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Moduli Stabilisation for the Dark Dimension Scenario

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Abstract

This thesis provides a moduli stabilisation mechanism for the Dark Dimension (DD) scenario within the framework of type IIB superstring theory. The DD scenario is a recent proposal which involves one large extra dimension with associated Kaluza-Klein (KK) scale around the cosmological constant scale, a tower of light sterile neutrinos and dark matter from massive KK gravitons. Realising a model with just one large extra dimension out of the six extra dimensions of string theory requires an anisotropic moduli stabilisation. This is achieved in the context of the type IIB Large Volume Scenario (LVS) which allows to obtain a low Kaluza-Klein scale thanks to the fact that the Calabi-Yau volume is exponentially large in string units. Anisotropy is realised by considering a Calabi-Yau threefold which is a K3 fibration over a $\mathbb{P}^1$ base. The volume of the K3 fibre is stabilised at relatively small values by perturbative corrections to the effective action, in particular string loops and higher derivative effects. This leaves an exponentially large volume of the 2-dimensional $\mathbb{P}^1$ base. Thanks to a hierarchical stabilisation of the complex structure moduli, one can then ensure that the $\mathbb{P}^1$ is deformed into an elongated cigar, leading to a model with just one large extra dimension. Interestingly, the desired anisotropy and low KK scale can be obtained for natural choices of the microscopic parameters.

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Chapter 1

Introduction

The last century has been, without any doubt, the most flourishing for physics. We have seen the birth of Special Relativity (SR) and Quantum Mechanics (QM) . The former has later been extended to General Relativity (GR), while SR and QM have been unified into a single coherent structure called Quantum Field Theory (QFT), as illustrated in Fig. 1.1. Since the 1960s, when QFT established itself as the best framework to describe nature at the quantum level, a countless number of physicists has tried to unify GR and QFT into a theory named quantum gravity (QG), but without any success. A QG theory is in fact necessary whenever we want to describe regions of space where quantum and gravitational effects are both non-negligible, such as the singularity of black holes or the moments immediately after the Big Bang. The issue is that the ultra-violet divergencies we meet when quantising gravity cannot be cured by the traditional renormalization methods employed for other QFTs, pure QG is not finite at 2-loops [1] and with matter coupling the divergences are already incurable at 1-loop [2]. Many ideas have since been suggested to try and find a way around this issue, some remaining in a QFT context and some changing paradigm completely.

This is where string theory comes into play. String theory, first proposed around the 1960s as a theory to explain the strong nuclear interaction but later discarded and replaced by QCD, came back to life in the middle of 1970s when people realized it contained in the spectrum a massless spin 2 particle, the graviton, the quantum of the gravitational interaction. On top of this the renormalization issues and UV divergencies of standard quantum gravity are gone in string theory. A strong hint for this can be found in the in the different structure between the Feynman diagrams of a QFT and the stringy diagrams of string theory. As the name suggests, string theory is a theory where the building blocks of nature are not point-like particles but rather strings, which can be both open and closed, hence the corresponding Feynman diagrams will look as "tube" diagrams. To exemplify this, and to explain the absence of UV divergencies, let us take a one loop Feynman diagram and the corresponding tube diagram from string theory as the one in Fig. 1.2. This diagram diverges in the UV in QG because the interaction

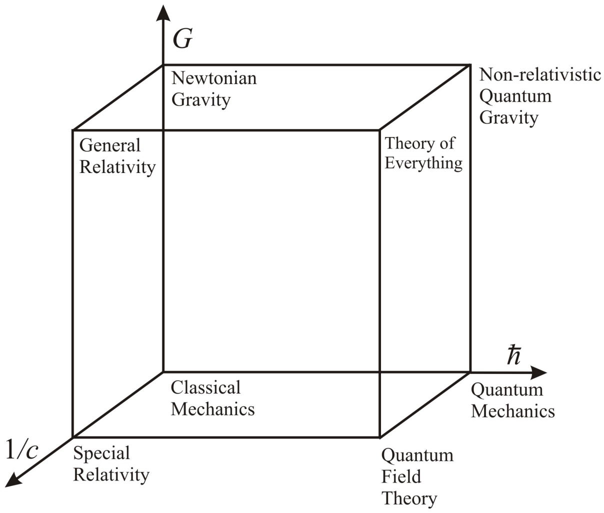

Figure 1.1: Okun's cube. Picture taken from [3].

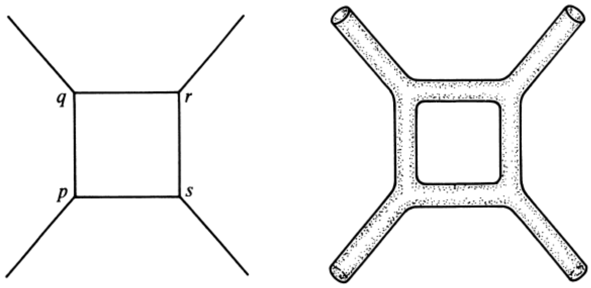

Figure 1.2: One-loop Feynman diagram with interaction points p, q, r, s on the left and the corresponding stringy diagram for closed strings on the right. Picture taken from [4].

vertices are well defined and when $p = q = r = s$ the propagators, which connect the vertices, explode. In the stringy one instead there is no notion of interaction vertex and hence we never reach the critical situation of $p = q = r = s$.

Since its dawn string theory has developed a lot, undergoing two "revolutions", leading to five consistent superstring theories interrelated by several dualities. Despite being different these show some common features, most importantly unbroken $\mathcal{N} = 1$ supersymmetry and the existence of extra dimensions. To be honest the $\mathcal{N} = 1$ supersymmetry is required by us as it is phenomenologically viable in the sense that is chiral and it brings with itself properties that result incredibly useful when applied to other physical contexts, such as the Standard Model (SM), which, despite its success, is plagued by several issues. Some examples, all solved by supersymmetry, are:

- Incompleteness problems:
  1. The SM does not describe the gauge coupling unification, while supersymmetry achieves this.
  2. The SM does not describe dark matter while supersymmetry gives a candidate for it.
- Technical problems:
  1. The SM cannot explain the measured value of the Higgs mass $m_H \simeq 125$ GeV. A way out that does not require fine tuning is invoking a new symmetry such as supersymmetry.
  2. The SM cannot explain the origin of the Higgs potential while supersymmetry provides a way to do that via radiative electroweak symmetry breaking.
  3. The SM predicts a cosmological constant $\Lambda_{\text{SM}} \sim M_{\text{Pl}}^4$ while current measurements yield $\Lambda \sim 10^{-122} M_{\text{Pl}}^4$. Supersymmetry can help in this case but does not solve the problem as it predicts $\Lambda_{\text{SUSY}} \sim 10^{-60} M_{\text{Pl}}^4$.

As we mentioned, the other common aspect of string theories is the existence of extra dimensions, an idea that already traces back to the 1920s with the proposal, by Kaluza and Klein, of a fifth dimension in order to unify gravity and electromagnetism. The internal consistency of the theory sets the total number of dimensions to 10 but since we observe only 4, 6 of them must be too small and compact to be detected by our current technology. The observable low-energy data, which in the light of this argument belong to an effective field theory, can then be derived by performing a compactification of the full ten-dimensional theory on the six extra dimensions. It is at this point that we impose our request on unbroken supersymmetry, that is we want to have $\mathcal{N} = 1$ supersymmetry on the $4D$ space surviving after the compactification. This greatly constrains the compact six-dimensional manifold.

Extra dimensions have several other effects on the effective field theory, most notably they give rise to a number of massless scalar particles called moduli. They represent a flat direction of the scalar potential in field space. In order not to spoil the observed phenomenology these particles must have a mass, so algorithms for moduli stabilisation, the procedure by which moduli acquire a mass, have been proposed. These algorithms consist in introducing correction to the scalar potential which mostly descend from corrections to the Kähler potential and the superpotential, since these two completely determine the scalar potential in $\mathcal{N} = 1$ theories, in order to lift the flat direction. We shall see that actually corrections are needed to stabilise only one class of moduli, the Kähler ones, whereas all the others, the complex structure moduli and the axiodilaton, are stabilised at tree level by the fluxes, which is just another word to describe gauge fields and their field strengths. The Kähler moduli stabilisation is performed under specific assumptions, the most famous ones being the KKLT scenario and the LVS scenario. They are quite different but the fundamental idea is the same: reach the stabilisation by targeting regions of field space where we have good control over the known corrections and can neglect the unknown ones.

This work is structured as follows:

- In Chapter 2 we review all the relevant background material. We start with bosonic string theory as well as its supersymmetric extension, with particular attention on type IIB superstring theory. We then investigate string compactifications, a bridge between the four-dimensional spacetime we observe and the ten-dimensional one predicted by superstring theory. Moreover, we discuss moduli stabilisation, especially in the context of type IIB, a key aspect of any superstring theory in order to have phenomenologically acceptable models.

- In Chapter 3 we present the main work of this thesis, an anisotropic moduli stabilisation mechanism for the dark dimension scenario. We first describe the Calabi-Yau geometry which features a K3 fibration over a $\mathbb{P}^1$ base. Subsequently, we show how perturbative corrections can fix the volume of the K3 fibre at small values and the volume of the $\mathbb{P}^1$ fibre at large values. Finally we argue that the $\mathbb{P}^1$ can be deformed into an elongated cigar via an appropriate stabilisation of the complex structure moduli.

Chapter 2

String Theory and its Low-Energy Limit

1 Bosonic String Theory and Superstring Theory

As was already mentioned in the introduction, string theory is able to unify in a very natural way general relativity and quantum field theory. To understand why that is the case we will start by reviewing bosonic string theory and in doing so we will meet its drawbacks, which will lead us to superstring theory. We refer to the books [4, 5, 6, 7, 8].

1.1 Bosonic String Theory

The main idea of string theory is that the building blocks of nature aren't actually point-like particles, but rather one-dimensional objects, strings indeed, that can be either open or closed. This simple idea makes the graviton emerge in the spectrum upon quantisation. In order to understand why we first need to perform a classical analysis.

Classical Theory

As a point-like particle sweeps a one-dimensional trajectory while moving in spacetime, the worldline, a one-dimensional string sweeps a two-dimensional surface, called worldsheet. For this reason, while a point particle in a D-dimensional spacetime can be described through D functions of the worldline parameter, $X^\mu(\tau)$, $\mu = 0, \dots, D-1$, for a string we need two parameters: $X^\mu(\tau, \sigma)$, $\mu = 0, \dots, D-1$. These maps from the worldsheet Σ to the (Minkowski) spacetime $X^\mu : \Sigma \to \mathbb{R}^{1,D-1}$ are called embedding functions and are usually denoted just as $X^\mu(\sigma^a)$, with $\sigma^a = \tau, \sigma$ and the index taking the values $a = 0, 1$.

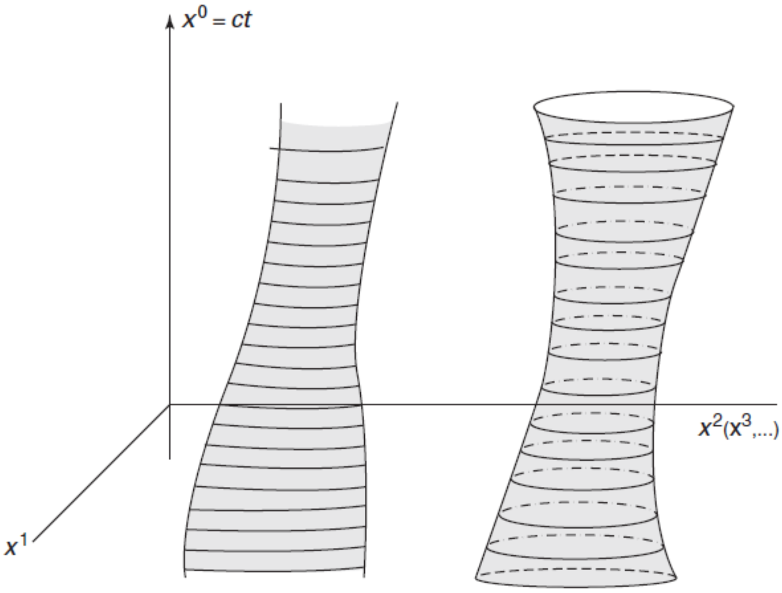

Figure 2.1: Worldsheet of an open (left) and a closed (right) string. Picture taken from [9].

The dynamics of a string are described by the Nambu-Goto action:

$$S_{NG} = -T \int dA. \tag{1.1}$$

Where dA is the world-sheet area element:

$$dA = \sqrt{-\det\left(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\right)}\, d^2\sigma. \tag{1.2}$$

With $d^2\sigma = d\sigma d\tau$.

The action (1.2) is just the stringy extension of the relativistic point particle action:

$$S = -m \int ds, \quad ds = \sqrt{-\frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \tau}\eta_{\mu\nu}}. \tag{1.3}$$

As in the case of the point particle, also for strings it is preferable to get rid of the square root appearing in (1.2). For this reason we introduce an auxiliary field, the worldsheet metric $h_{ab}(\sigma^a)$, in order to get Polyakov action:

$$S_{\mathrm{P}} = -\frac{T}{2}\int_\Sigma d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu. \tag{1.4}$$

Where $h = \det(h_{ab})$, and T is the tension of the string. The latter can also be expressed as $T = (2\pi\alpha')^{-1}$, with α' the Regge slope parameter, which is related to the string length via $\ell_s^2 = (2\pi)^2\alpha'$.

The action (1.4) enjoys several symmetries:

- Poincaré invariance, that is invariance under $SO(1, D-1)$:

$$\begin{aligned} X'^{\mu}(\tau,\sigma) &= \Lambda^{\mu}_{\nu} X^{\nu}(\tau,\sigma) + a^{\mu}, \\ h'_{ab}(\tau,\sigma) &= h_{ab}(\tau,\sigma). \end{aligned} \tag{1.5}$$

  Where $\Lambda \in SO(1, D-1)$ and a^{μ} is constant.

- Reparametrization invariance, that is invariance under a diffeomorphism mapping the old world-sheet coordinates σ^a into new ones $\sigma'^a(\tau,\sigma)$:

$$\begin{aligned} X'^{\mu}(\tau',\sigma') &= X^{\mu}(\tau,\sigma), \\ \frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b} h'_{cd}(\tau',\sigma') &= h_{ab}(\tau,\sigma). \end{aligned} \tag{1.6}$$

- Two-dimensional Weyl invariance, that is a rescaling of the world-sheet metric:

$$\begin{aligned} X'^{\mu}(\tau',\sigma') &= X^{\mu}(\tau,\sigma), \\ h'_{ab}(\tau,\sigma) &= e^{2\omega(\tau,\sigma)}\, h_{ab}(\tau,\sigma). \end{aligned} \tag{1.7}$$

  Where $\omega(\tau,\sigma)$ is an arbitrary function of the worldsheet coordinates.

The reparametrization symmetry, together with the Weyl symmetry, allows us to gauge fix all three degrees of freedom of the world-sheet metric and set it equal to the $(2D)$ Minkowski metric $\eta_{ab} = \mathrm{diag}(-,+)$. However this is only possible when the world-sheet has vanishing Euler characteristic [10].
Furthermore, the Weyl symmetry also implies tracelessness of the energy momentum tensor $T^a_a = 0$. Not only, by the equation of motion for h_{ab} one finds the whole energy momentum tensor vanishes $T_{ab} = T_{ba} = 0$.

$$T_{ab} \equiv \frac{-2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h_{ab}} \tag{1.8}$$

$$= \partial_a X^{\mu}\partial_b X_{\mu} - \frac{1}{2}h_{ab}h^{cd}\partial_c X^{\mu}\partial_d X_{\mu} 0 = 0. \tag{1.9}$$

All these properties of the energy momentum tensor descend from the worldsheet conformal symmetry enjoyed by the theory and will be crucial for the quantisation.

Let us assume we can set $h_{ab} = \eta_{ab}$ in (1.4), then we can derive the equation of motion for X^μ:

$$\partial_a \partial^a X^\mu = (\partial_\sigma^2 - \partial_\tau^2) X^\mu = 0. \tag{1.10}$$

To ensure stationarity of (1.4) under $X^\mu \to X'^\mu = X^\mu + \delta X^\mu$ the equation of motion is not enough, we must require the vanishing of the boundary term, given by:

$$-T \int d\tau \left[X'_\mu \delta X^\mu |_{\sigma=\pi} - X'_\mu \delta X^\mu |_{\sigma=0} \right]. \tag{1.11}$$

The choice that makes (1.11) vanish is not unique:

- In the case of closed strings we require the embedding functions to be periodic:

$$X^\mu(\tau, \sigma) = X^\mu = (\tau, \sigma + \pi). \tag{1.12}$$

- In the case of open strings we can impose either Neumann or Dirichlet boundary conditions.

  1. Neumann boundary conditions (NC) represent the case where no momentum is flowing through the ends of the strings:

  $$X'_\mu(\tau, \sigma) = 0, \quad \sigma = 0, \ \pi. \tag{1.13}$$

     Poincaré invariance is satisfied if such a choice is made for all $\mu = 0, \ldots D - 1$.

  2. Dirichlet boundary conditions (DC) represent the case where the ends of the string are fixed:

  $$\delta X^\mu(\tau, \sigma) = 0, \quad \sigma = 0, \ \pi. \tag{1.14}$$

     These conditions break Lorentz invariance so they might seem not physically sensible, but there are cases where they are unavoidable. More details can be found in the Appendix B.
     The current interpretation is that the ends of the string lie on a hyperplane called D-brane, short for Dirichlet-membrane. These objects turn out to be fundamental in string theory and even more in superstring theory, as they are necessary for the theory to be consistent at non perturbative level.

  It is not mandatory to have only one type of boundary condition for open strings, there might be circumstances where we have DC for $\mu = 1, \ldots, D - p - 1$ and NC for the remaining $p + 1$ coordinates.

We can now focus on solving (1.10). To do that it is convenient to introduce the light-cone coordinates $\sigma^\pm = \tau \pm \sigma$. This implies that derivatives ∂_τ and ∂_σ are now mixed and the metric is not diagonal anymore:

$$\begin{aligned}\partial_\pm =& \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right), \\ \eta =& -\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.\end{aligned} \tag{1.15}$$

Consequently also (1.10) changes and now looks like:

$$\partial_+\partial_- X^\mu = 0. \tag{1.16}$$

The energy momentum tensor (1.9) changes too. The vanishing of its trace reads as $T_{+-} = T_{-+} = 0$ and its own vanishing reads as:

$$\begin{aligned}T_{++} &= \partial_+ X^\mu \partial_+ X_\mu = 0, \\ T_{--} &= \partial_- X^\mu \partial_- X_\mu = 0.\end{aligned} \tag{1.17}$$

The general solution of (1.16) is:

$$X^\mu(\sigma^+, \sigma^-) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-). \tag{1.18}$$

Where X_L^μ and X_R^μ are the left moving and right moving parts of X^μ respectively.

To enforce the constraints (1.17) it is useful to expand (1.18) in Fourier modes. This will also allow us to perform the quantisation canonically, that is by quantising the oscillator modes. Once again we need to distinguish between open and closed strings.

- Closed strings.
  In the case of closed strings we have a periodicity constraint on X^μ (1.12). The Fourier expansion reads as:

$$X_R^\mu = \frac{1}{2}x^\mu + \alpha' p_\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-}, \tag{1.19}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \alpha' p_\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n}\widetilde{\alpha}_n^\mu e^{-2in\sigma^+}. \tag{1.20}$$

  Where x^μ and p^μ are the centre of mass and momentum of the string respectively. The α and $\widetilde{\alpha}$ modes are unrelated. By requiring reality of X_L^μ and X_R^μ it follows that both x^μ and p^μ must be real and that the positive and negative oscillators are conjugate of each other:

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^*, \quad \widetilde{\alpha}_{-n}^\mu = (\widetilde{\alpha}_n^\mu)^*. \tag{1.21}$$

  Furthermore (1.12) is satisfied only if:

$$\alpha_0^\mu = \widetilde{\alpha}_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu \tag{1.22}$$

- For open string we can have Neumann, Dirichlet or Neumann-Dirichlet boundary conditions:

  1. For Neumann conditions we have:

  $$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-i\pi\tau}\cos(n\sigma). \tag{1.23}$$

  With $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$.

  2. For Dirichlet conditions have:

  $$X^\mu(\tau,\sigma) = x_0^\mu + \frac{x_1^\mu - x_0^\mu}{\pi}\sigma + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-i\pi\tau}\sin(n\sigma). \tag{1.24}$$

  With $x_0^\mu = X^\mu(\tau, \sigma = 0)$, $x_1^\mu = X^\mu(\tau, \sigma = \pi)$ and $\alpha_0^\mu = \frac{1}{\sqrt{2\alpha'}\pi}(x_1^\mu - x_0^\mu)$, .

  3. For Neumann-Dirichlet conditions we have:

  $$X^\mu(\tau,\sigma) = x^\mu + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n}\alpha_n^\mu e^{-i\pi\tau}\cos(n\sigma). \tag{1.25}$$

The reality conditions are as in the case of the closed string. The main difference between the two possibilities is that for open strings we have just one type of oscillator mode. Indeed, while for closed string the left and right moving waves are independent, for open strings they combine into a stationary wave. This fact will have an important consequence on the spectrum of the theory.

Plugging the expansions (1.19) and (1.20) into (1.17) we can expand also the energy momentum tensor:

$$T_{--} = 4\alpha' \sum_{m\in\mathbb{Z}} L_m e^{-2in\sigma^-}, \tag{1.26}$$

$$T_{++} = 4\alpha' \sum_{m\in\mathbb{Z}} \widetilde{L}_m e^{-2in\sigma^+}. \tag{1.27}$$

Where we introduced the Virasoro generators:

$$\begin{aligned} L_m &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n, \\ \widetilde{L}_m &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \widetilde{\alpha}_{m-n}\cdot\widetilde{\alpha}_n. \end{aligned} \tag{1.28}$$

These generators classically obey a Witt algebra, but quantisation will change things.

The vanishing of the energy momentum tensor (1.17) implies that the Virasoro modes (1.28) should vanish, $L_m = \widetilde{L}_m = 0 \; \forall m \in \mathbb{Z}$. In particular, the condition $L_0 = \widetilde{L}_0 = 0$ allows us to express the (classical) mass of the string in terms of the oscillator modes as:

$$\begin{aligned} M^2 &= \frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n \quad \text{for open strings,} \\ M^2 &= \frac{2}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \widetilde{\alpha}_{-n}\widetilde{\alpha}_n \quad \text{for closed strings.} \end{aligned} \tag{1.29}$$

Canonical Quantisation

We are now ready to quantise the bosonic theory. There are different ways to achieve this goal, we will follow the old covariant quantisation (OCQ) method, that consists in promoting the Fourier modes to quantum operators and impose the constraints on the states. A more advanced approach is based on BRST symmetry, but since the final result is the same, we will limit ourselves with OCQ since it does not require to introduce the BRST machinery.

Since OCQ is a canonical approach, we need to compute $\Pi^\mu \equiv \delta S/\delta \dot{X}\mu = T\dot{X}^\mu$ and find the Poisson brackets for all the combinations of Π^μ and X^μ (at equal τ). They are all vanishing except for:

$$[P^\mu(\tau,\sigma), X^\nu(\tau,\sigma')]_{PB} = \eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{1.30}$$

Plugging the expansion (1.19) and (1.20) into (1.30) we can find the Poisson brackets for the modes α and $\widetilde{\alpha}$. Of course in the case for the open strings we only have the set of modes α. Then we perform the replacement $[\dots]_{PB} \to -i[\dots]$ and read off the commutation relations for α and $\widetilde{\alpha}$:

$$\begin{aligned} [\alpha^\mu_m, \alpha^\nu_n] = [\widetilde{\alpha}^\mu_m, \widetilde{\alpha}^\nu_n] &= m\eta^{\mu\nu}\delta_{m+n,0}, \\ [\alpha^\mu_m, \widetilde{\alpha}^\nu_n] &= 0. \end{aligned} \tag{1.31}$$

Since we promoted the α and $\widetilde{\alpha}$ modes to operators, their reality condition (1.21) now reads as:

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^\dagger \quad \text{and} \quad \widetilde{\alpha}^\mu_{-n} = (\widetilde{\alpha}^\mu_n)^\dagger\,. \tag{1.32}$$

Up to a redefinition of the modes, (1.31) describes the algebra of creation/annihilation operators of a harmonic oscillator. Therefore we can construct the spectrum just by acting with raising operators on the ground state. However the presence of $\eta^{\mu\nu}$ poses an obstacle: for $\mu = \nu = 0$ (1.31) is negative, implying the existence of negative norm states. Their presence indicates that the theory violates causality and unitarity. A specific choice of the spacetime dimension and of a constant (yet to be introduced) will solve this issue.

Since the α (and $\widetilde{\alpha}$) modes have been promoted to operators, we need to consider again the expression of the Virasoro generators (1.28). In particular we want the product $\alpha_{m-n} \cdot \alpha_n$ to be normal ordered:

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha_{m-n} \cdot \alpha_n : . \tag{1.33}$$

The same redefinition also holds for $\widetilde{L}_m$.

Using (1.31) and (1.33) one can check the Virasoro modes do not satisfy anymore a Witt algebra, but rather its central extension, called Virasoro algebra:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}. \tag{1.34}$$

Where $c = D$ is the central charge.
Moreover, due to (1.32), also (1.33) satisfies a reality condition, namely:

$$L_{-m} = (L_m)^\dagger . \tag{1.35}$$

Going back to (1.33) we can see that the only operator needing normal ordering is L_0. This introduces a normal ordering constant, the one mentioned above, that we denote by a. It follows that the classical constraint $L_0 = \widetilde{L}_0 = 0$, upon quantisation, is updated to $L_0 - a = \widetilde{L}_0 - a = 0$, whereas is still holds that $L_m = 0$ for $m > 0$. Of course these equalities must now be understood as operator equations, that is, given a physical state $|\phi\rangle$:

$$\begin{aligned} (L_0 - a)|\phi\rangle = (\widetilde{L}_0 - a)|\phi\rangle = 0, \\ L_m|\phi\rangle = \widetilde{L}_m|\phi\rangle = 0 \quad \text{for } m > 0. \end{aligned} \tag{1.36}$$

The operator $\widetilde{L}_0$ carries the same constant a as L_0. This is a consequence of the absence of gravitational anomalies on the worldsheet, which sets $a = \widetilde{a}$. Otherwise the diffeomorphism symmetry would be broken at the quantum level.

Taking the difference of the two members of (1.36) yields a level matching condition:

$$L_0 = \widetilde{L}_0. \tag{1.37}$$

Furthermore, the constant a also changes the mass formulas (1.29):

$$\begin{aligned} M^2 &= \frac{1}{\alpha'}(N - a) \quad \text{for open strings,} \\ M^2 &= \frac{4}{\alpha'}(N + \widetilde{N} - 2a) \quad \text{for closed strings.} \end{aligned} \tag{1.38}$$

Where we introduced the number operator:

$$N = \sum_{n>0} \alpha_{-n} \cdot \alpha_n. \tag{1.39}$$

$\widetilde{N}$ is defined analogously by replacing $\alpha \to \widetilde{\alpha}$.

As we mentioned above, imposing the absence of negative norm state fixes a and D. To actually compute them we first define spurious states. A state $|\psi\rangle$ is spurious if it satisfies $(L_0 - a)|\psi\rangle = 0$ and is orthogonal to all physical states, as we defined them in (1.36), so $\langle\phi|\psi\rangle = 0$. Such states can always be written in the form:

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}|\chi_n\rangle. \tag{1.40}$$

With $|\chi_n\rangle$ such that

$$(L_0 - a + n)|\chi_n\rangle = 0. \tag{1.41}$$

Actually, due to the Virasoro algebra (1.34), we can truncate the series (1.40) at $n = 2$.

If a state defined as (1.40) is also physical, namely it satisfies the further condition $L_m|\psi\rangle = 0$ for $m > 0$ on top of the other ones, then it is orthogonal not only to other physical states but also to itself. We refer to such states as null states. Despite being physical in the sense of (1.36), these states are unphysical because they decouple from physical processes. This can also be phrased as the Hilbert space of our theory being $\mathcal{H} = \mathcal{H}_{physical}/\mathcal{H}_{null}$.
We can build null states starting from spurious ones of the form:

$$|\psi\rangle = L_{-1}|\chi\rangle \tag{1.42}$$

With $|\chi\rangle$ a physical state obeying (1.41) with $n = 1$.

