Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE

Corso di Laurea in Matematica

Whitehead product

and configuration spaces

Tesi di Laurea in Topologia Algebrica

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Introduction

Algebraic topology is a branch of mathematics that makes use of algebraic tools to study the properties of topological spaces. The central idea is that certain properties do not depend on the exact "shape" of a space, but rather on topological invariants that can be classified through algebraic concepts. One such example of describing topological properties using algebraic tools is the *fundamental group*, whose elements are equivalence classes under homotopy of loops in the space, based at a fixed point. Specifically, two loops are considered equivalent if one can be continuously deformed into the other while keeping the base point fixed. Although the fundamental group gives valuable insights into distinguishing different topological spaces, it has limitations. For instance, it cannot distinguish between spheres S^n and S^m for $n \neq m$ and $n, m \geq 2$. This is one of the reasons that led to the generalization of the fundamental group: instead of considering maps from S^1 to the space in exam X, one considers maps from spheres of higher dimension S^n to X for $n \geq 2$.

These *higher homotopy groups*, usually denoted $\pi_n(X)$, are deeply connected to other constructions in algebraic topology, such as homology groups. For example, if X is connected, then the first homology group $H_1(X)$ is isomorphic to the abelianization of the fundamental group $\pi_1(X, x_0)$. More generally, the Hurewicz theorem states that the first nonzero homotopy group $\pi_n(X)$ of a simply-connected space X is isomorphic to the first nonzero homology group $H_n(X)$. Thus, the higher homotopy groups further characterize topological spaces, although they are generally much harder to compute. Notably, neither the excision property used in homology theory nor the Van Kampen theorem for π_1 hold for higher homotopy groups. However, a point of similarity with homology groups is that there exists a relative version of homotopy groups, which fits into a long exact sequence. Using this sequence along with other tools — such as *fiber bundles*, which generalize covering spaces and play a similar role for higher homotopy groups, as explained in detail by R.L. Cohen in [4] — we observe that homotopy groups exhibit complex and interesting behavior. Indeed, the *Hopf bundle* is a significant example that highlights the rich algebraic structure of the homotopy groups of spheres, which remain an active area of research to this day.

This is why the work of many mathematicians has focused on understanding the structure of higher homotopy groups of spaces. One approach to better understand these algebraic constructions is to define operations between them, such as the Pontryagin product in homology, the Massey product and Steenrod squares in cohomology, and the Samelson product for the homotopy groups of H-spaces. In this thesis, we analyze the *Whitehead product*, an operation on homotopy groups $\pi_k(X) \times \pi_h(X) \to \pi_{k+h-1}(X)$ introduced by J.H.C. Whitehead in 1941 and presented again in his work *Elements of Homotopy theory* [2], which endows the homotopy groups of a space with a graded Lie algebra structure.

As an application of the tools mentioned above, we give an example of how they can be used in the study of *configuration spaces*, denoted by $\mathbb{F}_k(M)$. Configuration spaces are useful in describing systems of points that move on a manifold without colliding. They are valuable for modeling the physical arrangements of particles in problems such as the N-body problem or, in applied fields like robotics, for motion planning of objects while avoiding collisions — such as for motion planning of the components of robot arms that must conform to a specific geometry. Additionally, configuration spaces have significant mathematical relevance, as shown by the contributions of E. Fadell to the study of their geometric and topological properties (many of which are present in his works [3] and [8]). For instance, configuration spaces are important tools in the study of braid groups (since the configuration space of k points in $\mathbb{R}^2$ is the classifying space of the pure braid group on k strands), for the homotopy classification of higher-dimensional links, and the description of iterated loop spaces.

This thesis is structured as follows. After some preliminary definitions in Chapter 1, in Chapter 2 we introduce the higher homotopy groups $\pi_n(X)$ of a topological space X and prove some of their initial properties. Then, we extend our discussion to relative homotopy groups associated with pairs of spaces (X, A), and show that they fit into a long exact sequence. Additionally, we introduce some computation techniques, focusing on how fiber bundles can be used to better understand the homotopy groups of the spaces involved. In Chapter 3 we shift focus to the Whitehead product $\pi_k(X) \times \pi_h(X) \to \pi_{k+h-1}(X)$, and observe some key properties such as biadditivity, graded commutativity and the Jacobi identity, which emphasize the connection between homotopy theory and algebraic structures. In Chapter 4 we give an introduction to the theory of configuration spaces of manifolds $\mathbb{F}_k(M)$, for which we construct a fiber bundle structure and sections. Finally, we focus on the configuration spaces in $\mathbb{R}^{n+1}$, for $n > 1$. Through the fundamental fiber sequence we show that their homotopy groups can be expressed as sums of the homotopy groups of a bouquet of spheres. We present generators of the groups and study their behavior under the action of the symmetric group, and conclude with the presentation of the Yang-Baxter relations.

Contents

Chapter 1

Definitions and notations

This section introduces the notation and provides several preliminary definitions that will be used throughout the thesis. These concepts are standard in the literature and are presented here for consistency and clarity in the subsequent analysis.

Definition 1.1 (Pair of spaces). *A pair of spaces is a pair* (X, A) *where* X *is a topological space and* A *a subspace of* X*. Given two pairs* (X, A) *and* (Y, B)*, we will dentote with* $f : (X, A) \to (Y, B)$ *a continuous function* $f : X \to Y$ *such that* $f(A) \subseteq B$*.*

If $A = \{x_0\}$, then the pair $(X, \{x_0\})$ is called a *pointed space* and is denoted by (X, x_0). Similarly, a map of pointed spaces $f : (X, x_0) \to (Y, y_0)$ is a function $f : X \to Y$ such that $f(x_0) = y_0$. We will also make use of *pointed pairs*, which are triples (X, A, x_0) where (X, A) is a topological pair and $x_0 \in A$.

Definition 1.2 (Homotopy between functions). *Given two topological spaces* X *and* Y *and given two continuous functions* $f, g : X \to Y$*, a homotopy between* f *and* g *is a continuous function* $H : X \times [0, 1] \to Y$ *such that* $H(x, 0) = f(x)$ *and* $H(x, 1) = g(x)$ *for all* $x \in X$*. We say that* f *is homotopic to* g *if there exist a homotopy between* f *and* g*, and in such case we write* $f \simeq g$*.*

If A is a subspace of X and the homotopy H is such that its restriction to A is independent of the second variable $t \in [0, 1]$, then we call it a *homotopy relative to* A, or simply a homotopy rel A. Moreover, a map that is homotopic to a constant map is said *nullhomotopic*.

Remark 1.3. We also define homotopies as a family of continuous functions $h_t : X \to Y$ for $t \in [0, 1]$ such that $h_0 = f$, $h_1 = g$ and the map $(x, t) \mapsto h_t(x)$ is continuous. The two definitions are equivalent and it can be proved setting $H(x, t) = h_t(x)$.

Definition 1.4 (Inverse path). *Given X a topological space, $x_0, x_1 \in X$ and $\gamma : [0,1] \to X$ a path in X such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$, we define the inverse path of γ as the path $\gamma^{op} : [0,1] \to X$, $\gamma^{op}(t) \coloneqq \gamma(1-t)$.*

Definition 1.5 (Covering space). *A covering space of a space X is a space $\tilde{X}$ with a map $p : \tilde{X} \to X$ such that each point $x \in X$ has an open neighborhood $U \subset X$ such that $p^{-1}(U)$ is a union of disjoint open subsets of $\tilde{X}$, each of which is mapped homeomorphically onto U by p.*

We recall some important properties of covering spaces, that will be useful later on. Proof of this properties can be found in [1, Chapter 1].

Proposition 1.6 (Homotopy lifting property). *Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$, and a map $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0 (i.e. $p \circ \tilde{f}_0 = f_0$), then there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.*

Proposition 1.7 (Lifting criterion). *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space and $f : (Y, y_0) \to (X, x_0)$ any map with Y connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

In particular, the lift $\tilde{f}$ always exists if Y is simply connected. Observe that the lifting criterion fails for spaces that are not locally path connected, as, for instance, in the following example.

Example 1.8. Let Y be the closed subset of $\mathbb{R}^2$ defined as follows:

$$Y = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \,\Big|\, x \in \left(0, \frac{1}{\pi}\right] \right\} \cup \{(0,y) \mid -1 \leq y \leq 1\} \cup C,$$

where C is a curve connecting $(0,0)$ and $(1,0)$ as in Figure 1.1. Note that Y is connected, but not locally path connected. Indeed, it is path connected, however in a small enough neighborhood of $(0,0)$, there are no paths connecting points of the form $(x, sin(1/x))$ with points in C. Let $f : Y \to S^1$ be the quotient map that collapses $\{(0,y) \mid -1 \leq y \leq 1\}$ to a point. Then, $f_*(\pi_1(Y)) \subseteq p_*(\pi_1(\mathbb{R}))$, since $\pi_1(Y) = 0$, however f does not admit lifts. Indeed, denote by $Z = Y \setminus \{0\} \times [-1,1]$, and consider $f|_Z$, homeomorphism between Z and $S^1 \setminus \{*\}$. By the lifting criterion, $f|_Z$ admits a lift s.t. $\tilde{f}|_Z : Z \to (i, i+1) \subset \mathbb{R}$ is a homeomorphism for all $i \in \mathbb{Z}$. If there existed a lift $\tilde{f}$ of f, then its restriction on Z would be $\tilde{f}|_Z$, by uniqueness of the lift. However, $\tilde{f}|_Z$ cannot be extended continuously to Y: indeed, if $\{y_n\}_{n=1}^{\infty}$ and $\{y'_n\}_{n=1}^{\infty}$ are sequences that tend to $(0,0)$ from the left and the right respectively, then $\tilde{f}(y_n) \xrightarrow{n\to\infty} i$ and $\tilde{f}(y'_n) \xrightarrow{n\to\infty} i+1$.

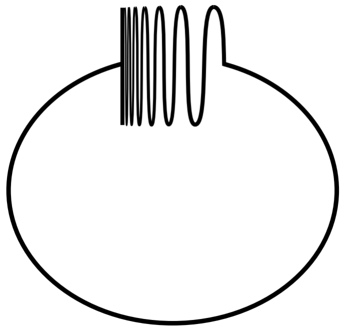

Figure 1.1: *The space Y shows that the lack of local path connectedness makes the lifting criterion fail. Figure from [1].*

A class of spaces that has proved to be very natural in the study of algebraic topology because of their intuitive behaviour with homotopy theory, among many other reasons, is the class of *CW complexes*, or *cell complexes*, which are defined as follows.

Definition 1.9 (CW complex)**.** *A topological space X is called a cell complex or CW complex if it is constructed this way:*

- *start with a discrte set X^0, whose points are called* 0*-cells;*
- *inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$ where each e^n_α is an open n-disk;*
- *one can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \bigcup_n X^n$. In the latter case, X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.*

Other important spaces widely studied in geometry, are *projective spaces.* We will make use of the complex projective line to better study the *Hopf bundle*, an important example analyzed in Section 2.3. Their definition is recalled here.

