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Quantum Gravity Effective Action: Corrections to Classical Metrics and Observational Tests

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Abstract

General relativity is non-renormalizable, meaning that we lack a complete quantum theory of gravity. However, working at energies below the Planck mass, which is the cutoff scale of quantum gravity, we can resort to the effective field theory approach, implemented here via the Barvinsky-Vilkovisky unique effective action. The latter has been used to derive quantum corrections to classical metrics, such as the Schwarzschild solution. In this thesis, we compute these corrections and extend the analysis to a static and electrically charged star modeled as a perfect fluid, considering distinct scenarios based on different applications of the perfect fluidity condition to the energy-momentum tensor components. Additionally, we explore gravastars and dark energy stars, proposed as compact objects alternative to classical Schwarzschild black holes, arguing that quantum-induced hairs in their metrics may allow us to experimentally distinguish between these objects. Finally, we examine gravitational lensing observables, namely the photon sphere radius and the deflection angle of bent light rays, and the gravitational redshift, to validate the predicted quantum corrections. These findings provide a framework for potential empirical tests to distinguish quantum-corrected metrics from classical ones, contributing to the broader understanding of quantum gravity phenomenology.

Contents

My subject is the quantum theory of gravitation. My interest in it is primarily in the relation of one part of nature to another. There's a certain irrationality to any work in gravitation, so it's hard to explain why you do any of it [...] But since I am among equally irrational men, I won't be criticized I hope for the fact that there is no possible, practical reason for making these calculations.

– Richard Feynman, ”Quantum Theory of Gravitation”

Chapter 1

Introduction

General relativity is an outstanding theory that completely changed our understanding of gravity and has survived many tests in its more than a century-long life. However, it was eventually realized that although it works extremely well when applied to astrophysical objects, problems start to arise both on very large and very small scales. When looking at galaxies, we encounter the well known problem of the rotation velocity of stars around the galactic disk [1, 2] and when looking at the Universe as a whole, we encounter the problem of accelerated expansion and dark energy in the ΛCDM model [3–8]. On the other hand, general relativity breaks down in the quantum regime. As a gauge theory it is non-renormalizable [9, 10] and we thus cannot quantize it as we do with the other fundamental interactions: electromagnetism, weak force and strong force.

Although a theory of quantum gravity would seem to be non-relevant for current everyday physics, as it becomes important at energies much bigger than those we deal with nowadays, there are many physical scenarios that we still don't fully comprehend and that could be made more clear by this theory. The most interesting of which are surely black hole singularities and the very first moments after the birth of the Universe, if not even the origin of the Universe itself. Besides, the fact that we can quantize the other three fundamentals interactions but not gravity, is in itself a good enough motivation for the most "irrational men" among us to pursue this subject.

Many attempts have been made in order to find the UV completion of general relativity. Even though we do not yet have the full theory, there is still something that can be said about quantum gravity. If we restrict ourselves to energies far below its typical energy scale, that is the Planck mass $M_p = 2.4 \times 10^{18}$ GeV, we can work in the effective field theory approach. The resulting effective action is the unique effective action in quantum gravity [11–14], which, in this work, we will refer to as the Barvinsky-Vilkovisky unique effective action. The effective action has been extensively studied: from solutions to the modified Einstein field equations one obtains from this action [15–25] to implications for the Standard Model and dark matter [26–32], many are the information and hints towards the full theory that we can learn.

In this thesis, we will add to this vast phenomenology by computing quantum corrected metrics as perturbative solutions of the modified Einstein equations, looking specifically at the charged star metric, modeled as a static perfect fluid, and at gravastars [33–36] and dark energy stars [37]. Furthermore, we will study implications for gravitational lensing of these and other modified metrics already present in the literature, finding deviations from some well known, classical results [38–41] in order to find possible experimental effects of the theory. In particular, these observables could allow us to experimentally distinguish classical black holes from gravastars or dark energy stars.

The outline of the thesis is the following.

Chapter 2

We briefly review the quantization of general relativity and how the theory is non-renormalizable. We then describe the concept of effective field theories (EFTs) and discuss the quantum gravitational EFT with the Barvinsky-Vilkovisky unique effective action.

Chapter 3

We first show how the gravitational effective action can be used to find corrections to known metrics solving perturbatively the modified Einstein equations. We report these corrections for the simple case of a dust ball, described by the interior Schwarzschild metric star, in the background field method.

Chapter 4

We generalize the result of the previous chapter to an electrically charged star. We consider the star to be made up of a perfect fluid. The energy-momentum tensor is now given by the sum of the proper matter and electromagnetic tensors. Therefore we distinguish two cases, depending on which tensor we impose the perfect fluidity condition on, namely that the spacelike eigenvalues of the energy-momentum tensor should all be equal everywhere: case I, where the condition is imposed only on the matter tensor; case II, where the condition is imposed on the whole tensor.

Chapter 5

After introducing the gravastar and the closely related dark energy star models as possible compact objects alternative to black holes, we argue that the existence of hairs in the quantum corrected metrics of these objects may allow us to experimentally distinguish them from classical Schwarzschild black holes.

Chapter 6

Finally, we turn to the computation of several observables in the framework of gravitational lensing, namely refractive indices in the optical-mechanical analogy in general relativity, the photon sphere radius, the bending of light rays and the gravitational redshift, with the aim of testing the validity of our calculations.

Chapter 7

We reserve this chapter for conclusions and future outlooks.

Chapter 2

The effective field theory of quantum gravity

2.1 The quantization of gravity

In the past century, Quantum Field Theory has been extremely successful in quantizing the electromagnetic, weak and strong interactions leading to the construction of the Standard Model. However, when one attempts to repeat the same quantization procedure for General Relativity, the theory turns out to be non-renormalizable. To better understand what this means, let us start by briefly reviewing the quantization procedure through the path integral formalism [42].

2.1.1 Path integrals

Path integrals give the transition amplitude for a generic field ϕ to go from an initial to a final configuration. This can be expressed as

$$\mathcal{A} = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}S[\phi]}, \tag{2.1}$$

where $\mathcal{D}\phi$ denotes the integration over all possible field configurations weighted by the action $S[\phi] = \int d^4x\mathcal{L}(\phi)$.

Observables are expressed in terms of correlation functions, that is normalized averages of the time-ordered product of a given number n of field operators on the vacuum:

$$\langle 0|\phi(x_1)\phi(x_2)\ldots\phi(x_n)|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi \, T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\}e^{\frac{i}{\hbar}S[\phi]}, \tag{2.2}$$

where $Z = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}S[\phi]}$ is the normalization constant and T the time-ordering operator. If we now introduce the generating functional:

$$Z[J] = \int \mathcal{D}\phi \; e^{\frac{i}{\hbar}[S[\phi]+\int d^4x J(x)\phi(x)]}, \tag{2.3}$$

where $J(x)$ is an external source, we can then express correlation functions as

$$\langle 0|\phi(x_1)\phi(x_2)\ldots\phi(x_n)|0\rangle = \frac{1}{Z[0]}\left(\frac{\hbar}{i}\right)^n \frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}. \tag{2.4}$$

For later convenience, it is useful to introduce the generating functional of connected correlation functions $W[J]$ defined in terms of the usual generating functional (2.3) as

$$Z[J] = e^{\frac{i}{\hbar}W[J]}. \tag{2.5}$$

Path integrals are in general very difficult if not impossible to solve analytically. However, for the free (i.e. non-interacting) theory, characterized by Gaussian integrals, these are exactly solvable. Once we turn on interactions, the resulting path integrals can be computed using perturbation theory around the free theory.

These integrals usually turn out to be divergent. Clearly, a divergent transition amplitude is physically meaningless. Therefore, to get rid of these divergences, the integrals need to be regularized using a given regularization scheme and then renormalized introducing a given number of counterterms in the interaction Lagrangian. If the number of necessary counterterms is finite then the theory is said to be *renormalizable*. As a general rule, if the coupling constant of a term appearing in the Lagrangian has negative mass dimension then the theory is non-renormalizable.

2.1.2 Background field method

General relativity can be quantized with the background field method [43]. That is, the metric $g_{\mu\nu}$ is split into a background metric $\bar{g}_{\mu\nu}$ and a small perturbation $h_{\mu\nu}$ as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}, \tag{2.6}$$

where $\kappa^2 = 8\pi G_N$. The perturbation is then quantized on top of the background metric, which is kept classical and preserves the gauge symmetry of the theory, that is diffeomorphism invariance. In this way, we can quantize general relativity while saving its symmetry.

However, the theory turns out to be non-renormalizable. This is not unexpected, as the coupling constant has a negative mass dimension: $[\kappa^2] = -2$. Divergences in general relativity emerge already at one-loop. In fact, although at this order the pure theory may be renormalized, when we consider interactions with scalar fields this is no longer true [9]. At the two-loop order, even the pure theory is already non-renormalizable [44].

2.2 Effective field theory

Even though general relativity is non-renormalizable, not all hope is lost. In terms of Planck units and setting $c = \hbar = 1$, we may rewrite the coupling constant of general relativity as $\kappa^{-2} = M_p^2$, with

$$M_p = \sqrt{\frac{\hbar\, c}{8\pi G_N}} = 2.4 \times 10^{18} \text{ GeV} \tag{2.7}$$

the Planck mass. As long as we are at energies far below M_p, we may approximate the full theory with an *effective field theory* or EFT. Energies relevant to modern day physics are below this value. For example, the highest energy ever reached at the Large Hadron Collider is of the order of 13.6×10^3 GeV [45].

The EFT is valid up to the given cutoff scale. Once we know this scale we can then divide the energy modes into two distinct categories: high energy modes, that is modes with energies above the cutoff scale and low energy modes, that is modes with energies below the cutoff scale. Since we work at low energies, high energy modes cannot be excited and thus need to be removed from initial and final configurations. They are therefore ”integrated out” in the path integral formalism. Formally, we may express this as

$$\int \mathcal{D}l \; e^{\frac{i}{\hbar}\Gamma[l]} = \int \mathcal{D}l \; \mathcal{D}h \; e^{\frac{i}{\hbar}S[l,h]}, \tag{2.8}$$

where h and l are respectively high and low energy modes, S is the action of the full theory and Γ is the effective action.

If we know the full theory then we can carry out this integration explicitly. However, even if the UV completion is unknown, we can still make predictions on the form of the effective action based on the symmetries of the theory, as all the extra terms we add on top of the low energy limit action must preserve its symmetries. Therefore for gravity, knowing that the cutoff scale is the Planck mass, the low energy theory is given by general relativity and its symmetry is that of diffeomorphism invariance, we expect an effective action of the form

$$\Gamma = \int d^4x\sqrt{-g}\left(\frac{M_p^2}{2}R + a_1\mathcal{R}^2 + \frac{a_2}{M_p^2}\mathcal{R}^3 + \frac{a_3}{M_p^4}\mathcal{R}^4 + \dots\right), \tag{2.9}$$

where $\mathcal{R}^n$ is any contraction of the product of n Riemann tensors and the Wilson coefficients a_i can only be determined from the UV completion of quantum gravity (see e.g. [46, 47]).

For gravity, the heavy modes we have to integrate out are the gravitons. Let us consider the full graviton action, whatever it may be:

$$S[g] = M_p^2 \int d^4x\sqrt{-g}\mathcal{L}(g). \tag{2.10}$$

The generating functional of connected correlation functions (2.5) is in this case

$$W[J] = -i\hbar \ln\left(\int \mathcal{D}g e^{\frac{i}{\hbar}\left(S[g] + \int d^4x\sqrt{-g}g_{\mu\nu}J^{\mu\nu}\right)}\right), \tag{2.11}$$

and taking the Legendre transform we get the effective action

$$\Gamma[\bar{g}] = W[J] - \int d^4x\sqrt{-g}\bar{g}_{\mu\nu}J^{\mu\nu}, \tag{2.12}$$

where

$$\bar{g}_{\mu\nu} = \frac{\delta W[J]}{\delta J^{\mu\nu}} = \langle g_{\mu\nu}\rangle \tag{2.13}$$

is the vacuum expectation value of the metric. We can then use the background field method with the metric (2.6):

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + M_p^{-1}h_{\mu\nu}, \tag{2.14}$$

to finally obtain

$$\Gamma[\bar{g}] = S[\bar{g}] + \frac{i}{2M_p}\,\mathrm{Tr}\left[\ln\left(\frac{\delta^2 S[\bar{g}]}{\delta g^2}\right)\right] + \mathcal{O}(M_p^{-2}), \tag{2.15}$$

which does not depend on the perturbation $h_{\mu\nu}$. We thus succesfully integrated out the gravitons.

This expansion may be interpreted as a loop expansion in Feynman diagrams: the leading order term is the on-shell graviton action corresponding to tree level diagrams, the next term contains one loop graviton diagrams and so on. In order to find corrections to the classical action we thus need to compute the graviton loops. To do so in such a way that the background metric may preserve its gauge freedom we can use DeWitt's mean-field method [43, 48, 49]. The resulting effective action is the Barvinsky-Vilkovisky unique effective action [11–14].

2.3 The Barvinsky-Vilkovisky unique effective action

The Barvinsky-Vilkovisky effective quantum gravitational action is given by the sum of a local and a non-local part:

$$\Gamma[g] = \Gamma_L[g] + \Gamma_{NL}[g]. \tag{2.16}$$

At second order in curvature, the local part reads

$$\Gamma_L = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R + c_1(\mu)R^2 + c_2(\mu)R_{\mu\nu}R^{\mu\nu} + c_3(\mu)R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \mathcal{O}(M_p^{-2})\right], \tag{2.17}$$

where the prefactors c_i are the Wilson coefficients and μ is the renormalization scale. The non-local part is instead

$$\Gamma_{NL} = -\int d^4x\sqrt{-g}\left[\alpha R\ln\left(\frac{\Box}{\mu^2}\right)R + \beta R_{\mu\nu}\ln\left(\frac{\Box}{\mu^2}\right)R^{\mu\nu} + \gamma R_{\mu\nu\alpha\beta}\ln\left(\frac{\Box}{\mu^2}\right)R^{\mu\nu\alpha\beta} + \mathcal{O}(M_p^{-2})\right], \tag{2.18}$$

	α	β	γ
Scalar	$5(6\xi - 1)^2$	-2	2
Fermion	-5	8	7
Vector	-50	176	-26
Graviton	250	-244	424

Table 2.1: Non-local Wilson coefficients for different fields. All numbers should be divided by $11520\pi^2$. ξ is the value of the non-minimal coupling for a scalar theory.

where $\Box := g_{\mu\nu}\nabla^\mu\nabla^\nu$. The action of the non-local operator $\ln(\Box/\mu^2)$ on radial functions is discussed in Appendix A. The last term of the local action (2.17), containing the contraction of two Riemann tensors, can be rewritten as a function of the Ricci tensor and Ricci scalar using the Gauss-Bonnet topological invariant:

$$\int d^4x\sqrt{-g}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}) = 32\pi^2\chi(\mathcal{M}), \tag{2.19}$$

where $\chi(\mathcal{M})$ is the Euler characteristic of the manifold. Being this a topological term, it does not affect the equations of motion. In this way we may simplify the local action to

$$\Gamma_L = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G_N} + \bar{c}_1 R^2 + \bar{c}_2 R_{\mu\nu}R^{\mu\nu}\right], \tag{2.20}$$

where $\bar{c}_1 = c_1 - c_3$ and $\bar{c}_2 = c_2 + 4c_3$.

The value of the Wilson coefficients of the local part is unknown, since we need the UV completion of the quantum gravity theory in order to be able to compute them. Bounds on these coefficients can be determined from the Eöt-Wash experiment [50], which is a Cavendish experiment looking at deviations from the Newtonian potential. The coefficients must be such that: $c_i \lesssim 10^{61}$. The value of those of the non-local part are instead calculable [11, 12, 51] in a gauge invariant manner and are listed in Tab. 2.1. The non-local coefficients depend on the type and number of fields that the graviton couples to. Denoting by N_s, N_f, N_v, N_g the number of scalar, fermionic, vector and graviton fields in the theory, we have in general

$$\alpha = N_s\alpha_s + N_f\alpha_f + N_v\alpha_v + N_g\alpha_g. \tag{2.21}$$

By varying the effective action with respect to the metric (see Appendix B), we find the equations of motion

$$G_{\mu\nu} + 16\pi G_N(H^L_{\mu\nu} + H^{NL}_{\mu\nu}) = 0, \tag{2.22}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{2.23}$$

is the usual Einstein tensor. The local part is given by

$$\begin{aligned} H^{L}_{\mu\nu} =&\bar{c}_1 \left(2RR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2 + 2 g_{\mu\nu} \Box R - 2\nabla_\mu \nabla_\nu R \right) \\ &+ \bar{c}_2 \left(2R^{\alpha}{}_{\mu} R_{\nu\alpha} - \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + \Box R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \Box R - \nabla_\alpha \nabla_\mu R^{\alpha}{}_{\nu} - \nabla_\alpha \nabla_\nu R^{\alpha}{}_{\mu} \right). \end{aligned} \tag{2.24}$$

The non-local part is

$$\begin{aligned} H^{NL}_{\mu\nu} =& -2\alpha \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R + g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) \ln\left(\frac{\Box}{\mu^2} \right) R \\ & - \beta \left(2\delta^{\alpha}_{(\mu} R_{\nu)\beta} - \frac{1}{2} g_{\mu\nu} R^{\alpha}{}_{\beta} + \delta^{\alpha}_{\mu} g_{\nu\beta} \Box + g_{\mu\nu} \nabla^\alpha \nabla_\beta - \delta^{\alpha}_{\mu} \nabla_\beta \nabla_\nu - \delta^{\alpha}_{\nu} \nabla_\beta \nabla_\mu \right) \ln\left(\frac{\Box}{\mu^2} \right) R^{\beta}{}_{\alpha} \\ & - 2\gamma \left(\delta^{\alpha}_{(\mu} R^{\beta}_{\nu)\sigma\tau} - \frac{1}{4} g^{\mu\nu} R^{\alpha\beta}{}_{\sigma\tau} + (\delta^{\alpha}_{\mu} g_{\nu\sigma} + \delta^{\alpha}_{\nu} g_{\mu\sigma}) \nabla^\beta \nabla_\tau \right) \ln\left(\frac{\Box}{\mu^2} \right) R_{\alpha\beta}{}^{\sigma\tau}. \end{aligned} \tag{2.25}$$

Note that variations of the $\ln(\Box/\mu^2)$ terms yield terms of higher order in curvature which can then be neglected at second order in the curvature expansion [52].

Solving these modified equations of motion could help us better understand gravitational phenomena and possibly solve some of the problems of general relativity. However, finding analytic solutions is indeed a very difficult task and therefore we must resort to some approximate methods. In the following chapter, using the background field method and perturbation theory, we see how we may solve these equations finding corrections to the Schwarzschild metric.

Chapter 3

Quantum gravitational corrections to the Schwarzschild metric

The simplest solution of the Einstein field equations is given by the Schwarzschild metric [53, 54], describing the spacetime outside a static spherically symmetric object. In this chapter we study the corrections that this metric gets using the modified equations of motions.

3.1 Quantum corrections to a star metric

Let us consider a static, homogeneous and isotropic star satisfying the Tolman-Volkoff-Oppenheimer equation [55, 56], with constant density

$$\rho(r) = \rho_0 \Theta(R_s - r) = \begin{cases} \rho_0 & \text{if } r < R_s, \\ 0 & \text{if } r > R_s, \end{cases} \tag{3.1}$$

where $\rho_0 > 0$ is a constant, R_s is the star radius and $\Theta(x)$ is the Heaviside step function. The solution to the Einstein equations inside the star ($r \leq R_S$) is the interior Schwarzschild metric [53, 54]:

$$\begin{aligned} ds^2 &= \left(3\sqrt{1 - \frac{2G_N M}{R_s}} - \sqrt{1 - \frac{2G_N M r^2}{R_s^3}}\right)^2 \frac{dt^2}{4} - \left(1 - \frac{2G_N M r^2}{R_s^3}\right)^{-1} dr^2 - r^2 d\Omega^2 \\ &= g^{\text{int}}_{\mu\nu} dx^\mu dx^\nu, \end{aligned} \tag{3.2}$$

where

$$M = 4\pi \int_0^{R_s} \rho\, r^2 dr = \frac{4}{3}\pi R_s^3 \rho_0 \tag{3.3}$$

is the total Misner-Sharp mass of the source. The corresponding pressure is

$$p(r) = \rho_0 \frac{\sqrt{1-\frac{2G_N M}{R_s}} - \sqrt{1-\frac{2G_N M r^2}{R_s^3}}}{\sqrt{1-\frac{2G_N M r^2}{R_s^3}} - 3\sqrt{1-\frac{2G_N M}{R_s}}} = \mathcal{O}(G_N). \tag{3.4}$$

Due to Birkhoff's theorem [57], the metric outside the star $(r > R_s)$ is the usual Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2G_N M}{r}\right) dt^2 - \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 = g^{\text{ext}}_{\mu\nu} dx^\mu dx^\nu. \tag{3.5}$$

We will solve the equations of motion (2.22) with the background field method, that is we consider perturbations of the above metrics of the form

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \tag{3.6}$$

where $g_{\mu\nu}$ is the classical background metric and the perturbation $h_{\mu\nu}$ is taken to be of order $\mathcal{O}(G_N)$. The equations of motion (2.22) then become

$$G^L_{\mu\nu}[h] + 16\pi G_N (H^L_{\mu\nu}[g] + H^{NL}_{\mu\nu}[g]) = 0, \tag{3.7}$$

where the linearized Einstein tensor is given by

$$\begin{aligned} 2G^L_{\mu\nu} = \Box h_{\mu\nu} - g_{\mu\nu}\Box h + \nabla_\mu \nabla_\nu h - \nabla_\mu \nabla^\beta h_{\nu\beta} - \nabla_\nu \nabla^\beta h_{\mu\beta} \\ + g_{\mu\nu} \nabla^\alpha \nabla^\beta h_{\alpha\beta} + 2R^\alpha{}_\mu{}^\beta{}_\nu h_{\alpha\beta}, \end{aligned} \tag{3.8}$$

and $H^L_{\mu\nu}[g]$ and $H^{NL}_{\mu\nu}[g]$ are given, respectively, by (2.24) and (2.25). Note that for a Schwarzschild black hole, since it is a vacuum solution of the Einstein equations and its Ricci scalar and Ricci tensor vanish, there are no corrections at second order in the Newton constant [15, 16].

Focusing now on the local corrections in (3.7), outside the star the background metric is the usual Schwarzschild vacuum solution, for which $R, R_{\mu\nu} = 0$ and therefore there are no corrections due to the local part. Inside the star instead these corrections are non-vanishing. However, they are of order $\mathcal{O}(G_N^3)$ and therefore will be neglected.

As for the non-local part, knowing that the Ricci scalar, Ricci tensor and Riemann tensor are all $\mathcal{O}(G_N)$, (2.25) simplifies to

$$\begin{aligned} H^{NL}_{\mu\nu} =& 2\alpha(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) \ln\left(\frac{\Box}{\mu^2}\right) R \\ &+ \beta(\delta^\alpha_\mu g_{\nu\beta}\Box + g_{\mu\nu}\nabla^\alpha\nabla_\beta - \delta^\alpha_\mu \nabla_\beta\nabla_\nu - \delta^\alpha_\nu \nabla_\beta \nabla_\mu) \ln\left(\frac{\Box}{\mu^2}\right) R^\beta{}_\alpha \\ &+ 2\gamma(\delta^\alpha_\mu g_{\nu\sigma} + \delta^\alpha_\nu g_{\mu\sigma})\nabla^\beta\nabla_\rho \ln\left(\frac{\Box}{\mu^2}\right) R_{\alpha\beta}{}^{\sigma\rho} + \mathcal{O}(G_N^3). \end{aligned} \tag{3.9}$$

Let us now express the Ricci scalar and Ricci tensor in terms of the energy-momentum tensor of the source:

$$R = -8\pi G_N T, \tag{3.10}$$

$$R_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \tag{3.11}$$

where, for the case of a perfect and isotropic fluid, we have

$$T = \rho_0 + \mathcal{O}(G_N), \tag{3.12}$$

$$T_{\mu\nu} = \delta^0_\mu \delta^0_\nu \rho_0 + \mathcal{O}(G_N). \tag{3.13}$$

Now that everything is expressed in terms of the energy density, we can use the results from Appendix A to find

$$8\pi G_N \ln\left(\frac{\Box}{\mu^2}\right)\rho = \frac{6 G_N M}{R_s^3} f(r) + \mathcal{O}(G_N^2), \tag{3.14}$$

where

$$f(r) = \begin{cases} -2\left[\gamma_E - 1 + \ln\left(\mu\sqrt{R_s^2 - r^2}\right)\right] & \text{if } r < R_s, \\ 2\frac{R_s}{r} - \ln\left(\frac{r+R_s}{r-R_s}\right) & \text{if } r > R_s, \end{cases} \tag{3.15}$$

with γ_E the Euler-Mascheroni constant. Note that the function f is not defined at $r = R_s$. As we shall see later, the results we get are valid only outside a small region of the size of the Planck length around the star radius. We can now plug (3.14) into (3.9) and from (3.7) we find

$$\begin{aligned} G^L_{\mu\nu} = {} & 192\pi(\alpha-\gamma)\frac{G_N^2 M}{R_s^3}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box) f(r) \\ & + 96\pi(\beta+4\gamma)\frac{G_N^2 M}{R_s^3}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box + \delta^0_\mu g_{\nu 0}\Box) f(r) + \mathcal{O}(G_N^3), \end{aligned} \tag{3.16}$$

where we used the non-local Gauss-Bonnet theorem [16] at second order in curvature to substitute $\alpha \to (\alpha - \gamma)$ and $\beta \to (\beta + 4\gamma)$ and the identity

$$(g_{\mu\nu}\nabla^0\nabla_0 - \delta^0_\mu \nabla_0\nabla_\nu - \delta^0_\nu \nabla_0 \nabla_\mu) f(r) = \mathcal{O}(G_N). \tag{3.17}$$

From (3.7), (3.8) and (3.16), we can solve for the components of the perturbation $h_{\mu\nu}$, imposing that this metric is spherically symmetric and time-independent as the background metric and using the gauge freedom to set $h_{\theta\theta} = 0$. We thus find the corrections $h_{\mu\nu} = \delta g^{\text{int}}_{\mu\nu}$ to the interior Schwarzschild solution [17]:

$$\delta g^{\text{int}}_{tt} = (\alpha+\beta+3\gamma)\frac{192\pi G_N^2 M}{R_s^3}\ln\left(\frac{R_s^2}{R_s^2 - r^2}\right) + \frac{C_1}{r} + C_2 + \mathcal{O}(G_N^3), \tag{3.18}$$

$$\delta g^{\text{int}}_{rr} = (\alpha-\gamma)\frac{384\pi G_N^2 M r^2}{R_s^3(R_s^2 - r^2)} + \frac{C_1}{r} + \mathcal{O}(G_N^3), \tag{3.19}$$

where the integration constants C_i must be set to zero if we require regularity in the origin at $r = 0$. Similarly we find the corrections $h_{\mu\nu} = \delta g^{\text{ext}}_{\mu\nu}$ to the exterior Schwarzschild solution:

$$\delta g^{\text{ext}}_{tt} = (\alpha + \beta + 3\gamma)\frac{192\pi G_N^2 M}{R_s^3}\left[2\frac{R_s}{r} + \ln\left(\frac{r - R_s}{r + R_s}\right)\right] + \frac{C_3}{r} + C_4 + \mathcal{O}(G_N^3), \tag{3.20}$$

$$\delta g^{\text{ext}}_{rr} = (\alpha - \gamma)\frac{384\pi G_N^2 M}{r(r^2 - R_s^2)} + \frac{C_3}{r} + \mathcal{O}(G_N^3), \tag{3.21}$$

where the integration constants C_i must be set to zero if we require asymptotic flatness, that is $\lim_{r\to\infty} \delta g^{\text{ext}}_{\mu\nu} = 0$.

