Alma Mater Studiorum · University of Bologna

School of Science
Department of Physics and Astronomy
Master Degree in Physics

From AdS/CFT to dS/CFT: Anomalous dimensions of bound states

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Academic Year 2023/2024

Ad Erika,
Alla mia famiglia,
A mio nonno.

Abstract

The de Sitter spacetime is of fundamental importance in the field of cosmology, as it serves to model both the primordial and current expansion of the Universe. The quantum theory of inflation posits that the anisotropies of the cosmic microwave background and of the large-scale structure originate from primordial quantum fluctuations, which are imprinted in correlators at the end of inflation. A holographic correspondence between the de Sitter bulk and its conformal boundary would provide a powerful computational tool and a deeper understanding of a quantum field theory embedded in de Sitter space. Nevertheless, while numerous useful results in the context of AdS/CFT have already been obtained, the dS case presents a distinctive behaviour due to the conformal boundary being a spacelike hypersurface at time infinity, which gives rise to an Euclidean dual CFT. One may circumvent this problem by exploiting a map between Schwinger-Keldysh boundary propagators in dS and those in Euclidean AdS. These are linked by an analytical continuation, which becomes a simple phase shift in the Mellin-Barnes representation. From this relation, it is possible to express Schwinger-Keldysh diagrams in dS at all orders in perturbation theory as a sum of Witten diagrams in EAdS. By utilising the EAdS spacetime as an intermediary, a perturbative dS/CFT correspondence can be constructed. Following the presentation of these core results, the thesis will proceed to present an original computation achieved via the aforementioned holographic approach. The objective is to compute the anomalous dimensions of double-trace operators of the boundary CFT, which are associated with exchanges of bound states in the bulk. This boundary quantity is a means of encoding information on the stability of bound states and their mass spectrum. It will be demonstrated that even such dS anomalous dimensions can be expressed in terms of the related anomalous dimensions in the EAdS.

Lo spaziotempo di de Sitter è di fondamentale importanza nel campo della cosmologia, in quanto modella sia l'espansione primordiale che quella attuale dell'Universo. La teoria quantistica dell'inflazione sostiene che le anisotropie della radiazione cosmica di fondo e della struttura su larga scala hanno origine da fluttuazioni quantistiche primordiali, che vengono impresse nei correlatori alla fine dell'inflazione. Una corrispondenza olografica tra il bulk del de Sitter e il suo contorno conforme fornirebbe un potente strumento di calcolo e una comprensione più approfondita per una QFT nello spaziotempo di de Sitter. Tuttavia, mentre sono già stati ottenuti numerosi risultati utili nel contesto dell'AdS/CFT, il caso dS presenta un comportamento peculiare dovuto al fatto che il confine conforme è un'ipersuperficie spaziale all'infinito temporale, che dà origine a una CFT euclidea. Si può aggirare questo problema sfruttando una mappa tra i propagatori di confine (nel formalismo di Schwinger-Keldysh) nel dS e quelli nell'AdS euclideo. Questi sono legati da una continuazione analitica, che diventa un semplice spostamento di fase nella rappresentazione di Mellin-Barnes. Da questa relazione è possibile esprimere

i diagrammi di Schwinger-Keldysh in dS, a tutti gli ordini perturbativi, come una somma di diagrammi di Witten nell'EAdS. Utilizzando lo spaziotempo EAdS come intermediario è possibile costruire una corrispondenza perturbativa dS/CFT. Dopo la presentazione di questi risultati fondamentali, la tesi seguirà con la presentazione di un calcolo originale ottenuto tramite il suddetto approccio olografico. L'obiettivo è quello di calcolare le dimensioni anomale degli operatori a doppia traccia della CFT di contorno, che sono associati agli scambi di stati legati nello spaziotempo interno. Questa quantità del contorno è un mezzo per codificare informazioni sulla stabilità degli stati legati e sul loro spettro di massa. Si dimostrerà che anche queste dimensioni anomale del dS possono essere espresse in termini delle relative dimensioni anomale dell'EAdS.

Contents

Introduction

Since its first presentation in 1998 by J. Maldacena [1], the AdS/CFT correspondence has gained increasing interest in various branches of Physics, from cosmology and quantum gravity to particle and condensed matter physics, constituting a very active field of research today.
The original idea of relating a string theory weakly coupled to gravity, embedded in the bulk of a 5-dimensional Anti-de Sitter spacetime, with a strongly coupled conformal field theory embedded in the boundary at spatial infinity, soon led to the idea that we were not dealing with a mere mathematical coincidence, but with a relationship of profound physical significance. In fact, this correspondence was soon interpreted as a realisation of the so-called **holographic principle**. Conceived by Gerard t'Hooft in 1993 [2] in relation to the study of the entropy of black holes, it consists of the hypothesis that the physical information contained in a volume of space is proportional to the surface area enclosing it, and consequently encoded in a boundary theory, just as in Maldacena's work. It suggests that quantum fields in the boundary somehow codify physics of one spatial dimension: the geometrical structure, the fields' propagation and the interaction between these two, i.e. gravity.
Over the past twenty-six years, the AdS/CFT correspondence has been enhanced with significant formal outcomes, expanded to encompass more comprehensive quantum field theories and spacetime dimensions, and applied in a variety of phenomenological contexts, particularly in the field of black hole physics. A sophisticated mathematical apparatus, founded upon the powerful techniques characteristic of conformal theories, has emerged alongside a multitude of implementations in physics of fundamental interactions.

The main limitation of AdS/CFT correspondence is the fact that it regards a spacetime with a negative cosmological constant. This is the opposite situation of two of the most important periods of our Universe's history: the current era of cosmological expansion and the inflationary one. In fact, in both cases, it is described (approximately) by a **de Sitter manifold**, i.e. an exponentially expanding spacetime due to a positive cosmological constant.
However, while the peculiar geometry of AdS allows an exact conformal bootstrap of

correlators of the boundary CFT, fundamentally based on the construction of a state-operator correspondence, the dS case presents a more challenging situation, where such conditions are still not achieved.
Further, an even more cumbersome and exotic property of a possible dS/CFT correspondence is the unitarity on the boundary. Indeed, the essential difference between dS and AdS regards the nature of the conformal invariant boundary: while in AdS it is located at spatial infinity and is a timelike hypersurface, in dS it corresponds to the infinite time limit, thus to a spacelike hypersurface. In other words, a possible dual CFT in the de Sitter case would have no Minkowskian time coordinate, being instead an Euclidean theory. For this reason the causality structure of the desitterian boundary dual theory is still unclear, needing further exploration.

While on one hand this is a big complication to the construction of a holographic theory describing our Universe, from the other it offers some unique and unexpected opportunities.
For first, dS/CFT would lead to a holographic reduction of the temporal coordinate: evolution in time would be encoded in a theory without time, or, from the opposite view point, time would be *emergent* [3]. Emergence of time is taken in account in various quantum gravity/cosmology theories, like the *no-boundary proposal*; however, it is still a speculative idea, lacking of a full and accurate description.
A second and already concrete advantage coming from the dS/CFT problematic geometrical structure, regards its astonishing and immediate application to **inflationary cosmology**.

As will be presented later on, inflationary models of the primordial Universe involve an approximate desitterian expansion which, due to a phase transition mechanism, spontaneously breaks at a point in time evolution [4]. Quantum fluctuations generated during inflation are frozen in time at its end, later becoming experimentally observable[1] in what is called the Hot Big Bang. This way we have a sort of window on the primordial era, which involved very high energy scales, of the order of 10^{14} GeV, provided that we are able to compute inflationary quantum correlators in the late-time limit.
For this reason, the dS complication of forcing the dual CFT to be in the future time limit is, in the inflation theory context, a benefit[2]. The dS/CFT approach will be therefore suited for inflationary applications.

This thesis presents a specific approach used to construct a (until now) perturbative dS/CFT, valid at all orders. It is based on exploit the already rich variety of AdS/CFT

[1] At least indirectly, due to transfer processes caused by interactions emerging in the Hot Big Bang.
[2] As said, in inflation the spacetime is never exactly dS and always has a finite time t^* in which the exponential evolution breaks. This hypersurface behaves as the future time one of the exact dS for an observer out of the inflationary era looking to the past. So the holographic approach can be still applied assuming $t^* \to +\infty$.

results, building a method to map a generic Feynman diagram of a QFT embedded in a dS_{d+1} spacetime to a finite sum of Witten diagrams[3] of the same QFT embedded in an Euclidean AdS_{d+1} [5]. This map is completely general and valid for whatever diagram, at all orders in perturbation theory, including also spin and derivative interactions. It proves itself to be a powerful method to compute even multiple-loop diagrams, while the direct perturbative approach in dS gives limited results up to 1-loop.

In chapter 1, the most important characteristics of desitterian quantum field theory will be presented: starting with the dS geometrical structure and peculiarities, with a point of view on the physical implications, we will proceed to derive the field behaviour both at classical and quantum levels.
After having explored the free theory kinematics, we will introduce interactions, showing the inappropriateness of usual scattering theory in the inflationary context. Finally, the *Schwinger-Keldysh formalism* will be fully presented.

Chapter 2 will be the core of the thesis, presenting the above mentioned dS $\leftrightarrow$ EAdS method. After having reviewed some notions of AdS/CFT, we will introduce the initial and simple relation linking the two bulk theories, deriving step by step its consequences. The final result will be a direct relation between propagators of the two theories, which could be applied directly to relate diagrams.
Then, we will focus to the study of exchange diagrams, deriving their mapping relation with different approaches, the most important being the *conformal bootstrap*. Finally, the relation with the boundary CFT will be treated.

In chapter 3 will present the application of the previously built holographic approach to derive an original result still lacking in current literature: the anomalous dimensions $\gamma^{\text{dS}}_{0,l}$ of double-trace operators $[OO]_{0,l}$, with null radial quantum number, induced by a dS 4-point diagram of an exchanged scalar massive field. These quantities characterise the interactive behaviour of bound states in the bulk, providing information to their stability and mass spectrum.
We will proceed first defining formally the anomalous dimension and presenting a method to extract it from the boundary CFT. At this point the map to EAdS will be useful, making us able to proceed with the full computation of its general formula.

[3] Witten diagrams are AdS Feynman diagrams with all the external legs on its conformal boundary.

Chapter 1

Quantum fields in de Sitter

The spacetime of de Sitter is of fundamental importance in inflationary cosmology, as it describes an empty Universe with a positive cosmological constant, which in slow-roll models makes a good approximation for the early moments of Universe evolution [4].
Unlike the exact dS spacetime, where the exponential expansion never ends, in slow-roll inflation it stops at a finite time as an effect of the potential of the *inflaton* scalar field ϕ, whose gravitational coupling drives the metric evolution. A typical inflationary action, involving a minimal coupling with gravity, is:

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(\frac{M_p^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) , \tag{1.0.1}$$

where $V(\phi)$ is the potential and $g_{\mu\nu}$ is assumed to be a FLRW metric:

$$ds^2 = -dt^2 + a^2(t) \left(d\chi^2 + \Phi\left(\chi^2\right) \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right) , \tag{1.0.2}$$

$$\Phi\left(\chi^2\right) = \begin{cases} \sinh^2\chi \, , & \chi \in [0, +\infty) \, , \quad \text{open} \\ \chi^2 \, , & \chi \in [0, +\infty) \, , \quad \text{flat} \\ \sin^2\chi \, , & \chi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \, , \quad \text{close} \end{cases} \tag{1.0.3}$$

with the form of $\Phi(\chi^2)$ defining the global topology.
Hubble parameter $H(t) := \frac{\dot{a}(t)}{a(t)}$ is then dynamical and satisfies the Freedman equation:

$$H^2(t) = \frac{1}{3} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) . \tag{1.0.4}$$

An exact de Sitter Universe has $H(t) = \text{const}$ and expands at a constant exponential rate. Deviations from dS are encoded in the *slow-roll parameter*

$$\varepsilon := -\frac{\dot{H}}{H^2} , \tag{1.0.5}$$

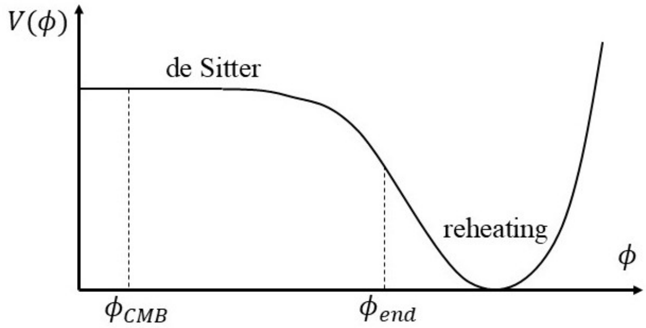

Figure 1.1: Slow-roll potential of the inflaton ϕ. de Sitter era is characterized by $V(\phi) \approx$ const and the formation of the quantum fluctuations which will generate the CMB anisotropies. These fluctuations are frozen in time evolution at ϕ_{end} due to exponential expansion. Finally, ϕ reaches a potential minimum and decays in SM particles.

with dS limit being at $\varepsilon \to 0$, dynamically verified when

$$\begin{aligned} \dot{\phi}^2 &\ll V(\phi) \ , \\ V(\phi) &\approx 0 \, . \end{aligned} \tag{1.0.6}$$

Then, in this case 1.0.4 implies $a(t) \approx \mathrm{e}^{Ht}$.
Inflation theory requires a desitterian era for an enough amount of time[1]. Slow-roll models satisfy this requirement via a potential of the type shown in Figure 1.1, having a slightly decreasing behaviour generating a long-time non-equilibrium dS era, followed by a minimum. When $\dot{\phi}^2 \approx V(\phi)$ inflation ends and ϕ is led to decay in Standard Model particles starting a process called *reheating*.

Thanks to the exponential expansion, quantum fluctuations of the inflaton (and of other possible extra fields) generated in the early stages are frozen at the end of inflation, creating the seeds for anisotropies we measure in Hot Big Bang remnants, like CMB or LSS[2].

[1] Enough time such that far points of the measured Hot Big Bang surface could have been in causal contact in the past.

[2] They refer respectively to the Cosmic Microwave Background, i.e. the last scattered electromagnetic radiation when Universe became transparent to it (recombination era), and to Large Scale Structures, such as the distributions of galaxies.

For this reason, a clear understanding and control of quantum evolution in the bulk of inflation, encoded in the **cosmological correlators**, is crucial to derive prediction for the cosmological observables, like the power spectra or the bispectra of the involved fields. An exact inflationary QFT needs to take into account the entire potential, which usually can be done through a numerical approach. Nevertheless, the dS limit is a good approximation for most models and, as will be shown in this work, allows a strong and promising control of the primordial fluctuations.

In this section, we begin by reviewing the geometry and symmetries of dS spacetime. We show then the peculiar behaviour of quantum fields embedded in it, like the past and future time limit, the instability and the unitary irreducible representations.
We will then introduce the in-in formalism, which is the key tool for computing cosmological correlators in perturbation theory and will allow us to build the dS/CFT in a later chapter.

1.1 De Sitter geometry

The most general definition of the $(d+1)$-dimensional de Sitter spacetime[3] (dS_{d+1}) is to represent it implicitly as an hyperboloid embedded in a $(d+2)$-dimensional Minkowski spacetime ($\mathcal{M}_{d+2}$) [6, 7]:

$$-\left(X^0\right)^2+\left(X^1\right)^2+\cdots+\left(X^{d+1}\right)^2=L^2\,. \tag{1.1.1}$$

For the ambient spacetime we assume the following signature convention for the metric:

$$ds^2=\eta_{MN}dX^M dX^N, \quad \eta_{MN}=\mathrm{diag}\,(-\;+\;...\;+\;+)\,, \quad M,N=0,\cdots,d+1\,. \tag{1.1.2}$$

The parameter L is called dS *radius* and is related to the expansion rate of the described spacetime. A 3D graphical representation of dS manifold for $d=2$ is shown in Figure 1.2.
Due to constraint 1.1.1, the $\mathcal{M}_{d+2}$ distance between two points can be given in terms of the $\mathcal{M}_{d+2}$ scalar product:

$$|X_1-X_2|^2=2L^2-2X_1\cdot X_2\,. \tag{1.1.3}$$

So it is common work with the **hyperbolic distance**:

$$P(X_1,X_2)=\frac{X_1\cdot X_2}{L^2}=\frac{\eta_{MN}X_1^M X_2^N}{L^2}\,, \tag{1.1.4}$$

[3]The choice of working with $(d+1)$ dimensions will be convenient in building the correspondence with a d-dimensional CFT embedded on the boundary of dS_{d+1}.

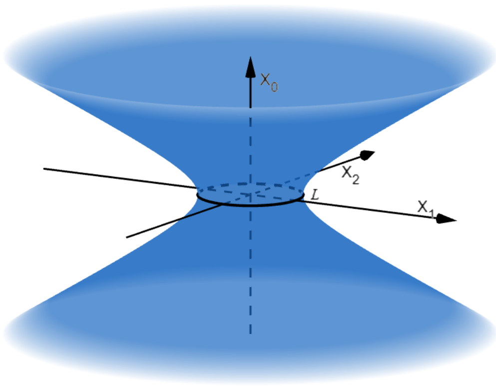

Figure 1.2: Three dimensional graphical representation of the dS_3 hyperboloid. L is the radius of the minimal S^2 spatial hypersurface (in closed slicing).

or the **geodesics distance** $D(X_1, X_2)$, defined such that:

$$P(X_1, X_2) \equiv \cos\frac{D(X_1, X_2)}{L}\,. \tag{1.1.5}$$

The manifold such defined is strongly symmetric: equation 1.1.1 is clearly invariant under the special orthochronous Lorentz group $SO(1, d+1)$, which is then inherited from $\mathcal{M}_{d+2}$ as the **isometry group** of dS_{d+1}. Also, each point of the manifold is left unchanged by a $SO(1, d)$ transformation, so that dS_{d+1} could be defined by $SO(1, d+1)/SO(1, d)$[4]. For this reason dS_{d+1} is a homogeneous and isotropic spacetime, which implies it to be *maximally symmetric* [8].

We can now prove dS represents an empty spacetime with a positive cosmological constant Λ.
For maximally symmetric spacetimes the curvature of the manifold is the same at each point and in any direction, i.e. the Ricci scalar R is a constant over dS_{d+1} and the Riemann tensor can be proven to have always the following form[5]:

$$R_{\alpha\beta\gamma\delta} = \frac{R}{d(d+1)}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right), \tag{1.1.6}$$

[4]This is in complete analogy with the Euclidean $(d+1)$-dimensional sphere, defined by $SO(d+2)/SO(d+1)$. Indeed, the two manifold are simply related by a Wick rotation of the X^0 coordinate, with L becoming the radius of the sphere.

[5]Assuming the following convention: $R^{\alpha}_{\beta\gamma\delta} = \partial_\gamma\Gamma^{\alpha}_{\delta\beta} - \partial_\delta\Gamma^{\alpha}_{\gamma\beta} + \Gamma^{\alpha}_{\gamma\lambda}\Gamma^{\lambda}_{\delta\beta} - \Gamma^{\alpha}_{\delta\lambda}\Gamma^{\lambda}_{\gamma\beta}$.

so the Ricci tensor is:

$$R_{\alpha\beta} = R^{\lambda}_{\alpha\lambda\beta} = \frac{R}{d+1} g_{\alpha\beta} \,, \tag{1.1.7}$$

so that curvature is completely coded by R.
We can now inspect the consequences for the Einstein tensor $G_{\alpha\beta}$:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = -\frac{d-1}{2\,(d+1)} R g_{\alpha\beta} \,, \tag{1.1.8}$$

but R is a constant and can be proven, using a parametrization of the metric, to be

$$R = \frac{d\,(d+1)}{2L^2} \,. \tag{1.1.9}$$

So $G_{\alpha\beta}$ clearly satisfies the Einstein equation for an empty Universe with a positive cosmological constant depending just on L:

$$G_{\alpha\beta} = -\Lambda g_{\alpha\beta} \,, \tag{1.1.10}$$

$$\Lambda = \frac{d\,(d-1)}{2L^2} \,. \tag{1.1.11}$$

The constraint 1.1.1 can be solved by a number of parametrizations of the X^M. Let's show the most important ones.

Global coordinates

A set of coordinates (t, θ_i) covering the whole hyperboloid is given by the following:

$$X^0 = L \sinh \frac{t}{L} \,, \qquad X^i = n_i L \cosh \frac{t}{L} \,, \qquad i = 1, \cdots, d+1 \,. \tag{1.1.12}$$

where $-\infty < t < +\infty$ and n_i is a unit vector which can be parametrized as follows[6]:

$$\begin{aligned}
n_1 &= \cos\theta_1, & -\frac{\pi}{2} < \theta_1 < \frac{\pi}{2} \,, \\
n_2 &= \sin\theta_1 \cos\theta_2, & -\frac{\pi}{2} < \theta_2 < \frac{\pi}{2} \,, \\
& & \cdots \\
n_{d-1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-2} \cos\theta_{d-1}, & -\frac{\pi}{2} < \theta_{d-1} < \frac{\pi}{2} \,, \\
n_d &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-1} \cos\theta_d, & -\pi < \theta_d \le \pi \,, \\
n_{d+1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-1} \sin\theta_d \,.
\end{aligned} \tag{1.1.13}$$

[6] Actually the coordinates (t, θ_i) can not be properly "global", covering the whole manifold, as for each spatial section the angular coordinates leave the poles uncovered, as can be seen from their definition intervals.

The structure of 1.1.13 reminds us of the usual spherical coordinates. Indeed the *global metric* of dS_{d+1} is:

$$ds^2 = -dt^2 + L^2 \cosh^2\left(\frac{t}{L}\right) d\Omega_d^2 \tag{1.1.14}$$

with

$$d\Omega_d^2 = \sum_{j=1}^{d}\left(\prod_{i=1}^{j-1}\sin^2\theta_i\right) d\theta_j^2 \tag{1.1.15}$$

which is the metric of a S^d space.

Then dS_{d+1} spacetime in global coordinates behaves as a d-dimensional spatial sphere which is contracting exponentially from $-\infty$ to $t = 0$, when it reaches a minimum volume proportional to L, then expanding exponentially from $t = 0$ to $+\infty$. The metric is of the type of a closed FLRW one and for this reason it is called the *closed slicing* of dS spacetime.

Conformal coordinates

It is useful to move to a set of coordinates which makes evident the causal structure of dS_{d+1}. To do this we start from 1.1.12 making this transformation:

$$\cosh^2\frac{t}{L} = \frac{1}{\cos^2\tau}\,, \tag{1.1.16}$$

where τ is defined in the interval $(-\pi/2, \pi/2)$. The *global conformal metric* is:

$$ds^2 = \frac{L^2}{\cos^2\tau}\left(-d\tau^2 + d\Omega_d^2\right)\,. \tag{1.1.17}$$

This is related via a conformal transformation to the Einstein static Universe metric: $d\tilde{s}^2 = -d\tau^2 + d\Omega_d^2$. Also, dS_{d+1} shares with the latter the same *cylindrical topology*: $\mathbb{R}\times S^d$.

This relation could be used for tracing a Penrose diagram of dS_{d+1}.

Focusing for now on the case $d > 2$, we can expand the spatial line element and consider just the θ_1 direction, getting the line element of a 2-dimensional Minkowski spacetime compactified to a square:

$$d\tilde{s}^2 = -d\tau^2 + d\theta_1^2\,, \qquad -\frac{\pi}{2} < \tau, \theta_1 < \frac{\pi}{2}\,. \tag{1.1.18}$$

Then, we can represent in a simple way the Penrose diagram of dS_{d+1} by a square where each point represents a S^{d-1} sphere, as it is depicted in Figure 1.3. Causality is expressed by null trajectories $d\tau = \pm d\theta_1$. The vertical edges at $\theta_1 = \pm\pi/2$ do not coincide as they represent worldlines of the North and South poles of the S^d sphere[7], so

[7]So, on these edges, the S^{d-1} sphere is shrunk to a point.

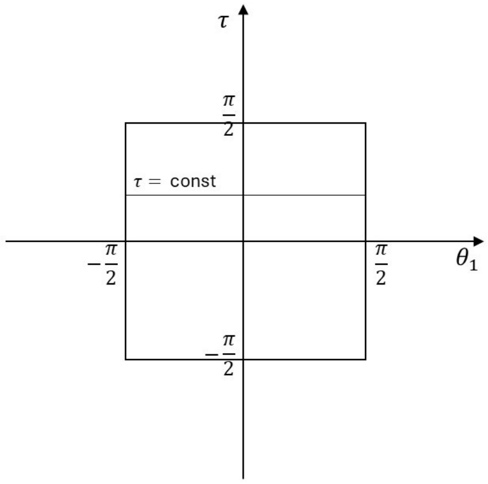

Figure 1.3: Penrose diagram of dS_{d+1} for $d > 2$. Each point of the square represents a S^{d-1} sphere, while a line at $\tau = \mathrm{const}$ represents a S^d sphere. The points $\theta_1 = \pm\pi/2$ are the two distinguished north and south poles of S^d.

the cylindrical topology is not manifest. It could be made evident selecting θ_d instead of θ_1, as in this case the definition interval of the spatial coordinate is $(-\pi, \pi]$ and the vertical edges $\theta_d = \pm\pi$ coincide. However we would get a non-flat metric, changing the causality structure of the diagram.

Instead, for $d = 2$ both the cylindrical topology is manifest and flatness is preserved, as in this case there is just one spatial coordinate and we get a rectangular Penrose diagram:

$$d\tilde{s}^2 = -d\tau^2 + \theta^2 \,, \quad -\frac{\pi}{2} < \tau < \frac{\pi}{2} \,, \quad -\pi < \theta \leq \pi \,. \tag{1.1.19}$$

Differently to Minkowski, de Sitter presents regions causally forbidden to observers; in other words, it presents *horizons.*
In $\mathcal{M}$ all the inertial observers, starting from whatever point of spacetime, can causally access to all the possible events. Indeed, after an enough amount of time, the past light cone of a Minkowskian inertial observer will cover an event before out of his reach, making possible the arriving of signals from it. on the other hand, going enough backward in time, the future light cone covered any possible future event, so that it could be causally affected by the observer.
In dS the situation is complicated by its exponential expansion and contraction: due to this, every possible observer, fixed on a certain space point, will be characterized by two particular regions of past and future dependence. Events out of these regions can not respectively affect or be affected by the observer.

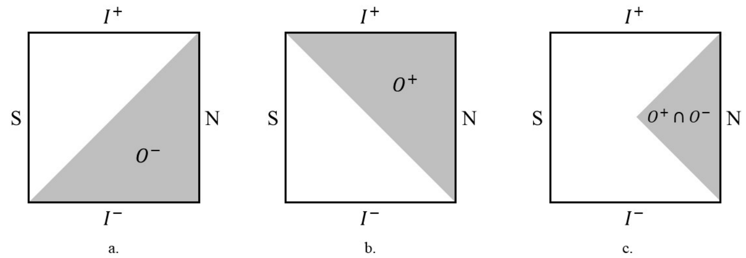

Figure 1.4: Causality for an observer at rest on N: *a.* region of past dependence, collecting events able to affect N; *b.* region of future dependence, collecting events N can influence; *c.* causal patch of N, collecting events N can communicate to.