Definition 1.10 (Complex projective line)**.** *Define the following equivalence relation $\sim$ on $\mathbb{C}^2$, such that $(z_1, z_2) \sim (w_1, w_2)$ if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ for which $(z_1, z_2) =$*

$\lambda(w_1, w_2)$. *Then the set* $\mathbb{C}^2/\sim$ *is called complex projective line and denoted* $\mathbb{CP}^1$. *Its elements are equivalence classes denoted by* $[z_1 : z_2]$. *Note that* $\mathbb{CP}^1 = \{[z : 1] : z \in \mathbb{C}\} \cup [1 : 0]$.

Finally, we will make extensive use of the following construction, thanks to which we can express many algebraic concepts, and that provides a useful tool, not only in homotopy theory, but in algebraic topology in general.

Definition 1.11 (Exact sequence)**.** *A sequence of group homomorphisms*

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

is said to be exact if $\ker \alpha_n = \operatorname{Im} \alpha_{n+1}$.

An exact sequence of the kind $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is called short exact sequence. In this case, exactness means that the map α is injective, β is surjective and $\ker \beta = \operatorname{Im} \alpha$. Recall the following important lemma.

Lemma 1.12 (Splitting lemma)**.** *If* A, B *and* C *are abelian groups that fit in a short exact sequence* $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$*, the following statements are equivalent.*

i) There is a homomorphism $p : B \to A$ *such that* $p \circ \alpha = \mathrm{id_A}$.

ii) There is a homomorphism $s : C \to B$ *such that* $\beta \circ s = \mathrm{id_C}$.

iii) There is a isomorphism $\varphi : B \to A \oplus C$ *making the diagram in Figure 1.2 commutative (where the maps in the lower row are the obvious ones).*

$$\begin{tikzcd}
 & & B \arrow[dr, "\beta"] \arrow[dd, "\varphi"] & & \\
0 \arrow[r] & A \arrow[ur, "\alpha"] \arrow[dr] & & C \arrow[r] & 0 \\
 & & A \oplus C \arrow[ur] & &
\end{tikzcd}$$

Figure 1.2: *Diagram for the splitting lemma. Figure from [1].*

Proof. For the implication iii) $\Rightarrow$ ii) and i), put $\iota_C : C \hookrightarrow A \oplus C$, $\pi_C : A \oplus C \to C$ and $\iota_A : A \hookrightarrow A \oplus C$, $\pi_A : A \oplus C \to A$. Define $s = \varphi^{-1} \circ \iota_C$ and $p = \pi_A \circ \varphi$, which are the homomorphisms required. Indeed $\beta \circ s = (\pi_C \circ \varphi) \circ (\varphi^{-1} \circ \iota_C) = \pi_C \circ \iota_C = id_C$ and $p \circ \alpha = (\pi_A \circ \varphi) \circ \alpha = \pi_A \circ \iota_A = id_A$.

For the implication i) $\Rightarrow$ iii) define $\varphi(x) = (p(x), \beta(x))$. As needed, this satisfies $\varphi(\alpha(y)) = (p(\alpha(y)), \beta(\alpha(y))) = (y, 0) = \iota_A(y)$ and $\pi_C(\varphi(x)) = \pi_C(p(x), \beta(x)) = \beta(x)$.

Finally, for the implication ii) $\Rightarrow$ iii) observe that $\beta(x - s(\beta(x))) = \beta(x) - \beta(s(\beta(x))) = \beta(x) - \beta(x) = 0$, thus $x - s(\beta(x)) \in A$, by identifying $x - s(\beta(x))$ with $\alpha^{-1}(x - s(\beta(x)))$ thanks to injectivity of α. Define $\varphi(x) = (x - s(\beta(x)), \beta(X))$. Then, $\pi_C \circ \varphi = \beta$ and $\varphi(\alpha(y)) = (y - s(\beta(\alpha(y))), \beta(\alpha(y))) = (y, 0) = \iota_A(y)$. $\square$

Chapter 2

Homotopy groups

In this chapter, we introduce the *homotopy groups* of a topological space X, $\pi_n(X)$, higher-dimensional analog of the fundamental group $\pi_1(X)$. In the study of algebraic topology they represent an important tool for classifying and analyzing topological spaces. While the fundamental group captures information about loops and connectedness of the space X, higher homotopy groups provide insight about maps from the higher dimensional sphere S^n to the space. After defining the higher homotopy groups and proving their first properties, we introduce their counterpart, *relative homotopy groups*, that extend the notion to pairs of spaces (X, A). These play a critical role in homotopy theory, since they fit into a long exact sequence of homotopy groups, similar to the one we know from homology theory. Finally, we introduce some elementary computation techniques, focusing in particular on those using the construction of a *fiber bundle* structure on a space X.

2.1 First properties

Let X be a topological space and $I = [0, 1]$. In analogy with the fundamental group of X, $\pi_1(X, x_0)$, we define the n-th homotopy groups as follows.

Definition 2.1 (Homotopy group). *For a space X with basepoint $x_0 \in X$ and $n > 0$, the n-th homotopy group $\pi_n(X, x_0)$ is the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$, where homotopies $H : I^n \times I \to X$ are required to satisfy $H(s, t) = x_0$ for $s \in \partial I^n$ for all $t \in I$.*

Remark 2.2. For the case $n = 0$, we take I^0 to be a point and ∂I^0 to be empty. This way, $\pi_0(X, x_0)$ is simply the set of path-components of X.

It is easy to recognize that for $n = 1$, one gets the definition of the fundamental group. In general, for $n \geq 1$, we define a group structure on $\pi_n(X, x_0)$ by generalizing the sum operation defined on the fundamental group as follows.

Definition 2.3. *When $n \geq 2$ and $f, g : (I^n, \partial I^n) \to (X, x_0)$, the sum of f and g is defined as*

$$(f+g)(s_1, s_2, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n), & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \cdots, s_n), & s_1 \in [1/2, 1] \end{cases}$$

Remark 2.4. This operation is well defined. Indeed, if $f \simeq f'$ and $g \simeq g'$ through homotopies F and G respectively, then $f + g \simeq f' + g'$ through the homotopy

$$H(s_1, s_2, \cdots, s_n, t) \coloneqq F(s_1, s_2, \cdots, s_n, t) + G(s_1, s_2, \cdots, s_n, t)$$

In order to prove further results, especially the fact that $\pi_n(X, x_0)$ as just defined is a group, it is handy to prove the following lemma first.

Lemma 2.5 (Domain shrinking)**.** *For all $f : (I^n, \partial I^n) \to (X, x_0)$ and for all $\varepsilon > 0$, f is homotopic to the map $f_\varepsilon : (I^n, \partial I^n) \to (X, x_0)$ defined as*

$$f_\varepsilon(s_1, s_2, \cdots, s_n) \coloneqq \begin{cases} x_0, & \textit{if} \quad \exists i : |s_i| < \varepsilon \quad \textit{or} \quad |s_i| > 1 - \varepsilon \\ f(\frac{s_1 - \varepsilon}{1 - 2\varepsilon}, \cdots, \frac{s_n - \varepsilon}{1 - 2\varepsilon}), & \textit{elsewhere} \end{cases}$$

Proof. The following homotopy:

$$F(s_1, s_2, \cdots, s_n, t) \coloneqq f_{t\varepsilon}(s_1, s_2, \cdots, s_n)$$

proves the lemma. □

For instance, Figure 2.1 gives a visual representation of the domain shrinkage for $n = 2$.

Theorem 2.6. *The set $\pi_n(X, x_0)$ with the sum of Definition 2.3 is a group. Moreover, for $n \geq 2$, $\pi_n(X, x_0)$ is abelian.*

Proof. Since only the first coordinate is involved in the sum operation, the same approach to proving that $\pi_1(X, x_0)$ is a group still holds for $\pi_n(X, x_0)$. The identity element is

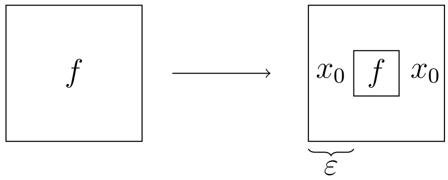

Figure 2.1: *Visual representation of the domain shrinking lemma for $n = 2$.*

the constant map sending I^n to x_0 and the inverse is given by $-f(s_1, s_2, \cdots, s_n) = f(1 - s_1, s_2, \cdots, s_n)$.

To prove that, for $n \geq 2$, $\pi_n(X, x_0)$ is abelian, we construct a homotopy between $f + g$ and $g + f$. By the shrinking domain lemma, one can shrink the domain of f and g to smaller subcubes of I^n and slide them past one another, while keeping the domains disjoint (which is possible for $n \geq 2$). Finally, using the domain shrinking lemma again, we enlarge the domains of f and g back to their original size.

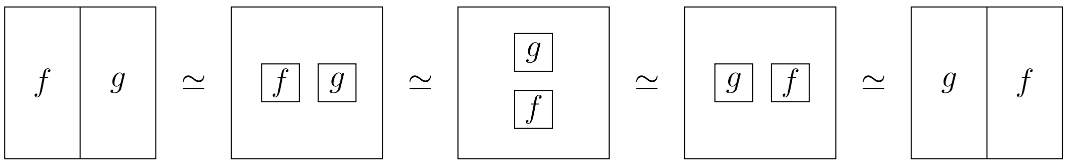

Figure 2.2: *Visual representation of the proof of abelianity of $\pi_n(X, x_0)$ for $n = 2$. Figure from [1].*

Figure 2.2 shows this process for $n = 2$. $\square$

Remark 2.7. The homotopy groups $\pi_n(X, x_0)$ can be defined as homotopy classes of maps $(S^n, s_0) \to (X, x_0)$, through homotopies of the same form $(S^n, s_0) \to (X, x_0)$. Indeed, maps $(I^n, \partial I^n) \to (X, x_0)$ are the same as maps from the quotient $I^n/\partial I^n = S^n$ to X sending the basepoint $s_0 = \partial I^n/\partial I^n$ to x_0. With this definition, the sum operation $f + g$ can be defined as follows: take the map $\pi : S^n \to S^n \vee S^n$ that collapses the equator S^{n-1} in S^n; consider the basepoint s_0 as the wedge point of the two n-spheres; compose π with the map $f \vee g$ that maps each copy of S^n into $f(S^n)$ and $g(S^n)$ respectively. The map $(f \vee g) \circ \pi$ is a representative of the element $[f + g] \in \pi_n(X, x_0)$.

We will prove that, in analogy with $\pi_1(X, x_0)$, if X is a path-connected space, the choice of different basepoints yields isomorphic homotopy groups. First, let us define the following.