The state (1.42), on top of being spurious, satisfies the physical condition (1.32), except for the mode L_1. For this reason we require:

$$L_1|\psi\rangle = L_1 L_{-1}|\chi\rangle = 2(a-1)|\chi\rangle \stackrel{!}{=} 0. \tag{1.43}$$

This fixes $a = 1$.
We can now fix the number of spacetime dimensions D by choosing another class of spurious states and following the same route as above. This yields $D = 26$. With such choices for a and D, the physical spectrum has no negative norm states.

Having the values of a and D at hand we can finally determine the spectrum of the theory from (1.38). We will limit ourselves to the first few levels only.

- For open strings we can construct excited states by acting on the ground state, which we denote as $|0;k\rangle$, with the operator α_n^μ with $n<0$.

  1. At $N=0$ we have the ground state with mass $\alpha' M^2=-1$. Such a state is called a tachyon.
  2. At $N=1$ we have the vector $\alpha_{-1}^\mu|0;k\rangle$ with mass $\alpha' M^2=0$. It is a photon-like state. Since it is massless it forms a vector representation of $SO(24)$.

- For closed strings we can construct excited states by acting on the ground state, which we denote again as $|0;k\rangle$, with the operator product $\alpha_n^\mu\,\widetilde{\alpha}_n^\nu$ with $n<0$. Notice that, for each level, we must have $N=\widetilde{N}$ due to the level matching condition (1.37).

  1. At $N=\widetilde{N}=0$ we have the tachyonic ground state with mass $\alpha' M^2=-4$.
  2. A $N=\widetilde{N}=1$ we have the tensor $\alpha_{-1}^\mu\widetilde{\alpha}_{-1}^\nu|0;k\rangle$ with mass $\alpha' M^2=0$. Since it is massless it forms a tensor representation of $SO(24)$ which can be decomposed into three irreducible representations:
     - A traceless symmetric rank 2 tensor, which we denote by $g_{\mu\nu}$. It is the graviton.
     - An antisymmetric rank 2 tensor, which we denote by $B_{\mu\nu}$. It is the Kalb-Ramond field.
     - A scalar, which we denote by Φ. It is the dilaton.

The Dilaton and the String Coupling

We close the paragraph on bosonic string theory with some comments on the fundamental role of the dilaton.

Let us start from the path integral associated to the Polyakov action (1.4):

$$\int \mathcal{D}X\mathcal{D}g\; e^{-S}. \tag{1.44}$$

The Minkowskian world-sheet metric h_{ab} entering (1.4) has been replaced here by a Euclidean one, so with signature $(+,+)$. In this way the path integral over the metrics is better defined and the traditional factor of $\exp(iS)$ is replaced by $\exp(-S)$.

The path integral (1.44) tells us how to sum over all the possible metrics, but we should also take into account the possible different worldsheet topologies. Indeed, if we think to the standard Feynman path integral, we have a sum over all the possible "histories" of a particle and the factor $\exp(iS/\hbar)$ acts as a weight for each history. In the string case also the worldsheet topology plays a role in this history. This means that the action S entering (1.44) cannot just be the Polyakov action. Rather we have:

$$\begin{aligned} S &= S_P + \lambda\chi \\ &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int_\Sigma d^2\sigma\sqrt{h}R. \end{aligned} \tag{1.45}$$

Where λ is a real number and χ is the Euler characteristic, in this case of the worldsheet, which we expressed in the second line via the Gauss-Bonnet theorem, with R the Ricci scalar. In principle there would also be a boundary term $\int_{\partial\Sigma}$ but it is convention to neglect it by considering closed strings' worldsheets, which indeed have no boundary. It is a good guess as χ is a topological invariant, hence it will only distinguish worldsheets with different topologies. Furthermore such a term is also allowed by the symmetries of the theory.

In two dimensions, as on a worldsheet, the $\lambda\chi$ term of (1.45) is purely topological hence it does not add any dynamics. Instead the resulting factor of $\exp(-\lambda\chi)$ will only act as a weight for the different topologies, which is exactly what we wanted. To understand the physical effect of this consider adding a handle to the worldsheet of a closed string, which physically corresponds to the emission and absorption of a closed string. Then, in light of the relation $\chi = 2 - 2g$, where the genus g can be thought of as the number of handles roughly speaking, we have that χ is reduced by 2 and hence a factor of $\exp(2\lambda)$ is added in the path integral. We can construct a similar example also for open strings where we add a strip to the worldsheet, corresponding to the emission and absorption of an open string. In this case a factor of $\exp(\lambda/2)$ is added to the path integral. Therefore we can give the interpretation of this factor in terms of coupling constant:

$$g_o^2 \sim g_c \sim e^\lambda. \tag{1.46}$$

We can finally come back to the dilaton. In particular we are interested to the string coupling to the dilaton as a background field. The action describing this coupling is:

$$S_\Phi = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{h}R\Phi(X). \tag{1.47}$$

This action would actually be part of a larger one including the coupling to the other massless fields of the spectrum, which also act as backgrounds, namely the graviton and the Kalb-Ramond field.

The action (1.47) is not invariant under Weyl transformations unless Φ is constant $\langle\Phi\rangle$. Under this assumption it is easy to notice that (1.47) is nothing but the $\lambda\chi$ term of (1.45) where $\langle\Phi\rangle$ plays the role of λ. This tells us that different values of λ correspond to different values of the background field.

Finally, in light of (1.46), we understand that the string coupling is not an independent parameter, but rather the expectation value of a field:

$$g_s \equiv g_c = e^{\langle \Phi \rangle}. \tag{1.48}$$

This is one of the distinctive aspects of string theory with respect to the more traditional field theory approaches where we can find "free" constants not determined by the theory itself.

1.2 Superstring Theory

If, on one hand, bosonic string theory is a good candidate for quantum gravity as it contains the graviton in its spectrum, on the other hand it is plagued by two main issues:

- The spectrum contains tachyons. Any theory containing such particles are unphysical because tachyons represent an instability of the vacuum. They sit on a maximum of the potential rather than a minimum.
- It does not contain fermions. Indeed only bosons arose in our spectrum and it cannot be otherwise since the operators $\alpha, \widetilde{\alpha}$ are vectors which always act on bosonic states.

Both these problems can be solved by the introduction of supersymmetry in the theory. There are two approaches to achieve such goal:

- The Ramond-Neveu-Schwarz (RNS) formalism, which is supersymmetric on the worldsheet.
- The Green-Schwarz (GS) formalism, which is supersymmetric in the spacetime.

These two approaches are equivalent, for this reason we choose to follow the first.
As in the bosonic case, before we can look at the spectrum we need a classical analysis.

Classical Theory

In order to realize a theory which is supersymmetric on the worldsheet we first introduce fermions and then check that supersymmetry is satisfied. We start from the Polyakov action (1.4), with $h_{ab} = \eta_{ab}$, and just add to it a Dirac action term for D massless free fermions $\psi^\mu(\tau, \sigma)$. These are classically anticommuting two-component Majorana spinors on the worldsheet and vectors in the D-dimensional spacetime. Setting $T = 1/\pi$:

$$S = \frac{1}{2\pi} \int d^2\sigma (\partial_a X^\mu \partial^a X_\mu + \bar{\psi}^\mu \rho^a \partial_a \psi_\mu). \tag{1.49}$$

Where $\bar{\psi} = \psi^\dagger i\rho^0$ is the Dirac conjugate of ψ and $\rho^a, a = 0, 1$, are two-dimensional Dirac matrices obeying to the Clifford algebra:

$$\{\rho^a, \rho^b\} = 2\eta^{ab}. \tag{1.50}$$

The action (1.49) is invariant under the supersymmetry transformations:

$$\begin{aligned} \delta X^\mu &= \bar{\varepsilon}\psi^\mu, \\ \delta\psi^\mu &= \rho^a\partial_a X^\mu\varepsilon. \end{aligned} \tag{1.51}$$

Where ε, the supersymmetry parameter, is a two-component Majorana spinor.

To deal with the mode expansion, which we will discuss later, it is convenient to actually work with the components of the spinors ψ^μ, which we denote by ψ^μ_A with $A = +, -$ the spinor index:

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}. \tag{1.52}$$

Plugging (1.52) into (1.49) and not displaying the Lorentz indices for simplicity, we get for the fermionic part only:

$$S_f = \frac{i}{\pi}\int d^2\sigma(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+). \tag{1.53}$$

Where $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$ and we chose to represent the Dirac matrices as:

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.54}$$

It follows from (1.53) that the equation of motion for the spinor now are:

$$\partial_+\psi_- = 0, \quad \partial_-\psi_+ = 0. \tag{1.55}$$

These are two-dimensional Weyl equations, which means that the spinors $\psi_\pm$ are Majorana-Weyl spinors.

We can now proceed in complete analogy with the bosonic case. We will derive, starting from the action (1.53), the Fourier expansion of the fields and canonically quantise their modes to find the spectrum and, like in the bosonic case, we will find negative norm states that can be removed by imposing super-Virasoro constraints. This is due to the superconformal symmetry enjoyed by the RNS string, the supersymmetric extension of the conformal symmetry we have in the bosonic theory. For this reason we first need to find the conserved currents of the action (1.53). These currents are the energy momentum tensor T_{ab} and the supercurrent J_{Aa}, coming from supersymmetry. We use light-cone coordinates so $a, b = +, -$. Their non-vanishing components are:

$$\begin{aligned} T_{++} &= \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_{+\mu}, \\ T_{--} &= \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_{-\mu}. \end{aligned} \tag{1.56}$$

And:

$$\begin{aligned} J_+ &= \psi_+^\mu \partial_+ X_\mu, \\ J_- &= \psi_-^\mu \partial_- X_\mu. \end{aligned} \tag{1.57}$$

Where we renamed the non-vanishing spinor component of J_{+A} and J_{-A} just J_+ and J_- respectively. Their conservation follows from the equations of motion (1.16) and (1.55) and their modes are the super-Virasoro generators, as we shall see later.

In order to understand the super-Virasoro constraints associated to these currents we need to impose equal τ anticommutation relations on the fermions $\psi_\pm^\mu$:

$$\{\psi_A^\mu(\tau,\sigma), \psi_B^\nu(\tau,\sigma')\} = \pi\eta^{\mu\nu}\delta_{AB}\delta(\sigma-\sigma'). \tag{1.58}$$

These relations hold together with the X^μ commutation relation (1.30) and, just like those ones, they are negative for $\mu = \nu = 0$. This means there are negative-norm states in the spectrum, which must be removed to have a physical theory. In the bosonic theory this was done using the Virasoro constraints $T_{++} = T_{--} = 0$, therefore we are now led to the super-Virasoro constraints:

$$T_{++} = T_{--} = J_+ = J_- = 0. \tag{1.59}$$

It is important to notice that we are basically postulating (1.59). In fact these can be obtained in the same way we derived the standard Virasoro constraints (1.17), however the discussion is much more complicated.

We are now ready to consider the boundary conditions we need to impose on the superstring and the consequent mode expansions. We will focus only on the fermionic part of (1.49), as for the bosonic part things work out like in the Section 1.1.
The variation of (1.53) leaves the boundary terms:

$$\delta S = \int d\tau \left(\psi_+\delta\psi_+ - \psi_-\delta\psi_-\right)|_{\sigma=\pi} - \left(\psi_+\delta\psi_+ - \psi_-\delta\psi_-\right)|_{\sigma=0}. \tag{1.60}$$

We want the action to be stationary, so (1.60) must be zero. This is ensured by an appropriate choice of boundary conditions and, as for bosonic strings, such choice is not unique. As usual we distinguish between open and closed strings:

- For open strings (1.60) is zero if at both ends of the string it holds that:

$$\psi^\mu_\pm = \pm\psi^\mu_\pm. \tag{1.61}$$

The sign choice is just a matter of convention for one of the ends. This means we are free to choose for instance $\psi^\mu_+|_{\sigma=0} = \psi^\mu_-|_{\sigma=0}$, but now the sign choice at the other end is relevant and there are two possibilities:

1. Ramond boundary conditions, corresponding to $\psi^\mu_+|_{\sigma=\pi} = \psi^\mu_-|_{\sigma=\pi}$, which lead to fermions in the spectrum. Such a choice implies the following mode expansion:

$$\begin{aligned}\psi^\mu_+(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d^\mu_n e^{-in(\tau+\sigma)},\\ \psi^\mu_-(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d^\mu_n e^{-in(\tau-\sigma)}.\end{aligned} \tag{1.62}$$

2. Neveu-Schwarz boundary conditions, corresponding to $\psi^\mu_+|_{\sigma=\pi} = -\psi^\mu_-|_{\sigma=\pi}$, which lead to bosons in the spectrum. Such a choice implies the following mode expansion:

$$\begin{aligned}\psi^\mu_+(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b^\mu_r e^{-ir(\tau+\sigma)},\\ \psi^\mu_-(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b^\mu_r e^{-ir(\tau-\sigma)}.\end{aligned} \tag{1.63}$$

- For closed strings (1.60) is zero if an (anti)periodicity condition holds:

$$\psi^\mu_\pm(\tau,\sigma) = \pm\psi^\mu_\pm(\tau,\sigma+\pi). \tag{1.64}$$

As in the bosonic case, for closed strings we have a left and a right moving sector and we can impose the periodicity (Ramond) or the antiperiodicity (Neveu-Schwarz) separately on them. This means that, for the right moving waves, we can have either:

$$\psi^\mu_-(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d^\mu_n e^{-2in(\tau-\sigma)}, \quad \text{or} \quad \psi^\mu_-(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} b^\mu_r e^{-2ir(\tau-\sigma)}. \tag{1.65}$$

And similarly for the left moving waves, we can have either:

$$\psi^\mu_+(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \widetilde{d}^\mu_n e^{-2in(\tau+\sigma)}, \quad \text{or} \quad \psi^\mu_+(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} \widetilde{b}^\mu_r e^{-2ir(\tau+\sigma)}. \tag{1.66}$$

Hence there are four possible combinations of left and right moving waves. To these correspond four different sectors: (NS, NS), (R,R), (NS, R), (R, NS). The first two lead to bosons in the spectrum, while the last two to fermions.

Canonical Quantisation

We can now move on to the canonical quantisation of the superstring. Plugging the expansions (1.62) and (1.63) into (1.58) (or (1.65) and (1.66) in the case of closed strings) we find the anticommutation relations for the modes d^μ_n and b^μ_r:

$$\{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0} \quad \text{and} \quad \{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0}. \tag{1.67}$$

These relations exist on top of the bosonic ones (1.31) and, as we anticipated, are negative in the time component. Once again, an appropriate choice of the number of the spacetime dimensions and of a normal ordering constant will remove the negative norm states.

The algebra (1.67), together with (1.31) tells us that we can build excited states applying the creation operators α_m, d_m and b_r (with $m, r < 0$) on the ground state. As usual the ground state is defined as the state annihilated by all destruction operators, however we need to pay attention to the sector we are considering. In fact we have:

$$\alpha^\mu_m|0\rangle_R = d^\mu_m|0\rangle_R = 0, \;\; m > 0 \;\; \text{in the R sector,} \tag{1.68}$$

$$\alpha^\mu_m|0\rangle_{NS} = b^\mu_r|0\rangle_{NS} = 0, \;\; m, r > 0 \;\; \text{in the NS sector.} \tag{1.69}$$

The ground state (1.69) is unique and corresponds to a spin 0 state, but the ground state (1.68) is not. In fact the action of the operators d^μ_0 does not change the mass of a state since they commute with the number operator (to be defined later). To understand what kind of particle does (1.68) describe we need to start from the algebra (1.67), which gives us that $\{d^\mu_0, d^\nu_0\} = \eta^{\mu\nu}$. This is, up to a factor of 2, the Clifford algebra (1.50). For this reason the R sector ground state, or to be more precise, the set of degenerate ground states, which we denote by $|a\rangle$, must form an irreducible representation of (1.50). Representation theory of Clifford algebras tells us that its irreducible representations correspond to spinors of $Spin(1,9)$, the double cover of $SO(1,9)$. This implies that $|a\rangle$ is a spacetime fermion and satisfies

$$d^\mu_0|a\rangle = \frac{1}{\sqrt{2}}\Gamma^\mu_{ba}|b\rangle. \tag{1.70}$$

Where a, b are spinor indices.
Higher states in the R sector can be built by acting on (1.68) with the creation operators α^μ_m and d^μ_m with $m < 0$, which are vectors. Hence the R sector contains fermions only.

As we anticipated, the modes of the conserved currents are the generators of the super-Virasoro algebra and we are now ready to see their expression. The generators associated to the energy momentum tensor are still the Virasoro generators and they now receive two contributions, one from the bosonic modes and one from the fermionic modes. The former still have the same structure as in bosonic string theory (1.33), the latter depends on the sector we are working in, as well as the supercurrent modes. Suppressing the Lorentz index for simplicity these are:

- NS sector:

$$L_m^{(f)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+1/2}\left(r+\frac{m}{2}\right) : b_{-r}\cdot b_{m+r} : \quad m\in\mathbb{Z}. \tag{1.71}$$

  And for the supercurrent:

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot b_{r+n} \quad r\in\mathbb{Z}+\frac{1}{2}. \tag{1.72}$$

  No normal ordering since the α and b modes commute.

- R sector:

$$L_m^{(f)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(n+\frac{m}{2}\right) : d_{-n}\cdot d_{n+m} : \quad m\in\mathbb{Z}. \tag{1.73}$$

  And for the supercurrent:

$$F_m = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot d_{m+n} \quad m\in\mathbb{Z}. \tag{1.74}$$

  No normal ordering since the α and d modes commute.

The algebra satisfied by these generators is the super-Virasoro algebra:

$$\begin{aligned} [L_m, L_n] &= (m-n)\, L_{m+n} + A(m)\delta_{m+n,0}, \\ [L_m, G_r] &= \left(\frac{m}{2}-r\right) G_{m+r}, \\ \{G_r, G_s\} &= 2L_{r+s} + B(m)\delta_{r+s,0}. \end{aligned} \tag{1.75}$$

Where $c = 3D/2$ is the central charge and we define:

$$\begin{aligned} A(m)_{NS} &\equiv \frac{c}{12}m(m^2-1), \\ B(m)_{NS} &\equiv \frac{c}{3}\left(r^2-\frac{1}{4}\right). \end{aligned} \tag{1.76}$$

Actually (1.75) is the algebra for the NS sector only, with $m, n\in\mathbb{Z}$ and $r, s\in\mathbb{Z}+1/2$. The R sector algebra is identical to this one upon replacing $G_r \to F_m, m\in\mathbb{Z}$ and (1.76) with:

$$\begin{aligned} A(m)_R &\equiv \frac{c}{12}m^3, \\ B(m)_R &\equiv \frac{c}{3}m^2 \end{aligned} \tag{1.77}$$

This mismatch between (1.76) and (1.77) can be removed by a redefinition of L_0 by a constant.

The physical state condition (1.36) is extended to include the new modes now. Given a physical state $|\phi\rangle$ we have

- NS sector:

$$\begin{aligned}(L_0 - a_{NS})|\phi\rangle &= 0,\\ L_m|\phi\rangle &= 0 \quad m>0,\\ G_r|\phi\rangle &= 0 \quad r>0.\end{aligned} \tag{1.78}$$

- R sector:

$$\begin{aligned}(L_0 - a_R)|\phi\rangle &= 0,\\ L_m|\phi\rangle &= 0 \quad m>0,\\ F_n|\phi\rangle &= 0 \quad n\geq 0.\end{aligned} \tag{1.79}$$

Where we tacitly assumed $n, m \in \mathbb{Z}$ and $r \in \mathbb{Z} + 1/2$ and we will continue to do so for the rest of this section. We also introduced normal ordering constants a_{NS}, a_R due to the normal ordering of L_0. However $a_R = 0$ due to $L_0 = F_0^2$ and $a_{NS} = 1/2$ by requiring that zero-norm spurious states, as we defined them in (1.40), are physical in the sense of (1.78). Continuing along these lines we can also fix the spacetime dimension $D = 10$, both in the NS and R sector.
As an example let us see the case of a_{NS}. We start from an NS sector state defined as:

$$|\psi\rangle \equiv G_{-1/2}|\chi\rangle. \tag{1.80}$$

With $|\chi\rangle$ such that:

$$G_{1/2}|\chi\rangle = G_{3/2}|\chi\rangle = \left(L_0 - a_{NS} + \frac{1}{2}\right)|\chi\rangle \equiv 0. \tag{1.81}$$

We want $|\psi\rangle$ to be physical, i.e. it must satisfy (1.78). The condition $(L_0 - a_{NS})|\psi\rangle = 0$ directly follows from the last equality of (1.81), so we just need $G_{1/2}|\psi\rangle = G_{3/2}|\psi\rangle = 0$, but also the latter of these two directly follows from (1.81). Thus we are only left with:

$$G_{1/2}|\psi\rangle = G_{1/2}G_{-1/2}|\chi\rangle = (2a_{NS} - 1)|\chi\rangle. \tag{1.82}$$

Where we used the algebra (1.75) and (1.81). Therefore (1.82) is zero if $a_{NS} = 1/2$.

Spectrum

With the normal ordering constant at our disposal in both sectors we can now express the mass as we did in the bosonic case $\alpha' M^2 = N - a$, or a "double copy" of it for closed strings in order to account for all four combinations of sectors. Of course the number operator N is not anymore (1.39) as it needs to count fermionic modes too:

$$\begin{aligned}N_{NS} &= \sum_{n>0} \alpha_{-n\mu}\cdot\alpha_n^\mu + \sum_{r>0} r b_{-r\mu}\cdot b_r^\mu - \frac{1}{2},\\ N_R &= \sum_{n>0}\left(\alpha_{-n\mu}\cdot\alpha_n^\mu + n d_{-n\mu}\cdot d_n^\mu\right).\end{aligned} \tag{1.83}$$

Having an expression for the mass, we can finally derive the spectrum of the theory:

- For the NS sector we can construct excited states by acting on the ground state, which we denote as $|0;k\rangle_{NS}$, with the operator α_n^μ with $n<0$ or with b_r^μ with $r<0$.

  1. At $N=0$ we have the ground state with mass $\alpha' M^2 = -1/2$. Once again we meet a tachyon in the spectrum but, in a moment, we shall see that it is now possible to get rid of it.
  2. The first excited state is not $\alpha_{-1}^\mu|0;k\rangle_{NS}$, as operators α_{-n}^μ and b_{-r}^μ increase $\alpha' M^2$ by n and r units respectively. For this reason the first excited state is actually $b_{-1/2}^\mu|0;k\rangle_{NS}$ with a mass $\alpha' M^2 = 0$. This state forms a massless spacetime vector representation of $SO(8)$ since $|0;k\rangle_{NS}$ is a spacetime scalar and $b_{-1/2}^\mu$ a spacetime vector.

- For the R sector we can construct excited states by acting on the ground state, which we denote as $|a;k\rangle_R$, with the operator α_n^μ and d_n^μ with $n<0$. As we anticipated, all these states are spacetime fermions since the ground state is a spacetime fermion itself and the creation operators are spacetime vectors.
  It is worthwhile to spend more words on the ground state $|a;k\rangle_R$. Due to supersymmetry we would expect it to have eight physical degrees of freedom since it should form a supersymmetry multiplet with $b_{-1/2}^\mu|0;k\rangle_{NS}$, which is a massless vector in ten dimensions and so has exactly eight propagating components. However a spinor in a D-dimensional spacetime has $2^{\lfloor \frac{D}{2} \rfloor}$ degrees of freedom. Hence for $D=10$ a spinor has 32 complex components, which are reduced to 16 real components once we impose Majorana-Weyl conditions [1]. To describe physical propagating degrees of freedom these remaining 16 components must satisfy the massless Dirac equation, which relates half of the spinor's components to the other half. Therefore a Majorana-Weyl spinor in $D=10$ dimensions has only 8 propagating physical modes, as required by supersymmetry. This also means that the $Spin(1,9)$ ground state is reduced to a $Spin(8)$ spinor with two possible (ten-dimensional) chiralities due to the Weyl condition we imposed.

The spectrum of the RNS superstring has two problems: it contains a tachyon and is not supersymmetric, e.g. there is no supersymmetric counterpart of the tachyon. So it looks like that we made no progress with respect to the bosonic string, even though we said at the beginning of this section that with supersymmetry the issues of bosonic string theory would be eliminated. Indeed we are missing a last piece to make the RNS

[1] To be precise the Majorana and Weyl conditions can be imposed simultaneously on a $Spin(p,q)$ spinor only if $p+q=0$ mod 8. In Minkowski spacetime $p=1$, so this conditions boils down to $d=2+8n,\ n\in\mathbb{N}$.

superstring a consistent theory: a truncation of the spectrum called GSO projection. To understand what this is let us introduce the G-parity operators for the NS and R sector:

$$G_{NS} = (-1)^{F_{NS}+1} = (-1)^{\sum_{r>0} b^{\mu}_{-r} b_{r\mu}+1}, \tag{1.84}$$

$$G_R = \Gamma_{11}(-1)^{\sum_{n>0} d^{\mu}_{-n} d_{n\mu}}. \tag{1.85}$$

Where F_{NS} counts the number of b-type excitations and Γ_{11} is the ten-dimensional version of γ_5:

$$\Gamma_{11} = \Gamma_0\Gamma_1 \dots \Gamma_9. \tag{1.86}$$

As such it enjoys the same properties of γ_5, namely $(\Gamma_{11})^2 = 1$ and $\{\Gamma_{11}, \Gamma^{\mu}\} = 0 \ \forall \mu = 0, \dots, 9$ and it can be used to define chirality projection operators. As in four dimensions, spinors obeying $\Gamma_{11}\psi = \pm\psi$ are said to have positive or negative chirality respectively.

The GSO projection consists in keeping only those NS-states that are invariant under the action of (1.84), which in this sense are equivalent to states with an odd number of b-type excitations. Other states must instead be removed. In the R sector we only keep states with positive or negative G-parity depending on the ground state chirality, which is just a matter of convention. Clearly the tachyon state $|0;k\rangle_{NS}$ of the NS sector has a negative G-parity while the first excited state $b^{\mu}_{-1/2}|0;k\rangle_{NS}$ has positive G-parity. Thus the GSO prescription tells us to remove the former and keep the latter. The tachyon is now gone from the spectrum and we have a hint of spacetime supersymmetry as the NS ground state is now a boson whereas the R ground state is a fermion. However an actual proof of supersymmetry can only be given in the GS setup of the superstring. Indeed the GS superstring is automatically spacetime supersymmetric and has no analogue of the GSO projection.
Despite it may look an ad hoc approach that we used just to make things work out nicely, the GSO projection has a solid theoretical foundation. In fact it can be interpreted as the requirement of one-loop modular invariance of the theory.

We have left behind the spectrum of the closed superstring on purpose. In fact, now that we have a consistent theory for open superstrings thanks to the GSO projection, we can build the closed superstring spectrum just by combining the open NS and R spectra, that is by combining left and right movers, after the projection. The possible combinations are R-R, R-NS, NS-R, NS-NS and we should also take into account the G-parity of the R sector states, depending on its ground state chirality. For this reason we are led to two different superstring theories called type IIA and type IIB. For the former, the left and right moving R sector ground states are chosen to have opposite chirality, whereas for the latter they are chosen to have the same chirality. We denote with $|+\rangle_R$ the positive chirality one and $|-\rangle_R$ the negative chirality one. The massless spectrum is summarized in Tab. 2.1.

Type IIA	**Type IIB**
$\|-\rangle_R \otimes \|+\rangle_R$	$\|+\rangle_R \otimes \|+\rangle_R$
$\widetilde{b}^{\mu}_{-1/2}\|0\rangle_{NS} \otimes b^{\nu}_{-1/2}\|0\rangle_{NS}$	$\widetilde{b}^{\mu}_{-1/2}\|0\rangle_{NS} \otimes b^{\nu}_{-1/2}\|0\rangle_{NS}$
$\widetilde{b}^{\mu}_{-1/2}\|0\rangle_{NS} \otimes \|+\rangle_R$	$\widetilde{b}^{\mu}_{-1/2}\|0\rangle_{NS} \otimes \|+\rangle_R$
$\|-\rangle_R \otimes b^{\nu}_{-1/2}\|0\rangle_{NS}$	$\|+\rangle_R \otimes b^{\nu}_{-1/2}\|0\rangle_{NS}$

Table 2.1: Spectra of type IIA and type IIB at the massless level.