Figure 1.4 shows this property for an observer located on the North pole of dS_{d+1}. I^+ and I^- represent the spacelike hypersurfaces respectively at future and past time infinity. Diagram a shows a light ray starting from S at I^- and reaching N exactly at I^+. If it starts at finite time it will never reach N. This light ray traces a horizon for the observer sit at rest on N, limiting its past dependence region O^-: inner events can affect N, while outer ones can not. Analogously, diagram b shows the future dependence region of the same observer, O^+, limited by the light ray from (I^-, N) to (I^+, S), which contains the events N can influence physically.
Then, intersection $O^+ \cap O^-$, shown in c, is the whole region in full causal contact with N, being possible for it to extract information from every point sending a physical signal and receiving a response. It is called the northern *causal patch* (or causal diamond). on the other hand, the southern causal patch is completely inaccessible to N.

Expanding and contracting Poincaré patches

The two sets of coordinates now presented have the property to cover two distinguished patches of whole dS_{d+1}: the expanding Poincaré patch (EPP) and the contracting one (CPP).

The EPP coordinates (t_+, x^i) for dS_{d+1} are the following:

$$
\begin{aligned}
X^0 &= L \sinh \frac{t_+}{L} + \frac{\delta_{ij} x^i x^j}{2L} \, \mathrm{e}^{\frac{t_+}{L}} \,, \\
X^i &= x^i \, \mathrm{e}^{\frac{t_+}{L}} \,, \qquad i = 1, \cdots, d \,, \\
X^{d+1} &= -L \cosh \frac{t_+}{L} + \frac{\delta_{ij} x^i x^j}{2L} \, \mathrm{e}^{\frac{t_+}{L}} \,.
\end{aligned}
\tag{1.1.20}
$$

with the definition intervals $-\infty < t_+ < +\infty$ and $-\infty < x^i < +\infty \;\; \forall i = 1, \cdots, d$.
The resulting *EPP metric* is:

$$
ds_+^2 = -dt_+^2 + \mathrm{e}^{\frac{2t_+}{L}} \delta_{ij} dx^i dx^j \,. \tag{1.1.21}
$$

Differently to the global case, which was a spherical foliation of dS_{d+1}, this metric manifests EPP is a *flat foliation*[8]. In particular, it is a flat FLRW type metric, with a scale factor $a(t_+) = \mathrm{e}^{\frac{t_+}{L}}$ growing exponentially in time. This behaviour justifies the name of *expanding* patch.
From 1.1.21 it is manifest that dS spacetime models the approximate exponential expansion proper of slow-roll inflationary models, with L coding the expansion rate, i.e. the **Hubble parameter**:

$$
H = \frac{1}{L} \,, \tag{1.1.22}
$$

which is $H \approx$ const in the slow-roll phase[4]. For $L \to +\infty$, $a(t_+) \to 1$ and we recover a Minkowskian spacetime.

It can be shown 1.1.20 does not cover the whole manifold noticing $-X^0 + X^{d+1} = -L \, \mathrm{e}^{\frac{t_+}{L}} \leq 0$. Then, in this parametrization it is always $X^0 \geq X^{d+1}$, so EPP covers just half of dS_{d+1} and is *geodesically incomplete*. Inspecting this patch using global conformal coordinates, we can visualize it in the Penrose diagram. Using 1.1.12, 1.1.13 and choosing $\theta_i = \pi/2$ for $i = 2, \cdots, d$ we have

$$
X^0 = L \sinh \frac{t}{L} \,, \qquad X^{d+1} = L \sin \theta_1 \cosh \frac{t}{L} \,, \tag{1.1.23}
$$

and the patch is defined by

$$
\tanh \frac{t}{L} \geq \sin \theta_1 \,. \tag{1.1.24}
$$

Using now 1.1.16:

$$
\sin \tau \geq \sin \theta_1 \,, \tag{1.1.25}
$$

which in the interval $(-\pi/2, \pi/2)$ implies $\tau \geq \theta_1$. Then EPP coincides with the O^+ patch of an observer at rest on the South pole.

[8]For this reason these are also called *planar coordinates*.

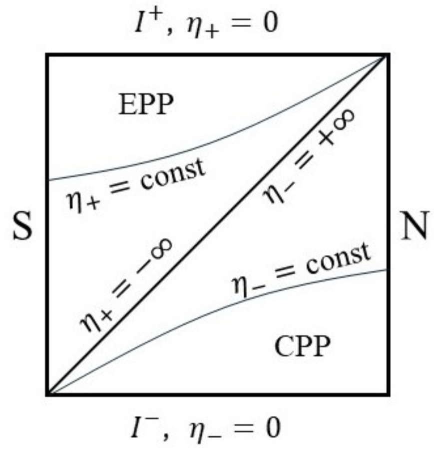

Figure 1.5: EPP and CPP are shown in the global Penrose diagram of dS_{d+1}, including various hypersurfaces at fixed conformal time. Future/past time infinity hypersurfaces $I^\pm$ correspond to $\eta_\pm \to 0$. The light ray from (I^-, S) to (I^+, N) traces the boundary between the two patches and corresponds to $\eta_\pm \to \mp\infty$.

Analogously, CPP coordinates are obtained just inverting the time direction, defining the new coordinate $t_- \equiv -t_+$:

$$
\begin{aligned}
X^0 &= -L \sinh \frac{t_-}{L} + \frac{\delta_{ij} x^i x^j}{2L}\, \mathrm{e}^{-\frac{t_-}{L}} \,, \\
X^i &= x^i\, \mathrm{e}^{-\frac{t_-}{L}} \,, \qquad i = 1, \cdots, d-1 \,, \\
X^d &= -L \cosh \frac{t_-}{L} + \frac{\delta_{ij} x^i x^j}{2L}\, \mathrm{e}^{-\frac{t_-}{L}} \,,
\end{aligned}
\tag{1.1.26}
$$

$$
ds_-^2 = -dt_-^2 + \mathrm{e}^{-\frac{2t_-}{L}}\, \delta_{ij} dx^i dx^j \,, \tag{1.1.27}
$$

which cover the other half[9] $X^0 \leq X^d$ and now the scale factor decreases exponentially in time. Again, they constitute a flat slicing of the spacetime.

Even these sets of coordinates have a correspondent set of conformal ones. Defining the conformal times

$$
\eta_\pm = \mp L\, \mathrm{e}^{\mp\frac{t_\pm}{L}} \,, \qquad -\infty < \eta_+ < 0 \,, \qquad 0 < \eta_- < +\infty \,, \tag{1.1.28}
$$

[9]This is the O^- region of the observer on N.

where again η_+ parametrises the EPP and η_- the CPP, we get again a metric conformal to $\mathcal{M}_{d+1}$:

$$d\tilde{s}^2_\pm = \left(\frac{L}{\eta_\pm}\right)^2 \left(-d\eta^2_\pm + \delta_{ij}dx^i dx^j\right) , \tag{1.1.29}$$

and the scale factor is now expressed in terms of $\eta_\pm$:

$$a(\eta_\pm) = \mp\frac{L}{\eta_\pm} = \mathrm{e}^{\pm\frac{t_\pm}{L}} . \tag{1.1.30}$$

Counter-intuitively, the two hypersurfaces given by the limits $\eta_\pm \to 0$ do not coincide: referring to the the Penrose diagram, $\eta_+ \to 0$ $(t_+ \to +\infty)$ corresponds to I^+, while $\eta_- \to 0$ $(t_- \to -\infty)$ to I^-. on the other hand, $\eta_\pm \to \mp\infty$ limits EPP and CPP and is the light-like hypersurface linking (I^-, S) to (I^+, N). All of this is depicted in Figure 1.5 together with intermediate lines of $\eta_\pm = \text{const}$.

In next chapters we will adopt the EPP conformal coordinates. This is motivated by two reasons: being our interest on modelling the inflationary era, we prefer to focus just on the exponentially expanding phase of dS, neglecting the CPP; also, EPP provides a flat slicing, which, while in an exact dS spacetime[10] is just a choice and is equivalent to other foliations, in the actual contest of inflation it is picked out as a preferred reference frame due to the matter content[11].
Moreover, as in inflation theory dS evolution is truncated at a finite time $\eta^* < 0$ and followed by an expansion era dominated by matter and radiation, a more realistic conformal diagram describing an inflationary spacetime is shown in Figure 1.6.

Static coordinates

Despite dS has not a global timelike Killing vector, it is possible to construct a set of coordinates covering just the causal patch with a invariance under time translation.
Let's take a set of $d+1$ parameters $\left(\bar{t}, r, \bar{\theta}_i\right)$, such that:

$$\begin{aligned} X^0 &= \sqrt{L^2 - r^2}\sinh\frac{\bar{t}}{L} , \\ X^i &= r n_i , \qquad i = 1, \cdots, d , \\ X^{d+1} &= \sqrt{L^2 - r^2}\cosh\frac{\bar{t}}{L} , \end{aligned} \tag{1.1.31}$$

[10]This requires the Universe to be completely *empty*.

[11]In particular, matter can be seen as a perturbation of dS spacetime content. Its average rest frame selects a specific foliation: in inflationary models this turns out to be the flat one (in agreement with experimental measurements).

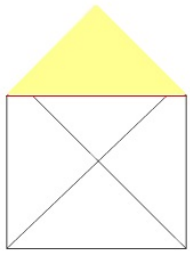

Figure 1.6: Penrose diagram of a model of truncated dS_{d+1} spacetime. The upper yellow part is a $\mathcal{M}_{d+1}$ spacetime that for ease of representation is depicted with a cylinder topology (doubling the radial coordinate).
In inflationary cosmology the truncation is due to a phase transition on a stable vacuum of the inflaton field.

where $-\infty < \bar{t} < +\infty\,,\ 0 \leq r < L$ and n_i is a unit vector representing a point of a S^{d-1} sphere and parametrized by $\bar{\theta}_i\,,\ i = 1, \cdots, d-1$, in analogy with 1.1.13.
The *static metric* is:

$$ds^2 = -\left(1 - \frac{r^2}{L^2}\right) d\bar{t}^2 + \left(1 - \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega^2_{d-1}\,. \tag{1.1.32}$$

This is a Schwarzschild-like metric: it presents an horizon for $r \to L$ and is invariant under $\bar{t}$ translations. With this choice of r's definition interval the patch covered is just the causal diamond of an observer at rest on N, with this latter given by $r = 0$. The Killing vector $\partial/\partial\bar{t}$ is timelike[12] and future pointing, then it allows to define time evolution. However, 1.1.31 describes a reference frame of a non-inertial observer at generic r position, or of an inertial one but at fixed $r = 0$. For this reason it is preferred to work with EPP (conformal) coordinates, which describe the point of view of an inertial observer.

Hyperbolic coordinates

Other than the closed and flat slicings of dS, it is possible to foliate it in hyperbolic spatial sections, getting an open FLRW metric. It is gained using the parametrization

[12]It becomes null for $r \to L$.

$\left(\hat{t}, \chi, \hat{\theta}_i\right)$:

$$\begin{aligned} X^0 &= L \sinh \frac{\hat{t}}{L} \cosh \chi \,, \\ X^i &= n_i L \sinh \frac{\hat{t}}{L} \sinh \chi \,, \qquad i = 1, \cdots, d \,, \\ X^{d+1} &= L \cosh \frac{\hat{t}}{L} \,, \end{aligned} \tag{1.1.33}$$

where $-\infty < \hat{t} < +\infty$, $0 < \chi < +\infty$ and n_i is a unit vector representing a point of a S^{d-1} sphere and parametrized by $\hat{\theta}_i$, $i = 1, \cdots, d-1$, in analogy with 1.1.13.
The *hyperbolic metric* is:

$$ds^2 = -d\hat{t}^2 + L^2 \sinh^2 \left(\frac{\hat{t}}{L}\right) \left(d\chi^2 + \sinh^2 \chi \, d\Omega_{d-1}^2\right) \,, \tag{1.1.34}$$

which is, as claimed, an open FLRW type with a scale factor $a\left(\hat{t}\right) = L \left|\sinh\left(\hat{t}/L\right)\right|$. Even these do not cover the whole dS_{d+1} manifold, as it is always $X^{d+1} > 0$.

1.2 Symmetries and representations

In previous section we have seen dS_{d+1} is a maximally symmetric spacetime and its isometry group is $SO\left(1, d+1\right)$ [9]. This is as well isomorphic to the symmetry group of an *Euclidean conformal field theory* (ECFT) on $\mathbb{R}^d$. Related Lie algebra $\mathfrak{so}\left(1, d+1\right)$ is defined by the basis of generators:

$$L_{AB} = -L_{BA} \,, \qquad A, B = 0, 1, \cdots, d+1 \,, \tag{1.2.1}$$

and their commutators:

$$\left[L_{AB}, L_{CD}\right] = \eta_{BC} L_{AD} - \eta_{AC} L_{BD} + \eta_{AD} L_{BC} - \eta_{BD} L_{AC} \,, \tag{1.2.2}$$

with $\eta_{AB} = \mathrm{diag}\left(- + \cdots +\right)$.

In the case of Euclidean CFT on $\mathbb{R}^d$, 1.2.1 can be physically interpreted decomposing them as follows [10]:

$$
\begin{aligned}
L_{ij} &= J_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i} \, , \\
L_{0,d+1} &= D = x^i \frac{\partial}{\partial x^i} \, , \\
L_{d+1,i} &= \frac{P_i + K_i}{2} \, , \\
L_{0,i} &= \frac{P_i - K_i}{2} \, ,
\end{aligned} \tag{1.2.3}
$$

$$
P_i = \frac{\partial}{\partial x^i} \, , \qquad K_i = 2x^i x^j \frac{\partial}{\partial x^j} - \mathbf{x}^2 \frac{\partial}{\partial x^i} \, ,
$$

where $i = 1, \cdots, d$, $\mathbf{x}^2 = \delta_{ij} x^i x^j$ and we have assumed a representation acting on $\mathbb{R}^d$ fields.
Here the J_{ij} are $\mathbb{R}^d$-*rotations*, P_i are $\mathbb{R}^d$-*translations*, D is the *dilation*, while the K_i are the *special conformal transformations* in $\mathbb{R}^d$. Commutation relations take the form:

$$
\begin{aligned}
&[D, P_i] = P_i \, , \\
&[D, K_i] = -K_i \, , \\
&[K_i, P_j] = 2\delta_{ij} D - 2M_{ij} \, , \\
&[M_{ij}, P_k] = \delta_{jk} P_i - \delta_{ik} P_j \, , \\
&[M_{ij}, K_k] = \delta_{jk} K_i - \delta_{ik} K_j \, , \\
&[M_{ij}, M_{kl}] = \delta_{jk} M_{il} - \delta_{ik} M_{jl} + \delta_{il} M_{jk} - \delta_{jl} M_{ik} \, ,
\end{aligned} \tag{1.2.4}
$$

and the remaining vanish.

General elements of $SO(1, d+1)$ can be then expressed via the exponential representation. Looking at 1.2.4 some non-trivial subgroups can be recognised: $\{M_{ij}\}$ form the subgroup of rotations $SO(d)$, $\{M_{ij}, P_k\}$ the Poincaré group in $\mathbb{R}^d$, $\{M_{ij}, P_k, D\}$ the group $SO(d+1)$, which is its maximal compact subgroup.

Killing vectors and conformal limit

It is useful to present $SO(1, d+1)$ infinitesimal generators in a representation acting on dS_{d+1} fields (then acting on the metric), i.e. in the form of Killing vector fields, whose action is defined via the Lie dragging. This makes manifest how a set coordinates could hide manifold's isometries[13]. Let's consider those of our main interest, the EPP ones[14].

[13] Of course a change of coordinates never break an isometry, but just transforms the Killing vector fields besides the metric. After the transformation, these could get a form with no immediate physical interpretation, making difficult to recognize isometries via a first look on the metric.

[14] From now on we will use the notation t instead of t_+ for the EPP time.

Being dS_{d+1} maximally symmetric, the number of Killing vector fields is $n = \frac{(d+1)(d+2)}{2}$ [8]. As 1.1.21 is a FLRW metric, so we expect it to be invariant under spatial $\mathbb{R}^d$ rotations and translations Killing vector fields J_{ij} and P_i defined in 1.2.3. Each component of these generators is meant to be a vector field on dS_{d+1}; also, they are a number of $d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$, so the remaining are $d + 1$.
To check this and derive the others it is convenient to start from the ambient coordinates X^M, where the 1.2.1 are simply represented by:

$$L_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}\,, \tag{1.2.5}$$

evidently satisfying the 1.2.2. The next step is to apply the 1.1.20, deriving the fields $\partial/\partial X^A$ in function of $\partial/\partial t$ and $\partial/\partial x^i$ and assembling the result.
The found Killing vectors are:

$$L_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i} \equiv J_{ij}\,, \tag{1.2.6}$$

$$L_{0,d+1} = L\frac{\partial}{\partial t} - x^i \frac{\partial}{\partial x^i}\,, \tag{1.2.7}$$

$$L_{d+1,i} = x^i \frac{\partial}{\partial t} - \frac{1}{2L}\left[\left(-\mathbf{x}^2 + L^2 - L^2\,\mathrm{e}^{-\frac{2t}{L}}\right)\frac{\partial}{\partial x^i} + 2x^i x^j \frac{\partial}{\partial x^j}\right]\,, \tag{1.2.8}$$

$$L_{0,i} = -x^i \frac{\partial}{\partial t} - \frac{1}{2L}\left[\left(\mathbf{x}^2 + L^2 + L^2\,\mathrm{e}^{-\frac{2t}{L}}\right)\frac{\partial}{\partial x^i} - 2x^i x^j \frac{\partial}{\partial x^j}\right]\,. \tag{1.2.9}$$

Summing and subtracting 1.2.8 and 1.2.9:

$$L_{d+1,i} + L_{0,i} = -L\frac{\partial}{\partial x^i} \propto P_i\,, \tag{1.2.10}$$

$$L_{d+1,i} - L_{0,i} = 2x^i \frac{\partial}{\partial t} + \left(\frac{\mathbf{x}^2}{L} + L\,\mathrm{e}^{-\frac{2t}{L}}\right)\frac{\partial}{\partial x^i} - \frac{2x^i x^j}{L}\frac{\partial}{\partial x^j}\,. \tag{1.2.11}$$

Except for 1.2.6 and 1.2.10, which are expected due to the FLRW structure, EPP Killing vector fields may appear far from the conformal ones 1.2.3. The difference is not a matter of group structure[15], but of chosen representation and coordinates. Instead, it is clear the lack of translation or boost symmetries along the EPP t coordinate.

It is very interesting to take the $t \to +\infty$ limit of the above Killing vector fields. In this case they are projected to the limit spacelike hypersurface I^+, described just by the x^i, $i = 1, \cdots, d$. On I^+, $\partial/\partial t$ vector components are suppressed; the non trivial results are:

$$L_{0,d+1} = -x^i \frac{\partial}{\partial x^i} \propto D\,, \tag{1.2.12}$$

[15] Indeed, in both cases, we have started from the same formal definition of the generators and they both do satisfy same commutation relation 1.2.2.

$$L_{d+1,i} - L_{0,i} = -\frac{1}{L}\left(2x^i x^j \frac{\partial}{\partial x^j} - \mathbf{x}^2 \frac{\partial}{\partial x^i}\right) \propto K_i \,. \tag{1.2.13}$$

So on the future time infinity limit we recover Killing vector fields proportional to 1.2.3, i.e. I^+ is a manifold with a d-dimensional Euclidean conformal symmetry. Then it is called the **conformal boundary** of dS_{d+1} expanding Poincaré patch. This will be the boundary used to construct the holographic dS/CFT correspondence.

Unitary irreducible representations

Unitary irreducible representations (UIRs) are very important to build a QFT, as enable to define particle states and construct Hilbert one-particle spaces using the Wigner's principle of classification, as commonly done in Minkowskian QFT [11, 9].

To impose unitarity, the representation of the Lie algebra must be anti-hermitian:

$$L^\dagger_{MN} = -L_{MN} \,. \tag{1.2.14}$$

Also, unitarity and non-compactness of $SO(1, d+1)$ impose every possible its UIRs to be infinite-dimensional. So we have to search for representations acting on elements of Hilbert spaces $|\psi(\mathbf{x}); i, j, \cdots\rangle$, where $i, j, \cdots$ are an indefinite number of extra discrete or continuous indices (e.g. the spin).
Spatial translation invariance allows always to get a state for general $\mathbf{x}$ starting from $|\psi(0); \cdots\rangle$, so we can classify UIRs focusing these latter.

Wigner classification starts from defining the *quadratic Casimir* $\mathcal{C}_2$:

$$\begin{aligned}\mathcal{C}_2 = \frac{1}{2}L_{MN}L^{MN} &= -D^2 + \frac{(P_i + K_i)^2}{4} - \frac{(P_i - K_i)^2}{4} + \frac{1}{2}J_{ij}J^{ij} \\ &= D(d - D) + P_i K_i + \frac{1}{2}J_{ij}J^{ij} \,,\end{aligned} \tag{1.2.15}$$

$\mathcal{C}_2$ commutes with all the $SO(1, d+1)$ generators, then different irreducible representations (IRs) are labeled by its eigenvalues c: $\mathcal{C}_2 |\psi(0); c; i, j, \cdots\rangle = c\,|\psi(0); c; i, j, \cdots\rangle$.
Then we have to analyse eigenvalues of the operators appearing in the right hand of 1.2.15 under the imposition of unitarity and irreducibility.

The term $\mathcal{C}_{SO(d)} := \frac{1}{2}J_{ij}J^{ij}$ is the quadratic Casimir of the compact $SO(d)$ subgroup. As before, $SO(d)$ IRs are classified by $\mathcal{C}_{SO(d)}$ eigenvalues, labeled by the highest weight vector of integers $\mathbf{s} = (s_1, s_2, \cdots, s_r)$, with $s_1 \geq s_2 \geq \cdots \geq |s_r|$ and $r = \left\lfloor \frac{d}{2} \right\rfloor$. However we are interest just on the single-row representation, i.e. when $\mathbf{s} = (s, 0, \cdots, 0),\ s \in \mathbb{N}$, which describes particles with integer **spin**. In this case the eigenvalues are equal to

$-s\,(s+d-2)$.
As searched IRs include spin, which is a finite-dimensional representation of $SO\,(d)$, Schur's lemma implies that, in such representation, every element of $SO\,(1,d+1)$ commuting with J_{ij} is a multiple of the identity operator on the spin index [10]. This is the case for $\mathcal{C}_{SO(d)}$, D and K_i, so in 1.2.15 we can set them equal to (or depending on) complex constant values, labeling $SO\,(1,d+1)$ IRs:

$$\mathcal{C}_{SO(d)} = -s\,(s+d-2)\,\mathbb{I}_{spin} + \cdots\,, \tag{1.2.16}$$

$$D = \bar{\Delta}\,\mathbb{I}_{spin} + \cdots\,, \tag{1.2.17}$$

$$K_i = \kappa_i\,\mathbb{I}_{spin} + \cdots\,, \tag{1.2.18}$$

where $\mathbb{I}_{spin}$ is the identity acting on the spin index and the dots indicate operators acting on the spatial wave function $\psi\,(\mathbf{x})$. $\bar{\Delta}$ is called the *scaling dimension* of the representation.

These values are constrained by group structure 1.2.4 and unitarity. The first impose all the κ_i to be null, due to the commutator

$$[D, K_i] = -K_i\,, \tag{1.2.19}$$

which in the spin part is

$$-\kappa_i\,\mathbb{I}_{spin} = \left[\bar{\Delta}\,\mathbb{I}_{spin}, \kappa_i\,\mathbb{I}_{spin}\right] = 0 \tag{1.2.20}$$

Defining the **conformal dimension** Δ such that $\bar{\Delta} = d-\Delta$, now $\mathcal{C}_2$ eigenvalues become:

$$c = \Delta\,(d-\Delta) - s\,(s+d-2)\;. \tag{1.2.21}$$

So $SO\,(1,d+1)$ UIRs are labelled by two constant values: $\Delta \in \mathbb{C},\ s \in \mathbb{N}$. We can start to build a one-particle Hilbert space starting from a *primary state* $|\Delta, 0\rangle_s$:

$$\begin{aligned} &D\,|\Delta,0\rangle_s = (d-\Delta)\,|\Delta,0\rangle_s \\ &K_i\,|\Delta,0\rangle_s = 0\,, \\ &\frac{1}{2}J_{ij}J^{ij}\,|\Delta,0\rangle_s = -s\,(s+d-2)\,|\Delta,0\rangle_s\;. \end{aligned} \tag{1.2.22}$$

This can be translated to a general $\mathbf{x}$ position:

$$|\Delta,\mathbf{x}\rangle_s = \mathrm{e}^{\mathbf{x}\cdot\mathbf{P}}\,|\Delta,0\rangle_s\;. \tag{1.2.23}$$

Finally, a general state is defined by a given wave function $\psi\,(\mathbf{x})$[16]:

$$|\psi\,(\mathbf{x})\,;\Delta\rangle_s = \int d^dx\,\psi\,(\mathbf{x})\,|\Delta,\mathbf{x}\rangle_s\;. \tag{1.2.24}$$

[16] To guarantee the consistency of the representation, i.e. that $SO\,(1,d+1)$ elements could actually act on the Hilbert space vectors, $\psi\,(\mathbf{x})$ should be a smooth function and satisfy: $\psi\,(\mathbf{x}) \xrightarrow{\mathbf{x}\to\infty} \frac{1}{\mathbf{x}^{2\Delta}}\sum_{n=0}^{\infty} C^{(n)}\left(\frac{\mathbf{x}}{\mathbf{x}^2}\right)$, where $C^{(n)}$ is a homogeneous polynomial of grade n.

From now on we will restrict our analysis to scalar representations ($s = 0$).

