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Quantum clocks and the measurability of spacetime distances

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Abstract

Standard quantum mechanics typically assumes the existence of an external, idealized reference frame to define states and observables. However, this "absolute" reference frame may be practically inaccessible or fundamentally unavailable becouse of symmetry principles, for instance general covariance in General Relativity. In such situations, a consistent quantum mechanical description must rely on relational observables, where certain subsystems serve as (quantum) reference frames for the others.

Within the framework of quantum reference frames (QRF), this thesis explores the limitations on measurements of spacetime distances, which are necessary to establish a physical reference frame. First, we investigate how the Heisenberg uncertainty principle imposes a fundamental limitation on Einstein's synchronization of distant quantum clocks, preventing the attainment of perfect synchronization. Then, we investigate the interplay between the precision of a quantum clock and its spatial localization, revealing a fundamental trade-off: improving the precision of time measurements inevitably increases the uncertainty in the clock's position, establishing a limitation in how precisely a quantum clock can serve as a reference for both space and time.

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Chapter 1

Introduction

1.1 Quantum reference frames - overview

The description of any physical system is ultimately tied to a specific reference frame, tipically considered as an abstract mathematical object that can, *in principle*, be realized by some concrete physical system. The prototypical example is an observer in general relativity carrying a clock and three rods along its word-line, establishing a local coordinate frame. When the absence of an "absolute reference frame" is imposed, for instance by a symmetry principle, considering their physical realizations - *material reference frames* - becomes a necessity. Without an absolute description of the system, we can only rely on a *relational* description of its subsystems with respect to one another. When the physical reference frame is a quantum system, it is referred to as a *quantum reference frames*. This scenario arises in various areas of physics and for multiple reasons. Here, we provide some examples.

The most evident examples is **General Relativity**, where "the absence of an absolute frame" is imposed by the symmetries of the theory, specfically the invariance under diffeomorphisms. In this context, space-time coordinates have no intrinsic physical meaning, as they are not directly observable. This idea is captured in Einstein's famous "hole argument", which we briefly review in Appendix (5.A). One way to give physical meaning to space-time events is to define them via "space-time coincidences", which are ultimately relations between physical systems [1]. Therefore, even in the absence of a fixed background, we can still formally define physical observables by treating a physical system as a reference frame, which itself is a dynamical system obeying the equations of the theory [2], [3]. Quantum reference frames thus become important in any non-perturbative approach to quantum gravity, where background independence is taken as fundamental principle to start with [4],[5],[1].

A different context where this operational approach to reference frames is relevant is **Quantum Information** [6],[7]. Here the absence of an external reference system is a practical concern: one must be able to formulate communication tasks between parties that do not share a common reference frames. The prototypical example is a communication of a classical bit between two parties using a spin-$\frac{1}{2}$ system: Alice prepares the system in the state $|z, \pm\rangle$ by a projective measurement along the z axis of her local Cartesian frame and then sends it to Bob. If Bob has no access to the same Cartesian frame, he cannot perform the same projective measurement, causing the communication task to fail. A way circumvent the lack of an external, classical frame is to use another quantum system as a reference: in this scenario, one can transmit the relational information between the two systems, which is *frame invariant*, meaning it is defined independently of any external frame. Within this context, quantum reference frames have been used to address issues related to superselection rules in quantum mechanics—postulated rules that prohibit the preparation of quantum states exhibiting coherence between eigenstates of certain observables. For instance, a superselection rule (SSR) for electric charge asserts the impossibility of preparing a coherent superposition of different charge eigenstates. It has been recognized [8], [7] that these rules are mathematically equivalent to the lack of a reference frame for the group generated by the observables (at least for compact groups, as U(1) for the charge), indicating that in principle all SSRs may be "lifted", once a proper reference frame for the degree of freedom in question is accessible. The common feature of all these applications in Quantum Information is that quantum reference frames, or reference frames in general, serve as a $resource$ to overcome physical restrictions (namely the lack of a frame, or equivalently a SSR), similarly to how entanglement is a resource under the restriction of LOCC (Local Operations and Classical Communications).

Combining these two different approaches, QRF have been used to address foundational issues aimed at removing the background structure inherent in standard quantum mechanics [9], [10]. On the one hand, the motivation stems from General Relativity and the principle of covariance. On the other hand, these external structures require the availability of classical reference frames, which imply a significant amount of resources that, from an operational standpoint, cannot be assumed. Additionally, one might invoke the "universality of quantum theory": since reference frames are physical systems, they should ultimately be subject to the laws of quantum theory. The fundamental question then becomes how the description of quantum systems $^-$ and ultimately of spacetime itself $-$ changes when the resources are limited to other quantum systems. Given that reference frames are such a fundamental concept in General Relativity, closely connected to principles like the equivalence principle and general covariance, quantum reference frames have been proposed to extend these principles to quantum theory [11], [9],[10],[12],[13]. The aim is not to find a theory of quantum gravity, but rather to explore the incompatibility between General Relativity and Quantum Mechanics from a fundamental perspective, seeking to understand which principles can theoretically coexist and which cannot.

More generally, an operational approach to reference frames has always been a tool for addressing problems in the **foundations of physics**. The prototypical example is Einstein's investigation of the concept of time using physical clocks, particularly "light clocks," which led to the development of Special Relativity. After the development of General Relativity on one hand, and quantum mechanics on the other, this operational viewpoint has been adopted to explore the limits of measurability of spacetime intervals and, ultimately, the definition of a physical reference frame. It is in this context that the concept of a $quantum\ clock$ first appeared, in the famous paper by Salecker and Wigner.[14]. The motivation was to study how the classical notion of reference frames $^-$ and ultimately of spacetime $^-$ is modified by quantum effects. This work inspired further investigations on the limitations to the measurability of spacetime distances, particularly in the quantum gravity community. For instance, when the gravitational interaction is included in the analysis, the Planck length emerges as a fundamental limitation to spacetime distances [15], which seems to be a model-independent feature of quantum gravity.

Quantum clocks are fundamental tools for investigating the concept of **time in quantum mechanics**, which is non-trivial even in a non-relativistic theory. The primary challenge is finding a theoretical representation for time measurements, as in general, physical [1] hamiltonians do not admit the existence of a self-adjoint time operator canonically conjugate to them, as first noted by Pauli. The problem becomes even more fundamental when attempting to canonically quantize a generally covariant theory, such as relativistic mechanics or, more importantly, General Relativity. In fact, the hamiltonian of a generally covariant theory is constrained to vanish in the absence of boundaries, leading to the famous Wheeler-de-Witt equation [1],[5]. As a consequence, in the quantum theory, it appears as if one obtains a 'frozen formalism' where physical states (of the spatial geometry and matter) do not evolve in time. The reason is that an external background time is not defined, and time evolution must be extracted relationally, which involves selecting some quantized degrees of freedom $-$a quantum clock $-$ to serve as an internal (quantum) reference frame for time, relative to which the remaining quantum degrees of freedom evolve [3], [16]. Even though a self-adjoint time operator is not physical, time measurements can still be represented as a Positive Operator Valued Measure (POVM) [17], which generalize the notion of quantum observable.

Quantum clocks and quantum reference frames have also been utilized to study phenomena at the interface between General Relativity and quantum mechanics in **experimentally accessible regimes**[2]. An example, is the theoretical study of quantum states of gravitational-source-masses, which are expected to give some informations about the quantum nature of gravity [11], [18]. The motivation for these studies comes from numerous proposals for tabletop experiments to probe the phenomenology of such objects[19],[20], in particular the entanglement induced by the gravitational field. The extension of quantum optomechanics (quantum optical control of solid-state mechanical devices) to levitated solids has opened up new ways of coherently controlling the motion of massive quantum objects in engineered potential landscapes[21], [22], thereby providing opportunities to test these phenomena experimentally. Another example is the study of quantum clocks within the context of General and Special Relativity, which has led to the prediction of novel phenomena involving time dilation in quantum systems [23], [24], and subsequent proposals for tabletop experiments to test these predictions [25].

[1] physical here means that the spectum should be bounded from below

[2] in contrast to many predicted phenomena in the high energy regime of QFT in curved ST, such as the Hawking effect, which seems to be far beyond our current experimental capabilities

For the sake of completeness, it is worth briefly mentioning the growing importance of quantum reference frames (QRFs) in algebraic approaches to Quantum Field Theory (**AQFT**) in curved spacetime (which also impacts perturbative approaches to quantum gravity) and in the **theory of measurements in QFT**. Notably, recent works [26],[27] have demonstrated that properly accounting for the role of a dynamical material frame (also referred to as an "observer") can promote the von Neumann algebra of observables in a given spacetime subregion from Type III to Type II. This advancement allows for the mathematically precise definition of entanglement entropy, a fundamental quantity in studying the connections between geometry and information in quantum gravity. Motivated by these findings, several subsequent studies from different research communities [28],[29] have aimed to further mathematically generalize and operationally/physically understand this topic. More generally, QRF seems to be a fundamental concept towards the developement of a theory of measurements in QFT [28], [30],[31], independently to the presence of constraints given by symmetry principles (such as spacetime symmetries).

1.2 Outline of the thesis

This thesis begins by introducing two fundamental concepts necessary for understanding quantum reference frames (QRFs) and quantum clocks: decoherence in quantum mechanics and covariant POVMs, which generalize the notion of quantum observables beyond self-adjoint operators.

In chapter (3) we introduce the formalism of QRFs. Specifically, we adopt the approach first developed in the context of quantum information theory [7],[32], where a QRF is more naturally understood as a resource to overcome the lack of an external reference frame, since it allows to describe quantum systems in terms of relational observables. After that, following [17], we introduce the formalism of quantum clocks, which are ultimately QRFs for the group of time translations. In particular, we will show how the introduction of a covariant POVM allows to consistently describe time as an observable also for real quantum system, where the hamiltonian is bounded from below. Finally, we present different models of quantum clocks that will be employed thoughout the thesis.

In chapter (4) we introduce the formalism of quantum clocks in general relativity, following [24]. Firstly, we show how a classical composite particle is a physical realization of an "*ideal clock*" in GR, meaning that its internal dynamics evolves with respect to the proper time determined by the background metric, as the "*clock hypothesis*" requires. The analysis is then extended to quantum clocks, and we discuss how this formalism has been used to predict a novel phenomenon at the intersection of quantum mechanics and general relativity: a universal decoherence mechanism due to time dilation in composite quantum systems.

Chapter (5) contains the original contributions of the thesis. In particular, we apply the theory QRFs and of quantum clocks in GR developed in the previous chapters in the context of foundations of physics, in particular to investigate the limitations on ST measurements in quantum mechanics, in the spirit of the early work by Salecker and Wigner [14].

In section (5.1), we investigate the limitations to the synchronization of distant clocks by means of Einstein's synchronization protocol, showing that a perfect synchronization of quantum clocks seems to be in fundamentally unachievable in the quantum regime.

In section (5.2), we investigate how the precision of time measurements in quantum clocks affects their localization in space and vice versa. We show that, considering the most natural choice of states, increasing the precision in time of a quantum clock leads to higher uncertainty in its spatial localization. Using the framework of QRFs developed in chapter (3), we explore the consequencies of this trade-off in defining both space and time coordinates of another quantum system. In particular, we show that the invariant position, which is the physical quantity we can use to describe a quantum system in the absence of an external reference frame, can never be sharply defined. The quantitative results obtained in this work are based on a specific model of the quantum clock. While this might be viewed as a partial result, it provides valuable insights into the interplay between time and space measurements in quantum systems and lays the groundwork for future investigations into more general principles.

1.3 Notation and conventions

States in a Hilbert space $|\psi\rangle \in \mathcal{H}$ are represented using ket notation.

Operators O acting on $\mathcal{H}$ are denoted by capital letters, while the corresponding lowercase letters represent their eigenvalues. In Chapter (4), capital letters are also used for the classical hamiltonian (H) and Lagrangian (L). Given an operator O, the calligraphic font $\mathcal{O}$ represents the corresponding "superoperator," which maps operators to operators, defined as $\mathcal{O}[\cdot] = O \cdot O^\dagger$.

For a group G, any representation acting on a Hilbert space $\mathcal{H}$ is denoted by $U(g)$, where $g \in G$. Elements of a covariant POVM with respect to the group G are denoted by $E(g)$, where $g \in G$. The superscript "tilde" ($\tilde{\cdot}$) is used to indicate G-invariant quantities, such as operators $\tilde{O}$ that satisfy $U(g)\tilde{O}U(g)^\dagger = \tilde{O}$.

When describing composite systems, particularly in Chapter (5.2), internal degrees of freedom are denoted by the subscript "c" (standing for "clock"), while kinematical degrees of freedom are denoted by the subscript "r" (standing for "rod" or "ruler"). For instance, the corresponding Hilbert spaces are denoted by $\mathcal{H}_c$ and $\mathcal{H}_r$, and the quantum states by ρ_r and ρ_c.

Chapter 2

Preliminary tools from Quantum Information Theory

In this chapter, we provide a brief introduction to some fundamental concepts from Quantum Information Theory that will be used throughout the thesis. Specifically, the first section discusses the decoherence mechanism, which is essential both for the theory of quantum reference frames (Chapter 3) and for understanding the time dilation effects in quantum clocks (Chapter 4). The second section offers a short review of the theory of measurements in quantum mechanics to introduce the concept of covariant POVMs, a critical tool for the theory of QRFs and the problem of time in quantum theory.

These sections serve as a basic introduction to the topics. For a more detailed and wide treatment of decoherence in quantum theory we refer to [33], while for POVM and covariant POVMs to [34], [35], [36] and [37].

2.1 Decoherence

Quantum coherence refers to a system's ability to maintain its quantum state and produce interference effects. Decoherence, on the other hand, describes the process by which quantum interference is suppressed due to entanglement with an environment. Essentially, when a quantum system interacts with its environment, its quantum coherence becomes "delocalized" into the entangled system-environment state. This effectively removes the coherence from our observations unless we can access the entire entangled state.

2.1.1 Interference effects and quantum coherence

Coherence between quantum states in superposition allows them to interfere, and this coherence can be quantified by *visibility* in interference experiments. Consider a Mach-Zehnder interferometer composed of two beam splitters and two mirrors (as shown in Fig. 2.A.1). The first beam splitter puts the incoming beam in a general superposition of states $|\gamma_+\rangle, |\gamma_-\rangle$, corresponding to the upper and lower paths, respectively. These can be viewed as forming an orthonormal basis of a two-dimensional Hilbert space:

$$|\psi\rangle = a\,|\gamma_+\rangle + be^{i\phi}\,|\gamma_-\rangle \tag{2.1}$$

where $\phi \in [0, 2\pi]$ and $a, b \in \mathbb{R}$ s.t. $a^2 + b^2 = 1$. The second beam splitter divides the incoming beam into a transmitted and a reflected part that have the same amplitude and a phase difference of $\frac{\pi}{2}$. [1] A brief review of how a generic beam splitter works is presented in appendix 2.A.

[1] The phase shift depends in general on the material. Here we adopt the conventional choice of a beam splitter with symmetrical phase shifts of $\pi/2$

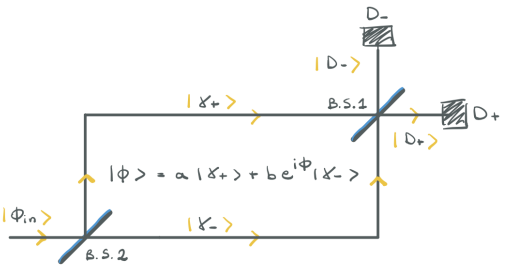

Figure 2.1: *A Mach-Zender interferometer consists of two beam splitters and two detectors $D_\pm$. The incoming wave is split into a superposition of the two paths $|\gamma_\pm\rangle$ and then recombined. The interference pattern is then observed performing a measurement with both detectors.*

Given this setup, the states right after the second beam splitter $\{|D_+\rangle, |D_-\rangle\}$, that correspond to the two possible detected states, can be expressed in terms of this basis as :

$$\begin{cases} |D_+\rangle = \frac{1}{\sqrt{2}}\big(|\gamma_+\rangle + i\,|\gamma_-\rangle\big) \\ |D_-\rangle = \frac{1}{\sqrt{2}}\big(|\gamma_-\rangle + i\,|\gamma_+\rangle\big) \end{cases} \tag{2.2}$$

where $i = e^{i\frac{\pi}{2}}$ is due to the phase shift. The probability of detecting the system in either D_+ or D_- gives us the interference pattern:

$$p_{D_\pm} = |\,\langle\psi|D_\pm\rangle\,|^2 = \frac{1}{2}|\big(a \pm ibe^{i\phi}\big)|^2 = \frac{1}{2} \pm ab\ \sin\phi \tag{2.3}$$

The visibility quantifies the coherence between the states $|\gamma_\pm\rangle$ and is defined as:

$$\mathcal{V} = \frac{max(p_{D_\pm}) - min(p_{D_\pm})}{max(p_{D_\pm}) + min(p_{D_\pm})} = 2ab \tag{2.4}$$

We can see that the visibility measures the contrast of the interference pattern (2.3) and is directly related to the magnitude of the off-diagonal terms of the system's density matrix in the $|\gamma_\pm\rangle$ basis:

$$\rho_\psi = \begin{pmatrix} a^2 & abe^{i\phi} \\ abe^{-i\phi} & b^2 \end{pmatrix} \tag{2.5}$$

This holds for a mixed states as well. In fact, considering a potentially mixed state, described by a positive semi-definite density matrix of the form

$$\rho = \begin{pmatrix} a^2 & ce^{i\phi} \\ c^*e^{-i\phi} & b^2 \end{pmatrix} \qquad \det(\rho) = a^2b^2 - |c|^2 \geq 0, \tag{2.6}$$

the probability to observe the system in a superposition D_+ or D_- is

$$p_{D_\pm} = tr\left[\rho P_{D_\pm}\right] = tr\left[\begin{pmatrix} a^2 & ce^{i\phi} \\ c^*e^{-i\phi} & b^2 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & \pm i \\ \mp i & 1 \end{pmatrix}\right] = \frac{1}{2} \pm |c|\sin(\phi + \alpha) \tag{2.7}$$

where $P_{D_\pm}$ are the projectors onto the basis $|D_\pm\rangle$. The visibility, as defined above, reads

$$\mathcal{V} = 2|c| \tag{2.8}$$

The visibility can also be interpreted as a physical quantity complementary to the *predictability* of the path taken by the system. If visibility quantifies the coherence of the system, predictability quantifies the which-way information (or, more generally, the information about which state the system is in). Consider a scenario where a measurement is performed on the system before the two amplitudes are recombined. The probability of observing the system in either of the two paths is given by:

$$p_+ = tr\left[\rho P_+\right] = a^2 \qquad p_- = tr\left[\rho P_-\right] = b^2 \tag{2.9}$$

where $P_\pm$ is the projector onto the states $|\gamma_\pm\rangle$. The predictability is defined as the difference between these two probabilities

$$\mathcal{P} = |p_+ - p_-| = |a^2 - b^2| \tag{2.10}$$

The complementarity between visibility and predictability is captured by the following relation

$$\mathcal{P}^2 + \mathcal{V}^2 = \left(a^2 + b^2\right)^2 - 4\left(a^2 b^2 - |c|^2\right) = 1 - 4\det(\rho) \leq 1 \tag{2.11}$$

where the disequality follows from the positivity of the determinant (2.6). In the case of pure states, where $det\left(\rho_\psi\right) = 0$, it becomes an equality

$$\mathcal{P}^2 + \mathcal{V}^2 = 1 \tag{2.12}$$

It states that non-zero predictability of the path that has been taken by the system, or "*which-way-information*", necessarily results in a non maximal visibility of the interference pattern. Viceversa, non-maximal which-way-information, guarantees the appearence of an interference pattern.

In simpler terms, when any information about the path is available, the visibility decreases, reflecting a loss of coherence in the system.

2.1.2 Which-path detectors and decoherence

It's important to note that the which-way information doesn't need to be stored directly in the system's degrees of freedom. In most cases, the system interacts with its environment, and the information about the system's state is shared between the system and the environment. This shared which-way information becomes potentially accessible via local measurements on the environment, which acts as a "which-way detector," without necessarily disturbing the system itself. The critical point is that even if this information is not accessed, the visibility will decrease as if it were.

This outcome can be understood through a simple analogy. Suppose Alice and Bob share a pair of entangled systems. According to the quantum mechanical no-signaling theorem, the probabilities for Bob's measurement results are unaffected by whether or not Alice chooses to perform a measurement on her system. Formally, this follows from the fact that all accessible information about a subsystem is entirely described by its reduced density matrix, which remains unchanged if local measurements are performed on other subsystems. In a similar fashion, the pattern of detections on a screen cannot distinguish between mere entanglement with other systems and actual use of those systems to detect which path was taken at the slits.

Environment in a pure state

To see this explicitly we consider the symplest case in which in addition to the system of interest $|\psi\rangle$ we have another system described by the (normalized) state $|w\rangle$, the "which-way-detector" and initially they are uncorrelated:

$$|\Psi\rangle_{sw} = |\psi_0\rangle_s \otimes |w_0\rangle_w, \qquad |\psi_0\rangle_s = a\,|\gamma_+\rangle_s + be^{i\phi}\,|\gamma_-\rangle_s \tag{2.13}$$

Suppose now that the interaction between the two depends on the path $\gamma_\pm$ that is taken. For instance we consider a unitary evolution of the whole system under a coupling-hamiltonian of the form:

$$H_{sw} = H_s \otimes H_w \tag{2.14}$$

such that $H_s\,|\gamma_\pm\rangle_s = \epsilon_\pm\,|\gamma_\pm\rangle_s$. The initial separable state evolves into an entangled state

$$|\Psi(t)\rangle_{sw} = a\,|\gamma_+\rangle_s\, e^{-it\epsilon_+ H_w}\,|w_0\rangle_w + be^{i\phi}\,|\gamma_-\rangle_S\, e^{-it\epsilon_- H_w}\,|w_0\rangle_w \tag{2.15}$$

$$\coloneqq a\,|\gamma_+\rangle_s\,|w_+\rangle_w + be^{i\phi}\,|\gamma_-\rangle_s\,|w_-\rangle_w \tag{2.16}$$

Suppose now we are only able to perform local measurements on our system, while we have no access to the "which-way-detector" $|w\rangle_w$. Formally the system is described by the reduced density matrix

$$\rho_s = tr_w\left[|\Psi\rangle\langle\Psi|_{sw}\right] = \begin{pmatrix} a^2 & abe^{i\phi}\,\langle w_+|w_-\rangle_w \\ abe^{-i\phi}\,\langle w_-|w_+\rangle_w & b^2 \end{pmatrix} \tag{2.17}$$

The interference pattern (2.7) is given by

$$p_{D_\pm} = \frac{1}{2} \pm ab|\,\langle w_+|w_-\rangle_w\,|\,sin\,(\phi+\alpha) \tag{2.18}$$

where α is the contribution to the phase due to the environment

$$\langle w_+|w_-\rangle_w = |\,\langle w_+|w_-\rangle_w\,|e^{i\alpha} \tag{2.19}$$

The visibility (4.56) of the interference pattern is

$$\mathcal{V} = 2ab\,|\,\langle w_+|w_-\rangle_w\,| \tag{2.20}$$

so, ultimaltely, it depends on the distinguishability between the "path-detector" states

$$\mathcal{D} = \sqrt{1 - |\,\langle w_+|w_-\rangle_w\,|^2} \tag{2.21}$$

We can consider the two extreme cases:

- if the two states are orthogonal $\langle w_+|w_-\rangle = 0$, one could in principle extract the full information about which-path has been taken by performing a local measurement in the $\{|w_+\rangle, |w_-\rangle\}$ basis. Correspondingly the state $|\Psi\rangle$ is maximally entangled and the reduced density matrix is maximally mixed.
- On the contrary, if the two states $\{|w_+\rangle = |w_-\rangle = |w_0\rangle\}$ are indistinguishable, performing a local measurement cannot give any information about the path. Correspondingly the state is separable and the reduced density matrix is pure [2].

The *visibility* of the reduced state of the system and the *distinguishability* between the entangled path detector's states resulting from the interaction are complementary quantities. In fact we can write

$$\mathcal{V}^2 + \mathcal{D}^2 = 1 - |\,\langle w_+|w_-\rangle\,|^2(1-4a^2b^2) = 1 - (1-\mathcal{D})^2\mathcal{P}^2 \leq 1 \tag{2.22}$$

where $\mathcal{P}$ is the predictability (2.10). The above result is the generalization of (2.11), which in fact is recovered in the case $\mathcal{D} = 0$, i.e. when the system $|w\rangle$ has no information about the path. In the case $\mathcal{P} = 0$ (for instance when the centre of mass is in a 50:50 superposition of the two paths) (2.22) becomes an equality

$$\mathcal{V}^2 + \mathcal{D}^2 = 1 \tag{2.23}$$

which holds only for pure states of the joint system. The conclusion is that quantum mechanics imposes a complementarity principle between the possibility of observing an interference pattern, quantified by the visibility, and the "which-way-information" that is *potentially available* from any other degree of freedom, regardless if a measurement is actually performed or not.

[2] the $purity$ of a state is quantified by $tr\,[\rho^2] \leq 1$ where the equality holds for pure states.

Environment in a mixed state

In appendix (2.11) we show that for a which path detector in a general mixed state, even if $\mathcal{P} = 0$, the following disequality holds:

$$\mathcal{V}^2 + \mathcal{D}^2 \leq 1 \tag{2.24}$$

Even if no which-way information is available (i.e., $\mathcal{D} = 0$), the visibility may still be less than 1 due to the mixed nature of the environment. For instance, consider an environment initially in a maximally mixed state:

$$\rho_{sw} = \rho_s \otimes \rho_w, \qquad \rho_w = \frac{1}{2}\left(|w_1\rangle\langle w_1| + |w_2\rangle\langle w_2|\right) \tag{2.25}$$

where the sysytem is in the usual coherent superposition of paths $\rho_s = |\psi_0\rangle\langle\psi_0|$. Now, consider the interaction hamiltonian (2.14) with

$$H_w |w_i\rangle = w_i |w_i\rangle, \qquad i = 1, 2 \tag{2.26}$$

The initial separable state evolves into

$$\rho_{sw}(t) = \frac{1}{2}\sum_{i,j=\pm} |\gamma_i\rangle\langle\gamma_j| \frac{1}{2}\left(e^{-\frac{i}{\hbar}t(\epsilon_i-\epsilon_j)w_1} |w_1\rangle\langle w_1| + e^{-\frac{i}{\hbar}t(\epsilon_i-\epsilon_j)w_2} |w_2\rangle\langle w_2|\right) \tag{2.27}$$

Thus, the reduced density matrix of the system reads

$$\rho_s(t) = \frac{1}{2}\sum_{i,j=\pm} |\gamma_i\rangle\langle\gamma_j| \frac{1}{2}\left(e^{-\frac{i}{\hbar}t(\epsilon_i-\epsilon_j)w_1} + e^{-\frac{i}{\hbar}t(\epsilon_i-\epsilon_j)w_2}\right) \tag{2.28}$$

This shows that the off-diagonal terms acquire a phase that is the average of the phases corresponding to the different states in the mixture. In other words, when the environment is in a mixed state, it acts as a classical source of noise, which can cause a reduction in visibility:

$$\mathcal{V} = 2\,|\rho(t)_{i\neq j}| = \left|\frac{1}{2}\left(e^{-\frac{i}{\hbar}t(\epsilon_i-\epsilon_j)w_1} + e^{-\frac{i}{\hbar}t(\epsilon_i-\epsilon_j)w_2}\right)\right| \leq 1 \tag{2.29}$$

Large environments and irreversibility

An important observation is that, in most cases, the environment is composed of a large number of microscopic constituents, which results in the environment having many degrees of freedom. For example, a quantum particle may interact with air molecules or the photons of thermal radiation.

Firstly, let's consider a pure state in the form

$$|\omega_0\rangle = \bigotimes_j |w_{0,j}\rangle_j, \qquad j = 1, ..., N \tag{2.30}$$

where N is the number of constituents. In this case, the visibility of the system would be proportional to

$$\mathcal{V} \sim |\langle w_+|w_-\rangle|^2 = \prod_{j=1}^{N} \left|\langle w_{+,j}|w_{-,j}\rangle_j\right|^2 \tag{2.31}$$

Thus, even if the single constituents cannot carry enough which path information to give decoherence , i.e. their states are almost indistinguishable $\langle w_{+,j}|w_{-,j}\rangle_j \simeq 1$, the presence of a huge number of constituents $N \gg 1$ interacting with the system always leads to decoherence.

Similarly, we can generalize the result we found for the simplest mixed state with only two degrees of dreedom (2.28) to a large environment with N degrees of freedom. For simplicity, we consider equally spaced energies $w_n = nw$, so that

$$\rho_s(t) = \frac{1}{2} \sum_{i,j=\pm} |\gamma_i\rangle\langle\gamma_j| \, \frac{1}{N} \sum_{n=1}^{N} e^{-\frac{i}{\hbar} t(\epsilon_i - \epsilon_j) w n} \tag{2.32}$$

So, in the limit of a very large $N >> 1$, the phases sum up to 0, resulting in a full decoherence of the system.

The only way to restore coherence would be to access all the degrees of freedom of the environment, which is usually impossible. Therefore, in most cases, decoherence is *irreversible for all practical purposes*.

2.1.3 Environment-induced superselection

From the previous analysis, it is clear that the basis in which the system decoheres is determined by the interaction (2.14) with the environment, or more generally, by the unitary evolution of the total system+environment state. Specifically, given a total hamiltonian:

$$H = H_s + H_e + H_{se} \tag{2.33}$$

where H_s and H_e govern the internal dynamics of the system and the environment, respectively, the basis that undergoes decoherence is given by the eigenstates of the part of the hamiltonian that acts on the system. These states are called "*pointer states*".

To understand this in practice, let's consider two scenarios.

Strong interaction - $H \approx H_{se}$

Consider a case where the energy scales associated with the system-environment interaction hamiltonian H_{se} are much larger than those of the self-hamiltonians H_s and H_e, so that the interaction dominates the system's evolution ($H \approx H_{se}$). In this scenario, the pointer states are the eigenstates of the interaction hamiltonian:

$$H_{se} \left(|s_i\rangle_s \otimes |e\rangle \right) = |s_i\rangle_s \otimes \left(H_e^{s_i} |e\rangle \right) \tag{2.34}$$

where $H_e^{s_i} \equiv \mathbb{I}_s \otimes H_e^{s_i}$ acts only on the environment. In fact, if the system before the interaction is in a generic superposition of such states

$$|\psi(0)\rangle_{se} = \left(\sum_i c_i |s_i\rangle_s \right) \otimes |e_0\rangle_e \, , \tag{2.35}$$

after the interaction it becomes an entangled state with the environment

$$|\psi(t)\rangle_{se} = e^{-\frac{i}{\hbar} H_{se} t} |\psi(0)\rangle_{se} = \sum_i c_i |s_i\rangle_s \otimes |e_i(t)\rangle \, , \qquad |e_i\rangle = e^{-\frac{i}{\hbar} H_e^{s_i} t} |e_0\rangle_e \tag{2.36}$$

Decoherence implies that the interference effects given by the coherent superposition of states of the basis $\{|s_i\rangle_s\}$ cannot be (fully) observed by only measuring the system.

In general, it is always possible to write an arbitrary interaction hamiltonian H_{se} in form of a diagonal decomposition of (unitary but not necessarily Hermitian) system and environment operators S_α and E_α

$$H_{se} = \sum_\alpha S_\alpha \otimes E_\alpha . \tag{2.37}$$

It follows that the pointer states are the non-degenerate (simultaneous) eigenstates of all operators S_α acting on the system, such that:

$$S_\alpha \left|s_i\right\rangle_s = \lambda_\alpha^i \left|s_i\right\rangle_s, \qquad \forall i \neq j : \lambda_\alpha^i \neq \lambda_\alpha^j \tag{2.38}$$

In this case any superposition of these states will get entangled with the environment. In fact, we see that (2.36) holds, with

$$\left|e_i(t)\right\rangle = e^{-\frac{i}{\hbar} t \sum_\alpha \lambda_\alpha^i E_\alpha} \left|e_0\right\rangle_e \tag{2.39}$$

Conversely, any set of degenerate eigenstates will generate a "decoherence-free subspace". In fact, given such set $\{\left|s_k\right\rangle_s\}$

$$S_\alpha \left|s_k\right\rangle_s = \lambda_\alpha \left|s_k\right\rangle_s, \qquad \forall k \tag{2.40}$$

then the subspace generated by this set is immune to decoherence, since a superpostition of these states does not get entangled with the environment

$$\left(\sum_k c_k \left|s_k\right\rangle_s\right) \otimes \left|e_0\right\rangle_e \rightarrow \left(\sum_k c_k \left|s_k\right\rangle_s\right) \otimes \left|e(t)\right\rangle_e \tag{2.41}$$

where

$$\left|e(t)\right\rangle_e = e^{-\frac{i}{\hbar} t \sum_\alpha \lambda_\alpha E_\alpha} \left|e_0\right\rangle_e . \tag{2.42}$$

Usually, the interaction term is just a tensor product of two observables. A particularly important example is the position operator X_s

$$H_{se} = X_s \otimes E_e, \tag{2.43}$$

where E_e is some operator actin on the environment's Hilbert space. In this case, the system undergoes decoherence in the position basis[3]. Since the force laws sorrounding tipycally depend on some power of the distance x^{-n}, this case in particularly important and it explains why we are often not able to observe spatial superpositions.

In general, the vast space of possible quantum-mechanical superpositions is reduced so much because the laws governing physical interactions depend on only a few physical quantities (position, momentum, charge, and the like). The appearance of classicality is therefore grounded in the structure of the physical laws governing the system–environment interactions. This process is called "environment-induced superselection".

Weak interaction - $H \approx H_s$

The opposite scenario arises when the scale of energies of the system's hamiltonian H_s is much higher than the scale of energies charachterizing the interaction $H \approx H_s$. In this case the pointer states are the eigenstates of H_s, so the observables that undergo decoherence are the constants of motion.

Notice that, even if the pointer-states are fully determined by the self- hamiltonian, since it completely dominates the evolution of the system, the environment plays the crucial role of continuously monitoring those states, which leads to decoherence.

In the case of nondegeneracy[4], the "pointer-observable" will be simply the energy of the system, thus leading to the environment-induced superselection of energy eigenstates for the system (i.e., eigenstates of the self-hamiltonian of the system).

[3] Since individual eigenstates of the position operator are not proper quantum states of physical objects, in practice one considers as pointer states narrow position-space wave packets

[4] As we highlighted above in eq. (2.41), any set of degenerate eigenstates of the hamiltonian would give rise to decoherence-free subspaces.

Intermediary regime

When the evolution of the system is governed by H_{se} and H_s in roughly equal strengths, the resulting preferred states will represent a compromise between the first two cases. For instance, in the frequently studied model of quantum Brownian motion [33] the interaction hamiltonian $H_{se} \sim X_s$ describes monitoring of the position of the system. However, the intrinsic dynamics induced by $H_s \sim P_s^2$ describes monitoring of the momentum. This combined influence of H_{se} and H_s results in the emergence of preferred states localized in phase space, i.e., in both position and momentum.

In general, the pointer states, which are the states that undergo decoherence, are the states of the system that are most robust under the evolution generated by the total hamiltonian of the system and the environment.

2.1.4 Summary

In conclusion, decoherence is a process by which quantum superpositions are suppressed due to interactions with the environment, resulting in the emergence of classical behavior.

A key concept to understand this process is that of entangled states. In quantum theory, while interactions are always local, the states resulting from the interactions can be non-local, meaning that the physical properties of a subsystem are now at least partially encapsulated in the non-local quantum correlations and therefore cannot be accessed only by local observations on the subsystem. This is a crucial difference with respect to classical mechanics, where the actual "nature" of the system is not modified by external interactions (as far as the perturbations on the system are small enough).

The specific states that undergo decoherence, called pointer states, are determined by the structure of the system-environment interaction and the dynamics of the system itself. This mechanism leads to environment-induced superselection, where certain observables (e.g., position, energy) become "classical" due to their robustness under the system's evolution in the presence of environmental monitoring.

The study of decoherence reveals why, in practice, we observe classical phenomena at macroscopic scales and how quantum information becomes effectively inaccessible once it is delocalized across the system-environment entanglement.

2.2 Covariant POVMs

The role of a quantum measurement on a state $\rho(\alpha)$, where α is a parameter of interest, is to provide a set measurement outcomes $(\chi_1, ..., \chi_N)$ from which one can infere α via classical methods of parameter estimation. Quantum mechanics then imposes fundamental restrictions which involve the parameter α and the operator A that generates the displacement of that parameter, which are known as uncertainty principles.

The standard example is the parameter x which labels the elements of the group of 1D translations and the generator of the group P, which is the self-adjoint momentum operator. In this case, to the parameter x we can associate a self-adjoint operator X and the fundamental limitation imposed by QM is the standard uncertainty principle $\sigma_{X_r}\sigma_{P_r} \geq \hbar/2$, which follows from the commutation relations between the two operators $[X, P] = i\hbar$.

However, a self-adjoint operator associated to the parameter of interest does not always exist. An example is the time t which parametrizes the group of time translations generated by the hamiltonian H. It's well known [38] that there cannot exist a self-adjoint operator T which is canonically conjugate to H (meaning that $[T, H] = i\hbar$), as it happens for position and momentum. In fact, Stone's theorem would imply the hamiltonian to be unbounded, and we know that a physical hamiltonians should be at least bounded from below. Similarly, there exist no self-adjoint operator associated to the phase of a single mode of radiation.

The question becomes then how to represent observables associated to parameters in a more general way and how to optimize them with respect to the fundamental bounds imposed by quantum mechanics. In the case there exists no self-adjoint operator associated to the parameter of interest, these fundamental bounds can be phrased in terms of *generalized-* or *parameter-based uncertainty principles* (see, for instance [36] and [37]). At a qualitative level, they give the limits on the precision to which a parameter - e.g. the elapsed time - may be determined via arbitrary data analysis of arbitrary measurements on N identically preparred

quantum systems.

2.2.1 Projective measurements

If a self-adjoint operator X exists, its eigenvectors form an orthonormal basis for the Hilbert space, and each possible outcome x of that measurement corresponds to one of the vectors $|x\rangle$ comprising the basis. Formally, a measurement associated to the parameter x can be described by means of a Projective Valued Measure (PVM), which is defined as follows: let $((\mathbb{X}), \mathrm{B}(\mathbb{X}))$ be a measurable space, meaning that $\mathbb{X}$ is a non-empty set, which corresponds to the set of possible results x, and $\mathrm{B}(\mathbb{X})$ the Borel σ-algebra of subsets of $\mathbb{X}$. Further, let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. A PVM is a map from the (measurable) space of possible results to the space of operators $P : \mathrm{B}(\mathbb{X}) \to \mathcal{B}(\mathcal{H})$ is defined through the following three measure properties :

1. $P(\mathcal{X})$ is an orthogonal projection for all $\mathcal{X} \subset \mathbb{X}$
2. $P(\mathbb{X}) = \mathbb{I}$ and $P(\emptyset) = 0$
3. σ-additivity: $E(\cup_i \mathcal{X}_i) = \sum_i E(\mathcal{X}_i)$ for any sequence $\{\mathcal{X}_i\}$ of disjoint sets in $\mathrm{B}(\mathbb{X})$
4. $P(\mathcal{X}_1 \cap \mathcal{X}_2) = P(\mathcal{X}_1)P(\mathcal{X}_2)$ for all $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{X}$

The second and the fourth properties show that if $\mathcal{X}_1$ and $\mathcal{X}_2$ are disjoint, the images $P(\mathcal{X}_1)P(\mathcal{X}_2) = P(\mathcal{X}_1 \cap \mathcal{X}_1) = P(\emptyset) = 0$ are orthogonal to each other.

The self adjoint operator X admits a spectral decomposition in terms of orthogonal projectors $P(x)$

$$X = \int P(dx)\; x \coloneqq \int dx\; P(x)\; x \tag{2.44}$$

where each $P(x)$ projects onto the the eigenspace associated to eigenvector $|x\rangle$ with eigenvalue x. The measurement postulate of QM for PVM states that

1. Given a state ρ, and a PVM P, the probability to obtain an outcome $x \in B(\mathbb{X})$ is given by the Born rule

$$p(x|\rho) = tr[P(x)\rho] \tag{2.45}$$

   Notice that the normalization condition $(2.)$ ensures it is a normalized probability distribution, i.e. $\int dx p(x|\rho) = 1$.

2. Given the outcome x, the post-measurement state is

$$\rho'_x = \frac{P(x)\rho P(x)}{p(x|\rho)} \tag{2.46}$$

An important property of PVM is their repeatability. In fact, if we perform the same measurement on the post-measurement state (2.46) we get the same outcome with probability 1

$$p(x|\rho') = tr[P(x)\rho'] = \frac{tr[P(x)\rho]}{p(x|\rho)} = 1 \tag{2.47}$$

and the state is unchanged

$$\rho'_x \to \frac{P(x)\rho' P(x)}{p(x|\rho')} = \rho' \tag{2.48}$$

2.2.2 POVMs

As mentioned earlier, self-adjoint operators associated with certain parameters may not exist. Moreover, many practical measurements are not repeatable, such as measuring the position of a photon using a silvered screen, which destroys the photon in the process. In such cases, a more general framework for measurement is needed.

The most general way to describe a measurement is by a Positive Operator Valued Measure (POVM), which formally is defined as follows: let $((\mathbb{X}), \mathrm{B}(\mathbb{X}))$ be a measurable space, meaning that $\mathbb{X}$ is a non-empty set, which corresponds to the set of possible results, and $\mathrm{B}(\mathbb{X})$ the Borel σ-algebra of subsets of $\mathbb{X}$. Further, let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. A POVM $E : \mathrm{B}(\mathbb{X}) \to \mathcal{B}(\mathcal{H})$ is defined through the following three measure properties :

1. Positivity (hence self-adjointness): $E(\mathcal{X}) \geq 0 \quad \forall \mathcal{X} \in \mathrm{B}(\mathbb{X})$
2. Normalization: $E(\mathbb{X}) = Id$
3. σ-additivity: $E(\cup_i \mathcal{X}_i) = \sum_i E(\mathcal{X}_i)$ for any sequence $\{\mathcal{X}_i\}$ of disjoint sets in $\mathrm{B}(\mathbb{X})$

Essentially, the requirement of $P(\mathcal{X})$ being orthogonal projectors is weakened to $E(\mathcal{X})$ being mere bounded positive operators. Thus, for POVM we still have Born's rule to describe measurement statistics: given a state $\rho \in \mathcal{B}(\mathcal{H})$ and a POVM with elements $E(x)$, the probability associated to a given outcome x is

$$p(x|\rho) = tr[E(x)\rho] \tag{2.49}$$

Then, the normalization property (2.) still guarantees that $p(x|\rho) = tr[E(x)\rho]$ is a normalized probability distribution $\int dg p(x|\rho) = 1$.

However, it there's no rule describing the post-measurement of the system. Hence POVMs can be viewed as a special case of the projective measurement formalism (PVMs), providing the symplest means by which one can study general measurement statistics, without the necessity for knowing the post-measurement state.

Example 2.2.1. Unsharp position measurement A generalization of the PVM measurement $P(x) = |x\rangle\langle x|$ associated to the position operator X is an unsharp measurement of position. One can consider a more general set of POVM elements corresponding to a convolution of the projectors $P(x)$ with some confidence measure μ

$$E^{\mu}(x) = \int d\mu(q) P(x+q). \tag{2.50}$$

For instance, one can choose a gaussian measure parametrized by the width δ:

$$E^{\delta}(x) = \int dq \, \frac{e^{-q^2/\delta^2}}{\sqrt{\pi}\delta} P(x+q). \tag{2.51}$$

Given a gaussian pure state with centre $x = 0$ and width σ

$$\rho = |e_\sigma\rangle\langle e_\sigma| \,, \qquad |e_\sigma\rangle = \int dx \, \frac{e^{-x^2/2\sigma^2}}{\pi^{\frac{1}{4}}\sqrt{\sigma}} |x\rangle \,, \tag{2.52}$$

the Born probability associated to the measurement x reads

$$p(x|\rho) = tr[E^{\delta}(x)\rho] = \frac{e^{-x^2/(\sigma^2+\delta^2)}}{\sqrt{\pi}\sqrt{\sigma^2+\delta^2}}. \tag{2.53}$$

2.2.3 Generalized measurement

Extending axiom (2.55) about the post-measurement states to POVMs is problematic. It is not possible to establish, in practice, in which state the system is after a measurement whose reading is $x \in \mathbb{X}$ if the observable is represented by a POVM $E(x) \in \mathcal{B}(\mathcal{H})$, and without further information. The extra data is assigned by decomposing each POVM element into generally non positive "measuring operators" $M(x) \in \mathcal{B}(\mathcal{H})$

$$E(\mathcal{X}) = M(\mathcal{X})^\dagger M(\mathcal{X}) \tag{2.54}$$

If so, the post-measurement state, if x is the outcome, is supposed to look like

$$\rho'_x = \frac{M(x)\rho M^\dagger(x)}{p(x|\rho)} \tag{2.55}$$

The set $M(\mathcal{X})$, $\mathcal{X} \subset \mathbb{X}$ is called generalized measurement, or weak measurement. The case of PVM is recovered when $E(\mathcal{X}) = M(\mathcal{X}) = P(\mathcal{X})$. In the general case, since $M(\mathcal{X})$ are not required to be positive, there are an infinite number of solutions to (2.54). In fact, given $M(\mathcal{X}) = \sqrt{E(\mathcal{X})}$, then any $M'(\mathcal{X}) \coloneqq VM(\mathcal{X})$ with $V \in \mathcal{B}(\mathcal{H})$ a unitary operator (such that $V^\dagger = V^{-1}$) is also a valid measuring operator. From the physical viewpoint, this implies that there are an infinite many experimental apparatuses that give the same probabilities for the outcomes.