Definition 2.8 (Change-of-basepoint transformation)**.** *Let X be a path-connected topological space and consider $x_0, x_1 \in X$. Given a path $\gamma : I \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$ and a map $f : (I^n, \partial I^n) \to (X, x_0)$, define the map $\tau_\gamma(f) \coloneqq \gamma f : (I^n, \partial I^n) \to (X, x_1)$, by shrinking the domain of f to a smaller cube inside I^n (similarly as in the domain shrinking lemma) and inserting the path γ in every radial segment between the inner and outer cube.*

It is easy to recognize that for $n = 1$ one gets the well-known change-of-basepoint transformation $\tau_\gamma : \pi_1(X, x_0) \to \pi_1(X, x_1)$, sending $[\alpha] \in \pi_1(X, x_0)$ to $[\gamma^{op} * \alpha * \gamma] \in \pi_1(X, x_1)$. This transformation yields an isomorphism between the fundamental groups, thus justifying the definition of $\pi_1(X) \coloneqq \pi_n(X, x_0)$, independent of the choice of $x_0 \in X$. Similarly, the following proposition holds.

Proposition 2.9. *The change-of-basepoint transformation $\tau_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_1)$ is a well-defined isomorphism. Thus, if X is path-connected, different choices of basepoint yield isomorphic homotopy groups, that may then be written simply as $\pi_n(X)$.*

Proof. The proposition can be easily proved following these steps. First, prove that τ_γ is a group homomorphism by showing that $\gamma(f + g) \simeq \gamma f + \gamma g$. Then, to prove that it is an isomoprhism, it suffices to find the homotopies $(\gamma\eta)f \simeq \gamma(\eta f)$ and $\mathbf{1}_{x_0} f \simeq f$, where $\mathbf{1}_{x_0}$ denotes the constant path at x_0. $\square$

Definition 2.10 (Action of π_1 on π_n)**.** *There exists a homomorphism $\pi_1(X, x_0) \to Aut(\pi_n(X, x_0))$ that maps $[\gamma] \in \pi_1(X, x_0)$ to τ_γ. This homomorphism is an action of π_1 on π_n.*

For $n = 1$, π_1 acts on itself by comjugation. For $n > 1$, the action makes the abelian group $\pi_n(X, x_0)$ into a module over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$, with module structure given by $(\sum_i n_i \gamma_i)\alpha = \sum_i n_i(\gamma_i \alpha)$ for $\alpha \in \pi_n(X, x_0)$.

Remark 2.11 (Functoriality of π_n). Any map $\varphi : (X, x_0) \to (Y, y_0)$ induces $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ defined as $\varphi_*([f]) = [\varphi f]$, which is a group homomorphism. It is also clear that $(\varphi\psi)_* = \varphi_* \psi_*$ and that $(id_X)_* = id_{\pi_n(X, x_0)}$. In particular, a homotopy equivalence of pointed spaces $(X, x_0) \simeq (Y, y_0)$ induces isomorphisms on all homotopy groups.

2.2 Relative homotopy groups

In this section, we extend the notion of homotopy groups to pairs of topological spaces, defining the relative homotopy groups $\pi_n(X, A, x_0)$, where $A \subseteq X$ and $x_0 \in A$. These groups capture the homotopy classes of maps that respect the subspace structure and play a fundamental role in the study of long exact sequences in algebraic topology.

Definition 2.12 (Relative homotopy group)**.** *Consider I^{n-1} as the face of I^n with the last coordinate $s_n = 0$. We define J^{n-1} as the union of the other faces of I^n, which is the closure of $\partial I^n \setminus I^{n-1}$. Let X be a topological space and $x_0 \in A \subseteq X$. For $n \geq 1$ we define $\pi_n(X, A, x_0)$ as the set of homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, where homotopies are of the same form.*

Observe that for $A = \{x_0\}$, we get $\pi_n(X, \{x_0\}, x_0) = \pi_n(X, x_0)$, thus homotopy groups can be seen as a special case of relative homotopy groups. The sum operation defined for $\pi_n(X, x_0)$ still works for $\pi_n(X, A, x_0)$. However, in the relative case, the last coordinate s_n cannot be used in the sum operation, thus $\pi_n(X, A, x_0)$ is a group only for $n \geq 2$ and it is abelian only for $n \geq 3$.

Remark 2.13. Similarly as for the absolute case, we can define $\pi_n(X, A, x_0)$ as the set of homotopy classes of maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$. Indeed, collapsing J^{n-1} turns $(I^n, \partial I^n, J^{n-1})$ into (D^n, S^{n-1}, s_0).

Remark 2.14. Any map $\varphi : (X, A, x_0) \to (Y, B, y_0)$ induces $\varphi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$, which is a group homomorphism for $n \geq 2$ and satisfies the same properties of the absolute case, but with homotopies of the form $(X, A, x_0) \to (Y, B, y_0)$.

Relative homotopy groups are important because they fit into a long exact sequence. We will prove this result applying repeatedly the following lemma, which gives good intuition about who trivial elements of $\pi_n(X, A, x_0)$ are.

Lemma 2.15 (Compression criterion)**.** *A map $f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represents zero in $\pi(X, A, x_0)$ if and only if it is homotopic rel S^{n-1} to a map with image contained in A.*

Proof. Assume that $f_1 : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ is the map with image contained in A homotopic rel S^{n-1} to f. Then $[f] = [f_1] \in \pi_n(X, A, x_0)$. However, since D^n retracts onto s_0, composing f_1 with the deformation retraction we get that $[f] = [f_1] = 0$.

Now assume that $[f] = 0$ and let $F : D^n \times I \to X$ be the homotopy between f and 0. Consider the family of maps $\{F|_{D^n \times \{t\} \cup S^{n-1} \times [0,t]}\}_{t \in I}$ with domains homeomorphic to D^n. By Remark 1.3, this family defines a homotopy stationary on S^{n-1} between $f = F|_{D \times \{0\}}$ and a map into A. □

Theorem 2.16. *For any $x_0 \in B \subset A \subset X$, let i and j be the inclusion maps $(A, B, x_0) \hookrightarrow (X, B, x_0)$ and $(X, B, x_0) \hookrightarrow (X, A, x_0)$, respectively. Define the boundary map ∂ as the map induced by the restriction of maps $(I^n, \partial I^n, J^{n-1})$ to I^{n-1}, which is a group homomorphism for $n \geq 2$. Then, the sequence*

$$\cdots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \to \cdots$$

is exact.

Proof. Exactness at $\pi_n(X, B, x_0)$. Observe that any map $(I^n, \partial I^n, J^{n-1}) \to (A, B, x_0)$ is in particular a map $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ with image contained in A. Thus, by the compression criterion, it represents zero in $\pi_n(X, A, x_0)$. This implies that $j_* i_* = 0$. We now check that $\ker j_* \subset \operatorname{Im} i_*$. Any $f : (I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$ that represents 0 in $\pi_n(X, A, x_0)$ is, by the compression criterion, homotopic rel ∂I^n to a map $g : (I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$ with image contained in A. Thus $[g]$ is actually an element of $\pi_n(A, B, x_0)$ such that $i_*([g]) = [f]$.

Exactness at $\pi_n(X, A, x_0)$. Again, we observe that the composition $\partial j_* = 0$ since the restriction of a map $(I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$ to $I^{n-1} \subset \partial I^n$ has image contained in B and is thus trivial as an element of $\pi_{n-1}(A, B, x_0)$, thanks to the compression criterion. To prove that $\ker \partial \subset \operatorname{Im} j_*$, let us consider any map $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ such that its restriction to I^{n-1} represents zero in $\pi_{n-1}(A, B, x_0)$. Then, by the compression criterion, $f|_{I^{n-1}}$ is homotopic rel ∂I^{n-1} to a map with image in B and let $F : I^{n-1} \times I \to A$ be said homotopy. Finally, tacking F onto f, we get a map $(I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$ that is homotopic to f (via homotopies of the form $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, that that tack increasingly longer initial segments of F).

Exactness at $\pi_n(A, B, x_0)$. The composition $i_* \partial = 0$ since maps $f : (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0)$ restricted to I^n are homotopic rel ∂I^n to a constant map, via f itself using the last coordinate as the second variable of the homotopy. The final step is to demonstrate that $\ker i_* \subset \operatorname{Im} \partial$. Let $f : (I^n, \partial I^n, J^{n-1}) \to (A, B, x_0)$ be a map that represents zero in $\pi_n(X, B, x_0)$. Then, there exists F nullhomotopy of f through maps $(I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$. The restriction of F to $I^{n-1} \times I$ lies in B, but reparametrizing the n-th and

$(n+1)$-th coordinates we construct a map $G : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ such that $\partial \circ G$ is homotopic to f. A visual intuition of this construction is offered in Figure 2.3. □

(a) *How F maps the faces of I^{n+1}.* **(b)** *How G maps the faces of I^{n+1}.*

Figure 2.3: *Visual intuiton for $n = 2$ of the construction in the proof of Theorem 2.16.*

Corollary 2.17 (Exact sequence of relative homotopy groups)**.** *The exact sequence of Theorem 2.16 when $B = \{x_0\}$ reduces to the exact sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots$$

for the pair (X, A, x_0).

This exact sequence continues on to

$$\cdots \to \pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0)$$

even where group structures are not defined: the image of each map is the "kernel" of the next, i.e. the counterimage of the base point.

Remark 2.18. With samll adjustments, it can be proved that there are change-of-basepoint isomorphisms for relative homotopy groups analogous to those seen in Proposition 2.9 for the absolute case, that satisfy the same basic properties.

2.3 Some methods of calculation

We now present some elementary methods for computing homotopy groups. In most cases, these tools do not consist of a direct computation of the groups, but rather give relations with the homotopy groups of different spaces. For instance, we prove that homotopy groups behave nicely with respect to product and covering spaces.