These equations can be simplified if we consider that astrophysical distances are many orders of magnitude bigger than the typical star radius. Therefore, in the limit $r \gg R_s$, the exterior metric corrections reduce to

$$\delta g^{\text{ext}}_{tt} = -(\alpha + \beta + 3\gamma)\frac{128\pi G_N^2 M}{r^3} + \mathcal{O}(G_N^3), \tag{3.22}$$

$$\delta g^{\text{ext}}_{rr} = (\alpha - \gamma)\frac{384\pi G_N^2 M}{r^3} + \mathcal{O}(G_N^3). \tag{3.23}$$

On the other hand, deep inside the star, that is in the $r \ll R_s$ limit, the interior corrections vanish:

$$\delta g^{\text{int}}_{tt} = \delta g^{\text{int}}_{rr} = \mathcal{O}(G_N^3). \tag{3.24}$$

An interesting feature of the metric corrections is the presence of ”quantum hair” [20, 58]. The no-hair theorem [59] states that stationary black hole solutions of the Einstein-Maxwell equations depend only on three parameters: mass, electric charge and angular momentum of the black hole; all the other informations about the interior of the star are lost during the formation of the black hole. If we look at the corrections (3.21) to the exterior Schwarzschild metric, we see that these depend on the density distribution of the dust ball: two stars with same mass but different density produce two different metric corrections. These corrections will survive when following the gravitational collapse of the star [24]. In general, because of the non-locality of the $\ln(\Box/\mu^2)$ operator, any exterior metric will carry some information about the interior of the star in the form of the extra metric terms. Therefore the presence of hair is a general feature of this framework.

Lastly, the horizon radius is shifted. The gravitational radius R_H of the system is given in general by the condition

$$g^{rr}(R_H) = 0. \tag{3.25}$$

For our case this implies

$$r - 2G_N M = -\frac{384\pi G_N^2 M(\alpha - \gamma)}{r^2}. \tag{3.26}$$

We solve this equation perturbatively, that is we first set the right-hand side to zero and solve the resulting equation, finding the zeroth-order solution

$$R_H = 2G_N M, \tag{3.27}$$

which is the Schwarzschild radius of the star. We then plug it on the right-hand side and obtain:

$$R_H = 2G_N M - \frac{96\pi}{M}(\alpha - \gamma). \tag{3.28}$$

From Tab. 2.1 we see that $\alpha - \gamma < 0$ for vectors, fermions, gravitons and also scalars in the case of minimal coupling $\xi = 0$. Therefore the resulting horizon radius is bigger than the usual Schwarzschild one, although this modification is subleading with respect to the classical result.

3.2 Divergence at the surface

Note that all the metric corrections diverge in the limit $\epsilon \equiv |r - R_s| \to 0^+$. This is because we are including higher derivatives of the metric while the metric is only once continuously differentiable. These divergences are of two types:

$$d_1 = \frac{G_N^2 M}{R_s^3} \ln\left(\frac{\epsilon}{R_s}\right), \tag{3.29}$$

$$d_2 = \frac{G_N^2 M}{R_s^2}\frac{1}{\epsilon}. \tag{3.30}$$

However, since we obtained these corrections solving the modified Einstein equations perturbatively in G_N, we should require that these terms are small with respect to the classical metric coefficients, namely:

$$V \sim \frac{G_N M}{r}. \tag{3.31}$$

In our units $G_N = l_p^2$, with $l_p = 1.62 \times 10^{-35}$ m the Planck length, therefore requiring $d_1 \lesssim V$ leads to

$$\frac{l_p^2}{R_s^2} \ln\left(\frac{|r - R_s|}{R_s}\right) \lesssim 1, \tag{3.32}$$

whereas for d_2:

$$\frac{l_p^2}{R_s|r - R_s|} \lesssim 1. \tag{3.33}$$

These two conditions are satisfied for $\epsilon \lesssim l_p$, since for a star we obviously have $R_s \gg l_p$. Therefore the metric corrections should be considered to apply only outside a layer of thickness $\epsilon \gtrsim l_p$ around the star surface.

Chapter 4

Quantum gravitational corrections to the Reissner-Nordström metric

In this section, after reviewing the corrections received by the Reissner-Nordström black hole found in [21], we will extend the result of the previous section for a static and spherically symmetric star considering also its electric charge. This is not merely a theoretical exercise, as astrophysical objects do indeed have charge, with an average charge-to-mass ratio of the order of 100 coulomb per solar mass [60]. For example, the Sun has an estimated charge of 154 Coulomb [61].

4.1 Reissner-Nordström black hole

The general line element describing a static and spherically symmetric object is

$$ds^2 = e^{\beta(r)}dt^2 - e^{\alpha(r)}dr^2 - r^2 d\Omega^2, \tag{4.1}$$

where the functions α and β depend only on the radial coordinate. The spacetime outside a static, spherically symmetric body with mass M and electric charge q is given by the Reissner-Nordström metric:

$$ds^2 = \left(1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2}\right) dt^2 - \left(1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{4.2}$$

Since we are dealing with a charged object, this metric is a solution of both the Einstein and Maxwell field equations. The Einstein equations are

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{4.3}$$

where now the energy-momentum tensor receives two contributions:

$$T_{\mu\nu} = M_{\mu\nu} + E_{\mu\nu}, \tag{4.4}$$

with $M_{\mu\nu}$ the proper matter energy-momentum tensor and $E_{\mu\nu}$ the electromagnetic energy-momentum tensor, defined in terms of the field-strength tensor $F_{\mu\nu}$ as

$$4\pi E_{\mu\nu} = \left(-F_{\mu\alpha}F_{\nu}{}^{\alpha} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{4.5}$$

The four Maxwell equations can be expressed in terms of $F_{\mu\nu}$:

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = \sqrt{-g}J^\mu, \quad F_{[\mu\nu,\lambda]} = 0, \tag{4.6}$$

where J^μ is the four-current density vector:

$$J^\mu = 4\pi\sigma U^\mu, \tag{4.7}$$

with σ the charge density and U^μ its four-velocity, with normalization $U^\mu U_\mu = 1$. Since we consider a static field, we can write its four-velocity as

$$U^\nu = (e^{-\beta/2}, 0, 0, 0). \tag{4.8}$$

However, outside the body there is no charge and the four-current vanishes in this case:

$$J^\mu = 0. \tag{4.9}$$

The electric field has only a radial component, thus the only non-vanishing components of $F_{\mu\nu}$ are

$$F_{01} = -F_{10} = \frac{Q(r)}{r^2}e^{(\alpha+\beta)/2}, \tag{4.10}$$

and, as a result, the only non-vanishing components of the electromagnetic energy-momentum tensor are

$$E_0^0 = E_1^1 = -E_2^2 = -E_3^3 = \frac{Q(r)^2}{8\pi r^4}. \tag{4.11}$$

Since the Einstein and Maxwell equations are coupled, it is possible that also $F_{\mu\nu}$ receives a correction. We can then define the function $\Omega(r)$ such that

$$F_{01} = \left[\frac{Q(r)}{r^2} + G_N^2\Omega(r)\right]e^{(\alpha+\beta)/2}, \tag{4.12}$$

which will then leave the equations of motion invariant as its contribution is subleading. Once we find the metric corrections $\delta g_{tt}^{\text{ext}}$ and $\delta g_{rr}^{\text{ext}}$ outside the star, the t component of (4.6) in the vacuum, that is

$$\partial_\nu(\sqrt{-g}F^{0\nu}) = 0, \tag{4.13}$$

will then become an equation in $\Omega(r)$:

$$4r\Omega(r) + q\left[\frac{d\delta g_{tt}^{\text{ext}}(r)}{dr} - \frac{d\delta g_{rr}^{\text{ext}}(r)}{dr}\right] + 2r^2\frac{d\Omega(r)}{dr} = 0. \tag{4.14}$$

Repeating the same procedure followed in Section 3, the quantum corrections to the metric are found to be [21]:

$$\delta g_{tt}^{\text{ext}} = -\frac{32\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{3}{2}\right)\right] + \mathcal{O}(G_N^3), \tag{4.15a}$$

$$\delta g_{rr}^{\text{ext}} = -\frac{64\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - 2\right)\right] + \mathcal{O}(G_N^3), \tag{4.15b}$$

while solving (4.14) we find

$$\Omega(r) = -\frac{16\pi q^3}{r^6}\left[\bar{c}_2 + (\beta+4\gamma)\left(2\ln(\mu r) + 2\gamma_E - 5\right)\right]. \tag{4.16}$$

In all these corrections there seems to be a dependence on the renormalization scale μ. However, the Wilson coefficients c_1, c_2 and c_3 are also dependent on the renormalization scale:

$$c_1(\mu) = c_1(\bar{\mu}) - \alpha \ln\left(\frac{\mu^2}{\bar{\mu}^2}\right), \tag{4.17}$$

$$c_2(\mu) = c_2(\bar{\mu}) - \beta \ln\left(\frac{\mu^2}{\bar{\mu}^2}\right), \tag{4.18}$$

$$c_3(\mu) = c_3(\bar{\mu}) - \gamma \ln\left(\frac{\mu^2}{\bar{\mu}^2}\right), \tag{4.19}$$

where $\bar{\mu}$ is some fixed scale where the effective theory is matched onto the full theory. Therefore inserting these in the metric corrections we see that the terms involving μ cancel out. This invariance with respect to the renormalization scale is a non trivial check of the validity of the calculations.

Many other interesting results for the quantum corrected Reissner-Nordström black hole can be found. For example, corrections to the Wald entropy [21] or to the charge and mass loss rate [22] have been computed. We will now extend this calculation to a charged star metric, giving a characterization of its interior and showing that, even in the large distance limit where the star becomes essentially point-like, the outside metric still depends on the interior distribution.

4.2 Interior charged star metric

Let us study a static and spherically symmetric perfect fluid charged distribution. The general line element for static spherically symmetric objects is

$$ds^2 = e^{\beta(r)}dt^2 - e^{\alpha(r)}dr^2 - r^2 d\Omega^2, \tag{4.20}$$

with $r \in [0, R_s]$, R_s being the star radius, and the functions α and β depending only on the radial coordinate. The functions $\beta(r)$ and $\alpha(r)$ satisfying the Einstein-Maxwell equations are given by [62]

$$e^{-\alpha(r)} = 1 - \frac{8\pi G_N}{r}\int_0^r \left(\rho + \frac{Q^2}{8\pi r^4}\right) r^2\, dr \tag{4.21a}$$

$$= 1 - \frac{2G_N(m+\epsilon)}{r} + G_N\frac{Q^2}{r^2}, \tag{4.21b}$$

$$\beta(r) = \int_0^r \frac{e^\alpha}{r}\left(1 - e^{-\alpha} - G_N\frac{Q^2}{r^2} + 8\pi G_N M_r^r\right) dr \tag{4.22a}$$

$$= -\alpha(r) + 8\pi G_N \int_0^r re^\alpha(\rho + M_1^1)\, dr. \tag{4.22b}$$

The functions Q, m and ϵ are defined as

$$Q(r) = 4\pi\int_0^r \sigma r^2 e^{\alpha/2} dr, \tag{4.23}$$

$$m(r) = 4\pi\int_0^r \rho r^2 dr, \tag{4.24}$$

$$\epsilon(r) = 4\pi\int_0^r \sigma r Q e^{\alpha/2} dr, \tag{4.25}$$

where ρ is the mass density and Q and m are, respectively, the charge and the mass inside a sphere of radius r. The total charge of the distribution is then

$$q = Q(R_s). \tag{4.26}$$

The Einstein-Maxwell equations for the components $M_2^2 = M_3^3$ reduce to

$$M_2^2 = M_3^3 = \frac{r}{2}\frac{dM_1^1}{dr} + \left(1 + \frac{1}{4}r\beta'\right) M_1^1 + \frac{1}{4}r\left(\rho\beta' - 2\sigma\frac{Q}{r^2}e^{\alpha/2}\right), \tag{4.27}$$

where the prime denotes derivatives with respect to the radial coordinate.

Outside the star the pressure, mass density and charge densitiy vanish and (4.21) and (4.22) reduce smoothly to the exterior Reissner-Nordström solution:

$$e^{-\alpha} = \left(1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2}\right), \quad e^\beta = \left(1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2}\right) e^C, \tag{4.28}$$

where C is the constant

$$C = 8\pi G_N \int_0^{R_s} re^\alpha(\rho + M_1^1) dr, \tag{4.29}$$

and M is the total gravitational mass of the distribution, given by the sum of the proper matter mass m and the mass equivalent of the electromagnetic energy distribution ϵ:

$$M = m(R_s) + \epsilon(R_s). \tag{4.30}$$

Perfect fluidity can be characterized by the requirement that the three spacelike eigenvalues of the energy-momentum tensor are equal everywhere. However, we have three different energy-momentum tensors and thus three different possibilities, depending on which tensor we choose to satisfy this condition:

I. Matter tensor, $M_{\mu\nu}$: we can find an explicit expression for the matter and charge density.

II. Total tensor, $T_{\mu\nu} = M_{\mu\nu} + E_{\mu\nu}$: in this case we have unspecified $\rho(r)$ and $Q(r)$ but can still find the metric.

III. Electromagnetic tensor, $E_{\mu\nu}$: from (4.11) we have that at the origin $Q(r) = 0$ and this would imply that the eigenvalues of $E_{\mu\nu}$ vanish everywhere.

We shall therefore focus on case I and II.

4.2.1 Case I: perfect fluidity requirement on matter tensor

We require that the matter energy-momentum tensor satisfies

$$M_1^1 = M_2^2 = M_3^3 = p(r). \tag{4.31}$$

Therefore (4.27) becomes

$$\frac{1}{2}rp' + \left(1 + \frac{1}{4}r\beta'\right)p + \frac{1}{4}r\left(\rho\beta' - 2\sigma\frac{Q}{r^2}e^{\alpha/2}\right) - p = 0. \tag{4.32}$$

From (4.22b) we have

$$\beta' = -\alpha' + 8\pi G_N[re^{\alpha}(\rho + p)], \tag{4.33}$$

and from (4.23)

$$Q' = 4\pi\sigma r^2 e^{\alpha/2}. \tag{4.34}$$

Upon insertion in (4.32) we get

$$Z' - \frac{1}{2}\alpha' Z + 4\pi G_N re^{\alpha}Z^2 - \frac{Q^2}{2\pi r^5} - \left(\rho + \frac{Q^2}{8\pi r^4}\right)' = 0, \tag{4.35}$$

where we defined

$$Z = \rho + p. \tag{4.36}$$

Equation (4.35) is a first-order ordinary differential equation quadratic in Z and is recognized as a Riccati equation [63], that is an equation of the form

$$y' = a(x)y + b(x)y^2 + c(x). \tag{4.37}$$

If a particular solution y_1 of a Riccati equation is known, the general solution of the equation is given by

$$y = y_1 + u. \tag{4.38}$$

In fact, substituting this solution in the general Riccati equation (4.37) we get

$$(y_1 + u)' = a(x)(y_1 + u) + b(x)(y_1 + u)^2 + c(x) \tag{4.39}$$
$$= a(x)y_1 + b(x)y_1^2 + c(x) + a(x)u + 2b(x)y_1 u + b(x)u^2. \tag{4.40}$$

Since y_1 is a solution, this equation reduces to

$$u' = b(x)u^2 + [2b(x)y_1 + a(x)]u, \tag{4.41}$$

which is a Bernoulli equation. Substituting $z = 1/u$ in the Bernoulli equation converts it to a linear differential equation which can be easily solved. If we now plug $y = y_1 + 1/z$ in the Riccati equation (4.37) we get the linear equation

$$z' + (a + 2by_1)z = -b. \tag{4.42}$$

Solutions of the Riccati equation are then of the form

$$y = y_1 + \frac{1}{z}, \tag{4.43}$$

with z solving (4.42). Going back to our equation, no particular solution of (4.35) has been found. Therefore we must make some assumptions in order to be able to solve it. A convenient simplification is to choose

$$\rho + \frac{Q^2}{8\pi r^4} = c, \tag{4.44}$$

with c a constant, so that the total energy density in the star interior is constant and this model is a generalization of the Schwarzschild interior solution. In this way, the last term in (4.35) vanishes. Note that, in order for the physical condition $\rho \geq 0$ to be satisfied, (4.44) implies

$$c \geq \frac{q^2}{8\pi R_s^4}. \tag{4.45}$$

Upon insertion of the assumption (4.44) in (4.21a), we get the simple expression

$$e^{-\alpha} = 1 - r^2/R^2, \tag{4.46}$$

where

$$\frac{1}{R^2} = \frac{8\pi G_N c}{3}. \tag{4.47}$$

Furthermore, since we use the metric signature $(+, -, -, -)$, it must be

$$R_s < R. \tag{4.48}$$

From (4.46) we find

$$\alpha' = \frac{2re^{\alpha}}{R^2}, \tag{4.49}$$

and thus the Riccati equation (4.35) simplifies to

$$Z' = \frac{1}{4}(2Z - 8\pi G_N R^2 Z^2)\alpha' + \frac{Q^2}{2\pi r^5}. \tag{4.50}$$

This is still a Riccati equation, for which a particular solution is difficult to determine. One way to further simplify it is to assume that the term $Q^2/(2\pi r^5)$ be proportional to α', so that the equation becomes separable:

$$\frac{Q^2}{2\pi r^5} = \frac{1}{4}A\alpha'. \tag{4.51}$$

The constant A can be determined in terms of the total charge $q = Q(R_s)$ as

$$A = \frac{q^2}{\pi R_s^6} R^2 \left(1 - \frac{R_s^2}{R^2}\right). \tag{4.52}$$

We note that, by equations (4.23) and (4.46), the assumption (4.51) is equivalent to specifying the charge density σ as

$$\begin{aligned} \sigma(r) &= \pm(A/16\pi R^2)^{1/2}(3 + r^2 e^{\alpha}/R^2) \\ &= \pm(A/16\pi R^2)^{1/2}(3 - 2r^2/R^2)(1 - r^2/R^2)^{-1}. \end{aligned} \tag{4.53}$$

Using (4.23), the charge distribution is then

$$Q(r) = \frac{r^3\sqrt{q^2(3 - 8\pi G_N R_s^2 c)}}{R_s^3\sqrt{3 - 8\pi G_N r^2 c}} = \frac{r^3\sqrt{q^2(1 - \frac{R_s^2}{R^2})}}{R_s^3\sqrt{1 - \frac{r^2}{R^2}}}. \tag{4.54}$$

The assumption we made allows us to separate (4.50) into

$$(A + 2Z - 8\pi G_N R^2 Z^2)^{-1} dZ = \frac{d\alpha}{4}, \tag{4.55}$$

which can now be integrated leading to

$$Z = \frac{1}{8\pi G_N R^2}\left[\frac{(n+1) - B(n-1)(1 - r^2/R^2)^{n/2}}{1 + B(1 - r^2/R^2)^{n/2}}\right], \tag{4.56}$$

where B is an integration constant and we defined n as

$$n = (1 + 8\pi G_N R^2 A)^{1/2}. \tag{4.57}$$

In order to determine B we use the conditions that on the boundary the pressure vanishes while the matter density is given by (4.44):

$$p(R_s) = 0, \quad \rho(R_s) = c - \frac{q^2}{8\pi R_s^4}. \tag{4.58}$$

Therefore we have that $Z(R_s) = \rho(R_s)$ which yields

$$B = [n + 1 - 8\pi G_N R^2 \rho(R_s)][n - 1 + 8\pi G_N R^2 \rho(R_s)]^{-1}(1 - R_s^2/R^2)^{-n/2}. \tag{4.59}$$

Having found $e^{-\alpha(r)}$, we can now determine the expression for $\beta(r)$. Let us rewrite (4.22b) as

$$\beta(r) = -\alpha(r) + 8\pi G_N J(r), \tag{4.60}$$

where we define:

$$J(r) = \int_0^r r e^{\alpha}(\rho + p) dr = \int_0^r r e^{\alpha} Z dr. \tag{4.61}$$

To evaluate this integral we express α' as in (4.49) and use (4.55):

$$\begin{aligned} J(r) &= \frac{1}{2} R^2 \int_0^r Z \alpha' dr \\ &= 2R^2 \int_{Z(0)}^{Z(r)} Z(A + 2Z - 8\pi G_N R^2 Z^2)^{-1} dZ. \end{aligned} \tag{4.62}$$

Performing the integral and using (4.56) we find

$$8\pi G_N J(r) = 2 \ln \left[\frac{1 + B(1 - r^2/R^2)^{n/2}}{1 + B} \right] - \frac{1}{2}(n + 1) \ln \left(1 - \frac{r^2}{R^2} \right). \tag{4.63}$$

Then, substituting in (4.60) and using the result (4.46) we finally get

$$e^{\beta} = \left[\frac{1 + B(1 - r^2/R^2)^{n/2}}{1 + B} \right]^2 \left(1 - \frac{r^2}{R^2} \right)^{-\frac{(n-1)}{2}}. \tag{4.64}$$

For the constant C in (4.29) we have

$$e^{C} = \left[\frac{1 + B(1 - R_s^2/R^2)^{n/2}}{1 + B} \right]^2, \tag{4.65}$$

which can be set to zero by a suitable rescaling of the time coordinate.