There are 64 states in all sectors, in particular:

- The R-R sector contains bosons for both theories since we tensored two spinors. For type IIA this yields a one form C_1, with 8 states, and a three-form C_3, with 56 states. For type IIB this yields a zero-form C_0, so a scalar with 1 state, a two-form C_2, with 28 states, and a four-form C_4, with 35 states. This follows because we can impose a self-duality condition on the field strength of C_4 in $D = 10$ which halves the number of states. Otherwise we would lose supersymmetry.
- The NS-NS sector is the same for the two theories. It contains a scalar called dilaton Φ, with 1 state, a two-form called Kalb-Ramond field B_2, with 28 states, and a traceless symmetric tensor called graviton $g_{\mu\nu}$, with 35 states.
- Both the NS-R and R-NS sector contain a spin 3/2 gravitino Ψ^M_α, with 56 states, and a spin 1/2 dilatino λ, with 8 states. The gravitini have opposite chirality in type IIA and same in type IIB.

The presence of two gravitini in the spectra implies that both type IIA and type IIB enjoy a $\mathcal{N} = 2$ supersymmetry in $D = 10$, corresponding to $\mathcal{N} = 8$ in $4D$. Any theory with $\mathcal{N} \geq 2$ cannot describe the phenomenology we observe, we shall see in the later sections how to get out of this impasse.

This is not the end of the story, there are in fact three more consistent superstring theories in $D = 10$:

- Type I superstring theory, which can be understood as projection of type IIB. Without delving too much into the details, consider the operator reversing the orientation of the worldsheet $\Omega\colon \sigma \to -\sigma$. This action swaps left and right movers of the world-sheet fields X^μ and ψ^μ, for this reason it is a symmetry only for type IIB where left-moving and right-moving fermions have the same chirality. If we gauge this symmetry we are left with type I superstring theory, whose spectrum is derived from the IIB one by keeping only those states that are symmetric under Ω. Type I superstrings are unoriented and enjoy an $\mathcal{N} = 1$ supersymmetry in $D = 10$.

- Heterotic string theory, which comes in two types based on the allowed gauge groups: $SO(32)$ and $E_8 \times E_8$.
  Heterotic string theories are realised by mixing the left-moving sector of bosonic string theory with the right-moving sector of superstring theory. Clearly there is a mismatch in the dimensionalities, which is fixed by compactifying the 16 extra dimensions of the bosonic string on a 16-torus. Imposing the absence of tachyons and modular invariance leads to the two heterotic superstring theories we mentioned. Also heterotic string theory has a $\mathcal{N} = 1$ supersymmetry in $D = 10$.

The five different theories we presented are related by a "web" of (non-)perturbative dualities shown in Fig 2.2.

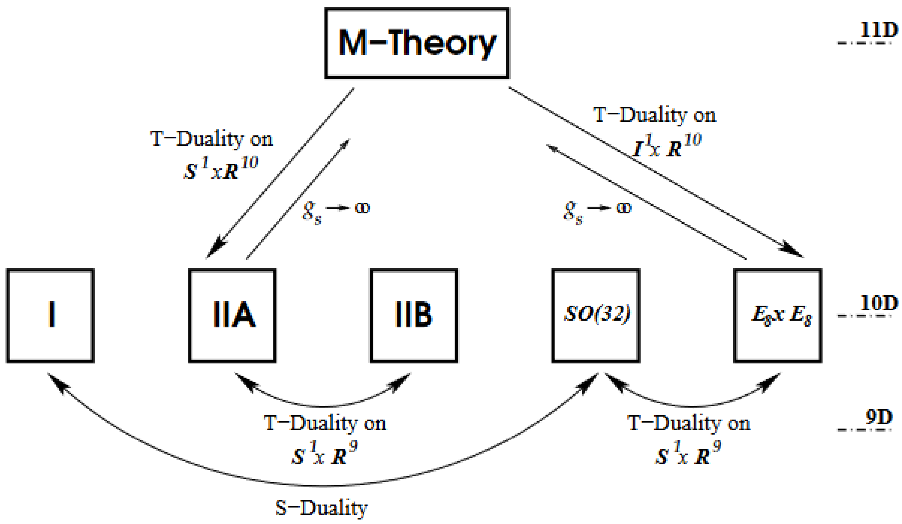

Figure 2.2: Web of dualities between the various string theories and M-theory. Picture taken from [11].

Type IIB String Theory

A few more comments on type IIB string theory are of order, as they will be needed later on. Indeed in Section 3 we will deal with moduli stabilisation in the context of type IIB. In particular we are interested in the action of type IIB supergravity, the low energy limit of type IIB string theory. We follow the notation of [12].

While for type IIA supergravity the action can be derived from the dimensional reduction of $11D$ supergravity, the low energy limit of M-Theory (see Fig. 2.2), for type IIB that is not possible. The issue is the self-dual field strength $\widetilde{F}_5$ associated to C_4 as it leads to a trivial kinetic term $\widetilde{F}_5 \wedge \star\widetilde{F}_5 = 0$. Therefore the action has been built so that it yields the correct equation of motion once we impose the self-duality constraint and it reads as:

$$\begin{aligned} S^{\text{IIB}}_{\text{bosonic}} =& S_{NS} + S_R + S_{CS} \\ =& \frac{1}{2\kappa_{10}^2}\int d^{10}X\sqrt{-G}e^{-2\Phi}\left(R^{(10)} + 4(\partial\Phi)^2 - \frac{1}{2}|H_3|^2\right) \\ &-\frac{1}{4\kappa_{10}^2}\int d^{10}X\sqrt{-G}\left(|F_1|^2 + |\widetilde{F}_3|^2 + \frac{1}{2}|\widetilde{F}_5|^2\right) \\ &-\frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3. \end{aligned} \tag{1.87}$$

Where $R^{(10)}$ is the ten-dimensional Ricci scalar and $\kappa_D^2 \equiv 8\pi G_D$, with G_D the D-dimensional Newton's constant. For $D = 10$, the case of our interest, this can be related to the string tension by comparing the worldsheet and supergravity action:

$$\kappa^2 = \frac{1}{2}(2\pi)^7(\alpha')^4. \tag{1.88}$$

Furthermore we have defined

$$H_3 \equiv dB_2, \tag{1.89}$$
$$F_p \equiv C_{p-1}, \tag{1.90}$$
$$\widetilde{F}_3 \equiv F_3 - C_0\wedge H_3 \tag{1.91}$$
$$\widetilde{F}_5 \equiv F_5 - \frac{1}{2}(C_2\wedge H_3 - B_2\wedge F_3), \tag{1.92}$$
$$|F_p|^2 \equiv F_p\wedge\star F_p. \tag{1.93}$$

Where $\widetilde{F}_5$ is the self-dual five form.

The action (1.87) is expressed, at least for the NS-NS sector, in the so-called string frame, where the Ricci scalar is accompanied by the prefactor $\exp(-2\Phi)$. This frame is convenient when we want to stress the perturbative aspect of the theory, but it is not for gravity-related aspects. In these cases it is better to work in the Einstein frame where the metric is related to the string frame one via rescaling:

$$G_{E,MN} \equiv e^{-\Phi/2}G_{MN}. \tag{1.94}$$

Furthermore it is also convenient to define specific field combinations:

$$G_3 \equiv F_3 - \tau H_3, \tag{1.95}$$
$$\tau \equiv C_0 + ie^{-\Phi}. \tag{1.96}$$

Using the Einstein frame metric (1.94) and these two new fields (1.95) (1.96), the action (1.87) takes the form:

$$\begin{aligned} S_{\text{IIB}} = & \frac{1}{2\kappa_{10}^2}\int d^{10}X\sqrt{-G_E}\left[R_E^{(10)} - \frac{|\partial\tau|^2}{2\left(\text{Im}(\tau)\right)^2} - \frac{|G_3|^2}{2\left(\text{Im}(\tau)\right)} - \frac{|\widetilde{F}_5|^2}{4}\right] \\ & -\frac{i}{8\kappa_{10}^2}\int\frac{C_4\wedge G_3\wedge\bar{G}_3}{\text{Im}\,(\tau)}. \end{aligned} \tag{1.97}$$

Clearly the action (1.87), as well as (1.97), is not supersymmetric as all the fermionic content is absent and the self-duality constraint is not embedded in it. This is not a problem as the field equations derived from it and the constraint are, however it is conventional not to give the fermionic part of the action since it is mostly used to construct classical solutions and classical solutions always have vanishing fermionic fields.

D-Branes

We already explained earlier that string theory isn't just a theory of strings, it also contains D-branes. Therefore we conclude this section with a few comments on these objects, again with an eye towards type IIB. Here we mostly follow [12]

In the simplest sense a Dp-brane is a hypersurface with p spatial dimension and solid argument for their existence, more than the one we gave previously, is via T-duality [2]. This already holds for bosonic string theory, but things are much more interesting in superstring theory where branes are stable as they couple to RR fields and hence carry a conserved charge. Indeed a Dp-brane is charged under C_{p+1} via the coupling:

$$S_{\text{CS}} = \mu_p\int_{\Sigma_{p+1}} C_{p+1}. \tag{1.98}$$

Where Σ_{p+1} is the brane world-volume and μ_p its charge. It is just the generalization to p dimensions of the action for a point particle charged under a gauge field.

As we illustrated earlier D-branes represent the surface on which the endpoints of open strings lie, whereas closed strings can propagate between them. Quantising these open strings generates a spectrum of bosonic and fermionic fields living on the brane. If we limit ourselves to massless ones, we find scalars describing the brane position, a gauge field and the corresponding superpartners. Hence (1.98) is not the complete brane action since we would like to describe those fields too and with a background given by the massless fields coming from string theory [3]. The contribution we need to add is made up of two parts:

[2] Details on T-duality can be found in Appendix B

[3] To be precise here we only think about type II or type I as the heterotic has no branes.

- A Dirac action term.
  A Dirac action is the higher-dimensional version of the Polyakov action (1.4) and describes an uncharged brane moving in a curved background with metric G_{MN}:

$$S_{\text{D}} = -T_{D_p} \int_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{-\det\left(G_{ab}\right)}. \tag{1.99}$$

  Where T_{D_p} is the brane tension and G_{ab} is the pullback of the metric:

$$G_{ab} \equiv \frac{\partial X^M}{\partial \sigma^a}\frac{\partial X^N}{\partial \sigma^b} G_{MN}. \tag{1.100}$$

- A Born-Infeld action term.
  A Born-Infeld action is a non-linear version of the Maxwell action:

$$S_{\text{BI}} = -Q_p \int d^{p+1}\sigma \sqrt{-\det\left(\eta_{ab} + 2\pi\alpha' F_{ab}\right)}. \tag{1.101}$$

  Where F_{ab} is the field strength of the gauge field and Q_p is a constant with the same dimension of the brane tension T_p.

Putting (1.99) and (1.101) together we get the Dirac-Born-Infeld action:

$$S_{\text{DBI}} = -g_s T_{D_p} \int_{\Sigma_{p+1}} d^{p+1}\sigma \; e^{-\Phi} \sqrt{-\det\left(G_{ab} + \mathcal{F}_{ab}\right)}. \tag{1.102}$$

Where $\mathcal{F}_{ab}$ is the field strength which includes the B field:

$$\mathcal{F}_{ab} \equiv B_{ab} + 2\pi\alpha' F_{ab}. \tag{1.103}$$

The quadratic term of (1.102) is a Maxwell term in the sense that it is $\frac{1}{g^2}\int d^{p+1}\sigma F_{ab}F^{ab}$. The prefactor must be the gauge coupling and we expect that $g \sim g_o \sim \sqrt{g_s}$ since the gauge field comes from open string excitations. Hence we can say $T_{D_p} \propto g_s^{-1}$ and via T-duality we can fix the missing factor:

$$T_{D_p} = \frac{1}{(2\pi)^p g_s (\alpha')^{(p+1)/2}}. \tag{1.104}$$

Finally we need to make a change in (1.98) due to the presence of the background. In fact now we have:

$$S_{\text{CS}} = i\mu_p \int_{\Sigma_{p+1}} \sum_n C_n \wedge e^{\mathcal{F}}. \tag{1.105}$$

Finally we have the D-brane action which is just the sum of (1.102) and (1.105):

$$S_{\text{brane}} = S_{\text{DBI}} + S_{\text{CS}} \tag{1.106}$$

2 Compactification of the Extra Dimensions

As we have just seen in Section 1, string theory predicts a number of dimension greater than the one we observe in nature. Clearly something must be done in this sense if we want string theory to be a realistic theory of nature and a very simple idea is that the six extra dimensions are compact and too small to be observed. For this reason we assume our ten-dimensional spacetime to be the product of a maximally symmetric spacetime, such as Minkowski, dS or AdS, and some compact six-dimensional space $M_{10} = M_4 \times X_6$. The requirement of partially unbroken supersymmetry greatly constraints the kind of manifold X_6 can be: it turns out that it must be a Calabi-Yau manifold (CY for short), a special kind of Kähler manifold with vanishing first Chern class, or, more generally, a manifold with $SU(3)$ structure. However manifold with $SU(3)$ structure that are not CY are much less studied, both in the mathematics and physics literature. For this reason we will focus on CY compactifications. These break 3/4 of the existing supersymmetry, meaning that they yield a $4D$ $\mathcal{N} = 1$ theory for type I and heterotic string theories and a $4D$ $\mathcal{N} = 2$ theory for type II string theories, which can be further broken to $\mathcal{N} = 1$ with orientifold projections.

In this section we deal with these aspects, the references are [5, 8, 13, 14] for the first part, while for the part on moduli space we also follow [12, 15, 16, 17]. The mathematical details can be found in the Appendix A.

2.1 Kaluza-Klein Compactifications

The feature of extra dimensions is not something new in physics, in fact in the 1920s Kaluza and Klein (KK for short) proposed the existence of a fifth dimension to try and unify gravity with electromagnetism. As a warm up we start from scalar fields in five dimensions to then move to gravity.

Let us assume that we have a $D = 5$ spacetime given by $\mathbb{R}^{1,3} \times S^1$ and a massless scalar field $\phi(x^M)$, $M = 0, \ldots, 4$. Its dynamics are described by the action:

$$S_{5D} = \int d^5x \; \partial_M \phi \partial^M \phi. \tag{2.1}$$

Since ϕ lives on S^1 it must be periodic along $x^4 \equiv y$, hence it can be Fourier expanded along that direction:

$$\phi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu)\; e^{iny/r}. \tag{2.2}$$

With r the radius of S^1

Plugging (2.2) into the equations of motion derived from (2.1) we find:

$$\partial_\mu \partial^\mu \phi_n - \left(\frac{n}{r}\right)^2 \phi_n = 0. \tag{2.3}$$

This equation is effectively a massive Klein-Gordon equation if we identify $m_n^2 = (n/r)^2$. To be more precise, we have an infinite number of KG equations, one for every n, which means that the compactification of a higher dimensional scalar field to $4D$ yields an infinite tower of massive modes, the KK tower. This is even more clear if we go back to (2.1) and integrate over S^1:

$$\begin{aligned} S_{5D} &= \int d^4x \; dy \left[\partial_\mu \phi_n \partial^\mu \phi_n - \left(\frac{n}{r}\right)^2 \phi_n^2 \right] \\ &= 2\pi r \int d^4x \left[\partial_\mu \phi_0 \partial^\mu \phi_0 + \dots \right] \end{aligned} \tag{2.4}$$

Upon integration the five-dimensional action has become, up to a prefactor, the sum of $4D$ scalar actions, one massless and all the other massive. In most cases we are only interested in the massless modes only so we can regard (2.4) as an EFT, with cutoff at $M_{KK} = 1/r$ and integrate out the heavy modes. In this case we talk about dimensional reduction rather than compactification. The problem with this EFT is that the zero mode ϕ_0 has no potential, as such it would mediate unobserved fifth forces. These kinds of fields are called moduli and the process of giving them a non-zero mass goes under the name of "moduli stabilisation". We will discuss this aspect in-depth later on, in Section 3.

Having developed an intuition for KK compactification in the case of scalars we can move on to gravity, which shares many of the same features. We start from the Einstein Hilbert action in $5D$:

$$S_{\text{EH}}^{5D} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-G} R^{(5)} \tag{2.5}$$

With G the determinant of the $5D$ metric G_{MN} and $R^{(5)}$ the $5D$ Ricci scalar.

We decompose the metric $G_{MN} = g_{\mu\nu} \oplus g_{\mu 4} \oplus g_{4\nu} \oplus g_{44}$ and hence perform the identifications $g_{\mu 4} = A_\mu$, $g_{4\nu} = A_\nu$ and $g_{44} = \phi$. Now we can parameterize the metric as:

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + \kappa_4^2 A_\mu A_\nu & \kappa_4 \phi^2 A_\mu \\ \kappa_4 \phi^2 A_\nu & \phi^2 \end{pmatrix} \tag{2.6}$$

As we did for the scalar we can Fourier expand the fields in (2.6) along the periodic direction. Once again we get an infinite tower of states and we integrate out all the massive ones. In this way we can derive the EFT describing the zero-modes only:

$$S_{\text{EFT}} = \int d^4x \sqrt{|g|} \phi_0 \left(\frac{R^{(4)}}{\kappa_4^2} + \phi_0^2 \frac{F_{\mu\nu}^0 F_0^{\mu\nu}}{4} + \frac{2\partial_\mu \phi_0 \partial^\mu \phi_0}{3\kappa_4^2 \phi^2} \right). \tag{2.7}$$

Where we defined:

$$\kappa_4^2 \equiv \frac{\kappa_5^2}{\text{Vol}} \tag{2.8}$$

With Vol the volume of the extra dimension(s).

As anticipated, we have a unified theory of gravity, electromagnetism and scalar fields which is affected by the same problem of moduli as the purely scalar one (2.4). This role is played by ϕ_0.

String compactifications share many characteristics with the simple KK one we just described, as we shall see in the next section. Indeed, as a simple example, consider again the EH action, but in string theory this time:

$$S_{\text{EH}}^{10D} = \frac{1}{2\kappa_{10}^2} \int_{M_{10}} d^{10}X \sqrt{-G} e^{-2\Phi} R^{(10)}. \tag{2.9}$$

Where we assume that the 10-dimensional spacetime is equipped with a metric G_{MN} such that:

$$G_{MN} dX^M dX^N = e^{-6u(x)} g_{\mu\nu} dx^\mu dx^\nu + e^{2u(x)} \hat{g}_{mn} dy^m dy^n \tag{2.10}$$

With $e^{u(x)}$ a mode describing size fluctuations of the compact manifold X_6. The factor of $e^{-6u(x)}$ for the first term is just a matter of convention.

The action (2.9) can be expanded as:

$$S_{\text{EH}}^{10D} = \frac{1}{2\kappa_{10}^2} \int_{M_4} d^4x \sqrt{-g} \int_{X_6} d^6y \sqrt{\hat{g}}\, e^{-2\Phi} \left(R^{(4)} + e^{-8u} \hat{R}^{(6)} + 12 \partial_\mu u \partial^\mu u \right). \tag{2.11}$$

The last term of this action is a kinetic term for the field $u(x)$, but the potential term is absent [4]: $u(x)$ is a modulus.
Furthermore, assuming that the dilaton is constant, from (2.11) we can read the four-dimensional effective EH action:

$$S_{\text{EH}}^{4D} = \frac{M_{\text{Pl}}^2}{2} \int_{M_4} d^4x \sqrt{-g} R^{(4)}. \tag{2.12}$$

Where we defined:

$$M_{\text{Pl}}^2 \equiv \frac{\text{Vol}(X_6)}{g_s^2 \kappa_{10}^2} \tag{2.13}$$

2.2 Calabi-Yau Compactifications

As already mentioned, we assume our spacetime to be a direct product $M_{10} = M_4 \times X_6$, where M_4 is any maximally symmetric spacetime and X_6 some compact space. We will refer to them as external and internal space respectively and we shall use the index rule $M = (\mu, m)$. This notation stands for: M is a $10D$ index of M_{10}, μ is a $4D$ index of

[4] We shall see later that, in string theory, the compact manifold X_6 must be Ricci-flat, hence the would-be potential term $e^{-8u}\hat{R}^{(6)}$ vanishes.

M_4 and m is a $6D$ index of X_6. Following it we denote coordinates on M_{10} with x^M, coordinates on M_4 with x^μ and coordinates on X_6 with y^m.
The hypothesis of a maximal symmetry implies that the Yang-Mills field strength F and the field strength H of the NS-NS two-form C_2 must be zero on M_4: $F_{\mu\nu} = F_{\mu n} = 0$ and $H_{\mu\nu\rho} = H_{\mu\nu p} = H_{\mu n p} = 0$. On top of this we also assume $H_{mnp} = 0$ and a constant dilation Φ for simplicity.

The statement of unbroken $\mathcal{N} = 1$ supersymmetry means that there exists a conserved supercharge Q_α, generating the supersymmetry transformation with supersymmetry parameter ε_α, which annihilates the vacuum $Q_\alpha|\Omega\rangle = 0$. This condition is equivalent to saying that, for any operator U:

$$\langle\Omega|[U, Q_\alpha]|\Omega\rangle = 0. \tag{2.14}$$

This equation is trivial only if U is a bosonic operator. If U is fermionic $\{U, Q_\alpha\} \propto \delta_\varepsilon U$ hence the unbroken supersymmetry condition boils down to:

$$\delta_\varepsilon U = 0. \tag{2.15}$$

In our case we are interested in U being the fermionic fields of string theory and we can limit ourselves to the gravitino only. Despite our focus up to now has been on type IIB string theory, and it will be the same in the following, this is most easily done in heterotic string theory so we briefly switch to it, still, the results we will find hold for all types of string theory. Indeed the gravitino variation in heterotic string theory at leading order is:

$$\delta\Psi_M = \nabla_M\varepsilon - \frac{1}{4}\mathbf{H}_M\varepsilon. \tag{2.16}$$

Where $\mathbf{H}_M$ is given by the contraction of antisymmetrized gamma matrices and $H_3 = dB_2 + \omega_3$, with ω_3 a particular combination of the spin connection and the Chern-Simons three-form. Their explicit expressions are not relevant as this H_3 field strength is the one we have set to zero above. In principle there would also be the dilatino and gaugino variations on top of the gravitino one, but these do not yield any interesting condition as they are already zero once we set $H_3 = F_2 = 0$ and Φ constant [5].

Having set $H_3 = 0$ and requiring $\delta\Psi_M = 0$, (2.16) tells us that ε is a Killing spinor, sometimes also called parallel spinor:

$$\nabla_M\varepsilon = 0. \tag{2.17}$$

[5] To be honest the gaugino variation yields a non-trivial condition since F has been set to zero only on M_4 but not on X_6. This condition reads as $\Gamma^{ij}F_{ij}\varepsilon = 0$, with F_{ij} the X_6 components of the field strength of the Yang-Mills gauge connection A on some vector bundle. It can be proved that this equation involving F classifies such bundle as a holomorphic vector bundle, that is, a vector bundle with holomorphic transition functions. It is possible to find a holomorphic gauge connection living on such bundle, thus the gauge field A can be considered holomorphic.

Since we assumed $M_{10} = M_4 \times X_6$ we can split it as $\varepsilon = \zeta(x) \otimes \eta(y)$ and perform the analysis separately on the external space M_4 and the internal one X_6:

- Equation (2.17) on the external space reduces to:

$$\nabla_\mu \zeta = 0. \tag{2.18}$$

  It then follows that:

$$[\nabla_\mu, \nabla_\nu]\,\zeta = \frac{1}{4} R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\zeta = 0. \tag{2.19}$$

  Where $\Gamma^{\rho\sigma} = [\Gamma^\rho, \Gamma^\sigma]/2$.

  The equation (2.19) goes by the name of integrability condition and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor on M_4. Now recall that we assumed maximal symmetry of M_4 in order to simplify (2.16), but, on the other hand, maximal symmetry also implies $R_{\mu\nu\rho\sigma} = (R/12)(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho})$, with R the Ricci scalar. This means that, out of the three possible choices for M_4, namely Minkowski, dS and AdS, supersymmetry imposes Minkowski.

- Equation (2.17) on the internal space reduces once again to the integrability condition:

$$[\nabla_m, \nabla_n]\,\eta = \frac{1}{4} R_{mnpq}\Gamma^{pq}\eta = 0. \tag{2.20}$$

  Clearly on X_6 we do not have anymore the assumption of maximal symmetry, but still (2.20) can be treated so as to yield $R_{mn} = 0$. The internal space X_6 must be Ricci-flat.

Actually (2.20) tell us much more than just $R_{mn} = 0$. In fact it also means that, due to $\mathcal{N} = 1$ supersymmetry, such a spinor must exist. This might sound tautological but the existence of a covariantly constant spinor poses stringent conditions on X_6. To understand why that is the case we have to question ourselves about the holonomy properties of the spinor η, that is, its behaviour under parallel transport around a closed loop.

The holonomy group of an orientable Riemannian N-dimensional manifold is $\text{Hol(M)} \subseteq SO(N)$, so for us $\text{Hol}(X_6) = SO(6)$. The holonomy group is not all of $SO(6)$ however, because a covariantly constant spinor satisfies $U\eta = \eta$ for $U \in \text{Hol}(X_6)$, just like vectors. Hence η is a singlet of $\text{Hol}(X_6)$. To understand which subgroup of $SO(6)$ satisfies this condition we note that at the level of Lie algebra $SO(6) \cong SU(4)$ and that $\eta \in SO(6)$ has eight real components that transform as the fundamental $\mathbf{4}$ and antifundamental $\bar{\mathbf{4}}$ representations of $SU(4)$, corresponding to the two possible chiralities. In other words we have the decomposition $\mathbf{8} = \mathbf{4} \oplus \bar{\mathbf{4}}$.

Now let us assume that η has a definite chirality, say positive, and focus on $\mathbf{4}$. This $\mathbf{4}$ representation decomposes under $SU(3)$ into a triplet and a singlet $\mathbf{4}_{SU(4)} = (\mathbf{3} \oplus \mathbf{1})_{SU(3)}$, which is exactly what we are looking for. Thus $\text{Hol}(X_6) = SU(3)$ implies the existence of one covariantly constant spinor of positive chirality and one of negative chirality, η_+ and $\eta_- = (\eta_+)^*$, which in turn imply unbroken $\mathcal{N} = 1$ supersymmetry in $4D$ (or $\mathcal{N} = 2$ for type II theories). This result classifies X_6 as a manifold with $SU(3)$ structure, proving it is actually a CY manifold requires a bit more work.

The relevance of the η spinor is even greater than what emerges from our discussion above. Indeed starting from η we can define two key bilinears: the Kähler form J and the holomorphic $(3,0)$-form Ω.

- Kähler form.
  To build the Kähler form we start from:

$$J_m{}^n = i\eta_+^\dagger \gamma_m{}^n \eta_+ = -i\eta_-^\dagger \gamma_m^n \eta_-. \tag{2.21}$$

  With a proper normalization of η, namely $\eta_\pm^\dagger \eta_\pm = 1$, one can prove that $J_m{}^n J_n{}^p = -\delta_m{}^p$, which classifies J as an almost complex structure. Furthermore any tensor constructed out of η will be covariantly constant because η itself is, thus $\nabla_m J_n{}^p = 0$. It follows that the Nijenhuis tensor associated to J is zero $N^p{}_{mn} = 0$ and therefore, by Newlander–Nirenberg theorem, J is actually a complex structure. This result promotes X_6 to a complex manifold.
  We can now define the actual Kähler form as:

$$\begin{aligned} J &= \frac{1}{2} J_{mn} dx^m \wedge dx^n, \\ &= i g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}. \end{aligned} \tag{2.22}$$

  With $J_{mn} = J_m{}^k g_{kn}$. In the second line we introduced complex coordinates and the Hermitian metric $g_{i\bar{j}}$.
  The $(1,1)$-form (2.22) is closed, i.e. $dJ = 0$, hence it really is the Kähler form and X_6 is a Kähler manifold.

- Holomorphic $(3,0)$-form.
  To build the holomorphic $(3,0)$-form we start from:

$$\Omega_{abc} = \eta_-^T \gamma_{abc} \eta_-. \tag{2.23}$$

  From this one we can define the actual $(3,0)$-form:

$$\Omega = \frac{1}{6} \Omega_{abc} dz^a \wedge dz^b \wedge dz^c. \tag{2.24}$$

  This form is closed, i.e. $d\Omega = 0$, but not exact as $\Omega \wedge \bar{\Omega} \propto \omega$, where ω is the volume form on X_6 which is not exact. Furthermore Ω is also unique up to a multiplicative constant.