Unitarity condition 1.2.14 translates the inner product structure:

$$\langle\psi_1(\mathbf{x});\Delta|\psi_2(\mathbf{y});\Delta\rangle = \int d^dx d^dy\, \psi_1^*(\mathbf{x}) \langle\Delta,\mathbf{x}|\Delta,\mathbf{y}\rangle \psi_2(\mathbf{y}) \ , \tag{1.2.25}$$

where the *kernel* $K_\Delta(\mathbf{x},\mathbf{y}) := \langle\Delta,\mathbf{x}|\Delta,\mathbf{y}\rangle$ is to be chosen such to satisfy unitarity in the form of differential equations. For example, one of these equations is:

$$\langle\Delta,\mathbf{x}|P_i^\dagger|\Delta,\mathbf{y}\rangle = -\langle\Delta,\mathbf{x}|P_i|\Delta,\mathbf{y}\rangle \quad \Longrightarrow \quad \left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)K_\Delta(\mathbf{x},\mathbf{y}) = 0\,. \tag{1.2.26}$$

The whole set of unitarity conditions constraints both the kernel form and the related allowed values of Δ.
For scalar particles, the results are two classes of possible values [5]:

1. **Principal series**: $\Delta = \frac{d}{2} + i\mu, \quad \mu \in \mathbb{R}$. In this case the inner product becomes the usual one:
$$\langle\psi_1;\Delta|\psi_2;\Delta\rangle = \int d^dx\, \psi_1^*(\mathbf{x})\,\psi_2(\mathbf{x}) \ . \tag{1.2.27}$$
In a free scalar theory this series describe massive particles, with the mass parametrized by μ. In this case the mass is given by:
$$m = \frac{1}{L}\sqrt{\Delta(d-\Delta)} = \frac{1}{L}\sqrt{\frac{d^2}{4}+\mu^2}\,, \tag{1.2.28}$$
so it is always $mL \geq \frac{d}{2}$, and the equality concerns the case of a *conformally coupled* scalar field.

2. **Complementary series**: $\Delta \in \mathbb{R}\,, \quad 0 \leq \Delta < \frac{d}{2}$. In this case the inner product (in Fourier representation) is:
$$\langle\psi_1;\Delta|\psi_2;\Delta\rangle = \int \frac{d^dp}{(2\pi)^d}\,\psi_1^*(\mathbf{p})\,\psi_2(\mathbf{p})\,|\mathbf{p}|^{2\bar{\Delta}-d}\,, \tag{1.2.29}$$
This case describes particles with $0 \leq mL < \frac{d}{2}$.

For $s \neq 0$, the range of Δ in the complementary series is $0 < \Delta < \frac{d-2}{2}$. Also, this more general case present a third class, the **discrete series**: $\Delta = \frac{d}{2} + i\mu\,, \quad \mu = \pm\frac{i}{2}(d-4+2(s-r))$.
This case describes so-called *partially massless* particles of spin s and depth r, but will not be further treated in this work.

1.3 Free scalar quantum fields

We are now ready to build a quantum field theory embedded on de Sitter spacetime. The simplest case is that of a **free massless scalar field**, which simply emerges in inflation theory considering quantum fluctuations over the vacuum and allow already to compute important cosmological observables, as the power spectrum, related to free propagators. The model can be enriched adding mass and self-interactions: in particular the cubic term is important for the computation of non-Gaussianities (encoded in 3-point correlation functions and *bispectra*). While adding a mass the theory remains free and the treatment is substantially analogous to the massless case[17], for interactions a perturbative formalism appropriate to the cosmological context will be needed. This will be done in the next section.

Now, we need to build a QFT on dS_{d+1} coherent with slow-roll models presented in the introduction of this chapter. We should, therefore, start from the classical inflationary action 1.0.1, generalizing it from 4 to $d+1$ dimensions and adapting it to the study of quantum fluctuations of ϕ over the approximate dS vacuum characterizing the expansion phase. This can be formalised using a background field method:

$$\phi\left(t,\mathbf{x}\right)=\bar{\phi}\left(t\right)+\varphi\left(t,\mathbf{x}\right)\,, \tag{1.3.1}$$

where $\bar{\phi}$ is the homogeneous inflaton background and φ is the perturbation to be quantized.

For now the coordinates $(t,\mathbf{x})$ have not been specified. However, due to the gravitational coupling, to proceed with the inflaton quantization it is necessary to fix the gauge of $g_{\mu\nu}$, i.e. the set of coordinates. A good choice of them allow to neglect the quantization of metric fluctuations $\delta g_{\mu\nu}$ in dS limit: indeed, choosing a spatially flat metric $g_{ij}=a^2\left(t\right)\delta_{ij}$, the coupling φ - $\delta g_{\mu\nu}$ vanishes for $\varepsilon\to 0$.
Finally, we can choose a conformal time coordinate η. The resulting action involves a free massless scalar field embedded on an exact dS_{d+1} spacetime, and the coordinates could be interpreted as a EPP or CPP ones[18]:

$$\begin{aligned}\mathcal{S}[\varphi]&=\int d\eta\, d^dx\,\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right)\\&=\int d\eta\, d^dx\; a^{d-1}\left(\eta\right)\left(\frac{1}{2}\dot{\varphi}^2-\frac{1}{2}\left|\nabla\varphi\right|^2\right)\,,\end{aligned} \tag{1.3.2}$$

where the metric is given by 1.1.29 and $g:=\det\left(g_{\mu\nu}\right)=-a^{2(d+1)}$. Recall the EPP scale factor is $a\left(\eta\right)=-\frac{L}{\eta}=-\frac{1}{H\eta}$.

[17]The only peculiarity will be to distinguish between the principal and complementary series.

[18]For the reason explained in 1.1, we choose the EPP coordinates. Also, we omit the "+" subscript on the time coordinate for ease of presentation.

As said mass and interactions can be added to the QFT maintaining effects of the potential on φ (so not only on the background $\bar{\phi}$). In particular, the mass term $-\frac{1}{2}m^2\varphi^2$ does not complicate the computation of both classical waves and quantum propagators. Then, it is convenient for ease of presentation to already consider it in the following. The updated action for a **massive** scalar field on dS_{d+1} is:

$$\begin{aligned} \mathcal{S}_m[\varphi] &= \int d\eta\, d^dx\, \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2\right) \\ &= \int d\eta\, d^dx\; a^{d-1}(\eta)\left(\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\left|\nabla\varphi\right|^2 - \frac{1}{2}a^2(\eta)\, m^2\varphi^2\right). \end{aligned} \tag{1.3.3}$$

Classical waves

For first it is necessary and instructive to study the classical behaviour of field perturbations. To construct a classical field theory on a curved spacetime we proceed similarly to the flat case [6]. From 1.3.3 we can derive the equation of motion:

$$\ddot{\varphi} + (d-1)\,\frac{\dot{a}}{a}\dot{\varphi} - \nabla^2\varphi + a^2(\eta)\, m^2\varphi^2 = 0\,. \tag{1.3.4}$$

In dS_{d+1} we can exploit translation invariance of spatial coordinates to work in Fourier representation. Using the ansatz $\varphi(\eta, \mathbf{x}) = \varphi_{\mathbf{k}}(\eta)\, \mathrm{e}^{\pm i\mathbf{k}\cdot\mathbf{x}}$, the equation of motion for a Fourier mode with momentum $\mathbf{k}$ is:

$$\ddot{\varphi}_{\mathbf{k}} - \frac{d-1}{\eta}\dot{\varphi}_{\mathbf{k}} + \left(k^2 + \frac{m^2L^2}{\eta^2}\right)\varphi_{\mathbf{k}} = 0\,, \tag{1.3.5}$$

with $k := |\mathbf{k}|$.
1.3.5 can be further simplified by the defining the *canonically-normalized* field $\varphi_{\mathbf{k}}(\eta) \equiv a(\eta)^{\frac{1-d}{2}}\, u_{\mathbf{k}}(\eta)$, obtaining an harmonic-oscillator-like wave equation:

$$\ddot{u}_{\mathbf{k}} + \left(k^2 - \frac{\lambda^2}{\eta^2}\right)u_{\mathbf{k}} = 0\,, \tag{1.3.6}$$

where $\lambda := i\sqrt{m^2L^2 - \frac{d^2-1}{4}}$. This equation is conceivable as a Klein-Gordon equation with a tachyonic and time-dependent "mass" term $i\lambda/\eta$, characterising the dS field kinematics.

The next step consists in analysing the time boundary behaviour, which, besides allowing us to choose appropriate boundary conditions (or vacuum state upon quantization), is interesting for an interpretation of starting and ending inflationary spacetimes.

At past infinity, $\eta \to -\infty$, the mass term tends to 0 and the previous reduces to a Minkowskian wave equation:

$$\ddot{u}_{\mathbf{k}} + k^2 u_{\mathbf{k}} = 0\,. \tag{1.3.7}$$

This behaviour is understood in terms of the Equivalence Principle. Indeed, in this limit, the comoving Hubble radius $R_H(\eta)$, measuring the distance (in comoving coordinates) at which relative expansions becomes superluminal, surpasses in scale any other wave mode:

$$R_H(\eta) := (aH)^{-1} = -\eta \overset{\eta\to-\infty}{\gg} k^{-1}\,. \tag{1.3.8}$$

In other words, at the beginning of exponential expansion, dS EPP behaves as a Minkowski spacetime and modes' sizes appear small compared to the physical reference scale of the system. Then we deal with a *UV limit* of the theory: relevant modes are at $k \to +\infty$. A general solution of 1.3.7 is the linear combination:

$$u_{\mathbf{k}}(\eta) = A\frac{1}{\sqrt{2k}}\,\mathrm{e}^{ik\eta} + B\frac{1}{\sqrt{2k}}\,\mathrm{e}^{-ik\eta}\,, \tag{1.3.9}$$

which in terms of $\varphi_{\mathbf{k}}$:

$$\varphi_{\mathbf{k}}(\eta) = \frac{A}{\sqrt{2k}}\left|\frac{\eta}{L}\right|^{\frac{d-1}{2}}\mathrm{e}^{ik\eta} + \frac{B}{\sqrt{2k}}\left|\frac{\eta}{L}\right|^{\frac{d-1}{2}}\mathrm{e}^{-ik\eta}\,, \tag{1.3.10}$$

where A and B are free complex parameters and we used a common normalisation convention[19]. Then, we expect the solutions of 1.3.6 and 1.3.5 to tend to these in the $k\eta \to -\infty$ limit.

At future infinity, $\eta \to 0$, the k^2 term can be neglected respect to the "mass" one[20]:

$$\ddot{u}_{\mathbf{k}} - \frac{\lambda^2}{\eta^2}u_{\mathbf{k}} = 0\,. \tag{1.3.11}$$

In this case dS_{d+1} is expanding very fast and R_H has shrunk, making relevant modes very large in size compared to it: $k^{-1} \gg -\eta$. In this case the theory has an *IR limit.* A non-normalised general solution is

$$u_{\mathbf{k}}(\eta) = C\,|\eta|^{\frac{1}{2}+i\mu} + D\,|\eta|^{\frac{1}{2}-i\mu}\,, \tag{1.3.12}$$

where $\mu = \sqrt{m^2L^2 - \frac{d^2}{4}}$. Returning to the original field $\varphi_{\mathbf{k}}$, we have

$$\varphi_{\mathbf{k}}(\eta) = \frac{C}{L^{\frac{d-1}{2}}}|\eta|^{\frac{d}{2}+i\mu} + \frac{D}{L^{\frac{d-1}{2}}}|\eta|^{\frac{d}{2}-i\mu}\,. \tag{1.3.13}$$

[19] This is motivated by the fact that $\frac{1}{\sqrt{2k}}\,\mathrm{e}^{\pm ik\eta}$ has Wronskian normalised to i.

[20] Except when $m^2L^2 = \frac{d^2-1}{4}$, in which case we have again equation 1.3.7 and resulting $\varphi_{\mathbf{k}}$ decays as $\propto \eta^{\frac{d-1}{2}} \xrightarrow{\eta\to 0} 0$.

We can see μ is exactly the representation index defined in previous section, returning the principal series for real values, the complementary for imaginary values. Also, we find the two terms in 1.3.13 scale with the conformal dimension Δ or the scaling dimension $\bar{\Delta}$ of a given UIR. These two types of future limit behaviour will be fundamental in the next chapters and are called respectively the **Dirichlet** and **Neumann boundary conditions**. Δ and $\bar{\Delta}$ are most often named with the following more intuitive notation:

$$\Delta^{\pm} := \frac{d}{2} \pm i\mu\,. \tag{1.3.14}$$

Classical waves in the principal series ($mL \geq d/2$) oscillate and decay in the future limit[21]; for the complementary series, when $0 < mL < d/2$, they just decay exponentially; the very exceptional case is for $mL = 0$, which presents a different boundary behaviour:

$$\varphi_{\mathbf{k}}\left(\eta\right) = \frac{C}{L^{\frac{d-1}{2}}}|\eta|^{d} + \frac{D}{L^{\frac{d-1}{2}}}|\eta|^{0} \to 0 + \mathrm{const}\,. \tag{1.3.15}$$

Therefore, massless scalar waves in the far future of dS_{d+1} are stretched to large scales and are **frozen in time**, not propagating. This is the most peculiar property of de Sitter QFT and it is at the basis of inflation theory: the spatial distribution of inflaton quantum fluctuations on the vacuum state, generated during inflation, are fixed at the end of it. At reheating, regions of higher energy density are seeds for gravitational collapse and cluster formation, making a strong link between CMB/LSS anisotropies and inflaton field ones.

Complete solutions of 1.3.6 are given by linear combinations of Bessel functions. Different combinations correspond to different choices of boundary conditions, i.e. of coefficients (A, B, C, D), which are related such to have just two degrees of freedom, being 1.3.6 a second order differential equation.
For example, solutions proportional to Bessel functions of the first type $J_{i\mu}\left(\eta\right)$ or of second type $Y_{i\mu}\left(\eta\right)$ correspond to null either C or D but non vanishing both A and B. Vice versa, specific linear combination of these latter, called first and second type *Hankel functions*[22], correspond to waves with vanishing either A or B but both C, D different from 0, thus representing single Minkowskian modes in the past limit, also known as **Bunch-Davies modes**.

Choosing boundary conditions consist, therefore, in fixing the two degrees of freedom defining a specific solution. One of them is fixed by normalisation, while the remaining involves a more physical choice, based for example in the interpretation of a specific limit, as happens for Bunch-Davies modes. This will be further discussed in next subsection.

[21]The oscillating behaviour is directly seen in non-conformal time t, as $\eta^{\pm i\mu} \propto \mathrm{e}^{\mp i\mu t}$.

[22]They are simply $H^{(1)}_{i\mu} := J_{i\mu} + iY_{i\mu}$ and $H^{(2)}_{i\mu} := J_{i\mu} - iY_{i\mu}$.

Quantization and vacuum choice

Then main difference of a field theory embedded on a curved spacetime respect to a usual flat one emerges upon quantization. In general, it is due to the different symmetry structure of the curved spacetime, which not always possesses Killing vectors of immediate physical interpretation or even does not have a globally timelike one, preventing the definition of an energy-like conserved quantity. This, in turn, is strictly related to the choice of the vacuum state, which in Minkowskian QFT is done choosing the one with minimum energy [12].
We have seen EPP does not provide a globally timelike Killing vector. In the next we will see the set of dS vacua and the physical reasons behind picking a specific one.

To proceed with the canonical quantization of φ the first step is to find the momentum conjugate field. It is convenient to keep working with the canonically-normalized $u\left(\eta\right) := a^{\frac{d-1}{2}}\left(\eta\right)\varphi\left(\eta\right)$. Then, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\dot{u}^2 - \frac{1}{2}\left|\nabla u\right|^2 + \frac{1}{2}\frac{d^2-1}{8}\frac{\ddot{a}}{a}u^2\,, \tag{1.3.16}$$

which returns the equation of motion 1.3.6. The momentum conjugate field is:

$$\pi\left(\eta\right) := \frac{\delta\mathcal{L}}{\delta\dot{u}}\left(\eta\right) = \dot{u}\left(\eta\right)\,. \tag{1.3.17}$$

Quantization is now straightforward. u and π are promoted to quantum operators and the following equal-time commutators are defined[23]:

$$\left[\hat{u}\left(\eta,\mathbf{x}_1\right),\hat{\pi}\left(\eta,\mathbf{x}_2\right)\right] = i\delta^{(d)}\left(\mathbf{x}_1-\mathbf{x}_2\right)\,, \tag{1.3.18}$$

$$\left[\hat{u}_{\mathbf{k}_1}\left(\eta\right),\hat{\pi}_{\mathbf{k}_2}\left(\eta\right)\right] = i\left(2\pi\right)^d\delta^{(d)}\left(\mathbf{k}_1+\mathbf{k}_2\right)\,. \tag{1.3.19}$$

Now $\hat{u}\left(\eta,\mathbf{x}\right)$ can be expanded in modes defining *creation* and *annihilation operators*:

$$\hat{u}\left(\eta,\mathbf{x}\right) = \int\frac{d^dk}{\left(2\pi\right)^d}\,\left(u^*_{\mathbf{k}}\left(\eta\right)\hat{a}_{\mathbf{k}} + u_{\mathbf{k}}\left(\eta\right)\hat{a}^\dagger_{-\mathbf{k}}\right)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{1.3.20}$$

where $u_{\mathbf{k}}\left(\eta\right)$ is a solution of 1.3.6 and $u^*_{\mathbf{k}}\left(\eta\right)$ is its complex conjugate. If we want to work in a convention coherent with common Minkowskian QFT, i.e. asking:

$$\left[\hat{a}_{\mathbf{k}_1},\hat{a}^\dagger_{\mathbf{k}_2}\right] = \left(2\pi\right)^d\delta^{(d)}\left(\mathbf{k}_1-\mathbf{k}_2\right)\,, \tag{1.3.21}$$

we must impose the *Wronskian condition* on the mode function scalar product:

$$\left(u_{\mathbf{k}}\left(\eta\right),u^*_{\mathbf{k}}\left(\eta\right)\right) := \partial_\eta u_{\mathbf{k}}(\eta)u^*_{\mathbf{k}}(\eta) - u_{\mathbf{k}}(\eta)\partial_\eta u^*_{\mathbf{k}}(\eta) = i\,, \tag{1.3.22}$$

[23] Of course the second is derived from the first going via a Fourier transforamtion of the spatial coordinates.

fixing the normalisation of $u_{\mathbf{k}}(\eta)$.

As explained in previous subsection, this condition leaves one boundary parameter unfixed. From a quantum point of view this means there is an ambiguity on the vacuum definition. This latter is defined as the state $|0\rangle$ such that:

$$\hat{a}_{\mathbf{k}} |0\rangle = 0 \qquad \forall \mathbf{k} \,. \tag{1.3.23}$$

Being $\hat{a}_{\mathbf{k}}$ defined in function of chosen mode function $u_{\mathbf{k}}$, it is clear different choice of this latter correspond to different vacua states (and, therefore, different "particle" states $|1_{\mathbf{k}_1}, 1_{\mathbf{k}_2}, \cdots\rangle$ built via creation operators). In usual flat QFT this choice is done picking the mode correspond to unique minimum Minkowskian energy (given by $E = \langle 0|\hat{H}|0\rangle$ where $\hat{H}$ is the Hamiltonian):

$$c_1 \frac{1}{\sqrt{2k}} \mathrm{e}^{iEt} + c_2 \frac{1}{\sqrt{2k}} \mathrm{e}^{-iEt} \xrightarrow{\text{minimum } E} \frac{1}{\sqrt{2k}} \mathrm{e}^{iEt} \,. \tag{1.3.24}$$

But, as explained, there is not an analogous derivation in dS, so we should base our choice on a different physical reference. The most reasonable idea is to pick the mode function/vacuum state which returns 1.3.24 in the UV limit $k\eta \to -\infty$. This is exactly the Bunch-Davies (BD) mode (with $A = 1$ and $B = 0$) defined previously, which is proportional to a second type Hankel function:

$$u_{\mathbf{k}}(\eta) = \sqrt{\frac{\pi}{4}}\, \mathrm{e}^{-i\frac{\pi}{4}(1+2i\mu)} (-\eta)^{\frac{1}{2}}\, H^{(2)}_{i\mu}(-k\eta) \,. \tag{1.3.25}$$

All others modes satisfying the Wronskian condition give different but physically acceptable vacua state. They constitute a one-parameter family of states, called *α-vacua*. These are relevant in cosmology, and one important line of open research tries to derive the formal and phenomenological differences among them, with the aim to select the true Universe vacuum [13].

Finally, it is important to remind that particle interpretation is not proper on a curved spacetime. In fact, in this context, particle's detection is an observer-dependent phenomenon: e.g., a BD vacuum $|0\rangle_{\mathrm{BD}}$ appear as a "particle" vacuum for an observer located at past time infinity, as it tends to coincide with the local Minkowski vacuum $|0\rangle_{\mathcal{M}_{-\infty}}$: ${}_{\mathrm{BD}}\langle 0|0\rangle_{\mathcal{M}_{-\infty}} = 1$ and ${}_{\mathrm{BD}}\langle 0|1_{\mathbf{k}_1}, 1_{\mathbf{k}_2}, \cdots\rangle_{\mathcal{M}_{-\infty}} = 0$ for every possible excited state.
But this is not anymore the case for an observer located at finite η or at I^+, whose local vacuum $|0\rangle_{\mathcal{M}_\eta}$ is much different from the BD one, giving non-null transition amplitudes with (some) excited states: ${}_{\mathrm{BD}}\langle 0|1_{\mathbf{k}_1}, 1_{\mathbf{k}_2}, \cdots\rangle_{\mathcal{M}_\eta} \neq 0$.

For the following of this work we will always assume the **Bunch-Davies vacuum**, omitting the "BD" subscript on it.

Free propagators

We can now finish the presentation of free QFT in dS presenting results for Wightman and Feynman propagators [14]. These are important for two reasons: allow to derive the most important cosmological observable, the **power spectrum**, related to distribution of temperature oscillations in the CMB; are the building blocks for the perturbative computation of n-point functions in interactive theories.

The Wightman propagators (or Wightman 2-point functions) are used to define the other type of propagators, such as retarded, advanced and Feynman ones. They are simply defined as:

$$W(x_1, x_2) := \langle 0|\hat{\varphi}(x_1)\hat{\varphi}(x_2)|0\rangle \ , \tag{1.3.26}$$

where $x_{1,2} := (\eta_{1,2}, \mathbf{x}_{1,2})$. This function obeys the Bessel equation 1.3.4 and is invariant under $SO((1, d+1)$ transformation, then it is a function of the normalised invariant hyperbolic distance $P\left(x_1, x_2\right)$. This in, in turn, can be usefully parametrized via:

$$\begin{aligned}\sigma\left(x_1, x_2\right) &:= \frac{1 + P\left(x_1, x_2\right)}{2} \\ &= \frac{1 + \cos\frac{D(x_1,x_2)}{L}}{2} \\ &= 1 + \frac{(\eta_1 - \eta_2)^2 - (\mathbf{x}_1 - \mathbf{x}_2)^2}{4\eta_1\eta_2} \ ,\end{aligned} \tag{1.3.27}$$

which takes values in $[0, 1]$, with $\sigma = 1$ when $x_1 = x_2$.

Under this, equation 1.3.4 becomes an Euler's hypergeometric differential equation:

$$\left[\sigma(1-\sigma)\,\partial_\sigma^2 - \left(\frac{d+1}{2}\right)(2\sigma - 1)\,\partial_\sigma - m^2L^2\right] W(\sigma) = 0. \tag{1.3.28}$$

The general solution is a linear combination of two Gaussian hypergeometric functions

${}_2F_1\left(a,b;c;z\right)^{24}$:

$$W(\sigma) = A\,{}_2F_1\left(\frac{d}{2}+i\mu,\frac{d}{2}-i\mu;\frac{d+1}{2};\sigma\right) + B\,{}_2F_1\left(\frac{d}{2}+i\mu,\frac{d}{2}-i\mu;\frac{d+1}{2};1-\sigma\right). \tag{1.3.31}$$

In the Bunch-Davies vacuum we have $B=0$ and

$$A = \frac{1}{L^{d-1}}\frac{\Gamma\left(\frac{d}{2}+i\mu\right)\Gamma\left(\frac{d}{2}-i\mu\right)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)}, \tag{1.3.32}$$

where this latter has been determined imposing a coincidence with flat space propagator in the short-distance limit $\sigma \to 1$. Thus, the Wightman dS_{d+1} propagator in BD vacuum is given by

$$W\left(\sigma\right) = \frac{1}{L^{d-1}}\frac{\Gamma\left(\frac{d}{2}+i\mu\right)\Gamma\left(\frac{d}{2}-i\mu\right)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)}\,{}_2F_1\left(\frac{d}{2}+i\mu,\frac{d}{2}-i\mu;\frac{d+1}{2};\sigma\right), \tag{1.3.33}$$

and it is singular in $\sigma = 1$.

Being W dependent on σ, which is symmetric under arguments permutation, it has lost the original operator ordering present in its definition 1.3.26. However, to define time-order propagators characterized by different singularity structures, we have to reintroduce operator ordering analytically continuing $W\left(\sigma\right)$ and exploiting the branch cut on $(1,+\infty)$ via a $i\varepsilon$ prescription ($\varepsilon > 0$). Using

$$\langle 0|\hat{\varphi}(\eta_1,\mathbf{x}_1)\hat{\varphi}(\eta_2,\mathbf{x}_2)|0\rangle = \lim_{\varepsilon\to 0}\langle 0|\hat{\varphi}(\eta_1 - i\varepsilon,\mathbf{x}_1)\hat{\varphi}(\eta_2 + i\varepsilon,\mathbf{x}_2)|0\rangle\ , \tag{1.3.34}$$

we can define:

$$\sigma_{\pm}\left(x_1,x_2\right) = 1 - \frac{(\mathbf{x}_1-\mathbf{x}_2)^2 - (\eta_1-\eta_2)^2 \mp i\,\mathrm{sgn}(\eta_1-\eta_2)\,\varepsilon}{4\eta_1\eta_2}, \tag{1.3.35}$$

then the new ordered Wightman functions are:

$$\langle 0|\hat{\varphi}(x_1)\hat{\varphi}(x_2)|0\rangle \equiv W(\sigma_-)\,, \tag{1.3.36}$$

[24]These are special functions defined by:

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{1.3.29}$$

where a,b,c are integers and $z\in\mathbb{C}$, $|z|<1$. The symbol $(m)_n$ denotes the *Pochhammer symbol*:

$$(m)_n = \begin{cases} 1 & \text{if } n=0 \\ m(m+1)\cdots(m+n-1) & \text{if } n>0\,. \end{cases} \tag{1.3.30}$$

It can be analytically continued to $|z|\geq 1$ with a branch cut on the $(1,+\infty)$ interval of the real axis.

$$\langle 0|\hat{\varphi}(x_2)\hat{\varphi}(x_1)|0\rangle \equiv W(\sigma_+)\,. \tag{1.3.37}$$

Thanks to the symmetry property ${}_2F_1\left(z^*\right) = {}_2F_1^*\left(z\right)\ \forall z \in \mathbb{C}$, we have the useful relation:

$$W(\sigma_+) = W^*(\sigma_-)\,, \tag{1.3.38}$$

thank to which in the following we can use the simpler notation $W\left(x_1, x_2\right)$ defined in 1.3.26, using the complex conjugate when the reverse time ordering is needed.
Starting from them it is possible to define the other propagators studied in QFT. In the next section we will use the *time-ordered* and *anti-time-ordered* propagators, respectively:

$$\langle 0|T\hat{\varphi}(x_1)\hat{\varphi}(x_2)|0\rangle = \theta(\eta_1 - \eta_2)\,W\left(x_1, x_2\right) + \theta(\eta_2 - \eta_1)\,W^*\left(x_1, x_2\right)\ , \tag{1.3.39}$$
$$\langle 0|\overline{T}\hat{\varphi}(x_1)\hat{\varphi}(x_2)|0\rangle = \theta(\eta_1 - \eta_2)\,W^*\left(x_1, x_2\right) + \theta(\eta_2 - \eta_1)\,W\left(x_1, x_2\right)\ , \tag{1.3.40}$$

where T and $\bar{T}$ are respectively the time and anti-time ordering operators. The first one is indicated by $\Pi\left(x_1, x_2\right)$, while the second is simply $\Pi^*\left(x_1, x_2\right)$. In the following we will call these Feynman or **bulk-to-bulk** propagators, describing causal propagation in the bulk of dS_{d+1}.