Summarising all the results, the interior metric is given by

$$ds^2 = \left[\frac{1 + B(1 - r^2/R^2)^{n/2}}{1 + B} \right]^2 \left(1 - \frac{r^2}{R^2} \right)^{-\frac{(n-1)}{2}} dt^2 - \left(1 - \frac{r^2}{R^2} \right)^{-1} dr^2 - r^2 d\Omega^2, \tag{4.66}$$

where the constants R, n and B are given by (4.47), (4.57) and (4.59). We see that in the zero charge limit we recover the interior Schwarzschild metric [64] in the form

$$ds^2 = 4\left(1 - \frac{2G_N M}{r}\right)\left(3\sqrt{1 - \frac{2G_N M}{R_s}} - 1\right)^{-2} dt^2 - \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{4.67}$$

To obtain the usual expression (3.2), we simply need to rescale the time as

$$t \to \left(3\sqrt{1 - \frac{2G_N M}{R_s}} - 1\right)\frac{t}{2}. \tag{4.68}$$

With this rescaling, the interior metric (4.66) is then

$$\begin{aligned} ds^2 = &\left[\frac{1 + B(1 - r^2/R^2)^{n/2}}{1 + B}\right]^2 \left(1 - \frac{r^2}{R^2}\right)^{-\frac{(n-1)}{2}} \left(3\sqrt{1 - \frac{2G_N M}{R_s}} - 1\right)^2 \frac{dt^2}{4} \\ &- \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2 d\Omega^2. \end{aligned} \tag{4.69}$$

The charge density $\sigma(r)$ is given by (4.53) and from (4.44), (4.46) and (4.51) we find the matter density

$$\rho(r) = c - \frac{1}{8} A R^{-2} r^2 (1 - r^2/R^2)^{-1}. \tag{4.70}$$

We see then that $\rho(0) = c$ and we thus rename this constant as ρ_0. Knowing the matter density and recalling that $Z = \rho + p$, the pressure can be found from (4.56). Note that the matter density and pressure reduce to the Schwarzschild ones in the zero charge limit. Moreover, since outside the star the matter density and pressure vanish and the charge distribution becomes that of a point-like source with total charge q, these quantities are better described by the distributions

$$\rho(r) = \begin{cases} \rho_0 - \frac{1}{8} A R^{-2} r^2 (1 - r^2/R^2)^{-1} & \text{if } r < R_s, \\ 0 & \text{if } r > R_s, \end{cases} \tag{4.71}$$

$$p(r) = \begin{cases} Z(r) - \rho(r) & \text{if } r < R_s, \\ 0 & \text{if } r > R_s, \end{cases} \tag{4.72}$$

$$Q(r) = \begin{cases} \frac{r^3 \sqrt{q^2 (3 - 8\pi G_N R_s^2 \rho_0)}}{R_s^3 \sqrt{3 - 8\pi G_N r^2 \rho_0}} & \text{if } r < R_s, \\ q & \text{if } r > R_s. \end{cases} \tag{4.73}$$

4.2.2 Case II: perfect fluidity requirement on total tensor

We require that the total energy-momentum tensor satisfies

$$T_1^1 = T_2^2 = T_3^3. \tag{4.74}$$

From (4.4) and (4.11) we find

$$M_2^2 = M_3^3 = M_1^1 - \frac{Q^2}{4\pi r^4}, \tag{4.75}$$

where $M_i^i = p(r)$. Thus (4.27) becomes

$$\frac{1}{2} r p' + \frac{1}{4} r \beta' (p + \rho) + \frac{Q^2}{4\pi r^4} - \frac{1}{2} \frac{\sigma Q}{r} e^{\alpha/2} = 0. \tag{4.76}$$

From (4.22b) we have

$$\beta' = -\alpha' + 8\pi G_N r e^{\alpha} (\rho + p), \tag{4.77}$$

and from (4.23)

$$Q' = 4\pi \sigma r^2 e^{\alpha/2}. \tag{4.78}$$

Upon insertion in (4.76) we get

$$Z' - \frac{1}{2}\alpha' Z + 4\pi G_N r e^{\alpha} Z^2 - \frac{Q^2}{2\pi r^5} - \left(\rho + \frac{Q^2}{8\pi r^4}\right)' = 0, \tag{4.79}$$

where

$$Z = \rho + p. \tag{4.80}$$

Equation (4.79) is a Riccati equation. As before, no particular solution has been found. Therefore we make again the assumption

$$\rho + \frac{Q^2}{8\pi r^4} = c, \tag{4.81}$$

with c a constant. Upon insertion of this assumption in (4.21a), we get the simple expression

$$e^{-\alpha} = 1 - r^2/R^2, \tag{4.82}$$

where

$$\frac{1}{R^2} = \frac{8\pi G_N c}{3}, \tag{4.83}$$

and since we use the metric signature $(+, -, -, -)$ then it must be:

$$R_s < R. \tag{4.84}$$

From (4.82) we find

$$\alpha' = \frac{2 r e^{\alpha}}{R^2}, \tag{4.85}$$

and thus the Riccati equation (4.79) simplifies to

$$Z^{-1} \left(1 - 4\pi G_N R^2 Z\right)^{-1} dZ = \frac{1}{2} d\alpha, \tag{4.86}$$

which integrates to

$$Z = \frac{1}{4\pi G_N R^2}\left[1 + B\left(1 - \frac{r^2}{R^2}\right)^{1/2}\right]^{-1}, \tag{4.87}$$

where the integration constant B is determined using the conditions that on the boundary the pressure vanishes while the matter density is given by (4.81):

$$p(R_s) = 0, \quad \rho(R_s) = c - \frac{q^2}{8\pi R_s^4}. \tag{4.88}$$

Therefore we have

$$B = [2 - 8\pi G_N R^2 \rho(R_s)][8\pi G_N R^2 \rho(R_s)]^{-1}(1 - R_s^2/R^2)^{-1/2}. \tag{4.89}$$

Having found $e^{-\alpha(r)}$, we can now determine the expression for $\beta(r)$. Let us rewrite (4.22b) as

$$\beta(r) = -\alpha(r) + 8\pi G_N J(r), \tag{4.90}$$

where again we define

$$J(r) = \int_0^r r e^{\alpha}(\rho + p) dr = \int_0^r r e^{\alpha} Z dr. \tag{4.91}$$

To evaluate this integral we express α' as in (4.85) and use (4.82):

$$J(r) = \frac{1}{2}R^2 \int_0^r Z\alpha' dr = R^2 \int_{Z(0)}^{Z(r)} \left(1 - 4\pi G_N R^2 Z\right)^{-1} dZ. \tag{4.92}$$

Performing the integral and using (4.87) we find

$$8\pi G_N J(r) = \alpha(r) + 2\ln\left[\frac{1 + B(1 - r^2/R^2)^{1/2}}{1 + B}\right]. \tag{4.93}$$

Then, substituting in (4.90) and using the result (4.82) we finally get

$$e^{\beta} = \left[\frac{1 + B(1 - r^2/R^2)^{1/2}}{1 + B}\right]^2. \tag{4.94}$$

For the constant C in (4.29) we have

$$e^{C} = \left[\frac{1 + B(1 - R_s^2/R^2)^{1/2}}{(1 + B)(1 - R_s^2/R^2)^{1/2}}\right]^2, \tag{4.95}$$

which can be set to zero by a suitable rescaling of the time coordinate.

Summarising all the results, the interior metric is given by

$$ds^2 = \left[\frac{1 + B(1 - r^2/R^2)^{1/2}}{1 + B}\right]^2 dt^2 - \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{4.96}$$

where the constants R and B are given by (4.83) and (4.89). As before, to obtain the usual expression for the interior Schwarzschild metric (3.2) in the zero charge limit, we need to rescale the time as in (4.68). With this rescaling, the interior metric (4.66) is then

$$\begin{aligned} ds^2 = \left[\frac{1 + B(1 - r^2/R^2)^{1/2}}{1 + B}\right]^2 &\left(3\sqrt{1 - \frac{2G_N M}{R_s}} - 1\right)^2 \frac{dt^2}{4} \\ &- \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2 d\Omega^2. \end{aligned} \tag{4.97}$$

With all these elements, we can now turn to the calculation of the metric corrections.

4.3 Quantum corrections to the charged star metric

As we already did for the Schwarzschild case, we solve the equations of motions (3.7) perturbatively at second order in the Newton constant G_N.

4.3.1 Case I: perfect fluidity requirement on matter tensor

For the charged star, besides the non-local term in the modified Einstein field equations, we need also to consider the local part, as this will now give a contribution. For the case I, the Ricci scalar and Ricci tensor are

$$R = -8\pi G_N T = -8\pi G_N(\rho - 3p), \tag{4.98}$$

$$\begin{aligned} R^t_t &= 8\pi G_N \left(\frac{\rho + 3p}{2} + \frac{Q^2}{8\pi r^4}\right), \\ R^r_r &= 8\pi G_N \left(\frac{p - \rho}{2} + \frac{Q^2}{8\pi r^4}\right), \\ R^\theta_\theta = R^\phi_\phi &= 8\pi G_N \left(\frac{p - \rho}{2} - \frac{Q^2}{8\pi r^4}\right), \end{aligned} \tag{4.99}$$

with ρ, p and Q given by (4.71), (4.72) and (4.73) respectively. The action of the $\ln(\Box/\mu^2)$ operator on the relevant quantities can then be computed as usual using the results of Appendix A.

For the interior Reissner-Nordström metric given by (4.66), to first order in G_N the Ricci scalar is

$$R = -8\pi G_N T = G_N \left(\frac{10q^2r^2 - 9q^2R_s^2 - 8\pi R_s^6\rho_0}{R_s^6} \right) + \mathcal{O}(G_N^2), \tag{4.100}$$

and the components of the Ricci tensor are

$$R_t^t = G_N \left(\frac{10q^2r^2 - 9q^2R_s^2 + 8\pi R_s^6\rho_0}{2R_s^6} \right) + \mathcal{O}(G_N^2), \tag{4.101a}$$

$$R_r^r = G_N \left(\frac{6q^2r^2 - 3q^2R_s^2 - 8\pi R_s^6\rho_0}{2R_s^6} \right) + \mathcal{O}(G_N^2), \tag{4.101b}$$

$$R_\theta^\theta = G_N \left(\frac{2q^2r^2 - 3q^2R_s^2 - 8\pi R_s^6\rho_0}{2R_s^6} \right) + \mathcal{O}(G_N^2), \tag{4.101c}$$

$$R_\phi^\phi = G_N \left(\frac{2q^2r^2 - 3q^2R_s^2 - 8\pi R_s^6\rho_0}{2R_s^6} \right) + \mathcal{O}(G_N^2). \tag{4.101d}$$

As for the exterior Reissner-Nordström metric, the matter density and pressure are zero and since the electromagnetic energy-momentum tensor is traceless then the Ricci scalar vanishes:

$$\tilde{R} = 0, \tag{4.102}$$

where we use the tilde notation to distinguish between the interior and exterior quantities, and the components of the Ricci tensor reduce to

$$\tilde{R}_t^t = \frac{q^2}{8\pi r^4}, \tag{4.103a}$$

$$\tilde{R}_r^r = \frac{q^2}{8\pi r^4}, \tag{4.103b}$$

$$\tilde{R}_\theta^\theta = -\frac{q^2}{8\pi r^4}, \tag{4.103c}$$

$$\tilde{R}_\phi^\phi = -\frac{q^2}{8\pi r^4}, \tag{4.103d}$$

with $q = Q(R_s)$ the total charge of the star. We can now proceed with the computation of the action of the $\ln(\Box/\mu^2)$ operator on all these quantities.

Interior metric

For the Ricci scalar we find

$$\begin{aligned} \ln\left(\frac{\Box}{\mu^2}\right) R = \frac{G_N q^2}{3R_s^6} \bigg[r^2 \left(\frac{110}{3} - 20\gamma_E \right) + 2R_s^2(-14 + 9\gamma_E) - 2(10r^2 - 9R_s^2) \bigg] \\ + 16\pi G_N \rho_0 \Big[\gamma_E - 1 + \ln\left(\mu\sqrt{R^2 - r^2}\right) \Big] + \mathcal{O}(G_N^2), \end{aligned} \tag{4.104}$$

and for the components of the Ricci tensor

$$\ln\left(\frac{\Box}{\mu^2}\right) R^t_t = \frac{G_N q^2}{r^4 R_s^6}\Bigg\{ r^2\Big[R_s^4 + r^4\left(\frac{55}{3} - 10\gamma_E\right) + r^2 R_s^2(-14 + 9\gamma_E)\Big] + R_s^6 \ln\left(\frac{R_s^2 - r^2}{R_s^2}\right) - r^4\Big(10r^2 - 9R_s^2\Big)\ln\Big(\mu\sqrt{R_s^2 - r^2}\Big)\Bigg\} - 8\pi G_N \rho_0 \Big[\gamma_E - 1 + \ln\Big(\mu\sqrt{R^2 - r^2}\Big)\Big] + \mathcal{O}(G_N^2), \tag{4.105a}$$

$$\ln\left(\frac{\Box}{\mu^2}\right) R^r_r = \frac{G_N q^2}{r^4 R_s^6}\Bigg\{ r^2\Big[R_s^4 + r^4(11 - 6\gamma_E) + 3r^2 R_s^2(-2 + \gamma_E)\Big] + R_s^6 \ln\left(\frac{R_s^2 - r^2}{R_s^2}\right) - 3r^4(2r^2 - R_s^2)\ln\Big(\mu\sqrt{R_s^2 - r^2}\Big)\Bigg\} + 8\pi G_N \rho_0 \Big[\gamma_E - 1 + \ln\Big(\mu\sqrt{R_s^2 - r^2}\Big)\Big] + \mathcal{O}(G_N^2), \tag{4.105b}$$

$$\ln\left(\frac{\Box}{\mu^2}\right) R^\theta_\theta = \frac{G_N q^2}{r^4 R_s^6}\left\{ r^2\left[-R_s^4 + r^4\left(\frac{11}{3} - 2\gamma_E\right) + r^2 R_s^2(-4 + 3\gamma_E)\right] - R_s^6 \ln\left(\frac{R_s^2 - r^2}{R_s^2}\right) - r^4\left(2r^2 - 3R_s^2\right)\ln\Big(\mu\sqrt{R_s^2 - r^2}\Big)\right\} + 8\pi G \rho_0 \Big[\gamma_E - 1 + \ln\Big(\mu\sqrt{R_s^2 - r^2}\Big)\Big] + \mathcal{O}(G_N^2), \tag{4.105c}$$

$$\ln\left(\frac{\Box}{\mu^2}\right) R^\phi_\phi = \frac{G_N q^2}{r^4 R_s^6}\left\{ r^2\left[-R_s^4 + r^4\left(\frac{11}{3} - 2\gamma_E\right) + r^2 R_s^2(-4 + 3\gamma_E)\right] - R_s^6 \ln\left(\frac{R_s^2 - r^2}{R_s^2}\right) - r^4\left(2r^2 - 3R_s^2\right)\ln\Big(\mu\sqrt{R_s^2 - r^2}\Big)\right\} + 8\pi G \rho_0 \Big[\gamma_E - 1 + \ln\Big(\mu\sqrt{R_s^2 - r^2}\Big)\Big] + \mathcal{O}(G_N^2). \tag{4.105d}$$

We can now solve for the components of the perturbation metric $h_{\mu\nu}$, imposing that it is spherically symmetric and time-independent as the background metric and using the gauge freedom to set $h_{\theta\theta} = 0$. We thus find the corrections $h_{\mu\nu} = \delta g^{\text{int}}_{\mu\nu}$ to the interior Reissner-Nordström metric:

$$
\begin{aligned}
\delta g_{tt}^{\text{int}} = \frac{32\pi G_N^2 q^2}{3r^4 R_s^6}\Bigg[& -27r^4 R_s^2(\alpha-\gamma)\ln\left(\frac{R_s^2-r^2}{R_s^2}\right) + 3(\beta+4\gamma)R_s^6\ln\left(\frac{R_s^2-r^2}{R_s^2}\right) \\
& + 30\bar{c}_1 r^6 + 60(\alpha-\gamma)r^6\ln\left(\mu\sqrt{R_s^2-r^2}\right) \\
& + 10r^6(\alpha-\gamma)(-11+6\gamma_E) + 3r^2R_s^4(\beta+4\gamma)\Bigg] \\
& - (\alpha+\beta+3\gamma)256\pi^2 G_N^2\rho_0\ln\left(\frac{R_s^2-r^2}{R_s^2}\right) + \frac{C_1}{r} + C_2 + \mathcal{O}(G_N^3),
\end{aligned} \tag{4.106}
$$

$$
\begin{aligned}
\delta g_{rr}^{\text{int}} = \frac{64\pi G_N^2 q^2}{3r^4(R_s^2-r^2)R_s^6}\Bigg\{ & \alpha r^6\left(80r^2 - 60r^2\gamma_E - 83R_s^2 + 60R_s^2\gamma_E\right) \\
& + \beta(R_s^2-r^2)\left(6r^4R_s^2 + 3r^2R_s^4 - 43r^6 + 30r^6\gamma_E\right) \\
& + \gamma\left[12r^4R_s^4 + 12r^2R_s^6 + r^8(92-60\gamma_E) + r^6R_s^2(-113+60\gamma_E)\right] \\
& + 15(2\bar{c}_1+\bar{c}_2)r^6(R_s^2-r^2) + 3(R_s^2-r^2)\Bigg[(\beta+4\gamma)R_s^6\ln\left(\frac{R_s^2-r^2}{R_s^2}\right) \\
& + 10(2\alpha+\beta+2\gamma)\ln\left(\mu\sqrt{R_s^2-r^2}\right)\Bigg]\Bigg\} \\
& + (\alpha-\gamma)\frac{512\pi^2G_N^2\rho_0 r^2}{R_s^2-r^2} + \frac{C_1}{r} + \mathcal{O}(G_N^3),
\end{aligned} \tag{4.107}
$$

where the integration constants C_i must be set to zero if we require regularity at the origin. In the $r \ll R_s$ limit we find indeed that these corrections are regular:

$$
\delta g_{tt}^{\text{int}} = -\frac{16\pi G_N^2 q^2(\beta+4\gamma)}{R_s^4} + \mathcal{O}(G_N^3), \tag{4.108}
$$

$$
\delta g_{rr}^{\text{int}} = \frac{96\pi G_N^2 q^2(\beta+4\gamma)}{R_s^4} + \mathcal{O}(G_N^3). \tag{4.109}
$$

Exterior metric

For the Ricci scalar we find

$$
\begin{aligned}
\ln\left(\frac{\Box}{\mu^2}\right)\tilde{R} = \frac{G_N q^2}{3rR_s^6}&\left[(60r^2R_s - 34R_s^3) - 3r(10r^2-9R_s^2)\ln\left(\frac{r+R_s}{r-R_s}\right)\right] \\
&+ 8\pi G_N\rho_0\left[-\frac{2R_s}{r} + \ln\left(\frac{r+R_S}{r-R_s}\right)\right] + \mathcal{O}(G_N^2),
\end{aligned} \tag{4.110}
$$

and for the components of the Ricci tensor

$$\ln\left(\frac{\Box}{\mu^2}\right)\tilde{R}^t_t = -\frac{G_N q^2}{r^4 R_s^6}\Bigg[-10r^5 R_s + \frac{17}{3}r^3 R_s^3 + 2rR_s^5 + R_s^6(-3+2\gamma_E)$$
$$+ 2R_s^6\ln(\mu r) + \frac{1}{2}\left(10r^6 - 9r^4 R_s^2 - 2R_s^6\right)\ln\left(\frac{r+R_s}{r-R_s}\right)\Bigg] \tag{4.111a}$$
$$- 4\pi G_N \rho_0\left[-\frac{2R_s}{r} + \ln\left(\frac{r+R_s}{r-R_s}\right)\right] + \mathcal{O}(G_N^2),$$

$$\ln\left(\frac{\Box}{\mu^2}\right)\tilde{R}^r_r = -\frac{G_N q^2}{r^4 R_s^6}\Bigg[-6r^5 R_s + r^3 R_s^3 + 2rR_s^5 + R_s^6(-3+2\gamma_E)$$
$$+ 2R_s^6\ln(\mu r) + \frac{1}{2}(6r^6 - 3r^4 R_s^2 - 2R_s^6)\ln\left(\frac{r+R_s}{r-R_s}\right)\Bigg] \tag{4.111b}$$
$$- 4\pi G\rho_0\left[2\frac{R_s}{r} - \ln\left(\frac{r+R_s}{r-R_s}\right)\right] + \mathcal{O}(G_N^2),$$

$$\ln\left(\frac{\Box}{\mu^2}\right)\tilde{R}^\theta_\theta = \frac{G_N q^2}{(6r^4 R_s^6)}\Bigg[12r^5 R_s - 14r^3 R_s^3 + 12rR_s^5 + 6R_s^6(-3+2\gamma_E)$$
$$+ 12R_s^6\ln(\mu r) - 3(2r^6 - 3r^4 R_s^2 + 2R_s^6)\ln\left(\frac{r+R_s}{r-R_s}\right)\Bigg] \tag{4.111c}$$
$$- 4\pi G_N\rho_0\left[2\frac{R_s}{r} - \ln\left(\frac{r+R_s}{r-R_s}\right)\right] + \mathcal{O}(G_N^2),$$

$$\ln\left(\frac{\Box}{\mu^2}\right)\tilde{R}^\phi_\phi = \frac{G_N q^2}{(6r^4 R_s^6)}\Bigg[12r^5 R_s - 14r^3 R_s^3 + 12rR_s^5 + 6R_s^6(-3+2\gamma_E)$$
$$+ 12R_s^6\ln(\mu r) - 3(2r^6 - 3r^4 R_s^2 + 2R_s^6)\ln\left(\frac{r+R_s}{r-R_s}\right)\Bigg] \tag{4.111d}$$
$$- 4\pi G_N\rho_0\left[2\frac{R_s}{r} - \ln\left(\frac{r+R_s}{r-R_s}\right)\right] + \mathcal{O}(G_N^2).$$

We find the corrections $h_{\mu\nu} = \delta g^{\text{ext}}_{\mu\nu}$ to the exterior Reissner-Nordström metric to be:

$$\begin{aligned}\delta g^{\text{ext}}_{tt} = \frac{-32\pi G_N^2 q^2}{r^4 R_s^6}\Big\{&2r^3 R_s(10r^2 + R_s^2)(\alpha - \gamma) + 2rR_s^5(\beta + 4\gamma)\\ &+ R_s^6(\beta + 4\gamma)(-3 + 2\gamma_E) + R_s^6\bar{c}_2 + 2R_s^6(\beta + 4\gamma)\ln(\mu r)\\ &+ \left[10r^6(\alpha - \gamma) - 9r^4 R_s^2(\alpha - \gamma) + R_s^6(\beta + 4\gamma)\right]\ln\left(\frac{r - R_s}{r + R_s}\right)\Big\}\\ &+ 256\pi^2 G_N^2 \rho_0(\alpha + \beta + 3\gamma)\left[\frac{2R_s}{r} + \ln\left(\frac{r - R_s}{r + R_s}\right)\right] + \frac{C_3}{r} + C_4 + \mathcal{O}(G_N^3),\end{aligned} \tag{4.112}$$

$$\begin{aligned}\delta g^{\text{ext}}_{rr} = \frac{-64\pi G_N^2 q^2}{3r^4 R_s^6(r^2 - R_s^2)}\Big\{&3\bar{c}_2 R_s^6(r^2 - R_s^2) + \alpha r^3(60r^4 R_s - 20r^2 R_s^3 - 37R_s^5)\\ &+ \beta(r^2 - R_s^2)\left(30r^5 R_s + 10r^3 R_s^3 + 6rR_s^5 + 6R_s^6\gamma_E - 12R_s^6\right) + 3\gamma\left[20r^7 R_s\right.\\ &\left.-20r^5 R_s^3 + 7r^3 R_s^5 - 8rR_s^7 + 8R_s^5(\gamma_E - 2)(r^2 - R_s^2)\right]\\ &+ 3(r^2 - R_s^2)\left[(10\alpha + 5\beta + 10c)r^6 + (\beta + 4\gamma)R_s^6\right]\ln\left(\frac{r - R_s}{r + R_s}\right)\\ &+ 6(r^2 - R_s^2)(\beta + 4\gamma)R_s^6\ln(\mu r)\Big\} + (\alpha - \gamma)\frac{512\pi^2 G_N^2 R_s^3\rho_0}{r(r^2 - R_s^2)} + \frac{C_3}{r} + \mathcal{O}(G_N^3),\end{aligned} \tag{4.113}$$

where the integration constants C_i must be set to zero if we require asymptotic flatness. By looking in (4.112) at the coefficients in front of the $\ln[(r - R_s)/(r + R_s)]$ terms, we see that there is an explicit dependence on the matter density $\rho_0 = 3M/(4\pi R_s^3)$ and quadratically on the charge density $\sigma_0 = 3q/(4\pi R_s^3)$.

In the $r \gg R_s$ limit:

$$\begin{aligned}\delta g^{\text{ext}}_{tt} = &-\frac{1280\pi G_N^2 q^2(\alpha - \gamma)}{3rR_s^3} - \frac{64\pi G_N^2}{3r^3 R_s}(\alpha + \beta + 3\gamma)(3q^2 + 8\pi R_s^4\rho_0)\\ &- \frac{32\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta + 4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{3}{2}\right)\right] + \mathcal{O}(G_N^3),\end{aligned} \tag{4.114a}$$

$$\begin{aligned}\delta g^{\text{ext}}_{rr} = &-\frac{1280\pi G_N^2 q^2(\alpha - \gamma)}{3rR_s^3} + \frac{64\pi G_N^2}{r^3 R_s}(\alpha - \gamma)\left(3q^2 + 8\pi R_s^4\rho_0\right)\\ &- \frac{64\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta + 4\gamma)\left(\ln(\mu r) + \gamma_E - 2\right)\right] + \mathcal{O}(G_N^3).\end{aligned} \tag{4.114b}$$

From (4.14), using the corrections (4.114), we find

$$\Omega(r) = \frac{32\pi q}{3r^5 R_s}(\beta + 4\alpha)\left(3q^2 + 8\pi R_s^4\rho_0\right) - \frac{16\pi q^3}{r^6}\left[\bar{c}_2 + (\beta + 4\gamma)\left(2\ln(\mu r) + 2\gamma_E - 5\right)\right]. \tag{4.115}$$

Note that, because of the non-locality of the corrections, we have a dependence on the star radius R_s. If we consider the black hole limit by naively sending the star radius to the Schwarzschild radius, the exterior corrections will induce a shift in the horizon radius, which is found from

$$g^{rr}(R_H) = 0, \tag{4.116}$$

leading to

$$\begin{aligned} r^2 - 2G_N M r + G_N q^2 = &-\frac{384\pi G_N^2(\alpha-\gamma)}{r}\left(\frac{q^2}{2R_s}+M\right)+\frac{1280\pi G_N^2 q^2(\alpha-\gamma)r}{3R_s^3} \\ &+\frac{64\pi G_N^2 q^2}{r^2}\left[\bar{c}_2+2(\beta+4\gamma)\left(\ln(\mu r)+\gamma_E-2\right)\right], \end{aligned} \tag{4.117}$$

which we wrote in this way in order to have on the left-hand side the classical equation for the horizon radius of a Reissner-Nordström black hole. The resulting equation is a quartic with the presence of $\ln(\mu r)$ terms, which can't be solved analytically. Therefore, in order to find an analytical solution, we solve this equation perturbatively. That is, we first solve it setting the right-hand side to zero, obtaining the zeroth-order solution

$$R_H = G_N M \pm \sqrt{G_N^2 M^2 - G_N q^2}. \tag{4.118}$$

We are interested in the outer horizon and since we work in the approximation in which $q^2 \ll G_N M^2$, we can expand (4.118) as

$$R_H \simeq 2G_N M - \frac{q^2}{2M} + \mathcal{O}(q^4). \tag{4.119}$$

We then plug this result on the right-hand side of (4.117). Since we are treating the additional terms as a perturbation of the classical result, in the logarithms we keep only the $2G_N M$ term. We can now set $R_H = 2G_N M$. Solving the resulting equation we finally find the modified horizon radius:

$$\begin{aligned} R_H = &\, 2G_N M - \frac{q^2}{2M} - 96\pi(\alpha-\gamma)\left(\frac{1}{M}-\frac{7q^2}{36G_N M^3}\right) \\ &+\frac{8\pi q^2}{G_N M^3}\left[\bar{c}_2+2(\beta+4\gamma)(\ln(2G_N M)+\gamma_E-2\right]+\mathcal{O}(q^4)+\mathcal{O}(\hbar^2), \end{aligned} \tag{4.120}$$

where by $O(\hbar^2)$ we mean all those terms which are quadratic in the local and non-local coefficients, for example $\alpha \cdot \beta$ or $\alpha \cdot \bar{c}_2$.