The holomorphicity of (2.24), following from its covariant conservation, implies that we can locally write $\Omega_{abc} = f(z)\epsilon_{abc}$. Therefore the Ricci $(1,1)$-form is:

$$\mathfrak{R} = -i\partial\bar{\partial}\log\left(\det(g_{k\bar{l}})\right) = -i\partial\bar{\partial}\log\left(\|\Omega\|^2\right). \tag{2.25}$$

Where we defined

$$\|\Omega\|^2 = \frac{1}{6}\Omega_{abc}\bar{\Omega}^{abc}. \tag{2.26}$$

The Ricci form (2.25) is always closed on Kähler manifolds and in this specific case is also exact via the $\partial\bar{\partial}$-lemma since the argument of the logarithm is a globally defined scalar. It follows that $\mathfrak{R}$ belongs to the trivial cohomology class and hence the first Chern class of X_6 vanishes $c_1(X_6) = 0$. This final result classifies X_6 as a Calabi-Yau three-fold.

The volume of our CY three-fold can be expressed using either of these two forms we constructed. This can be guessed from the fact that J is a $(1,1)$-form and Ω a $(3,0)$-form, hence $J \wedge J \wedge J$ and $\Omega \wedge \bar{\Omega}$ will both be $(3,3)$-forms as the volume form. The missing prefactor can be guessed starting from the complex one-dimensional case, at least for J:

$$\mathcal{V} = \int J. \tag{2.27}$$

Indeed in $1D$ we have $J = ig_{z\bar{z}}dz \wedge d\bar{z} = 2g_{z\bar{z}}dx \wedge dy = \sqrt{g}dx \wedge dy$. Now we can move to the n-dimensional case, start from (2.22) and use $\sqrt{g} = 2^n \det g_{i\bar{j}}$, which in the case of $n = 3$ tells us that:

$$\mathcal{V} = \frac{1}{6}\int_{X_6} J \wedge J \wedge J. \tag{2.28}$$

Whereas for $\Omega \wedge \bar{\Omega}$ we just have to expand:

$$\begin{aligned}\Omega \wedge \bar{\Omega} =& -\frac{i}{36} J \wedge J \wedge J \left(\Omega_{i_1 i_2 i_3}\bar{\Omega}_{\bar{j}_1\bar{j}_2\bar{j}_3} g^{i_1\bar{j}_1} g^{i_2\bar{j}_2} g^{i_3\bar{j}_3}\right) \\ =& -i\|\Omega\|^2 d\mathcal{V}.\end{aligned} \tag{2.29}$$

Where in the second line we used (2.26) and (2.28).

Calabi-Yau manifolds have several other characterizing properties and the one we are the most interested in is its Betti numbers, which are summarized in the Hodge diamond in Fig. 2.3.
As we can see the only independent Hodge numbers are $h^{1,1}$ and $h^{1,2}$ as all the others are either fixed or related by different dualities, namely:

- By Serre duality $h^{p,q} = h^{(n-p),(n-q)}$.

$$
\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}
=
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Figure 2.3: Generic Hodge diamond for a complex three-dimensional manifold on the left and Hodge diamond for a CY three-fold on the right.

- By Hodge duality $h^{p,q} = h^{q,p}$.
- The two dualities above also imply Poincaré duality $h^{p,q} = h^{(n-q),(n-p)}$.
- The remaining unfixed Hodge numbers are $h^{0,0}$, which is 1 for any compact connected Kähler manifold and $h^{0,1} = h^{1,0}$, which is 0, assuming X_6 has a trivial fundamental group, that is to say, X_6 is simply-connected.

2.3 Moduli Space of Calabi-Yau Three-Folds

The Hodge numbers we showed above do not characterize a CY manifold completely as some of them are related by deformations of the parameters describing their shape and size, parameters which go by the name of moduli. In the light of Yau's theorem that parameter space, called moduli space, is the space of Ricci-flat Kähler metrics. This is the goal of this section: to understand which kind of deformations preserve the Ricci flatness condition. The physical interest in this branch stems from the fact the space parameterized by vacuum expectation values of massless scalar fields is itself a moduli space. Such fields are called moduli and we already met them when discussing KK compactifications.
Since we only consider CY three-folds from now on, unless specified otherwise, we rename $X_6 \to X$.

As we mentioned above studying moduli spaces means studying which kind of deformations, metric deformations in particular, leave the Ricci-flatness condition untouched. Let us consider therefore a Ricci-flat Riemannian manifold (M, g) and a metric deformation $g \to g + \delta g$. Our request is that:

$$R_{\mu\nu}(g) = 0 \;\text{ and }\; R_{\mu\nu}(g + \delta g) = 0. \tag{2.30}$$

Obviously, given that g is Ricci-flat, any metric related to it via a diffeomorphism is also Ricci-flat, for this reason we are not interested in the δg generated by diffeomorphism. These can be eliminated by a suitable gauge condition such as $\nabla^\mu \delta g_{\mu\nu} = 0$. Now we can expand (2.30) at linear order in δg and use the Ricci-flatness of g to obtain the Lichnerowicz equation:

$$\nabla^\rho \nabla_\rho \delta g_{\mu\nu} + 2R_{\mu\ \nu}^{\ \rho\ \sigma} \delta g_{\rho\sigma} = 0. \tag{2.31}$$

If we promote (M, g) to a Kähler manifold it turns out that we can split the solutions to (2.31) in two types: $\delta g_{i\bar{j}}$ and δg_{ij}, as it can be understood from the index structure of the metric and of the Riemann tensor. The former represent a different choice of Kähler class, or, simply put, a deformation of the Kähler form, while the latter represent a deformation of the complex structure. These solutions can be expanded as:

$$\delta g_{i\bar{j}} = i v^a(x)\, (\omega_a)_{i\bar{j}}\,, \quad a = 1, \dots, h^{(1,1)}, \tag{2.32}$$

$$\delta g_{ij} = \frac{i}{\|\Omega\|^2} \bar{z}^k(x)\, (\bar{\chi}_k)_{i\bar{i}\bar{j}}\, \Omega^{\bar{i}\bar{j}}{}_j, \quad k = 1, \dots, h^{(1,2)}. \tag{2.33}$$

Where ω_a are harmonic (1,1)-forms on X, forming a basis of $H^{(1,1)}(X, \mathbb{C})$ and the $\bar{\chi}_k$ are a basis of $H^{(2,1)}(X, \mathbb{C})$.

The expansions (2.32), (2.33) also feature v^a and $\bar{z}^k$, which we call Kähler moduli and complex structure moduli respectively. Indeed they are the four-dimensional moduli fields we have been talking about and appear in the effective theory as massless scalar fields. They act as coordinates on the moduli space, which for CY three-folds is rather easy, at least locally, simply being the direct product of the two different spaces parameterized by the two different kinds of moduli:

$$\mathcal{M} = \mathcal{M}_{CS}^{h^{(1,2)}} \times \mathcal{M}_K^{h^{(1,1)}}. \tag{2.34}$$

Where $\mathcal{M}_{CS}^{h^{(1,2)}}$ is the complex $h^{(1,2)}$-dimensional moduli space spanned by the complex structure deformations $\bar{z}^k$ and $\mathcal{M}_K^{h^{(1,1)}}$ is the real $h^{(1,1)}$-dimensional moduli space spanned by the Kähler form deformations v^a.

From purely geometric considerations we found the existence of $h^{(1,1)} + 2h^{(2,1)}$ real moduli, but now physics comes into play. Indeed, upon compactification, we find additional massless scalar degrees of freedom coming from the internal components of (NS, NS) two-form field B_2. To understand why consider the kinetic term of a generic p-form field:

$$\int d^{10}x \sqrt{-g}\; F_{p+1} \wedge \star F_{p+1}. \tag{2.35}$$

Where F_{p+1} is the field strength of the p-form field, so in our case it will be $H_3 = dB_2$. From (2.35) we can derive the equation of motion:

$$\Delta_{10} B_2 = d \star dB_2 = 0. \tag{2.36}$$

Where Δ_{10} is the ten-dimensional Laplacian, which can be split as $\Delta_{10} = \Delta_4 + \Delta_6$ since we compactify on $M_4 \times X$. It follows that the massless four-dimensional fields are given by zero modes of the internal Laplacian Δ_6, so by harmonic forms. This is most easily seen for a scalar field ϕ:

$$(\Delta_4 + \Delta_6)\,\phi = \left(\Delta_4 + m^2\right)\phi = 0. \tag{2.37}$$

We assumed that ϕ is eigenfunction of Δ_6 with eigenvalue m^2.

By Hodge theorem the harmonic (p, q)-forms are in a one-to-one correspondence to the elements of the Dolbeaut cohomology groups $H^{(p,q)}(X)$, thus the zero modes are counted by Betti numbers. In particular the compactification of B_2 yields 1 two-form, 0 one-forms and $h^{(1,1)}$ zero-forms. If we use a hat "ˆ" to denote $10D$ fields the expansion reads as:

$$\hat{B}_2 = B_2(x) + b^a(x)\omega_a. \tag{2.38}$$

Where $B_2(x)$ denotes the four-dimensional two-form and $b^a(x)$ are the scalar moduli.

The zero-forms in (2.38) combine with the $h^{(1,1)}$ Kähler deformations, for a total of $h^{(1,1)}$ complex massless scalar fields in $4D$. Thus string theory complexifies the Kähler form which is instead real from a purely geometric perspective $J \to \mathcal{J} = B + iJ$. This is also reflected in the expansion of the deformation (2.32) that becomes:

$$\left(\delta B_{i\bar{j}} + i\delta g_{i\bar{j}}\right) = t^a\omega_a, \quad \omega_a \in H^{(1,1)}(X, \mathbb{C}). \tag{2.39}$$

This expansion is effectively an expansion of the Kähler form itself $\mathcal{J}$, whose variations give the massless fields we just mentioned.

We could now derive the derive the Kähler potential on the moduli space (2.34), but we defer it to the next paragraph.

Type IIB on Calabi-Yaus and Orientifolding

If we now specialize the above discussion to the case of type IIB we can expand in a similar fashion to (2.38) also the remaining fields, namely C_2 and C_4, since C_0 and Φ are scalars:

$$\hat{C}_2 = C_2(x) + c^a(x)\omega_a, \tag{2.40}$$

$$\hat{C}_4 = D_2^a(x) \wedge \omega_a + V^K(x) \wedge \alpha_K - U_K(x) \wedge \beta^K + \theta_a(x)\widetilde{\omega}^a, \quad K = 0, \ldots, h^{(1,2)}. \tag{2.41}$$

As we can see the expansion of C_4 is more complicated than the other ones, as it presents, on top of the two-form D_2 and the scalar $\theta_a(x)$ also one-form contributions, namely V^K and U_K. We also introduced (α_K, β^K) which are real harmonic three-forms forming a

symplectic basis of $H^3(X,\mathbb{Z})$ [6] and $\widetilde{\omega}^a$ which are harmonic $(2,2)$-forms forming a basis for $H^{(2,2)}(X,\mathbb{C})$ and are dual to the ω^a $(1,1)$-forms.
Furthermore the self-duality of the field strength $\widetilde{F}_5$ associated to C_4 allows us to eliminate half of the degrees of freedom of C_4. It is convention to set to zero D_2^a and U_K.

These fields can be organized into $\mathcal{N}=2$ multiplets as in Tab. (2.2).

Type of multiplet	**Number of multiplets**	**States of multiplet**
Gravity multiplet	1	$(g_{\mu\nu}, V^0)$
Vector multiplets	$h^{(2,1)}$	(V^k, z^k)
Hypermultiplets	$h^{(2,1)}$	(t^a, b^a, c^a, ρ_a)
Double tensor multiplet	1	(B_2, C_2, C_0, Φ)

Table 2.2: Spectra of type IIB compactified on a Calabi-Yau three-fold at the massless level. The double tensor multiplet can be treated as another hypermultiplet if we dualize the B_2 and C_2 fields to scalar fields.

It is a general result in the context of $\mathcal{N}=2$ theories that, if we restrict ourselves only to vector multiplets and hypermultiplets, the resulting scalar manifold $\mathcal{M}$ has a strong characterization. It is in fact the direct product of a quaternionic Kähler manifold [7], spanned by the scalars of the hypermultiplets, and of a special Kähler manifold [8], spanned by the scalars z^k of the vector multiplet. Therefore the scalar manifold is:

$$\mathcal{M} = \mathcal{M}_Q^{4(h^{(1,1)}+1)} \times \mathcal{M}_{SK}^{2h^{(1,2)}}. \tag{2.42}$$

The relevance of this result is that for CY compactification this space has a submanifold, product of two special Kähler manifolds, that coincide with the true moduli space (2.34). We will recall this fact in a moment.

Clearly any $\mathcal{N}=2$ theory cannot describe a realistic model of nature due to the absence of fermions in chiral representations, therefore we would like to further break the supersymmetry to $\mathcal{N}=1$. In this sense the models are more promising if we include, on top of the fluxes, also D-branes, as the resulting theory are enriched, to the point of

[6] A symplectic basis is such that $\int_X \alpha_K \wedge \beta^L = \delta_L^K$ and $\int_X \alpha_K \wedge \alpha_L = \int_X \beta^K \wedge \beta^L = 0$.

[7] Despite the name, quaternionic Kähler manifold are not Kähler. They are instead Riemannian $4n$-manifolds with holonomy group $Sp(n)Sp(1) \subset SO(4n)$.

[8] Special Kähler manifold are Kähler manifolds where the Kähler potential can be written in terms of a single holomorphic function, called prepotential $\mathcal{F}$.

containing the Standard Model for example. However consistency of flux compactifications requires the cancellation of tadpoles and the presence of D-branes gives a positive contribution to the gravitational one, due to their positive tension. A way out is via orientifold planes which carry a negative tension. This aspect will be made more clear in the following section. Coincidentally orientifolds are also what we need to break the $\mathcal{N}=2$ supersymmetry of type II theories compactified on CY three-folds to $\mathcal{N}=1$.
An orientifold action can be understood as a transformation that includes the worldsheet parity operator Ω. We already met in when talking about type I string theory and, in order not to confuse it with the holomorphic $(3,0)$-form Ω, we now rename it $\Omega \rightarrow \Omega_{WS}$. The kind of orientifolds we are interested in are of the form:

$$\mathcal{O} = (-1)^{F_L} \Omega_{WS} \sigma^*. \tag{2.43}$$

Where $(-1)^{F_L}$ is the left-moving sector worldsheet fermion number, σ is a holomorphic isometric involution that reverses the sign of Ω, but leaves the metric and complex structure untouched and σ^* is its pullback.
Orientifold planes are the (hyper-)planes consisting of fixed points of an orientifold action and, in the specific case of (2.43), these planes can have 3 or 7 dimensions, hence the name O3/O7-planes.

Under the action of (2.43) the Dolbeaut cohomology groups split into two eigenspaces $H^{(p,q)} = H_+^{(p,q)} \oplus H_-^{(p,q)}$, denoting the even and odd eigenspaces respectively, with dimensionality $h_+^{(p,q)}$ and $h_-^{(p,q)}$. The properties of the involution σ impose a set of relation on the Hodge numbers:

- $h_\pm^{(1,1)} = h_\pm^{(2,2)}$ due to the commutativity between σ^* and the Hodge star $\star$.
- $h_\pm^{(2,1)} = h_\pm^{(1,2)}$ due to the holomorphicity of σ.
- $h_+^{(3,0)} = h_+^{(0,3)} = 0$ and $h_-^{(3,0)} = h_-^{(0,3)} = 1$ due to the action of σ^* on Ω: $\sigma^*\Omega = -\Omega$.
- $h_+^{(0,0)} = h_+^{(3,3)} = 1$ and $h_-^{(0,0)} = h_-^{(0,0)} = 0$ due to the fact that $\Omega \wedge \bar{\Omega}$ is invariant under σ^*.

This obviously affects all the expansions we have seen. Starting from (2.32) and (2.33) we now have:

$$\mathcal{J} = t^{a_+}(x)\omega_{a_+}, \quad a_+ = 1, \ldots, h_+^{(1,1)}, \tag{2.44}$$

$$\delta g_{ij} = \frac{i}{\|\Omega\|^2} \bar{z}^{k_-} \, (\bar{\chi} k_-)_{i\bar{i}\bar{j}} \, \Omega_j^{\bar{i}\bar{j}}, \quad k = 1, \ldots h_-^{(1,2)}. \tag{2.45}$$

With ω_{a_+} a basis of $H_+^{(1,1)}$ and $\bar{\chi}_{k_-}$ a basis of $H_-^{(1,2)}$. Therefore the Kähler form deformations surviving after orientifolding correspond to elements in the even eigenspace whereas the surviving complex structure deformations correspond to elements in the odd eigenspace. The moduli t^a of (2.44) geometrically represent the volume of two-cycles inside X.

The expansion (2.44) allows us to give a new expression for the volume (2.28):

$$\begin{aligned}\mathcal{V} &= \frac{1}{6}\int \mathcal{J}\wedge\mathcal{J}\wedge\mathcal{J} \\ &= \frac{1}{6}t^i t^j t^k \int \omega_i\wedge\omega_j\wedge\omega_k \\ &\equiv \frac{1}{6}\kappa_{ijk}t^i t^j t^k.\end{aligned} \tag{2.46}$$

Where we defined the triple intersection numbers $\kappa_{ijk} \equiv \int \omega_i\wedge\omega_j\wedge\omega_k$, which count how many times a curve intersects with itself. We also dropped the $\pm$ index for readability.

Now we consider the expansions (2.38), (2.40) and (2.41) that become:

$$\hat{B}_2 = b^{a_-}(x)\omega_{a_-},\ \hat{C}_2 = c^{a_-}(x)\omega_{a_-},\quad a_- = 1,\dots,h_-^{(1,1)}, \tag{2.47}$$

$$\hat{C}_4 = D_2^{a_+}(x)\wedge\omega_{a_+} + V^{K_+}\wedge\alpha_{K_+} + U_{K_+}\wedge\beta^{K_+} + \theta_{a_+}\widetilde{\omega}^{a_+},\quad K_+ = 1,\dots,h_+^{(1,2)}. \tag{2.48}$$

With ω_{a_-} a basis of $H_-^{(1,1)}$, $\widetilde{\omega}^{a_+}$ a basis of $H_+^{(2,2)}$, dual to ω_{a_+}, and $(\alpha_{K_+},\beta^{K_+})$ a real symplectic basis of $H_+^{(3)}$. Once again the self-duality of $\widetilde{F}_5$ removes half of the degrees of freedom of $\hat{C}_4$.

The fields can now be arranged into $\mathcal{N}=1$ multiplets as we anticipated. They can be found in Tab. 2.3.

Type of multiplet	**Number of multiplets**	**States of multiplet**
Gravity multiplet	1	$g_{\mu\nu}$
Vector multiplets	$h_+^{(2,1)}$	V^{K_+}
Chiral Multiplets	$h_-^{(2,1)}$	z^{k_-}
	1	(Φ, C_0)
	$h_-^{(1,1)}$	(b^{a_-}, c^{a_-})
Chiral/linear multiplets	$h_+^{(1,1)}$	(t^{a_+}, θ_{a_+})

Table 2.3: Spectra of type IIB compactified on a Calabi-Yau three-fold at the massless level after orientifolding.

At this point some natural questions we can ask ourselves are if we can find coordinates on the moduli space (2.34), if there is a metric and if they are Kähler and hence if we can find a Kähler potential.
Let us start from the first of these questions. A previous result comes in handy now: the fact that the scalar manifold (2.42) has a special Kähler component spanned by the complex structure moduli z, means that those moduli are already "good" Kähler coordinates. The other ones are not obvious and are given by:

$$S \equiv C_0 + ie^{-\Phi}, \tag{2.49}$$

$$G^a \equiv c^a - \tau b^a, \tag{2.50}$$

$$T_a \equiv \frac{1}{2}\kappa_{abc}t^b t^c + i\theta_a + \frac{1}{4}e^{\Phi}\kappa_{abc}G^b\left(G-\bar{G}\right)^c. \tag{2.51}$$

Again we dropped the $\pm$ index for readability, but it can be easily restored by looking at the index structure of the expansions above.
S (2.49) is usually referred to as axiodilaton, it is exactly the same one we have met in Sec. 1 under the name of τ. To avoid confusion with the four-cycles volumes, to be introduced in a moment, we now call it S.

We can get a better understanding of the T coordinate (2.51) if we drop the G contribution, something that can always be done via an appropriate choice of orientifold projection such that $h_-^{(1,1)} = 0$, and recall that the t^i describe the volume of the two-cycles in X. These are related to the four-cycles volumes by:

$$\begin{aligned} \tau_i &\equiv \frac{\partial \mathcal{V}}{\partial t^i} \\ &= \frac{1}{2}\kappa_{ijk}t^j t^k. \end{aligned} \tag{2.52}$$

Where in the second line we used (2.46).

It follows that now (2.51) can be written as:

$$T_i = \tau_i + i\theta_i. \tag{2.53}$$

This expression tells us that T_i is the complexification of the four-cycle volume τ_i by the θ_i, which can be expressed as:

$$\theta_i = \int_{\hat{D}_i} C_4. \tag{2.54}$$

With the $\{D_i\}_{i=1,\ldots,h_+^{(1,1)}}$ effective divisors, so four-cycles, forming a basis of $H^{(1,1)}(X,\mathbb{Z})$ and $\{\hat{D}_i\}$ are its Poincaré duals forming a basis of $H_4(X,\mathbb{Z})$.

Concerning the other questions, it turns out that now the moduli space is Kähler and given by a direct product:

$$\mathcal{M} = \mathcal{M}_{CS}^{h_-^{(1,2)}} \times \mathcal{M}_K^{h^{(1,1)}+1}. \tag{2.55}$$

In particular $\mathcal{M}_K^{h^{(1,1)}+1}$ is Kähler and $\mathcal{M}_{CS}^{h_-^{(1,2)}}$ is special Kähler.

Having a direct product in (2.55) signifies that the associated Kähler potential does not mix the complex structure moduli z with the other moduli. Indeed we have that the total Kähler potential is:

$$K = K_{\rm CS}(z,\bar{z}) + K_{\rm K}(S,T,G). \tag{2.56}$$

Where:

$$K_{\rm CS} = -\ln\left(-i\int_X \Omega(z)\wedge\bar{\Omega}(\bar{z})\right), \tag{2.57}$$

$$K_{\rm K} = -\ln\left(-i(S-\bar{S})\right) - 2\ln\left(\mathcal{V}\right). \tag{2.58}$$

Hence also a Kähler metric can be found for the moduli space (2.55). Actually the standard way to proceed is finding first the metric and then express it in terms of the Kähler potential $g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K$, so that we can read K from there.

The splitting (2.56) is coherent with the moduli space direct product structure (2.55), however, for practical reasons, it is useful to further split (2.58) as follows:

$$\begin{aligned} K_{\rm K} &\equiv K_{\rm dil} + K_{\rm K\ddot{a}hl} \\ &\equiv -\ln\left(-i(S-\bar{S})\right) - 2\ln\left(\mathcal{V}\right). \end{aligned} \tag{2.59}$$

3 Moduli Stabilisation

In the previous section we have learnt that moduli, scalar particles with no potential, arise whenever we compactify a string theory on a CY three-fold. This fact constitutes a problem as massless moduli would mediate unobserved long-range "fifth forces". Indeed moduli couple gravitationally to ordinary matter and the associated particle exchange can generate forces. The range of such forces is $\mathcal{O}(1/m)$, with m the mass of the modulus. Experimental tests on gravity at the sub-millimeter scale [18] impose severe constraints on the range and hence on the mass of the modulus as it must satisfy $m > \mathcal{O}(10^{-3})$ eV. For this reason "stabilising" the moduli, that is, giving them a potential, is of utmost importance. This can be done with the introduction of fluxes and corrections to the tree level Kähler potential (2.56) and to the superpotential.
Furthermore, despite being scalar particles, part of the intuition we have for "standard" scalars, such as the Higgs boson, does not carry over to moduli, as they present several distinctive features. Some of the most notable ones are:

- Moduli are not charged under the Standard Model gauge fields.
- The couplings of moduli carry factors of M_{Pl}^{-1}. To be more precise moduli that determine local properties, such as the volume of specific cycles, of the compactification carry couplings of M_S^{-1} while moduli that determine global properties of the compactification, such as the total volume, carry couplings of M_{Pl}^{-1}. The two scales can be related using (1.88) and (2.13):

$$M_S = \frac{g_s M_{\text{Pl}}}{\sqrt{4\pi\mathcal{V}}}. \tag{3.1}$$

  So if the compactification volume is very large the two scales are widely separated. In cosmological contexts we are most interested in moduli carrying a suppression of M_{Pl} as they survive the longest, even more if we combine this with the fact that they are uncharged under the SM fields.
- The notion of zero VeV for moduli is ill-defined. Indeed we think about the VeV of moduli fields as parameters for the compactification, a good example is the dilaton whose VeV sets the string coupling g_s (1.48). There is no preferred value for such VeV and hence also the notion of zero VeV is ill-defined.

In this section we discuss the procedure of moduli stabilisation just mentioned. As was already commented in the introduction we will be working in type IIB, since it is the best understood among the five string theories for what concerns moduli stabilisation. We follow the references [12, 19, 20, 21].

We start by studying so-called flux compactifications on CY orientifolds where the F_3 and H_3 fluxes, defined at the end of Sec. 1, are turned on. Not only, they also satisfy a Dirac quantisation condition:

$$\frac{1}{4\pi^2\alpha'}\int_{\Sigma_3} F_3 \in \mathbb{Z}, \quad \frac{1}{4\pi^2\alpha'}\int_{\Sigma_3} H_3 \in \mathbb{Z}. \tag{3.2}$$

Where Σ_3 is some three-cycle.
We also assume that local sources such as D-branes are present.

We shall see that the fluxes alone are enough to stabilise the axiodilaton and the complex structure moduli. The Kähler moduli, however, will require more work.
To understand why that is the case we take a so-called warped metric which is the most general metric compatible with maximal symmetry and Poincaré invariance of the four-dimensional spacetime:

$$ds^2 = g_{MN}dx^M dx^N = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}g_{mn}dy^m dy^n \tag{3.3}$$

Where g_{mn} is a Riemannian metric, but not necessarily a CY one due to the presence of fluxes that break the Ricci flatness. Only under specific assumptions, that we will state later, g_{mn} is related to the CY metric, the one of the vacuum configuration, via a conformal transformation. We also have $A(y)$: it is the warp factor, function of the coordinates of the internal manifold only due to Poincaré invariance.
Not only, Poincaré invariance also fixes to zero the "external" components, so the ones on $\mathbb{R}^{1,3}$, of G_3 (1.95) (recall that we renamed the axiodilaton $\tau \to S$), while the self-dual five form $\widetilde{F}_5$ must take the form:

$$\widetilde{F}_5 = (1 + \star_{10}) d\alpha(y) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{3.4}$$

Where $\alpha(y)$ is a scalar function on X and it will turn out to be related to $A(y)$.

The $10D$ Einstein equation associated to (3.3) upon tracing reads as:

$$\Delta_6 e^{4A} = \frac{e^{8A}}{2\mathrm{Im}(S)} |G_3|^2 + e^{-4A} \left(|\partial\alpha|^2 + |\partial e^{4A}|^2 \right) + 2\kappa_{10}^2 e^{2A} \mathcal{J}_{\mathrm{loc}}. \tag{3.5}$$

Where $\mathcal{J}_{\mathrm{loc}}$ contains all the "local" contributions, that is to say contributions from local sources and can be expressed in terms of the energy-momentum tensor. Hence if such sources are absent $\mathcal{J}_{\mathrm{loc}} = 0$ and, in that case, the only possible solution is the trivial one since the left hand side of (3.5) is a total derivative while the right hand side is strictly positive. To achieve a non-trivial solution we must have sources of negative $\mathcal{J}_{\mathrm{loc}}$, such as orientifold planes.

The presence of local sources also contributes to the Bianchi identity for the $\widetilde{F}_5$ flux:

$$d\widetilde{F}_5 = H_3 \wedge F_3 + 2\kappa_{10}^2 T_{D_3} \rho_{D_3}^{\mathrm{loc}}. \tag{3.6}$$

Where $\rho_{D_3}^{\mathrm{loc}}$ is the D3-brane charge density and T_{D_3} its tension.

Integrating (3.6) yields the tadpole-cancellation condition:

$$\frac{1}{2\kappa_{10}^2 T_{D_3}} \int_X H_3 \wedge F_3 + Q_3^{\mathrm{loc}} \equiv Q_{D_3}^{\mathrm{flux}} + Q_{D_3}^{\mathrm{loc}} = 0. \tag{3.7}$$

So once more we see the need for negative contributions of D3-brane charge, $Q_{D_3}^{\mathrm{loc}}$, as the solutions of interest for moduli stabilisation require $Q_{D_3}^{\mathrm{flux}} > 0$.

