag{3.20}$$

such that the right-hand side is finite.

That (3.20) defines a norm is something that can be verified. Also, such a norm represents the case $k = 0$ of (3.7). In this respect, it is worth mentioning that functions for which (3.20) holds are called square-integrable. An important identity for such functions, based on (3.20), is given by the following lemma.

Lemma 3.2 *Let $f, g \in C(\mathbb{R}^n)$ be such that their square is integrable. Then,*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \|f\|_{L^2(\mathbb{R}^n)} \, \|g\|_{L^2(\mathbb{R}^n)} . \tag{3.21}$$

Now, to have an idea of what properties the function space corresponding to the norm (3.20) should have, a sequence of functions $f_m(x)$, for the case $n = 1$, can be constructed as

$$f_m(x) = \begin{cases} x^m & if \quad x \in [-1, 1] \\ x^{-m} & if \quad x \notin [-1, 1], \end{cases} \tag{3.22}$$

which, for $m = 1, 2, ...$, represents a sequence of continuous, square integrable functions. So, it can be calculated that

$$\begin{aligned} \|f_m\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}} |f_m(x)|^2 dx = \int_{-\infty}^{-1} x^{-2m} dx + \int_{-1}^{1} x^{2m} dx + \int_{1}^{\infty} x^{-2m} dx, \\ &= \left[\frac{x^{-2m+1}}{1-2m}\right]_{-\infty}^{-1} + \left[\frac{x^{2m+1}}{1+2m}\right]_{-1}^{1} + \left[\frac{x^{-2m+1}}{1-2m}\right]_{1}^{\infty}, \\ &= \frac{2}{2m+1} + \frac{2}{2m-1}. \end{aligned}$$

This shows that $\|f_m\|^2_{L^2(\mathbb{R}^n)}$ converges to zero as $m \to \infty$. But, this is intriguing as, considering the point-wise convergence of f_m, $f_m(x) \to 1$ for $x = 1$, $f_m(x) \to 0$ for $x \notin \{-1, 1\}$, and for $x = -1$, $f_m(x)$ does not converge at all. However, $\|f_m\|^2_{L^2(\mathbb{R}^n)}$ converges to zero in all of these cases. Hence, it seems that considering (3.20) as a norm on a space of functions implies that the limiting function should always be thought of as being zero. Another issue is that even though f_m is continuous, the point-wise limit is not. As it turns out, the space corresponding to the norm (3.20), and hence the norm (3.7), should not be a space of functions, but instead a space of equivalence classes of functions, where two functions f and g are equivalent if the set on which they differ is of measure zero. In this respect, it becomes clear that the basic concepts of measure and integration theory are relevant in discussing such spaces [21].

To further see the need for spaces of equivalence classes, instead of spaces of functions, and to get to know the space that gives a Banach space when equipped with (3.20), the following definition is needed.

Definition 3.10 The class of Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$, such that $|f|^2$ is integrable, is denoted $\mathcal{L}^2(\mathbb{R}^n)$.

Based on this definition, it follows that the $\|\cdot\|_{L^2(\mathbb{R}^n)}$ norm can be extended to $f \in \mathcal{L}^2(\mathbb{R}^n)$ as

$$\|f\|_{L^2(\mathbb{R}^n)} = \left(\int |f|^2 d\mu_n\right)^{1/2},$$

where μ_n is the Lebesgue measure. Moreover, from the properties of the Lebesgue integral, it follows that if f is measurable, and the set A on which f is non-zero is such that $\mu_n(A) = 0$ (of measure zero), then

$$\int |f| d\mu_n = 0.$$

As a result, in order for $\|\cdot\|_{L^2(\mathbb{R}^n)}$ to be a meaningful norm, it has to be the case that functions $f \in \mathcal{L}^2(\mathbb{R}^n)$ that are only non-zero on a set of measure zero are considered to be zero. This motivates introducing an equivalence relation on the space $\mathcal{L}^2(\mathbb{R}^n)$ as follows: Two functions f and g are equivalent, $f \sim g$, if the set A on which $f \neq g$ is such that $\mu_n(A) = 0$. Consequently, a corresponding function space, denoted $L^2(\mathbb{R}^n)$, is defined as follows.

Definition 3.11 $L^2(\mathbb{R}^n)$ is defined to be the set of equivalence classes $[f]$ of functions f such that

$$f \in \mathcal{L}^2(\mathbb{R}^n).$$

In a similar fashion, the norm $\|\cdot\|_{L^2(\mathbb{R}^n)}$ can be extended to the space $L^2(\mathbb{R}^n)$ as

$$\|[f]\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)},$$

where it can be shown that such a definition is well-defined. For example, if $[f] = [g]$, then $f = g$, except for a set of measure zero. But, from the above considerations, this implies that $\|f\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)}$. Similarly, it can be shown that for $[f] \in L^2(\mathbb{R}^n)$, $\|\cdot\|_{L^2(\mathbb{R}^n)}$ indeed defines a norm. Taking all of this into account, the following theorem can be proved.

Theorem 3.3 *The space $L^2(\mathbb{R}^n)$ equipped with $\|\cdot\|_{L^2(\mathbb{R}^n)}$ is a Banach space.*

3.2.4 Sobolev spaces and weak solutions

After looking into $L^2(\mathbb{R}^n)$ spaces, it remains the task of identifying the corresponding space, of possibly equivalence classes, of the norm (3.7), or equivalently (3.15). As long as the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is concerned, both of (3.7) and (3.15) can be shown to define a norm. However, the issue is that equipping $\mathcal{S}(\mathbb{R}^n)$ with the $\|\cdot\|_{H^k(\mathbb{R}^n)}$, for a non-negative integer k, results in a metric space that is not complete. So, something more general needs to be considered, as was indicated above.

To start tackling the problem, it is first worth remarking that the space $\mathcal{S}(\mathbb{R}^n)$ can be viewed as a linear subspace of $L^2(\mathbb{R}^n)$. Moreover, for any $f \in \mathcal{S}(\mathbb{R}^n)$, it is the case that

$$\|\partial^\alpha f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{H^k(\mathbb{R}^n)}, \tag{3.23}$$

for $|\alpha| \leq k$. This indicates that if $\{f_m\}$ is a Cauchy sequence of Schwartz functions with respect to $\|\cdot\|_{H^k(\mathbb{R}^n)}$, namely for every $\epsilon > 0$ there is an M such that for $m, l \geq M$

$$\|f_m - f_l\|_{H^k(\mathbb{R}^n)} \leq \epsilon,$$

then $\{\partial^\alpha f_m\}$ is a Cauchy sequence with respect to $\|\cdot\|_{L^2(\mathbb{R}^n)}$ for $|\alpha| \leq k$. From theorem 3.3, it follows that there is a function $f^\alpha \in L^2(\mathbb{R}^n)$ such that $\partial^\alpha f_m \to f^\alpha$ with respect to $\|\cdot\|_{L^2(\mathbb{R}^n)}$, where the notation $[\,\cdot\,]$ for equivalence classes is dropped for simplicity. Could this indicate anything for the function to which $\{f_m\}$ converges with respect to $\|\cdot\|_{H^k(\mathbb{R}^n)}$, given a complete metric space? This would be the case if f^α can be thought of as $\partial^\alpha f$, as this would imply that

$$f_m \to f$$

with respect to $\|\cdot\|_{H^k(\mathbb{R}^n)}$, where $f \equiv f^0$. However, it is not clear if this is possible because f could not be even continuous. So, it seems that there is need for generalizing the concept of differentiability to account for this case. This can be done as follows: For $\phi, g \in \mathcal{S}(\mathbb{R}^n)$, employing integration by parts gives

$$\int_{\mathbb{R}^n} \phi \partial^\alpha g dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \partial^\alpha \phi g dx.$$

Furthermore, from the fact that $\phi, f^\alpha \in L^2(\mathbb{R}^n)$, it follows that $f^\alpha \phi$ is integrable, based on (3.21). Hence,

$$\left| \int_{\mathbb{R}^n} \phi f^\alpha dx - \int_{\mathbb{R}^n} \phi \partial^\alpha f_m dx \right| \leq \int_{\mathbb{R}^n} |(f^\alpha - \partial^\alpha f_m)\phi| dx \leq \|f^\alpha - \partial^\alpha f_m\|_{L^2(\mathbb{R}^n)} \|\phi\|_{L^2(\mathbb{R}^n)} \to 0,$$

where (3.21) has been used. In a similar fashion,

$$\int_{\mathbb{R}^n} f_m \partial^\alpha \phi dx \to \int_{\mathbb{R}^n} f \partial^\alpha \phi dx.$$

Consequently, it follows that

$$\int_{\mathbb{R}^n} \phi \partial^\alpha f_m dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \partial^\alpha \phi f_m dx \to (-1)^{|\alpha|} \int_{\mathbb{R}^n} \partial^\alpha \phi f dx,$$

which yields

$$\int_{\mathbb{R}^n} \phi f^\alpha dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \partial^\alpha \phi f dx. \tag{3.24}$$

Hence, a generalization of the concept of differentiability for $f \in L^2(\mathbb{R}^n)$ can be motivated as in the following definition.

Definition 3.12 Let k be a non-negative integer. A function $f \in L^2(\mathbb{R}^n)$ is said to be k-times L^2-weakly differentiable if for every multi-index α such that $|\alpha| \leq k$, there is a function $f^\alpha \in L^2(\mathbb{R}^n)$ such that for every $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \phi f^\alpha dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \partial^\alpha \phi f dx.$$

In this context, f^α is referred to as the α-th weak derivative of f.

By appealing to measure and integration theory, it can be shown that the weak derivative f^α is well-defined. Based on this, it follows that the norm $\|\cdot\|_{H^k(\mathbb{R}^n)}$ for a k-times L^2-weakly differentiable function f can be expressed as

$$\|f\|_{H^k(\mathbb{R}^n)} = \left(\sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |f^\alpha|^2 dx \right)^{1/2}. \tag{3.25}$$

Taking all the above considerations into account, it follows that if $\{f_m\}$ defines a sequence of Schwartz functions which is a Cauchy sequence with respect to $\|\cdot\|_{H^k(\mathbb{R}^n)}$, then there exists a k-times L^2-weakly differentiable function f such that $\|f - f_m\|_{H^k(\mathbb{R}^n)}$ converges to zero. This motivates the following definition.

Definition 3.13 A Sobolev space, denoted $H^k(\mathbb{R}^n)$, is the set of k-times L^2-weakly differentiable functions f such that there is a sequence $f_m \in \mathcal{S}(\mathbb{R}^n)$ with $\|f - f_m\|_{H^k(\mathbb{R}^n)} \to 0$.

Then, the following theorem is in order.

Theorem 3.4 *The Sobolev space $H^k(\mathbb{R}^n)$ equipped with the norm $\|\cdot\|_{H^k(\mathbb{R}^n)}$ is a Banach space.*

To complete the picture, it is of interest to know if corollary 3.1 holds for $f \in H^k(\mathbb{R}^n)$. This is shown by the following theorem.

Theorem 3.5 *Let k and m be non-negative integers such that $m > k + n/2$. Then, there is a constant C, depending on k, n and m such that for all $f \in H^k(\mathbb{R}^n)$*

$$\|f\|_{C_b^k(\mathbb{R}^n,\mathbb{C})} \leq C \|f\|_{H^m(\mathbb{R}^n)} . \tag{3.26}$$

Given that a member $f \in H^k(\mathbb{R}^n)$ represents an equivalence class, the meaning of the previous theorem may be a little bit obscure. Equation (3.26) implies that for an equivalence class, there is one function which is also in $C_b^k(\mathbb{R}^n, \mathbb{C})$, and for which the inequality holds. The justification for this formula relies on the fact that if two continuous functions differ, they do so on a set of positive measure.

As a consequence of theorem 3.4, and specifying to the case $n = 3$, it follows that the natural spaces of the initial data f and g for the system (3.4) are

$$f \in H^{k+1}(\mathbb{R}^3), \qquad g \in H^k(\mathbb{R}^3).$$

As a result, the sequence of approximations for the solution u should be set up in the space

$$C\{[0, T],\ H^{k+1}(\mathbb{R}^3)\},$$

which is a Banach space, according to theorem 3.1. On the other hand, it would be nice if a relation between this space and the notion of classical differentiability can be found. This is achieved by the following lemma.

Lemma 3.3 *Assume $u \in C\{[0, T], H^k(\mathbb{R}^3)\}$ where $k > 3/2$. Then, u is continuous. In other words, $u \in C\{[0, T] \times \mathbb{R}^3\}$.*

Based on this, if $u \in C[(T_-, T_+), H^{k+1}(\mathbb{R}^3)]$ is a solution to (3.4) with $k > 3/2$, then it follows that

$$u \in C^1[(T_-, T_+) \times \mathbb{R}^3].$$

However, trying to make sense of this in light of the wave equation

$$u_{tt} - \Delta u = 0, \tag{3.27}$$

it may seem that such a solution is not well-defined. In fact, this motivates the following definition.

Definition 3.14 A function $u \in C^1[(T_-, T_+) \times \mathbb{R}^3]$ is called a weak solution to equation (3.27) if for every $\phi \in C_0^\infty[(T_-, T_+) \times \mathbb{R}^3]$

$$\int_{(T_-,T_+)\times\mathbb{R}^3} [\phi_{tt} - \Delta\phi]u = 0.$$

Chapter 4

Asymptotics of cosmological linear systems of wave equations

In this chapter, a method, based on [48], to analyze the asymptotic behaviour of linear systems of wave equations on a given cosmological background is discussed. In particular, the case of interest is that of **weakly silent**, **balanced** and **convergent** equations, as it will be explained later. The reason for this specification stems from the fact that the coefficients of the equations need to be constrained in some sense before being able to extract any information regarding their asymptotics. In this respect, the cosmological background, for which such an analysis is possible, is first specified, along with its properties. Then, other conditions necessary for reducing the equations to a suitable form, which permits analysis, are explained.