We finish this section deriving the power spectrum. First of all, it is defined via the Fourier transform of the 2-point correlation function at equal time. Fixing, for simplicity, one spatial coordinate to the origin, this function is:

$$\langle 0|\hat{\varphi}(\eta, \mathbf{x})\hat{\varphi}(\eta, 0)|0\rangle = \int \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\ \langle 0|\hat{\varphi}_{\mathbf{k}}(\eta)\hat{\varphi}_{\mathbf{k}'}(\eta)|0\rangle\ \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{1.3.41}$$

Using the following decomposition, where[25] $f_{\mathbf{k}} := a\left(\eta\right)^{\frac{1-d}{2}}\,u_{\mathbf{k}}$:

$$\hat{\varphi}_{\mathbf{k}}(\eta) = f^*_{\mathbf{k}}(\eta)\,\hat{a}_{\mathbf{k}} + f_{\mathbf{k}}(\eta)\,\hat{a}^\dagger_{-\mathbf{k}}\,, \tag{1.3.42}$$

we get

$$\langle 0|\hat{\varphi}_{\mathbf{k}}(\eta)\hat{\varphi}_{\mathbf{k}'}(\eta)|0\rangle = (2\pi)^d\delta^{(d)}(\mathbf{k} + \mathbf{k}')\ |f_{\mathbf{k}}(\eta)|^2\ . \tag{1.3.43}$$

Therefore, we can recognize the *power spectrum*:

$$P_\varphi\left(\mathbf{k}\right) \equiv |f_{\mathbf{k}}\left(\eta\right)|^2\,. \tag{1.3.44}$$

So it is the modulus squared of the *BD mode function.* This is actually dependent only on $k = |\mathbf{k}|$, thanks to dS spatial isotropy, so in the following we will use $f_k(\eta)$ and $u_k(\eta)$ instead of $f_{\mathbf{k}}(\eta)$ and $u_{\mathbf{k}}(\eta)$.

[25] This mode function was called $\varphi_{\mathbf{k}}$ in previous sections, but here we change its name to distinguish it from the Fourier mode operator $\hat{\varphi}_{\mathbf{k}}$, which is actually a combination of the mode function and its complex conjugate.

1.4 Schwinger-Keldysh formalism

The introduction of interaction terms in the potential of quantum fluctuations $V(\varphi)$ – like self-interactions (φ^3, φ^4); terms with derivatives ($\partial_\mu\varphi\partial^\mu\varphi$); interactions with other fields $\left(\varphi^2\chi^2,\, \varphi\bar{\psi}\psi\right)$ – needs a perturbative approach analogous to the Feynman diagrams used in flat QFT.
This latter is specifically built with the final aim of computing scattering matrices, in view of an application on particle physics which has historically been the main bench test of quantum field theory [15]. In particular, common Feynman diagrams, by means of the Wick theorem, allows a perturbative expansion of the Dyson Formula, which in turns provides n-point *time-ordered* correlators of the interactive theory. Finally, via the LSZ Formula, these give the transition amplitude between two *asymptotically free in* and *out* particle states, or, in other terms, the S-matrix:

$$S_{\beta\alpha} = \langle\beta| T\, \mathrm{e}^{-i\int_{-\infty}^{+\infty} dt\, H_{\mathrm{int}}} |\alpha\rangle \ , \tag{1.4.1}$$

where $|\alpha\rangle$, $|\beta\rangle$ are asymptotically free particle states.

However, this common approach is useless in cosmology. In the previous case it is meaningful to impose free mode boundary conditions both in future e past infinity, as in Minkowski spacetime can always be designed, in principle, a scattering experiment with preparation of the initial state and measure of the final one, till they are enough far away in time and space.
Instead, in dS the time dependence of the Hamiltonian makes impossible to set proper asymptotic free wave packets: spacetime expansion acts as background field interacting with φ everywhere and in every moment, impossible to decouple even asymptotically, and producing a continuous and endless production of particles. Things are exacerbated by the fact that a single observer in dS does not have access to a complete Cauchy surface.
But even assuming to define a formally correct approach to scattering in dS, it would be phenomenologically and experimentally useless as we would never have the possibility of preparing an initial states in inflation.

Rather than this, we can assume the hypothesis Universe started at a certain initial vacuum state and then study its possible evolution. Therefore, a more interesting and better posed goal in inflationary physics, than S-matrix, is the expectation value of an observable O at a certain time η in the chosen initial vacuum $|\Omega\rangle$ of the interacting theory[26]:

$$\langle O(\eta)\rangle := \langle\Omega|O(\eta)|\Omega\rangle \ . \tag{1.4.2}$$

[26] O can be a single operator or a product of operators at same time instant. It is possible to extend the formalism to product of operators at different times just inserting evolution operators between them, but they are note much used in cosmology.

In the context of inflation, such functions are called *cosmological correlators*, while the related approach is called **Schwinger-Keldysh** or **in-in formalism**.

The Master Formula

The defined function 1.4.2 is very similar to a correlation function $\langle\Omega|T\hat{\varphi}(x_1)\hat{\varphi}(x_2)\cdots\hat{\varphi}(x_n)|\Omega\rangle$ in flat QFT. The key difference is subtle: these latter assume the bra $\langle\Omega|$ to be a *future* infinity state and $|\Omega\rangle$ a *past* one, so in this case they are related to the free theory vacuum $|0\rangle$ as:

$$|\Omega\rangle \propto F_{\mathcal{M}}\left(t_0,-\infty\right)|0\rangle\ , \qquad \langle\Omega| \propto \langle 0|\, F_{\mathcal{M}}\left(+\infty,t_0\right) \neq \left[F_{\mathcal{M}}\left(t_0,-\infty\right)|0\rangle\right]^{\dagger}\ , \tag{1.4.3}$$

where, assuming interaction picture, $F_{\mathcal{M}}$ is the time evolution operator (TEO) of states (in Minkowskian QFT) and t_0 is the reference time at which $|\Omega\rangle \equiv |0\rangle$.
on the other hand, 1.4.2 is a common expectation value, so both $\langle\Omega|$ and $|\Omega\rangle$ are considered at **past** infinity.

To derive a formula for 1.4.2 suitable for perturbation theory it is convenient to work in *interaction picture*, as commonly done in flat QFT. Splitting the total Hamiltonian in a free and an interaction parts $\mathcal{H}_{tot} = \mathcal{H}_0 + \mathcal{H}_{int}$, observables in interaction picture are defined as:

$$O_I(\eta) := U_0^{\dagger}(\eta,\eta_0)\, O(\eta_0)\, U_0(\eta,\eta_0)\,, \qquad U_0\left(\eta,\eta_0\right) = \mathrm{e}^{-i\mathcal{H}_0(\eta-\eta_0)}\,, \tag{1.4.4}$$

where U_0 is the TEO of the free theory, leading the evolution of observables in interaction pictures.
Being, instead, U the TEO of observables in the Heisenberg picture (or of states in the Schrödinger one), we have:

$$\begin{aligned}\langle\Omega|O(\eta)|\Omega\rangle &= \langle\Omega|U^{\dagger}(\eta,\eta_0)\, O(\eta_0)\, U(\eta,\eta_0)|\Omega\rangle \\ &= \langle\Omega|F^{\dagger}(\eta,\eta_0)O_I(\eta)F(\eta,\eta_0)|\Omega\rangle\ ,\end{aligned} \tag{1.4.5}$$

with F defined as:

$$F(\eta,\eta_0) := U_0^{\dagger}(\eta,\eta_0)U(\eta,\eta_0)\,, \tag{1.4.6}$$

implying the differential equation

$$\frac{dF}{d\eta}\left(\eta,\eta_0\right) = -i\mathcal{H}_{int}^{I}\left(\eta\right)F\left(\eta,\eta_0\right)\ , \tag{1.4.7}$$

where $\mathcal{H}_{int}^{I}$ is $\mathcal{H}_{int}$ in interaction picture. This is equivalent to the integral equation

$$F(\eta,\eta_i) = T\,\mathrm{e}^{\left(-i\int_{\eta_i}^{\eta} d\eta'\, \mathcal{H}_{\mathrm{int}}^{I}(\eta')\right)}\,, \tag{1.4.8}$$

where T is the time ordering operator. It is clear $F(\eta, \eta_0)$ leads the evolution of states in interaction picture: $|\psi_I(\eta)\rangle = F(\eta, \eta_0)\,|\psi_I(\eta_0)\rangle$.

Choosing η_0 such that $|\Omega_I(\eta_0)\rangle \equiv |\Omega\rangle$ and using 1.4.5 we have:

$$\begin{aligned}\langle O(\eta)\rangle &= \langle\Omega|O(\eta)|\Omega\rangle = \langle\Omega_I(\eta)|O_I(\eta)|\Omega_I(\eta)\rangle \\ &= \langle\Omega|\overline{T}\,\mathrm{e}^{i\int_{\eta_0}^{\eta} d\eta'\,\mathcal{H}^I_{\mathrm{int}}(\eta')}O_I(\eta)T\,\mathrm{e}^{-i\int_{\eta_0}^{\eta} d\eta'\,\mathcal{H}^I_{\mathrm{int}}(\eta')}|\Omega\rangle\ , \quad \forall\,\eta\,.\end{aligned} \tag{1.4.9}$$

The next step consists in finding a suitable relation between $|\Omega\rangle$ and the free theory BD vacuum $|0\rangle$, valid at least in the $\eta \to -\infty$ limit. We already know $|0\rangle$ does not minimize globally $\mathcal{H}_0$ (it is neither an eigenvector of it); nevertheless, it reproduces asymptotically the Minkowski vacuum, so we can assume it a minimum energy eigenvector of $\mathcal{H}_0$ in the past infinity. An analogous argument could be done for $|\Omega\rangle$ as minimum energy eigenvector of $\mathcal{H}_{tot}$ for $\eta \to -\infty$.
Taking now a complete set of $\mathcal{H}_{tot}$ past-infinity approximate eigenvectors $|N\rangle$, including $|\Omega\rangle$, we can express $|0\rangle$ proceeding as:

$$\begin{aligned}\lim_{\eta_0\to-\infty}\mathrm{e}^{-i\mathcal{H}_{tot}(\eta-\eta_0)}\,|0\rangle &= \sum_N \mathrm{e}^{-iE_N(\eta-\eta_0)}\,|N\rangle\,\langle N|0\rangle \\ &= \mathrm{e}^{-iE_\Omega(\eta-\eta_o)}\,|\Omega\rangle\,\langle\Omega|0\rangle + \sum_{N'\neq\Omega}\mathrm{e}^{-iE_{N'}(\eta-\eta_0)}\,|N'\rangle\,\langle N'|0\rangle\ ,\end{aligned}$$

where $\mathcal{H}_{tot}$ is assumed to be evaluated also near to past infinity. To get rid of the $N' \neq \Omega$ terms we can slightly rotate η in the complex:

$$\eta \mapsto \eta^- := \eta(1 - i\varepsilon)\,, \quad \varepsilon > 0\,, \tag{1.4.10}$$

by which a decaying exponential factor $\mathrm{e}^{\varepsilon E_N \eta_0}$ appear in front of each term of 1.4.10. Reversing the expression for $|\Omega\rangle$:

$$\begin{aligned}|\Omega\rangle &= \lim_{\eta_0^-\to-\infty}\left[\frac{\mathrm{e}^{iE_\Omega(\eta^- - \eta_0^-)}}{\langle\Omega|0\rangle}\left(\mathrm{e}^{-i\mathcal{H}_{tot}(\eta^- - \eta_0^-)}\,|0\rangle - \sum_{N'\neq\Omega}\mathrm{e}^{-iE_{N'}\left(\eta^- - \eta_0^-\right)}\,|N'\rangle\,\langle N'|0\rangle\right)\right] \\ &= \lim_{\eta_0^-\to-\infty}\frac{\mathrm{e}^{iE_\Omega(\eta^- - \eta_0^-)}}{\langle\Omega|0\rangle}\,\mathrm{e}^{-i\mathcal{H}_{tot}(\eta^- - \eta_0^-)}\,|0\rangle\ ,\end{aligned} \tag{1.4.11}$$

where the vanishing of the $N' \neq \Omega$ terms is due to E_Ω being the minimum eigenvalue. So, the searched relation is:

$$\lim_{\eta_0^-\to-\infty}\mathrm{e}^{-i\mathcal{H}_{tot}(\eta^- - \eta_0^-)}\,|\Omega\rangle = \lim_{\eta_0^-\to-\infty}\frac{\mathrm{e}^{-i\mathcal{H}_{tot}(\eta^- - \eta_0^-)}}{\langle\Omega|0\rangle}\,|0\rangle\ . \tag{1.4.12}$$

Applying this to 1.4.9, together with the prescription 1.4.10, and choosing $\eta_0 \to -\infty$ we get:

$$\langle O(\eta)\rangle = \frac{1}{|\langle\Omega|0\rangle|^2}\,\langle 0|\overline{T}\,\mathrm{e}^{i\int_{-\infty^+}^{\eta} d\eta'\,\mathcal{H}^I_{int}(\eta')} O_I(\eta) T\,\mathrm{e}^{-i\int_{-\infty^-}^{\eta} d\eta'\,\mathcal{H}^I_{int}(\eta')}|0\rangle\,, \tag{1.4.13}$$

where $-\infty := -\infty\,(1 \pm i\varepsilon)$. The prefactor $|\langle\Omega|0\rangle|^{-2}$ can be easily fixed assuming the identity as operator, gaining:

$$\langle\Omega|\Omega\rangle = \frac{\langle 0|F^\dagger F|0\rangle}{|\,\langle\Omega|0\rangle\,|^2} = \frac{\langle 0|0\rangle}{|\,\langle\Omega|0\rangle\,|^2} \implies |\,\langle\Omega|0\rangle\,|^2 = \frac{\langle 0|0\rangle}{\langle\Omega|\Omega\rangle} = 1. \tag{1.4.14}$$

Therefore we have finally derived the **Master Formula** of the Schwinger-Keldysh formalism:

$$\langle O(\eta)\rangle = \langle 0|\overline{T}\,\mathrm{e}^{i\int_{-\infty^+}^{\eta} d\eta'\,\mathcal{H}^I_{int}(\eta')} O_I(\eta) T\,\mathrm{e}^{-i\int_{-\infty^-}^{\eta} d\eta'\,\mathcal{H}^I_{int}(\eta')}|0\rangle\,. \tag{1.4.15}$$

It is the analogous of the Dyson Formula used in scattering theory, which expresses in-out states transition amplitudes via free correlation functions. The above, instead, gives in-in expectation values and is used in all phenomenological situations where these are preferred respect to the scattering approach, like, other than cosmology, for thermal QFT or when the vacuum of the theory is a non-equilibrium one. The difference with 1.4.1 from an intuitive point of view is that here the BD vacuum $|0\rangle$ is evolved from past infinity to η via the interaction Hamiltonian, then $O_I(\eta)$ is applied on this resulting state, to be after evolved back to past infinity and finally evaluated with the original $|0\rangle$. The integration contour of η' proceeding on the complex plane as $-\infty^- \to \eta \to -\infty^+$ is called **in-in contour**.

As for the Dyson one, also the Master Formula can be directly applied in perturbation theory, expanding the two integrals in powers of the coupling constants of the various interaction terms. The leading order in $\mathcal{H}_{int}$ is:

$$\langle O(\eta)\rangle^{(1)} = 2\,\mathrm{Im}\left(\int_{-\infty}^{\eta} d\eta'\ \langle 0|O(\eta)\mathcal{H}_{int}(\eta')|0\rangle\right)\,, \tag{1.4.16}$$

while the second order is

$$\begin{aligned}\langle O(\eta)\rangle^{(2)} = &\int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta} d\eta''\ \langle 0|\mathcal{H}_{int}(\eta')O(\eta)\mathcal{H}_{int}(\eta'')|0\rangle \\ &- 2\,\mathrm{Re}\left(\int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta'} d\eta''\ \langle 0|\mathcal{H}_{int}(\eta')\mathcal{H}_{int}(\eta'')O(\eta)|0\rangle\right)\,.\end{aligned} \tag{1.4.17}$$

Feynman diagrams and Feynman rules

After having perturbatively expanded the time-ordered exponentials in 1.4.15, substituting $\mathcal{H}_{int}$ terms and going in Fourier representation, we get n-point correlation functions of the free theory to be computed. This can be afforded using the Wick theorem: a n-point correlator $\langle 0|\hat{\varphi}_{\mathbf{k}_1}(\eta_1), \hat{\varphi}_{\mathbf{k}_2}(\eta_2), \cdots, \hat{\varphi}_{\mathbf{k}_n}(\eta_n)|0\rangle$ is equal to the sum over all possible combinations of Wick contractions between the n field operators. A Wick contraction is defined as:

$$\contraction{}{\varphi}{_{\mathbf{k}}(\eta)}{\varphi}\varphi_{\mathbf{k}}(\eta)\varphi_{\mathbf{k}'}(\eta') = f_k^*(\eta)f_k(\eta')\delta^{(d)}(\mathbf{k}+\mathbf{k}') \equiv W_k(\eta,\eta')\delta^{(d)}(\mathbf{k}+\mathbf{k}')\,, \tag{1.4.18}$$

where $W_k(\eta,\eta')$ is the Fourier transform, along spatial coordinates, of the Wightman propagator 1.3.26.

The next step of the in-in formalism construction are completely analogous to perturbative flat QFT. Indeed, thanks to Wick decomposition, it is possible to trace a set of **Feynman rules** for a diagrammatic representation of the terms constituting the perturbative expansion[27]. The resulting **in-in Feynman diagrams** are distinguished from the scattering ones due to the reference time instant η^* of the correlator $\langle O(\eta^*)\rangle$. As we are using in-in formalism in the context of inflation, we are interested on correlators defined on the reheating surface: for this reason η^* is here called *final* time instant[28]. Another important difference is that, while in scattering theory free propagators are all of the Feynman type, here diagrams are built with both Feynman and Wightman ones. Finally, we can apply Fourier transformation for the spatial coordinates thanks to their translation symmetry, but not on the temporal one, so these diagrams are partially in position space.

Given a cosmological correlator at final time η^* and the interaction Hamiltonian $\mathcal{H}_{int}$, a connected Feynman diagram belonging to its perturbative expansion is constructed following these rules:

1. Consider a plane with η coordinate on the vertical axis, running along the complete in-in contour $(-\infty^-, -\infty^+)$, while spatial coordinates on the horizontal one.
2. Trace an horizontal line at $\eta = \eta^*$, representing the final time surface Σ and dividing

[27] In particular we will focus just on the connected part of in-in correlators, which is that involving interaction processes.

[28] However this name could be misleading, as in-in formalism is very general and does not require at all to be strictly applied to compute correlators on "final" hypersurfaces, like commonly done in inflation theory (where exponential expansion is truncated at a finite time). In fact, we could assume to work in an exact dS which has no finite future time boundary: in this case η^* would be just an arbitrary time instant on which the correlator is defined and has not any "final" behaviour.

the in-in contour in two branches: the "+" one for $(-\infty^-, \eta^*)$; the "−" one for $(\eta^*, -\infty^+)$.

3. Trace the diagram, imposing external legs to end on Σ, while interaction vertexes stay on the bulk.

4. For each vertex placed on one of the two branches, it must be considered another diagram with that vertex placed on the other branch.

5. Lines ending on or crossing Σ represent Wightman propagators $W_k(\eta_1, \eta_2)$, with the assumption $\eta_1 > \eta_2$; lines entirely contained in + represent time-ordered Feynman propagators[29] $\Pi_k(\eta_1, \eta_2)$, while those in − represent the anti-time-ordered ones $\Pi_k^*(\eta_1, \eta_2) = \Pi_k(\eta_2, \eta_1)$.

6. Each vertex in + provides a factor $-i\mathcal{V}$ and an integration $\int_{-\infty^-}^{\eta^*} d\eta\, a^d(\eta)$, while those in − a factor $i\mathcal{V}^\dagger$ and an integration $\int_{-\infty^+}^{\eta^*} d\eta\, a^{d+1}(\eta)$, where the integration variable η is the vertex temporal coordinate and $\mathcal{V}$ is the functional derivative of the Hamiltonian interaction term times a Dirac delta for total spatial momentum conservation.

7. Include an integration over each loop momentum.

8. Divide by the symmetry factors of the diagram.

An important and general property of this machinery is that the sum of two diagram being one the reflection of the other respect Σ is simply given by two times the real part of one of them (which is, of course, the same for both).Examples of in-in Feynman diagrams are shown in Figure 1.7.

Although formally well defined and very general, this perturbative approach is of limited power in the case of dS quantum field theory. The limitation is mainly due to the analytical complexity of desitterian propagators, which is further increased even in the simplest integrations involved in computations of Feynman diagrams. Most of the times these diagrams are not analytically computable and quite always involves special functions, forcing the use of numerical solutions. Also, difficulties increase with higher orders: at the present day, the perturbative approach on dS does not give analytical results beyond one-loop order. Other possible factors of complexity are the spin and mass of involved particles.
These are the main practical reasons under the research of a different and less direct

[29] Fourier space Feynman propagators, here considered, are defined as in 1.3.39, but with $W_k(\eta_1, \eta_2)$ instead of $W(x_1, x_2)$

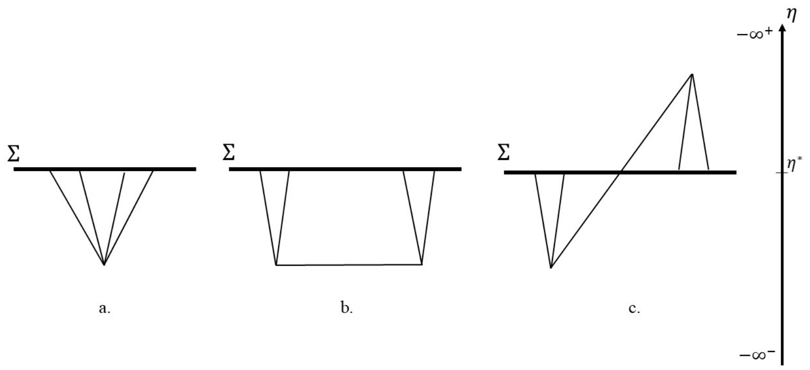

Figure 1.7: Examples of in-in Feynman diagrams. Diagram a. shows a contact diagram of a φ^4 interaction with the vertex in the + branch; b. shows an exchange diagram of a φ^3 interaction with both vertexes in +; c. is the same exchange diagram but with a vertex in + and the other in $-$ branch.

approach, which could circumvent the direct computation of Feynman diagrams. Fortunately, de Sitter spacetime is already provided with a property with a strong computational power, that is its conformal symmetry. It is well known that conformal field theories allow a good control of correlators at a non-perturbative level, completely fixing 3-point functions just by conformal symmetry (the so-called **conformal bootstrap**) [10]. The holographic approach goes exactly in this direction, exploiting the conformal symmetry on the dS_{d+1} time boundary.

Example of application

To both show as the above Feynman rules actually work and how they are very limited to produce analytical results, in this subsection we will present an application to a scalar theory with a cubic interaction, approaching perturbatively a 3-point cosmological correlator.
Schwinger-Keldysh 3- point functions, computed on the reheating hypersurface, are the main interest in the study of inflationary interactions (also known as "*cosmological collider physics*"), as they express the leading order in the non-Gaussianities of cosmological observables. While fields' power spectra, i.e. 2-point functions, encode just the Gaussian behaviour, cosmological 3-point functions are a measure of the dynamics in the first stages of Universe history [16].
In phenomenology the 3-point function is expressed in the **bispectrum** B_φ, defined such

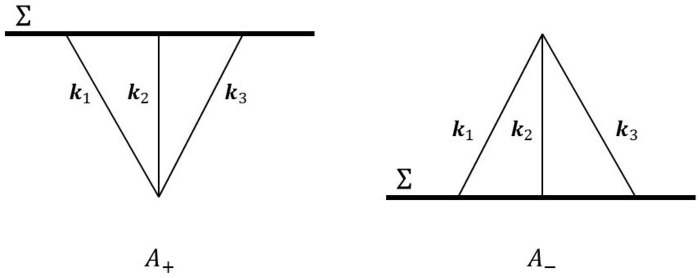

Figure 1.8: The two contact diagrams constituting the leading order of $\langle\varphi_{\mathbf{k}_1}(\eta^*)\varphi_{\mathbf{k}_2}(\eta^*)\varphi_{\mathbf{k}_3}(\eta^*)\rangle$ in a scalar theory with a cubic interaction. These are related by a reflection respect to Σ, so their sum is equal to $2\mathrm{Re}\,(A_\pm)$.

as:

$$\langle\varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\varphi_{\mathbf{k}_3}\rangle \equiv (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{(2\pi^2)^2}{(k_1k_2k_3)^2}B_\varphi(k_1,k_2,k_3)\,. \tag{1.4.19}$$

The full action with a cubic interaction is:

$$\mathcal{S} = \int d\eta\, d^dx\left(-\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{g}{3!}\varphi^3\right)\,. \tag{1.4.20}$$

Thank to the absence of interactions with derivatives, we have $\mathcal{H}_{int} = \mathcal{L}_{int} = -\frac{g}{3!}\varphi^3$ and the vertex is simply the interaction coupling constant $\mathcal{V} = g$. For simplicity we assume $d = 3$.
From 1.3.25 we derive the BD mode function:

$$f_k(\eta) = L^{-1}\sqrt{\frac{\pi}{4}}\,\mathrm{e}^{-i\frac{\pi}{4}(1+2i\mu)}(-\eta)^{\frac{3}{2}}\,H^{(2)}_{i\mu}(-k\eta)\,, \tag{1.4.21}$$

necessary to derive Wightman propagators.