One can show that performing a weak measurement on a system $\mathcal{H}_s$ is equivalent to a unitary evolution of a larger system $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$ and a subsequent projective meauserement on $\mathcal{H}_e$, that can be thought as the environment. Thus, weak measurement are the right framework to use when describing an open system which does not undergo a unitary evolution and a measurement which is not repeatable, hence not projective.

Example 2.2.2. Unsharp position weak-measurement Consider the previous example of an unsharp position measurement (2.2.1): we can decompose it into measuring operators $M^\delta(x) \in \mathcal{B}(\mathcal{H})$ defined as $M^\delta(x) = \sqrt{E(x)}$

$$M^\delta(x) = \int dq \, \frac{e^{-q^2/2\delta^2}}{\pi^{\frac{1}{4}}\sqrt{\delta}} P(x+q) \tag{2.56}$$

Now, given the same pure gaussian state ρ (2.2.1), the Born probability associated to the measurement x reads

$$p(x|\rho) = tr[{M^\delta}^\dagger{}_x M^\delta x \, \rho] = \frac{e^{-x^2/(\sigma^2+\delta^2)}}{\sqrt{\pi}\sqrt{\sigma^2+\delta^2}} \tag{2.57}$$

while the post measurement state reads

$$\rho'_x = \frac{M^\delta(x)\rho M^{\delta\dagger}(x)}{p(x|\rho)} \coloneqq \frac{|e_{\sigma'}\rangle\langle e_{\sigma'}|}{p(x|\rho)} \tag{2.58}$$

where

$$|e_{\sigma'}\rangle = \sqrt{\frac{1}{\sqrt{\pi}\sigma'}} \int dq \, \exp\left\{-\frac{(q - \frac{x\sigma^2}{\sigma^2+\delta^2})^2}{2\sigma'^2}\right\} |q\rangle \tag{2.59}$$

and $\sigma' = \sigma^2\delta^2/(\sigma^2+\delta^2)$. In the limit $\delta \to 0$ we have a sharp measurement, in fact $\sigma' \to 0$ and $|e_{\sigma'}\rangle \to |x\rangle$.

The name weak measurement reflects the fact that, with respect to PVM, they perturb less the system. The price to pay is that the information about the parameter $x \in \mathbb{X}$ that can be extracted is less.

2.2.4 Covariant POVMs

In physics we are mostly interested in the case in which the space of possible results $\mathbb{X}$ is in 1-1 correspondance with the elements of a group G. This happens, for instance, when we look at the transformations generated by a self adjoint operator: from Stone's theorem it follows that any self-adjoint operators J generates a one-dimensional group G whose unitary representation on the Hilbert space we are considering is $U := e^{-igJ}$ for all $g \in G$ [5]. Standard examples include the momentum operator P, which generates the group of spatial translations, and the hamiltonian H, which generates the group of time translations. Their unitary representations are the translation operator $U(x) = e^{-ixP}$ and the evolution operator $U(t) = e^{-itH/\hbar}$, respectively.

More generally, given a group G we consider the space of possible results $\mathbb{X}$ to be a transitive G-space, meaning that the space is endowed with a group action $U : G \times \mathbb{X} \to \mathbb{X}$ that satisfies

1. transitivity: all different configurations are connected by a group element, so that $\mathbb{X}$ is an *orbit* under the action of G $-$ $\mathbb{X}$: $\forall x, y \in \mathbb{X}, \quad \exists g \in G : U(g, x) = y$
2. identity element e $-$ $U(e, x) = x, \quad \forall x \in \mathbb{X}$
3. associativity $-$ $\forall x, y \in \mathbb{X}, \ g \in G, \quad U(h, U(g, x)) = U(gh, x)$

This means there exists a one-to-one map $g \to x_g$, which assigns to each group element a distinct configuration. Notice that, while the identity element e of the group G is uniquely determined, we are free to choose any "origin" $x_e \in \mathbb{X}$ to represent it. Once this choice is made (say, $e \to x_e$), the one-to-one map is realized as the orbit of x_e under the group G:

$$x_g = U(g, x_e) \tag{2.60}$$

Assuming that $\mathcal{H}$ is a unitary representation $U(g)$ of G, then a POVM $E : \mathrm{B}(\mathbb{X}) \to \mathcal{B}(\mathcal{H})$ is *covariant* if

$$U^\dagger(h) E(g) U(h) = E(gh), \quad \forall\, g, h \in G \tag{2.61}$$

This property ensures that the probability distribution does not change in shape if the system we measure is transformed by a group action but is shifted accordingly

$$p(g|h) = tr\left[E(g)\rho_h\right] = tr\left[E(g)U(h)\rho_e U^\dagger(h)\right] = tr\left[E(gh)\rho\right] = p(gh|e) \tag{2.62}$$

A practicle example could be g being the set of possible configurations of a Cartesian QRF, which are rotated by the group $G = SO(3)$. Then $E(g)$ represents a measurement on the QRF whose outcomes give information about the orientation of the QRF. The covariance property tells us that we can equivalently rotate actively the system, or rotate passively the measurement apparatus, obtaining the same probability distribution. Notice that the covariance property is implicit in the case of "sharp observables", i.e. when a self-adjoint operator is associated to the parameter. For instance, the position operator $X = \int dx\, x\, |x\rangle\langle x|$ is covariant under translations

$$e^{iaP} X e^{-iaP} = \int dx\, x e^{iaP} |x\rangle\langle x|\, e^{-iaP} = \int dx\, x\, |x - a\rangle\langle x - a| \tag{2.63}$$

$$= \int dx (x + a)\, |x\rangle\langle x| = X + a \tag{2.64}$$

Given the infinitely many possible covariant observables, the next step is to find the optimal ones in relation to the fundamental bounds imposed by quantum mechanics, or more generally, with respect to some "optimality criteria", which usually vary depending on the purpose of the measurement. Further details on this topic can be found in [36], [37], and related works. Another commonly required property for optimal POVMs is that their elements be one-dimensional projectors (or rank-1) onto generally unnormalized states:

[5] More precisely, $g \in \mathbb{G} \subset \mathbb{R}$ where $\mathbb{G}$ should denote the set of values necessary to parametrize the group. For simplicity, we directly write $\mathbb{G} \equiv G$.

$$E(g) = |\psi(g)\rangle\langle\psi(g)| \tag{2.65}$$

The normalization property of the POVM then implies the states $|\psi(g)\rangle$ to form a complete or an overcomplete basis, depending if they are orthogonal or not.

The general way [39],[7] to construct a covariant POVM is to start from some arbitrary "origin" $E(e)$ and define

$$E(g) = U^\dagger(g)E(e)U(g) \tag{2.66}$$

Requiring the POVM elements to be rank-1 projectors $E(e) = |\psi(e)\rangle\langle\psi(e)|$, the above equation reads

$$E(g) = U^\dagger(g)\,|\psi(e)\rangle\langle\psi(e)|\,U(g) \tag{2.67}$$

The "origin" E_e must be chosen such that the covariance property is satisfied.

Concerning the theory of QRFs, the existance of a covariant POVM allows for a meaningful probabilistic and operational interpretation of G-invariant observables (3.66) and for the definition of a recovery operation (3.3).

Example 2.2.3. Galilei-Group. An important symmetry group for the physics at low energies is the Galilei group of translations and boosts in an Euclidean space. We consider the one dimensional case $Gal(1) \simeq \mathbb{R} \times \mathbb{R}$, where the group is parametrized by two real numbers $(x, v) \in \mathbb{R}^2$. The action of the group is

$$(x, v) \cdot (x', v') = (x + (v + v')t, v + v') \tag{2.68}$$

As a physical system, we consider a single particle with mass m in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$. This system forms a projective unitary representation of the Galilei group. The reason it is a projective representation, rather than a linear one, is that the generator of boosts, $K_m = Pt - mX$, does not commute with the generator of translations, P:

$$[P, K_m] = i\hbar m \tag{2.69}$$

The group action is realized by

$$U_m(x, v) = e^{-\frac{i}{\hbar}(xP + vK_m)} = e^{-\frac{i}{\hbar}(Px_t - mvX)} = e^{\frac{i}{\hbar}(x+vt)\frac{mv}{2}} e^{-\frac{i}{\hbar}x_t P} e^{+\frac{i}{\hbar}vmX} \tag{2.70}$$

where we use the BHC formula and we define $x_t = x + vt$. Thus, composing two Galilei transformations results in

$$U_m(x, v)U_m(x', v') = e^{-\frac{i}{\hbar}m\frac{v'x - xv'}{2}} e^{-\frac{i}{\hbar}\left(P(x_t + x'_t) - m(v+v')X\right)} = e^{-\frac{i}{\hbar}m\frac{v'x - xv'}{2}} U_{m;x+x',v+v'} \tag{2.71}$$

while the action on position and momentum eigenstates reads

$$U_m(x, v)\,|x_0\rangle = e^{\frac{i}{\hbar}mv\left(x_0 + \frac{x}{2} + \frac{vt}{2}\right)}\,|x_0 + x + vt\rangle \tag{2.72}$$

$$U_m(x, v)\,|p_0\rangle = e^{\frac{i}{\hbar}(x+vt)(\frac{mv}{2} - p_0)}\,|p_0 + mv\rangle \tag{2.73}$$

One can construct a covariant POVM starting from a coherent state in phase space, which is a Gaussian state centered in (x_0, p_0)

$$|\psi_{x_0,p_0}\rangle = \int dy\ \psi_{x_0,p_0}(y)\,|y\rangle\,, \quad \psi_{x_0,p_0}(y) = N\exp\left\{-\frac{(y - x_0)^2}{4\sigma^2} + i\frac{p_0\,(y - x_0)}{\hbar}\right\} \tag{2.74}$$

Choosing these states as origin $|\psi(e)\rangle \equiv |\psi_{x_0,p_0}\rangle$, they are transformed into other coherent states with shifted average position and momentum, namely

$$U_{m;x,v}\,|\psi_{x_0,p_0}\rangle = e^{\frac{i}{\hbar}mv\left(\frac{x}{2}+\frac{vt}{2}+x_0\right)}\,|\psi_{x_0+x+vt,p_0+mv}\rangle \coloneqq e^{\frac{i}{\hbar}mv\left(\frac{x}{2}+\frac{vt}{2}+x_0\right)}\,|m;x,v\rangle \tag{2.75}$$

$$U_{m;x',v'}\,|m;x,v\rangle = e^{i\phi(v,v',x,x')}\,|m;x+x',v+v'\rangle \tag{2.76}$$

where $\phi(v,v',x,x')$ is the total phase resulting from composing transformations of the form (2.75). Different configuration states $|m;x,v\rangle$ correspond to gaussians in phase space with the same spread but different "centre" (x_0+x+vt, p_0+mv). It's then clear they cannot be orthogonal

$$\langle m;v,x|m;v'x'\rangle \sim \int dy\ \tilde{\psi}^*_{x_0+x+vt,p_0+mv}(y)\tilde{\psi}_{x_0+x'+v't,p_0+mv'}(y) \neq \delta(x-x')\delta(v-v') \tag{2.77}$$

.
Nevertheless, coherent states are known to form an overcomplete basis. We can check this directly:

$$\begin{aligned}\int dvdx\,|m;v,x\rangle\langle m;v,x| &= \int dvdx\left(\int dydy'\ \tilde{\psi}_{x_0+x+vt,p_0+mv}(y)\tilde{\psi}^*_{x_0+x+vt,p_0+mv}(y')\,|y\rangle\langle y'|\right)\\ &= \int d(x_0)d(p_0)\left(\int dydy'\ \tilde{\psi}_{x_0,p_0}(y)\tilde{\psi}^*_{x_0,p_0}(y')\,|y\rangle\langle y'|\right)\\ &= \frac{N^2}{m}\int dydy'\,|y\rangle\langle y'|\ \int d(x_0)d(p_0)\ e^{-\frac{(y-x_0)^2}{4\sigma^2}}e^{+i\frac{p_0(y-x_0)}{\hbar}}e^{-\frac{(y'-x_0)^2}{4\sigma^2}}e^{-i\frac{p_0(y'-x_0)}{\hbar}}\end{aligned} \tag{2.78}$$

where from the first to the second line we make a change of variables $(x,v)\to(x+vt,mv)$. The integration over p_0 leads to a $2\pi\delta(y-y')$, hence:

$$\int dvdx\,|m;v,x\rangle\langle m;v,x| = \frac{2\pi}{m}\int dy\,|y\rangle\langle y|\ \ N^2\int d(x_0)\ e^{-\frac{(y-x_0)^2}{2\sigma^2}} = \frac{2\pi}{m}\int dy\,|y\rangle\langle y| = \frac{2\pi}{m}\,\mathbb{I} \tag{2.79}$$

Appendix

2.A Beam splitters

At a classical level a beam splitter divides an incoming wave of complex amplitude $\mathcal{E}_i$ [6] into a reflected and a transmitted part, with amplitudes rsp. $\mathcal{E}_r = r\mathcal{E}_i$ and $\mathcal{E}_t = t\mathcal{E}_i$ such that, for a lossless beam splitter, $|r|^2+|t|^2 = 1$. Between the transmitted and the reflected beam there is a relative phase θ, that is constrained by the conservation of energy. To see why let's assume t to be a real number and $r = \rho e^{i\theta}$.

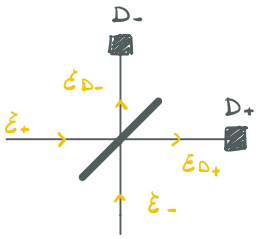

Figure 2.A.1: *Synchronization in the c.o.m. frame*

A beam splitter with 2 incoming beams $\mathcal{E}_+$ and $\mathcal{E}_-$ (as in fig. 2.A.1) can be described by the following matrix equation

$$\begin{pmatrix}\mathcal{E}_{D_+}\\ \mathcal{E}_{D_-}\end{pmatrix} = \begin{pmatrix} t & r'e^{i\theta'}\\ re^{i\theta} & t'\end{pmatrix}\begin{pmatrix}\mathcal{E}_+\\ \mathcal{E}_-\end{pmatrix} \coloneqq \mathbb{M}\begin{pmatrix}\mathcal{E}_{D_+}\\ \mathcal{E}_{D_-}\end{pmatrix} \tag{2.80}$$

Considering a lossless beam splitter, the conservation of energy requires $|\mathcal{E}_+|^2+|\mathcal{E}_-|^2 = |\mathcal{E}_{D_+}|^2+|\mathcal{E}_{D_-}|^2$, or in matrix form:

$$\mathbb{M}^\dagger\mathbb{M} = \begin{pmatrix} r'^2+t^2 & \left(rte^{i\theta'}+r't'e^{-i\theta}\right)\\ \left(r't'e^{-i\theta'}+rte^{i\theta}\right) & r^2+t'^2\end{pmatrix} = \mathbb{I} \tag{2.81}$$

This gives $r'^2+t^2 = 1$ and $r^2+t'^2 = 1$ that, combined with the condition $r^2+t^2 = 1$, gives $r = r'$ and $t = t'$. Therefore the condition on the phase shift is

$$0 = rt\left(e^{i\theta'}+e^{-i\theta}\right) \iff \cos\frac{\theta+\theta'}{2} = 0 \iff \theta+\theta' = \pi \tag{2.82}$$

The specific values of the coefficients and phase shifts depend in general on the material. In the following we assume a 50:50 splitting with $r = t$ and a symmetric phase shift $\theta = \theta' = \pi/2$.

A QFT description is obtained by replacing these complex amplitudes with the corresponding creation operators. The crucial difference is that even the incoming field is in the vacuum, that classically corresponds to $\mathcal{E}_i = 0$, the corresponding creation operator must be taken into account. In fact, consider a single incoming beam $\mathcal{E}_+$. Naively, since $\mathcal{E}_- = 0$, we'd write

[6] we focus on a single plane wave mode of given polarization

$$\hat{a}_{D_+}{}^\dagger = t\hat{a}_+{}^\dagger \qquad \hat{a}_{D_-}{}^\dagger = r\hat{a}_+{}^\dagger \tag{2.83}$$

However this gives the wrong commutation relations

$$\left[\hat{a}_{D_+}, \hat{a}_{D_+}{}^\dagger\right] = |t|^2 \left[\hat{a}_+, \hat{a}_+{}^\dagger\right] = |t|^2 < 1 \tag{2.84}$$

The reason is that the vacuum state of the field $\mathcal{E}_-$ must be taken into account. In fact, the correct description is

$$\begin{cases} \hat{a}_{D_+}{}^\dagger = t\hat{a}_+{}^\dagger + r'\hat{a}_-{}^\dagger \\ \\ \hat{a}_{D_-}{}^\dagger = r\hat{a}_+{}^\dagger + t'\hat{a}_-{}^\dagger \end{cases} \tag{2.85}$$

The commutation rules read

$$\begin{cases} \left[\hat{a}_{D_+}, \hat{a}_{D_+}{}^\dagger\right] = |t|^2 + |r'|^2 \\ \\ \left[\hat{a}_{D_-}, \hat{a}_{D_+}{}^\dagger\right] = tr^* + r't'^* \end{cases} \tag{2.86}$$

and they are preserved if $|t|^2 + |r'|^2 = 1$ and $tr^* + r't'^* = 0$ that, toghether with $|r|^2 + |t|^2 = 1$ implies $|r| = |r'|$ and $|t| = |t'|$. Notice that this is the same condition imposed by the conservation of energy in eq. (2.81). Considering then a 50:50 beam splitter with symmetric phase shifts we have the following relations

$$\begin{cases} \hat{a}_{D_+}{}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}_+{}^\dagger + i\hat{a}_-{}^\dagger\right) \\ \\ \hat{a}_{D_-}{}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}_-{}^\dagger + i\hat{a}_+{}^\dagger\right) \end{cases} \tag{2.87}$$

This permits to compute the probabilities of detecting a given initial quantum state in the detectors. For instance, considering an initial state

$$\hat{a}_+{}^\dagger \left|0\right\rangle \coloneqq \left|\gamma_+\right\rangle \tag{2.88}$$

we obtain

$$\left|\gamma_+\right\rangle = \frac{1}{\sqrt{2}}\left(\hat{a}_{D_+}{}^\dagger - i\hat{a}_{D_-}{}^\dagger\right)\left|0\right\rangle \coloneqq \frac{1}{\sqrt{2}}\left(\left|D_+\right\rangle - i\left|D_-\right\rangle\right) \tag{2.89}$$

that gives 50:50 probability for the two detectors to click. Considering a Mach-Zender interferometer, consisting of two beam splitters and two mirrors, one could obtain the state

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\gamma_+\right\rangle + e^{i\Phi}\left|\gamma_-\right\rangle\right) \tag{2.90}$$

for instance by letting a single-beam go through the first beam splitter. Here Φ denotes any contribtion to the phase difference given by the geometrical setup or different dynamics. Thus, one gets

$$\left|\Psi\right\rangle = \frac{1}{2}\left(\left|D_+\right\rangle\left(1 - ie^{i\Phi}\right) + \left|D_-\right\rangle\left(e^{i\Phi} - i\right)\right) \tag{2.91}$$

that means $p_{D_\pm} = \frac{1}{2}\left(1 \pm \sin(\Phi)\right)$, which coincides with (2.3). In the limit of low-energies, where QFT effects are negligible, one can see the 1 particle sectors $\left\{\left|\gamma_+\right\rangle, \left|\gamma_-\right\rangle\right\}$ as the orthogonal states of a two-dimensional Hilbert space.

2.B Complementarity principle for mixed states of the environment

We consider the system's degrees of freedom to be in a coherent susperposition of the two paths, while the environment is a generic mixed state:

$$\rho(0) = |\psi(0)\rangle\langle\psi(0)|_s \otimes \rho_e \tag{2.92}$$

where

$$|\psi(0)\rangle_s = \frac{1}{\sqrt{2}}\left(|\gamma_+\rangle_s + |\gamma_-\rangle_s\right) \tag{2.93}$$

We are interested in the case in which the evolution is in the form

$$\rho(t) = \frac{1}{2}\sum_{i,j=\pm} |\gamma_i(t)\rangle\langle\gamma_j(t)|_s \cdot \left(U_i(t)\rho_e U_j(t)^\dagger\right) \tag{2.94}$$

The visibility (4.56) is (twice) the absolute value of the non diagonal terms of the reduced density matrix, i.e.

$$\mathcal{V} = |\rho(t)_{i\neq j}| = \left|tr_e\left[U_+(t)\rho_e U_-(t)^\dagger\right]\right| \tag{2.95}$$

and using the spectral decomposition of ρ_e gives

$$\mathcal{V} = \left|\sum_k D_k tr_e\left[U_i(t)\,|k\rangle\langle k|\,U_j(t)^\dagger\right]\right| = \left|\sum_k D_k\,\langle k|\,U_+(t)U_-(t)^\dagger\,|k\rangle\right| \tag{2.96}$$

There are different measures of dinstinguishability for mixed states. The most natural choise is the trace-distance, defined as:

$$\Delta(\rho,\sigma) = \frac{1}{2}tr_e\left[|\rho-\sigma|\right] \tag{2.97}$$

that in the case of $\rho, \hat{\sigma}$ being pure states, it reduces to the usual definition of distinguishability, namely:

$$\mathcal{D}(|\psi\rangle, |\phi\rangle) = \sqrt{1-\left(\langle\psi|\phi\rangle\right)} \tag{2.98}$$

In our case it reads

$$\mathcal{D} = \Delta\left(\rho_e^+(t), \rho_e^-(t)\right) = \frac{1}{2}tr_e\left[\left|U_+(t)\rho_e U + (t)^\dagger - U_-(t)\rho_e U_-(t)^\dagger\right|\right] \tag{2.99}$$

Again it's useful to use the spectral decomposition of ρ_e:

$$\mathcal{D} = \frac{1}{2}tr_e\left[\left|\sum_k D_k\left(U_+(t)\,|k\rangle\langle k|\,U_+(t)^\dagger - U_-(t)\,|k\rangle\langle k|\,U_-(t)^\dagger\right)\right|\right] \tag{2.100}$$

In the sum over k we apply the triangular inequality for trace-class norm

$$tr\left[|A+B|\right] \leq tr\left[|A|\right] + tr\left[|B|\right] \tag{2.101}$$

which gives

$$\mathcal{D} \leq \frac{1}{2}\sum_k D_k tr_e\left[\left|\left(U_+(t)\left|k\right\rangle\!\left\langle k\right|U_+(t)^\dagger - U_-(t)\left|k\right\rangle\!\left\langle k\right|U_-(t)^\dagger\right)\right|\right] = \tag{2.102}$$

$$= \sum_k D_k \Delta\left(\left|k_-(t)\right\rangle\!\left\langle k_-(t)\right|, \left|k_-(t)\right\rangle\!\left\langle k_-(t)\right|\right) \tag{2.103}$$

Finally, we can use the fact that the trace-norm for pure states reduces to (2.98), so that

$$\mathcal{D} = \sum_k D_k \sqrt{1 - \left|\left\langle k\right|U_+(t)^\dagger U_-(t)\left|k\right\rangle\right|^2} \tag{2.104}$$

Now we can look at

$$\mathcal{V}^2 + \mathcal{D}^2 \leq \sum_{jk} D_j D_k \left(\sqrt{1-|u_k|^2}\sqrt{1-|u_j|^2} + \frac{1}{2}\left(u_k u_j^* + u_k^* u_j\right)\right) \leq \sum_{jk} D_j D_k = tr\left[\rho_e\right]^2 = 1 \tag{2.105}$$

where we defined the complex coefficients $u_k = \left\langle k\right|U_+(t)^\dagger U_-(t)\left|k\right\rangle$ which always satisfy $|u_k|^2 \leq 1$.

Chapter 3

Quantum reference frames

In this section we review the theory of quantum reference frames developed firstly in the context of Quantum Information theory [7], [32] based on the so called "*incoherent group average*". This framework is different in many aspects from the so called "perspective-neutral-approach", developed mainly in the context of quantum gravity [17] and foundations of physics [9], that instead makes use of the "coherent group average" to represent the lack of an external frame. This is the most natural approach when dealing with (globally) constrained system obeying a Wheeler-de-Witt-like equation:

$$J\,|\psi\rangle = 0 \tag{3.1}$$

where J can be in general the generator of some symmetry group G, so that the quantum states are invariant under the group action. A noticeable example is canonical General Relativity, since the hamiltonian of a generally covariant theory is constrained to vanish in the absence of boundaries ([1],[5]). In this case, in eq. (3.1) $J \equiv H$ is the hamiltonian of the system, and this reflects the invariance under time reparametrization.

The space of physical pure states $|\tilde{\psi}\rangle$ can be defined via the so called "coherent group averaging"

$$|\tilde{\psi}\rangle = \int_G dg\; U(g)\,|\psi\rangle \tag{3.2}$$

where $U(g) = e^{-igJ}$ is the (unitary) group action on the Hilbert space of the system and dg is the Haar measure. Conceptually, this approach sums over all possible group transformations, or "orientations" of the reference frame, without destroying the system's coherence. This ensures that eq. (3.1) is satisfied, or, equivalently, that physical pure states are *G-invariant*

$$U(g)|\tilde{\psi}\rangle = |\tilde{\psi}\rangle \tag{3.3}$$

The perspective-neutral approach, or Dirac quantization (see e.g. [5]), is physically relevant when $|\tilde{\psi}\rangle$ describes the entire universe, as in general relativity, or when the system's total "charge" (such as total momentum) vanishes with respect to an external observer. For instance, in [40], this formalism is applied to a system of N particles constrained to have zero total linear and angular momentum, reflecting invariance under translations and rotations. This scenario is meaningful either when the N-particle system represents a closed system (the "whole universe") or when the total momentum of the system vanishes with respect to an external observer.

In contrast, the quantum information (QI) approach provides a more general framework where the systems under consideration can be part of a larger environment and may exhibit correlations with it. Instead of focusing on invariant pure states, as in equation (3.1), the QI approach uses invariant density operators $\tilde{\rho}$, which are defined via the "incoherent group averaging":

$$\tilde{\rho} = \int_G dg\; U(g)\rho U^{\dagger}(g) \tag{3.4}$$

so that

$$U(g)\tilde{\rho}U^{\dagger}(g) = \tilde{\rho} \tag{3.5}$$

where $U(g)$ is the group action on the Hilbert space of the system . The key distinction is that eq. 3.4 describes an incoherent mixture of group actions, which does not preserve the coherence of the state ρ in general. In group-theoretic terms, the QI approach does not require specifying the value of the total charge (e.g., total linear or angular momentum). Restricting to the "zero-charge sector", the two approaches are equivalent (this is highlighted, for instance, in [41]).

In the following, we will adopt the quantum information approach, wherein quantum reference frames are treated as a resource to address the lack of information regarding the relationship between a quantum system and the external reference frame used to describe it.

3.1 QRFs as a resource

The concept of reference frame is implicit in the definition of any quantum state. For instance, the wave-function of a quantum particle in the position representation $\psi(x)$ is defined with respect to some Cartesian reference frame, or a single-mode coherent state $|\alpha\rangle$ is defined with respect to some reference frame for the phase $arg(\alpha)$.

We focus the case in which a reference frame can be associated to a group G, meaning that the different configurations of the reference frames can be associated to a group element $g \in G$. For instance a reference frame for the orientation in space $(\hat{x}, \hat{y}, \hat{z})$ can be associated to the group $SO(3)$ of rotations. Moreover, we assume that G can act on the (Hilbert space associated to the) quantum system we are describing via a unitary representation U.

To understand in which sense a reference frame is a resource it's useful to consider a practicle example: a party, say Alice, wants to communicate some information to another party, say Bob, by encoding some information on a quantum system using her local reference frame A and then sending the system to B. If Bob has no access to the same reference frame, the communication is compromised.

To see why, let's consider the case of a spin-$\frac{1}{2}$ system, which is an irreducible representation of the group $SU(2)$, whose elements we denote by Ω. Alice preprares the system in a pure state $\rho = \frac{1}{2}\left(1 + \vec{n}\cdot\vec{\sigma}\right) \in \mathcal{H}_s$, where the vector $|\vec{n}| = 1$ is defined with respect to her local Cartesian reference frame A (and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the 2x2 Pauli matrices). For instance, she performs a projective measurement of the spin along her z-axis, so that ρ will be an eigenstate of the spin-z operator J_z, i.e. $\vec{n} = \pm\hat{z}$. In this way Alice encodes in the system a classical-bit of information $\{+ \equiv yes, - \equiv no\}$. To access this information, one just has to perform the $same$ projective measurement.

Suppose now that Bob has a differenct local Cartesian frame B to define the state of the same system. If the relation between the two frames is known, for instance $\vec{n}' = R(\Omega)\vec{n}$, where $R(\Omega)$ is some rotation matrix, then also the relation between the states is known. Namely, in the frame B the state preparred by Alice will be:

$$\rho' = U(\Omega)\rho U^{\dagger}(\Omega) \coloneqq \mathcal{U}(\Omega)[\rho] \tag{3.6}$$

where now $U(\Omega)$ is the unitary representation of $SU(2)$. This means that Bob can rotate the state, perform a measurement of the spin-z in his frame and extract the BIT. [1]

However, if the relation between the frame is unknown, the state preparred by Alice will be described by Bob as an incoherent mixture of such states

$$\tilde{\rho} = \frac{1}{4\pi}\int_{SU(2)} d\Omega \;\; U(\Omega)\rho U(\Omega)^{\dagger} = \frac{1}{4\pi}\int_{SU(2)} d\Omega \;\; \mathcal{U}(\Omega)[\rho] \tag{3.7}$$

[1] Alternatively, he can perform a "passive" rotation on the measurement apparatus, i.e. on his reference frame, then perform a measurement along the z-axis. If we call the projective measurement $\Pi_{z_A,\pm}$ with respect to Alice's frame, Bob can perform the rotated measurement $\Pi_{z_B,\pm} = U^{\dagger}(\Omega)\Pi_{z_A,\pm}U(\Omega)$

where $d\Omega$ is the invariant Haar measure which is uniform over the group $SU(2)$, hence represents complete unknowledge of the relation between the frames. The normalization factor is given by the volume of the group $|G| = \int d\Omega = 4\pi$. In this scenario, the "physical" states that Alice can prepare in order to communicate to Bob are only states in the form (3.7), which are invariant under the action of the group $SU(2)$, i.e. under any reference frame transformation.

The same reasoning applies when one considers the meaningful local operations that Alice can perform on ρ in order to communicate with Bob. Suppose that Alice applies some operation [2] on her state ρ

$$\mathcal{V}[\rho] = V\rho V^{\dagger} \tag{3.8}$$

which is defined with respect to her frame A. In order to describe the same operation in his local frame B, Bob must know the relation between the two frames. As before, if the frames are related by a rotation $\vec{n}' = R(\Omega)\vec{n}$, he can describe the process by rotating the state ρ' in B to ρ in A, performing the same operation $\mathcal{V}$, and then transform back to his frame B. The resulting state is

$$\mathcal{U}(\Omega)^{\dagger} \circ \mathcal{V} \circ \mathcal{U}(\Omega)\,[\rho] = U(\Omega)^{\dagger}\left[V\left(U(\Omega)\rho U(\Omega)^{\dagger}\right)V^{\dagger}\right]U(\Omega) \tag{3.9}$$

However, if the relation between the two frames is unknown, Bob would describe it as an incoherent mixture of operations of this form, i.e.

$$\tilde{V}[\cdot] = \int_{SU(2)} d\Omega\, \mathcal{U}(\Omega)^{\dagger} \circ \mathcal{V} \circ \mathcal{U}(\Omega)\,[\cdot] \tag{3.10}$$

This means that the only meaningful operations that Alice can perform on ρ in order to communicate with Bob are in the form (3.10), which is invariant under rotations.

This example makes clear that the lack of a reference frame for a group G imposes physical restrictions on states and operations. In particular, the only meaningful states and operations are those invariant under the group action, as it happens in the presence of a symmetry principle. It follows that a reference frame can serve as a resource to overcome these restrictions.

3.1.1 Lack of a reference frame

Now we generalize the previous example. In general, some restrictions on the group G are required. Firstly, as in the previous example we deal with integrations on G, for which we use the Haar measure. For a general group, there exist both a left- and a right- invariant measure, which are unique up to a positive multiplicative constant . In the following, we restrict ourselves to unimodular groups[3], where these two measure coincide $\mu_{dg} \coloneqq dg$, so that

$$\int d(gh) = \int d(hg) = dg \tag{3.11}$$

Then, in order to ensure normalizability of invariant states one should restrict to compact groups, so that $|G| < \infty$. However, as we will see in the next sections, there's often no need to directly deal with invariant states, so this restriction is not strictly necessary. As we will see explicitly, the advantage of compact groups lies in their complete reducibility, which allows for a very simple physical intepretation of the G-twirl operation.

Finally, we assume that the system $\mathcal{H}$ carries a unitary representation U of the group G, defined by:

$$U(g)U(h) = U(gh) \tag{3.12}$$

[2] Formally, quantum operations - or quantum channels - are described by completely-positive-trace-preserving-maps (CPTP). A detailed treatement of this topics can be found in [34].

[3] Most of the Lie groups used in physics are unimodular. For instance, any compact, discrete, Abelian locally compact group is unimodular, as the general- and special- linear group $GL(N)$, $SL(N)$. Thus, unimodularity is not a severe restriction for most physical purposes.

In the absence of a reference frame for the group G, the "physical" states are those invariant under the (unitary) action of the group, i.e. satisfying the following equation:

$$\mathcal{U}(g)\left[\tilde{\rho}\right] = U(g)\tilde{\rho}U^{\dagger}(g) = \tilde{\rho}, \qquad \forall g \in G. \tag{3.13}$$

Mathematically, they are represented by an *incoherent group average* over all group elements, or "*G-twirl*", of any state ρ

$$\tilde{\rho} = \frac{1}{|G|}\int_G dg\; U(g)\rho U^{\dagger}(g) \coloneqq \mathcal{G}[\rho], \tag{3.14}$$

where $|G|$ is the volume of the group, which ensures $tr[\tilde{\rho}] = 1$. In fact, using the invariance of the group measure (3.11)

$$\mathcal{U}(g)\left[\tilde{\rho}\right] = \frac{1}{|G|}\int_G dh\; U(g)U(h)\rho U^{\dagger}(h)U^{\dagger}(g) = \frac{1}{|G|}\int_G dh\; U(gh)\rho U^{\dagger}(gh) = \frac{1}{|G|}\int_G dh\; U(h)\rho U^{\dagger}(h) = \tilde{\rho}, \tag{3.15}$$

hence (3.13) is satisfied. Moreover, it's clear that the G-twirl operation is a projection, meaning that $\mathcal{G}\circ\mathcal{G} = \mathcal{G}$ [4]:

$$\mathcal{G}\circ\mathcal{G}[\cdot] = \frac{1}{|G|^2}\int_G dg\;\int_G dg\; U(g)U(h)[\cdot]U^{\dagger}(h)U^{\dagger}(g) = \frac{\int_G dg}{|G|}\;\frac{1}{|G|}\int_G dg\; U(g)[\cdot]U^{\dagger}(g) = \mathcal{G}. \tag{3.16}$$

Similarly, physical operations are those commuting with the group operations $\mathcal{U}$

$$\mathcal{U}^{\dagger}\circ\tilde{\mathcal{E}}\circ\mathcal{U}[\cdot] = \tilde{\mathcal{E}}[\cdot], \qquad \forall g \in G. \tag{3.17}$$

Mathematically, they are represented by the "$super - G - twirl$":

$$\tilde{\mathcal{E}}[\cdot] = \int_G dg\,\mathcal{U}^{\dagger}(g)\circ\mathcal{E}\circ\mathcal{U}(g)[\cdot] \tag{3.18}$$

The extension to superoperator allows to reach immediately a similar conclusion about which measurements can be considered "physical". The most general measurement on a quantum system is represented by a POVM (see 2.2.2) , which is a set of positive operators $\{E_k\}$ satisfying $\sum_k E_k = I$ [5]. The probability that a measurement on a state ρ has the outcome k is $p_k = tr[\rho E_k]$. We can always associate the POVM to a set of operations $\{\mathcal{E}_k\}$ such that $E_k = {\mathcal{E}^{\dagger}}_k[I] = {M_k}^{\dagger}M_k$, where M_k is a generalized measurement associated to the POVM (see 2.54). We can now apply the previous result to the superoperators ${\mathcal{E}^{\dagger}}_k[I]$. In absence of a reference frame for the group G, it must be represented by

$$\tilde{\mathcal{E}}^{\dagger}{}_k[I] = \int_G dg\,\mathcal{U}^{\dagger}(g)\circ{\mathcal{E}^{\dagger}}_k\circ\mathcal{U}(g)[I] = \int_G dg\,\mathcal{U}^{\dagger}(g)\circ{\mathcal{E}^{\dagger}}_k[I] \tag{3.19}$$

It follows that the physically meaningful measurements are of the form

$$\tilde{E}_k = \int_G dg\,\mathcal{U}^{\dagger}(g)[E_k] = \int_G dg\; U(g)^{\dagger}E_kU(g), \tag{3.20}$$

which ensures the POVM to be G-invariant.

To sum up, if the description of a system lacks of a reference frame associated to a group G, the state of the system, operations and measurements on the system must be represented in the form (3.14), (3.18) and (3.20) respectively, which are invariant under the group action.

[4] One can also show that the G-twirl is a CPTP map [7], hence a proper quantum operation.

[5] Without loss of generality, we consider here measurements defined on discrete values spaces. In (2.2.2) more detailed definitions are given in the case of continuous values spaces and possibly infinite dimensional input- and output systems.

3.1.2 Relational degrees of freedom

From the previous section, it is clear that the lack of a reference frame imposes strong restrictions on the physical degrees of freedom of a system, where "physical" refers to degrees of freedom accessible via local measurements.

To understand precisely what these physical degrees of freedom are, it's useful to analyze the G-twirling operation from a group-theoretical perspective. In the following, we will present the main results for compact groups, where the complete reducibility allows for a straightforward analysis.

In general, since the Hilbert space of the system is a unitary representation of a compact group G, it can be decomposed as a direct sum of the so called "charge-sectors" $\mathcal{H}_q$, labelled by some index q, each of which carries an inequivalent irreducible representation of the group

$$\mathcal{H} = \bigoplus_q \mathcal{H}_q \tag{3.21}$$

Each charge sector can be further decomposed into a tensor product

$$\mathcal{H}_q = \mathcal{M}_q \otimes \mathcal{N}_q \tag{3.22}$$

of a subsystem $\mathcal{M}_q$, called *gauge* space or *color* space, on which the group acts irreducibly, and a subsystem $\mathcal{N}_q$ on which the group acts trivially. The subsystem $\mathcal{N}_q$ is called *multiplicity*- or *flavour space* and its dimension correspond to the number of times the same irreducible representation q appears in the decomposition.

Intuitively, the $gauge$ degrees of freedom in $\mathcal{M}_q$ are defined with respect to an external reference system, and they change under reference frame transformations (group actions). In contrast, the $multiplicity$ degrees of freedom do not. These degrees of freedom can be thought of as relational, meaning they describe relationships between subsystems of the physical system, and thus, they are invariant under reference frame transformations.

This becomes clear with some concrete examples. Let's consider the group $SU(2)$, whose irreducible representations, labelled by an half-integer number j, are $2j+1$ dimensional spaces spanned by the common eigenbasis $\{|j,m\rangle\}_{m=-j}^{j}$ of one of the three generators of the group, say J_z, and the Casimir J^2.

Example 3.1.1. - Two spin 1/2. A composite system of two spin-$\frac{1}{2}$ particles can be decomposed in the triplets- and singlet-spaces $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ which are respectively a three- and one-dimensional irreducible representation labelled by the total spin of the system, which is an invariant quantity under the action of the group. The Hilbert-space decomposition reads

$$\mathcal{H} = \left(\mathcal{H}^{\frac{1}{2}}\right)^{\otimes 2} = \mathcal{M}_1 \oplus \mathcal{M}_0 \tag{3.23}$$

In this case each irreducible representation $\mathcal{M}_j$, labelled by the total spin $j = 1, 0$, appears in the decomposition with multiplicity 1, hence the multiplicity spaces $\mathcal{N}_j = \mathbb{C}$ are 1 dimensional and can be omitted.

Example 3.1.2. - Three spin 1/2. A composite system of 3 three spin-$\frac{1}{2}$ particles can be decomposed as $\frac{1}{2} \otimes \left(1 \oplus 0\right) = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$. The corresponding Hilbert-space decomposition reads

$$\mathcal{H} = \left(\mathcal{H}^{\frac{1}{2}}\right)^{\otimes 3} = \mathcal{H}_{\frac{3}{2}} \oplus \mathcal{H}_{\frac{1}{2}} = \mathcal{M}_{\frac{3}{2}} \oplus \left(\mathcal{M}_{\frac{1}{2}} \otimes \mathbb{C}^2\right) \tag{3.24}$$

In this case the irreducible representation $j = \frac{1}{2}$ appears in the decomposition twice, hence the multiplicity space is 2-dimensional $\mathcal{N}_{\frac{1}{2}} = \mathbb{C}^2$. A basis for the space $\mathcal{H}_{\frac{1}{2}}$ is $|j,m,\lambda\rangle = \left|\frac{1}{2}, \pm\frac{1}{2}, \pm 1\right\rangle$, where λ is the multiplicity-index. The action of the group SU(2) corresponds to a rotation of the representation with respect to some reference frame. This means that what is physically transformed are the eigenvalues m of J_z, which correspond to the projections of the angular momentum along a specific axis of the reference

frame. Since m depends on the external reference frame, it corresponds to the $gauge$ degree of freedom. On the contrary, the $multiplicity$ degrees of freedom λ are not modified by a physical rotation. In other worlds, the group $SU(2)$ acts irreducibly on the space $\mathcal{M}_{\frac{1}{2}}$, i.e. mixing the states $\{\left|\frac{1}{2}, m\right\rangle\}$, while it acts trivially on the space $\mathcal{N}_{\frac{1}{2}}$, i.e. leaving the states $\left|\frac{1}{2}, \lambda\right\rangle$ untouched.

Given this decomposition of a generic Hilbert space, the G-twirl takes a particularly simple form that clarifies its physical meaning. We state it as a theorem and the proof can be found in [7]:

Theorem 1. The action of the G-twirl $\mathcal{G}$ in terms of the decomposition
is given by

$$\mathcal{G} = \sum_q \left(\mathcal{D}_{\mathcal{M}_q} \otimes \mathcal{I}_{\mathcal{N}_q}\right) \circ \mathcal{P}_q \tag{3.25}$$

where $\mathcal{P}_q$ is the superoperator associated with projection into the charge sector q, that is, $\mathcal{P}_q[\rho] = \Pi_q \rho \Pi_q$ with Π_q the projection onto $\mathcal{H}_q = \mathcal{M}_q \otimes \mathcal{N}_q$, while $\mathcal{D}_{\mathcal{M}}$ denotes the trace-preserving operation that takes every operator on the Hilbert space $\mathcal{M}$ to a constant times the identity operator on that space - this corresponds to complete decoherence (see 2.1). $\mathcal{I}$ denotes the identity map over operators in the space $\mathcal{N}$.

The operation $\mathcal{G}$ thus induces complete decoherence on the subspace $\mathcal{M}_q$, while acting as the identity on the decoherence-free - or multiplicity - subspace $\mathcal{N}_q$. Intuitively, the $gauge$ degrees of freedom are tied to the external reference frame and are lost when the frame is absent, leading to decoherence, while the relational degrees of freedom (in $\mathcal{N}_q$) remain accessible.

Consider a state $|q; m, \lambda\rangle$, where q is the total charge, and m and λ denote the $gauge$ ($\mathcal{M}_q$) and multiplicity ($\mathcal{N}_q$) degrees of freedom, respectively. If Alice prepares the subsystem $\mathcal{M}_q$ in a coherent superposition of basis states, say $\sum_m c_m |q; m, \lambda\rangle$, which are defined with respect to her local reference frame, Bob will not be able to access this superposition state unless the reference frame is shared. From Theorem (1), the absence of a reference frame implies that Bob will represent the subsystem $\mathcal{M}_q$ as a maximally mixed state (i.e. in the form $\sim \sum_m |q; m, \lambda\rangle\langle q; m, \lambda|$), corresponding to complete ignorance. However, Alice is free to prepare the subsystem $\mathcal{N}_q$ in any state, as this subspace is insensitive to the frame orientation (group action).

Again, let's consider the practicle example of the group $SU(2)$ of rotations, whose elements are denoted by Ω.

Example 3.1.3. - One transmitted QBIT. If the system is a single spin-$\frac{1}{2}$, $\mathcal{H} = \mathcal{M}_{\frac{1}{2}}$ is already an irreducible representation of the group. The $SU(2)$-$twirl$ is equivalent to the completely depolarizing operation

$$\mathcal{G} = \mathcal{D}_{\frac{1}{2}} \tag{3.26}$$

This means that whatever state ρ is preparred by Alice with respect to her local reference frame for the orientation, Bob will receive a completely mixed state

$$\tilde{\rho} = \mathcal{G}[\rho] = \frac{1}{2}\mathbb{I} \tag{3.27}$$

which means he cannot infer anything about the state preparred by Alice from the outcome of any measurement. In other words, Alice cannot communicate any information to Bob by using a single QBIT.