4.1 Equations and background cosmological models

4.1.1 Equations of interest

The main objective of this chapter is to investigate the asymptotics of linear systems of wave equations given by the following form

$$\Box_g u + Xu + \zeta u = f, \tag{4.1}$$

where $\Box_g u := \mathrm{div}(\mathrm{grad}\, u)$ is the wave operator on a given Lorentzian manifold (M, g); X represents a smooth $(m \times m)$-matrix of vector fields on M with coefficients that can be complex, hence $X \in \mathbf{M}_m(\mathbb{C})$; ζ is a smooth $\mathbf{M}_m(\mathbb{C})$-valued function on M; and f is a smooth $\mathbb{C}^m$-valued function on M. In this setting, the goal is to analyze the asymptotic behaviour of solutions $u : M \to \mathbb{C}^m$. It is noteworthy that the complex setting is considered here for technical reasons related to the arguments of the method. However, it is the real setting which is of relevance regarding the results of this thesis. Interest in equations of the form (4.1) stems from the fact that similar equations arise upon linearizing the Einstein's equations, as can be seen from equation (2.119). Even though the system (4.1) does not represent the most general class of equations in this respect, it is still a general class to consider.

As was the case in chapter 2, the Lorentz manifolds of interest are globally hyperbolic ones with closed spatial sections. In particular, $M := \bar{M} \times I$, where $\bar{M}$ is a closed manifold, and $I = (t_-, t_+)$ is an open interval. This is the setting of interest in cosmology, where non-stationary solutions to the field equations play a central role. The asymptotic regimes $t \to t_\pm$ correspond to either a cosmological singularity, whether a big bang or big crunch, or the expanding direction. This can be contrasted with the asymptotically flat setting, which is based on static and/or stationary solutions, and which is of interest in the study of isolated systems, such as galaxies or black holes.

As mentioned above, system (4.1) represents quite a general class of equations. So, a necessary first step is to constraint the underlying manifolds of interest. This can be achieved by restricting the metric g to the following form

$$g = g_{00}(t)dt \otimes dt + g_{0i}(t)dt \otimes dx^i + g_{i0}(t)dx^i \otimes dt + g_{ij}(t)dx^i \otimes dx^j + \sum_{r=1}^{R} a_r^2(t)g_r, \quad (4.2)$$

where $R \in \mathbb{Z}$, $g_{00}, g_{0i}, g_{ij}, a_r \in C^\infty(I, \mathbb{R})$, and the summation convention is invoked. Specifically, $-g_{00}(t)$ and $a_r(t), r = 1, ..., R$, take values only in the interval $(0, \infty)$, and $g_{ij}(t),\ i, j = 1, ..., d$, represent the components of a positive definite matrix for all $t \in I$. Moreover, $(M_r, g_r), r = 1, ..., R$, represent a family of closed Riemannian manifolds.

Based on (4.2), the system (4.1) can be expressed as

$$-g^{00}(t)u_{tt} - \sum_{j,l=1}^{d} g^{jl}(t)\partial_j\partial_l u - 2\sum_{l=1}^{d} g^{0l}(t)\partial_l\partial_t u - \sum_{r=1}^{R} a_r^{-2}(t)\Delta_{g_r} u + \alpha(t)u_t + \sum_{j=1}^{d} X^j(t)\partial_j u + \zeta(t)u = f, \quad (4.3)$$

where $0 \leq d \in \mathbb{Z}$, and Δ_{g_r} is the Laplace-Beltrami operator on (M_r, g_r). The functions f and u are smooth functions from $M = \bar{M} \times I$ to $\mathbb{C}^m$, where

$$\bar{M} := \mathbb{T}^d \times M_1 \times ... \times M_R; \quad (4.4)$$

the differential operators ∂_i are the standard vector fields on $\mathbb{T}^d$; ∂_t denotes differentiation with respect to t; and $\alpha, X^i, \zeta \in C^\infty[I, \mathbf{M}_m(\mathbb{C})]$.

It is worth mentioning that going from (4.1) to (4.3) involves restrictions not only on the underlying metric g, as expressed by (4.2), but also on X and ζ. In particular, the coefficients of (4.3) depend only on t, which is a simple situation, as the equations become separable in this case. Such a choice is also relevant in the context of cosmology, where one basic assumption is that of spatial homogeneity. Another important observation regarding (4.3) is that there are two operations that leave these equations invariant. First, multiplying the system by a strictly positive function of t. Such an operation is equivalent to conformally rescaling the metric by a factor depending only on t. Second, changing the time coordinate. Both of these operations will be employed to put the underlying metric in a preferred form.

4.1.2 Cosmological backgrounds of interest

Due to interest in the class of equations given by (4.3), the relevant class of manifolds is restricted according to the following definition.

Definition 4.1 A separable cosmological model manifold is a Lorentz manifold (M, g) such that $M = \bar{M} \times I$, where $I = (t_-, t_+)$ is an open interval, $0 \leq d, R \in \mathbb{Z}$, $\bar{M}$ is given by (4.4), and

$$g = g_{00}(t)dt \otimes dt + g_{0i}(t)dt \otimes dx^i + g_{i0}(t)dx^i \otimes dt + g_{ij}(t)dx^i \otimes dx^j + \sum_{r=1}^{R} a_r^2(t)g_r, \quad (4.5)$$

where the $g_{\gamma\beta}$, with $\gamma, \beta \in \{0, ..., d\}$, and a_r depend only on t; the (M_r, g_r) are closed Riemannian manifolds; $g_{\gamma\beta}(t) = g_{\beta\gamma}(t)$ for all t; $g_{00}(t) < 0$ for all t; $a_r(t) > 0$ for all t; and $g_{ij}(t)$ represent the components of a positive definite matrix for all t.

Due to the importance of the metrics, second fundamental forms and volumes induced on the spatial hypersurfaces $\bar{M}_t := \bar{M} \times \{t\}$, the following definition is in order.

Definition 4.2 Let (M, g) be a separable cosmological model manifold. Then, the metric and second fundamental form of $\bar{M}_t$ are denoted by $\bar{g}_t = \bar{g}(t)$ and $\bar{k}_t = \bar{k}(t)$, respectively. The trace $\mathrm{tr}_{\bar{g}}\bar{k}$ is referred to as the mean curvature of $\bar{M}_t$. Moreover, $V(t) := \mathrm{vol}_{\bar{g}(t)}(\bar{M})$ denotes the volume of $\bar{M}$ with respect to $\bar{g}(t)$.

Recalling equation (2.9), it is follows immediately that

$$K_\mu{}^\mu \equiv \mathrm{tr}_{\bar{g}}\bar{k} = \theta.$$

Also, denoting the unit timelike normal to $\bar{M}_t$ as U (which was previously denoted $\mathbf{n}$), a standard result is that [46]

$$U(\mathrm{ln}V) = \mathrm{tr}_{\bar{g}}\bar{k}, \quad (4.6)$$

hence the expansion of a congruence represents the fractional change in the volume of the spatial hypersurfaces in the direction of the unit timelike normal.

So, from now on, focus is going to be on separable cosmological models. For these spacetimes, two interesting asymptotic regimes can be identified, namely expanding directions and big bang/big crunch singularities. The first regime is characterized by the volume of the hypersurfaces $V \to \infty$, whereas the second is indicated by $V \to 0$. For these two cases, it is natural to assume that the derivative of the volume in the direction of the unit normal is non-zero, at least asymptotically. In this way, there would be a strict monotonicity of the volume. From equation (4.6), this is equivalent to requiring that the mean curvature is either strictly positive or strictly negative, asymptotically. Hence, the class of separable cosmological models is further restricted to one of the cases of the following definition.

Definition 4.3 Let (M, g) be a separable cosmological model manifold.

- If $V(t) \to \infty$ as $t \to t_+^{(-)}$ and there is a $t_0 \in I$ such that $U(V) > 0$ for $t \geq t_0$, then (M, g) is said to be future expanding.
- If $V(t) \to \infty$ as $t \to t_-^{(+)}$ and there is a $t_0 \in I$ such that $U(V) < 0$ for $t \leq t_0$, then (M, g) is said to be past expanding.
- If $V(t) \to 0$ as $t \to t_+^{(-)}$ and there is a $t_0 \in I$ such that $U(V) < 0$ for $t \geq t_0$, then (M, g) is said to have big crunch asymptotics.
- If $V(t) \to 0$ as $t \to t_-^{(+)}$ and there is a $t_0 \in I$ such that $U(V) > 0$ for $t \leq t_0$, then (M, g) is said to have big bang asymptotics.

As a result of the definition, if (M, g) is a separable cosmological model which is future expanding, then reversing the time coordinate gives a corresponding separable model which is past expanding, and vice versa. Same thing also holds for big crunch and big bang asymptotics. For this reason, it is enough to focus on separable models that are either future expanding, or have big crunch asymptotics.

It turns out that the restrictions imposed in definition 4.3 are still not sufficient, and hence they need to be complemented in order to derive asymptotic information regarding equations (4.3). One suitable way to achieve this is by imposing conditions on the second fundamental form $\bar{k}$. For this purpose, it is more convenient to consider $\bar{k}$ as a map from $T\bar{M}$ to itself. This is obtained by raising one index of $\bar{k}$ using the induced metric $\bar{g}$, which gives the so-called Weingarten map or shape operator $\bar{K}$. From the definition, it follows that $\bar{K}_i^j = \bar{k}_i^j$ with respect to local coordinates, and

$$\mathrm{tr}\bar{K} = \mathrm{tr}_{\bar{g}}\bar{k}.$$

If the assumptions of definition 4.3 hold true, then this implies that the following normalized quantity

$$\bar{K}/\text{tr}\bar{K},$$

known as the expansion normalized Weingarten map, is well-defined, at least asymptotically. Focus on this quantity is motivated by the fact that, in many relevant situations, $\text{tr}\bar{K}$ tends to zero or $\pm\infty$, asymptotically. Specifically, the case of interest is when $\bar{K}/\text{tr}\bar{K}$ converges, a situation referred to as the convergent setting. In addition, it turns out to be necessary to impose conditions on the normal derivative of $\text{tr}_{\bar{g}}\bar{k}$. For a solution to the Einstein's equations, such a condition is usually satisfied by combining the convergence assumption, the energy constraint, the Raychaudhuri equation and, possibly, energy conditions. However, in the context of this discussion, the underlying manifold (M, g) is not assumed to be a solution to the field equations, and hence there is need to impose conditions on $U[\text{tr}_{\bar{g}}\bar{k}]$.

A suitable way to implement the convergent setting is that of imposing boundedness. For this purpose, the notation of the following definition is needed.

Definition 4.4 Let (M, g) be a separable cosmological model manifold and let ϱ be a fixed Riemannian metric on $\bar{M}$. Then, (M, g) is said to have future bounded geometry if there is a constant $0 \leq C \in \mathbb{R}$ and a $t_0 \in I$ such that $|\text{tr}_{\bar{g}}\bar{k}| > 0$ and

$$|\bar{k}|_{\bar{g}}/|\text{tr}_{\bar{g}}\bar{k}| + |U[(\text{tr}_{\bar{g}}\bar{k})^{-1}]| \leq C,$$

for all $t \geq t_0$. If, in addition, there is a 2-tensor field A of mixed type, a constant $a \in \mathbb{R}$ and constants $0 \leq C, \eta \in \mathbb{R}$ such that

$$|\bar{K}/\text{tr}\bar{K} - A|_{\varrho} + |U[(\text{tr}_{\bar{g}}\bar{k})^{-1}] - a| \leq C\exp[-\eta|\ln \mathrm{V(t)}|],$$

for all $t \geq t_0$, then (M, g) is said to be future convergent.

Based on the previous remark about operations leaving the system (4.3) invariant, the notions of the previous definition can be re-expressed in a more convenient form through the following lemma.

Lemma 4.1 *Let (M, g) be a separable cosmological model manifold and let ϱ be a fixed Riemannian metric on $\bar{M}$. Assume that (M, g) is future expanding and has future bounded geometry. Introduce the metric $\hat{g} := (\text{tr}_{\bar{g}}\bar{k})^2 g$ and the time coordinate*

$$\tau(t) := \ln\frac{V(t)}{V(t_0)}.$$

Then, the interval $[t_0, t_+)$ in t-time corresponds to $[0, \infty)$ in τ-time. Moreover, $\hat{g}$ is well-defined on $[0, \infty)$ in τ-time, and can be expressed such that the lapse function $N = 1$.

If $\breve{g}$ and $\hat{k}$ are the induced metric and second fundamental form on constant τ-hypersurfaces by $\hat{g}$, then there is a constant $0 < C \in \mathbb{R}$ such that $|\hat{k}|_{\breve{g}} \leq C$ for all $\tau \geq 0$.

Assuming, in addition to the above, that (M, g) is future convergent, then there is a 2-tensor field $\hat{A}$ on $\bar{M}$ of mixed type such that

$$|\hat{K} - \hat{A}|_{\varrho} \leq Ce^{-\eta\tau},$$

for all $\tau \geq 0$, where $0 < C, \eta \in \mathbb{R}$ and $\hat{K}$ is obtained from $\hat{k}$ by raising one index up using $\breve{g}$.

It is worth remarking that a similar result can be given in the case of big crunch asymptotics.

Now, lemma 4.1 motivates the following important definition.

Definition 4.5 A canonical separable cosmological model manifold is a separable cosmological model manifold such that the interval $I = (t_-, t_+)$ contains $[0, \infty)$ and $N = 1$.

Two important remarks follow. First, to obtain a canonical separable cosmological model manifold, it is sufficient to assume that (M, g) is future expanding; to conformally rescale the metric to $\hat{g} = g(\text{tr}_{\bar{g}}\bar{k})^2$; and to change the time coordinate according to lemma 4.1. Second, there are many examples of cosmological solutions that are canonical in the sense of definition 4.5. For example, the de Sitter spacetime given by the following metric

$$g_{dS} = -dt \otimes dt + \sum_{i=1}^{d} e^{2Ht} dx^i \otimes dx^i, \tag{4.7}$$

on $\mathbb{T}^d \times \mathbb{R}$ is a canonical separable model, where $0 < H \in \mathbb{R}$ and $3 \leq d \in \mathbb{Z}$. It represents a solution to the Einstein's vacuum equations with a positive cosmological constant given by $\Lambda = d(d-1)H^2/2$. Also, from the fact that [25]

$$\bar{k} = \frac{1}{2}\mathcal{L}_U \bar{g},$$

it follows that $\bar{k} = H\bar{g}$. Hence,

$$\bar{K} = H\text{Id}_{T\mathbb{T}^d},$$

which shows that the solution is also future convergent.