In this example, we aim to compute a 3-point cosmological correlator at leading order (tree level): $\langle\varphi_{\mathbf{k}_1}(\eta^*)\varphi_{\mathbf{k}_2}(\eta^*)\varphi_{\mathbf{k}_3}(\eta^*)\rangle^{(1)}$.
It consists in just the two diagrams A_+ and A_- shown in Figure 1.8: they are called "contact" diagrams as represent the correlation of three excitations of the field φ, at the instant η^*, which have been in contact in the past or will be in the future. The reflection symmetry respect to Σ allows to compute just one of the two diagrams, being $A_+ + A_- = 2\mathrm{Re}\,(A_\pm)$.

Then, we proceed computing A_+:

$$
\begin{aligned}
A_+ &= -iV \int_{-\infty^-}^{\eta^*} d\eta\, a^4(\eta)\, W_{k_1}(\eta^*,\eta) W_{k_2}(\eta^*,\eta) W_{k_3}(\eta^*,\eta) \\
&= -ig\, f^*_{k_1}(\eta^*) f^*_{k_2}(\eta^*) f^*_{k_3}(\eta^*) \int_{-\infty^-}^{\eta^*} d\eta\, a^4(\eta)\, f_{k_1}(\eta) f_{k_2}(\eta) f_{k_3}(\eta) \\
&= -\frac{i\pi^3 g}{64L^2}\, \mathrm{e}^{3\pi \mathrm{Re}(\mu)} \left(-\eta^*\right)^{\frac{9}{2}} \prod_{i=1}^{3} H^{(2)*}_{i\mu}\left(-k_i\eta^*\right) \int_{-\infty^-}^{\eta^*} d\eta\, \left(-\eta\right)^{\frac{1}{2}} \prod_{i=1}^{3} H^{(2)}_{i\mu}\left(-k_i\eta\right) .
\end{aligned}
\tag{1.4.22}
$$

Here we face the first computational complication: the above formula involves an integration of three Hankel functions and it is of difficult analytical solution. There exists a solution in terms of Appell's function F_4 [17], which is however a special function still not completely studied. There are, instead, particular values of μ for which the diagram is of easier treatment. For example, computations simplify much if we assume a mixed interaction with two scalar fields conformally coupled to gravity and one scalar of generic mass. Even simpler cases are those corresponding to a massless or conformally coupled one[30]. We will now approach this latter case, which is the simplest one.

The condition for a conformally coupled field in dS_{d+1} is just $i\mu = \frac{1}{2}$. In this case 1.4.21 reduces to the much more simpler:

$$
f_k(\eta) = L^{-1}(-\eta)\frac{\mathrm{e}^{ik\eta}}{\sqrt{2k}} \,, \tag{1.4.23}
$$

giving the integral:

$$
A_+ = -\frac{ig\,\left(\eta^*\right)^3}{8L^2\, k_1k_2k_3} \int_{-\infty^-}^{\eta^*} \frac{d\eta}{\eta}\, \mathrm{e}^{i(k_1+k_2+k_3)\eta} = -\frac{ig\,\left(\eta^*\right)^3}{8L^2\, k_1k_2k_3} \log(i\left(k_1+k_2+k_3\right)\eta_*) \,, \tag{1.4.24}
$$

which can be easily calculated via a Wick rotation $\eta \to i\eta$. Note, also, that it presents a logarithmic IR divergence for $\eta^* \to 0$, making compulsory the introduction of a cut-off at finite final time. This singularity is removed when we pass to the complete correlator taking the real part of the diagram. Recalling logarithm's branch cut, we have the final result:

$$
\langle \varphi_{\mathbf{k}_1}(\eta^*) \varphi_{\mathbf{k}_2}(\eta^*) \varphi_{\mathbf{k}_3}(\eta^*) \rangle^{(1)} = \frac{\pi g}{8L^2} \frac{(\eta^*)^3}{k_1k_2k_3} \,. \tag{1.4.25}
$$

[30] In these cases we assume all the three fields of the correlator to be of the same type.

Chapter 2

Building dS/CFT

The AdS/CFT duality is able to describe the AdS_{d+1} bulk processes through boundary correlators of a CFT_d theory, which are determined by conformal bootstrap. Being the conformal boundary of AdS is a timelike hypersurface, its correspondent CFT is a usual casual theory, with one time direction.
on the other hand, the fundamental difference of dS/CFT with the previous case regards exactly the causal structure of the conformal boundary. This latter is the hypersurface gained taking the future infinite limit in the time direction of the expanding Poincaré patch. It is clearly a spacelike one, constraining the boundary CFT to be an Euclidean theory[1]. In this chapter we will analyse the properties of this boundary theory starting from its holographic construction.

A way to bypass (at least perturbatively) the explained difficulties for a direct dS/CFT consists in exploiting an intermediate map between in-in Feynman diagrams in dS_{d+1} and Witten diagrams in the Euclidean AdS_{d+1} [5]. As seen, the latter enjoys a well established relation to the ECFT counterpart, making possible a perturbative holographic description of quantum field theory in dS.

2.1 Mapping propagators

The before mentioned map is built expressing the main ingredient of in-in diagrams, i.e. free dS propagators, in terms of EAdS propagators. To achieve this goal, it will be crucial to exploit a powerful mathematical tool which is technical core of this construction: the

[1] In other words, the time direction of the ECFT will be imaginary.

Mellin transform. It exploits the dilatation symmetry, which in (A)dS[2] substitute the time translation one, representing most of the involved functions in a Fourier-like manner, simplifying all the computations.
After this step we will be able to apply the relation to contact and exchange diagrams, which are the most important ones in cosmological QFT.

Note: spinning representations. In the previous introductory chapters we have neglected spin (which is allowed for UIRs) for ease of presentation. In the next it is necessary to reintroduce it for both completeness and non-triviality of the results. Thus are here shown the relevant modifications to the notation due to spin[3] $J \neq 0$.
The mass of the representation in the dS case is

$$m^2 L^2 = \Delta^+ \Delta^- + J\,, \tag{2.1.1}$$

while in AdS

$$m^2 R^2 = -\left(\Delta^+ \Delta^- + J\right)\,, \tag{2.1.2}$$

and it is still $\Delta^\pm = \frac{d}{2} \pm i\mu$. In both cases the future limit behaviour of the spinning field φ_J is:

$$\lim_{\lambda \to 0} \varphi_J(\lambda, \mathbf{x}) = \chi^+(\mathbf{x})\lambda^{\Delta^+ - J} + \chi^-(\mathbf{x})\lambda^{\Delta^- - J}, \tag{2.1.3}$$

where λ is either η or z, and $\chi^\pm$ are respectively the Dirichlet and Neumann boundary conditions' spatial components.

AdS/CFT correspondence

Before proceeding with the construction of the dS − EAdS map, it is useful to review some important properties and results regarding QFT in AdS which will be useful later [18].
The AdS$_{d+1}$ spacetime is defined as the set of points of $\mathcal{M}_{d+2}$ satisfying the constraint:

$$-\left(X^0\right)^2 + \left(X^1\right)^2 + \cdots - \left(X^{d+1}\right)^2 = -R^2\,, \tag{2.1.4}$$

where R is the AdS$_{d+1}$ radius. In analogy to the dS case, it is evident that the AdS$_{d+1}$ symmetry group coincides with the group leaving invariant 2.1.4: $SO\,(2, d)$. This is as well the Lorenztian conformal symmetry group. In global conformal coordinates:

[2] In the following the adjective "Euclidean", or prefix "E", could be sometime omitted for ease of presentation. Nevertheless, we will always work in Euclidean signature of AdS spacetime, except if explicitly specified.

[3] In the following we use J and not s for the spin to not make confusion with the Mellin variable.

$-\infty < t < +\infty$, $0 \leq r < \frac{\pi}{2}$, Ω_{d-1}, where the latter indicates the set of angular coordinates of a sphere S^{d-1}. The global metric then is:

$$ds^2 = \frac{R^2}{\cos^2 r}\left(-dt^2 + dr^2 + \sin^2 r\, d\Omega_{d-1}^2\right) , \tag{2.1.5}$$

and describes a cylinder directed along time t, sliced in hyperbolic spatial hypersurfaces[4].

To embed a QFT in AdS_{d+1} it is convenient to work in Euclidean signature Wick rotating X^{d+1}, being careful to perform later an inverse Wick rotation to return to the original Lorentzian AdS_{d+1}. The EAdS_{d+1} manifold satisfies then:

$$-\left(X^0\right)^2 + \left(X^1\right)^2 + \cdots + \left(X^{d+1}\right)^2 = -R^2 , \quad X^0 > 0 . \tag{2.1.6}$$

Here the symmetry group is again $SO\,(1, d+1)$, being the Euclidean conformal group. In this case we have an analogous of dS_{d+1} EPP coordinates but with an Euclidean signature:

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + d\mathbf{x}^2\right) , \tag{2.1.7}$$

with $z > 0$, called the *radial coordinate*, and $\mathbf{x} \in \mathbb{R}^d$.

The limit $z \to +\infty$ corresponds to the conformal boundary. Taking this limit for all the radial coordinates in propagators or correlators gives functions with a structure satisfying conformal symmetry. For example, taking a QFT in (E)AdS with a cubic interaction term, the two point function is:

$$\langle \varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2)\rangle = \frac{1}{(\mathbf{x}_1 - \mathbf{x}_2)^{2\Delta}} + O(g^2) , \tag{2.1.8}$$

where Δ is the scaling dimension of φ. This is exactly the usual form a conformal 2-point correlator of a CFT_d theory.
The 3-point functions takes also a CFT_d form:

$$\langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}} + O(g^3) , \tag{2.1.9}$$

where $x_{ij} := ||\mathbf{x}_i - \mathbf{x}_j||$ and C_{123} is a *structure constant* proportional to the cubic vertex.

In a general CFT the form of correlators involving four or more fields is not completely fixed by conformal symmetry. The 4-point function is in general:

$$\langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\rangle = \frac{\mathcal{A}(u,v)}{(x_{12}^2)^{\Delta}(x_{34}^2)^{\Delta}} , \tag{2.1.10}$$

[4] In addition, the hyperbolic foliation is the only allowed in AdS.

where $\mathcal{A}(u,v)$ is an unconstrained function of the two harmonic ratios:

$$u := \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,. \tag{2.1.11}$$

Nevertheless, another peculiarity of conformal field theories it that the 4-point functions can be expanded in a series of functions called **conformal blocks**, encoding contributions to the correlators brought by operators of the theory [10]:

$$\langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\rangle = \sum_k C_{12k}C_{k34}G_{\Delta_k,l_k}(u,v)\,. \tag{2.1.12}$$

In a CFT and in (E)AdS the above series converges. This is proven thanks to very crucial feature of these theories: the **state-operator correspondence**. In few words, it consists in the possibility of building a map between states and local operators in the theory, which depends on the spacetime geometry where the theory is embedded. In the case of flat CFT$_d$ each state corresponds to a time constant sphere of $\mathbb{R}^{d-1}$ centred on the origin. Shrinking the sphere very near to the origin corresponds to taking a $t \to -\infty$ limit, imposes a localized boundary condition and, therefore, define a local operator correspondent to the original state.
Expression 2.1.12 can be expressed in terms of local operators, taking the name of *operator product expansion* (OPE).

Such a construction is possible in (E)AdS if we take a time constant half sphere shrinking to the conformal boundary at spatial infinity, thanks to the hyperbolic spatial geometry.
An analogous method would not work for dS, as if we push a time constant hypersurface in the conformal boundary direction, it will grow instead of shrink, making impossible the definition of a local operator on I^+. Until now a desitterian state-operator correspondence is lacking, so without the guarantee of convergence of the conformal block, dS can not still enjoy a *non-perturbative* holographic correspondence. Nevertheless, the dS $\leftrightarrow$ EAdS method here presented allows dS QFT to borrow the AdS/CFT results at all orders in perturbation theory.

Analytic continuation

The map which is at the core of this dS-EAdS QFT relation is built simply starting from a geometrical relation, which is the *analytic continuation* between the dS$_{d+1}$ EPP metric and the EAdS$_{d+1}$ one[5], via Wick rotation of, respectively, the temporal and radial

[5]As in the previous chapter, we proceed working in conformal coordinates.

coordinates and their radii:

$$z = -\eta e^{\pm\frac{i\pi}{2}}, \qquad R = -iL, \tag{2.1.13}$$

$$ds^2_{\mathrm{dS}} = \frac{L^2}{\eta^2}\left(-d\eta^2 + d\mathbf{x}^2\right) \longleftrightarrow ds^2_{\mathrm{EAdS}} = \frac{R^2}{z^2}\left(dz^2 + d\mathbf{x}^2\right), \tag{2.1.14}$$

where the $\pm$ in the z transformation is related to which one of the $\pm$ branches of the in-in contour we are working with. Being now clear the relation between L and R, henceforth we will assume $L = 1$ as it will not be relevant anymore.
At this point there are two possible ways of proceeding: one consists in finding a relation between boundary correlators in EAdS$_{d+1}$ and Schwinger-Keldysh boundary correlators of dS$_{d+1}$, which is the one we will follow; the second, instead, finds a relation between the same EAdS$_{d+1}$ functions and dS$_{d+1}$ wavefunction coefficients, defined in the so-called "*wavefunction of the Universe*" approach [19]. The two methods are equivalent as one could compute dS boundary correlators via wavefunction coefficients expectation values [20].

The above relation can be immediately applied to bulk-to-boundary propagators in (A)dS, which are defined taking the infinite limit of the temporal (radial) coordinate of one of the two arguments of a bulk-to-bulk propagator. Then, an (A)dS bulk-to-boundary propagator for a field of conformal dimension Δ and spin J is:

$$K^{\mathrm{(A)dS}}_{\Delta,J}\left(\lambda, \mathbf{x}; \mathbf{x}'\right) := \lim_{\lambda'\to 0} \Pi^{\mathrm{(A)dS}}_{\Delta,J}\left(\lambda, \mathbf{x}; \lambda', \mathbf{x}'\right), \tag{2.1.15}$$

where again $\lambda = \eta, z$ and Δ could be either $\Delta^{\pm}$. $K^{\mathrm{dS}}_{\Delta,s}$ reduces to the Wightman propagator, being one of the two extremes fixed on the final surface of the in-in contour[6], nevertheless paying attention to which branch the free extreme is positioned. For this reason we define the more accurate notation for $K^{\mathrm{dS}}_{\Delta,J}$ propagators:

$$K^{+}_{\Delta,J}\left(\eta, \mathbf{x}; \mathbf{x}'\right) := \lim_{\eta'\to 0} W\left(\eta, \mathbf{x}; \eta', \mathbf{x}'\right), \qquad K^{-}_{\Delta,J}\left(\eta, \mathbf{x}; \mathbf{x}'\right) := \lim_{\eta'\to 0} W^{*}\left(\eta, \mathbf{x}; \eta', \mathbf{x}'\right). \tag{2.1.16}$$

Under the condition of setting the Bunch-Davies vacuum for all fields, the analytic continuation relates bulk-to-boundary propagators of the two theories [21]:

$$K^{\pm}_{\Delta,J}\left(\eta, \mathbf{x}; \mathbf{x}'\right) = c^{\text{dS-AdS}}_{\Delta}\ \mathrm{e}^{\mp\Delta\frac{i\pi}{2}} K^{\mathrm{AdS}}_{\Delta,J}\left(-\eta\, \mathrm{e}^{\pm\frac{i\pi}{2}}, \mathbf{x}; \mathbf{x}'\right), \tag{2.1.17}$$

where the exponential factor is typical of a tensorial dilatation transformation and $c^{\mathrm{dS-AdS}}_{\Delta}$ changes the normalisation in going from EAdS$_{d+1}$ to dS$_{d+1}$:

$$c^{\text{dS-AdS}}_{\Delta} = \frac{C^{\mathrm{dS}}_{\Delta,J}}{C^{\mathrm{AdS}}_{\Delta,J}} = \frac{\Gamma\left(\frac{d}{2} - \Delta\right)\Gamma\left(\Delta - \frac{d}{2} + 1\right)}{2\pi} = \frac{1}{2}\csc\left(\left(\tfrac{d}{2} - \Delta\right)\pi\right), \tag{2.1.18}$$

[6] It is important to stress that from now on the final surface of the in-in contour coincides with the conformal boundary I^{+}.

with $C^{(\mathrm{A})\mathrm{dS}}_{\Delta,J}$ being the normalization coefficients of $K^{(\mathrm{A})\mathrm{dS}}_{\Delta,J}$:

$$C^{\mathrm{dS}}_{\Delta,J} = \frac{\Delta+J-1}{\Delta-1}\frac{\Gamma\left(\frac{d}{2}-\Delta\right)\Gamma(\Delta)}{2\pi^{\frac{d+1}{2}}}\,, \tag{2.1.19}$$

$$C^{\mathrm{AdS}}_{\Delta,J} = \frac{\Delta+J-1}{\Delta-1}\frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}+1\right)}\,. \tag{2.1.20}$$

In order to extend this relation to bulk-to-bulk propagators, we need a bridge between them and the bulk-to-boundary ones. This is provided by **harmonic functions** $\Omega^{\mathrm{AdS}}_{\nu,J}$, which are the analogue of Wightman functions in AdS, thus satisfying the homogeneous wave equation:

$$\left(\nabla^2-m^2\right)\Omega^{\mathrm{AdS}}_{\nu,J}(x_1,x_2) = \left(\nabla^2+\left(\frac{d}{2}+i\nu\right)\left(\frac{d}{2}-i\nu\right)+J\right)\Omega^{\mathrm{AdS}}_{\nu,J}(x_1,x_2) = 0\,, \tag{2.1.21}$$

where ∇^2 is the EAdS_{d+1} Laplacian. Also, they are trace-less, have a null divergence and form an orthogonal basis parametrized by ν and J. From the case $J=0$ it is evident their analogy to Wightman functions [22]:

$$\Omega^{\mathrm{AdS}}_{\nu,0}(x_1,x_2) = \frac{1}{\Gamma(i\nu)\Gamma(-i\nu)}\frac{\Gamma\left(\frac{d}{2}+i\nu\right)\Gamma\left(\frac{d}{2}-i\nu\right)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)}\,{}_2F_1\left(\frac{d}{2}+i\nu,\frac{d}{2}-i\nu;\frac{d+1}{2};\sigma_{\mathrm{AdS}}\right)\,, \tag{2.1.22}$$

where, similarly to 1.3.27, σ_{AdS} is:

$$\sigma_{\mathrm{AdS}} = 1-\frac{(z_1+z_2)^2+(\mathbf{x}_1-\mathbf{x}_2)^2}{4z_1z_2}\,. \tag{2.1.23}$$

These functions can be written in terms of $K^{\mathrm{AdS}}_{\Delta,J}$ via the so-called *split representation* [23]:

$$\Omega^{\mathrm{AdS}}_{\nu,J}(z_1,\mathbf{x}_1;z_2,\mathbf{x}_2) = \frac{\nu^2}{\pi}\int d^dx'\,K^{\mathrm{AdS}}_{\frac{d}{2}+i\nu,J}(z_1,\mathbf{x}_1;\mathbf{x}')\,K^{\mathrm{AdS}}_{\frac{d}{2}-i\nu,J}(z_2,\mathbf{x}_2;\mathbf{x}')\,, \tag{2.1.24}$$

whose name is motivated by the fact the two bulk-to-boundary correlators are integrated over a contact point on the boundary.
In turn, harmonic functions can be used to express bulk-to-bulk EAdS_{d+1} propagators in the form of a *spectral decomposition*. This relation must be distinguished for the two $\Delta^{\pm}$ conditions. For the Dirichlet:

$$\Pi^{\mathrm{AdS}}_{\Delta_+,J}(x_1,x_2) = \int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\left(\Delta^+-\frac{d}{2}\right)^2}\,\Omega^{\mathrm{AdS}}_{\nu,J}(x_1,x_2)+\text{contact}\,, \tag{2.1.25}$$

with "contact" expressing a series of *contact terms*, i.e. contributions in which the propagator shrinks to a point, arising from harmonic functions with inferior spin or from additional derivatives interactions[7] [21].
For the Neumann we have:

$$\Pi^{\text{AdS}}_{\Delta_-,J}(x_1,x_2) = \frac{2\pi i}{\mu}\,\Omega^{\text{AdS}}_{\mu,J}(x_1,x_2) + \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2+\left(\Delta^+ - \frac{d}{2}\right)^2}\,\Omega^{\text{AdS}}_{\nu,J}(x_1,x_2) + \text{contact}\,. \tag{2.1.26}$$

Put together, 2.1.25 and 2.1.26 provide a inverse relation for $\Omega^{\text{AdS}}_{\nu,J}$ in function of $\Pi^{\text{AdS}}_{\Delta,J}$:

$$\Omega^{\text{AdS}}_{\nu,J}(x_1,x_2) = \frac{i\nu}{2\pi}\left(\Pi^{\text{AdS}}_{\frac{d}{2}+i\nu,J}(x_1,x_2) - \Pi^{\text{AdS}}_{\frac{d}{2}-i\nu,J}(x_1,x_2)\right)\,. \tag{2.1.27}$$

Using now 2.1.17 in 2.1.24, it is straightforward to analytically continue $\Omega^{\text{AdS}}_{\nu,J}$ to dS_{d+1}, paying attention to distinguish the in-in branches for each of the two points of the harmonic function, so defining the new **harmonic function in dS**:

$$\begin{aligned}\Omega^{\pm,\hat{\pm}}_{\nu,J}(\eta_1,\mathbf{x}_1;\eta_2,\mathbf{x}_2) :=\, & c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}\, c^{\text{dS-AdS}}_{\frac{d}{2}-i\nu}\, \mathrm{e}^{\mp i\frac{\pi}{2}\left(\frac{d}{2}+i\nu\right)}\, \mathrm{e}^{\hat{\mp} i\frac{\pi}{2}\left(\frac{d}{2}-i\nu\right)} \\ & \cdot\, \Omega^{\text{AdS}}_{\nu,J}\left(-\eta_1\,\mathrm{e}^{\pm i\frac{\pi}{2}},\mathbf{x}_1;-\eta_2\,\mathrm{e}^{\hat{\pm} i\frac{\pi}{2}},\mathbf{x}_2\right)\,.\end{aligned} \tag{2.1.28}$$

The relation with in-in bulk-to-boundary propagators is analogous to 2.1.24:

$$\Omega^{\pm,\hat{\pm}}_{\nu,J}(\eta_1,\mathbf{x}_1;\eta_2,\mathbf{x}_2) = \frac{\nu^2}{\pi}\int d^dx'\,K^{\pm}_{\frac{d}{2}+i\nu,J}(\eta_1,\mathbf{x}_1;\mathbf{x}')\,K^{\hat{\pm}}_{\frac{d}{2}-i\nu,J}(\eta_2,\mathbf{x}_2;\mathbf{x}')\,. \tag{2.1.29}$$

Using directly 2.1.28 with the EAdS spectral decomposition would involve also the two exponential factors in the integration. A much powerful method to approach this step requires the introduction of a new mathematical object.

The Mellin-Barnes representation

The Mellin-Barnes representation is the analogous of the Fourier one when, along a direction (or coordinate), instead of translation invariance there is a scale invariance.
Besides the physical meaning of momentum space, the Fourier transform $\mathcal{F}$ is a mathematical tool of natural application when a physical system is invariant under translation

[7]So, when inserted in Keldysh/Witten diagrams, they generate contact diagrams involving fields of inferior spin or derivative interactions.

of one or more coordinates. Taken a generic function of single variable $f(x) \in L^2(\mathbb{R}, dx)$, the direct and inverse Fourier transforms are:

$$f(k) = \mathcal{F}\{f(x)\} := \int_{-\infty}^{+\infty} dx\, f(x)\, \mathrm{e}^{-ikx}\,, \tag{2.1.30}$$

$$f(x) = \mathcal{F}^{-1}\{f(k)\} := \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, f(k)\, \mathrm{e}^{+ikx}\,. \tag{2.1.31}$$

In particular, the last one takes also the name of *Fourier representation* of $f(x)$. The completely delocalized wave functions $e^{\pm ikx}$ absorb the position dependence and form an orthonormal basis of eigenvectors of translation generator $P_x = -i\partial_x$ of eigenvalues $\pm k$. If the physical system is symmetric under translations, integrations involving translation invariant measures dx in momentum space are simply reduced to Dirac deltas expressing momentum conservation:

$$\int d^dx\; \mathrm{e}^{-i\mathbf{x}\cdot\sum_{i=1}^n \mathbf{k}_i} = (2\pi)^d\, \delta^{(d)}\left(\sum_{i=1}^n \mathbf{k}_i\right)\,. \tag{2.1.32}$$

In this manner, in addition to make explicit important conservation properties, we can significantly simplify analytical computations, for example avoiding to deal with Bessel functions for free propagators in Minkowskian QFT.

We have seen dS and (E)AdS present a dilatation symmetry along respectively the temporal and radial coordinates[8], so the Mellin-Barnes representation is the most appropriate framework to treat functions of them.
The *Mellin transform* $\mathcal{M}$ of a function $f(z) \in L^2\left((0,+\infty)\,, \frac{dz}{z}\right)$ is defined as[9] [24]:

$$f(s) = \mathcal{M}\{f(z)\} := \int_0^{+\infty} \frac{dz}{z}\, f(z)\, z^{2s-\frac{d}{2}}\,, \tag{2.1.33}$$

where and its inverse, which is as well the *Mellin-Barnes representation* of $f(z)$, is:

$$f(z) = \mathcal{M}^{-1}\{f(s)\} := \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i}\, 2\, f(s)\, z^{-\left(2s-\frac{d}{2}\right)}\,, \tag{2.1.34}$$

which are normalized to get the identity when applied one after the other. The **Mellin variable** s is imaginary and is the analogue of momentum k. This time the integration measure $\frac{dz}{z}$ is scale invariant, making $\mathcal{M}$ preserving this symmetry. Monomials $z^{\pm\left(2s-\frac{d}{2}\right)}$ take the place of Fourier modes, constituting an orthonormal basis of eigenvectors of

[8] At least in global coordinates

[9] This definition directly adapt to functions of the EAdS_{d+1} z coordinate; in the case of function of η the interval of definition and integration is $(-\infty, 0)$.

dilatations generator $D = z\partial_z$, of eigenvalues $\pm\left(2s - \frac{d}{2}\right)$. Finally, in scale invariant integrals of Mellin transformed functions, the z-like variables decouple and produce Dirac deltas conserving s, due to the property:

$$\int_0^\infty \frac{dz}{z^{d+\bar{d}+1}}\, z^{-\sum_{i=1}^n\left(2s_i-\frac{d}{2}\right)} = 2\pi i\,\delta\left(d+\bar{d}+\sum_{i=1}^n\left(2s_i-\frac{d}{2}\right)\right)\,, \tag{2.1.35}$$

where in general there should be considered an extra scaling dimension $\bar{d}$, due to tensorial or derivative vertex interactions[10].