4.3.2 Case II: perfect fluidity requirement on total tensor

Let us now look at case II. For the interior Reissner-Nordström metric given by (4.96), to first order in G_N the Ricci scalar is

$$R = -8\pi G_N T = G_N \left(\frac{-3q^2 - 8\pi R_s^4 \rho_0}{R_s^4} \right) + \mathcal{O}(G_N^2), \tag{4.121}$$

and the components of the Ricci tensor are

$$R_t^t = G_N \left(\frac{-3q^2 + 8\pi R_s^4 \rho_0}{2R_s^4} \right) + \mathcal{O}(G_N^2), \tag{4.122a}$$

$$R_r^r = G_N \left(\frac{-q^2 - 8\pi R_s^4 \rho_0}{2R_s^4} \right) + \mathcal{O}(G_N^2), \tag{4.122b}$$

$$R_\theta^\theta = G_N \left(\frac{-q^2 - 8\pi R_s^4 \rho_0}{2R_s^4} \right) + \mathcal{O}(G_N^2), \tag{4.122c}$$

$$R_\phi^\phi = G_N \left(\frac{-q^2 - 8\pi R_s^4 \rho_0}{2R_s^4} \right) + \mathcal{O}(G_N^2). \tag{4.122d}$$

Note that, as a check for the validity of the calculation for the model of case II, we correctly have that $T_1^1 = T_2^2 = T_3^3$. The exterior is still the usual Reissner-Nordström metric, with Ricci scalar and tensor given by (4.102) and (4.103) respectively.

Interior metric

The action of the $\ln(\Box/\mu^2)$ operator on the Ricci scalar is

$$\ln\left(\frac{\Box}{\mu^2}\right) R = \frac{2G_N\left(3q^2 + 8\pi R_s^4 \rho_0\right)}{R_s^4} \left[\gamma_E - 1 + \ln\left(\mu\sqrt{R_s^2 - r^2}\right)\right] + \mathcal{O}(G_N^2), \tag{4.123}$$

and on the components of the Ricci tensor

$$\begin{aligned}\ln\left(\frac{\Box}{\mu^2}\right) R_t^t = {}& \frac{G_N q^2}{r^4 R_s^4}\left[r^2 R_s^2 + 3r^4(\gamma_E - 1) + R_s^4 \ln\left(\frac{R_s^2 - r^2}{R_s^2}\right) + 3r^4 \ln\left(\mu\sqrt{R^2 - r^2}\right)\right] \\ & - 8\pi G\rho_0 \left[\gamma_E - 1 + \ln\left(\mu\sqrt{R_s^2 - r^2}\right)\right] + \mathcal{O}(G_N^2),\end{aligned} \tag{4.124a}$$

$$\begin{aligned}\ln\left(\frac{\Box}{\mu^2}\right) R_i^i = {}& \frac{G_N q^2}{r^4 R_s^4}\left[r^2 R_s^2 + r^4(\gamma_E - 1) + R_s^4 \ln\left(\frac{R_s^2 - r^2}{R_s^2}\right) + r^4 \ln\left(\mu\sqrt{R^2 - r^2}\right)\right] \\ & + 8\pi G\rho_0 \left[\gamma_E - 1 + \ln\left(\mu\sqrt{R_s^2 - r^2}\right)\right] + \mathcal{O}(G_N^2),\end{aligned} \tag{4.124b}$$

with $i = r, \theta, \phi$. We can now solve for the components of the perturbation metric $h_{\mu\nu}$, imposing that it is spherically symmetric and time-independent as the background metric and using the gauge freedom to set $h_{\theta\theta} = 0$. We thus find the corrections $h_{\mu\nu} = \delta g^{\text{int}}_{\mu\nu}$ to the interior Reissner-Nordström metric:

$$\begin{aligned}\delta g^{\text{int}}_{tt} = & \frac{32\pi G_N^2 q^2}{r^4 R_s^4}\left[(\beta+4\gamma)r^2R_s^2 - [3(\alpha-\gamma)r^4 - (\beta+4\gamma)R_s^4]\ln\left(\frac{R_s^2-r^2}{R_s^2}\right)\right] \\ & - (\alpha+\beta+3\gamma)256\pi^2 G_N^2\rho_0 \ln\left(\frac{R_s^2-r^2}{R_s^2}\right) + \frac{C_1}{r} + C_2 + \mathcal{O}(G_N^3),\end{aligned} \tag{4.125}$$

$$\begin{aligned}\delta g^{\text{int}}_{rr} = & \frac{64\pi G_N^2 q^2}{r^4(R_s^2-r^2)R_s^4}\left[(\beta+4\gamma)R_s^4 + (\beta+4\gamma)R_s^4(R_s^2-r^2)\ln\left(\frac{R_s^2-r^2}{R_s^2}\right)\right. \\ & \left. + (3\alpha+\beta+\gamma)r^6\right] + (\alpha-\gamma)\frac{512\pi^2 G_N^2\rho_0 r^2}{R_s^2-r^2} + \frac{C_1}{r} + \mathcal{O}(G_N^3),\end{aligned} \tag{4.126}$$

where the integration constants C_i must be set to zero if we require regularity in the origin at $r = 0$. In the $r \ll R_s$ limit we find indeed that these corrections are regular:

$$\delta g^{\text{int}}_{tt} = -\frac{16(\beta+4\gamma)\pi G_N^2 q^2}{R_s^4} + \mathcal{O}(G_N^3), \tag{4.127}$$

$$\delta g^{\text{int}}_{rr} = \frac{32(\beta+4\gamma)\pi G_N^2 q^2}{R_s^4} + \mathcal{O}(G_N^3). \tag{4.128}$$

Exterior metric

For the Ricci scalar we find

$$\ln\left(\frac{\Box}{\mu^2}\right)\tilde{R} = \frac{G_N(3q^2+8\pi R_s^4\rho_0)}{R_s^4}\left[\frac{2R_s}{r} + r\ln\left(\frac{r-R_s}{r+R_s}\right)\right] + \mathcal{O}(G_N^2), \tag{4.129}$$

and for the components of the Ricci tensor

$$
\begin{aligned}
\ln\left(\frac{\Box}{\mu^2}\right)\tilde{R}^t_t = &-\frac{G_N q^2}{2r^4R_s^4}\Bigg[6r^3R_s + 4rR_s^3 + 2R_s^4(-3+2\gamma) + 4R_s^4\ln(\mu r) \\
&+ (3r^4 + 2R_s^4)\ln\left(\frac{r-R_s}{r+R_s}\right)\Bigg] \\
&+ 4\pi G_N\rho_0\left[\frac{2R_s}{r} + \ln\left(\frac{r-R_s}{r+R_s}\right)\right] + \mathcal{O}(G_N^2),
\end{aligned} \tag{4.130a}
$$

$$
\begin{aligned}
\ln\left(\frac{\Box}{\mu^2}\right)\tilde{R}^i_i = &-\frac{G_N q^2}{2r^4R_s^4}\Bigg[2r^3R_s + 4rR_s^3 + 2R_s^4(-3+2\gamma) + 4R_s^4\ln(\mu r) + \\
&(r^4 + 2R_s^4)\ln\left(\frac{r-R_s}{r+R_s}\right)\Bigg] \\
&- 4\pi G_N\rho_0\left[\frac{2R_s}{r} + \ln\left(\frac{r-R_s}{r+R_s}\right)\right] + \mathcal{O}(G_N^2),
\end{aligned} \tag{4.130b}
$$

with $i = r, \theta, \phi$. We can now solve for the components of the perturbation metric $h_{\mu\nu}$, imposing that it is spherically symmetric and time-independent as the background metric and using the gauge freedom to set $h_{\theta\theta} = 0$. We thus find the corrections $h_{\mu\nu} = \delta g^{\text{ext}}_{\mu\nu}$ to the exterior Reissner-Nordström metric:

$$
\begin{aligned}
\delta g^{\text{ext}}_{tt} = &-\frac{32\pi G_N^2 q^2}{r^4R_s^4}\Bigg\{R_s^4[\bar{c}_2 + (\beta+4\gamma)(2\ln(\mu r) + 2\gamma - 3)] - 6(\alpha-\gamma)r^3R_s \\
&+ 2(\beta+4\gamma)rR_s^3 - \left[3(\alpha-\gamma)r^4 - (\beta+4\gamma)R_s^4\right]\ln\left(\frac{r-R_s}{r+R_s}\right)\Bigg\} \\
&+ (\alpha+\beta+3\gamma)256\pi^2G_N^2\rho_0\left[\frac{2R_s}{r} + \ln\left(\frac{r-R_s}{r+R_s}\right)\right] + \frac{C_3}{r} + C_4 + \mathcal{O}(G_N^3),
\end{aligned} \tag{4.131}
$$

$$
\begin{aligned}
\delta g^{\text{ext}}_{rr} = &-\frac{64\pi G_N^2 q^2}{r^4R_s(r^2-R_s^2)}\Bigg\{R_s(r^2-R_s^2)[\bar{c}_2 + (\beta+4\gamma)(2\ln(\mu r) + 2\gamma - 4)] \\
&- 3(\alpha-\gamma)r^3 - 2(\beta+4\gamma)rR_s^2 + (\beta+4\gamma)R_s^4\ln\left(\frac{r-R_s}{r+R_s}\right)\Bigg\} \\
&+ (\alpha-\gamma)\frac{512\pi^2G_N^2R_s^3\rho_0}{r(r^2-R_s^2)} + \frac{C_3}{r} + \mathcal{O}(G_N^3),
\end{aligned} \tag{4.132}
$$

where the integration constants C_i must be set to zero if we require asymptotic flatness, that is $\lim_{r\to\infty}\delta g^{\text{int}}_{\mu\nu} = 0$.

In the $r \gg R$ limit:

$$\begin{aligned}\delta g_{tt}^{\text{ext}} = &-\frac{64\pi G_N^2}{3r^3 R_s}\left(3q^2 + 8\pi R_s^4 \rho_0\right)(\alpha + \beta + 3\gamma)\\ &- \frac{32\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta + 4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{3}{2}\right)\right] + \mathcal{O}(G_N^3),\end{aligned} \tag{4.133a}$$

$$\begin{aligned}\delta g_{rr}^{\text{ext}} = &\frac{64\pi G_N^2}{r^3 R_s}(\alpha - \gamma)\left(3q^2 + 8\pi R_s^4 \rho_0\right)\\ &- \frac{64\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta + 4\gamma)\left(\ln(\mu r) + \gamma_E - 2\right)\right] + \mathcal{O}(G_N^3).\end{aligned} \tag{4.133b}$$

The quantum correction $\Omega(r)$ obtained by solving (4.14) in this case is the same as (4.115) for case I.

For the horizon radius calculation, we can repeat the same steps showed in Section (4.3.1) and find

$$\begin{aligned}R_H = &2G_N M - \frac{q^2}{2M} - 96\pi(\alpha - \gamma)\left(\frac{1}{M} - \frac{4q^2}{3G_N M^3}\right)\\ &+ \frac{8\pi q^2}{G_N M^3}\left[\bar{c}_2 + 2(\beta + 4\gamma)(\ln(2G_N M) + \gamma_E - 2\right] + \mathcal{O}(q^4) + \mathcal{O}(\hbar^2),\end{aligned} \tag{4.134}$$

which, besides the prefactor of the term proportional to $q^2/(G_N M^3)$, is the same as the result (4.120) for case I.

A few comments are now in order. First of all we note that all these corrections reduce to the Schwarzschild corrections in the zero charge limit once we substitute ρ_0 with $3M/4\pi R_s^3$, as we expect. Of course being in the non-extremal case in which $q^2 \ll G_N M^2$, for a real star these contributions will be subleading with respect to the ones proportional to the proper matter mass.

Again, there seems to be a dependence on the renormalization scale μ, which however is removed by the Wilson coefficients c_1, c_2 and c_3.

By looking at the exterior corrections in the $r \gg R_s$ limit and in particular at the r^{-3} terms, we clearly see the contribution of the electromagnetic energy to the mass (4.30). In fact from (4.24) and (4.25) we have that in both case I and II the total mass is

$$M = \frac{4}{3}\pi R_s^3 \rho_0 + \frac{q^2}{2R_s}. \tag{4.135}$$

Divergences at the surface

The interior and exterior metric corrections, for both case I and II, are divergent in the limit $\epsilon \equiv |r - R_s| \to 0^+$. This is due to the fact that we are including higher order derivatives of the metric, which is instead only once continuously differentiable. For both cases these divergences are of the type

$$d_1 = \frac{G_N^2 M}{R_s^3} \ln\left(\frac{\epsilon}{R_s}\right), \tag{4.136a}$$

$$d_2 = \frac{G_N^2 q^2}{R_s^4} \ln\left(\frac{\epsilon}{R_s}\right), \tag{4.136b}$$

$$d_3 = \frac{G_N^2 M}{R_s^2} \frac{1}{\epsilon}, \tag{4.136c}$$

$$d_4 = \frac{G_N^2 q^2}{R_s^3} \frac{1}{\epsilon}, \tag{4.136d}$$

whereas only for the interior corrections of case I there is also

$$\frac{G_N^2 q^2}{R_s^4} \ln\left(\frac{R_s \epsilon}{\bar{\mu}}\right). \tag{4.137}$$

Since we have obtained the metric corrections by solving the modified Einstein equations perturbatively in G_N, the divergences (4.136) coming from the exterior corrections of order $\mathcal{O}(G_N^2)$ must be small compared to the usual metric coefficients of order $\mathcal{O}(G_N)$, which are

$$V_M \sim \frac{G_N M}{R_s}, \quad V_q \sim \frac{G_N q^2}{R_s^2}. \tag{4.138}$$

In our units $G_N = l_p^2$, therefore requiring $d_1 \lesssim V_M$ and $d_2 \lesssim V_q$ we have

$$\frac{l_p^2}{R_s^2} \ln\left(\frac{|r - R_s|}{R_s}\right) \lesssim 1, \tag{4.139}$$

whereas from $d_3 \lesssim V_M$ and $d_4 \lesssim V_q$:

$$\frac{l_p^2}{R_s |r - R_s|} \lesssim 1. \tag{4.140}$$

All these conditions are satisfied if $\epsilon \equiv |r - R_s| \lesssim l_p$, since for a star we obviously have $R_s \gg l_p$. Therefore our results for the metric corrections have to be considered only outside a layer of thickness ϵ around R_s.

Chapter 5

Quantum hair in dark energy stars

5.1 Gravastars

The final state reached by a star with large enough mass after gravitational collapse is a black hole. Black holes are peculiar in many aspects, among the most bizarre of which are the formation of the event horizon and of the singularity, a place where tidal forces diverge and general relativity breaks down. If during the collapse a trapped surface, that is a region from which not even light can escape, is formed and if the matter making up the star satisfies the strong energy condition, namely:

$$\rho + \sum_{i=1}^{3} p_i \geq 0, \tag{5.1}$$

then a singularity will form, as stated by the Hawking-Penrose theorem [65]. Another interesting aspect of black holes is the Hawking radiation, that is radiation emitted from the near horizon region which eventually leads to the evaporation of the black hole [66].

The presence of the event horizon and of the singularity are problematic not only from a gravitational point of view but also from a quantum information one, as they lead to the well known information paradox [67, 68]. Moreover, from the definition of the event horizon, the radiation emitted at the horizon radius would be infinitely redshifted. Therefore, if the Hawking radiation produced by the black hole were to propagate and reach an observer with a finite frequency, following the process backwards we would have that the radiation started with an arbitrarily large energy, reaching trans-Planckian values. Thus, the backreaction of the emitted radiation cannot be neglected, as it disrupts the geometry of the black hole. Another problem worth mentioning is that the entropy of the black hole, as given by the Bekenstein-Hawking formula [69]:

$$S_{BH} = \frac{Ak_B}{4\hbar G_N}, \tag{5.2}$$

where A is the horizon surface, far exceeds the entropy of a typical star [70, 71].

A model which attempted to take the backreaction into account considers the black hole as immersed in a Hawking radiation atmosphere, with an equation of state $p = \omega\rho$ [72]. The resulting entropy of this fluid is

$$S = 4\frac{\omega + 1}{7\omega + 1}S_{BH}, \tag{5.3}$$

reducing to the Bekenstein-Hawking one for $\omega = +1$. However, there is still the problem of trans-Planckian energies close to the horizon.

A different proposal is that as the star collapses, the quantum vacuum undergoes a phase transition at or near the location where the event horizon is expected to form, similar to the quantum liquid-vapor critical point of an interacting Bose fluid [73, 74]. The interior of the critical surface at the horizon is sustained by a fluid with equation of state $p = -\rho$, equivalent to the cosmological vacuum dark energy in Einstein's equations. Therefore the interior can be described by the de Sitter spacetime:

$$ds^2 = (1 - H^2r^2)dt^2 - (1 - H^2r^2)^{-1}dr^2 - r^2d\Omega^2, \tag{5.4}$$

where $H^2 = \frac{8\pi G_N\rho}{3} = \frac{\Lambda}{3}$, Λ being the cosmological constant and the horizon is located at H^{-1}. We note also that trying to match an interior de Sitter solution to an exterior Schwarzschild one is not a novelty and several attempts have been made [75, 76].

All these motivations lead to the search for objects that could substitute black holes. To be good candidates for this job, these objects must concentrate as much mass as possible in a radius $R_s \gtrsim 2G_M M$ while avoiding the formation of an event horizon and of the singularity. Based on these and on the considerations above, Mazur and Mottola proposed the model of a "gravitational vacuum star", also called *gravastar* [35, 36]. Gravastars are composed of three distinguished regions:

- an interior de Sitter region with equation of state $p = -\rho$;
- a shell of "non-inflationary" material [77] with $p = +\rho$;
- an exterior Schwarzschild region;

with two infinitesimally thin layers at the junction surfaces. The strong energy condition (5.1) holds for all known types of matter and radiation. However, this is not the case for the cosmological vacuum dark energy, for which

$$\rho + \sum_{i=1}^{3} p_i = -2\rho < 0. \tag{5.5}$$

Since the interior of a gravastar doesn't satisfy the strong energy condition, this allows us to get rid of the singularity. Moreover, since the position of the shell is such that the gravastar radius is greater than $2G_N M$ and smaller than H^{-1}, we also get rid of the event horizon and thus of

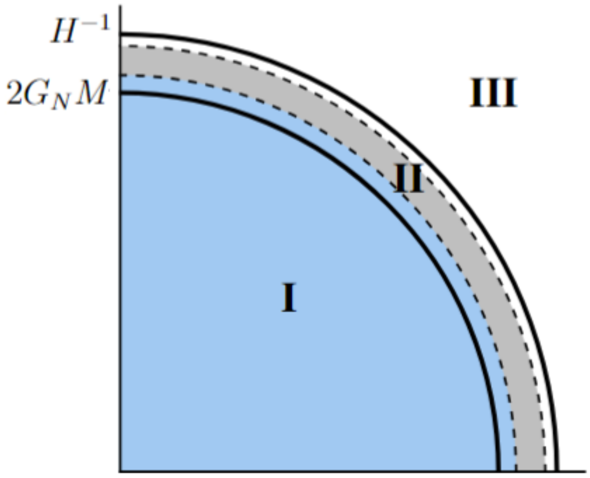

Figure 5.1: Representation of the three-layer dark energy star model. Region I in blue is the interior dark energy region ($p = \omega\rho,\ \omega < -1/3$), region II in gray and delimited by the two dashed lines is the thin shell ($p = +\rho$), region III in white is the exterior Schwarzschild region ($p = \rho = 0$). The thick lines correspond to the Schwarzschild and de Sitter horizons. The star radius is such that $2G_NM \lesssim R_s \lesssim H^{-1}$.

the information paradox. Lastly, the entropy of the gravastar, given only by the entropy of the shell, is found to be much smaller than the Bekenstein-Hawking one.

Gravastar models have been extensively studied throughout the years. Mazur and Mottola [35] showed that gravastars are thermodynamically stable. Visser and Wiltshire [34] studied the dynamic stability against radial perturbations of a simplified model, where the thick shell of matter and the two junction surfaces are combined in a single infinitesimal junction surface at $r \gtrsim 2G_NM$, showing that stability regions do exist. In [78] the stability analysis was extended to generic thin-shell gravastars. In [79] it was found that models where the two thin layers were replaced by a continuous pressure profile require that the matter of the shell cannot be a perfect fluid because of the presence of anisotropic pressures. Moreover, it was also shown that these models allow for a higher compactness than that allowed by the Buchdahl limit [80], which prescribes that a ball of dust with radius R_s and mass M must satisfy the condition $R_s > \frac{9}{8}(2G_NM)$ in order to be stable. In [33], studying the Buchdahl limit for spherically symmetric stars with constant density, it was shown that an interior solution with a negative pressure can already emerge in the classical theory of general relativity.

5.1.1 Observational tests

Since the exterior metric of a gravastar is the same as that of a black hole down to the length scale of the shell, it is very difficult to tell them apart experimentally. For example, being the radius of a gravastar arbitrarily close to its Schwarzschild radius, the light it emits will be largely redshifted, to the point that a gravastar is essentially indistinguishable from a black hole if we look at electromagnetic radiation only. The astonishing images of the (possibly) black holes Sgr A* at the center of the Milky Way [81–86] and of M87* at the center of the galaxy Messier 87 [87–90] lack the angular resolution to resolve the near horizon region and essentially observe the electromagnetic radiation of the photon sphere (see Section 6.4) around it.

Nonetheless, there are several proposed observational tests to differentiate between gravastars and black holes [91]. For example, in [92] the stability against axial perturbations was studied and it was found that the quasinormal modes eigenfrequencies of the two objects are, indeed, different: even though we can always choose the thickness and compactness of the gravastar in such a way that it has the same oscillation frequency of a black hole with the same mass, the decay time of the oscillations will differ. Therefore, the gravitational radiation produced by the oscillation of a gravastar can be used to distinguish it from a black hole. Although the literature on these observational tests is extensive and growing larger [93–97], as of now the question regarding the existence of gravastars remains unanswered.

5.2 Dark energy stars

An extension of the single thin-shell gravastar model [34] can be found in the concept of *dark energy stars* [37], where the de Sitter interior is generalized to a region governed by the equation of state $p = \omega\rho$, with $\omega < -1/3$. The motivation for this resides in the fact that it has been observed how the Universe is currently undergoing a phase of accelerated expansion [3–8]. The main proposal to explain this phenomenon is that of *dark energy*, a cosmic fluid parametrized exactly by an equation of state with $\omega < -1/3$. Current observations suggest that the value of ω is close to -1, therefore dark energy may be identified with the vacuum gravitational constant Λ of the de Sitter spacetime (5.4). It is then natural to extend the interior de Sitter region to a generic dark energy one.

Summarising, we consider a three layer star (see Fig. 5.1) with:

I. an interior dark energy region, with equation of state $p = \omega\rho$ and $\omega < -1/3$;

II. a single thin shell $p = +\rho$, with a radius R_s such that $2G_N M \lesssim R_s \lesssim H^{-1}$, in order to avoid the formation of an horizon;

III. an exterior Schwarzschild region, $\rho = p = 0$.

We take the interior energy density to be constant. The total mass of the star is given not only by the de Sitter vacuum but receives also a contribution from the thin shell, therefore we parametrize the energy density as $\rho = k\rho_0$, where $\rho_0 = \frac{3M}{4\pi R_s^3}$, with M the total mass, and $k \lesssim 1$. The interior metric is thus [37]

$$ds^2 = \left(1 - \frac{2G_N kM}{R_s^3} r^2\right)^{-(1+3\omega)/2} dt^2 - \left(1 - \frac{2G_N kM}{R_s^3} r^2\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{5.6}$$

In the remainder of this section we will compute the quantum gravitational corrections to the dark energy star metric, showing how the external metric carries information on the interior distribution.

5.3 Quantum corrections to the dark energy star metric

Following the same procedure of Section 3, it is straightforward to compute the metric corrections, in the limit where the shell is infinitesimally thin. For the interior we find

$$\begin{aligned}\delta g_{tt}^{\text{int}} &= [\alpha + \beta + 3\gamma - 3\omega(\alpha - \gamma)] \frac{192\pi G_N^2 kM}{R_s^3} \ln\left(\frac{R_s^2}{R_s^2 - r^2}\right) \\ &\quad + \frac{C_1}{r} + C_2 + \mathcal{O}(G_N^3), \end{aligned} \tag{5.7}$$

$$\delta g_{rr}^{\text{int}} = [(\alpha - \gamma) - \omega(3\alpha + \beta + \gamma)] \frac{384\pi G_N^2 M r^2}{R_s^3 (R_s^2 - r^2)} + \frac{C_1}{r} + \mathcal{O}(G_N^3), \tag{5.8}$$

where the integration constants C_i must be set to zero if we require regularity at the origin. Similarly for the exterior we find

$$\begin{aligned}\delta g_{tt}^{\text{ext}} &= [\alpha + \beta + 3\gamma - 3\omega(\alpha - \gamma)] \frac{192\pi G_N^2 kM}{R_s^3} \left[2\frac{R_s}{r} + \ln\left(\frac{r - R_s}{r + R_s}\right)\right] \\ &\quad + \frac{C_3}{r} + C_4 + \mathcal{O}(G_N^3), \end{aligned} \tag{5.9}$$

$$\delta g_{rr}^{\text{ext}} = [(\alpha - \gamma) - \omega(3\alpha + \beta + \gamma)] \frac{384\pi G_N^2 kM}{r(r^2 - R_s^2)} + \frac{C_3}{r} + \mathcal{O}(G_N^3), \tag{5.10}$$

where the integration constants C_i must be set to zero if we require asymptotic flatness.