4.1.3 Silent metrics

Regarding the asymptotics of equations (4.3), it is important to remark that they are largely dependent on the asymptotic behaviour of the underlying metric, assuming conformal rescaling and change of time coordinate have been done, according to lemma 4.1. In this respect, three types of behaviours can be distinguished, namely asymptotically silent equations, transparent equations or equations with a dominant noisy spatial direction. Out of these three, it is the silent setting that is going to be investigated further in this chapter.

To introduce the notion of silence, it is important to remark first that in many relevant cases, the corresponding metric can be asymptotically expressed as

$$g = -dt \otimes dt + \sum_{r=1}^{R} \alpha_r^2 e^{2\beta_r t} g_r, \tag{4.8}$$

where $0 < \alpha_r \in \mathbb{R}$, $\beta_r \in \mathbb{R}$, and (M_r, g_r) are closed Riemannian manifolds (where the $\mathbb{T}^d$-part of equation (4.5) can be thought of to be included here). Based on (4.8), the following definition can be given.

Definition 4.6 Consider a mertic of the form (4.8). If all the β_r are strictly positive, then the metric is said to be silent.

To understand the motive behind the name, let g be a silent metric, and let γ be a causal curve such that $g(\dot{\gamma}, \dot{\gamma}) \leq 0$. Reparametrizing γ, it can be expressed as

$$\gamma(t) = (\bar{\gamma}_1(t), ..., \bar{\gamma}_R(t), t),$$

where $\bar{\gamma}_r$ takes values in M_r. Using this, the assumption of causality gets expressed as

$$-1 + \sum_{r=1}^{R} \alpha_r^2 e^{2\beta_r t} |\dot{\bar{\gamma}}_r(t)|_{g_r}^2 \leq 0,$$

which implies that $|\dot{\bar{\gamma}}_r(t)|_{g_r} \leq \alpha_r^{-1} e^{-\beta_r t}$. This indicates that $\bar{\gamma}_r(t)$ converges to a point $p_r \in M_r$. For λ another causal curve, the corresponding points $q_r \in M_r$ are also obtained. Typically, there exists one $r \in \{1, ..., R\}$ such that $p_r \neq q_r$. So, for two such curves, there is a t_1 such that for $t \geq t_1$, it is not possible to send information from $\gamma(t)$ to λ. In other words, no future-directed causal curve starting at $\gamma(t)$ intersects λ, and hence there is silence, asymptotically. It is noteworthy that this asymptotic behaviour is related to the important notion of horizons in standard cosmology [41].

4.2 The notion of balance

4.2.1 Motivation: Pathological behaviour of solutions

In the previous section, the highest order coefficients of equation (4.3), namely coefficients of the Lorentz metric, have been constrained to yield a canonical form for the underlying manifold, according to definition 4.5. These restrictions represent a large part of the necessary conditions to be imposed at the moment. Later, additional assumptions will be made, in particular regarding the $g^{0i}(t)$ coefficients.

In this respect, it is interesting to see what kind of conditions are needed for the lower order coefficients, namely the rest of coefficients of equation (4.3). For the matrix-valued functions α and ζ, this can be seen by considering the spatially homogeneous version of the equation, with $f = 0$, as

$$u_{tt} + \alpha(t)u_t + \zeta(t)u = 0. \tag{4.9}$$

Assuming there are constants $C_{cu}, \gamma, \eta_{cu} > 0$ and matrices $\alpha_\infty, \zeta_\infty \in \mathbf{M}_m(\mathbb{C})$ such that

$$\left\|\gamma^{-1}e^{-\gamma t}\alpha(t) - \alpha_\infty\right\| + \left\|\gamma^{-2}e^{-2\gamma t}\zeta(t) - \zeta_\infty\right\| \leq C_{cu}e^{-\eta_{cu}t},$$

for all $t \geq 0$, and defining the following matrix

$$A_\infty := \begin{pmatrix} 0 & \mathrm{Id}_m \\ -\zeta_\infty & -\alpha_\infty \end{pmatrix}, \tag{4.10}$$

where one of $\alpha_\infty, \zeta_\infty$ is assumed to be non-zero, then, as it will become clear in subsection 4.3.4, solutions to (4.9) behave as $\exp[\kappa_1 e^{\gamma t}]$ as $t \to \infty$, where κ_1 is the largest real part of an eigenvalue of A_∞. So, if $\kappa_1 > 0$, then solutions grow super exponentially, and if $\kappa_1 < 0$, solutions decay super exponentially (or grow super exponentially to the past). The case $\kappa_1 = 0$, instead, represents a borderline.

From applying lemma 4.1, it follows that the volume of the spatial hypersurfaces $\bar{M}_t$ goes like $e^{\pm t}$, where the exponentially decaying case is that of big crunch asymptotics. As a result, it turns out that the natural length scale in this context is $e^{\pm t/D}$, where $D := \dim\bar{M}$. Based on this, exponential growth or decay of solutions is considered natural, whereas super exponential growth or decay is viewed as pathological. Hence, it is necessary to avoid the case of α and ζ growing exponentially. Also, it turns out that the case of α and ζ growing polynomially typically yields faster than exponential growth or decay for solutions. So, this other case must be avoided. Consequently, the norms $\|\alpha\|$ and $\|\zeta\|$ have to be bounded to the future to be able to obtain the desired results.

What about the coefficients of spatial derivatives X^i? To have an idea of what happens when these are left unrestricted, the following equation can be considered

$$u_{tt} - e^{-4t}u_{\theta\theta} + e^{-t}u_\theta + u_t + u = 0, \tag{4.11}$$

on $\mathbb{S}^1 \times \mathbb{R}$. One strategy to attack this equation is by arguing that the coefficients of spatial derivatives are exponentially decaying. Hence, they can be neglected, asymptotically, and the equation is approximated by the resulting ODE. Based on this, it can be argued that, in general, solutions should decay exponentially. However, based on an analysis of the corresponding energy

$$E_u(t) := \frac{1}{2}\int_{\mathbb{S}^1} [|u_t(\theta,t)|^2 + e^{-4t}|u_\theta(\theta,t)|^2 + |u(\theta,t)|^2]d\theta,$$

it can be shown that $\dot{E}_u(t) \leq e^t E_u(t)$, which implies a growth of solutions going like $\exp(e^t)$. As it turns out, the main reason for this super exponential growth is related to the following quantity

$$Y^1 = \frac{X^1(t)}{|g^{11}(t)|^{1/2}},$$

growing exponentially, where $X^1 = e^{-t}$ and $g^{11} = e^{-4t}$. To have a more geometrical insight into the interpretation of this quantity, it is useful first to remark that the underlying metric of equation (4.11) is given by

$$g = -dt \otimes dt + e^{4t}d\theta \otimes d\theta, \tag{4.12}$$

which, in its turn, defines the following induced metric on $\mathbb{S}^1$

$$\bar{g} = e^{4t}d\theta \otimes d\theta,$$

for each $t \in \mathbb{R}$. Denoting the vector field $e^{-t}\partial_\theta$ as $\boldsymbol{\chi}$, it follows that $\bar{g}(\boldsymbol{\chi},\boldsymbol{\chi}) = e^{2t}$, which is not small, even though it is exponentially decaying. Defining an orthonormal coordinate system $\{e_0, e_1\} := \{\partial_t,\ e^{-2t}\partial_\theta\}$, it follows that

$$\boldsymbol{\chi} = e^t e_1 = Y^1 e_1.$$

As it will be shown later, equation (4.11) provides an example of what is referred to as an unbalanced equation, which is the reason why the logic of dropping the exponentially decaying spatial derivatives did not work. This is an important notion, as equations of this type appear frequently in analyzing cosmological singularities. In such a context, it is relevant to know when dropping spatial derivatives can be justified.

4.2.2 Geometrical conditions and balanced equations

As was motivated above, to ensure that solutions to (4.3) do not behave, asymptotically, in a pathological manner, it is necessary to complement the previously mentioned restrictions on the coefficients of the equation. At the same time, it is also convenient to express such conditions in a geometrical form, instead of analytical one. So, turning again to the highest order coefficients, it is worth recalling first that the metric corresponding to a canonical separable cosmological model can be expressed as

$$g = -dt \otimes dt + g_{ij}(t)(\chi^i(t)dt + dx^i) \otimes (\chi^j(t)dt + dx^j) + \sum_{r=1}^{R} a_r^2(t) g_r, \tag{4.13}$$

where $\chi^i(t) \equiv g^{0i}(t)$ represent the components of the shift vector field. Such a metric induces on $\bar{M}$ the corresponding metric

$$\bar{g} = g_{ij} dx^i \otimes dx^j + \sum_{r=1}^{R} a_r^2 g_r, \tag{4.14}$$

where the summation convention has been used in both (4.13) and (4.14). Considering the components χ^i first, it is worth remembering that the shift vector field is related to the freedom of choosing a certain foliation of the spacetime. So, it may seem that a convenient choice is to set these components to zero. However, this turns out to be sometimes a problematic choice. On the other hand, it is also not convenient to have a shift vector field with a large norm. Hence, a suitable way to achieve this is implemented in the following definition.

Definition 4.7 Let (M, g) be a canonical separable cosmological model manifold. If there is an $0 < \eta_{\text{sh},0} \leq 1$ such that

$$g(\partial_t, \partial_t) \leq -\eta_{\text{sh},0}^2, \tag{4.15}$$

for all $t \geq 0$, then ∂_t is said to be future uniformly timelike. If, in addition, there is a $1 \leq k \in \mathbb{Z}$ and an $0 < \eta_{\text{sh},k} \in \mathbb{R}$ such that

$$\sum_{l=1}^{k} |\mathcal{L}_U^l \chi|_{\bar{g}} \leq \eta_{\text{sh},k}, \tag{4.16}$$

for all $t \geq 0$, then the shift vector field is said to be C^k-future bounded.

Two remarks are in order. First, equation (4.15) implies the bound $\chi_i \chi^i \leq 1 - \eta_{\text{sh},0}^2$. Second, equation (4.16) can be expressed in terms of $\dot{\chi}$ given that

$$\mathcal{L}_U \chi = \mathcal{L}_{\partial_t} \chi.$$

Regarding constraints on the coefficients g_{ij} and a_r of (4.13), it is more convenient to formalize them in terms of conditions on the second fundamental form $\bar{k}(t)$ of the spatial hypersurfaces $\bar{M}_t$. From (4.14), it follows that

$$\bar{k} = \frac{1}{2} \sum_{i,j=1}^{d} \partial_t g_{ij} dx^i \otimes dx^j + \sum_{r=1}^{R} \frac{\dot{a}_r}{a_r} a_r^2 g_r, \tag{4.17}$$

$$\mathcal{L}_U \bar{k} = \frac{1}{2} \sum_{i,j=1}^{d} \partial_t^2 g_{ij} dx^i \otimes dx^j + \sum_{r=1}^{R} \left[\frac{\ddot{a}_r}{a_r} + \left(\frac{\dot{a}_r}{a_r} \right)^2 \right] a_r^2 g_r. \tag{4.18}$$

In terms of these, conditions on $\bar{k}$ can be given as in the following definition.

Definition 4.8 Let (M, g) be a canonical separable cosmological model manifold. Let, moreover, $0 \leq k \in \mathbb{Z}$. If there is a constant $0 < C_k \in \mathbb{R}$ such that

$$\sum_{l=0}^{k} |\mathcal{L}_U^l \bar{k}|_{\bar{g}} \leq C_k, \tag{4.19}$$

for all $t \geq 0$, then the second fundamental form is said to be C^k-future bounded.

In the notation of the previous definition, it is worth remarking that

$$|\bar{k}|_{\bar{g}} := (\bar{g}^{im} \bar{g}^{jn} \bar{k}_{ij} \bar{k}_{mn})^{1/2}.$$

Regarding conditions on the lower order coefficients, it is necessary first to express, as done before, the X^i coefficients in terms of a matrix of vector fields $\boldsymbol{\chi}$ on $\bar{M}$ as

$$\boldsymbol{\chi} := X^i(t) \partial_i. \tag{4.20}$$

Then, the following notation can be introduced

$$|\boldsymbol{\chi}|_{\bar{h}} = \left(\sum_{\varsigma \in \mathfrak{S}} \sum_{i,j=1}^{d} \bar{h}_{ij} \varsigma_i \left\| X^i \right\| \cdot \varsigma_j \left\| X^j \right\| \right)^{1/2}, \tag{4.21}$$

where $\mathfrak{S}$ denotes the set of elements of $\mathbb{R}^d$ whose components are either plus or minus one. Based on this, a definition follows.

Definition 4.9 Consider (4.3). Assume the associated metric to be such that (M, g) is a canonical separable cosmological model manifold. Define $\boldsymbol{\chi}$ by (4.20). If there is $0 \leq k \in \mathbb{Z}$ and a $0 < C_k \in \mathbb{R}$ such that

$$\sum_{l=0}^{k} |\mathcal{L}_U^l \boldsymbol{\chi}|_{\bar{h}} \leq C_k, \tag{4.22}$$

for all $t \geq 0$, then $\boldsymbol{\chi}$ is said to be C^k-future bounded.

In a similar fashion, $\|\alpha(t)\|, \|\dot{\alpha}(t)\|, \|\zeta(t)\|$ and $\|\dot{\zeta}(t)\|$ need to be bounded to the future, as was discussed before.

Now, a basic notion of balance of equations, which prevents solutions from growing super exponentially, can be defined as follows.

