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ON SPIN COHERENCE AND MINIMUM LENGTH IN RECOMBINATION EXPERIMENTS

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Abstract

In this work, we have inquired about the effects of the minimum length hypothesis on the recombination of a delocalized mass. The question stems from a class of gedankenexperiments involving the gravitational interaction of such a delocalized system with another distant body, while the former is being recombined. These experiments are aimed at investigating potential violations of causality if gravity is quantum in nature. We focus specifically on the case of the system being split in a superposition of spatially separated spin states, which bears a strong resemblance to the Humpty-Dumpty effect studied by Schwinger and collaborators. We find that by taking the physical distance between the two parts of the system to be bounded from below by a minimum value (usually of the order of the Planck length), the ability to restore coherence by reconstructing the original spin state is bound by the same conditions found in previous works on the subject where, to avoid tensions between causality and complementarity, emission of gravitons has been invoked as a mean to carry away the coherence of the state. We also find that the velocities experienced by the delocalized system during recombination can approach the speed of light when the aforementioned bounds are reached and, therefore, we propose a relativistic generalization, finding a more complex dependence on mass and velocity for the spin coherence. We conclude by positing an intriguing equivalence between graviton emission and minimal length in accounting for the expected loss of coherence.

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1 Introduction

The idea of witnessing[1] the quantum properties of a field through its ability to entangle spatially delocalized states goes back to a proposal by Bohr. The idea is that in order to preserve both causality and complementarity, quantized radiation, able to carry away part or all quantum coherence from the state by means of entangling with the emitting body, must be taken into account.
An in-detail analysis of this early proposal, as well as a first analysis of the gravitational case, was given in the work of Baym & Ozawa [2] and further analysis on the gravitational case was developed later by Mari et al. [3].

These apparent tensions between the principle of complementarity and causality seem to suggest superluminal signaling to be possible. Resolutions to this tension have been proposed[2], involving the emission of quanta of the electromagnetic/gravitational field, and, thus, these experiments have been suggested as indirect proofs of quantum carriers for these fields. In the development of the gravitational version of the experiment, it has been noted that the necessity of invoking gravitational radiation to avoid paradoxes is sometimes debatable, for example in the case in which, after recombination, the coherence of the state is measured by interferometric means, i.e. by observing an interference pattern [5]. In such a case a minimum length, below which distances cannot be resolved or defined, is enough to resolve the paradox. The reason is that, as we will show, the interference fringe spacing goes below the Planck length under the same conditions for which graviton emission has been postulated. To resolve the paradox, this same minimum length has been invoked in the form of vacuum fluctuations of the gravitational field.

The minimum-length argument, however, apparently does not apply once one considers any other setup that does not rely on a physical interference pattern to be measured. For example, in the setup we have explored in this work, using a Stern-Gerlach apparatus, a definite spin state along an axis can be brought into a superposition of orthogonal, spatially separated, spin states along another axis. At least in principle, this state can later be recombined into a localized state with the initial spin reconstructed (the difficulty in doing so is called the Humpty-Dumpty effect [6]). The only important difference between this setup and the former is how coherence is measured. The threshold on recombination time, the mass and other parameters needed to avoid the possible insurgence of superluminal signaling remain the same. Our work quantitatively investigates the degree of spin coherence at the merging of two delocalized wavepackets and, particularly, its value in the regime in which graviton emission has been proposed. As a means to introduce the minimum length hypothesis in a consistent way we introduce and discuss the q-metric: an effective metric, resulting from a transformation of the usual spacetime metric, whose

[1]See [1] for details on the problem of what is meant by "witnessing".
[2]As an example, specifically the one whose reasoning we will follow, see Belenchia et al. [4]

geodesic distances have a lower bound, successfully implementing this way the finite length in a Lorentz-covariant way.
We find that, in this minimum-length scenario, spin coherence is suppressed in the same conditions that have been previously attributed to a mass-quadrupole moment variation large enough to prompt graviton emission.

The work is structured as follows:

- We begin in 2 by introducing the gedankenexperiment and deriving the current situation regarding its implications. We show the equivalence between the conditions for the emission of quantized radiation and the disappearance of interference fringes both non-relativistically and relativistically.
- Proceeding to 3, we give a brief introduction to the q-metric and some of its results, leaving the details of the calculations to the appendices A and B.
- 4 shows a derivation of the Humpty-Dumpty effect and its connection with this work, showing how the dynamics of the gedankenexperiment can translate into practice.
- In 5, we calculate the spin coherence of merging wavepackets using Gaussian packets, we go to the finite length limit for their separation, and confront the results with the conditions derived in 2.
- In 6, we introduce Lorentz-invariant wavepackets whose non-relativistic limit are the packets from the preceding section. We find a closed-form relativistic generalization of the results from the preceding section and make some observations.
- Finally, in the appendices A and B, we introduce the general theory of bitensors and use them to derive the q-metric mentioned above.

2 Electrically and gravitationally mediated entanglement

2.1 The electric case

On one side we have Alice (A) holding a charged massive particle of mass m_A and charge q_A in a delocalized superposition, with a separation of $2d$ between the paths. The state has been split adiabatically as to preserve coherence and held like this from a distant past. At time $t = 0$ Alice starts recombining the two particles, she will complete the process in a time which we shall denote as T_A. At a distance from Alice, Bob (B) holds another charged body of mass m_B and charge q_B in a narrow trap such that it is not displaced by any field. Again, at $t = 0$, Bob chooses to release or not release the trap

and let his particle evolve freely under the influence of the field of A. [3]
Bob will then measure the position of his particle at a time T_B. If throughout the whole experiment, Alice and Bob are spacelike separated at a distance D, (i.e. causally disconnected) we have that [4]:

$$T_B < T_A < D,$$

where the first inequality is necessary for the experiment to make sense, indeed Bob needs to resolve which-path before the particle ultimately merges into one single path. A seemingly paradoxical situation then arises: Since the electromagnetic field is quantum, Bob's particle feels a superposition of fields implying distinct evolutions. If Bob can resolve the deflection of the trajectory of his particle between the two paths, he will gain which-path information on Alice's state. But, being Alice and Bob spacelike separated, no signal can reach Alice in time T_B. As Bob gains which-path information, complementarity tells us that the coherence of Alice's state must be lost. At the same time, if Alice can (or can't) coherently recombine her state, she will know that Bob didn't (or did) release his trap. Hence an apparent protocol for superluminal signaling arises where, depending on the choice of Bob to release or not release his trap, Alice will be able to get one bit of information by discriminating between restored coherence or lost coherence on her recombined state, regardless of what measurement she does (be it a measure of spin, interference pattern or anything else).
Let us go in more detail and see how this tension is resolved:

The system can initially be described as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|L\rangle_A |\varphi_L\rangle + |R\rangle_A |\varphi_R\rangle \right) \otimes |\psi_0\rangle_B \,, \tag{2.1}$$

where $|L\rangle_A$ ($|R\rangle_A$) is the left (right) component of Alice's particle (including possible spin degrees of freedom), $|\varphi_L\rangle$ ($|\varphi_R\rangle$) is the state of the electromagnetic field entangled with the left (right) component of A and $|\psi_0\rangle_B$ is the ground state of Bob's particle in the trap.
After a certain evolution time and assuming Bob did release his trap, the state would become entangled:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|L\rangle_A |\varphi_L\rangle |\psi_L\rangle_B + |R\rangle_A |\varphi_R\rangle |\psi_R\rangle_B \right).$$

And any which-path knowledge on Bob's side would collapse Alice's part, thus destroying the coherence of her state.
To resolve the paradox, for what concerns Bob's part of the experiment, we will make

[3] For the electromagnetic case we will ignore gravitational effects. When the time comes to talk about the gravitational version of the experiment we will set $q_a = q_B = 0$

[4] Throughout this paper $c = \hbar = 1$ unless explicitly stated

use of the concept of vacuum fluctuations. These fluctuations are, for the electric field, on a scale $\sim R$, in the order of magnitude [7]

$$\Delta E \sim \frac{1}{R^2},$$

and using Newton's law the displacement of a particle under an electric field [5].

$$m\ddot{x} = qE \rightarrow x = \frac{q}{2m}Et^2 + v_0 t + x_0.$$

Hence these fluctuations imply an uncertainty in position

$$\Delta x = \frac{q}{2m}\Delta(Et^2),$$

which over a timescale $\sim R$, as above, yields $\Delta x \sim \frac{q}{m}$. [6]
Given this, we will need the separation between the paths of Bob's particle, δx, to obey the condition

$$\delta x > \frac{q_B}{m_B}, \tag{2.2}$$

in order to say that Bob can, even in principle, acquire which-path information on Alice's system.
As for the field produced on Alice's part, in the time Bob performs his task (T_B), his side will only feel the static electric field produced by the superposition A. Given the assumption that $d \ll D$, the sensitivity to the electric field that Bob needs to discriminate the two paths is

$$E_{rel} \propto q_A \left(\frac{1}{(D-d)^2} - \frac{1}{(D+d)^2} \right) \sim 2d\frac{q_A}{D^3},$$

which is no less than the dipole contribution to the electric field in the multipole expansion.
The displacement B will undergo in a time T_B given this field is

$$\delta x \sim \frac{q_B}{m_B} E_{rel} T_B^2 = 2d\frac{q_B q_A}{m_B}\frac{T_B^2}{D^3}, \tag{2.3}$$

which, inserted in (2.2), yields the condition

$$2dq_A\frac{T_B^2}{D^3} > 1. \tag{2.4}$$

[5] We are assuming a non-relativistic setting, this should be justified (at least for the moment) assuming large masses and slow motion

[6] This is also known as the "charge radius". Reinstating constants it becomes $\frac{\hbar q}{mcq_P}$ with $q_P = \sqrt{\hbar c}$ the Planck charge. Note that for $q > 1$ this is larger than the Compton wavelength localization limit $\lambda_C \sim \frac{1}{m}$

Now, assuming the electromagnetic field to be quantized (which of course is, by now, a known fact), we can look for conditions on Alice's recombination process such that no such quanta (photons) are emitted, thus ensuring the recombination process is coherent.[7]

The energy flux radiated from a time-dependent electric dipole is known to be proportional to $(\ddot{\mathcal{D}})^2$, where $\mathcal{D}$ is the time-dependent dipole moment (in our case $\mathcal{D} = 2dq_A$). We can then estimate the total energy radiated in the time T_A as

$$W \sim \left(\frac{\mathcal{D}}{T_A^2}\right)^2 T_A.$$

Estimating the energy of emitted photons to be the peak value of their energy distribution, i.e. $\omega \sim \frac{1}{T_A}$, yields for the number of emitted photons the value

$$N \sim \frac{W}{\omega} = \left(\frac{\mathcal{D}_A}{T_A^2}\right)^2 < 1.$$

The last inequality is the requirement that no photons be emitted, hence

$$\mathcal{D}_{\mathcal{A}} < T_A.$$

Using these results, we can now solve the tension between causality and complementarity mentioned earlier. In the case of interest, namely for spacelike separated A and B ($T_B < T_A < D$), there emerge two possibilities: $\mathcal{D}_{\mathcal{A}} < T_A$ and $\mathcal{D}_{\mathcal{A}} > T_A$. If $\mathcal{D}_{\mathcal{A}} < T_A$, Alice will be able to recombine her state coherently without emitting photons but, given that $\mathcal{D}_{\mathcal{A}} < T_A < D$, we have that both $\mathcal{D}_{\mathcal{A}} < D$ and $T_B < D$ hold. Then by inequality (2.4) Bob will be unable to gain which-path information, no matter what he does. On the other hand, if $\mathcal{D}_{\mathcal{A}} > T_A$, Alice's system will emit radiation and consequently entangle with it. Coherence will be lost and, while Bob can obtain which-path information, he is not the cause for the loss of coherence in A, hence no superluminal signaling can be deduced from this.

[7] This follows from the LOCC argument (local operations and classical communication), whereby if we want to assume local interactions through mediating fields, entanglement cannot be generated by classical mediators. For this reason, the experiment discussed in this work has been proposed as a witness for the quantization of the EM/gravitational field. Emission of classical radiation would at most accumulate a total relative phase corresponding, for example, to a shift in an interference pattern produced during the recombination (similar to what is observed in the Aharonov-Bohm effect). Quantum radiation on the other hand, having quantum degrees of freedom of its own, can generate entanglement and decohere a state, thus eliminating the possibility of observing an interference pattern in the previous example. Still, it has been argued that if one considers interactions to be nonlocal (yet causal), as in the absorber theory of electromagnetism of Wheeler and Feynman [8], i.e., we exclude mediators of the interaction, one can anyway recover the results above, this shows that entanglement alone, even if displaying quantum features of the field, might in principle not be enough to deduce the existence of (quantum) mediators [9].

Other scenarios arise by allowing either $T_A > D$, or $T_B > D$, or both. These do not lead to any apparent paradoxes and are not of interest in the present work. Analyses of such cases can still be found in the references.

2.2 The gravitational case

Following Belenchia et. al [4], the extension of the experiment to the gravitational case is straightforward, for this purpose, we set $q_A = q_B = 0$ and consider the gravitational Newtonian potential between the two parties. The problem of localizability of a particle in the context of general relativity is hindered by the absence of a fixed background, being the metric itself the gravitational field. A well-defined concept is, then, the relative distance between two bodies, this being a physically significant quantity independent of the reference frame [8].
Positions in spacetime are expected to fluctuate by an amount of order

$$\Delta x \sim l_P.$$

This can be derived through arguments involving little and elementary assumptions, but the result has been argued by many theories and estimates in the landscape of quantum gravity.[9]

So now the threshold for Bob to gain significant which-path information is for the displacement of his particle to respect the following inequality

$$\delta x > l_P.$$

Now, as far as graviton emission on Alice's part is concerned, previous works have replaced the charge dipole with the mass dipole term. This has been corrected by [4], observing that in the case of gravitational multipole expansions, the dipole term must vanish by the principle of conservation of center of mass. We cannot ignore the fact that Alice will have used her lab[10] to give the paths a relative motion. This will, necessarily, shift the lab in the opposite direction, such that both trajectories have effectively the same center of mass (i.e. they cannot be used to distinguish the two trajectories and gain which-path information). Qualitatively one may write state (2.1) as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|L\rangle_A |LAB_L\rangle |\varphi_L\rangle + |R\rangle_A |LAB_R\rangle |\varphi_R\rangle \right) \otimes |\psi_0\rangle_B ,$$

[8] We will construct and discuss a useful tool to construct general relativity using distances between points in appendix A and derive a metric endowed with a concept of minimal localization in section B.

[9] The argument of vacuum fluctuations has also been used as a further claim that the experiment, if successful, would be witnessing the gravitational field as a quantum field subject to vacuum fluctuations. Hossenfelder [10] derives the minimum length argument from many different perspectives, some of which do not require the gravitational field to be quantum in nature, hence weakening this claim.

[10] Here laboratory refers abstractly to whatever body Alice is using to perform the recombination, be it a Stern-Gerlach apparatus, the earth attached to it, etc. We will return to this point shortly.

where the inclusion of the LAB states, which move in the opposite direction of the particle in each of the two superposed cases, balances the mass distribution to cancel any dipole moment.
The next relevant term to distinguish the paths is the quadrupole[11] contribution to the field $\frac{\mathcal{Q}_{\mathcal{A}}}{D^4}$. Equation (2.3) has the gravitational equivalent

$$\delta x \sim \frac{\mathcal{Q}_{\mathcal{A}}}{D^4} T_B^2.$$

The differences from the electric version of the experiment are clear: firstly, the localization limit described above does not depend on any property of system A and secondly, by the usual correspondence of inertial and gravitational mass, this result is independent of m_B.
Which-path discrimination by Bob now requires the following to be true. Setting the gravitational constant $G = 1$, such that $l_P = 1$:

$$\delta x > \Delta x \rightarrow \frac{\mathcal{Q}_A}{D^4} T_B^2 > 1.$$

This can be rewritten as

$$\frac{\mathcal{Q}_A}{D^2} > \frac{D^2}{T_B^2}. \tag{2.5}$$

And, if causality is respected ($T_B > D$), this sets the condition $\mathcal{Q}_A > D^2$ for Bob to distinguish the paths.
But, since our focus is on the case of spacelike separation ($T_B < D$), let us proceed in this case: For what concerns Alice's emission of quantized gravitational radiation (photons), the emitted energy in this case is

$$W \sim \left(\frac{\mathcal{Q}_A}{T_A^3}\right)^2 T_A,$$

and just as before, we take the mean energy of emitted quanta to be of order $\sim \frac{1}{T_A}$ and get, for the number of gravitons

$$N \sim \left(\frac{Q_A}{T_A^2}\right)^2 < 1 \longrightarrow \mathcal{Q}_A < T_A^2, \tag{2.6}$$

where the second inequality is required for no such gravitons to be emitted, i.e. it is the condition for coherent recombination of A. These conditions solve the paradox entirely, in the same way as for the electromagnetic case: if condition (2.6) holds, Alice will be able to recohere her particle, but since then $\mathcal{Q}_{\mathcal{A}} < T_A^2 < D^2$ and $T_B < D$, the condition

[11] The contribution of the lab to the quadrupole moment can now be reasonably neglected if $m_A \ll M$ (M is the mass of the lab).

for Bob to distinguish the two paths (2.5) cannot be met. On the other hand, if $\mathcal{Q}_\mathcal{A} > T_A^2$ Alice's particle will emit and decohere, rendering any action by Bob inconsequential for the means of communicating between them.
It might be of interest now to give an explicit expression for the quadrupole moment concerning both Alice's and Bob's parts of the experiment. In the work by Rydving, Aurell, and Pikovski [5], it is argued that the quadrupole moment for the superposed paths is not able to distinguish them and that it is the octupole moment that decides the distinguishability condition for Bob. We will follow the line of reasoning of Pesci [11], which using a more general case of mass distribution for A, restores the quadrupole moment as the lowest order contribution in both processes.
The gravitational potential is

$$V(\mathbf{r}) = -\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|},$$

where $\mathbf{r}$ and $\mathbf{r}'$ are vector quantities. This has the following explicit multipole expansion, assuming all particles lay on the same line as is the case in this setup

$$V(x) \approx -\frac{M+m}{x} - \frac{\mathcal{D}}{x^2} - \frac{\mathcal{Q}}{2x^3} - \frac{\mathcal{O}}{6x^4} - ...$$

where M is the mass of the lab and m is the mass of Alice's particle.
If the mass m is at position x_m, setting the origin at the center of mass position x_{CM} gives the displacement for the lab x_M

$$x_{CM} = \frac{Mx_M + mx_m}{M+m} \equiv 0 \longrightarrow x_M = -\frac{m}{M}x_m = -\eta x_m,$$

where quite generally $\eta \ll 1$.
Given a Dirac delta mass distribution $\rho_m = m\delta(x - x_m)$ for the particle, its quadrupole moment can be easily evaluated:

$$\mathcal{Q}_m(x_m) = \int 2x^2 \rho_m dx = 2mx_m^2.$$

On the other hand, considering a general mass distribution for the lab ρ_M, its quadrupole moment without the particle is

$$\mathcal{Q}_M(x_m) = \int 2x^2\rho_M dx = \int 2(x - x_M + x_M)^2 \rho_M dx = 2\int (x - x_M)^2 \rho_M dx + 2\eta m x_m^2.$$

If we consider the quite general case in which the center of mass of the lab and that of the particle, taken by themselves, do not coincide with the center of mass of the whole system (the origin), but retain instead some offset (x_m in the notation used until now)[12], we can write for the two paths taken by the particle in the delocalized state their respective quadrupole moments:

$$\mathcal{Q}_d = \mathcal{Q}_m(x_m - d) + \mathcal{Q}_M(x_m - d),$$
$$\mathcal{Q}_{-d} = \mathcal{Q}_m(x_m + d) + \mathcal{Q}_M(x_m + d).$$

While for the recombined state

$$\mathcal{Q}_F = \mathcal{Q}_m(x_m) + \mathcal{Q}_M(x_m).$$

The relevant quantities for the experiment are the difference between the quadrupole moments of the paths $|\mathcal{Q}_d - \mathcal{Q}_{-d}|$ as far as Bob's side is concerned, and the variation between delocalized and localized state ($|\mathcal{Q}_F - \mathcal{Q}_{-d}|$ and $|\mathcal{Q}_F - \mathcal{Q}_d|$) as far as Alice's part is concerned. Neglecting terms of order $O(\eta)$ and greater, we obtain

$$\mathcal{Q}_F - \mathcal{Q}_{-d} = -2md^2 - 4mx_m d = -2md^2 - \mathcal{Q},$$
$$\mathcal{Q}_F - \mathcal{Q}_d = -2md^2 + 4mx_m d = -2md^2 + \mathcal{Q},$$

where we defined $\mathcal{Q} = 2mx_m d$. From these, we can consequently obtain

$$\mathcal{Q}_d - \mathcal{Q}_{-d} = -8mx_m d = -2\mathcal{Q}.$$

It is now clear that, if we take $x_m \gg d$ (while necessarily requiring $x_m \ll D$), all these quantities can be taken to be of the same order, i.e.

$$|\mathcal{Q}_F - \mathcal{Q}_{-d}| \approx |\mathcal{Q}_F - \mathcal{Q}_d| \approx |\mathcal{Q}_d - \mathcal{Q}_{-d}| \approx \mathcal{Q}.$$

This proves that the same quantity representing the quadrupole moment can be used to describe both sides of the experiment.

[12] for clearer understanding see figure 1.

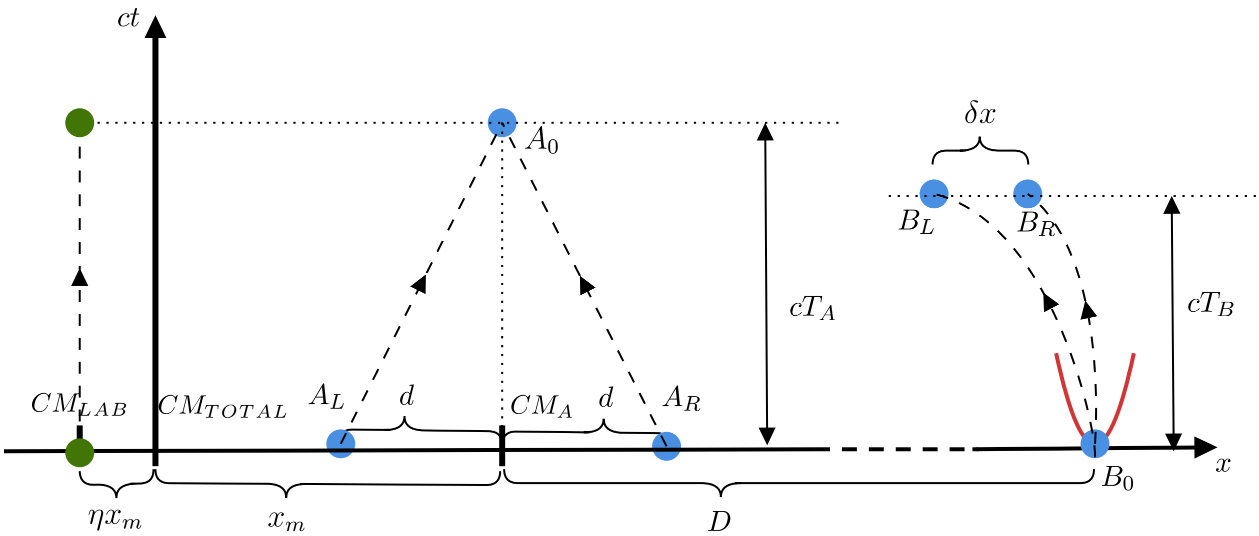

Figure 1: Schematic visualization of the experimental setup. The L and R subscripts here strictly refer to the components of systems A and B being on the left or right of each other. The entangled pairs are $A_L \leftrightarrow B_R$ and $A_R \leftrightarrow B_L$. Nothing here is to scale.

Inserting this explicit value in our condition for graviton emission[13] results in

$$T_A < \sqrt{\mathcal{Q}_A} \sim \sqrt{2m_A x_m d},$$

and rearranging this equation while also reinserting factors of $\hbar$ and c gives

$$T_A < \sqrt{\frac{2x_m}{d}} \sqrt{\frac{m_A}{m_p}} \frac{d}{c}.$$

Here m_p is the Planck mass. The last factor $\frac{d}{c}$ clearly acts as a lower bound on T_A, imposed by causality itself.
A crucial point follows now by giving a better definition of what we call the “lab”: Until now, we considered the lab to quantitatively describe the vanishing of a net mass dipole contribution to the experiment, and to calculate higher order multipoles by accounting for all the masses involved in the process (this is as usual a consequence of gravity being only attractive, being there no negative mass). Now, we set a boundary to this concept of laboratory by imposing that what is involved in the experiment is no more than what is causally connected to it. What this means is

$$T_A > 2x_m,$$

[13]Be careful: (2.6) is the condition for coherent recombination, i.e. no graviton emission. The condition for graviton emission will have the inverse inequality sign.

such that the portion of the lab involved in the experiment be in causal contact with particle A.
Now, our inequality can be recast as

$$T_A < \sqrt{\frac{2x_m}{d}}\sqrt{\frac{m_A}{m_p}}\frac{d}{c} < \sqrt{\frac{cT_A}{d}}\sqrt{\frac{m_A}{m_p}}\frac{d}{c}. \tag{2.7}$$

Or, in other terms

$$\frac{cT_A}{d} < \sqrt{\frac{cT_A}{d}}\sqrt{\frac{m_A}{m_p}},$$

which requires $m_A > m_p$ ($\frac{cT_A}{d} > 1$ for causality). Now we found that, for graviton emission to be possible in such a scenario, the m_A needs to be greater than the Planck mass ($\sim 10^{-8}$ kg), very big for current delocalization experiments. The last condition simplifies to

$$cT_A < \frac{m_A}{m_p}d, \tag{2.8}$$

and, taking $2x_m = T_A$, i.e. the largest possible value for x_m, the first and second inequality in (2.7) are equivalent, and this last condition becomes the strongest condition for emission.
Before moving on we wish to emphasize that two conditions have been used to resolve the apparent paradox: the minimum length requirement for Bob's part, and the quantized radiation requirement for Alice's part. As noted in footnote 9, the minimum length requirement does not strictly imply, by itself, the existence of quantum mediators but simply that the field is subject to quantum fluctuations.
As for the quantized radiation requirement, it should be noted that in the proposed resolution of the paradox, quantum emission is sufficient to resolve the paradox, and necessary (in the sense that classical radiation would not suffice). Yet, other settings can be envisaged in which the loss of coherence, taken alone, can find alternative explanations (see, again, [1] section 6), including, as we mentioned above, the case without any mediators at all. We will now show the equivalence between emission and impossibility of fringe discrimination for the interference-type measurement and, later, give a quantitative calculation for the case of spin-type measurements.

2.3 Equivalence of conditions for graviton emission and fringe disappearance

Alice's check of coherence after recombination can be done in more than one way, if this is done interferometrically, then the interference pattern would be observed over repeated runs of the experiment, and the emergence of fringes or absence thereof would account for the maintained or lost coherence of the state. It is clear that if the fringe spacing were to shrink below the Planck length l_p, the visibility of the interference pattern is to be considered lost. Following [5], from geometric optics considerations, the fringe spacing is given by

$$\delta = \lambda \frac{L}{d},$$

where L is the distance traveled to the detection screen on which the recombination pattern is checked. This can be written as $L = vT_A$, where v is the velocity of the particle and $\lambda = \frac{h}{p} = \frac{h}{m_A v}$ is the de Broglie wavelength of the particle. Hence, requiring δ to be greater than the Planck length we obtain

$$\delta = \frac{hT_A}{m_A d} \sim l_p \frac{m_p}{m_A} \frac{cT_A}{d} > l_p \longrightarrow cT_A > \frac{m_A}{m_p} d. \tag{2.9}$$

Remarkable! This is equivalent to condition (2.8), which we found for the onset of graviton emission. This formula has been given in [5], where the authors conclude, however, that fringe disappearance is a stronger condition than emission for the avoidance of the paradox. The reason is that the configuration considered in our calculation for the multipole moments is more general than that considered in [5] [14]. Here, we found that as long as the experiment is done through interferometric means, the two conditions are the same (in the maximal case $2x_m = T_A$, the fringe condition is stronger otherwise). This is quite intriguing: In this particular setting, the experiment shows that the loss of coherence could be explained both through quantized gravitational radiation and fringe disappearance from ideal limit spatial resolution. They set in concomitantly in producing decoherence. In the next section, we will go into further detail about another possible procedure for Alice to check the coherence of her state: namely a measure of spin, as suggested in [12]. This excludes the previous argument, as there are no fringes to be observed. Yet, is there some argument bringing the finite minimal length scale l_p to still resolve the paradox as quantized radiation does?