Far away from the star, that is in the $r \gg R_s$ limit, the exterior metric corrections reduce to

$$\delta g_{tt}^{\text{ext}} = -[\alpha + \beta + 3\gamma - 3\omega(\alpha - \gamma)] \frac{128\pi G_N^2 kM}{r^3} + \mathcal{O}(G_N^3), \tag{5.11}$$

$$\delta g_{rr}^{\text{ext}} = [(\alpha - \gamma) - \omega(3\alpha + \beta + \gamma)] \frac{384\pi G_N^2 kM}{r^3} + \mathcal{O}(G_N^3), \tag{5.12}$$

whereas deep inside the star, that is in the $r \ll R_s$ limit, the interior corrections vanish:

$$\delta g_{tt}^{\text{int}} = \delta g_{rr}^{\text{int}} = \mathcal{O}(G_N^3). \tag{5.13}$$

These corrections also apply to the gravastar model when $\omega = -1$. Note also that for $\omega = 0$, i.e. dust, we recover the corrections to the Schwarzschild star of Section 3. We thus see explicitly the presence of quantum hairs: an outside oberserver can recover informations about the interior fluid's equation of state. As a last remark, because of the divergences in the limit $\epsilon \equiv |r - R_s| \to 0^+$, the metric corrections only apply outside a layer of thickness $\epsilon \gtrsim l_p$ around the star surface. Therefore the corrected metric cannot be used to study the stability of the model in the Israel–Lanczos–Sen junction condition formalism [98–100], which aims to find the equilibrium position of the freely moving transition layer at R_s. Moreover, even though the horizon radius is now shifted

$$R_H = 2G_N M - \frac{96k\pi}{M}[(\alpha - \gamma) - \omega(3\alpha + \beta + \gamma)], \tag{5.14}$$

the extra terms are subleading with respect to the classical result and won't affect the stability of the star.

We can now turn to the study of gravitational lensing as a way to find observables aiming to test the validity of our calculations. In particular, because of the presence of quantum hairs in the metric outside the gravastar/dark energy star and of the absence of second order corrections to the Schwarzschild black hole, these observational tests could allow us to experimentally tell these objects apart from one another.

Chapter 6

Observational tests in gravitational lensing

Since the metric components are not measurable by themselves, it is interesting to find some observables which could, in principle at least, allow us to detect the effects produced by the different metric corrections found so far. Of particular interest in this regard is then gravitational lensing. Gravitational lensing is not only of great historical value in physics, since it has been one of the very first tests of general relativity [101], but is also relevant from an experimental point of view, as many phenomena related to the bending of light have been observed.

By *gravitational lensing* we mean the collection of all the effects caused by a gravitational field on the propagation of electromagnetic radiation (see e.g. Fig. 6.1). The gravitational field is characterized by a metric with Lorentzian signature describing the spacetime manifold while the radiation is described in terms of rays, that is the lightlike geodesics of the metric. Therefore the mathematical description of gravitational lensing reduces to the study of lightlike geodesics in a 4-dimensional spacetime manifold and thus allows us to directly connect our metric corrections to measurements. We will now briefly review some of the basic concepts of this framework and then procede to compute several observables.

6.1 Celestial sphere

In a generic spacetime $(\mathcal{M}, g)$, the past light cone of an observer at a given event P_O, that is a point in space and time, is outlined by the lightlike geodesics departing from P_O into the past. The observer can detect only those signals generated by a (pointlike) source moving along a worldline γ_s intersecting its past light cone. For every past oriented lightlike geodesic λ departing from P_O and intersecting γ_s, an image of the source will be produced on the observer's sky. The observer's sky or *celestial sphere* S_O is the set of all lightlike directions at P_O [40]. Given the velocity U_O of the observer at P_O we can then identify the celestial sphere S_O as a subset of the tangent space $T_{P_O}\mathcal{M}$:

$$S_O = \{V \in T_{P_O}\mathcal{M} | g(V, V) = 0 \quad \text{and} \quad g(V, U_O) = 1\}. \tag{6.1}$$

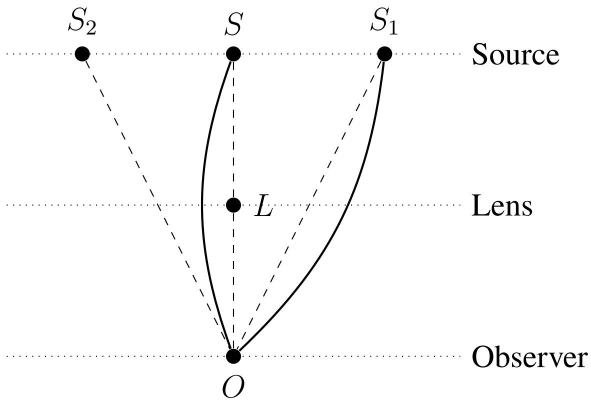

Figure 6.1: Representation of how light rays emitted by a source S are bent by a gravitational lens L as they travel towards an observer O.

By definition of the exponential map, every affinely parametrized geodesic $s \to \lambda(s)$ satisfies $\lambda(s) = \exp\Big(s\dot{\lambda}(0)\Big)$. Therefore, the past light cone of P_O is the image of the map

$$(s, V) \to \exp(sV), \tag{6.2}$$

defined on a subset of $]0, \infty[\times S_O$.

It is useful to introduce coordinates on the observer's past light cone by choosing an orthonormal tetrad $\{e_0, e_1, e_2, e_3\}$ with $e_0 = -U_O$ at the observation event. This allows us to parametrize the points on the observer's celestial sphere with spherical coordinates (Θ, Φ):

$$V = \sin\Theta\cos\Phi e_1 + \sin\Theta\sin\Phi e_2 + \cos\Theta e_3 + e_0. \tag{6.3}$$

Therefore the map (6.2) maps each (s, Θ, Φ) to a spacetime point. We can then let the observation event move along the observer's worldine, parametrized by the proper time τ, and map each (s, Θ, Φ, τ) to a spacetime point. In terms of generic coordinates $x^\mu = (x^0, x^1, x^2, x^3)$ on the spacetime manifold, this map can be formally expressed as

$$x^\mu = F^\mu(s, \Theta, \Phi, \tau). \tag{6.4}$$

6.2 The optical-mechanical analogy in general relativity

For static and spherically symmetric metrics, as is the case for the metrics studied so far, it can be useful to express the generic line element not in terms of the standard spherical coordinates (t, r, θ, ϕ):

$$ds^2 = f(r)c_0^2dt^2 - g(r)^{-1}dr^2 - r^2d\Omega^2, \tag{6.5}$$

but rather in terms of the isotropic radius r', which is introduced in order to treat the 3-dimensional space metric similarly to the Euclidean one:

$$\begin{aligned} ds^2 &= \Psi^2(r')c_0^2 dt^2 - \Gamma^{-2}(r')(dr'^2 + r'^2 d\Omega^2) \\ &= \Psi^2(r')c_0^2 dt^2 - \Gamma^{-2}(r')dl^2. \end{aligned} \tag{6.6}$$

Throughout this section we will keep explicit the dependence on the vacuum speed of light c_0 as we will deal with propagation of particles and light rays in terms of optical mechanics. The isotropic speed of light $c(r') = |dl/dt|$ can be found from $ds^2 = 0$, leading to

$$c(r') = |dl/dt| = c_0 \Gamma(r')\Psi(r'). \tag{6.7}$$

The effective refractive index is therefore

$$n = \frac{c_0}{c(r')} = \Psi(r')^{-1}\Gamma(r')^{-1}. \tag{6.8}$$

Light trajectories in a gravitational field, i.e. lightlike geodesics, may be calculated using the effective refractive index in the geometrical optics formalism [39]. A convenient approach is the "$F = ma$" formulation [102, 103], in which optical rays obey an equation similar to Newton's law:

$$\frac{d^2\vec{r'}}{dA^2} = \frac{1}{2}\vec{\nabla}(n^2 c_0^2), \tag{6.9}$$

where $\vec{r'}(A)$ is the position along the light ray parametrized by A. We will now show the analogy between trajectories in general relativity (governed by the least action principle), geometrical optics (governed by Fermat's principle) and classical mechanics (governed by Hamilton's principle). This will allow us to write down the equations of motion for massive and massless particles in general relativity in analogy to the equations of motion of Newtonian mechanics, which will be then used to investigate various phenomena.

6.2.1 Transformation of the geodesic condition

Let us consider the geodesic condition for particle trajectories:

$$\delta \int_{(t_1,\vec{x}_1)}^{(t_2,\vec{x}_2)} ds = 0, \tag{6.10}$$

where the variation is taken over the path of integration between two fixed spacetime points, $(t_1, \vec{x}_1)$ and $(t_2, \vec{x}_2)$. For the isotropic line element (6.6) we have

$$\delta \int_{(t_1,\vec{x}_1)}^{(t_2,\vec{x}_2)} \Psi c_0 \left(1 - \frac{v^2 n^2}{c_0^2}\right)^{1/2} = 0. \tag{6.11}$$

If we define the effective Lagrangian:

$$L(x_i, \dot{x}_i) = -c_0^2 \Psi \left(1 - \frac{v^2 n^2}{c_0^2}\right)^{1/2}, \tag{6.12}$$

we clearly see an analogy between (6.11) and Hamilton's principle, where $\dot{x}_i \equiv dx_i/dt$ and $v^2 = \sum_{i=1}^{3} (dx_i/dt)^2$ and we multiplied by an extra c_0 for later convenience. The canonical momenta are then

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = \Psi n^2 \dot{x}_i \left(1 - \frac{v^2 n^2}{c_0^2}\right)^{1/2}, \tag{6.13}$$

and the Hamiltonian is

$$H = \sum_{i=1}^{3} p_i \dot{x}_i - L = c_0^2 \Psi \left(1 - \frac{v^2 n^2}{c_0^2}\right)^{-1/2}, \tag{6.14}$$

or, in terms of the momentum

$$H = c_0^2 \left(\Psi^2 + \frac{p^2}{n^2 c_0^2}\right)^{1/2}, \tag{6.15}$$

where $p = |\vec{p}|$. Since there is no time dependence (as we work in static spacetimes) the Hamiltonian is a constant of motion. From Hamilton's principle:

$$\delta \int_{(t_1, \vec{x}_1)}^{(t_2, \vec{x}_2)} L \, dt = 0, \tag{6.16}$$

we can then obtain Maupertuis's principle:

$$\delta \int_{\vec{x}_1}^{\vec{x}_2} \sum_{i=1}^{3} p_i \dot{x}_i \, dt = \delta \int_{\vec{x}_1}^{\vec{x}_2} n^2 v^2 \Psi \left(1 - \frac{v^2 n^2}{c_0^2}\right)^{-1/2} dt = 0, \tag{6.17}$$

where now the variation is over a path of integration with fixed space endpoints and conserved energy along the path while the times at the endpoints are not fixed. If we substitute in (6.17) the right-hand side of (6.14) for H, in order to restrict the varied paths to those satisfying the energy constraint, we finally obtain

$$\delta \int_{\vec{x}_1}^{\vec{x}_2} n^2 v \, dl = 0, \tag{6.18}$$

where $dl = v dt = \left(\sum_{i=1}^{3} dx_i^2\right)^{1/2}$. We clearly see how Fermat's principle, at the basis of geometrical optics, and Maupertius' principle, at the basis of classical mechanics as long as the force can be derived from a velocity-independent potential, are simply a special case of (6.18):

Relativistic gravitational mechanics	Geometrical optics (Fermat)	Classical mechanics (Maupertuis)
$\delta \int n^2 v \, dl = 0$	$\delta \int n \, dl = 0$	$\delta \int v \, dl = 0$

Dealing with light instead of massive particle requires some care but the end result is the same. In this case the trial curves among which the lightlike geodesics extremizing a given functional are to be singled out are all the past-pointing lightlike curves from the event P_O to γ_s. By parametrizing these curves with a generic past-oriented parameter and assigning to each curve the parameter at which it arrives at the observer, we can define the *arrival time functional* T that has to be extremized. By definition a spacetime is stationary if it admits a timelike Killing vector field $\vec{\xi}$. If $\vec{\xi}$ is complete and if there are no closed timelike curves then the spacetime must be a product $\mathcal{M} \simeq \mathbb{R} \times \widehat{\mathcal{M}}$, with $\widehat{\mathcal{M}}$ a 3-manifold and $\vec{\xi}$ parallel to the $\mathbb{R}$-line [104]. Denoting by t the projection from $\mathcal{M}$ to $\mathbb{R}$ and choosing local coordinates $x = (x^1, x^2, x^3)$ on $\widehat{\mathcal{M}}$ we may rewrite the metric (6.6) as

$$ds^2 = e^{2\phi(x)}[c_0^2 dt^2 - \gamma_{ij}(x) dx^i dx^j], \quad \text{with} \quad i = 1, 2, 3 \tag{6.19}$$

where $e^{2\phi} = \Psi^2$ and $\gamma_{ij} = (\Gamma\Psi)^{-2}\delta_{ij} = n^2\delta_{ij}$ is called the *Fermat metric*. The factor $e^{2\phi(x)}$ won't affect the lightlike geodesics apart from their parametrization and the light rays' paths are determined only by the metric γ_{ij}. If we assume that the observation event P_O takes place at $t = 0$, then the arrival time for each trial curve λ from P_O to γ_s is equal to to the travel time and we may write the arrival time functional as

$$T(\lambda) = \int_{l_1}^{l_2} \sqrt{\gamma_{ij}(x) \frac{dx^i}{dl} \frac{dx^j}{dl}} dl = \int_{l_1=0}^{l_2} n(x) dl. \tag{6.20}$$

Therefore lightlike geodesics in a curved spacetime can be treated equivalently to light propagating in a medium with a suitable refractive index.

6.2.2 Equation of motion

Let us parametrize the particle trajectories by a parameter A, which will later be suitably chosen in order to simplify the equations of motion. We can then rewrite (6.18) as

$$\delta \int_{\vec{x}_1}^{\vec{x}_2} n^2 v \left| \frac{d\vec{r'}}{dA} \right| dA = 0, \tag{6.21}$$

where $\left| d\vec{r'}/dA \right| = [\sum_{i=1}^3 (dx_i/dA)^2]^{1/2}$. Consider now an infinitesimal displacement $\vec{w}(A)$ from the true path $\vec{r'}(A)$, such that it vanishes at the end points $\vec{r'} = \vec{x}_1, \vec{x}_2$. We thus have

$$\delta \int n^2 v \left| \frac{d\vec{r'}}{dA} \right| dA = \int \delta(n^2 v) \left| \frac{d\vec{r'}}{dA} \right| dA + \int n^2 v \left(\delta \left| \frac{d\vec{r'}}{dA} \right| \right) dA + \int n^2 v \left| \frac{d\vec{r'}}{dA} \right| \delta dA. \tag{6.22}$$

For the first term on the right-hand side, since it depends only on $\vec{r'}$, to first order in $\vec{w}$ its variation is

$$\delta(n^2 v) = \vec{\nabla}(n^2 v) \cdot \vec{w}. \tag{6.23}$$

For the second term, since a variation in the path will also induce a variation in the parameter A, we may write

$$\delta\left|\frac{d\vec{r'}}{dA}\right| = \left|\frac{d\vec{r'} + d\vec{w}}{dA + \delta dA}\right| - \frac{d\vec{r'}}{dA} = \frac{d\vec{r'}}{dA} \cdot \frac{d\vec{w}}{dA}\left|\frac{d\vec{r'}}{dA}\right|^{-1} - \left|\frac{d\vec{r'}}{dA}\right|\frac{\delta dA}{dA}, \tag{6.24}$$

again to first order in the variation. Plugging (6.23) and (6.24) in (6.22) we get

$$\delta \int n^2 v\left|\frac{d\vec{r'}}{dA}\right| dA = \int \left[\left|\frac{d\vec{r'}}{dA}\right| \vec{\nabla}(n^2 v) \cdot \vec{w} + n^2 v\left|\frac{d\vec{r'}}{dA}\right|^{-1} \frac{d\vec{r'}}{dA} \cdot \frac{d\vec{w}}{dA}\right] dA. \tag{6.25}$$

Integrating the last term by parts and remembering that $\vec{w}$ vanishes at the endpoints, we finally find the differential equation of motion that particle trajectories must satisfy:

$$\left|\frac{d\vec{r'}}{dA}\right| \vec{\nabla}(n^2 v) - \frac{d}{dA}\left(n^2 v\left|\frac{d\vec{r'}}{dA}\right|^{-1} \frac{d\vec{r'}}{dA}\right) = 0. \tag{6.26}$$

Since A is a generic parameter, we may choose it in such a way that it simplifies the equation of motion and makes explicit the analogy with Newtonian mechanics. We thus define it as

$$\left|\frac{d\vec{r'}}{dA}\right| \equiv n^2 v. \tag{6.27}$$

It is also useful to have an explicit relation between the stepping parameter A and the time t. If in (6.27) we substitute

$$\left|\frac{d\vec{r'}}{dA}\right| = \left|\frac{d\vec{r'}}{dt}\right| \frac{dt}{dA} = v\frac{dt}{dA} \tag{6.28}$$

we find

$$dA = dt/n^2. \tag{6.29}$$

The choice (6.27) simplifies (6.26) to

$$\frac{d^2\vec{r'}}{dA^2} = \vec{\nabla}\left(\frac{1}{2} n^2 v\right). \tag{6.30}$$

This equation is therefore a generalization of the $F = ma$ optics formula (6.9): the left-hand side is the second derivative of the position with respect to the chosen parameter whereas the right-hand side is a force expressed as the gradient of a potential energy. Moreover, the total energy, written as the sum of the kinetic energy $\frac{1}{2}\left|d\vec{r'}/da\right|^2$ and potential energy $-n^4v^2/2$, vanishes because of (6.27):

$$\frac{1}{2}\left|\frac{d\vec{r'}}{dA}\right|^2 - \frac{1}{2}n^4v^2 = 0. \tag{6.31}$$

These equations hold for both massive and massless particles, choosing $v(\vec{r'})$ appropriately:

$$v = \begin{cases} c_0 n^{-1} & \text{for light,} \\ c_0 n^{-1}\left(1 - \frac{c_0^4\Psi^2}{H^2}\right)^{1/2} & \text{for particles.} \end{cases} \tag{6.32}$$

In dealing with particle trajectories through (6.30) and (6.31) we can thus use the very well known methods of Newtonian mechanics. Furthermore, we may also write an exact general-relativistic formula by analogy to the classical one for particle motion in static and velocity-independent potentials once we substitute

$$t \to A, \quad U \to -n^4v^2/2, \quad E \to 0. \tag{6.33}$$

We stress that this formalism applies in the *isotropic* coordinate system. To go back to standard coordinates we can then simply transform the results thus obtained.

6.3 Refractive indices

In the previous section we dealt with objects whose outside metric is either the Schwarzschild or Reissner-Nordström one. Before dealing with the quantum corrected metrics, let us first review how to compute the refractive index for the classical case [39, 40]. We will then compute the refractive index for the corrected metrics as a small variation of the respective classical result.

The general line element (6.5) in standard coordinates (t, r, θ, ϕ) for the Reissner-Nordström metric is

$$ds^2 = \left(1 - \frac{2G_NM}{r} + \frac{G_Nq^2}{r^2}\right)dt^2 - \left(1 - \frac{2G_NM}{r} + \frac{G_Nq^2}{r^2}\right)^{-1}dr^2 - r^2d\Omega^2. \tag{6.34}$$

In isotropic coordinates (t, r', θ, ϕ) the metric will take the form (6.6), with $\Psi(r')$ and $\Gamma(r')$ to be determined. To find the transformation relating standard coordinates to the isotropic ones, we equate the angular and radial part of these two metrics, finding

$$r^2 = \Gamma^{-2}(r')r'^2, \tag{6.35}$$

$$g(r)^{-1}\,dr^2 = \Gamma^{-2}(r')\,dr'^2, \tag{6.36}$$

where

$$g(r) = 1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2}. \tag{6.37}$$

Dividing the second equation by the first to eliminate Γ we get

$$\frac{1}{\sqrt{g(r)}}\frac{dr}{r} = \frac{dr'}{r'} \tag{6.38}$$

and integrating we find the relation

$$2r' = (r - G_N M) + \sqrt{r^2 - 2G_N M r + G_N q^2}, \tag{6.39}$$

where we imposed that at large radial distances the two coordinates are equal. The inverse transformation is

$$r = r' + G_N M + \frac{G_N^2 M^2 - G_N q^2}{4r}. \tag{6.40}$$

Having found the isotropic radius in terms of the standard one, we can compute Γ from (6.35), whereas by a direct comparison of the two metrics (6.34) and (6.6) we have $\Psi^2(r') = f(r')$:

$$\Psi^2(r') = \left(1 - \frac{G_N^2 M^2 - G_N q^2}{4r'^2}\right)^2 \left(1 + \frac{G_N M}{r'} + \frac{G_N^2 M^2 - G_N q^2}{4r'^2}\right)^{-2}, \tag{6.41}$$

$$\Gamma^2(r') = \left(1 + \frac{G_N M}{r'} + \frac{G_N^2 M^2 - G_N q^2}{4r'^2}\right)^2, \tag{6.42}$$

leading to a refractive index

$$n(r') = \left(1 + \frac{G_N M}{r'} + \frac{G_N^2 M^2 - G_N q^2}{4r'^2}\right)^2 \left(1 - \frac{G_N^2 M^2 - G_N q^2}{4r'^2}\right)^{-1}. \tag{6.43}$$

Let us define $u \equiv 1/r$ and $u' \equiv 1/r'$. When transforming from one coordinate to the other, it is useful to use the following relations:

$$du' = n\, du \qquad \text{or} \qquad dr' = \Gamma \Psi^{-1}\, dr, \tag{6.44}$$

$$u' = \Gamma^{-1} u \qquad \text{or} \qquad r' = \Gamma r. \tag{6.45}$$

Since we will deal with the extra terms coming from the quantum corrected metric components expanding to second order in G_N and q, it is useful to repeat this analysis in this expansion. Let us start with the expansion and integration of (6.38), giving the relation between the standard and isotropic radial coordinates:

$$\ln(r') = \ln(r) - \frac{G_N M}{r} + \frac{G_N q^2}{4r^2} - \frac{3G_N^2 M^2}{4r^2} + \frac{G_N^2 M q^2}{r^3} + \mathcal{O}(G_N^3) + \mathcal{O}(q^4). \tag{6.46}$$

If we now exponentiate and expand, we find

$$r' \simeq r - G_N M + \frac{G_N q^2}{4r} - \frac{G_N^2 M^2}{4r} + \frac{G_N^2 M q^2}{4r^2}, \tag{6.47}$$

which coincides with the expansion of (6.39). Inverting the transformation and expanding we have

$$r \simeq r' + G_N M - \frac{G_N q^2}{4r'} + \frac{G_N^2 M^2}{4r'}, \tag{6.48}$$

which reproduces exactly (6.40). The resulting refractive index in terms of the isotropic radius (6.47) is then

$$n(r') = 1 + \frac{2G_N M}{r'} - \frac{3G_N q^2}{4r'^2} + \frac{7G_N^2 M^2}{4r'^2} - \frac{G_N^2 M q^2}{r'^3} + \mathcal{O}(G_N^3) + \mathcal{O}(q^4), \tag{6.49}$$

which coincides with the expansion of (6.43). Unfortunately, since the quantum corrected metric components are often very complicated, once we find the isotropic radius as a function of the standard one, we won't always be able to invert this relation. Therefore it is also useful to express the refractive index in terms of the standard coordinate, which upon expanding, yields

$$n(r) = 1 + \frac{2G_N M}{r} - \frac{3G_N q^2}{4r^2} + \frac{15G_N^2 M^2}{4r^2} - \frac{3G_N^2 M q^2}{r^3} + \mathcal{O}(G_N^3) + \mathcal{O}(q^4). \tag{6.50}$$

Let us now compute the refractive indices of our quantum corrected metrics.

6.3.1 Quantum corrected Schwarzschild star

We consider the metric corrections in the large distance limit, which is well justified since observed effects due to lensing are typically occurring on astrophysical distances.