Definition 4.10 Consider (4.3). Assume the associated metric to be such that (M, g) is a canonical separable cosmological model manifold. Let $0 \leq k \in \mathbb{Z}$. If ∂_t is future uniformly timelike; there is a constant $0 < C_k \in \mathbb{R}$ such that

$$\sum_{l=0}^{k} [\|\partial_t^l \alpha(t)\| + \|\partial_t^l \zeta\|] \leq C_k,$$

for $t \geq 0$; $\boldsymbol{\chi}$ is C^k-future bounded; the shift vector field is C^{k+1}-future bounded; and the second fundamental form is C^k-future bounded, then (4.3) is said to be C^{k+1}-balanced.

It is worth remarking that conditions on the metric involve $k+1$ derivatives, whereas those on the lower order coefficients involve k derivatives. Moreover, in relation to the previous definition, it can be shown that for a C^1-balanced equation, the basic energy, defined by

$$\begin{aligned} \mathfrak{E}_{bas}[u](t) = \frac{1}{2} \int_{\bar{M}} \Big(& |u_t(\cdot, t)|^2 + \sum_{i=1}^{m} [g^{kl}(t) \partial_k u_i(\cdot, t) \partial_l u_i^*(\cdot, t) \\ & + \sum_{r=1}^{R} a_r^{-2}(t) |\text{grad}_{g_r} u_i(\cdot, t)|_{g_r}^2] + |u(\cdot, t)|^2 \Big) \mu_B, \end{aligned} \tag{4.23}$$

cannot grow faster than exponentially, where * denotes the complex conjugate, and

$$\mu_B := dx \wedge \mu_{g_1} \wedge \ldots \wedge \mu_{g_R},$$

with dx being the standard volume form on $\mathbb{T}^d$, and μ_{g_r} the volume form associated with the Riemannian manifold (M_r, g_r).

In spite of the fundamental role played by the concept of balance, it only excludes pathological behaviour of solutions. To be able to say something more regarding the asymptotics of equation (4.3), it is necessary to restrict this class of equations even more.

4.3 Asymptotics of weakly silent, balanced and convergent equations

4.3.1 The silent setting

As mentioned before, equations of the form (4.3) such that the coefficients of spatial derivatives decay exponentially turn out to be relevant for cosmology. For example, when considering the polarized $\mathbb{T}^3$-Gowdy symmetric spacetimes, the main part of the Einstein's equations takes the form of a linear scalar wave equation as

$$P_{tt} - e^{-2t} P_{\theta\theta} = 0, \tag{4.24}$$

on $\mathbb{S}^1 \times \mathbb{R}$, where $t \to \infty$ corresponds to the big bang singularity of the model. In general, it is often thought that the field equations, whether in the direction of the singularity or the expanding direction, can be approximated by similar equations. Then, in order to analyze the asymptotics of the corresponding solutions, it is argued, heuristically, that terms that decay exponentially can be dropped, as they become insignificant asymptotically, resulting in one ODE for each spatial point. For the case of equation (4.24), this results in

$$P_{tt} = 0,$$

which is solved to give $P(\theta, t) = v(\theta)t + \psi(\theta)$. Such a result is to be compared with the corresponding one obtained by applying the method of asymptotic analysis at the end of this chapter.

In this respect, it is important to remark that such a line of reasoning is also relevant for the working of the BKL conjecture. However, as it is clear from the analysis of equation (4.11), such a strategy does not always work, in particular for unbalanced equations. Hence, this motivates investigating balanced equations with exponentially decaying coefficients of spatial derivatives. It is in this context that a mathematical justification for the above heuristic arguments can be looked for.

Similar to the balance condition, it is also desirable to cast the other condition in a geometrical form. As it turns out, exponential decay of the coefficients of spatial derivatives is related to having lower bounds on the second fundamental form. This is expressed in the following definition.

Definition 4.11 Consider (4.3). Assume the associated metric to be such that (M, g) is a canonical separable cosmological model manifold. If ∂_t is future uniformly timelike and there is a $0 < \mu \in \mathbb{R}$ and a continuous non-negative $\mathfrak{e} \in L^1([0, \infty))$ such that

$$\bar{k} \geq (\mu - \mathfrak{e})\bar{g}, \qquad |\chi|_{\bar{g}} \cdot |\dot{\chi}|_{\bar{g}} \leq \mathfrak{e}, \tag{4.25}$$

for all $t \geq 0$, then (4.3) is said to be C^1-silent.

This notion of silence is similar to the one introduced in relation to the metrics (4.8). In fact, assuming that $\gamma : J \to M$ is a future-directed inextendible causal curve in (M, g), where $J = (s_-, s_+)$, and that the assumptions of definition (4.11) are satisfied, then it follows that the $\bar{M}$-coordinate of γ, namely $\bar{\gamma}$, converges to a point $\bar{p}[\gamma]$ as $s \to s_+$, whereas the t-coordinate $\gamma^0(s)$ goes to infinity. Consequently, if two future-directed inextendible causal curves $\gamma_i : J_i \to M$, $i = 1, 2$ satisfy $\bar{p}[\gamma_1] \neq \bar{p}[\gamma_2]$, then there are $s_i \in J_i$ such that

$$J^+[\gamma_1(s_1)] \cap J^+[\gamma_2(s_2)] = \varnothing,$$

where $J^+(\gamma)$ denotes the causal future of γ [14].

Now, assuming both the conditions of silence and balance, equations (4.3) can be put into a form that admits asymptotic analysis. In particular, if (4.3) are C^1-silent; $\boldsymbol{\chi}$ is C^0-future bounded; and there are $\alpha_\infty, \zeta_\infty \in \mathbf{M}_m(\mathbb{C})$ and $0 < \eta_{mn}, C_{mn} \in \mathbb{R}$ with the property that

$$\|\alpha(t) - \alpha_\infty\| + \|\zeta(t) - \zeta_\infty\| \leq C_{mn} e^{-\eta_{mn} t}, \tag{4.26}$$

for all $t \geq 0$, then the following terms of the equation

$$\chi^i \partial_i \partial_t u, \qquad X^i \partial_i u, \qquad (\alpha - \alpha_\infty) u_t, \qquad (\zeta - \zeta_\infty) u,$$

can be ignored, along with the second spatial derivatives of u. In this case, solutions of (4.3) can be compared to solutions of the corresponding system

$$\partial_t \begin{pmatrix} v \\ v_t \end{pmatrix} = A_\infty \begin{pmatrix} v \\ v_t \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{4.27}$$

where as before

$$A_\infty := \begin{pmatrix} 0 & \mathrm{Id}_m \\ -\zeta_\infty & -\alpha_\infty \end{pmatrix}. \tag{4.28}$$

Equation (4.27) is a system of constant coefficient equations, ignoring the second term on the right-hand side. In particular, it represents a system of ODEs at each spatial point of $\bar{M}$, given that the spatial dependence has not been eliminated from the solution u. In fact, this is the system that is going to be considered later in deriving asymptotic information for equations (4.3). But first, to justify such a use of system (4.27), and to be able to see the relation between the above conditions and the important notions of weak silence, balance and convergence, it is necessary to decompose the equation into the corresponding Fourier modes.

4.3.2 Fourier decomposition of the equations

Based on the fact that the underlying manifold of interest is separable, according to definition 4.1, it follows correspondingly that equations (4.3) are also separable. Indeed, behaviour of the solutions can be better understood by looking into the individual modes. To this end, it is worth reviewing first some background on the Laplace-Beltrami operator on a closed Riemannian manifold, before proceeding to the relevant case. For this purpose, there is the following theorem.

Theorem 4.1 *Let (M, g) be a closed Riemannian manifold and Δ_g be the associated Laplace-Beltrami operator. Then, the eigenvalues of Δ_g consist of a sequence λ_i, $0 \leq i \in \mathbb{Z}$ such that $0 = \lambda_0 > \lambda_1 > \ldots$ and $\lambda_i \to -\infty$ as $i \to \infty$. Moreover, if $\mathcal{E}_i$ is the eigenspace corresponding to λ_i, then $\mathcal{E}_i$ is finite dimensional and consists of smooth functions. In particular, $\mathcal{E}_0$ is the set of constant functions. Finally, the set of eigenfunctions of Δ_g is a basis for $L^2(M)$.*

In relation to the previous theorem, it is important to remark that since each of the eigenspaces is finite dimensional, there is a sequence $0 = \bar{\lambda}_0 \geq \bar{\lambda}_1 \geq \ldots$, for $0 \leq i \in \mathbb{Z}$, and corresponding orthonormal eigenfunctions $\phi_i \in C^\infty(M)$ such that the sequence ϕ_i is a basis for $L^2(M)$. Moreover, there is a corresponding sequence $0 \leq \nu_0 \leq \nu_1 \leq \ldots$ such that $\bar{\lambda}_i = -\nu_i^2$, where $\nu_i \to \infty$ as $i \to \infty$.

For (M, g) an oriented Riemannian manifold, the volume form μ_g associated with the metric g defines a positive measure λ_g such that

$$\int_M f \lambda_g = \int_M f \mu_g,$$

for all $f \in C_0(M, \mathbb{C})$, based on the Riesz representation theorem [52]. In particular, if (M, g) is closed (in addition to being connected and oriented), then the following inner product can be defined

$$\langle u, v\rangle_g := \int_M uv^* \lambda_g,$$

for $u, v \in L^2(M)$, which turns the space $L^2(M)$ into a complex Hilbert space. Based on this, for $u \in L^2(M)$ and $0 \le i \in \mathbb{Z}$, the following decomposition can be defined

$$\hat{u}(i) := \langle u, \phi_i\rangle_g,$$

which is nothing other than the Fourier transform of u. Then, the fact that ϕ_i represent a basis for $L^2(M)$ can be expressed as

$$\int_M |u|^2 \lambda_g = \sum_{i=0}^{\infty} |\hat{u}(i)|^2,$$

for all $u \in L^2(M)$. If $u \in C^\infty(M, \mathbb{C})$ and $s \in \mathbb{R}$, then the following norm can be defined

$$\|u\|_{(s)} := \left(\sum_{i=0}^{\infty} \langle \nu_i\rangle^{2s} |\hat{u}(i)|^2 \right)^{1/2}, \tag{4.29}$$

where $\langle \cdot \rangle := (1+|\cdot|^2)^{1/2}$. If $s = k$ a non-negative integer, then this norm is equivalent to the Sobolev norm $H^k(M)$, as can be seen from (3.15). However, $\| \cdot \|_{(s)}$ will be referred to as a Sobolev norm in what follows.

To consider the case related to equations (4.3), the Laplace-Beltrami operator Δ_{g_r}, $r = 1, 2, ..., R$, on (M_r, g_r) is considered first. Applying the previous remarks gives orthonormal eigenfunctions $\phi_{r,i}$, $0 \le i \in \mathbb{Z}$, and corresponding eigenvalues $-\nu_{r,i}^2$. In addition, for $n \in \mathbb{Z}^d$ and $0 \le i_r \in \mathbb{Z}$, the following set can be defined

$$\iota := (n, i_1, i_2, ..., i_R). \tag{4.30}$$

Then, the set of all such ι is denoted $\mathcal{I}_B$. For each $\iota \in \mathcal{I}_B$, there is a unique ν given by

$$\nu(\iota) := (n, \nu_{1,i_1}, \nu_{2,i_2}, ..., \nu_{R,i_R}). \tag{4.31}$$

This indicates that

$$\nu_T(\iota) = n, \qquad \nu_{T,j}(\iota) = n_j, \qquad \nu_{r,i_r}(\iota) = \nu_{r,i_r}.$$

For $x \in \mathbb{T}^d$, $p_r \in M_r$, $r = 1, ..., R$, and $p = (x, p_1, ..., p_R)$, the following functions ϕ_ι can be defined

$$\phi_\iota(p) := (2\pi)^{-d/2} e^{in\cdot x} \phi_{1,i_1}(p_1)\phi_{2,i_2}(p_2) \cdots \phi_{R,i_R}(p_R), \tag{4.32}$$

for $\iota \in \mathcal{I}_B$. These represent eigenfunctions for the operator

$$\Delta_T + \Delta_1 + ... + \Delta_R,$$

with an eigenvalue $-|\nu(\iota)|^2$, where Δ_T represents the Laplacian on $\mathbb{T}^d$.

In a similar fashion, the volume form μ_B on $\bar{M}$ induces a positive measure λ_B, which renders $L^2(\bar{M})$ a complex Hilbert space. The corresponding inner product is defined by

$$\langle u, v\rangle_B := \int_{\bar{M}} uv^* \lambda_B,$$

for $u, v \in L^2(\bar{M})$. It should be clear from orthonormality that for $\iota_a, \iota_b \in \mathcal{I}_B$

$$\langle \phi_{\iota_a}, \phi_{\iota_b}\rangle_B = 0,$$

for $\iota_a \neq \iota_b$. Similarly, $\hat{u}(\iota)$ is defined as

$$\hat{u}(\iota) := \langle u, \phi_\iota\rangle_B, \tag{4.33}$$

for $u \in L^2(\bar{M})$. From the fact that $e^{in\cdot x}$, $n \in \mathbb{Z}^d$, is a basis for $L^2(\mathbb{T}^d)$, and ϕ_{r,i_r} is a basis for $L^2(M_r)$, $r = 1, ..., R$, it can be shown that

$$\int_{\bar{M}} |u|^2 \lambda_B = \sum_{\iota \in \mathcal{I}_B} |\hat{u}(\iota)|^2,$$

for $u \in C^\infty(\bar{M}, \mathbb{C})$. Hence, ϕ_ι represent a basis for $L^2(\bar{M})$. For $u \in C^\infty(\bar{M}, \mathbb{C})$ and $s \in \mathbb{R}$, it follows that

$$\|u\|_{(s)} := \left(\sum_{\iota \in \mathcal{I}_B} \langle \nu(\iota)\rangle^{2s} |\hat{u}(\iota)|^2 \right)^{1/2} \tag{4.34}$$

represents a Sobolev norm.