Example 3.1.4. - Two transmitted QBITS. If the system is composed by two spin-$\frac{1}{2}$ particles, it's Hilbert space can be decomposed as

$$\mathcal{H} = \left(\mathcal{H}^{\frac{1}{2}}\right)^{\otimes 2} = \mathcal{M}_1 \oplus \mathcal{M}_0 \tag{3.28}$$

where the irreducible representations correspond to the two possible values of the total angular momentum $j = 0, 1$, which is a quantity that does not depend on the orientation. Whatever state ρ is preparred by Alice with respect to her local frame, the state received by Bob will be

$$\tilde{\rho} = \mathcal{G}[\rho] = p_{j=1}\big(\frac{1}{3}\Pi_{j=1}\big) + p_{j=1}\big(\frac{1}{3}\Pi_{j=1}\big) \tag{3.29}$$

This means that if Alice prepares the state in some coherent superposition of the triplet states $\{|J, M\rangle = |1, \{0, \pm 1\}\rangle\}$, Bob cannot infer anything about it, since the basis states are only defined with respect to her frame. The subspace $\mathcal{M}_1$ undergoes a complete decoherence. However the value of the total spin J does not depend on the reference frame. Hence Bob is able to distinguish between the triplet and singlet subspaces $|1, \{0, \pm 1\}\rangle \in \mathcal{M}_1$ and $\mathcal{M}_0 = |0, 0\rangle$, by performing a projective measurement onto the two subspaces $\{\Pi_1, \Pi_0\}$. Thus, Alice can encode a classical bit of information to Bob by using two QBITS.

Example 3.1.5. - Three transmitted QBITS. If the system is composed by three spin-$\frac{1}{2}$ particles, it's Hilbert space can be decomposed as

$$\mathcal{H} = \left(\mathcal{H}^{\frac{1}{2}}\right)^{\otimes 3} = \mathcal{H}_{\frac{3}{2}} \oplus \mathcal{H}_{\frac{1}{2}} = \mathcal{M}_{\frac{3}{2}} \oplus \left(\mathcal{M}_{\frac{1}{2}} \otimes \mathbb{C}^2\right) \tag{3.30}$$

Whatever state ρ is preparred by Alice with respect to her local frame, the state received by Bob will be

$$\tilde{\rho} = \mathcal{G}[\rho] = p_{j=\frac{3}{2}}\big(\frac{1}{4}\Pi_{j=\frac{3}{2}}\big) + p_{j=\frac{1}{2}}\big(\frac{1}{2} I_{\mathcal{M}_{\frac{1}{2}}} \otimes \rho_{\mathcal{N}_{\frac{1}{2}}}\big) \tag{3.31}$$

where $\rho_{\mathcal{N}_{\frac{1}{2}}} = \frac{1}{p_{j=\frac{1}{2}}} tr_{\mathcal{M}_{\frac{1}{2}}}\left[\Pi_{\frac{1}{2}} \rho \Pi_{\frac{1}{2}}\right]$ is the reduced density matrix of the invariant subsystem $\mathcal{N}_{\frac{1}{2}}$.

Again, if Alice prepares the system in any superposition within the *gauge subsystems* $\mathcal{M}_{\frac{3}{2}}$ and $\mathcal{M}_{\frac{1}{2}}$, Bob will not be able to infer anything about it. However, as before, since the total angular momentum is a gauge-invariant quantity, Bob can distinguish if Alice prepared the system in a spin $j = \frac{3}{2}$ or $j = \frac{1}{2}$ state, which corresponds to a classical bit. The difference here is the presence of the decoherence-free subsystem $\mathcal{N}_{\frac{1}{2}}$. Whatever state $\rho_{\mathcal{N}_{\frac{1}{2}}}$ Alice prepares, it is frame-independent, so Bob can access all the information Alice stores in it. Since $\mathcal{N}_{\frac{1}{2}}$ is a 2-dimensional subspace, it can store another classical bit. Thus, Alice can encode two classical bits of information using three qubits.

To sum up, every quantum state is defined with respect to some reference frame, which in turn is associated with a group G. However, not all the information about the system depends on the reference frame. Some quantities, like the total charge q, are invariant under group actions (i.e., gauge-invariant). Additionally, for each charge q, there is a subspace $\mathcal{N}_q$ of the Hilbert space $\mathcal{H}_q$, called the $multiplicity$ space, which is also invariant under group action. Thus, any state within this subspace is independent of the reference frame.

In conclusion, even if a reference frame for G is not accessible, the system can still contain meaningful information, i.e., information accessible via local measurements. This meaningful information is independent of the reference frame and can be seen as the *relational information* about the constituents or subsystems of the physical system.

3.2 Perfect and unperfect QRFs

To overcome the lack of a shared frame, Alice can use another quantum system $\mathcal{H}_r$ as a reference token, or quantum reference frame (QRF), which she also sends to Bob. The role of the QRF is to simulate Alice's classical reference frame. Its states $\rho_g \in \mathcal{B}(\mathcal{H}_r)$ should represent the different possible configurations $g \in G$. If this is achieved, the degrees of freedom that were previously considered $gauge$ (defined only relative to an external, inaccessible frame) become $relational$, as they can now be defined with respect to the states of the QRF, which Bob can access.

Before discussing how this procedure works practically, we must first understand how an idealized classical frame can be represented by a quantum system. The first step is to specify what we mean by an idealized classical frame.

3.2.1 Ideal reference frame

Intuitively, what we usually mean by reference frame is an idealized system that *maximally breaks the symmetry* of G. Formally, if $\mathbb{X} \neq \emptyset$ is the set of possible configurations of the system, one requires that

1. $\mathbb{X}$ is endowed with a group action $U: \ G \times \mathbb{X} \ \rightarrow \mathbb{X}$ which is transitive and free
2. $\mathbb{X}$ is isomorphic to the group G, so there exists a 1-1 and invertible map $g \rightarrow x_g$ that to each group element assigns a distinct configuration.
3. these configurations are physically distinguishable

A transitive action implies that every configuration is connected to another by a group element, so $\mathbb{X}$ forms an $orbit$ under the action of G:

$$transitive: \quad \forall x, y \in \mathbb{X}, \quad \exists g \in G: \ U(g, x) = y \tag{3.32}$$

while a free action free means that this group element is unique or, equivalently, that there are no fixed points

$$free: \quad \forall g \in G, \ g \neq e, \ x \in \mathbb{X}, \quad U(g, x) \neq x \tag{3.33}$$

If these conditions are met, an isomorphism $\mathbb{X} \simeq G$ can be defined. Notice that, while the identity element e of the group G is uniquely determined, we are free to choose any "origin" $x_e \in \mathbb{X}$ to represent it. Once this choice is made (say, $e \rightarrow x_e$), the isomorphism is realized as the orbit of x_e under the group G:

$$x_g = U(g, x_e) \tag{3.34}$$

Example 3.2.1. - Cartesian frame. A Cartesian frame provides a practical example. Its configurations form a homogeneous $E^+(3)$-space for the group $E^+(3)$ of rotations and translations. Here, any configuration can be chosen as the identity, x_e, and other configurations are related by transformations $gx_e = x_g$.

The properties of an ideal reference frame are challenging to reproduce using a physical system, even classically. Real configurations are often not perfectly distinguishable due to finite experimental resolution or intrinsic limitations (such as non-orthogonality of quantum states). Additionally, the configuration space $\mathbb{X}$ is often "smaller" than the group G, so that the map $\mathbb{X} \rightarrow G$ may not be invertible. This implies that the action of G may be transitive but not free.

Example 3.2.2. - Periodic clock. A real clock used as a reference for the group of time translations provides a relevant example. On the one hand, real clocks always have a finite resolution, hence they cannot distinguish between all time states t with infinite precision. Additionally, they are usually periodic (e.g. 12 hours for standard clocks), so that transformations of the form $t \rightarrow t + n(12h)$ with $n \in \mathbb{Z}$ act as the identity. This reflects the presence of a non-trivial isotropy subgroup, meaning that the group action is not free.

3.2.2 Distinguishability - covariant POVM

Now, we can characterize a quantum reference frame, which is, in essence, a physical quantum system. The primary differences between a quantum and classical system are that the possible configurations are elements of the Hilbert space $\mathcal{H}$ and that the distinguishability between these configurations is defined by the distinguishability of quantum states.

Operationally, the distinguishability of quantum states is described by the Born rule. Thus, to define a quantum reference frame, we require a covariant POVM (details in 2.61), associated with group elements $E(g) \in \mathcal{B}(\mathcal{H}_r)$, such that the probability of measuring g for a state ρ_r is given by:

$$tr_r \left[\rho_r E(g)\right] = p(g) \tag{3.35}$$

In the next section we will see that a covariant POVM allows for a meaningful operational interpretation of G-invariant observables.

Now, consider a 1-1 map between G-configurations and quantum states $g \to \rho_g$, which takes the place of the idealized configuration state $x_g \in \mathcal{X}$. The distinguishability between different configuration states is given by the Born probability

$$tr_r[E(h)\rho_g] = p(h,g) \tag{3.36}$$

Perfect distinguishability corresponds to a probability distribution which is sharply peaked around g, i.e $p(h,g) = \delta(gh)$ [6].

3.2.3 Perfect QRFs

A perfect quantum reference frame for a group G is a quantum system $\mathcal{H}$ carrying a unitary representation U that is able to reproduce all the properties of an idealized classical frame. Specifically, it satisfies the following criteria:

1. the group G is isomorphic to a set of states[7] $\{\rho_g, g \in G\} \subseteq \mathcal{H}$, so there exists a 1-1 and invertible map $G \ni g \to \rho_g \in \mathcal{H}$ that to each group element assigns a distinct quantum state
2. the action U on these states is transitive and free: $\quad \forall g, g' \in G, \exists!\ h \in G: \quad \rho_g = U(h)\rho_{g'}U^\dagger(h)$
3. the configuration-states are perfectly distinguishable: one can define a covariant POVM $E : G \to \mathcal{B}(\mathcal{H})$ such that $tr[E(g)\rho_h] = \delta(gh) \quad \forall g, h \in G,\ g \neq h$

The delta function on G is defined by $\int dg\delta(g)f(g) = f(e)$, where e is the identity element and f any continuous function. In this case, the set of configuration-states $\{\rho_g,\, g \in G\}$ constitutes a principle homogeneous G-space, exactly as the space of configurations $\mathbb{X}$ of an idealized frame.

Since in physics we are mostly interested in Lie groups, we usually need an infinite and uncountable set of orthogonal states. In practice, a perfect QRF is often realized as the space of square-integrable functions over G, i.e., $\mathcal{H} = \mathcal{L}^2(G, dg)$, where dg is the Haar measure of the group. Any state in this space can be expanded in the orthonormal basis $|g\rangle$

$$|\chi\rangle = \int_G dg\ \chi(g)\,|g\rangle \tag{3.37}$$

on which the group acts regularly[8]

$$U(h)\,|g\rangle = |hg\rangle\,, \qquad U(h)\,|\chi\rangle = \int_G dg\ \chi(h^{-1}g)\,|g\rangle \tag{3.38}$$

Ultimately, the resolution of the identity

$$\int_G dg\ \ |g\rangle\langle g| = \mathbb{I} \tag{3.39}$$

naturally defines a covariant POVM 2.2 whose elements are

$$E(g) = U^\dagger(g)\,|e\rangle\langle e|\,U(h) = \left|g^{-1}\right\rangle\left\langle g^{-1}\right|, \qquad \int_G dg\ E(g) = \mathbb{I} \tag{3.40}$$

In fact, they satisfy the covariance property (2.61)

$$U^\dagger(h)E(g)U(h) = U^\dagger(h)\left|g^{-1}\right\rangle\left\langle g^{-1}\right|U(h) = \left|h^{-1}g^{-1}\right\rangle\left\langle h^{-1}g^{-1}\right| = E(gh) \tag{3.41}$$

[6] or δ_{gh} in the case of a discrete group.

[7] In general, these states could be not proper vectors in the Hilbert space, but distributions in the sense of rigged Hilbert spaces [35], as in the example below. To keep this exposition as simple as possible, we will omit these technical details.

[8] Here we conventionally choose the left-regular action, however choosing the right-regular action leads to the same results.

Example 3.2.3. A practicle example is a single particle $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ as QRF for the one-dimensional translation group in space. One can define an isomorphism between configurations and states $x \to |x\rangle$. The unitary action of the group is generated by the momentum operator P and it corresponds to a (left-)regular action

$$U_x = e^{-\frac{i}{\hbar}xP} |x_0\rangle = |x_0 + x\rangle \tag{3.42}$$

The configuration states form a complete basis

$$\int dx \ |x\rangle\langle x| = \mathbb{I} \tag{3.43}$$

so one can construct a covariant POVM out of $E_0 = |0\rangle\langle 0|$, whose elements are projectors over the position eigenbasis $E(x) = U^\dagger(x) E_0 U(x) = |-x\rangle\langle -x|$

3.2.4 Imperfect QRFs

As noted in the definition of an ideal reference frame (3.2.1), a reference frame typically fails to be perfect in two primary ways: either the action of G on the space of configurations is not free, or the different configurations are not physically distinguishable. An imperfect QRF is therefore defined as a quantum system $\mathcal{H}$ that carries a unitary representation U of G and satisfies the following properties:

1. there exists a 1-1 map $\quad G \ni g \to \rho_g \in \mathcal{H}$ between the group G and a set of quantum states;
2. the action U on these states is transitive (but not necessarily free): $\quad \forall g, g' \in G, \exists\, h \in G : \quad \rho_g = U(h)\rho_{g'}U^\dagger(h)$;
3. the configuration-states are not perfectly distinguishable; nevertheless one can define a covariant POVM $E : G \to \mathcal{B}(\mathcal{H})$ such that $\quad tr[E(g)\rho_h] = p(g,h) \ \ \forall g, h \in G$, where $p(g,h)$ is some probability distribution on G.

The most common example of an imperfect QRF is a quantum system whose configuration states are coherent states $\rho_g = |\psi(g)\rangle\langle\psi(g)|$. The group still acts regularly

$$\mathcal{U}_h[\rho_g] = U(h) \, |\psi(g)\rangle\langle\psi(g)| \, U^\dagger(h) = |\psi(hg)\rangle\langle\psi(hg)| = \rho_{hg} \tag{3.44}$$

however the states are not orthogonal. Nevertheless, they form a overcomplete basis

$$\frac{1}{\mu}\int_G dg \ |\psi(g)\rangle\langle\psi(g)| = \mathbb{I} \tag{3.45}$$

where μ is some normalization constant [9]. This naturally defines a covariant POVM whose elements are:

$$E(g) = U^\dagger(g) \, |\psi(e)\rangle\langle\psi(e)| \, U(h) = \left|\psi(g^{-1})\right\rangle\left\langle\psi(g^{-1})\right|, \qquad \frac{1}{\mu}\int_G dg \ E(g) = Id \tag{3.46}$$

In fact, they satisfy

$$U^\dagger(h)E(g)U(h) = U^\dagger(h) \left|\psi(g^{-1})\right\rangle\left\langle\psi(g^{-1})\right| U(h) = \left|\psi(h^{-1}g^{-1})\right\rangle\left\langle\psi(h^{-1}g^{-1})\right| = E(gh) \tag{3.47}$$

[9] If the group is non-compact, the required normalization constant could be infinite. There's no general way to treat this issue, so each case must be treated separately. An example can be found in [42].

Example 3.2.4. A practicle example is a single particle $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ as QRF for the one dimensional Galilei group $Gal(1) \simeq \mathbb{R} \times \mathbb{R}$ of translations and boosts, for which we constructed a POVM in (2.2.3). In that example, we show that a single particle with a fixed mass m is a unitary-projective representation of $Gal(1)$

$$U_m(x,v) = e^{-\frac{i}{\hbar}(xP+vK_m)} = e^{-\frac{i}{\hbar}(Px_t - mvX)} = e^{\frac{i}{\hbar}(x+vt)\frac{mv}{2}} e^{-\frac{i}{\hbar}x_t P} e^{+\frac{i}{\hbar}vmX} \tag{3.48}$$

where $x_t = x + vt$. It's action on eigenstates of position and momentum reads

$$\begin{aligned} U_m(x,v)\,|x_0\rangle &= e^{\frac{i}{\hbar}mv\left(x_0+\frac{x}{2}+\frac{vt}{2}\right)}\,|x_0 + x + vt\rangle \\ U_m(x,v)\,|p_0\rangle &= e^{\frac{i}{\hbar}(x+vt)(\frac{mv}{2}-p_0)}\,|p_0 + mv\rangle \end{aligned} \tag{3.49}$$

To have a perfect QRF for the group we need orthogonal states that transform under the group regularly

$$U_m(x',v')\,|m;x,v\rangle \sim |m;x+x',v+v'\rangle \tag{3.50}$$

Intuitively, one would need a state which is at the same time eigenstate of position, serving as a perfect QRF for translations, and eigenstate of momentum, serving as perfect QRF for boosts. Given a single-particle Hilbert space, the best one can do is to consider coherent/gaussian states in phase space, which saturates Heisenberg uncertainty principle $\sigma_{X_r}\sigma_{P_r} = \hbar/2$.

$$|\psi_{x_0,p_0}\rangle = \int dy\ \psi_{x_0,p_0}(y)\,|y\rangle\,, \quad \psi_{x_0,p_0}(y) = N\exp\left\{-\frac{e^{-2r}\,(y-x_0)^2}{4\sigma^2} + i\frac{p_0\,(y-x_0)}{\hbar}\right\} \tag{3.51}$$

We show in example (2.2.3) that, choosing a gaussian state as a seed state,

$$|\psi(0,0)\rangle := |\psi_{x_0,p_0}\rangle = \int dy\ \psi_{x_0,p_0}(y)\,|y\rangle \tag{3.52}$$

one can construct a meaningful covariant POVM

$$\int dvdx\,|m;v,x\rangle\langle m;v,x| = \frac{2\pi}{m}\int dy\,|y\rangle\langle y|\ \ N^2\int d(x_0)\ e^{-\frac{(y-x_0)^2}{2\sigma^2}} = \frac{2\pi}{m}\int dy\,|y\rangle\langle y| = \frac{2\pi}{m}\,\mathbb{I} \tag{3.53}$$

where

$$U_m(x,v)\,|\psi_{x_0,p_0}\rangle = e^{\frac{i}{\hbar}mv\left(\frac{x}{2}+\frac{vt}{2}+x_0\right)}\,|\psi_{x_0+x+vt,p_0+mv}\rangle := e^{\frac{i}{\hbar}mv\left(\frac{x}{2}+\frac{vt}{2}+x_0\right)}\,|m;x,v\rangle \tag{3.54}$$

In [41] a perfect QRF for the (centrally extended) Galilei group is constructed explicitly, showing that, physically, it corresponds to two single-particles, one serving as a QRF for translations and the other serving as a QRF for boosts.

Another important example is a spin-j sytem used as a QRF for the rotation group $SU(2)$, that can be found in [32]. As far as the Hilbert space is finite dimensional $d = 2j+1$, the system cannot provide a full angular resolution, the reason being that we do not have an infinite number of orthogonal states $|\Omega\rangle$, $\Omega \in SU(2)$. A perfect QRF is recovered in the limit of $j \to \infty$.

Other examples of non-perfect QRF are given in the section on quantum clocks (3.6).

3.3 Recovery operation

Let's consider that Alice prepares the quantum system in some state $\rho_s \in \mathcal{B}(\mathcal{H}_s)$ and the quantum reference frame in a "fiducial" state $\rho_r \equiv \rho_{r;e} \in \mathcal{B}(\mathcal{H}_r)$ that takes the role of her classical reference frame in the configuration $x_e \in \mathbb{X}$. In order to distinguish the two, let's call U the unitary action of G on $\mathcal{H}_s$ and T the unitary action on $\mathcal{H}_r$.

If the composite state of the system and the QRF preparred by alice is $\rho = \rho_r \otimes \rho_s$, Bob will describe it as a uniform average, or $G\text{-}twirl$, over all $g \in G$ that possibly connect his frame with Alice's:

$$\tilde{\rho} = \mathcal{G}[\rho_r \otimes \rho_s] = \frac{1}{|G|} \int dg\ \mathcal{T}(g)[\rho_r] \otimes \mathcal{U}(g)[\rho_s] \tag{3.55}$$

Again, this state is invariant with respect to the action of the group G, reflecting Bob's complete ignorance about the reference frame. However, as anticipated above, Bob can now try to access the information he misses via a measurement on the QRF and eventually recover the state ρ_s preparred by Alice. This procedure is called "recovery operation" [32], and consists of a few steps:

- Alice preparres the reference frame in some "fiducial" state $\rho_{r;e} = |\psi(e)\rangle\langle\psi(e)|$, representing the configuration e of her local frame for the group G.
- Bob makes a measurement on the QRF, which was previously preparred by Alice, by means of a covariant POVM $E(g)$ for the group G.
- Depending on the measurement outcome g, Bob orients the system accordingly via a unitary $\mathcal{U}(g)$. Then he discards both the QRF and the measurement result.

Explicitly, the "recovery" operation has the form

$$\begin{aligned}
\rho'_s &= tr_r\left[\int dg\,(E(g)\otimes\mathcal{U}(g))\,[\tilde{\rho}]\right] \\
&= \frac{1}{|G|}\int dh \int dg\ tr_r\left[E(g)\mathcal{T}(h)[\rho_{r;e}]\right]\cdot\mathcal{U}_{gh}[\rho_s] \\
&= \frac{1}{|G|}\int dh \int dg\ tr_r\left[\mathcal{T}(h)^{\dagger}\,[E_g]\,\rho_{r;e}\right]\cdot\mathcal{U}_{gh}[\rho_s] \\
&= \frac{1}{|G|}\int dh \int dg\ tr_r\left[E(gh)\rho_{r;e}\right]\cdot\mathcal{U}_{gh}[\rho_s] \\
&= \frac{\int dh}{|G|}\int dg\ tr_r\left[E_g\rho_{r;e}\right]\cdot\mathcal{U}_g[\rho_s] \\
&= \int dg\ p(g)\cdot\mathcal{U}(g)[\rho_s]
\end{aligned} \tag{3.56}$$

where we use the covariance property of the POVM (2.2) and the invariance of the Haar measure. ρ'_s represents the state of the system in the absence of an external frame, after the QRF has been used as a resource to overcome the restrictions. In Fig. (3.3.1), we provide a simplified pictorial representation of the recovery operation.

Notice that the success of the procedure is determined by the Born probability

$$p(g) \equiv p(g,e) = tr_r\left[E(g)\rho_{r;e}\right] \tag{3.57}$$

so, ultimately, by how good is the QRF as a resource. If the quantum system $\mathcal{H}_r$ is a perfect QRF, then $p(g,e) = \delta(ge)$ and

$$\rho'_s = \int dg\ \delta(ge)\cdot\mathcal{U}(g)[\rho_s] = \rho_s \tag{3.58}$$

This means Bob has access to all the degrees of freedom of the system. Conversely, if $\mathcal{H}_r$ is an imperfect QRF, Bob will recover:

$$\rho'_s = \int d\mu(g)\,\mathcal{U}(g)[\rho_s], \qquad d\mu(g) = dg\ p(g) \tag{3.59}$$

This corresponds to a weighted-G-twirl, where instead of a uniform average over all possible group elements (as in the full G-twirl of eq. (3.14), the average is weighted by the Born probability $p(g) = tr_r\left[E(g)\rho_{r;e}\right]$. The closer $p(g)$ is to a uniform distribution, the less perfect the QRF, and the more mixed the system's state becomes.

An important point is that the recovery operation results in a normalizable state $\rho'_s \in \mathcal{B}(\mathcal{H}_r)$, even when the group is non-compact. This is prooved in [43], where the G-twirl is first defined on a compact subgroup of G, and then extended to the full group after the recovery operation. A key assumption for this result is the existence of a covariant POVM for the group G.

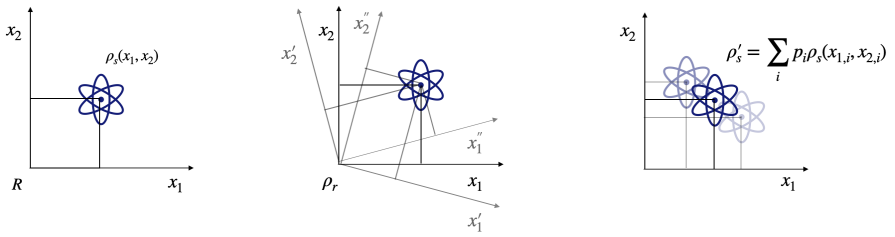

Figure 3.3.1: *A pictorial representation of the recovery operation. On the left, a quantum state ρ_s defined with respect to a classical, idealized reference frame $\vec{x} = (x_1, x_2)$, for instance two rods to define a Cartesian frame in 2D. In the centre, this frame is replaced by a quantum system $\rho_{r;e}$, a QRF, that we depict as a generic superposition of "configuration" states $\vec{x}^i$. On the right, the "recovered state" ρ'_s, defined through a POVM measurement on the QRF. The measurement yields a Born probability $p_i = tr[E_{\vec{x}^i}\ \rho_{r;e}]$. If this probability distribution p_i is not sharply peaked around a specific configuration $\vec{x}$, the recovered state ρ'_s will differ from ρ_s and acquire mixedness due to the quantum uncertainty in the QRF configuration.*

3.3.1 Physical meaning

The detailed group-theoretical analysis that we presented in (3.1.2) only holds for compact groups, which are completely reducible. For instance, in [32], a spin-j system is used as a QRF for the rotation group $SU(2)$ and the authors work out the explicit form of the effective decoherence after the recovery operation.

In the case of non-compact groups one can do a much simpler analysis, which however leads to a similar physical result: the system undergoes decoherence in the eigenbasis of the generators of the transformation.

For instance, considering the group of translations in 1D, the decoherence occurs in the eigenbasis of the momentum operator, meaning that Alice cannot prepare states in a coherent superposition of momentum-eigenstates, since these are only defined with respect to her local reference frame. If the state preparred by Alice is a generic coherent superposition:

$$\rho_s = \int dpdp'\ \phi(p)\phi^*(p')\,|p\rangle\langle p'| \tag{3.60}$$

The recovery operation (3.3) leads to

$$\rho'_s = \int dpdp'\ \phi(p)\phi^*(p') \int dx p(x) e^{-ix(p-p')}\,|p\rangle\langle p'| \tag{3.61}$$

where $p(x) = tr_r[\rho_r\ \ |x\rangle\langle x|]$. The diagonal elements are unaffected, since $\int dx p(x) = 1$:

$$\rho'_s(p,p) = |\phi_p|^2\,; \tag{3.62}$$

however the off-diagonal terms are multiplied by a factor, potentially reducing coherence:

$$\rho'_s(p,p') = \phi(p)\phi^*(p') \left(\int dx p(x) e^{-ix(p-p')} \right) \tag{3.63}$$

In the extreme cases, if $p(x) = \delta(x - x_0)$, corresponding to a sharp measurement (see (2.2.1)), no decoherence occurs, while if $p(x) = 1$ (a uniform distribution), the off-diagonal elements vanish, resulting in a loss of information about the system's position relative to the reference frame.

In [32] the authors also show that, in the case of a compact group G, the recovery operation can be equivalently represented as a measurement on the *relational degrees of freedom* (what we called multiplicicy- or flavour-space) of the system and the QRF.

A similar conclusion can be reached for a generic (unimodular) group G by rephrasing the recovery operation in terms of relational (G-invariant) observables.

3.4 Relational observables

Bob's operations O_s on the recovered state ρ'_s can be recast in terms of G-invariant operations $\tilde{O}$ on the G-invariant state $\tilde{\rho}$. Consider an operator $O_s \in \mathcal{L}(\mathcal{H}_s)$, where $\mathcal{L}(\mathcal{H})$ denotes the set of linear operators acting on $\mathcal{H}$. Here, the observable O_s could be either a self-adjoint operator or a more general POVM. One has the following correspondance:

$$\begin{aligned} tr_s\left[O_s\ \rho'_s\right] &= \int dg\ tr_r\Big[E(g)\rho_r\Big] tr_s\Big[O_s \mathcal{U}_g\left[\rho_s\right]\Big] \\ &= tr_{rs}\left[\rho_r \otimes \rho_s \int dg\ E(g) \otimes \mathcal{U}_g^\dagger\left[O_s\right]\right] \\ &= tr_{rs}\left[\tilde{\rho} \int dg\ E(g) \otimes \mathcal{U}_g^\dagger\left[O_s\right]\right] \coloneqq tr_{rs}\left[\tilde{\rho}\ \tilde{O}\right], \end{aligned} \tag{3.64}$$

where in the last line we use the fact that the expectation value of G-invariant POVMs over invariant or non-invariant states is the same, as it follows from the invariance of the measure dg and the covariance property of the POVM (2.2)

$$\begin{aligned} & tr_{rs}\left[\tilde{\rho} \int dg\ E(g) \otimes \mathcal{U}_g^\dagger\left[O_s\right]\right] \\ = tr_{rs}\left[\frac{1}{|G|} \int dh\ \mathcal{T}(h)[\rho_r] \otimes \mathcal{U}_h[\rho_s] \int dg\ \mathcal{T}(g)^\dagger\left[E(e)\right] \otimes \mathcal{U}(g)^\dagger\left[O_s\right]\right] \\ = tr_{rs}\left[\rho_r \otimes \rho_s \cdot \frac{1}{|G|} \int dh \int dg\ \mathcal{T}(gh)^\dagger\left[E(e)\right] \otimes \mathcal{U}(gh)^\dagger\left[O_s\right]\right] \\ = tr_{rs}\left[\rho_r \otimes \rho_s \cdot \frac{\int dh}{|G|} \int dg\ \mathcal{T}(g)^\dagger\left[E(e)\right] \otimes \mathcal{U}_g^\dagger\left[O_s\right]\right]. \end{aligned} \tag{3.65}$$

The conclusion is that the non-invariant operation O_s Bob performs on ρ'_s corresponds to a G-invariant operation $\tilde{E}$ on the G-invariant composite state $\tilde{\rho}$. The G-invariant operator (or Dirac operator), which could be either a POVM or an observable, reads:

$$\tilde{O} = \int dg\ E(g) \otimes \mathcal{U}_g^\dagger\left[O_s\right] = \int_G dg\ \mathcal{T}^\dagger(g) \otimes \mathcal{U}^\dagger(g)\big[E(e) \otimes O_s\big] \tag{3.66}$$

This is a uniform average over all possible group transformations, applied to the operator $E(e) \otimes O_s$, and it represents Bob's ignorance about the group G with respect to which it is defined.

Operators in this form constitute the so-called *relational algebra*, which is, in turn, part of[10] the G-invariant (or physical) algebra in the absence of an external reference frame for the group G. The existence of a covariant POVM allows for a meaningful operational interpretation of these invariant observables.

This becomes clear when the POVM elements are one-dimensional projectors $E(e) = |\psi(e)\rangle\langle\psi(e)|$:

[10]The full invariant/physical algebra is in general richer, since any quantum system could contain degrees of freedom which are G-invariant di per se, called $multiplicity$ degrees of freedom (see the previous discussion 3.1.2).

$$\tilde{O} = \int_G dg\, \mathcal{T}^\dagger(g) \otimes \mathcal{U}^\dagger(g) \big[|\psi(e)\rangle\langle\psi(e)| \otimes O_s \big] \tag{3.67}$$

We can interpret this as the group average of the operator $|\psi(e)\rangle\langle\psi(e)| \otimes O_s$, which is the system's observable O_s conditioned on the QRF being in the pure state $|\psi(e)\rangle\langle\psi(e)|$. This is the essence of a relational observable.

Physically, eq.(3.72) tells us that the observables that Bob can effectively access to are the $relational$ observables between the system and the QRF, which do not depend on any external frame of reference.

Example 3.4.1. Let's consider the simplest example of two single particles $\mathcal{H}_r = \mathcal{H}_s = \mathcal{L}^2(\mathbb{R})$, where $\mathcal{H}_r$ is used as QRF for the one dimensional group of translations (see the previous example 3.2.3). As system's observable we consider the self-adjoint position operator $X_s = \int dx\, x\, |x\rangle\langle x|_s$. The corresponding Dirac observable is

$$\tilde{X} = \int dx\, \mathcal{U}^\dagger(x) \big[|0\rangle\langle 0|_r \otimes X_s \big] \tag{3.68}$$

where $\mathcal{U}(x)[\cdot] = U(x)[\cdot]U^\dagger(x)$ and

$$U(x) = e^{-ixP_r} \otimes e^{-ixP_s} \tag{3.69}$$

is the translation operator acting both on the system and the QRF. We then have:

$$\tilde{X} = \int dx\ |-x\rangle\langle -x|_r \otimes (X_s + x) = X_s \otimes Id_r - Id_s \otimes X_r \equiv X_s - X_r \tag{3.70}$$

This is the $relative\ position$ between the system and the QRF.

Notice that eq. 3.72 toghether with eq. 3.3 shows how, in general, the relative algebra $\tilde{O}$ does not contain the same information of the system's algebra O_s in the presence of an external frame

$$tr_s\left[O_s\ \rho'_s\right] = tr_{rs}\left[\rho_r \otimes \rho_s\ \tilde{O}\right] \tag{3.71}$$

Physically, to access this information now we need to perform a measurement on the QRF as well. This means that the accessible information with- or without an external frame coincide only if the QRF is perfect, in which case $\rho_s = \rho'_s$ and therefore

$$tr_s\left[O_s\ \rho_s\right] = tr_{rs}\left[\rho_r \otimes \rho_s\ \tilde{O}\right] \tag{3.72}$$

3.5 Summary

To sum up, standard quantum mechanics assumes an external, idealized reference frame to define states and observables. However, this "absolute" frame may be either practically unavailable, as shown in the quantum-information-theoretic approach, or in principle unavailable due to the presence of symmetries. In such cases, a well-defined quantum mechanical description relies on relational observables (3.66), where some subsystems serve as (quantum) reference frames for the others.

This relational description does not, in general, align with the standard one, as all information must be extracted relative to the QRF. If the QRF is imperfect, meaning it lacks the degrees of freedom to fully simulate an absolute reference frame, subsystems experience decoherence, indicating that some information is lost.

3.6 Quantum Clocks

The concept of time undergoes significant changes as we move towards more fundamental physical theories. In non-relativistic mechanics time is considered absolute, flowing at the same rate for all physical system. Special relativity introduces the first major shift: time flows at the same rate only for inertial observers moving at the same velocity, rendering the concept of simultaneity no longer absolute. Mathematically the unique time t is replaced by a three-parameter family of times $t_{\vec{v}}$. This means there exists a different notion of time for each Lorentz observer. Nevertheless, for each of these observers, the dynamics can be expressed as evolution in a single time parameter $t_{\vec{v}}$, which is a global observable with measurable metric properties [11].

General Relativity radically changes this view: the only metric and measurable notion of time is the *proper time*, which is a local quantity determined by the gravitational field, which is itself a dynamical entity:

$$d\tau = \sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu} \tag{3.73}$$

The dynamics can still be expressed as evolution in a single *time coordinate* x^0 which, however, lacks metric properties and it's not directly observable: expressing time evolution in terms of an external background time is physically meaningless. Instead, time evolution must be extracted in a relational manner (see also chapter 5 and the "hole argument" in appendix 5.A).

Einstein's equations determine how observable quantities evolve relative to one another; therefore one can always choose one of them, say T_c, and express the dynamics of the other quantities as function of it. This notion of time is called *clock-time*. Depending on the physical scenario, one can consider a matter clock in an operational laboratory situation, as well as the dynamical size of the universe in a cosmological setting, which can serve as "cosmic time" [12].

This fact gives rise to the so called "*problem of time*", which originates from early research on quantum gravity. Mathematically, the hamiltonian of a generally covariant theory is constrained to vanish in the absence of boundaries, leading to the famous Wheeler-de-Witt equation [1],[5]

$$H\,|\psi\rangle = 0 \tag{3.74}$$

This equation reflects the symmetry under time parametrization: an external parameter x^0 is physically maningless.

Assuming the principle of general covariance, time evolution must be extracted relationally, which involves selecting some quantized degrees of freedom - a quantum clock - to serve as internal reference frame for time, relative to which the remaining quantum degrees of freedom evolve [3], [16]. Thus, a meaningful way to represent time observables in quantum theory is necessary.

An operational approach to time encounters several difficulties in quantum theory. The primary challenge, as first noted by Pauli [38], is that there cannot exist a self-adjoint operator T_c canonically conjugate to a bounded, self-adjoint hamiltonian H_c. This observation was refined by Unruh and Wald [44], who showed that for realistic quantum clock, with a bounded hamiltonian, if a self-adjoint time-operator exists, Schrödinger equations leads to a non zero probability of running backward, meaning that other observables would thereby appear to be multivalued at a given reading of the clock.

In this section, we introduce the concept of the quantum time observable, which overcomes the limitations of Pauli's theorem by relaxing the assumption that an observable must be represented by a self-adjoint operator, and adopting instead the more general mathematical of POVM (Positive Operator-Valued Measure), which is standard in quantum information theory (see (2.2.2)). This allows us to define time observables with a probabilistic interpretation, even when working with bounded hamiltonians. We will introduce this framework, following [17], then we will analyze different examples of quantum clocks, ranging from systems with continuous spectra (ideal clocks) to more realistic discrete-spectrum clocks (like spin systems).

[11] With metric properties we mean that time intervals can be compared to each other.

[12] However, clock time behaves as a "good" clock only in certain states or for a limited duration, lacking general temporal globality. For instance the size of the universe ceases to be a good time variable when the universe recollapses, as far as real matter clocks usually have a periodic nature.

3.6.1 Quantum time-observable

In principle, any dynamical quantum system $\mathcal{H}$ can be considered as a QRF for time, or $quantum\ clock$. In particular, we consider as group of time translations the one dimensional group G generated by its hamiltonian H_c [13], whose unitary representation on $\mathcal{H}$ is $U(t) = \exp\{-\frac{i}{\hbar}tH_c\}$. As we will analyze in the following paragraphs, depending on the hamiltonian's spectrum, the group G can be either compact or non-compact.

Instead of looking for a self-adjoint time observable, which in general does not exist, we look for a more general *covariant time-POVM* $E : \mathrm{B}(G) \to \mathcal{B}(\mathcal{H})$ [14], which satisfies

$$\mu \int_G dt\ E(t) = \mathbb{I} \qquad U(t')^\dagger E(t) U(t') = E(t+t') \tag{3.75}$$

where μ is some normalization constant. Furthermore, searching for an optimal POVM (see (2.2)), we require its elements to consist of one dimensional projectors over G-states which are in general not normalizable nor orthogonal

$$E(t) = U(t)^\dagger \left|0\right\rangle\!\left\langle 0\right| U(t) = \left|-t\right\rangle\!\left\langle -t\right| \tag{3.76}$$

where μ is some normalization constant. The "seed state" $|0\rangle$ is arbitrary: the only constraint is the covariance property which, in turns, implies

$$U(t)\left|0\right\rangle = \left|t\right\rangle \tag{3.77}$$

As we show in appendix (3.A) the POVM can be described by a single real number only if the spectrum of H is non degenerate. We then focus on the simplest cases of a non-degenerate spectrum of H which is either fully continuous σ_c or fully discrete σ_d.

Seed state

Considering a fully continuous spectrum, the most general seed state is

$$\left|0\right\rangle = \int_{\sigma_c} d\epsilon\ \phi(\epsilon)\left|\epsilon\right\rangle \tag{3.78}$$

where $|\phi_\epsilon|$ is an arbitrary complex function. The covariance property (3.75) requires

$$\mu \int dt \int_{\sigma_c} d\epsilon d\epsilon'\ e^{-\frac{i}{\hbar}t(\epsilon-\epsilon')}\phi_\epsilon \phi'^*_\epsilon \left|\epsilon\right\rangle\!\left\langle\epsilon'\right| = 2\pi\mu \int_{\sigma_c} d\epsilon\ \left|\phi(\epsilon)\right|^2 \left|\epsilon\right\rangle\!\left\langle\epsilon\right| = \mathbb{I} \tag{3.79}$$

To have an identity on the whole Hilbert space $\mathcal{H} = span_{\sigma_c}\{|\epsilon\rangle\}$ it is required that $|\phi(\epsilon)|^2 = 1$ or, equivalently $\phi_\epsilon = e^{ig(\epsilon)}$, where $g(\epsilon)$ is an arbitrary real function. [15] In this case the normalization constant is fixed to $\mu = \frac{1}{2\pi}$.

Notice that a similar analysis can be done in the case of a discrete spectrum, where the most general seed state would be

$$\left|0\right\rangle = \sum_{\sigma_d} c_i \left|\epsilon_i\right\rangle \tag{3.80}$$

and c_i in principle an arbitrary complex number.

[13] any self-adjoint operator J, by Stone's theorem, generates a one-dimensional group, whose unitary action is $U(g) = \exp\{-igJ\}$

[14] Formally, the map has domain on the space of real values $\mathbb{G} \in G$ that parametrize the group.

[15] As it is shown in ([17]), this reflects the fact that in a classical theory a canonical time observable is defined up to an arbitrary real function of the hamiltonian.

Time operator

One can define a *time-operator* as the first-moment-operator of the covariant time observable E

$$T_c = \mu \int_G dt\ t\, |t\rangle\langle t| \tag{3.81}$$

which, in general, is symmetric but not self-adjoint, since the clock's states are not necessarily orthogonal. This allows to overcome the main issue with a time-observable in quantum theory: a self-adjoint time observable is just a special case in which the time-states $|t\rangle$ form a complete basis of $\mathcal{H}$, which occurs only for unphysical hamiltonians, as we show below.

3.6.2 Continuous spectrum clock

The spectral decomposition of a hamiltonian with a continuous (non deg.) spectrum is

$$H_c = \int_{\sigma_c} d\epsilon\ \epsilon\, |\epsilon\rangle\langle\epsilon| \tag{3.82}$$

Non-compact group

First of all, the one dimensional group G generated by an hamiltonian with continuous spectrum is necessarily non compact and $G = \mathbb{R}$. This is becouse if the group is compact, there exists some group element, parametrized by some t_{max}, such that

$$U(t_{max}) = U(0) = e^{i\phi}\, \mathbb{I}, \qquad \phi \in [0, 2\pi) \tag{3.83}$$

where the phase $\phi \in$ account that states are rays in the Hilbert space. Then, the spectral decomposition

$$U(t) = \int_{\sigma_c} d\epsilon\ e^{-\frac{i}{\hbar}\epsilon t}\, |\epsilon\rangle\langle\epsilon|\,, \qquad \mathbb{I} = \int_{\sigma_c} d\epsilon\ \ |\epsilon\rangle\langle\epsilon| \tag{3.84}$$

leads to

$$\forall \epsilon \in \sigma_c: \quad e^{-\frac{i}{\hbar}\epsilon t_{max}} = e^{i\phi} \quad \Longleftrightarrow \quad \forall \epsilon \in \sigma_c: \quad \epsilon = \frac{\phi}{t_{max}} + \frac{2\pi}{t_{max}} m_\epsilon, \quad m_\epsilon \in \mathbb{Z} \tag{3.85}$$

which requires the spectrum to be discrete and rational.

Clock states

The most general clock state reads

$$|t\rangle = \int_{\sigma_c} d\epsilon e^{-\frac{i}{\hbar}\epsilon t} e^{ig(\epsilon)}\, |\epsilon\rangle \tag{3.86}$$

and the overlap of two clock states is given by

$$\langle t|t'\rangle = \int_{\sigma_c} d\epsilon\ e^{i\epsilon(t-t')} = \chi(t-t') \tag{3.87}$$

where we define

$$\chi(x) = \begin{cases} 2\pi\delta(x), & \sigma_c = \mathbb{R} \\ e^{i\epsilon_{min} x}\left(\pi\delta(x) + i\mathbf{P}(\frac{1}{x})\right), & \sigma_c = (\epsilon_{min}, \infty) \\ i\frac{e^{i\epsilon_{min} x} - e^{i\epsilon_{max} x}}{x}, & \sigma_c = (\epsilon_{min}, \epsilon_{max}) \end{cases} \tag{3.88}$$

Where $\mathbf{P}$ denotes the Cauchy principal value.

Ideal clock

From eq. (3.87) it follows that the clock states chosen above are orthogonal only if the spectrum is unbounded $\sigma_c = \mathbb{R}$. In this case the POVM corresponds to a PVM (Projection Valued Measure)(2.2.1) and the time operator T_c is self-adjoint and canonically conjugate to the hamiltonian H_c. In fact we have

$$U(t)^\dagger\, T_c\, U(t) = \mu \int_{\mathbb{R}} dt'\; t' \left(U(t)^\dagger\, |t'\rangle\langle t'|\, U(t)\right) = \mu \int_{\mathbb{R}} dt'\; t' \left(|t'-t\rangle\langle t'-t|\right) = \tag{3.89}$$

$$= \mu \int_{\mathbb{R}} dt'\; t'\, |t'\rangle\langle t'| + t\; \mu \int_{\mathbb{R}} dt'\;\; |t'\rangle\langle t'| = T_c + t\; \mathbb{I} \tag{3.90}$$

and differentiating with respect to t we obtain the canonical commutation relations

$$[T_c, H_c] = i\; \mathbb{I} \tag{3.91}$$

The states in the representation of the canonical variables are in fact related by a Fourier transform

$$|h\rangle = \frac{1}{2\pi} \int_{\mathbb{R}} dt e^{-\frac{i}{\hbar} g(h)} e^{iht}\, |t\rangle\,, \qquad |t\rangle = \int_{\mathbb{R}} d\epsilon e^{ig(h)} e^{-\frac{i}{\hbar}\epsilon t}\, |h\rangle \tag{3.92}$$

We refer to these quantum clocks, which are perfect QRFs with $\mathcal{H} \simeq \mathcal{L}^2(\mathbb{R})$ and with a self-adjoint time operator, as *ideal* clocks.