The quantum corrected metric components are

$$f(r) = 1 - \frac{2GM}{r} - \frac{(\alpha + \beta + 3\gamma)128\pi G_N^2 M}{r^3}, \tag{6.51}$$

$$g(r) = 1 - \frac{2GM}{r} + \frac{(\alpha - \gamma)384\pi G_N^2 M}{r^3}. \tag{6.52}$$

The resulting isotropic coordinate is

$$r' \simeq r - G_N M - \frac{G_N^2 M^2}{4r} + \frac{(\alpha - \gamma)64\pi G_N^2 M}{r^2}, \tag{6.53}$$

and the inverse transformation is

$$r \simeq r' + G_N M + \frac{G_N^2 M^2}{4r'^2} - \frac{(\alpha - \gamma)64\pi G_N^2 M}{r'^2}. \tag{6.54}$$

The refractive index in isotropic coordinates is thus

$$n(r') = 1 + \frac{2G_N M}{r'} + \frac{7G_N^2 M^2}{4r'^2} + \frac{64(\beta + 4\gamma)\pi G_N^2 M}{r'^3} + \mathcal{O}(G_N^3), \tag{6.55}$$

whereas in standard coordinates it becomes

$$n(r) = 1 + \frac{2G_N M}{r} + \frac{15G_N^2 M^2}{4r^2} + \frac{64(\beta + 4\gamma)\pi G_N^2 M}{r^3} + \mathcal{O}(G_N^3). \tag{6.56}$$

6.3.2 Quantum corrected dark energy star

The quantum corrected metric components are

$$f(r) = 1 - \frac{2GM}{r} - \frac{[\alpha + \beta + 3\gamma - 3\omega(\alpha - \gamma)]\, 128\pi G_N^2 M}{r^3}, \tag{6.57}$$

$$g(r) = 1 - \frac{2GM}{r} + \frac{[(\alpha - \gamma - \omega(3\alpha + \beta + \gamma))]\, 384\pi G_N^2 M}{r^3}. \tag{6.58}$$

The isotropic coordinate is found to be

$$r' \simeq r - G_N M - \frac{G_N^2 M^2}{4r} + \frac{[(\alpha - \gamma - \omega(3\alpha + \beta + \gamma)]64\pi G_N^2 M}{r^2}, \tag{6.59}$$

and the inverse transformation is

$$r \simeq r' + G_N M + \frac{G_N^2 M^2}{4r'^2} - \frac{[(\alpha - \gamma - \omega(3\alpha + \beta + \gamma)]64\pi G_N^2 M}{r'^2}. \tag{6.60}$$

The resulting refractive index in isotropic coordinates is

$$n(r') = 1 + \frac{2G_N M}{r'} + \frac{7G_N^2 M^2}{4r'^2} + \frac{64(\beta + 4\gamma)(1+\omega)\pi G_N^2 M}{r'^3} + \mathcal{O}(G_N^3), \tag{6.61}$$

whereas in standard coordinates it becomes

$$n(r) = 1 + \frac{2G_N M}{r} + \frac{15G_N^2 M^2}{4r^2} + \frac{64(\beta + 4\gamma)(1+\omega)\pi G_N^2 M}{r^3} + \mathcal{O}(G_N^3). \tag{6.62}$$

6.3.3 Quantum corrected Reissner-Nordström black hole

For the quantum corrected Reissner-Nordström black hole we have

$$f(r) = 1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2} - \frac{32\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta + 4\gamma)\left(\ln(\mu r) + \gamma - \frac{3}{2}\right)\right], \tag{6.63}$$

$$g(r) = 1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2} - \frac{64\pi G_N^2 q^2}{r^4}\Big[\bar{c}_2 + 2(\beta + 4\gamma)\left(\ln(\mu r) + \gamma - 2\right)\Big]. \tag{6.64}$$

The resulting isotropic coordinate is

$$r' \simeq r - G_N M + \frac{G_N q^2}{4r} - \frac{G_N^2 M^2}{4r} + \frac{G_N^2 M q^2}{4r^2} - \frac{8\pi G_N^2 q^2}{r^3}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma - \frac{7}{4}\right)\right]. \quad (6.65)$$

As anticipated earlier, we can't analytically invert this relation in order to find Γ as a function of the isotropic radius and thus compute the refractive index $n(r')$. Therefore, we can only express the refractive index in terms of the standard coordinate, which upon expansion yields

$$n(r) = 1 + \frac{2G_N M}{r} - \frac{3G_N q^2}{4r^2} + \frac{15 G_N^2 M^2}{4r^2} - \frac{3G_N^2 M q^2}{r^3} + \frac{24\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma - \frac{19}{12}\right)\right] + \mathcal{O}(G_N^3) + \mathcal{O}(q^4). \quad (6.66)$$

6.3.4 Quantum corrected charged star

We report here the metric components for the quantum corrected Reissner-Nordström star for the case I in the $r \gg R_s$ limit:

$$f(r) = 1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2} - \frac{1280\pi G_N^2 q^2(\alpha-\gamma)}{3rR_s^3} - \frac{64\pi G_N^2}{r^3}\left(\frac{q^2}{2R_s} + M\right)(\alpha+\beta+3\gamma) - \frac{32\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{3}{2}\right)\right] + \mathcal{O}(G_N^3), \quad (6.67a)$$

$$g(r) = 1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2} - \frac{1280\pi G_N^2 q^2(\alpha-\gamma)}{3rR_s^3} + \frac{64\pi G_N^2}{r^3}\left(\frac{q^2}{2R_s} + M\right)(\alpha-\gamma) - \frac{64\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - 2\right)\right] + \mathcal{O}(G_N^3). \quad (6.67b)$$

The resulting isotropic coordinate is

$$r' \simeq r - G_N M + \frac{G_N q^2}{4r} - \frac{G_N^2 M^2}{4r} + \frac{G_N^2 M q^2}{4r^2} + \frac{64\pi G_N^2 M(\alpha-\gamma)}{r^2}\left(\frac{q^2}{2R_s} + M\right) - \frac{640\pi G_N^2 q^2(\alpha-\gamma)}{3R_s^3} - \frac{8\pi G_N^2 q^2}{r^3}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma - \frac{7}{4}\right)\right]. \quad (6.68)$$

Again, this transformation is not invertible and we can only express the refractive index in terms of the standard coordinate:

$$n(r) = 1 + \frac{2G_N M}{r} - \frac{3G_N q^2}{4r^2} + \frac{15G_N^2 M^2}{4r^2} - \frac{3G_N^2 M q^2}{r^3} + \frac{64\pi G_N^2(\beta+4\gamma)}{r^3}\left(\frac{q^2}{2R_s} + M\right)$$
$$+\frac{1280\pi G_N^2 q^2(\alpha-\gamma)}{3rR_s^3} + \frac{24\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma - \frac{19}{12}\right)\right] + \mathcal{O}(G_N^3) + \mathcal{O}(q^4). \tag{6.69}$$

For case II the results are the same with the only exception that the term proportional to R_s^{-3} vanishes.

6.4 Photon sphere

We will now analyze the presence of photon spheres [41], that is sphere made up of photons moving in circular orbits around the source, for the metric of interest, motivated by the fact that we have images of the photon spheres around the black holes Sgr A* at the center of the Milky Way [81–86] and M87* at the center of the galaxy Messier 87 [87–90]. We start by reviewing the classical results for the Schwarzschild and Reissner-Nordström metrics and procede to compute the photon sphere radius for the quantum corrected metrics as a small modification of the classical result.

The generic line element (6.5) for static and spherically symmetric spacetimes (setting now $c_0 = 1$) has two Killing vectors: $\vec{k} = \vec{\partial}_t$ associated to invariance under time translations and $\vec{n} = \vec{\partial}_\phi$ associated to invariance under rotations around the z-axis. Defining $u^\mu \equiv dx^\mu/d\lambda$ as the photon four-momentum, we have the two integral of motions

$$E = -k_\mu u^\mu = f(r)\frac{dt}{d\lambda}, \tag{6.70}$$

$$L = n_\mu u^\mu = r^2\sin^2(\theta)\frac{d\phi}{d\lambda}. \tag{6.71}$$

Without loss of generality, we can restrict motion to be on the equatorial plane $\theta = \pi/2$. Using then the condition $g_{\mu\nu}u^\mu u^\nu = 0$ we find

$$\frac{f(r)}{g(r)}\left(\frac{dr}{d\lambda}\right)^2 + V(r,E,L) = 0, \tag{6.72}$$

where the effective potential is defined as

$$V(r,E,L) = f(r)\frac{L^2}{r^2} - E^2. \tag{6.73}$$

Since we are interested in circular orbits we must impose

$$\frac{dr}{d\lambda} = 0, \quad \frac{d^2 r}{d\lambda^2} = 0, \tag{6.74}$$

which translate into the conditions for the potential

$$V(r_p) = 0, \quad V'(r_p) = 0. \tag{6.75}$$

When $V'(r_p) = 0$, a geodesic of the γ metric that starts tangent to the sphere at $r = r_p$ remains in it. This will create the so called *photon sphere* at r_p that gives raise to a gravitational lensing generating infinitely-many images. Solving the first equation (6.75) for the impact parameter $b \equiv L/E$ and then plugging it into the second equation we find that the latter is satisfied when

$$f'(r_p) r_p - 2f(r_p) = 0, \tag{6.76}$$

which is equivalent to

$$\frac{d}{dr}\left(\frac{r_p}{\sqrt{f(r_p)}}\right) = 0. \tag{6.77}$$

It can be shown that any spherically symmetric and static spacetime with an horizon at r_H and which is asymptotically flat must have a light sphere at a radius between the horizon radius and infinity [105].

For the classical Reissner-Nordström metric we have

$$f(r) = 1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2}, \tag{6.78}$$

leading to the equation

$$r^2 - 3G_N M r + 2G_N q^2 = 0, \tag{6.79}$$

which has two solutions:

$$r_p = \frac{3G_N M \pm \sqrt{9G_N^2 M^2 - 8q^2}}{2}. \tag{6.80}$$

We are of course interested in the outer solution and since we deal with the non-extremal case for which $q^2 \ll GM^2$, we can expand in the charge finding

$$r_p = 3G_N M - \frac{2q^2}{3M^2} + \mathcal{O}(q^4). \tag{6.81}$$

In the zero charge limit we recover the Schwarzschild solution $r_p = 3G_N M$.

6.4.1 Quantum corrected uncharged star

The Schwarzschild black hole doesn't receive any correction at second order in curvature, although it does at third order [15]. The photon sphere radius for the corrected black hole has already been studied [106], therefore we shall focus only on the quantum corrected star of Section 3. For this latter case we have, considering an observer far away from the star:

$$f(r) = 1 - \frac{2G_N M}{r} - \frac{(\alpha+\beta+3\gamma)128\pi G_N^2 M}{r^3}, \tag{6.82}$$

leading to the equation

$$r^3 - 3G_N M r^2 - (\alpha+\beta+3\gamma)320\pi G_N^2 M = 0. \tag{6.83}$$

Similarly to what we did for the computation of the horizon radius in the previous sections, we solve this equation perturbatively around the classical result $r_p = 3G_N M$. We first recast it as

$$r - 3G_N M = (\alpha+\beta+3\gamma)\frac{320\pi G_N^2 M}{r^2}, \tag{6.84}$$

and solve setting the right-hand side to zero, finding $r_p = 3G_N M$. We then plug this result on the right-hand side and solve the whole equation, thus getting the modified photon sphere radius

$$r_p = 3G_N M + (\alpha+\beta+3\gamma)\frac{320\pi}{9M}. \tag{6.85}$$

Similarly, for the dark energy star we find

$$r_p = 3G_N M + [\alpha+\beta+3\gamma-3\omega(\alpha-\gamma)]\frac{320\pi k}{9M}. \tag{6.86}$$

6.4.2 Quantum corrected Reissner-Nordström black hole

For the quantum corrected Reissner-Nordström black hole we have

$$f(r) = 1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2} - \frac{32\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{3}{2}\right)\right], \tag{6.87}$$

leading to

$$r^2 - 3G_N M r + 2G_N q^2 = \frac{96\pi G_N^2 q^2}{r^2}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{5}{3}\right)\right]. \tag{6.88}$$

Setting $r = 3G_N M$ on the right-hand side and solving the resulting equation we find

$$r_p = 3G_N M - \frac{2q^2}{3M} + \frac{32\pi q^2}{9GM^3}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{5}{3}\right)\right] + \mathcal{O}(q^4), \tag{6.89}$$

where, as in the classical case, we expanded in the limit $q^2 \ll GM^2$.

6.4.3 Quantum corrected charged star

The quantum corrected charged star is more complicated and requires some care in order to give a sensible result. We start with the simple case II of Section 4.3.2, that is when we impose the perfect fluid condition on the whole energy-momentum tensor: $T_1^1 = T_2^2 = T_3^3$. From (4.133a) we have

$$f(r) = 1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2} - \frac{64\pi G_N^2}{r^3}\left(\frac{q^2}{2R_s} + M\right)(\alpha+\beta+3\gamma) - \frac{32\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{3}{2}\right)\right], \quad (6.90)$$

where we used $\rho_0 = 3M/(4\pi R_s^3)$ and fixed $R_s = 2G_N M$, since for the photon sphere to be observable the radius of the star has to be in the range $2G_N M < R_s < 3G_N M$. The equation we need to solve is then

$$r^4 - 3G_N M r^3 + 2G_N q^2 r^2 - (\alpha+\beta+3\gamma)80\pi\left(4G_N^2 M + \frac{G_N q^2}{M}\right) r - 96\pi G_N^2 q^2\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{5}{3}\right)\right] = 0. \quad (6.91)$$

Following the same steps as before we eventually find

$$r_p = 3G_N M + \frac{2q^2}{3M} + (\alpha+\beta+3\gamma)\frac{320\pi}{9}\left(\frac{1}{M} + \frac{25}{36}\frac{q^2}{GM^3}\right) + \frac{32\pi q^2}{9GM^3}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{5}{3}\right)\right] + \mathcal{O}(q^4). \quad (6.92)$$

For case I, where we impose perfect fluidity on the matter tensor alone, that is $M_1^1 = M_2^2 = M_3^3$, from (4.114a) we have

$$f(r) = 1 - \frac{2G_N M}{r} + \frac{G_N q^2}{r^2} - \frac{1280\pi G_N^2 q^2(\alpha-\gamma)}{3rR_s^3} - \frac{64\pi G_N^2}{r^3}\left(\frac{q^2}{2R_s} + M\right)(\alpha+\beta+3\gamma) - \frac{32\pi G_N^2 q^2}{r^4}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{3}{2}\right)\right], \quad (6.93)$$

leading to

$$r^4 - \left[3G_N M + (\alpha-\gamma)\frac{80\pi q^2}{G_N M^3}\right] r^3 + 2G_N q^2 r^2 - (\alpha+\beta+3\gamma)80\pi\left(4G_N^2 M + \frac{G_N q^2}{M}\right) r - 96\pi G_N^2 q^2\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{5}{3}\right)\right] = 0. \quad (6.94)$$

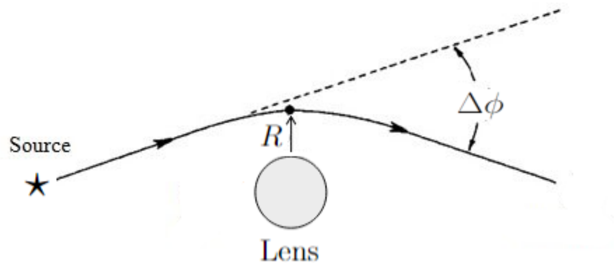

Figure 6.2: Gravitational bending of light rays passing near a massive object, where $\Delta\phi$ is the total deflection angle and R is the distance of closest approach to the origin.

Thus we find

$$r_p = 3G_N M + \frac{2q^2}{3M} + \frac{80\pi q^2}{GM^3}\left(\alpha - \frac{31}{81}\gamma\right) + (\alpha+\beta+3\gamma)\frac{320\pi}{9}\left(\frac{1}{M} + \frac{53}{36}\frac{q^2}{GM^3}\right) + \frac{32\pi q^2}{9GM^3}\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln(\mu r) + \gamma_E - \frac{5}{3}\right)\right] + \mathcal{O}(q^4). \quad (6.95)$$

As a final remark on photon spheres, we note that while they are perfectly possible for black holes (as recently observed for Sgr A* and M87*), gravastars and dark energy stars, the existence of regular stars as those analyzed in Section 3 and 4 with a radius in the range $2G_N M < R_s < 3G_N M$ is dubious [107].

6.5 Bending of light rays

Light rays passing near a massive object will be bent by an angle ϕ with respect to their original trajectory (see Fig. 6.2). Going back to orbits on the equatorial plane, we may write the general relativistic expression for the deflection angle in analogy to the classical result [38]:

$$\phi = L_0 \int \frac{1}{r'^2[2(E-U) - L_0^2/r'^2]^{1/2}} dr' \to L \int \frac{1}{r'^2[n^4 v^2 - L^2/r'^2]^{1/2}} dr' \quad (6.96)$$

where $L_0 = r'^2 d\phi/dt$ and $L = r'^2 d\phi/dA$ are related through (6.29) as $L = n^2 L_0$. Switching to $u' = 1/r'$, we may rewrite this result as

$$L^2\left[\left(\frac{du'}{d\phi}\right)^2 + u'^2\right] - n^4 v^2 = 0, \quad (6.97)$$

and using the expression for the massive particle velocity (6.32) we find

$$\left(\frac{du'}{d\phi}\right)^2 + u'^2 - \frac{c_0^2}{L^2} n^2 \left(1 - \frac{c_0^4 \Psi^2}{H^2}\right) = 0. \quad (6.98)$$

The same equation holds for photons once we send $H \to \infty$. We can now go back to standard coordinates using (6.44) and (6.45):

$$\left(\frac{du}{d\phi}\right)^2 + u^2\Psi^2 - \frac{c_0^2}{L^2}n^2\left(1 - \frac{c_0^4\Psi^2}{H^2}\right) = 0. \tag{6.99}$$

Considering now the general Reissner-Nordström metric and differentiating with respect to ϕ we get

$$\frac{d^2u}{d\phi^2} + u - \frac{G_N M c_0^6}{L^2H^2} = -\frac{G_N q^2 c_0^6}{L^2H^2}u + 3G_N M u^2 - 2G_N q^2 u^3. \tag{6.100}$$

Since we are interested in the deviation of light rays, we can now send $H \to \infty$ in (6.100), leading to

$$\frac{d^2u}{d\phi^2} + u = 3G_N M u^2 - 2G_N q^2 u^3, \tag{6.101}$$

and solve this differential equation perturbatively, i.e. we first set the right-hand side to zero and obtain the zeroth-order solution

$$u = \frac{\sin\phi}{R}, \tag{6.102}$$

where R is the distance of closest approach to the origin. We can now plug this result in the right-hand side of (6.101) and then solve to obtain the first-order solution

$$u = \frac{\sin\phi}{R} + \frac{3G_N M}{2R^2}\left(1 + \frac{1}{3}\cos 2\phi\right) + \frac{3G_N q^2}{4R^3}\phi\cos\phi - \frac{G_N q^2}{16R^3}\sin 3\phi. \tag{6.103}$$

At large distances $r \to \infty$, $u \to 0$ and the deflection angle becomes small $\phi \to \phi_\infty$. We can thus expand (6.103) as

$$0 = \frac{\phi_\infty}{R} + \frac{2G_N M}{R^2} + \frac{9q^2\phi_\infty}{16R^3}, \tag{6.104}$$

and the total deflection $\Delta\phi_\infty = 2|\phi_\infty|$ is:

$$\Delta\phi_\infty = \frac{4G_N M}{R}\left(1 - \frac{9G_N q^2\phi_\infty}{16R^2}\right) + \mathcal{O}(q^4), \tag{6.105}$$

where we expanded in the limit $q^2 \ll G_N M^2$. We will now repeat these calculations for the quantum corrected metrics of interest.

6.5.1 Quantum corrected Schwarzschild star

As we are interested in the $r \to \infty$ limit, we shall consider the quantum correction (3.20) in the $r \gg R_s$ limit, therefore we need to substitute

$$\Psi^2 = 1 - 2G_N M u - (\alpha + \beta + 3\gamma)128\pi G_N^2 M u^3, \tag{6.106}$$

in (6.99) in the $H \to \infty$ limit. Differentiating with respect to ϕ we get the differential equation

$$\frac{d^2u}{d\phi^2} + u = 3G_N M u^2 + 320\pi G_N^2 M(\alpha + \beta + 3\gamma)u^4. \tag{6.107}$$

Plugging the zeroth-order solution (6.102) on the right-hand side we can solve the resulting equation analytically thus getting the first-order solution

$$u = \frac{\sin\phi}{R} + \frac{3G_N M}{2R^2}\left(1 + \frac{1}{3}\cos 2\phi\right) + \frac{120\pi G_N^2 M(\alpha+\beta+3\gamma)}{R^4}\left(1 + \frac{4}{9}\cos 2\phi\right) - \frac{8\pi G_N^2 M(\alpha+\beta+3\gamma)}{3R^4}\cos 4\phi. \tag{6.108}$$

In the large distance limit this equation reduces to

$$0 = \frac{\phi_\infty}{R} + \frac{2G_N M}{R^2} + \frac{512\pi G_N^2 M(\alpha+\beta+3\gamma)}{3R^4}, \tag{6.109}$$

and the total deflection $\Delta\phi_\infty = 2|\phi_\infty|$ is

$$\Delta\phi_\infty = \frac{4G_N M}{R} + \frac{1024\pi G_N^2 M(\alpha+\beta+3\gamma)}{3R^3}. \tag{6.110}$$

6.5.2 Quantum corrected dark energy star

Working again in the $r \gg R_s$ limit, we need to substitute

$$\Psi^2 = 1 - 2G_N M u - [\alpha + \beta + 3\gamma - 3\omega(\alpha - \gamma)]128\pi G_N^2 k M u^3 \tag{6.111}$$

in (6.99) in the $H \to \infty$ limit. Differentiating with respect to ϕ we get the differential equation

$$\frac{d^2u}{d\phi^2} + u = 3G_N M u^2 + 320\pi G_N^2 k M[\alpha + \beta + 3\gamma - 3\omega(\alpha - \gamma)]u^4. \tag{6.112}$$

Plugging the zeroth-order solution (6.102) on the right-hand side we can solve the resulting equation analytically, finding the first-order solution

$$u = \frac{\sin\phi}{R} + \frac{3G_N M}{2R^2}\left(1 + \frac{1}{3}\cos 2\phi\right) + \frac{120\pi G_N^2 k M[\alpha+\beta+3\gamma-3\omega(\alpha-\gamma)]}{R^4}\left(1 + \frac{4}{9}\cos 2\phi\right) - \frac{8\pi G_N^2 k M[\alpha+\beta+3\gamma-3\omega(\alpha-\gamma)]}{3R^4}\cos 4\phi. \tag{6.113}$$

In the large distance limit this equation reduces to

$$0 = \frac{\phi_\infty}{R} + \frac{2G_N M}{R^2} + \frac{512\pi G_N^2 k M[\alpha+\beta+3\gamma-3\omega(\alpha-\gamma)]}{3R^4}, \tag{6.114}$$

and the total deflection $\Delta\phi_\infty = 2|\phi_\infty|$ is

$$\Delta\phi_\infty = \frac{4G_N M}{R} + \frac{1024\pi G_N^2 k M[\alpha+\beta+3\gamma-3\omega(\alpha-\gamma)]}{3R^3}. \tag{6.115}$$

6.5.3 Quantum corrected Reissner-Nordström black hole

In this case we have

$$\Psi^2 = 1 - 2G_N M u + G_N q^2 u^2 - 32\pi G_N^2 q^2 u^4 \left[\bar{c}_2 + 2(\beta + 4\gamma)\left(\ln\left(\frac{\mu}{u}\right) + \gamma_E - \frac{3}{2}\right)\right], \quad (6.116)$$

leading to the differential equation

$$\frac{d^2u}{d\phi^2} + u = 3G_N M u^2 - 2G_N q^2 u^3 + 96\pi G_N^2 q^2 u^5 \left[\bar{c}_2 + 2(\beta + 4\gamma)\left(\ln\left(\frac{\mu}{u}\right) + \gamma_E - \frac{5}{3}\right)\right]. \quad (6.117)$$

Comparing (6.117) to (6.101), we treat the extra terms of order $\mathcal{O}(G_N^2)$ as a small perturbation with respect to the $\mathcal{O}(G_N)$ terms. Therefore, in order to solve this differential equation, we plug in the $\mathcal{O}(G_N^2)$ terms the zeroth-order solution (6.102) where now the deflection angle ϕ is kept fixed at $\phi_\infty = -2G_N M/R$. We work in this approximation because otherwise, if we were to keep ϕ free in the logarithmic term, we would not get an analytical solution. Moreover, since we are interested in the large distance limit, the terms proportional to u^5 and $u^5 \ln(\mu/u)$ will behave similarly in the $u \to 0$ limit, therefore all these terms should be treated democratically and evaluated on the classical solution. The result is then

$$u = \frac{\sin\phi}{R} + \frac{3G_N M}{2R^2}\left(1 + \frac{1}{3}\cos 2\phi\right) + \frac{3G_N q^2}{4R^3}\phi\cos\phi - \frac{G_N q^2}{16R^3}\sin 3\phi - \frac{96\pi G_N^2 q^2}{R^5}\left\{\bar{c}_2 + 2(\beta + 4\gamma)\left[-\ln\left(\frac{1}{\mu R}\sin\left(-\frac{2G_N M}{R}\right)\right) + \gamma_E - \frac{5}{3}\right]\right\}\left[\sin\left(\frac{2G_N M}{R}\right)\right]^5, \quad (6.118)$$

which, at first order in G_N, reduces to the classical result (6.103). In the large distance limit:

$$0 = \frac{\phi_\infty}{R} + \frac{2G_N M}{R^2} + \frac{9q^2\phi_\infty}{16R^3} - \frac{96\pi G_N^2 q^2}{R^5}\{\bar{c}_2 + 2(\beta + 4\gamma) \times \left[-\ln\left(\frac{1}{\mu R}\sin\left(\frac{2G_N M}{R}\right)\right) + \gamma_E - \frac{5}{3}\right]\}\left[\sin\left(\frac{2G_N M}{R}\right)\right]^5, \quad (6.119)$$

and the total deflection angle is

$$\Delta\phi_\infty = \frac{4G_N M}{R} - \frac{9G_N^2 M q^2}{4R^3} - \frac{192\pi G_N^2 q^2}{R^5}\left\{\bar{c}_2 + 2(\beta + 4\gamma) \times \left[-\ln\left(\frac{1}{\mu R}\sin\left(\frac{2G_N M}{R}\right)\right) + \gamma_E - \frac{5}{3}\right]\right\}\left[\sin\left(\frac{2G_N M}{R}\right)\right]^5 + \mathcal{O}(q^4). \quad (6.120)$$

6.5.4 Quantum corrected charged star

For the electrically charged star we need to be more careful in the calculations. We start with case I, for which we have

$$\Psi^2 = 1 - 2G_N M u + G_N q^2 u^2 - \frac{128\pi G_N^2(\alpha+\beta+3\gamma)}{r^3}\left(\frac{q^2}{2R_s} + M\right) - \frac{1280\pi G_N^2 q^2(\alpha-\gamma)u}{3R_s^3} - 32\pi G_N^2 q^2 u^4\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln\left(\frac{\mu}{u}\right) + \gamma_E - \frac{3}{2}\right)\right], \quad (6.121)$$

leading to the differential equation

$$\frac{d^2u}{d\phi^2} + u = 3G_N M u^2 - 2G_N q^2 u^3 + 320\pi G_N^2(\alpha+\beta+3\gamma)\left(\frac{q^2}{2R_s} + M\right) + \frac{640\pi G_N^2 q^2(\alpha-\gamma)}{R_s^3}u^2 + 96\pi G_N^2 q^2 u^5\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln\left(\frac{\mu}{u}\right) + \gamma_E - \frac{5}{3}\right)\right]. \quad (6.122)$$