Lastly, in view of a relativistic discussion in a later section, we note that, from (2.9), the condition for fringe disappearance becomes

$$\frac{m_A}{m_p} > \frac{cT_A}{d} > 1,$$

where it is clear that masses below the Planck mass violate it, this means the fringe spacing will never go below the Planck length for such masses. Also, as we approach the Planck mass, the velocity of the particle in question becomes (strongly) relativistic. The

[14] It would be, though, in the very specific case in which we prepared the state such that $x_m = 0$ after recombination. This case, however, appears to be too specific after all, to the point that it might be questioned whether it could be prepared at all.

de Broglie wavelength for a relativistic particle is $\lambda_R = \frac{h}{m\gamma v}$. After some manipulation, the previous condition generalizes to

$$\gamma\frac{m_A}{m_p} > \frac{cT_A}{d} > 1,$$

which can be satisfied for any value of mass m_A given the right velocity. We expect, therefore, that our relativistic calculations will exhibit a threshold behavior at the Planck momentum scale, but not necessarily at the Planck mass scale only.

3 Minimum length and the q-metric

We have had a hint at how a minimum length, be it from vacuum fluctuations of a linearized quantum gravity, or any other more or less complex theory, can render the interference pattern undetectable even with the most ideal measurement conditions imaginable, i.e. one with a resolution up to this minimum length (presumably of the order of the Planck length l_p).
The background on which most theories are based is the metric of spacetime and this metric is what defines distances between events. One way to implement this concept of a minimum discernible distance, independently of the model taken to infer its existence, is to construct a metric with the property of rendering two events indiscernible once their classically expected separation reaches the aforementioned lower bound. Hence, what we are describing, is a metric that would coincide with the usual metric as the separation between events grows large, while strongly deviating from it once the limiting lower bound on geodesic distances is approached. This has been achieved, in its latest development, in the work [13] for time-like and space-like separations. We describe this work briefly here as an introduction, leaving a detailed derivation in the appendix.

To build such a metric, mathematical objects called bitensors come in handy: These are objects with two indices, referring to two distinct points, that transform as tensors in each index separately. Specifically, we make use of the Synge world function, defined as half the squared geodesic length between the two points

$$\Omega = \frac{1}{2}\sigma^2$$

In a convex spacetime, meaning one where geodesics between any two points are unique, this quantity completely characterizes the metric of this spacetime. [15]
Then, modifying the world function into a modified version of itself, having the desired limit at coincidence is what we are after:

$$\Omega \longrightarrow S(\Omega) \qquad \text{such that} \qquad \lim_{\Omega\to 0} S = \frac{1}{2}\epsilon l_0^2.$$

[15]See section on coincidence limits in appendix A.

Here, we called the minimum length a generic l_0, and $\epsilon = g_{\mu\nu}t^\mu t^\nu$ (t^μ is the tangent to the geodesic) is the signature of the geodesic characterizing its spacelike or timelike nature. We do assume that the modified world function only depends on the original world function.
The requirement that distances on large scales are unaffected is simply $\lim_{\Omega\to\infty} S(\Omega) = \Omega$.
The world function's defining equation is sometimes referred to as its Hamilton-Jacobi equation and reads

$$g^{\mu\nu}\Omega_\mu\Omega_\nu = 2\Omega,$$

To look for a modified metric $q_{\mu\nu}$ with the properties required above for the modified geodesic distance, we require the following to hold:

$$q^{\mu\nu}S_\mu S_\nu = 2S. \tag{3.1}$$

Previous work on the development of the q-metric (a summary can be found in the appendix) has suggested for it a form known in the literature as a disformal transformation of the classical metric, i.e.

$$q^{\mu\nu} = A(\Omega)g^{\mu\nu} + \epsilon B(\Omega)t^\mu t^\nu,$$

where $t^\mu = \frac{dx^\mu}{d\sigma}$. Condition (3.1) yields

$$A + B = \Omega\frac{S'^2}{S}.$$

Since the functions defining the new metric are two, we will need a second condition to fully specify it. This requirement comes from imposing that the coincidence limit of Green's functions of free relativistic particles be modified from being of order $\sim \frac{1}{\sigma^2}$, to $\sim \frac{1}{\sigma^2+l_0^2}$. In other words, we require that the modified Green's function $G_q(\Omega) = G_g(S(\Omega))$ be a solution to

$$\Box_q G_q(\Omega) = 0 \qquad \text{given} \qquad \Box_g G_g(\Omega) = 0,$$

where $\Box_q$ is the D'Alembertian derived from the q-metric. We require this in all maximally symmetric spacetimes, i.e. the ones in which we know the Green's function to depend only on the squared geodesic distance.
In general d-dimensional spacetime, one finds:

$$B = \Omega\frac{S'^2}{S} - \frac{S}{\Omega}\left(\frac{\bar{\Delta}}{\Delta}\right)^{-\frac{2}{d-1}},$$

where we introduced the Van Vleck-Morette Δ determinant and its modified equivalent $\bar{\Delta}$ (see appendix A).
The q-metric is then finally complete:

$$q_{\mu\nu} = \frac{S}{\Omega}\left(\frac{\bar{\Delta}}{\Delta}\right)^{-\frac{2}{d-1}} g_{\mu\nu} + \epsilon\left[\Omega\frac{S'^2}{S} - \frac{S}{\Omega}\left(\frac{\bar{\Delta}}{\Delta}\right)^{-\frac{2}{d-1}}\right] t_\mu t_\nu,$$

and its inverse, such that $q_{\mu\alpha}q^{\alpha\nu} = \delta_\mu^\nu$, reads

$$q^{\mu\nu} = \frac{\Omega}{S}\left(\frac{\bar{\Delta}}{\Delta}\right)^{\frac{2}{d-1}} g^{\mu\nu} + \epsilon\left[\frac{1}{\Omega}\frac{S}{S'^2} - \frac{\Omega}{S}\left(\frac{\bar{\Delta}}{\Delta}\right)^{\frac{2}{d-1}}\right] t^\mu t^\nu.$$

Given the abrupt discontinuity from $-l_0$ to l_0 (or vice versa, depending on conventions used) in passing from the coincidence limit of a spacelike geodesic to that of a timelike geodesic, geodesics on the light cone (i.e. null geodesics) require a separate treatment. Since we will not use them in this work. The form of this metric for light-like separations is [14]

$$q_{\mu\nu} = \frac{\tilde{\lambda}^2}{\lambda^2}\left(\frac{\bar{\Delta}}{\Delta}\right)^{-\frac{2}{d-2}} g_{\mu\nu} - \left[\frac{d\tilde{\lambda}}{d\lambda} - \frac{\tilde{\lambda}^2}{\lambda^2}\left(\frac{\bar{\Delta}}{\Delta}\right)^{-\frac{2}{d-2}}\right](l_\mu m_\nu + m_\mu l_\nu)$$

This result is not obtained by modifying the geodesic length itself, like for spacelike/timelike intervals, because as we know lightlike geodesics have zero proper distance along them, in any case. What is modified is the distance along the null geodesic as measured by a canonical observer. This distance defines an affine parameter λ, and we map it to a modified affine parameter $\tilde{\lambda} = \tilde{\lambda}(\lambda)$, such that $\tilde{\lambda} \to L_0$ when $\lambda \to 0$. The two null vectors used for the construction, which enter the final form of the metric, are $l^\mu = \frac{dx^\mu}{d\lambda}$ and $m^\mu = v^\mu - \frac{1}{2}l^\mu$, with v the velocity of a canonical observer such that $v_\mu l^\mu = 1$.
The consequences of this metric on areas and volumes have been explored in [15], where A. Perri finds, in particular, for null surface elements a finite area value in the coincidence limit around a base point. This limiting area value being of $4\pi l_0^2$. This result is particularly relevant for the study of black hole horizon area variations, since one can conclude that these variations must come in discrete value according to this bound.

One may check that geodesics distances are indeed modified, e.g. for timelike separations, using the fact that

$$t_\mu dx^\mu = t_\mu t^\mu d\sigma = \epsilon d\sigma$$

σ being geodesic distance. We can compute

$$\begin{aligned} d\sigma_q^2 = q_{\mu\nu}dx^\mu dx^\nu &= A\eta_{\mu\nu}dx^\mu dx^\nu + \epsilon B t_\mu t_\nu dx^\mu dx^\nu \\ &= (A + \epsilon B)d\sigma^2 \end{aligned} \tag{3.2}$$

We choose to use the mostly-minus convention ($\eta_{\mu\nu} = diag(1, -1, ...)$) so that for a timelike trajectory $\epsilon = 1$ and, therefore, $A + \epsilon B = \Omega\frac{S'^2}{S}$. Now, from (3.2), we can integrate to find the modified spacetime interval lengths:

$$\int d\sigma_q = \int \sqrt{\Omega\frac{S'^2}{S}}d\sigma = \int \Omega\frac{S'^2}{S}\frac{d\Omega}{\sqrt{2\Omega}} = \int \frac{S'^2(\Omega)}{2S(\Omega)}d\Omega = \sqrt{2S},$$

where we used the fact that $\Omega = \frac{1}{2}\sigma^2$ and that, by definition, $\sqrt{2S} = s$ with s the modified geodesic length, whose main property is the fact that $\lim_{\sigma\to 0} s = L_0$.

We will make use of this property of geodesic distances in the q-metric, as our objective is to determine the effects of this minimum finite length on GIE (gravitationally induced entanglement) experiments, and specifically on the maintenance of spin coherence during recombination. We hope to extend the results from the interference fringe spacing, described in the previous section, to cases where no such fringes are present, while a measurement of spin is performed instead.

4 Spin coherence and the Humpty Dumpty effect

The problem of splitting a beam of particles, polarized in a definite spin state, into a superposition of macroscopically spatially separated beams using a Stern-Gerlach apparatus has been studied before in a set of three papers by Englert, Schwinger, and Scully [16] [6] [17]. We quickly review their general treatment of this effect, since it will bear a strong resemblance to our results in the previous sections.

A particle with mass m and magnetic moment $\vec{\mu} = \gamma 2\hat{\vec{S}} = \gamma\vec{\sigma}$ (where $\vec{\sigma}$ is the vector of Pauli matrices), moving in a static inhomogeneous magnetic field $\vec{B}(r)$, obeys a Hamiltonian of the form

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} - \vec{\mu}\cdot\vec{B}(r),$$

and the Heisenberg equations of motion follow simply

$$\begin{aligned} \dot{\vec{r}}(t) &= \frac{\vec{p}(t)}{m} \\ \dot{\vec{p}}(t) &= \nabla(\vec{\mu}(t)\cdot\vec{B}(r(t)) = \gamma\vec{\nabla}\vec{\sigma}\cdot\vec{B}(r(t)). \end{aligned} \tag{4.1}$$

We will use a second quantization formalism for the spin operators. The advantage of this will be clear soon. The single particle spin operator can be written in second quantization formalism in terms of the annihilation and creation operators as follows:

$$\hat{\vec{S}}(t) = \frac{1}{2}\hat{a}^\dagger(t)\vec{\sigma}\hat{a}(t) \qquad or \qquad S^k(t) = \frac{1}{2}\sum_{i,j=\pm}\sigma^k_{ij}a^\dagger_i a_j,$$

where the annihilation and creation operators a and $a^\dagger$ follow the usual commutation relations

$$[a_i, a^\dagger_j] = \delta_{ij}, \qquad\qquad [a_i, a_j] = [a^\dagger_i, a^\dagger_j] = 0.$$

The time evolution of an operator in the Heisenberg picture is

$$\frac{d\hat{O}(t)}{dt} = i[\hat{H}, \hat{O}(t)] + \frac{\partial\hat{O}(t)}{\partial t}.$$

Consequently, we may calculate the time variation of the creation and annihilation operators:

$$\frac{d\hat{a}(t)}{dt} = i\left[-\gamma\hat{a}^\dagger\vec{\sigma}\cdot\vec{B}(\vec{r})\hat{a},\hat{a}\right]$$
$$= -i\gamma[\hat{a}^\dagger,\hat{a}]\vec{\sigma}\cdot\vec{B}(\vec{r})\hat{a}$$
$$= i\gamma\vec{\sigma}\cdot\vec{B}(\vec{r})\hat{a}, \quad (4.2)$$

and, similarly, one derives the conjugate equation

$$\frac{d\hat{a}^\dagger(t)}{dt} = -i\gamma\hat{a}^\dagger\vec{\sigma}\cdot\vec{B}(\vec{r}).$$

It is easily verified that

$$\frac{d(\hat{a}^\dagger(t)\hat{a}(t))}{dt} = 0,$$

at any time, meaning the number of spin $\frac{1}{2}$ is conserved, which will be important later. Meanwhile for the spatial variables, using (4.1), and writing the initial values $\vec{r}(0) = \vec{r}_0$ and $\vec{p}(0) = \vec{p}_0$:

$$\vec{p}(t) = \vec{p}_0 + \int_0^t dt'\gamma\vec{\nabla}\left[\vec{B}(\vec{r}(t'))\cdot\hat{a}^\dagger(t')\hat{\sigma}\hat{a}(t')\right],$$

$$\vec{r}(t) = \vec{r}_0 + \frac{1}{m}\int_0^t dt''\vec{p}(t'')$$
$$= \vec{r}_0 + \frac{\vec{p}_0}{m}t + \int_0^t dt''\int_0^{t''} dt'\gamma\vec{\nabla}\left[\vec{B}(\vec{r}(t'))\cdot\hat{a}^\dagger(t')\hat{\sigma}\hat{a}(t')\right], \quad (4.3)$$

and by careful analysis of the integration region in the plane $t' - t''$, we can use the following simplification of the integrals

$$\int_0^t dt'\int_0^{t'} dt'' f(t'') = \int_0^t dt''\int_{t''}^t dt' f(t'') = \int_0^t dt'(t-t')f(t').$$

Then, (4.3) becomes

$$\vec{r}(t) = \vec{r}_0 + \frac{\vec{p}(t)}{m}t - \frac{1}{m}\int_0^t dt'\gamma\vec{\nabla}\left[\vec{B}(\vec{r}(t'))\cdot\hat{a}^\dagger(t')\hat{\sigma}\hat{a}(t')\right]t'.$$

Consider now a single spin $\frac{1}{2}$ state $|\psi\rangle$, such that

$$\langle\psi|\,\vec{S}(0)\,|\psi\rangle = \frac{1}{2}\langle\psi|\,\hat{a}^\dagger(0)\vec{\sigma}\hat{a}(0)\,|\psi\rangle = \frac{1}{2}.$$

We want to calculate the quantity

$$\langle\psi|\,\vec{S}(t)\,|\psi\rangle = \frac{1}{2}\,\langle\psi|\,\hat{a}^\dagger(t)\vec{\sigma}\hat{a}(t)\,|\psi\rangle\,,$$

at a time t after the particle has exited the Stern-Gerlach interferometer. To do this, we apply (4.2) to this state:

$$\begin{aligned}\frac{d\hat{a}(t)}{dt}\,|\psi\rangle &= i\gamma\vec{\sigma}\cdot\vec{B}(\vec{r}(t))\hat{a}(t)\,|\psi\rangle\\ &= i\gamma\vec{\sigma}\cdot\vec{B}\left(\vec{r}_0+\frac{\vec{p}_0}{m}t+\frac{1}{m}\int_0^t dt'\gamma(t-t')\vec{\nabla}\left[\vec{B}(\vec{r}(t'))\cdot\hat{a}^\dagger(t')\hat{\sigma}\hat{a}(t')\right]\right)\hat{a}(t)\,|\psi\rangle\\ &= i\gamma\vec{\sigma}\cdot\vec{B}\left(\vec{r}_0+\frac{\vec{p}_0}{m}t\right)\hat{a}(t)\,|\psi\rangle\,,\end{aligned}$$

where the last step used the fact that, being $|\psi\rangle$ a single spin $\frac{1}{2}$ state, $\hat{a}(t)\,|\psi\rangle$ will be a null spin $\frac{1}{2}$ state. Consequently, any further application of $\hat{a}(t)$ from the field $\vec{B}$ will be zero. This is the advantage of the second quantization formalism.
The next step is separating the scales of the problem into macro- and microscopic so that we can expand the field in these distinct scales. Firstly, we write

$$\vec{r}_0+\frac{\vec{p}_0}{m}t = \langle\vec{r}_0\rangle+\frac{\langle\vec{p}_0\rangle}{m}t+(\vec{r}_0-\langle\vec{r}_0\rangle)+\frac{\vec{p}_0-\langle\vec{p}_0\rangle}{m}t,$$

where terms in angled brackets represent macroscopic averages, while the other terms are the microscopic fluctuations. Now the field can be expanded as

$$\begin{aligned}\vec{B}\left(\vec{r}_0+\frac{\vec{p}_0}{m}t\right) &\approx \vec{B}\left(\langle\vec{r}_0\rangle+\frac{\langle\vec{p}_0\rangle}{m}t\right)+\left((\vec{r}_0-\langle\vec{r}_0\rangle)+\frac{\vec{p}_0-\langle\vec{p}_0\rangle}{m}t\right)\vec{\nabla}\vec{B}\left(\langle\vec{r}_0\rangle+\frac{\langle\vec{p}_0\rangle}{m}t\right)\\ &= \vec{B}(t)+\left((\vec{r}_0-\langle\vec{r}_0\rangle)+\frac{\vec{p}_0-\langle\vec{p}_0\rangle}{m}t\right)\vec{\nabla}\vec{B}(t).\end{aligned}$$

Finally, the $\hat{a}$ equation of motion becomes

$$\frac{d\hat{a}(t)}{dt}\,|\psi\rangle = \left[i\gamma\vec{\sigma}\cdot\vec{B}(t)+i\gamma\left((\vec{r}_0-\langle\vec{r}_0\rangle)+\frac{\vec{p}_0-\langle\vec{p}_0\rangle}{m}t\right)\vec{\nabla}\vec{B}(t)\vec{\sigma}\right]\hat{a}(t)\,|\psi\rangle\,,$$

and the conjugate equation is easy to guess.
Let us now choose $\vec{B}$ such that only the x component $B_x = B$ plays a role here[16]. Let

[16] One will, rightfully, inquire about the property of magnetic fields: $\nabla\cdot B = 0$. This approximation is justified by moving the y-dependence to time dependence, meaning $\frac{\partial}{\partial y} = \frac{1}{v}\frac{\partial}{\partial t}$. Given the motion is considered constant and along y. Plus other careful considerations. For more details see [6] and [18].

us further set $\langle x\rangle = \langle p_x\rangle = 0$, and denote $\frac{\partial B_x}{\partial x} = B'$ and $p_x = p$. Then the differential equation above simplifies to (we also remove the subscripts indicating initial values):

$$\frac{d\hat{a}(t)}{dt}|\psi\rangle = i\gamma\sigma_x \cdot \Big[B(t) + \Big(x + \frac{p}{m}t\Big) B'(t)\Big]\, \hat{a}(t)\, |\psi\rangle\,.$$

This can be integrated, with the only problem being the fact that the operator on the right-hand side doesn't commute with itself. This is, luckily, not a problem since the commutator of x and p_x results at most in a constant phase factor, which will cancel in any expectation value we take. We thus ignore it, to yield the following solution:

$$\hat{a}(t) = U(\sigma_x)\hat{a}(0), \qquad U(\sigma_x) = \exp\{i\sigma_x\varphi\}\exp\Big\{i\sigma_x\Big[x\Delta p - p\Delta x + p\frac{\Delta p t}{m}\Big]\Big\},$$

where

$$\begin{aligned} \varphi &= \gamma B(t), \\ \Delta p &= \gamma\int_0^t dt' B'(t'), \\ \Delta x &= \frac{\gamma}{m}\int_0^t dt'(t-t')B'(t'). \end{aligned} \tag{4.4}$$

$U(\sigma_x)$ is unitary, i.e. $U^\dagger(\sigma_x) = U^{-1}(\sigma_x) = U(-\sigma_x)$. We can now find the effects of the apparatus on spin states:
The most obvious effect comes from measuring the x-component of spin:

$$\begin{aligned} \langle\psi|\,\hat{S}_x(t)\,|\psi\rangle &= \langle\psi|\,\hat{a}^\dagger(t)\frac{\hat{\sigma}_x}{2}\hat{a}(t)\,|\psi\rangle \\ &= \langle\psi|\,\hat{a}^\dagger(0)U^{-1}(\sigma_x)\frac{\hat{\sigma}_x}{2}U(\sigma_x)\hat{a}(0)\,|\psi\rangle \\ &= \langle\psi|\,\hat{a}^\dagger(0)\frac{\hat{\sigma}_x}{2}\hat{a}(0)\,|\psi\rangle \\ &= \langle\psi|\,\hat{S}_x(0)\,|\psi\rangle\,. \end{aligned}$$

Next, we will measure the spin along the z-axis. For this purpose, let us show that for a generic operator $\hat{O}$ that commutes with $\vec{\sigma}$[17]

$$\exp\Big\{i\sigma_x\hat{O}\Big\}\sigma_z = \sigma_z\sigma_z\exp\Big\{i\sigma_x\hat{O}\Big\}\sigma_z = \sigma_z\exp\Big\{-i\sigma_x\hat{O}\Big\},$$

[17] This can be derived by expanding the exponential in its series form and using the fact that $\sigma_i^{2n} = 1$ and therefore $\sigma_i^{2n+1} = \sigma_i$.

then

$$
\begin{aligned}
\langle\psi|\,\hat{S}_z(t)\,|\psi\rangle &= \langle\psi|\,\hat{a}^\dagger(t)\frac{\hat{\sigma}_z}{2}\hat{a}(t)\,|\psi\rangle \\
&= \frac{1}{2}\,\langle\psi|\,\hat{a}^\dagger(0)U^{-1}(\sigma_x)\hat{\sigma}_z U(\sigma_x)\hat{a}(0)\,|\psi\rangle \\
&= \frac{1}{4}\,\langle\psi|\,\hat{a}^\dagger(0)\hat{\sigma}_z U(2\sigma_x)\Big[\,|{\leftarrow}\rangle\,\langle{\leftarrow}|+|{\rightarrow}\rangle\,\langle{\rightarrow}|\,\Big]\hat{a}(0)\,|\psi\rangle \\
&= \frac{1}{4}\,\langle\psi|\,\hat{a}^\dagger(0)\hat{\sigma}_z\Big[\,|{\leftarrow}\rangle\,U(-2)\,\langle{\leftarrow}|+|{\rightarrow}\rangle\,U(2)\,\langle{\rightarrow}|\,\Big]\hat{a}(0)\,|\psi\rangle \\
&= \frac{1}{4}\,\langle\psi|\,\hat{a}^\dagger(0)\Big[\,|{\rightarrow}\rangle\,U(-2)\,\langle{\leftarrow}|+|{\leftarrow}\rangle\,U(2)\,\langle{\rightarrow}|\,\Big]\hat{a}(0)\,|\psi\rangle \\
&= \frac{1}{4}\Big[\,\langle\psi|\,\hat{a}^\dagger(0)\,|{\leftarrow}\rangle\,U(2)\,\langle{\rightarrow}|\,\hat{a}(0)\,|\psi\rangle+\langle\psi|\,\hat{a}^\dagger(0)\,|{\rightarrow}\rangle\,U(-2)\,\langle{\leftarrow}|\,\hat{a}(0)\,|\psi\rangle\,\Big] \\
&= \frac{1}{2}\,\mathrm{Re}\big\{\langle\psi|\,\hat{a}^\dagger(0)\,|{\leftarrow}\rangle\,U(2)\,\langle{\rightarrow}|\,\hat{a}(0)\,|\psi\rangle\big\} \\
&= \frac{1}{2}\,\mathrm{Re}\bigg\{\int\psi^*_{\leftarrow}(x)\,\langle x|\,U(2)\,|x\rangle\,\psi_{\rightarrow}(x)dx\bigg\} \\
&= \frac{1}{2}\,\mathrm{Re}\bigg\{\int\psi^*_{\leftarrow}(p)\,\langle p|\,U(2)\,|p\rangle\,\psi_{\rightarrow}(p)dp\bigg\},
\end{aligned}
\tag{4.5}
$$

where, in the third step, we inserted the completeness relation in the spin-x basis

$$
\frac{1}{2}\Big[\,|{\leftarrow}\rangle\,\langle{\leftarrow}|+|{\rightarrow}\rangle\,\langle{\rightarrow}|\,\Big]=\mathbb{I},
$$

which is composed of eigenstates of σ_x.
Using the Zassenhaus formula, one can rearrange the evolution operator in the following way:

$$
\begin{aligned}
U(2) &= \exp\{2i\varphi\}\exp\bigg\{2i\left[x\Delta p-p\Delta x+p\frac{\Delta pt}{m}\right]\bigg\} \\
&= e^{2i\varphi}e^{ix\Delta p}e^{-2i\left(\Delta x-\frac{\Delta p}{m}t\right)p}e^{ix\Delta p}.
\end{aligned}
$$

Taking the inner product of this operator with the left and right components of the initial spin state, as derived in (4.5), we see this operator corresponds to a chain of known operations:

$$
\begin{aligned}
\langle\psi|\,\hat{S}_z(t)\,|\psi\rangle &= \frac{1}{2}\,\mathrm{Re}\bigg\{e^{2i\varphi}\int\psi^*_{\leftarrow}(p)e^{ix\Delta p}e^{-2i\left(\Delta x-\frac{\Delta p}{m}t\right)p}e^{ix\Delta p}\psi_{\rightarrow}(p)dp\bigg\} \\
&= \frac{1}{2}\,\mathrm{Re}\bigg\{\int\psi^*_{\leftarrow}(p+\Delta p)e^{-2i\left(\Delta x-\frac{\Delta p}{m}t\right)p}\psi_{\rightarrow}(p-\Delta p)dp\bigg\}\cos 2\varphi \qquad (4.6) \\
&= \frac{1}{2}\,\mathrm{Re}\bigg\{\int\Big(e^{ip\left(\Delta x-\frac{\Delta p}{m}t\right)}\psi_{\leftarrow}(p+\Delta p)\Big)^*e^{-ip\left(\Delta x-\frac{\Delta p}{m}t\right)}\psi_{\rightarrow}(p-\Delta p)dp\bigg\}\cos 2\varphi,
\end{aligned}
$$

where we first recognized the boost operators and then the shift operators. Another equivalent way to express this result in position space is

$$\langle\psi|\,\hat{S}_z(t)\,|\psi\rangle = \\ = \frac{1}{2}\,\mathrm{Re}\bigg\{\int \psi^*_\leftarrow(x + \Big(\Delta x - \frac{\Delta p}{m}t\Big))e^{2ix\Delta p}\psi_\rightarrow(x - \Big(\Delta x - \frac{\Delta p}{m}t\Big))dx\bigg\}\cos 2\varphi.$$

Repeating the process for the expectation value of S_y produces a similar result, with the imaginary part in place of the real part (and a sine in place of the cosine as a consequence).
This result depends only on the initial state, and the degree to which spin coherence can be maintained depends on the magnitude of the total spin vector expectation value $\langle\psi|\,\vec{S}(t)\,|\psi\rangle$.
Let us make a specific example, choosing a minimum uncertainty (Gaussian) initial state, polarized along the positive spin-z-axis:

$$\langle p|\psi_0\rangle = A\exp\{-\alpha p^2\}\begin{pmatrix}1\\0\end{pmatrix}. \tag{4.7}$$

Whose projections along the positive and negative x-spin axis are easy to find.[18]
Applying result (4.6) to this state results in no less than

$$\langle\psi|\,\hat{S}_z(t)\,|\psi\rangle = \frac{1}{2}\exp\bigg\{-\frac{(\Delta x - \frac{\Delta p}{m}t)^2}{2\alpha} - 2\alpha(\Delta p)^2\bigg\}\cos 2\varphi. \tag{4.8}$$

This result serves to show how one can, at least in principle, use a Stern-Gerlach interferometer to split and recombine a beam of spin-polarized particles. The degree of coherence maintained during the recombination depends on the precision with which one can control the magnetic field and the magnetic field gradient of the apparatus. Further analysis on this can be found in the main article [6]. As for our purposes, we take this as a further justification of our previous results, and as a starting point for our discussions on the impact of a minimum length scale on the restoring of spin coherence.