Proposition 2.19. *If $\{X_i\}_{i\in I}$ is an arbitrary collection of path-connected spaces, then there are isomorphism $\pi_n(\prod_{i\in I} X_i) \simeq \prod_{i\in I} \pi_n(X_i)$ for all n.*

Proof. Let $p_j : \prod_{i\in I} X_i \to X_j$ be the projection to the j-th factor and define $\phi : \pi_n(\prod_{i\in I} X_i) \to \prod_{i\in I} \pi_n(X_i)$ as $\phi([f]) := ([p_i \circ f])_{i\in I}$. This is well-defined, since if $f \simeq g$ then clearly $p_i \circ f \simeq p_i \circ g$. Observe that, by the universal property of the product, any map $f : S^n \to \prod_{i\in I} X_i$ corresponds to a unique collection of maps $f_i : S^n \to X_i$, for $i \in I$. Similarly, homotopies $S^n \times I \to \prod_{i\in I} X_i$ correspond to a unique collection of homotopies $S^n \times I \to X_i$. Thus, ϕ is a bijection. To prove that it is also a group homomorphism it suffices to note that $[p_i \circ f] = [(p_i)_*(f)]$ and each $(p_i)_*$ is a group homomorphism. □

Proposition 2.20. *A covering space projection $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ induces isomorphisms $p_* : \pi_n(\tilde{X}, \tilde{x_0}) \to \pi_n(X, x_0)$ for all $n \geq 2$.*

Proof. Let $\tilde{f}$ be the representative of an element in $\ker p_*$. Then, there exists a homotopy of $f := p \circ \tilde{f}$ with a constant map. By Proposition 1.6, we can lift this homotopy to a homotopy between $\tilde{f}$ and the constant map in $\tilde{x_0}$, thus, p_* is injective. Let us now consider a representative $f : (S^n, s_0) \to (X, x_0)$ of an element of $\pi_n(X, x_0)$. By Proposition 1.7, since S^n is simply connected for $n \geq 2$, there exists a lift $\tilde{f} : (S^n, s_0) \to (\tilde{X}, \tilde{x_0})$ of f, which proves surjectivity. □

Example 2.21. In order to compute $\pi_i(S^1 \vee S^n)$ for $i \geq 2$, let us consider its universal cover: $\mathbb{R}$ with a sphere S^n_k attached at every integer point $k \in \mathbb{R}$. The universal cover is clearly homotopy equivalent to $\bigvee_k S^n_k$, thus $\pi_i(S^1 \vee S^n) \simeq \pi_i(\bigvee_k S^n_k)$ for all $i \geq 2$.

Example 2.22. Knowing that $\mathbb{R}$ is the universal cover of S^1, we can conclude that $\pi_i(S^1)$ is trivial for $i \geq 2$, since $\mathbb{R}$ is contractible.

Another important tool in the computation of homotopy groups are *fiber bundles*. Similarly to homology theory, one would wish that a short exact sequence of spaces $A \hookrightarrow B \to X/A$ would give rise to a long exact sequence of homotopy groups. Unfortunately, this does not happen, since excision doesn't hold for homotopy groups (we offer an example of its failure later in Example 2.32). However, fiber bundles offer a more "homogeneous" kind of sequence, which in specific cases gives rise to a long exact sequence of homotopy groups. Let us start by defining the homotopy lifting property, that, as seen in Proposition 1.6, holds for covering maps $p : \tilde{X} \to X$.

Definition 2.23. *A map $p : E \to B$ is said to have the homotopy lifting property with respect to a space X if, given a homotopy $H : X \times I \to B$ and a lift $\tilde{h}$ of $H(-, 0)$, there exists a homotopy $\tilde{H} : X \times I \to E$ which lifts H.*

A particular case of the homotopy lifting property is the *homotopy lifting property for a pair* (X, A), where the lifted homotopy $\tilde{H}$ is also required to extend $\tilde{K} : A \times I \to E$, a given lift of the restriction of H to A.

Definition 2.24 (Fibration)**.** *A fibration is a map $p : E \to B$ having the homotopy lifting property with respect to all spaces X. A map $p : E \to B$ that satisfies the homotopy lifting property for CW-complexes is called Serre fibration.*

Example 2.25. A simple example of fibration is the projection $B \times F \to B$. Indeed, let $H : X \times I \to B$ be a homotopy and $\tilde{h} : X \to B \times F$ be a lift of $H(-, 0)$. Then define $\tilde{H} : X \times I \to B \times F$ as $\tilde{H}(x, t) := (H(x, t), pr_2(\tilde{h}(x)))$, where $pr_2 : B \times F \to F$ is the projection on the second factor.

Theorem 2.26. *Let $p : E \to B$ be a Serre fibration over a path-connected space B. Choose base points $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then, the map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$ and the following long exact sequence holds:*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0.$$

Proof. Let us start by proving that p_* is surjective. Take any map $f : (I^n, \partial I^n) \to (B, b_0)$, representative of an element in $\pi_n(B, b_0)$. We see f as a homotopy of maps $I^{n-1} \to B$. Observe that $f|_{I^{n-1}}(I^{n-1}) = b_0$, thus we take the constant map at x_0, $\mathbf{1}_{x_0}$, as a lift of $f|_{I^{n-1}}$. Then, by the homotopy lifting property, there exists a homotopy $\tilde{f} : I^n \to E$ extending $\mathbf{1}_{x_0}$. Observing that $p \circ \tilde{f} = f$ and $f(\partial I^n) = b_0$ we conclude that $\tilde{f}(\partial I^n) \subset F$. Thus, we constructed a map of pairs $\tilde{f} : (I^n, \partial I^n) \to (E, F)$ that represents $[\tilde{f}] \in \pi_n(E, F)$, whose image under p_* is $[f] \in \pi_n(B, b_0)$.

Now we prove that p_* is injective. Take $f : (I^n, \partial I^n) \to (E, F)$, representative of an element in $\ker p_*$ and let us prove it that it represents zero in $\pi_n(E, F)$. Observe that $p \circ f : (I^n, \partial I^n) \to (B, b_0)$ is nullhomotopic (since $p_*([f]) = 0$) and let $F : (I^n, \partial I^n) \times I \to (B, b_0)$ be the homotopy with the constant map in b_0. By the homotopy lifting property, there exists a lift $\tilde{F} : I^n \times I \to E$ of F that extends f. Observe that $p \circ \tilde{F}(\partial I^n) = F(\partial I^n) = b_0$, thus $\tilde{F}(\partial I^n \times I) = \mathbf{1}_{b_0}$, the constant map at b_0. Therefore, $\tilde{F}$ is a homotopy between

f and a map with image contained in F and by the compression criterion we conclude that f represents zero in $\pi_n(E, F)$.

If we consider the long exact sequence for the pair (E, F) and plug in $\pi_n(B, b_0)$ in place of $\pi_n(E, F, x_0)$ we get the exact sequence of the thesis. The map in the sequence becomes the composition $\pi_n(E, x_0) \to \pi_n(E, F, x_0) \xrightarrow{p_*} \pi_n(B)$, that is just p_*. The ending of the sequence $\pi_0(F) \to \pi_0(E) \to 0$, comes from the hypothesis that B is path-connected. Indeed, a path in E from any arbitrary point $x \in E$ to F can be obtained by lifting a path in B from $p(x)$ to b_0. $\square$

Definition 2.27 (Fiber bundle)**.** *A continuous map $p : E \to B$ is a fiber bundle on E with fiber F if for every point $b \in B$ there is a neighborhood U and a homeomorphism $h : p^{-1}(U) \to U \times F$, that makes the diagram in Figure 2.4 commute (where pr_1 is the projection on the first factor). The fiber bundle is usually written as $F \to E \to B$, to emphasize what the fiber is. The space E is called total space, B base space and h a local trivialization of the bundle.*

$$\begin{tikzcd}
& p^{-1}(U) \arrow[dl, "h"'] \arrow[d, "p"] \\
U \times F \arrow[r, "pr_1"'] & U
\end{tikzcd}$$

Figure 2.4: *Diagram of a fiber bundle $F \to E \to B$.*

Note that, thanks to commutativity of the diagram, we know that each fiber $F_b := p^{-1}(b)$ is homeomorphic to $\{b\} \times F$ via the homeomorphism h. Thus, fibers are arranged locally as in $B \times F$, but not necessarily globally. Also note that since the first coordinate of h is p, the local trivialization is determined by its second coordinate: a map $p^{-1}(u) \to F$ which is a homeomorphism on each fiber F_b.

Remark 2.28. Fiber bundles satisfy the homotopy lifting property with respect to CW-complexes, thus they are Serre fibrations and we can apply Theorem 2.26. A proof can be found in [4, Chapter 4].

Example 2.29. A simple example of fiber bundles are covering spaces. Indeed, a fiber bundle with discrete fiber F, is a covering space.

Example 2.30. Another example of fiber bundle is the Möbius band $E = (I \times [-1, 1])/\sim$, where $\sim$ is the equivalence relation $(0, y) \sim (1, -y)$. Indeed, it is a bundle over S^1 with fiber an interval, with $p : E \to S^1$ the map induced by the projection $I \times [-1, 1] \to I$ and fiber $[-1, 1]$. Visually, take the base space S^1 to be a circle running along the center of the strip. An open neighborhood U of a point $p(e) \in S^1$ is an arc on the circle. The preimage of this arc $p^{-1}(U)$ is a slice of the band E, which is homeomorphic to a slice of a cylinder, $U \times [-1, 1]$ (the homeomorphism, simply adds some "twisting").

Example 2.31. An important example of fiber bundle is the *Hopf bundle*, $S^1 \hookrightarrow S^3 \to S^2$. Since the spheres are low dimension, we can explicitly see how this bundle works. Consider S^3 as a subset of $\mathbb{C}^2$, i.e. $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, and identify S^2 with $\mathbb{CP}^1$, the complex projective line. Define the projection $p : S^3 \to \mathbb{CP}^1 \cong S^2$ as $p(z_1, z_2) := [z_1 : z_2]$. Observe that two points $(z_1, z_2), (w_1, w_2) \in S^3$ have the same image under p if and only if there exists $\lambda \in \mathbb{C}$ such that $(z_1, z_2) = \lambda(w_1, w_2)$. Since

$$1 = |z_1|^2 + |z_2|^2 = |\lambda|^2(|w_1|^2 + |w_2|^2) = |\lambda|^2$$

we have that $|\lambda| = 1$. Therefore,

$$p^{-1}([z_1 : z_2]) = \{\lambda(z_1, z_2) : \lambda \in \mathbb{C}, |\lambda| = 1\} = \{e^{i\theta}(z_1, z_2) : -\pi < \theta \leq \pi\}$$

which is a great circle on S^3. With this, we have shown that the preimage of any point in the base space S^2 is homeomorphic to a circle S^1, the fiber. It is left to prove that, for any point in the base space, there exist a neighborhood U and a homeomorphism h that make the diagram in Figure 2.4 commute. Consider the cover of $\mathbb{CP}^1 \cong S^2$ given by $\mathbb{CP}^1 \setminus [0 : 1] \cong S^2 \setminus \{0\}$ and $\mathbb{CP}^1 \setminus [1 : 0] \cong \mathbb{R}^2$. The following homeomorphisms, $\psi_1 : S^2 \setminus \{0\} \times S^1 \to p^{-1}(S^2 \setminus \{0\})$ and $\psi_2 : \mathbb{R}^2 \times S^1 \to p^{-1}(\mathbb{R}^2 \times S^1)$, defined as

$$\psi_1(\mu, \theta) = \begin{cases} \left(\frac{e^{i\theta}}{\sqrt{1+\frac{1}{|\mu|^2}}}, \frac{e^{i\theta}}{\mu\sqrt{1+\frac{1}{|\mu|^2}}} \right), & \mu \neq \infty \\ \left(e^{i\theta}, 0\right), & \mu = \infty \end{cases}$$

and

$$\psi_2(\mu, \theta) = \left(\frac{\mu e^{i\theta}}{\sqrt{1 + |\mu|^2}}, \frac{e^{i\theta}}{\sqrt{1 + |\mu|^2}} \right)$$

where $\mu \in S^2$ corresponds to the extended complex number z_1/z_2, do just what we wanted, as shown in [5].