Some comments on (3.7) are definitely needed. The expression we derived hides all the intricacies inside of $Q_{D_3}^{\mathrm{loc}}$ and we might naively think that only D3-branes contribute to such charge, however it turns that is not true. If we allow for a rich brane setup with D7-branes and O3/O7-planes on top of D3-branes, as usually happens, things become much more complicated. Indeed not only D3-branes and O3-planes contribute to it but also D7-branes and O7-planes. The contribution coming from D3-branes can easily understood from (1.98): it is due to the C_4 field which couples to D3-branes. The other contributions are not so obvious tough. As we already explained all orientifold planes give a negative charge contribution, while D7-branes carry both a positive and a negative one. The former is due to gauge fields living on the brane world-volume while the latter is of geometrical origin. Thus, in full generality, the D3 tadpole cancellation condition is usually expressed as [22]:

$$N_{D3} + \frac{N_{\text{flux}}}{2} + N_{\text{gauge}} = \frac{N_{O3}}{4} + \frac{\chi(O7)}{12} + \sum_a \frac{N_a(\chi(D_a) + \chi(D'_a))}{48}. \tag{3.8}$$

Where N_{flux} is the contribution of the fluxes F_3, H_3, N_{gauge} is the contribution of the gauge fields living on the D7-branes, N_{D3} and N_{O3} are the number of D3-branes and O3-planes respectively and N_a is the number of D7-branes wrapping some suitable divisors, denoted by D_a, and their image under the orientifold involution D'_a.

The expression (3.7), and hence (3.8), only holds in the case of D3-branes, that is to say, when we consider the local charge associated to D3-branes. If we consider the case of D7-branes things are much simpler as we only have the contributions coming from D7-branes and O7-planes, all the other ones, including the fluxes H_3 and F_3, are gone. The D7 tadpole cancellation condition is simply [22]:

$$\sum_a N_a([D_a] + [D'_a]) = 8[\text{O7}] \tag{3.9}$$

The factor of 8 on the right hand side follows from the fact that one single D7-brane carries a charge of $+1$, while one single O7-plane carries a charge of -8.

We can finally go back to our discussion on moduli stabilisation and combine (3.4), (3.5) and (3.6) to find:

$$\begin{aligned}\Delta_6\left(e^{4A} - \alpha\right) = \frac{e^{8A}}{24\text{Im}(S)}|iG_3 - \star_6 G_3|^2 + e^{-4A}|\partial\left(e^{4A} - \alpha\right)|^2 \\ + 2\kappa_{10}^2 e^{2A}\left(\mathcal{J}_{\text{loc}} - \mathcal{Q}_{\text{loc}}\right).\end{aligned} \tag{3.10}$$

Where we defined $\mathcal{Q}_{\text{loc}} \equiv T_{D_3}\rho^{\text{loc}}_{D_3}$.

Once again the left hand side of (3.10) integrates to zero. Instead, for what concerns the right hand side, the kind of localized sources that have been studied the most satisfy a BPS-type bound:

$$\mathcal{J}_{\text{loc}} \geq \mathcal{Q}_{\text{loc}}. \tag{3.11}$$

This condition is saturated only by D3-branes, O3-planes and D7-branes wrapping four-cycles, so if we assume the presence these three local sources only, then (3.10) yields two further conditions, namely:

$$e^{4A} = \alpha, \tag{3.12}$$

$$G_3 = i \star_6 G_3. \tag{3.13}$$

Equation (3.13) tells us that G_3 must be imaginary self-dual (ISD for short), hence solutions of this type are called ISD solutions. Among all possible configuration, ISD ones have been one of the most studied due to the wealth of properties they enjoy [23]:

- The internal metric $e^{-2A(y)}g_{mn}$ of (3.3) is conformally CY.
- The size of X is a modulus.
- Fluxes alone give a mass to the complex structure moduli and to the axiodilaton.

To understand why the last of these three properties is true consider the F-term scalar potential for a supersymmetric theory [9]:

$$V = e^K \left[\sum_{i,\bar{j}=T,S,U} K^{i\bar{j}} D_i W \bar{D}_{\bar{j}} \bar{W} - 3|W|^2 \right]. \tag{3.14}$$

Where K is the tree level Kähler potential, $K^{i\bar{j}}$ is the associated inverse metric, $D_i W = \partial_i W + W \partial_i K$ is the Kähler covariant derivative and W is the superpotential.

We already know the Kähler potential is (2.56) as it was given in Sec 3. What we are missing is the superpotential, even though it doesn't really matter for what we want to prove here. It was proved in [24] that (3.13) can be derived from the Gukov-Vafa-Witten superpotential:

$$W = \int_X G_3 \wedge \Omega. \tag{3.15}$$

Now we can understand why the non-Kähler moduli are stabilised by the fluxes alone. In fact the sum in (3.14) is taken over all moduli however the Kähler potential enjoys a no-scale structure, meaning that:

$$\sum_{i,\bar{j}=T} K^{i\bar{j}} \partial_i K \partial_{\bar{j}} K = 3. \tag{3.16}$$

Thus, when we sum over the Kähler moduli in (3.14), the term $-3|W|^2$ cancels off exactly with the one coming from the covariant derivative due to (3.16). Therefore, for the Kähler moduli only $V \geq 0$, whose minimum is necessarily at 0. This means that, while the axiodilaton and the complex structure moduli are stabilised "classically", i.e.

[9]To be consistent with the literature we rename the complex structure moduli $z \to U$.

without the introduction of any correction but just thanks to the background fluxes, for the Kähler moduli we need to add corrections.

It also follows that the scalar potential reduces to:

$$V = e^K \left(\sum_{i,\bar{j}=S,U} K^{i\bar{j}} D_i W \bar{D}_{\bar{j}} \bar{W} \right), \tag{3.17}$$

and thus the complex structure moduli and the axiodilaton can be stabilised by solving:

$$D_U W = 0, \tag{3.18}$$

$$D_S W = 0. \tag{3.19}$$

Notice that the minimum is not necessarily supersymmetric as the condition for unbroken supersymmetry is that all the F-terms vanish, $D_I W = 0$ with I representing all the moduli, and we might have $D_{T_i} W \neq 0$.

From now on we will always consider the complex structure moduli and the axiodilaton to be stabilised, consequently the superpotential (3.15) will be a constant which we denote by:

$$W_0 = \left\langle \int_X G_3 \wedge \Omega \right\rangle. \tag{3.20}$$

3.1 Quantum Effects

As we just explained, we managed to stabilise the complex structure moduli and the axiodilaton. We left behind the Kähler moduli so in this paragraph we will explain how to add corrections in order to stabilise them.

The corrections, affecting the tree level Kähler potential (2.56) and the tree level superpotential (3.15), can be formally expressed as:

$$K = K_{\text{tree}} + K_{\text{p}} + K_{\text{np}}, \quad W = W_{\text{tree}} + W_{\text{np}}. \tag{3.21}$$

So K_{tree} receives both perturbative and non-perturbative of corrections while W_{tree} only non-perturbative ones, but we will neglect K_{np} in our discussions as it is currently poorly understood.
The reason for the absence of perturbative corrections, the α' ones specifically, to W_{tree} must be searched in the Peccei-Quinn (PQ for short) shift symmetry of the axion. Even though we did not mention it, we already met an axion in Sec. 2 when we introduced the "good" coordinates for the moduli spaces, it is the θ entering the definition of (2.53). Indeed an axion [10] a is a pseudoscalar field that enjoys the PQ shift symmetry $a \to a + const$

[10] The QCD axion is the most well-known example of axion, but it is not the only one. Indeed we will use the term axion to denote several particles that need not couple to QCD.

and, in the context of string compactifications, they emerge from the compactification of p-form fields over p-cycles. Indeed we expressed θ_i as such in (2.54), but there are more such as the B_2 and the C_2 axions given by the integral of those two-forms on some two-cycle.
Having a clear definition of what an axion is, we can now understand the origin of the PQ shift symmetry: it follows from the gauge invariance of the $10D$ p-form field and, since the string worldsheet carries no R-R charge, this symmetry is not broken at any order in α' but it is broken non-perturbatively by instanton effects. This result in turn implies that the superpotential W can only depend non-perturbatively on the Kähler moduli and $W_{\mathrm{p}} = 0$. Indeed the superpotential must be holomorphic so it can only depend on the T_i and any (non-trivial) polynomial in the T_i is invariant under the axion shift symmetry.
Perturbative corrections do not come only in the form of α' corrections however, we also have the g_s ones. Still the statement $W_{\mathrm{p}} = 0$ does not change due to non-renormalization theorems, proved in the context of generic supersymmetric theories [25, 26]. Hence the superpotential receives no perturbative corrections at all.

We can now go back to (3.21) and see the explicit form of the corrections.

- Perturbative corrections to the Kähler potential.
  One of the first perturbative correction to (2.56) that has been derived is the BBHL one [27]. It follows from an $(\alpha')^3$ correction in the $10D$ theory and in the $4D$ effective theory takes the form:
  $$K_{\alpha'^3} = K_0 - 2\ln\left(\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right). \tag{3.22}$$
  Where K_0 contains the other contributions to (2.56) not affected by the correction, namely K_{dil} (2.59) and K_{CS} (2.57). We also introduced the topological quantity ξ defined as:
  $$\xi = -\frac{\chi\left(X\right)\zeta(3)}{2\left(2\pi\right)^3}. \tag{3.23}$$
  With $\chi(X)$ the Euler characteristic of X and ζ the Riemann zeta function.

  There exist other perturbative corrections related to spacetime loop effects, such as the g_s corrections. We defer their description to the next section as they will play a key role in our model of moduli stabilisation.

- Non-perturbative corrections to the superpotential.
  Non-perturbative corrections to the superpotential to the superpotential can come either from gaugino condensation or from Euclidean D3-branes or from Euclidean D(-1)-branes, even though this last possibility is less considered in the literature.
  $$W_{\mathrm{np}} = W_{\lambda\lambda} + W_{\mathrm{ED3}} + W_{\mathrm{ED}(-1)} \tag{3.24}$$

We start from the first of the three by considering a stack of N D7-branes wrapping some four-cycle Σ_4. A Yang-Mills field lives on the worldvolume of the branes and the associated action is:

$$S = \frac{1}{2g_7^2} \int_{\Sigma_4} d^4\sigma \sqrt{g_{\text{ind}}} e^{-4A(y)} \int d^4x \sqrt{-g}\, \text{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right]. \tag{3.25}$$

Where g_7 is the coupling constant of the Yang-Mills theory living on the worldvolume, whereas the four-dimensional coupling for the same theory, g, can be expressed as:

$$g^2 = \frac{8\pi^2}{T_{D_3}\mathcal{V}_4}. \tag{3.26}$$

With $\mathcal{V}_4$ the warped volume of the four-cycle, that is to say taking into account the factor $e^{-4A(y)}$ too:

$$\mathcal{V}_4 \equiv \int_{\Sigma_4} d^4\sigma \sqrt{g_{\text{ind}}} e^{-4A(y)}. \tag{3.27}$$

Under some topological assumptions on Σ_4, simply put it must have no deformations corresponding to charged matter field, the theory resulting from dimensional reduction is a $\mathcal{N} = 1$ pure super Yang-Mills. At low energies we have gaugino condensation which produces a non-perturbative superpotential:

$$|W_{\lambda\lambda}| \propto \exp\left(-\frac{8\pi^2}{Ng^2}\right) = \exp\left(-\frac{T_{D_3}\mathcal{V}_4}{N}\right) \tag{3.28}$$

From (2.52) we know that $\mathcal{V}_4$ is proportional to the real part of a corresponding Kähler modulus, thus we can write:

$$W_{\lambda\lambda} = \mathcal{A}\, e^{-aT}. \tag{3.29}$$

Where $a = 2\pi/N$ and the prefactor $\mathcal{A}$ is a one-loop Pfaffian that only depends on the complex structure moduli and the position of the branes.

We can now turn to the second contribution of (3.24), W_{ED3}, which has a very similar structure to (3.29). This term is generated when a four-cycle Σ_4 is wrapped by Euclidean D3-branes, a special kind of instantonic contribution to the path integral whose action has a real part proportional to the volume of the $(p+1)$-cycle wrapped by the brane, rather than by a D7-brane. The superpotential that gets generated is:

$$W_{\text{ED3}} = \mathcal{A}\, e^{-aT}. \tag{3.30}$$

Here $a = 2\pi$ and once again $\mathcal{A}$ is a one-loop Pfaffian does not depend on the Kähler moduli but only on the complex structure one and on the D-branes positions.

Finally we can have Euclidean D(-1) branes contributing to the superpotential and in this case:

$$W_{\text{ED}(-1)} = \mathcal{O}\left(e^{-\pi\tau}\right) \tag{3.31}$$

We can now move on to the actual stabilisation of the Kähler moduli. There are several proposals to achieve this goal, but we will limit ourselves to the best established ones, the Kachru-Kallosh-Linde Trivedi scenario (KKLT for short) and the Large Volume scenario (LVS for short). They will be described in detail in the following paragraphs, but it is useful to summarize here their characterizing features:

- The KKLT scenario constructs a competition between the flux superpotential (3.15) and the non perturbative superpotential (either (3.29) or (3.30)) by making the former small via an appropriate choice of the fluxes.

- The LVS scenario constructs a competition between the perturbative $(\alpha')^3$ correction (3.22) to the Kähler potential and the non-perturbative correction to the superpotential (either (3.29) or (3.30)) by working in a region of (Kähler) moduli space where some cycles are larger than others. At the minimum $\mathcal{V} \gg 1$, hence the name of Large Volume Scenario, allowing us to neglect unknown corrections as long as they are subleading in $\mathcal{V}$.

3.2 KKLT Scenario

The KKLT scenario has first been proposed in [28] and, as we mentioned above, neglects the perturbative corrections while focusing only on the non-perturbative ones. We first integrate out the axiodilaton S and the complex structure moduli U so that the low energy theory will only depend on Kähler moduli. Assuming there are $h^{(1,1)} = h^{(1,1)}_+$ Kähler moduli T_i then the full superpotential is:

$$W = W_0 + \sum_{i=1}^{h^{(1,1)}_+} \mathcal{A}_i \, e^{-a_i T_i}. \tag{3.32}$$

Where W_0 denotes the constant flux superpotential (3.20). Clearly a competition between the two terms requires $W_0 \ll 1$. Such configurations have indeed been constructed reaching up to $|W_0| \sim 10^{-95}$ [29] and a detailed description on how to realize them can be found in [19].

Given a generic tree level Kähler potential K then the scalar potential coming from (3.32) is:

$$\begin{aligned}\delta V_{np} = e^K K^{i\bar{j}} \Big[& a_j \mathcal{A}_j a_{\bar{i}} \bar{\mathcal{A}}_{\bar{i}} e^{\left(-a_j T_j + a_{\bar{i}} \bar{T}_{\bar{i}}\right)} \\ & - \left(a_j \mathcal{A}_j e^{-a_j T_j} \bar{W} \partial_{\bar{i}} K + a_{\bar{i}} \bar{\mathcal{A}}_{\bar{i}} e^{-a_{\bar{i}} \bar{T}_{\bar{i}}} W \partial_j K \right) \Big]\end{aligned} \tag{3.33}$$

Now we assume that $h^{(1,1)}_+ = 1$ for simplicity so that $K = -2\ln(\mathcal{V})$ with $\mathcal{V} = \left(T + \bar{T}\right)^{3/2}$. Recalling that $\text{Re}(T) = \tau$ (2.51) and setting the axion θ to the minimum, we find:

$$\delta V_{np} = \frac{a\mathcal{A}e^{-a\tau}}{2\tau^2} \left[\mathcal{A}e^{-a\tau} \left(1 + \frac{1}{3} a\tau \right) + W_0 \right]. \tag{3.34}$$

The minimum is supersymmetric and can be found from $D_T W = 0$, which yields:

$$W_0 = -\mathcal{A}e^{-a\tau}\left(1 + \frac{2}{3}a\tau\right). \tag{3.35}$$

The stabilised modulus $\langle\tau\rangle$ is found by solving this equation:

$$\langle\tau\rangle \sim \frac{1}{a}\ln\left(|W_0|^{-1}\right) + \ldots \ . \tag{3.36}$$

Where "..." represent omitted corrections to the solution of (3.35) that are subleading for $W_0 \ll 1$.
Since the KKLT scenario requires $W_0 \ll 1$ we also have $\tau \gg 1$. This allows us to use the tree level Kähler potential neglecting the perturbative corrections (such as (3.22)) as they are strongly suppressed.

It can be proved that, at the minimum, the scalar potential is negative. Therefore we have a supersymmetric, since $D_T W = 0$, AdS minimum and, if we want to describe phenomenologically consistent models, we should add uplifting terms to (3.34).

3.3 Large Volume Scenario

LVS has first been proposed in [30] and its starting point is the same as the KKLT one, that is, we integrate out the complex structure moduli and the axio-dilaton to achieve a low energy theory depending only on the Kähler moduli. However, differently from KKLT, LVS requires at least two moduli, as we will clarify soon.

We already mentioned that LVS makes use both of the perturbative (3.22) and non-perturbative, either (3.29) or (3.30), to achieve stabilisation. Hence the superpotential has the same structure as in the KKLT scenario (3.32) and the same holds for the associated scalar potential (3.33). Therefore we are just missing the scalar potential associated to the $(\alpha')^3$-corrected Kähler potential (3.22) and to the constant flux superpotential W_0:

$$\delta V_{\alpha'} = 3\hat{\xi}e^K \frac{\hat{\xi}^2 + 7\hat{\xi}\mathcal{V} + \mathcal{V}^2}{(\mathcal{V} - \hat{\xi})(2\mathcal{V} + \hat{\xi})^2} W_0^2 \approx \frac{3}{4}\hat{\xi}W_0^2\frac{1}{\mathcal{V}^3}. \tag{3.37}$$

Where we defined $\hat{\xi} = \xi/g_s^{3/2}$ and in the second line we assumed $\mathcal{V} \gg \hat{\xi}$, according to the LVS philosophy.

Now we can put (3.33) and (3.37) together to have the full LVS scalar potential:

$$\begin{aligned}\delta V_{np} + \delta V_{\alpha'} = e^K \Bigg\{ & K^{i\bar{j}}\Big[a_j\mathcal{A}_j a_{\bar{i}}\bar{\mathcal{A}}_{\bar{i}} e^{\left(-a_j T_j + a_{\bar{i}}\bar{T}_{\bar{i}}\right)} \\ & - \left(a_j\mathcal{A}_j e^{-a_j T_j}\bar{W}\partial_{\bar{i}}K + a_{\bar{i}}\bar{\mathcal{A}}_{\bar{i}}e^{-a_{\bar{i}}\bar{T}_{\bar{i}}}W\partial_j K\right)\Big] + \frac{3}{4}\hat{\xi}W_0^2\frac{1}{\mathcal{V}}\Bigg\}.\end{aligned} \tag{3.38}$$

For $\mathcal{V} \to \infty$ the last term, (3.37), dominates over (3.33). The competition that we want to create between these two terms can be achieved if some cycles are exponentially smaller than others, something that clearly requires $h_+^{(1,1)} > 1$. To implement it we denote by τ_s the small cycles and work in a region of the Kähler moduli space where:

$$\mathcal{V} \to \infty, \quad a_s \tau_s = \ln(\mathcal{V}). \tag{3.39}$$

In this way we are ensured that the exponentials $e^{-a_s T_s}$ of (3.33) are not subleading with respect to the $1/\mathcal{V}$ of (3.37).

A prototypical example of LVS constructions is represented by the "Swiss-cheese" Calabi-Yau manifolds where the volume takes the form:

$$\mathcal{V} = \alpha F_{3/2}(\tau_b) - \beta G_{3/2}(\tau_s). \tag{3.40}$$

Where $\alpha, \beta > 0$ are positive constants and $F_{3/2}, G_{3/2}$ are homogeneous function of degree $3/2$ in their arguments, that are respectively the N_b big cycles τ_b and the $N_s = h_+^{(1,1)} - N_b$ small ones τ_s. The small cycles represent holes in the CY, hence the name of these models. The easiest non-trivial example is obtained for $\alpha = \beta = 1$ and $h_+^{(1,1)} = 2$ so that:

$$\mathcal{V} = \tau_b^{3/2} - \tau_s^{3/2}. \tag{3.41}$$

To stabilise these two moduli we consider the BBHL correction (3.22) to the Kähler potential and a non-perturbative one, either (3.29) or (3.30), carried by the small cycle only:

$$\begin{aligned} K =& K_0 - 2\ln\left(\left(\tau_b^{3/2} - \tau_s^{3/2}\right) + \frac{\xi}{2g_s^{3/2}}\right), \\ W =& W_0 + \mathcal{A}_s e^{-a_s T_s}. \end{aligned} \tag{3.42}$$

If we omit numerical factors, take the limit $\tau_b \gg \tau_s$ and set the axion to the minimum, then the scalar potential reads as:

$$V_{\text{LVS}} \simeq \frac{1}{\mathcal{V}} a_s^2 \mathcal{A}_s^2 \sqrt{\tau_s} e^{-2a_s\tau_s} - \frac{1}{\mathcal{V}^2} a_s \mathcal{A}_s |W_0| \tau_s e^{-a_s\tau_s} + \frac{1}{\mathcal{V}^3}\hat{\xi}|W_0|^2. \tag{3.43}$$

Now we can minimize the potential to find stabilised $\langle\mathcal{V}\rangle$ and $\langle\tau_s\rangle$:

$$\frac{\partial V_{\text{LVS}}}{\partial \tau_s} = 0, \tag{3.44}$$

$$\frac{\partial V_{\text{LVS}}}{\partial \mathcal{V}} = 0. \tag{3.45}$$

From (3.44) we can find the volume:

$$
\begin{aligned}
\langle \mathcal{V} \rangle =& \frac{2|W_0|\sqrt{\tau_s}}{a_s \mathcal{A}_s} \left(\frac{1 - a_s\tau_s}{1 - 2a_s\tau_s} \right) e^{a_s\tau_s} \\
\simeq& \, e^{a_s\tau_s}.
\end{aligned} \tag{3.46}
$$

So the volume is exponentially large, as we wanted it be. To have an exact expression we need also $\langle \tau_s \rangle$ of course and we can obtain it from (3.45) using (3.46):

$$
\begin{aligned}
\langle \tau_s \rangle =& \left\{ \frac{3\hat{\xi}}{4} \left(\frac{(1 - 2a_s\tau_s)^2}{(1 - a_s\tau_s)(-a_s\tau_s)} \right) \right\}^{2/3} \\
\sim& \frac{\xi^{2/3}}{g_s}.
\end{aligned} \tag{3.47}
$$

In the second line we used the definition of $\hat{\xi} = \xi / g_s^{3/2}$. Clearly this result makes sense as long as $\xi > 0 \Leftrightarrow \chi(X) < 0$. For a generic CY three-fold it holds that:

$$
\chi = \sum_{p=0}^{6} (-1)^p b_p = 2(h^{(1,1)} - h^{(2,1)}). \tag{3.48}
$$

So, as long as $h^{(1,1)} < h^{(2,1)}$, (3.47) is a sensible result and, since the string coupling is small $g_s \ll 1$ in the perturbative regime, the volume (3.46) is indeed large $\langle \mathcal{V} \rangle \gg 1$.

It can be checked that, at the minimum, the potential (3.43) is negative, hence what we are describing is a non-supersymmetric, as the F-terms are non-vanishing, AdS minimum. Therefore, if we want to describe phenomenologically consistent models, we should add uplifting terms to (3.43).
Furthermore, as we can see from our simple model, the LVS scenario does not require to fine-tune to extremely small values the flux superpotential W_0, allowing instead for more natural choices $W_0 \sim \mathcal{O}(1 - 10)$.

Chapter 3

Moduli Stabilisation for the Dark Dimension Scenario

After reviewing the main features of the most promising moduli stabilisation mechanisms, we focus now on the recently proposed Dark Dimension (DD for short) Scenario [31]. We first present a brief description of this scenario, and we then describe how to realise moduli stabilisation for this model.

1 Brief Review of the Dark Dimension Scenario

The dark dimension scenario [31] is a proposal, descending from the Swampland program [32] combined with experimental data, for the existence of a mesoscopic fifth dimension which also furnishes a candidate for dark matter [33] hence unifying dark matter and dark energy. Indeed starting from the Distance/Duality conjecture [34] and from the smallness of the cosmological constant we are naturally led to the existence of a tower of light states and of a fifth dimension with size $l \sim \Lambda^{-1/4} \sim 10^{-6}m$. From here a series of interesting phenomenological implications, such as the existence of a new energy scale and the identification of gravitons of the dark dimension with dark matter, follow naturally as we shall see below.

The Distance/Duality Conjecture

As we just mentioned the DD proposal originates, in the context of the Swampland project, from the Distance/Duality conjecture, which we now state following [35]:

Conjecture 1. Suppose to have a field theory coupled to gravity and to denote its moduli space by $\mathcal{M}$. This space is parameterized by the VeVs of fields ϕ^i with no potential. Starting from any point $P \in \mathcal{M}$ there exists another point $Q \in \mathcal{M}$ such that the geodesic distance between the two, $d(P,Q)$, is infinite. It is then conjectured that there exists an infinite tower of states, with mass scale m such that:

$$m(Q) \sim m(P)\, e^{-\alpha d(P,Q)}, \tag{1.1}$$

with α a positive constant of order one in Planck units.

Therefore at large distance in field space we find a tower of exponentially light states with mass scale m. This tower is weakly coupled and leads to a dual description of the theory. There exists also another characteristic scale of the tower, the species scale $\Lambda_{\rm sp}$, where the local QFT description breaks down and quantum gravity effects become relevant. In the weak coupling limit $\Lambda_{\rm sp}$ can actually be interpreted as the string scale M_S (3.1) [36].
So far only two possibilities have been suggested for the microscopic origin of such tower:

- A tower of string excitation modes. In this case $\Lambda_{\rm sp} \sim m$. At this scale the theory is still weakly coupled to gravity, but the QFT description breaks down due to higher spin states.
- A KK tower. This is a signal of decompactification: $n \geq 1$ extra dimensions open at the scale m and the QFT description holds until $\Lambda_{\rm sp}$ is reached. In this case the species scale is a new fundamental scale corresponding to a higher dimensional Planck scale:
$$M_S \equiv \Lambda_{\rm sp} = m^{\frac{n}{n+2}} M_{\rm Pl}^{\frac{2}{2+n}}. \tag{1.2}$$

After its proposal, the Distance/Duality conjecture has been carefully studied and extended to other contexts, such as the AdS one, yielding the AdS distance conjecture [37]:

Conjecture 2. Consider a quantum gravity theory in a d-dimensional AdS space with cosmological constant Λ. Then there is an infinite tower of states with mass scale m which, in the limit $\Lambda \to 0$, goes like:

$$m \sim |\Lambda|^{\alpha}, \tag{1.3}$$

where α is a positive constant of order one in Planck units.
Furthermore, if the space is supersymmetric AdS, we also have $\alpha = 1/2$.

In this sense we can regard $\log(1/|\Lambda|)$ as a natural distance in field space.

Even though the AdS distance conjecture has been proposed in the context of AdS space, the Swampland argument holds both in the dS and AdS case, hence we can assume that an analogue of such conjecture also holds in the dS case. In this way (1.3) can also be viewed as a solution to the cosmological constant problem as $\Lambda \sim m^{1/\alpha}$ goes to zero when $m \to 0$. Since m is the mass scale of the tower of light states this result can also be understood as the fact that heavy states do not contribute to the cosmological constant. We can make this statement more quantitative by saying that from an EFT perspective $\Lambda = \Lambda_0 + A\, m^{1/\alpha}$, with Λ_0 the contribution of heavy states to the cosmological constant. By the Distance/Duality conjecture we expect $\Lambda_0 = 0$.

In [37] it was also argued that it must exist an upper bound for α, namely $\alpha \leq \frac{1}{2}$, for consistency with the Higuchi bound [1] [38]. On the other hand we also have reason to believe in the existence of a lower bound on α:

$$\frac{1}{d} \leq \alpha \leq \frac{1}{2}. \tag{1.4}$$

Indeed, since the tower of light states has a mass scale m, we expect a one-loop contribution to the scalar potential coming from such tower that scales as $V \sim m^d$. To have a dS space we would also need other contributions on top of this one, but still the net dependence will go as m^d due to the one-loop contribution. Hence any higher power of m, corresponding to a weaker potential, would require a magical cancellation of the m^d term.