5 Wavepacket approach

5.1 Non-relativistic Gaussian packets

Let us first describe the recombination process of a delocalized mass in the non-relativistic regime. From here on, we will ignore the presence of Bob's particle and focus on the

[18] Simply use $\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}$.

delocalized particle only. To do so, recall that by choosing a representation for the Pauli matrices such that σ_z is diagonal:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we can define the projectors along the spin z-axis as

$$\mathbb{P}_\uparrow = \frac{1+\sigma_z}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{P}_\downarrow = \frac{1-\sigma_z}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and along the spin x-axis

$$\mathbb{P}_\rightarrow = \frac{1+\sigma_x}{2} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbb{P}_\leftarrow = \frac{1-\sigma_x}{2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Here, $\uparrow$ ($\downarrow$) represents positive (negative) spin along z, while $\rightarrow$ ($\leftarrow$) represents positive (negative) spin along x. The Pauli matrices are related to the spin operator matrices by the relation $S_i = \frac{\hbar}{2}\sigma_i$. In this basis, the Pauli spinors for positive and negative z-axis spin are

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and using the projectors, we may decompose a positive spin state in the z direction in terms of x-axis Pauli spinors:

$$\mathbb{P}_\rightarrow |\uparrow\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|\rightarrow\rangle,$$

and, similarly,

$$\mathbb{P}_\leftarrow |\uparrow\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}|\leftarrow\rangle.$$

So that the properly normalized positive and negative x-oriented spin states are

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |\leftarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, the sought-after decomposition takes the form

$$\mathbb{P}_\rightarrow |\uparrow\rangle + \mathbb{P}_\leftarrow |\uparrow\rangle = (\mathbb{P}_\rightarrow + \mathbb{P}_\leftarrow)|\uparrow\rangle = |\uparrow\rangle = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}(|\rightarrow\rangle + |\leftarrow\rangle),$$

and, similarly, for a z-axis negative spin state

$$|\downarrow\rangle = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}(|\rightarrow\rangle - |\leftarrow\rangle).$$

This said, imagine we had a localized state, represented by a Gaussian wavepacket with a positive (negative) spin along the z direction: we can write such a state as

$$\langle x|\psi_I\rangle |\uparrow (\downarrow)\rangle = A \exp\left(ik_0 x - \frac{x^2}{4\alpha}\right)|\uparrow (\downarrow)\rangle .$$

where $A = \left(\frac{1}{2\pi\alpha}\right)^{\frac{1}{4}}$ and k_0 represents the initial wavenumber (momentum) of the packet. The Hilbert space containing this state can be written as a tensor product of the spatial Hilbert space of square-integrable functions and the spin Hilbert space:

$$\mathcal{H} = \mathcal{H}_{space} \otimes \mathcal{H}_{spin}.$$

Upon sending this state through an x-oriented Stern-Gerlach apparatus, the x-directed spins that make up our spin-z eigenstate will delocalize, entangling with the spatial part of the wavefunction into two distinct (albeit the spatial parts, being Gaussian, will be slightly overlapping) Gaussian functions. We also impart each packet a momentum ($k_L, k_R \geq 0$), such that they move towards each other. For the most general case, we choose a state such as

$$|\psi_I\rangle |\uparrow (\downarrow)\rangle \longrightarrow \left|\psi_F^{\pm}\right\rangle = |\psi_L\rangle |\rightarrow\rangle \pm |\psi_R\rangle |\leftarrow\rangle ,$$

where we condensed both cases in the notation $|\uparrow (\downarrow)\rangle$. From here, the $+$ sign refers to an initially spin up $|\uparrow\rangle$ polarized state, while the $-$ sign to an initially spin down $|\downarrow\rangle$ polarized state. Explicitly:

$$\begin{aligned}\left\langle x\middle|\psi_F^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}\left[Ae^{ik_L(x+d_L)-\frac{(x+d_L)^2}{4\alpha}}|\rightarrow\rangle \pm Ae^{i\varphi-ik_R(x-d_R)-\frac{(x-d_R)^2}{4\alpha}}|\leftarrow\rangle\right]\\ &= \frac{1}{2}\left[Ae^{ik_L(x+d_L)-\frac{(x+d_L)^2}{4\alpha}}\begin{pmatrix}1\\1\end{pmatrix} \pm Ae^{i\varphi-ik_R(x-d_R)-\frac{(x-d_R)^2}{4\alpha}}\begin{pmatrix}1\\-1\end{pmatrix}\right],\end{aligned}$$

where we also added a relative phase φ between the two packets for later use. Its conjugate transpose is

$$\left\langle\psi_F^{\pm}\middle|x\right\rangle = \frac{1}{2}\left[Ae^{-ik_L(x+d_L)-\frac{(x+d_L)^2}{4\alpha}}\begin{pmatrix}1 & 1\end{pmatrix} \pm Ae^{-i\varphi+ik_R(x-d_R)-\frac{(x-d_R)^2}{4\alpha}}\begin{pmatrix}1 & -1\end{pmatrix}\right].$$

Upon applying the spin operator, which in matrix form reads

$$\hat{S}_z = \frac{1}{2}\sigma_z = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},$$

to our ket, its effect is flipping each spin state and adding a multiplicative factor of $\frac{\hbar}{2}$ ($\frac{1}{2}$ in our units). We then obtain

$$\hat{S}_z\left|\psi_F^{\pm}\right\rangle = \frac{1}{4}\left[Ae^{ik_L(x+d_L)-\frac{(x+d_L)^2}{4\alpha}}\begin{pmatrix}1\\-1\end{pmatrix} \pm Ae^{i\varphi-ik_R(x-d_R)-\frac{(x-d_R)^2}{4\alpha}}\begin{pmatrix}1\\1\end{pmatrix}\right].$$

We can now calculate the expectation value of $\hat{S}_z$ by taking the internal product of these last two expressions. Notice only terms mixing right and left components survive the vector products. Leaving us with

$$\langle \psi_F^\pm | \hat{S}_z | \psi_F^\pm \rangle = \pm\frac{1}{2} \left(\langle \psi_R | \psi_L \rangle + \langle \psi_L | \psi_R \rangle \right) = \pm \operatorname{Re} \left\{ \langle \psi_R | \psi_L \rangle \right\},$$

which is, explicitly,

$$\langle \hat{S}_z \rangle = \pm\frac{1}{2} \frac{1}{\sqrt{2\pi\alpha}} \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} \exp\Big[-\frac{(x+d_L)^2}{4\alpha} - \frac{(x-d_R)^2}{4\alpha} \right.$$
$$\left. - ik_L(x+d_L) - ik_R(x-d_R) + i\varphi \Big] \right\} dx.$$

The argument of the exponential:

$$-\frac{(x+d_L)^2}{4\alpha} - \frac{(x-d_R)^2}{4\alpha} - ik_L(x+d_L) - ik_R(x-d_R) + i\varphi,$$

can be expanded in powers of x as

$$-\frac{x^2}{2\alpha} - \left(\frac{d_L}{2\alpha} - \frac{d_R}{2\alpha} + i(k_L+k_R) \right) x - \left(\frac{d_L^2}{4\alpha} + \frac{d_R^2}{4\alpha} - ik_R d_R + ik_L d_L - i\varphi \right)$$
$$\equiv -(ax^2 + bx + c).$$

The integral of such a Gaussian function is well-known to be

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx+c)} dx = \sqrt{\frac{\pi}{a}} \exp\left\{ \frac{b^2}{4a} - c \right\}.$$

Putting everything together and using the fact that $e^{i\vartheta} + e^{-i\vartheta} = 2\cos(\vartheta)$, after some algebra, one obtains

$$\langle \hat{S}_z \rangle = \pm\frac{1}{2} \exp\left\{ -\frac{(d_L+d_R)^2}{8\alpha} - \frac{\alpha}{2}(k_L+k_R)^2 \right\} \times$$
$$\times \cos\left\{ -\frac{1}{2}(k_R-k_L)(d_L+d_R) - \varphi \right\}.$$

We recognize in this expression the relative distance between the peaks of the Gaussians $D \equiv d_L + d_R$ and the relative momentum $K \equiv k_L + k_R$. For the moment, let us set $\varphi = 0$. The result then becomes

$$\langle \hat{S}_z \rangle = \pm\frac{1}{2} \exp\left\{ -\frac{D^2}{8\alpha} - \frac{\alpha K^2}{2} \right\} \cos\left(-\frac{D}{2}(k_R - k_L) \right). \tag{5.1}$$

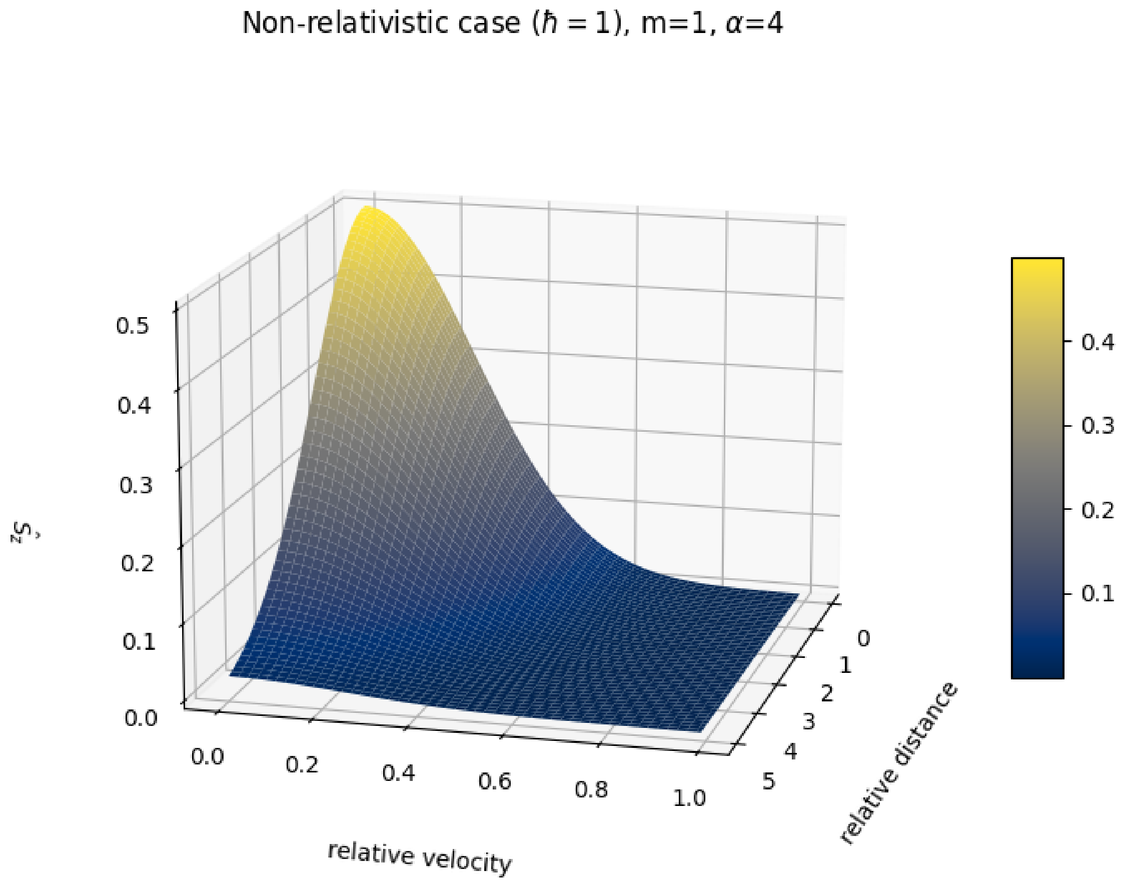

Figure 2: $\hat{S}_Z$ expectation value for non-relativistic motion in the symmetric case ($d_L = d_R = d$ and $k_L = k_R = k$). Then, relative velocity stands for $\frac{2k}{m}$ and relative distance stands for $2d$.

Here, we can see that there is an exponential damping of the initial state spin expectation value, due to the overlap in both coordinate and momentum space, and that only relative distance and motion play a role in this. The oscillating factors of cosines and sines can be made to vanish with a suitable choice of relative phase φ, rather than setting it to zero as we did. These oscillating terms are of little importance to our question, which is related mainly to the loss of spin coherence associated with the merging of the packets.

This result can be shown, by explicitly evaluating the dynamics of the superposed wave packets, to not depend on time. Or in other words, to hold at any point in time. This can be traced back to the property of the non-relativistic dispersion relation of not affecting the Gaussian form of the packets while they broaden in time. As well as the fact that the spin operators commute with the free Hamiltonian, and are therefore trivially conserved in time.

5.2 Spin coherence and graviton emission

Let us consider our latest result (5.1) under symmetric conditions, i.e. $d_L = d_R = d_0$ and $k_R = k_L = k_0$. For clarity, we define the relative distance $D_0 = 2d_0$ and relative momentum $K_0 = 2k_0$. This yields

$$|\langle \hat{S}_z \rangle| = \frac{1}{2} \exp\left\{ -\frac{D_0^2}{8\alpha} - \alpha \frac{K_0^2}{2} \right\}.$$

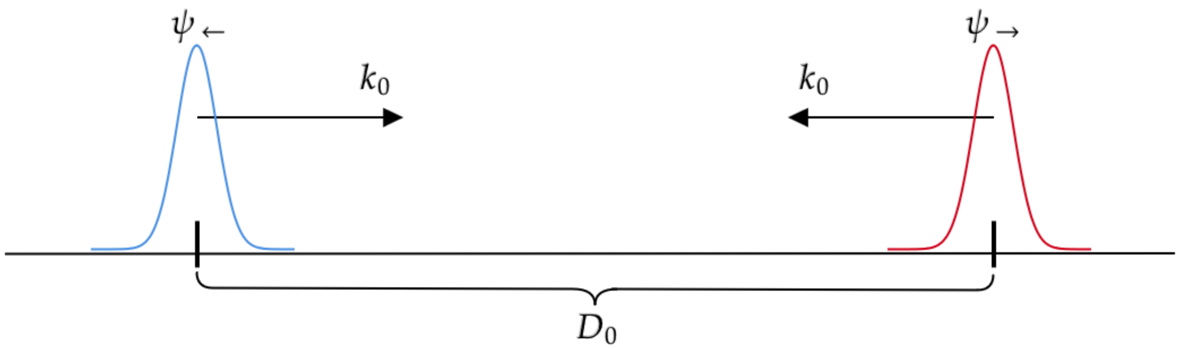

Figure 3: Schematic representation of the simplified case. We assume constant relative velocity and Gaussian shape for our packets.

This expression is very similar to the Humpty-Dumpty effect from chapter 4 (see, in particular, (4.8)). We are making the following simplifying assumptions, which we will discuss throughout this chapter:

1. At time $t = 0$ the state is delocalized, with a distance D_0 between the two components of the delocalized state. Recombination from the initial distance D_0 to the final distance l_0 (with $l_0 \ll D_0$) occurs in a time T_A.

2. The state at the time $t = T_A$ of recombination can be described as the superposition of two Gaussians, with relative distance l_0 and relative momentum K_0.

3. The relative momentum K_0 can be roughly estimated from the average relative velocity during recombination: $v = \frac{D_0}{T_A} \longrightarrow K_0 = m_A \frac{D_0}{T_A}$.

We will discuss the analogy with the Humpty-Dumpty effect in relation to our simplifying assumptions later on in this chapter. Let us check how this result holds up to the previous discussion concerning the GIE experiment and introduce the minimum length effect into it.
First of all, we take a snapshot of the packets at the time of recombination. Here, under the minimum length hypothesis, we can not have $D(t) \to 0$, as in the ordinary metric, but we have instead $D(t) \to l_0$, as contemplated in the qmetric. Using (5.1) with $k_R = k_L \approx K_0/2$ in the last instant, this gives

$$|\langle \hat{S}_z \rangle| = \frac{1}{2} \exp\left\{ -\frac{l_0^2}{8\alpha} - \alpha \frac{K_0^2}{2} \right\}. \tag{5.2}$$

Next, find the value of the Gaussian width α that maximizes $|\langle \hat{S}_z \rangle|$, which is straightforwardly the value such that

$$\frac{\partial}{\partial \alpha} \left(-\frac{l_0^2}{8\alpha} - \alpha \frac{K_0^2}{2} \right) = \frac{l_0^2}{8\alpha^2} - \frac{K_0^2}{2} = 0,$$

which is satisfied by $\alpha_0 = \frac{l_0}{2K_0}$.
Now, with this condition, the value for our spin is

$$|\langle \hat{S}_z \rangle_{\alpha_0}| = \frac{1}{2} \exp\left\{ -\frac{l_0 K_0}{2} \right\}.$$

Any other value of α implies greater decoherence.

Were we, instead, to send $D(t) \to 0$ (i.e. $l_0 = 0$), we would have no loss of coherence with this most favorable value for α_0, but we need to be careful since this also implies $\alpha_0 \to 0$. Going back to (5.2) we see, anyhow, that a reasonably small α gives $|\langle \hat{S}_z \rangle| \approx 1/2$. Here is where the minimum length condition, described with the backup of the q-metric, shows its effect.

Applying our third assumption 3, the spin expectation value may be written as

$$|\langle \hat{S}_z \rangle_{\alpha_0}| = \frac{1}{2} \exp\left\{ -l_0 \frac{m_A D_0}{2T_A} \right\} = \frac{\hbar}{2} \exp\left\{ -\frac{l_0}{2l_p} \frac{m_A}{m_p} \frac{D_0}{cT_A} \right\} = \frac{\hbar}{2} \exp\{-C\}, \tag{5.3}$$

where, in the last steps, we reintroduced factors of $\hbar$ and c, and subsequently factors of the Planck mass and length. Remembering the condition for graviton emission derived earlier in the context of the GIE experiment (2.8):

$$cT_A < \frac{m_A}{m_p} D_0 \longrightarrow \frac{m_A}{m_p} \frac{D_0}{cT_A} > 1,$$

we see that the same conditions for graviton emission are here conditions for which the spin coherence of a delocalized spin state is strongly suppressed in (5.3). Or, at the very least, coherence is not fully restored, even when considering the most optimal value for the position uncertainty parameter α.

The factor $\frac{l_0}{2l_p}$ is assumed to be of order ~ 1, then the argument of the exponential reduces to

$$C \propto \frac{m_A}{m_p}\frac{D_0}{cT_A} = \frac{m_A}{m_p}\frac{v_A}{c} < \frac{m_A}{m_p} \tag{5.4}$$

And if we set a threshold for the loss of spin coherence at a suppression of order $\sim \frac{1}{e}$, then this requires

$$C \gtrsim 1 \tag{5.5}$$

in case $m_A < m_p$ we have trivially from (5.4)

$$C \propto \frac{m_A}{m_p}\frac{v_A}{c} < \frac{m_A}{m_p} < 1$$

such that if m_A is lighter than the Planck mass, no suppression below $\frac{1}{e}$ is possible. This coincides with what we showed in the previous chapter, resulting in no graviton emission being possible under the same conditions. In addition, if we take $m_A > m_p$, in order to satisfy (5.5) we need:

$$\frac{m_A}{m_p}\frac{v_A}{c} \gtrsim 1 \longrightarrow v_A \gtrsim \frac{c}{\left(\frac{m_A}{m_p}\right)}$$

We can then conclude the following: below the Planck mass, no suppression of spin coherence can happen to the degree we talked about above. Above the Planck mass, our condition requires larger and larger velocities between the two packets the closer m_A is to m_p, while this requirement on velocity becomes smaller as $m_A \gg m_p$ (see figure 4). Interestingly, condition (5.5) can also be written as

$$C \propto \frac{k}{k_p} \gtrsim 1$$

where k is the relative momentum, while $k_p = m_p c \approx 6.5249\, kg\frac{m}{s}$ is the Planck momentum.

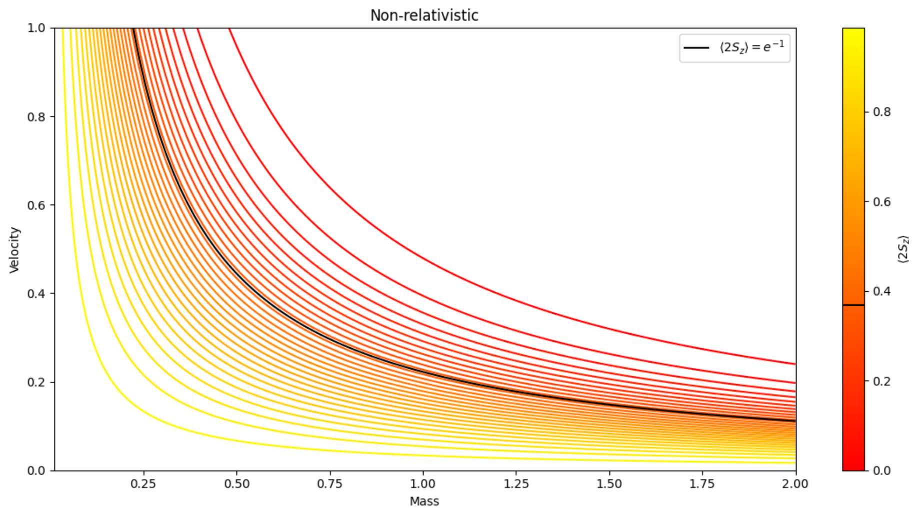

Figure 4: Contour plot of equation (5.3) as a function of particle mass and velocity. Notice decoherence below $\frac{1}{2e}$ does not set in below a value of the order of the Planck mass.

In section 2, we have invoked causality in the conditions for avoiding the paradox. If we were now to impose a condition such as $v < c$, we would see a lower bound on the spin value, which hardly makes sense. Furthermore, in the conditions for suppression of coherence just discussed, we have shown that as we go from below to above the Planck mass, the velocities required to bridge between these two regimes approach the speed of light, prompting a relativistic description. For this reason, in the next chapter, we will repeat the calculation using a relativistic generalization of our wavepackets, where we expect causality to emerge naturally. We will thus check how this result holds at the highest velocities that have been considered (which is any, as long as causality is respected).

5.3 Comparison with Humpty Dumpty

We wish to compare the simplified wavepacket approach of this chapter and the Humpty Dumpty effect from 4. To do so, in analogy with the gedankenexperiments discussed in 2, we need to describe the recombination of a delocalized state, split adiabatically in a distant past.

We begin with a split version of the state (4.7):

$$\psi(k) = \frac{A}{2} e^{-\alpha k^2 + ikd} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{A}{2} e^{-\alpha k^2 - ikd} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

describing the two packets with orthogonal spin being centered at $x = \pm d$.
Using the following projections:

$$\psi_{\leftarrow}(p) = Ae^{-\alpha k^2 + ikd}, \qquad \psi_{\rightarrow}(p) = Ae^{-\alpha k^2 - ikd},$$

where $|A|^2 = \sqrt{\frac{2\alpha}{\pi}}$.
We apply (4.6) to this state. After another straightforward Gaussian integral, we get the following time-dependent value for the spin coherence:

$$\langle \hat{S}_z \rangle = \frac{1}{2} \exp\left\{ -\frac{1}{2\alpha} \left(d + \Delta x - \frac{\Delta p}{m} t \right)^2 - 2\alpha (\Delta p)^2 \right\}.$$

As we know, Δx and Δp are functions of time t and of the magnetic field gradient $B'(t)$, as given in (4.4). Thus, we assume that with a careful choice of $B(t)$[19], one can drive the recombination with the assumptions of this chapter, i.e. with constant velocity. It would be appropriate to include a stopping phase for a realistic description of the dynamics of such an experiment.

6 Relativistic wavepackets

6.1 Lorentz-invariant Gaussian packet

Our objective now is to find a relativistic generalization for the calculation performed in the previous section.
We remind the reader that, throughout this work, we use the "mostly-minus" metric convention, i.e.

$$\eta_{\mu\nu} = diag(1, -1, -1, -1).$$

Starting with the usual 1-dimensional non-relativistic Gaussian wavepacket in the momentum representation:

$$\langle k|\psi\rangle = \left(\frac{2\alpha}{\pi} \right)^{\frac{1}{4}} \exp\left\{ -\alpha(k-p)^2 + ikd \right\}.$$

[19] An example of such a choice, though not the one we need for our purposes, is shown graphically in [16]. More appropriate conditions, specifically magnetic field gradient pulses for the splitting, reversing, and stopping of the beam, with free propagation between pulses, have been discussed in [19].

Note that we now refer to the mean momentum as p to later avoid confusion with the 0-th component of relativistic momentum.
This form already suggests a way of writing a relativistic version of it by generalizing the quantities involved to their Lorentz-invariant counterparts, i.e.

$$\langle k|d,p\rangle \propto \exp\{-\alpha(k+p)_\mu(k+p)^\mu + ik^\mu d_\mu\},$$

which, now, includes 3 spatial dimensions, the number of dimensions did not matter to the result in the non-relativistic case, as the axes orthogonal to the motion just integrate out but, as we will see, they do make a difference here.
The factors of $k^\mu + p^\mu$ may initially seem not intuitive; however, they can be shown to represent the relativistic, minimum-uncertainty, and Lorentz-covariant generalization of the Gaussian packet described above. We also adopted a new convention to name these Lorentz invariant states, labeling them by their central position d and momentum p, meanwhile $k^0 = E_k = \sqrt{\vec{k}^2 + m^2}$ and $p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$ are the relativistic energies, while the initial displacement vector d^μ and the momentum vectors k^μ are of the form

$$d^\mu = \begin{pmatrix} t_0(=0) \\ \vec{d} \end{pmatrix} \qquad\qquad k^\mu = \begin{pmatrix} E_k \\ \vec{k} \end{pmatrix},$$

as shown in parenthesis, the initial time displacement will be set to zero since it is of no use to our discussion. We can write this in a more compact form by using the following:

$$(k+p)_\mu(k+p)^\mu = (E_k + E_p)^2 - (\vec{k}+\vec{p})^2 = 2m^2 + 2k_\mu p^\mu,$$

and hence, the wavepacket will take the form

$$\langle k|d,p\rangle = N \exp\big\{-2\alpha(m^2 + k_\mu p^\mu) + ik_\mu d^\mu\big\}.$$

The factor $e^{-2\alpha m^2}$ can be absorbed in the normalization constant N, which we shall evaluate next.
Notice that the state can also be recast in the form

$$\begin{aligned}\langle k|d,p\rangle &= N \exp\{-2\alpha k_\mu p^\mu + ik_\mu d^\mu\} \\ &= N \exp\{ik_\mu(d^\mu + 2\alpha i p^\mu)\},\end{aligned}$$

which shows an interesting property: the state could be labeled by one single quantity, namely $d^\mu + 2\alpha i p^\mu$.