Applying Theorem 2.26 to the Hopf bundle, we get the following long exact sequence:

$$\cdots \to \pi_n(S^1) \to \pi_n(S^3) \to \pi_n(S^2) \to \pi_{n-1}(S^1) \to \cdots \to \pi_0(S^3) \to 0$$

and thus we have that $\pi_n(S^3) \simeq \pi_n(S^2)$ for $n \geq 3$. Moreover, thanks to cellular approximation ([1, Corollary 4.9]) it can be proved that $\pi_n(S^k) = 0$ for $n < k$, that combined with the information given by the long exact sequence, proves that $\pi_2(S^2) = \mathbb{Z}$. Another important consequence is that $\pi_3(S^2) = \mathbb{Z}$, an early example of how higher homotopy groups of spheres are not always trivial. This led to a broader investigation of homotopy groups of spheres, which remains an active area of research in topology.

Example 2.32. We show an example of failure of excision for homotopy groups, as we know it from homology theory (reference for homology theory can be found in [1, Chapter 3]). If excision held for homotopy groups too, it would lead to a contradiction for the pair (D^2, S^1): we would get that $\pi_n(D^2, S^1) \simeq \pi_n(D^2/S^1, *)$ (in analogy with [1, Proposition 2.22]). However, since D^2 is contractible, from the long exact sequence of homotopy for the pair (D^2, S^1) we get $\pi_n(D^2, S^1) \simeq \pi_n(S^1)$ which, by Example 2.22, is trivial for $n \geq 2$. On the other hand, $\pi_n(D^2/S^1, *) \simeq \pi_n(S^2)$ and from the long exact sequence for the Hopf bundle we get that $\pi_2(S^2) \simeq \mathbb{Z}$. In conclusion, we have found that $\pi_2(D^2, S^1) \simeq \{0\}$ and $\pi_2(D^2/S^1, *) \simeq \mathbb{Z}$, which is in contrast with excision.

An important result concerning the homotopy sequence of a fiber bundle arises in the case where the bundle admits a section.

Definition 2.33 (Section of a fiber bundle)**.** *If $p : E \to B$ is a fiber bundle structure on E with fiber F, a section of the fiber bundle is a continuous map $\sigma : B \to E$ such that $p(\sigma(x)) = x$ for all $x \in B$.*

Proposition 2.34. *Let $F \to E \xrightarrow{p} B$ be a fiber bundle with a section $\sigma : B \to E$. Then, for $n > 1$, $\pi_n(E) = \pi_n(F) \oplus \pi_n(B)$.*

Proof. The existence of a section grants that p_* is surjective, therefore the map $\pi_n(B) \to \pi_{n-1}(F)$ has to be the zero map, by exactness of the sequence. This also means that the map $\pi_n(F) \to \pi_n(E)$ is injective, again by exactness. Thus, we get the short exact sequence

$$0 \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to 0,$$

and the result is granted by the splitting lemma (Lemma 1.12). □

Chapter 3

Whitehead product

The Whitehead product is a bilinear operation on homotopy groups that plays a crucial role in the study of higher homotopy structures. The goal of this chapter is to give an introduction to the key properties of the Whitehead product, groundwork for its algebraic applications. After presenting its definition, we analyze the cases in which one or both factors lie in $\pi_1(X, x_0)$. Then we state some of its properties: biadditivity, graded commutativity and the Jacobi identity. Its construction provides an important bridge between homotopy theory and algebraic structures, enabling the definition of a graded Lie algebra on the homotopy groups of a topological space.

3.1 Definition

Remark 3.1. Note that the product of spheres $S^k \times S^h$ can be obtained from the wedge sum $S^k \vee S^h$ by attaching a $(k+h)$-cell. Let $\phi : S^{k+h-1} \to S^k \vee S^h$ be the attaching map.

Definition 3.2 (Whitehead product)**.** *Let $f : (S^k, *) \to (X, x_0)$ and $g : (S^h, *) \to (X, x_0)$ be the representatives of elements $\alpha = [f] \in \pi_k(X, x_0)$ and $\beta = [g] \in \pi_h(X, x_0)$, respectively. Then the map $h : S^{k+h-1} \xrightarrow{\phi} (S^k \vee S^h, *) \xrightarrow{f \vee g} (X, x_0)$ is the representative of an element $[\alpha, \beta] \coloneqq [h] \in \pi_{k+h-1}(X, x_0)$, called the Whitehead product of α and β.*

Remark 3.3. Observe that $S^{k+h-1} \simeq \partial I^{k+h} = \partial I^k \times I^h \cup_{\partial I^k \times \partial I^h} I^k \times \partial I^h$ and that $\partial I^{k+h}/(\partial I^k \times \partial I^h) \simeq I^h/\partial I^h \cup_* I^k/\partial I^h \simeq S^h \vee S^k$, where on each hypercube I^i we fix the standard orientation induced by $\mathbb{R}^i$. Therefore, the map h can be defined similarly as

$$h(x,y) := \begin{cases} f(x), & \text{if} \quad x \in I^k \quad \text{and} \quad y \in \partial I^h \\ g(y), & \text{if} \quad x \in \partial I^k \quad \text{and} \quad y \in I^h \end{cases}.$$

Proposition 3.4. *The Whitehead product* $[-,-] : \pi_k(X,x_0)\times\pi_h(X,x_0) \to \pi_{k+h-1}(X,x_0)$ *is well-defined, that is* $[\alpha,\beta]$ *only depends on the homotopy classes* α *and* β*.*

Proof. Consider f and f' two representatives of the same class $\alpha \in \pi_k(X,x_0)$ and g and g' two representatives of the element $\beta \in \pi_h(X,x_0)$. Let $F : (S^k,*) \times I \to (X,x_0)$ and $G : (S^h,*) \times I \to (X,x_0)$ be basepoint preserving homotopies between f, f' and g, g', respectively. Then, $F \vee G : (S^k \vee S^h) \times I \to (X,x_0)$,

$$F \vee G(x,t) = \begin{cases} F(x,t), & \text{if} \quad x \in S^k \\ G(x,t), & \text{if} \quad x \in S^h \end{cases}$$

is a homotopy between $f \vee g$ and $f' \vee g'$. Thus, the composition on the first factor with ϕ, $(F \vee G)(\phi(y),t)$ gives us a homotopy between h and h'. Thus $[\alpha,\beta] \in \pi_{k+h-1}(X,x_0)$ does not depend on the choice of representative for α and β. □

Remark 3.5. The Whitehead product $(\alpha,\beta) \mapsto [\alpha,\beta]$ is clearly a natural operation, meaning that, given a map $\varphi : (X,x_0) \to (Y,y_0)$, then

$$\varphi_*([\alpha,\beta]) = [\varphi_*(\alpha),\varphi_*(\beta)].$$

Let us consider the case $[-,-] : \pi_1(X,x_0) \times \pi_1(X,x_0) \to \pi_1(X,x_0)$. Then $\partial(I \times I)$ is the boundary of the unit square $I^2 \subset \mathbb{R}^2$, with clockwise orientation and with the origin as basepoint. Consider $f : (I,\partial I) \to (X,x_0)$ and $g : (I,\partial I) \to (X,x_0)$ representatives of $\alpha,\beta \in \pi_1(X,x_0)$ respectively. Then, using Remark 3.3, it is clear that $[\alpha,\beta] = \alpha\beta\alpha^{-1}\beta^{-1} \in \pi_1(X,x_0)$, the commutator of the two elements. Compare with the visual representation in Figure 3.1.

Now consider the case $k = 1 < h$, $[-,-] : \pi_k(X,x_0) \times \pi_h(X,x_0) \to \pi_{k+h-1}(X,x_0)$. In Figure 3.2 we show the case $k = 1, h = 2$. Then $\partial(I \times I^h)$ is the boundary of the hypercube I^{h+1}, oriented coherently with $\{1\} \times I^h$ and thus discoherently with $\{0\} \times I^h$. Observe that, as per Remark 3.3, the point $h(x,y)$ is independent of $y \in \partial I^h$, thus the map $h|_{I\times\{y\}}(x) = h(x,y)$ represents $\alpha \in \pi_1(X,x_0)$ for all $y \in \partial I^h$. Moreover, the map $h|_{\{1\}\times I^h} = g$, thus represents $\beta \in \pi_h(X,x_0)$. Note that $J := I \times \partial I^h \cup \{1\} \times I^h$ is homeomorphic to I^h and give it orientation coherent with $\{1\} \times I^h$. It is easy to

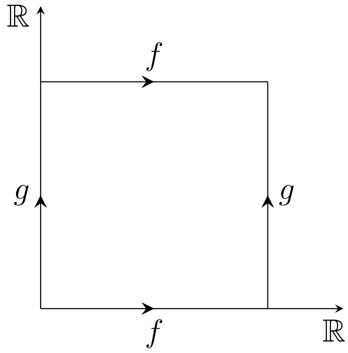

Figure 3.1: *The Whitehead product for $k = h = 1$ is the commutator of the elements. Image from [2].*

recognize that, beacuse of how $h|_{I\times\{y\}}$ behaves on $I \times \partial I^h$, then $h|_J$ coincides exactly with the construction of $\tau_\alpha(h|_{I\times\{y\}})$ defined in Definition 2.8. Since the map $h|_{\{0\}\times I^h} = g$ represents $\beta \in \pi_h(X, x_0)$, then we can conclude that $[\alpha, \beta] = \tau_\alpha(\beta) - \beta$.

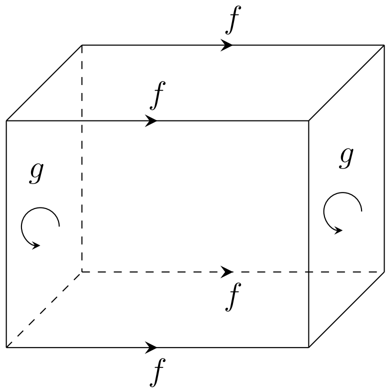

Figure 3.2: *The Whitehead product for $k = 1, h = 2$ is $\tau_\alpha(\beta) - \beta$.*

The following theorem provides a characterization of when the Whitehead product of two maps is trivial.