Then, working in both Mellin and Fourier representations, the former for the scale invariant coordinate, the latter for translation invariant ones, we derive that each n-points vertex, in Keldysh or Witten diagrams, the integral over η or z is simply solved in a conservation delta function like:

$$2\pi i\,\delta\left(d+\bar{d}+\left(2s_1-\frac{d}{2}\right)+\ldots+\left(2s_n-\frac{d}{2}\right)\right)(2\pi)^d\,\delta^{(d)}(\mathbf{k}_1+\ldots+\mathbf{k}_n)\,. \tag{2.1.36}$$

Bulk-to-bulk propagators

Now we can approach the relations 2.1.17 and 2.1.29 in Mellin space.
Bulk-to-boundary propagators are represented as:

$$K^{\mathrm{AdS}}_{\Delta,J}(z,\mathbf{k}) = \int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}\,K^{\mathrm{AdS}}_{\Delta,J}(s,\mathbf{k})\,z^{-2s+\frac{d}{2}}\,, \tag{2.1.37}$$

$$K^{\pm}_{\Delta,J}(\eta,\mathbf{k}) = \int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}\,K^{\pm}_{\Delta,J}(s,\mathbf{k})\,(-\eta)^{-2s+\frac{d}{2}}\,, \tag{2.1.38}$$

where we name s the *external* Mellin variable, to make clear it refers to a boundary leg. For the harmonic function there are two Mellin variables:

$$\Omega^{\mathrm{AdS}}_{\nu,J}(z,\mathbf{k};\bar{z},\bar{\mathbf{k}}) = \int_{-i\infty}^{+i\infty}\frac{du}{2\pi i}\frac{d\bar{u}}{2\pi i}\,\Omega^{\mathrm{AdS}}_{\nu,J}(u,\mathbf{k};\bar{u},\bar{\mathbf{k}})\,z^{-2u+\frac{d}{2}}\,\bar{z}^{-2\bar{u}+\frac{d}{2}}\,, \tag{2.1.39}$$

$$\Omega^{\pm,\hat{\pm}}_{\nu,J}(\eta,\mathbf{k};\bar{\eta},\bar{\mathbf{k}}) = \int_{-i\infty}^{+i\infty}\frac{du}{2\pi i}\frac{d\bar{u}}{2\pi i}\,\Omega^{\pm,\hat{\pm}}_{\nu,J}(u,\mathbf{k};\bar{u},\bar{\mathbf{k}})\,(-\eta)^{-2u+\frac{d}{2}}\,(-\bar{\eta})^{-2\bar{u}+\frac{d}{2}}\,, \tag{2.1.40}$$

where this time u, $\bar{u}$ are called *internal* Mellin variables, as they are conjugated to bulk points.

[10] In this work we can always assume $\bar{d}=0$.

The analytic continuation of bulk-to-boundary propagators is Mellin space reduces to a simple phase shift:

$$K^{\pm}_{\Delta,J}(s,\mathbf{k}) = c^{\text{dS-AdS}}_{\Delta}\, \mathrm{e}^{\mp\left(s+\frac{1}{2}\left(\Delta-\frac{d}{2}\right)\right)\pi i}\, K^{\text{AdS}}_{\Delta,J}(s,\mathbf{k})\,. \tag{2.1.41}$$

To express the split representation of harmonic functions in a suitable way for further applications, we can use both Fourier and Mellin transforms. Now 2.1.24 and 2.1.29 become:

$$\Omega^{\text{AdS}}_{\nu,J}(u,\mathbf{p};\bar{u},-\mathbf{p}) = \frac{\nu^2}{\pi} K^{\text{AdS}}_{\frac{d}{2}+i\nu,J}(u,\mathbf{p})\, K^{\text{AdS}}_{\frac{d}{2}-i\nu,J}(\bar{u},-\mathbf{p})\,, \tag{2.1.42}$$

$$\Omega^{\pm,\hat{\pm}}_{\nu,J}(u,\mathbf{p};\bar{u},-\mathbf{p}) = \frac{\nu^2}{\pi} K^{\pm}_{\frac{d}{2}+i\nu,J}(u,\mathbf{p})\, K^{\pm}_{\frac{d}{2}-i\nu,J}(\bar{u},-\mathbf{p})\,, \tag{2.1.43}$$

having used a Dirac delta on momenta. It is worth noting they are factorised in u and $\bar{u}$.
Eventually, even for harmonic functions in Mellin space we have simple phase shifting relation between dS and EAdS ones:

$$\Omega^{\pm,\hat{\pm}}_{\nu,J}(u,\mathbf{p};\bar{u},-\mathbf{p}) = c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}\, c^{\text{dS-AdS}}_{\frac{d}{2}-i\nu}\, \mathrm{e}^{\mp\left(u+\frac{i\nu}{2}\right)\pi i}\, \mathrm{e}^{\hat{\mp}\left(\bar{u}-\frac{i\nu}{2}\right)\pi i}\, \Omega^{\text{AdS}}_{\nu,J}(u,\mathbf{p};\bar{u},-\mathbf{p}). \tag{2.1.44}$$

We are now ready to work on bulk-to-bulk propagators, preparing them for dS-EAdS relation, which will be the basis to relate diagrams.
For first, let's Fourier and Mellin transform $\Pi^{\text{AdS}}_{\Delta^{\pm},J}(\eta,\mathbf{x};\bar{\eta},\bar{\mathbf{x}})$:

$$\Pi^{\text{AdS}}_{\Delta^{\pm},J}(z,\mathbf{k};\bar{z},\bar{\mathbf{k}}) = \int_{-i\infty}^{+i\infty} \frac{du}{2\pi i}\frac{d\bar{u}}{2\pi i}\, \Pi^{\text{AdS}}_{\Delta^{\pm},J}(u,\mathbf{k};\bar{u},\bar{\mathbf{k}})\, z^{-2u+\frac{d}{2}}\, \bar{z}^{-2\bar{u}+\frac{d}{2}}\,. \tag{2.1.45}$$

Inserting this and 2.1.42 in the spectral decomposition of both Dirichlet and Neumann propagators, 2.1.25-2.1.26, and integrating on ν, we get a common simple expression:

$$\Pi^{\text{AdS}}_{\Delta^{\pm},J}(u,\mathbf{p};\bar{u},-\mathbf{p}) = \csc\left(\pi\left(u+\bar{u}\right)\right)\, \omega_{\Delta^{\pm}}(u,\bar{u})\, \Gamma(i\mu)\, \Gamma(-i\mu)\, \Omega^{\text{AdS}}_{\mu,J}(u,\mathbf{p};\bar{u},-\mathbf{p})\,. \tag{2.1.46}$$

The term $\csc\left(\pi\left(u+\bar{u}\right)\right)$, being the only which couples u and $\bar{u}$, is interpreted to account for contact terms contributing to the exchanged propagator.
The factor $\omega_{\Delta^{\pm}}$ is a projector to one of the two $\Delta^{\pm}$ boundary conditions:

$$\omega_{\Delta^{\pm}}(u,\bar{u}) = 2\sin\left(\pi\left(u\mp\frac{i\mu}{2}\right)\right)\sin\left(\pi\left(\bar{u}\mp\frac{i\mu}{2}\right)\right)\,, \tag{2.1.47}$$

so, a linear combination $\alpha\omega_{\Delta^+}+\beta\omega_{\Delta^-}$ can be used to define propagators with general boundary conditions:

$$\begin{aligned}\Pi^{\text{AdS}}_{\alpha\Delta^{+}+\beta\Delta^{-},J}(u,\mathbf{p};\bar{u},-\mathbf{p}) = {}& \csc\left(\pi\left(u+\bar{u}\right)\right)\left(\alpha\omega_{\Delta^{+}}(u,\bar{u})+\beta\omega_{\Delta^{-}}(u,\bar{u})\right)\\ &\cdot\Gamma(i\mu)\Gamma(-i\mu)\,\Omega^{\text{AdS}}_{\mu,J}(u,\mathbf{p};\bar{u},-\mathbf{p})\,.\end{aligned} \tag{2.1.48}$$

Now, using 2.1.44 inside 2.1.48, we obtain the definition of a (Mellin transformed) dS_{d+1} bulk-to-bulk propagator **with general boundary condition** (b.c.) [25], so to not be confused with the Bunch-Davies Feynman propagator:

$$\begin{aligned}\Pi^{\pm,\hat{\pm}}_{\alpha\Delta^{+}+\beta\Delta^{-},J}(u,\mathbf{p};\bar{u},-\mathbf{p}) = &\csc\left(\pi(u+\bar{u})\right)\left[\alpha^{\pm,\hat{\pm}}\omega_{\Delta^{+}}(u,\bar{u})+\beta^{\pm,\hat{\pm}}\omega_{\Delta^{-}}(u,\bar{u})\right]\\ &\cdot\Gamma(i\mu)\Gamma(-i\mu)\,\Omega^{\pm\hat{\pm}}_{\mu,J}(u,\mathbf{p};\bar{u},-\mathbf{p})\,,\end{aligned} \tag{2.1.49}$$

which in terms of $\Pi^{\mathrm{AdS}}_{\Delta^{\pm},J}$ is:

$$\begin{aligned}\Pi^{\pm,\hat{\pm}}_{\alpha\Delta^{+}+\beta\Delta^{-},J}(u,\mathbf{p};\bar{u},-\mathbf{p}) = &\, c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}\, c^{\text{dS-AdS}}_{\frac{d}{2}-i\nu}\ \mathrm{e}^{\mp\left(u+\frac{i\mu}{2}\right)\pi i}\ \mathrm{e}^{\hat{\mp}\left(\bar{u}-\frac{i\mu}{2}\right)\pi i}\\ &\cdot\left[\alpha^{\pm,\hat{\pm}}\Pi^{\mathrm{AdS}}_{\frac{d}{2}+i\mu,J}(u,\mathbf{p};\bar{u},-\mathbf{p})+\beta^{\pm,\hat{\pm}}\Pi^{\mathrm{AdS}}_{\frac{d}{2}-i\mu,J}(u,\mathbf{p};\bar{u},-\mathbf{p})\right]\,.\end{aligned} \tag{2.1.50}$$

But all this construction is based on the assumption of choosing the BD vacuum[11] on dS_{d+1}, so we have to find appropriate values for $\alpha^{\pm,\hat{\pm}}$ and $\beta^{\pm,\hat{\pm}}$. To do this, we exploit two relations found in[14, 21], linking dS_{d+1} (Bunch-Davies) Wightman/Feynman propagators and EAdS_{d+1} harmonic functions:

$$\begin{aligned}W^{\mathrm{dS}}_{\mu,J}(\eta,\mathbf{p};\bar{\eta},-\mathbf{p}) \equiv \Pi^{\pm\mp}_{\mu,J}(\eta,\mathbf{p};\bar{\eta},-\mathbf{p}) =&\\ =\Gamma(+i\mu)\Gamma(-i\mu)\,\Omega^{\mathrm{AdS}}_{\mu,J}&\left(-\eta\,\mathrm{e}^{\pm\frac{i\pi}{2}},\mathbf{p};-\bar{\eta}\,\mathrm{e}^{\hat{\pm}\frac{i\pi}{2}},-\mathbf{p}\right)\,,\end{aligned} \tag{2.1.51}$$

$$\begin{aligned}\Pi^{\mathrm{dS}}_{\mu,J}(\eta,\mathbf{p};\bar{\eta},-\mathbf{p}) \equiv \Pi^{\pm\pm}_{\mu,J}(\eta,\mathbf{p};\bar{\eta},-\mathbf{p}) =&\\ =\Gamma(+i\mu)\Gamma(-i\mu)\Big[&\theta(\bar{\eta}-\eta)\,\Omega^{\mathrm{AdS}}_{\mu,J}\left(-\eta\,\mathrm{e}^{\pm\frac{i\pi}{2}},\mathbf{p};-\bar{\eta}\,\mathrm{e}^{\hat{\pm}\frac{i\pi}{2}},-\mathbf{p}\right)\\ &+\theta(\eta-\bar{\eta})\,\Omega^{\mathrm{AdS}}_{\mu,J}\left(-\bar{\eta}\,\mathrm{e}^{\hat{\mp}\frac{i\pi}{2}},\mathbf{p};-\eta\,\mathrm{e}^{\mp\frac{i\pi}{2}},-\mathbf{p}\right)\Big]\,,\end{aligned} \tag{2.1.52}$$

which, going in Mellin space and using 2.1.27, after some computations give:

$$\begin{aligned}\alpha^{\pm\pm} &= \frac{1}{c^{\text{dS-AdS}}_{\frac{d}{2}-i\mu}}\,\mathrm{e}^{\pm\pi\mu}, \quad \beta^{\pm\pm} = \frac{1}{c^{\text{dS-AdS}}_{\frac{d}{2}+i\mu}}\,\mathrm{e}^{\mp\pi\mu},\\ \alpha^{\pm\mp} &= \frac{1}{c^{\text{dS-AdS}}_{\frac{d}{2}-i\mu}}\,\mathrm{e}^{\mp\pi\mu}, \quad \beta^{\pm\mp} = \frac{1}{c^{\text{dS-AdS}}_{\frac{d}{2}+i\mu}}\,\mathrm{e}^{\mp\pi\mu},\end{aligned} \tag{2.1.53}$$

[11] Indeed, as bulk-to-bulk propagators are solutions of the inhomogeneous (with a Dirac delta source) equation of motion, which is a second order differential equation, they can have just two degrees of freedom: one is fixed by normalization, the other by vacuum choice. As BD vacuum fixes the early-time behaviour as a Minkowskian limit, the late-time behaviour is fixed to be a specific linear combination of Dirichlet and Neumann modes.

or, more compactly:

$$\alpha^{\hat{\pm}} := \alpha^{\pm\hat{\pm}} = \frac{1}{c^{\text{dS-AdS}}_{\frac{d}{2}-i\mu}}\,\mathrm{e}^{\hat{\pm}\pi\mu}, \tag{2.1.54}$$

$$\beta^{\hat{\pm}} := \beta^{\pm\hat{\pm}} = \frac{1}{c^{\text{dS-AdS}}_{\frac{d}{2}+i\mu}}\,\mathrm{e}^{\hat{\mp}\pi\mu}\,. \tag{2.1.55}$$

The final relation between dS_{d+1} and EAdS_{d+1} bulk-to-bulk propagators is:

$$\begin{aligned}\Pi^{\pm\hat{\pm}}_{\mu,J}(\eta,\mathbf{p};\bar{\eta},-\mathbf{p}) = {} & c^{\text{dS-AdS}}_{\frac{d}{2}+i\mu}\;\mathrm{e}^{\mp\frac{i\pi}{2}\left(\frac{d}{2}+i\mu\right)}\;\mathrm{e}^{\hat{\mp}\frac{i\pi}{2}\left(\frac{d}{2}+i\mu\right)}\Pi^{\text{AdS}}_{\frac{d}{2}+i\mu,J}\left(-\eta\,\mathrm{e}^{\pm\frac{i\pi}{2}},\mathbf{p};-\bar{\eta}\,\mathrm{e}^{\hat{\pm}\frac{i\pi}{2}},-\mathbf{p}\right)\\ & +(\mu\to-\mu)\,,\end{aligned} \tag{2.1.56}$$

which is is symmetric under $\Delta^+ \leftrightarrow \Delta^-$. Or, in Mellin space:

$$\begin{aligned}\Pi^{\pm\hat{\pm}}_{\mu,J}(u,\mathbf{p};\bar{u},-\mathbf{p}) = {} & c^{\text{dS-AdS}}_{\frac{d}{2}+i\mu}\;\mathrm{e}^{\mp\left(u+\frac{i\mu}{2}\right)\pi i}\;\mathrm{e}^{\hat{\mp}\left(u+\frac{i\mu}{2}\right)\pi i}\Pi^{\text{AdS}}_{\frac{d}{2}+i\mu,J}\left(u,\mathbf{p};\bar{u},-\mathbf{p}\right)\\ & +(\mu\to-\mu)\,,\end{aligned} \tag{2.1.57}$$

So, while $\Pi^{\text{AdS}}_{\alpha\Delta^++\beta\Delta^-,J}$ can be defined in a general linear combination of Dirichlet and Neumann boundary conditions, for $\Pi^{\pm\hat{\pm}}_{\mu,J}$ this freedom is fixed by the choice of the vacuum and its relation with the former depends on the branches on which its extremes are located.
The importance of this result stays in the possibility of use it in perturbation theory, directly substituting it in Schwinger-Keldysh diagrams, which are therefore related to a sum of EAdS Witten diagrams [25]. These, in turn, are related to conformal boundary correlators of an ECFT, leading to a perturbative holographic dS/CFT correspondence.

2.2 Exchange diagrams

In subsection 1.4 we have shown an application of perturbative QFT in dS to compute a cubic contact diagram in the expansion of the 3-point correlator. When we move to 4-point correlators a new type of diagrams, rich of physical information, appear: the **exchange diagrams**. These are characterized by the propagation of a virtual particle in the bulk, providing then information on the mass spectrum of an interactive theory.
They are a good trial bench for the above construction, involving both bulk-to-boundary and bulk-to-bulk propagators. Furthermore, from the boundary ECFT perspective they are not trivial, being related to the conformal block expansion.

Example with scalar fields

Here we present, as an example, an application of the dS $-$ EAdS diagrams relation for an exchange diagram of scalar fields via a cubic interaction without derivatives. We can assume, without exceedingly complications, five different scalar field: we name them $\varphi_i\,,\ i=1,\cdots,4$ for the external legs, φ for the exchanged one. The cubic interactions could also be distinguished as $g_{12\varphi}\varphi_1\varphi_2\varphi$ and $g_{34\varphi}\varphi_3\varphi_4\varphi$.
Working in Mellin and Fourier spaces, the considered Witten diagram in a general boundary condition is:

$$
\begin{aligned}
&A^{\mathrm{AdS}}_{\alpha\Delta^{+}+\beta\Delta^{-},0}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,s_3,\mathbf{k}_3,s_4,\mathbf{k}_4)=\\
&\qquad=g_{12\varphi}\,2\pi i\,\delta\left(-\frac{d}{2}+2s_1+2s_2+2u\right)(2\pi)^d\delta^{(d)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{p})\\
&\qquad\cdot g_{34\varphi}\,2\pi i\,\delta\left(-\frac{d}{2}+2\bar{u}+2s_3+2s_4\right)(2\pi)^d\delta^{(d)}(\mathbf{k}_3+\mathbf{k}_4-\mathbf{p})\\
&\cdot K^{\mathrm{AdS}}_{\Delta_1,0}(s_1,\mathbf{k}_1)\,K^{\mathrm{AdS}}_{\Delta_2,0}(s_2,\mathbf{k}_2)\,\Pi^{\mathrm{AdS}}_{\alpha\Delta^{+}+\beta\Delta^{-},0}(u,\mathbf{p};\bar{u},-\mathbf{p})\,K^{\mathrm{AdS}}_{\Delta_3,0}(s_3,\mathbf{k}_3)\,K^{\mathrm{AdS}}_{\Delta_4,0}(s_4,\mathbf{k}_4)\,,
\end{aligned}
\tag{2.2.1}
$$

while the in-in diagrams, in all the four possible combination respect to the vertexes positions in the two branches of the in-in contour, are:

$$
\begin{aligned}
&A^{\pm\hat{\pm}}_{\mu,0}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,s_3,\mathbf{k}_3,s_4,\mathbf{k}_4)=\\
&\qquad=g_{12\varphi}\,2\pi i\,\delta\left(-\frac{d}{2}+2s_1+2s_2+2u\right)(2\pi)^d\delta^{(d)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{p})\\
&\qquad\cdot g_{34\varphi}\,2\pi i\,\delta\left(-\frac{d}{2}+2\bar{u}+2s_3+2s_4\right)(2\pi)^d\delta^{(d)}(\mathbf{k}_3+\mathbf{k}_4-\mathbf{p})\\
&\cdot K^{\pm}_{\Delta_1,0}(s_1,\mathbf{k}_1)\,K^{\pm}_{\Delta_2,0}(s_2,\mathbf{k}_2)\,\Pi^{\pm\hat{\pm}}_{\mu,0}(u,\mathbf{p};\bar{u},-\mathbf{p})\,K^{\hat{\pm}}_{\Delta_3,0}(s_3,\mathbf{k}_3)\,K^{\hat{\pm}}_{\Delta_4,0}(s_4,\mathbf{k}_4)\,.
\end{aligned}
\tag{2.2.2}
$$

Using the machinery developed in the previous section, the problem has to be approached using relations 2.1.41, 2.1.57 to express each of the $A^{\pm\hat{\pm}}_{\mu,0}$ in terms of the $A^{\mathrm{AdS}}_{\Delta^{\pm},0}$. After the substitutions, recalling $\Delta^{\pm}=\frac{d}{2}\pm i\mu$, 2.2.2 becomes:

$$
\begin{aligned}
&A^{\pm\hat{\pm}}_{\mu,0}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,s_3,\mathbf{k}_3,s_4,\mathbf{k}_4)=\\
&\qquad=\left(\prod_{i=1}^{4}c^{\mathrm{dS-AdS}}_{\Delta_i}\right)c^{\mathrm{dS-AdS}}_{\frac{d}{2}+i\mu}\;\mathrm{e}^{\mp i\frac{\pi}{2}\left(\Delta_1+\Delta_2+\frac{d}{2}+i\mu-d\right)}\;\mathrm{e}^{\hat{\mp}i\frac{\pi}{2}\left(\Delta_3+\Delta_4+\frac{d}{2}+i\mu-d\right)}\\
&\qquad\cdot A^{\mathrm{AdS}}_{\Delta^{+},0}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,s_3,\mathbf{k}_3,s_4,\mathbf{k}_4)+(\mu\to-\mu)\;.
\end{aligned}
\tag{2.2.3}
$$

We recall that the full in-in diagram is composed by the sum of all the above $\pm,\hat{\pm}$ combinations, i.e. $A^{\mathrm{dS}}_{\mu,0}=\sum_{\pm,\hat{\pm}}(\pm i)\,(\hat{\pm}i)A^{\pm\hat{\pm}}_{\mu,0}$, where the$(\pm i)\,(\hat{\pm}i)$ coefficients are proper of in-in Feynman diagrams in Mellin space. The aim is, therefore, to express $A^{\text{dS}}_{\mu,0}$ as a sum of Witten diagrams. This involves a sum of eight Witten diagrams, then recollected in a linear combination of the two with Dirichlet/Neumann boundary conditions:

$$
\begin{aligned}
A^{\text{dS}}_{\mu,0}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,s_3,\mathbf{k}_3,s_4,\mathbf{k}_4) = 4\left(\prod_{i=1}^{4} c^{\text{dS}-\text{AdS}}_{\Delta_i}\right) c^{\text{dS}-\text{AdS}}_{\frac{d}{2}+i\mu} \\
\cdot \sin\left[\left(-d+\Delta_1+\Delta_2+\Delta^+\right)\frac{\pi}{2}\right] \sin\left[\left(-d+\Delta_3+\Delta_4+\Delta^+\right)\frac{\pi}{2}\right] \\
\cdot A^{\text{AdS}}_{\Delta^+,0}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,s_3,\mathbf{k}_3,s_4,\mathbf{k}_4) + (\mu\to-\mu) \,.
\end{aligned}
\tag{2.2.4}
$$

Factorisation via cutting rules

It is worth to use this example to show a general property of both in-in and Witten diagrams, made evident by the use of the Mellin representation. This regard the possibility of *factorise* such diagrams in a product of smaller ones putting on-shell internal propagators, i.e. via *cutting rules*.

The on-shell limit of the EAdS propagator 2.1.46, in the two $\Delta^\pm$ boundary conditions, can be taken via the usual cutting (or Cutkosky) rules, i.e. taking the discontinuity[12] of it respect to the momentum squared p^2. The final result is:

$$
\text{Disc}_{p^2}\left[\Pi^{\text{AdS}}_{\Delta^\pm,J}(u,\mathbf{p};\bar{u},-\mathbf{p})\right] = \omega_{\Delta^\pm}(u,\bar{u})\,\Gamma(i\mu)\,\Gamma(-i\mu)\,\Omega^{\text{AdS}}_{\mu,J}(u,\mathbf{p};\bar{u},-\mathbf{p})\,, \tag{2.2.6}
$$

which is equal to 2.1.46 divided by the $\csc\left(\pi\left(u+\bar{u}\right)\right)$ factor. We can see it is manifestly factorised in u and $\bar{u}$, considering 2.1.47 and the split representation of the harmonic function 2.1.42. Then, in EAdS$_{d+1}$ the on-shell $\Pi^{\text{AdS}}_{\Delta^\pm,J}$ are proportional to a decoupled product of two bulk-to-boundary propagators, so having an analogous of the split representation for them. For general linear combinations of them, having a boundary condition factor $\omega_{(\alpha,\beta)} := \alpha\omega_{\Delta^+} + \beta\omega_{\Delta^-}$, the on-shell propagator is not factorised and is given by 2.2.6 with $\omega_{\Delta^\pm}$ replaced by $\omega_{(\alpha,\beta)}$.
This applies also for dS$_{d+1}$ Wightman and Feynman propagators: using 2.1.49, the result is

$$
\text{Disc}_{p^2}\left[\Pi^{\pm\hat{\pm}}_{\mu,J}(u,\mathbf{p};\bar{u},-\mathbf{p})\right] = \omega_{BD}(u,\bar{u})\,\Gamma(i\mu)\Gamma(-i\mu)\,\Omega^{\pm\hat{\pm}}_{\mu,J}(u,\mathbf{p};\bar{u},-\mathbf{p})\,, \tag{2.2.7}
$$

where $\omega_{BD}(u,\bar{u}) := \alpha^{\hat{\pm}}\omega_{\Delta^+}(u,\bar{u}) + \beta^{\pm}\omega_{\Delta^-}(u,\bar{u})$ is the projector on the BD vacuum.