Proceeding to evaluate N, we impose normalization on the state

$$\begin{aligned}
1 &= \langle d,p|d,p\rangle \\
&= \int \frac{d^3k}{2E_k} \langle d,p|k\rangle \langle k|d,p\rangle \\
&= |N|^2 \int d^4k dE_k \vartheta(E_k)\delta(E_k^2 - (\vec{k}^2 + m^2))e^{-4\alpha k_\mu p^\mu} \\
&= |N|^2 \int d^4k dE_k \vartheta(E_k)\delta(E_k^2 - (\vec{k}^2 + m^2))e^{-4\alpha m\sqrt{\vec{k}^2+m^2}} \\
&= |N|^2 \int_{-\infty}^{\infty} \frac{d^3k}{2E_k} \exp\left\{-4\alpha m\sqrt{\vec{k}^2 + m^2}\right\} \\
&= 2\pi m|N|^2 \int_m^{\infty} dE_k e^{-4\alpha m E_k}\sqrt{\frac{E^2}{m^2} - 1} \\
&= 2\pi m^2|N|^2 \int_1^{\infty} dt e^{-4\alpha m^2 t}\sqrt{t^2 - 1} \\
&= |N|^2 \frac{\pi}{2\alpha} K_1\left[4\alpha m^2\right],
\end{aligned}$$

where

$$K_1\left[z\right] = z\int_1^{\infty} dt e^{-zt}\sqrt{t^2 - 1},$$

is the modified Bessel function of the second kind of order 1. We did the following: In the third line, we restored the manifest Lorentz invariant form of the integral measure, now k is not on-shell anymore (this condition is enforced by the δ function). Meanwhile, p is still on-shell and we may perform a boost to its rest frame such that $p^\mu \to \Lambda p^\mu = (m, \vec{0})$. Given that the measure is Lorentz-invariant, we arrive at the fourth line. Next, we put k on-shell again, then change variable from k to E_k using spherical coordinates, change variable again from $\frac{E_k}{m}$ to t, and finally recognize one of the integral forms of the modified Bessel function K_1.
Finally, we have

$$\langle k|d,p\rangle = \sqrt{\frac{2\alpha}{\pi}} \frac{1}{\sqrt{K_1\left[4\alpha m^2\right]}} \exp\{-2\alpha k_\mu p^\mu + ik_\mu d^\mu\}. \tag{6.1}$$

Notice that this does indeed have the correct non-relativistic limit, but care needs to be taken. The argument of the exponential reduces, using the non-relativistic limit

$$E_k \approx m + \frac{\vec{k}^2}{2m}, \qquad E_p \approx m + \frac{\vec{p}^2}{2m},$$

to

$$-2\alpha m^2 - \alpha(\vec{k} - \vec{p})^2 - i\vec{k}\cdot\vec{d}.$$

Meanwhile, for the normalization constant, the limit we need to take is $m \to \infty$ or, more reasonably, $\alpha m^2 \to \infty$[20]. The following limit is a well-known property of the Bessel functions K_n:

$$\lim_{z\to\infty} K_n\left[z\right] = \sqrt{\frac{\pi}{2z}}e^{-z}. \tag{6.2}$$

Then, in the aforementioned limit

$$\lim_{\alpha m^2\to\infty} N = \sqrt{2m}\left(\frac{2\alpha}{\pi}\right)^{\frac{3}{4}} e^{2\alpha m^2},$$

and the non-relativistic limit of the wavepacket is

$$\langle k|d,p\rangle \to \sqrt{2m}\left(\frac{2\alpha}{\pi}\right)^{\frac{3}{4}} \exp\Big\{-\alpha(\vec{k}-\vec{p})^2 - i\vec{k}\cdot\vec{d}\Big\},$$

which we immediately recognize as the non-relativistic packet used in the previous section (in the momentum basis), apart from a constant factor $\sqrt{2m}$, and the different dimensionality.

A closed form of this packet in the coordinate basis can be derived. Because we will not need it, we limit ourselves to quickly showing it here. For this purpose and later use, we will need the "master integral": [21]:

$$\int \frac{d^3k}{2E_k} e^{k_\mu \Xi^\mu} = 2\pi m \frac{K_1\left[m|\Xi|\right]}{|\Xi|}, \tag{6.3}$$

where $|\Xi| = \sqrt{\Xi_\mu \Xi^\mu}$ in the case of Ξ^μ timelike. [22]

Using this and $\langle x|k\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-ik_\mu x^\mu}$, we can calculate:

$$\langle x|d,p\rangle = \int \frac{d^3k}{2E_k} \langle x|k\rangle \langle k|d,p\rangle = \sqrt{\frac{\alpha m^2}{\pi^2}} \frac{1}{\sqrt{K_1\left[4\alpha m^2\right]}} \frac{K_1\left[m|\Xi|\right]}{|\Xi|},$$

[20] This is a very reasonable limit: in our units, $\frac{1}{m} = \lambda_C$ where λ_C is the Compton wavelength of the particle. Then, $\alpha m^2 = \frac{\alpha}{\lambda_C^2}$. Requiring this to be greater than one then just amounts to requiring the spread of the particle to be greater than its Compton wavelength ($\sqrt{\alpha} > \lambda_C$). It is well known that, when trying to localize a particle to within its λ_C, the energy uncertainty reaches order m and hence, leads to the regime of pair-creation. At such a point, relativistic quantum mechanics would have to leave its place to quantum field theory.

[21] Reference [20] is useful, as it contains proof of this integral for any dimension $D \geq 2$, as well as more detailed calculations.

[22] For real values of $\Xi^\mu \Xi_\mu$, timelike just refers to this number being positive. But this might in general be complex-valued. In this case, it implies the magnitude of the complex value to be positive and we may choose a branch cut on the negative real axis such that, for $-\pi \leq \theta \leq \pi$ and $z \geq 0$, we define $\sqrt{z} = \sqrt{re^{i\theta}} \equiv \sqrt{r}e^{i\frac{\theta}{2}}$.

where

$$|\Xi| = \sqrt{\Xi^\mu \Xi_\mu} = \sqrt{4\alpha^2 m^2 - (x-d)^\mu (x-d)_\mu + 4i\alpha p^\mu (x-d)_\mu}.$$

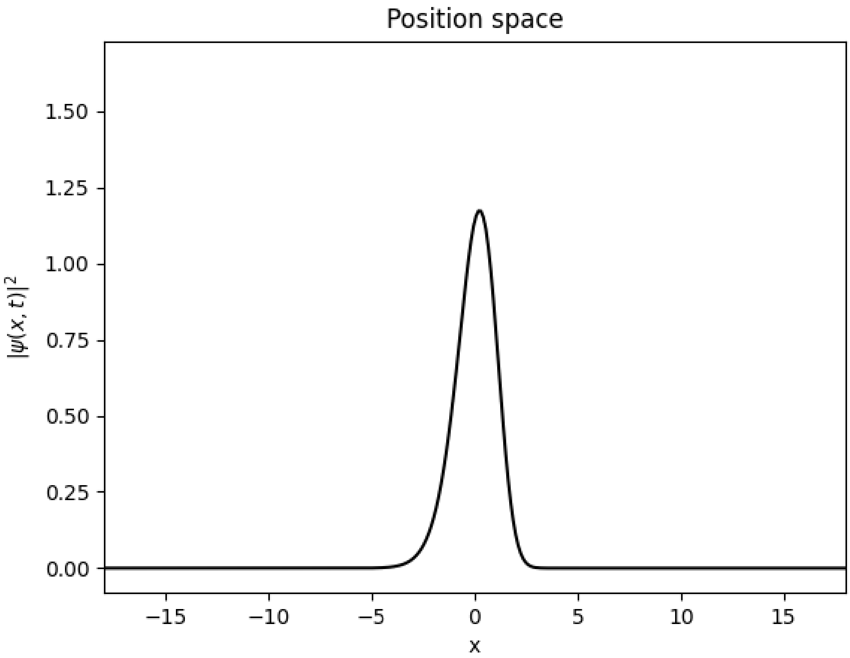

Figure 5: Example of Lorentz-invariant packet moving to the right. Notice the asymmetry and overall deviation from a Gaussian shape.

6.2 Boosting a spinor

As for the spinorial part of the relativistic case, we will need to solve for the Dirac equation. Let us begin by finding solutions to the Dirac equation, which in the mostly minus convention is:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

where we are using the Dirac representation for the gamma matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

Here, $\mathbb{1}$ is the 2x2 identity matrix, σ^i are the Pauli matrices and the zeroes fill in whatever missing blocks.

Moving to momentum representation by using $k_\mu = i\partial_\mu \rightarrow k^\mu = i\eta^{\mu\nu}\partial_\nu$, leads to

$$(\gamma^\mu k_\mu - m)\psi = 0.$$

Then, in the rest frame of the spinor, the three-momentum $\vec{k} = 0$ and $k^0 = E = \pm m$. Therefore the application of momentum and energy operators to the spinor gives the following results:

$$\hat{k}\psi = \vec{k}\psi = 0 \longrightarrow -i\vec{\nabla}\psi = \vec{0}.$$

Hence, ψ does not depend on the coordinates $\vec{x}$ and

$$\hat{E}\psi = E\psi \longrightarrow i\partial_t\psi = \pm m\psi, \tag{6.4}$$

where the positive sign stands for a Dirac particle, while the negative sign stands for its antiparticle. In this frame it is easy to find solutions to the Dirac equation:

$$(\gamma^0 p - m)\psi = 0 \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \frac{m}{E} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix},$$

as $\frac{m}{E} = \pm 1$, we have for the positive energy solution

$$\psi_1 = \psi_1, \psi_2 = \psi_2, \psi_3 = -\psi_3, \psi_4 = -\psi_4.$$

Hence the third and fourth components must vanish. A similar procedure applies to the negative sign solutions. We get the results

$$u(\vec{k} = 0) = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}, \qquad v(\vec{k} = 0) = \begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix},$$

which, by virtue of equation (6.4), gives the solutions

$$\psi_+ = u(0)e^{-imt}, \qquad \psi_- = v(0)e^{imt}.$$

The last thing we need to do is normalize these solutions. Using a standard normalization we write

$$u(0)^\dagger u(0) = a^* a + b^* b = 2m.$$

To define our initial state, we will not need solutions of the form $v(0)$ since they represent negative energy states. In the present discussion, since we are working in the regime of relativistic quantum mechanics, we will only need the positive energy solutions u to describe the spinors of interest in the recombination experiments. To generalize what was done in the non-relativistic case, we will focus on the following spinors:

$$|\rightleftarrows\rangle = \sqrt{m} \begin{pmatrix} 1 \\ \pm 1 \\ 0 \\ 0 \end{pmatrix} = \sqrt{2m} \begin{pmatrix} \chi_\pm \\ 0 \end{pmatrix}, \tag{6.5}$$

where $\chi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm 1\end{pmatrix}$. We now need to boost these spinors to arbitrary momentum. Spinors transform under a specific representation of the Lorentz group:

$$\psi \to S(\Lambda)\psi, \qquad \Lambda = \exp\left\{\frac{1}{2}\omega_{\mu\nu}L^{\mu\nu}\right\},$$

where $L^{\mu\nu}$ are the generators of the Lorentz transformation.
Following [21], we find the form of $S(\Lambda)$ to be

$$S(\Lambda) = \exp\left\{-\frac{i}{4}\omega_{\mu\nu}\Sigma^{\mu\nu}\right\},$$

where $\Sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.
The coefficients ω_{0i} are the boost parameters (rapidity) for boosts in the i-th spatial dimension, while the ω_{ij} are the parameters (angles) for rotations in the i-j plane. Let us focus on computing explicitly the operator for a boost in the x-direction since that is what we will need. Given that the quantity $\omega_{\mu\nu}$, as well as $\Sigma^{\mu\nu}$, are antisymmetric, we need

$$S(\Lambda_x) = \exp\left\{-\frac{i}{2}\zeta\Sigma^{01}\right\},$$

$\omega_{01} = \zeta = \tanh^{-1}(-v)$ being the rapidity for a boost of velocity v in the negative x-direction. Given that $\gamma^0\gamma^1 = \alpha^1$ and the anticommutation property of the gamma matrices, we get

$$[\gamma^0, \gamma^1] = 2\gamma^0\gamma^1 = 2\alpha^1 \to \Sigma^{01} = i\alpha^1 \to S(\Lambda_x) = \exp\left\{\frac{\zeta}{2}\alpha^1\right\},$$

and we can show the following:

$$\begin{aligned}
\exp\left\{\frac{\zeta}{2}\alpha^1\right\} &= \sum_{n=0}^{\infty}\left(\frac{\zeta}{2}\right)^n \frac{(\alpha^1)^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{\zeta}{2}\right)^{2n} \frac{(\alpha^1)^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\left(\frac{\zeta}{2}\right)^{2n+1} \frac{(\alpha^1)^{2n+1}}{(2n+1)!} \\
&= \mathbb{1}\sum_{n=0}^{\infty}\left(\frac{\zeta}{2}\right)^{2n} \frac{1}{(2n)!} + \alpha^1\sum_{n=0}^{\infty}\left(\frac{\zeta}{2}\right)^{2n+1} \frac{1}{(2n+1)!} \\
&= \mathbb{1}\cosh\left(\frac{\zeta}{2}\right) + \alpha^1\sinh\left(\frac{\zeta}{2}\right).
\end{aligned}$$

Now, using the hyperbolic trigonometric identities

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{1}{2}(1+\cosh(x))}, \qquad \sinh\left(\frac{x}{2}\right) = \sqrt{\frac{1}{2}(\cosh(x)-1)}.$$

Use now the fact that, in our case, $\zeta = \tanh^{-1}(-v) = -\tanh^{-1}(v)$ and $\zeta = \cosh^{-1}(\gamma)$ to derive

$$\tanh\left(\frac{\zeta}{2}\right) = \frac{\sinh\left(\frac{\zeta}{2}\right)}{\cosh\left(\frac{\zeta}{2}\right)} = \sqrt{\frac{\cosh\zeta - 1}{\cosh\zeta + 1}} = \sqrt{\frac{\gamma - 1}{\gamma + 1}}.$$

Some more relations we need are $\gamma = \frac{E}{m}$, $\gamma\beta = \frac{k}{m}$, and $\beta = \frac{k}{E}$. So that we can show

$$\cosh\left(\frac{\zeta}{2}\right) = \sqrt{\frac{1}{2}(\gamma + 1)} = \sqrt{\frac{E+m}{2m}},$$

and

$$\tanh\left(\frac{\zeta}{2}\right) = \sqrt{\frac{E-m}{E+m}} = \frac{k}{E+m}.$$

This allows us to write the final result for our boost operator

$$S(\Lambda_x) = \cosh\left(\frac{\zeta}{2}\right)\begin{pmatrix} 1 & 0 & 0 & \tanh\left(\frac{\zeta}{2}\right) \\ 0 & 1 & \tanh\left(\frac{\zeta}{2}\right) & 0 \\ 0 & \tanh\left(\frac{\zeta}{2}\right) & 1 & 0 \\ \tanh\left(\frac{\zeta}{2}\right) & 0 & 0 & 1 \end{pmatrix}$$

$$= \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 & 0 & 0 & \frac{k}{E+m} \\ 0 & 1 & \frac{k}{E+m} & 0 \\ 0 & \frac{k}{E+m} & 1 & 0 \\ \frac{k}{E+m} & 0 & 0 & 1 \end{pmatrix}.$$

Meanwhile, the spatial part of the spinor (the plane wave factor) gets transformed simply to arbitrary momentum as

$$e^{-imt} = e^{-iEt} = e^{-ipx^0} \to e^{-ik_\mu x^\mu}.$$

Let us continue by applying the boost we just found to our choice of spinor (6.5), yielding the boosted version of it:

$$S(\Lambda_x)\left|\rightleftarrows\right\rangle = \left|\rightleftarrows, k\right\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{E+m}}\begin{pmatrix} E+m \\ \pm(E+m) \\ \pm k \\ k \end{pmatrix}$$

$$= \frac{1}{\sqrt{E+m}}\begin{pmatrix} (E+m)\chi_\pm \\ k(\sigma_x\chi_\pm) \end{pmatrix} = \frac{1}{\sqrt{E+m}}\begin{pmatrix} (E+m)\chi_\pm \\ \pm k\chi_\pm \end{pmatrix},$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the first Pauli matrix.

So far so good. The generalization to four-component spinors of the spin operators is[23]

$$\hat{S}_i = \frac{1}{2}\gamma^0\gamma^i\gamma^5 = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix},$$

[23]For the generalization of the spin operator to the relativistic regime, see the "Pauli-Lubanski" vector.

such that the value of spin along the z-direction on a spinor $u(k)$ is given by $\frac{1}{2}\bar{u}(k)\gamma^3\gamma^5 u(k)$. This means that applying the spin along the z-direction operator to the boosted spinors results in

$$\hat{S}_z \left|\rightleftarrows, k\right\rangle = \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{E+m}}\begin{pmatrix} E+m \\ \mp(E+m) \\ \pm k \\ -k \end{pmatrix} = \frac{1}{2}\frac{1}{\sqrt{E+m}}\begin{pmatrix} (E+m)\chi_{\mp} \\ \pm k\chi_{\mp} \end{pmatrix}.$$

Now it is easy to check the following identities:

$$\frac{1}{2E}\left\langle \rightarrow, -k\right| \hat{S}_z \left|\rightarrow, k\right\rangle = 0 \qquad \frac{1}{2E}\left\langle \rightarrow, -k\right| \hat{S}_z \left|\leftarrow, k\right\rangle = \frac{1}{2}$$
$$\frac{1}{2E}\left\langle \leftarrow, -k\right| \hat{S}_z \left|\leftarrow, k\right\rangle = 0 \qquad \frac{1}{2E}\left\langle \leftarrow, -k\right| \hat{S}_z \left|\rightarrow, k\right\rangle = \frac{1}{2},$$

which shows that, as in the non-relativistic case, only cross terms will contribute to this spin measurement.
It is of no surprise that boosting spin-x eigenstates along the x-axis doesn't bring any new results, since in Dirac theory the spin along the direction of motion is conserved and, therefore, doesn't vary with boosts in that direction. One may ask what would happen if the merging happened along the y-axis or a generic direction in the x-y plane. Although we won't proceed with the calculation here, it can be shown that the result we are interested in does not vary: the peculiar quantity that is the $\hat{S}_Z$ operator, sandwiched between orthogonal spin states along any axis perpendicular to z, moving relatively to each other in such a symmetric fashion, is in fact always equal to what we just found.

6.3 Computing $\langle \hat{S}_z \rangle$

We now use what was found in the previous two sections to generalize our result to the relativistic regime. We construct the state using equation (6.1) respectively for the left and right components of the delocalized particle:

$$\langle k|\phi_L\rangle = \langle k|-d_L, k_L\rangle = \frac{1}{\sqrt{2}}\sqrt{\frac{2\alpha}{\pi}}\frac{1}{\sqrt{K_1\left[4\alpha m^2\right]}}\exp\Big\{-2\alpha E_k E_{k_L} + 2\alpha\vec{k}\cdot\vec{k}_L + iE_k t_0 + i\vec{k}\cdot\vec{d}_L\Big\}$$

$$\langle k|\phi_R\rangle = \langle k|d_R, -k_R\rangle = \frac{1}{\sqrt{2}}\sqrt{\frac{2\alpha}{\pi}}\frac{1}{\sqrt{K_1\left[4\alpha m^2\right]}}\exp\Big\{-2\alpha E_k E_{k_R} - 2\alpha\vec{k}\cdot\vec{k}_R + iE_k t_0 - i\vec{k}\cdot\vec{d}_R\Big\},$$

where we set the zeroth (time) components of the displacements (d_L and d_R) of the packets to the same value t_0 (this can safely be set to 0, but it will cancel either way). Our state is, then,

$$|\psi\rangle = |\phi_L\rangle \frac{|\rightarrow, k\rangle}{\sqrt{2E_k}} + |\phi_R\rangle \frac{|\leftarrow, -k\rangle}{\sqrt{2E_k}}.$$

Calculating the spin expectation value, while keeping in mind the orthogonality of the spin states derived previously:

$$\langle\psi| \hat{S}_z |\psi\rangle = \frac{1}{2}(\langle\phi_L|\phi_R\rangle + \langle\phi_R|\phi_L\rangle) = \mathrm{Re}\{\langle\phi_L|\phi_R\rangle\}.$$

Using the completeness relation

$$\mathbb{1} = \int |k\rangle \frac{d^3k}{2E_k} \langle k| ,$$

allows us to calculate

$$\langle\psi| \hat{S}_z |\psi\rangle = \mathrm{Re}\bigg\{\int \frac{d^3k}{2E_k} \langle\phi_L|k\rangle \langle k|\phi_R\rangle\bigg\}.$$

Let us focus on the argument of the integral:

$$\begin{aligned}
&\langle\phi_L|k\rangle \langle k|\phi_R\rangle \\
&= \frac{\alpha}{\pi} \frac{1}{K_1\left[4\alpha m^2\right]} \exp\Big\{ - 2\alpha\left[E_k(E_{k_L} + E_{k_R}) - k(k_L - k_R)\right] - ik(d_L + d_R)\Big\} \\
&= \frac{\alpha}{\pi} \frac{1}{K_1\left[4\alpha m^2\right]} \exp\Big\{k^\mu \left(iD_\mu - 2\alpha K_\mu\right) \Big\},
\end{aligned}$$

where we defined the constant 4-vectors

$$D^\mu = \begin{pmatrix} 0 \\ d_L + d_R \\ 0 \\ 0 \end{pmatrix}, \qquad K^\mu = \begin{pmatrix} E_{k_L} + E_{k_R} \\ k_L - k_R \\ 0 \\ 0 \end{pmatrix}.$$

Now we can evaluate our integral using the formula (6.3), first we will need

$$\begin{aligned}
|\Xi|^2 = \Xi^\mu \Xi_\mu &= (iD_\mu - 2\alpha K_\mu)(iD^\mu - 2\alpha K^\mu) \\
&= -D_\mu D^\mu + 4\alpha^2 K_\mu K^\mu - 4\alpha i K^\mu D_\mu \\
&= (d_L + d_R)^2 + 4\alpha^2(E_{k_L} + E_{k_R})^2 - 4\alpha^2(k_L - k_R)^2 + 4\alpha i(d_L + d_R)(k_L - k_R),
\end{aligned}$$

and thus the integral will result in

$$\langle\psi| \hat{S}_z |\psi\rangle = \frac{1}{2} \frac{4m\alpha}{K_1\left[4\alpha m^2\right]} \mathrm{Re}\bigg\{ \frac{K_1\left[m|\Xi|\right]}{|\Xi|} \bigg\}.$$

This can be proven to reduce to (5.1) by taking the correct limit, we will showcase this in the simple case of interest to us: the symmetric case, where we choose $d_L = d_R = d$ and $k_L = k_R = k$. The argument of K_1 becomes real, and the Bessel function itself is real too as a consequence. The result simplifies to

$$\begin{aligned} \langle\psi| \hat{S}_z |\psi\rangle &= \frac{1}{2} \frac{4m\alpha}{K_1\left[4\alpha m^2\right]} \frac{K_1\left[2m\sqrt{d^2+4\alpha^2 E_k^2}\right]}{2\sqrt{d^2+4\alpha^2 E_k^2}} \\ &= \frac{1}{K_1\left[4\alpha m^2\right]} \frac{K_1\left[2\alpha m^2\sqrt{\frac{d^2}{\alpha^2 m^2}+4\gamma^2}\right]}{\sqrt{\frac{d^2}{\alpha^2 m^2}+4\gamma^2}}, \end{aligned} \tag{6.6}$$

where γ is the Lorentz factor $\gamma^2 = \frac{1}{1-v^2} = 1 + \frac{k^2}{m^2}$.
Interestingly, taking the limit $v \ll 1$ is not enough to recover the non-relativistic limit. This is because the condition $\alpha m^2 \gg 1$ is also needed. As we noted earlier, this condition is nothing more than requiring the position uncertainty of the wavepacket to be much greater than the Compton wavelength of the particle. This is necessary to ensure that we remain in the domain of quantum mechanics (be it relativistic or not) and don't cross into quantum field theory, where the concept of single particle doesn't hold. Another way to interpret this limit is that, due to the uncertainty principle, once one tries to localize a particle below its Compton wavelength, the uncertainty in energy grows beyond m, enabling pair creation.
Let us take this limit, while not assuming anything regarding the velocity itself. Using the limit (6.2) again, we know the normalization constant goes to

$$\lim_{\alpha m^2 \to \infty} \frac{1}{K_1\left[4\alpha m^2\right]} = \sqrt{\frac{8\alpha m^2}{\pi}} e^{4\alpha m^2},$$

while the main part can be expanded using the aforementioned limit plus

$$\sqrt{\frac{d^2}{\alpha^2 m^2}+4\gamma^2} \to 2\gamma + \frac{d^2}{4\alpha^2 m^2 \gamma} + \ldots$$

which leads to

$$\lim_{\alpha m^2 \to \infty} K_1\left[2\alpha m^2\sqrt{\frac{d^2}{\alpha^2 m^2}+4\gamma^2}\right] = \sqrt{\frac{\pi}{8\alpha m^2 \gamma}} \exp\left\{-4\alpha m^2 \gamma - \frac{d^2}{2\alpha\gamma}\right\}.$$

Thus the final limit is

$$\begin{aligned} \lim_{\alpha m^2 \to \infty} \langle\psi| \hat{S}_z |\psi\rangle &= \frac{1}{2}\gamma^{-\frac{3}{2}} \exp\left\{-\frac{d^2}{2\alpha\gamma} - 4\alpha m^2(\gamma - 1)\right\} \\ &= \frac{1}{2}\gamma^{-\frac{3}{2}} \exp\left\{-\frac{d^2}{2\alpha\gamma} - 4\alpha m(E_k - m)\right\} = \langle \hat{S}_z \rangle_R, \end{aligned} \tag{6.7}$$

where $E_k - m$ is the relativistic kinetic energy.

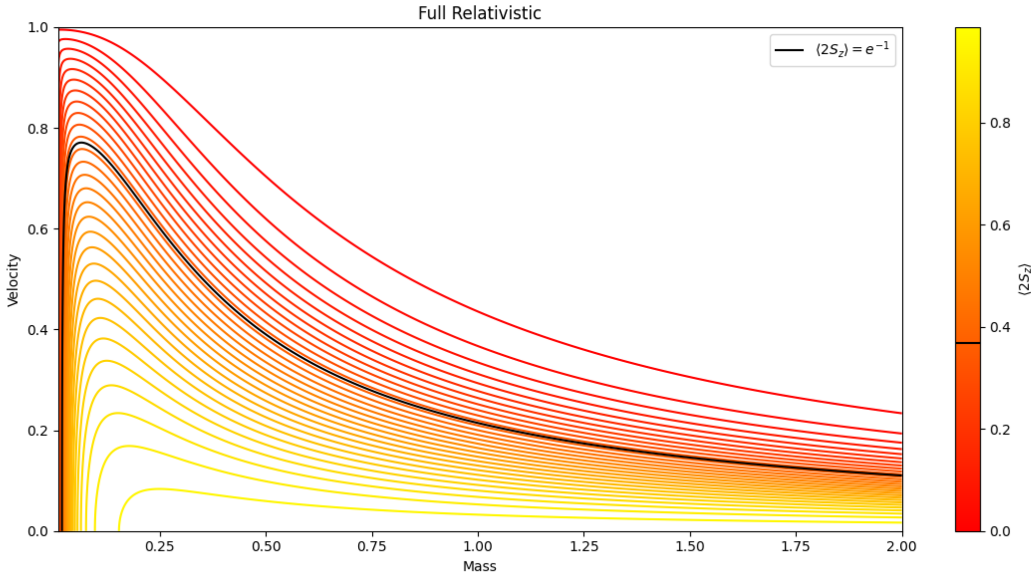

Figure 6: Contour plot of equation (6.6) as a function of particle mass velocity in the lower distance limit $2d \to l_0$, notice the massless limit goes inevitably to zero and the same can be said about the ultra-relativistic limit independently of the mass.