3.6.3 Discrete spectrum clock

The spectral decomposition of the hamiltonian is

$$H_c = \sum_j \epsilon_j\, |\epsilon_j\rangle\langle\epsilon_j| \tag{3.93}$$

Compact group

In the case of a (non deg.) discrete spectrum σ_d one is forced to consider a compact group G. The reason is that if the group is non compact, we've shown above that $G = \mathbb{R}$, in which case the completeness relation diverges. In fact, given the clock states

$$|t\rangle = e^{-\frac{i}{\hbar} t H_c}\, |0\rangle = \sum_{j\in\sigma_d} e^{-\frac{i}{\hbar}\epsilon_j t} e^{ig(\epsilon_j)}\, |\epsilon_j\rangle\,, \tag{3.94}$$

one has:

$$\mu \int_G dt\, |t\rangle\langle t| = \mu \sum_{i,j} |\epsilon_i\rangle\langle\epsilon_j|\, e^{i(g(\epsilon_i)-g(\epsilon_j)} \left(\int_G dt e^{-\frac{i}{\hbar} t(\epsilon_i-\epsilon_j)}\right) \tag{3.95}$$

$$= \mu \sum_{i,j} |\epsilon_i\rangle\langle\epsilon_j|\, e^{i(g(\epsilon_i)-g(\epsilon_j)} \left(\delta_{ij}\, |G|\right) = \mu\, |G|\,. \tag{3.96}$$

Therefore we can restrict to a unitary generator $U(t) = e^{-\frac{i}{\hbar} t H_c}$ which is periodic with smallest period τ, so that $\mathbb{I} = e^{-\frac{i}{\hbar}\tau H_c}$, $t \in [0, \tau]$ and the normalization constant is fixed to $\mu = \frac{1}{|G|} = \frac{1}{\tau}$. Moreover, all eigenvalues can be written as $\epsilon_j = j\frac{2\pi}{\tau}$ with $j \in \mathbb{Z}$.

Clock states

The inner product between clock states (3.94) reads

$$\langle t|t'\rangle = \sum_j e^{-\frac{i}{\hbar}(t-t')\frac{2\pi}{\tau}j}, \tag{3.97}$$

and it yields $\tau\delta(t-t')$ only if all integers are present in the eigenvalue spectrum [16], e.g. $\sigma_d = \mathbb{Z}$. Therefore, both if the hamiltonian's spectrum is continuous or discrete, one can have orthogonal clock states (i.e. a perfect QRF) only if it is unbounded.

Considering a general discrete spectrum, one has:

$$T_c = \frac{1}{\tau}\int_0^\tau dt\; t\,|t\rangle\langle t| = \frac{1}{\tau}\sum_i |\epsilon_i\rangle\langle\epsilon_i| \int_0^\tau dt\; t + \frac{1}{\tau}\sum_{i\neq j}|\epsilon_i\rangle\langle\epsilon_j| \int_0^\tau dt\; te^{-\frac{i}{\hbar}t(\epsilon_i-\epsilon_j)}e^{i(g(\epsilon_i)-g(\epsilon_j))} = \tag{3.98}$$

$$= \frac{\tau}{2}\mathbb{I} + i\sum_{i\neq j}\frac{e^{i(g(\epsilon_i)-g(\epsilon_j))}}{\epsilon_i-\epsilon_j}|\epsilon_i\rangle\langle\epsilon_j|\,, \tag{3.99}$$

so we find that the clock state are not eigenstates of the time operator

$$T_c\,|t\rangle = \frac{\tau}{2}\,|t\rangle + i\sum_{i\neq j}\frac{e^{ig(\epsilon_i)}}{\epsilon_i-\epsilon_j}e^{-\frac{i}{\hbar}\epsilon_j t}\,|\epsilon_i\rangle \tag{3.100}$$

The commutation relation between T and H_c reads

$$[T_c, H_c] = -i\sum_{i\neq j} e^{ig(\epsilon_i)-g(\epsilon_j)}\,|\epsilon_i\rangle\langle\epsilon_j| \tag{3.101}$$

and, substituting

$$|\tau\rangle\langle\tau| = \sum_{i,j} e^{ig(\epsilon_i)-g(\epsilon_j)}\,|\epsilon_i\rangle\langle\epsilon_j| = \mathbb{I} + \sum_{i\neq j} e^{ig(\epsilon_i)-g(\epsilon_j)}\,|\epsilon_i\rangle\langle\epsilon_j|\,, \tag{3.102}$$

one gets:

$$[T_c, H_c] = i\,(\mathbb{I} - |\tau\rangle\langle\tau|)\,. \tag{3.103}$$

So in general the standard commutation relations are satisfied only on a subspace of the whole Hilbert space. The relation between the clock's states and the energy eigenstates now reads:

$$|t\rangle = \sum_{i\in\sigma_d} e^{-\frac{i}{\hbar}\epsilon_i t}e^{ig(\epsilon_i)}\,|\epsilon_i\rangle \qquad |\epsilon_i\rangle = \frac{1}{\tau}\int_0^\tau dt e^{-\frac{i}{\hbar}g(\epsilon_i)}e^{-\frac{i}{\hbar}t\epsilon_i}\,|t\rangle\,. \tag{3.104}$$

[16] as follows from the serie reprentation of the dirac delta, which is $\delta(x-a) = \sum_{k=-\infty}^{+\infty} e^{-\frac{i}{\hbar}k(x-a)}$

3.6.4 Examples

In this section, we provide examples of some of the most widely used models for a quantum clock.

Example 3.6.1. - Ideal clock. The simplest realization an unbounded and continuous spectrum, where $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$, is a single ultra-realtivistic particle, for which

$$H_c = cP = c \int_{\mathbb{R}} p \, |p\rangle\langle p| \tag{3.105}$$

the clock's states are fourier transform of momentum eigenstates, i.e. position eigenstates

$$|x\rangle = \int_{\mathbb{R}} dp e^{-\frac{i}{\hbar}px} \, |p\rangle \tag{3.106}$$

where we fixed the arbitrary phase $g(p) = 0$. The time operator is proportional to the position operator

$$T_c = \frac{X}{c} + a\mathbb{I} \tag{3.107}$$

where the arbitrary constant a can be fixed in order to have $\langle T_c \rangle (0) = 0$.

This example represents the archetype of an ideal quantum clock, where the time operator T_c is self-adjoint, and the clock states are orthogonal. This orthogonality allows for perfect time measurements.

Example 3.6.2. - Salecker-Wigner-Peres clock. A common example of a discrete spectrum clock a spin-j system with a $2j + 1$ dimensional hilbert space. The hamiltonian is

$$H_c = \sum_{m=0}^{d-1} m\omega \, |m\rangle\langle m| \tag{3.108}$$

with $m \in \mathbb{N}$ and the general clock's states are

$$|t\rangle = \mu \sum_{m=0}^{d-1} e^{-i\omega m t} \, |m\rangle \tag{3.109}$$

We see that, in general, they are not orthogonal:

$$\langle t|t'\rangle = \mu^2 \sum_{m=0}^{d-1} e^{-i2\pi \frac{m}{d}(t-t')} \tag{3.110}$$

However, there's a finite subset of them $t_i = \frac{2\pi}{\omega d} i$, $i = 0, ..d-1$ which are orthogonal and form a proper basis of the Hilbert space

$$|t_i\rangle = \mu \sum_{m=0}^{d-1} e^{-i2\pi \frac{m}{d} i} \, |m\rangle \tag{3.111}$$

In fact, fixing $\mu = \frac{1}{\sqrt{d}}$

$$\langle t_i|t_j\rangle = \mu^2 \sum_{m=0}^{d-1} e^{-i2\pi \frac{m}{d}(i-j)} = \delta_{i,j} \tag{3.112}$$

This is usually referred as $time \; basis$. The optimal POVM and the clock operator can be constructed using this basis:

$$\begin{cases} E_i = |t_i\rangle\langle t_i| \quad \rightarrow \quad \sum_{i=0}^{d-1} E_i = \sum_{m,n=0}^{d-1} |n\rangle\langle m| \, \frac{1}{d} \sum_{i=0}^{d-1} e^{-i2\pi \frac{\omega}{d}(n-m)i} = \mathbb{I} \\ T_c = \sum_{i=0}^{d-1} t_i \, |t_i\rangle\langle t_i| \end{cases} \tag{3.113}$$

This is an example of an imperfect QRF (3.2.4) for the compact group G generated by the hamiltonian (3.108).

The time operator does not obey canonical commutation relations, since:

$$[T_c, H_c] = \frac{1}{\sqrt{d}} \sum_{i,m=0}^{d-1} t_i m\omega \ \left(|t_i\rangle\langle m| \, \langle t_i|m\rangle - |m\rangle\langle t_i| \, \langle m|t_i\rangle \right) \tag{3.114}$$

For instance, one immediately see that

$$\langle n| \, [T_c, H_c] \, |n\rangle = \langle t_i| \, [T_c, H_c] \, |t_i\rangle = 0 \tag{3.115}$$

It turns out [45] that the canonical commutation relation can be approximately satisfied, with an exponentially small error in the clock's dimension, if one restricts to the space spanned by complex gaussian superposition of clock states.

$$|\psi(m_0)\rangle = \sum_{i=0}^{d-1} G(m_0, \sigma) \, |t_i\rangle \tag{3.116}$$

where $G(m_0, \sigma)$ is a complex gaussian distribution. These states are referred as $quasi\text{-}ideal$ clock states.

In contrast, to the previous example, the Salecker-Wigner-Peres clock represents a more common real-world example of a discrete spectrum clock. Unlike the ultra-relativistic particle, this clock is imperfect because its hamiltonian spectrum is discrete, leading to non-orthogonal clock states. This imperfection reflects the general limitation that many quantum systems face: considering real systems, with discrete energy levels or a continuous but bounded energy spectrum, time cannot be measured perfectly, and the covariant time-POVMs are less precise.

Example 3.6.3. - QBIT-phase clock. A particular case of the previous example is a spin $j = 1/2$ or 2-level system, which is the symplest model of a quantum clock. The hamiltonian is

$$H_c = \frac{\hbar\omega}{2} \, (\mathbb{I} - \sigma_z) \tag{3.117}$$

and the clock's states

$$|t\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + e^{-i\omega t} \, |1\rangle \right) \tag{3.118}$$

with $t \in [0, \frac{2\pi}{\omega}]$. Physically they can be preparred by putting an atom in a coherent superposition $|\gamma_+\rangle$ of its two hyperfine energy levels, then letting it evolve freely. The optimal covariant POVM is:

$$E(t) = \frac{\omega}{\pi} \, |t\rangle\langle t| \, , \tag{3.119}$$

while the corresponding time operator is:

$$T_c = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} dt \ t \, |t\rangle\langle t| = \frac{1}{\omega} \, (\pi\mathbb{I} + \sigma_y) \, . \tag{3.120}$$

Therefore one has

$$[T_c, H_c] = \frac{1}{2}[\sigma_z, \sigma_y] = -i\sigma_x, \tag{3.121}$$

so the canonical commutation relations are satisfied only on the subspace generated by the state $|\gamma_-\rangle$, which is the eigenstate of σ_x with eigenvalue -1. One can also see this by looking at the scalar product of clock's states: they are orthogonal only if the time elapsed is $t - t' = \frac{\pi}{\omega}$

$$\langle t|t'\rangle = \frac{1}{2}\left(1 + e^{-i\omega t(t'-t)}\right) = e^{-i\omega\frac{(t'-t)}{2}}\cos\left(\omega\frac{(t'-t)}{2}\right). \tag{3.122}$$

Physically this reflects the fact that one can extract the information about the time elapsed only when the initial state has evolved into a perfectly distinguishable state. Starting form the state $|\gamma_+\rangle$ one has then exact information about the time elapsed after a half-period $t_\perp = \frac{\pi}{\omega}$, when the state is $|\gamma_-\rangle$. Therefore the "orthogonalization time" $t_\perp$, which is inversely proportional to the energy spacing $\Delta E = \hbar\omega$, determines the clock's precision.

This statement can be made even more precise. Operationally, the precision in time ΔT can be defined as the shortest timescale we are able to observe. Thus, given any dynamical observable O, by means of which we "observe" the flow of time, we can define ΔT as the time required by the expectation value $\langle O\rangle$ to change of a standard deviation σ_O^2. In our case, consider the observable T_c of eq. (3.120), so that we define precision in time measurements as:

$$\Delta T = \frac{\sigma_{T_c}}{d\langle T_c\rangle/dt} \tag{3.123}$$

Using $\sigma_y|0\rangle = -i|1\rangle$ and $\sigma_y|1\rangle = i|0\rangle$ we can easily compute

$$\begin{aligned}\langle T_c\rangle &= \frac{1}{\omega}(\pi + \sin(\omega t))\\ \langle T_c^2\rangle &= \frac{1}{\omega^2}\left(1 + \pi^2 + 2\pi\sin(\omega t)\right)\end{aligned} \tag{3.124}$$

So we can read out the standard deviation

$$\sigma_{T_c}^2 = \langle T_c^2\rangle - \langle T_c\rangle^2 = \frac{1}{\omega^2}\left(1 - \sin^2(\omega t)\right) = \frac{1}{\omega^2}\cos^2(\omega t) \tag{3.125}$$

Thus, the "precision" of the clock (3.123) is

$$\Delta T = \frac{1}{\omega} = \frac{\hbar}{\Delta E} \tag{3.126}$$

This limitation is overcome in practice by considering a sample of clocks and performing a large serie of measurements. The information about the time elapsed is extracted by looking at the distribution of the outcomes:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\Omega t}|1\rangle) = \frac{1}{2}((1 + e^{-i\omega t})|\gamma_+\rangle + (1 - e^{-i\omega t})|\gamma_-\rangle) \tag{3.127}$$

so one has $p_+(t) = |\langle +|\psi(t)\rangle|^2 = \cos^2(\frac{\omega t}{2})$ and similarly $p_-(t) = |\langle -|\psi(t)\rangle|^2 = \sin^2(\frac{\omega t}{2})$.

The QBIT-phase clock offers the simplest and most accessible model of a quantum clock, involving a two-level system. Its simplicity makes it an excellent pedagogical example, showing how even a basic system can act as a quantum clock. This example reinforces the concept that even for two-level systems, time can be represented using a POVM, although the states are non-orthogonal and measurement precision is limited. The QBIT clock also illustrates a key principle: the precision of quantum clocks is tied to the system's "energy spread" of clock's states (see chapter 5.2).

3.6.5 Summary

One aspect of the "problem of time" in quantum theory is the difficulty of representing time as an observable, since a self-adjoint time operator only exists for unphysical systems with an unbounded spectrum hamiltonian. This issue is resolved by defining the time observable using a more general covariant POVM (see 2.61), which includes the unphysical operator as a special case.

A quantum clock can then be defined as a generic quantum system in a dynamical state, i.e., in a non-trivial superposition of energy eigenstates. The broader the superposition, i.e. the larger the $energy\ spread$, the more distinguishable the clock states become through the covariant time POVM.

An infinitely precise clock, or ideal clock, corresponds to a Fourier transform of energy eigenstates, which in turn requires an unbounded spectrum. Only in this ideal case can one define sharp time measurements (PVMs) capable of distinguishing every instant of time.

In the more general case, a clock is represented by a finite, often discrete, superposition of energy eigenstates. The states of these clocks are not fully distinguishable, so one must rely on unsharp measurements (or POVM, see 2.2.2) with finite resolution.

Appendix

3.A Time POVM for a non-degenerate spectrum

Consider a degenerate hamiltionian

$$H_c \left|\epsilon, \alpha\right\rangle = \epsilon \left|\epsilon, \alpha\right\rangle, \qquad \mathbb{I} = \sum_{\epsilon,\alpha} \left|\epsilon, \alpha\right\rangle\!\left\langle\epsilon, \alpha\right| \tag{3.128}$$

From eq. (3.77) follows that

$$\left|t\right\rangle = U(t) \left|0\right\rangle \tag{3.129}$$

$$\left\langle\epsilon\alpha, t|\epsilon\alpha, t\right\rangle = e^{-\frac{i}{\hbar}\epsilon t} \left\langle\epsilon\alpha, 0|\epsilon\alpha, 0\right\rangle \tag{3.130}$$

Now we can ask whether it is possible to satisfy the completeness property of the POVM:

$$\int \mu \left|t\right\rangle\!\left\langle t\right| = \sum_{h,h'} \int dt \mu e^{-\frac{i}{\hbar}t(h-h')} \left(\sum_{\alpha,\alpha'} \left|\epsilon\alpha\right\rangle \left\langle\epsilon\alpha|0\right\rangle \left\langle 0|h'\alpha'\right\rangle \left\langle h'\alpha'\right| \right) \tag{3.131}$$

Integration over dt yields a $\delta_{h,h'}$, which simplifies the previous expression to

$$\int \mu \left|t\right\rangle\!\left\langle t\right| = \sum_{h} \left(\sum_{\alpha,\alpha'} \left|\epsilon\alpha\right\rangle \left\langle\epsilon\alpha|0\right\rangle \left\langle 0|\epsilon\alpha'\right\rangle \left\langle\epsilon\alpha'\right| \right) \tag{3.132}$$

The only way to get the identity is requiring

$$\left\langle\epsilon\alpha|0\right\rangle \left\langle 0|\epsilon\alpha'\right\rangle = \delta_{\alpha,\alpha'} \qquad \forall h \tag{3.133}$$

which can only be satisfied if the spectrum is non-degenerate. Only in this case one can hope to find an "optimal" measurement in terms of a POVM described by a single real number t.

Chapter 4

Quantum clocks in general relativity

To date, all experiments in regimes where both quantum mechanics and general relativity are relevant are fully compatible with the non-metric, Newtonian gravity—despite its thorough disproof for classical systems. For instance, the first experiment measuring the effect of gravity on a quantum system was performed by Colella, Overhauser, and Werner (COW) [46]. In this experiment, single neutrons travel in a superposition of paths at different heights above the Earth, and thus experience different gravitational potentials. The superposed wavefunctions acquire a relative phase, which is measured from the interference pattern when the amplitudes are recombined. Such effects are fully accounted for by a non-relativistic Schrödinger equation with a Newtonian gravitational potential. At the same time, in all tests of general relativity so far, the relevant observations have been explained within the laws of classical physics. A good example is the Pound-Rebka experiment [47], where gravitational redshift on gamma rays was measured. This observation can be fully described within classical electromagnetism in curved spacetime. Thus, the regime where both classical general relativity and quantum mechanics are at play has not yet been subjected to experimental verification.

The most common approach to investigate effects where both quantum mechanics and general relativity are crucial is to look at high-energy quantum field theory in curved spacetime, predicting phenomena like Hawking radiation or the effects of spacetime curvature on ultrarelativistic particles. However, these phenomena lie beyond current experimental capabilities.

A parallel approach is to search for testable effects on Earth, which means in the accessible regime of weak gravitational fields and low-velocities, i.e. where only the first special- and general-relativistic corrections to Newtonian mechanics play a role. This allows to work in the so called one-particle regime, where laboratory experiments can directly test quantum systems in superpositions.

One verified relativistic effect in this regime is time dilation, although it has not yet been tested when quantum effects are also relevant. The theoretical framework of quantum clocks allows us to explore whether new [1] physical effects emerge when general relativistic time dilation interacts with quantum systems that exhibit features such as coherence or entanglement.

In this chapter, we introduce the framework of quantum clocks in the low-velocity and weak-gravitational field regime, following the work in [24]. We explore how time dilation couples internal and external degrees of freedom in a quantum clock, leading to decoherence of the clock's position. Extending this to composite systems with dynamical internal degrees of freedom (e.g., the vibrational modes of molecules) time dilation provides a universal decoherence mechanism.

4.1 Composite particles as ideal clocks

A fundamental concept in general relativity is that of an ideal clock, which satisfies the "clock hypothesis." According to this hypothesis, a clock correlates with the metric field such that its time-readings coincide with the proper time predicted by the theory. In other words, ideal clocks measure proper time along their trajectories.

[1] here "new" means that these effects should not be explainable in the framework of classic relativistic clocks, nor in that of quantum clocks in a Newtonian spacetime.

The simplest model for a material clock in general relativity is a point-like particle with an internal structure that serves as a reference for time evolution. Examples include the internal constituents of a molecule, the structure of an atom, or the spin of a quantum particle. Under certain approximations, this simple model describes a physical realization of an ideal clock in General Realtivity.

4.1.1 hamiltonian

Consider a composite particle in an arbitrary coordinate system, following a wordline $x^\mu(t)$ (where t is an arbitrary parameter) and four-momentum $p_\mu(x) = (p_0(x), p_j(x))$, where $H = cp_0$ is the energy component. In the coordinates x'^μ at which the particle is at rest, the four-momentum is $p'_\mu(x') = (p'_0(x'), 0)$, where cp'_0 is the total rest energy, including both the rest mass-energy mc^2 and the internal (or $clock's$-) hamiltonian H_c, describing the dynamics of the internal degrees of freedom.

$$cp'_0 = H_{rest} = mc^2 + H_c \coloneqq Mc^2 \tag{4.1}$$

Since the scalar product $p_\mu p^\mu$ is a coordinate-invariant quantity, it follows that $p_0^{'2} g^{00} = p_\mu p^\mu$. Hence the energy component in an arbitrary reference frame reads

$$p_0 = \sqrt{\frac{1}{g^{00}} \left(-p_i p_j g^{ij} - g'_{00} p_0^{'2} \right)} \tag{4.2}$$

Substituting $g'_{00} = -1$ and considering a static metric, where $g_{00} = 1/g_{00}$, we get the total hamiltonian for a composite particle

$$H = \sqrt{-g_{00} \left(c^2 p_i p_j g^{ij} + H_{rest}^2 \right)} \tag{4.3}$$

4.1.2 Lagrangian

We now turn to the Lagrangian formalism. Using canonical variables (x^i, p_i), $i = 1, ...3$ for the kinematical degrees of freedom and (q^k, w_k), $k = 1, ..., N$ for the internal N degrees of freedom, the Lagrangian is

$$L = \sum_{j=1}^{3} p_j \frac{\partial H}{\partial p_j} + \sum_{k=1}^{N} w_k \frac{\partial H}{\partial w_k} - H \tag{4.4}$$

Notice that the dependence on the internal degrees of freedom is only in $H_{rest} = mc^2 + H_c$. The explicit computation is reported in appedix (4.A). The result is

$$L = \dot{\tau} L_{rest}, \qquad \dot{\tau} = \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} = \frac{\partial H}{\partial H_{rest}} \tag{4.5}$$

where $\dot{x}^\mu = \frac{\partial H}{\partial p_\mu}$ and, using the definition of rest hamiltonian 4.1, we have

$$L_{rest} = \sum_{k=1}^{N} w_k \frac{\partial H_{rest}}{\partial w_k} - H_{rest} = \sum_{k=1}^{N} w_k \frac{\partial H_c}{\partial w_k} - H_c - mc^2 \coloneqq L_c - mc^2 \tag{4.6}$$

This is the generalization of the standard lagrangian of a structureless particle $L = -\dot{\tau} mc^2$.

Substituting this result into the equations of motion for the internal degrees of freedom

$$\dot{q}^k = \frac{\partial H}{\partial w_k} = \frac{\partial H}{\partial H_{rest}} \frac{\partial H_{rest}}{\partial w_k} = \dot{\tau} \frac{\partial H_{rest}}{\partial w_k} \quad \Longleftrightarrow \quad \frac{dq^k}{d\tau} = \frac{\partial H_{rest}}{\partial w_k} \tag{4.7}$$

This shows that the internal degrees of freedom evolve with respect to the proper time τ, regardless of the external coordinates. In other worlds, one has

$$L\big(x, \frac{dx}{xt}, q, \frac{dq}{d\tau}\big) = \sqrt{-g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu}\ L_{rest}\big(q^k, \frac{dq^k}{d\tau}\big) \equiv \dot{\tau} L_{rest} \tag{4.8}$$

4.1.3 Time dilation - clock's hypothesis

Eq. 4.7 shows that the evolution of the internal degrees of freedom with respect to an arbitrary parameter t is "redshifted", or "time dilated", by a factor $\dot{\tau} = \frac{d\tau}{dt}$. More generally, given any internal observable O_i, its evolution with respect to the time parameter t is given by

$$\frac{dO_i}{dt} = \{O_i, H\} = \{O_i, H_{rest}\}\frac{\partial H}{\partial H_{rest}} = \{O_i, H_{rest}\}\dot{\tau} \equiv \{O_i, H_{rest}\dot{\tau}\} \tag{4.9}$$

where $\{\cdot,\cdot\}$ are the classical Poisson brackets and we use the fact that $O_i = O_i(q^k, \dot{q}^k)$ only depends on the internal degrees of freedom, so that only the internal hamiltonian H_{rest} contributes.

This shows that, for an observer using a time parameter t, the system's internal evolution is time-dilated by a universal factor $\dot{\tau}$, or equivalently, the internal hamiltonian is redshifted by the same factor. Time dilation can thus be understood as the result of interactions between internal (H_{rest}) and external ($\dot{\tau}$) degrees of freedom, which ultimately arises from the general relativistic description of a composite system.

By clarifying this relationship, we can now explore how two systems following different trajectories in spacetime experience different amounts of proper time, leading to observable differences in their internal clock's evolution. This insight is crucial for understanding time dilation in quantum systems.

For simplicity, we consider the internal "velocity" to be approximately constant, so that

$$\frac{dO_i}{d\tau} = \{O_i, H_{rest}\} \approx v_{O_i} \tag{4.10}$$

Then, the above expression shows that the internal degrees of freedom of the system moving along a world line γ evolve as if the time elapsed during this evolution was the proper lenght of the world line $\tau_\gamma = \int_\gamma d\tau$

$$\Delta_\gamma O_i = \int_\gamma \dot{O}_i dt = v_{O_i}\int_\gamma d\tau = v_{O_i}\tau_\gamma \tag{4.11}$$

Thus, considering two different trajectories, one has

$$\int_{\gamma_1} \dot{O}_i dt - \int_{\gamma_2} \dot{O}_i dt = v_{O_i}\left(\tau_{\gamma_1} - \tau_{\gamma_2}\right) \tag{4.12}$$

In this precise sense we can regard a composite particle as an effective realization of an ideal clock in General Relativity, which satisfies the "clock's hypothesis."

4.1.4 Limit of a system of N relativistic particles

In [24], it is shown how a composite particle with the hamiltonian in equation (4.3) can be derived as the limit of a system of N relativistic particles, which are sufficiently localized in spacetime. At a qualitative level, the condition required is that a single coordinate chart must exist, covering all the regions occupied by the N particles, in which the metric is approximately flat. This happens when the relative distances between the particles are sufficiently small, such that variations of the metric over their spatial extension can be neglected.

In this case, one can assign a single position degree of freedom to the center-of-mass of the system, which describes a single world line. Strictly speaking, it is not necessary for all constituents to follow exactly the same world line or move with the same velocity. Rather, the key requirement is that there is no considerable time dilation between the particles.

This provides, on one hand, a condition for how localized the particles must be, so that variations of the metric are negligible, and on the other hand, a condition on the scale of the total energy/momentum of the system. More details can be found in the cited reference.

4.2 Low-energy hamiltonian

A relativistic description of a single quantum particle is meaningful only at low energies, in the so-called "one-particle regime", i.e. where the effects of quantum field theory (QFT) related to particle creation and annihilation are negligible. In this regime, an effective description of the dynamics can be given by a Taylor expansion of the hamiltonian (4.3) in powers of (p/mc) and (ϕ/c^2), where ϕ is the gravitational field.

Considering a static metric, the hamiltonian in a generic coordinate system reads:

$$H = \sqrt{-g_{00}\left(c^2 p_i p_j g^{ij} + Mc^2\right)} \tag{4.13}$$

where $Mc^2 = mc^2 + H_c$ and H_c is the internal hamiltonian. First, we expand in powers of p/mc up to second order

$$H = \sqrt{-g_{00}}\left(Mc^2 + \frac{p^2}{2M}\right) \tag{4.14}$$

In the regime of weak gravitational fields, we describe the metric in the second-order post-Newtonian form, which can be derived from a Taylor expansion of Schwarzschild's metric in isotropic coordinates. This is shown in Appendix (4.B). Choosing the coordinates of a static observer at rest at $r = \infty$, the post-Newtonian metric reads

$$ds^2 \approx -c^2 dt^2\left(1 + 2\frac{\phi(r)}{c^2} + 2\frac{\phi(r)^2}{c^4}\right) + \left(1 - 2\frac{\phi(r)}{c^2}\right)\left(dr^2 + r^2 d\Omega^2\right) \tag{4.15}$$

Expanding the metric further

$$\sqrt{-g_{00}} \approx \left(1 + \frac{\phi(r)}{c^2} + \frac{\phi(r)^2}{2c^4}\right) \tag{4.16}$$

$$p^2 = p_i p_j g^{ij} = p^2\left(1 + 2\frac{\phi(r)}{c^2}\right) \tag{4.17}$$

and neglecting terms of order $p^2\phi(r)^2/c^4$, p^4/c^4, $\phi(r)^2/c^4$ or higher, the hamiltonian becomes

$$H = Mc^2 + M\phi(r) + \frac{M\phi(r)^2}{2c^2} + \frac{p^2}{2M} - \frac{1}{8}\frac{p^4}{M^3c^2} + \frac{3}{2}\frac{p^2\phi(r)}{Mc^2} \tag{4.18}$$

Then, if the energy scale of the internal dinamics H_c is much smaller then the rest-mass mc^2 energy, we can expand the internal mass/energy $M = m + H_c/c^2$ in powers of H_c/c^2. At first order, the hamiltonian reads:

$$H = H_{cm} + H_c\left(1 + \frac{\phi(r)}{c^2} - \frac{p^2}{2m^2c^2}\right) \tag{4.19}$$

where H_{cm} is the hamiltonian of the "centre of mass":

$$H_{cm} = mc^2 + m\phi(r) + \frac{m\phi(r)^2}{2c^2} + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \frac{3}{2}\frac{p^2\phi(r)}{mc^2} \tag{4.20}$$

The main result is the coupling between the internal and external degrees of freedom corresponding to the special and general relativistic time dilation factor [2], which (at lowest order) reads:

[2]Formally, to have an exact correspondance with the time dilation factor, which is given in terms of the configuration space variables, one should adopt the "Routhian" formalism, which is a partial Legendre transform of the Lagrangian w.r.t. a subset of the degrees of freedom. In particular, one can use the hamiltonian formalism for the internal d.o.f. and the Lagrangian formalism for the external d.o.f. More details about this topic can be found in [24]

$$\dot{\tau} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \approx \left(1 + \frac{\phi(r)}{c^2} - \frac{\dot{x}^2}{2c^2}\right). \tag{4.21}$$

The physical interpretation is clear: in an arbitrary "external" reference frame with time coordinate t, the time scale of the internal dynamics will be subjected to the time dilation effect or, equivalently, the internal energy will be redshifted. Notice that this coupling is entirely independent on the nature of the binding/internal energies, which leads to the universality of time dilation and redshift: it affects all "clocks" in the same way, irrespectively of their specific composition.

4.2.1 Non-relativistic limit

The fully classical limit cannot be taken just by neglecting all terms of order $\mathcal{O}(1/c^2)$ or higher. In fact, this would be inconsistent with the experimental evidence of time dilation in composite systems with an internal dynamics in a regime of very slow velocities and weak gravitational fields. For instance, the special-relativistic time dilation, given by the factor $(1 - \frac{v^2}{2c^2})$, has been observed using Alluminium ions moving at velocities of the order of few meters per seconds. Similarly the gravitational time dilation has been observed on clocks separated by a distance of the order $h \sim 30cm$, where the time dilation factor is $(1 + \frac{gh}{c^2})$ [48].

The reason is that the major role is played by the scale of energy of the internal dynamics. The only consistent non-relativistic limit is, in fact, to consider the dynamical part of the internal energy H_c [3] to be also small, such that the hamiltonian reduces to

$$H \approx mc^2 + m\phi(r) + \frac{p^2}{2m} \tag{4.22}$$

Only this limit is consistent with a Newtonian spacetime with an absolute, global time coordinate t. [4]

4.2.2 Quantization

In classical relativity the distinction between rest mass and rest energy is purely conventional and is dictated by the energy scale: we separate the contribution from the "frozen" degrees of freedom to the dynamical ones. However, when considering a quantum system, the contribution of the dynamical degrees of freedom is determined by the internal state of the clock. For each state $|e_i\rangle$ corresponding to an internal energy e_i the total energy energy will be $mc^2 + e_i$. Thus, by linearity, the "rest-mass" M becomes an operator that acts on the internal degrees of freedom by means of the internal hamiltonian

$$m \to M \coloneqq m\, \mathbb{I}_c + \frac{H_c}{c^2} \tag{4.23}$$

where quantities in capital letters are, from now on, understood as operators. With this choice, mc^2 is the mass-energy associated with the internal ground state $|e_0\rangle$ for which $H_c\,|e_0\rangle = 0$.

A rigorous derivation of the hamiltonian operator can be provided as a low-energy limit of a QFT in a curved background. In [24] and [**hamiltonian-from-QFT**] it is shown that one can start from a composite bosonic quantum field ψ_I, $I = 1, ..., N$ with action:

$$S = \int d^4x\, \sqrt{-g} \left(\sum_I\, g^{\mu\nu} \partial_\mu \psi_I \partial_\nu \psi_I + \sum_{I,J} M^2_{IJ} c^2 \psi_I \psi_J \right) \tag{4.24}$$

The Lagrangian can be diagonalized, obtaining the Klein-Gordon equation for each mode $\tilde{\psi}_a$

[3] Formally, one could split the contribution of the internal energy into a dynamical part, played by the hamiltonian H_c, and a static one E_{static}. The non-relativistic limit is then $H_c + E_{static} \approx E_{static}$.

[4] This is important when one looks at the symmetries of the theory. In fact, the Galilean group is recovered only in this limit, whereas considering dynamical degrees of freedom requires its central extension, leading to the superselection rule for the mass in QM. Details about this topic can be found in chapter 3.2 of [24].

$$\left(g^{\mu\nu}D_\mu D_\nu - m_a^2c^2\right)\tilde{\psi}_a^2 = 0 \tag{4.25}$$

where D_μ is the covariant derivative. In the low-energy limit, when particle creation and annihilation effects are negligible, one can treat the field as a single particle in first quantization, approximating the field's components as $\tilde{\psi}_a = \exp\{i\left(c^2S_0(x) + S_1(x) + c^{-2}S_2(x) + ..\right)\}$, representing the internal state of a composite particle. In this approximation, the Schrödinger equation is obtained and so the corresponding hamiltonian. In particular, considering a post-newtonian background at order $\mathcal{O}(\phi(r)/c^2)$, the hamiltonian reads:

$$H = Mc^2 + M\phi(X) + \frac{1}{2}M\frac{\phi(X)^2}{c^2} + \frac{P^2}{2M} - \frac{P^4}{8M^3c^2} + \frac{3}{2Mc^2}\left(\phi(X)P^2 + [P\phi(X)]P + \frac{1}{2}[P^2\phi(X)]\right) \tag{4.26}$$

which has the same form of the classical hamiltonian (4.18), except for the terms which are coupled in P and $\phi(X)$, for which a direct canonical quantization leads to ordering ambiguities.

Then, if we consider the energy scale of the internal dinamics H_c to be much smaller then the rest-mass-energy $\langle H_c\rangle \ll mc^2$, we can expand (4.26) in powers of H_c/c^2. At first order we obtain

$$H = H_{cm} + H_c\left(1 + \frac{\phi(X)}{c^2} - \frac{P^2}{2m^2c^2}\right) \tag{4.27}$$

where

$$H_{cm} = mc^2 + m\phi(X) + \frac{1}{2}m\frac{\phi(X)^2}{c^2} + \frac{P^2}{2m} - \frac{P^4}{8m^3c^2} + \frac{3}{2mc^2}\left(\phi(X)P^2 + [P\phi(X)]P + \frac{1}{2}[P^2\phi(X)]\right)$$

As in the classical case, the above dynamics is fully compatible with the understanding of the hamiltonian as a generator of time translations and with general relativistic understanding of time. H_c is a generator of time translations with respect to the proper time τ and describes the dynamics of internal states in the rest frame. To see this, we notice that the Schrödinger equation in the rest frame should be of the form:

$$i\hbar\frac{d}{d\tau} = H_c \quad \Longleftrightarrow \quad i\hbar\frac{d}{dt} = H_c\dot{\tau} \tag{4.28}$$

Now, consider the hamiltonian (4.27). In the rest frame, where $H_{cm} \sim \mathbb{I}$, one has $H = H_c\dot{\tau}(X,P)$, where the time dilation factor is expressed in term of phase space variables, instead of the usual configuration space variables $X, \dot{X}$. Thus, the Schrödinger equation reads

$$i\hbar\frac{d}{dt} = H_c\dot{\tau}(X,P) \tag{4.29}$$

In appendix (4.C) it is shown, following [23], how the same coupling can be obtained in a fully field-theoretic description, considering a sufficiently localized scalar field in a curved spacetime.

4.3 Universal decoherence due to gravitational time dilation

In the regime of low-velocities and weak gravitational fields, the hamiltonian of a composite particle reads (at lowest order):

$$H = H_{cm} + H_c\left(1 + \frac{\phi(X)}{c^2} - \frac{P^2}{2m^2c^2}\right) \tag{4.30}$$

The interaction term between internal and kinematical degrees of freedom is entirely independent of the nature of the binding/internal energies and this leads to the universality of the resulting (special- and general-relativistic) time dilation (or redshift): the clock's degrees of freedom evolve at a rate $\dot{\tau}(x,\dot{x})$ that depends

on its trajectory in spacetime. Different world lines correspond to a different time evolution, which is a prediction of General Relativity that has been experimentally tested in various scenarios.

With this model at hand, one can consider a novel situation, where a composite quantum system follows a superposition of two different trajectories. Quantum theory allows to assign a quantum state to the external as well as the internal degrees of freedom of any composite system. Then, superposition principle of quantum theory and relativistic time dilation imply that in this situation the clock runs in a superposition of two different proper times. This novel phenomena can in principle be tested, for instance, in an interferometric experiment.

In general, the "universal coupling" present in the hamiltonian (4.30) results in entanglement between the external and internal degrees of freedom. This entanglement inevitably causes a loss of coherence in the center of mass of a generic composite system. Thus, considering an interferometric experiment where the system is placed in a superposition of two different paths, we expect a decoherence mechanism (see 2.1) to manifest in testable effects on the visibility pattern.

In particular, we consider the following scenario, depicted in fig. (4.3.1): the kinematical degrees of freedom of a composite quantum system are put in a $50/50$ coherent superposition of two different paths $\gamma_{\pm}$, for instance two arms of a Mach Zender interferometer, at different heights on the Hearth, where the gravitational field takes two different values. In (2.1) we considered the same scenario, where the system becomes entangled with a generic environment that acts as a which-path detector, leading to decoherence. In this case, the role of the environment is played by the internal degrees of freedom and the generic interaction (2.14) corresponds here to the time dilation factor coupled to the internal hamiltonian.

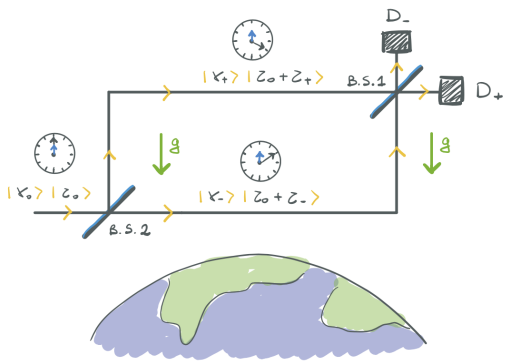

Figure 4.3.1: *Gedanken-experiment of a quantum clock following a superposition of two paths $\gamma_{\pm}$ corresponding to different values of the gravitational field $\Delta\phi \approx gh$ where h is the vertical distance between the paths. Due to time dilation the internal degrees of freedom evolve into different states $|\tau_0 + \tau_{\pm}\rangle$ depending on the path, which can act as "which-way-detectors". The decoherence mechanism implies that the more distinguishable these states are, the more visibility of the interference pattern will be reduced (see (2.22).*

4.3.1 Pure clock's state

We consider the composite system after the first beam splitter to be in a separable pure state, in the form

$$|\Psi(0)\rangle = |\gamma_{sup}\rangle_r \otimes |\tau_0\rangle_c\,, \qquad |\gamma_{sup}\rangle_r = \frac{1}{\sqrt{2}}\Big(|\gamma_+\rangle_r + |\gamma_-\rangle_r\Big) \tag{4.31}$$

where $\gamma_{\pm}$ denotes the two paths, subjected to different gravitational fields. The subscripts c and r denote, respectively, the internal and external degrees of freedom and stand for "$clock$" and "rod" (or "$ruler$")[5] For simplicity, we rewrite the hamiltonian (4.30) as

$$H = H_{cm} + H_c \otimes D \tag{4.32}$$

[5]This notation will be more meaningful in chapter 5.2, where the kinematical and internal degrees of freedom serve, respectively, as a reference frame for space (rod) and time (clock).

where $D = D(X, P)$ is the time dilation factor. Furthermore, we consider a semiclassical approximation so that all terms in the H, apart from the internal hamiltonian H_c, can be substituted with their expectation values along the interferometric paths, which are fixed functions. Specifically, the position operator takes the two values corresponding to the different heights of the paths $X \to x_\pm$, while we consider the momentum to be the same $P \to p$. Additionally, we assume the accelleration the particle undergoes along the "vertical" direction to be the same along the two paths, meaning that that time dilation is only given by the difference in the gravitational field along the "horizontal" paths. Under these assumptions, we have:

$$H_{cm} \left|\gamma_\pm\right\rangle_r \coloneqq E^\pm_{cm} \left|\gamma_\pm\right\rangle_r, \qquad D \left|\gamma_\pm\right\rangle_r \coloneqq D_\pm \left|\gamma_\pm\right\rangle_r \tag{4.33}$$

where

$$E^\pm_{cm} = mc^2 + \frac{p^2}{2m} + m\phi(x_\pm) - \frac{p^4}{8m^3c^2} + \frac{1}{2}m\frac{\phi(x_\pm)^2}{c^2} + \frac{3}{2mc^2}\phi(x_\pm)p^2 \tag{4.34}$$

$$D_\pm = \left(1 + \frac{\phi(x_\pm)}{c^2} - \frac{p^2}{2m^2c^2}\right) \tag{4.35}$$

The initial separable state evolves into :

$$\begin{aligned} &\left|\Psi(t)\right\rangle = e^{-\frac{i}{\hbar}tH} \left|\Psi(0)\right\rangle \\ &= \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}tE^+_{cm}} \left|\gamma_+\right\rangle_r e^{-\frac{i}{\hbar}tD_+H_c} \left|\tau_0\right\rangle_c + e^{i\varphi}e^{-\frac{i}{\hbar}tE^-_{cm}} \left|\gamma_-\right\rangle_r e^{-\frac{i}{\hbar}tD_-H_c} \left|\tau_0\right\rangle_c\right) \\ &\coloneqq \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}tE^+_{cm}} \left|\gamma_+\right\rangle_r \left|\tau_0 + \tau_+\right\rangle_c + e^{i\varphi}e^{-\frac{i}{\hbar}tE^-_{cm}} \left|\gamma_-\right\rangle_r \left|\tau_0 + \tau_+\right\rangle_c\right) \end{aligned} \tag{4.36}$$

where φ is a generic additional contribution to the phase difference accumulated along the paths which depends on the experimental setup. The reduced density matrix of the centre of mass degrees of freedom is

$$\rho_r(t) = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\varphi}e^{-\frac{i}{\hbar}t\Delta E_{cm}} \left\langle\tau_0\right| e^{-\frac{i}{\hbar}\Delta\tau H_c} \left|\tau_0\right\rangle_c \\ e^{i\varphi}e^{\frac{i}{\hbar}t\Delta E_{cm}} \left\langle\tau_0\right| e^{i\Delta\tau H_c} \left|\tau_0\right\rangle_c & 1 \end{pmatrix} \tag{4.37}$$

where

$$\Delta\tau = \tau_+ - \tau_- = t(D_+ - D_-) \tag{4.38}$$

is the difference between the proper time accumulated alongh the different paths and $\Delta E_{cm} = E^+_{cm} - E^-_{cm}$. From (4.37) one can read out the interference pattern and the visibility, which is twice the magnitude of the off-diagonal elements (see 2.8):

$$\begin{cases} p_{D_\pm} = \frac{1}{2} \pm \frac{1}{2}\left|\left\langle\tau_0\right| e^{-\frac{i}{\hbar}\Delta\tau H_c} \left|\tau_0\right\rangle_c\right| \sin(t\Delta E_{cm} + \alpha + \varphi) \\ \mathcal{V} = \left|\left\langle\tau_0\right| e^{-\frac{i}{\hbar}\Delta\tau H_c} \left|\tau_0\right\rangle_c\right| \end{cases} \tag{4.39}$$

where α is defined by

$$\left\langle\tau_0\right| e^{-\frac{i}{\hbar}\Delta\tau H_c} \left|\tau_0\right\rangle_c = \left|\left\langle\tau_0\right| e^{-\frac{i}{\hbar}\Delta\tau H_c} \left|\tau_0\right\rangle_c\right| e^{i\alpha} \tag{4.40}$$

To derive quantitative statements we need to choose a particular model for the quantum clock.