[12]The discontinuity for of a generic function $f(z)$ is defined as:

$$
\text{Disc}_z\left[f(z)\right] = \frac{i}{2} f\left(\mathrm{e}^{i\pi}z\right) - f\left(\mathrm{e}^{-i\pi}z\right)\,. \tag{2.2.5}
$$

It is clear that applying the cutting rules to an exchange diagrams, computing the discontinuity respect to the internal line, we maintain the above splitting structure, with a boundary condition factor multiplying a product of "split" diagrams. For the example above, we have:

$$\begin{aligned}\mathrm{Disc}_{p^2}\left[A^{\mathrm{AdS}}_{\alpha\Delta^+ + \beta\Delta^-,0}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2;s_3,\mathbf{k}_3,s_4,\mathbf{k}_4)\right] &= \frac{\Gamma(1+i\mu)\Gamma(1-i\mu)}{\pi}\omega_{(\alpha,\beta)}(u,\bar{u}) \\ &\cdot F^{\mathrm{AdS}}_{\Delta_1,\Delta_2,\Delta^+}(s_1,\mathbf{k}_1,s_2,u,\mathbf{p})F^{\mathrm{AdS}}_{\Delta^-,\Delta_3,\Delta_4}(\bar{u},-\mathbf{p},s_3,\mathbf{k}_3,s_4,\mathbf{k}_4)\,,\end{aligned} \tag{2.2.8}$$

$$\begin{aligned}\mathrm{Disc}_{p^2}\left[A^{\pm\hat{\pm}}_{\mu,0}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2;s_3,\mathbf{k}_3,s_4,\mathbf{k}_4)\right] &= (\pm i)(\hat{\pm} i)\,\frac{\Gamma(1+i\mu)\Gamma(1-i\mu)}{\pi}\omega_{BD}(u,\bar{u}) \\ &\cdot F^{\pm}_{\Delta_1,\Delta_2,\Delta^+}(s_1,\mathbf{k}_1,s_2,u,\mathbf{p})F^{\hat{\pm}}_{\Delta^-,\Delta_3,\Delta_4}(\bar{u},-\mathbf{p},s_3,\mathbf{k}_3,s_4,\mathbf{k}_4)\,,\end{aligned} \tag{2.2.9}$$

where we have defined the 3-point contact diagrams:

$$\begin{aligned}F^{\bullet}_{\Delta_i,\Delta_j,\Delta^k}(s_i,\mathbf{k}_i,s_j,\mathbf{k}_j,u,\mathbf{p}) = &- g_{ij\phi}\,(2\pi i)\,\delta\left(-\frac{d}{2}+2s_i+2s_j+2u\right)(2\pi)^d\delta^{(d)}(\mathbf{k}_i+\mathbf{k}_j+\mathbf{p}) \\ &\cdot K^{\bullet}_{\Delta_i,0}(s_i,\mathbf{k}_i)K^{\bullet}_{\Delta_j,0}(s_j,\mathbf{k}_j)K^{\bullet}_{\Delta^k,0}(u,\mathbf{p})\,,\end{aligned} \tag{2.2.10}$$

with $i,j=1,\cdots,4$; $k=\pm$ and $\bullet=\mathrm{AdS},\pm,\hat{\pm}$.
This useful decomposition is represented in Figure 2.1. The discontinuity of the full dS diagram is again the sum:

$$\begin{aligned}\mathrm{Disc}_{p^2}&\left[A^{\mathrm{dS}}_{\mu,0}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2\ ;s_3,\mathbf{k}_3,s_4,\mathbf{k}_4)\right] = \\ &= \sum_{\pm\hat{\pm}}(\pm i)(\hat{\pm} i)\,\mathrm{Disc}_{p^2}\left[A^{\pm\hat{\pm}}_{\mu,0}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2;s_3,\mathbf{k}_3,s_4,\mathbf{k}_4)\right]\,.\end{aligned} \tag{2.2.11}$$

Hence, a general cut (EA)dS exchange diagram, in Mellin-Barnes representation, is equal to a linear combination of products of split diagrams, factorised in the internal Mellin variables.
Instead, the original diagrams in Mellin space are simply re-obtained multiplying 2.2.8 and 2.2.11 for the overall factor $\csc\left(\pi\left(u+\bar{u}\right)\right)$, which reintroduce a coupling between u and $\bar{u}$. Now, performing an inverse Mellin transform, to return in position space, involves a possibly non-trivial integral, which is the analogue of the flat QFT *dispersion formula*.

Conformal bootstrap

The factorisation result, derived via cutting rules in the special case of only scalar fields, can be proven in a more general and meaningful way for exchanges of spinning particles[13].

[13]The following derivation can be further applied to theories involving derivative interactions.

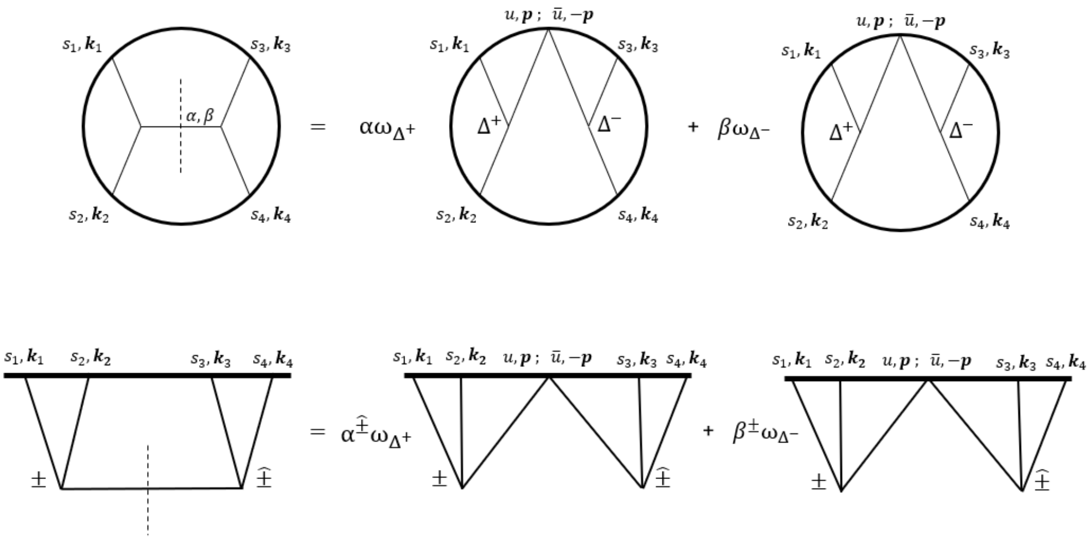

Figure 2.1: Cutting rules for (EA)dS$_{d+1}$ exchange diagrams. Thanks to the split representation of harmonic functions, they can be expressed as linear combinations of 3-point diagrams.

This different approach proceeds to derive the form of bulk exchanges just from assuming the boundary structure of the theory to be conformal [26]. We do not need then to solve the bulk QFT, as done in the previous chapter where we started from the dS$_{d+1}$ wave equation and directly computed an in-in diagram via integration over conformal time. Building the computation of (EA)dS$_{d+1}$ diagrams from ECFT$_d$ solutions, the *conformal bootstrap* institutes the essence of the holographic correspondence.
In addition, this common approach to the two theories will allow to establish a general relation between dS$_{d+1}$ and EAdS$_{d+1}$ exchange diagrams.

We have seen that 4-point correlators of a general (E)CFT can be expanded in terms of *conformal blocks*. These contributions are UIRs of the conformal group $SO(1, d+1)$. In the holographic perspective, if a ECFT$_d$ is interpreted as a theory embedded on the conformal boundary of a (EA)dS$_{d+1}$ spacetime, then such conformal blocks could be seen as bulk exchanges of particles on a certain channel. They satisfy the operatorial representation of the Casimir equation 1.2.15, with $\mathcal{C}_2$ assumed to be a constant. This can be seen as an eigenvalue differential equation of the quadratic Casimir operator of $SO(1, d+1)$.
Working in embedding coordinates $X^M, P^N \in \mathcal{M}_{d+2}$, with the convention that X represents a point of the (EA)dS$_{d+1}$ bulk while P a point of its conformal boundary, the Casimir equation for a **s**-channel contribution of a ECFT$_d$ 4-point boundary correlator

is:

$$\left(\frac{1}{2}L^{MN}L_{MN}-\mathcal{C}_2\right)G(P_1,P_2,P_3,P_4)=0\,, \tag{2.2.12}$$

where we recall the eigenvalues' parametrization $\mathcal{C}_2=\Delta^+\Delta^- - J\,(J+d-2)$ and the differential operators L_{MN} are assumed to act on $P_{1,2}$.
General solutions of it are:

$$G(P_i)=\alpha\, G_{\Delta^+,J}(P_i)+\beta\, G_{\Delta^-,J}(P_i)\,, \tag{2.2.13}$$

with α,β being free coefficients and $G_{\Delta^\pm,J}$ the conformal block:

$$G_{\Delta^\pm,J}(P_i)\sim P_{12}^{\Delta^\pm-J-\Delta_1-\Delta_2/2}, \tag{2.2.14}$$

in the $(P_1-P_2)^2\to 0$ limit.

The link with bulk exchanges comes formally from the result that the Casimir equation applied to a general bulk (EA)dS$_{d+1}$ field $\varphi_J\,(X)$ reduces to the homogeneous wave equation [27]:

$$\left(\frac{1}{2}L^{MN}L_{MN}-\mathcal{C}_2\right)\varphi_J\,(X)=0\implies\left(\nabla^2-m^2\right)\varphi_J\,(X)=0\,. \tag{2.2.15}$$

The homogeneous equation eigenvectors are the harmonic functions $\Omega^{(\mathrm{AdS}),\pm\hat{\pm}}_{\mu,J}$. In Mellin space, these can be independently related, via 2.2.6, to on-shell bulk-to-bulk propagators with $\Delta^\pm$ boundary conditions[14]. In turn, we have seen it is possible to obtain general boundary conditions via linear combinations of the $\omega_{\Delta^\pm}$ projectors, and then on-shell exchange diagram just multiplying by the four bulk-to-boundary propagators.

This proves that the request of conformal symmetry and the choice of boundary conditions impose the form of bulk exchanges in terms of (EA)dS$_{d+1}$ harmonic functions. A whole exchange diagram $A^{\mathrm{AdS},\pm\hat{\pm}}$ results to be a function of four points P_i $i=1,\cdots,4$ of the conformal boundary written in terms of solutions of the Casimir equation (in Mellin-Barnes representation). This suggests that we could express $A^{\mathrm{AdS},\pm\hat{\pm}}$ in terms of conformal blocks, instead of harmonic functions. The conformal bootstrap, as well the holographic approach, resides in this change of ingredients to compute a diagram: instead of using a bulk function, to be computed solving the wave equation , we use a solution of the conformal Casimir equation 2.2.12 applied on the boundary. Choosing a certain b.c. for the Casimir eigenfunction 2.2.13, corresponds to choose a related b.c. for a solution of the bulk homogeneous wave equation; in Mellin space, the b.c. is encoded in the projector prefactor $\omega_{(\alpha,\beta)}$ multiplying the harmonic function.

[14]This result, based on the spectral decomposition 2.1.46, is still independent by factorisation, i.e. split representation, of harmonic functions in bulk-to-boundary propagators.

To put at work this method, we need now to find a boundary dual to harmonic functions $\Omega^{(\mathrm{AdS}),\pm\hat{\pm}}_{\mu,J}$, miming their properties.
Heuristically, if we want to reproduce all the properties of the previous section results 2.2.8-2.2.11, the remaining element is the on-shell diagram's factorizability. This descend from the split representation of harmonic functions. Thus, we can base the research of their duals imposing an analogous of split representation for elements of 2.2.13. The linear combination satisfying this is given by [28]:

$$\mathcal{F}_{\mu,J}(P_i) = \frac{1}{2}\left(G_{\Delta^+,J}(P_i) + \frac{N_{\Delta^-,J}}{N_{\Delta^+,J}} G_{\Delta^-,J}(P_i)\right)\,, \tag{2.2.16}$$

where $\mathcal{F}_{\mu,J}$ is called **conformal partial wave** (CPW) and is the only single-valued linear combination of conformal blocks in Euclidean CFT. The coefficients are given by:

$$N_{\Delta^\pm,J} = \frac{\Gamma(\Delta^\pm - 1)}{\Gamma\left(\frac{\Delta^\pm-\Delta^\mp}{2}\right)} \frac{\Gamma\left(\frac{\Delta^\pm+J}{2} - a\right)\Gamma\left(\frac{\Delta^\pm+J}{2} + a\right)\Gamma\left(\frac{\Delta^\pm+J}{2} - b\right)\Gamma\left(\frac{\Delta^\pm+J}{2} + b\right)}{2\pi^2\Gamma(\Delta^\pm + J - 1)\Gamma(\Delta^\pm + J)}\,. \tag{2.2.17}$$

This is the dual of the bulk harmonic function and has an analogous split representation:

$$\mathcal{F}_{\mu,J}\left(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_3,\mathbf{x}_4\right) \propto \int d^dx'\, \mathcal{F}_{\Delta_1,\Delta_2,\Delta^+}\left(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}'\right) \mathcal{F}_{\Delta^-,\Delta_3,\Delta_4}\left(\mathbf{x}',\mathbf{x}_3,\mathbf{x}_4\right)\,, \tag{2.2.18}$$

or in Fourier representation:

$$\mathcal{F}_{\mu,J}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{p};-\mathbf{p},\mathbf{k}_3,\mathbf{k}_4) \propto F_{\Delta_1,\Delta_2,\Delta^+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{p}) F_{\Delta^-,\Delta_3,\Delta_4}(-\mathbf{p},\mathbf{k}_3,\mathbf{k}_4)\,, \tag{2.2.19}$$

where $F_{\Delta_i,\Delta_j,\Delta^k}$ are ECFT_d 3-point boundary correlators[15], while the normalization is arbitrary and can be chosen, for example, to reproduce the factorization of the bulk exchange. As the conformal symmetry group $SO\,(1,d+1)$, which acts both on (EA)dS$_{d+1}$ and on ECFT$_d$, completely constraints the 3-point diagrams, it is clear that the $F_{\Delta_i,\Delta_j,\Delta^k}$ coincide with the related contact diagrams of the two bulk theories. Going in Mellin space and acting with the b.c. projectors on the $\mathcal{F}_{\mu,J}$ we can get general boundary conditions.

Using this peculiar functions to construct a factorisation in the bulk requires gradual steps. For first, we start relating them to on-shell exchange diagram in EAdS$_{d+1}$.
Going to Fourier space, as in EAdS we are free to choose whatever b.c. for the exchanged particle, we can assume one with the CPW boundary condition, such that the discontinuity of the diagram amplitude is equal to the CPW itself:

$$\mathrm{Disc}_{p^2}\left[A^{\mathrm{AdS}}_{CPW}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{p};-\mathbf{p},\mathbf{k}_3,\mathbf{k}_4\right)\right] = \mathcal{F}^{\mathrm{AdS}}_{\mu,J}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{p};-\mathbf{p},\mathbf{k}_3,\mathbf{k}_4\right)\,, \tag{2.2.20}$$

[15]We omitted spin indexes for ease of presentation.

where the factorization is immediate as inherited from the CPW:

$$\mathcal{F}^{\text{AdS}}_{\mu,J}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{p};-\mathbf{p},\mathbf{k}_3,\mathbf{k}_4\right)=F^{\text{AdS}}_{\Delta_1,\Delta_2,\Delta^+,J_1,J_2,J}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{p}\right)\times F^{\text{AdS}}_{\Delta^-,\Delta_3,\Delta_4,J_4,J_3,J}\left(-\mathbf{p},\mathbf{k}_3,\mathbf{k}_4\right)\,. \tag{2.2.21}$$

Here the $F^{\text{AdS}}_{\Delta_i,\Delta_j,\Delta^k,J_i,J_j,J}$ are *3-point contact Witten diagrams.* They can be in turn bootstrapped, as conformal symmetry completely fixes 3-point correlators up to normalization. This latter is fixed via the cubic coupling constant g of the EAdS_{d+1} bulk QFT.

To extend the relation 2.2.20 to a general b.c., the crucial tool, as in previous section, is the Mellin-Barnes representation. Then we can act with the general projector 2.1.47 on the CPW, to project it on $\Delta^{\pm}$ boundary conditions and form linear combinations of them:

$$\begin{aligned}&\text{Disc}_{p^2}\left[A^{\text{AdS}}_{\alpha\Delta^++\beta\Delta^-,J}\left(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,u,\mathbf{p};\bar{u},-\mathbf{p},s_3,\mathbf{k}_3,s_4,\mathbf{k}_4\right)\right]=\\&=\frac{\Gamma\left(1+i\mu\right)\Gamma\left(1-i\mu\right)}{\pi}\left(\alpha\omega_{\Delta^+}\left(u,\bar{u}\right)+\beta\omega_{\Delta^-}\left(u,\bar{u}\right)\right)\\&\quad\mathcal{F}^{\text{AdS}}_{\mu,J}\left(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,u,\mathbf{p};\bar{u},-\mathbf{p},s_3,\mathbf{k}_3,s_4,\mathbf{k}_4\right)\,,\end{aligned} \tag{2.2.22}$$

which is in agreement with 2.2.8.

Moving to dS_{d+1} is now straightforward, as it needs just to select the BD b.c.:

$$\begin{aligned}&\text{Disc}_{p^2}\left[A^{\pm,\hat{\pm}}_{\mu,J}\left(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,u,\mathbf{p};\bar{u},-\mathbf{p},s_3,\mathbf{k}_3,s_4,\mathbf{k}_4\right)\right]=\\&=\frac{\Gamma\left(1+i\mu\right)\Gamma\left(1-i\mu\right)}{\pi}\left(\alpha^{\hat{\pm}}\omega_{\Delta^+}\left(u,\bar{u}\right)+\beta^{\pm}\omega_{\Delta^-}\left(u,\bar{u}\right)\right)\\&\quad\mathcal{F}^{\pm,\hat{\pm}}_{\mu,J}\left(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,u,\mathbf{p};\bar{u},-\mathbf{p},s_3,\mathbf{k}_3,s_4,\mathbf{k}_4\right)\,,\end{aligned} \tag{2.2.23}$$

where the CPW has a new normalisation adapted to dS 3-point contact diagrams:

$$\begin{aligned}\mathcal{F}^{\pm\hat{\pm}}_{\mu,J}\left(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,u,\mathbf{p};\bar{u},-\mathbf{p},s_3\mathbf{k}_3,s_4\mathbf{k}_4\right)&=F^{\pm}_{\Delta_1,\Delta_2,\Delta^+,J_1,J_2,J}\left(s_1\mathbf{k}_1,s_2\mathbf{k}_2,u,\mathbf{p}\right)\\&\cdot F^{\hat{\pm}}_{\Delta^-,\Delta_3,\Delta_4,J_4,J_3,J}\left(\bar{u},-\mathbf{p},s_3\mathbf{k}_3,s_4\mathbf{k}_4\right)\,.\end{aligned} \tag{2.2.24}$$

Using the analytic continuation of bulk-to-boundary propagators 2.1.41, in Mellin space they are related to the $F^{\text{AdS}}_{\Delta_i,\Delta_j,\Delta^k,J_i,J_j,J}$ as:

$$\begin{aligned}F^{\pm}_{\Delta_1\Delta_2\Delta_3}\left(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,s_3,\mathbf{k}_3\right)=&\left(\prod_{j=1}^{3}c^{\text{dS-AdS}}_{\Delta_j}\right)e^{\mp\left(\frac{d}{4}+\frac{1}{2}\sum_{j=1}^{3}\left(\Delta_j-\frac{d}{2}\right)\right)\pi i}\\&\cdot F^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}\left(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,s_3,\mathbf{k}_3\right)\,.\end{aligned} \tag{2.2.25}$$

Now these on-shell exchange functions have to be related to their off-shell version. While the former is related to solutions of the (EA)dS_{d+1} homogeneous wave equation,

the latter, instead, to the inhomogeneous one. Then, the physical difference between on-shell and full exchange functions is the *locality* condition, introduced by the source of the inhomogeneous equation.
Computing the dispersion formula in Mellin space:

$$f(a) = \int \frac{dz}{\pi} \frac{1}{z-a} \mathrm{Disc} f(z) \,, \tag{2.2.26}$$

we find that again this relation is encoded in the prefactor $\csc\left(\pi\left(u+\bar{u}\right)\right)$, as expected, thus expressing the locality of the bulk QFTs.

As we have general formulas relating $A^{\mathrm{AdS}}_{\alpha\Delta^+ + \beta\Delta^-,J}$ and $A^{\pm,\hat{\pm}}_{\mu,J}$ to the respectively 3-point contact diagrams, a relation between these two exchange diagrams involves a comparison on (EA)dS$_{d+1}$ contact diagrams normalizations.
Eventually, after summing over $\pm, \hat{\pm}$ combinations, the searched relation is:

$$A^{\mathrm{dS}}_{\mu,J} = \frac{\lambda^{\mathrm{dS}}_{\Delta_1\Delta_2\Delta^+}\lambda^{\mathrm{dS}}_{\Delta^+\Delta_3\Delta_4}}{\lambda^{\mathrm{AdS}}_{\Delta_1\Delta_2\Delta^+}\lambda^{\mathrm{AdS}}_{\Delta^+\Delta_3\Delta_4}} c^{\text{dS-AdS}}_{\Delta^+} A^{\mathrm{AdS}}_{\Delta^+,J} + \frac{\lambda^{\mathrm{dS}}_{\Delta_1\Delta_2\Delta^-}\lambda^{\mathrm{dS}}_{\Delta^-\Delta_3\Delta_4}}{\lambda^{\mathrm{AdS}}_{\Delta_1\Delta_2\Delta^-}\lambda^{\mathrm{AdS}}_{\Delta^-\Delta_3\Delta_4}} c^{\text{dS-AdS}}_{\Delta^-} A^{\mathrm{AdS}}_{\Delta^-,J} \,, \tag{2.2.27}$$

where the coefficients come from the 3-point diagrams normalisation and from the exponential prefactors in 2.2.25. Their expressions are:

$$\lambda^{\mathrm{AdS}}_{\Delta_1\Delta_2\Delta_3} = -g \prod_{j=1}^{3} \frac{1}{2\Gamma\left(\Delta_j - \frac{d}{2} + 1\right)} \,, \tag{2.2.28}$$

where g is the coupling constant of the cubic interaction;

$$\lambda^{\mathrm{dS}}_{\Delta_1\Delta_2\Delta_3} = \lambda^{\mathrm{AdS}}_{\Delta_1\Delta_2\Delta_3} \times 2 \left(\prod_{j=1}^{3} c^{\text{dS-AdS}}_{\Delta_j}\right) \sin\left[\left(\frac{d}{4} + \frac{1}{2}\sum_{j=1}^{3}\left(\Delta_j + J_j - \frac{d}{2}\right)\right)\pi\right] \,. \tag{2.2.29}$$

Therefore, the total coefficients are:

$$\begin{aligned}\frac{\lambda^{\mathrm{dS}}_{\Delta_{1,3}\Delta_{2,4}\Delta^\pm}}{\lambda^{\mathrm{AdS}}_{\Delta_{1,3}\Delta_{2,4}\Delta^\pm}} =& 2\, c^{\text{dS-AdS}}_{\Delta_{1,3}}\, c^{\text{dS-AdS}}_{\Delta_{2,4}}\, c^{\text{dS-AdS}}_{\Delta^\pm} \\ & \cdot \sin\left[\left(\frac{-d + \Delta_{1,3} + \Delta_{2,4} + \Delta^\pm + J_{1,3} + J_{2,4} + J}{2}\right)\pi\right] \,,\end{aligned} \tag{2.2.30}$$

recalling the change in normalisation $c^{\mathrm{dS-AdS}}_{\Delta}$ is given by 2.1.18.
We can note 2.2.27 to be symmetric under $\Delta^+ \leftrightarrow \Delta^-$ and that for $J, J_i = 0$ it recovers the previous result 2.2.4. The relation is graphically illustrated in Figure 2.2.

This relation is just a simple example of how the analytic continuation can map dS QFT to the EAdS one. It could be further applied to loop diagrams or even to whatever

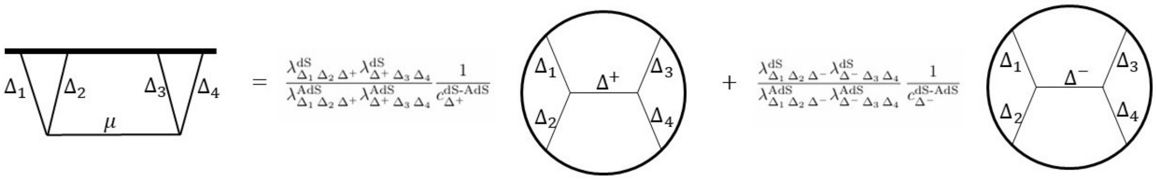

Figure 2.2: Relation between (EA)dS$_{d+1}$ exchange diagrams: an exchange Schwinger-Keldysh diagram in dS$_{d+1}$ is equal to the linear combination of Dirichlet/Neumann exchange Witten diagrams in EAdS$_{d+1}$, where the exchanged states are related via $\Delta^{\pm} = \frac{d}{2} \pm i\mu$.

order in perturbation theory. The general approach is that seen in the subsection 2.2, i.e. to work in Mellin space and to substitute dS bulk-to-boundary and bulk-to-bulk propagators via relations 2.1.41 and 2.1.57.
In addition, the possibility of conformal bootstrap introduces a third framework, the ECFT$_d$, for computing the same objects, which has a wealth of strong results, often valid even non-perturbatively, typical of conformal field theories.

Exchanges in dS/CFT

With regard to the mere relationship between the dS QFT and its dual ECFT, it is important to emphasize the contribution made by the dS $-$ EAdS map here constructed. Indeed, we have seen the possibility of bootstrapping via conformal symmetry the exchange diagrams $A^{\pm,\hat{\pm}}_{\mu,J}$ using the CPWs $\mathcal{F}^{\pm,\hat{\pm}}_{\mu,J}$, solutions of the Casimir equation of an ECFT$_d$ properly normalized, proportional (in Mellin space) to the product of two 3-point contact diagrams. In the whole previous derivation we never had the need of the above mentioned map to reach the wanted results. In fact, it has been primarily a relevant but collateral result of computing dS diagrams, or at most a useful shortcut.
We want now to show the how this map enforces a holographic dS/CFT, surmounting some problems of a direct approach.

The fact that in-in dS diagrams can be generally expressed as linear combinations of Witten diagrams, allows to inherit some of the strong holographic properties of this latter.
A first simpler implication is that, using 2.2.27, a dS$_{d+1}$ exchange diagram can expanded in conformal blocks via the Euclidean AdS/CFT expansion. From what we have seen in previous subsection, conformal blocks $G_{\Delta^{\pm},J}$ are proportional to on-shell exchange amplitudes in the $\Delta^{\pm}$ b.c., which could be interpreted as them to be the boundary

dual of a specific particle exchange contributing to a 4-point boundary correlator. The proportionality is:

$$A^{\text{AdS}}_{\Delta^\pm,J}\Big|_{\text{on-shell}} = \frac{\lambda^{\text{AdS}}_{\Delta_1\Delta_2\Delta^+}\lambda^{\text{AdS}}_{\Delta_3\Delta_4\Delta^+}}{C^{\text{AdS}}_{\Delta^\pm,J}} G_{\Delta^\pm,J}\,, \tag{2.2.31}$$

where $C^{\text{AdS}}_{\Delta^\pm,J}$ is the same of 2.1.20.