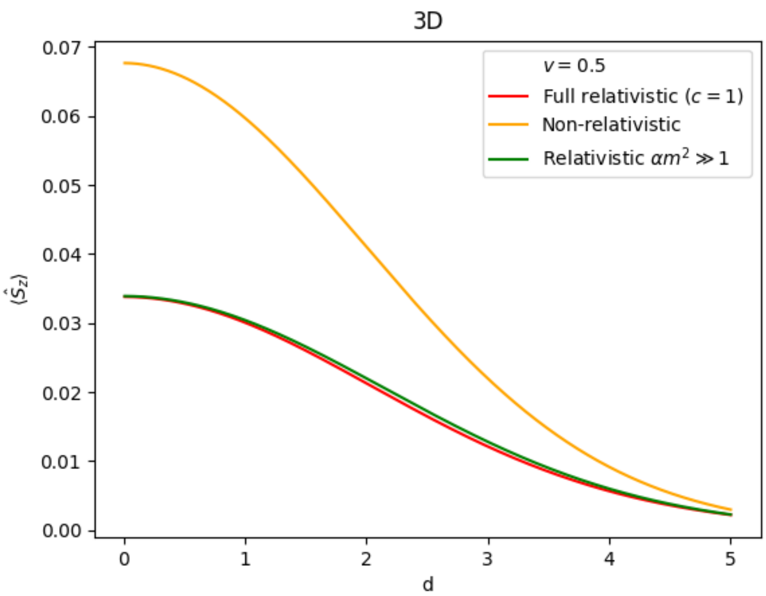

Figure 7: Example plot of our results as a function of half-separation d at fixed $v = 0.5$. The limit $\alpha m^2 \gg 1$ is almost indistinguishable from the full result (6.6).

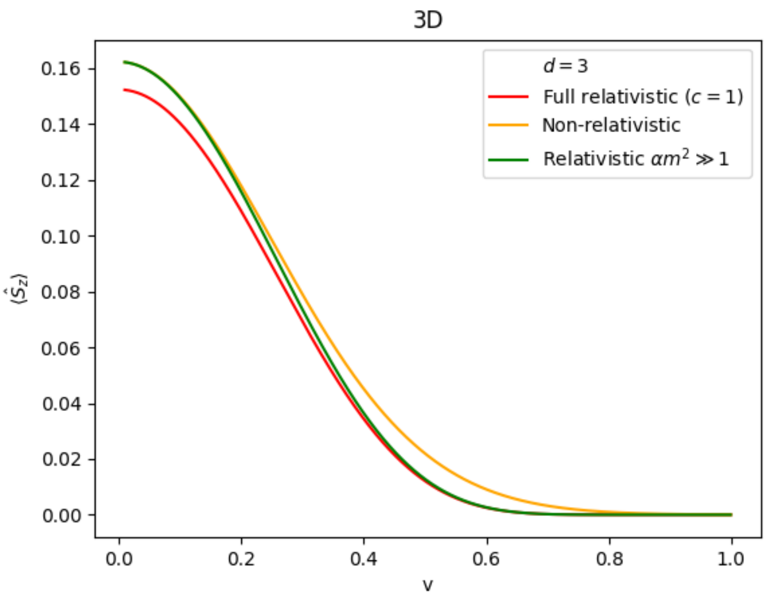

Figure 8: Similarly to the above picture, example plot of our results as a function of velocity v at fixed $d = 3$.

By taking the low velocity approximation $\gamma \to 1 + \frac{v^2}{2}$, or equivalently $\gamma \to 1 + \frac{k^2}{2m^2}$, the non-relativistic result follows trivially. This result retains the relativistic speed while enforcing the single-particle regime of relativistic quantum mechanics. Notice the additional suppression factor $\gamma^{-\frac{3}{2}}$ in front of the result.

Next, notice first of all that the $v \to c$ ($\gamma \to \infty$) limit makes now more sense:

$$\lim_{\gamma \to \infty} \langle \hat{S}_z \rangle_R = 0.$$

Let us check how this compares to the non-relativistic result. Given the following chain of inequalities:

$$\gamma = \frac{1}{\sqrt{1 - v^2}} = 1 + \frac{v^2}{2} + \frac{3v^4}{8} + ... > 1 + \frac{v^2}{2} > 1.$$

We compare the arguments of the two exponentials:

$$-\frac{d^2}{2\alpha\gamma} - 4\alpha m^2(\gamma - 1) < -\frac{d^2}{2\alpha} - 2\alpha(mv)^2.$$

This can be manipulated into

$$\gamma^2 - \gamma\left(1 + \frac{v^2}{2} + \frac{d^2}{8\alpha^2 m^2}\right) + \frac{d^2}{8\alpha^2 m^2} > 0.$$

We wish to neglect factors of $\frac{d^2}{8\alpha^2 m^2}$. This can be done if

$$\frac{d^2}{8\alpha^2 m^2} = \frac{1}{8}\frac{d^2}{\alpha}\frac{1}{\alpha m^2} \ll 1.$$

Given we are working in the regime where $\alpha m^2 \gg 1$, if $\frac{d^2}{\alpha} \gtrsim 1$ (the other way around would mean our state is initially overlapping) we can reasonably approximate the inequality to

$$\gamma > 1 + \frac{v^2}{2},$$

which is always true. Thus we can write

$$\langle \hat{S}_z \rangle_R < \gamma^{\frac{3}{2}} \langle \hat{S}_z \rangle_R = \frac{1}{2} \exp\left\{-\frac{d^2}{2\alpha\gamma} - 4\alpha m^2(\gamma - 1)\right\} < \langle \hat{S}_z \rangle_{NR},$$

and conclude that the relativistic correction will not enhance spin coherence, but rather suppress it even further. Were we to minimize this result with respect to the width α, as done in the non-relativistic case, we would find a value

$$\alpha_0 = \frac{d}{2\sqrt{2}m}\frac{1}{\sqrt{\gamma(\gamma - 1)}}$$

This value can only be used with caution though, since we can see by multiplying by m^2 that

$$\alpha_0 m^2 = \frac{md}{2\sqrt{2}} \frac{1}{\sqrt{\gamma(\gamma-1)}}$$

But since we have previously derived our result by using $\alpha m^2 \gg 1$, this α_0 is only valid as long as this condition is also respected. Nevertheless, let us see where this leads us. The spin expectation value becomes:

$$\langle \hat{S}_z \rangle_R(\alpha_0) = \frac{1}{2}\gamma^{-\frac{3}{2}} \exp\left\{-\sqrt{2}(2d)m\sqrt{\frac{\gamma-1}{\gamma}}\right\}$$

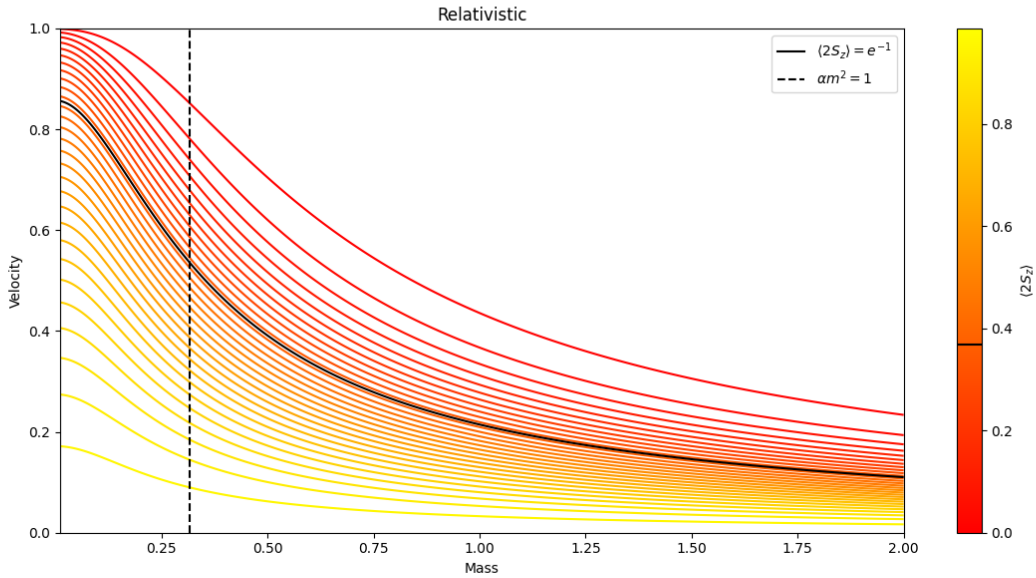

Figure 9: Contour plot of equation (6.7) as a function of particle mass and velocity in the lower distance limit $2d \to l_0$. This results validity steps in for values of mass above the dashed line. Notice the factor $\gamma^{-\frac{3}{2}}$ causes suppression of the spin coherence due to relativistic velocities for any value of mass.

Now, as before, we send $2d \to 1 (= l_p)$, this gives

$$\begin{aligned} \lim_{2d\to 1} \langle \hat{S}_z \rangle_R(\alpha_0) = \frac{1}{2}\gamma^{-\frac{3}{2}} \exp\left\{-\sqrt{2}m\sqrt{\frac{\gamma-1}{\gamma}}\right\} &< \frac{1}{2}\gamma^{-\frac{3}{2}} \exp\left\{-\sqrt{2}m\right\} \\ &< \frac{1}{2} \exp\left\{-\sqrt{2}m\right\} \end{aligned} \tag{6.8}$$

and we see that the mass $\frac{m}{m_p}$ is an upper bound limiting the amount of spin coherence maintained or lost in (6.8). However, it should be noted that relativistic velocities give an increasingly strong suppression (see figure 9).

GIE experiments are usually meant to be performed using massive particles, such as nanodiamonds or similar objects. As it is generally not in the scope of the practical realization of these experiments to achieve relativistic relative velocities between the components of the delocalized state, our analysis here was mainly meant as a proof of concept: massive particles can, in principle, achieve relativistic velocities. It is good practice to see how this would affect the recombination, and look for unexpected setups and parameter choices that may produce interesting scenarios.

7 Conclusions

In the present thesis, we have explored the behaviour of spin coherence in the merging of a delocalized single-particle state. This state is modeled by two spatially separated wavepackets. We have done this in the context of a well-known gedankenexperiment in which the gravitational field, sourced by a delocalized particle, is probed at a distance while the particle is recombined. This setup has been noted to bring some tension between the principles of complementarity and causality if gravity is quantum mechanical in its nature. Focusing mainly on the delocalized state, rather than the probe particle, we found under what conditions spin coherence is maintained during the recombination process.

In doing so (this being the specific piece of research characterizing this work), we have tried to study the effects of the existence of a universal lower limit value on physical distances. We have done this using the effective metric, known as q-metric, which precisely implements the existence of such a minimum length, as applied to the separation of the two wavepacket components. Using this and under some simplifying assumptions, namely that:

- The recombination occurs in a time T_A.
- The state at the moment of recombination can be described as the superposition of two Gaussians, characterized by their relative distance and momentum.
- The relative momentum can be estimated from the average relative velocity during the recombination.

In this way, the level of decoherence we get is the same required to avoid faster-than-light communication in the aforementioned experiment (where the decoherence arises from graviton emission). The formula we get exhibits some analogy with the formula providing the residual decoherence one gets when recombining, in the spin basis, a particle with

finite experimental resolution in position and momentum (the so-called Humpty-Dumpty effect [6]).
We found that, for the delocalized particle, the Planck mass acts as a pivotal value of mass in the phenomenon. We also noted that, both in the case of interferometric recombination (measurement of fringe pattern) and in the case of spin coherence measurement, the required recombination time (within the bounds such that the delocalized particle and the probe are causally disconnected) can be such that the relative motion between the two components can become relativistic. For this, we proposed a quick order of magnitude computation of interference fringe spacing in the case of relativistic momenta, and a relativistic generalization of our Gaussian wavepackets for the discussion on spin coherence. We managed to obtain a closed-form equation for the spin coherence in terms of Bessel functions which, again, assuming a finite lower bound on separation, produces similar or slightly stronger bounds in the mass-velocity plane for the maintenance of spin coherence.

All in all, spin coherence is a fragile object, and the question of recombining a delocalized spin state coherently is heavily influenced by its mass and velocity. Real- and gedankenexperiments, aimed at studying the possible quantum gravitational effects of such coherent delocalized states, require masses big enough to witness their gravitational pull on another distant body and possibly relativistic velocities. Our analysis shows that a finite-length assumption might be sufficient to avoid the tension between complementarity and causality mentioned above. by causing the expected decoherence on the state, i.e. with no need to invoke graviton emission. What is quite intriguing, however, is that the effects arising from minimum length and graviton emission happen together, i.e. strong decoherence from minimum length does set in right when the emission of gravitons is expected to take place.

A Theory of bitensors

A.1 Synge's world function

In this section, we construct and derive some general results from the theory of bitensors, the main objects we will use in constructing the q-metric. Let us begin with the world function introduced by Synge.[24]
Take the following integral

$$I(v) = \frac{1}{2}(u_1 - u_0)\int_{u_0}^{u_1} g_{\mu\nu}\frac{\partial x^\mu}{\partial u}\frac{\partial x^\nu}{\partial u}du, \tag{A.1}$$

along $v =$ constant curves. Also, consider two vector fields, defined as

$$U^\mu = \frac{\partial x^\mu}{\partial u}, \qquad V^\mu = \frac{\partial x^\mu}{\partial v},$$

and it follows that

$$\frac{DU^\mu}{Dv} = \frac{DV^\mu}{Du}, \tag{A.2}$$

where the operator $\frac{D}{D\lambda}$ stands for the covariant derivative with respect to a parameter λ,

$$\frac{DT^\mu}{D\lambda} = \left[\frac{\partial T^\mu}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}T^\alpha\right]\frac{dx^\beta}{d\lambda}.$$

Then, (A.1) can be written as:

$$I(v) = \frac{1}{2}\Delta u\int_{u_0}^{u_1} g_{\mu\nu}U^\mu U^\nu du,$$

where $\Delta u = (u_1 - u_0)$. Using (A.2), we can differentiate the integral

$$\begin{aligned}\frac{dI(v)}{dv} = \Delta u\int_{u_0}^{u_1} g_{\mu\nu}U^\mu\frac{DV^\nu}{Du}du = \\ \Delta u\int_{u_0}^{u_1}\frac{\partial}{\partial u}\Big(g_{\mu\nu}U^\mu V^\nu\Big)du - \Delta u\int_{u_0}^{u_1} g_{\mu\nu}\frac{DU^\mu}{Du}V^\nu du = \\ \Delta u\Big[g_{\mu\nu}U^\mu V^\nu\Big]_{u_0}^{u_1} - \Delta u\int_{u_0}^{u_1} g_{\mu\nu}\frac{DU^\mu}{Du}V^\nu du.\end{aligned} \tag{A.3}$$

If the endpoints of the curve are fixed, say we call them A_0 and A_1, at these points $V^\mu = 0$ by virtue of (A.2), and we are left with

$$\frac{dI}{dv} = -\Delta u\int_{u_0}^{u_1} g_{\mu\nu}\frac{DU^\mu}{Du}V^\nu du.$$

[24] From Synge's book [22]

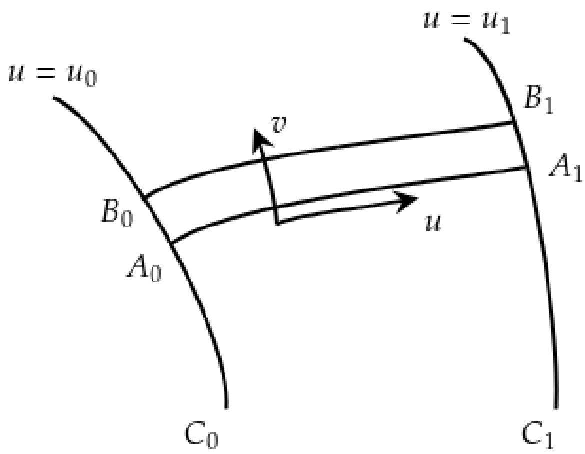

Figure 10: A graphical representation of our u-v parameterized curves

A geodesic is defined as the curve (we assume it to be unique) that gives a stationary value to I for a variation that leaves the endpoints fixed. Hence we want

$$\frac{dI}{dv} = 0,$$

for arbitrary V^ν (except at the endpoints). Then we get that a geodesic must satisfy

$$\frac{DU^\mu}{Du} = \frac{D}{Du}\frac{dx^\mu}{du} = \frac{d^2x^\mu}{du^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{du}\frac{dx^\beta}{du} = 0. \tag{A.4}$$

This equation has a first integral of the form

$$g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = \epsilon k^2,$$

or, equivalently

$$ds = kdu, \tag{A.5}$$

where k is a constant and ϵ indicates whether the geodesic is null, timelike, or spacelike by taking, respectively, the values 0, -1, and 1 (this depends on the chosen convention). Every geodesic allows for a class of parameters, called affine parameters, for which it satisfies equation (A.4). Affine parameters are related through one another by linear transformations: $u' = au + b$. Any other parameterization modifies eq. (A.4) with a term proportional to the tangent vector $t^\alpha = \frac{\partial x^\alpha}{\partial u}$. Unless the geodesic is a null one, an affine parameter for which $k = 1$ can be found. From (A.5) then $ds = du$. Thus the parameter can be associated with proper time, and the geodesic equation (A.4) becomes

$$\frac{D}{Ds}\frac{dx^\mu}{ds} = \frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = \frac{dt^\mu}{ds} + \Gamma^\mu{}_{\alpha\beta}t^\alpha t^\beta = 0.$$

To interpret the meaning of I, we note that for a non-null curve $x^\alpha = x^\alpha(u)$ with $u_0 \leqslant u \leqslant u_1$, we can choose a parameter u' using the linear transformation property between affine parameters. Let us take it such that

$$u' = \frac{\Delta u}{L} s + u_0,$$

where s is the geodesic distance of the current point from $u = u_0$, while L is the value of s at $u = u_1$. (A.1) then becomes

$$I(v) = \frac{1}{2}\epsilon L^2. \tag{A.6}$$

And, in the case of a null curve, this equates to $0 = 0$. Hence our variational principle turns into

$$\delta(L^2) = 0 \Longrightarrow \delta L = \delta(\int ds) = 0,$$

which is the usual stationary action principle for a free particle.
Consider now figure 10 and equation (A.3). If we suppose the two curves to be geodesics and let u be an affine parameter on each one of them running between u_0 and u_1, eq. (A.3) reduces, in virtue of (A.4), to

$$\frac{dI}{dv} = \Delta u \left[g_{\mu\nu} U^\mu v^\nu \right]_{u_0}^{u_1},$$

which, in terms of variations gives

$$\delta I = \Delta u \left[g_{\mu\nu} U^\mu \delta x^\nu \right]_{u_0}^{u_1}. \tag{A.7}$$

This is now a function of the coordinates of the endpoints, say $x^{\mu'}$ for A_0 and x^μ for A_1. Additionally, (A.7) leads us to the derivatives of I:

$$\frac{\partial I}{\partial x^\mu} = \Delta u \left(g_{\mu\nu} \frac{dx^\nu}{du} \right)_{A_1}. \tag{A.8}$$

An identical calculation results in the derivative with respect to the other end's coordinates:

$$\frac{\partial I}{\partial x^{\mu'}} = -\Delta u \left(g_{\mu\nu} \frac{dx^\nu}{du} \right)_{A_0}. \tag{A.9}$$

These are general results for any geodesic, if the curve is non-null we get the special cases

$$\frac{\partial I}{\partial x^\mu} = L t_\mu, \qquad \frac{\partial I}{\partial x^{\mu'}} = -L t_{\mu'},$$

where t_μ and $t_{\mu'}$ are unit tangent vectors to the geodesic, respectively at A_1 and A_0, while L is the length $|A_0 A_1|$.

What was done until now will now be used to construct and discuss what is referred to as Synge's world function.
Let $P'(x')$ and $P(x)$ be two events joined by a geodesic Γ, described by equations $x^\mu = \xi^\mu(u)$, where u is an affine parameter. We suppose the geodesic to be unique and rewrite the integral (A.1), now taken along Γ:

$$\Omega(P', P) = \Omega(x', x) = \frac{1}{2}\Delta u \int_{u_0}^{u_1} g_{\mu\nu} U^\mu U^\nu du, \qquad U^\mu = \frac{d\xi^\mu}{du}, \tag{A.10}$$

which is independent of the particular affine parameter that is used. This object, which is a function of the eight variables $x^\mu, x^{\mu'}$, is what we will refer to as the world function.

Since along a geodesic we know $\frac{DU^\mu}{Du} = 0$, we have that $g_{\mu\nu}U^\mu U^\nu$ is a constant along Γ, and the world function becomes

$$\Omega(x', x) = \frac{1}{2}(\Delta u)^2 \left[g_{\mu\nu}U^\mu U^\nu\right]_\Gamma . \tag{A.11}$$

One can rescale and shift u to make $u_0 = 0$ and $u_1 = 1$, to yield $\Delta u = 1$. Then, as in eq. (A.6)

$$\Omega(x, x') = \frac{1}{2}\epsilon L^2, \qquad L = \int_{P'}^{P} ds.$$

This shows that in flat space there is a coordinate system for which

$$\Omega(x, x') = \frac{1}{2}\eta_{\mu\nu}(x^{\mu'} - x^\mu)(x^{\nu'} - x^\nu).$$

Ω is a biscalar, it is invariant under coordinate transformations, both in the coordinate system $x^{\mu'}$ at P' and the coordinate system x^μ at P, independently so.

Let us now refer back to a generic 2-point invariant I, all results will apply specifically for Ω as well.

For convenience, we will denote covariant derivatives of I by subscripts while omitting the usual semicolon notation, e.g. $I_\mu = I_{;\mu}$. Derivatives can be taken with respect to both coordinate systems of the points on which the object has a dependence, for example

$$I_{\mu'} = \frac{\partial I}{\partial x^{\mu'}}, \qquad I_{\mu'\nu'} = \frac{\partial I_{\mu'}}{\partial x^{\nu'}} - \Gamma(P')^{\lambda'}{}_{\mu'\nu'} I_{\lambda'}$$
$$I_\mu = \frac{\partial I}{\partial x^\mu}, \qquad I_{\mu\nu} = \frac{\partial I_\mu}{\partial x^\nu} - \Gamma(P)^{\lambda}{}_{\mu\nu} I_\lambda.$$

Another important relation, which will be needed later and can be proved by an argument found in [22], is the fact that derivatives with respect to primed and unprimed indices commute, hence the following holds

$$I_{\dots\mu'\nu\dots} = I_{\dots\nu\mu'\dots}. \tag{A.12}$$

These are then examples of 2-point tensors. Mixed index derivatives act no different:

$$I_{\mu'\nu\lambda} = \frac{\partial}{\partial x^\lambda} I_{\mu'\nu} - \Gamma^\rho{}_{\nu\lambda} I_{\mu'\rho}.$$

Subscripts can then be raised at P' through the action of $g^{\mu'\nu'}$, and at P through $g^{\mu\nu}$. As for the world function, by (A.8) and (A.9) we have

$$\Omega_{\mu'} = -\Delta u U_{\mu'}, \qquad \Omega_\mu = \Delta u U_\mu. \tag{A.13}$$

And, again, if Γ is not null, choosing the affine parameter such that $du = ds$, we have the special case

$$\Omega_{\mu'} = -L t_{\mu'}, \qquad \Omega_\mu = L t_\mu,$$

with t_μ the unit tangent vector to Γ at P, and $t_{\mu'}$ the one at P'.
Equation (A.13) yields

$$g^{\mu\nu}\Omega_\mu\Omega_\nu = \Delta u^2 g^{\mu\nu} U_\mu U_\nu.$$

And by (A.11), we obtain the important partial differential equations obeyed by the world function

$$g^{\mu\nu}\Omega_\mu\Omega_\nu = 2\Omega, \qquad g^{\mu'\nu'}\Omega_{\mu'}\Omega_{\nu'} = 2\Omega. \tag{A.14}$$

Note the following properties of the derivatives of Ω:

$$\Omega_{\mu'\nu'} = \Omega_{\nu'\mu'}, \qquad \Omega_{\mu\nu} = \Omega_{\nu\mu}, \tag{A.15}$$

$$\Omega_{\mu'\nu'\ldots} = \Omega_{\nu'\mu'\ldots}, \qquad \Omega_{\mu\nu\ldots} = \Omega_{\nu\mu\ldots}, \tag{A.16}$$

where (A.16) holds only when the subscripts being exchanged are the ones adjacent to Ω. Also ... indicates a set of unchanged indices.

A.2 Coincidence limits of the world function

The notation for the limit $P \longrightarrow P'$ (which implies $x^\mu \longrightarrow x^{\mu'}$), is the following

$$\lim_{P\to P'} \Omega_{\ldots} = [\Omega\ldots].$$

The coincidence limit better not depend on the path taken between the two points, a formal argument (as long as power series expansions are valid, which puts constraints on the metric) is the following.
From the geodesic equation, one has:

$$\frac{dU^\mu}{du} = -\Gamma^\mu{}_{\alpha\beta} U^\alpha U^\beta, \qquad U^\mu = \frac{dx^\mu}{du}.$$

This suggests the following expansion for sufficiently close P and P':

$$x^\mu|_P = x^\mu|_{P'} + \left.\frac{dx^\mu}{du}\right|_{P'} \delta u + \left.\frac{d^2x^\mu}{du^2}\right|_{P'} (\delta u)^2 + ...$$
$$= x^{\mu'} + U^{\mu'}u_1 - \frac{1}{2}\Gamma^{\mu'}{}_{\alpha'\beta'}U^{\alpha'}U^{\beta'}u_1^2 + ..., \tag{A.17}$$

which is an expansion of the coordinate point P in terms of the affine parameter u and the tangent vector pointing towards P'. Note $\delta u = u_1 - u_0 \equiv u_1$.
The term $U^{\mu'}U^{\nu'}u_1^2$ can be seen as

$$\frac{dx^{\mu'}}{du}\frac{dx^{\nu'}}{du}\delta u\delta u \approx \delta x^{\mu'}\delta x^{\nu'} \approx (x^\mu - x^{\mu'})(x^\nu - x^{\nu'}).$$

And (A.17) can be inverted as

$$u_1U^{\mu'} = (x^\mu - x^{\mu'}) + \frac{1}{2}\Gamma^{\mu'}{}_{\alpha'\beta'}(x^\alpha - x^{\alpha'})(x^\beta - x^{\beta'}) + ...,$$

defining $\xi^\mu = x^\mu - x^{\mu'}$, we can write

$$u_1U^{\mu'} = \xi^\mu + \frac{1}{2}\Gamma^{\mu'}{}_{\alpha'\beta'}\xi^\alpha\xi^\beta + ...$$

Using (A.11), we thus write

$$2\Omega(x, x') = u_1^2 g_{\mu'\nu'}\xi^\mu\xi^\nu + A_{\mu'\nu'\lambda'}\xi^\mu\xi^\nu\xi^\lambda + ...$$

The coefficients of this series are functions of $g_{\mu'\nu'}$ and its derivatives, hence Ω appears as an analytic function, and the coincidence limits should be independent of the path. In this limit, we will drop the primes on the base point.