Theorem 3.6. *Let $\alpha \in \pi_k(X, x_0), \beta \in \pi_h(X, x_0)$. A map $d : S^k \times S^h \to X$ is said to have type (α, β) if and only if $d|_{S^k\times\{*\}}$ represents $\alpha \in \pi_k(X)$ and $d|_{\{*\}\times S^h}$ represents $\beta \in \pi_h(X)$. There exists a map $d : S^k \times S^h \to X$ of type (α, β) if and only if $[\alpha, \beta] = 0$.*

Proof. Suppose there exists such a map d and let f and g be representatives of α and β respectively. Denote by $i = i_1 \vee i_2 : S^k \vee S^h \to S^k \times S^h$ the inclusion map. Note that

$(f \vee g) = d \circ i$ and thus $(f \vee g) \circ \phi = d \circ i \circ \phi$. However, $i \circ \phi$ is homotopically trivial, because it attaches a contractible disk D^{k+h}, thus $[\alpha, \beta] = 0$.

Conversely, if $[\alpha, \beta] = 0$, then it means that $(f \vee g) \circ \phi : S^{k+h-1} \to X$ is homotopically trivial, so it extends to a map $d' : D^{k+h} \to X$. Then, by the universal property of the quotient, there exists a map $d : S^k \times S^h \to X$ such that $d' = d \circ p$ and $f \vee g = d \circ i$, where $p : D^{k+h} \to S^k \times S^h$ is the map resulting from the attachment along ϕ. □

3.2 Graded Lie algebra structure

Definition 3.7 (Lie algebra)**.** *A Lie algebra is a vector space L over a field $\mathbb{K}$ with a binary operation $[-,-] : L \times L \to L$ called Lie bracket that satisfies the following properties:*

- *bilinearity: $[ax + by, z] = a[x, z] + b[y, z]$ and $[z, ax + by] = a[z, x] + b[z, y]$ for all $a, b \in \mathbb{K}, x, y, z \in L$;*
- *antisimmetry: $[x, y] = -[y, x]$ for all $x, y \in L$;*
- *Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.*

Observe that if the field $\mathbb{K}$ does not have characteristic 2, then antisimmetry implies $[x, x] = -[x, x]$ for all $x \in L$, from which we get the following alternating property: $[x, x] = 0$ for all $x \in L$. Conversely, if we assume the alternating property, by bilinearity we can derive antisimmetry as follows: for $x, y \in L$

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

Definition 3.8 (Graded Lie algebra)**.** *A graded Lie algebra is a vector space L over a field $\mathbb{K}$ with a gradation $L = \bigoplus_{i \in \mathbb{Z}} L_i$ and a bilinear map $[-,-] : L \times L \to L$ that satisfies the following properties, graded analogues of those of a standard Lie bracket:*

- *compatibility with the gradation: $[L_i, L_j] \subseteq L_{i+j}$;*
- *graded antisymmetry: $[x, y] = -(-1)^{ij}[y, x]$ for all $x \in L_i$ and $y \in L_j$;*
- *graded Jacobi identity: $(-1)^{ik}[x, [y, z]] + (-1)^{ij}[y, [z, x]] + (-1)^{kj}[z, [x, y]] = 0$ for all $x \in L_i, y \in L_j$ and $z \in L_k$.*

We now observe that the Whitehead product endows $\pi_*(X) = \bigoplus_{i\geq 0}\pi_{i+1}(X)$ with a structure of graded Lie algebra.

Proposition 3.9 (Biadditivity)**.** *For $h > 1$, if $\alpha \in \pi_k(X, x_0)$ and $\beta_1, \beta_2 \in \pi_h(X, x_0)$, then $[\alpha, \beta_1 + \beta_2] = [\alpha, \beta_1] + [\alpha, \beta_2]$. Analogously, For $k > 1$, if $\alpha_1, \alpha_2 \in \pi_k(X, x_0)$ and $\beta \in \pi_h(X, x_0)$, then $[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta]$.*

Proof. Observe that a representative of $[\alpha, \beta_1 + \beta_2]$ is the map

$$S^{k+h-1} \xrightarrow{\phi} S^k \vee S^h \xrightarrow{\pi^h} S^k \vee S^h \vee S^h \xrightarrow{f\vee g_1 \vee g_2} X$$

where π^h is the map that collapses the equator of S^h and $S^k \vee S^h \vee S^h$ is the bouquet of the spheres. But this can also be seen as

$$S^{k+h-1} \xrightarrow{\pi^{k+h-1}} S^{k+h-1} \vee S^{k+h-1} \xrightarrow{\phi\vee\phi} (S^k \vee S^h) \vee (S^k \vee S^h) \xrightarrow{(f\vee g_1)\vee(f\vee g_2)} X$$

which represents $[\alpha, \beta_1] + [\alpha, \beta_2]$. □

Proposition 3.10 (Graded commutativity)**.** *If $\alpha \in \pi_k(X, x_0)$ and $\beta \in \pi_h(X, x_0)$, then $[\alpha, \beta] = (-1)^{kh}[\beta, \alpha]$.*

Proof. Consider the map $T : S^k \times S^h \to S^h \times S^k$ that interchanges the factors, which has degree $(-1)^{kh}$. We have defined a representative h of $[\alpha, \beta]$ as the map

$$h : S^{k+h-1} \xrightarrow{\phi} (S^k \vee S^h, *) \xrightarrow{f\vee g} (X, x_0).$$

Composing with T we get a map

$$S^{k+h-1} \xrightarrow{\phi} (S^k \vee S^h, *) \xrightarrow{T} (S^h \vee S^k, *) \xrightarrow{g\vee f} (X, x_0)$$

which is a representative of $[\beta, \alpha]$. In conclusion,

$$[\alpha, \beta] = (f \vee g) \circ \phi = (g \vee f) \circ T \circ \phi \simeq (-1)^{kh}(g \vee f) \circ \phi = (-1)^{kh}[\beta, \alpha].$$

□

Remark 3.11 (Jacobi identity). Consider $\alpha \in \pi_k(X, x_0)$, $\beta \in \pi_h(X, x_0)$ and $\gamma \in \pi_m(X, x_0)$, where $k, h, m > 1$. Then the following modified form of the Jacobi identity holds:

$$(-1)^{km}[[\alpha, \beta], \gamma] + (-1)^{hk}[[\beta, \gamma], \alpha] + (-1)^{mh}[[\gamma, \alpha], \beta] = 0.$$

Two proofs are given in [7]: the first makes use of a generalized version of the Whitehead product and is more geometrical; the second makes use of the theory of torus homotopy groups, which are constructed from the homotopy groups via groups extensions determined by the Whitehead product.

Note that the indexing of homotopy groups is shifted by 1 with respect to the graduation, i.e. $L_i = \pi_{i+1}(X)$, so that compatibility of the Whitehead product with the graduation is granted. Indeed,

$$[L_i, L_j] = [\pi_{i+1}, \pi_{j+1}] \subseteq \pi_{(i+1)+(j+1)-1} = \pi_{i+j+1} = L_{i+j}.$$

Chapter 4

Configuration spaces

In this chapter, we give an introduction to the theory of configuration spaces. We begin by examining the configuration spaces of manifolds $\mathbb{F}_k(M)$, giving a fiber bundle structure of these spaces and proving that sections can be constructed to better understand their homotopy groups. In the second part of the chapter, we focus on configuration spaces of $\mathbb{R}^{n+1}, n > 1$, and we analyze the fundamental fiber sequence that gives us information about their homotopical properties. In particular, we will show that the homotopy groups of configuration spaces can be expressed as sums of the homotopy groups of a bouquet of spheres, leading to a richer understanding of their algebraic structure. We will identify their generators and how they behave under the action of the symmetric group, in order to state the Yang-Baxter (YB) relations.

4.1 Definition and properties of $\mathbb{F}_k(M)$

Definition 4.1 (Configuration space)**.** *Let M be a connected n-manifold without boundary. The configuration space of k particles in M is the space*

$$\mathbb{F}_k(M) = \{(x_1, x_2, \cdots, x_k) \in M^k \quad | \quad x_i \neq x_j \quad \forall i \neq j\}.$$

Note that it can be regarded as the space of k noncolliding particles in M.

For $r < k$, there is a natural projection $proj_{k,r} : \mathbb{F}_k(M) \to \mathbb{F}_r(M)$ defined by $proj_{k,r}(x_1, x_2, \cdots, x_k) \coloneqq (x_1, x_2, \cdots, x_r)$. We will prove that these projections define a fiber bundle structure on $\mathbb{F}_k(M)$ with fiber defined as follows. For $(x_1, \cdots, x_k) = x \in \mathbb{F}_k(M)$, let $Q_i^x = \{x_1, \cdots, x_i\}$, where $1 \leq i \leq k$. For a fixed basepoint $q = (q_1, \cdots, q_k) \in$

$\mathbb{F}_k(M)$, we denote Q_r^q simply as Q_r. Then, the fiber is $\mathbb{F}_{k-r,r}(M) := \mathbb{F}_{k-r}(M \setminus Q_r)$. Another tool necessary for the proof is provided by the following lemma:

Lemma 4.2. *Denote with D^m the closed disk in $\mathbb{R}^m$ of radius 1 centered at the origin, and with $\mathring{D}^m$ its interior. Let $G_0(D^m)$ be the group of homeomorphisms of D^m to itself that leave the boundary pointwise fixed. There is a map $\gamma : \mathring{D}^m \to G_0(D^m)$ such that:*

- $\gamma(x)(x) = 0$, *for all* $x \in \mathring{D}^m$;
- $\gamma(x)(y) = y$, *for all* $y \in \partial D^m$.

Proof. Observe that the following maps $g : \mathring{D}^m \to \mathbb{R}^m$ and $h : \mathbb{R}^m \to \mathring{D}^m$

$$g(y) = \frac{y}{1 - |y|}, \qquad h(x) = \frac{x}{1 + |x|},$$

are homeomorphisms and $g^{-1} = h$. Define $\gamma' : \mathring{D}^m \times D^m \to D^m$ to be the map such that

$$\gamma'(x, y) = \begin{cases} h(g(y) - g(x)), & \forall y \in \mathring{D}^m \\ y, & \forall y \in \partial D^m. \end{cases}$$

Now, let $\gamma : D^m \to G_0(D^m)$ be the map that takes $x \in \mathring{D}^m$ to the homeomorphism $y \mapsto \gamma'(x, y)$. □

Definition 4.3 (Stable homeomorphism)**.** *Denote by $Top(M)$ the group of homeomorphisms $\varphi : M \to M$. A homeomorphism φ is said to be stable if it is a homeomorphism of M fixed outside some proper closed subset of M. The stable homeomorphisms generate a subgroup of $Top(M)$, denoted by $Top_s(M)$.*

Corollary 4.4. *If U is a proper open subset of M with closure $\overline{U}$ homeomorphic to D^m in such a way that $\mathring{D}^m$ corresponds to U and 0 to $x_0 \in U$, then Lemma 4.2 implies that there is a map $\gamma : U \to Top_s(M)$ such that:*

- $\gamma(x)(x) = x_0$, $\forall x \in U$;
- $\gamma(x)(y) = y$, $\forall x \in U$ *and* $y \in M \setminus U$.