In order to solve this equation in such a way that it is in agreement with the results of the previous sections, we keep terms of order $\mathcal{O}(u^5)$ fixed on the classical result $u = \sin(\phi_\infty)/R$, with $\phi_\infty = -2G_N M/R$ while evaluating all the other terms on the right-hand side of (6.122) on the zeroth-order solution (6.102). We thus find

$$\begin{aligned} u = {} & \frac{\sin\phi}{R} + \frac{3G_N M}{2R^2}\left(1 + \frac{1}{3}\cos 2\phi\right) + \frac{3G_N q^2}{4R^3}\phi\cos\phi - \frac{G_N q^2}{16R^3}\sin 3\phi \\ & + \frac{320\pi G_N^2 q^2(\alpha-\gamma)}{R^2 R_s^3}\left(1 + \frac{1}{3}\cos 2\phi\right) + \frac{120\pi G_N^2(\alpha+\beta+3\gamma)}{R^4}\left(\frac{q^2}{2R_s} + M\right)\left(1 + \frac{4}{9}\cos 2\phi\right) \\ & - \frac{8\pi G_N^2(\alpha+\beta+3\gamma)}{3R^4}\left(\frac{q^2}{2R_s} + M\right)\cos 4\phi \\ & - \frac{96\pi G_N^2 q^2}{R^5}\left\{\bar{c}_2 + 2(\beta+4\gamma)\left[-\ln\left(\frac{1}{\mu R}\sin\left(-\frac{2G_N M}{R}\right)\right) + \gamma_E - \frac{5}{3}\right]\right\}\left[\sin\left(\frac{2G_N M}{R}\right)\right]^5, \end{aligned} \quad (6.123)$$

and expanding we get

$$\begin{aligned} 0 = {} & \frac{\phi_\infty}{R} + \frac{2G_N M}{R^2} + \frac{9q^2\phi_\infty}{16R^3} + \frac{1280\pi G_N^2 q^2(\alpha-\gamma)}{3R^2 R_s^3} + \frac{512\pi G_N^2(\alpha+\beta+3\gamma)}{3R^4}\left(\frac{q^2}{2R_s} + M\right) \\ & - \frac{96\pi G_N^2 q^2}{R^5}\left\{\bar{c}_2 + 2(\beta+4\gamma)\left[-\ln\left(\frac{1}{\mu R}\sin\left(-\frac{2G_N M}{R}\right)\right) + \gamma_E - \frac{5}{3}\right]\right\}\left[\sin\left(\frac{2G_N M}{R}\right)\right]^5. \end{aligned} \quad (6.124)$$

The total deflection angle is thus

$$\Delta\phi_\infty = \frac{4G_N M}{R} - \frac{9G_N^2 M q^2}{4R^3} + \frac{1024\pi G_N^2(\alpha+\beta+3\gamma)}{3R^3}\left(\frac{q^2}{2R_s}+M\right) + \frac{2560\pi G_N^2 q^2(\alpha-\gamma)}{3RR_s^3}$$
$$- \frac{192\pi G_N^2 q^2}{R^5}\left\{\bar{c}_2 + 2(\beta+4\gamma)\times\left[-\ln\left(\frac{1}{\mu R}\sin\left(\frac{2G_N M}{R}\right)\right)+\gamma_E-\frac{5}{3}\right]\right\}$$
$$\times\left[\sin\left(\frac{2G_N M}{R}\right)\right]^5 + \mathcal{O}(G_N^3) + \mathcal{O}(q^4). \tag{6.125}$$

For case II we have

$$\Psi^2 = 1 - 2G_N M u + G_N q^2 u^2 - \frac{128\pi G_N^2(\alpha+\beta+3\gamma)}{r^3}\left(\frac{q^2}{2R_s}+M\right)$$
$$- 32\pi G_N^2 q^2 u^4\left[\bar{c}_2 + 2(\beta+4\gamma)\left(\ln\left(\frac{\mu}{u}\right)+\gamma_E-\frac{3}{2}\right)\right], \tag{6.126}$$

which is equal to (6.121) without the $(\alpha-\gamma)$ term. Therefore all the previous results apply also in this case once we set terms proportional to $(\alpha-\gamma)$ to zero.

6.6 Gravitational redshift

In geometrical optics, when a light ray travels from a region with a given refractive index n_1 to a region with a different index n_2, its velocity and wavelength change but not its frequency. Therefore we have the general relation

$$\lambda(\vec{r}'_1)n(\vec{r}'_1) = \lambda(\vec{r}'_2)n(\vec{r}'_2), \tag{6.127}$$

where λ is the wavelength. We can easily extend this to gravitational redshift. Consider the coordinate distance $|\Delta\vec{r}'_s|$ between two successive crests or valleys of a light wave as emitted by a source at $\vec{r}'_s$. Now let $|\Delta\vec{r}'_o|$ be the coordinate distance between two successive crests or valleys of the same light wave as measured by an observer at $\vec{r}'_2$. In analogy with geometrical optics we may write:

$$|\Delta\vec{r}'_s|n(\vec{r}'_s) = |\Delta\vec{r}'_o|n(\vec{r}'_o). \tag{6.128}$$

From (6.6), the physical wavelength λ is given by

$$\lambda(\vec{r}') = \Gamma^{-1}(\vec{r}')|\Delta\vec{r}'| \tag{6.129}$$

and hence we have

$$\Gamma(\vec{r}'_s)\lambda(\vec{r}'_s)n(\vec{r}'_s) = \Gamma(\vec{r}'_o)\lambda(\vec{r}'_o)n(\vec{r}'_o), \tag{6.130}$$

which, using (6.8), reduces to

$$\lambda(\vec{r}'_s)\Psi^{-1}(\vec{r}'_s) = \lambda(\vec{r}'_o)\Psi^{-1}(\vec{r}'_o). \tag{6.131}$$

For any astrophysical application, the observer is far away from the source thus $\Psi(\vec{r}'_o) \approx 1$ and the redshift reduces to

$$z \equiv \frac{\lambda_o - \lambda_s}{\lambda_s} = \Psi^{-1}(\vec{r}'_s) - 1. \tag{6.132}$$

Therefore measuring the gravitational redshift of light and quantity related to it, e.g. the luminosity distance, may also allow us to test the validity of the metric corrections.

Chapter 7

Conclusions and outlook

The aim of this thesis was, on one hand, to extend the computation of quantum gravitational corrections to a wider class of metrics and, on the other hand, to find observables in order to experimentally test the theory. In the first part of the thesis, we used the linearized Einstein equations, obtained from the Barvinsky-Vilkovisky unique effective action, to compute corrections at second order in curvature to the metric of a charged star, modeled as a perfect fluid, and to gravastars and dark energy stars.

We then proceeded to study implications of the metric corrections for the field of gravitational lensing, analysing how the modified metrics affect the propagation of light rays by looking at deviations from the classical results for the photon sphere radius, the deflection angle of light rays and the gravitational redshift. This was motivated by the large amount of observations of lensing phenomena, among which stand out the observation of the photon ring around the (possibly) black holes Sgr A* and M87*. These corrected observables, if measured, could lead to further constraints on the value of the Wilson coefficients. Moreover, because of the presence of quantum hairs in the metric outside of gravastars and dark energy stars, which are instead absent for black holes as these do not receive corrections at second order in curvature, the observables we calculated may allow us to experimentally distinguish black holes from these other compact objects, adding to the existing literature of the tests proposed with this aim.

A natural extension of this work is to study rotating objects, in particular Kerr black holes, as these more closely describe astrophysical black holes. However, considerably more work will need to be done in this case. First of all, as the Kerr metric is not spherically symmetric, one has to generalize the action of the $\ln(\Box/\mu^2)$ operator on functions depending on both the radial and angular coordinates. Furthermore, the system of differential equations for the perturbation $h_{\mu\nu}$ one gets from the linearized Einstein equations will now be a system of coupled differential equations with derivatives with respect to both the radial and angular coordinate. Once these problems are solved, one can study the broad class of rotating objects filling the Universe, from regular stars to exotic objects such as rotating gravastars.

Another continuation would be to look at the third order expansion, both in the Newton constant expansion performed in computing metric corrections and in the curvature expansion in the unique effective action. However, as already second order effects are much smaller then the usual terms one finds from general relativity alone, a huge improvement in the experimental technology is needed in order to be able to detect these effects.

What the complete theory of quantum gravity is remains elusive. In this thesis we set out to explore some phenomenological implications of the Barvinsky-Vilkovisky unique effective action, in the hope that, one day, the results here obtained can help to test and eventually shine some light on the theory of quantum gravity.

Appendix A

Dealing with $\ln\left(\frac{\Box}{\mu^2}\right)$

In this appendix we show how to calculate the action of the non-local operator $\ln(\Box/\mu^2)$ on time-independent and spherically symmetric functions $f(r)$, where $r = |\vec{x}|$:

$$\ln\left(\frac{\Box}{\mu^2}\right) f(r). \tag{A.1}$$

We shall distinguish between two cases [17]:

a) if $\exists\, \epsilon > 0$ such that $f(r') = 0$ for $|r' - r| < \epsilon$, we find the result:

$$\ln\left(\frac{\Box}{\mu^2}\right) f(r) = \frac{1}{r}\int_0^\infty \left(\frac{r'}{r+r'} - \frac{r'}{|r-r'|}\right) f(r')dr'; \tag{A.2}$$

b) otherwise, if $r > 0$, $f(r) \neq 0$ and $\exists\, \epsilon > 0$ such that $f(r')$ is smooth for $|r' - r| < \epsilon$, then we find:

$$\ln\left(\frac{\Box}{\mu^2}\right) f(r) = \frac{1}{r}\int_0^\infty \frac{r'}{r+r'} f(r')dr' - \lim_{\epsilon\to 0^+}\left\{\frac{1}{r}\int_0^{r-\epsilon}\frac{r'}{r-r'}f(r')dr' + \frac{1}{r}\int_{r+\epsilon}^\infty \frac{r'}{r'-r}f(r')dr' + 2f(r)[\gamma_E + \ln(\mu\epsilon)]\right\}, \tag{A.3}$$

where γ_E is the Euler-Mascheroni constant and μ the renormalization scale.

Let us start by exploiting the time independence of the function f to express it in terms of its Fourier transform $\hat{f}$:

$$\ln\left(\frac{\Box}{\mu^2}\right) f(r) = \int \frac{d^3k}{(2\pi)^3} \ln\left(\frac{k^2}{\mu^2}\right) e^{i\vec{k}\cdot\vec{x}} \hat{f}(\vec{k}), \tag{A.4}$$

where $k = |\vec{k}|$. Next, using the spherical symmetry of f we may write $\vec{x} = (0, 0, r)$ without loss of generality so that

$$\begin{aligned}
\ln\left(\frac{\Box}{\mu^2}\right) f(r) &= \frac{1}{(2\pi)^2}\int_0^\infty dk\, k^2 \int_{-1}^{+1} d(\cos\theta) \ln\left(\frac{k^2}{\mu^2}\right) e^{ikr\cos\theta} \hat{f}(k) \\
&= \frac{1}{\pi^2 r}\int_0^\infty dk\, k \ln\left(\frac{k^2}{\mu^2}\right) \sin(kr) \hat{f}(k).
\end{aligned} \tag{A.5}$$

We can then Fourier transform back by using the relation between the Fourier and Hankel transforms for spherically symmetric functions in 3 dimensions [108]:

$$k^{1/2}\hat{f}(k) = (2\pi)^{3/2}\int_0^\infty r^{3/2} f(r) J_{1/2}(kr) dr, \tag{A.6}$$

where $J_{1/2}(kr) = \sqrt{\frac{2}{\pi kr}}\sin(kr)$ is a Bessel function. The Hankel transform of order ν of a function $f(r)$ is defined as

$$\hat{f}_\nu(k) = \int_0^\infty f(r) J_\nu(kr) r dr, \tag{A.7}$$

where J_ν is a Bessel function of the first kind of order $\nu \geq -1/2$.

In order to understand (A.6) let us consider the 3-dimensional Fourier transform of a generic function $f(\vec{r})$ defined as

$$\hat{f}(\vec{k}) = \int_{\mathbb{R}^3} f(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d\vec{r}. \tag{A.8}$$

Using the decomposition of plane waves into spherical harmonics $Y_{l,m}$:

$$e^{-i\vec{k}\cdot\vec{r}} = (2\pi)^{3/2}(kr)^{-1/2}\sum_{l=0}^{+\infty}(-i)^l J_{1/2+l}(kr)\sum_{m=-l}^{l} Y_{l,m}(\Omega_{\vec{k}}) Y^*_{l,m}(\Omega_{\vec{r}}), \tag{A.9}$$

where $\Omega_{\vec{k}}$ and $\Omega_{\vec{r}}$ are the sets of all the spherical angles in the $\vec{k}$-space and $\vec{r}$-space, we can then write the Fourier transform (A.8) in spherical coordinates as

$$\hat{f}(\vec{k}) = (2\pi)^{3/2}(k)^{-1/2}\sum_{l=0}^{+\infty}(-i)^l \sum_{m=-l}^{l} Y_{l,m}(\Omega_{\vec{k}}) \int_0^\infty J_{1/2+l}(kr) r^{3/2} dr \int f(\vec{r}) Y^*_{l,m}(\Omega_{\vec{r}}) d\Omega_{\vec{r}}. \tag{A.10}$$

If we now expand $f(\vec{r})$ and $\hat{f}(\vec{k})$ in spherical harmonics as

$$f(\vec{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{l,m}(r) Y_{l,m}(\Omega_{\vec{r}}), \quad \hat{f}(\vec{k}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} F_{l,m}(k) Y_{l,m}(\Omega_{\vec{k}}), \tag{A.11}$$

then (A.10) simplifies to (A.6) for $l = m = 0$.

We can now proceed to plug (A.6) into (A.5):

$$\begin{aligned}
\ln\left(\frac{\Box}{\mu^2}\right) f(r) &= \frac{4}{\pi r}\int_0^\infty dk \int_0^\infty dr' \ln\left(\frac{k}{\mu}\right)\sin(kr)\sin(kr')r'f(r') \\
&= \frac{1}{\pi r}\int_0^\infty dk \int_0^\infty dr' \lim_{\delta\to 0^+}\Big\{ f(r')r'\ln\left(\frac{k}{\mu}\right) \\
&\quad \times e^{-\delta k}[e^{ik(r-r')} + e^{-ik(r-r')} - e^{ik(r+r')} - e^{-ik(r+r')}]\Big\} \\
&= \frac{\mu}{\pi r}\int_0^\infty dr' \lim_{\delta\to 0^+}\int_0^\infty dq f(r')r'\ln(q) \\
&\quad \times e^{-\delta\mu q}[e^{i\mu q(r-r')} + e^{-i\mu q(r-r')} - e^{i\mu q(r+r')} - e^{-i\mu q(r+r')}],
\end{aligned} \tag{A.12}$$

where we substituted the momentum for $q = k/\mu$ and swapped the limit with the integration over the momentum in the last line. For $\mathrm{Re}(\alpha) > 0$ we have in general:

$$\int dq \ln(q) e^{-\alpha q} = -\frac{1}{\alpha}[\gamma_E + \ln(\alpha)], \tag{A.13}$$

thus we find

$$\begin{aligned}
\ln\left(\frac{\Box}{\mu^2}\right) f(r) &= \frac{\mu}{\pi r}\int_0^\infty dr' f(r')r' \lim_{\delta\to 0^+}\Big\{ -\frac{\gamma_E + \ln[\delta\mu - i\mu(r-r')]}{\delta\mu - i\mu(r-r')} - \frac{\gamma_E + \ln[\delta\mu + i\mu(r-r')]}{\delta\mu + i\mu(r-r')} \\
&\quad + \frac{\gamma_E + \ln[\delta\mu - i\mu(r+r')]}{\delta\mu - i\mu(r+r')} + \frac{\gamma_E + \ln[\delta\mu + i\mu(r+r')]}{\delta\mu + i\mu(r+r')}\Big\} \\
&= \frac{1}{\pi r}\int_0^\infty dr' f(r')r' \lim_{\delta\to 0^+}\Big[-\frac{\gamma_E + \ln(\mu R_-) - i\phi_-}{\delta - i(r-r')} - \frac{\gamma_E + \ln(\mu R_-) + i\phi_-}{\delta + i(r-r')} \\
&\quad + \frac{\gamma_E + \ln(\mu R_+) - i\phi_+}{\delta - i(r+r')} + \frac{\gamma_E + \ln(\mu R_+) + i\phi_+}{\delta + i(r+r')}\Big],
\end{aligned} \tag{A.14}$$

where $R_\pm = \sqrt{\delta^2 + (r \pm r')^2}$, $\phi_\pm = \arctan[(r \pm r')/\delta]$ and we used the property that for a complex number $z = x + iy = \sqrt{x^2+y^2}e^{i\theta}$ the logarithm is

$$\ln(z) = \ln\left(\sqrt{x^2+y^2}\right) + i\arctan\left(\frac{y}{x}\right). \tag{A.15}$$

We see that the last two terms are regular and we can take the limit $\delta \to 0^+$ directly whereas the first two terms contain a pole at $r = r'$. Here is where we have to distinguish between the two cases mentioned before:

a) since $f(r') = 0$ in a neighborhood of r, there is no pole and we find the result (A.2);

b) for $f(r) \neq 0$ but bounded and sufficiently smooth we may rewrite (A.14) as

$$\begin{aligned}\ln\left(\frac{\Box}{\mu^2}\right) f(r) =& \frac{1}{r}\int_0^\infty dr' \frac{r' f(r')}{r+r'} - \lim_{\epsilon\to 0^+} \frac{1}{r}\left\{\int_0^{r-\epsilon} dr' \frac{r' f(r')}{|r-r'|} + \int_{r+\epsilon}^\infty dr' \frac{r' f(r')}{|r-r'|}\right. \\ &\left. + \frac{1}{\pi}\int_{r-\epsilon}^{r+\epsilon} dr' f(r') r' \lim_{\delta\to 0^+}\left[\frac{\gamma_E + \ln(\mu R_-) + i\phi_-}{\delta + i(r-r')} + \frac{\gamma_E + \ln(\mu R_-) - i\phi_-}{\delta - i(r-r')}\right]\right\} \\ =& \frac{1}{r}\int_0^\infty \frac{r' f(r')}{r+r'} dr' - \frac{1}{r}\lim_{\epsilon\to 0^+}\left[\int_0^{r-\epsilon} \frac{r' f(r')}{r-r'} dr' + \int_{r+\epsilon}^\infty \frac{r' f(r')}{r'-r} dr'\right] + L_1,\end{aligned} \tag{A.16}$$

where $0 < \delta < \epsilon$ before the limits are taken. Let us now focus on the last integral, namely:

$$L_1 \equiv -\frac{1}{\pi r}\lim_{\epsilon\to 0^+}\int_{r-\epsilon}^{r+\epsilon} dr' f(r') r' \lim_{\delta\to 0^+}\left[\frac{\gamma_E + \ln(\mu R_-) + i\phi_-}{\delta + i(r-r')} + \frac{\gamma_E + \ln(\mu R_-) - i\phi_-}{\delta - i(r-r')}\right]. \tag{A.17}$$

By swapping the limit with the integral and defining a contour around the pole at $r' = r$ we find

$$\begin{aligned}L_1 =& -\frac{1}{\pi r}\lim_{\epsilon\to 0^+}\left\{\lim_{\delta\to 0^+}\int_\pi^{2\pi} i\epsilon e^{it} dt (r+\epsilon e^{it}) f(r+\epsilon e^{it})\right. \\ &\times\left.\left[\frac{\gamma_E + \ln\left(\mu\sqrt{\delta^2+\epsilon^2 e^{2it}}\right) - i\arctan\left(\frac{\epsilon e^{it}}{\delta}\right)}{\delta - i\epsilon e^{it}} + \frac{\gamma_E + \ln\left(\mu\sqrt{\delta^2+\epsilon^2 e^{2it}}\right) + i\arctan\left(\frac{\epsilon e^{it}}{\delta}\right)}{\delta + i\epsilon e^{it}}\right]\right\}.\end{aligned} \tag{A.18}$$

Using the fact that f is locally smooth we can Taylor expand it as $f(r+\epsilon e^{it}) = f(r) + \mathcal{O}(\epsilon)$. Therefore:

$$\begin{aligned}L_1 =& -\frac{f(r)}{\pi}\lim_{\epsilon\to 0^+}\left[\lim_{\delta\to 0k}\int_\pi^{2\pi} i e^{i\epsilon t} dt \frac{\gamma_E + \ln\left(\mu\sqrt{\delta^2+\epsilon^2 e^{2it}}\right) - i\arctan\left(\frac{\epsilon e^{it}}{\delta}\right)}{\delta - i\epsilon e^{it}} + \mathcal{O}(\epsilon)\right] \\ & -\frac{f(r)}{\pi}\lim_{\epsilon\to 0^+}\left[\lim_{\delta\to 0^+}\int_\pi^{2\pi} i e^{i\epsilon t} dt \frac{\gamma_E + \ln\left(\mu\sqrt{\delta^2+\epsilon^2 e^{2it}}\right) + i\arctan\left(\frac{\epsilon e^{it}}{\delta}\right)}{\delta + i\epsilon e^{it}} \mathcal{O}(\epsilon)\right] \\ =& -\frac{4f(r)}{\pi}\lim_{\epsilon\to 0^+}\left[\lim_{\delta\to 0^+}\arctan\left(\frac{\epsilon}{\delta}\right)\left[\gamma_E + \ln\left(\mu\sqrt{\delta^2+\epsilon^2}\right) + \mathcal{O}(\epsilon)\right]\right] \\ =& -2f(r)\left[\gamma_E + \ln(\mu\epsilon)\right].\end{aligned} \tag{A.19}$$

With this calculation, (A.16) reproduces the end result (A.3).

Appendix B

Variational formulae

In this appendix we report some variational formulae useful for the calculation of the modified equations of motion (2.22).

The Riemann curvature tensor is defined as

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{B.1}$$

The Riemann tensor depends only on the Christoffel symbols, therefore its variation can be calculated as

$$\delta R^{\rho}_{\sigma\mu\nu} = \partial_\mu \delta\Gamma^{\rho}_{\nu\sigma} - \partial_\nu \delta\Gamma^{\rho}_{\mu\sigma} + \delta\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\sigma} - \delta\Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\rho}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\sigma}. \tag{B.2}$$

Since $\delta\Gamma$ is the difference of two connections, it is a tensor and we can thus calculate its covariant derivative:

$$\nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\sigma\lambda}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\mu\lambda}\delta\Gamma^{\rho}_{\nu\sigma}. \tag{B.3}$$

We thus see that the variation of the Riemann tensor is given by the difference of two such terms:

$$\delta R^{\rho}_{\sigma\mu\nu} = \nabla_\mu(\delta\Gamma^{\rho}_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^{\rho}_{\mu\sigma}). \tag{B.4}$$

From the definition of the Ricci tensor it then follows that

$$\delta R_{\mu\nu} \equiv \delta R^{\rho}_{\mu\rho\nu} = \nabla_\rho(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\rho\mu}). \tag{B.5}$$

As for the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{B.6}$$

we can write its variation as

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\sigma(g^{\mu\nu}\delta\Gamma^{\sigma}_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^{\rho}_{\rho\mu}), \tag{B.7}$$

where we used the previous result for the variation of the Ricci tensor and pushed the metric inside the covariant derivative, which we are allowed to do since $\nabla_\rho g^{\mu\nu} = 0$. The last term is a total derivative and thus only yields a boundary term when integrated since the variation of the metric $\delta g^{\mu\nu}$ vanishes at infinity, leaving us with

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu}. \tag{B.8}$$

However, when the variation of the Ricci scalar is multiplied by other terms, we cannot neglect the term $g^{\mu\nu}\delta R_{\mu\nu}$. From the definition of the Christoffel symbol in terms of the metric, its variation is

$$\begin{aligned}
\delta\Gamma^\lambda_{\mu\nu} &= \delta g^{\lambda\rho} g_{\rho\alpha}\Gamma^\alpha_{\mu\nu} + \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu \delta g_{\nu\rho} + \partial_\nu \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\nu}\right) \\
&= \frac{1}{2} g^{\lambda\rho}\left(\nabla_\mu \delta g_{\nu\rho} + \nabla_\nu \delta g_{\mu\rho} - \nabla_\rho \delta g_{\mu\nu}\right) \\
&= -\frac{1}{2}\left(g_{\nu\alpha}\nabla_\mu \delta g^{\alpha\lambda} + g_{\mu\alpha}\nabla_\nu \delta g^{\alpha\lambda} - g_{\mu\alpha} g_{\nu\beta}\nabla^\lambda \delta g^{\alpha\beta}\right),
\end{aligned} \tag{B.9}$$

where we used

$$\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta}\delta g^{\alpha\beta}. \tag{B.10}$$

We then have

$$\begin{aligned}
g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} &= -\nabla_\alpha \delta g^{\alpha\lambda} + \frac{1}{2} g_{\alpha\beta}\nabla^\lambda \delta g^{\alpha\beta}, \\
g^{\mu\nu}\delta\Gamma^\rho_{\rho\mu} &= -\frac{1}{2} g_{\alpha\beta}\nabla^\nu \delta g^{\alpha\beta}.
\end{aligned} \tag{B.11}$$

From these and (B.5) we finally find

$$g^{\mu\nu}\delta R_{\mu\nu} = -\nabla_\mu\nabla_\nu \delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu}. \tag{B.12}$$

Bibliography

[1] Y. Sofue and V. Rubin, "Rotation curves of spiral galaxies," *Ann. Rev. Astron. Astrophys.*, vol. 39, pp. 137–174, 2001. doi:10.1146/annurev.astro.39.1.137.