Using (A.10) and (A.13), we know

$$[\Omega] = [\Omega_{\mu'}] = [\Omega_\mu] = 0,$$

which also implies

$$[\Omega^{\mu'}] = [\Omega^\mu] = 0.$$

Another important equation comes from differentiating eq. (A.14):

$$\Omega_\mu = \Omega^\nu{}_\mu\Omega_\nu. \tag{A.18}$$

Now divide by Δu, then use (A.13):

$$\frac{\Omega_\mu}{\Delta u} = \frac{\Omega_\nu}{\Delta u}\Omega^\nu{}_\mu \longrightarrow U_\mu = U_\nu\Omega^\nu{}_\mu.$$

Taking the coincidence limit of this equation, considering it must be independent of the limit U_μ takes (because of path independence), one obtains

$$[\Omega^\nu{}_\mu] = \delta^\nu_\mu \longrightarrow [\Omega_{\mu\nu}] = g_{\mu\nu}.$$

The coincidence limit can be equivalently taken by letting $P \to P'$, or by letting $P' \to P$, hence we may exchange primed indices with unprimed ones in all our results, e.g.

$$[\Omega_{\mu'\nu'}] = g_{\mu\nu}.$$

As for mixed indices, we will need to invoke Synge's rule (A.24)

$$[\Xi_{PQ';\mu}] + [\Xi_{PQ';\mu'}] = [\Xi_{PQ'}]_{;\mu},$$

which will be proven at the end of this section. Thus we obtain, as an example that we will need later for the world function's second mixed derivative:

$$[\Omega_{\mu\nu'}] = [\Omega_\mu]_{;\nu} - [\Omega_{\mu\nu}] \longrightarrow [\Omega_{\mu\nu'}] = -g_{\mu\nu} = [\Omega_{\mu'\nu}].$$

While more coincidence limits can be extracted by further differentiation of (A.18):

1. $\Omega_\mu = \Omega_\nu \Omega^\nu{}_\mu$
2. $\Omega_{\mu\nu} = \Omega_{\lambda\nu}\Omega^\lambda{}_\mu + \Omega_\lambda \Omega^\lambda{}_{\mu\nu}$
3. $\Omega_{\mu\nu\rho} = \Omega_{\lambda\nu\rho}\Omega^\lambda{}_\mu + \Omega_{\lambda\nu}\Omega^\lambda{}_{\mu\rho} + \Omega_{\lambda\rho}\Omega^\lambda{}_{\mu\nu} + \Omega_\lambda\Omega^\lambda{}_{\mu\nu\rho}$
4. $\Omega_{\mu\nu\rho\sigma} = \Omega_{\lambda\nu\rho\sigma}\Omega^\lambda{}_\mu + \Omega_{\lambda\nu\rho}\Omega^\lambda{}_{\mu\sigma} + \Omega_{\lambda\nu\sigma}\Omega^\lambda{}_{\mu\rho} + \Omega_{\lambda\nu}\Omega^\lambda{}_{\mu\rho\sigma}$
   $+ \Omega_{\lambda\rho\sigma}\Omega^\lambda{}_{\mu\nu} + \Omega_{\lambda\rho}\Omega^\lambda{}_{\mu\nu\sigma} + \Omega_{\lambda\sigma}\Omega^\lambda{}_{\mu\nu\rho} + \Omega_\lambda\Omega^\lambda{}_{\mu\nu\rho\sigma}.$

Coincidence of equation number 2 above gives nothing of worth. On the other hand, coincidence of number 3 gives

$$[\Omega_{\mu\nu\rho}] = -[\Omega_{\rho\nu\mu}].$$

And by this property and the symmetry on the first two indices given above, we see that

$$[\Omega_{\mu\nu\rho}] = 0.$$

The last equation in the list, at coincidence, gives

$$[\Omega_{\nu\mu\rho\sigma}] + [\Omega_{\rho\mu\nu\sigma}] + [\Omega_{\sigma\mu\nu\rho}] = 0. \tag{A.19}$$

Recall the commutation rule for covariant derivatives is

$$T_{\mu;\nu;\rho} - T_{\mu;\rho;\nu} = R^\lambda{}_{\mu\nu\rho} T_\lambda,$$

and

$$T_{\mu\nu;\rho;\sigma} - T_{\mu\nu;\sigma;\rho} = R^{\lambda}{}_{\mu\rho\sigma} T_{\lambda\nu} + R^{\lambda}{}_{\nu\rho\sigma} T_{\mu\lambda}.$$

Using this last property on $\Omega_{\mu\nu\rho\sigma}$, and taking the coincidence limit, also using the symmetries of the Riemann tensor leads to

$$[\Omega_{\mu\nu\rho\sigma}] = [\Omega_{\mu\nu\sigma\rho}].$$

Next, applying the first covariant derivative commutation rule we stated on $\Omega_{\mu\nu\rho}$, differentiating and taking the coincidence limit gives

$$[\Omega_{\mu\nu\rho\sigma}] - [\Omega_{\mu\rho\nu\sigma}] = R_{\sigma\mu\nu\rho} = -R_{\mu\sigma\nu\rho}.$$

Exchange ρ and σ in this expression and add it to itself to get

$$2[\Omega_{\mu\nu\rho\sigma}] - [\Omega_{\mu\rho\nu\sigma}] - [\Omega_{\mu\sigma\rho\nu}] = -\left(R_{\mu\sigma\nu\rho} + R_{\mu\rho\nu\sigma}\right).$$

And lastly, add expression (A.19) with the first two indices swapped on each factor. This yields

$$[\Omega_{\mu\nu\rho\sigma}] = S_{\mu\nu\rho\sigma} = -\frac{1}{3}\left(R_{\mu\sigma\nu\rho} + R_{\mu\rho\nu\sigma}\right), \tag{A.20}$$

where S is the symmetrized Riemann tensor, which contains the same information as R by the relation

$$R_{\mu\nu\rho\sigma} = -\left(S_{\mu\rho\nu\sigma} - S_{\mu\sigma\nu\rho}\right).$$

A.3 Proof of Synge's rule

To prove Synge's rule, we start with a mixed index bitensor

$$\Xi_{PQ'} = \Xi_{i_1 \ldots i_p j'_1 \ldots j'_q},$$

where P and Q' are multi-indices of the p and q indices respectively, as shown.
Next, take p vectors on the geodesic Γ, parallel transported to the point P and q vectors, parallel transported on the base point P' along the geodesic:

$$A^P = A_1^{i_1} \ldots A_p^{i_p}, \qquad B^{Q'} = B_1^{j'_1} \ldots B_q^{j'_q}.$$

Now we can form a biscalar, parametrized on Γ by the parameter u in such a way that $u = u_0$ at P' and $u = u_1$ at P:

$$H(u_0, u_1) = \Xi_{PQ'} A^P B^{Q'}, \tag{A.21}$$

then, for small $\Delta u = u_1 - u_0$, to first order, we can write

$$H(u_0, u_1) = H(u_0, u_0) + \Delta u \left(\frac{\partial H}{\partial u_1}\right)_{u_1 = u_0} = H(u_0, u_0) + \Delta u \left[\Xi_{PQ';\mu}\right]_{P'} U^{\mu'} (A^P B^{Q'})|_{P'}. \tag{A.22}$$

And the same reasoning can be applied by starting from P:

$$H(u_0,u_1) = H(u_1,u_1) + (u_0-u_1)\left(\frac{\partial H}{\partial u_0}\right)_{u_0=u_1} = H(u_1,u_1) - \Delta u\,[\Xi_{PQ';\mu'}]_P\,U^{\mu}(A^P B^{Q'})|_P. \tag{A.23}$$

Subtracting (A.23) from (A.22), we obtain

$$\frac{H(u_1,u_1)-H(u_0,u_0)}{\Delta u} = \left([\Xi_{PQ';\mu}]_{P'}\,U^{\mu'}(A^P B^{Q'})|_{P'} + [\Xi_{PQ';\mu'}]_P\,U^{\mu}(A^P B^{Q'})|_P\right).$$

And taking the limit as $u_0 \to u_1$, we find

$$\frac{dH(u)}{du} = \left([\Xi_{PQ';\mu}] + [\Xi_{PQ';\mu'}]\right) U^{\mu} A^P B^Q.$$

Now, note that from the coincidence limit of equation (A.21), we get

$$H(u) = [\Xi_{PQ'}]\,A^P B^Q \longrightarrow \frac{dH(u)}{du} = [\Xi_{PQ'}]_{;\mu}\,U^{\mu} A^P B^Q.$$

Finally, being the vectors A and B and the tangent to the geodesic U arbitrary, we obtain the desired result

$$[\Xi_{PQ';\mu}] + [\Xi_{PQ';\mu'}] = [\Xi_{PQ'}]_{;\mu}\,. \tag{A.24}$$

A.4 The parallel propagator

We now introduce on the geodesic Γ an orthonormal basis e^{μ}_{a}, which is parallel transported along the geodesic. The Latin indices indicate the frame spanned by this basis. We have the following for normal local coordinate systems:

$$g_{\mu\nu} e^{\mu}_{a} e^{\nu}_{b} = \eta_{ab},$$

and since the basis vectors are parallel transported along the geodesic, they satisfy

$$e^{\mu}_{a;\nu} t^{\nu} = 0,$$

where, again, t^{μ} are the unit tangent vectors to the geodesic. By defining the cotetrad

$$e^{a}_{\mu} = \eta^{ab} g_{\mu\nu} e^{\nu}_{b},$$

we obtain the completeness relation

$$g_{\mu\nu} = \eta_{ab} e^{a}_{\mu} e^{b}_{\nu}. \tag{A.25}$$

Tetrad and cotetrad relate as such on Γ:

$$e^{a}_{\mu} e^{\mu}_{b} = \delta^{a}_{b}, \qquad e^{a}_{\nu} e^{\mu}_{a} = \delta^{\mu}_{\nu}.$$

Any vector field can be decomposed on Γ in this basis, e.g.

$$A^\mu = A^a e^\mu_a \longrightarrow A^a = A^\mu e^a_\mu.$$

If, then, A^μ is parallel transported along Γ, and so is our basis, the coefficients A^a are constant along the geodesic. This vector can be written as

$$A^\mu(x) = (A^{\nu'}(x')e^a_{\nu'})e^\mu_a.$$

Then, by defining

$$\Pi^\mu_{\nu'} = e^a_{\nu'} e^\mu_a, \tag{A.26}$$

we can write

$$A^\mu(x) = \Pi^\mu_{\nu'} A^{\nu'}(x'). \tag{A.27}$$

The role of $\Pi^\mu_{\nu'}$ is to take a vector at x' and parallel transport it to x. Since the metric tensor is covariantly constant and satisfies, therefore, the geodesic equation, it is automatically parallel transported on Γ. This implies

$$g_{\alpha\beta} = \Pi^{\alpha'}_\alpha \Pi^{\beta'}_\beta g_{\alpha'\beta'}.$$

From (A.8) and (A.9), we can see that

$$e^\alpha_{a;\beta}\Omega^\beta = e^\alpha_{a;\beta}\Delta u t^\beta = \Delta u e^\alpha_{a;\beta}\frac{dx^\beta}{du} = 0,$$

at x, and

$$e^{\alpha'}_{a;\beta'}\Omega^{\beta'} = \Delta u e^{\alpha'}_{a;\beta'}\frac{dx^{\beta'}}{du} = 0,$$

at x'.
These are the geodesic equations for the tetrad, which are zero since the basis vectors e are parallel transported. From these last expressions, it is clear that

$$\Pi^\alpha_{\alpha';\beta}\Omega^\beta = \Pi^\alpha_{\alpha';\beta'}\Omega^{\beta'} = 0. \tag{A.28}$$

Also note that, if t^μ is the tangent to Γ, we have $t^\alpha = \Pi^\alpha_{\alpha'} t^{\alpha'}$ and hence, again using (A.8) and (A.9), we notice that

$$\Omega_\alpha = -\Pi^{\alpha'}_\alpha \Omega_{\alpha'}, \qquad \Omega_{\alpha'} = -\Pi^\alpha_{\alpha'}\Omega_\alpha.$$

The coincidence limit of the parallel propagator follows from (A.27):

$$[\Pi^\alpha_{\beta'}] = \delta^{\alpha'}_{\beta'},$$

while other coincidence limits can be extracted by further differentiation of (A.28) and by then taking coincidence limits, the results are [23]:

$$[\Pi^{\alpha}_{\beta';\gamma}] = [\Pi^{\alpha}_{\beta';\gamma'}] = 0,$$

and

$$[\Pi^{\alpha}_{\beta';\gamma;\delta}] = -\frac{1}{2}R^{\alpha'}{}_{\beta'\gamma'\delta'}, \qquad [\Pi^{\alpha}_{\beta';\gamma;\delta'}] = \frac{1}{2}R^{\alpha'}{}_{\beta'\gamma'\delta'},$$
$$[\Pi^{\alpha}_{\beta';\gamma';\delta}] = -\frac{1}{2}R^{\alpha'}{}_{\beta'\gamma'\delta'}, \qquad [\Pi^{\alpha}_{\beta';\gamma';\delta'}] = \frac{1}{2}R^{\alpha'}{}_{\beta'\gamma'\delta'}. \tag{A.29}$$

A.5 Near-coincidence expansion of bitensors

To find a series expansion approximation for a generic bitensor $\Xi_{\mu'\nu'}(x, x')$ near coincidence, we write it as a series in the quantity $\Omega^{\alpha'}$, since it is the closest analog to the flat spacetime event distance $(x' - x)^{\alpha}$. A general form of such an expansion for a rank two tensor at point x' takes the form

$$\Xi_{\mu'\nu'}(x, x') = A_{\mu'\nu'} + A_{\mu'\nu'\rho'}\Omega^{\rho'} + \frac{1}{2}A_{\mu'\nu'\rho'\sigma'}\Omega^{\rho'}\Omega^{\sigma'} + ...,$$

where the coefficients $A_{...}$ are regular tensors at the base point x', clearly symmetric in the indices ρ' and σ'. To find their form, we differentiate and take coincidence limits of this expression, hence to zeroth order we get

$$A_{\mu'\nu'} = [\Xi_{\mu'\nu'}].$$

Differentiating once and taking the limit gives

$$A_{\mu'\nu'\alpha'} = [\Xi_{\mu'\nu'\alpha'}] - A_{\mu'\nu';\alpha'}.$$

Repeating this process once more gives the second-order expansion coefficient

$$A_{\mu'\nu'\alpha'\beta'} = [\Xi_{\mu'\nu'\alpha'\beta'}] - A_{\mu'\nu';\alpha';\beta'} - A_{\mu'\nu'\alpha';\beta'} - A_{\mu'\nu'\beta';\alpha'}.$$

In the case of a mixed-indices type of bitensor $\Xi_{\mu'\nu}$, the parallel propagator comes in to help, we may write

$$\tilde{\Xi}_{\mu'\nu'} = \Pi^{\nu}_{\nu'}\Xi_{\mu'\nu},$$

and expand this. Then, invert the relation with the inverse of the parallel propagator before taking the usual coincidence limits. We thus get the following expansion

$$\Xi_{\mu'\nu} = \Pi^{\nu'}_{\nu}\left(B_{\mu'\nu'} + B_{\mu'\nu'\rho'}\Omega^{\rho'} + \frac{1}{2}B_{\mu'\nu'\rho'\sigma'}\Omega^{\rho'}\Omega^{\sigma'} + ...\right). \tag{A.30}$$

The coincidence limit of this equation gives, simply

$$B_{\mu'\nu'} = [\Xi_{\mu'\nu}].$$

Differentiating once gives

$$\Xi_{\mu'\nu;\alpha'} = \Pi^{\nu'}_{\nu;\alpha'}(...) + \Pi^{\nu'}_{\nu}[B_{\mu'\nu';\alpha'} + B_{\mu'\nu'\rho';\alpha'}\Omega^{\rho'} + \mu'\nu'\rho'\Omega^{\rho'}{}_{\alpha'} + \\ + B_{\mu'\nu'\rho'\sigma'}\Omega^{\rho'}\Omega^{\sigma'}{}_{\alpha'} + \frac{1}{2}B_{\mu'\nu'\rho'\sigma';\alpha'}\Omega^{\rho'}\Omega^{\sigma'}],$$

where the ... in parenthesis refers to the whole expression in parenthesis in eq. (A.30). By what we know about the coincidence limits of the various quantities, we get easily that

$$B_{\mu'\nu'\alpha'} = [\Xi_{\mu'\nu;\alpha'}] - B_{\mu'\nu';\alpha'}.$$

After further differentiation, the terms not vanishing in the limit now will include a term proportional to the Riemann tensor from the second formula in (A.29):

$$B_{\mu'\nu'\alpha'\beta'} = [\Xi_{\mu'\nu;\alpha';\beta'}] + \frac{1}{2}B_{\mu'\lambda'}R^{\lambda'}{}_{\nu'\alpha'\beta'} - B_{\mu'\nu';\alpha';\beta'} - B_{\mu'\nu'\alpha';\beta'} - B_{\mu'\nu'\beta';\alpha'}.$$

The same method with two parallel propagators can be applied to a tensor with two unprimed indices. The procedure is the same, hence we only report the results here: The expansion takes the form

$$\Xi_{\mu\nu} = \Pi^{\nu'}_{\nu}\Pi^{\mu'}_{\mu}\left(C_{\mu'\nu'} + C_{\mu'\nu'\rho'}\Omega^{\rho'} + \frac{1}{2}C_{\mu'\nu'\rho'\sigma'}\Omega^{\rho'}\Omega^{\sigma'} + ...\right).$$

The coefficients follow from the usual procedure:

$$C_{\mu'\nu'} = [\Xi_{\mu\nu}],$$
$$C_{\mu'\nu'\rho'} = [\Xi_{\mu\nu;\rho'}] - C_{\mu'\nu';\rho'},$$
$$C_{\mu'\nu'\rho'\sigma'} = [\Xi_{\mu'\nu;\alpha';\beta'}] + \frac{1}{2}C_{\mu'\lambda'}R^{\lambda'}{}_{\nu'\rho'\sigma'} + \frac{1}{2}C_{\lambda'\nu'}R^{\lambda'}{}_{\mu'\rho'\sigma'} \\ - C_{\mu'\nu';\alpha';\beta'} - C_{\mu'\nu'\alpha';\beta'} - C_{\mu'\nu'\beta';\alpha'}.$$

A.6 The Van Vleck-Morette determinant

Another important object in our study will be the so-called Van Vleck-Morette determinant. It is a biscalar defined as follows [23]:

$$\Delta(x,x') = \det\left\{\Delta^{\mu'}{}_{\nu'}(x,x')\right\}, \qquad \Delta^{\mu'}{}_{\nu'}(x,x') = -\Pi^{\mu'}_{\mu}(x,x')\Omega^{\mu}{}_{\nu'}(x,x').$$

And we will now proceed to write it in a more recognizable form, as it is cited in many other texts. We begin using the inverse of equation (A.25), namely

$$g^{\alpha\beta} = \eta^{ab} e_a^\alpha e_b^\beta.$$

Then, by taking the determinant and using the fact that it is a multiplicative map, we get

$$\det\left\{g^{\alpha\beta}\right\} = \det\left\{\eta^{ab} e_a^\alpha e_b^\beta\right\} = \det\left\{\eta^{ab}\right\} \det\{e_a^\alpha\} \det\left\{e_b^\beta\right\} = -e^2,$$

where e is the determinant of the basis vectors coefficients e_a^α. Calling g the determinant of the metric and recalling that $\det\{A^{-1}\} = \frac{1}{\det\{A\}}$, we obtain

$$\frac{1}{\det\{g_{\alpha\beta}\}} = -e^2 \longrightarrow e = \frac{1}{\sqrt{-g}}. \tag{A.31}$$

The same procedure applied to the inverse completeness relation at P', namely $g^{\alpha'\beta'} = \eta^{ab} e_a^{\alpha'} e_b^{\beta'}$, results in

$$e' = \frac{1}{\sqrt{-g'}}, \tag{A.32}$$

where primed quantities are evaluated at point P'. Now, consider equation (A.26), written as

$$\Pi^\alpha_{\alpha'} = \eta^{ab} g_{\alpha'\beta'} e_a^\alpha e_b^{\beta'},$$

and take the determinant, to yield

$$\det\{\Pi^\alpha_{\alpha'}\} = -g' e e',$$

which becomes, by equations (A.31) and (A.32),

$$\det\{\Pi^\alpha_{\alpha'}\} = \frac{\sqrt{-g'}}{\sqrt{-g}}.$$

Now, take the definition of the VVD and apply what we just found

$$\Delta(x, x') = \det\left\{-\Pi^\alpha_{\alpha'} \Omega^\alpha_{\beta'}\right\} = \det\{\Pi^\alpha_{\alpha'}\} \det\{-g^{\alpha\gamma}\Omega_{\gamma\beta'}\} = -\frac{\det\{-\Omega_{\gamma\beta'}\}}{\sqrt{-g'}\sqrt{-g}},$$

which is the desired result.

From the coincidence limits of the world function and the parallel propagator derived in the previous sections, it is clear that

$$[\Delta^{\alpha'}_{\beta'}] = \delta^{\alpha'}_{\beta'}, \qquad [\Delta] = 1.$$

The near coincidence expansion of the VVD follows from the expansion (A.30) applied to the world function, given by

$$\Omega_{\mu'\nu} = \Pi^{\nu'}_{\nu}\left(B_{\mu'\nu'} + B_{\mu'\nu'\rho'}\Omega^{\rho'} + \frac{1}{2}B_{\mu'\nu'\rho'\sigma'}\Omega^{\rho'}\Omega^{\sigma'} + ...\right),$$

with the coefficients

$$B_{\mu'\nu'} = [\Omega_{\mu'\nu}] = -g_{\mu'\nu'},$$
$$B_{\mu'\nu'\alpha'} = [\Omega_{\mu'\nu;\rho'}] = 0,$$
$$B_{\mu'\nu'\rho'\sigma'} = [\Omega_{\mu'\nu;\rho';\sigma'}] + \frac{1}{2}B_{\mu'\lambda'}R^{\lambda'}{}_{\nu'\rho'\sigma'} = \frac{1}{3}\left(R_{\nu'\rho'\sigma'\mu'} - \frac{1}{2}R_{\mu'\nu'\rho'\sigma'}\right),$$

where the first two results follow from well-known facts, while the third follows from a combination of the commutation rule (A.12) and the use of Synge's rule (A.24) on the known coincidence limits of Ω (A.20):

$$[\Omega_{\mu'\nu\rho'\sigma'}] = [\Omega_{\nu\mu'\rho'\sigma'}] = -S_{\nu'\sigma'\rho'\mu'} = \frac{1}{3}\left(R_{\nu'\rho'\sigma'\mu'} + R_{\nu'\mu'\sigma'\rho'}\right).$$

And so, we finally get the near coincidence expansion of the mixed-index second derivative of the world function:

$$\Omega_{\mu'\nu} = -\Pi^{\nu'}_{\nu}\left(g_{\mu'\nu'} + \frac{1}{6}R_{\nu'\rho'\mu'\sigma'}\Omega^{\rho'}\Omega^{\sigma'}\right) + ...$$

Then, using the fact that $\Pi^{\nu}_{\alpha'}\Pi^{\nu'}_{\nu} = \delta^{\nu'}_{\alpha'}$, we immediately get the expansion for the VVD, starting with

$$\Delta^{\alpha'}_{\mu'} = -\Pi^{\alpha'\nu}\Omega_{\mu'\nu} = \delta^{\alpha'}_{\mu'} + \frac{1}{6}R^{\alpha'}{}_{\rho'\mu'\sigma'}\Omega^{\rho'}\Omega^{\sigma'} + ...$$

And recalling the approximation for a small magnitude matrix $\mathbf{A}$:

$$\det\{\mathbf{I} + \mathbf{A}\} = 1 + \mathrm{Tr}\{\mathbf{A}\} + O(\mathbf{A}^2),$$

we obtain

$$\Delta = 1 + \frac{1}{6}R_{\rho'\sigma'}\Omega^{\rho'}\Omega^{\sigma'} + ...$$

We now set out to derive an important differential equation satisfied by Δ [24]. We begin with equation (A.14), written in the form

$$\Omega = \frac{1}{2}\Omega^{\mu}\Omega_{\mu},$$

and derive both sides. First with respect to x^{ν} and then with respect to $x^{\sigma'}$, obtaining

$$\Omega_{\nu\sigma'} = \Omega_{\mu\nu\sigma'}\Omega^{\mu} + \Omega_{\mu\nu}\Omega^{\mu}{}_{\sigma'}.$$

Now, contract both sides with $-\Pi^{\nu}_{\nu'}$ and note that $\Omega_{\mu\nu\sigma'} = \Omega_{\nu\sigma'\mu}$, thanks to the symmetries (A.15) and (A.12). Then,

$$\Delta_{\nu'\sigma'} = -\Pi^{\nu}_{\nu'}\Omega_{\nu\sigma'\mu}\Omega^{\mu} + \Delta_{\mu\nu'}\Omega^{\mu}{}_{\sigma'}.$$

Next, integrate the first term on the right-hand side by parts:

$$\Delta_{\nu'\sigma'} = \left([-\Pi^{\nu}_{\nu'}\Omega_{\nu\sigma'}]_{;\mu} - \Pi^{\nu}_{\nu';\mu}\Omega_{\nu\sigma'}\right)\Omega^{\mu} + \Delta_{\mu\nu'}\Omega^{\mu}{}_{\sigma'}.$$

The second term in brackets is zero due to (A.28), this leaves us with

$$\Delta_{\nu'\sigma'} = \Delta_{\nu'\sigma';\mu}\Omega^{\mu} + \Delta_{\mu\nu'}\Omega^{\mu}{}_{\sigma'}.$$

Now, multiply both sides by the inverse of the VVD, namely $(\Delta^{-1})^{\nu'}_{\alpha'}$, defined by the fact that

$$\left(\Delta^{-1}\right)^{\nu'}_{\alpha'}\Delta^{\alpha'}_{\mu'} = \delta^{\nu'}_{\mu'},$$

to obtain

$$g_{\alpha'\sigma'} = \left(\Delta^{-1}\right)^{\nu'}_{\alpha'}\Delta_{\nu'\sigma';\mu}\Omega^{\mu} + g_{\mu\alpha'}\Omega^{\mu}{}_{\sigma'}.$$

Now contract with $g^{\alpha'\sigma'}$ and, given that we are working in d space-time dimensions, we arrive to

$$d = \left(\Delta^{-1}\right)^{\nu'\sigma'}\Delta_{\nu'\sigma';\mu}\Omega^{\mu} + \Omega^{\alpha'}{}_{\alpha'}.$$

We now invoke Jacobi's formula:

$$\frac{\delta(\det\{\mathbf{A}\})}{\det\{\mathbf{A}\}} = \delta(\ln\det\{\mathbf{A}\}) = \mathrm{Tr}\big\{\mathbf{A}^{-1}\delta\mathbf{A}\big\}.$$

And recognize that we do have the trace in question in our expression, we then finally write

$$d = \Delta^{-1}\Delta_{;\mu}\Omega^{\mu} + \Omega^{\mu}{}_{\mu} = \ln\Delta_{;\mu}\Omega^{\mu} + \Omega^{\mu}{}_{\mu}. \qquad \text{(A.33)}$$

Another compact form of this equation, which may be useful, comes from multiplying by Δ:

$$d\Delta = \Delta_{;\mu}\Omega^{\mu} + \Omega^{\mu}{}_{\mu}\Delta = \left(\Delta\Omega^{\mu}\right)_{;\mu}.$$

The last thing we wish to derive are the relations I1 and I2 involving the VVD stated in [13], namely

$I1: \ \nabla_{\mathbf{t}}\ln\Delta = \frac{d-1}{|2\Omega|^{\frac{1}{2}}} - K,$

$I2: \ \nabla_{\mathbf{t}}\nabla_{\mathbf{t}}\ln\Delta = -\frac{d-1}{|2\Omega|} + K_{\mu\nu}K^{\mu\nu} + R_{\mu\nu}t^{\mu}t^{\nu},$

where

$$K_{\mu\nu} = t_{\nu;\mu} = \left[\frac{\Omega_\nu}{|2\Omega|^{\frac{1}{2}}}\right]_{;\mu} = \frac{\Omega_{\nu\mu}}{|2\Omega|^{\frac{1}{2}}} - \epsilon\frac{\Omega_\nu\Omega_\mu}{|2\Omega|^{\frac{3}{2}}}.$$

From which follows

$$K = K^\mu_\mu = \frac{\Omega^\mu{}_\mu}{|2\Omega|^{\frac{1}{2}}} - \epsilon\frac{2\Omega}{|2\Omega|^{\frac{1}{2}}} = \frac{\Omega^\mu{}_\mu - 1}{\sigma},$$

where we used a notation such that $|2\Omega|^{\frac{1}{2}} = (2\epsilon\Omega)^{\frac{1}{2}} = \sqrt{\epsilon\sigma^2} \equiv \sigma$.
Using this and equation (A.13), equation $I1$ and (A.33) are clearly equivalent. As for equation $I2$, we again take (A.33) and covariantly differentiate it. We obtain

$$\ln\Delta_{;\mu;\nu}\Omega^\mu + \ln\Delta_{;\mu}\Omega^\mu{}_\nu = -\Omega^\mu{}_{\mu\nu}.$$

By multiplying both sides by Ω^ν, this becomes

$$\ln\Delta_{;\mu;\nu}\Omega^\mu\Omega^\nu + \ln\Delta_{;\mu}\Omega^\mu = -\Omega^\mu{}_{\mu\nu}\Omega^\nu,$$

and, using (A.33) on the second term, gives

$$\ln\Delta_{;\mu;\nu}\Omega^\mu\Omega^\nu = -d + \Omega^\mu{}_\mu - \Omega^\mu{}_{\mu\nu}\Omega^\nu.$$

One can easily show the following two relations hold:

$$\Omega^\mu{}_{\mu\nu}\Omega^\nu = \Omega^\mu{}_\mu - \Omega^{\mu\nu}\Omega_{\mu\nu} - R_{\lambda\nu}\Omega^\lambda\Omega^\nu,$$

and

$$K_{\mu\nu}K^{\mu\nu} = \frac{\Omega_{\mu\nu}\Omega^{\mu\nu} - 1}{|2\Omega|}.$$

Finally, combining these last 3 equations and using (A.13), again gives equation $I2$.