Now, for equation (4.3), taking the inner product (with respect to $\langle \cdot, \cdot \rangle_B$) with ϕ_ι yields the following

$$\begin{gathered} \ddot{z}(\iota, t) + \mathfrak{g}^2(\iota, t) z(\iota, t) - 2\sum_{l=1}^{d} i n_l g^{0l}(t) \dot{z}(\iota, t) + \sum_{l=1}^{d} i n_l X^l(t) z(\iota, t) \\ + \alpha(t)\dot{z}(\iota, t) + \zeta(t) z(\iota, t) = \hat{f}(\iota, t), \end{gathered} \tag{4.35}$$

where

$$z(\iota,t) := \langle u(\cdot,t), \phi_\iota\rangle_B, \tag{4.36}$$

$$\mathfrak{g}(\iota,t) := \left(\sum_{j,l=1}^{d} n_l n_j g^{jl}(t) + \sum_{r=1}^{R} a_r^{-2}\nu_{r,i_r}^2\right)^{1/2}, \tag{4.37}$$

$$\hat{f}(\iota,t) := \langle f(\cdot,t), \phi_\iota\rangle_B. \tag{4.38}$$

To simplify the notation a little bit, the dependence on ι and t can be omitted, along with employing the summation convention. This gives

$$\ddot{z} + \mathfrak{g}^2 z - 2in_l g^{0l}\dot{z} + in_l X^l z + \alpha\dot{z} + \zeta z = \hat{f}. \tag{4.39}$$

As it will be relevant for subsequent discussion, the following two quantities are introduced

$$\sigma(\iota,t) := \frac{n_l g^{0l}(t)}{\mathfrak{g}(\iota,t)}, \qquad X(\iota,t) := \frac{n_l X^l}{\mathfrak{g}(\iota,t)}. \tag{4.40}$$

In terms of these, equation (4.39) can be recast as

$$\ddot{z} + \mathfrak{g}^2 z - 2i\sigma\mathfrak{g}\dot{z} + iX\mathfrak{g}z + \alpha\dot{z} + \zeta z = \hat{f}. \tag{4.41}$$

4.3.3 Weakly silent, balanced and convergent equations

As indicated at the beginning of this chapter, the restricted class of equations, based on (4.3), for which asymptotics can be analyzed is that of weakly silent, balanced and convergent equations. To define these notions, and to see their relation to the previously discussed concepts of balance and silence, the following quantity is defined

$$\ell(\iota,t) := \ln[\mathfrak{g}(\iota,t)], \tag{4.42}$$

based on the Fourier decomposition of the equation. Using this and (4.40), a definition can be given as follows.

Definition 4.12 Consider (4.3). Assume that the associated metric to be such that (M,g) is a canonical separable cosmological model manifold. For $0 \neq \iota \in \mathcal{I}_B$, define $\mathfrak{g}(\iota,t)$ by (4.37), σ and X by (4.40), and ℓ by (4.42). If there is a constant C_{coeff} such that

$$|\sigma(\iota,t)| + \|X(\iota,t)\| + \|\alpha(t)\| + \|\zeta(t)\| \leq C_{\text{coeff}}, \tag{4.43}$$

for all $0 \neq \iota \in \mathcal{I}_B$ and all $t \geq 0$, then equations (4.3) are said to be weakly balanced. If there is a constant $\mathfrak{b}_s > 0$ and a continuous non-negative function $\mathfrak{e} \in L^1([0,\infty))$ such that

$$\dot{\ell}(\iota, t) \leq -\mathfrak{b}_s + \mathfrak{e}(t), \tag{4.44}$$

for all $0 \neq \iota \in \mathcal{I}_B$ and all $t \geq 0$, then equations (4.3) are said to be weakly silent. In this case, a quantity denoted T_{ode} is defined as follows. If $c_{\mathfrak{e}} := \|\mathfrak{e}\|_1$ and $\mathfrak{g}(\iota, 0) \leq e^{-c_{\mathfrak{e}}}$, then $T_{ode} := 0$. If $\mathfrak{g}(\iota, 0) > e^{-c_{\mathfrak{e}}}$, then T_{ode} is defined as the first $t \geq 0$ such that $\mathfrak{g}(\iota, t) = e^{-c_{\mathfrak{e}}}$. Finally, if there are $\alpha_\infty, \zeta_\infty \in \mathbf{M}_m(\mathbb{C})$ and constants $0 < \eta_{mn}$ and C_{mn} such that

$$\|\alpha(t) - \alpha_\infty\| + \|\zeta(t) - \zeta_\infty\| \leq C_{mn} e^{-\eta_{mn} t},$$

for all $t \geq 0$, then (4.3) are said to be weakly convergent.

Two remarks are in order. First, the reason these conditions are termed weak is that they only involve bounds on the quantities of interest, and not the quantities and their first derivatives, as exemplified by definition 4.10. Second, sometimes it is convenient to impose, in addition to the above assumptions, that there are constants C_{der}, $\beta_{\mathrm{der}} > 0$ such that

$$|\sigma(\iota, t)| + \|X(\iota, t)\| \leq C_{\mathrm{der}} e^{-\beta_{\mathrm{der}} t}, \tag{4.45}$$

for all $0 \neq \iota \in \mathcal{I}_B$ and all $t \geq 0$.

To see the relation of these formal definitions to previous notions, the following two lemmas are needed.

Lemma 4.2 *Let (M, g) be a canonical separable cosmological model manifold. Assume, moreover, that there is an $0 < \eta_{sh,0} \leq 1$ such that (4.15) holds for all $t \geq 0$, and that there are a $0 < \mu \in \mathbb{R}$ and a continuous non-negative function $\mathfrak{e} \in L^1([0,\infty))$ such that*

$$\bar{k} \geq (\mu - \mathfrak{e})\bar{g},$$
$$|\chi|_{\bar{g}} \cdot |\dot{\chi}|_{\bar{g}} \leq \mathfrak{e},$$

for all $t \geq 0$. Then, there is a continuous non-negative function $\mathfrak{e}_{\mathfrak{g}} \in L^1([0,\infty))$ such that

$$\dot{\ell}(\iota, t) \leq -\mu + \mathfrak{e}_{\mathfrak{g}}(t), \tag{4.46}$$

for all $t \geq 0$ and all $0 \neq \iota \in \mathcal{I}_B$. Moreover, $\mathfrak{e}_{\mathfrak{g}} := 2\eta_{sh,0}^{-2}\mathfrak{e}$.

Lemma 4.3 *Consider (4.3). Assume that the associated metric to be such that* (M, g) *is a canonical separable cosmological model manifold. Let* $\bar{h}$ *be the induced metric by (4.14) on* $\mathbb{T}^d$*. Assume, moreover, that there is an* $0 < \eta_{sh,0} \leq 1$ *such that (4.15) holds for all* $t \geq 0$*. Then*

$$|\sigma(\iota, t)| \leq \eta_{sh,0}^{-1} |\chi(t)|_{\bar{g}},$$

$$\|X(\iota, t)\| \leq \eta_{sh,0}^{-1} |\boldsymbol{\chi}|_{\bar{h}},$$

for all $t \geq 0$ *and* $0 \neq \iota \in \mathcal{I}_B$*.*

Consequently, the conditions of weak silence, balance and convergence turn out to be just a reformulation of the previous conditions of C^1-silence, C^0-future-boundedness of $\boldsymbol{\chi}$, and the convergence of $\alpha(t)$ and $\zeta(t)$. Also, it is noteworthy that, based on (4.44) and (4.46), $\mu = \mathfrak{b}_s$.

Now, to better understand how these conditions relate equations (4.3) to the system of ODEs (4.27), it is first worth remarking that equation (4.44) can be integrated to yield

$$\mathfrak{g}(t) \leq \mathfrak{g}(0) e^{-\mathfrak{b}_s t} e^{c_\mathfrak{e}}, \tag{4.47}$$

for $t \geq 0$. Under the assumption that $\mathfrak{g}(0) \leq e^{-c_\mathfrak{e}}$, this simplifies to

$$\mathfrak{g}(t) \leq e^{-\mathfrak{b}_s t}, \tag{4.48}$$

for $t \geq 0$. However, in the other, more general case, equation (4.44) yields

$$\mathfrak{g}(t) \leq \exp\bigl(-\mathfrak{b}_s \bar{t}\bigr), \tag{4.49}$$

for $t \geq T_{ode}$, where $\bar{t} := t - T_{ode}$. Hence, this motivates dividing the time interval $[0, \infty)$ into $[0, T_{ode}]$ and $[T_{ode}, \infty)$, such that during the second interval, which is known as the ODE regime, $\mathfrak{g}(t) \leq 1$. Based on (4.49), the condition of weak balance can re-expressed as

$$|n_l g^{0l}(t)| + \|n_l X^l(t)\| \leq C_{\text{coeff}} e^{-\mathfrak{b}_s \bar{t}}, \tag{4.50}$$

for $t \geq T_{ode}$. Rewriting equation (4.39) as

$$\ddot{z} + \mathfrak{g}^2 z - 2i n_l g^{0l} \dot{z} + i n_l X^l z + \alpha_\infty \dot{z} + \zeta_\infty z + (\alpha - \alpha_\infty)\dot{z} + (\zeta - \zeta_\infty) z = \hat{f},$$

and introducing additional variables, the equation can be recast as a first order system

$$\dot{v} = A_\infty v + A_{\text{rem}}(t) v + F, \tag{4.51}$$

where

$$v = \begin{pmatrix} z \\ \dot{z} \end{pmatrix}, \qquad A_{\text{rem}}(t) = \begin{pmatrix} 0 & 0 \\ \mathfrak{g}^2 + in_l X^l + (\zeta - \zeta_\infty) & -2in_l g^{0l} + (\alpha - \alpha_\infty) \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ \hat{f} \end{pmatrix}, \tag{4.52}$$

and A_∞ is defined by (4.28). Based on (4.49), (4.50) and (4.26), the matrix $A_{\text{rem}}(t)$ has the important property that

$$\|A_{\text{rem}}(t)\| \leq C_{\text{rem}} e^{-\beta_{\text{rem}} \bar{t}}, \tag{4.53}$$

for $t \geq T_{ode}$, where $\beta_{\text{rem}} = \min\{\mathfrak{b}_s, \eta_{mn}\}$, and C_{rem} only depends on C_{coeff} and C_{mn}. In case (4.45) is also satisfied, the definition of β_{rem} is replaced by

$$\beta_{\text{rem}} = \min\{2\mathfrak{b}_s, \mathfrak{b}_s + \beta_{der}, \eta_{mn}\}.$$

To be able to analyze the asymptotics of solutions to equation (4.51), it is important first to decide under what conditions the system can be approximated by the corresponding system $\dot{v} = A_\infty v + F$, asymptotically. For this purpose, the equation should be cast in a form that allows to isolate the leading order asymptotic behaviour of its various terms.

4.3.4 ODEs with exponentially decaying error terms

Based on the discussion of the previous two subsections, the system (4.3) can be formulated as

$$\dot{v}(t) = B(t)v(t) + F(t),$$

for $t \geq T_{ode}$, where $B(t) = A_\infty + A_{\text{rem}}(t)$, and $A_{\text{rem}}(t)$ satisfies (4.53). In this context, A_∞ represents the dominant part, while $A_{\text{rem}}(t)$ is an error term. However, it is important not to forget that A_{rem} depends on the frequency ι, and that typically $\|A_{\text{rem}}(0)\| \to \infty$ as $|\nu(\iota)| \to \infty$. Hence, it is not an immediate result that this term is negligible, asymptotically, because it decays exponentially.

So, to understand the conditions under which solutions to the system $\dot{v} = A_\infty v + F$ better approximate solutions to the equation (4.51), and to analyze their asymptotic behaviour, it is important first to recall some background on the Jordan normal form of a general matrix with complex entries.

Let $A \in \mathbf{M}_n(\mathbb{C})$, $1 \leq n \in \mathbb{Z}$. Then, there exists a matrix $T \in \mathbf{GL}_n(\mathbb{C})$ such that

$$J := T^{-1}AT,$$

known as the Jordan normal form of A, is a block diagonal matrix. The matrix T consists of the generalized eigenvectors of A. Denoting SpA the set of eigenvalues of the matrix A, then the generalized eigenspace E_λ, corresponding to $\lambda \in \mathrm{Sp}A$, is defined as

$$E_\lambda := \ker(A - \lambda \mathrm{Id}_n)^{n_\lambda},$$

where $1 \leq n_\lambda \in \mathbb{Z}$.

The matrices on the diagonal of J are called Jordan blocks. These are square matrices of the following form

$$J_{\lambda,d} = \begin{pmatrix} \lambda & 1 & 0 & ... & 0 & 0 \\ 0 & \lambda & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & 0 & ... & 0 & \lambda \end{pmatrix},$$

where $\lambda \in \mathrm{Sp}A$, and $d = n_\lambda$ is the dimension of the block. From the fact that

$$(J_{\lambda,d} - \lambda \mathrm{Id}_d)^d = 0,$$

it follows that the generalized eigenspace of $J_{\lambda,d}$ is simply $\mathbb{C}^d$. In addition, the matrix exponential $\exp[J_{\lambda,d}\, t]$ can be evaluated as

$$\exp[J_{\lambda,d}\, t] = \exp[(J_\lambda - \lambda \mathrm{Id}_d)t + \lambda \mathrm{Id}_d t] = e^{\lambda t}\exp[J_{0,d}\, t] = e^{\lambda t}\sum_{k=0}^{\infty}\frac{1}{k!}J_{0,d}^k\, t^k = e^{\lambda t}\sum_{k=0}^{d-1}\frac{1}{k!}J_{0,d}^k\, t^k, \tag{4.54}$$

where $J_{0,d} = J_{\lambda,d} - \lambda \mathrm{Id}_d$, and the fact that $J_{0,d}^d = 0$ has been employed. Consequently, for $v \in \mathbb{C}^d$, the leading order asymptotic term of $\exp[J_{\lambda,d}\, t]v$ as $t \to \infty$ behaves like

$$t^{d-1}\exp[\mathrm{Re}\{\lambda\}t],$$

where $\exp[\mathrm{Im}\{\lambda\}t]$ contributes only as a rotation. To see how this is related to the matrix exponential $\exp[At]$, it is convenient to order the Jordan blocks as

$$J = \{J_{\lambda_1,d_1}, J_{\lambda_2,d_2}, ..., J_{\lambda_m,d_m}\},$$

where $\mathrm{Re}\{\lambda_i\} \geq \mathrm{Re}\{\lambda_{i+1}\}$, and $d_i \geq d_{i+1}$ whenever $\mathrm{Re}\{\lambda_i\} = \mathrm{Re}\{\lambda_{i+1}\}$. So, from the fact that

$$e^{At} = Te^{Jt}T^{-1} = T\{\exp[J_{\lambda_1,d_1}\, t], \exp[J_{\lambda_2,d_2}\, t], ..., \exp[J_{\lambda_m,d_m}\, t]\}T^{-1},$$

it follows that the leading order term of $e^{At}v$, for a generic $v \in \mathbb{C}^n$, has a time dependence of $t^{d_1-1}\exp[\text{Re}\{\lambda_1\}t]$ as $t \to \infty$. This motivates introducing the following notation.