QBIT-clock

The symplest model for a quantum clock is a two level system (see 3.6.3) where the clock's states are coherent superpositions of internal-energy's eigenstates

$$\left|\tau_0\right\rangle_c = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_c + e^{-\frac{i}{\hbar}\omega\tau_0} \left|1\right\rangle_c\right), \qquad \left|\tau_0 + \tau\right\rangle_c = e^{-\frac{i}{\hbar}\tau H_c} \left|\tau_0\right\rangle_c = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_c + e^{-i\omega(\tau_0+\tau)} \left|1\right\rangle_c\right) \tag{4.41}$$

and $\hbar\omega$ is the difference between the two energy levels. The scalar product between clocks state reads

$$\langle t|t'\rangle_c = \frac{1}{2}\left(1 + e^{-i\omega(t'-t)}\right) = e^{-i\frac{\omega(t'-t)}{2}}\cos\left(\frac{\omega(t'-t)}{2}\right) \tag{4.42}$$

Specifically, we are interested in

$$\langle \tau_0|\tau_0+\Delta\tau\rangle_c = e^{-\frac{i}{\hbar}\frac{\hbar\omega\Delta\tau}{2}}\cos\left(\frac{\omega\Delta\tau}{2}\right) = e^{-\frac{i}{\hbar}\langle H_c\rangle\Delta\tau}\cos\left(\frac{\omega\Delta\tau}{2}\right) \tag{4.43}$$

where $\langle H_c\rangle = \hbar\omega/2$ is the average internal energy. The probability (4.39) to observe interference and the visibility read

$$\begin{cases} p_{D_\pm} = \frac{1}{2} \pm \frac{1}{2}\left|\cos\left(\frac{\omega\Delta\tau}{2}\right)\right| \ \sin\left(\frac{t\Delta E_{cm}}{\hbar} + \frac{\langle H_c\rangle\Delta\tau}{\hbar} + \varphi\right) \\ \mathcal{V} = \left|\cos\left(\frac{\omega\Delta\tau}{2}\right)\right| \end{cases} \tag{4.44}$$

The physical interpretation becomes clear when we express this result in terms of the resolution of the clock $t_\perp = \frac{\pi}{\omega}$ (see 3.6.3), which is the time it takes to evolve a clock-state into an orthogonal (i.e., perfectly distinguishable) state:

$$\mathcal{V} = \left|\cos\left(\frac{\pi}{2}\frac{\Delta\tau}{t_\perp}\right)\right| \tag{4.45}$$

When the difference in the proper time accumulated is comparable to its resolution $\Delta\tau \approx t_\perp$, the two superposed clock's internal states are nearly orthogonal, hence distinguishable. The clock's internal state can serve as a measurement device that "knows" which path the particle takes, affecting the overall coherence in the system.

Considering any periodic quantum clock with a discrete spectrum (see 3.6.3), this decoherence effect will also be periodic: the which-way information is periodically stored into orthogonal clock states. Importantly, this periodicity is advantageous for experimental verification since it allows the differentiation of the time dilation effect from other factors, such as visibility losses due to experimental imperfections or other environmental decoherence mechanisms, which typically follow an exponential decay $\sim e^{-t/T_d}$.

For completeness, in appendix (4.D) we explicitly compute $\Delta\tau$ and ΔE_{cm} for this specific scenario. The result is

$$\begin{cases} p_{D_\pm} = \frac{1}{2} \pm \frac{1}{2}\left|\cos\left(t\frac{\omega}{2}\frac{\Delta\phi}{c^2}\right)\right| \ \sin\left(\frac{t\Delta\phi}{\hbar}\left(m + \frac{\langle H_c\rangle}{c^2} + \frac{\bar{E}_{GR}}{c^2}\right) + \varphi\right) \\ \mathcal{V} = \left|\cos\left(t\frac{\omega}{2}\frac{\Delta\phi}{c^2}\right)\right| \end{cases} \tag{4.46}$$

where $\Delta\phi = \phi(x_+) - \phi(x_-) \approx hg$ corresponds to the gravitational potential difference. Notice that the phase has three contributions: $\Phi_N = \frac{t}{\hbar}\frac{m\Delta\phi}{}$ comes from the newtonian coupling, $\Phi_{GR} = \frac{t\bar{E}_{GR}}{\hbar}$ comes from its general-relativistic correction and $\alpha = \frac{\Delta\tau\langle H_c\rangle}{\hbar} = \frac{t\Delta\phi\langle H_c\rangle}{c^2\hbar}$ comes from the time-dilation effect on the quantum clock's internal dynamics. A plot of the predicted visibility pattern is depicted in fig. (4.3.2)

It's important to stress that the main result of this Gedanken-experiment is the drop in visibility (decoherence effect), which can only be explained assuming that the time dilation effect, as predicted by GR, applies to quantum systems in superposition. In particular it's crucial that the internal energy contributes to the total mass as an operator

$$m \to m + \frac{H_c}{c^2} = M \tag{4.47}$$

Only in this case the total hamiltonian (5.22) contains an interaction term that couples internal and external degrees of freedom, resulting in an entangled state and, consequently, in a decoherence effect.

On the contrary, a semiclassical version of the mass-energy equivalence would result in a semiclassical coupling of the form

$$m \;\rightarrow\; m + \frac{\langle H_c \rangle}{c^2}, \qquad H = H_{cm} + \langle H_c \rangle \left(1 + \frac{\phi(X)}{c^2} - \frac{P^2}{2m^2c^2}\right) \tag{4.48}$$

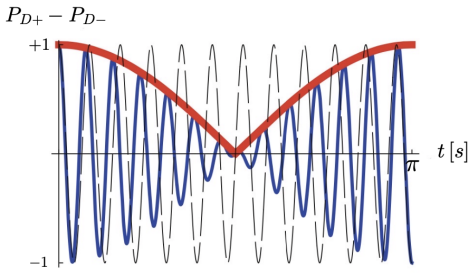

Figure 4.3.2: *Plot of the difference between the probabilities $p_{D_\pm}$ (4.39) to find the particle in the detectors $D_\pm$ of the Mach Zender interferometer (4.3.1) as a function of the t for which the particle travels in a superposition of two different trajectories at constant heights. The dashed, black line corresponds to the same function in the absence of the internal degrees of freedom, so it has no modulation in the visibility and the frequency is given by the newtonian potential with the general relativistic corrections to the centre-of-mass dynamics, that in the main text are referref to as $\Phi_N + \Phi_{GR}$. Adding the clock's degrees of freedom results in the blue line, i.e. in an additional phase shift α and in the periodic modulation of the visibility, that is captured by the red line. The values are chosen so that $\alpha = \frac{\Delta\phi\omega}{2c^2} = 1Hz$. While the effects on relative phase can be understood in terms of classical corrections to the Newtonian potential, the visibility modulation can only be explained if the internal degrees of freedom of the quantum clock undergo general relativistic time dilation, as a consequence of the coupling term in the total hamiltonian (4.32). Fig. taken from [24].*

This coupling (Eq. 4.48) cannot create entanglement, hence it gives a different prediction on the visibility. However it provides the same prediction for the phase of the visibility pattern. In fact, the phase difference resulting from the newtonian potential and the time dilation effect in eq. (4.46) reads:

$$\Phi_N + \alpha = \frac{1}{\hbar} t \left(m + \frac{\langle H_c \rangle}{c^2}\right) \Delta\phi \tag{4.49}$$

Both this and the other general relativistic contribution Φ_{GR} could be fully explained by a modified Newtonian potential in Euclidean spacetime, i.e. considering additional terms in the coupling with the rest-mass of the particle $\sim m\big(V_N + V_{add}\big)$ [6].

The key takeaway is therefore the distinction between a fully quantum mechanical treatment (where the internal energy is an operator and induces entanglement) versus a semiclassical approach (which only shifts the energy levels without introducing entanglement). Observing a drop in visibility would simultaneously disprove non-relativistic, Newtonian gravity as well as the classical description of the world lines of the clocks.

Experimental realization

To have an idea about the possibility of realizing such a Gedanken experiment, we consider the best current atomic clock, for which $\omega \sim 10^{15} Hz$. To observe a loss-and-revival in the modulation of the visibility for a

[6] More generally the phase acquired by a state during its time evolution is proportional to the action along the trajectory on which the system moves. In the presence of any position dependent potential $V(x)$ the phase dependens on the trajectory $\Phi_\gamma \sim \int_\gamma dt V(x(t))$. A phase difference than arise whenevere there's a potential difference along the two paths, even if alongh the single paths the potential is uniform: $\Delta\Phi = t(V(\gamma_+) - V(\gamma_-))$, as in the case we considered above. In the case of an electromagnetic potential this is the well-known Ahronov-Bohm effect, and an analogous effect is present for a Newtonian gravitational field, which results in the first contribution to the phase, i.e. $\Phi_N = t\frac{m\Delta\phi}{\hbar}$

two level-system (4.45) one needs to accumulate a proper time difference of $\Delta\tau = 2t_\perp$ along the two path, which means

$$th \sim 2t_\perp \frac{c^2}{g} = 2\pi \frac{c^2}{\omega g} \sim 10s \cdot m \tag{4.50}$$

For instance one should be ablo to maintain a coherent superposition for $10s$ with a spatial separation of $1m$. This seems to be still a great challenge: for neutrons, one of the best achievements is a separation of $h \sim 10^{-2}$ m with a coherence time of $t \sim 10^{-4}$ [49]. To implement our "clock" in neutron interferometry one can use spin precession in a strong, homogeneous magnetic field. However such quantum-clock could reach frequencies only up to $\omega \sim 10^9 Hz$, which is still a few orders of magnitude lower than necessary for the observation of full decoherence due to a proper time difference. One could also try to measure only the initial drop in the visibility, without waiting for a full orthogonalization of the clocks state, but in this case we would loose the advantage of a periodicity of the signal. A more detailed analysis of the experimental feasibility of this experiment can be found in [24].

4.3.2 General clock's state

The previous result can be generalized to an arbitrary clock state, not necessarily pure. This allows us to consider much more general physical systems as clocks, for instance the vibrational degrees of freedom of a molecule. One considers the centre-of-mass degrees of freedom to be in a coherent susperposition of the two paths, while the clock to be in a possibly mixed state

$$\rho(0) = |\gamma_{sup}\rangle\langle\gamma_{sup}|_r \otimes \rho_c, \qquad |\gamma_{sup}\rangle_r = \frac{1}{\sqrt{2}}\left(|\gamma_+\rangle_r + |\gamma_-\rangle_r\right) \tag{4.51}$$

As before we consider a semiclassical approximation of the system's motion [7], so that:

$$H\,|\gamma_\pm\rangle_r = \left(H_{cm} + H_c \otimes D\right)|\gamma_\pm\rangle_r = \left(E^\pm_{cm} + H_c \otimes D^\pm\right)|\gamma_\pm\rangle_r \tag{4.52}$$

The evolution of the the system reads

$$\rho(t) = \frac{1}{2}\sum_{i,j=\pm} |\gamma_i(t)\rangle\langle\gamma_j(t)|_r \cdot \left(e^{-\frac{i}{\hbar}\tau_{\gamma_i}H_c}\rho_c e^{\frac{i}{\hbar}\tau_{\gamma_j}H_c}\right), \tag{4.53}$$

where $\tau_{\gamma_i} \coloneqq \int_{\gamma_i} d\tau = tD^i$, with $i = +, -$, and

$$|\gamma_i(t)\rangle_r = e^{-\frac{i}{\hbar}tE^i_{cm}}\,|\gamma_i(0)\rangle_r\,. \tag{4.54}$$

The coupling causes entanglement between internal and external degrees of freedom. Explicitly, The reduced density matrix of the centre of mass degrees of freedom reads

$$\begin{aligned}\rho_r(t) &= \frac{1}{2}\sum_{i,j=\pm} |\gamma_i(t)\rangle\langle\gamma_j(t)|_r \cdot tr_c\left[e^{-\frac{i}{\hbar}\tau_{\gamma_i}H_c}\rho_c e^{\frac{i}{\hbar}\tau_{\gamma_j}H_c}\right] \\ &= \frac{1}{2}\sum_{i,j=\pm} |\gamma_i(t)\rangle\langle\gamma_j(t)|_r \cdot tr_c\left[e^{-\frac{i}{\hbar}\Delta\tau_{ij}H_c}\rho_c\right].\end{aligned} \tag{4.55}$$

The visibility is twice the magnitude of the off diagonal terms, hence

$$\mathcal{V} = |\rho_r(t)_{i\neq j}| = \left|tr_c\left[e^{-\frac{i}{\hbar}\Delta\tau H_c}\rho_c\right]\right| = \left|\left\langle e^{-\frac{i}{\hbar}\Delta\tau H_c}\right\rangle_c\right| \tag{4.56}$$

which is the generalization of (4.39). It shows that decoherence occurs whenever a proper time difference is present, i.e. $\Delta\tau \neq 0$, and whenever the internal state is not a pure eigenstate of internal energy, in which

[7] we consider semiclassical paths constrained to have coordinates $z_+ = R + h$, $z_- = R$, $p_\pm = p$

case $\left\langle e^{-\frac{i}{\hbar}\Delta\tau H_c}\right\rangle_c$ would be just a phase factor (the clock would be "frozen" in that eigenstate, without evolving at all). Recall that in (2.1) we show that when the environment is in a classical mixture and gets entangled with the system, it acts essentially as a classical source of noise, which can lead to decoherence, even the mixed state itself does not carry which path information.

This generalization to arbitrary clock states emphasizes that the decoherence is a universal effect, applying to any system that experiences time dilation, whether in a pure or mixed state. It's important because many real-world systems are not in pure states, and yet they still undergo decoherence.

Thermal states

A physically interesting case is a system with N internal independent degrees of freedom, each of them in a thermal state at temperature T. This could describe for instance the normal modes of the vibrational degrees of freedom of a molecule.

Although a detailed analysis is out of the scope of this thesis, this case is of outmost importance to understand the role that this universal decoherence mechanism can have in the physics of every-day life, i.e. considering macroscopic systems with with $N \sim 10^{23}$ in the energy regime of classical GR in its weak-field limit. Therefore we briefly present and comment the main results, following [24].

It is considered the case of N harmonic oscillators with charachteristic frequency ω_i in a thermal state at a temperature T. The visibility (4.56) reads

$$\mathcal{V} = \prod_i \left| \frac{1 - e^{-\beta_T \hbar\omega_i}}{1 - e^{-(\beta_T + \frac{i\Delta\tau}{\hbar})\hbar\omega_i}} \right| \tag{4.57}$$

The key observations are the following:

- Periodicity: The visibility is still a product of periodic functions, hence the visibility undergoes a sequece of "loss and revivals". However revivals of the visibility depend on the lowest frequency mode. Considering that for bulk matter the frequency of the lowest phononic mode decreases with the linear size of the system, the revival time becomes increasingly long for larger systems. For instance, considering a macroscopic system with $N \sim 10^{23}$ with typical frequencies $\omega \sim 500Hz$ and a superposition size $h \sim 1mm$ the visibility is below 0.01 already after $2s$ and the revival time is $t_{rev} = \frac{2\pi}{\omega}\frac{c^2}{gh} \sim 10^{17}$ s, which is the scale of the age of the Universe.
- Dependence of the number of modes: The more modes the system comprises, the faster the visibility becomes negligible with increasing the superposition size h or the time it lasts t, and the shorter is the duration of the revival peaks. This is even more clear in the high temperature limit, where the std. deviation of the energy of each harmonic oscillator is $\Delta H = k_B T$ does not depend on the individual frequency, the visibility can be approximated to

$$\mathcal{V} \approx \left(1 - \left(\frac{ghtk_BT}{\hbar c^2}\right)^2\right)^{\frac{N}{2}} \approx e^{-\left(\frac{t}{t_{dec}}\right)^2}, \qquad t_{dec} \coloneqq \sqrt{\frac{2}{N}}\frac{\hbar c^2}{k_B T gh} \tag{4.58}$$

  where $gh = \Delta\phi$ is the difference in the gravitational field (at first order).

  For example, consider a system at a temperature $T = 300K$ and with frequencies $\omega_i \sim 10^{13}Hz$, which is the typical magnitude for molecular vibrational modes. For a single-mode $N = 1$, the visibility drops to $\frac{1}{2}$ for the time dilation $\Delta\tau = 10^{-12}s$. For the time dilation caused dominantly by the gravitational potential difference $\Delta\phi = gh$, which reads $\Delta\tau = \frac{ght}{c^2}$ with $g \approx 10m/s^2$ and $h = 1mm$ the laboratory time t_{dec} necessary to observe such a visibility drop is $t_{dec} = 107s$ (116 days). It turns out that, already for a system with $N = 10^{18}$, the decoherence time is $t_{dec} \approx 1.7 \cdot 10^{-5}s$. Macroscopic object with $N = 10^{23}$ constituents have decoherence time of order of few microseconds.
- Experimental challenges: An experimental verification of the decoherence induced by time dilation for macroscopic particles encounters many challenges. The first, is to bring such systems in superposition of paths with a suffient spatial separation. One of the largest molecules with which interference

has been observed which has the number of normal modes $N = 2424$ [50] one finds $t_{dec} \approx 3.5 \cdot 10^4 s$ ($\sim 10h$). Furthermore one has to suppress any other decoherence mechanism, as the scattering with surrounding molecules and thermal radiation or the emission of thermal radiation by the system. This requires such an experiment to be performed at liquid Helium temperatures and in ultra-high-vacuum. However, the simple model of N independent harmonic oscillator used for the composition of the system, necessary to estimate the time dilation decoherence rate, is very crude and we expect it to break down at these low temperatures. Given a specific system, the time dilation decoherence could be estimated more accurately by measurement of the internal energy fluctuations through the heat capacity. Nevertheless, the rapid developments in controlling large quantum systems for quantum metrology and for testing wave function collapse models will inevitably come to the regime where the time-dilation-induced decoherence predicted here will be of importance. In the long run, experiments on Earth will have to be specifically designed to avoid this gravitational effect on quantum coherence.

4.3.3 Summary

In this chapter, we have explored the theoretical framework of quantum clocks within the context of General Relativity, showing that time dilation effects, which have been tested experimentally in numerous classical systems, affects quantum systems in superposition in a unique way. In particular, the gravitational and velocity-induced time dilation factor results in a coupling between internal and external degrees of freedom which, in turn, can lead to a universal decoherence mechanism in composite quantum systems. The fact that the decoherence depends on the difference in the accumulated proper times along the superposed paths has far-reaching implications for experiments that probe the intersection of quantum mechanics and general relativity.

In particular, the Mach-Zehnder-type interferometric experiments discussed provide a conceptual setup for testing the decoherence effects induced by time dilation. Such experiments could probe whether quantum clocks, in superpositions of different trajectories, experience gravitational time dilation in a manner consistent with general relativistic predictions. The results of these experiments would not only provide empirical validation of the theoretical framework but also test the universality of time dilation and gravitational redshift at the quantum level. In fact, the above analysis shows that, while the phase shift due to time dilation can be explained within a semi-classical framework, the modulation of the visibility, leading to decoherence, requires a full quantum mechanical treatment.

Finally, we have shown that in the case of macroscopic systems with many internal degrees of freedom, such as molecules or larger objects, the decoherence rate increases drastically. As any decoherence mechanism (2.1), the one caused by time dilation explains the emergence of classicality without any modification of quantum mechanics[8]. On the one hand time dilation provides an ideal decoherence mechanism in the context of reaserch on the quantum-to-classical transition, since it affects all systems and is present in any general-relativistic background. Importantly, this effect can already play a significant role already classical GR in its a weak-field limit. Thus, on the other hand, it could have significant implications for quantum technologies, such as quantum computing and quantum communication, where maintaining coherence is crucial.

[8]This is in contrast to the "collapse-models" approaches, that usually require beakdown of unitarity or include stochastic fluctuations of the metric.

Appendix

4.A Lagrangian of a composite particle

The Legendre transform of the hamiltonian (4.3) reads

$$\begin{aligned} L &= p_j \frac{\partial H}{\partial p_j} + w_k \frac{\partial H}{\partial w_k} - H \\ &- \frac{1}{H} g_{00} c^2 p^i p_i - \frac{1}{H} g_{00} \frac{\partial H}{\partial H_{rest}} \frac{\partial H_{rest}}{\partial w_k} w^k - H \\ &= -\frac{1}{H} g_{00} \left(c^2 p^i p_i + H_{rest} L_{rest} - H_{rest}^2 - \frac{H^2}{g_0 0} \right) \\ &= -\frac{1}{H} g_{00} H_{rest} L_{rest} \end{aligned} \tag{4.59}$$

where $L_{rest} = \frac{\partial H_{rest}}{\partial w_k} w^k$ and $\dot{x}^i = \frac{\partial H}{\partial p_i} = -\frac{1}{H} g_{00} c^2 p^i$. Then, we can write

$$\frac{H_{rest}^2}{H^2} = -\frac{1}{c^2 g_{00}} \left(c^2 g_{00} + \dot{x}^i \dot{x}_i \right) = \frac{1}{c^2} g_{\mu\nu} \dot{x}^\mu \dot{x}_\nu \tag{4.60}$$

where we set the time coordinate to $x^0 = ct$. Thus, one reaches (4.5):

$$\begin{aligned} L &= L_{rest} \left(-\frac{1}{H} g_{00} H_{rest} \right) = L_{rest} \frac{\partial H}{\partial H_{rest}} \\ &= L_{rest} \frac{1}{c} \sqrt{-g_{\mu,\nu} \dot{x}^\mu \dot{x}^\nu} = L_{rest} \dot{\tau} \end{aligned} \tag{4.61}$$

where the proper time τ is defined by the line element

$$c^2 d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu \tag{4.62}$$

4.B Post-newtonian metric

Starting from Schwarschield spacetime in spherical coordinates

$$ds^2 = c^2 dt^2 \left(1 + 2\frac{\phi(r)}{c^2} \right) - dr^2 \left(1 + 2\frac{\phi(r)}{c^2} \right)^{-1} - r^2 d\Omega^2 \tag{4.63}$$

where $\phi(r) = -\frac{Gm}{c^2}$ is the Newtonian potential and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. In order to obtain its isotropic form one performs the following change of variables

$$\begin{cases} r = \omega \left(1 + \frac{Gm}{2c^2 \omega} \right), \qquad dr = d\omega \left(1 - (\frac{Gm}{2c^2 \omega})^2 \right) \\ \left(1 + 2\frac{\phi(r)}{c^2} \right) = \left(1 - 2\frac{Gm}{c^2 r} \right) = \left(\frac{1 - \frac{Gm}{2\omega c^2}}{1 - \frac{Gm}{2\omega c^2}} \right)^2 = \left(\frac{1 + \frac{\phi(\omega)}{2c^2}}{1 - \frac{\phi(\omega)}{2c^2}} \right)^2 \end{cases} \tag{4.64}$$

which yields:

$$ds^2 = c^2 dt^2 \left(\frac{1+\frac{\phi(r)}{2c^2}}{1-\frac{\phi(r)}{2c^2}}\right)^2 - \left(\frac{1-\frac{\phi(r)}{2c^2}}{1+\frac{\phi(r)}{2c^2}}\right)^2 \left(dr^2 + r^2 d\Omega^2\right) \tag{4.65}$$

The following Taylor expansion

$$\left(\frac{1+\frac{x}{2}}{1-\frac{x}{2}}\right)^2 = 1 + 2x + 2x^2 + o(x^3) \tag{4.66}$$

gives, up to order $o\left((\frac{\phi}{c^2})^2\right)$,

$$ds^2 = c^2 dt^2 \left(1 + 2\frac{\phi(r)}{c^2} + 2\frac{\phi(r)^2}{c^4} + o\left((\frac{\phi(r)}{c^2})^3\right)\right) - \left(1 - 2\frac{\phi(r)}{c^2} + o\left((\frac{\phi(r)}{c^2})^2\right)\right)\left(dr^2 + r^2 d\Omega^2\right) \tag{4.67}$$

which is the post-Newtonian metric (4.15).

4.C Coupling hamiltonian in QFT

We consider a Klein-Gordon field of mass m localized in a "small" region of space V, such that the gravitational field can be considered approximately constant. Thus, considering a static gravitational field, one has

$$g_{\mu\nu}(x) \approx g_{\mu\nu}(x'), \qquad \forall x, x' \in V \tag{4.68}$$

The action of a Klein-Gordon field $\psi(x)$ in a generic spacetime (in $c = 1$ units) is

$$S = -\frac{1}{2}\int_V d^4x \, \sqrt{-g(x)} \left(g^{\mu\nu}(x)\partial_\mu\psi(x)\partial_\nu\psi(x) + m^2\psi(x)^2\right) \tag{4.69}$$

Now we consider the region V to be at fixed height above the Earth, hence the Newtonian limit of Schwarschield's metric: $g_{ij} = \delta_{ij}$ and $g_{00}(x) = -(1 + 2\phi(x))$, where $\phi(x)$ the gravitational potential. Substituting $g = g_{00}(x)$ and $g^{00} = 1/g_{00}(x)$ the action reads

$$S = -\frac{1}{2}\int_V d^4x \, \sqrt{-g_{00}(x)} \left(\frac{(\partial_t\psi(x))^2}{g_{00}(x)} + |\nabla\psi(x)|^2 + m^2\psi(x)^2\right) \tag{4.70}$$

From $S = \int_V dt \; L$ one can read out the Lagrangian

$$L = \frac{1}{2}\int_V d^3x \, \left(\frac{(\partial_t\psi(x))^2}{\sqrt{-g_{00}(x)}} - \sqrt{-g_{00}(x)}\left(|\nabla\psi(x)|^2 + m^2\psi(x)^2\right)\right) \tag{4.71}$$

The hamiltonian is the Legendre transform

$$H = \int_V d^3x \, \pi(x)\partial_t\psi(x) - L \tag{4.72}$$

where the conjugate momentum is

$$\pi(x) = \frac{\delta L}{\delta(\partial_t\psi(x))} = \frac{\partial_t\psi(x)}{\sqrt{-g_{00}(x)}} \tag{4.73}$$

So the hamiltonian reads

$$H = \frac{1}{2}\int_V d^3x\sqrt{-g_{00}(x)}\left(\pi(x)^2 + |\nabla\psi(x)|^2 + m^2\psi(x)^2\right) \tag{4.74}$$

In the above approximation, i.e. restricting to a volume where the gravitational field is approximately constant, we get

$$H = \sqrt{-g_{00}(\overline{x})}\left(\frac{1}{2}\int_V d^3x\left(\pi(x)^2 + |\nabla\psi(x)|^2 + m^2\psi(x)^2\right)\right) \coloneqq \sqrt{-g_{00}(\overline{x})}H_0 \tag{4.75}$$

where $\overline{x}$ is some point in V and H_0 is the hamiltonian is the usual Klein-Gordon hamiltonian in a finite volume in Minkowski spacetime. The factor in front of H_0 is the same redshift - or time-dilation - factor we found for the hamiltonian of a classical composite particle (4.3): all energies measured in an external reference frame (t, x) are redshiftd with respect to those measured in the frame $(\tau, 0)$, where τ is the proper time (in this case the "comoving" frame with respect to the volume V we are considering). The special-relativistic contribution from the "external" momentum is recovered by moving to an arbitrary coordinate system.

4.D Proper time difference for a QBIT clock in a superposition of different heights

In this section we compute explicitly the proper time difference and the difference in the centre of mass energy between the two superposed path in the scenario depicted in Fig. (4.3.1). In particular, we consider for simplicity semiclassical paths with fixed vertical coordinates $z_+ = R + h$, $z_- = R$ and the same momentum $p_\pm = p$. In this case, the time dilation effect is only given by the gravitational field

$$\Delta\tau = \int_{\gamma_+} dt\left(1 - \frac{p_+^2}{2m^2c^2} + \frac{\phi(x_+)}{c^2}\right) - \int_{\gamma_-} dt\left(1 - \frac{p_-^2}{2m^2c^2} + \frac{\phi(x_-)}{c^2}\right) \tag{4.76}$$

$$= t\left(\frac{\phi(R+h) - \phi(R)}{c^2}\right) = t\frac{\Delta\phi}{c^2} \tag{4.77}$$

The earth's gravitational field can be expanded at first order $\phi(R+h) \approx \phi(R) + h\phi'(R) = \phi(R) + hg$, where $g = \frac{G_N M}{R^2}$ and M the mass of the earth. Thus, one has

$$\Delta\tau = t\left(\frac{gh}{c^2}\right) \tag{4.78}$$

where t is the total "time of flight". Similarly, we can compute the difference in the centre of mass energy, which, neglecting the derivatives in the gravitational field, reads

$$H_{cm} = mc^2 + m\phi(X) + \frac{P^2}{2m} - \frac{P^4}{8m^3c^2} + \frac{m\phi(X)^2}{2c^2} + \frac{3}{2mc^2}\phi(X)P^2 \tag{4.79}$$

Under the above approximations, one has

$$H_{cm} = mc^2 + m\phi(x_\pm) + \frac{m\phi(x_\pm)^2}{2c^2} + \frac{3}{2mc^2}\phi(x_\pm)p^2 \tag{4.80}$$

$$\Phi_{GR_\pm} = \frac{1}{\hbar}\int_\gamma dt\frac{\phi(x_\pm)}{c^2}\left(mc^2 + \frac{m\phi(x_\pm)}{2c^2} + \frac{3p^2}{2mc^2}\right) \tag{4.81}$$

Thus, we reach the terms of eq. (4.46):

$$\Delta\Phi_{GR} = \Phi_{GR_+} - \Phi_{GR_-} = \frac{t}{\hbar}\frac{\Delta\phi}{c^2}\left(mc^2 + \frac{m\overline{\phi}}{c^2} + \frac{3p^2}{2mc^2}\right) \coloneqq \frac{t}{\hbar}\frac{\Delta\phi}{c^2}\left(mc^2 + \overline{E}_{GR}\right) \tag{4.82}$$

where $\overline{\phi} = \big(\phi(R+h) + \phi(R)\big)/2$ is the average gravitational field. The second and third contributions are the general relativistic corrections to newtonian dynamics of the centre of mass, that we denote with $\overline{E}_{GR}$.

Chapter 5

Quantum limitations on the measurements of ST distances

A common foundational approach to physics is to assume key principles from different theories and investigate their consequences, often combining them through thought experiments. A prototypical example is Einstein's development of Special Relativity. By combining the relativity principle from Galilean mechanics with the constancy of the speed of light from electromagnetism, simple thought experiments involving light signals, clocks and "rods", led to the discovery of the contraction of lengths, time dilation, and the relativity of simultaneity, which implied conceptual changes to the notions of space and time. Similarly, by combining Heisenberg's uncertainty principle $\Delta x \Delta p \geq \hbar/2$ from quantum mechanics with the relativistic energy relation $E^2 = p^2c^2 + m^2c^4$, one can reveal the limitations of the single-particle concept. For a relativistic particle, where $E \sim p$, the uncertainty relation $\Delta x \Delta E/c \geq \hbar/2$ implies that if we push the spatial resolution beyond the Compton wavelength $\Delta x \leq \hbar/mc$, the uncertainty in energy becomes larger than the particle's rest mass $\Delta E/2mc^2 \geq 1$. This straightforward calculation highlights that relativistic quantum mechanics has to face a resolution limit, beyond which a multiparticle theory is needed, i.e. Quantum Field Theory (QFT).

In the same spirit, an operational approach to reference frames in quantum theory has been adopted in the literature [14],[51],[15], to explore how the operational notions of space and time might change when combining foundational principles of quantum theory and general relativity. This intersection of principles allows us to question our classical assumptions about spacetime structure and measurements.

In quantum mechanics one specifies a space-time point by its coordinates, assuming a fixed, background structure. However, according to the principles of General Relativity (GR), in particular diffeomorphism invariance, space-time points have no physical meaning by themselves as they are not directly observable. The only way to define space-time events and their relations is through space-time coincidences, i.e., intersections between world-lines (see Appendix 5.A).

A common approach to defining events in spacetime involves a latticework of point-like clocks. The main assumption is that these clocks correlate with the metric field such that their readings coincide with the proper time predicted by the theory — this is known as the "clock hypothesis". If this hypothesis holds true [1], a physical reference frame can be established over an extended region by arranging an array of material clocks on a spacelike hypersurface. Spacetime can then be split into the (proper-) time measured by the clocks and the spacelike distance between nearby clocks. This picture allows to define a physically meaningful notion of space-time point: the clock records an event (ST coincidence) and labels it using the (local proper-)time read when the event occurs. The spatial coordinates are defined via measurement of the spacelike distances between the nearby clocks.

In the following sections, we will explore some limitations on the measurability of space and time that arise when considering the quantum nature of clocks. In particular, in the first section, inspired by the famous work by Salecker and Wigner [14], we investigate the limitation on Einstein's synchonization of distant clocks, showing that if one considers quantum clocks as composite systems with internal degrees of free-

[1] for instance, in the previous chapter (4), we show that, in the low-energy regime, composite systems are a physical realization of ideal clocks, satisfying the clock hypothesis.

dom, perfect synchronization seems to be unachievable.

In the second section, we investigate how the precision of time measurements in quantum clocks affects their localization in space, showing that increasing the precision in time leads to a loss of spatial localization. This trade-off emerges from putting together fundamental aspects from QM and SR. According to QM, a very precise clock means a very large uncertainty in its energy [52]. According to SR energy is equivalent to mass. Combining these two aspects, it follows that a very precise clock has a large uncertainty in the mass, implying a large uncertainty in its time evolution.

5.1 Measurability of space-like distances and clock-synchronization

As explained in the introductory part, the measurement of distances between events in spacetime (ST) is essential for making the definition of a coordinate system physically meaningful. While in a classical theory it is reasonably assumed that these measurements can be carried out without restrictions, quantum mechanics, in particular Heisenberg's uncertainty principle, inevitably imposes fundamental limitations on them. This issue was first pointed out by Salecker and Wigner [14].

In this section, we will first review the simple arguments used by Salecker and Wigner, who showed that considering the quantum nature of clocks leads to fundamental limitations on the measurement of space-like distances. The first novel part of the thesis consists on analyzing the consequencies of this limitation on the synchronization of distant clocks. Specifically, we will discuss how Einstein's synchronization of two quantum clocks, when in a separable state, is inherently limited by the uncertainty in their relative distance. Finally, we will explore whether entanglement between the positions of quantum clocks can be used to overcome these limitations. We will show that, if one considers quantum clocks as composite systems with internal degrees of freedom, perfect synchronization is not achievable in this way.

5.1.1 Space-like distances

The prototypical procedure for the measurement of spacetime distances involves a clock, a mirror and a light signal: the light signal is sent from the clock to the mirror and reflected back, so that the coincidences between the trajectories of light and the two massive objects define physical spacetime events. By measuring of the elapsed proper time of the clock T one measures also the spacelike distance $L = Tc/2$ between these two events. This setup essentially reduces the measurement of spatial distances to that of proper-time intervals, relying on the constancy of the speed of light. This setup is sketched in Fig. (5.1.1) below.

Notice that, in a general spacetime, this measurement procedure is clearly restricted to distances which are small compared to the curvature and to a static metric, otherwise the very notion of definite distance between two events looses its meaning. Only in this case the measurement of spacelike-distances can be reduced to the measurement of time intervals [14].

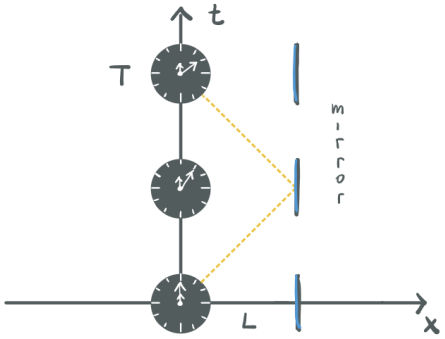

Figure 5.1.1: *Prototypical setup to measure the spacelike distance between two events in flat spacetime. A light signal is sent from a clock to a mirror and reflected back. The events are the coincidences of the wordline of the light signal with the woldlines of the clock and the mirror. By measuring the elapsed proper time T of the clock one also measures the spacelike distance $L = T/2c$.*

Under these restrictions, in the classical theory the spacelike distance between the clock and the mirror can be measured with arbitrary precision, if the clock has arbirtary accuracy. However, if one considers the clock to be a quantum system, its position and momentum are subjected to quantum uncertainty and this inevitably reflects on an uncertainty on the measurement. In particular, given an uncertainty σ_X on the position, due to the uncertainty principle the uncertainty on the velocity is, in the best scenario, of the order $\hbar/(2\sigma_X m)$. If the average time of flight of the light signal is $T = 2L/c$, the total uncertainty in position is:

$$\delta x \sim \sigma_X + \frac{\hbar}{2\sigma_X m} T = \sigma_X + \frac{\hbar L}{\sigma_X c m} = \sigma_X + \frac{L\lambda_c}{\sigma_X} \gtrsim \sqrt{L\lambda_c} \tag{5.1}$$

where $\lambda_c = \hbar/mc$ is the Compton wavelenght of the particle. The disequality follows from finding the minimum of δx as function of σ_{X_r}, which corresponds to $\sigma_X = \sqrt{\lambda_c L}$. This shows that even in optimal cases, quantum mechanics sets a minimum uncertainty in measuring space-like distances. The fact that the uncertainty grows with the distance reflects the spread of the wavefunction with time of a free particle (see also chapter 5.2).

Another way to look at (5.1) is that a classical clock, or in general a classical measurement device [2], is infinitely massive.

Operational approach in quantum gravity

To better understand the implications of this line of reasoning, it is insightful to introduce gravity, even though it will be neglected in the following section. General relativity imposes further constraints on the problem, as a sufficiently massive body will collapse into a black hole, preventing any signals from escaping to an observer. Specifically, according to general relativity, if the mass of a body of finite size exceeds a critical value, an event horizon forms, such that no signals can escape to the outside region. Therefore, the radial size of the body must be larger than the Schwarzschild radius r_S, given by:

$$r \geq r_S = \frac{2Gm}{c^2} \quad \rightarrow \quad m \leq \frac{c^2 r}{2G} = \frac{\hbar r}{c L_p^2} \tag{5.2}$$

Where $L_p = \sqrt{G\hbar/c^3}$ is the Planck lenght. This shows that, as far as the body has a finite (radial) extension r, the concept of a classical reference frame in the presence of gravity, represented by a physical system with a limited mass, seems to be incompatible with the same concept in quantum mechanics. The only theoretical solution is to consider an infinitely extended body $r \to \infty$.

Thus, even at this level of reasoning, combining quantum mechanics and general relativity reveals that the classical notion of a reference frame becomes fundamentally inadequate.

In [15] the mass bound (5.2) is combined qith the previously derived quantum mechanical bound (5.1), obtaining:

$$\delta x \geq L_p \sqrt{\frac{L}{r}} \tag{5.3}$$

This result indicates that there is a minimum uncertainty in the measurements of distances of the order of the Planck length, which seems to be a model independent feature of quantum gravity.

5.1.2 Clock's synchronization

The synchronization of distant clocks in a Minkowski spacetime can be achieved using Einstein's protocol. Consider two clocks at relative rest separated by a distance $2L$, and a laboratory with a source of light in the midpoint. Einstein's synchronization consists of sending two perfectly time-correlated light signals [3] from

[2] where classical means it can in principle perform measurements with arbitrary accuracy, at least if the observed quantity is itself classical

[3] Tight time correlations of photon pairs, generated from spontaneous parametric down-conversion (SPDC), are used in practice to achieve distant clock's synchronization. See, for instance, [53], [54]

the midpoint between them: whatever is their relative velocity with respect to the source, in the rest frame of the two clocks the time of flight of the photons is the same $\tau_1 = \tau_2 = L/c$ (assuming the constancy and isotropy of the speed of light). Thus, when the light signals reach the two clocks, they can set their initial proper times $\tau_1 = \tau_2 = \tau_0$, thereby achieving synchronization.

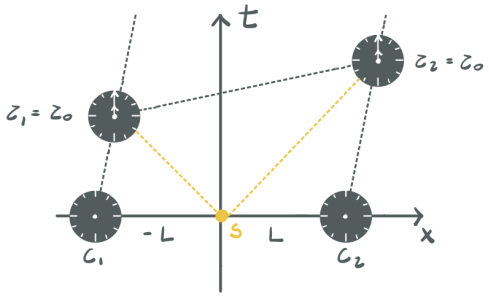

Figure 5.1.2: *Einstein's syncrhonization protocol. When the source S lies a the midpoint between the two clocks C_1, C_2, two light pulses are emitted. If the clocks are at relative rest, when they are reached by the pulses, they can set their initial (proper) time to the same value $\tau_1 = \tau_2 = \tau_0$ to achieve synchronization.*

However, once the quantum nature of the clocks is considered, the uncertainty in their positions leads to limitations in the synchronization process. One can think to prepare the first clock at rest $p_{0,1} = 0$ and at position $x_{0,1} = -L$, and similarly the second with $p_{0,1} = 0$ at a fixed distance $x_{0,2} = L$. However the phase space variables of the clocks (x_i, p_i), $i = 1, 2$ are subjected to quantum fluctuations $(\delta x_i, \delta p_i)$.

At $T = 0$ a light signal is emitted from the source. While the signal is travelling, the clocks move with a velocity $v = p/m$ which is 0 on average, but has some uncertainty $\delta v = \delta p/m$. Thus, to find the time T the light takes to reach the clocks we have to solve for

$$T = \frac{x + Tp/m}{c} \quad \Longleftrightarrow \quad T = \frac{x}{c - \frac{p}{m}} \tag{5.4}$$

By simple error propagation, one finds

$$\delta T^2 = \left(\frac{x_0}{m(c - p_0/m)^2} \right)^2 \delta p^2 + \left(\frac{1}{(c - p_0/m)} \right)^2 \delta x^2 \tag{5.5}$$

Thus, considering $x_0 = L$ and $p_0 = 0$ one has

$$\delta T^2 = \left(\frac{L}{mc^2} \right)^2 \delta p^2 + \frac{\delta x^2}{c^2} \tag{5.6}$$

Considering the best scenario of gaussian wavepackets we have $\delta x = \sigma_X = \frac{\hbar}{2\delta p}$, hence:

$$\delta T^2 = \left(\lambda_c \frac{L}{2c} \right)^2 \frac{1}{\sigma_X^2} + \frac{\sigma_X^2}{c^2} \tag{5.7}$$

where $\lambda_c = \hbar/mc$ is the Compton wavelenght. Its minimum corresponds to $\sigma_X^2 = \frac{\lambda_c L}{2}$, and by inserting it into (5.7) we find

$$\delta T^2 \geq \frac{\lambda_c L}{c^2} \tag{5.8}$$

The same bound can be found by considering the bound on spacelike distances found by Salecker and Wigner and dividing it by the speed of light:

$$\delta x^2_{min} = \lambda_c L \iff \delta T^2_{min} = \frac{\lambda_c L}{c^2} \tag{5.9}$$

If the clocks are regarded as quantum systems, perfect synchronization is not achievable in principle. In other words, a simultaneity surface $t = const$ cannot be operationally defined in Minkowski spacetime when the uncertainty principle is considered as a fundamental limit.

5.1.3 Thought experiment for a perfect synchronization

In the previous analysis we considered the two clocks to be preparred, from the laboratory frame, in a separable state of the form

$$|\Psi\rangle = |\psi_{p_0=0,x_0=-L}\rangle_1 \otimes |\psi_{p_0=0,x_0=L}\rangle_1 \tag{5.10}$$

where $|\psi_{p_0,x_0}\rangle_i$ are, in the best scenario, gaussian wavepackets centered in phase space at (p_0, x_0).

The question arises whether entanglement between the two clocks can be used to overcome this fundamental limit. In principle, perfect synchronization would require the two clocks to be in the same inertial frame and the signal to start exactly from the midpoint between them. One way to approach this is to consider a maximally entangled Einstein-Podolsky-Rosen (EPR) state, where the relative velocity $v = \frac{p_1}{m_1} - \frac{p_2}{m_2} = 0$ and the total position $X = x_1 + x_2 = 0$ vanish without uncertainty. For two quantum systems with the same, fixed mass m, this condition can be achieved with an EPR state of the form:

$$\begin{aligned} |\Psi\rangle &= \int dx_1 \int dx_1 \; \delta(x_1 + x_2) \, |x_1\rangle_1 \, |x_2\rangle_2 = \\ &= \int dp_1 \int dp_2 \; \delta(p_1 - p_2) \, |p_1\rangle_1 \, |p_2\rangle_2 \end{aligned} \tag{5.11}$$

These correlations in continuous variables are not a pure idealization. For instance in ([55]) a scheme for preparing this EPR state of distantly separated trapped atoms is proposed, in which the correlation is transferred from propagating light fields to the atoms via a set of interactions in a cavity. Thus, even if perfect correlation as in eq. (5.11) is unphysical, these states can be very well approximated in reality.