A.7 Further readings

More coincidence limits and expansions, as well as an explanation of the point-splitting regularization method involving bitensors, can be found in [25].

B The Q-metric: a deeper look

B.1 Introduction

The classical description of gravity in general relativity relies on local quantities derived from the metric or the fields that represent sources of matter and energy. In trying to implement a quantum-compatible description of space-time, these local metric or matter/energy fields might not make much sense given what we know about the quantum realm. As a matter of fact, many (if not all) proposals of quantum theories of gravity agree on the impossibility of localizing an event with infinite precision to a point, which, in the theory of the q-metric, is effectively translated by the endowing of space-time with a finite zero-point length. Having no complete framework of quantum gravity, the fundamental reason for the emergence of this finite scale at coincidence is not known. Nevertheless, its consequences can be explored using the usual tools of differential geometry, as long as the theory can be kept covariant. Thus explained why we discussed bitensors at length: the metric theory of gravity can be reconstructed through the use of bitensors given that, as we have seen, quantities such as the metric tensor, the Riemann tensor, its contractions, etc., can be recovered in the limit of coincidence of such bitensors. It should, then, be immediate to apply a modification to these limits in which an exact localization of events becomes impossible.
Let us begin by finding the relation between the usual metric $g_{\mu\nu}$ and our q-metric. We begin by citing previous results: The position space propagator for a relativistic particle in flat space-time can be evaluated using the path integral approach. The action is:[25]

$$A = -m\int_{x'}^{x} d\tau = -m\int_{x'}^{x} \sqrt{\eta_{\mu\nu}dx^{\mu}dx^{\nu}} = -m\int_{u_0}^{u_1} du\sqrt{\eta_{\mu\nu}t^{\mu}t^{\nu}} = -m\sqrt{\epsilon}\Delta u,$$

which can be written, in our notation, as

$$A = -m\sqrt{\epsilon\sigma^2}.$$

Then the propagator is, symbolically

$$G(x;x') = \sum_{x'}^{x} \exp\Big\{-im\sqrt{\epsilon\sigma^2}\Big\},$$

which can be evaluated to yield

$$G(x;x') = \frac{1}{4\pi^2}\frac{-im}{\sqrt{\epsilon\sigma^2}}K_1(im\sqrt{\epsilon\sigma^2}),$$

where K_1 is the modified Bessel function of first order and $\sigma^2 = \eta_{\alpha\beta}(x^{\alpha'} - x^{\alpha})(x^{\beta'} - x^{\beta})$. [27] suggests a modification to the Feynman propagator of a massive scalar field, due

[25] Calculations can be found in [26]

to the introduction of a zero-point length L_0, through shifting the geodesic length σ to $\sqrt{\sigma^2 + L_0^2}$. But, more generally, we might just introduce our modified geodesic length without assuming the form of the function describing it by the substitution $\sigma \to s$:

$$G(x; x') = \frac{1}{4\pi^2} \frac{-im}{\sqrt{\epsilon s}} K_1(im\sqrt{\epsilon s}), \tag{B.1}$$

Another result is the leading order divergent part of the Hadamard form of the propagator in arbitrary curved space being of the form [28]

$$G_S(x, x') \propto \frac{\sqrt{\Delta}}{(\Omega(x, x') + i\epsilon)^{\frac{d-2}{2}}} + ...$$

and, interestingly enough, equation (B.1), when generalized to curved space-time, does in fact gain a factor of $\sqrt{\Delta}$ as can be read in [29]. This shows that to be general and include curvature effects in the modified propagator, the q-metric will have to include, in some way, the VVD, which accounts for the expansion of geodesic congruences, i.e. curvature.

B.2 Construction

We wish to construct a modified metric such that the form of the 2-point function (B.1) arises naturally as its kernel. We start here with the simple shift of magnitude L_0 in the geodesic distance and, later on, generalize to arbitrary modifications. To achieve this, we denote the initial, usual coordinates with primes and follow [27]. Start in flat space-time and rotate to Euclidean signature. The line element is, in Cartesian and hyperspherical coordinates:

$$d\tau^2 = \sum_i dx_i^2 = dR^2 + R^2 d\Theta_{(3)}^2,$$

where $d\Theta_{(3)}^2$ is the line element on a 3-sphere of radius R.
Note geodesic distances from a point are given by the radial distance R. For this reason, from now on, we identify R with the geodesic length σ and write

$$d\tau^2 = d\sigma^2 + \sigma^2 d\Theta_{(3)}^2. \tag{B.2}$$

We now apply the simplest (or leading order) prescription for a zero-point minimum length by the Pythagorean addition of such length scale

$$\sigma \to \bar{\sigma} = \sqrt{\sigma^2 + \epsilon L_0^2},$$

where ω is the modified geodesic distance. The differentials are related in the following way:

$$d\bar{\sigma} = \frac{\partial \bar{\sigma}}{\partial \sigma} d\sigma = \frac{\sigma}{\sqrt{\sigma^2 + \epsilon L_0^2}} d\sigma \longrightarrow d\bar{\sigma}^2 = \frac{\sigma^2}{\sigma^2 + \epsilon L_0^2} d\sigma^2.$$

Hence the line element above can be rewritten as

$$d\tau^2 = \left(1 + \epsilon\frac{L_0^2}{\sigma^2}\right) d\bar{\sigma}^2 + \sigma^2 d\Theta_{(3)}^2.$$

Subtracting the last term on both sides yields

$$d\tau^2 - \sigma^2 d\Theta_{(3)}^2 = d\sigma^2 = \left(1 + \epsilon\frac{L_0^2}{\sigma^2}\right) d\bar{\sigma}^2.$$

It is then clear, by (B.2), that

$$d\sigma = \sqrt{1 + \epsilon\frac{L_0^2}{\sigma^2}} d\bar{\sigma}.$$

This can be integrated to obtain the transformation between the geodesic lengths:

$$\sigma = T\bar{\sigma}, \qquad T = \sqrt{1 + \epsilon\frac{L_0^2}{\sigma^2}} = \sqrt{1 + \frac{L_0^2}{2\epsilon\Omega}},$$

where we used the fact that $\sigma^2 = 2\Omega$, and we will alternate between the geodesic length and the world function depending on what is more convenient. The factor ϵ is trivially 1 in Euclidean space, but it will be needed when we rotate back to Lorentzian metric. Notice the transformation is singular in the coincidence limit, where $\sigma \to 0$, which is to be expected. The transformation we just performed corresponds to the following when applied to a Cartesian set of coordinates:

$$\sum_i x_i^2 \to T^2 \sum_i x_i^2,$$

and for the single coordinates, we have

$$x_i \to y_i = T x_i.$$

Now, we wish to incorporate this transformation in the metric. To do this, we impose that the geodesic interval element, as measured in the new metric using the old coordinates, be equal to the one measured in the flat metric using the transformed coordinates, this means

$$q_{\mu\nu} dx^\mu dx^\nu = g_{\alpha\beta} dy^\alpha dy^\beta,$$

which in turn implies

$$q_{\mu\nu}(x) = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(y(x)),$$

where $g_{\alpha\beta}(x) = \delta_{\alpha\beta}$ in euclidean space. Since $y^\mu = Tx^\mu$, this requires us to compute

$$q_{\mu\nu}(x) = \frac{\partial(Tx^\alpha)}{\partial x^\mu}\frac{\partial(Tx^\beta)}{\partial x^\nu}\delta_{\alpha\beta}. \tag{B.3}$$

Clearly, we have

$$\frac{\partial(Tx^\alpha)}{\partial x^\mu} = \frac{\partial T}{\partial x^\mu}x^\alpha + T\delta^\alpha_\mu.$$

And, recalling that $\Omega_\mu = \sigma t_\mu = \sqrt{2\epsilon\Omega}t_\mu$, and that in this geometry the normalized tangent vector to the geodesic is, simply, $t_\mu = \frac{\delta_{\mu\nu}x^\nu}{\sqrt{2\epsilon\Omega}} = \frac{x_\mu}{\sigma}$, we can write

$$\frac{\partial T}{\partial x^\mu} = \frac{\partial \sigma^2}{\partial x^\mu}\partial_{\sigma^2}T = 2\Omega_\mu\partial_{\sigma^2}T = -\epsilon\frac{L_0^2}{\sigma^4}T^{-1}\Omega_\mu,$$

and it follows that

$$(\partial_\mu T x^\alpha) = -\epsilon\frac{L_0^2}{\sigma^4}T^{-1}\Omega_\mu\sigma t^\alpha = -\epsilon\frac{L_0^2}{\sigma^4}T^{-1}\sqrt{\epsilon\sigma^2}^2 t_\mu t^\alpha = -\frac{L_0^2}{\sigma^2}T^{-1}t_\mu t^\alpha.$$

Hence we can finally write

$$\frac{\partial(Tx^\alpha)}{\partial x^\mu} = T\delta^\alpha_\mu - \frac{L_0^2}{\sigma^2}T^{-1}t_\mu t^\alpha.$$

Plugging what we know in (B.3), we obtain

$$q_{\mu\nu}(x) = T^2\delta_{\mu\nu} - \epsilon\left[\frac{L_0^2}{\sigma^2}\left(2\epsilon - \frac{L_0^2}{\sigma^2}T^{-2}\right)\right]t_\mu t_\nu,$$

which can be cast in the following, more familiar form

$$q_{\mu\nu}(x) = T^2\delta_{\mu\nu} - \frac{L_0^2}{\sigma^2}\left(\frac{2+\epsilon\frac{L_0^2}{\sigma^2}}{1+\epsilon\frac{L_0^2}{\sigma^2}}\right)t_\mu t_\nu = T^2\delta_{\mu\nu} - \epsilon(T^2 - T^{-2})t_\mu t_\nu.$$

Rotating back to Lorentzian signature now only requires us to allow for ϵ to take both values ± 1, to accommodate for the fact that $t^\mu t_\mu = \epsilon = \pm 1$, depending on the spacelike/timelike character of the tangent vector respectively.
Finally, this leaves us with

$$q_{\mu\nu}(x) = T^2\eta_{\mu\nu} - \epsilon(T^2 - T^{-2})t_\mu t_\nu, \tag{B.4}$$

where, now, $T = \sqrt{1+\epsilon\frac{L_0^2}{\sigma_\eta^2}}$, and σ_η is the Minkowski geodesic length.

The inverse is easily found to be

$$q^{\mu\nu}(x) = (T^{-1})^2 g^{\mu\nu} + \epsilon(T^2 - T^{-2})t^\mu t^\nu.$$

This can be proven either by direct computation or by a general result which we will state later.

B.3 Disformal transformations

Equation (B.4) is essentially a disformal transformation in the biscalar Ω. A disformal transformation is a generalization of a conformal transformation that redefines measures differently along the gradient of a particular scalar field, the world function in this case. Unlike conformal transformations, disformal transformations are not isotropic and therefore do not preserve angles. Such a transformation was shown by Bekenstein [30] to be the most general relation between two geometries in a theory (one describing the gravitational geometry, the other describing the dynamics of matter) which is compatible with the principles of weak equivalence and causality. Unlike the simpler conformal transformation, due to the anisotropy of the modification of scales it induces, this sort of transformation does affect the shape of light cones, opening up the possibility of, e.g., the shrinking of light cones at small scales or other phenomena which have been suggested in the context of quantum gravity [31].

A general disformal transformation takes the form:

$$q_{\mu\nu} = A g_{\mu\nu} + \epsilon B t_\mu t_\nu. \tag{B.5}$$

And the inverse has the general form [32]

$$q^{\mu\nu} = A^{-1} g^{\mu\nu} - \epsilon \frac{A^{-1}B}{A - 2\epsilon I B} t^\mu t^\nu,$$

with

$$I = -\frac{1}{2} g^{\mu\nu} t_\mu t_\nu = -\frac{\epsilon}{2},$$

gives

$$q^{\mu\nu} = A^{-1} g^{\mu\nu} - \epsilon \frac{A^{-1}B}{A + B} t^\mu t^\nu = A^{-1} g^{\mu\nu} + \epsilon C t^\mu t^\nu,$$

where for the moment we consider t_μ as the tangent vector to a generic scalar field ϕ, hence

$$t_\mu = \frac{\partial_\mu \phi}{\sqrt{\epsilon g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi}} \longrightarrow g^{\mu\nu} t_\mu t_\nu = \epsilon = \pm 1. \tag{B.6}$$

We can further calculate the magnitude of these vectors in the disformal metric:

$$q_{\mu\nu} t^\mu t^\nu = \epsilon (A + B).$$

Then deduce the following:

$$\frac{q_{\mu\nu} t^\mu t^\nu}{g_{\mu\nu} t^\mu t^\nu} = A + B,$$

which suggests the introduction of the following vectors of unit magnitude in this frame:

$$T_\mu = (A + B)^{\frac{1}{2}} t_\mu, \qquad T^\mu = q^{\mu\nu} t_\nu = (A + B)^{-\frac{1}{2}} t^\mu,$$

which respect the relation

$$q^{\mu\nu}T_\mu T_\nu = \epsilon.$$

It's important to notice that the Lorentzian signature of the metric has to be conserved. If the metric is such that no sign-inversion in its terms is present, we should impose, at all points, that the following holds:

$$q_{00} = Ag_{00} + \epsilon B t_0 t_0 > 0. \tag{B.7}$$

Since the function B may be zero at some point, this also implies the condition $A > 0$ (clearly $g_{00} > 0$).
We now wish to find the induced metric of a hypersurface orthogonal to the vector field T_μ (also called its "first fundamental form"), this is, in analogy with the case for the t_μ in the usual metric:

$$\bar{h}_{\mu\nu} = q_{\mu\nu} - \epsilon T_\mu T_\nu = Ah_{\mu\nu}. \tag{B.8}$$

Thus we found that the induced metrics on such hypersurfaces, as seen in the usual metric and the disformally related one, are conformally related to each other.

The "second fundamental form" or extrinsic curvature is, in the usual metric:

$$K_{\mu\nu} = h^\alpha_\mu \nabla_\alpha t_\nu = (\delta^\alpha_\mu - \epsilon t_\mu t^\alpha)\nabla_\alpha t_\nu = \nabla_\mu t_\nu - \epsilon a_\nu t_\mu,$$

where $a_\nu = t^\alpha \nabla_\alpha t_\nu$ is the acceleration of the tangent vector t_μ along the hypersurface. This vanishes if the tangent vector satisfies the geodesic equation, which will be our case later on. But for the moment, we remain general and keep the acceleration term.

The extrinsic curvature in the disformal metric can thus be written as:

$$\bar{K}_{\mu\nu} = \bar{h}^\alpha_\mu \bar{\nabla}_\alpha T_\nu = \bar{\nabla}_\mu T_\nu - \epsilon \bar{a}_\nu T_\mu,$$

where $\bar{a}_\nu = T^\alpha \bar{\nabla}_\alpha T_\nu$, and the covariant derivative in the disformal frame acts as:

$$\bar{\nabla}_\alpha T_\beta = \partial_\alpha T_\beta - \bar{\Gamma}^\gamma_{\alpha\beta} T_\gamma.$$

Now, to find the relation between the Christoffel symbols in the two frames, we recall the usual form of the connection:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}\right),$$

and compare it with the connection in the disformal frame we need:

$$\bar{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{2} q^{\alpha\delta} \left(\partial_\gamma q_{\delta\beta} + \partial_\beta q_{\delta\gamma} - \partial_\delta q_{\beta\gamma}\right).$$

A relation between the two can be easily found by adding and subtracting the relevant connections to complete the derivatives into their covariant form, i.e.

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} = \frac{1}{2} q^{\alpha\delta} \Big[\left(\partial_{\gamma} q_{\delta\beta} - \Gamma^{\mu}_{\gamma\delta} q_{\beta\mu} - \Gamma^{\mu}_{\gamma\beta} q_{\delta\mu} \right) + \Gamma^{\mu}_{\gamma\delta} q_{\beta\mu} + \Gamma^{\mu}_{\gamma\beta} q_{\delta\mu} + ... \\ = \frac{1}{2} q^{\alpha\delta} \left(\nabla_{\gamma} q_{\delta\beta} + \nabla_{\beta} q_{\delta\gamma} - \nabla_{\delta} q_{\beta\gamma} \right) + \Gamma^{\alpha}_{\gamma\beta}.$$

Given this, we will need a few identities. Start by expanding the following term:

$$\nabla_{\gamma} q_{\delta\beta} = \nabla_{\gamma} \left(A g_{\delta\beta} + \epsilon B t_{\delta} t_{\beta} \right) = (\partial_{\gamma} A) g_{\delta\beta} + \epsilon \left((\partial_{\gamma} B) t_{\delta} t_{\beta} + B (t_{\delta} \nabla_{\gamma} t_{\beta} + t_{\beta} \nabla_{\gamma} t_{\delta}) \right),$$

and use this to find some equations that we will need, firstly

$$t^{\gamma} \nabla_{\gamma} q_{\delta\beta} = g_{\delta\beta} \partial_t A + \epsilon (\partial_t B) t_{\delta} t_{\beta} + \epsilon B (t_{\delta} a_{\beta} + t_{\beta} a_{\delta}), \tag{B.9}$$

and

$$t^{\beta} \nabla_{\gamma} q_{\delta\beta} = t_{\delta} \partial_{\gamma} (A + B) + \epsilon B (t_{\delta} t^{\beta} \nabla_{\gamma} t_{\beta} + \epsilon \nabla_{\gamma} t_{\delta}) = t_{\delta} \partial_{\gamma} (A + B) + B \nabla_{\gamma} t_{\delta}, \tag{B.10}$$

where we used the fact that

$$t^{\delta} \nabla_{\gamma} t_{\delta} = \nabla_{\gamma} (t^{\delta} t_{\delta}) - t_{\delta} \nabla_{\gamma} t_{\delta} = -t^{\delta} \nabla_{\gamma} t_{\delta},$$

which therefore vanishes.
We are now ready to find the following expression:

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} T_{\alpha} = (A + B)^{\frac{1}{2}} \Gamma^{\alpha}_{\beta\gamma} t_{\alpha} + \frac{1}{2} (A + B)^{\frac{1}{2}} q^{\alpha\delta} t_{\alpha} \left(\nabla_{\gamma} q_{\delta\beta} + \nabla_{\beta} q_{\delta\gamma} - \nabla_{\delta} q_{\beta\gamma} \right).$$

Using the fact that $q^{\alpha\delta} t_{\alpha} = (A + B)^{-1} t^{\delta}$, this becomes:

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} T_{\alpha} = (A + B)^{\frac{1}{2}} \Gamma^{\alpha}_{\beta\gamma} t_{\alpha} + \frac{1}{2} (A + B)^{-\frac{1}{2}} \left(t^{\delta} \nabla_{\gamma} q_{\delta\beta} + t^{\delta} \nabla_{\beta} q_{\delta\gamma} - t^{\delta} \nabla_{\delta} q_{\beta\gamma} \right),$$

and the term in parenthesis is what we need the relations in (B.9) and (B.10) for. The result for these terms is:

$$t^{\delta} \nabla_{\gamma} q_{\delta\beta} + t^{\delta} \nabla_{\beta} q_{\delta\gamma} - t^{\delta} \nabla_{\delta} q_{\beta\gamma} = \\ B(K_{\gamma\beta} + K_{\beta\gamma}) + t_{\beta} \partial_{\gamma} (A + B) + t_{\gamma} \partial_{\beta} (A + B) - \epsilon (\partial_t B) t_{\gamma} t_{\beta} - g_{\gamma\beta} \partial_t A.$$

The last piece we need is:

$$\partial_{\beta} T_{\gamma} = \partial_{\beta} \left[(A + B)^{\frac{1}{2}} t_{\gamma} \right] = \frac{1}{2} (A + B)^{-\frac{1}{2}} \partial_{\beta} (A + B) t_{\gamma} + (A + B)^{\frac{1}{2}} \partial_{\beta} t_{\gamma}.$$

After putting everything together, we finally find:

$$\bar{\nabla}_\beta T_\gamma = \partial_\beta T_\gamma - \bar{\Gamma}^\alpha_{\beta\gamma} T_\alpha = (A+B)^{\frac{1}{2}} \nabla_\beta t_\gamma - \frac{1}{2}(A+B)^{-\frac{1}{2}} \Big(B(K_{\gamma\beta} + K_{\beta\gamma}) + + t_\beta \partial_\gamma (A+B) - \epsilon(\partial_t B) t_\gamma t_\beta - g_{\gamma\beta}\partial_t A \Big).$$

The next step is adding and subtracting a factor of $\epsilon(\partial_t A) t_\gamma t_\beta$. After some simplifications and recalling the form of the induced metric ($h_{\mu\nu} = g_{\mu\nu} - \epsilon t_\mu t_\nu$), we obtain:

$$\bar{\nabla}_\beta T_\gamma = (A+B)^{\frac{1}{2}} \Big\{ \nabla_\beta t_\gamma - \frac{1}{2}(A+B)^{-1} \Big[B(K_{\gamma\beta} + K_{\beta\gamma}) - (\partial_t A) h_{\gamma\beta} - \big(\epsilon t_\gamma \partial_t (A+B) - \epsilon \partial_\gamma (A+B) \big) \Big] \Big\}.$$

We can, then, easily find that the acceleration of the tangent vectors in the disformal frame is:

$$T^\beta \bar{\nabla}_\beta T_\gamma = \bar{a}_\gamma = a_\gamma + \frac{1}{2}(A+B)^{-1} \left(t_\gamma \partial_t (A+B) - \epsilon \partial_\gamma (A+B) \right),$$

where we used the fact that $h_{\gamma\beta} t^\beta = 0$ and $t^\beta K_{\gamma\beta} = 0 = t^\beta K_{\beta\gamma}$.

Note that the second term vanishes if the coefficients of the disformal transformation are taken to be functions of the scalar field ϕ only (as in [33]). This is because, using (B.6), we obtain

$$\partial_t = t^\alpha \partial_\alpha = \frac{\phi^\alpha}{\sqrt{\epsilon\phi^2}} \partial_\alpha = \frac{\phi^2}{\sqrt{\epsilon\phi^2}} \partial_\phi = \sqrt{\frac{\phi^2}{\epsilon}} \partial_\phi = \epsilon \sqrt{\epsilon\phi^2} \partial_\phi,$$

where we used the notation $\partial_\alpha \phi = \phi_\alpha$, $\phi_\alpha \phi^\alpha = \phi^2$ and $\frac{\partial}{\partial\phi} = \partial_\phi$.[26]
Now it should be clear that, in such a case,

$$\epsilon \partial_\gamma (A+B) = \epsilon \partial_\phi (A+B) \phi_\gamma = \epsilon \sqrt{\epsilon\phi^2} t_\gamma \partial_\phi (A+B) = t_\gamma \partial_t (A+B).$$

Either way, we will continue with the general case, especially since we shall now see that in the expression for the extrinsic curvature, these terms cancel.

We are now in the condition of finding the disformal frame extrinsic curvature, we begin by defining:

$$\bar{K}_{\mu\nu} = \bar{\nabla}_\mu T_\nu - \epsilon \bar{a}_\nu T_\mu.$$

[26] We also note that $\frac{1}{\sqrt{\epsilon}} = \epsilon\sqrt{\epsilon}$ is valid in any but the null case.

After plugging in what we found for the individual terms on the right-hand side and some basic manipulation, the terms discussed above indeed cancel and we are left with

$$\begin{aligned}\bar{K}_{\mu\nu} = (A+B)^{\frac{1}{2}}\left[\nabla_\mu t_\nu - \epsilon a_\nu t_\mu - \frac{1}{2}(A+B)^{-1}\left(B(K_{\mu\nu}+K_{\nu\mu}) - (\partial_t A)h_{\mu\nu}\right)\right] \\ = (A+B)^{\frac{1}{2}}\left[K_{\mu\nu} - \frac{1}{2}(A+B)^{-1}\left(B(K_{\mu\nu}+K_{\nu\mu}) - (\partial_t A)h_{\mu\nu}\right)\right].\end{aligned}$$

If we assume the scalar field is hypersurface orthogonal, by Froebenius' theorem this implies the following condition holds:

$$t_{[\alpha}\nabla_\gamma t_{\beta]} = t_\gamma\nabla_\beta t_\alpha - t_\beta\nabla_\gamma t_\alpha + t_\alpha\nabla_\gamma t_\beta - t_\gamma\nabla_\alpha t_\beta + t_\beta\nabla_\alpha t_\gamma - t_\alpha\nabla_\beta t_\gamma = 0,$$

which, once contracted with t^γ on both sides, yields

$$K_{\alpha\beta} = K_{\beta\alpha}.$$

This assumption also implies the field t^μ is irrotational, i.e. $\epsilon^{\alpha\beta\gamma\delta}t_\beta\nabla_\gamma t_\delta = 0$.