Predicting the Dark Dimension

We now have all the tools to understand how the idea of the DD arises from the Distance/Duality conjecture together with experimental data.

A tower of light states, as the one arising from the Distance/Duality conjecture, would cause deviations from Newton's law at the energy scale of the tower m, but there are stringent experimental bounds on it down to $\sim 50\,\mu m$ [39], implying that we must have:

$$m \gtrsim 25\ meV. \tag{1.5}$$

Now, since $\Lambda \sim 10^{-122} M_{\rm Pl}^4$, or $\Lambda^{1/4} \simeq 2.3\ meV$, we must have $m \gtrsim \Lambda^{1/4}$, otherwise deviations from Newton's law should have been observed already. Therefore, in the light of (1.3) and (1.4), Swampland bounds and experimental bounds are consistent only if $d = 4$:

$$m \sim \Lambda^{1/4}, \tag{1.6}$$

roughly coinciding with the neutrino scale. This is an important result as it rules out the stringy origin of the tower of light states, leaving us only with the possibility of a KK origin. Indeed, as we explained above, the stringy origin has $\Lambda_{\rm sp} \sim m$, hence at m the QFT description must break down, but we are able to describe physics above the neutrino scale. Thus the only possibility is $m < \Lambda_{\rm sp}$ as happens in the KK tower

[1] The Higuchi bound is a bound on the mass of states in dS spacetime: M^2 cannot be between 0 and $2\Lambda/3$ for spin 2 states because otherwise negative norm states would appear.

scenario.

Having established that the microscopic origin of the tower of light modes is a KK tower, we must determine the number of extra dimensions that decompactify, that is to say the number of large extra dimensions. This can be done by looking at the data of astrophysical experiments, in fact, if extra dimensions exist, new decay modes via the emission of KK modes become possible and such decays would leave marks in neutron stars and supernovae explosions [40]. The most stringent bound, derived from the heating of neutron stars due to a cloud of gravitationally trapped KK gravitons [40], give [40, 41]:

- For one single extra dimension $m^{-1} \sim l < 44\ \mu m$.
- For more than one extra dimension $m^{-1} \sim l < 1.6 \cdot 10^{-4}\ \mu m$.

Clearly the case of more than one decompactified extra dimension is not compatible with (1.6), leaving us with the possibility of one single large extra dimension. This is what we call the Dark Dimension. Having n we can also find a value for the new mass scale M_S from (1.2): $M_S \sim 10^8 - 10^9\ GeV$.

We can further refine the result on the size of the extra dimension by taking (1.3) and updating it to $\Lambda^{1/4} \sim \lambda m$, for some parameter λ to be determined. It can be argued that:

$$m^{1/2} \leq \lambda^4 \lesssim 1 \Leftrightarrow \Lambda^{-2/9} \lesssim l \leq \Lambda^{-1/4}. \tag{1.7}$$

The lower bound is justified by the fact that we want to preserve a scaling of the type $\Lambda \sim m^4$: if we had $\lambda \sim m$ the scaling of the cosmological constant would be $\sim m^5$, hence $\lambda \sim m^{1/2}$ is the lowest scaling we can have for λ while also having $\Lambda \sim m^4$. The upper bound can be explained recurring to the argument that led us to (1.4), namely the one-loop contribution to the vacuum energy. Indeed this loop will contribute to the scalar potential as $V \sim \lambda_{\text{Casimir}} m^4$, with λ_{Casimir} some constant that, for instance, can be computed in the case of a circle compactification going from five to four dimensions $\lambda_{\text{Casimir}} \simeq 10^{-5}$. Taking into account that $\Lambda \sim \lambda m$ we can expect that $\lambda \leq \lambda_{\text{Casimir}}^{1/4} \sim 10^{-1}$. Indeed starting from λ_{Casimir} we find a size estimate $l_{\text{Casimir}} \simeq 7.4\ \mu m$ which we can plug into (1.7) to find $10^{-3} \lesssim \lambda \lesssim 10^{-1}$ and finally:

$$0.1\ \mu m \lesssim l \lesssim 10\ \mu m. \tag{1.8}$$

Phenomenological Implications of a Dark Dimension

As we just reviewed the DD scenario is the hypothesis, justified by Swampland conjectures and experimental data, of the existence of an extra dimension with large size (1.8). As such the DD scenario shares many features with theories involving extra dimensions, however the scales are different. In this paragraph we quickly list some of the most compelling phenomenological consequences of accepting a DD scenario.

- We have already seen that there is a new scale at which new physics must be present, $M_S \sim 10^8 - 10^9\ GeV$.
- It was already mentioned that the DD scale $m \sim \Lambda^{1/4}$ coincides with the neutrino scale. Hence the tower of light states may be identified with a tower of sterile right-handed neutrinos.
- One of the possible ways to relax the Hubble tension problem is assuming the presence of early dark energy, dark energy which is already active at the time of the matter-radiation equality, and by introducing a new scalar field that couples to the neutrino mass term [42]. These assumptions find a natural embedding in the DD scenario where the new scalar is the radion, a field parametrizing the size of the large extra dimension, and the coupling comes from the KK tower being coupled to the metric.
- In [33] it was shown that the DD scenario naturally leads to spin 2 KK excitations of the graviton in the dark dimension, called dark gravitons, as a dark matter candidate. They are emitted at a temperature of $T \sim 4\ GeV$ from SM fields and decay to lower KK modes as time goes on, while also decreasing their mass which today is estimated to be $m_{\rm DG} \lesssim 100\ keV$. Dark gravitons can also decay to SM fields affecting the CMB, hence, requiring consistency with the observed data, [43] put constraints on the allowed parameter space of the DD scenario. Their results are in agreement with those of [31, 33] that we just reviewed.

2 Calabi-Yau Threefold

2.1 Generic Features of $h^{(1,1)} = 3$ Models

In order to perform moduli stabilisation we first need the toric data to build a CY, from that we can then derive the expression for the volume and proceed with the stabilisation. For this reason we first illustrate generic features of $h^{(1,1)} = 3$ models and then specify those to the case of our interest.

Following [22], the kind of CY three-fold we are interested in for this work is a so-called "weak Swiss-cheese", where the volume takes the form:

$$\mathcal{V} = f_{3/2}(\tau_j) - \sum_{i=1}^{N_{small}} \lambda_i \tau_i, \quad j = 1, \ldots, N_{big}. \tag{2.1}$$

Where $h^{(1,1)} = N_{big} + N_{small}$, $f_{3/2}(\tau_j)$ is a degree 3/2 function and λ_i are constants.

The volume form is generally derived from the intersection polynomial, that in the case of $h^{(1,1)} = 3$ takes the form:

$$I_3 = aD_f D_b^2 + bD_s^3. \tag{2.2}$$

Where a, b are integers and the labels s, b, f stand for small, base and fibre respectively. Indeed by Oguiso theorem [44], stating that whenever the intersection polynomial is linear in a particular divisor D_i, then D_i is either a K3 or a $\mathbb{T}^4$ fibration over a $\mathbb{P}^1$ base, we understand that D_f is either a K3 or $\mathbb{T}^4$ fibre [2], hence the label. It can also be proved that D_s is a shrinkable del Pezzo divisor that admits non-perturbative effects.

Now recall that the general structure of the volume of a CY is given by [3] (2.28):

$$\mathcal{V} = \frac{1}{6}\int_X J \wedge J \wedge J. \tag{2.3}$$

Where J is the Kähler (1,1)-form on X which can be expanded on a basis of divisors $\{D_i\} \in H^{1,1}(X, \mathbb{Z})$ as $J = \sum_{i=1}^{h^{1,1}} D_i t^i$, with t^i the 2-cycles volumes. Hence it follows that:

$$\mathcal{V} = \frac{1}{6} t^i t^j t^k \int_X D_i \wedge D_j \wedge D_k \equiv \frac{1}{6} t^i t^j t^k \kappa_{ijk}. \tag{2.4}$$

Where κ_{ijk} are the triple intersection numbers. These can be read as the coefficients of each term of (2.11).

From (2.4) we can express the four-cycle volumes τ_i as in (2.52):

$$\tau_i \equiv \frac{\partial \mathcal{V}}{\partial t^i} = \frac{1}{2}\int_X J \wedge J \wedge D_i = \frac{1}{2}\kappa_{ijk} t^j t^k. \tag{2.5}$$

Therefore we now expand the Kähler form on the basis $\{D_f, D_b, D_s\}$ as:

$$J = t_f D_b + t_b D_f + t_s D_s. \tag{2.6}$$

Thus:

$$\mathcal{V} = \frac{a}{2} t_f^2 t_b + \frac{b}{6} t_s^3, \tag{2.7}$$

and:

$$\tau_b = \frac{\partial \mathcal{V}}{\partial t_f} = a t_b t_f, \quad \tau_f = \frac{\partial \mathcal{V}}{\partial t_b} = \frac{a}{2} t_f^2, \quad \tau_s = \frac{\partial \mathcal{V}}{\partial t_s} = \frac{b}{2} t_s^2. \tag{2.8}$$

So finally:

$$\mathcal{V} = \frac{1}{\sqrt{2a}} \tau_b \sqrt{\tau_f} - \sqrt{\frac{2}{9b}} \tau_s^{3/2}. \tag{2.9}$$

[2]We will only focus on the K3 case here.

[3]We are now using J instead of $\mathcal{J}$ for consistency with the literature, but it sill is the complexified Kähler form.

2.2 Explicit Model

The specific model we will work with is taken from [22] and the CY three-fold is described by the following data:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
6	0	0	1	1	1	0	3
8	0	1	1	1	0	1	4
8	1	0	1	0	1	1	4
	dP_8	NdP_{10}	SD_1	NdP_{15}	NdP_{13}	K3	SD_2

The Hodge numbers are Hodge numbers $(h^{(2,1)}, h^{(1,1)}) = (99, 3)$ and hence the Euler number is $\chi(X) = 2(h^{(1,1)} - h^{(2,1)}) = -192$.

The Stanley-Reisner ideal reads as:

$$\mathrm{SR} = \{x_1x_5,\ x_1x_6x_7,\ x_2x_3x_4,\ x_2x_6x_7,\ x_3x_4x_5\}. \tag{2.10}$$

The intersection polynomial in the basis of divisors $\{D_1, D_6, D_7\}$ is:

$$I_3 = D_1^3 + 9D_7^2D_1 - 3D_7D_1^2 + 18D_7^2D_6 + 81D_7^3. \tag{2.11}$$

While the second Chern class is:

$$c_2(X) = -2D_1^2 + 2D_1D_6 - 2D_6^2 + \frac{2}{3}D_6D_7 + \frac{4}{3}D_7^2. \tag{2.12}$$

We can express the remaining divisors in term of our basis $\{D_1, D_6, D_7\}$ as:

$$\begin{aligned} D_2 &= D_6 - D_1, & D_3 &= \frac{1}{3}\left(D_7 - D_6\right), \\ D_4 &= \frac{1}{3}\left((D_7 - D_6 - 3D_1\right), & D_5 &= \frac{1}{3}\left(D_7 - 4D_6 + 3D_1\right). \end{aligned} \tag{2.13}$$

Therefore moving to our specific model we expand the Kähler form on the basis of divisors as:

$$J = t_1D_1 + t_6D_6 + t_7D_7. \tag{2.14}$$

So finally making use of (2.11) we can express the volume of our CY as:

$$\mathcal{V} = \frac{27}{2}t_7^3 + 9t_7^2t_6 + \frac{9}{2}t_7^2t_1 - \frac{3}{2}t_7t_1^2 + \frac{1}{6}t_1^3. \tag{2.15}$$

Now we can compute the four-cycles volumes:

$$\tau_1 = \frac{1}{2}\left(t_1 - 3t_7\right)^2, \quad \tau_6 = 9t_7^2, \quad \tau_7 = \frac{3}{2}\left(27t_7^2 - t_1^2 + 12t_7t_6 + 6t_7t_1\right). \tag{2.16}$$

Therefore the volume (2.20) can also be expressed as:

$$\mathcal{V} = \frac{1}{6}\left(\sqrt{\tau_6}(\tau_7 - 2\tau_6 + 3\tau_1) - 2\sqrt{2}\tau_1^{3/2}\right) = t_6\tau_6 + \frac{2}{3}\tau_6^{3/2} - \frac{\sqrt{2}}{3}\tau_1^{3/2}. \tag{2.17}$$

This expression for the volume can be further simplified if we notice that the element D_7 of our basis of divisors can be replaced $D_x = D_7 - 2D_6 + 3D_1$. Indeed the intersection polynomial (2.11) reduces to a simpler structure that resembles the one of (2.2):

$$I_3 = D_1^3 + 18D_6D_x^2. \tag{2.18}$$

By Oguiso theorem D_6 is the K3 fibre, as we can also read from the table of the toric data, and D_1 is a dP_8 surface. For this reason we rename $D_1 \to D_{\mathrm{dP}_8}$ and $D_6 \to D_{\mathrm{K3}}$. Accordingly we expand the Kähler form on this new basis as:

$$J = t_s D_{\mathrm{dP}_8} + t_1 D_{\mathrm{K3}} + t_2 D_x. \tag{2.19}$$

Making reference to (2.6) we can identify $t_1 \leftrightarrow t_b$ and $t_2 \leftrightarrow t_f$.

Finally the volume reads as:

$$\begin{aligned}\mathcal{V} &= 9t_1t_2^2 + \frac{1}{6}t_s^3 = t_1\tau_1 - \frac{\sqrt{2}}{3}\tau_s^{3/2} \\ &= \frac{1}{6}\sqrt{\tau_1}\tau_2 - \frac{\sqrt{2}}{3}\tau_s^{3/2} \\ &\simeq \frac{1}{6}\sqrt{\tau_1}\tau_2.\end{aligned} \tag{2.20}$$

Where in the last line we just used the LVS assumption of $\mathcal{V} \gg 1$ while for the first and second equality we used:

$$t_s = -\sqrt{2}\sqrt{\tau_s}, \quad t_1 = \frac{\tau_2}{6\sqrt{\tau_1}}, \quad t_2 = \frac{1}{3}\sqrt{\tau_1}. \tag{2.21}$$

By Oguiso theorem we can also understand (2.20) as $\mathcal{V} \simeq \tau_1 t_1 = \mathrm{Vol}(K3)\cdot\mathrm{Vol}(\mathbb{P}^1)$. Therefore we can picture our CY manifold as in Fig. 3.1.

Now we briefly recall the ideas we met in Sec. 3. Assuming that the axiodilaton and the $h^{1,2}$ complex structure moduli are already stabilised thanks to the background fluxes, we are left with 3 unstabilised Kähler moduli and the volume (2.20) due to the no-scale structure of the tree level Kähler potential. Out of them, the small 4-cycle τ_s and the volume itself can be stabilised by the introduction of α'^3 corrections to the Kähler

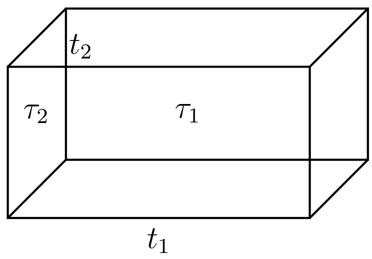

Figure 3.1: Pictorial representation as a direct product (instead of a proper fibration) of the CY manifold we consider for our model. We shall "forget" it later as we will deform away from it.

potential (3.22) and non-perturbative ones to the superpotential:

$$K_{\alpha'^3} = K_0 - 2\ln\left(\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right) \tag{2.22}$$

$$W = W_0 + \sum_{i=1}^{N_{small}} A_i e^{-a_i T_i}. \tag{2.23}$$

From (3.22) and (2.23) we can compute the F-term scalar potential, which is not flat anymore along $\mathcal{V}$ and τ_s, and stabilise these two moduli:

$$\langle\tau_s\rangle \sim \frac{\xi^{2/3}}{g_s}, \tag{2.24}$$

$$\langle\mathcal{V}\rangle \sim e^{a_s\langle\tau_s\rangle}. \tag{2.25}$$

Since we are working in the small coupling limit, namely $g_s \ll 1$, the volume is stabilised at large values, as per required by LVS.

However there still is a flat direction of the potential in the (τ_1, τ_2)-plane, given by the ratio of those two cycles. Indeed we began with three Kähler moduli and we only stabilised one. This can be easily seen by performing a canonical normalisation of the kinetic Lagrangian [45]. Neglecting the small cycle in (2.20), such Lagrangian reads as:

$$\mathcal{L}_{kin} = \frac{\partial^2 K}{\partial T_i \partial \bar{T}_{\bar{j}}} \partial_\mu T^i \partial^\mu \bar{T}^{\bar{j}} = \frac{1}{\tau_1^2}\partial_\mu\tau_1\partial^\mu\tau_1 + \frac{2}{\tau_2^2}\partial_\mu\tau_2\partial^\mu\tau_2. \tag{2.26}$$

Where we have set to zero the corresponding axions θ_i.

We can put (2.26) into canonical form if we define τ_1, τ_2 in terms of two new fields ϕ, χ:

$$\tau_1 = e^{\sqrt{\frac{1}{3}}\chi + \sqrt{\frac{2}{3}}\phi}, \tag{2.27}$$

$$\tau_2 = e^{\sqrt{\frac{1}{3}}\chi - \frac{1}{\sqrt{6}}\phi}. \tag{2.28}$$

So finally:

$$\mathcal{V} \simeq \sqrt{\tau_1}\tau_2 = e^{\frac{\sqrt{3}}{2}\chi}, \tag{2.29}$$

$$u \equiv \frac{\tau_1}{\tau_2} = e^{\sqrt{\frac{3}{2}}\phi}. \tag{2.30}$$

From (2.30) we can see that the flat direction that needs to be stabilised is ϕ, which is equivalent to the ratio τ_1/τ_2. However it turns out that it is preferable to stabilise τ_1 directly and then infer τ_2 from (2.20) as we will see later.

3 Anisotropic Moduli Stabilisation

3.1 Kähler Moduli Stabilisation

All we are missing to perform the stabilisation are the correction to V_{LVS} which lift the flat direction. Following ideas presented in [46], where the made use of string loop and poly-instanton corrections to achieve an anisotropic stabilisation of the moduli, the kind of corrections we introduce are string loop corrections and higher derivative corrections. The former were first computed in [47] for toroidal orientifolds (with $\mathcal{N} = 1, 2$) and, from there, inferred for smooth CY orientifolds in [48], while a derivation of the former can be found in [49]. We follow the conventions of [50] where these corrections take the following form:

- String loop corrections.
  These corrections come in two different kinds: KK ones and winding ones. The former, of order $(\alpha')^2 g_s^2$, is due to the exchange of closed strings carrying KK momentum between D-branes, while the latter, of order $(\alpha')^4 g_s^2$, is due to the exchange of winding strings between intersecting D-branes.

  These two corrections affect the Kähler potential:

$$\delta K_{g_s}^{KK} = g_s \sum_i \frac{C_i^{KK} t_\perp^i}{\mathcal{V}}, \tag{3.1}$$

$$\delta K_{g_s}^{W} = \sum_{ij} \frac{C_{ij}^{W}}{\mathcal{V} t_{ij}^{\cap}}. \tag{3.2}$$

  Where C_i^{KK} and C_{ij}^W are functions of the complex structure moduli, which become constant once such moduli have been stabilised, and are expected to be of order $\sim \mathcal{O}(1)$. The $t_\perp^i, t_{ij}^\cap$ denote respectively the 2-cycles transverse to the D-branes and the 2-cycles where the D-branes intersect.

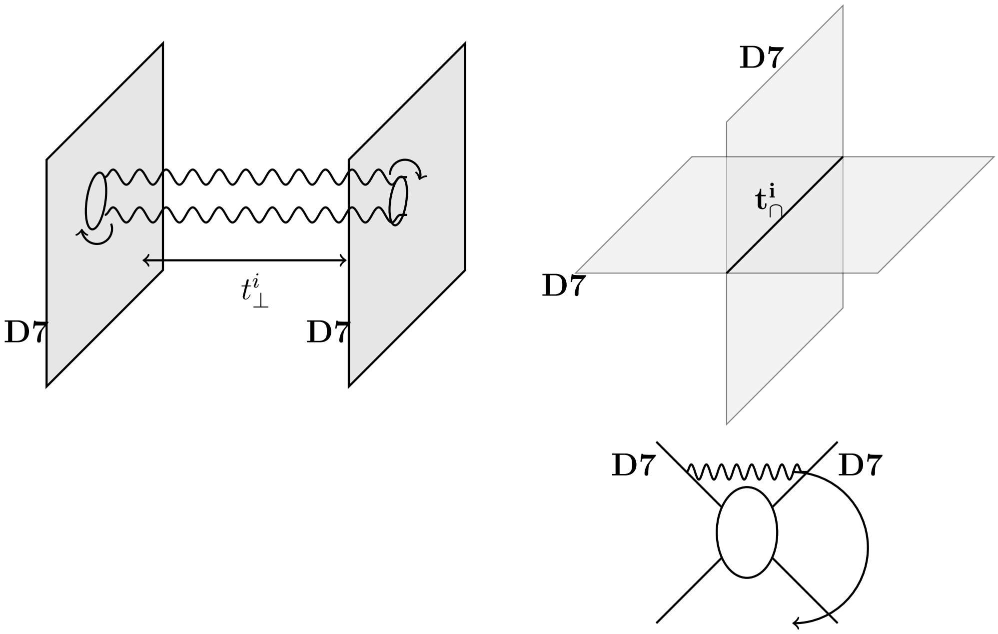

Figure 3.2: Physical realization of the string loop corrections. On the left we have two D7-branes exchanging an open string that makes a loop, which is effectively the same as the exchange of a closed string. On the right we have two intersecting D7-branes and, below, a zoom of a string extending between them that can wind around the "throat" formed by the D-branes. Stacks of multiple D-branes are also possible in both cases.

Then (3.1), (3.2) in turn imply that:

$$\delta V_{g_s}^{KK} = \frac{g_s^3}{2}\frac{|W_0|^2}{\mathcal{V}^2}\sum_{i,j} C_i^{KK} C_j^{KK} K^0_{ij}, \tag{3.3}$$

$$\delta V_{g_s}^{W} = -\, g_s \frac{|W_0|^2}{\mathcal{V}^3}\sum_{ij} \frac{C_{ij}^W}{t_\cap^{ij}} \tag{3.4}$$

Where K^0_{ij} is the tree-level Kähler potential and $t^{ij}_\cap$ is computed as:

$$t^{ij}_\cap = \int_X J \wedge D_i \wedge D_j. \tag{3.5}$$

It is important to notice that, despite (3.1) is a correction of order $(\alpha')^2 g_s^2$, it becomes of order $(\alpha')^4 g_s^4$ when we move to the potential (3.3) due to an extended no-scale structure. As such in can be effectively considered as a 2-loop KK correction. This fact will be crucial later on.

The extended no-scale structure we just mentioned can be easily understood in the case of a single modulus, τ. In this case $\mathcal{V} = \tau^{3/2}$ and:

$$K_{tree} = -\,2\ln(\mathcal{V}) = -3\ln(\tau)\,, \tag{3.6}$$

$$\delta K^{KK}_{g_s} \simeq \frac{Ct}{\mathcal{V}} = \frac{C}{\tau} \tag{3.7}$$

Therefore $K_{tot} = -3\ln(\tau) + C/\tau$, that, on the other hand, can also be expressed as:

$$\begin{aligned} K_{tot} &= -\,3\ln(\tau-\lambda) = -3\ln\left(\tau\left(1-\frac{\lambda}{\tau}\right)\right) \\ &= -\,3\ln(\tau) - 3\ln\left(1-\frac{\lambda}{\tau}\right) \overset{\lambda/\tau\ll 1}{\simeq} -3\ln(\tau) + \frac{3\lambda}{\tau} + o\left(\frac{1}{\tau^2}\right) \end{aligned} \tag{3.8}$$

By an appropriate choice of λ we get back the other form of K_{tot} and we are left with two-loop correction $o\,(1/\tau^2)$.
The extended no-scale structure property (3.8) is just a consequence of the fact that the correction (3.7) can be put inside of the logarithm.
This line of reasoning cannot be applied to (3.2) due to its different τ-scaling: $\delta K^W_{g_s} \simeq 1/(\mathcal{V}t) \sim 1/\tau^2$. Hence $\delta K^W_{g_s}$ is indeed a one-loop correction.

- Higher derivative corrections.
  These corrections, differently from the string loop ones, cannot be described starting from the Kähler potential or the superpotential. They are direct corrections to the scalar potential coming from higher derivative operators and can be thought of as a subset of the $(\alpha')^3$ order corrections. In particular we focus on the four derivative terms so that:

$$\delta V_{F^4} = -\frac{\sqrt{g_s}}{4}\frac{|W_0|^4}{\mathcal{V}^4}\Pi_i t^i\,, \tag{3.9}$$

where λ is a combinatorial factor, whose numerical value has been computed to be $\lambda = -3.5 \cdot 10^{-4}$ for the case of a single modulus in [51], where the authors also argue that it is expected to remain small for more moduli too. Indeed this coefficient, as well as the rest of the correction, can be understood as coming from a tensor coupling to four derivatives, as it is an F^4 correction, which always carries a factor of $c \cdot \zeta(3)/(2\pi)^4$, with c some other combinatorial factor, which is $11/384$ for a single modulus [52]:

$$\delta V_{F^4} = e^{-2K}\mathcal{T}^{ij\bar{k}\bar{l}} D_i W D_j W \bar{D}_{\bar{k}}\bar{W}\bar{D}_{\bar{l}}\bar{W}. \tag{3.10}$$

With:

$$\mathcal{T}_{ij\bar{k}\bar{l}} = \frac{c\cdot\zeta(3)}{(2\pi)^4\mathcal{V}^{8/3}g_s^{3/2}}\int_X c_2(X)\wedge J. \tag{3.11}$$

From now on, we will take $\lambda = -|\lambda|$.

The Π_i's are topological quantities which depend on the second Chern class as follows:

$$\Pi_i \equiv \int_X c_2(X) \wedge D_i. \tag{3.12}$$

Where $c_2(X)$ is the second Chern class of X and D_i are the divisors associated to the 2-cycles t^i. Neglecting the t_s cycle, we only have two divisors, τ_1 and τ_2. The τ_1 divisor is a $K3$ surface with $\Pi_1 = 24$ [53].
Recalling that t_2 is associated to the divisor $D_x = D_7 + 2D_6 - 3D_1$ (2.19) we can also compute Π_2 by making use of its definition (3.12) and (2.12): $\Pi_2 = 96$.