Definition 4.13 For $B \in \mathbf{M}_n(\mathbb{C})$, if

$$\kappa_{max}(B) := \sup\{\text{Re}\lambda \mid \lambda \in \text{Sp}B\}, \qquad \kappa_{min}(B) := \inf\{\text{Re}\lambda \mid \lambda \in \text{Sp}B\}.$$

Then, $\text{Rsp}B$, the real eigenvalue spread of B, is defined by $\text{Rsp}B := \kappa_{max}(B) - \kappa_{min}(B)$. In addition, if $\kappa \in \{\text{Re}\lambda \mid \lambda \in \text{Sp}B\}$, then $d_{max}(B, \kappa)$ is defined to be the largest dimension of a Jordan block corresponding to an eigenvalue of B with a real part κ. Finally, if $\kappa \notin \{\text{Re}\lambda \mid \lambda \in \text{Sp}B\}$, then $d_{max}(B, \kappa) := 1$.

So, it follows by construction that $\kappa_{max}(A) = \text{Re}\{\lambda_1\}$ and $d_{max}(A, \kappa_{max}) = d_1$.

Now, turning to the point of when the term $A_{\text{rem}}(t)v$ in equation (4.51) can be dropped, it is important to remark that, under the assumption

$$\|F\|_{A_\infty} := \int_0^\infty e^{-\kappa_1 s}|F(s)|ds < \infty, \tag{4.55}$$

where $\kappa_1 := \kappa_{max}(A_\infty)$, solutions $v(t)$ to (4.51) do not grow faster than the fastest growing solutions to the equation $\dot{v} = A_\infty v$. In other words,

$$|v(t)| \leq C\langle t - T_{ode}\rangle^{d_1-1}e^{\kappa_1(t-T_{ode})}, \tag{4.56}$$

for $t \geq T_{ode}$, where $d_1 := d_{max}(A_\infty, \kappa_1)$. It is noteworthy that condition (4.55) is just the statement that also F does not grow faster than the fastest growing solutions to $\dot{v} = A_\infty v$. From the fact that equation (4.51) can be expressed as

$$\frac{d}{dt}\left(e^{-A_\infty t}v - \int_{T_{ode}}^t e^{-A_\infty s}F(s)ds\right) = e^{-A_\infty t}A_{\text{rem}}(t)v(t), \tag{4.57}$$

and taking the previous observations into consideration, it follows that the right-hand side decays exponentially, and hence the term inside the parenthesis converges exponentially, if $\text{Rsp}A_\infty < \beta_{\text{rem}}$, or in other words if

$$\kappa_{min}(A_\infty) > \kappa_{max}(A_\infty) - \beta_{\text{rem}}. \tag{4.58}$$

To get an insight into what this condition implies, it is worth remarking that β_{rem} represents the threshold between the leading order part of solutions to $\dot{v} = A_\infty v$, which can be distinguished from the error resulting from neglecting $A_{\text{rem}}(t)v$, and remainder part which cannot. Given that the leading order behaviour is proportional to $\kappa_{max}(A_\infty)$, it is then necessary to focus on solutions with $\text{Re}\{\lambda_i\}$ such that

$$\text{Re}\{\lambda_i\} > \kappa_{max}(A_\infty) - \beta_{rem}. \tag{4.59}$$

So, equation (4.58) indicates that all the eigenvalues of $\exp[A_\infty t]v$, for some $v \in \mathbb{C}^{2m}$, correspond to a leading order behaviour in this case. On the other hand, if $\text{Rsp}A_\infty \geq \beta_{rem}$, this could imply an exponential growth for the right-hand side of (4.57). Hence, to be able to proceed further, it is important to consider only eigenvalues λ_i such that (4.59) is satisfied. This is summarized by the following lemma.

Lemma 4.4 *Given $1 \leq n \in \mathbb{Z}$, $B \in \mathbf{M}_n(\mathbb{C})$ and $0 < \beta \in \mathbb{R}$, there is a $T \in \mathbf{GL}_n(\mathbb{C})$ such that $T^{-1}BT$ has the following properties:*

$$T^{-1}BT = \text{diag}\{B_a, B_b\},$$

where $B_a \in \mathbf{M}_{n_a}(\mathbb{C})$; $B_b \in \mathbf{M}_{n_b}(\mathbb{C})$; $1 \leq n_a \in \mathbb{Z}$ and $0 \leq n_b \in \mathbb{Z}$ are such that $n_a+n_b = n$; and B_a and B_b consist of Jordan blocks. Moreover, $\text{Rsp}B_a < \beta$; $\kappa_{max}(B_a) = \kappa_{max}(B)$; and $\kappa_{max}(B_b) \leq \kappa_{max}(B) - \beta$ (assuming $n_b \geq 1$).

Consequently, for $T \in \mathbf{GL}_{2m}(\mathbb{C})$, it follows that

$$T^{-1}A_\infty T = \text{diag}\{A_{\infty,a}, A_{\infty,b}\},$$

where the blocks $A_{\infty,a}, A_{\infty,b}$ satisfy the properties mentioned in the lemma, with $\beta = \beta_{\text{rem}}$. Denoting $w(t) := T^{-1}v(t)$, and letting w_a and w_b be the components of w corresponding to the blocks $A_{\infty,a}$ and $A_{\infty,b}$, respectively, then equation (4.51) can be decomposed as

$$\dot{w}_a(t) = A_{\infty,a}w_a(t) + [T^{-1}A_{\text{rem}}(t)v(t)]_a + [T^{-1}F(t)]_a, \tag{4.60}$$

$$\dot{w}_b(t) = A_{\infty,b}w_b(t) + [T^{-1}A_{\text{rem}}(t)v(t)]_b + [T^{-1}F(t)]_b. \tag{4.61}$$

For equation (4.60), same arguments as before can be applied to obtain the leading order behaviour. However, for the system (4.61), it is only possible to have an estimate of the solutions. As mentioned previously, this is related to the fact that the second term on the right-hand side could grow faster than the fastest growing solutions to $\dot{w}_b = A_{\infty,b}w_b(t)$.

It is noteworthy that the previous decomposition of the matrix A_∞ can be done already on the level of the generalized eigenspaces of the matrix. For this, the following notation is needed.

Definition 4.14 Given $1 \leq n \in \mathbb{Z}$, $B \in \mathbf{M}_n(\mathbb{C})$, $0 < \beta \in \mathbb{R}$, and let n_a and n_b be the integers obtained by appealing to lemma 4.4. n_a and n_b are referred to as the dimensions of the first and second subspaces (respectively) of the β, B-decomposition of $\mathbb{C}^n$. If T is obtained as in lemma 4.4, $E_a := T(\mathbb{C}^{n_a} \times \{0\}^{n_b})$, $E_b := T(\{0\}^{n_a} \times \mathbb{C}^{n_b})$, then $E_a(E_b)$ is referred to as the first (second) generalized eigenspace in the β, B-decomposition of $\mathbb{C}^n$. In other words, $E_a(E_b)$ is the direct sum of the generalized eigenspaces of B corresponding to eigenvalues in $\mathrm{Sp}B_a$ ($\mathrm{Sp}B_b$).

So, looking into the first generalized eigenspace of A_∞ is equivalent, on the transformed, Jordan block side, to focusing on vectors $v \in \mathbb{C}^{2m}$ such that the components corresponding to Jordan blocks J_{λ_i,d_i} with $\mathrm{Re}\lambda_i \leq \kappa_{max}(A_\infty) - \beta_{\mathrm{rem}}$ vanish.

As a final remark, it is noteworthy that if $B \in \mathbf{M}_n(\mathbb{R})$, then $\lambda \in \mathrm{Sp}B \Rightarrow \lambda^* \in \mathrm{Sp}B$. Moreover, $n_\lambda = n_{\lambda^*}$, so that $v \in E_\lambda$ implies that $v^* \in E_{\lambda^*}$. In this case, the spaces E_a and E_b can be thought of as subspaces of $\mathbb{R}^n$.

4.3.5 Deriving and specifying asymptotics

After investigating the behaviour of solutions to the system (4.51) in the ODE regime, it is worth remarking that during the interval $[0, T_{ode}]$, known as the oscillatory regime, less information is available. Due to the assumption of weak balance, it is at least guaranteed that the energy associated with the solutions does not grow faster than exponentially. However, the estimate of the energy turns out to be in the form of $\langle \nu(\iota) \rangle^{s_0}$ times the initial energy, where $s_0 \geq 0$. From the fact that the energy $\mathcal{E}_s$, $s \in \mathbb{R}$, for a given solution z of equation (4.35) is defined by

$$\mathcal{E}_s(\iota, t) := \frac{1}{2} \langle \nu(\iota) \rangle^{2s} [|\dot{z}(\iota, t)|^2 + \mathfrak{g}^2(\iota, t)|z(\iota, t)|^2 + |z(\iota, t)|^2], \tag{4.62}$$

it follows that this estimate involves loss of regularity. In other words, initial data have to be estimated in the Sobolev space H^{s+s_0} to obtain an estimate regarding asymptotics in the H^s space.

Combining the previous remarks with the analysis of solutions in the ODE regime, the asymptotics of solutions to equation (4.3) can be derived according to the following lemma.

Lemma 4.5 *Assume that (4.3) is weakly silent, weakly balanced and weakly convergent. Assume, moreover, that f is a smooth function such that for every $s \in \mathbb{R}$,*

$$\|f\|_{A,s} := \int_0^\infty e^{-\kappa_1 \tau} \|f(\cdot, \tau)\|_{(s)} d\tau < \infty \tag{4.63}$$

holds, where $\kappa_1 := \kappa_{max}(A_\infty)$ and A_∞ is defined in (4.28). Let $\beta_{\mathrm{rem}} := \min\{\mathfrak{b}_s, \eta_{mn}\}$, where $\mathfrak{b}_s$ and η_{mn} are the constants appearing in the definition of weak silence and weak convergence (definition 4.12), respectively. Let, moreover, E_a be the first generalized eigenspace in the $\beta_{\mathrm{rem}}, A_\infty$-decomposition of $\mathbb{C}^{2m}$. Then, there are constants C, N and

$s_{\rm hom}, s_{\rm ih} \geq 0$, *depending only on the coefficients of equation (4.3) such that the following holds. Given a smooth solution u to (4.3), there is a unique $V_\infty \in C^\infty(\bar{M}, E_a)$ such that*

$$\begin{aligned} & \left\| \begin{pmatrix} u(\cdot,t) \\ u_t(\cdot,t) \end{pmatrix} - e^{A_\infty} V_\infty - \int_0^t e^{A_\infty(t-\tau)} \begin{pmatrix} 0 \\ f(\cdot,\tau) \end{pmatrix} d\tau \right\|_{(s)} \\ \leq & C\langle t\rangle^N e^{(\kappa_1-\beta_{\rm rem})t}(\|u_t(\cdot,0)\|_{(s+s_{\rm hom})} + \|u(\cdot,0)\|_{(s+s_{\rm hom}+1)} + \|f\|_{A,s+s_{\rm ih}}), \end{aligned} \tag{4.64}$$

holds for $t \geq 0$ and all $s \in \mathbb{R}$. Moreover,

$$\|V_\infty\|_{(s)} \leq C(\|u_t(\cdot,0)\|_{(s+s_{\rm hom})} + \|u(\cdot,0)\|_{(s+s_{\rm hom}+1)} + \|f\|_{A,s+s_{\rm ih}}). \tag{4.65}$$

Three remarks are in order. First, if, in addition to the assumptions of the lemma, the estimate (4.45) is satisfied, then the definition of $\beta_{\rm rem}$ is replaced by

$$\beta_{\rm rem} = \min\{2\mathfrak{b}_s, \mathfrak{b}_s + \beta_{\rm der}, \eta_{mn}\}. \tag{4.66}$$

Second, the estimate (4.65) can be interpreted as indicating that the map from initial data to asymptotic data, as represented by V_∞, is continuous with respect to the C^∞-topology. Third, it is not difficult to see that

$$e^{A_\infty} V_\infty + \int_0^t e^{A_\infty(t-\tau)} \begin{pmatrix} 0 \\ f(\cdot,\tau) \end{pmatrix} d\tau,$$

is just a solution to the system (4.27).

Given a solution to the equation, lemma 4.5 provides a method to calculate the asymptotics. However, it is also interesting to be able to specify asymptotics that correspond to a certain solution. For this purpose, it is necessary first to impose an upper bound on $\bar{k}(t)$, or a lower bound on $\ell(\iota,t)$. The reason for this is that when going backwards in time from initial data at $t = T_{ode}$, specified by asymptotic data in the ODE regime, to initial data at $t = 0$, it is important that the energy does not grow faster than exponentially. This is implemented in the following lemma.