Theorem 4.5. *For $r < k$, the projection $proj_{k,r} : \mathbb{F}_k(M) \to \mathbb{F}_r(M)$ defines a fiber bundle structure on $\mathbb{F}_k(M)$ with fiber $\mathbb{F}_{k-r,r}(M)$.*

Proof. Fix a point $x = (x_1, \cdots, x_r) \in \mathbb{F}_r(M)$ and choose mutually disjoint neighborhoods $U_1, \cdots, U_r \subset M$ such that $x_i \in U_i \cong \mathring{D}^m$ for all $i = 1, \cdots, r$. For each i, let $\gamma_i : U_i \to Top_s(M)$ be the map of Corollary 4.4. Identify the fiber of $proj_{k,r}$ at x with $\mathbb{F}_{k-r}(M \setminus Q_r^x)$ and observe that $U \coloneqq U_1 \times \cdots \times U_r$ is an open neighborhood of x. Define the local trivialization $\phi'_U : U \times \mathbb{F}_{k-r}(M \setminus Q_r^x) \to proj_{k,r}^{-1}(U)$ as

$$\phi'_U(x'_1, \cdots, x'_r; y) = (x'_1, \cdots, x'_r, (\gamma_1(x'_1)^{-1} \circ \cdots \circ \gamma_r(x'_r)^{-1})(y))$$

and consider the diagram in Figure 4.1 where the unnamed map is the natural projection. Clearly, ϕ'_U is a homeomorphism since every coordinate component is a home-

$$\begin{tikzcd}
 & proj_{k,r}^{-1}(U) \arrow[dl, "\phi'_U"'] \arrow[d, "proj_{k,r}"] \\
U \times \mathbb{F}_{k-r}(M \setminus Q_r^x) \arrow[r] & U
\end{tikzcd}$$

Figure 4.1: *Diagram for the bundle $proj_{k,r} : \mathbb{F}_k(M) \to \mathbb{F}_r(M)$.*

omorphism. To check that it is well-defined, let $y \in \mathbb{F}_{k-r}(M \setminus Q_r^x)$. Then, for $1 \leq j \leq k-r$, $y_j \notin \{x_1, \cdots, x_r\}$ and $\gamma_j(x_j)^{-1}$ only maps x_j to x'_j. Thus, if $y_j \in U_j$, then $\gamma_j(x'_j)^{-1}(y_j) \neq x'_j$. Therefore, by composing all $\gamma_j(x'_j)^{-1}$ and using it acting on y, $(\gamma_1(x'_1)^{-1} \circ \cdots \circ \gamma_r(x'_r)^{-1})(y) \in \mathbb{F}_{k-r}(M \setminus Q_r^{x'})$ and as a consequence, $(x'_1, \cdots, x'_r, (\gamma_1(x'_1)^{-1} \circ \cdots \circ \gamma_r(x'_r)^{-1})(y)) \in proj_{k,r}^{-1}(U)$. Note that ϕ'_U depends on U and on the choice of x. Now fix a basepoint $q = (q_1, \cdots, q_r) \in \mathbb{F}_r(M)$. Then, $\mathbb{F}_{k-r}(M \setminus Q_r^x)$ is homeomorphic to $\mathbb{F}_{k-r}(M \setminus Q_r)$. Such a homeomorphism can be constructed as follows: for every pair (q_i, x_i) consider disjoint neighborhoods $U_i \cong \mathring{D}^m$ containing q_i, x_i; consider the stable homeomorphism α_i that maps x_i to q_i, whose existence is granted by Corollary 4.4; $\alpha = (\alpha_1, \cdots, \alpha_r)$ is the homeomorphism required. Now, $\phi_U = \phi'_U \circ (1 \times \alpha)$ is a trivialization with fixed fiber $\mathbb{F}_{k-r,r}(M)$, i.e., independent of x. This argument proves the theorem. □

Finally, we state and sketch the proof of an important lemma that will be fundamental in the following discussion.

Lemma 4.6. *Let M be a connected manifold of dimension $n \geq 2$, and $Q_r = \{q_1, \cdots, q_r\} \subset M$, $r \geq 1$ be a set of r distinct points. Then, the projection*

$$proj_{k,1} : \mathbb{F}_k(M \setminus Q_r) \to M \setminus Q_r, \quad (x_1, \cdots, x_k) \mapsto x_1$$

admits a section.

Proof. Choose any point $a \in M \setminus Q_r$, and denote by L a simple arc from a to q_1, avoiding q_i for all $i \geq 2$. Then, $L \setminus Q_1$ is a closed subset of $M \setminus Q_r$ homeomorphic to the half-line $[0, +\infty) \subset \mathbb{R}$. Let $g_i : L \setminus Q_1 \to L \setminus Q_1$ denote a translation along $L \setminus Q_1$, such that $(x, g_1(x), \cdots, g_{k-1}(x)) \in \mathbb{F}_k(L \setminus Q_1)$. It can be proved that there is a retraction $\rho : M \setminus Q_r \to L \setminus Q_1$, thus the map $\sigma : M \setminus Q_r \to \mathbb{F}_k(M \setminus Q_r)$,

$$x \mapsto (x, g_1(\rho(x)), \cdots, g_{k-1}(\rho(x)))$$

is a section. $\square$

4.2 The case $\mathbb{F}_k(\mathbb{R}^{n+1})$

In this section we will study the configuration space $\mathbb{F}_k(\mathbb{R}^{n+1})$, when $n > 1$. We establish some additional notation that will be used throughout. Denote by e the unit vector $(1, 0, \cdots, 0) \in \mathbb{R}^{n+1}$ and put

$$q_1 = (0, \cdots, 0), \quad q_i = q_1 + 4(i-1)e, \quad Q_i = \{q_1, \cdots, q_i\}, i \geq 1, \quad Q_0 = \emptyset,$$

for $1 \leq i \leq k$. For the sake of brevity, we will denote $\mathbb{F}_k = \mathbb{F}_k(\mathbb{R}^{n+1})$, $\mathbb{F}_{k-r,r} = \mathbb{F}_{k-r,r}(\mathbb{R}^{n+1})$ and $\mathbb{R}^{n+1}_r = \mathbb{R}^{n+1} \setminus Q_r$.

Remark 4.7. By Section 3.2, the Whitehead product

$$\pi_p(\mathbb{F}_k) \times \pi_q(\mathbb{F}_k) \to \pi_{p+q-1}(\mathbb{F}_k)$$

turns $\pi_*(\mathbb{F}_k)$ into a graded Lie algebra.

Remark 4.8. Note that the map $\phi : \mathbb{R}^{n+1} \times \mathbb{F}_{k-1,1} \to \mathbb{F}_k$ defined as $\phi(x_1, (q_1, x_2, \cdots, x_k)) = (x_1, x_1 + x_2, \cdots, x_1 + x_k)$ induces a trivialization of the bundle

$$\mathbb{F}_{k-1,1} \to \mathbb{F}_k \to \mathbb{R}^{n+1},$$

thanks to Theorem 4.5. Since $\mathbb{R}^{n+1}$ is contractible, the long exact sequence of Theorem 2.26, grants us that $\pi_n(\mathbb{F}_k) = \pi_n(\mathbb{F}_{k-1,1})$.

$$\mathcal{F}_k : \left\{ \begin{tikzcd} \mathbb{F}_{k,0} \arrow[d] & \cdots \arrow[l] & \mathbb{F}_{k-r,r} \arrow[l] \arrow[d] & \mathbb{F}_{k-r-1,r+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[d] & \mathbb{F}_{2,k-2} \arrow[l] \arrow[d] & \mathbb{F}_{1,k-1} = \mathbb{R}^{n+1}_{k-1} \arrow[l] \\ \mathbb{R}^{n+1} & \cdots & \mathbb{R}^{n+1}_{r} & \mathbb{R}^{n+1}_{r+1} & \cdots & \mathbb{R}^{n+1}_{k-2} & \end{tikzcd} \right.$$

Figure 4.2: *The fundamental fiber sequence.*

Definition 4.9 (Fundamental fiber sequence)**.** *The sequence of fiber bundles $\mathcal{F}_k$ in Figure 4.2 is called fundamental fiber sequence. The vertical maps $p_r : \mathbb{F}_{k-r,r} \to \mathbb{R}^{n+1}_r$ are the projections such that $(q_1, \cdots, q_r, x_{r+1}, \cdots, x_k) \mapsto x_{r+1}$.*

Observe that for each r we have $\mathbb{R}^{n+1}_r \simeq (S^n \vee \cdots \vee S^n) = (S^n)^{\vee r}$. Moreover, by Lemma 4.6, the vertical maps admit sections. Immediate consequences are the following.

Remark 4.10. For $n > 1$, $\mathbb{F}_{k-r,r}$ is simply connected. Indeed, fix $n > 1$ and let us prove it proceeding by induction on $k-r$. For $k-r = 1$, it holds that $\pi_1(\mathbb{F}_{1,k-1}) = \pi_1(\mathbb{R}^{n+1}_{k-1}) = 0$. Then, assuming $\pi_1(\mathbb{F}_{k-r-1,r+1}) = 0$, from the long exact sequence of the fundamental fiber sequence, we get that $\pi_1(\mathbb{F}_{k-r,r}) = \pi_1(\mathbb{F}_{k-r-1,r+1}) \oplus \pi_1(\mathbb{R}^{n+1}_r) = 0$.

Theorem 4.11. *There is an isomorphism*

$$\pi_*(\mathbb{F}_k) \simeq \bigoplus_{r=1}^{k-1} \pi_*(\mathbb{R}^{n+1}_r) \simeq \bigoplus_{r=1}^{k-1} \pi_*((S^n)^{\vee r}).$$

Proof. The theorem is a simple consequence of the homotopy long exact sequence of a fibration in Theorem 2.26 and the fundamental fiber sequence in Figure 4.2. □

Our goal is to find generators that lead to a simple description of the Whitehead product

$$\pi_*(\mathbb{F}_k) \times \pi_*(\mathbb{F}_k) \to \pi_*(\mathbb{F}_k).$$

Definition 4.12. *For $1 \leq s \neq r \leq k$, let*

$$\alpha'_{rs} : S^n \to \mathbb{F}_{k-r+1,r-1} \subset \mathbb{F}_k$$

be the map where

$$\xi \in S^n \mapsto (q_1, \ldots, q_{r-1}, q_s + \xi, q_r, \ldots, q_{k-1}),$$

with S^n being the unit sphere in $\mathbb{R}^{n+1}$. Denote the homotopy class $[\alpha'_{rs}] \in \pi_n(\mathbb{F}_k)$ by α_{rs}.