We can finally apply these ideas to our model, the one with volume (2.20). The Kähler metric, entering (3.3), is:

$$K^0_{ij} = \begin{pmatrix} \frac{1}{\tau_1^2} & 0 \\ 0 & \frac{2}{\tau_2^2} \end{pmatrix} \tag{3.13}$$

Thus the corrections (3.3),(3.4) and (3.9) take the form:

$$\begin{aligned} V_{\text{correction}} &= \delta V_{g_s}^{KK} + \delta V_{g_s}^{W} + \delta V_{F^4} \\ &= \frac{g_s^3}{2}\frac{|W_0|^2}{\mathcal{V}^2}\left[\frac{(C_1^{KK})^2}{\tau_1^2} + \frac{\tau_1}{18}\frac{(C_2^{KK})^2}{\mathcal{V}^2}\right] - \frac{g_s}{6}\frac{|W_0|^2}{\mathcal{V}^3}\frac{C_{12}^W}{\sqrt{\tau_1}} \\ &+ \frac{\sqrt{g_s}}{4}\frac{|\lambda||W_0|^4}{\mathcal{V}^4}\left[\Pi_1\frac{\mathcal{V}}{\tau_1} + \frac{\Pi_2}{3}\sqrt{\tau_1}\right]. \end{aligned} \tag{3.14}$$

We replaced any τ_2 with τ_1 using (2.20) so that we can now take $\partial_{\tau_1} V_{\text{correction}}$ and stabilise τ_1. However, before this, it is important to notice that the first term of (3.14) is not subdominant with respect to $V_{LVS} \propto \mathcal{V}^{-3}$, when we would expect it to be since (3.14) is made up of correction terms. A solution to the issue is setting $C_1^{KK} = 0$, a choice that can be readily justified by an argument mentioned previously. Indeed we have seen that (3.3) is effectively a 2-loop correction due to the extended no-scale structure of (3.1). For this reason we should regard our total Kähler potential as carrying a 2-loop correction too $K_{tot} = -3\ln(\tau) + C/\tau + K_{\text{2-loop}}$, but we cannot be ensured that $K_{\text{2-loop}}$ does not enjoy a no-scale property too, so we should move to $K_{\text{3-loop}}$ and so on. Following this idea it is safe to assume not only $C_1^{KK} = 0$ but also $C_2^{KK} = 0$ as they both descend from KK corrections. This solution can also be justified via a specific choice of brane wrappings, for example if all D7-branes intersect each other and there are no D3-branes [54].
We can now stabilise τ_1:

$$\frac{\partial V_{\text{correction}}}{\partial \tau_1} = \frac{A}{\mathcal{V}^3}\sqrt{\tau_1} - \frac{B}{\mathcal{V}^3} + \frac{C}{\mathcal{V}^4}\tau_1^{3/2} = 0\,, \tag{3.15}$$

where we defined:

$$\begin{aligned} A &= \frac{g_s}{12} C_{12}^W |W_0|^2, \\ B &= \frac{\sqrt{g_s}}{4} |\lambda||W_0|^4 \Pi_1, \\ C &= \frac{\sqrt{g_s}}{24} |\lambda||W_0|^4 \Pi_2. \end{aligned} \tag{3.16}$$

Now we define $x = \sqrt{\tau_1}$, $p = A\mathcal{V}/C$, $q = B\mathcal{V}/C$ so that we can put (3.15) into the form:

$$x^3 + px - q = 0. \tag{3.17}$$

Whose general solution is:

$$x = \frac{2^{1/3}\left[9q + \sqrt{12p^3 + 81q^2}\right]^{2/3} - 2 \cdot 3^{1/3} p}{6^{2/3}\left[9q + \sqrt{12p^3 + 81q^2}\right]^{1/3}}. \tag{3.18}$$

The large volume limit, $\mathcal{V} \gg 1$, corresponds to $p \gg 1$ and $q \gg 1$. In this limit (3.18) simplifies to:

$$x \simeq \frac{q}{p} \qquad \Leftrightarrow \qquad \tau_1 = \left(\frac{q}{p}\right)^2 = \left(\frac{B}{A}\right)^2 = \left(3\frac{|\lambda||W_0|^2\Pi_1}{C_{12}^W\sqrt{g_s}}\right)^2. \tag{3.19}$$

All we have to do now is choose specific values for the parameters. Actually we have already done it for most of them previously, but we left out W_0. Scans over many different vacua in type IIB, such as [55], show that the typical values are $W_0 \sim \mathcal{O}(1-10)$. Calling N the number of stacked D-branes and taking $\chi(X) = -192$, we choose our parameters as follows:

Parameters	Stabilised Moduli
$W_0 = 7 \quad N = 1$ $C_{12}^W = 1 \quad \xi \simeq 0.46$ $\lvert\lambda\rvert = 3.5 \cdot 10^{-4} \quad \Pi_1 = 24$ $g_s = 0.1 \quad \Pi_2 = 96$	$\langle \mathcal{V} \rangle \simeq 2.3 \cdot 10^{16}$ $\langle \tau_1 \rangle \simeq 15.2$ $\langle \tau_2 \rangle \simeq 3.6 \cdot 10^{16}$
$W_0 = 7 \quad N = 5$ $C_{12}^W = 1 \quad \xi \simeq 0.46$ $\lvert\lambda\rvert = 3.5 \cdot 10^{-4} \quad \Pi_1 = 24$ $g_s = 0.1 \quad \Pi_2 = 96$	$\langle \mathcal{V} \rangle \simeq 1.9 \cdot 10^{3}$ $\langle \tau_1 \rangle \simeq 15.2$ $\langle \tau_2 \rangle \simeq 2.9 \cdot 10^{3}$

Table 3.1: Different values for the parameters and corresponding values of the stabilised moduli. Depending on the context in which we want to perform the stabilisation we might consider one possibility or the other. For example applications to inflation require $\mathcal{V} \sim 10^3$, but to match with the string scale predicted by the DD scenario we want $\mathcal{V} \sim 10^{15} - 10^{16}$.

We could have also neglected the last term of (3.15) because of the stronger suppression caused by $\mathcal{V}^{-4}$. Indeed in this case $\tau_1 = (B/A)^2 \simeq 15.2$. Either way the moduli are stabilised at the minimum, in fact:

$$\frac{\partial^2 V_{\text{correction}}}{\partial^2 \tau_1} = \frac{1}{\mathcal{V}^3 \tau_1^3} \left(2B - \frac{3}{2} A \sqrt{\tau_1} \right) = \frac{1}{\mathcal{V}^3 \tau_1^3} \left(\frac{1}{2} B \right) > 0. \tag{3.20}$$

This holds since $B \propto |\lambda| > 0$, which is therefore a necessary condition. Reintroducing the $\mathcal{V}^{-4}$ term does not spoil the result due to its stronger suppression.

We can see a first contact between the moduli stabilisation just described and the DD scenario by computing the string scale starting from the data in Tab. 3.1. In fact we have seen that the DD scenario predicts $M_S \sim 10^8 - 10^9$ GeV (1.2) and indeed, making use of (3.1) with $\mathcal{V} \simeq 2.3 \cdot 10^{16}$, we find exactly $M_S \sim 10^8$ GeV.

3.2 Complex Structure Stabilisation and Deformation of the $\mathbb{P}^1$ Base

With the moduli stabilisation just described we reached a situation where the $\mathbb{P}^1$ base is much larger than the K3 fibre, hence our CY manifold presents two large dimensions. The DD scenario however tells us that only one extra dimension must be large so, in order to be consistent with this prediction, we must now deform the $\mathbb{P}^1$ base so that one dimension is much larger than the other (we can picture it as an S^2 being deformed into a long cigar).

The size of the $\mathbb{P}^1$ base is controlled by the holomorphic $(3,0)$-form integrated on some appropriate three-cycles which we must construct. In order to understand why, it is useful first to re-write the weight system as follows:

	x_1	x_6	x_2	x_3	x_4	x_5	x_7
6	0	0	0	1	1	1	3
8	0	1	1	1	1	0	4
6	-1	0	1	1	2	0	3
	dP_8	K3	NdP_{10}	SD_1	NdP_{15}	NdP_{13}	SD_2

The first two lines show a K3 fibration over a $\mathbb{P}^1$ base. Furthermore the K3 is a double cover of a $\mathbb{P}^2$.
If we had only the first two lines, without the first column, we would have a K3 fibration where the fibre has at most ADE singularities, i.e. it never splits into two divisor components. The third line shows a blow up of a dP_8: this is obtained first by generating dP_8 singularities over the point $x_2 = 0$ in the $\mathbb{P}^1$ base, and then blowing up such singularity. After this we obtain a K3 fibration in which the generic fibre (the fibre of a generic point in the $\mathbb{P}^1$ base) is a K3 surface, while over one point of the $\mathbb{P}^1$ it splits into two surfaces, one of which is the dP_8.

It is also convenient to blow down the dP_8 so that we work with less projective coordinates and the weight system simplifies to:

	x_6	x_2	x_3	x_4	x_5	x_7
6	0	0	1	1	1	3
8	1	1	1	1	0	4

From here we read that the projective coordinates for the $\mathbb{P}^1$ base are x_2 and x_6.

From these data we can also derive the CY equation by writing all possible monomials that keep X smooth:

$$\begin{aligned} x_7^2 = x_3^6\, Q_1(x_2,x_6) + x_3^4 x_4^2 Q_2(x_2,x_6) + x_4^6 Q_3(x_2,x_6) \\ + x_3^4 x_5^2 x_2^2 Q_4(x_2,x_6) + x_5^6 x_2^6 Q_5(x_2,x_6) + c_0 x_4^3 x_5^3 x_6^5 \end{aligned} \tag{3.21}$$

Where $Q_i(x_2,x_6)$ are homogeneous polynomial of degree 2 in the coordinates x_2, x_6. From (3.21) are finally able to build the aforementioned three-cycles as follows:

- Choose $Q_1(x_2,x_6) = x_6^2 - x_2^2$ and all the other Q_i's different from Q_1.

- Take the two points $P_1: x_6 = x_2$ and $P_2 : x_6 = -x_2$. Up to the $\mathbb{P}^1$ $\mathbb{C}^*$-rescaling, one can set $P_1 : x_6 = x_2 = 1$ and $P_2 : x_6 = -x_2 = 1$. Over these points the K3 fibre develops an A_1 singularity along $x_7 = x_4 = x_5 = 0$. In fact, near these points $x_3 \neq 0$ and we can gauge fix the $\mathbb{P}^2$ $\mathbb{C}^*$-action by setting $x_3 = 1$; neglecting the terms that goes to zero quicker that the remaining ones, the K3 looks like:

$$x_7^2 = Q_2(\pm 1, 1)x_4^2 + Q_4(\pm 1, 1)x_5^2. \tag{3.22}$$

- If we move away the points P_1 and P_2, we add the term $Q_1(\pm 1, 1)$ to the equation (3.22), deforming the A_1-singularity (the K3 fibre becomes smooth). The deformation occurs by blowing up an S^2. This sphere shrinks to zero size on top of P_1 and P_2 generating the A_1 singularity.

- We now take a path γ in $\mathbb{P}^1$ that connects P_1 and P_2, such that over all its points the K3 fibre is smooth, except over P_1 and P_2.

- The fibration of the S^2 over the γ gives a three-cycle with the topology of S^3. Its size is controlled by $\int_{S^3} \Omega_3$, with Ω_3 the holomorphic (3,0)-form of the CY threefold.

Its symplectic dual 3-cycle can be constructed as follows:

- Over each point of γ the K3 is smooth and will have several two-cycles that intersect S^2 at one point. These two cycles will have non-trivial monodromies along closed path in the $\mathbb{P}^1$ base, that make them trivial two-cycles in the CY threefold X.

- There will however be a closed loop ℓ in the $\mathbb{P}^1$ base such that it intersects γ in one point and there is a two-cycle Σ_2 that has trivial monodromy along ℓ and intersects S^2 once.

- One can construct the three-cycle Σ_3 as the fibration of Σ_2 along ℓ.

The intersection between the two three-cycles is

$$S^3 \cdot \Sigma_3 = 1. \tag{3.23}$$

These two three-cycles are shown in Fig. 3.3.

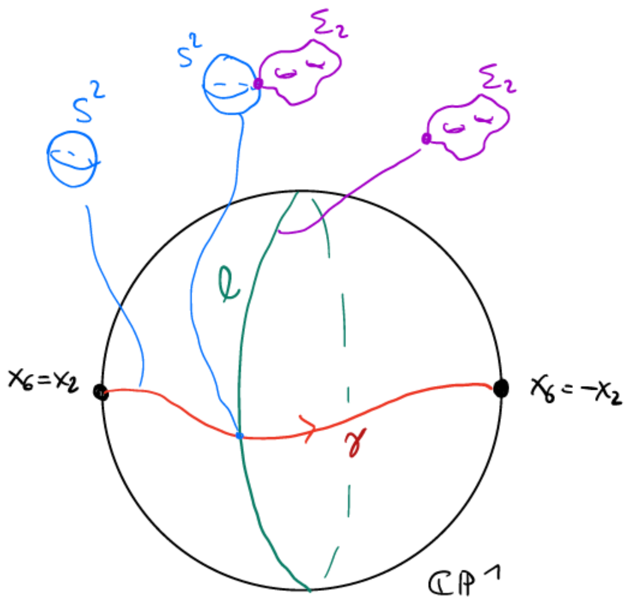

Figure 3.3: The three-cycles controlling the size of the $\mathbb{P}^1$ base.

Now we can consider a region in the complex structure moduli space where Ω is such that:

$$\frac{\int_{S^3}\Omega}{\int_{\Sigma^3}\Omega} \gg 1. \tag{3.24}$$

On the other hand the K3 two-cycles are all of comparable size and smaller with respect to the $\mathbb{P}^1$ base. In this region, while S^2 and Σ_2 are of comparable small size, the relation (3.24) forces γ to be much longer than ℓ so the large $\mathbb{P}^1$ base takes the form with one direction much bigger that the others, as shown in Fig. 3.4.

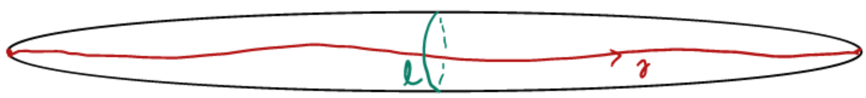

Figure 3.4: The deformed $\mathbb{P}^1$ base and the corresponding three-cycles.

We can give a physical interpretation of what (3.24) means starting from the quantisation condition for the fluxes (3.2) and the tadpole cancellation condition (3.7). Indeed (3.2) can also be understood as the fluxes being elements of a cohomology group: $H_3, F_3 \in H^3(X,\mathbb{Z})$. Hence, recalling that the symplectic basis (α_K, β^K), introduced in Sec. 3, is exactly a basis of $H^3(X, \mathbb{Z})$, we have the following expansions:

$$F_3 = m^K_{RR}\alpha_K - e_{RRK}, \tag{3.25}$$

$$H_3 = m^K\alpha_K - e_K\beta^K. \tag{3.26}$$

Where we set $\ell_s = 1$ and $m^K_{RR}, m^K, e_{RRK}, e_K$ are a set of integers.

Thus the tadpole cancellation condition (3.7) takes the form:

$$e_K m^K_{RR} - m^K e_{RRK} + Q_{\text{loc}} = 0. \tag{3.27}$$

Therefore, if we know Q_{loc}, which means knowing the specifics of the brane setup, we are able to find integers that saturate (3.27).

In the context of moduli stabilisation it is also common to define the A-period and the B-period. These are just the integral of the $(3,0)$-form Ω on three-cycles, respectively called A-cycle and B-cycle, that are the Poincaré duals of the symplectic basis (α_K, β^K):

$$X^I = \int_{A^I}\Omega, \tag{3.28}$$

$$F_I = \int_{B_I}\Omega. \tag{3.29}$$

Therefore it also follows that:

$$\Omega = X^I\alpha_I - F_I\beta^I. \tag{3.30}$$

In our case these three-cycles are exactly S^3 and Σ^3.

Finally, recalling that the flux superpotential (3.15) involves Ω as well as H_3 and F_3 through the definition of G_3 (1.95), we are able to express (3.24) as the ratio of functions of the integers $m^K_{RR}, m^K, e_{RRK}, e_K$:

$$\frac{\int_{S^3}\Omega}{\int_{\Sigma^3}\Omega} = \frac{U(m^K, m^K_{RR})}{Z(e_K, e_{RR\ K})}. \tag{3.31}$$

Chapter 4

Conclusions

In this thesis we presented a model for moduli stabilisation where the Kähler moduli are stabilised at very different values, yielding a compactification where the extra dimensions have different sizes, a picture that reproduces nicely the DD scenario. This result was achieved via g_s and F^4 corrections to the tree level Kähler potential.
We started in Chapter 2 with a review of string theory, both the bosonic one and its supersymmetric extension which turns out to be necessary if we want to describe fermions and get rid of tachyons, highlighting their spectra in order to show why string theory is not only a QG candidate, but a promising one, as was already hinted at in Chapter 1 by comparing Feynman and stringy diagrams. We focused especially on type IIB, describing its low energy supergravity action as well as the contribution of D-branes.
All five superstring theories predict ten spacetime dimensions, six of which must be unobserved if we want to make contact with the observations. We dealt with this aspect in Chapter 2 where we proved that the requirement of $\mathcal{N} = 1$ supersymmetry in four dimensions severely constrains the geometrical and topological properties of the six extra dimensions, forcing them to be a CY three-fold. At that point we had the tools to discuss deformations of CY manifolds. These are parametrized by scalar fields called moduli, hence giving rise to the concept of moduli space, and are generated whenever we compactify a theory with extra dimensions. Once again we focused on type IIB and its spectrum upon compactification, both in the un-orientifolded and orientifolded case, the latter being necessary if we want type IIB compactifications to have $\mathcal{N} = 1$ supersymmetry.
We then moved on to discuss the physical implications of moduli fields in the context of ISD compactifications. Indeed moduli spoil the phenomenology of a theory generating unobserved forces due to their masslessness, an issue that can be cured using fluxes for the complex structure moduli while for the Kähler moduli we need quantum corrections due to the no-scale structure. These are perturbative for the Kähler potential and non-perturbative for the flux superpotential. We then showed in the context of the KKLT scenario and LVS, the most successful paradigms for moduli stabilisation, how such

corrections can lift the flat directions of the scalar potential, hence giving a mass to the moduli.
Finally, in Chapter 3 we presented an explicit model for moduli stabilisation which naturally leads to an anisotropic compactification. This was achieved introducing one-loop g_s corrections, of KK and W type, as well as higher derivative F^4 corrections. The former descend from corrections to the Kähler potential, of order $(\alpha')^2 g_s^2$ for the KK ones and of order $(\alpha')^4 g_s^2$ for the W ones. The latter, a subset of the $(\alpha')^3$ corrections, affect directly the scalar potential and descend from a tensorial coupling. We also proved that the g_s corrections of KK type enjoy an extended no-scale structure, making them two-loop corrections effectively.
Knowing the general structure of the corrections, we could apply those ideas to a specific model: starting from toric data that describe the CY manifold, we derived the volume, the tree-level Kähler potential, the intersection two-cycles and the topological numbers Π_i. This led us to a correction to the leading order LVS scalar potential, which is not subdominant due to the presence of a $\mathcal{V}^{-2}$ factor carried by the KK corrections. Here the extended no-scale structure came in our rescue. Since the g_s corrections of KK type are effectively two-loop corrections, we should take into account all two-loop corrections to the tree-level Kähler potential, let us denote them by $K_{2-\mathrm{loop}}$. However we cannot be ensured that $K_{2-\mathrm{loop}}$ does not enjoy a no-scale structure property too. Hence we should move to $K_{3-\mathrm{loop}}$ and so on. This allowed us to set to zero all the KK corrections [1] and have a correction to the leading order scalar potential which is indeed subdominant. From there we performed moduli stabilisation and obtained naturally anisotropic results for the four-cycle volumes where the $\mathbb{P}^1$ base of our fibered CY is much larger than the K3 fibre itself.
Finally, we concluded by deforming the $\mathbb{P}^1$ base. Indeed the DD scenario predicts one single large extra dimension, but from the stabilisation we found two. In order to achieve that result we started from the equation describing the CY manifold and constructed two three-cycles, S^3 and Σ^3, which control the shape of the $\mathbb{P}^1$ base once we integrate the holomorphic $(3,0)$-form Ω on them. The deformation was achieved by limiting ourselves in a region of the complex structure moduli space where $\int_{S^3}\Omega/\int_{\Sigma^3}\Omega \gg 1$. We gave a physical interpretation of this result starting from the quantisation condition for the fluxes and the tadpole cancellation condition.

[1] As we mentioned in the relative section, this can also be justified via a specific choice of brane wrappings.

Appendix A

Mathematical Preliminaries

In this appendix we want to present a few details on the mathematical background needed throughout the thesis. The standard books [4, 5, 8, 12, 13] have chapters dedicated to these aspects, while a more in-depth treatment can be found in [56, 57].

1 Complex Geometry

Definition 1. A complex manifold M is a real manifold of dimensionality $n = 2m$ such that the charts are $\varphi_\alpha \colon U_\alpha \to \mathbb{C}^k$, $U_\alpha \subset M$, and the transition functions are holomorphic.

Complex manifolds can also be defined starting from a $(1, 1)$-tensor field J called almost complex structure. This tensor field can be thought of as map [1] $J \colon TM \to TM$ satisfying a "manifold version" of the Cauchy–Riemann equations. Given a basis of TM $\{(\partial/\partial x_\mu, \partial/\partial y_\mu)\}_{\mu=1,\dots,m}$:

$$J\left(\frac{\partial}{\partial x_\mu}\right) = \frac{\partial}{\partial y_\mu}, \quad J\left(\frac{\partial}{\partial y_\mu}\right) = -\frac{\partial}{\partial x_\mu}. \tag{A.1}$$

From here one can check that $J^2 = -\mathrm{Id}_{TM}$, hence the action of J corresponds to the multiplication by $\pm i$ roughly speaking.
It can be proved that, thanks to the Cauchy–Riemann relations, the action of J is independent of the chart and hence we can always express it as:

$$J = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix} \tag{A.2}$$

[1] Actually we should first define J pointwise as $J_P \colon T_P M \to T_P M$ but since the components of J_P are constant at any point (A.2) we can then define J as we have done here.

We can further extend the definition of J to the complexified tangent space $TM^{\mathbb{C}} = \{X + iY \mid X, Y \in TM\}$ as $J(X+iY) = JX + iJY$ and (A.2) is updated to:

$$J = \begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix} \tag{A.3}$$

The usual expression of the almost complex structure using (anti-)holomorphic indices is:

$$J_\mu{}^\nu = i\delta_\mu{}^\nu, \quad J_{\bar\mu}{}^{\bar\nu} = -i\delta_{\bar\mu}{}^{\bar\nu} \tag{A.4}$$

There will be vectors that are eigenvectors of J. Take for example $Z = Z^\mu \partial/\partial z^\mu$, with $\{\partial/\partial z^\mu\}$ a holomorphic basis of $TM^{\mathbb{C}}$, then $JZ = iZ$. Similarly if $Z' = Z'^\mu \partial/\partial \bar z^\mu$m with $\{\partial/\partial \bar z^\mu\}$ an anti-holomorphic basis of $TM^{\mathbb{C}}$ then $JZ' = -iZ'$. Hence we can split $TM^{\mathbb{C}}$ into even and odd eigenspaces:

$$TM^{\mathbb{C}} = TM^+ \oplus TM^-. \tag{A.5}$$

With

$$TM^\pm = \{Z \in TM^{\mathbb{C}} \mid JZ = \pm iZ\}. \tag{A.6}$$

TM^+ is spanned by $\{\partial/\partial z^\mu\}$ and TM^- by $\{\partial/\partial \bar z^\mu\}$.
Now we can uniquely decompose any $Z \in TM^{\mathbb{C}}$ as $Z = Z^+ + Z^-$, $Z^\pm \in TM^\pm$. If $Z = Z^+$ only then Z is a holomorphic vector and if $Z = Z^-$ only then Z is an anti-holomorphic vector.

As we anticipated we can define a complex manifold in terms of this tensor field J.

Definition 2. Given a differentiable manifold M of dimension m the pair (M, J) is called almost complex manifold of dimension $n = 2m$ if J is an almost complex structure.

Almost complex manifolds can be promoted to complex manifolds if the Nijenhuis tensor associated to J vanishes. This is also stated as J being integrable.

Definition 3. Given two vector fields $X, Y \in \mathcal{X}(M)$, the set of all vector fields on M, the Nijenhuis tensor is defined as:

$$N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]. \tag{A.7}$$

Given a coordinate basis $\{e_\mu = \partial/\partial x^\mu\}$ its component expression is:

$$N(X,Y) = X^\kappa Y^\nu \left[-J_\lambda{}^\mu(\partial_\nu J_\kappa{}^\lambda) + J_\lambda{}^\mu(\partial_\kappa J_\nu{}^\lambda) - J_\kappa{}^\lambda(\partial_\lambda J_\nu{}^\mu) + J_\nu{}^\lambda(\partial_\lambda J_\kappa{}^\mu)\right] e_\mu. \tag{A.8}$$

Or simply:

$$N^\lambda_{\mu\nu} = J_\mu{}^\sigma \partial_{[\sigma} J_{\nu]}{}^\lambda - J_\nu{}^\sigma \partial_{[\sigma} J_{\mu]}{}^\lambda. \tag{A.9}$$

We can now state the Newlander-Nirenberg theorem:

Theorem 1. Given an almost complex manifold (M, J), if J is integrable then the manifold M is a complex manifold.

1.1 Differential Forms on Complex Manifolds

We can now start discussing differential forms on complex manifolds and the cohomology groups built from them.

Definition 4. Given two real r-forms α_r and β_r, we can define a complex r-form as the sum:

$$\gamma_r \equiv \alpha_r + i\beta_r. \tag{A.10}$$

Its conjugate form is $\bar{\gamma}_r = \alpha_r - i\beta_r$.

We denote the vector space of complex r-forms by $\Omega^r_{\mathbb{C}}(M)$. This vector space can be decomposed into holomorphic and anti-holomorphic components, just like $TM^{\mathbb{C}}$ (A.5).

$$\Omega^k_{\mathbb{C}} = \bigoplus_{k=r+s} \Omega^{r,s}. \tag{A.11}$$

Where $\Omega^{r,s}(M)$ is the vector space of (r,s)-forms.

Definition 5. An (r,s)-form is a complex differential form with r holomorphic indices and s anti-holomorphic indices.

An element $\gamma_{r,s} \in \Omega^{r,s}(M)$ can be expanded as:

$$\gamma_{r,s} = \frac{1}{r!s!}\gamma_{\mathbf{M}\bar{\mathbf{N}}} dz^{\mathbf{M}} \wedge d\bar{z}^{\bar{\mathbf{N}}}. \tag{A.12}$$

With the convention $\gamma_{\mathbf{M}\bar{\mathbf{N}}} = \gamma_{\mu_1 \ldots \mu_r \bar{\nu}_1 \ldots \bar{\nu}_s}$ and $dz^{\mathbf{M}} \wedge d\bar{z}^{\bar{\mathbf{N}}}$ a basis for (r,s)-forms that follows the same notation:

$$dz^{\mathbf{M}} \wedge d\bar{z}^{\bar{\mathbf{N}}} \equiv dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_r} \wedge d\bar{z}^{\bar{\nu}_1} \wedge \cdots \wedge d\bar{z}^{\bar{\nu}_s}. \tag{A.13}$$

Any complex k-form can be decomposed into a sum of (r,s)-forms coherently with the decomposition (A.11):

$$\gamma_k = \sum_{k=r+s} \gamma_{r,s}. \tag{A.14}$$

A natural question is if we can extend the concept of exterior derivative d to complex forms. The answer is yes and what we are looking for are the Dolbeaut operators. There are two such operators:

- The operator $\partial\colon \Omega^{r,s} \to \Omega^{r+1,s}$ whose action on $\gamma_{r,s} \in \Omega^{r,s}$ is:

$$\partial\gamma_{r,s} = \left(\frac{\partial}{\partial z^{\alpha}}\gamma_{\mathbf{M}\bar{\mathbf{N}}}\right) dz^{\alpha} \wedge dz^{\mathbf{M}} \wedge d\bar{z}^{\bar{\mathbf{N}}}. \tag{A.15}$$

- The operator $\bar{\partial}\colon \Omega^{r,s} \to \Omega r, s+1$ whose action on $\gamma_{r,s} \in \Omega^{r,s}$ is:

$$\bar{\partial}\gamma_{r,s} = \left(\frac{\partial}{\partial \bar{z}^{\bar{\alpha}}}\gamma_{\mathbf{M}\bar{\mathbf{N}}}\right) d\bar{z}^{\bar{\alpha}} \wedge dz^{\mathbf{M}} \wedge d\bar{z}^{\bar{\mathbf{N}}}. \tag{A.16}$$

It can be proved that these operators obey:

$$d =\partial + \bar{\partial}, \tag{A.17}$$

$$\partial^2 =\bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0. \tag{A.18}$$

Given $\gamma_{r,0} \in \Omega^{r,0}$, this is called holomorphic $(r,0)$-form if and only if $\bar{\partial}\gamma_{r,0} = 0$. Similarly $\gamma_{0,s} \in \Omega^{0,s}$ is an anti-holomorphic $(0,s)$-form if and only if $\partial\gamma_{0,s} = 0$. If these relations hold for generic (r,s)-forms, namely $\bar{\partial}\gamma_{r,s} = 0$ or $\partial\gamma_{r,s} = 0$, we say that $\gamma_{r,s}$ is $\bar{\partial}$-closed or ∂-closed respectively, as happens in the case of real forms. Not only, we can also define exact forms: $\gamma_{r,s}$ is ∂-exact if $\gamma_{r,s} = \partial\delta_{r-1,s}$ and is $\bar{\partial}$-exact if $\gamma_{r,s} = \bar{\partial}\delta_{r,s-1}$

Having the Dolbeaut operators (A.15), (A.16) at our disposal we can define the Dolbeaut cohomology groups

Definition 6. The (r,s)-th Dolbeaut cohomology group is defined as [2] :

$$H^{r,s}_{\bar{\partial}}(M,\mathbb{C}) \equiv Z^{r,s}_{\bar{\partial}}(M)/B^{r,s}_{\bar{\partial}}(M). \tag{A.19}$$

Where $Z^{r,s}_{\bar{\partial}}(M)$ is the set of $\bar{\partial}$-closed (r,s)-forms on M and $B^{r,s}_{\bar{\partial}}(M)$ is the set of $\bar{\partial}$-exact (r,s)-forms on M.