Perfectly correlated EPR states correspond to complete knowledge about the relative velocity $\tilde{v}$ and the total position x_T, but maximum uncertainty about their canonically-conjugate variables, respectively the relative position $\tilde{x} = x_1 - x_2$ and total velocity $v_T = v_1 + v_2$.

Formally, one can describe a system of two (non-interacting) particles with the phase space variables given by following symplectic map:

$$\begin{aligned} x_1 \to x_1 - x_2 \coloneqq \tilde{x} \qquad & p_1 \to \frac{p_1 - p_2}{2} \coloneqq \tilde{p} \\ x_2 \to \frac{x_1 + x_2}{2} \coloneqq x_T \qquad & p_2 \to p_1 + p_2 \coloneqq p_T \end{aligned} \tag{5.12}$$

The canonically conjugate variables are now the relative position and momentum $\{\tilde{x}, \tilde{p}\} = 1$ and the total position and momentum $\{x_T, p_T\} = 1$. Therefore one can in principle arbitrarily squeeze the states along x_T and $\tilde{p}$, which are commuting variables, but paying the price to increase the uncertainty on their conjugate variables $\tilde{r}$ and p_T.

In fact, the EPR state of eq. (5.11) can then defined as the limit of Gaussian states in these variables, with infinite squeezing (delta function) along the total position x_T but maximum uncertainty (uniform distribution) along the relative position $\tilde{x}$:

$$\delta(\frac{x_1 + x_2}{2}) = \delta(x_T) = \lim_{s \to \infty} \left(\sqrt{\frac{1}{\sqrt{\pi} e^s}} \exp\left\{ -\frac{x_T^2}{2e^{2s}} + \frac{i}{\hbar} x_T p_{T,0} \right\} \quad \sqrt{\frac{e^s}{\sqrt{\pi}}} \exp\left\{ -\frac{(\tilde{x} - \tilde{x}_0)^2}{2e^{-2s}} \right\} \right) \tag{5.13}$$

This corresponds, in terms of the conjugate variables, to infinite squeezing along the relative momentum $\tilde{p}$ but maximum uncertainty for the total momentum p_T:

$$\delta(\frac{p_1-p_2}{2}) \sim \delta(\tilde{p}) = \lim_{s\to\infty} \left(\sqrt{\frac{e^s}{\sqrt{\pi}\hbar}} \exp\left\{ -\frac{(p_T-p_{T,0})^2}{2\hbar^2 e^{-2s}} \right\} \quad \sqrt{\frac{\hbar}{\sqrt{\pi}e^s}} \exp\left\{ -\frac{\tilde{p}^2}{2\hbar^2 e^{+2s}} - \frac{i}{\hbar}\tilde{p}\tilde{x}_0 \right\} \right) \tag{5.14}$$

The uncertainty in the time of flight of the photons (5.7 can be expressed in terms of the relative velocity and total position:0

$$\delta T \sim \frac{\delta\tilde{p}}{m}\frac{\tilde{x}_0}{c^2} + \frac{\delta x_T}{c} \tag{5.15}$$

where $\frac{\delta\tilde{p}}{m} = \delta\tilde{v}$ is the uncertainty in the clocks' velocity and $\tilde{x}_0/c$ is the average time of flight. The difference with respect to the case of separable state is that δT depends on the variances of two commuting variables that, in principle, can be "squeezed" jontly. Hence, in the case of fixed-mass systems, perfect synchronization is in principle achievable through this entanglement-based protocol.

In a hypothetical scenario, one could measure the position of the second clock a time t_M after the signal has been emitted. If the clock is found at position x_M such that $\frac{x_M}{c} < t_M$ than the other clock will be at position $-x_M$ and will be synchronized with the measured clock.

Dynamical mass

Considering real quantum clocks with internal degrees of freedom, the situation changes. A quantum clock is a composite system where the mass becomes a dynamical entity (see (4.23). In this case, the relative momentum and the relative velocity no longer coincide, and the simple synchronization protocol outlined earlier no longer applies.

The previous entangled EPR state, incorporating internal degrees of freedom, reads:

$$|\Psi\rangle = \int dp_1 \int dp_2\ \delta(p_1-p_2)\,|p_1\rangle_1\,|p_2\rangle_2 \left(\sum_i c_i\,|m_i\rangle_1\right)\left(\sum_j c_j\,|m_j\rangle_2\right) \tag{5.16}$$

where, without loss of generality, we consider a discrete internal energy spectrum (see (3.6.3) and for simplicity we define $m_i = m + \epsilon_i/c^2$, where m is the rest mass and ϵ_i are the discrete eigenvalues of the internal hamiltonian of the clocks (see 4.27).

In this case, velocities are no longer maximally correlated, so perfect synchronization is not possible. To achieve synchronization one would need maximal correlations between the relative velocities of the clocks, rather than their momenta:

$$v_1 - v_2 = \frac{p_1}{m_1} - \frac{p_2}{m_2} = 0 \to p_1 m_2 - p_2 m_1 = 0 \tag{5.17}$$

This leads us to consider a different canonical transformation in phase space:

$$\begin{aligned} x_1 \to x_1 - x_2 &\coloneqq \tilde{x} \qquad & p_1 \to \frac{m_2 p_1 - m_1 p_2}{m_1+m_2} &\coloneqq \mu\tilde{v} \\ x_2 \to \frac{m_1 x_1 + m_2 x_2}{x_1+x_2} &\coloneqq x_{CM} \qquad & p_2 \to p_1 + p_2 &\coloneqq p_T \end{aligned} \tag{5.18}$$

that corresponds to the usual splitting between centre-of-mass and relative degrees of freedom. Now the conjugate variables are the relative velocity and the relative position $(\tilde{x}, \mu\tilde{v})$, the c.o.m. position and the total momentum (x_{CM}, p_T). To achieve perfect synchronization one should consider wavefunctions with both infinite squeezing along the relative velocity $\mu\tilde{v}$ and along the total position x_T. However, this cannot be achieved, since they are not commuting variables. This is clear once we express

$$x_T = x_1 + x_2 = 2x_{CM} + \tilde{x}(\frac{m_2 - m_1}{m_1 + m_2}) \tag{5.19}$$

and $\{\tilde{x}, \mu\tilde{v}\} = 1$. In other words, the uncertainty in the time of flight of light depends on the variances of two non-commuting variables

$$\delta T \sim \delta\tilde{v}\frac{\tilde{x}_0}{c^2} + \frac{\delta x_T}{c} \tag{5.20}$$

As a result, even in principle, perfect synchronization cannot be achieved with quantum clocks that have internal degrees of freedom. Notice that the error in the synchronization ultimately depends on how large is the uncertainty on the internal energy states. In fact, considering the average mass-energy to be the same $m_{1,0} = m_{2,0}$, one has, by error progagation:

$$\delta x_T \sim \tilde{x}_0 \frac{\delta m_T}{m_{T,0}} \tag{5.21}$$

where $m_T = m_1 + m_2$.

This concludes the discussion on how quantum mechanical effects limit the synchronization of distant clocks. While perfect synchronization seems to be in principle achievable with fixed-mass EPR states, the inclusion of internal degrees of freedom in quantum clocks introduces fundamental limitations that make perfect synchronization unachievable.

5.2 Joint measurability of space and time

A classical material clock can continuously measure time with arbitrary accuracy while having well-defined relative positions in space. The precision in time measurements of the clock has no impact on the precision in the measurement of the spatial (or spacelike) distance with respect to the nearby physical systems. In this section, we investigate whether this is true also in a quantum setting, i.e. when the clocks are regarded as quantum systems.

When considering a quantum clock (see 3.6), the precision of time measurements is closely related to its internal energy spectrum. Clock's states are coherent superpositions of internal energy eigenstates, and the broader this spectrum, the more distinguishable the clock states become, leading to higher temporal precision.

However, including the external (or kinematical) degrees of freedom of the clock, a critical trade-off arises. As discussed in section (5.22), the internal dynamics of quantum clocks inevitably influence their external dynamics due to the energy-mass equivalence principle. A quantum clock in a superposition of internal energies is also in a superposition of masses. Given that the mass determines the external motion, an uncertainty in the internal energy spectrum leads to an uncertainty in position, as the clock evolves in time. This seems to imply a fundamental duality: as a quantum clock improves its time precision, it becomes less effective as a spatial reference (a "rod"). In this way, a single quantum clock used as both a spatial and temporal reference frame exhibits an unavoidable trade-off between spatial localization and temporal accuracy.

A suited framework to investigate this fundamental issue is a single quantum clock used as a QRF (see 3) both for space and time translations. Operationally, this corresponds to restricting to the minimum amount of resources to define the space and time coordinates for another quantum system. In this minimalistic approach, the clock serves as both a temporal and spatial reference, forcing us to confront the inevitable trade-off between precision in these two domains.

In this section, we will first analyze the dynamics of two different states of a quantum clock with respect to an external reference frame for spacetime coordinates. This analysis will help us understand the physical principles underlying the trade-off between spatial and temporal precision. We will then consider the quantum clock as a *spacetime QRF* for another quantum system, examining the implications of this trade-off on the invariant (i.e., physical) algebra (see 5.56), ultimately demonstrating that the relative position of the system w.r.t. the QRF can never be sharply defined.

5.2.1 States of a spacetime QRF

Throughout the whole section, we denote as "*spacetime QRF*" the internal, together with the external degrees of freedom of a quantum clock, i.e. $\mathcal{H}_{cr} = \mathcal{H}_r \otimes \mathcal{H}_c$, where "$c$" stands for "clock", while "r" stands for "ruler" (or "rod").

The trade-off between spatial localization and time precision arises naturally from the dynamics of a relativistic spacetime QRF. We consider a composite particle in a Minkowski spacetime in the low-energy (or one-particle) regime, where $p \leq mc$. At order $\mathcal{O}((p/mc)^2)$, the hamiltonian (see 4.26) reads:

$$H_{cr} = \frac{P_r^2}{2M} + Mc^2 \tag{5.22}$$

where $M = m_r + H_c/c^2$ and H_c is the hamiltonian of the internal degrees of freedom. This form of the hamiltonian highlights how changes in the internal energy spectrum directly impact the external dynamics. Specifically, the uncertainty in the internal energy H_c corresponds to an uncertainty in the effective mass M, which influences how the clock propagates in space.

To study the dynamics of a spacetime-QRF, a choice for its states has to be made. In the following, we will choose states that, on the one hand, permit us to easily perform computations and, on the other hand, allow for a full control over the key parameters underlying the trade-off, namely the standard deviation of the position operator $\sigma^2_{X_r}$ and of the internal hamiltonian $\sigma^2_{H_c}$.

External degrees of freedom - phase-space MUS

The most natural choice for the state of the "ruler" (external d.o.f.) is a coherent state, which is a gaussian state both in momenta and position space and it is the minimum uncertainty state (MUS) for these variables (i.e. in phase space), meaning it saturates the Heisenberg uncertainty principle

$$\sigma_X^2 \sigma_{P_r}^2 - \sigma_{X_r P_r}^2 \geq \frac{1}{4} \left| \langle [X_r, P_r] \rangle \right|^2 \tag{5.23}$$

These states are gaussians centered around a peak momentum, say p_0, independent of the mass of the particle.

Considering a spacetime QRF, i.e. including the clock's degrees of freedom, MUS for position and momentum must be separable states of the form

$$|\psi\rangle^P = \int dp \; \psi_{p_0,x_0}(p) \, |p\rangle_r \otimes \int d\epsilon \; \phi_{\epsilon_0,\tau_0}(\epsilon) \, |\epsilon\rangle_c \tag{5.24}$$

Here, $\psi_{p_0,x_0}(p)$ is a (normalized) Gaussian in phase space centered around (p_0, x_0), while $\phi_{\epsilon_0}(\epsilon)$ can be in principle any complex function in (ϵ, τ) space. We will make a specific choice later on.

At a qualitative level, we can see that for each mass/energy $m_\epsilon = m + \epsilon/c^2$, which are the eigenvalues of the relativistic mass operator $M = m + H_c/c^2$, there is a common peak momentum p_0. This implies a different peak velocity $v_{0,\epsilon} = \frac{p_0}{m_\epsilon}$, leading to different rates of evolution in configuration space:

$$x(t)_\epsilon = v_{0,\epsilon} t = \frac{p_0}{m_\epsilon} t \tag{5.25}$$

Initially the composite particle is localized within the spread $\sigma_{X_r} = \hbar/2\sigma_{P_r}$. However, as the system evolves, each "branch," corresponding to a different internal energy (or mass), evolves at different velocities, causing the system to delocalize further in space, as depicted in fig. (5.2.1). We will make this qualitative statement more precise in the following section (5.2.2).

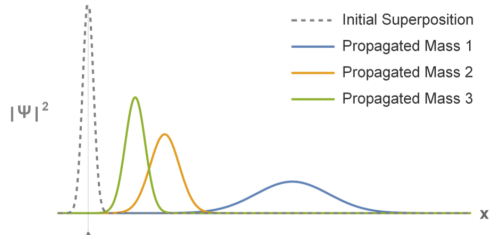

Figure 5.2.1: *Evolution of a gaussian wavepacket in an equal superposition of three internal energy eigenstates (in natural units, with $\hbar = c = 1$). Initial states at $t = 0$ (dashed, grey lines) and final states at $t = 5$ (solid, coloured lines). For each (integer) value of the mass-energy, the wavepacket evolves with a different peak velocity. The result is a larger spread in position space. The figure is taken from [56].*

External degrees of freedom - configuration-space MUS

An alternative choice is to start with an entangled state, such that for each mass in the superposition, we have the same peak velocity v_0 and hence a different peak momentum $p_{0,\epsilon} = m_\epsilon v_0$. In this way each branch evolves with the same rate in configuration space $\delta x(t)_\epsilon = v_0 t$ and the dynamical coherent spread in position is the same as for a particle with a fixed mass.

In appendix. (5.B.2) we show, following [56], that these states are the MUS in configuration space, meaning that they minimize the Heisenberg uncertainty principle between position and velocity

$$\sigma_{X_r}^2 \sigma_{V_r}^2 - \sigma_{X_r V_r}^2 \geq \frac{1}{4} \left| \langle [X_r, V_r] \rangle \right|^2 \tag{5.26}$$

where $V_r = -i/\hbar\,[X_r, H_{cr}]$.

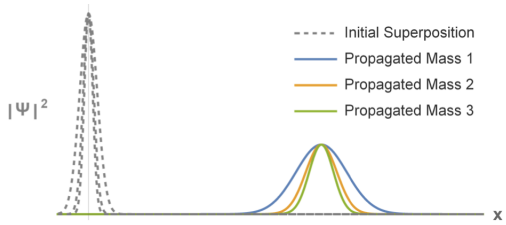

Figure 5.2.2: *Evolution of "generalized gaussian" wavepackets, which have minimum uncertainty in configuration space. Each packet corresponding to a different relativistic mass evolves with the same peak velocity and spread at the same rate. Moreover, as we show in Appendix (5.B.2), these wavepackets remain in general more localized under a free evolution with respect to the standard gaussian-wavepackets. The figure is taken from [56].*

Explicitly, they read

$$|\psi\rangle^C = \int d\epsilon \int dp\; \psi_{v_0,x_0}(p,\epsilon)\phi_{\epsilon_0,\tau_0}(\epsilon)\;\; |\psi_p\rangle_c \otimes |\epsilon\rangle_r \tag{5.27}$$

where each (normalized) gaussian wavefunction in phase space depends on the internal energy

$$\psi_{v_0,x_0}(p,\epsilon) = \left(\frac{1}{2m_\epsilon\hbar\Omega}\right)^{\frac{1}{4}} \exp\left\{-\frac{1}{2m_\epsilon\hbar\Omega}\left(p - m_\epsilon v_0\right)^2 - \frac{i}{\hbar}\left(p - m_\epsilon v_0\right)x_0\right\} \tag{5.28}$$

Here Ω is an arbitrary constant with the dimension of a frequency, which determines the initial spread. Unlike standard MUS in phase space, the peak velocity v_0 is independent of the mass, allowing each "branch" to evolve at the same rate. However, the price to pay is that the initial position and momentum spreads now depend on the mass:

$$\sigma^2_{x,m_\epsilon} = \frac{\hbar}{2m_\epsilon\Omega} \iff \sigma^2_{p,m_\epsilon} = \frac{m_\epsilon\hbar\Omega}{2} \tag{5.29}$$

In Appendix. (5.B.2), we make a more detailed comparison between these two states, showing that MUS in configuration space are the most appropriate description of a relativistic quantum clock, whose trajectories in space-time are as localized as quantum mechanics allows.

We emphasize that, considering real clocks with a finite and discrete spectrum, these states are in principle not hard to realize in practice. In fact, as the ground state of a harmonic potential for a massive particle is a Gaussian with squared width $\sigma^2 \propto 1/m$, a particle in a superposition of internal mass-energies, cooled down to the motional ground state of a harmonic trap that has a fixed frequency, would be prepared exactly in a configuration-space MUS, with initial velocity given by the velocity of the trap in the laboratory reference frame [56], [57].

Clock's degrees of freedom

We recall from (3.6) that an infinitely precise clock, or ideal clock, corresponds to an internal hamiltonian H_c with an unbounded spectrum which allows to construct perfectly distinguishable time-states in the form

$$|\tau\rangle_c = \int d\epsilon\; e^{-\frac{i}{\hbar}(\epsilon-\epsilon_0)\tau}\,|\epsilon\rangle_c \tag{5.30}$$

Furthermore, these states are eigenstates of the time operator T_c which is self-adjoint:

$$T_c = \int d\tau\;\tau\;\; |\tau\rangle\langle\tau|_c = \int d\tau\;\tau\;\left(\int d\epsilon d\epsilon' e^{-\frac{i}{\hbar}\tau(\epsilon-\epsilon')}\right)|\epsilon\rangle\langle\epsilon'| \tag{5.31}$$

It's easy to check that the standard deviation of the time operator over these time-states vanishes

$$\sigma^2_{T_c} = \left\langle T_c^2 \right\rangle_c - \left\langle T_c \right\rangle_c^2 = 0 \tag{5.32}$$

Considering the above MUS states of eqs. 5.24 and 5.27, this corresponds to the choice $\phi_{\epsilon_0}(\epsilon) = e^{-\frac{i}{\hbar}\tau_0\epsilon}$.

In the following analysis, we still consider an unbounded internal energy spectrum. However, instead of ideal-clock's states, which are fourier transform of energy eigenstates as in eq. (5.30) and, consequently, are eigenstates of the (self-adjoint) time operator T_c, we consider Gaussian wavepackets

$$|\tau\rangle_c = \int d\epsilon\ \phi_{\epsilon_0}(\epsilon) e^{-\frac{i}{\hbar}\tau\epsilon} |\epsilon\rangle_c\,, \qquad \phi_{\epsilon_0}(\epsilon) = \left(\frac{1}{2\pi\sigma^2_{H_c}}\right)^{\frac{1}{4}} \exp\left\{-\frac{(\epsilon-\epsilon_0)^2}{4\sigma^2_{H_c}}\right\} \tag{5.33}$$

The ideal-clock states are recovered in the limit of an infinite energy spread $\sigma_{H_c} \to \infty$. This choice has two main advantages.

- First of all the energy spread $\sigma^2_{H_c}$ consistutes an "effective" spectrum, which "simulates" a bounded (hence physical) spectrum while allowing us to more easily perform integrations.
- More importantly, we can still consider the self-adjoint time operator T_c of eq. (5.31) and use $\sigma^2_{H_c}$ as a parameter for the clock's precision in time measurements.

We will compute the standard deviation of the time operator explicitly in the next section. In general, we can see that the internal-energy spread $\sigma^2_{H_c}$ determines the distinguishability between clock states:

$$\langle\tau|\tau'\rangle = \int d\epsilon\, |\phi_{\epsilon_0}(\epsilon)|^2\, e^{-i\epsilon(\tau-\tau')} = \left(\frac{2\sigma^2_{H_c}}{\pi\hbar^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{(\tau-\tau')^2}{2\left(\hbar^2/4\sigma^2_{H_c}\right)}\right\} \tag{5.34}$$

which is a Gaussian in τ space with standard deviation $\hbar^2/4\sigma^2_{H_c}$. Thus, we see that perfectly distinguishable clock's states occur only with an infinite energy spread $\sigma_{H_c} \to \infty$, in which case $\langle\tau|\tau'\rangle = \delta(\tau-\tau')$.

5.2.2 External description

In this section, we explore how the dynamics of a quantum clock affects the precision of measurements of space and time with respect to an external reference frame, which corresponds to coordinates (x, t). Specifically, we want to study how the precision of time and position measurements evolves over time due to the coupling between the internal and external degrees of freedom of the relativistic hamiltonian (5.22).

The scope of this analysis is to understand the physical principles underlying the trade-off between spatial and temporal precision and to characterize it quantitatively.

Dynamics

We consider now the dynamics of both MUS states. The standard (uncorrelated) gaussian state reads

$$|\psi\rangle^P = \int dp\ \psi_{p_0}(p)\, |p\rangle_r \otimes \int d\epsilon\ \phi_{\epsilon_0}(\epsilon)\, |\epsilon\rangle_c \tag{5.35}$$

where $\psi_{p_0}(p)$ is a Gaussian in phase space centered around $(p_0, x_0 = 0)$, and similarly, $\phi_{\epsilon_0}(\epsilon)$ is the Gaussian in (ϵ, τ) space of Eq. (5.33), with $\tau_0 = 0$. The time evolution of this state under the hamiltonian (Eq. 5.22) is given by

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}tH_{cr}}\, |\Psi\rangle = \int d\epsilon dp\ \phi_{\epsilon_0}(\epsilon)\psi_{p_0}(p) e^{-\frac{i}{\hbar}E_{cr}(p,\epsilon)t}\, |p\rangle_r\, |\epsilon\rangle_c \tag{5.36}$$

where, considering $m_\epsilon = m + \epsilon/c^2$, we have :

$$E_{cr}(p,\epsilon) = \frac{p^2}{2m_\epsilon} + \epsilon \tag{5.37}$$

Similarly, the MUS in configuration space reads

$$|\psi\rangle^C = \int d\epsilon \int dp\ \psi_{v_0,x_0}(p,\epsilon)\phi_{\epsilon_0}(\epsilon)\ \ |\psi_p\rangle_c \otimes |\epsilon\rangle_r \tag{5.38}$$

where $\psi_{v_0,x_0}(p,\epsilon)$, for each ϵ, is again a Gaussian in phase space (see 5.28) but centered around $(x_0, m_\epsilon v_0)$, while $\phi_{\epsilon_0}(\epsilon)$ is the same gaussian wavefunction we have above. The time evolution of this state under the hamiltonian (Eq. 5.22) is given by

$$|\Psi(t)\rangle^C = e^{-\frac{i}{\hbar}tH_{cr}}|\Psi\rangle^C = \int d\epsilon dp\ \phi_{\epsilon_0}(\epsilon)\psi_{v_0,x_0}(p,\epsilon)e^{-\frac{i}{\hbar}E_{cr}(p,\epsilon)t}|p\rangle_r|\epsilon\rangle_c \tag{5.39}$$

Variance of the position operator - phase space MUS

We can compute the variance of the position operator $X_r = \int dx\ x\,|x\rangle\langle x|_r$ from the reduced density matrix of the "rod" (kinematical degrees of freedom). Considering MUS in phase space (5.36) up to order $\mathcal{O}((H_c/m_rc^2)^4)$, and fixing for simplicity $\epsilon_0 = 0$ [4], the variance evolves as follows (details in Appendix (5.C):

$$\sigma^P_{X_r}(t)^2 = \sigma^2_{X_r} + \left(\frac{\sigma_{P_r}}{m_r}t\right)^2 + 3\left(\frac{\sigma_{P_r}}{m_r}t\right)^2\frac{\sigma^2_{H_c}}{m_r^2c^4} + \left(\frac{p_0t}{m_r}\right)^2\frac{\sigma^2_{H_c}}{m_r^2c^4} \tag{5.40}$$

We see that:

- the first two terms represent the the standard spread of the Gaussian wavefunction $\sigma^2_{X_r} + \left(\frac{\sigma_{P_r}}{m_r}t\right)^2$, where $\sigma_{X_r} = \hbar/2\sigma_{P_r}$.

At second order $\mathcal{O}((H_c/m_rc^2)^2)$, two contributions arise due to the internal energy spread:

- the first contribution $3\left(\frac{\sigma_{P_r}}{m_r}t\right)^2\left(\frac{\sigma^2_{H_c}}{m_r^2c^4}\right)$, which is present even if $p_0 = 0$, comes from the second-order correction to the mass in the standard spread of the Gaussian wavefunction;
- the second contribution $\left(\frac{p_0t}{m_r}\right)^2\frac{\sigma^2_{H_c}}{m_r^2c^4}$, only present when $p_0 \neq 0$, accounts for the fact that different "mass branches" propagate at different velocities, as depicted in fig. (5.2.1) .

The result is consistent with error propagation analysis, where the position variance grows quadratically in time: considering $X(t) = X + \frac{P}{M}t$ one has

$$\delta X(t)^2 \sim \delta X_0^2 + \left(\delta P_0\left(\frac{1}{M^2}\right)_0 t\right)^2 + \left(\left(\frac{P}{M^2}\right)_0\delta Mt\right)^2 \tag{5.41}$$

where the subscript O_0 denotes the expectation value of the observable O.

Variance of the position operator - configuration space MUS

Considering configuration space-space MUS of Eq. 5.39, the variance in position reads (details in Appendix 5.D):

$$\sigma^C_{X_r}(t)^2 = \frac{\hbar}{2m_r\Omega}\left(1+\Omega^2t^2\right)\left(1+\frac{\sigma^2_{H_c}}{m_r^2c^4}\right) \tag{5.42}$$

[4]The effect of a non-zero average internal energy is just a shift of the rest mass $m_r \to m_r + \epsilon_0/c^2$, which does not modify the results of this section. This is highlighted in Appendix (5.C).

We recall from eq. 5.28 (details in appendix 5.B.2) that

$$\sigma^2_{X_r,l.o.} = \hbar/2m_r\Omega, \qquad \sigma^2_{V_r,l.o.} = \frac{\sigma^2_{P_r,l.o.}}{m_r^2} = \sigma^2_{X_r,l.o.}\Omega^2 \tag{5.43}$$

are the initial spread in the position and in the velocity at lowest order. Thus, we can rewrite the above equation as follows:

$$\sigma^C_{X_r}(t)^2 = \left(\sigma^2_{X_r,l.o.} + \frac{\sigma^2_{P_r,l.o.}}{m_r^2}t^2\right)\left(1 + \frac{\sigma^2_{H_c}}{m_r^2c^4}\right) \tag{5.44}$$

In this form, it's clear that:

- the first two terms $\sigma^2_{X_r,l.o.} + \frac{\sigma^2_{P_r,l.o.}}{m_r^2}t^2$ represent the standard spread of the Gaussian wavepacket, with the difference that now the initial spread in position and velocity both depend on the mass ;
- at second order $\mathcal{O}((H_c/m_rc^2)^2)$, only one contribution arise due to the internal energy spread, which is the correction to the standard spread of the Gaussian wavepacket. This correction is present regardless of the average velocity v_0.

The crucial difference with respect to the standard gaussian states, is that there's no dynamical term proportional to the average velocity or momentum (last term of eq. 5.40), reflecting the fact that each "mass branch" propagates with the same average velocity v_0, as depicted in fig. (5.2.2).

This result is also consistent with error propagation analysis: considering $X(t) = X + Vt$ one has $\delta X(t)^2 \sim \delta X_0^2 + (\delta V_0 t)^2$,where both δX_0 and δV_0 depend on the relativistic mass.

Variance of the time operator - phase space MUS

Similarly, from the reduced state of the clock (internal d.o.f.) one can compute the variance of the (self-adjoint) time operator $T_c = \int d\tau\ \tau\, |\tau\rangle\langle\tau|_c$. For phase space MUS the variance reads (details in Appendix (5.E)):

$$\sigma^P_{T_c}(t)^2 = \frac{\hbar^2}{4}\frac{1}{\sigma^2_{H_c}} + t^2\frac{\sigma^2_{P_r^2}}{4m_r^4c^4} \tag{5.45}$$

We see that:

- the first factor is the "intrinsic precision" of the clock, $\sigma^P_{T_c}(0)^2 = \hbar^2/4\sigma^2_{H_c}$ determines the distinguishability between clock's states and is inversely proportional to the energy spread.
- At fourth order $\mathcal{O}((P/mc)^4)$ a contribution arises from the dynamical coupling with the momentum.

Again, this is what we get from a simple analysis via error propagation, considering $\tau(t) = \tau_0 + t\Delta_P$, where $\Delta_P = 1 - P^2/2m^2c^2$, one has $\delta\tau(t)^2 = \delta\tau_0^2 + t^2\delta(\Delta_P)^2 = \delta\tau_0^2 + t^2\left(\delta P^2/2m^2c^2\right)^2$.

Variance of the time operator - configuration space MUS

In Appendix (5.F) we compute the variance of the time operator for the MUS state in configuration space, which reads

$$\sigma^C_{T_c}(t)^2 = \frac{\hbar^2}{4\sigma^2_{H_c}} + \frac{1}{2}\frac{\hbar}{m_rc^2\Omega}\frac{v_0^2}{c^2} + t^2\frac{\sigma^2_{P_r^2}}{4m_r^4c^4} \tag{5.46}$$

- The first two terms constitute the intrinsic precision of the clock, which is the initial variance of the time operator:

$$\sigma^C_{T_c}(0)^2 = \frac{\hbar^2}{4\sigma^2_{H_c}} + \frac{1}{2}\frac{\hbar}{m_r c^2 \Omega}\frac{v_0^2}{c^2} \tag{5.47}$$

  This shows that, although in the case of configuration-space MUS a non vanishing average velocity does not affect the spread in position (see eqs. 5.40 and 5.42), it does have an impact on clock's precision.

- The dynamical term $\sim t^2$ is formally equivalent to (5.45). The difference is in the variance of the momentum operator, as we highlighted previously (see 5.28).

Trade-off

The trade-off is captured by considering the variance in the time operator and in the position operator toghether and the relation between the spread of the wavefunctions. Considering a MUS in phase space, one has

$$\begin{cases} \sigma^P_{X_r}(t)^2 = \sigma^2_{X_r} + \left(\frac{\sigma_{P_r}}{m_r}t\right)^2 + 3\left(\frac{\sigma_{P_r}}{m_r}t\right)^2 \frac{\sigma^2_{H_c}}{m_r^2 c^4} + \left(\frac{p_0 t}{m_r}\right)^2 \frac{\sigma^2_{H_c}}{m_r^2 c^4} \\ \\ \sigma^P_{T_c}(t)^2 = \frac{\hbar^2}{4}\frac{1}{\sigma^2_{H_c}} + t^2 \frac{\sigma^2_{P_r^2}}{4 m_r^4 c^4} \end{cases} \tag{5.48}$$

Similarly, considering a MUS in configuration space,

$$\begin{cases} \sigma^C_{X_r}(t)^2 = \left(\sigma^2_{X_r,l.o.} + \frac{\sigma^2_{P_r,l.o.}}{m_r^2}t^2\right)\left(1 + \frac{\sigma^2_{H_c}}{m_r^2 c^4}\right) \\ \\ \sigma^C_{T_c}(t)^2 = \frac{\hbar^2}{4\sigma^2_{H_c}} + \frac{1}{2}\frac{\hbar}{m_r c^2 \Omega}\frac{v_0^2}{c^2} + t^2 \frac{\sigma^2_{P_r^2}}{4 m_r^4 c^4} \end{cases} \tag{5.49}$$

We see that, to have a perfect clock one should both have an infinite intrinsic precision $\sigma_{H_c} \to \infty$ which means an infinite internal-energy spread, and a negligible spread in the (squared) momentum $\sigma_{P_r^2} \approx 0$. Both conditions imply an infinite spread in position. A pictorial representation of the trade-off is given in fig. (5.2.3) below.

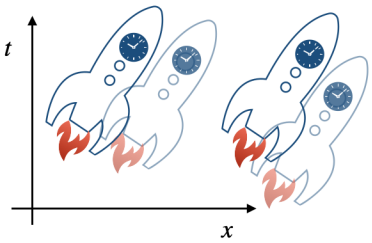

Figure 5.2.3: *Pictorial representation of the trade-off between precision in time measurements and localizability of quantum clocks, depicted as a space-ship that moves in space carrying a clock which measures proper time along its worldline. The more localized is the clock in space, represented by the x axis, the more the uncertainty in the time readings, represented by the "spread" in the t axis.*

In particular, we emphasize that even when the average momentum (or velocity) of the clock vanishes $p_0 = v_0 = 0$, still the variance in position has a term that is proportional to $\sigma^2_{H_c}$, which becomes bigger the more precise the clock. This term is due to the (second order) relativistic correction to the rest mass of the clock, and it reads

$$\begin{cases} \sigma^P_{X_r}(t)^2 \geq 3\left(\frac{\sigma_{P_r}}{m_r}t\right)^2 \frac{\sigma^2_{H_c}}{m_r^2c^4} \\ \\ \sigma^C_{X_r}(t)^2 \geq \left(\frac{\sigma^2_{P_r}}{m_r}t\right) \frac{\sigma^2_{H_c}}{m_r^2c^4} \end{cases} \tag{5.50}$$

where, in the second expression, we identified $\sigma^2_{P_r} \equiv \sigma^2_{P_r,l.o.} = \frac{\hbar m_r \Omega}{2}$. Thus, substituting $\sigma^2_{H_c}\sigma^2_{T_c}(t) \geq \hbar^2/4$ [5] we have, for both MUS states:

$$\sigma_{X_r}(t)^2\sigma_{T_c}(t)^2 \geq \left(\frac{\hbar}{2m_rc}\frac{\sigma_{P_r}}{m_rc}t\right)^2 \tag{5.51}$$

The r.h.s. is characterized by three terms:

1. the Compton wavelenght of the clock $\frac{\hbar}{mc}$: this is the fundamental quantum limit to spatial localization in the one-particle regime. It represents the smallest resolvable length scale for a particle of mass m (see the introduction to the chapter 5).

2. The ratio $\frac{\sigma_{P_r}}{mc}$ between the (lowest-order) initial uncertainty in momentum and the maximum allowed momentum in the regime, i.e. mc. This ratio indicates the degree of localization of the particle in momentum space.

3. Linear dependence on time t: the uncertainty in position increases linearly with time as the wavepacket spreads during its dynamical evolution.

Eq. (5.51) shows that, independently on its average velocity, the more a quantum clock is precise in time-keeping $\sigma_{T_c} \to 0$, the higher its spatial delocalization $\sigma_{X_r} \to \infty$, and viceversa. This trade-off is a fundamental feature of quantum systems in relativistic settings, where the energy-mass equivalence principle creates an unavoidable coupling between time and spatial measurements.

[5] Ultimately, this follows from the commutation relations $[T_c, H_c] = i\hbar$, which holds in the case of a self-adjoint time operator. Details can be found in chapter (3.6).

5.2.3 Relational description

In this section we explore the consequences of the trade-off in the context of QRFs. In particular, we want to describe the evolution of a quantum system $\mathcal{H}_s$ with fixed mass m_s using a quantum clock as a QRF to define its spacetime coordinates.

Using a relational, or G-invariant description [6], the state $\rho_s(x,t)$ and observables $O(x,t)$ of the system at different times are defined relative to the state of the spacetime QRF. However, the previous analysis suggests that a single quantum clock cannot serve as a perfect reference frame for both position and time (see eq. 5.2.3). This has important consequences on how a states and observables of another system can be described in the absence of an external, idealized reference frame.

In the following, we will formalize this "Gedanken-experiment" using the framework of QRFs of chapter (3). In particular, by defining a covariant POVM (see 2.61), we will first compute explicitly the invariant position $\tilde{X}$, showing that the coupling between the internal and external degrees of freedom of the quantum clock leads to non-trivial additional terms compared to the standard "relative position" obtained with a perfect QRF. Then, via the recovery operation (see 3.72), we will explicitly compute the variance of $\tilde{X}$ in the case of a phase-space-MUS for the quantum clock.

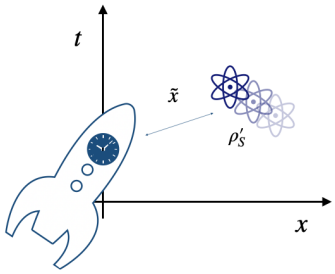

Figure 5.2.4: *Pictorial representation of the relational description of a spacetime QRF (space-ship) and a quantum system. ρ'_s (see eq. (5.54) is the state of the system whose space and time coordinates are defined via (POVM) measurements with respect to the QRF. Thus, the uncertainty in space and time measurements reflects on the relational states and observables of the system. In particular, we focus on the relative position $\tilde{X}$ of eq. (5.65).*

Quantum clock as a QRF

In the following section, we denote with $U_{rc}(x,t)$ and $U_s(x,t)$ the unitary representation of the group of spatial and time translations of, respectively, the spacetime QRF, composed by the rod and the clock $\mathcal{H}_{rc} = \mathcal{H}_r \otimes \mathcal{H}_c$, and the system $\mathcal{H}_f$. The corresponding superoperator is denoted by the calligraphic font $\mathcal{U}(x,t)$, which acts on operators: $U(x,t)\,[\cdot] = U(x,t)\,[\cdot]\,U^\dagger(x,t)$.

Concerning the spacetime QRF, time translations are generated by the relativistic hamiltonian H_{cr} of eq. (5.22), while the system we consider does not have internal dynamical degrees of freedom, so that its hamiltonian reads simply $H_s = P_s^2/2m_s$. The group of spatial translations is generated from the linear momenta, P_r and P_s. Explicitly, the representations we use read

$$\begin{aligned} U_{rc}(t) &= \exp\left\{-\frac{i}{\hbar}tH_{cr}\right\}, \qquad U_r(x) = \exp\left\{-\frac{i}{\hbar}xP_r\right\} \\ U_s(t) &= \exp\left\{-\frac{i}{\hbar}tH_s\right\}, \qquad U_s(x) = \exp\left\{-\frac{i}{\hbar}xP_s\right\} \end{aligned} \tag{5.52}$$

Let us assume that Alice has access to an external reference frame (x,t), which she uses to prepare both the quantum system and the quantum clock in the state:

$$\rho_{rc} \otimes \rho_s \tag{5.53}$$

[6] In our case G is the group of translations in space and time.

If Bob lacks access to this external frame, he describes the system S via the "$recovered\ state$" (see 3.3)

$$\rho'_s = \int dtdx\ p(x,t) \otimes \mathcal{U}_s(x,t)[\rho_s] \tag{5.54}$$

where $p(x,t) = tr_{rc}\left[E_{x,t}\rho_{rc}\right]$ is the Born probability resulting from a (covariant POVM) measurement $E_{x,t}$ (see 2.61) on the QRF's state preparred by Alice. As we show in (3.3.1), if the QRF ρ_{rc} is not perfect, the state ρ_s undergoes decoherence in the momentum basis, increasing uncertainty in its relative position.

Correspondingly, given any observable $O_s \in \mathcal{L}(\mathcal{H}_s)$[7] defined with respect to Alice's external reference frame, Bob describes it through the $relational\ observable$ $\tilde{O} \in \mathcal{L}(\mathcal{H}_s \otimes \mathcal{H}_{rc})$

$$\tilde{O} = \int dxdt\ E_{x,t} \otimes \mathcal{U}^{\dagger}_{x,t}\left[O_s\right] \tag{5.55}$$

This correspondance is expressed in tems of Born probabilities, as we shown in (3.72):

$$tr_s\left[O_s\ \rho'_s\right] = tr_{rc,s}\left[\rho_{rc} \otimes \rho_s \int dxdt\ E_{x,t} \otimes \mathcal{U}^{\dagger}_{x,t}\left[O_s\right]\right] \tag{5.56}$$

This equation shows that measurements of relational (or G-invariant) observables correspond to measurements of the corresponding system's observables on the "recovered" state.

Covariant POVM

The generators of space and time translations acting on the quantum clock's Hilbert space $\mathcal{H}_r \otimes \mathcal{H}_c$ are, respectively, the total momentum P_r and the low-energy hamiltonian H_{cr} (5.22). A covariant POVM can be constructed starting from an arbitrary POVM element (the "seed") $E_{0,0}$ and considering its "orbit" under the group we are considering (see 2.61). In our case:

$$E_{x,t} = \mathcal{U}^{\dagger}_r(x)\mathcal{U}^{\dagger}_{rc}(t)\left[E_{0,0}\right] \tag{5.57}$$

where the group actions are defined above in eq. (5.52). The "seed" element must be chosen so that the normalization condition is satisfied. In the case of a perfect reference frame, one would choose

$$E_{0,0} = |x=0\rangle\langle x=0|_r \otimes |\tau=0\rangle\langle\tau=0|_c \tag{5.58}$$

Then, its orbit is given by POVM elements in the form:

$$E_{x,t} = \mathcal{U}^{\dagger}_r(x)\mathcal{U}^{\dagger}_{rc}(t)\left[E_{0,0}\right] = \int dpdp' e^{-\frac{i}{\hbar}x(p-p')}\ |p\rangle\langle p'| \otimes \int d\epsilon d\epsilon' e^{-\frac{i}{\hbar}t\Delta_p(\epsilon-\epsilon')}\ |\epsilon\rangle\langle\epsilon'| \tag{5.59}$$

where $\Delta_p = \left(1 - \frac{p^2}{2m^2c^2}\right)$ is the time dilation factor. The normalization condition reads:

$$\mathbb{I}_r \otimes \mathbb{I}_c = \int dxdt\ E_{x,t} = \int dp\ |p\rangle\langle p|_r \otimes \int d\epsilon d\epsilon'\ e^{it\Delta_p(\epsilon-\epsilon')}\ |\epsilon\rangle\langle\epsilon'|_c \tag{5.60}$$

where we used the fact that integrating over x yields $\delta(p-p')$. Now, changing variables $t\Delta_p \to t$, we can integrate over time to get a $\delta(\epsilon-\epsilon')$. The result is

$$\int dxdt\ E_{x,t} = \int dp\ \frac{1}{\Delta_p}\ |p\rangle\langle p|_r \otimes \int d\epsilon\ |\epsilon\rangle\langle\epsilon|_c = \int dp\ \frac{1}{\Delta_p}\ |p\rangle\langle p|_r \otimes \mathbb{I}_c \neq \mathbb{I}_r \otimes \mathbb{I}_c \tag{5.61}$$

We see that the normalization condition is not satisfied. We have different possibilities:

1. We restrict the Hilbert space to the states $|\psi\rangle_{cr} \in \mathcal{H}_{cr}$ that satisfy

$$\left(\int dp\ \frac{1}{\Delta_p}\ |p\rangle\langle p|_r \otimes \mathbb{I}_c\right)|\psi\rangle_{cr} \approx |\psi\rangle_{cr} \tag{5.62}$$

   Physically, we are requiring the time dilation factor Δ_p to be negligible. For instance we could restrict to gaussian states in momentum space with standard deviation $\sigma_{P_r} \ll mc$, so that the condition is approximately satisfied.

[7] Here $\mathcal{L}(\mathcal{H})$ denotes the set of linear operators acting on the Hilbert space $\mathcal{H}$.

2. We consider the following POVM elements

$$E_{0,0} = \{|0\rangle\langle 0|_r , \Delta_{P_r}\} \otimes |0\rangle\langle 0|_c \tag{5.63}$$

where $\{A, B\} = 1/2(AB + BA)$ is the anticommutator. Physically, we can think at this POVM as measuring the proper time of the clock $\tau \approx t\Delta_p$, instead of the "external" time t. Notice that, since it's a measuement in phase space, rather than in position space only, it cannot be sharp. Formally, we see that $|0\rangle\langle 0|_r$ and Δ_{P_r} are non commuting. Following the above computation, we can easily check that, with this choice, the normalization condition is satisfied.

In the following, we consider the second possibility, which leads to a covariant POVM defined over the whole Hilbert space $\mathcal{H}_r \otimes \mathcal{H}_c$

$$E_{x,t} = \mathcal{U}_r^\dagger(x)\mathcal{U}_{rc}^\dagger(t)\left[\{|0\rangle\langle 0|_r , \Delta_{P_r}\} \otimes |0\rangle\langle 0|_c\right] \tag{5.64}$$

Relative position

With a covariant POVM we can define the relative algebra (3.66). In particular we are interested in the relative position of the system with respect to the QRF, which reads

$$\tilde{X} = \int dxdt\ E_{x,t} \otimes \mathcal{U}_s^\dagger(x,t)[X_s] \tag{5.65}$$

where the POVM is (5.64). This operator is computed explicitly in Appendix (5.G) and it reads:

$$\begin{aligned}\tilde{X} = X_s \otimes \mathbb{I}_r \otimes \mathbb{I}_c + \frac{P_s}{m_s} \otimes \frac{1}{\Delta_{P_r}} \otimes T_c \\ -\mathbb{I}_s \otimes X_r \otimes \mathbb{I}_c - \mathbb{I}_s \otimes P_r \frac{1}{\Delta_{P_r}} \otimes \{T_c, \frac{1}{M}\}\end{aligned} \tag{5.66}$$

where $\{A, B\} = \frac{1}{2}(AB + BA)$ is the anticommutator. To understand the physical meaning we can notice that, in the limit in which the "coupling term" $H_c \otimes \Delta_{P_r}$ in the hamiltonian vanishes, the invariant position reduces to

$$\begin{aligned}\tilde{X} = X_s \otimes \mathbb{I}_r \otimes \mathbb{I}_c + \frac{P_s}{m_s} \otimes T_c \\ -\mathbb{I}_s \otimes X_r \otimes \mathbb{I}_c - \mathbb{I}_s \otimes \frac{P_r}{m_r} \otimes T_c\end{aligned} \tag{5.67}$$

which corresponds to the relative position of the system with respect to the "rod", both evolving with the time of the clock.