Proposition 4.13. *The elements $\{\alpha_{rs} | 1 \leq s < r \leq k\}$ generate the group $\pi_n(\mathbb{F}_k)$.*

Proof. Observe that the composite maps

$$S^n \xrightarrow{\alpha'_{rs}} \mathbb{F}_{k-r+1,r-1} \xrightarrow{proj_{k-r+1,1}} \mathbb{R}^{n+1}_{r-1}$$

are embeddings, and put $S_{rs} = (proj_{k-r+1,1} \circ \alpha'_{rs})(S^n)$. Note that, for each pair r, s such that $1 \le s < r \le k$, the restriction of the vertical fibration

$$p_r : \mathbb{F}_{k-r+1,r-1} \to \mathbb{R}^{n+1}_{r-1}$$

in the fundamental fiber sequence (Figure 4.2) to the embedded sphere $S_{rs} \subset \mathbb{R}^{n+1}_{r-1}$ admits a section induced by the map α'_{rs}. As already observed, $\mathbb{R}^{n+1}_{r-1} \simeq (S^n)^{\vee r-1} \simeq S_{r1} \vee \cdots \vee S_{rr-1}$. Thus, by Theorem 4.11, we obtain a decomposition

$$\pi_n(\mathbb{F}_k) \cong \bigoplus_{r=2}^{k} \pi_n(S_{r1} \vee \cdots \vee S_{rr-1}).$$

It is a result in [2, Chapter XI, §8] that $\pi_n(S_{r1} \vee \cdots \vee S_{rr-1})$ is generated by the elements α_{rs} with $1 \le s < r$. Therefore, it follows that the group $\pi_n(\mathbb{F}_k)$ is generated by the elements listed in the proposition. □

With the following result, we show that the elements α_{rs}, for $r > s$, determine those when $r < s$.

Proposition 4.14. *If* $1 \le s < r \le k$, *then* $\alpha'_{sr} = (-1)^{n+1}\alpha'_{rs}$.

Proof. Define $p_{ij} : \mathbb{F}_k \to \mathbb{F}_2$ as $(x_1, x_2, \cdots, x_k) \mapsto (x_i, x_j)$. We easily verify that

$$(p_{ij})_*(\alpha_{rs}) = \begin{cases} \alpha_{21} & (r, s) = (i, j) \\ 0 & (r, s) \ne (i, j). \end{cases}$$

Indeed, $p_{rs} \circ \alpha'_{rs}(\xi) = (q_s, q_s + \xi) = \alpha'_{21}(\xi)$ and $p_{ij} \circ \alpha'_{rs}(\xi) = (x_i, x_j)$ for all $\xi \in S^n$. Since $\{\alpha_{rs} \mid 1 \le s < r \le k\}$ generate the group $\pi_n(\mathbb{F}_k)$, we see that

$$\alpha_{sr} = \sum_{1 \le j < i \le k} c_{ij}\alpha_{ij},$$

where $c_{ij} \in \mathbb{Z}$. Applying the maps p_{ij} for all $j < i$, one sees immediately that the left side of the equation goes to zero unless $(i, j) = (r, s)$ or (s, r). Hence, the same is true of the right-hand side of the equation, and we conclude that $\alpha_{sr} = \pm\alpha_{rs}, \quad s < r$. To determine the sign, consider the projection $\phi : \mathbb{F}_2 \to S^n$, where $(x_1, x_2) \mapsto (x_2 - x_1)/||x_2 - x_1||$.

Observe that $\phi \circ \alpha_{21}(\xi) = \frac{q_1+\xi-q_1}{\|q_1+\xi-q_1\|} = \xi$, whereas $\phi \circ \alpha_{12}(\xi) = \frac{q_1-q_1-\xi}{\|q_1-q_1-\xi\|} = -\xi$. Thus, the morphism ϕ_* induced on the homotopy groups takes α_{21} to the class $\iota_n \in \pi_n(S^n)$ of the identity map of S^n and α_{12} to the class of the antipodal map, which has degree $(-1)^{n+1}$. $\square$

Other useful properties arise from the study of the natural action of the symmetric group Σ_k on k letters on the homotopy groups of $\mathbb{F}_k$.

Theorem 4.15. *For $\sigma \in \Sigma_k$, the elements in $\{\alpha_{rs} | 1 \leq s < r \leq k\}$ satisfy $\sigma_* \alpha_{rs} = \alpha_{\sigma r \sigma s}$.*

Proof. Since the symmetric group Σ_k is generated by the transpositions $\tau_t = (t, t+1)$, it suffices to prove the theorem for $\sigma = \tau_t$. There are four cases to consider.

i) Assume that $\tau_t(r) \neq r$ and $\tau_t(s) = s$. Thus, $r \in \{t, t+1\}$ and $s \notin \{t, t+1\}$, meaning $s < t$. If $r = t$, observe that

$$\begin{aligned}\tau_t \alpha'_{ts}(\xi) &= \tau_t(q_1, \cdots, q_s, \cdots, q_{t-1}, q_s + \xi, q_t, \cdots) \\ &= (q_1, \cdots, q_{t-1}, q_t, q_s + \xi, q_{t+1}, \cdots).\end{aligned}$$

Thus, $(\tau_t)_*(\alpha_{ts}) = \alpha_{t+1s}$. Moreover, since τ_t is an involution, this also implies that $(\tau_t)_*(\alpha_{t+1s}) = \alpha_{ts}$.

ii) Assume that $\tau_t(r) = r$ and $\tau_t(s) \neq s$. Thus, $r \notin \{t, t+1\}$ and $s \in \{t, t+1\}$, meaning $r > t+1$. If $s = t$, observe that

$$\begin{aligned}\tau_t \alpha'_{rt}(\xi) &= \tau_t(q_1, \cdots, q_t, q_{t+1}, \cdots, q_{r-1}, q_t + \xi, q_r, \cdots) \\ &= (q_1, \cdots, q_{t+1}, q_t, \cdots, q_{r-1}, q_t + \xi, q_r, \cdots).\end{aligned}$$

Hence, using a homotopy that permutes q_t with q_{t+1} and takes the unit sphere centered at q_{t+1} to the unit sphere centered at q_t, we see that $(\tau_t)_*(\alpha_{rt}) = \alpha_{rt+1}$. Again, since τ_t is an involution, this also implies that $(\tau_t)_*(\alpha_{rt+1}) = \alpha_{rt}$.

iii) Assume that $\tau_t(r) \neq r$ and $\tau_t(s) \neq s$. Since $s < r$, $r = t+1$ and $s = t$. To prove that $\tau_t \alpha_{t+1t} = (-1)^{n+1} \alpha_{t+1t}$, observe that

$$\begin{aligned}\tau_t \alpha'_{t+1t}(\xi) &= \tau_t(q_1, \cdots, q_t, q_t + \xi, q_{t+1}, \cdots, q_{k-1}) \\ &= (q_1, \cdots, q_t + \xi, q_t, q_{t+1}, \cdots, q_{k-1}).\end{aligned}$$

The homotopy $(\xi, u) \mapsto (\cdots, q_{t-1}, q_t + (1-u)\xi, q_t - u\xi, q_{t+q}, \cdots)$ shows that $\tau_t \alpha'_{t+1t} = \alpha'_{t+1t} \circ a$, where $a : S^n \to S^n$ is the antipodal map, which gives the result we wanted.

iv) Finally, assume that $\tau_t(r) = r$ and $\tau_t(s) = s$. Thus, $r \notin \{t, t+1\}$ and $s \notin \{t, t+1\}$.Note that

$$\begin{aligned}\tau_t \alpha'_{rs}(\xi) &= \tau_t(q_1, \cdots, q_s, \cdots, q_{r-1}, q_s + \xi, q_{r+1}, \cdots) \\ &= (q_1, \cdots, q_{r-1}, q_s + \xi, q_r, \cdots),\end{aligned}$$

since neither r nor s is equal to t or $t+1$. Therefore, $(\tau_t)_*(\alpha_{rs}) = \alpha_{rs}$.

□

This tells us that the elements $\{\alpha_{rs} \in \pi_n(\mathbb{F}_k) | 1 \le s < r \le k\}$ constitute a set of generators of the group $\pi_n(\mathbb{F}_k)$ that are invariant set-wise under the action of Σ_k. We are now ready to see how these generators interact through the Whitehead product, through the Yang-Baxter relations.

Theorem 4.16 (Y-B relations)**.** *For all $\sigma \in \Sigma_k$, the following identities hold:*

i) $[\alpha_{\sigma 2 \sigma 1}, \alpha_{\sigma 3 \sigma 1} + \alpha_{\sigma 3 \sigma 2}] = 0$*, for* $k \ge 3$*,*

ii) $[\alpha_{\sigma 2 \sigma 1}, \alpha_{\sigma 4 \sigma 3}] = 0$*, for* $k \ge 4$*,*

in $\pi_*(\mathbb{F}_k)$*.*

Proof. Note that, by Remark 3.5, $\sigma_*[\alpha, \beta] = [\sigma_* \alpha, \sigma_* \beta]$. Then, thanks to Theorem 4.15, it suffices to prove that

i) $[\alpha_{21}, \alpha_{31} + \alpha_{32}] = 0$, for $k \ge 3$;

ii) $[\alpha_{21}, \alpha_{43}] = 0$, for $k \ge 4$.

To prove Statement i), consider the map $\phi : S^n \times S^n \to \mathbb{F}_k$ such that

$$(\xi_1, \xi_2) \mapsto (q_1, q_1 + \xi_1, q_1 + 5\xi_2, q_3, \cdots).$$

Then, using the same demonstrative technique as in the proof of Proposition 4.14, it is easy to verify that $[\phi|_{S^n \times \{*\}}] = \alpha_{21}$ and $[\phi|_{\{*\} \times S^n}] = \alpha_{31} + \alpha_{32}$. By Theorem 3.6, this proves the statement. The same approach holds for proving Statement ii), considering the map $\psi : S^n \times S^n \to \mathbb{F}_k$ under which

$$(\xi_1, \xi_2) \mapsto (q_1, q_1 + \xi_1, q_2, q_3 + \xi_2, q_4, \cdots).$$

Indeed, $[\psi|_{S^n \times \{*\}}] = \alpha_{21}$ and $[\psi|_{\{*\} \times S^n}] = \alpha_{43}$, and, again, Theorem 3.6 ends the proof.

□

The relation in Statement i) describes a cluster of three distinct particles bound together in motion, where one is at the center, while the other two circle around it in two different orbits. On the other hand, the relation in Statement ii), describes a formation of two pairs of distinct particles where the particles of each pair form a 2-body system, but the two systems move independently of each other.

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