Following this assumption, our previous result simplifies greatly:

$$\bar{K}_{\mu\nu} = (A+B)^{-\frac{1}{2}}\left[AK_{\mu\nu} + \frac{1}{2}(\partial_t A)h_{\mu\nu}\right],$$

and it follows easily, using the same methods used before in this section, that the extrinsic curvature scalar is

$$\bar{K} = q^{\mu\nu}\bar{K}_{\mu\nu} = (A+B)^{-\frac{1}{2}}\left[K + \frac{d-1}{2}\partial_t \ln A\right].$$

We can also find the variation along the tangent to our scalar field, in the disformal frame, of the extrinsic curvature scalar. For this purpose, we start by writing

$$\bar{\nabla}_T\bar{K} = T^\alpha\bar{\nabla}_\alpha\bar{K} = (A+B)^{-\frac{1}{2}}t^\alpha\partial_\alpha\bar{K},$$

which is easily found to be

$$\bar{\nabla}_T\bar{K} = \frac{1}{2}\partial_t(A+B)^{-1}\left[K + \frac{d-1}{2}\partial_t \ln A\right] + (A+B)^{-1}\left[\partial_t K + \frac{d-1}{2}\partial_t^2 \ln A\right].$$

Given the objects derived until now, we can remarkably derive the Ricci scalar for the conformal frame, appealing to the Gauss-Codazzi equation (for a derivation see section 3.5 of [34]), which reads

$$\begin{aligned}R &= R_\Sigma + \epsilon(K^2 - K_{\mu\nu}K^{\mu\nu}) + 2\epsilon\nabla_\alpha(t^\beta\nabla_\beta t^\alpha - t^\alpha\nabla_\beta t^\beta) \\ &= R_\Sigma - \epsilon(K^2 + K_{\mu\nu}K^{\mu\nu}) + 2\epsilon\nabla_\alpha a^\alpha - 2\epsilon\nabla_t K,\end{aligned}$$

where, in the second line, we used the fact that $t^\beta \nabla_\beta t^\alpha = a^\alpha$, and that

$$\nabla_\alpha (t^\alpha \nabla_\beta t^\beta) = (\nabla_\alpha t^\alpha)(\nabla_\beta t^\beta) + \nabla_t \nabla \beta t^\beta = K^2 + \nabla_t K.$$

Hence all we need to do to find the disformal frame Ricci scalar is translate all quantities as

$$\bar{R} = \bar{R}_\Sigma - \epsilon(\bar{K}^2 + \bar{K}_{\mu\nu}\bar{K}^{\mu\nu}) + 2\epsilon \bar{\nabla}_\alpha \bar{a}^\alpha - 2\epsilon \bar{\nabla}_T \bar{K}.$$

The only thing we are missing in order to put this object together is the disformal induced Ricci scalar on Σ. Luckily, given that the induced metric on the hypersurfaces are conformally related (equation (B.8)), as long as A is constant on the hypersurface Σ, this is conformally related to the Einstein frame induced Ricci scalar, hence

$$\bar{R}_\Sigma = A^{-1} R_\Sigma.$$

Another object which may be useful is the traceless symmetric part of the extrinsic curvature tensor. In the disformal frame, this is

$$\bar{\sigma}_{\mu\nu} = \bar{K}_{\mu\nu} - \frac{1}{d-1} \bar{h}_{\mu\nu} \bar{K},$$

which, using equation (B.8), quickly becomes

$$\bar{\sigma}_{\mu\nu} = \frac{A}{\sqrt{A+B}} \left(K_{\mu\nu} - \frac{1}{d-1} h_{\mu\nu} K \right) = \frac{A}{\sqrt{A+B}} \sigma_{\mu\nu}.$$

It is then easy to show that the following holds:

$$\bar{\sigma}^2_{\mu\nu} = q^{\alpha\mu} q^{\beta\nu} \bar{\sigma}_{\mu\nu} \bar{\sigma}_{\alpha\beta} = (A+B)^{-1} \sigma^2_{\mu\nu}.$$

Lastly, let us state some properties of the disformal frame:
From the definition of the connection coefficients, it is easy to see that these are symmetric in their lower indices, just like in the original frame, i.e. $\bar{\Gamma}^\alpha_{\mu\nu} = \bar{\Gamma}^\alpha_{\nu\mu}$. Given this fact, it can be proven, through a lengthy calculation, that the disformal metric respects the condition of metric compatibility in the disformal frame, namely

$$\bar{\nabla}_\mu q_{\alpha\beta} = 0,$$

and, in very much the same way as in the normal frame, the product rule for this derivative holds. We will need the form of the wave operator for a scalar field in the disformal frame. Recall that in the Einstein frame the following is true:

$$\Box \varphi = g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi = g^{\mu\nu} \nabla_\mu \partial_\nu \varphi = \nabla_\mu \partial^\mu \varphi = \partial_\mu \partial^\mu \varphi + \Gamma^\mu_{\alpha\mu} \partial^\alpha \varphi,$$

and it can be found easily that

$$\Gamma^{\mu}_{\alpha\mu} = \frac{1}{2} g^{\mu\beta} \partial_\alpha g_{\mu\beta} = \frac{1}{\sqrt{-g}} \partial_\alpha \sqrt{-g},$$

where the second equality follows from the matrix identity

$$\frac{d}{d\lambda} \log \det \mathbf{M} = \mathrm{Tr}\left(\mathbf{M}^{-1} \frac{d}{d\lambda} \mathbf{M} \right).$$

Then, it is clear that for a scalar field the d'Alembertian takes the form:

$$\Box \varphi = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} g^{\alpha\beta} \partial_\beta \varphi \right).$$

Much of the same procedure applies in the disformal frame, we write

$$\Box^q \varphi = q^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \varphi = \bar{\nabla}_\mu \bar{\varphi}^\mu = \partial_\mu \bar{\varphi}^\mu + \bar{\Gamma}^{\mu}_{\alpha\mu} \bar{\varphi}^\alpha,$$

where $\bar{\varphi}^\mu = q^{\mu\nu} \partial_\nu \varphi$.
And from the definition, it is easy to see that:

$$\bar{\Gamma}^{\mu}_{\alpha\mu} = \Gamma^{\mu}_{\alpha\mu} + \frac{1}{2} q^{\mu\delta} \nabla_\alpha q_{\mu\delta},$$

and the second term on the right-hand side is the one we will need to evaluate.
Expanding the metric terms and using the same methods used until now in the calculations of this section we arrive at

$$q^{\mu\delta} \nabla_\alpha q_{\mu\delta} = d \partial_\alpha \ln A + A^{-1} B \left(\partial_\alpha \ln \frac{B}{A+B} \right) = (d-1) \partial_\alpha \ln A + \partial_\alpha (A+B),$$

where d is the number of space-time dimensions. Finally, putting it all together, we find:

$$\begin{aligned} \bar{\Gamma}^{\mu}_{\alpha\mu} &= \partial_\alpha \ln \sqrt{-g} + \partial_\alpha \ln A^{\frac{d-1}{2}} + \partial_\alpha \ln (A+B)^{\frac{1}{2}} \\ &= \partial_\alpha \ln \sqrt{(-g) A^{d-1} (A+B)} = \partial_\alpha \ln \sqrt{-q}, \end{aligned} \quad \text{(B.11)}$$

and we will see that q is indeed the determinant of the disformal metric. As such, we will need to require it to be non-vanishing for the inverse metric to be well defined.[27]
Finally, the d'Alembertian operator in the disformal frame reads

$$\Box_q \varphi = \partial_\mu \bar{\varphi}^\mu + (\partial_\mu \ln \sqrt{-q}) \bar{\varphi}^\mu = \frac{1}{\sqrt{-q}} \partial_\mu \left(\sqrt{-q} q^{\mu\nu} \partial_\nu \varphi \right). \quad \text{(B.12)}$$

[27]Notice that both this requirement and requirement (B.7) are satisfied in the special case derived in section B.2

B.4 The q-metric for arbitrary curved spacetime

We have seen, at the beginning of this section, that the imposition of a minimal length in space-time generates a new metric, disformally related to the usual metric. But we were missing a key ingredient, there are reasons to believe that our previous derivation should be missing corrections due to curvature and can, at most, approximate the desired effects in an arbitrary curved spacetime. Let us proceed from here: since the precise way distances should be modified on small scales depends on a full theory of quantum gravity, we now want to extend the construction of the q-metric to arbitrary functions of the world function (or, equivalently, of the geodesic distance). That being said, we now consider a transformation of the world function of the type

$$\Omega \to S_{L_0}(\Omega),$$

where L_0 is the zero-point length.
The only constraints we will require from the function $S_{L_0}(\Omega)$ are

- Zero-point length assumption $S_{L_0}(0) = \epsilon \frac{L_0^2}{2}$,
- Identity $S_0(\Omega) = \Omega$,
- $\left[\frac{S_{L_0}(\Omega)}{S'^2_{L_0}(\Omega)}\right]_{\Omega=0} < \infty$,

where a prime indicates differentiation with respect to Ω. The last condition will become clear during the derivation.
Taking what we learned in the previous section, we now take the disformal transformation in the form (B.5) as our ansatz. To determine the coefficients A, B, and C we first require the modified geodesic length to satisfy the defining equation for the world function (A.14) in the q-metric, which then becomes

$$g^{\mu\nu}\Omega_\mu\Omega_\nu = 2\Omega \longrightarrow q^{\mu\nu}S_\mu S_\nu = 2S. \tag{B.13}$$

Note that, just as indices for the world function mean differentiation, also for S we write $\frac{\partial S}{\partial x^\alpha} = S_\alpha$.

Now, we will need the relations

$$t^\alpha = \frac{\Omega^\alpha}{\sqrt{2\epsilon\Omega}},$$

and

$$S_\mu = \partial_\mu S = \frac{\partial S}{\partial \Omega}\Omega_\mu = (\partial_\Omega S)\Omega_\mu = S'\Omega_\mu,$$

where we also took the occasion to showcase some of the notation which will be used later. Using these and plugging the inverse q-metric in (B.13), we obtain

$$\left(A^{-1}g^{\mu\nu} + \epsilon C\frac{\Omega^\mu}{\sqrt{2\epsilon\Omega}}\frac{\Omega^\nu}{\sqrt{2\epsilon\Omega}}\right)S'^2\Omega_\mu\Omega_\nu = 2S,$$

and, using (A.14), this yields the result

$$A^{-1} + C = Q = \frac{1}{\Omega}\frac{S}{S'^2}. \tag{B.14}$$

Let us now state the relations between the factors we have for later use, they are

- $Q = A^{-1} + C = \frac{1}{A+B}$,
- $C = -\frac{A^{-1}B}{A+B} = -A^{-1}BQ = Q - A^{-1}$,
- $B = -\frac{AC}{A^{-1}+C} = -ACQ^{-1}$.

That said, the q-metric will be completely determined by the next condition, which involves the propagators of fields and hence the d'Alembertian $\Box$. The condition will be explained later. For now, let us derive the q-metric wave operator.

B.5 The q-metric d'Alembertian

Recall a general property of matrix determinants, namely the matrix determinant lemma, which states that

$$\det\left\{\mathbf{M} + \mathbf{u}\mathbf{v}^T\right\} = \det\{\mathbf{M}\}\left(\mathbf{1} + \mathbf{v}^T\mathbf{M}^{-1}\mathbf{u}\right).$$

Then, it is clear that for our metric the following holds:

$$q = \det\{q_{\mu\nu}\} = \det\{Ag_{\mu\nu}\}\left(1 + \epsilon BA^{-1}g^{\mu\nu}t_\mu t_\nu\right) = A^{d-1}\left(A+B\right)g = \frac{A^{d-1}}{Q}g. \tag{B.15}$$

Here, d is the number of space-time dimensions and g is the determinant of $g_{\mu\nu}$. Note that this is the same result obtained before (see (B.11)). Also, let us define

$$\xi = \frac{A^{\frac{d-1}{2}}}{Q^{\frac{1}{2}}}.$$

We will need the q-metric d'Alembertian to take the usual form, as in eq. (B.12).

$$\Box_q = \frac{1}{\sqrt{-q}}\partial_\mu\left(\sqrt{-q}q^{\mu\nu}\partial_\nu\right),$$

which, using (B.15) and the metric itself, we break into two terms as

$$\Box_q = \frac{1}{\xi\sqrt{-g}}\partial_\mu\left[\xi\sqrt{-g}A^{-1}g^{\mu\nu}\partial_\nu\right] + \frac{\epsilon}{\xi\sqrt{-g}}\partial_\mu\left[\xi\sqrt{-g}Ct^\mu t^\nu\partial_\nu\right] = \Box_{q,1} + \epsilon\Box_{q,2}.$$

So, let us begin with the first term

$$\Box_{q,1} = \frac{1}{\xi\sqrt{-g}}\partial_\mu\left[\xi\sqrt{-g}A^{-1}\partial^\mu\right].$$

After derivation and some basic manipulation, this becomes:

$$\Box_{q,1} = \frac{\partial_\mu \xi}{\xi} A^{-1}\partial^\mu + \partial_\mu A^{-1}\partial^\mu + A^{-1}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu),$$

which can be further simplified to

$$\begin{aligned}\Box_{q,1} &= (\partial_\mu \ln \xi) A^{-1}\partial^\mu - A^{-2}\partial_\mu A \partial^\mu + A^{-1}\Box_g \\ &= A^{-1}\left[\partial_\mu \ln \xi - \partial_\mu \ln A\right]\partial^\mu + A^{-1}\Box_g.\end{aligned}$$

Notice, by the definition of ξ, that the following is true:

$$\partial_\mu \ln \xi = \partial_\mu \left(\ln A^{\frac{d-1}{2}} - \ln Q^{\frac{1}{2}}\right) = \frac{d-1}{2}\partial_\mu \ln A - \frac{1}{2}\partial_\mu \ln Q, \tag{B.16}$$

and as such, we can continue and finally obtain

$$\Box_{q,1} = \frac{1}{2}A^{-1}\left[(d-3)\partial_\mu \ln A - \partial_\mu \ln Q\right]\partial^\mu + A^{-1}\Box_g.$$

We now move on to the second term

$$\Box_{q,2} = \frac{1}{\xi\sqrt{-g}}\partial_\mu\left[\xi\sqrt{-g}Ct^\mu\partial_t\right] = \frac{1}{\xi\sqrt{-g}}\partial_\mu\left[\xi\sqrt{-g}(Q-A^{-1})t^\mu\partial_t\right],$$

where $\partial_t = t^\alpha\partial_\alpha$ is the directional derivative along the tangent vector t.
After derivation and some simplifications, this becomes

$$\begin{aligned}\Box_{q,2} = (\partial_\mu \ln \xi)(Q-A^{-1})t^\mu\partial_t + \frac{1}{\sqrt{-g}}(\partial_\mu\sqrt{-g})(Q-A^{-1})t^\mu\partial_t + \\ + \partial_\mu(Q-A^{-1})t^\mu\partial_t + (Q-A^{-1})(\partial_\mu t^\mu)\partial_t + (Q-A^{-1})\partial_t^2.\end{aligned}$$

By applying (B.16), and recalling that

$$\frac{\partial_\mu\sqrt{-g}}{\sqrt{-g}} = \Gamma^\alpha{}_{\alpha\mu},$$

and using the form of the covariant derivative we arrive at the result

$$\begin{aligned}\Box_{q,2} = \frac{d-1}{2}Q\partial_t \ln A\partial_t - \frac{d-3}{2}A^{-1}\partial_t \ln A\partial_t + \\ \frac{1}{2}A^{-1}\partial_t \ln Q\partial_t + (Q-A^{-1})\nabla_\mu t^\mu\partial_t + \frac{1}{2}\partial_t Q\partial_t + (Q-A^{-1})\partial_t^2.\end{aligned}$$

Putting the two terms back together, we get the following expression for the box operator:

$$\begin{aligned}\Box_q = \Box_{q,1} + \epsilon\Box_{q,2} = A^{-1}\left[\Box_g + \frac{d-3}{2}\left(\partial_\mu \ln A \partial^\mu - \epsilon\partial_t \ln A \partial_t\right)\right] + \\ + \epsilon(Q - A^{-1})\left[\nabla_\mu t^\mu \partial_t + \partial_t^2\right] + \frac{\epsilon}{2}\partial_t Q \partial_t + \\ + \epsilon\frac{d-1}{2} Q\partial_t \ln A \partial_t + \frac{1}{2}A^{-1}\left[\epsilon\partial_t \ln Q \partial_t - \partial_\mu \ln Q \partial^\mu\right].\end{aligned}$$

Lastly, if the coefficients A and B only depend on the world function Ω, we can show that the two terms in the last parenthesis cancel each other:

$$\begin{aligned}\epsilon\partial_t \ln Q \partial_t = \epsilon t^\mu(\partial_\mu \ln Q) t^\nu \partial_\nu = \epsilon(\ln Q)' t^\mu \Omega_\mu t^\nu \partial_\nu = \\ = \epsilon(\ln Q)'\frac{\Omega^\mu}{\sqrt{2\epsilon\Omega}}\Omega_\mu \frac{\Omega^\nu}{\sqrt{2\epsilon\Omega}}\partial_\nu = \Omega_\nu(\ln Q)'\partial_\nu = \partial^\nu \ln Q \partial_\nu.\end{aligned}$$

And the very same argument applies to the second term in the first parenthesis. This is the first step towards our next goal: finding a simpler version of this operator, restricted to maximally symmetric spaces. At the moment, we have the following form of the q-metric d'Alembertian:

$$\Box_q = \Box_{q,1} + \epsilon\Box_{q,2} = A^{-1}\Box_g + \epsilon(Q - A^{-1})\left[\nabla_\mu t^\mu \partial_t + \partial_t^2\right] + \frac{\epsilon}{2}\partial_t Q \partial_t + \epsilon\frac{d-1}{2}Q\partial_t \ln A \partial_t,$$

which can be further simplified into

$$\begin{aligned}\Box_q &= A^{-1}\Box_g + \epsilon(Q - A^{-1})\left[\nabla_\mu t^\mu \partial_t + \partial_t^2\right] + \frac{\epsilon}{2}\left((d-1)Q\partial_t \ln A \partial_t + Q\partial_t \ln Q \partial_t\right) \\ &= A^{-1}\Box_g + \epsilon(Q - A^{-1})\left[\nabla_\mu t^\mu \partial_t + \partial_t^2\right] + \epsilon Q \partial_t \ln(A^{\frac{d-1}{2}}Q^{\frac{1}{2}})\partial_t,\end{aligned} \tag{B.17}$$

where, as a sanity check, we gladly recognize the d'Alembertian deriving from a conformal transformation in the conformal limit, i.e. when $B = 0$ and consequently $Q = A^{-1}$.

B.6 The d'Alembertian in maximally symmetric spaces

The wave operator in maximally symmetric spaces takes a simpler form. This is because we will have the 2-point function and the various coefficients of the metric depend only on the geodesic distance. Given this assumption, We will take the Green's function to be $G = G(\Omega)$ (a scalar) and see how the standard box operator becomes. Let us begin by applying the d'Alembertian to G:

$$\Box_g G(\Omega) = \nabla_\mu \nabla^\mu G = \nabla_\mu \partial^\mu G = \nabla_\mu\left[\Omega^\mu G'\right] = \Omega^\mu{}_\mu G' + 2\Omega G''.$$

We finally get to use the identity (A.33), involving the Van Vleck determinant in the form

$$\Omega^\mu{}_\mu = d - (\ln\Delta)_{;\mu}\Omega^\mu = d - 2\Omega(\ln\Delta)',$$

and, with this, we continue:

$$\begin{aligned}\Box_g G(\Omega) &= (d - 2\Omega(\ln\Delta)')\, G' + 2\Omega G'' \\ &= 2\Omega\left[\left(\frac{d}{2\Omega} - (\ln\Delta)'\right)\partial_\Omega + \partial_\Omega^2\right] G,\end{aligned}$$

and by employing the trick

$$\frac{d}{2\Omega} = \partial_\Omega \ln\Omega^{\frac{d}{2}},$$

we achieve the desired result, namely

$$\Box_g G(\Omega) = 2\Omega\left[\partial_\Omega \ln(\Omega^{\frac{d}{2}}\Delta^{-1})\partial_\Omega + \partial_\Omega^2\right] G.$$

For clarity, we will write

$$\Box_g^{MSS} = 2\Omega\left[\partial_\Omega \ln(\Omega^{\frac{d}{2}}\Delta^{-1})\partial_\Omega + \partial_\Omega^2\right], \tag{B.18}$$

which will be useful later. Now we turn our attention back to the q-metric d'Alembertian and find its form in maximally symmetric spaces. By using the same chain rule tricks and substitutions seen above, and assuming the coefficients of the metric depend on Ω only, we find the following simplified form of the relevant terms:

① $\partial_t Q \partial_t = 2\epsilon\Omega Q' \partial_\Omega$,

② $\partial_t \ln A \partial_t$ same as above with $Q \longleftrightarrow \ln A$,

③ $\partial_t^2 = 2\epsilon\Omega\left[\partial_\Omega^2 + \partial_\Omega\left(\ln\Omega^{\frac{1}{2}}\right)\partial_\Omega\right]$,

④ $(\nabla_\mu t^\mu)\partial_t = 2\epsilon\Omega\partial_\Omega\left[\ln\left(\Omega^{\frac{d-1}{2}}\Delta^{-1}\right)\right]\partial_\Omega$.

Then, it is easy to realize that

$$\epsilon(③ + ④) = \Box_g^{MSS}.$$

Next, we go back to the general case q-metric d'Alembertian (B.17) and plug-in was just found. After some straightforward simplifications, we achieve

$$\Box_q^{MSS} = Q\left[\Box_g^{MSS} + \Omega\partial_\Omega\left[\ln(A^{d-1}Q)\right]\partial_\Omega\right].$$

Now, use expression (B.18) to yield

$$\Box_q^{MSS} = 2Q\Omega\left[\partial_\Omega^2 + \partial_\Omega\left[\ln(\Omega^{\frac{d}{2}}\Delta^{-1}A^{\frac{d-1}{2}}Q^{\frac{1}{2}})\right]\partial_\Omega\right].$$

B.7 Determination of the parameters

The final requirement we will use to fully determine the general case for the q-metric is the modification of Green's functions for a scalar propagator in a maximally symmetric space. We suppress the MSS in the superscript in this section, but this restriction should be understood. We require that the modified Green's function $G_q(\Omega) = G_g(S(\Omega))$ be a solution to

$$\Box_q G_q(\Omega) = 0 \qquad \text{given} \qquad \Box_g G_g(\Omega) = 0.$$

We proceed to evaluate $\Box_g G_g$, and impose the condition $[\Box_g G_g(\Omega)]_{\Omega=S} = 0$:

$$[\Box_g G_g(\Omega)]_{\Omega=S} = \Box_g G_g(S) = 2S\left[\partial_S^2 + \partial_S\left[\ln\left(S^{\frac{d}{2}}\bar{\Delta}^{-1}\right)\right]\partial_S\right]G_g(S),$$

where $\bar{\Delta}$ is the VVD, with the substitution $\Omega \longleftrightarrow S_{L_0}$.
Now, by the chain rule, we know

- $\partial_S = \frac{1}{S'}\partial_\Omega$,
- $\partial_S^2 = \left(\frac{1}{S'}\right)^2[\partial_\Omega^2 - \partial_\Omega \ln(S')\partial_\Omega]$,

which leads us to

$$\Box_g G_g(S) = 2\frac{S}{S'^2}\left[\partial_\Omega^2 + \partial_\Omega\left[\ln\left(\frac{S^{\frac{d}{2}}\bar{\Delta}^{-1}}{S'}\right)\right]\partial_\Omega\right]G_g(S).$$

Imposing the condition that this vanishes, we obtain the following differential equation:

$$\partial_\Omega^2 G_g(S) = \partial_\Omega\left[\ln\left(\frac{S'\bar{\Delta}}{S^{\frac{d}{2}}}\right)\right]\partial_\Omega G_g(S). \tag{B.19}$$

Now we will impose the condition that $\Box_q G_q(\Omega) = 0$. To do so, begin by writing

$$\Box_q G_q(\Omega) = \Box_q G_g(S) = 2Q\Omega\left[\partial_\Omega^2 + \partial_\Omega\left[\ln\left(\Omega^{\frac{d}{2}}\Delta^{-1}Q^{\frac{1}{2}}A^{\frac{d-1}{2}}\right)\right]\partial_\Omega\right]G_g(S) = 0,$$

which in turn immediately implies

$$\partial_\Omega^2 G(S) + \partial_\Omega\left[\ln\left(\Omega^{\frac{d}{2}}\Delta^{-1}Q^{\frac{1}{2}}A^{\frac{d-1}{2}}\right)\right]\partial_\Omega G_g(S) = 0.$$

Using eq. (B.19), this becomes

$$\partial_\Omega\left[\ln\left(\frac{\bar{\Delta}}{\Delta}A^{\frac{d-1}{2}}\left(\frac{\Omega}{S}\right)^{\frac{d}{2}}S'Q^{\frac{1}{2}}\right)\right]\partial_\Omega G_g(S) = 0.$$

Plugging in our expression for k, namely (B.14), this condition becomes

$$\frac{d-1}{2}\partial_\Omega \ln\left(\frac{A\Omega}{S}\left(\frac{\bar{\Delta}}{\Delta}\right)^{\frac{2}{d-1}}\right) = 0.$$

The solution is easy to find and reads

$$\frac{A\Omega}{S}\left(\frac{\bar{\Delta}}{\Delta}\right)^{\frac{2}{d-1}} = \alpha \longrightarrow A = \alpha\frac{S}{\Omega}\left(\frac{\bar{\Delta}}{\Delta}\right)^{-\frac{2}{d-1}},$$

where α is a constant of integration which is fixed by requiring that $A = 1$ when L_0 is set to zero, or equivalently when $S_{L_0=0}(\Omega) = \Omega$. This implies $\alpha = 1$, and so we finally have our result.
The relations between our coefficients help us find all we need to finally give the final form of the q-metric:

$$A^{-1} + C = Q = \frac{1}{\Omega}\frac{S}{S'^2} \longrightarrow C = \frac{1}{\Omega}\frac{S}{S'^2} - \frac{\Omega}{S}\left(\frac{\bar{\Delta}}{\Delta}\right)^{\frac{2}{d-1}},$$

and

$$B = -ACQ^{-1} = \Omega\frac{S'^2}{S} - \frac{S}{\Omega}\left(\frac{\bar{\Delta}}{\Delta}\right)^{-\frac{2}{d-1}}.$$

The q-metric, then, has the final general form:

$$q_{\mu\nu} = \frac{S}{\Omega}\left(\frac{\bar{\Delta}}{\Delta}\right)^{-\frac{2}{d-1}} g_{\mu\nu} + \epsilon\left[\Omega\frac{S'^2}{S} - \frac{S}{\Omega}\left(\frac{\bar{\Delta}}{\Delta}\right)^{-\frac{2}{d-1}}\right] t_\mu t_\nu,$$

and its inverse

$$q^{\mu\nu} = \frac{\Omega}{S}\left(\frac{\bar{\Delta}}{\Delta}\right)^{\frac{2}{d-1}} g^{\mu\nu} + \epsilon\left[\frac{1}{\Omega}\frac{S}{S'^2} - \frac{\Omega}{S}\left(\frac{\bar{\Delta}}{\Delta}\right)^{\frac{2}{d-1}}\right] t^\mu t^\nu.$$

As expected, the q-metric is singular in the limit $\Omega \to 0$ and reduces to the regular metric $g_{\mu\nu}$ in the limit of large separations $\Omega \to \infty$, regime in which $S(\Omega) \to \Omega$. It is also not uniquely determined by the event x, as the usual metric would be, because it explicitly depends on the choice of base point x'. It is therefore a bitensor and a non-local object (by construction). Also, notice that the simple case at the beginning of this section, which did not include the contribution of the VVD, can be recovered by setting $S = \Omega + \epsilon\frac{L_0^2}{2}$ and $\Delta = 1$.
Keep in mind that setting $\Delta = 1$ in any case but for flat space-time is dangerous. In general, the q-metric might yield a non-zero curvature even if the starting geometry is flat. The value of the VVD in maximally symmetric spaces is known and is [13]

- $\Delta^{-\frac{1}{d-1}} = \frac{\sin\frac{\sqrt{2\epsilon\Omega}}{a}}{\frac{\sqrt{2\epsilon\Omega}}{a}}$ for positive curvature,
- $\Delta^{-\frac{1}{d-1}} = 1$ for zero curvature,
- $\Delta^{-\frac{1}{d-1}} = \frac{\sinh\frac{\sqrt{2\epsilon\Omega}}{a}}{\frac{\sqrt{2\epsilon\Omega}}{a}}$ for positive curvature,

where a is the radius of curvature. The modified VVD, $\bar{\Delta}$, will take the same form with the exchange of $\Omega \Longleftrightarrow S_{L_0}$.

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