Thus, we notice that the relativistic coupling in the hamiltonian (5.22) leads to non-trivial terms in the invariant algebra of the system and the quantum clock. In particular,

- Since T_c measures the $proper\ time$, it is corrected by the time dilation factor $T_c \to T_c/\Delta_{P_r}$ [8];
- The dynamics of the "rod" is determined by the dynamical mass operator $M = m_r\mathbb{I}_c + H_c/c^2$.

At first order in $\mathcal{O}((H_c/m_rc^2)^n)$ the invariant position reads

$$\begin{aligned}\tilde{X} = X_s \otimes \mathbb{I}_r \otimes \mathbb{I}_c + \frac{P_s}{m_s} \otimes \frac{1}{\Delta_{P_r}} \otimes T_c \\ -\mathbb{I}_s \otimes X_r \otimes \mathbb{I}_c - \mathbb{I}_s \otimes \frac{P_r}{m_r}\frac{1}{\Delta_{P_r}} \otimes T_c \\ +\mathbb{I}_s \otimes \frac{P_r}{m_r}\frac{1}{\Delta_{P_r}} \otimes \{T_c, \frac{H_c}{m_rc^2}\}\end{aligned} \tag{5.68}$$

[8] Recall, that $1/\Delta_{P_r}$ corresponds to the Taylor expansion of the γ factor. See for instance eq. 4.21.

Recovery operation

The correspondance between "recovery operation" and invariant algebra of eq. (5.56) allows us to directly compute the variance of the relative position $\tilde{X}$ on a chosen state of the system+QRF. In fact, one has

$$tr_s\left[X_s\ \rho'_s\right] = tr_{rc,s}\left[\left(\rho_{rc}\otimes\rho_s\right)\tilde{X}\right] \tag{5.69}$$

Physically, $\tilde{X}$ corresponds to the position of the system X_s if, lacking of an external reference frame, one uses a QRF to define its spacetime coordinates. Instead of computing the directly the variance of the invariant operator $\tilde{X}$, we can equivalently compute the variance of X_s on the "recovered state" ρ'_s.

For simplicity, we consider the system to also preparred at rest in a pure gaussian state centered around some fixed distance $(x_{s,0}, p_{s,0}) = (x_0, 0)$:

$$\begin{aligned} \left|\psi_0\right\rangle_s &= \int dy\ \ \psi_{0,x_0}(y)\left|y\right\rangle_s \\ U_s(x,t)\left|\psi_0\right\rangle_s = \left|\psi(x+x_0,t)\right\rangle_s &= \int dy\ \ \psi_{0,x+x_0}(y,t)\left|y\right\rangle_s \end{aligned} \tag{5.70}$$

where $\psi_{x,0}(y,t)$ is the initial gaussian after a free evolution for a time t, so its std.dev. is

$$\sigma^2_{X_s}(t) = \sigma^2_{X_s} + \left(\frac{\sigma_{P_s}t}{m_s}\right)^2 \tag{5.71}$$

To compute the variance of X_s, we are interested in the following expectation values:

$$\left\langle\tilde{X}\right\rangle = \int dtdx\ \ p(x,t)\ \ tr_s[\rho_s(x,t)X_s] = \int dtdx\ \ p(x,t)(x+x_0) \tag{5.72}$$

$$\begin{aligned} \left\langle\tilde{X}^2\right\rangle &= \int dtdx\ \ p(x,t)\ \ tr_s[\rho_s(x,t)X_s^2] = \int dtdx\ \ p(x,t)\left[(x+x_0)^2+\sigma^2_{X_s}(t)\right] \\ &= \int dtdx\ \ p(x,t)\left[(x+x_0)^2+\sigma^2_{X_s}+\frac{\sigma^2_{P_s}}{m_s^2}t^2\right] \end{aligned} \tag{5.73}$$

Phase space MUS

First, we consider the quantum clock to be preparred in a pure (standard) gaussian state at the origin x_0 of Alice's frame. To avoid the dynamical delocalization which is typical of quantum clocks in these gaussian-states (see previous section), we consider $\langle P_r\rangle = p_0 = 0$

$$\rho_{rc} = \left|0,0\right\rangle\!\left\langle 0,0\right|_{rc}, \qquad \left|0,0\right\rangle_{rc} = \int dp\psi_{0,x_0}\left|p\right\rangle_r \otimes \int d\epsilon\phi_{\epsilon_0}\left|\epsilon\right\rangle_c \tag{5.74}$$

In appendix. (5.H) we compute explicitly the relevant integrals. The result is:

$$\begin{aligned} \int dxdt\ p(x,t)\ t^2 &= \left\langle\frac{\sigma^2_{T_c}}{\Delta^2_{P_r}}\right\rangle \\ \int dxdt\ p(x,t)\ x &= 0 \\ \int dxdt\ p(x,t)\ x^2 &= \frac{\hbar^2}{4m_r^2c^2} + \sigma^2_{X_r} + \left\langle\frac{P_r^2}{M^2}\frac{\sigma^2_{T_c}}{\Delta^2_{P_r}}\right\rangle + \left\langle\frac{\hbar^2}{4M^2c^2}\frac{1}{\Delta_{P_r}}\right\rangle \end{aligned} \tag{5.75}$$

where the expectation values are taken over the clock's wavepackets (5.74) and $\Delta_{P_r} = (1-\frac{P_r^2}{2m_r^2c^2})$ is the special-relativistic time dilation factor. The variance of the relative position reads

$$\begin{aligned} \sigma^2_{\tilde{X}} = \sigma^2_{X_r} + \sigma^2_{X_s} + \frac{\hbar^2}{4m_r^2c^2} + \left\langle\frac{\hbar^2}{4M^2c^2}\frac{1}{\Delta_{P_r}}\right\rangle \\ + \frac{\sigma^2_{P_s}}{m_s^2}\left\langle\frac{\sigma^2_{T_c}}{\Delta^2_{P_r}}\right\rangle + \left\langle\frac{P_r^2}{M^2}\frac{\sigma^2_{T_c}}{\Delta^2_{P_r}}\right\rangle \end{aligned} \tag{5.76}$$

Apart from the relativistic corrections due to the dime tilation factor $\sim \Delta^n_{P_r}$, we can read out the following contributions:

1. $\sigma^2_{X_r} + \sigma^2_{X_s}$ - are the spatial spreads of the initial wavefunction of the sistem and the clock;

2. $\frac{\hbar^2}{4m_r^2c^2}$ - is the squared Compton wavelenght of the clock with only the rest mass. As it is clear from the computation (see Appendix 5.H), this term comes from the fact that the POVM (5.64) involves two non commuting operators, i.e. the projector $|x=0\rangle\langle x=0|_r$ and the dilation factor $\Delta_{P_r} = (1 - \frac{P_r^2}{2m_r^2c^2})$;

3. $\left\langle \frac{1}{4}\frac{\hbar^2}{M^2c^2} \right\rangle$ - is the "average" squared Compton wavelength of the clock with the relativistic mass. Importantly, this term is not tied to the specific choice of clock state; instead, it is due to the unitary evolution generated by the relativistic hamiltonian (5.22). Thus, we can argue that this term, written as a generic expectation value as above, is completely independent on the choice of the states (details in Appendix 5.H).

4. $\frac{\sigma^2_{P_s}}{m_s^2}\sigma^2_{T_c}$ and $\left\langle \frac{P_r^2}{M^2} \right\rangle \sigma^2_{T_c}$ - represent the usual spread of the wavefunction. However, instead of the quadratic dependence on an external time-parameter t (see Eq. 5.40), we have the uncertainty in the time readings of the clock.

Now, for our choice of states, where $\langle P_r \rangle = 0$, we can substitute $\left\langle P_r^2 \right\rangle = \sigma^2_{P_r}$. Neglecting for simplicity the relativistic corrections due to the time dilation factors, the standard deviation of $\tilde{X}$ reads

$$\begin{aligned}\sigma^2_{\tilde{X}} = \sigma^2_{X_r} + \sigma^2_{X_s} + \frac{\hbar^2}{4m_r^2c^2} + \left\langle \frac{\hbar^2}{4M^2c^2} \right\rangle \\ + \frac{\sigma^2_{P_s}}{m_s^2}\sigma^2_{T_c} + \left\langle \frac{\sigma^2_{P_r}}{M^2} \right\rangle \sigma^2_{T_c}\end{aligned} \tag{5.77}$$

The important things to notice are:

1. In the absence of an external ideal frame, the time that defines the evolution is given by a measurement on the quantum clock. Thus, the uncertainty in the clock readings σ_{T_c} propagates to the relative position between the system and the clock. This is the "specular" side of the trade-off: the less precise the clock, the higher the uncertainty in the system's dynamics so the more indefinite its position.

2. Even if $p_0 = 0$, so that the dynamical delocalization does not play any role, the spread in the internal energy is still significant. In particular, we focus on [9]:

$$\sigma^2_{\tilde{X}} \geq \left\langle \frac{\hbar^2}{4M^2c^2} \right\rangle = \left(\frac{1}{4}\frac{\hbar^2}{m_r^2c^2}\right)\left(1 + 3\frac{\sigma^2_{H_c}}{m_r^2c^4}\right) \tag{5.78}$$

   This term represents the Compton wavelength of the clock, with a second-order relativistic correction to the rest mass. Importantly, it arises purely from the unitary evolution under the relativistic hamiltonian (5.22), making it independent of the specific choice of states. This shows that an infinitely precise clock ($\sigma_{H_c} \to \infty$) leads to an indefinite position as well.

Configuration space - MUS

Similarly, Alice can prepare the clock in a configuration space MUS:

$$\rho_{rc} = |0,0\rangle\langle 0,0|_{rc}\,, \qquad |0,0\rangle_{rc} = \int d\epsilon \int dp\ \psi_{v_0,x_0=0}(p,\epsilon)\phi_{\epsilon_0}(\epsilon)\ \ |p\rangle_c \otimes |\epsilon\rangle_r \tag{5.79}$$

In appendix. (5.I) we compute explicitly the relevant integrals. The result is:

$$\begin{aligned}\int dxdt\ p(x,t)\ t^2 = \left\langle \frac{\sigma^2_{T_c}(v_0)}{\Delta^2_{P_r}} \right\rangle \\ \int dxdt\ p(x,t)\ x = 0 \\ \int dxdt\ p(x,t)\ x^2 = \frac{\hbar^2}{4m_r^2c^2} + \left\langle \sigma^2_{X_r}(M) \right\rangle + \left\langle \frac{P_r^2}{M^2}\frac{\sigma^2_{T_c}(v_0)}{\Delta^2_{P_r}} \right\rangle + \left\langle \frac{1}{4}\frac{\hbar^2}{M^2c^2}\frac{1}{\Delta_{P_r}} \right\rangle\end{aligned} \tag{5.80}$$

[9] As before, we set for simplicity the average energy $\epsilon_0 = 0$.

where we denoted $\sigma^2_{T_c}(v_0) = \frac{\hbar^2}{4\sigma^2_{H_c}} + \frac{\hbar v_0^2}{2\Omega m_e c^4}$ and $\sigma^2_{X_r}(M) = \langle \hbar/2M\Omega \rangle$ (see 5.29). Thus, the variance reads

$$\sigma^2_{\tilde{X}} = \left\langle \sigma^2_{X_r}(M) \right\rangle + \sigma^2_{X_s} + \frac{\hbar^2}{4m_r^2c^2} + \left\langle \frac{\hbar^2}{4M^2c^2} \frac{1}{\Delta_{P_r}} \right\rangle + \frac{\sigma^2_{P_s}}{m_s^2} \left\langle \frac{\sigma^2_{T_c}(v_0)}{\Delta^2_{P_r}} \right\rangle + \left\langle \frac{P_r^2}{M^2} \frac{\sigma^2_{T_c}(v_0)}{\Delta^2_{P_r}} \right\rangle \tag{5.81}$$

We see that the result is formally equivalent to phase space MUS, but with a different clock's precision $\sigma^2_{T_c}(v_0)$ and initial delocalization $\sigma^2_{X_r}(M)$. This is consistent with or previous fundings (see 5.42 and 5.46).

We highlight that also in this case the variance of the relative position $\tilde{X}$ comprises the term

$$\sigma^2_{\tilde{X}} \geq \left\langle \frac{\hbar^2}{4M^2c^2} \right\rangle = \left(\frac{1}{4} \frac{\hbar^2}{m_r^2c^2} \right) \left(1 + 3 \frac{\sigma^2_{H_c}}{m_r^2c^4} \right) \tag{5.82}$$

As we argued previously, this term is independent of the state of the clock, since it comes from the dynamical evolution, i.e. from the relativistic hamiltonian (5.22).

Trade-off

From Eqs. (5.78) and (5.82), we can argue that the following inequality is always satisfied for any state of the clock:

$$\sigma^2_{\tilde{X}} \geq \left\langle \frac{\hbar^2}{4M^2c^2} \right\rangle \geq \frac{1}{4} \frac{\hbar^2}{m_r^2c^2} \frac{\sigma^2_{H_c}}{m_r^2c^4} \tag{5.83}$$

Substituting $\sigma^2_{H_c} \geq \frac{\hbar^2}{4\sigma^2_{T_c}}$[10], we obtain

$$\sigma_{\tilde{X}} \sigma_{T_c} \geq \left(\frac{1}{4} \frac{\hbar^2}{m_r^2c^2} \right) = \left(\frac{\lambda_{\mathcal{C}}}{2} \right) \left(\frac{\lambda_{\mathcal{C}}}{2c} \right), \tag{5.84}$$

where $\lambda_{\mathcal{C}}$ denotes the Compton wavelength $\hbar/m_r c$. This inequality expresses the trade-off between the effectiveness of a single quantum clock as a resource for defining both spatial and temporal coordinates of another quantum system. We see that as the precision of (proper) time measurements increases (i.e., $\sigma_{T_c} \to 0$), the uncertainty in the relative position $\tilde{X}$ with respect to another quantum system also increases. Thus, the more precisely the clock measures time, the less precisely it can measure space-like distances.

While we claim that the physical phenomenon underlying this trade-off is general—stemming from the universal coupling between internal and external degrees of freedom of the spacetime QRF (see (5.22))—the quantitative analysis presented in this section is tied to a specific model of the clock, namely one with a continuous and unbounded spectrum hamiltonian. Nevertheless, we believe this analysis lays the groundwork for more comprehensive and general investigations in the future.

[10] We stress that this follows from the commutation relations $[T_c, H_c] = i\hbar$, which holds in the case of a self-adjoint time operator.

Appendix

5.A Hole argument

According to the principles of General Relativity, spacetime points defined in terms of coordinates have no physical meaning. The only meaningful way to define ST points is in terms of ST coincidences, which are the only events that can be verified by observations. In Einstein's words ([58]) :

> "In fact, if events consist in the motion of material points, then ultimately nothing would be observable but the meeting of two or more of these points. Moreover, the result of our measuring are nothing but verifications of such meetings of the material points of our measuring device with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point-events happening at the same place and at the same time."

This is the main physical consequence of a theory to be general covariant. The way Einstein reached this conclusion is the famous "Hole argument", that can be summarized as follows ([1]).

Consider 2 ST points x_A, x_B in a region without matter ("hole") at which the metric field is different, for instance flat at x_A but not on x_B:

$$\eta_{\mu\nu} = g_{\mu\nu}(x_A) \neq g_{\mu\nu}(x_B) \tag{5.85}$$

Now consider an active diffeomorphism ϕ that maps x_A into x_B, i.e. $\phi(x_A) = x_B$. The metric field transforms accordingly:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(y) = \frac{\partial x^\alpha(y)}{\partial y^\mu}\frac{\partial x^\beta(y)}{\partial y^\nu} g_{\alpha\beta}(x) \tag{5.86}$$

Diffeomorphism invariance implies:

$$g_{\mu\nu}(x_A) = g'_{\mu\nu}(\phi(x_A)) \tag{5.87}$$

The crucial point is that diffeomorphism can be equivalently seen as an active transformation, which is given by the pull-back of the metric (as any other field) $\tilde{g} = \phi^* g$, defined by

$$\tilde{g}(y) = g(\phi^{-1}(y)) \tag{5.88}$$

Diffeomorphism invariance then also implies

$$\tilde{g}(x) = g_{\mu\nu}(\phi^{-1}(x)) = g'_{\mu\nu}(x) \tag{5.89}$$

The apparent paradox is the following: both $\tilde{g}$ and g are solutions of the same equations of motions, since the active transformation is equivalent to a passive change of coordinates, that clearly leave the equations of motion invariant; however if we compare them at the spacetime points x_A and x_B they have different properties. In fact, combining (5.85), and we find

$$\tilde{g}_{\mu\nu}(x_B) = \tilde{g}_{\mu\nu}(\phi(x_A)) = g'_{\mu\nu}(\phi(x_A)) = g_{\mu\nu}(x_A) \neq g_{\mu\nu}(x_B) \tag{5.90}$$

Equivalently, in terms of the curvature, which is a physical (gauge-invariant) quantity,

$$\tilde{R}(x_B) = \tilde{R}(\phi(x_A)) = R(x_A) \neq R(x_B) \tag{5.91}$$

Assuming general covariance to be a fundamental principle, this implies that ST points have no physical meaning.

To solve the apparent paradox one has to recognize that the only way to give physical meaning to spacetime points is to consider the coincidences of trajectories, which are themselves determined by metric and therefore transform under active diffeomorphisms.
Consider in fact the same situation but with some matter in the hole. In particular consider two particles a and b, whose worldlines $(x_a(\tau_a), x_b(\tau_b))$ intersect at the spacetime point B. Now, if we apply the same diffeomorphism ϕ, the particle's world lines $(x_a(\tau_a), x_b(\tau_b))$ are no longer solutions of the field equations, since the gravitational field is now $\tilde{g}$. To map solutions into solutions, also the particles worldlines must be transformed under the active diffeomorphism, so that now the intersection is in $A = \phi^{-1}(B)$

$$\tilde{x}_a(\tau_a) = \phi^{-1}((x_a(\tau_a)), \qquad \tilde{x}_b(\tau_b) = \phi^{-1}((x_b(\tau_b)) \tag{5.92}$$

Instead of asking how is the metric field at A, we can ask how it is at the coincidence of the 2 worldlines. The difference is that this spacetime event is not an unphysical label, but it is a physical event determined by the field's equations. Formally, assuming the intersection to be at some value $\tau_a = \tau_b = 0$, we find

$$\tilde{R}\big|_{inters.} = \tilde{R}(\tilde{x}_a(0)) = R(\phi(\tilde{x}_a(0)) = R(\phi \circ \phi^{-1}(x_a(0))) = R(x_a(0)) = R\big|_{inters.} \tag{5.93}$$

A general covariant theory is compatible with determinism only assuming that spacetime points have no physical meaning by themselves. The physical reality cannot be described in terms of particles and fields on a background spacetime: particles and fields (including the gravitational field) can only be localized with respect to one another.

5.B MUS in configuration space

Given a pair of non-commuting observables $X, Y \in \mathcal{L}(\mathcal{H})$ with $[X, Y] = i\hat{C}$, a necessary and sufficient condition for a state $|\psi\rangle \in \mathcal{H}$ to be a MUS for the pair is that it is a solution of the eigenvalue equation

$$(X - i\lambda Y)\,|\psi\rangle = (x - i\lambda y)\,|\psi\rangle \tag{5.94}$$

where x, y are the expectation values of the pair and $\lambda \in \mathbb{R}$ is a real number whose magnitude is given by $|\lambda| = \sigma_{\psi,A}/\sigma_{\psi,B}$ and the sign is the opposite of $c =< \hat{C} >_\psi$. The proof can be found in many standard QM books, for instance in [59].

5.B.1 MUS in phase space - standard coherent states

For the MUS in phase space one we substitute $X \to X, Y \to P$ and $\hat{C} = c \to \hbar > 0$, which implies $\lambda < 0$. Defining for convenience $-\lambda := \frac{2\sigma^2}{\hbar} e^{2r}$, where for completeness we include a "squeezing" parameter $r \in \mathbb{R}$, the eigenvalue equation reads

$$\left(X + i\frac{2\sigma^2}{\hbar} e^{2r} P\right)|\psi\rangle = \left(x_0 + i\frac{2\sigma^2}{\hbar} e^{2r} p_0\right)|\psi\rangle \tag{5.95}$$

To find the solution in position and momentum space we consider the corresponding representations of X and P

$$\left(x + 2\sigma^2 e^{2r}\partial_x\right)\psi(x) = \left(x_0 + i\frac{2\sigma^2}{\hbar}e^{2r}p_0\right)\psi(x) \iff \partial_x\psi(x) = \left(-e^{-2r}\frac{(x-x_0)}{2\sigma^2} + i\frac{p_0}{\hbar}\right)\psi(x) \tag{5.96}$$

$$\left(i\hbar\partial_p + i\frac{2\sigma^2}{\hbar}e^{2r}p\right)\psi(p) = \left(x_0 + i\frac{2\sigma^2}{\hbar}e^{2r}p_0\right)\psi(p) \iff \partial_p\psi(p) = \left(-e^{2r}\frac{2\sigma^2}{\hbar^2}(p-p_0) - i\frac{x_0}{\hbar}\right)\psi(p) \tag{5.97}$$

whose solution is the std. gaussian with squeezing parameter r.

$$\begin{aligned}\psi(x) &= N\exp\left\{-\frac{e^{-2r}\,(x-x_0)^2}{4\sigma^2} + i\frac{p_0\,(x-x_0)}{\hbar}\right\} \\ \psi(p) &= N\exp\left\{-\frac{e^{2r}\,(p-p_0)^2}{\hbar^2/\sigma^2} - i\frac{x_0\,(p-p_0)}{\hbar}\right\}\end{aligned} \tag{5.98}$$

5.B.2 MUS in configuration space

We repeat the same procedure but now, instead of momentum, we look at the velocity operator, that is defined by the hamiltonian via

$$V = \dot{X} = -\frac{i}{\hbar}\left[X, H_{cr}\right] \tag{5.99}$$

We focus on the low-energy and low-velocity regime, where the hamiltonian of a composite particle is [56]

$$H_{cr} = \frac{P^2}{2M} + Mc^2 \tag{5.100}$$

This gives

$$V = \frac{P}{M} \tag{5.101}$$

Notice that P and X act on the "rod" Hilbert space, while $M = m + \frac{H_c}{c^2}$ acts on the clock's one, so they commute. The commutation relation reads

$$[X, V] = \frac{i\hbar}{M} \tag{5.102}$$

To find the MUS one proceeds as in the std.case: we substitute $X \to X$, $Y \to \hat{v}$ and $\hat{C} \to \frac{\hbar}{M}$, and again, considering positive masses $c = \frac{\hbar}{m}$ we have again $\lambda < 0$. Defining $-\lambda \coloneqq \frac{e^{2r}}{\Omega}$, the eigenvalue equation reads

$$\left(X + ie^{2r}\frac{\hat{v}}{\Omega}\right)|\psi\rangle = \left(x_0 + ie^{2r}\frac{v_0}{\Omega}\right)|\psi\rangle \tag{5.103}$$

Now we can expand the states and the mass operator in the clock's Hilbert space. Without loss of generality, we consider a discrete spectrum $\{|\epsilon_i\rangle\}$ with $i \in \mathbb{Z}$ and, for simplicity, we denote the different relativistic masses with a single integer number $m_i = m + \epsilon_i/c^2 \to m \in \mathbb{Z}$. Thus, we have

$$|\Psi\rangle = \sum_m \phi_m\,|\psi_m\rangle \otimes |m\rangle \tag{5.104}$$

$$M = \sum_m m\hat{\Pi}_m \tag{5.105}$$

so we get an eigenvalue equation for each mass sector

$$\left(X + ie^{2r}\frac{P}{m\Omega}\right)|\psi_m\rangle\,|m\rangle = \left(x_0 + ie^{2r}\frac{v_0}{\Omega}\right)|\psi_m\rangle\,|m\rangle \tag{5.106}$$

The solutions in position space can be found by solving for

$$\begin{gathered}\left(x + e^{2r}\frac{\hbar}{m\Omega}\partial_x\right)\psi_m(x) = \left(x_0 + ie^{2r}\frac{v_0}{\Omega}\right)\psi_m(x) \\ \Longleftrightarrow \\ \partial_x\psi_m(x) = \left(-e^{-2r}\frac{m\Omega}{\hbar}(x - x_0) + \frac{i}{\hbar}mv_0\right)\psi_m(x)\end{gathered} \tag{5.107}$$

and similarly in momentum space

$$\begin{gathered}\left(i\hbar\partial_p + ie^{2r}\frac{p}{m\Omega}\right)\psi_m(p) = \left(x_0 + ie^{2r}\frac{v_0}{\Omega}\right)\psi_m(p) \\ \Longleftrightarrow \\ \partial_p\psi_m(p) = \left(-e^{+2r}\frac{1}{m\Omega}(p - p_0) - \frac{i}{\hbar}x_0\right)\psi_m(p)\end{gathered} \tag{5.108}$$

The solutions are again gaussian wavepackets

$$\psi_m(x) = N_m \exp\left\{-e^{-2r}\frac{m\Omega}{2\hbar}\,(x - x_0)^2 + \frac{i}{\hbar}(x - x_0)\,(mv_0)\right\} \tag{5.109}$$

$$\psi_m(p) = N_m \exp\left\{-e^{+2r}\frac{1}{2m\hbar\Omega}\,(p - mv_0)^2 - \frac{i}{\hbar}\,(p - mv_0)\,x_0\right\} \tag{5.110}$$

that coincides with (5.28). The crucial difference w.r.t. the standard MUS in phase space is that

1. the initial position/momentum spread depends on the mass

$$\sigma^2_{x,m} = \frac{\hbar}{2m\Omega} \iff \sigma^2_{p,m} = \frac{m\hbar\Omega}{2} \tag{5.111}$$

2. the peak momentum depends on the mass $p_0 = mv_0$, while peak velocity does not.

Physically this means that for each mass in the superposition the wavepackets moves with the same velocity, rather than with the same momentum, so that they evolve with the same rate.

5.B.3 Time evolution

The time evolution of these states is easily found in momentum space, where the hamiltonian (5.22) is diagonal and the evolution results in a phase factor. The wavefunction in position space is then found by a fourier transform. The result is well known so we will just present it without derivation.

MUS in phase space

The MUS in phase space, which is initially uncorrelated, evolves into an entangled state via the hamiltonian (5.100)

$$e^{-\frac{i}{\hbar}tH_{cr}}\left(|\psi(x,t)\rangle \otimes \sum_m \phi_m\,|m\rangle\right) = \sum_m \psi^P_m(x,t)\phi_m\,|x\rangle\,|m\rangle \tag{5.112}$$

where

$$\psi_m^P(x,t) = \left[\sqrt{2\pi\sigma_{X_r}^2\left(1+\frac{i\hbar t}{2m\sigma_{X_r}^2}\right)}\right]^{-\frac{1}{2}} \exp\left\{-\frac{\left(x-\frac{p_0}{m}t\right)^2}{4\sigma_{X_r}^2\left(1+\frac{i\hbar t}{2m\sigma_{X_r}^2}\right)}\right\}\exp\left\{\frac{i}{\hbar}xp_0\right\}\exp\left\{-\frac{i}{\hbar}mc^2t\left(1+\frac{p_0^2}{2m^2c^2}\right)\right\} \tag{5.113}$$

is the standard time-evolved wavepacket for a fixed mass "m", except for the rest-mass energy, that gives an additional phase factor. Computing the probability distribution

$$\left|\psi_m^P(x,t)\right|^2 = \left[2\pi\sigma_{X_r}^2\left(1+\left(\frac{\hbar t}{2m\sigma_{X_r}^2}\right)^2\right)\right]^{-\frac{1}{2}} \exp\left\{-\frac{\left(x-\frac{p_0}{m}t\right)^2}{2\sigma_{X_r}^2\left(1+\left(\frac{\hbar t}{2m\sigma_{X_r}^2}\right)^2\right)}\right\} \tag{5.114}$$

we can read out the spread in position for each mass eigenvalue

$$\sigma_{X_r,m}(t)^2 = \sigma_{X_r}^2\left[1+\left(\frac{\hbar t}{2m\sigma_{X_r}^2}\right)^2\right] \tag{5.115}$$

We notice that the dynamical part of the spread depends on the mass, while the initial spread $\sigma_{X_r}(0)^2 = \sigma_{X_r}^2$ does not. The full variance in position can be computed directly from the reduced density matrix of the kinematical degrees of freedom (tracing out the internal d.o.f.)

$$\rho_r^P(t) = \sum_m |\phi_m|^2 \left(\psi_m^P(x,t)\,|x\rangle\langle x'|\,\psi_m^P(x',t)^*\right) \tag{5.116}$$

So the average position reads

$$\langle X_r\rangle^P(t) = \sum_m |\phi_m|^2\,\overline{x}_m \tag{5.117}$$

where $\overline{x}_m = \frac{p_0 t}{m}$ is the average position for each mass in the superposition. Similarly we have

$$\left\langle X_r^2\right\rangle^P(t) = \sum_m |\phi_m|^2\,\overline{x^2}_m \tag{5.118}$$

where $\overline{x^2}_m = \sigma_{X_r,m}(t)^2 + \overline{x}_m^2$. Therefore the variance $\sigma_{X_r}^2(t) = \left\langle X_r^2\right\rangle^P(t) - \langle X_r\rangle^P(t)^2$ reads

$$\sigma_{X_r}^P(t)^2 = \sum_m |\phi_m|^2\,\sigma_{X_r,m}(t)^2 + \left[\sum_m |\phi_m|^2\,\overline{x}_m^2 - \left(\sum_m |\phi_m|^2\,\overline{x}_m\right)^2\right] \tag{5.119}$$

The physical meaning is clear once we separate the two contributions

1. the first one is present even when $p_0 = 0$

$$\sigma_{X_r}^2 + \sum_m |\phi_m|^2\left(\frac{\hbar t}{2m\sigma_{X_r}}\right)^2 = \sigma_{X_r}^2 + \sum_m |\phi_m|^2\left(\frac{\sigma_{P_r}}{m}t\right)^2 \tag{5.120}$$

   and corresponds to the usual spread of a (free) gaussian wavepacket, modified by the dynamical relativistic mass.

2. the second one is present only if the initial velocity is non-zero

$$p_0^2 t^2\left[\sum_m |\phi_m|^2\,\frac{1}{m^2} - \left(\sum_m |\phi_m|^2\,\frac{1}{m}\right)^2\right] \tag{5.121}$$

   and reflects the fact that each mass propagates at a different velocity, as shown in Fig. (5.2.1).

MUS in configuration space

For the time evolution of the MUS in configuration space (5.110) we can simply substitute $p_0 \to mv_0$, $\sigma^2_{X_r} \to \sigma^2_{x,m} = \frac{\hbar}{2m\Omega}$ to the previous result. This gives

$$\psi^C_m(x,t) = \left[\sqrt{2\pi \frac{\hbar}{2m\Omega}\left(1+i\Omega t\right)}\right]^{-\frac{1}{2}} \exp\left\{-\frac{m\Omega}{2\hbar}\frac{\left(x-v_0t\right)^2}{\left(1+i\Omega t\right)}\right\} \exp\left\{\frac{i}{\hbar}xmv_0\right\} \exp\left\{-\frac{i}{\hbar}mc^2t\left(1+\frac{v_0^2}{2c^2}\right)\right\} \tag{5.122}$$

Computing the probability distribution

$$\left|\psi^C_m(x,t)\right|^2 = \left[2\pi \frac{\hbar}{2m\Omega}\left(1+\Omega^2t^2\right)\right]^{-\frac{1}{2}} \exp\left\{-\frac{m\Omega}{\hbar}\frac{\left(x-v_0t\right)^2}{\left(1+\Omega^2t^2\right)}\right\} \tag{5.123}$$

we can read out the spread in position for each mass eigenvalue

$$\sigma^C_{x,m}(t)^2 = \frac{\hbar}{2m\Omega}\left(1+\Omega^2t^2\right) \tag{5.124}$$

The total spread in position can be computed from the reduced density matrix of the kinematical degrees of freedom:

$$\rho_r(t)^C = \sum_m \left|\phi_m\right|^2 \left(\psi^c_m(x,t)\left|x\right\rangle\!\left\langle x'\right| \psi^C_m(x',t)^*\right) \tag{5.125}$$

Since the avg. position $\overline{x}_m(t) = v_0t,\ \forall m$ does not depend on the mass, we can immediately conclude

$$\sigma^C_x(t)^2 = \sum_m \left|\phi_m\right|^2 \frac{\hbar}{2m\Omega}\left(1+\Omega^2t^2\right) \tag{5.126}$$

We see that the dependence on the mass is only given by the initial spread $\sigma^C_x(0)^2$ and there's no dynamical delocalization. This reflects the fact that ach mass propagates at the same velocity, as shown in Fig. (5.2.2).

Comparison

From the above analysis, we underline two main differences between the free evolution of the two MUS states.

1. First of all, as one can argue just looking at the form of the wavefunctions, the MUS in phase space are subjected to a mass-dependent delocalization, since they have a different velocity $\frac{p_0}{m}$ for each mass in the superposition. On the contrary, MUS in configuration space spread out at the same rate.

2. Secondly, we can see that, even when the mass-dependent delocalization does not play a role, the MUS in configuration space remain in general more localized. Considering a finite superposition of masses, we can take two different MUS with the same initial spread:

$$\sigma^C_{X_r}(0)^2 = \sum_m \frac{\hbar}{2m\Omega} = \sigma^P_{X_r}(0)^2 = \sigma^2_{X_r} \tag{5.127}$$

Thus, comparing the dynamical spread, we find

$$\begin{aligned}\sigma^C_{X_r}(t)^2 - \sigma^P_{X_r}(t)^2 &= \left(\sigma^2_{X_r}\Omega^2t^2 - \sum_m \frac{\hbar^2}{4\sigma^2_{X_r}m^2}t^2\right) \\ &= \frac{t^2\Omega^2}{\sigma^2_{X_r}}\left(\sigma^4_{X_r} - \sum_m \frac{\hbar^2}{4m^2}\right)\end{aligned} \tag{5.128}$$

Substituting (5.127), we have

$$\sigma^C_{X_r}(t)^2 - \sigma^P_{X_r}(t)^2 = \frac{t^2\Omega^2}{\sigma^2_{X_r}}\left(\left(\sum_m \frac{\hbar}{2m}\right)^2 - \sum_m\left(\frac{\hbar}{2m}\right)^2\right) \geq 0 \tag{5.129}$$

5.C Variance of the position operator - MUS in phase space

The reduced density matrix of the "rod" (kinematical d.o.f.) can be computed by tracing out the internal degrees of freedom from the pure state (5.36)

$$\begin{aligned}\rho_r(t) &= \int d\epsilon \left|\phi_{\epsilon_0}(\epsilon)\right|^2 \int dp dp' \psi_{p_0}(p) e^{-itE_{p,\epsilon}}\ \psi_{p_0}(p) e^{+itE_{p',\epsilon}} \left|p\right\rangle\!\left\langle p'\right| \\ &= \int d\epsilon \left|\phi_{\epsilon_0,\tau_0}(\epsilon)\right|^2\ \left|\psi_{p_0,x_0,m_\epsilon}\right\rangle\!\left\langle\psi_{p_0,x_0,m_\epsilon}\right|_r\end{aligned} \tag{5.130}$$

where $|\psi_{p_0,x_0,m_\epsilon}\rangle\langle\psi_{p_0,x_0,m_\epsilon}|$ is the standard gaussian wavefunction centered in (p_0, x_0) evolved in time by the hamiltonian with eigenvalues $E_{p,\epsilon} = \frac{p^2}{2m_\epsilon}$, where $m_\epsilon = m_r + \epsilon/c^2$ is the relativistic mass. As in the main text, we set the initial average position to $x_0 = 0$. In position space, the gaussian wavefunction reads

$$\psi_{m_\epsilon}(x,t) = \left[\sqrt{2\pi\sigma^2_{X_r}\left(1 + \frac{i\hbar t}{2m_\epsilon\sigma^2_{X_r}}\right)}\right]^{-\frac{1}{2}} \exp\left\{-\frac{\left(x - \frac{p_0}{m_\epsilon}t\right)^2}{4\sigma^2_{X_r}\left(1 + \frac{i\hbar t}{2m_\epsilon\sigma^2_{X_r}}\right)}\right\} \exp\left\{\frac{i}{\hbar}xp_0\right\} \exp\left\{-\frac{i}{\hbar}\frac{p_0^2}{2m_\epsilon}\right\} \tag{5.131}$$

where $\sigma_{X_r} = \frac{\hbar}{2\sigma_{P_r}}$ is the initial spread. This is the standard gaussian wavepacket in the Schrödinger picture, where the rest mass m is substituted by the relativistic mass m_ϵ. Notice that the average position $\overline{x}_\epsilon(t) = \frac{p_0}{m_\epsilon}t$ is different for each mass/energy in the superposition. From the probability distribution

$$\left|\psi_{m_\epsilon}(x,t)\right|^2 = \left[2\pi\sigma^2_{X_r}\left(1 + \left(\frac{\hbar t}{2m_\epsilon\sigma^2_{X_r}}\right)^2\right)\right]^{-\frac{1}{2}} \exp\left\{-\frac{\left(x - \frac{p_0}{m_\epsilon}t\right)^2}{2\sigma^2_{X_r}\left(1 + \left(\frac{\hbar t}{2m_\epsilon\sigma^2_{X_r}}\right)^2\right)}\right\} \tag{5.132}$$

we can read out the spread in position for each mass m_ϵ:

$$\sigma_{X_r,m_\epsilon}(t)^2 = \sigma^2_{X_r}\left[1 + \left(\frac{\hbar t}{2m_\epsilon\sigma^2_{X_r}}\right)^2\right] = \sigma^2_{X_r} + \left(\frac{\sigma_{P_r}}{m_\epsilon}t\right)^2 \tag{5.133}$$

This is the standard spread in time of a gaussain wavepacket, where the time dependence is due to the uncertainty in the velocity [11]. The variance of the position operator can be computed directly considering its first and second moment

$$\begin{aligned}\left\langle X_r\right\rangle(t) &= \int d\epsilon \left|\phi_{\epsilon_0,\tau_0}(\epsilon)\right|^2 \overline{x}_\epsilon(t) \\ \left\langle X_r^2\right\rangle(t) &= \int d\epsilon \left|\phi_{\epsilon_0,\tau_0}(\epsilon)\right|^2 \overline{x^2}_\epsilon(t)\end{aligned} \tag{5.134}$$

where $\overline{x^2}_\epsilon(t) = \sigma_{X_r,m_\epsilon}(t)^2 + \overline{x}_\epsilon(t)^2$ and $\overline{x}_\epsilon(t)$ is the average position for each mass in the superposition. Therefore the variance $\sigma^2_{X_r}(t) = \left\langle X_r^2\right\rangle(t) - \left\langle X_r\right\rangle^2(t)$ reads

$$\begin{aligned}\sigma_{X_r}(t)^2 &= \int d\epsilon\ \left|\phi_{\epsilon_0,\tau_0}(\epsilon)\right|^2 \left(\sigma_{X_r,m_\epsilon}(t)^2 + \overline{x}_\epsilon(t)^2\right) - \left(\int d\epsilon\ \left|\phi_{\epsilon_0,\tau_0}(\epsilon)\right|^2 \overline{x}_\epsilon\right)^2 \\ &= \sigma^2_{X_r} + \int d\epsilon \left|\phi_{\epsilon_0,\tau_0}\right|^2 \left(\frac{\sigma_{P_r}}{m_\epsilon}t\right)^2 + \left[\int d\epsilon \left|\phi_{\epsilon_0,\tau_0}\right|^2 \left(\frac{p_0 t}{m_\epsilon}\right)^2 - \left(\int d\epsilon \left|\phi_{\epsilon_0,\tau_0}\right|^2 \frac{p_0 t}{m_\epsilon}\right)^2\right]\end{aligned} \tag{5.135}$$

[11] Intuitively, an uncertainty in the velocity δv corresponds to an uncertainty in position $\delta x = \delta v t$

Expanding m_ϵ at first order in ϵ/c^2 we have

$$\sigma^P(x)(t)^2 = \sigma^2_{X_r} + \left(\frac{\sigma_{P_r}}{m_r}t\right)^2 \left\langle \left(1 - \frac{\epsilon}{m_r c^2}\right)^2 \right\rangle + \left(\frac{p_0 t}{m_r}\right)^2 \left[\left\langle \left(1 - \frac{\epsilon}{mc^2}\right)^2 \right\rangle - \left\langle \left(1 - \frac{\epsilon}{m_r c^2}\right) \right\rangle^2 \right] \tag{5.136}$$

We see that at this order the only modification is a correction due to the avg. internal energy

$$\sigma^P(x)(t)^2 = \sigma^2_{X_r} + \left(\frac{\sigma_{P_r}}{m_r}t\right)^2 \left(1 - 2\frac{\epsilon_0}{m_r c^2}\right) \tag{5.137}$$

A dependence on the spread in the internal energy only appears considering an order higher, namely up to $\mathcal{O}((\epsilon/mc^2)^2)$, in which case the variance reads

$$\sigma_{X_r}(t)^2 = \sigma^2_{X_r} + \left(\frac{\sigma_{P_r}}{m_r}t\right)^2 \left(1 - 2\frac{\epsilon_0}{m_r c^2} + 3\frac{\sigma^2_{H_c} + \epsilon_0^2}{m_r^2 c^4}\right) + \left(\frac{p_0 t}{m_r}\right)^2 \frac{\sigma^2_{H_c}}{m_r^2 c^4} \tag{5.138}$$

Notice that the average internal energy can be re-absorbed in the rest mass, using the second order $\mathcal{O}((\frac{H_c}{m_r c^2})^2)$ expansion

$$\frac{1}{m_r^n}\left(1 - n\frac{\epsilon_0}{m_r c^2} + \frac{n(n+1)}{2}\frac{\epsilon_0^2}{m_r^2 c^2}\right) \simeq \left(\frac{1}{m_r + \epsilon_0/c^2}\right)^n \tag{5.139}$$

Thus, the above equation reads

$$\sigma_{X_r}(t)^2 = \sigma^2_{X_r} + \left(\frac{\sigma_{P_r}}{m_r + \epsilon_0/c^2}t\right)^2 + 3\left(\frac{\sigma_{P_r}}{m_r}t\right)^2 \left(\frac{\sigma^2_{H_c}}{m_r^2 c^4}\right) + \left(\frac{p_0 t}{m_r}\right)^2 \frac{\sigma^2_{H_c}}{m_r^2 c^4} \tag{5.140}$$

For simplicity, we can set $\epsilon_0 = 0$ without affecting the results of this section. This gives eq. (5.40) of the main text.

5.D Variance of the position operator - MUS in configuration space

For the time evolution of the MUS in configuration space (5.28) we can simply substitute $p_0 \to m v_0$, $\sigma^2_{X_r} \to \sigma^2_{x,m} = \frac{\hbar}{2m\Omega}$ to the standard gaussian-wavepacket (5.131). This gives

$$\psi^C_{m_\epsilon}(x,t) = \left[\sqrt{2\pi\frac{\hbar}{2m_\epsilon\Omega}(1 + i\Omega t)}\right]^{-\frac{1}{2}} \exp\left\{-\frac{m_\epsilon\Omega}{2\hbar}\frac{(x - v_0 t)^2}{(1 + i\Omega t)}\right\} \exp\left\{\frac{i}{\hbar}x m_\epsilon v_0\right\} \exp\left\{-\frac{i}{\hbar}m_\epsilon\frac{v_0^2}{2}t\right\} \tag{5.141}$$

Computing the probability distribution

$$\left|\psi^C_{m_\epsilon}(x,t)\right|^2 = \left[2\pi\frac{\hbar}{2m_\epsilon\Omega}\left(1 + \Omega^2 t^2\right)\right]^{-\frac{1}{2}} \exp\left\{-\frac{m_\epsilon\Omega}{\hbar}\frac{(x - v_0 t)^2}{(1 + \Omega^2 t^2)}\right\} \tag{5.142}$$

we can read out the spread in position for each internal-energy eigenvalue

$$\sigma^C_{X_r,m_\epsilon}(t)^2 = \frac{\hbar}{2m_\epsilon\Omega}\left(1 + \Omega^2 t^2\right) \tag{5.143}$$

The total spread in position is computed from the reduced density matrix of the "rod"

$$\rho^C_r(t) = \int d\epsilon \left|\phi_{\epsilon_0}(\epsilon)\right|^2 \ |\psi_{v_0,x_0,m_\epsilon}\rangle\langle\psi_{v_0,x_0,m_\epsilon}|_r \tag{5.144}$$

Since the avg. position $\overline{x}_m(t) = v_0 t$ does not depend on the mass, the spread in position is given by

$$\sigma^C_{X_r}(t)^2 = \int d\epsilon \ |\phi_{\epsilon_0}(\epsilon)|^2 \, \sigma^C_{X_r,m_\epsilon}(t)^2 = \int d\epsilon \ |\phi_{\epsilon_0}(\epsilon)|^2 \, \frac{\hbar}{2m_\epsilon \Omega} \left(1 + \Omega^2 t^2\right) \tag{5.145}$$

Expanding m_ϵ in (5.147) up to order $\mathcal{O}((\epsilon/mc^2)^2)$ gives

$$\sigma^C_{X_r}(t)^2 = \frac{\hbar}{2m_r \Omega} \left(1 + \Omega^2 t^2\right) \left(1 - \frac{\epsilon_0}{m_r c^2} + \frac{\sigma^2_{H_c}}{m_r^2 c^4}\right) \tag{5.146}$$

Finally, using (5.139), we can absorb the average energy into the rest mass:

$$\sigma^C_{X_r}(t)^2 = \frac{\hbar}{2(m_r + \epsilon_0/c^2)\Omega} \left(1 + \Omega^2 t^2\right) + \frac{\hbar}{2m_r \Omega} \left(1 + \Omega^2 t^2\right) \left(\frac{\sigma^2_{H_c}}{m_r^2 c^4}\right) \tag{5.147}$$

By setting for simplicity $\epsilon_0 = 0$, we recover eq. (5.42) of the main text.