We can now extend Hodge theory, first developed for real manifolds, to complex manifolds just by generalizing the Hodge operator $\star$ so that it can act on the complexified tangent space. Thus we can define an inner product involving $\star$:

$$(\alpha, \beta) = \int_M \alpha \wedge \star\bar{\beta}, \tag{A.20}$$

and from here we define $\partial^\dagger$ and $\bar{\partial}^\dagger$, the adjoints of ∂ and $\bar{\partial}$, by means of:

$$(\alpha, \partial\beta) = (\partial^\dagger\alpha, \beta), \quad (\alpha, \bar{\partial}\beta) = (\bar{\partial}^\dagger\alpha, \beta). \tag{A.21}$$

Hence one can prove that the operators (A.15), (A.16) and their adjoints are related by (A.21):

$$\partial^\dagger = -\star\bar{\partial}\star, \tag{A.22}$$

$$\bar{\partial}^\dagger = -\star\partial\star\,. \tag{A.23}$$

[2]It is convention to work with the $\bar{\partial}$ operator but we could define $H^{r,s}_{\partial}(M,\mathbb{C})$ in complete analogy as we just need to take complex conjugation to move from ∂ to $\bar{\partial}$.

Finally we can define the Laplace operators, that come in two different kinds:

$$\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial, \quad \Delta_{\bar\partial} = \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial. \tag{A.24}$$

The forms annihilated by $\Delta_{\bar\partial}$ [3] are called $\bar\partial$-harmonic forms and we denote the set of all such (r,s)-forms by $\mathcal{H}^{r,s}_{\bar\partial}(M)$. The utility of this construction comes from Hodge theorem which states that

$$H^{r,s}_{\bar\partial}(M,\mathbb{C}) \cong \mathcal{H}^{r,s}_{\bar\partial}(M) \tag{A.25}$$

It follows that Hodge numbers also represent the dimensionality of Dolbeaut cohomology groups:

$$h^{(r,s)} = \dim\left(H^{r,s}_{\bar\partial}(M,\mathbb{C})\right). \tag{A.26}$$

1.2 Kähler Manifolds

In string theory Calabi-Yau manifolds play a key role, as we have seen in Sec. 2. These are just a special kind of Kähler manifolds so it is worth spending some time to describe their main features. Not only, Kähler manifolds are also relevant whenever studying general supersymmetric theories.

In order to define a Kähler manifold we first need to define what is an Hermitian metric:

Definition 7. A Hermitian metric on a complex manifold M is a Riemannian metric $g\colon TM \times TM \to \mathbb{R}$ such that for any $X, Y \in TM$:

$$g(JX, JY) = g(X,Y). \tag{A.27}$$

The pair (M, g) is called Hermitian manifold.

Any complex manifold admits a Hermitian metric by direct construction. Indeed given a Riemannian metric g on a complex manifold M we can define an associated Hermitian metric by means of:

$$\hat{g}(X,Y) \equiv \frac{1}{2}\left[g(X,Y) + g(JX,JY)\right] \tag{A.28}$$

Furthermore, as a consequence of the definition (A.27), the purely holomorphic and purely anti-holomorphic components of g vanish:

$$g_{\mu\nu} = g\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = g\left(J\frac{\partial}{\partial z^\mu}, J\frac{\partial}{\partial z^\nu}\right) = -g\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = -g_{\mu\nu}. \tag{A.29}$$

The same line of reasoning can be used to prove $g_{\bar\mu\bar\nu} = 0$, hence g can be expressed as:

$$g = g_{\mu\bar\nu} dz^\mu \otimes d\bar{z}^{\bar\nu} + g_{\bar\mu\nu} d\bar{z}^{\bar\mu} \otimes dz^\nu. \tag{A.30}$$

[3] Again we focus on the $\bar\partial$ only for consistency with the construction of the Dolbeaut cohomology groups.

Starting from this metric we can define a $(1,1)$-form called Kähler form of the Hermitian manifold:

$$\begin{aligned}\omega \equiv & - J_{\mu\bar{\nu}} dz^{\mu} d\bar{z}^{\bar{\nu}} \\ = & \, i g_{\mu\bar{\nu}} dz^{\mu} d\bar{z}^{\bar{\nu}}.\end{aligned} \tag{A.31}$$

With $J_{\mu\bar{\nu}} = g_{\mu\bar{\lambda}} J_{\nu}{}^{\bar{\lambda}} = -i g_{\mu\bar{\nu}}$.

Definition 8. A Kähler manifold is a Hermitian manifold with closed Kähler form:

$$d\omega = 0 \tag{A.32}$$

This statement is equivalent to saying that a complex manifold M of complex dimension m is Kähler if its holonomy group is $\mathrm{Hol}(M) = U(m)$.

Kähler metrics, that are metrics on Kähler manifolds, can always be express locally in terms of a scalar function K called Kähler potential:

$$g_{\mu\bar{\nu}} = \partial_{\mu}\partial_{\bar{\nu}} K. \tag{A.33}$$

Having a metric we can also define all the quantities we are used to from Riemannian geometry. For instance the only non-vanishing components of the Riemann tensor are:

$$R^{\mu}{}_{\nu\rho\bar{\sigma}} = -\partial_{\bar{\sigma}} \Gamma^{\mu}_{\nu\rho}. \tag{A.34}$$

Where $\Gamma^{\mu}_{\nu\rho} = g^{\mu\bar{\sigma}} \partial_{\nu} g_{\rho\bar{\sigma}}$ is the only non-vanishing component of the Christoffel symbols, together with the conjugate $\Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\rho}}$.

From the Riemann tensor (A.34) we can find the Ricci tensor:

$$\begin{aligned}R_{\nu\bar{\sigma}} \equiv & \, R^{\mu}{}_{\mu\nu\bar{\sigma}} \\ = & - \partial_{\bar{\sigma}} \Gamma^{\mu}_{\nu\mu} \\ = & - \partial_{\bar{\sigma}} \partial_{\nu} \ln \mathfrak{g}.\end{aligned} \tag{A.35}$$

Where we used the Jacobi's formula $g^{\mu\bar{\sigma}} \partial_{\nu} g_{\rho\bar{\sigma}} = \partial_{\nu} \ln \det(g_{\rho\bar{\sigma}})$ and defined $\mathfrak{g} \equiv \det(g_{\rho\bar{\sigma}})$.

Finally, out of the Ricci tensor (A.35), we can build the Ricci $(1,1)$-form. This is defined locally as:

$$\begin{aligned}\mathfrak{R} \equiv & \, i R_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\bar{\nu}} \\ = & \, i \partial\bar{\partial} \ln \mathfrak{g}.\end{aligned} \tag{A.36}$$

It is easy to check that the Ricci form we just defined is real using (A.18)

$$\bar{\mathfrak{R}} = -i \overline{\partial\bar{\partial}} \ln \mathfrak{g} = -i \bar{\partial}\partial \ln \mathfrak{g} = i \partial\bar{\partial} \ln \mathfrak{g} = \mathfrak{R}. \tag{A.37}$$

Not only, it is also closed due to (A.17) and (A.18). It is useful to state the closure also in the following terms:

$$\mathfrak{R} = \frac{1}{2} d \left(\bar{\partial} - \partial \right) \ln \mathfrak{g}. \tag{A.38}$$

Where we used another relation between the exterior derivative d and the Dolbeaut operators, namely $d(\bar{\partial} - \partial) = 2\partial\bar{\partial}$. Clearly $\mathfrak{R}$ is closed since $d^2 = 0$.
Indeed now with (A.38) we can easily see that $\mathfrak{R}$ is not exact as $\mathfrak{g}$ is not a scalar and $(\bar{\partial} - \partial) \ln \mathfrak{g}$ is not globally defined.

The existence of a closed and non-exact form naturally leads us to define a cohomology class:

Definition 9. The first Chern class is the de Rham cohomology class of the Ricci form:

$$c_1(M) \equiv \frac{1}{2\pi} [\mathfrak{R}]. \tag{A.39}$$

We conclude this paragraph on Kähler manifolds with the definition of a Calabi-Yau manifold:

Definition 10. A Calabi-Yau k-fold is a compact [4] manifold M of complex dimension k that is simply connected [5] and satisfies the following equivalent conditions:

1. M has a Kähler metric and holonomy group $\mathrm{Hol}(M) = SU(k)$.
2. There exists a nowhere-vanishing $(k, 0)$-form on M.
3. M has a Kähler metric with vanishing Ricci tensor.
4. M has vanishing first Chern class $c_1(M) = 0$.

Indeed it was first conjectured by Calabi and later proved by Yau that, given a compact Kähler manifold X with $c_1(X) = 0$, there exists a Ricci flat metric Kähler metric on X [58].

2 Bundles and Characteristic Classes

A bundle, abbreviated form for fibre bundle, is an object that locally look like the product of two manifolds, called base and fibre.

[4] There exist non-compact CY k-folds, such as the 1-fold $\mathbb{C}$ or the 2-folds $\mathbb{C}^2$ and $\mathbb{C} \times T^2$, however we are only interested in the compact one for physical application, so we limit ourselves to those ones.

[5] As in the case of compactness, we can allow for non-simply connected CY k-folds, but we are not physically interested in them. Furthermore, in this case, conditions 1 and 3 are not equivalent anymore.

Definition 11. A differentiable fibre bundle (E, B, F, G, π) is a structure consisting of [6]:

- A differentiable manifold E called total space.
- A differentiable manifold B called base space.
- A differentiable manifold F called fibre.
- A surjection $\pi\colon E \to B$ called projection. For any $p \in B$, the pre-image $\pi^{-1}(p) \equiv F_p$ is called fibre at p.
- A Lie group G, called structure group, which acts on F from that left, that is to say:

$$\begin{aligned} L_g\colon G \times F \to& F, \\ (g, f) \to& L_g(f) \equiv gf. \end{aligned} \tag{A.40}$$

- A set of open coverings $\{U_i\}$, $U_i \subset M$ $\forall i$ with a diffeomorphism:

$$\phi_i\colon U_i \times F \to \pi^{-1}(U_i), \tag{A.41}$$

  such that:

$$\pi \circ \phi_i(p, f) = p. \tag{A.42}$$

  The map ϕ_i is called local trivialization.
  We can also define $\phi_{i,p} \equiv \phi_i(p, \cdot)\colon F \to F_p$ which is still a diffeomorphism.

- Finally on $U_i \cap U_j \neq \emptyset$ we ask that the map:

$$t_{ij}(p) \equiv \phi_{i,p}^{-1} \circ \phi_{j,p}\colon F \to F \tag{A.43}$$

  is an element of G. In this way ϕ_i and ϕ_j are related by the smooth map $t_{ij}\colon U_i \cap U_j \to G$ as follows:

$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f). \tag{A.44}$$

  The map t_{ij} is called transition function (see Fig. A.1).

We denote bundles by $E \xrightarrow{\pi} B$.

[6] To be honest what define here is something known as coordinate bundle in the mathematical literature. Indeed the definition of fibre bundle must be independent of the specific covering $\{U_i\}$ we choose. Fibre bundles are than nothing but an equivalence class of coordinate bundles. Still, as physicists, we can confuse the two concepts as we always make use of some specific covering.

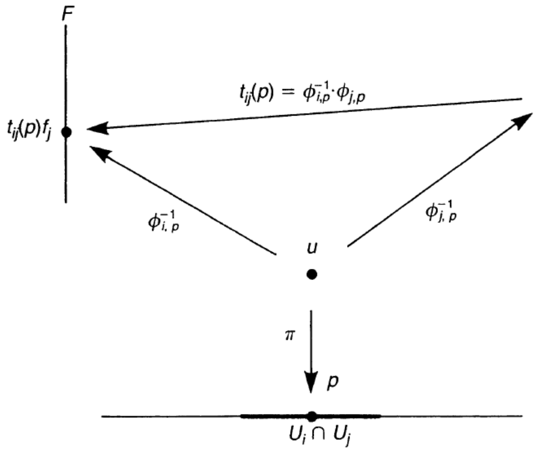

Figure A.1: On the overlap of two covers $U_i \cap U_j$ two elements, $f_i, f_j \in F$ are mapped, via $\phi_{i,u}$ and $\phi_{j,u}$, to $u \in \pi^{-1}(p)$, with $p \in U_i \cap U_j$. These elements of F are related by the transition functions as $f_i = t_{ij}(p) f_j$. Picture taken from [56].

Some comments are definitely needed to have a deeper grasp of the definition. The main idea is that for each point of the base B we have a fibre F and B can be covered by charts U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times F$. In this sense we say that the bundle can be locally trivialized as it is reduced to a product. Then, at a point p in the non-empty intersection of two different covers U_i and U_j we have the map $(\phi_i^{-1}\phi_j)_p \colon F \to F$ which defines the action on F of an element of the structure group G. This tells us that two different local trivializations are related by the action of G on the fibre, in particular via the transition functions. If all the transition functions are the identity the bundle is called trivial bundle as is just a direct product $B \times F$.
A typical example of fibre bundles are real vector bundles (see Fig. (A.2)) of rank k of a manifold B. These are fibre bundles for which the fibre F is $\mathbb{R}^k$ and the structure group is a subgroup of $GL(k, \mathbb{R})$. When $k = 1$ we talk about line bundles. A special case of vector bundles are tangent bundles for which the fibre at $p \in B$ is the tangent space at p, T_pB, hence the full tangent bundle can be regarded as the collection of all the tangent spaces to B.
Vector bundles of rank k can also be complex, in which case the fibre is $\mathbb{C}^k$ and the structure group is a subgroup of $GL(k, \mathbb{C})$.
Another relevant example are principal bundles, also called G-bundles. These are just fibre bundles where the fibre F is equal to the structure group G.

Definition 12. Given a fibre bundle $E \xrightarrow{\pi} B$ we define the (global) section of E over B the smooth map:

$$s \colon B \to E \tag{A.45}$$

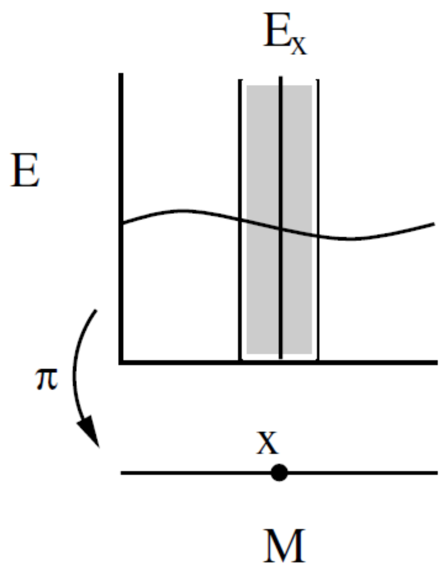

Figure A.2: A vector bundle E and the projection π to the base manifold, here denoted by M. $E_x \equiv \pi^{-1}(x)$ is the fibre over $x \in M$, whereas the gray region represents $\pi^{-1}(U_x)$, with $x \in U_x$. Finally the wavy line represents a section of E. Picture taken from [15]

such that

$$\pi \circ s = \mathrm{Id}_M. \tag{A.46}$$

We shall talk about local sections if the above definition only holds for $U \subset B$.

A good intuition for sections comes from the case of tangent bundles. Indeed a section of a tangent bundle is nothing but a vector field on the base space B.

As happens for standard Riemannian manifolds, it is possible to develop a theory of parallel transport also for fibre bundles but we will not delve in those aspects in order not to get too technical as these appendices are just meant to be an overview. Still it is worth to give one single piece of information as it is deeply connected with physics. If we want to define parallel transport on fibre bundles, in particular we focus on principal bundles, we must of course introduce the concept of covariant derivative and connection. Connections on bundles are already known to physicists under the name of gauge fields and the associated field strength is the curvature. This equivalence can be extended to vector bundles too.

Chern Class

Given a fibre F, a structure group G and a base space B, there are several ways, related to the choice of transition functions, to construct a fibre bundle with these elements.

Therefore a natural question can be how many bundles we can build once we have those data and how much they differ from the trivial bundle $B \times F$. Characteristic classes, a way of associating cohomology groups to the base space B, measure exactly the non-triviality (or the twisting) of a bundle, so how much the global structure deviates from the product structure holding locally. It is possible to construct many different characteristic classes but here we only focus on the Chern class as it is the one relevant for this thesis.

Definition 13. Given a complex vector bundle $E \xrightarrow{\pi} M$ with gauge connection $\mathfrak{A}$ and field strength $\mathfrak{F}$ we define the total Chern class as:

$$c(\mathfrak{F}) \equiv \det\left(I + \frac{i\mathfrak{F}}{2\pi}\right). \tag{A.47}$$

The total Chern class can be expanded as sum of single Chern classes and these are what we usually refer to when we talk about Chern classes:

$$c(\mathfrak{F}) = 1 + c_1(\mathfrak{F}) + c_2(\mathfrak{F}) + \dots \ . \tag{A.48}$$

Where $c_j(\mathfrak{F}) \in \Omega^{2j}(M)$ is the j-th Chern class.

On a m-dimensional manifold M all the Chern classes $c_j(\mathfrak{F})$ with $2j > m$ vanish identically. Furthermore it can be proved that c_j is closed, hence it defines a cohomology class $[c_j(\mathfrak{F})] \in H^{2j}(M)$. Indeed we tacitly implied this result in (A.39), which is another possible definition of the first Chern class.

If we specialize to the case of a Kähler manifold with a Kähler metric g we can explicitly construct the Chern class following [8]. First define a $(1,1)$-form Θ:

$$\Theta_i^j = g^{j\bar{p}} R_{i\bar{p}k\bar{l}} dz^k \wedge d\bar{z}^{\bar{z}}. \tag{A.49}$$

Hence define the total Chern class as:

$$\begin{aligned} c(M) \equiv \ \det\left(I + \frac{it}{2\pi}\Theta\right)\bigg|_{t=1} &= 1 + \sum_i c_i(M) \\ = \ \left(1 + t\phi_1(g) + t^2\phi_2(g) + \dots\right)|_{t=1}. \end{aligned} \tag{A.50}$$

We have that:

- $d\phi_i(g) = 0$ and $[\phi_i(g)] \in H^{(i,i)}_{\text{Dolbeaut}}(M, \mathbb{C}) \cap H^{2i}_{\text{DeRham}}(M, \mathbb{R})$.
- $[\phi_i(g)]$ is independent of g.
- $\phi_i(g)$ represents $c_i(M)$.

Now we can express $c_(M)$ in terms of the Ricci form $\mathfrak{R}$, verifying (A.39). We just need $\phi_1(g)$:

$$\begin{aligned}\phi_1(g) = \frac{i}{2\pi}\Theta^i_i =& \frac{i}{2\pi}R_{k\bar{l}}dz^k \wedge d\bar{z}^{\bar{l}} \\ =& \frac{1}{2\pi}\mathfrak{R}.\end{aligned} \tag{A.51}$$

In the first line we just used the definition of Θ (A.49) while in the second line we used (A.36).
This is exactly (A.39).

Appendix B

T-Duality

In this appendix we want to briefly describe T-duality, a way to relate different types of string theory that also holds for the bosonic one and furnishes a solid basis for the existence of D-branes. All the standard books [4, 5, 8, 12, 13] have chapters dedicated to it.

Since T-duality can already be understood in bosonic string theory we start from it for simplicity. Consider bosonic string theory, closed for the moment, compactified on a circle of radius R, that is to say we assume our space-time to be of the form $\mathbb{R}^{1,24} \times S^1$. For definiteness we choose X^{25} as the compactified direction, which therefore must satisfy a periodicity condition:

$$x^{25}(\tau, \sigma + \pi) = x^{25}(\tau, \sigma) + 2\pi R W, \quad W \in \mathbb{Z}. \tag{B.1}$$

Where W is the winding number, counting, with sign, how many times the string winds around the circle S^1.

The periodicity condition (B.1) changes the Fourier expansion of X^{25} with respect to the other coordinates $X^\mu, \mu = 0, \ldots, 24$ for which (1.18) still holds. Indeed we have to add a linear term in σ so that (B.1) is respected:

$$X^{25}(\tau, \sigma) = x^{25} + 2\alpha' p^{25}\tau + 2RW\sigma + \ldots \ . \tag{B.2}$$

With "..." containing the oscillator term that is unaffected by the compactification.

Since we have one compact dimension, the momentum along that direction must be quantised, in our case p^{25}. Recalling that a wave function contains a factor $\exp(ix_\mu p^\mu)$ we can find the quantisation condition. Indeed if x^{25} is increased by $2\pi R$, which corresponds to a trip around the circle, the wave function should go back to its original value, hence we must have:

$$p^{25} = \frac{K}{R}, \quad K \in \mathbb{Z}. \tag{B.3}$$

With K the Kaluza-Klein excitation number.

With this information we can now go back to (B.2) and split it as (1.18):

$$X^{25}(\tau,\sigma) = X_L^{25}(\tau+\sigma) + X_R^{25}(\tau-\sigma), \tag{B.4}$$

where

$$X_R^{25}(\tau-\sigma) = \frac{1}{2}\left(x^{25} - \widetilde{x}^{25}\right) + \left(\alpha'\frac{K}{R} - WR\right)(\tau-\sigma) + \ldots, \tag{B.5}$$

$$X_L^{25}(\tau+\sigma) = \frac{1}{2}\left(x^{25} + \widetilde{x}^{25}\right) + \left(\alpha'\frac{K}{R} + WR\right)(\tau+\sigma) + \ldots. \tag{B.6}$$

The constant $\widetilde{x}^{25}$ is irrelevant as it cancels in the sum.

In the light of (1.22) and (1.19) and (1.20) the above expansion can also be expressed as:

$$X_R^{25}(\tau-\sigma) = \frac{1}{2}\left(x^{25} - \widetilde{x}^{25}\right) + \sqrt{2\alpha'}\alpha_0^{25}(\tau-\sigma) + \ldots, \tag{B.7}$$

$$X_L^{25}(\tau+\sigma) = \frac{1}{2}\left(x^{25} + \widetilde{x}^{25}\right) + \sqrt{2\alpha'}\widetilde{\alpha}_0^{25}(\tau+\sigma) + \ldots. \tag{B.8}$$

Thus we have the following identifications:

$$\sqrt{2\alpha'}\alpha_0^{25} = \alpha'\frac{K}{R} - WR, \tag{B.9}$$

$$\sqrt{2\alpha'}\widetilde{\alpha}_0^{25} = \alpha'\frac{K}{R} + WR. \tag{B.10}$$

The compactification affects only partially the physical on shell condition (1.36) in the sense that those equation still hold, but they are updated due to the compactification on S^1 along the 25-th dimension. In particular $L_0 = 1$ and $\widetilde{L}_0 = 1$, where $1 = a$ is the normal ordering constant, now take the form:

$$\frac{1}{2}\alpha' M^2 = (\widetilde{\alpha}_0^{25})^2 + 2N_L - 2 = (\alpha_0^{25})^2 + 2N_R - 2. \tag{B.11}$$

Taking the sum and difference of the left and right hand sides of this equation and using (B.9) and (B.10) we find:

$$N_R - N_L = WK, \tag{B.12}$$

$$\alpha' M^2 = \alpha'\left[\left(\frac{K}{R}\right)^2 + \left(\frac{WR}{\alpha'}\right)^2\right] + 2N_R + 2N_L - 4. \tag{B.13}$$

These two equations are invariant under the exchange $W \leftrightarrow K$ as long as we also send $R \to \widetilde{R} = \alpha'/R$. This is the symmetry called T-duality. It signifies that a compactification on a circle of radius R and on a circle of radius $\widetilde{R}$ are physically equivalent and it can be proved to be true for the full interacting theory, at least at the perturbative level. These statements also hold for superstring and they hold non-perturbatively as well.
Furthermore, in the light of (B.9) and (B.10) we can also read the T-duality transformation as:

$$\alpha_0 \to -\alpha_0 \quad \text{and} \quad \widetilde{\alpha}_0 \to \widetilde{\alpha}_0. \tag{B.14}$$

Where we dropped the index 25 for simplicity.

Thus, from (B.5) and (B.6) and using (B.14), we see that T-duality actually flips the sign of the right moving part of X^{25} while leaving the left part untouched:

$$X_R \to -X_R \quad \text{and} \quad X_L \to X_L. \tag{B.15}$$

Therefore (B.4) will become:

$$X(\tau,\sigma) \to \widetilde{X}(\tau,\sigma) = X_L(\tau+\sigma) - X_R(\tau-\sigma), \tag{B.16}$$

whose expansion is

$$\widetilde{X}(\tau,\sigma) = \widetilde{x} + 2\alpha' \frac{K}{R}\sigma + 2RW\tau + \dots. \tag{B.17}$$

So, with respect to (B.2), τ and σ have been interchanged and x has been replaced by $\widetilde{x}$ that parametrizes the dual circle, of radius $\widetilde{R}$, with periodicity $2\pi\widetilde{R}$. The dual momentum is quantised according to $\widetilde{p} = W/\widetilde{R} = RW/\alpha'$.

We can now see how D-branes arise from T-duality by applying it to open strings. To do that recall the discussion about possible boundary conditions for open strings where we learnt that the only boundary conditions compatible with Poincaré invariance in all directions are Neumann boundary conditions for all the components of X^μ: $\partial_\sigma X^\mu(\tau,\sigma) = 0$ for $\sigma = 0, \pi$. A natural question can then be what happens if we apply a T-duality transformation to open strings. Clearly we expect to find something different with respect to the case of closed strings as open strings cannot wind and hence do not carry any winding number.
We start from the Fourier expansion of the coordinates for open strings (1.23). We may also set $\alpha' = 1/2$ in order to get rid of numerical factors. Since we want to apply a T-duality transformation it is useful to split the expansion (1.23) into left and right movers as we do for closed strings:

$$X_R(\tau-\sigma) = \frac{x-\widetilde{x}}{2} + \frac{1}{2}p(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau-\sigma)}, \tag{B.18}$$

$$X_RL(\tau-\sigma) = \frac{x+\widetilde{x}}{2} + \frac{1}{2}p(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau+\sigma)}. \tag{B.19}$$

At this point we compactify on a circle of radius R and perform the T-duality transformation via (B.15). In this way the dual coordinate in the 25-th direction will have the following expansion:

$$\widetilde{X}(\sigma,\tau) = X_L - X_R = \widetilde{x} + p\sigma + \sum_{n\neq 0} \frac{1}{n}\alpha_n e^{-in\tau}\sin(n\sigma). \tag{B.20}$$

This is exactly the expansion (1.24) for an open string with Dirichlet boundary conditions. T-duality maps Neumann boundary conditions into Dirichlet ones, and vice versa, in the compactified directions. In particular the boundary conditions for $\widetilde{X}$ read as:

$$\widetilde{X}(\tau,0) = \widetilde{x} \quad \text{and} \quad \widetilde{X}(\tau,\pi) = \widetilde{x} + \frac{\pi K}{R} = \widetilde{x} + 2\pi K\widetilde{R}. \tag{B.21}$$

To recap we have seen that T-duality transforms open bosonic strings with Neumann boundary conditions compactified on a circle of radius R into bosonic open strings with Dirichlet boundary conditions compactified on a circle of radius $\widetilde{R}$. The ends of the dual string are attached to the hyperplane $\widetilde{X} = \widetilde{x}$ and this is what we call a D-brane. We want to stress that this isn't just a mathematical artifact, D-branes are physical objects. When needed we can also specify the dimensionality of the brane as Dp-brane, with p the number of spatial dimensions of the brane. Hence a Dp-brane has $p+1$ dimensions in total.

The analysis we illustrated is interesting but affected by a major flaw, the existence of a tachyon in the bosonic string spectrum. The issue lies in the fact that $M^2 < 0$ suggests that we are not studying the theory at the minimum of the potential. We need to look for the true vacuum of the theory. Thus it has been proposed that D-branes decay into radiation, implying that D-branes are not stable objects in bosonic string theory. This problem is solved, once again, if we move to superstring theory where, as we already explained in Sec. 1, D-branes carry conserved charges, by coupling to $p+1$ form fields, ensuring their stability.
If we specialize to type II superstrings, it can be proved by a generalization of the Maxwell theory, that not all values of p are allowed for Dp-branes:

- In type IIA we can only have Dp-branes with p even: $p = 0, 2, 4, 6, 8$. It is customary to leave D8-branes out of this list as they would couple would couple to a nine-form gauge field but such a field is non-dynamical and hence it does not appear in the spectrum.

- In type IIB we can only have Dp-branes with p odd: $p = 1, 3, 5, 7, 9$. Actually it turns out there are also D(-1)-branes, called D-instantons. These objects are localized in time and space and only make sense in the Euclidean theory.

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