[2] R. H. Wechsler and J. L. Tinker, "The connection between galaxies and their dark matter halos," *Annual Review of Astronomy and Astrophysics*, vol. 56, pp. 435–487, Sep. 2018. doi:10.1146/annurev-astro-081817-051756.

[3] A. G. Riess *et al.*, "Observational evidence from supernovae for an accelerating universe and a cosmological constant," *Astron. J.*, vol. 116, pp. 1009–1038, 1998. doi:10.1086/300499.

[4] S. Perlmutter, M. S. Turner, and M. White, "Constraining dark energy with type ia supernovae and large-scale structure," *Physical Review Letters*, vol. 83, no. 4, pp. 670–673, Jul. 1999. doi:10.1103/physrevlett.83.670.

[5] A. G. Riess and et al., "The farthest known supernova: Support for an accelerating universe and a glimpse of the epoch of deceleration," *The Astrophysical Journal*, vol. 560, no. 1, pp. 49–71, Oct. 2001. doi:10.1086/322348.

[6] P. J. E. Peebles and B. Ratra, "The cosmological constant and dark energy," *Reviews of Modern Physics*, vol. 75, no. 2, pp. 559–606, Apr. 2003. doi:10.1103/revmodphys.75.559.

[7] G. Hinshaw *et al.*, "Nine-year wilkinson microwave anisotropy probe (wmap) observations: Cosmological parameter results," *The Astrophysical Journal Supplement Series*, vol. 208, no. 2, p. 19, Sep. 2013. doi:10.1088/0067-0049/208/2/19.

[8] Planck Collaboration, "Planck 2015 results - XIII. Cosmological parameters," *Astronomy & Astrophysics*, vol. 594, A13, 2016. doi:10.1051/0004-6361/201525830.

[9] G. 't Hooft and M. J. G. Veltman, "One loop divergencies in the theory of gravitation," *Ann. Inst. H. Poincare A Phys. Theor.*, vol. 20, pp. 69–94, 1974.

[10] M. H. Goroff and A. Sagnotti, "The ultraviolet behavior of einstein gravity," *Nuclear Physics B*, vol. 266, no. 3, pp. 709–736, 1986, ISSN: 0550-3213. doi:10.1016/0550-3213(86)90193-8.

[11] A. O. Barvinsky and G. A. Vilkovisky, "The generalized Schwinger-DeWitt technique and the unique effective action in quantum gravity," *Physics Letters B*, vol. 131, no. 4, pp. 313–318, 1983, ISSN: 0370-2693. doi:10.1016/0370-2693(83)90506-3.

[12] A. O. Barvinsky and G. A. Vilkovisky, "The generalized Schwinger-DeWitt technique in gauge theories and quantum gravity," *Physics Reports*, vol. 119, no. 1, pp. 1–74, 1985, ISSN: 0370-1573. doi:10.1016/0370-1573(85)90148-6.

[13] A. O. Barvinsky and G. A. Vilkovisky, "Beyond the Schwinger-DeWitt technique: Converting loops into trees and in-in currents," *Nuclear Physics B*, vol. 282, pp. 163–188, 1987, ISSN: 0550-3213. doi:10.1016/0550-3213(87)90681-X.

[14] A. O. Barvinsky and G. A. Vilkovisky, "Covariant perturbation theory (II). Second order in the curvature. General algorithms," *Nuclear Physics B*, vol. 333, no. 2, pp. 471–511, 1990, ISSN: 0550-3213. doi:10.1016/0550-3213(90)90047-H.

[15] X. Calmet and B. K. El-Menoufi, "Quantum corrections to schwarzschild black hole," *European Physical Journal C*, vol. 77, no. 4, p. 243, Apr. 2017. doi:10.1140/epjc/s10052-017-4802-0.

[16] X. Calmet, "Vanishing of quantum gravitational corrections to vacuum solutions of general relativity at second order in curvature," *Physics Letters B*, vol. 787, pp. 36–38, 2018, ISSN: 0370-2693. doi:10.1016/j.physletb.2018.10.040.

[17] X. Calmet, R. Casadio, and F. Kuipers, "Quantum gravitational corrections to a star metric and the black hole limit," *Physical Review D*, vol. 100, no. 8, p. 086 010, 2019. doi:10.1103/PhysRevD.100.086010.

[18] X. Calmet, R. Casadio, and F. Kuipers, "Quantum corrected equations of motion in the interior and exterior schwarzschild spacetime," *Phys. Rev. D*, vol. 102, no. 2, Jul. 2020. doi:10.1103/PhysRevD.102.026018.

[19] X. Calmet and F. Kuipers, "Quantum gravitational corrections to the entropy of a schwarzschild black hole," *Phys. Rev. D*, vol. 104, no. 6, Sep. 2021.

[20] X. Calmet, R. Casadio, S. D. H. Hsu, and F. Kuipers, "Quantum hair from gravity," *Physical Review Letters*, vol. 128, no. 11, p. 111 301, 2022. doi:10.1103/PhysRevLett.128.111301.

[21] R. Campos Delgado, "Quantum gravitational corrections to the entropy of a reissner–nordström black hole," *The European Physical Journal C*, vol. 82, no. 3, p. 179, 2022. doi:10.1140/epjc/s10052-022-10232-0.

[22] R. Campos Delgado, "Quantum gravitational corrected evolution equations of charged black holes," *Journal of Holography Applications in Physics*, no. Online First, 2023. doi:10.22128/jhap.2023.643.1037.

[23] X. Calmet, S. D. H. Hsu, and M. Sebastianutti, "Quantum gravitational corrections to particle creation by black holes," *Physics Letters B*, vol. 841, Jun. 2023. doi:10.1016/j.physletb.2023.137820.

[24] X. Calmet, R. Casadio, S. D. H. Hsu, and F. Kuipers, "Quantum hair during gravitational collapse," *Physical Review D*, vol. 108, no. 8, p. 086 012, 2023. doi:10.1103/PhysRevD.108.086012.

[25] X. Calmet, R. Casadio, and M. Sebastianutti, "Quantum gravitational corrections in cosmology," Aug. 2024. arXiv: 2408.08229 [hep-th].

[26] J. A. R. Cembranos, "Dark matter from R^2 gravity," *Physical Review Letters*, vol. 102, no. 14, p. 141 301, Apr. 2009, ISSN: 1079-7114. doi:10.1103/physrevlett.102.141301.

[27] X. Calmet, S. Capozziello, and D. Pryer, "Gravitational effective action at second order in curvature and gravitational waves," *The European Physical Journal C*, vol. 77, no. 9, p. 589, Sep. 2017, ISSN: 1434-6052. doi:10.1140/epjc/s10052-017-5172-3.

[28] X. Calmet and B. Latosh, "Three waves for quantum gravity," *The European Physical Journal C*, vol. 78, p. 205, Mar. 2018, ISSN: 1434-6052. doi:10.1140/epjc/s10052-018-5707-2.

[29] X. Calmet and B. Latosh, "Dark matter in quantum gravity," *The European Physical Journal C*, vol. 78, no. 6, p. 520, Jun. 2018. doi:10.1140/epjc/s10052-018-6005-8.

[30] X. Calmet and F. Kuipers, "Bounds on very weakly interacting ultra light scalar and pseudoscalar dark matter from quantum gravity," *The European Physical Journal C*, vol. 80, no. 8, p. 781, Aug. 2020. doi:10.1140/epjc/s10052-020-8350-7.

[31] X. Calmet and F. Kuipers, "Theoretical bounds on dark matter masses," *Physics Letters B*, vol. 814, p. 136 068, Mar. 2021. doi:10.1016/j.physletb.2021.136068.

[32] X. Calmet and F. Kuipers, "Implications of quantum gravity for dark matter," *International Journal of Modern Physics D*, vol. 30, no. 14, p. 2 142 004, Dec. 2021.

[33] P. O. Mazur and E. Mottola, "Surface tension and negative pressure interior of a non-singular 'black hole'," *Classical and Quantum Gravity*, vol. 32, no. 21, p. 215 024, Oct. 2015. doi:10.1088/0264-9381/32/21/215024.

[34] M. Visser and D. L. Wiltshire, "Stable gravastars—an alternative to black holes?" *Classical and Quantum Gravity*, vol. 21, no. 4, p. 1135, Jan. 2004. doi:10.1088/0264-9381/21/4/027.

[35] P. O. Mazur and E. Mottola, "Gravitational condensate stars: An alternative to black holes," *Universe*, vol. 9, no. 2, p. 88, Feb. 2023, ISSN: 2218-1997. doi:10.3390/universe9020088.

[36] E. Mottola, "Gravitational vacuum condensate stars," in *Springer Series in Astrophysics and Cosmology*. Springer Nature Singapore, 2023, pp. 283–352, ISBN: 9789819915965. doi:10.1007/978-981-99-1596-5_8.

[37] F. S. N. Lobo, "Stable dark energy stars," *Classical and Quantum Gravity*, vol. 23, no. 5, pp. 1525–1541, Feb. 2006, ISSN: 1361-6382. doi:10.1088/0264-9381/23/5/006.

[38] H. Goldstein, C. P. Poole, and J. L. Safko, *Classical Mechanics*. Addison Wesley, 2002.

[39] J. Evans, K. K. Nandi, and A. Islam, "The optical-mechanical analogy in general relativity: Exact Newtonian forms for the equations of motion of particles and photons," *General Relativity and Gravitation*, vol. 28, no. 4, pp. 413–439, 1996, ISSN: 1572-9532. doi:10.1007/BF02105085.

[40] V. Perlick, "Gravitational lensing from a spacetime perspective," *Living Reviews in Relativity*, vol. 7, no. 1, Dec. 2004. doi:10.12942/lrr-2004-9.

[41] S. E. Gralla, D. E. Holz, and R. M. Wald, "Black hole shadows, photon rings, and lensing rings," *Physical Review D*, vol. 100, no. 2, p. 024 018, Jul. 2019. doi:10.1103/PhysRevD.100.024018.

[42] M. D. Schwartz, *Quantum Field Theory and the Standard Model*. Cambridge University Press, Mar. 2014, ISBN: 978-1-107-03473-0.

[43] B. S. DeWitt, "Quantum Theory of Gravity. II. The Manifestly Covariant Theory," *Phys. Rev.*, vol. 162, pp. 1195–1239, 5 Oct. 1967. doi:10.1103/PhysRev.162.1195.

[44] M. H. Goroff and A. Sagnotti, "The ultraviolet behavior of einstein gravity," *Nuclear Physics B*, vol. 266, no. 3, pp. 709–736, 1986, ISSN: 0550-3213. doi:10.1016/0550-3213(86)90193-8.

[45] CERN, *Accelerator report: Crescendo at the lhc following first stable beams at 6.8 tev*, 2023. [Online]. Available: https://home.cern/news/news/accelerators/accelerator-report-crescendo-lhc-following-first-stable-beams-68-tev.

[46] L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lüst, and A. Riotto, "Aspects of quadratic gravity: Aspects of quadratic gravity," *Fortschritte der Physik*, vol. 64, no. 2–3, pp. 176–189, Feb. 2016, ISSN: 0015-8208. doi:10.1002/prop.201500100.

[47] X. Calmet, E. Kiritsis, F. Kuipers, and D. Lust, "Matching the Vilkovisky-DeWitt Effective Action of Quantum Gravity to String Theory," 2024. arXiv: 2408.15146.

[48] B. S. DeWitt, "Quantum Theory of Gravity. I. The Canonical Theory," *Phys. Rev.*, vol. 160, pp. 1113–1148, 5 Aug. 1967. doi:10.1103/PhysRev.160.1113.

[49] B. S. DeWitt, "Quantum Theory of Gravity. 3. Applications of the Covariant Theory," *Phys. Rev.*, vol. 162, pp. 1239–1256, Dec. 1967. doi:10.1103/PhysRev.162.1239.

[50] C. D. Hoyle *et al.*, "Submillimeter tests of the gravitational inverse-square law," *Phys. Rev. D*, vol. 70, no. 4, p. 042 004, Aug. 2004. doi:10.1103/PhysRevD.70.042004.

[51] J. F. Donoghue and B. K. El-Menoufi, "Nonlocal quantum effects in cosmology: Quantum memory, nonlocal flrw equations, and singularity avoidance," *Physical Review D*, vol. 89, no. 10, p. 104 062, 2014. doi:10.1103/PhysRevD.89.104062.

[52] J. F. Donoghue and B. K. El-Menoufi, "Covariant non-local action for massless QED and the curvature expansion," *Journal of High Energy Physics*, vol. 2015, no. 10, p. 44, 2015, ISSN: 1029-8479. doi:10.1007/JHEP10(2015)044.

[53] K. Schwarzschild, "On the gravitational field of a mass point according to einstein's theory," *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin (Mathematische Physik)*, vol. 1916, pp. 189–196, 1916. arXiv: physics/9905030.

[54] R. M. Wald, *General Relativity*. University of Chicago Press, 1984. doi:10.7208/chicago/9780226870373.001.0001.

[55] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology*. Oxford: Clarendon Press, 1934.

[56] J. R. Oppenheimer and G. M. Volkoff, "On Massive Neutron Cores," *Phys. Rev.*, vol. 55, pp. 374–381, 4 Feb. 1939. doi:10.1103/PhysRev.55.374.

[57] G. D. Birkhoff and R. E. Langer, *Relativity and Modern Physics*. Harvard University Press, 1923.

[58] X. Calmet and S. D. H. Hsu, *Quantum hair in electrodynamics and gravity*, 2022. arXiv: 2209.12798 [hep-th].

[59] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. W. H. Freeman, 1973.

[60] J. Bally and E. R. Harrison, "The electrically polarized universe," *Astrophysical Journal*, vol. 220, p. 743, 1978. doi:10.1086/155961.

[61] P. I. Vysikaylo and M. A. Korotkova, "Determination of the sun's charge by the parameters of heavy ions in the solar wind," *Journal of Physics: Conference Series*, vol. 1009, no. 1, p. 012 020, 2018. doi:10.1088/1742-6596/1009/1/012020.

[62] P. S. Florides, "The complete field of a general static spherically symmetric distribution of charge," *Il Nuovo Cimento*, vol. 42 A, no. 3, pp. 343–359, 1977. doi:10.1007/BF02862400.

[63] S. Bittanti, A. J. Laub, and C. Wilems, *The Riccati Equation* (Communications and Control Engineering Series). Springer-Verlag, 1991.

[64] J. L. Synge, *Relativity: The General Theory* (North-Holland Series in Physics). North-Holland Publishing Company, 1960.

[65] S. W. Hawking and R. Penrose, "The singularities of gravitational collapse and cosmology," *Proceedings of the Royal Society A*, vol. 314, pp. 529–548, 1970. doi:10.1098/rspa.1970.0021.

[66] S. W. Hawking, “Particle creation by black holes,” *Communications in Mathematical Physics*, vol. 43, no. 3, pp. 199–220, 1975, ISSN: 1432-0916. doi:10.1007/BF02345020.

[67] S. W. Hawking, “Breakdown of predictability in gravitational collapse,” *Physical Review D*, vol. 14, no. 10, pp. 2460–2473, Nov. 1976. doi:10.1103/PhysRevD.14.2460.

[68] S. W. Hawking, “The unpredictability of quantum gravity,” *Communications in Mathematical Physics*, vol. 87, pp. 395–415, 1982. doi:10.1007/BF01206031.

[69] J. D. Bekenstein, “Black holes and entropy,” *Physical Review D*, vol. 7, no. 8, pp. 2333–2346, Apr. 1973. doi:10.1103/PhysRevD.7.2333.

[70] L. M. Martyushev and S. N. Zubarev, “Entropy production of main-sequence stars,” *Entropy*, vol. 17, no. 2, pp. 658–668, 2015, ISSN: 1099-4300. doi:10.3390/e17020658.

[71] L. M. Martyushev and S. N. Zubarev, “Entropy production of stars,” *Entropy*, vol. 17, no. 6, pp. 3645–3655, 2015, ISSN: 1099-4300. doi:10.3390/e17063645.

[72] G. ’t Hooft, “The self-screening hawking atmosphere — a different approach to quantum black hole microstates,” *Nuclear Physics B - Proceedings Supplements*, vol. 68, no. 1, pp. 174–184, 1998, Strings ’97, ISSN: 0920-5632. doi:10.1016/S0920-5632(98)00151-0.

[73] G. Chapline, E. Hohlfeld, R. B. Laughlin, and D. I. Santiago, “Quantum phase transitions and the breakdown of classical general relativity,” *International Journal of Modern Physics A*, vol. 18, pp. 3587–3590, 2003. doi:10.1142/S0217751X03016380. eprint: gr-qc/0012094.

[74] G. Chapline, *Dark energy stars*, 2005. arXiv: astro-ph/0503200 [astro-ph].

[75] I. Dymnikova, “Vacuum nonsingular black hole,” *General Relativity and Gravitation*, vol. 24, no. 3, pp. 235–242, 1992. doi:10.1007/BF00760226.

[76] V. P. Frolov, M. A. Markov, and V. F. Mukhanov, “Black holes as possible sources of closed and semiclosed worlds,” *Physical Review D*, vol. 41, no. 2, pp. 383–394, Jan. 1990. doi:10.1103/PhysRevD.41.383.

[77] E. Poisson and W. Israel, “Structure of the black hole nucleus,” *Classical and Quantum Gravity*, vol. 5, no. 12, p. L201, Dec. 1988. doi:10.1088/0264-9381/5/12/002.

[78] P. Martin-Moruno, N. M. Garcia, F. S. N. Lobo, and M. Visser, “Generic thin-shell gravastars,” *Journal of Cosmology and Astroparticle Physics*, vol. 2012, no. 03, p. 034, Mar. 2012, ISSN: 1475-7516. doi:10.1088/1475-7516/2012/03/034.

[79] C. Cattoen, T. Faber, and M. Visser, “Gravastars must have anisotropic pressures,” *Classical and Quantum Gravity*, vol. 22, no. 20, pp. 4189–4202, Sep. 2005, ISSN: 1361-6382. doi:10.1088/0264-9381/22/20/002.

[80] H. A. Buchdahl, "General relativistic fluid spheres," *Physical Review*, vol. 116, no. 4, pp. 1027–1034, Nov. 1959. doi:10.1103/PhysRev.116.1027.

[81] K. A. et al. (Event Horizon Telescope Collaboration), "First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way," *The Astrophysical Journal Letters*, vol. 930, no. 2, p. L12, May 2022. doi:10.3847/2041-8213/ac6674.

[82] K. A. et al. (Event Horizon Telescope Collaboration), "First Sagittarius A* Event Horizon Telescope Results. III. Imaging of the Galactic Center Supermassive Black Hole," *The Astrophysical Journal Letters*, vol. 930, no. 2, p. L14, May 2022. doi:10.3847/2041-8213/ac6429.

[83] K. A. et al. (Event Horizon Telescope Collaboration), "First Sagittarius A* Event Horizon Telescope Results. IV. Variability, Morphology, and Black Hole Mass," *The Astrophysical Journal Letters*, vol. 930, no. 2, p. L15, May 2022. doi:10.3847/2041-8213/ac6736.

[84] K. A. et al. (Event Horizon Telescope Collaboration), "First Sagittarius A* Event Horizon Telescope Results. VI. Testing the Black Hole Metric," *The Astrophysical Journal Letters*, vol. 930, no. 2, p. L17, May 2022. doi:10.3847/2041-8213/ac6756.

[85] K. A. et al. (Event Horizon Telescope Collaboration), "First Sagittarius A* Event Horizon Telescope Results. VII. Polarization of the Ring," *The Astrophysical Journal Letters*, vol. 964, no. 2, p. L25, Mar. 2024. doi:10.3847/2041-8213/ad2df0.

[86] K. A. et al. (Event Horizon Telescope Collaboration), "First Sagittarius A* Event Horizon Telescope Results. VIII. Physical Interpretation of the Polarized Ring," *The Astrophysical Journal Letters*, vol. 964, no. 2, p. L26, Mar. 2024. doi:10.3847/2041-8213/ad2df1.

[87] K. A. et al. (Event Horizon Telescope Collaboration), "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole," *The Astrophysical Journal Letters*, vol. 875, no. 1, p. L1, Apr. 2019. doi:10.3847/2041-8213/ab0ec7.

[88] K. A. et al. (Event Horizon Telescope Collaboration), "First M87 Event Horizon Telescope Results. IV. Imaging the Central Supermassive Black Hole," *The Astrophysical Journal Letters*, vol. 875, no. 1, p. L4, Apr. 2019. doi:10.3847/2041-8213/ab0e85.

[89] K. A. et al. (Event Horizon Telescope Collaboration), "First M87 Event Horizon Telescope Results. VI. The Shadow and Mass of the Central Black Hole," *The Astrophysical Journal Letters*, vol. 875, no. 1, p. L6, Apr. 2019. doi:10.3847/2041-8213/ab1141.

[90] K. A. et al. (Event Horizon Telescope Collaboration), "First M87 Event Horizon Telescope Results. VII. Polarization of the Ring," *The Astrophysical Journal Letters*, vol. 910, no. 1, p. L12, Jan. 2021. doi:10.3847/2041-8213/abe71d.

[91] V. Cardoso and P. Pani, “Testing the nature of dark compact objects: A status report,” *Living Reviews in Relativity*, vol. 22, no. 1, p. 4, 2019, ISSN: 1433-8351. doi:10.1007/s41114-019-0020-4.

[92] C. B. M. H. Chirenti and L. Rezzolla, “How to tell a gravastar from a black hole,” *Classical and Quantum Gravity*, vol. 24, no. 16, pp. 4191–4206, Jul. 2007. doi:10.1088/0264-9381/24/16/013.

[93] A. E. Broderick and R. Narayan, “Where are all the gravastars? limits upon the gravastar model from accreting black holes,” *Classical and Quantum Gravity*, vol. 24, no. 3, pp. 659–666, 2007. doi:10.1088/0264-9381/24/3/009.

[94] P. Pani, E. Berti, V. Cardoso, Y. Chen, and R. Norte, “Gravitational wave signatures of the absence of an event horizon: Nonradial oscillations of a thin-shell gravastar,” *Physical Review D*, vol. 80, no. 12, p. 124 047, Dec. 2009. doi:10.1103/PhysRevD.80.124047.

[95] V. Cardoso, E. Franzin, and P. Pani, “Is the gravitational-wave ringdown a probe of the event horizon?” *Physical Review Letters*, vol. 116, no. 17, p. 171 101, Apr. 2016. doi:10.1103/physrevlett.116.171101.

[96] V. Cardoso, S. Hopper, C. F. B. Macedo, C. Palenzuela, and P. Pani, “Gravitational-wave signatures of exotic compact objects and of quantum corrections at the horizon scale,” *Physical Review D*, vol. 94, p. 084 031, 8 Oct. 2016. doi:10.1103/PhysRevD.94.084031.

[97] J. L. Rosa, D. S. J. Cordeiro, C. F. B. Macedo, and F. S. N. Lobo, “Observational imprints of gravastars from accretion disks and hot spots,” *Physical Review D*, vol. 109, no. 8, p. 084 002, Apr. 2024. doi:10.1103/physrevd.109.084002.

[98] W. Israel, “Singular hypersurfaces and thin shells in general relativity,” *Il Nuovo Cimento B*, vol. 44, no. 1, pp. 1–14, 1966. doi:10.1007/BF02710419.

[99] K. Lanczos, “Flächenhafte verteilung der materie in der einsteinschen gravitationstheorie,” *Annalen der Physik*, vol. 379, no. 14, pp. 518–540, 1924. doi:10.1002/andp.19243791403.

[100] N. Sen, “Über die grenzbedingungen des schwerefeldes an unstetigkeitsflächen,” *Annalen der Physik*, vol. 378, no. 5–6, pp. 365–396, 1924. doi:10.1002/andp.19243780505.

[101] F. W. Dyson, A. S. Eddington, and C. Davidson, “A determination of the deflection of light by the sun’s gravitational field, from observations made at the total eclipse of may 29, 1919,” *Philosophical Transactions of the Royal Society A*, vol. 220, p. 291, 1920. doi:10.1098/rsta.1920.0009.

[102] J. Evans and M. Rosenquist, “F = ma optics,” *American Journal of Physics*, vol. 54, no. 10, pp. 876–883, 1986.

[103] J. Evans, “Simple forms for equations of rays in gradient-index lenses,” *American Journal of Physics*, vol. 58, no. 8, pp. 773–778, 1990.

[104] S. Harris, “Conformally stationary spacetimes,” *Classical and Quantum Gravity*, vol. 9, no. 7, p. 1823, Jul. 1992.

[105] W. Hasse and V. Perlick, “Gravitational lensing in spherically symmetric static spacetimes with centrifugal force reversal,” *General Relativity and Gravitation*, vol. 34, no. 3, pp. 415–433, Mar. 2002, ISSN: 1572-9532. doi:10.1023/a:1015384604371.

[106] G. Lambiase, R. C. Pantig, and A. Övgün, “Traces of quantum gravitational correction at third-order curvature through the black hole shadow and particle deflection at the weak field limit,” 2024. arXiv: 2406.18292 [gr-qc].

[107] B. R. Iyer, C. V. Vishveshwara, and S. V. Dhurandhar, “Ultracompact $(\mathbf{R} < 3\mathbf{M})$ objects in general relativity,” *Classical and Quantum Gravity*, vol. 2, no. 2, p. 219, Mar. 1985. doi:10.1088/0264-9381/2/2/013.

[108] J. E. Avery, *Hyperspherical harmonics and their physical applications*. World Scientific, 2017, ISBN: 978-981-322-930-3.