5.E Variance of the time operator - phase space MUS

The reduced density matrix of the clock (internal d.o.f.) can be computed by tracing out the kinematical degrees of freedom from the pure state (5.35)

$$\begin{aligned} \rho_c(t) &= \int dp \, |\psi_{p_0}(p)|^2 \int d\epsilon d\epsilon' \phi_{\epsilon_0}(\epsilon) \phi_{\epsilon_0}(\epsilon') e^{-it\Delta_p(\epsilon - \epsilon')} \, |\epsilon\rangle\langle\epsilon'| \\ &= \int dp \, |\psi_{p_0}(p)|^2 \, |\tau_0 + t\Delta_p\rangle\langle\tau_0 + t\Delta_p| \end{aligned} \tag{5.148}$$

Here we expanded

$$E(p, \epsilon) = \frac{p^2}{2m_\epsilon} + \epsilon \approx \frac{p^2}{2m_r} + \epsilon(1 - \frac{p^2}{2m_r^2 c^2}, \tag{5.149}$$

so $\Delta_p = \left(1 - \frac{p^2}{2m_r^2 c^2}\right)$ is the time dilation factor; then we defined

$$|\tau_0 + t\Delta_p\rangle = \int d\epsilon \ \phi_{\epsilon_0}(\epsilon) e^{-it\Delta_p \epsilon} \, |\epsilon\rangle_c \tag{5.150}$$

that is the fourier transform of the gaussian states in the energy space. The reduced state of the clock (5.E) is a mixture of clock states $|t\Delta_p\rangle$ weighted by the gaussian distribution in momentum space. For the variance of the self-adjoint time operator

$$T_c = \int d\tau \ \tau \, |\tau\rangle\langle\tau| = \int d\epsilon d\epsilon' \ \tau e^{-i\tau(\epsilon - \epsilon')} \, |\epsilon\rangle\langle\epsilon'| \tag{5.151}$$

we compute explicitly its first and second moment

$$\langle T_c \rangle = tr_c[\rho_c(t) T_c] = \int dp \, |\psi_{p_0}(p)|^2 \int d\tau \tau \left|\tilde{\phi}(\tau - t\Delta_p)\right|^2 \tag{5.152}$$

$$\langle T_c^2 \rangle = tr_c[\rho_c(t) T_c^2] = \int dp \, |\psi_{p_0}(p)|^2 \int d\tau \tau^2 \left|\tilde{\phi}(\tau - t\Delta_p)\right|^2 \tag{5.153}$$

where

$$\tilde{\phi}(\tau - t\Delta_p) = \int d\epsilon \ \phi_{\epsilon_0}(\epsilon) e^{-i(t\Delta_p - \tau)\epsilon} \sim \exp\left\{-\frac{(\tau - t\Delta_p)^2}{4\sigma^2_{T_c}}\right\} \tag{5.154}$$

Considering $\phi_{\epsilon_0}(\epsilon)$ to be a gaussian state with std.dev $\sigma^2_{H_c}$, $\tilde{\phi}(\tau - t\Delta_p)$ will be a gaussian centered in $\tau_0 = t\Delta_p$ with std.dev. $\sigma^2_{T_c} = \hbar^2/4\sigma^2_{H_c}$, as it follows from the canonical commutation relations. Thus, one has

$$\langle T_c \rangle = \int dp \, |\psi_{p_0}(p)|^2 \, \overline{\tau} = \int dp \, |\psi_{p_0}(p)|^2 \, t\Delta_p = t \, \langle \Delta_{P_r} \rangle \tag{5.155}$$

$$\langle T_c^2 \rangle = \int dp \, |\psi_{p_0}(p)|^2 \, \overline{\tau^2} = \int dp \, |\psi_{p_0}(p)|^2 \left(\sigma^2_{T_c} + (t\Delta_p)^2 \right) = \sigma^2_{T_c} + t^2 \, \langle \Delta^2_{P_r} \rangle \tag{5.156}$$

It follows that relavistic effects on the precision of the clock vanish at order $\mathcal{O}((p/mc)^2)$. We then need to work at a precision higher than before, i.e. considering a time dilation factor

$$\Delta_{P_r} = \left(1 - \frac{P_r^2}{2m^2c^2} + \frac{3P_r^4}{8m^4c^4} \right) \tag{5.157}$$

At this order, one has the variance reads

$$\langle \Delta_{P_r} \rangle^2 = \left(1 - \frac{\langle P_r^2 \rangle}{2m^2c^2} + \frac{3}{8} \frac{\langle P_r^4 \rangle}{m^4c^4} \right)^2 = 1 - \frac{\langle P_r^2 \rangle}{m^2c^2} + \frac{3}{4} \frac{\langle P_r^4 \rangle}{m^4c^4} + \frac{\langle P_r^2 \rangle^2}{4m^4c^4} \tag{5.158}$$

$$\langle \Delta^2_{P_r} \rangle = \left\langle 1 - \frac{P_r^2}{m^2c^2} + \frac{P_r^4}{m^4c^4} \right\rangle = 1 - \frac{\langle P_r^2 \rangle}{m^2c^2} + \frac{\langle P_r^4 \rangle}{m^4c^4} \tag{5.159}$$

$$\sigma^P_T(t)^2 = \langle T_c^2 \rangle - \langle T_c \rangle = \sigma^2_{T_c} + t^2 \frac{< P_r^4 > - < P_r^2 >^2}{4m^4c^4} = \sigma^2_{T_c} + t^2 \frac{\sigma^2_{P_r^2}}{4m^4c^4} \tag{5.160}$$

5.F Variance in time - configuration space MUS

The reduced density matrix of the clock reads (using the same expansion as in 5.E):

$$\rho_c(t) = \int d\epsilon d\epsilon' \int dp \; \psi_{v_0,x_0}(p,\epsilon) \psi_{v_0,x_0}(p,\epsilon')^* \; \phi_{\epsilon_0}(\epsilon) \phi_{\epsilon_0}(\epsilon') e^{-\frac{i}{\hbar}(\epsilon - \epsilon')t\Delta_p} \, |\epsilon\rangle\langle\epsilon'|_c \,, \tag{5.161}$$

where we recall

$$\psi_{v_0,x_0}(p,\epsilon) = \left(\frac{1}{2m_\epsilon \hbar \Omega} \right)^{\frac{1}{4}} \exp\left\{ -\frac{1}{2m_\epsilon \hbar \Omega} \left(p - m_\epsilon v_0 \right)^2 - \frac{i}{\hbar} \left(p - m_\epsilon v_0 \right) x_0 \right\}. \tag{5.162}$$

To compute clock's variance, we need:

$$\begin{aligned} tr_c \left[T_c^n \rho_c(t) \right] = \int d\epsilon d\epsilon' \int dp \; \psi_{v_0,x_0}(p,\epsilon') \psi_{v_0,x_0}(p,\epsilon)^* \; \phi_{\epsilon_0}(\epsilon) \phi_{\epsilon_0}(\epsilon') \cdot \\ \cdot \int d\tau \; (t\Delta_p - \tau)^n e^{-\frac{i}{\hbar}(\epsilon - \epsilon')\tau} \end{aligned} \tag{5.163}$$

where we changed variables $t\Delta_p - \tau \to \tau$. Now we make use of the following identities

$$\int dt \; t e^{\frac{i}{\hbar} te} = -i\hbar \partial_e \delta(e), \qquad \int dt \; t^2 e^{\frac{i}{\hbar} te} = -\hbar^2 \partial_e^2 \delta(e) \tag{5.164}$$

so that we can first change variables $(\epsilon, \epsilon') \to (\frac{\epsilon + \epsilon'}{2}, \epsilon - \epsilon') := (E, e)$ and then we can integrate by parts. For instance, we have

$$tr_c\left[T_c\rho_c(t)\right] = \int d\epsilon d\epsilon' \int dp\ \psi_{v_0,x_0}(p,\epsilon')\psi_{v_0,x_0}(p,\epsilon)^*\ \phi_{\epsilon_0}(\epsilon)\phi_{\epsilon_0}(\epsilon')\left(\tau\delta(\epsilon-\epsilon') - i\hbar\partial_e\delta(e)\right) \tag{5.165}$$

By expressing $\epsilon = E + e/2$ and $\epsilon' = E - e/2$ we can easily integrate by parts. In particular, we deal with

$$(-i\hbar)^n\ \partial_e^n\Big(\psi_{v_0,x_0}(p,E+e/2)\psi_{v_0,x_0}(p,E-e/2)^*\ \phi_{\epsilon_0}(E+e/2)\phi_{\epsilon_0}(E-e/2)^*\Big)\delta(e) \tag{5.166}$$

For simplicity, we consider $\tau_0 = x_0 = 0$ so that the gaussians are fully real. We can check that

$$-i\hbar\ \partial_e\Big(\psi_{v_0,x_0}(p,E+e/2)\psi_{v_0,x_0}(p,E-e/2)\ \phi_{\epsilon_0}(E+e/2)\phi_{\epsilon_0}(E-e/2)\Big)\delta(e) = 0 \tag{5.167}$$

While we have

$$(-i\hbar)^2\ \partial_e^2\Big(\psi_{v_0,x_0}(p,E+e/2)\psi_{v_0,x_0}(p,E-e/2)\ \phi_{\epsilon_0}(E+e/2)\phi_{\epsilon_0}(E-e/2)\Big)\delta(e) \tag{5.168}$$

$$= \frac{\hbar^2}{4}\left(\frac{1}{\sigma_{H_c}^2} + \frac{2p^2}{\hbar\Omega m_E^3c^4} - \frac{1}{m_E^2c^4}\right) \tag{5.169}$$

where the last term comes from the normalization factor of $\psi_{v_0,x_0}(p,\epsilon)$. Thus, the expectation values read

$$tr_c\left[T_c\rho_c(t)\right] = \int d\epsilon \int dp\ \left|\psi_{v_0,x_0}(p,\epsilon')\right|^2\ \left|\phi_{\epsilon_0}(\epsilon)\right|^2 t\Delta_p = t\left\langle\Delta_{P_r}\right\rangle \tag{5.170}$$

and

$$tr_c\left[T_c^2\rho_c(t)\right] = \int d\epsilon \int dp\ \left|\psi_{v_0,x_0}(p,\epsilon')\right|^2\ \left|\phi_{\epsilon_0}(\epsilon)\right|^2 \left(t^2\Delta_p^2 + \frac{\hbar^2}{4}\left(\frac{1}{\sigma_{H_c}^2} + \frac{2p^2}{\hbar\Omega m_\epsilon^3c^4} - \frac{1}{m_\epsilon^2c^4}\right)\right) \tag{5.171}$$

Now, inserting $\left\langle p^2\right\rangle = p_0^2 + \sigma_{P_r}^2 = m_\epsilon^2v_0^2 + \frac{m_\epsilon\hbar\Omega}{2}$ in the above expression gives:

$$tr_c\left[T_c^2\rho_c(t)\right] = t^2\left\langle\Delta_p^2\right\rangle + \frac{\hbar^2}{4\sigma_{H_c}^2} + \left\langle\frac{1}{2}\frac{\hbar}{Mc^2\Omega}\frac{v_0^2}{c^2}\right\rangle \tag{5.172}$$

The variance in time then reads

$$\sigma_{T_c}^C(t)^2 = \frac{\hbar^2}{4\sigma_{H_c}^2} + \frac{1}{2}\left\langle\frac{\hbar}{Mc^2\Omega}\right\rangle\frac{v_0^2}{c^2} + t^2\left(\left\langle\Delta_p^2\right\rangle - \left\langle\Delta_p\right\rangle^2\right) \tag{5.173}$$

Following the calculations we did for phase-space MUS, we see that the dynamical corrections appears at order $\sim p^4/m^4c^4$. The difference is that the initial time precision has a correction given by the average velocity:

$$\sigma_{T_c}^C(t)^2 = \frac{\hbar^2}{4\sigma_{H_c}^2} + \frac{1}{2}\left\langle\frac{\hbar}{Mc^2\Omega}\right\rangle\frac{v_0^2}{c^2} + t^2\frac{\sigma_{P_r^2}^2}{4m_r^4c^4} \tag{5.174}$$

This is expected, since the configuration-space MUS are entangled from the beginning, so the clock's kinematics has an effect on the internal degrees of freedom. Since the second term is already of order $\sim c^{-4}$ and moreover we can assume $v_0/c \ll 1$, we can approximate the relativistic mass to the rest mass. Thus, we recover Eq. (5.46) in the main text.

5.G Invariant position

The invariant position (5.65), using the POVM (5.64), reads

$$\tilde{X} = \int dxdt\, \mathcal{U}_r^\dagger(x)\mathcal{U}_{rc}^\dagger(t)\left[\{|0\rangle\langle 0|_r, \Delta_{P_r}\} \otimes |0\rangle\langle 0|_c\right] \otimes \mathcal{U}_s^\dagger(x,t)[X_s] \tag{5.175}$$

Firstly, we focus on spatial translations, which act only on the kinematical degrees of fredom. We compute

$$\begin{aligned}&\int dx\, \mathcal{U}_r^\dagger(x)\mathcal{U}_s^\dagger(x)\left[\{|0\rangle\langle 0|_r, \Delta_{P_r}\} \otimes X_s\right] = \\ &= \int dx \int dpdp'\ \ |p\rangle\langle p'|_r\ \frac{1}{2}\left(\Delta'_p + \Delta_p\right) e^{\frac{i}{\hbar}x(p-p')} \otimes (X_s + x)\end{aligned}$$

Now we use

$$\int dx\ xe^{\frac{i}{\hbar}xq} = -i\hbar\partial_q\delta(q) \tag{5.176}$$

and we change variables $(p,' p) \to (p + q/2, p - q) \coloneqq (Q, q)$. This gives

$$\begin{aligned}&\int dp\ \Delta_p\, |p\rangle\langle p|_r \otimes X_s + \\ &-i\hbar \int dQdq\ \ |Q + q/2\rangle\langle Q - q/2|_r\, \frac{1}{2}\left(\Delta_{Q+q/2} + \Delta_{Q-q/2}\right)\left(\partial_q\delta(q)\right) \otimes \mathbb{I}_s\end{aligned} \tag{5.177}$$

Now we express

$$\begin{aligned}|Q + q/2\rangle_r &= e^{+\frac{i}{\hbar}\frac{q}{2}X_r}\, |Q\rangle \\ \langle Q - q/2|_r &= \langle Q|\, e^{+\frac{i}{\hbar}\frac{q}{2}X_r}\end{aligned} \tag{5.178}$$

Integrating by parts, when the derivative acts on these terms we get the position operator, while the action on the time dilation factors vanish. Expicitly, we get

$$\int dp\ \Delta_p\, |p\rangle\langle p|_r \otimes X_s - \int dp\ \{X_r, \Delta_p\, |p\rangle\langle p|_r\} \otimes \mathbb{I}_s \tag{5.179}$$

where $\{A, B\} = \frac{1}{2}(AB + BA)$ is the anticommutator. Notice that, in the limit in which $\Delta_p = 1$, we recover the usual relative position $X_s - X_r$.

Similarly, we can compute the time evolution, which acts through 5.22 on both the internal and external degrees of freedom.

$$\int dt\, \mathcal{U}_{rc}^\dagger(t)\mathcal{U}_s^\dagger(t)\left[\int dp\ \Delta_p\, |p\rangle\langle p|_r \otimes |0\rangle\langle 0|_c \otimes X_s - \int dp\ \{X_r, \Delta_p\, |p\rangle\langle p|_r\} \otimes |0\rangle\langle 0|_c \otimes \mathbb{I}_s\right] \tag{5.180}$$

We consider the two terms separately

1. The action on the system's Hilbert space reads

$$\mathcal{U}_s^\dagger(t)[X_s] = X_s + t\frac{P_s}{m_s} \tag{5.181}$$

While the action on the clock, by expanding $|0\rangle_c = \int d\epsilon\ \ |\epsilon\rangle$, reads

$$\mathcal{U}_{rc}^{\dagger}(t)\left[\int dp\ \Delta_p\,|p\rangle\langle p|_r\otimes|0\rangle\langle 0|_c\right]$$
$$=\int dp\ \Delta_p\,|p\rangle\langle p|_r\otimes\int dt\int d\epsilon d\epsilon'\ e^{\frac{i}{\hbar}(\epsilon-\epsilon')t\Delta_p}\,|\epsilon\rangle\langle\epsilon'|$$
$$=\int dp\ \Delta_p\,|p\rangle\langle p|\otimes\int dt\ \ |t\Delta_p\rangle\langle t\Delta_p| \tag{5.182}$$

All toghether, reads

$$\int dp\ \Delta_p\,|p\rangle\langle p|\otimes\int dt\ \ |t\Delta_p\rangle\langle t\Delta_p|\otimes\left(X_s+t\frac{P_s}{m_s}\right) \tag{5.183}$$

By a change of variables $t\Delta_p\to t$ we reach

$$X_s\otimes\mathbb{I}_r\otimes\mathbb{I}_c+\frac{P_s}{m_s}\otimes\frac{1}{\Delta_{P_r}}\otimes T_c \tag{5.184}$$

2. For the second part of 5.180 we have to compute

$$-\int dt\,\mathcal{U}_{rc}^{\dagger}(t)\ \left[\int dp\,\{X_r,\Delta_p\,|p\rangle\langle p|_r\}\otimes|0\rangle\langle 0|_c\otimes\mathbb{I}_s\right]$$
$$=-\int dt\ \int dp\ \int d\epsilon d\epsilon'\ \left(\{X_r,\Delta_p\,|p\rangle\langle p|_r\}+(\frac{p}{m_\epsilon}+\frac{p}{m'_\epsilon})t\Delta_p\,|p\rangle\langle p|_r\right)\otimes|\epsilon\rangle\langle\epsilon'|_c\,e^{-\frac{i}{\hbar}t\Delta_p(\epsilon-\epsilon')} \tag{5.185}$$

The computation is analogous to the previous one: we first change time variable $t\Delta_p\to t$ and we use

$$\int dt\ te^{\frac{i}{\hbar}te}=-i\hbar\partial_e\delta(e) \tag{5.186}$$

then we change variables $(\epsilon,\epsilon')\to(\epsilon+\epsilon'/2,\epsilon-\epsilon')\coloneqq(E,e)$ and we integrate by parts. In particular, expressing

$$|\epsilon\rangle\to\left|E+\frac{e}{2}\right\rangle=e^{\frac{i}{\hbar}\frac{e}{2}T_c}\,|E\rangle \tag{5.187}$$

when the derivative acts on these terms, the time operator appears. Thus, we have

$$-X_r\otimes\mathbb{I}_c-\int dp\ \int d\epsilon\frac{1}{\Delta_p}\,|p\rangle\langle p|_r\ \ \frac{p}{m_\epsilon}\{T_c,|\epsilon\rangle\langle\epsilon|_c\}$$
$$\coloneqq-X_r\otimes\mathbb{I}_c-\frac{P_r}{\Delta_{P_r}}\{T_c,\frac{1}{M}\}$$

The final result reads

$$\tilde{X}=X_s\otimes\mathbb{I}_r\otimes\mathbb{I}_c+\frac{P_s}{m_s}\otimes\frac{1}{\Delta_{P_r}}\otimes T_c$$
$$-\mathbb{I}_s\otimes X_r\otimes\mathbb{I}_c-\mathbb{I}_s\otimes P_r\frac{1}{\Delta_{P_r}}\otimes\{T_c,\frac{1}{M}\} \tag{5.188}$$

that coincides with (5.65).

5.H Recovery operation - phase space MUS

As initial initial state of the quantum clock we consider a separable state (MUS in phase space):

$$|0,0\rangle_{rc}^{P} = \int dp \psi_{p_0,x_0=0}(p) \, |p\rangle_r \otimes \int d\epsilon \phi_{\epsilon_0}(\epsilon) \, |\epsilon\rangle_c \tag{5.189}$$

where both wavepackets, in momentum and internal energy space, are gaussian. Explicitly

$$\psi_{p_0}(p) = \left(\frac{1}{2\pi\sigma_{P_r}^2}\right)^{\frac{1}{4}} \exp\left\{-\frac{(p-p_0)^2}{4\sigma_{P_r}^2}\right\}, \qquad \psi_{\epsilon_0}(\epsilon) = \left(\frac{1}{2\pi\sigma_{H_c}^2}\right)^{\frac{1}{4}} \exp\left\{-\frac{(\epsilon-\epsilon_0)^2}{4\sigma_{H_c}^2}\right\} \tag{5.190}$$

The "orbit of $|0,0\rangle_{rc}$ under the group of space and time translations is given by the states

$$|x,t\rangle_{rc}^{P} = U_{rc}(t)U_r(x) \, |0,0\rangle_{rc}^{P} = \int dp \int d\epsilon \psi_{p_0}(p) e^{-\frac{i}{\hbar}xp} e^{-\frac{i}{\hbar}tE_p} \, |p\rangle_r \otimes \phi_{\epsilon_0}(\epsilon) e^{-\frac{i}{\hbar}t\Delta_p\epsilon} \, |\epsilon\rangle_c \tag{5.191}$$

where the group action is defined in the main text (5.52) and $\Delta_p = (1 - \frac{p^2}{2m_r^2c^2})$ is the time dilation factor. The POVM measurment (5.64) performed by Bob is given by

$$E_{x,t} = \mathcal{U}_r^{\dagger}(x)\mathcal{U}_{rc}^{\dagger}(t)\Big[\{|0\rangle\langle 0|_r \, , \Delta_{P_r}\} \otimes |0\rangle\langle 0|_c\Big] \tag{5.192}$$

Thus, the Born probability reads

$$\begin{aligned} p(x,t) &= tr_{rc}\Big[E_{x,t} \;\; |0,0\rangle\langle 0,0|_{rc}\Big] \\ &= \int dp dp' \; \frac{1}{2}\big(\Delta_p + \Delta_{p'}\big)\psi_{p_0}(p)\psi_{p_0,x_0}(p')^* e^{-\frac{i}{\hbar}x(p-p')} e^{-\frac{i}{\hbar}t(E(p)-E(p'))} \; \cdot \\ &\quad \cdot \int d\epsilon d\epsilon' \phi_{\epsilon_0,\tau_0}(\epsilon)\phi_{\epsilon_0,\tau_0}(\epsilon')^* e^{-\frac{i}{\hbar}t(\epsilon\Delta_p - \epsilon'\Delta_{p'})} \end{aligned} \tag{5.193}$$

and we can check it's a normalized probability distribution, i.e. $\int dxdt \; p(x,t) = 1$.

To compute the integrals we are interested in, we will often use the following relations:

$$\begin{aligned} \int dt \; te^{-\frac{i}{\hbar}te} &= -i\hbar\partial_e\delta(e) & \int dx \; xe^{-\frac{i}{\hbar}xq} &= -i\hbar\partial_q\delta(q) \\ \int dt \; t^2e^{-\frac{i}{\hbar}te} &= -\hbar^2\partial_e^2\delta(e) & \int dx \; x^2e^{-\frac{i}{\hbar}xq} &= -\hbar^2\partial_q^2\delta(q) \end{aligned} \tag{5.194}$$

1. We start from the average time squared, in which case we can directly integrate over x, which gives a $\delta(p-p')$:

$$\begin{aligned} &\int dxdt \; p(x,t) \; t^2 \\ &= \int dp \; |\psi_{p_0,0}(p)|^2 \int dt \; \left(\frac{t}{\Delta_p}\right)^2 \int d\epsilon \int d\epsilon' \; \phi_{\epsilon_0,\tau_0}(\epsilon)\phi^*_{\epsilon_0,\tau_0}(\epsilon') e^{-\frac{i}{\hbar}t(\epsilon-\epsilon')} \\ &= \int dp \; |\psi_{p_0,0}(p)|^2 \int dt \; \left(\frac{t}{\Delta_p}\right)^2 \left|\tilde{\phi}_{\epsilon_0,\tau_0}(t)\right|^2 = \left\langle \frac{\sigma_{T_c}^2}{\Delta_{P_r}^2} \right\rangle \end{aligned}$$

   where we define the time-wavefunction as the fourier tranform $\tilde{\phi}_{\epsilon_0,0}(t) = \int d\epsilon \; \phi_{\epsilon_0}(\epsilon)^2 e^{-\frac{i}{\hbar}t\epsilon}$.

2. The average position reads

$$\begin{aligned} &\int dxdt \; p(x,t) \; x \\ &= \int dp dp' \; \frac{1}{2}\big(\Delta_p + \Delta_{p'}\big)\psi_{p_0}(p)\psi_{p_0,0}(p')^* e^{-\frac{i}{\hbar}t(E(p,\epsilon)-E_(p',\epsilon'))} \; \int dx \; xe^{-\frac{i}{\hbar}x(p-p')} \; \cdot \\ &\quad \cdot \int d\epsilon d\epsilon' \phi_{\epsilon_0}(\epsilon)\phi_{\epsilon_0,0}(\epsilon')^* e^{-\frac{i}{\hbar}t(\epsilon-\epsilon')} \end{aligned} \tag{5.195}$$

where $E(p,\epsilon) = \frac{p^2}{2m_\epsilon}$. Now we use (5.208) so that, changing variables $(p,p') \to (\frac{p+p'}{2}, p-p')$ and integrating by parts, we reach

$$= \int dp\ |\psi_{p_0,0}(p)|^2 \int d\epsilon \int d\epsilon'\ \phi_{\epsilon_0}(\epsilon)\phi^*_{\epsilon_0}(\epsilon') \quad \left[\frac{p}{2\Delta_p}\left(\frac{1}{m_\epsilon} + \frac{1}{m_{\epsilon'}}\right)\right] \int dt\ te^{-\frac{i}{\hbar}t(\epsilon-\epsilon')} \tag{5.196}$$

Finally, we use again (5.208), we change variables $(\epsilon,\epsilon') \to (\frac{\epsilon+\epsilon'}{2}, \epsilon-\epsilon') := (E,e)$ and we integrate by parts. We see that

$$\partial_e\left(\phi_{\epsilon_0,0}(E+e/2)\phi^*_{\epsilon_0,0}(E+e/2)\left(\frac{1}{m_{E+e/2}} + \frac{1}{m_{E-e/2}}\right)\right) = 0 \tag{5.197}$$

Thus, we have

$$\int dxdt\ p(x,t)\ x = 0 \tag{5.198}$$

3. Similarly, we can compute

$$\begin{aligned} &\int dxdt\ p(x,t)\ x^2 \\ &= \int dpdp'\ \frac{1}{2}(\Delta_p + \Delta_{p'})\psi_{p_0}(p)\psi_{p_0,0}(p')^* e^{-\frac{i}{\hbar}t(E(p,\epsilon)-E_(p',\epsilon'))} \int dx\ x^2 e^{-\frac{i}{\hbar}x(p-p')}\ . \\ &\qquad \cdot \int d\epsilon d\epsilon' \phi_{\epsilon_0}(\epsilon)\phi_{\epsilon_0,0}(\epsilon')^* e^{-\frac{i}{\hbar}t(\epsilon-\epsilon')} \end{aligned} \tag{5.199}$$

using (5.208) changing variables $(p,p') \to (\frac{p+p'}{2}, p-p')$ and integrating twice by parts we get

$$\begin{aligned} &= \int dp\ |\psi_{p_0,0}(p)|^2 \int dt \int d\epsilon \int d\epsilon'\ \phi_{\epsilon_0}(\epsilon)\phi^*_{\epsilon_0}(\epsilon') e^{-\frac{i}{\hbar}t\Delta_p(\epsilon-\epsilon')}. \\ &\cdot \left[\frac{\hbar^2}{4m_r^2c^2} + \frac{\hbar^2}{4\sigma^2_{P_r}}\Delta_p + i\frac{\hbar t}{4}\big(\frac{1}{m_\epsilon} - \frac{1}{m_{\epsilon'}}\big)\Delta_p + \left(t\big(\frac{p}{2m_\epsilon} + \frac{p}{2m_{\epsilon'}}\big)\right)^2 \Delta_p\right] \end{aligned} \tag{5.200}$$

Notice that the last two terms come from the unitary evolution of the clock $\sim \exp\{-\frac{i}{\hbar}tE(p,\epsilon)\}$, hence they do not strictly depend on the clock's initial state.

By changing variables $t\Delta_p \to t$, and $(\epsilon,\epsilon') \to (\frac{\epsilon+\epsilon'}{2}, \epsilon-\epsilon')$ and using (5.208), one has

$$\begin{aligned} &= \int dp\ |\psi_{p_0,0}(p)|^2 \int d\epsilon\ |\phi_{\epsilon_0}(\epsilon)|^2 \left[\frac{\hbar^2}{4m_r^2c^2} + \sigma^2_{X_r} + \frac{1}{\Delta_p}\frac{\hbar^2}{4m_\epsilon^2c^2} + \frac{p^2}{m_\epsilon^2}\frac{\sigma^2_{T_c}}{\Delta_p^2} - \frac{1}{2}\frac{\hbar^2p^2}{m_\epsilon^4c^4}\frac{1}{\Delta_p^2}\right] \\ &\approx \left\langle \frac{1}{4}\frac{\hbar^2}{m_r^2c^2}\frac{1}{\Delta_{P_r}}\right\rangle + \sigma^2_{X_r} + \left\langle \frac{P_r^2}{M^2}\frac{\sigma^2_{T_c}}{\Delta^2_{P_r}}\right\rangle + \left\langle \frac{1}{4}\frac{\hbar^2}{M^2c^2}\frac{1}{\Delta_{P_r}}\right\rangle \end{aligned} \tag{5.201}$$

where, in the second line, we neglected the term $\sim p^2/c^4$. To sum up, we have the following results

$$\begin{aligned} \int dxdt\ p(x,t)\ t^2 &= \left\langle \frac{\sigma^2_{T_c}}{\Delta^2_{P_r}}\right\rangle \\ \int dxdt\ p(x,t)\ x &= 0 \\ \int dxdt\ p(x,t)\ x^2 &= \frac{\hbar^2}{4m_r^2c^2} + \sigma^2_{X_r} + \left\langle \frac{P_r^2}{M^2}\frac{\sigma^2_{T_c}}{\Delta^2_{P_r}}\right\rangle + \left\langle \frac{1}{4}\frac{\hbar^2}{M^2c^2}\frac{1}{\Delta_{P_r}}\right\rangle \end{aligned} \tag{5.202}$$

that coincides with eq. (5.2.3) in the main text

5.I Recovery operation - configuration space MUS

We consider the clock to be preparred in a configuration space MUS:

$$|0,0\rangle^C = \int d\epsilon \int dp\ \psi_{v_0,x_0=0}(p,\epsilon)\phi_{\epsilon_0}(\epsilon)\ \ |\psi_p\rangle_c \otimes |\epsilon\rangle_r \tag{5.203}$$

where we recall that

$$\psi_{v_0,x_0}(p,\epsilon) = \left(\frac{1}{2m_\epsilon\hbar\Omega}\right)^{\frac{1}{4}} \exp\left\{-\frac{1}{2m_\epsilon\hbar\Omega}\left(p-m_\epsilon v_0\right)^2 - \frac{i}{\hbar}\left(p-m_\epsilon v_0\right)x_0\right\} \tag{5.204}$$

The "orbit of $|0,0\rangle_{rc}$ under the group of space and time translations is given by the states

$$|x,t\rangle^C_{rc} = U_{rc}(t)U_r(x)\,|0,0\rangle^C_{rc} = \int dp \int d\epsilon \psi_{v_0,0}(p,\epsilon)e^{-\frac{i}{\hbar}xp}e^{-\frac{i}{\hbar}tE_p}\,|p\rangle_r \otimes \phi_{\epsilon_0}(\epsilon)e^{-\frac{i}{\hbar}t\Delta_p\epsilon}\,|\epsilon\rangle_c \tag{5.205}$$

The POVM measurment (5.64) performed by Bob is given by

$$E_{x,t} = \mathcal{U}_r^\dagger(x)\mathcal{U}_{rc}^\dagger(t)\Big[\{|0\rangle\langle 0|_r\,,\Delta_{P_r}\}\otimes|0\rangle\langle 0|_c\Big] \tag{5.206}$$

Thus, the Born probability reads

$$\begin{aligned} p(x,t) &= tr_{rc}\Big[E_{x,t}\ \ |0,0\rangle\langle 0,0|^C_{rc}\Big] \\ &= \int dpdp'\ \frac{1}{2}\big(\Delta_p+\Delta_{p'}\big)\psi_{v_0,0}(p,\epsilon)\psi^*_{v_0,0}(p',\epsilon')e^{-\frac{i}{\hbar}x(p-p')}e^{-\frac{i}{\hbar}t(E(p)-E_(p'))}\ \cdot \\ &\qquad \cdot \int d\epsilon d\epsilon' \phi_{\epsilon_0}(\epsilon)\phi^*_{\epsilon_0}(\epsilon')e^{-\frac{i}{\hbar}t(\epsilon\Delta_p-\epsilon'\Delta_{p'})} \end{aligned} \tag{5.207}$$

and we can check it's a normalized probability distribution, i.e. $\int dxdt\ p(x,t) = 1$. Again, we will often use the following relations:

$$\begin{aligned} \int dt\ te^{-\frac{i}{\hbar}te} &= -i\hbar\partial_e\delta(e) & \int dx\ xe^{-\frac{i}{\hbar}xq} &= -i\hbar\partial_q\delta(q) \\ \int dt\ t^2e^{-\frac{i}{\hbar}te} &= -\hbar^2\partial_e^2\delta(e) & \int dx\ x^2e^{-\frac{i}{\hbar}xq} &= -\hbar^2\partial_q^2\delta(q) \end{aligned} \tag{5.208}$$

1. We start from the average time squared, in which case we can directly integrate over x, which gives a $\delta(p-p')$:

$$\begin{aligned} &\int dxdt\ p(x,t)\ t^2 \\ &= \int dp\ \int d\epsilon d\epsilon'\ \Delta_p\psi_{v_0,0}(p,\epsilon)\psi^*_{v_0,0}(p,\epsilon')e^{-\frac{i}{\hbar}t(E(p,\epsilon)-E_(p,\epsilon'))}\ \cdot \\ &\qquad \cdot \phi_{\epsilon_0}(\epsilon)\phi^*_{\epsilon_0}(\epsilon')\int dt\ t^2e^{-\frac{i}{\hbar}t(\epsilon-\epsilon')} \end{aligned}$$

   where $E(p,\epsilon) = p^2/2m_\epsilon$. The computation is similar to the one in (5.F). We use (**??**) and we integrate twice by parts:

$$\begin{aligned} &= \int dp\ \int d\epsilon d\epsilon'\ \Delta_p\ -\hbar^2\partial_e^2\Big(\psi_{v_0,0}(p,E+e/2)\psi^*_{v_0,0}(p,E-e/2)\cdot \\ &\cdot e^{-\frac{i}{\hbar}t\left(E(p,E+e/2)-E(p,E-e/2)\right)}\phi_{\epsilon_0,0}(E+e/2)\phi^*_{\epsilon_0,0}(E-e/2)\Big)\delta(e) \\ &= \int dp\int d\epsilon\ \left|\psi_{v_0,0}(p,\epsilon)\right|^2\left|\phi_{\epsilon_0}(\epsilon)\right|^2\ \frac{\hbar^2}{4}\left(\frac{1}{\sigma^2_{H_c}}+\frac{2p^2}{\hbar\Omega m_\epsilon^3c^4}-\frac{1}{m_\epsilon^3c^4}\right) \end{aligned} \tag{5.209}$$

Then, substituting $\langle p^2 \rangle = p_0^2 + \sigma_{P_r}^2 = m_\epsilon^2 v_0^2 + \frac{m_\epsilon \hbar \Omega}{2}$ we see that the last term vanishes and we get

$$\int dxdt\ p(x,t)\ t^2 = \frac{\hbar^2}{4\sigma_{H_c}^2} + \frac{\hbar v_0^2}{2\Omega m_\epsilon c^4} \tag{5.210}$$

which coincides with our previous fundings.

2. The average position reads

$$\begin{aligned} &\int dxdt\ p(x,t)\ x \\ &= \int d\epsilon d\epsilon' \int dpdp'\ \frac{1}{2}\big(\Delta_p + \Delta_{p'}\big)\psi_{v_0,0}(p,\epsilon)\psi^*_{v_0,0}(p',\epsilon')e^{-\frac{i}{\hbar}x(p-p')}e^{-\frac{i}{\hbar}t\left(E(p)-E(p')\right)} \\ &\qquad \phi_{\epsilon_0}(\epsilon)\phi^*_{\epsilon_0}(\epsilon')e^{-\frac{i}{\hbar}t(\epsilon-\epsilon')}\int dx\ xe^{-\frac{i}{\hbar}x(p-p')} \end{aligned} \tag{5.211}$$

Now we use (5.208) so that, changing variables $(p,p') \to (\frac{p+p'}{2}, p-p')$ and integrating by parts, we reach

$$= \int dp\ |\psi_{v_0,0}(p,\epsilon)|^2 \int d\epsilon \int d\epsilon'\ \phi_{\epsilon_0}(\epsilon)\phi^*_{\epsilon_0}(\epsilon')\ \ \left[\frac{p}{2}\left(\frac{1}{m_\epsilon} + \frac{1}{m_{\epsilon'}}\right)\right]\int dt\ te^{-\frac{i}{\hbar}t(\epsilon-\epsilon')} \tag{5.212}$$

Finally, we use again (5.208), we change variables $(\epsilon,\epsilon') \to (\frac{\epsilon+\epsilon'}{2}, \epsilon-\epsilon') := (E,e)$ and we integrate by parts. We see that

$$\partial_e\left(\phi_{\epsilon_0,0}(E+e/2)\phi^*_{\epsilon_0,0}(E+e/2)\left(\frac{1}{m_{E+e/2}} + \frac{1}{m_{E-e/2}}\right)\right) = 0 \tag{5.213}$$

Thus, we have

$$\int dxdt\ p(x,t)\ x = 0 \tag{5.214}$$

3. Similarly, we can compute

$$\begin{aligned} &\int dxdt\ p(x,t)\ x \\ &= \int d\epsilon d\epsilon' \int dpdp'\ \frac{1}{2}\big(\Delta_p + \Delta_{p'}\big)\psi_{v_0,0}(p,\epsilon)\psi^*_{v_0,0}(p',\epsilon')e^{-\frac{i}{\hbar}x(p-p')}e^{-\frac{i}{\hbar}t(E(p)-E_(p'))}\ . \\ &\qquad \cdot\ \phi_{\epsilon_0}(\epsilon)\phi^*_{\epsilon_0}(\epsilon')e^{-\frac{i}{\hbar}t(\epsilon-\epsilon')}\int dx\ x^2e^{-\frac{i}{\hbar}x(p-p')} \end{aligned} \tag{5.215}$$

using (5.208) changing variables $(p,p') \to (\frac{p+p'}{2}, p-p')$ and integrating twice by parts we get

$$\begin{aligned} &= \int dp\ |\psi_{v_0,0}(p,\epsilon)|^2 \int dt \int d\epsilon \int d\epsilon'\ \phi_{\epsilon_0}(\epsilon)\phi^*_{\epsilon_0}(\epsilon')e^{-\frac{i}{\hbar}t\Delta_p(\epsilon-\epsilon')}. \\ &\cdot\left[\frac{\hbar^2}{4m_r^2c^2} + \frac{\hbar^2}{4\sigma_{P_r}^2}\Delta_p + i\frac{\hbar t}{4}\big(\frac{1}{m_\epsilon} - \frac{1}{m_{\epsilon'}}\big)\Delta_p + \left(t\big(\frac{p}{2m_\epsilon} + \frac{p}{2m_{\epsilon'}}\big)\right)^2\Delta_p\right] \end{aligned} \tag{5.216}$$

where now $\sigma_{P_r}^2 = m_\epsilon \hbar\Omega/2$. Notice that the last two terms come from the unitary evolution of the clock $\sim \exp\big\{-\frac{i}{\hbar}tE(p,\epsilon)\big\}$, hence they do not strictly depend on the clock's initial state.

By changing variables $t\Delta_p \to t$, and $(\epsilon, \epsilon') \to (\frac{\epsilon+\epsilon'}{2}, \epsilon - \epsilon')$ and using (5.208), one has

$$= \int dp \ |\psi_{v_0,0}(p,\epsilon)|^2 \int d\epsilon \ |\phi_{\epsilon_0}(\epsilon)|^2 \left[\frac{\hbar^2}{4m_r^2c^2} + \frac{\hbar}{2m_\epsilon \Omega} + \frac{1}{\Delta_p}\frac{\hbar^2}{4m_\epsilon^2 c^2} + \frac{p^2}{m_\epsilon^2}\frac{\sigma^2_{T_c}(v_0)}{\Delta_p^2} - \frac{1}{2}\frac{\hbar^2 p^2}{m_\epsilon^4 c^4}\frac{1}{\Delta_p^2} \right]$$
$$\approx \left\langle \frac{1}{4}\frac{\hbar^2}{m_r^2c^2}\frac{1}{\Delta_{P_r}} \right\rangle + \left\langle \sigma^2_{X_r}(M) \right\rangle + \left\langle \frac{P_r^2}{M^2}\frac{\sigma^2_{T_c}(v_0)}{\Delta^2_{P_r}} \right\rangle + \left\langle \frac{1}{4}\frac{\hbar^2}{M^2c^2}\frac{1}{\Delta_{P_r}} \right\rangle \tag{5.217}$$

where, in the second line, we neglected the term $\sim p^2/c^4$ and we denoted $\sigma^2_{T_c}(v_0) = \frac{\hbar^2}{4\sigma^2_{H_c}} + \frac{\hbar v_0^2}{2\Omega m_\epsilon c^4}$ and $\sigma_{X_r}(M) = \hbar/2M\Omega$. To sum up, we have the following results

$$\int dxdt \ p(x,t) \ t^2 = \left\langle \frac{\sigma^2_{T_c}(v_0)}{\Delta^2_{P_r}} \right\rangle$$
$$\int dxdt \ p(x,t) \ x = 0$$
$$\int dxdt \ p(x,t) \ x^2 = \frac{\hbar^2}{4m_r^2c^2} + \left\langle \sigma^2_{X_r}(M) \right\rangle + \left\langle \frac{P_r^2}{M^2}\frac{\sigma^2_{T_c}(v_0)}{\Delta^2_{P_r}} \right\rangle + \left\langle \frac{1}{4}\frac{\hbar^2}{M^2c^2}\frac{1}{\Delta_{P_r}} \right\rangle \tag{5.218}$$

that coincides with eq. (5.2.3) in the main text

Chapter 6

Conclusion and Outlook

In this thesis, we use the framework of quantum reference frames (QRF) and quantum clocks in the context of foundations of physics, in particular at the intersection of quantum mechanics (QM) and general relativity (GR). The principles of diffeomorphism invariance in GR imposes an operational approach to reference frames in spacetime: spacetime coordinates are physically meaningful only if their corresponding reference frame is a physical system, obeying the dynamical equations of the theory. Extending this principle to QM introduces fundamental limitations that challenge classical assumptions about space and time.

Inspired by the work of Salecker and Wigner, we begin in section (5.1.2) by examining the synchronization of distant quantum clocks. We found that, unlike in the classical case, perfect synchronization is inherently unattainable when considering the quantum nature of clocks, particularly their internal degrees of freedom. Quantum uncertainties prevents clocks from achieving absolute simultaneity, introducing a fundamental limitation to the concept of simultaneous spacetime slices.

Then, in section (5.2), we investigate how the precision of time measurements affects the spatial localization of quantum clocks. The relativistic coupling between a clock's internal energy and its external motion implies an unavoidable trade-off: improving temporal precision leads to greater uncertainty in spatial localization, and vice versa. This trade-off, which emerges from combining fundamental aspects of quantum mechanics and special relativity, shows that a quantum clock cannot simultaneously serve as both an infinitely precise temporal and a perfectly localized spatial reference frame. The broader the clock's energy spectrum, the more precise its time measurements, but this also results in a larger spatial spread, revealing an interconnectedness between the operational notions of space and time in quantum systems.

While we claim that the physical phenomenon underlying this trade-off is general — stemming from the universal coupling between internal and external degrees of freedom of the spacetime QRF — the quantitative analysis we present is tied to a specific model of the clock, namely one with a continuous and unbounded spectrum hamiltonian. Nevertheless, we believe this analysis lays the groundwork for more comprehensive and general investigations in the future.

Overall, our analysis demonstrates that when the principles of quantum mechanics and relativity are combined, the classical conception of spacetime and reference frames becomes fundamentally inadequate. Quantum clocks do not exhibit the idealized classical behavior of continuous, precise timekeeping or perfect spatial localization, suggesting the need for a fundamental rethinking of reference frames and spacetime.

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