Lemma 4.6 *Assume that (4.3) is weakly silent, weakly balanced and weakly convergent. Assume, moreover, that $f = 0$ and that there is a constant $\mathfrak{b}_{\rm low} > 0$ and a non-negative continuous function $\mathfrak{e}_{\rm low} \in L^1([0,\infty))$ such that*

$$\dot{\ell}(\iota,t) \geq -\mathfrak{b}_{\rm low} - \mathfrak{e}_{\rm low}(t), \tag{4.67}$$

for all $t \geq 0$ *and all* $0 \neq \iota \in \mathcal{I}_B$. *Let* A_∞ *be defined by (4.28) and let* $\beta_{\mathrm{rem}} = \min\{\mathfrak{b}_s, \eta_{mn}\}$, *where* $\mathfrak{b}_s$ *and* η_{mn} *are the constants appearing in (4.44) and (4.26), respectively. Finally, let* E_a *be the first generalized eigenspace in the* $\beta_{\mathrm{rem}}, A_\infty$*-decomposition of* $\mathbb{C}^{2m}$. *Then, there is an injective map*

$$\Phi_\infty : C^\infty(\bar{M}, E_a) \to C^\infty(\bar{M}, \mathbb{C}^{2m}),$$

such that the following holds. First,

$$\|\Phi_\infty(\chi)\|_{(s)} \leq C\|\chi\|_{(s+s_\infty)}, \tag{4.68}$$

for all $s \in \mathbb{R}$ *and all* $\chi \in C^\infty(\bar{M}, E_a)$, *where the constants* C *and* $s_\infty \geq 0$ *only depend on* C_{coeff}, C_{mn}, $\mathfrak{b}_s$, $\mathfrak{b}_{\mathrm{low}}$, η_{mn}, A_∞, $\|\mathfrak{e}\|_1$, $\|\mathfrak{e}_{\mathrm{low}}\|_1$, $g^{ij}(0)$, $i, j = 1, ..., d$, *and* $a_r(0)$, $r = 1, ..., R$. *Secondly, if* $\chi \in C^\infty(\bar{M}, E_a)$ *and* u *is the solution to (4.3) (with* $f = 0$*) such that*

$$\begin{pmatrix} u(\cdot, 0) \\ u_t(\cdot, 0) \end{pmatrix} = \Phi_\infty(\chi), \tag{4.69}$$

then

$$\left\| \begin{pmatrix} u(\cdot, t) \\ u_t(\cdot, t) \end{pmatrix} - e^{A_\infty t}\chi \right\|_{(s)} \leq C\langle t\rangle^N e^{(\kappa_1 - \beta_{\mathrm{rem}})t}(\|u_t(\cdot, 0)\|_{(s+s_{\mathrm{hom}})} + \|u(\cdot, 0)\|_{(s+s_{\mathrm{hom}}+1)}), \tag{4.70}$$

for all $t \geq 0$ *and all* $s \in \mathbb{R}$, *where the constants* C, N *and* s_{hom} *have the same dependence as in the case of lemma 4.5. Finally, if* $E_a = \mathbb{C}^{2m}$ *(i.e. if* $\mathrm{Rsp}A_\infty < \beta_{\mathrm{rem}}$*), then* Φ_∞ *is surjective.*

Three remarks are in order. First, by combining (4.68), (4.69) and (4.70), it follows that the Sobolev norms of $u(\cdot, 0)$ and $u_t(\cdot, 0)$ on the right-hand side of (4.70) can be replaced by a suitable Sobolev norm of the asymptotic data χ. Second, similar to (4.65), the estimate (4.68) implies that the map from asymptotic data to initial data is continuous with respect to the C^∞-topology. Hence, based on both lemmas 4.5 and 4.6, it follows that when $E_a = \mathbb{C}^{2m}$, there is a homeomorphism between initial data and asymptotic data.

Third, even though the lemma is stated for the case $f = 0$, it can be extended to the inhomogeneous case by combining it with lemma 4.5. For this purpose, assume that $\|f\|_{A,s} < \infty$, and let u_{part} be the solution to (4.3) with initial data $u_{\mathrm{part}}(\cdot, 0) = 0$. Also, let $V_{\mathrm{part},\infty} \in C^\infty(\bar{M}, E_a)$ be such that (4.64) holds with u and V_∞ replaced with u_{part} and $V_{\mathrm{part},\infty}$, respectively. Given a general $V_\infty \in C^\infty(\bar{M}, E_a)$, denote u_{hom} the solution to

(4.3) with $f = 0$ and corresponding initial data $u_{\text{hom}}(\cdot, 0) = \Phi_\infty(V_\infty - V_{\text{part},\infty})$. Hence, it follows that $u := u_{\text{hom}} + u_{\text{part}}$ is a solution to (4.3) that satisfies (4.70).

Now, it is convenient to have a flavour of how lemmas 4.5 and 4.6 can be used to derive and specify asymptotics for a given equation, before applying them to the main equations of interest in the next chapter. For this purpose, equation (4.24) can be considered. In this case, the metric g_{ass} associated with the equation is given by

$$g_{\text{ass}} = -dt \otimes dt + e^{2t} d\theta \otimes d\theta,$$

on $\mathbb{S}^1 \times \mathbb{R}$. As a result

$$\bar{g}_{\text{ass}} = e^{2t} d\theta \otimes d\theta, \qquad \bar{k} = \bar{g}_{\text{ass}}, \qquad \chi = 0.$$

In addition, $\boldsymbol{\chi} = 0$, $\alpha = 0$ and $\zeta = 0$. Consequently, the equation is C^1-silent (or weakly silent) with $\mu = 1$. Moreover, the estimate (4.26) is satisfied with $\alpha_\infty = \zeta_\infty = 0$, and $\eta_{mn} = 1$. Hence, all the necessary assumptions of lemma 4.5 are fulfilled. Given that the estimate (4.45) is also satisfied with $\beta_{\text{der}} = 1$, it follow that $\beta_{\text{rem}} = 1$.

The matrix A_∞ is expressed as

$$A_\infty = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which is of the form of a Jordan block corresponding to an eigenvalue $\lambda = 0$ with a multiplicity $n_\lambda = 2$. So, $\kappa_1 = 0$, and E_a coincides with $\mathbb{R}^2$. So, lemmas 4.5 and 4.6 can be combined to yield a homeomorphism between initial data and asymptotic data.

Based on the previous remarks on Jordan blocks, the matrix exponential $e^{A_\infty t}$ is simply calculated as

$$e^{A_\infty t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

So, for $\psi = (\psi_\infty, v_\infty)^t \in C^\infty(\mathbb{S}^1, \mathbb{R}^2)$, it follows that for a given solution P

$$\left\| \begin{pmatrix} P(\cdot, t) \\ P_t(\cdot, t) \end{pmatrix} - \begin{pmatrix} \psi_\infty + v_\infty t \\ v_\infty \end{pmatrix} \right\|_{(s)} \leq C_s \langle t \rangle^N e^{-t}, \tag{4.71}$$

where C_s is allowed to depend on s and the solution in this case. This result agrees with the one obtained previously by dropping the second spatial derivatives, the reason being the fact that equation (4.24) is also balanced.

Chapter 5

Asymptotic analysis of the systems of interest

In this chapter, lemmas 4.5 and 4.6 are applied to analyze the asymptotics of the systems of interest, namely scalar linear perturbations of a flat FL background with a perfect fluid and a cosmological constant, according to (2.119), and a massive scalar field on the same background, according to (2.131), in the direction of the initial singularity.

5.1 Past asymptotics of scalar linear perturbations of a FL background

Recalling that, based on equation (2.116), the relevant matter model in the direction of the singularity of a FL model is a radiation fluid with $\omega = \frac{1}{3}$, then equation (2.119) gets expressed as

$$\Psi'' - \frac{1}{3}\Delta\Psi + \frac{4}{\eta}\Psi' + \frac{4}{3}\eta^2\Lambda\Psi = 0, \tag{5.1}$$

where Δ is the Laplace-Beltrami operator on $\mathbb{T}^3$. Now, based on remarks made in the previous chapter, the time coordinate can be changed as

$$\eta = e^{-\tau} \Rightarrow \frac{d}{d\eta} = -e^{\tau}\frac{d}{d\tau}, \tag{5.2}$$

so that $\tau \to \infty$ corresponds to $\eta \to 0$. Invoking this, equation (5.1) becomes

$$\begin{aligned} &e^{\tau}\frac{d}{d\tau}\left[e^{\tau}\Psi_{\tau}\right] - \frac{1}{3}\Delta\Psi - 4e^{2\tau}\Psi_{\tau} + \frac{4}{3}\Lambda e^{-2\tau}\Psi \\ &= e^{2\tau}\Psi_{\tau\tau} + e^{2\tau}\Psi_{\tau} - \frac{1}{3}\Delta\Psi - 4e^{2\tau}\Psi_{\tau} + \frac{4}{3}\Lambda e^{-2\tau}\Psi, \end{aligned} \tag{5.3}$$

which yields

$$\Psi_{\tau\tau} - \frac{1}{3}e^{-2\tau}\Delta\Psi - 3\Psi_\tau + \frac{4}{3}\Lambda e^{-4\tau}\Psi = 0. \tag{5.4}$$

The metric associated with (5.4) is given by

$$g_{\text{ass}} = -d\tau \otimes d\tau + 3e^{2\tau}\sum_{i=1}^{3} dx^i \otimes dx^i, \tag{5.5}$$

on $\mathbb{T}^3 \times \mathbb{R}$, and hence

$$\bar{g}_{\text{ass}} = 3e^{2\tau}\sum_{i=1}^{3} dx^i \otimes dx^i, \qquad \bar{k} = \bar{g}_{\text{ass}}, \qquad \chi = 0. \tag{5.6}$$

Also, $\boldsymbol{\chi} = 0$, $\alpha(\tau) = -3$ and $\zeta(\tau) = \frac{4}{3}\Lambda e^{-4\tau}$. So, the equation is C^1-silent with $\mu = 1$, and the estimate (4.26) is satisfied with $\alpha_\infty = -3$, $\zeta_\infty = 0$, and $\eta_{mn} = 4$. As a result, lemma 4.5 can be applied. In this respect, the estimate (4.45) is also satisfied with $\beta_{\text{der}} = 1$. Hence, $\beta_{\text{rem}} = 2$.

The matrix A_∞ is given as

$$A_\infty = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}, \tag{5.7}$$

which has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 3$, both with $n_\lambda = 1$. Thus, $\kappa_1 = 3$. The generalized eigenspace E_{λ_1} is spanned by

$$\begin{pmatrix} x^1 \\ 0 \end{pmatrix},$$

where $x^1 \in \mathbb{R}$. Similarly, E_{λ_2} is spanned by

$$\begin{pmatrix} x^1 \\ 3x^1 \end{pmatrix}.$$

So, choosing $x^1 = 1$, the matrix T and its inverse can be given by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}. \tag{5.8}$$

Consequently, the matrix exponential $e^{A_\infty\tau}$ is calculated as

$$e^{A_\infty \tau} = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{3\tau} \end{pmatrix} \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3}(e^{3\tau} - 1) \\ 0 & e^{3\tau} \end{pmatrix}. \tag{5.9}$$

Given that $\mathrm{Rsp}A_\infty > \beta_{\mathrm{rem}}$, it follows that $E_a = E_{\lambda_2}$, and V_∞ is expressed as

$$V_\infty = \begin{pmatrix} \psi_\infty \\ 3\psi_\infty \end{pmatrix},$$

for $\psi_\infty \in C^\infty(\mathbb{T}^3, \mathbb{R})$. Hence, applying the estimate (4.64) for a given smooth solution Ψ of (5.4) gives

$$\left\| \begin{pmatrix} \Psi(\cdot, \tau) \\ \Psi_\tau(\cdot, \tau) \end{pmatrix} - \begin{pmatrix} \psi_\infty e^{3\tau} \\ 3\psi_\infty e^{3\tau} \end{pmatrix} \right\|_{(s)} \leq C_s \langle \tau \rangle^N e^\tau, \tag{5.10}$$

where, as indicated before, C_s is allowed to depend on s and the solution.

As a result, there is a blow-up of scalar linear perturbations that goes like $1/\eta^3$ for $\eta \to 0$. Moreover, the cosmological constant Λ does not have any effect on the asymptotic behaviour of Ψ in the direction of the initial singularity. In fact, the time dependence of the blow-up agrees with the one deduced from solving equation (5.4) for $\Lambda = 0$, as shown in [42].

From equation (5.10) and the previous remark, the corresponding asymptotic behaviour of the relative energy density perturbations of the perfect fluid can be determined, through (2.120), as

$$\frac{\delta \rho_{1\ell}}{\rho_0} = \frac{2}{3} e^\tau \Delta \psi_\infty + 4\psi_\infty e^{3\tau}, \tag{5.11}$$

for $\tau \to \infty$.

5.2 Past asymptotics of the Klein-Gordon equation on a FL background

Specifying the matter model of the background to a radiation fluid as before, equation (2.131) gets expressed as

$$\phi'' - \Delta\phi + \frac{2}{\eta}\phi' + \eta^2 m^2 \phi = 0, \tag{5.12}$$

where $\Delta\phi \equiv \sum_{i=1}^3 \phi_{ii}$. Employing the change of time coordinate of (5.2), this equation becomes

$$\phi_{\tau\tau} - e^{-2\tau}\Delta\phi - \phi_\tau + e^{-4\tau}m^2\phi = 0. \tag{5.13}$$

Proceeding as before, the associated metric with (5.12) is given by

$$g_{\text{ass}} = -d\tau \otimes d\tau + e^{2\tau}\sum_{i=1}^{3} dx^i \otimes dx^i, \tag{5.14}$$

on $\mathbb{T}^3 \times \mathbb{R}$. Consequently,

$$\bar{g}_{\text{ass}} = e^{2\tau}\sum_{i=1}^{3} dx^i \otimes dx^i, \qquad \bar{k} = \bar{g}_{\text{ass}}, \qquad \chi = 0. \tag{5.15}$$

Also, $\boldsymbol{\chi} = 0$, $\alpha(\tau) = -1$ and $\zeta(\tau) = e^{-4\tau}m^2$. So, equation (5.13) is C^1-silent with $\mu = 1$, and the estimate (4.26) is satisfied with $\alpha_\infty = -1$, $\zeta_\infty = 0$, and $\eta_{mn} = 4$. As a result, lemma 4.5 can be applied, as before. In this respect, the estimate (4.45) is also satisfied with $\beta_{\text{der}} = 1$. Hence, $\beta_{\text{rem}} = 2$.

The matrix A_∞ is given by

$$A_\infty = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \tag{5.16}$$

The eigenvalues in this case are $\lambda_1 = 0$ and $\lambda_2 = 1$, both with $n_\lambda = 1$. Thus, $\kappa_1 = 1$. As the case with the previous analysis, the generalized eigenspace E_{λ_1} is spanned by

$$\begin{pmatrix} x^1 \\ 0 \end{pmatrix},$$

where $x^1 \in \mathbb{R}$. On the other hand, E_{λ_2} is spanned by

$$\begin{pmatrix} x^1 \\ x^1 \end{pmatrix}.$$

Again, choosing $x^1 = 1$, the matrices T and T^{-1} are expressed as

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{5.17}$$

Using (5.17), the matrix exponential $e^{A_\infty \tau}$ can be calculated as

$$e^{A_\infty \tau} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^\tau \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^\tau - 1 \\ 0 & e^\tau \end{pmatrix}. \tag{5.18}$$

One difference with respect to the previous case is that $\mathrm{Rsp} A_\infty < \beta_{\mathrm{rem}}$. So, $E_a = \mathbb{R}^2$, and the asymptotic data V_∞ are expressed as

$$V_\infty = \begin{pmatrix} \phi_\infty \\ v_\infty \end{pmatrix},$$

for $\phi_\infty,\ v_\infty \in C^\infty(\mathbb{T}^3, \mathbb{R})$. This also indicates, by appealing to lemma 4.6, that the map between initial data and asymptotic data is a homeomorphism with respect to the C^∞-topology.

Applying equation (4.64) for a given smooth solution ϕ yields

$$\left\| \begin{pmatrix} \phi(\cdot,\tau) \\ \phi_\tau(\cdot,\tau) \end{pmatrix} - \begin{pmatrix} \phi_\infty + (e^\tau - 1)v_\infty \\ e^\tau v_\infty \end{pmatrix} \right\|_{(s)} \leq C_s \langle \tau \rangle^N e^{-\tau}, \tag{5.19}$$

where, as above, C_s is allowed to depend on s and the solution.

Similar to the cosmological constant, it turns out that the mass m does not affect the asymptotics of the scalar field in the direction of the initial singularity. This agrees with both the results of [5] and [2], where the latter investigates the past asymptotics of equation (5.13) with $m = 0$.

Based on (5.19), the energy density of the scalar field (2.124) is asymptotically expressed as

$$\rho_\phi = \frac{1}{2}[e^{6\tau} v_\infty^2 - e^{2\tau}\Delta\phi_\infty - (e^\tau - 1)e^{2\tau}\Delta v_\infty], \tag{5.20}$$

for $\tau \to \infty$.

Conclusion

In this thesis, past asymptotics of two scalar linear wave equations on a flat FL cosmological background, coupled to a perfect fluid, have been analyzed using the method of [48]. For the first equation, the blow-up profile of scalar linear perturbations of this background has been derived in the presence of a cosmological constant Λ. In this respect, it is shown by estimate (5.10) that neither the size nor the sign of Λ affects the blow-up of perturbations. For the second equation, the blow-up of a massive scalar field on the same background has been investigated. Similar to the first case, the effect of mass turns out to be negligible, asymptotically, according to (5.19). This irrelevance of both the cosmological constant and mass suggests that the blow-up pattern in the direction of the initial singularity of the FL solution is entirely determined by the scale factor, and not by other parameters, in the two discussed cases.

In general, studying systems of the form (4.3) on cosmological backgrounds turns out to be relevant for another important issue, namely investigating the stability of these backgrounds. By linearizing the Einstein's equations for a given solution, the method of the previous chapter can be employed to analyze the asymptotics, whether to the past or future, of the resulting equations of perturbations. If a certain asymptotic feature of the background solution is retained by the perturbations, then it is considered to be stable, at least on the linear level. For example, turning again to the first case of interest, it results that the flat FL model with a radiation fluid is past linearly stable due to the blow-up of perturbations. However, it is important not to forget that the situation could be different in the full non-linear setting. In fact, by considering anisotropic cosmological solutions, it turns out that this same background is past non-linearly unstable, which represents a strong motivation to go beyond the assumption of isotropy when studying the cosmic singularity [51]. Even though the stability of many cosmological solutions of interest has already been established on the non-linear level, for example [22, 53, 50, 20, 11, 43], most of these results is proven either for the vacuum case, or in the presence of matter, but with symmetry assumptions. Hence, asymptotic analysis of these more general situations in the linear regime can help to further extend the non-linear results to more realistic models of the universe. Moreover, an understanding of the linearized system helps to inform how the gauge for the full Einstein-matter system can be better fixed [48].

References

[1] Ralph Abraham, Jerrold E Marsden, and Tudor Ratiu. *Manifolds, tensor analysis, and applications*. Vol. 75. Springer Science & Business Media, 2012.

[2] Artur Alho, Grigorios Fournodavlos, and Anne T Franzen. "The wave equation near flat Friedmann–Lemaître–Robertson–Walker and Kasner Big Bang singularities". In: *Journal of Hyperbolic Differential Equations* 16.02 (2019), pp. 379–400.

[3] Paul T Allen and Alan D Rendall. "Asymptotics of linearized cosmological perturbations". In: *Journal of Hyperbolic Differential Equations* 7.02 (2010), pp. 255–277.

[4] Vladimir I Arnold. *Ordinary differential equations*. Springer Science & Business Media, 1992.

[5] Alain Bachelot. "Wave asymptotics at a cosmological time-singularity: Classical and quantum scalar fields". In: *Communications in Mathematical Physics* 369.3 (2019), pp. 973–1020.

[6] James M Bardeen. "Gauge-invariant cosmological perturbations". In: *Physical Review D* 22.8 (1980), p. 1882.

[7] John D Barrow and Frank J Tipler. "Analysis of the generic singularity studies by Belinskii, Khalatnikov, and Lifschitz". In: *Physics Reports* 56.7 (1979), pp. 371–402.

[8] VA Belinskii and IM Khalatnikov. *EFFECT OF SCALAR AND VECTOR FIELDS ON THE NATURE OF THE COSMOLOGICAL SINGULARITY*. Tech. rep. Inst. of Theoretical Physics, Leningrad, 1972.

[9] Vladimir A Belinskii, Isaac M Khalatnikov, and Evgeny M Lifshitz. "Oscillatory approach to a singular point in the relativistic cosmology". In: *Advances in Physics* 19.80 (1970), pp. 525–573.

[10] Vladimir A Belinskii and Isaak Markovich Khalatnikov. "On the nature of the singularities in the general solution of the gravitational equations". In: *Sov. Phys. JETP* 29.5 (1969), pp. 911–917.

[11] Florian Beyer and Todd A Oliynyk. "Past stability of FLRW solutions to the Einstein-Euler-scalar field equations and their big bang singularites". In: *arXiv preprint arXiv:2308.07475* (2023).

[12] Oleg I Bogoyavlensky. *Methods in the qualitative theory of dynamical systems in astrophysics and gas dynamics*. Springer, 1985.

[13] Joan Centrella and Richard A Matzner. "Plane-symmetric cosmologies". In: *Astrophysical Journal, Part 1, vol. 230, June 1, 1979, p. 311-324.* 230 (1979), pp. 311–324.

[14] Yvonne Choquet-Bruhat. *General relativity and the Einstein equations*. Oxford university press, 2009.

[15] C Barry Collins. "More qualitative cosmology". In: *Communications in Mathematical Physics* 23 (1971), pp. 137–158.

[16] Spiros Cotsakis and Alexander P Yefremov. *100 years of mathematical cosmology: Models, theories, and problems, Part A*. 2022.

[17] John Earman. "Lambda: The constant that refuses to die". In: *Archive for History of Exact Sciences* 55.3 (2001), pp. 189–220.

[18] George FR Ellis, Roy Maartens, and Malcolm AH MacCallum. *Relativistic cosmology*. Cambridge University Press, 2012.

[19] Henk van Elst, Claes Uggla, and John Wainwright. "Dynamical systems approach to G2 cosmology". In: *Classical and Quantum Gravity* 19.1 (2001), p. 51.

[20] Grigorios Fournodavlos, Igor Rodnianski, and Jared Speck. "Stable Big Bang formation for Einstein's equations: The complete sub-critical regime". In: *Journal of the American Mathematical Society* 36.3 (2023), pp. 827–916.

[21] Avner Friedman. *Foundations of modern analysis*. Courier Corporation, 1982.

[22] Helmut Friedrich. "On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure". In: *Communications in Mathematical Physics* 107 (1986), pp. 587–609.

[23] Robert Geroch. "Singularities". In: *Relativity: Proceedings of the Relativity Conference in the Midwest, held at Cincinnati, Ohio, June 2–6, 1969*. Springer. 1970, pp. 259–291.

[24] Robert Geroch and Lee Lindblom. "Is perturbation theory misleading in general relativity?" In: *Journal of mathematical physics* 26.10 (1985), pp. 2581–2588.

[25] Eric Gourgoulhon. *3+ 1 formalism in general relativity*. Springer, 2012.

[26] Robert H Gowdy. "Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: topologies and boundary conditions". In: *Annals of Physics* 83.1 (1974), pp. 203–241.

[27] Andrés Franco Grisales. “Asymptotics of Solutions to Silent Wave Equations”. In: *arXiv preprint arXiv:2310.03582* (2023).

[28] Alan H Guth. “Inflationary universe: A possible solution to the horizon and flatness problems”. In: *Physical Review D* 23.2 (1981), p. 347.

[29] Stephen W Hawking and George FR Ellis. *The large scale structure of space-time.* Cambridge university press, 1973.

[30] CG Hewitt and J Wainwright. “Dynamical systems approach to titled Bianchi cosmologies: Irrotational models of type V”. In: *Physical Review D* 46.10 (1992), p. 4242.

[31] Conrad G Hewitt and John Wainwright. “Orthogonally transitive G2 cosmologies”. In: *Classical and Quantum Gravity* 7.12 (1990), p. 2295.

[32] James Isenberg and Vincent Moncrief. “Asymptotic behavior of the gravitational field and the nature of singularities in Gowdy spacetimes”. In: *Annals of Physics* 199.1 (1990), pp. 84–122.

[33] Hideo Kodama and Misao Sasaki. “Cosmological perturbation theory”. In: *Progress of Theoretical Physics Supplement* 78 (1984), pp. 1–166.

[34] EP Liang. “Dynamics of primordial inhomogeneities in model universes”. In: *Astrophysical Journal, vol. 204, Feb. 15, 1976, pt. 1, p. 235-250.* 204 (1976), pp. 235–250.

[35] Evgeny M Lifshitz and Isaak M Khalatnikov. “Investigations in relativistic cosmology”. In: *Advances in Physics* 12.46 (1963), pp. 185–249.

[36] Woei Chet Lim et al. “Asymptotic isotropization in inhomogeneous cosmology”. In: *Physical Review D* 69.10 (2004), p. 103507.

[37] MAH MacCallum. “A class of homogeneous cosmological models III: asymptotic behaviour”. In: *Communications in Mathematical Physics* 20 (1971), pp. 57–84.

[38] Karim A Malik and David R Matravers. “A concise introduction to perturbation theory in cosmology”. In: *Classical and Quantum Gravity* 25.19 (2008), p. 193001.

[39] Karim A Malik and David Wands. “Cosmological perturbations”. In: *Physics Reports* 475.1-4 (2009), pp. 1–51.

[40] Charles W Misner. “Mixmaster universe”. In: *Physical Review Letters* 22.20 (1969), p. 1071.

[41] Viatcheslav Mukhanov. *Physical foundations of cosmology.* Cambridge university press, 2005.

[42] Viatcheslav F Mukhanov, Hume A Feldman, and Robert Hans Brandenberger. “Theory of cosmological perturbations”. In: *Physics reports* 215.5-6 (1992), pp. 203–333.

[43] Hans Oude Groeniger, Oliver Petersen, and Hans Ringström. "Formation of quiescent big bang singularities". In: *arXiv e-prints* (2023), arXiv–2309.

[44] P James E Peebles et al. "The case for the relativistic hot big bang cosmology". In: *Nature* 352.6338 (1991), pp. 769–776.

[45] Lawrence Perko. *Differential equations and dynamical systems*. Vol. 7. Springer Science & Business Media, 2013.

[46] Eric Poisson. *A relativist's toolkit: the mathematics of black-hole mechanics*. Cambridge university press, 2004.

[47] Hans Ringström. "Asymptotic expansions close to the singularity in Gowdy spacetimes". In: *Classical and quantum gravity* 21.3 (2004), S305.

[48] Hans Ringström. "Linear systems of wave equations on cosmological backgrounds with convergent asymptotics". In: *arXiv preprint arXiv:1707.02803* (2017).

[49] Hans Ringström. "Non-linear wave equations". In: *Course Notes, KTH* (2005).

[50] Hans Ringström. *On the topology and future stability of the universe*. OUP Oxford, 2013.

[51] Hans Ringström. "Wave equations on silent big bang backgrounds". In: *arXiv preprint arXiv:2101.04939* (2021).

[52] W Rudin. "Real and Complex Analysis McGraw-Hill Singapore". In: (1986).

[53] Jared Speck. "The nonlinear future stability of the FLRW family of solutions to the Euler–Einstein system with a positive cosmological constant". In: *Selecta Mathematica* 18.3 (2012), pp. 633–715.

[54] Claes Uggla et al. "Past attractor in inhomogeneous cosmology". In: *Physical Review D* 68.10 (2003), p. 103502.

[55] John Wainwright. "A dynamical systems approach to Bianchi cosmologies: orthogonal models of class A". In: *Classical and Quantum Gravity* 6.10 (1989), p. 1409.

[56] John Wainwright and George Francis Rayner Ellis. *Dynamical systems in cosmology*. 1997.

[57] Robert M Wald. *General relativity*. University of Chicago press, 2010.