Instead, a full exchange diagram presents, other than the previous on-shell term, presents a infinite sum of conformal blocks related to exchanges of excited states, e.g. bound states, with increasing scaling dimension. From the CFT perspective, these states correspond to the so-called **double-trace operators**. Also, they encode the contact terms of the full-exchange. The general formula is:

$$\begin{aligned} A^{\text{AdS}}_{\Delta^\pm,J} =& \frac{\lambda^{\text{AdS}}_{\Delta_1\Delta_2\Delta^\pm}\lambda^{\text{AdS}}_{\Delta_3\Delta_4\Delta^\pm}}{C^{\text{AdS}}_{\Delta^\pm,J}} G_{\Delta^\pm,J} \\ &+ \sum_{J'=0}^{J}\sum_{n=0}^{\infty} {}^{(\mathbf{s})}a_{n,J'} G_{\Delta_1+\Delta_2+2n+J',J'} + \sum_{J'=0}^{J}\sum_{n=0}^{\infty} {}^{(\mathbf{s})}b_{n,J'} G_{\Delta_3+\Delta_4+2n+J',J'}\,. \end{aligned} \tag{2.2.32}$$

Then, the dS_{d+1} conformal block expansion is:

$$\begin{aligned} A^{\text{dS}}_{\mu,J} =& \frac{\lambda^{\text{dS}}_{\Delta_1\Delta_2\Delta^+}\lambda^{\text{dS}}_{\Delta_3\Delta_4\Delta^+}}{C^{\text{dS}}_{\Delta^+,J}} G_{\Delta^+,J} + \frac{\lambda^{\text{dS}}_{\Delta_1\Delta_2\Delta^-}\lambda^{\text{dS}}_{\Delta_3\Delta_4\Delta^-}}{C^{\text{dS}}_{\Delta^-,J}} G_{\Delta^-,J} \\ &+2\left(\prod_{i=1}^{4} c^{\text{dS-AdS}}_{\Delta_i}\right)\sin\left(\left(\frac{-d+2J+\Delta_1+\Delta_2+\Delta_3+\Delta_4+J_1+J_2+J_3+J_4}{2}\right)\pi\right) \\ &\cdot\left[\sum_{J'=0}^{J}\sum_{n=0}^{\infty} {}^{(\mathbf{s})}a_{n,J'} G_{\Delta_1+\Delta_2+2n+J',J'} + \sum_{J'=0}^{J}\sum_{n=0}^{\infty} {}^{(\mathbf{s})}b_{n,J'} G_{\Delta_3+\Delta_4+2n+J',J'}\right]. \end{aligned} \tag{2.2.33}$$

Analogous **t** and **u** channels conformal block expansion, in both theories, can be derive [25].

Another important feature transmitted to dS/CFT comes from EAdS boundary correlators being single-valued. Not by accident, we have already encountered this property for CPWs of the ECFT_d: indeed, thanks to their single-valuedness, these boundary correlators can be expanded along a basis of CPWs of the boundary theory[16].
Expanding perturbatively boundary correlators of the mapped dS theory, these maintain the single-valuedness as well the CPW expansion. A general expression for a CFT 4-point function expanded in CPWs is:

$$\langle O_1O_2O_3O_4\rangle = (\text{non-normalisable}) + \sum_J \int_{-\infty}^{+\infty} d\nu\, a_J(\nu)\,\mathcal{F}_{\nu,J}\,, \tag{2.2.34}$$

[16]Equivalently, it is clear they can be expanded in harmonic functions, being also these single-valued.

where the second term is normalisable and depends by the *spectral density* $a_J(\nu)$. If we translate this expansion in conformal blocks, we find the poles of $a_J(\nu)$ give the OPE coefficients. The analyticity of this functions is therefore crucial for the existence of the OPE, and thus of convergent conformal block expansion [29].

For example, in the normalisable b.c., the exchange amplitude $A^{\text{AdS}}_{\Delta^+,J}$ CPW expansion is:

$$A^{\text{AdS}}_{\Delta^+,J} = \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2 + \left(\Delta^+ - \frac{d}{2}\right)^2} \frac{\nu^2}{\pi} \mathcal{F}^{\text{AdS}}_{\nu,J} + \text{contact}\,, \tag{2.2.35}$$

while to obtain the one for the non-normalisable we could use the relation 2.1.26. The final result is:

$$A^{\text{AdS}}_{\Delta^-,J} = 2\mu i \mathcal{F}^{\text{AdS}}_{\mu,J} + A^{\text{AdS}}_{\Delta^+,J}\,, \tag{2.2.36}$$

where we can see it provides the non-normalisable part of the Neumann exchange.
In this case the spectral density is a meromorphic function also at a **non-perturbative level**, guaranteeing the exactness of the AdS/CFT correspondence.

The dS $-$ EAdS map we have built guarantees $a_J(\nu)$ is meromorphic at all orders in perturbation theory in the dS case, but not at non-perturbative level, with no guaranties that the conformal block expansion 2.2.33 converges. Ultimately, using again relation 2.2.27, the CPW expansion for the dS_{d+1} exchange diagram is:

$$\begin{aligned} A^{\text{dS}}_{\mu,J} = \left(\prod_{i=1}^{4} c^{\text{dS-AdS}}_{\Delta_i}\right) \Bigg[& -8\mu i\, c^{\text{dS-AdS}}_{\Delta^+} \sin\left(\frac{-d + \Delta_1 + \Delta_2 + \Delta^- + J_1 + J_2 + J}{2}\pi\right) \\ & \sin\left(\frac{-d + \Delta_3 + \Delta_4 + \Delta^- + J_3 + J_4 + J}{2}\pi\right) \mathcal{F}^{\text{AdS}}_{\mu,J} \\ & +2 \sin\left(\frac{-d + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + J_1 + J_2 + J_3 + J_4 + 2J}{2}\pi\right) \\ & \cdot \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2 + \left(\Delta^+ - \frac{d}{2}\right)^2} \frac{\nu^2}{\pi} \mathcal{F}^{\text{AdS}}_{\nu,J} \Bigg]\,. \end{aligned} \tag{2.2.37}$$

Chapter 3

Anomalous dimensions of double-trace operators in dS

In the previous chapter we have seen the conformal bootstrap method at work to compute dS_{d+1} exchange diagrams and derive them in terms of the analogous Witten diagrams, finding the core result 2.2.27. Successively, thanks to such relation, we have managed to express the desitterian exchange in a conformal block expansion valid in perturbation theory. Even if we do not have state-operator correspondence, it is worth to analyze this expansion, trying to interpret the double-trace operators as dual to exchanges of bound states in the bulk. Such application of the dS/CFT machinery is of great importance for inflationary phenomenology, as could provide information on the inflationary particle physics: from the mass spectrum, to the stability of bound state or resonances or, more generally, about inflationary kinematics and dynamics.

In this chapter we will address the original computation of a specific quantity characterising double-trace[1]operators in dS/CFT: the **anomalous dimension** $\gamma_{l,n}$. In particular, the specific case we will consider regards double-trace operators contributing (in the boundary perspective) to an exchange of a massive scalar field by mediated by a non-derivative interaction.

We will start revising the concept of anomalous dimension and the method we will use to derive it perturbatively . This latter will be based on the $\mathrm{dS} - \mathrm{EAdS}$ map: analogously to the diagram itself, we will prove that even the anomalous dimension $\gamma_{l,n}^{\mathrm{dS}}$ of the dS_{d+1} theory can be expressed as a sum of EAdS_{d+1} ones, $\gamma_{l,n}^{\mathrm{AdS}}$, with different boundary conditions.

[1]These are also known in literature as double-twist operators.

3.1 Anomalous dimensions

We have seen in section 2.1 that conformal symmetry fixes a 4-point correlators in terms of conformal blocks, which, from the bulk point of view, are related to on-shell exchanges. The conformal block expansion can be done along **s**, **t** or **u** channels, without changing the resulting correlator. This property is called **crossing symmetry** and constraints the shape of conformal block.

For example, the $\mathbf{t}-\mathbf{s}$ channels crossing symmetry for a correlator of scalar primary operators $\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle$, with equal conformal dimension Δ, is expressed by the equation:

$$\underbrace{u^{\Delta}\left(1+\sum_{\tau',l'} a_{\tau',l'}G_{\tau',l'}(v,u)\right)}_{\mathbf{t}\,\text{channel}} = \underbrace{v^{\Delta}\left(1+\sum_{\tau,l} a_{\tau,l}G_{\tau,l}(u,v)\right)}_{\mathbf{s}\,\text{channel}}, \tag{3.1.1}$$

where here conformal blocks are functions of the cross ratios:

$$u := \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,. \tag{3.1.2}$$

Each conformal block term, indicated by spin l and *twist*[2] $\tau := \Delta - l$, include a series composed by the lowest weight primary operator with such data, plus its descendant operators.

In the expansion, among the primary operators we find composite operators of the form

$$[OO]_{n,l} = O\partial_{i_1}\dots\partial_{i_\ell}\left(\partial^2\right)^n O + \dots, \tag{3.1.3}$$

where the $\dots$ represent terms of a similar schematic form required to make the operator primary and $i_a = 1,\dots,d$. These operators are referred to in the literature as *double-trace operators*. In (EA)dS$_{d+1}$ space, these operators correspond to bound states of the single particle dual to the operator O, with n and l being relative radial and angular momentum quantum numbers. The scaling dimension of the double-trace operators (3.1.3) takes the form

$$\Delta_{[OO]_{n,l}} = 2\Delta + 2n + l + \gamma_{n,l}\,, \tag{3.1.4}$$

where $\gamma_{n,l}$ is the *anomalous dimension*. In a free CFT it is always $\gamma_{n,l} \equiv 0$, so it depends by the interactions involved: in the (EA)dS$_{d+1}$ case they are strictly related to the dynamical behaviour of spacetime.

[2] For ease of presentation, referring to the conformal blocks, here we use l instead of J for the spin, and we substitute the conformal dimension with the twist.

In fact, for $l \to \infty$ double-trace operators represent states asymptotic to free bound states [30]. In this limit, in fact, the CFT behaves as mean field theory, and the twist of the double-trace operator considered above remains finite and tends to:

$$\tau^{(0)}_{n,l} = 2\Delta + 2n \,, \tag{3.1.5}$$

while the coefficient $a_{n,l} := a_{\tau_{n,l},l}$ of the conformal block related to $[OO]_{n,l}$ tends to [31]:

$$a^{(0)}_{n,l} = \frac{2^l(-1)^n(\Delta)^2_n\left(-\frac{d}{2}+\Delta+1\right)^2_n(n+\Delta)^2_l}{l!\,n!\left(\frac{d}{2}+l\right)_n(d-2n-2\Delta)_n(l+2n+2\Delta-1)_l\left(-\frac{d}{2}+l+n+2\Delta\right)_n} \,. \tag{3.1.6}$$

Perturbing this limit in orders of $\left(\frac{1}{l}\right)$, the twist acquires an anomalous dimension factor $\tau_{n,l} = \tau^{(0)}_{n,l} + \gamma_{n,l}$, which is:

$$\gamma_{n,l} = -\frac{c^{(0)}_n}{\mathfrak{J}^{\tau'}}\left(1+\sum_{k=1}^{\infty}\frac{c^{(k)}_n}{\mathfrak{J}^{2k}}\right) \,, \tag{3.1.7}$$

where $\mathfrak{J}$ is called conformal spin and is defined by:

$$\mathfrak{J}^2 = \left(l+\frac{\tau_{n,l}}{2}\right)\left(l+\frac{\tau_{n,l}}{2}-1\right) \,, \tag{3.1.8}$$

and the coefficients $c^{(k)}_n$ encode the perturbative corrections in function of n.

In the following section we will compute the anomalous dimension of double-trace operators (3.1.3) induced by the exchange of a particle in de Sitter space. To this end, one needs to determine the conformal block expansion of an exchange diagram in the crossed channel and the method we shall follow is based on the computation of **crossing kernels** of conformal partial waves.

Being the AdS/CFT correspondence exact, we can directly apply these anomalous dimensions' results for AdS exchanges, with an immediate bulk interpretation. Combining this with the dS$-$EAdS map presented in the previous chapter, it could be derived an exact expression for the anomalous dimensions of double-trace operators related to a dS$_{d+1}$ exchange, which has never been presented in literature. To avoid many mathematical technicalities, we will focus to the simpler case of four scalar identical fields exchanging a scalar massive field, and we will limit to compute the anomalous dimensions for $n = 0$.

3.2 Crossing kernel method

Let's present now the *crossing kernel method*, which will allow us to derive a closed analytical expression for anomalous dimensions of double-trace operators induced by a single-particle exchange. We will start from a desitterian exchange diagram of four scalar fields, exchanging a scalar field with conformal dimension Δ'. We have already presented its **s** channel conformal block expansion in 2.2.33, in terms of EAdS conformal blocks. Now we need to present the **t** channel exchange one; this can be further expanded in both **t** and **s** channel conformal blocks. Naming the diagram[3] as ${}^{(\mathbf{t})}A^{\mathrm{dS}}_{\Delta',0}(v,u)$, the former expansion is[4] [32]:

$$\begin{aligned} {}^{(\mathbf{t})}A^{\mathrm{dS}}_{\Delta',0}(v,u) = {}^{(\mathbf{t})}a_{\Delta'}\, G_{\Delta'}(v,u) + \sum_{n=0}^{\infty} {}^{(\mathbf{t})}a_{\Delta_1+\Delta_2+2n}\, G_{\Delta_1+\Delta_2+2n}(v,u) \\ + \sum_{n=0}^{\infty} {}^{(\mathbf{t})}b_{\Delta_3+\Delta_4+2n}\, G_{\Delta_3+\Delta_4+2n}(v,u)\,, \end{aligned} \tag{3.2.1}$$

where the $(\mathbf{t})$ subscript indicate the expansion channel and we omitted the spin subscript for conformal block and their coefficients. The first term represents the on-shell Δ' exchange.
The **s** channel expansion is:

$$\begin{aligned} {}^{(\mathbf{t})}A^{\mathrm{dS}}_{\Delta',0}(v,u) = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} {}^{(\mathbf{s})}a_{\Delta_1+\Delta_2+2n+l}\, G_{\Delta_1+\Delta_2+2n+l}(u,v) \\ + \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} {}^{(\mathbf{s})}b_{\Delta_3+\Delta_4+2n+l}\, G_{\Delta_3+\Delta_4+2n+l}(u,v)\,, \end{aligned} \tag{3.2.2}$$

where in this case are involved spinning double-trace operators.
These last two expressions are valid for $\Delta_1+\Delta_2-\Delta_3-\Delta_4 \neq 2n\,,\ n\in\mathbb{Z}$. But we are interested in the case in which all the boundary operators are identical, so this condition is not satisfied. In this case we have [33]:

$$\begin{aligned} {}^{(\mathbf{t})}A^{\mathrm{dS}}_{\Delta',0}(v,u) = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} {}^{(\mathbf{s})}a_{2\Delta+2n+l}\, G_{2\Delta+2n+l}(u,v) \\ + \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} {}^{(\mathbf{s})}a^{(0)}_{2\Delta+2n+l}\; {}^{(\mathbf{s})}\gamma^{\mathrm{dS}}_{n,l}\, \partial G_{2\Delta+2n+l}(u,v)\,, \end{aligned} \tag{3.2.3}$$

[3]To label the conformal dimension of the exchanged field, instead of μ here we use the equivalent index Δ', being they related by $\Delta' = \frac{d}{2}+i\mu$, as seen in section 1.2.

[4]Again, in the terms inside the sum over n, the index $2\Delta+2n$ is not here expressing the actual conformal dimensions but just indicating the conformal block.

where in the *derivative conformal block*, ∂G, the derivative acts respect to its total conformal dimension. ${}^{(\mathbf{t})}A^{\text{dS}}_{\Delta',0}$ can be expanded in the $u \to 0$ limit [30]:

$$\left(\frac{u}{v}\right)^{\Delta} {}^{(\mathbf{t})}A^{\text{dS}}_{\Delta',0}(v,u) = u^{\Delta} \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{{}^{(\mathbf{s})}\gamma^{\text{dS}}_{n,l}}{2} a^{(0)}_{2\Delta+2n+l} \, f_{2\Delta+2n,l}(v) \log u \right. \\ \left. + \frac{{}^{(\mathbf{s})}\gamma^{\text{dS}}_{n,l}}{2} a^{(0)}_{2\Delta+2n+l} \, \partial_n f_{2\Delta+2n,l}(v) + a_{2\Delta+2n+l} \, f_{2\Delta+2n,l}(v) + O(u) \right) , \tag{3.2.4}$$

where $f_{2\Delta+2n,l}(v)$ is the called the *collinear conformal block*:

$$f_{\tau,l}(v) := \lim_{u \to 0} u^{-\tau/2} G_{\tau,l}(u,v) = (1-v)^l \, {}_2F_1\left(\frac{\tau+2l}{2}, \frac{\tau+2l}{2}, \tau+2l; 1-v\right) . \tag{3.2.5}$$

Assuming now $n = 0$, to find the anomalous dimension we have to determine the coefficient of the $u^{\Delta} \log u$ term, naming it $\alpha(v)$:

$$\alpha(v) := \sum_{l=0}^{\infty} \frac{{}^{(\mathbf{s})}\gamma^{\text{dS}}_{0,l}}{2} a^{(0)}_{2\Delta+l} \, f_{2\Delta,l}(v) \, . \tag{3.2.6}$$

Once again, the crucial ingredient in the derivation will be the Mellin-Barnes representation. This will allow to derive coefficient 3.2.6 via the computation of a residue of the Mellin integrand. To implement it we have to Mellin transform directly the exchange diagram. This must satisfy the following structure imposed by conformal symmetry:

$${}^{(\mathbf{t})}A^{\text{dS}}_{\Delta',0}(v,u) = \frac{\mathcal{A}(u,v)}{(x_{12}^2)^{\Delta}(x_{34}^2)^{\Delta}} \, , \tag{3.2.7}$$

then, the Mellin-Barnes representation is [34][5]:

$$\mathcal{A}(u,v) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \frac{dt}{2\pi i} \, \rho(s,t) \, M(s,t) \, u^{\frac{t}{2}} v^{-\frac{s+t}{2}} \, , \tag{3.2.8}$$

$$\rho(s,t) := \Gamma\left(\frac{2\Delta - t}{2}\right)^2 \Gamma\left(-\frac{s}{2}\right)^2 \Gamma\left(\frac{t+s}{2}\right)^2 , \tag{3.2.9}$$

where the integrand has been appropriately factorised such that $\rho(s,t)$, called Mellin measure, encodes constraints imposed by conformal symmetry, while the function $M(s,t)$ encodes the dynamics of the diagram. This latter is called *Mellin amplitude* and is a

[5] Actually, in [34], the Mellin amplitude is defined as a representation of a full 4-point conformal correlators. Nevertheless, for the following treatment, it is just important that it satisfies the structure 3.2.7, which already the case for an exchange diagram.

peculiar version of the Mellin-Barnes representation suited for functions of harmonic ratios in CFT.
We can see that for $t = 2\Delta$ the integrand of 3.2.8, call it I, has a pole, due to the first gamma function, while the u factor gets the searched power, u^{Δ}. Instead, a log term in a residue can be generated by a double pole, which is the case $t = 2\Delta$, thus:

$$I_{t=2\Delta} = \cdots + 2u^{\Delta} \log u \left(\Gamma\left(-\frac{s}{2}\right)^2 \Gamma\left(\frac{s+2\Delta}{2}\right)^2 M\left(s, 2\Delta\right) v^{-\frac{s}{2}-\Delta} \right) + \cdots . \tag{3.2.10}$$

Note that $M\left(s, 2\Delta\right)$ coincides with the Mellin amplitude related to $\alpha(v)$:

$$\begin{aligned} \alpha(v) &= \int_{-i\infty}^{+i\infty} \frac{ds}{4\pi i} \, \tilde{\rho}(s) \, \alpha(s) \, v^{-\frac{s+2\Delta}{2}} , \\ \tilde{\rho}(s) &= \Gamma\left(-\frac{s}{2}\right)^2 \Gamma\left(\frac{s+2}{2}\Delta\right)^2 . \end{aligned} \tag{3.2.11}$$

It is clear, comparing 3.2.11 with 3.2.10, that $\alpha(s) \equiv M\left(s, 2\Delta\right)$.
Now the problem consists in extracting a single anomalous dimension from the sum in 3.2.6 and computing the Mellin amplitude.

Extraction of $\gamma^{(\mathbf{s})}_{0,l}$

A good starting point to achieve this goal is to exploit a property of the collinear conformal blocks. It has been proved that they can be expanded in terms of a set of orthogonal functions $Q_{\tau,l}(s)$ [34]:

$$f_{\tau,l}(v) = \int_{-i\infty}^{+i\infty} \frac{ds}{4\pi i} v^{-\frac{s+\tau}{2}} \rho_{\tilde{\Delta}}(s,\tau) Q_{\tau,l}(s) . \tag{3.2.12}$$

This functions are expressed in terms of Hanh polynomials $Q^{(a,b,c,d)}_l(s)$ as [35]:

$$Q_{\tau,l}(s) = (-1)^l l! \left(\mathfrak{N}^{(\tau,\tau+\tau_1-\tau_2-\tau_3+\tau_4,-\tau_1+\tau_2,\tau_3-\tau_4)}_l \right)^{-1} Q^{(\tau,\tau+\tau_1-\tau_2-\tau_3+\tau_4,-\tau_1+\tau_2,\tau_3-\tau_4)}_l(s) . \tag{3.2.13}$$

where $\mathfrak{N}^{(\tau,\tau+\tau_1-\tau_2-\tau_3+\tau_4,-\tau_1+\tau_2,\tau_3-\tau_4)}_l$ is a normalization prefactor composed by gamma functions and the τ_i are the twists of the external legs.
Hahn polynomial are orthogonal respect to the *Mellin-Barnes scalar product* [36]:

$$\langle f(s) g(s) \rangle_{a,b,c,d} = \int_{-i\infty}^{+i\infty} \frac{ds}{4\pi i} \Gamma\left(\frac{s+a}{2}\right) \Gamma\left(\frac{s+b}{2}\right) \Gamma\left(\frac{c-s}{2}\right) \Gamma\left(\frac{d-s}{2}\right) f(s) \, g(s) , \tag{3.2.14}$$

and are normalized by:

$$\left\langle Q_l^{(a,b,c,d)}(s)Q_n^{(a,b,c,d)}(s)\right\rangle = \delta_{l,n} \\ \cdot \frac{(-1)^n n^4 n!\,\Gamma\left(\frac{a+c}{2}+n\right)\Gamma\left(\frac{a+d}{2}+n\right)\Gamma\left(\frac{b+c}{2}+n\right)\Gamma\left(\frac{b+d}{2}+n\right)}{\left(\frac{a+b+c+d}{2}+n-1\right)_n\Gamma\left(\frac{a+b+c+d}{2}+2n\right)}, \tag{3.2.15}$$

where the coefficient to the Kronecker delta is exactly $\mathfrak{N}_l^{(a,b,c,d)}$.
Using 3.2.12, the Mellin amplitude is:

$$\alpha(s) = \sum_{l=0}^{\infty} \frac{{}^{(\mathbf{s})}\gamma_{0,l}^{\mathrm{dS}}}{2} a_{2\Delta+l}^{(0)}\, Q_{2\Delta,l}(s)\,. \tag{3.2.16}$$

The orthogonality allows to extract a generic coefficient of the above sum, containing the anomalous dimension for fixed l. Using $\tau_i = \tau = \Delta\,,\ \forall i = 1,\cdots,4$, the involved Hahn polynomial is $Q_l^{(2\Delta,2\Delta,0,0)}(s)$. For ease of presentation, we rename the normalization as $N_l := \mathfrak{N}_l^{(\tau,\tau+\tau_1-\tau_2-\tau_3+\tau_4,-\tau_1+\tau_2,\tau_3-\tau_4)}$, and compute the scalar product:

$$\begin{aligned}\left\langle \alpha(s), Q_l^{(2\Delta,2\Delta,0,0)}(s)\right\rangle &= \int_{-i\infty}^{+i\infty}\frac{ds}{4\pi i}\tilde{\rho}\left(s,2\Delta\right)\alpha(s)Q_l^{(2\Delta,2\Delta,0,0)} \\ &= \sum_{s=0}^{\infty}\frac{{}^{(\mathbf{s})}\gamma_{0,l}^{\mathrm{dS}}}{2}a_{2\Delta+s}^{(0)}\frac{(-1)^s s!}{N_s} \\ &\quad \underbrace{\cdot\int_{-i\infty}^{+i\infty}\frac{ds}{4\pi i}\Gamma\left(-\frac{s}{2}\right)^2\Gamma\left(\frac{s+2\Delta}{2}\right)^2\left(Q_l^{(2\Delta,2\Delta,0,0)}\right)^2}_{\delta_{ll}N_l} \\ &= \frac{{}^{(\mathbf{s})}\gamma_{0,l}^{\mathrm{dS}}}{2}a_{2\Delta+l}^{(0)}(-1)^l l!\,.\end{aligned} \tag{3.2.17}$$

Recalling that $\alpha(s) \equiv M\left(s,2\Delta\right)$, the anomalous dimension is given by:

$$\frac{1}{2}{}^{(\mathbf{s})}\gamma_{0,l}^{\mathrm{dS}}\,a_{2\Delta+l}^{(0)} = \frac{(-1)^l}{l!}\int_{-i\infty}^{+i\infty}\frac{ds}{4\pi i}\,\tilde{\rho}\left(s,2\Delta\right)\,M\left(s,2\Delta\right)\,Q_l^{(2\Delta,2\Delta,0,0)}\,. \tag{3.2.18}$$

The OPE coefficient in the mean field limit is given by the general expression 3.1.6, with $a_{2\Delta+l}^{(0)} \equiv a_{0,l}^{(0)}$:

$$a_{2\Delta+l}^{(0)} = \frac{2^l(\Delta)_l^2}{l!(2\Delta+l-1)_l}\,, \tag{3.2.19}$$

gaining the final formula:

$${}^{(\mathbf{s})}\gamma_{0,l}^{\mathrm{dS}} = \frac{(-1)^l(2\Delta+l-1)_l}{2^{l-1}(\Delta)_l^2}\int_{-i\infty}^{+i\infty}\frac{ds}{4\pi i}\,\tilde{\rho}\left(s,2\Delta\right)\,M\left(s,2\Delta\right)\,Q_l^{(2\Delta,2\Delta,0,0)}\,. \tag{3.2.20}$$

Mellin amplitude

We need to find an expression for $M(s, 2\Delta)$ in terms of known, or at least computable, objects of the boundary ECFT or of the bulk de Sitter QFT. By 3.2.8, it is defined as the Mellin transform of the exchange diagram ${}^{(\mathbf{t})}A^{\text{dS}}_{\Delta',0}(v,u)$.
On the other hand, we already know that ${}^{(\mathbf{t})}A^{\text{dS}}_{\Delta',0}(v,u)$ can be expressed in terms of the Witten diagrams ${}^{(\mathbf{t})}A^{\text{AdS}}_{\Delta^{\pm},0}(v,u)$ via the map 2.2.27[6]. As this is a linear combination, it is clear that it extends trivially to Mellin amplitudes:

$$ {}^{(\mathbf{t})}A^{\text{dS}}_{\Delta',0}(s,t) = \underbrace{\frac{\lambda^{\text{dS}}_{\Delta\Delta\Delta^+}\lambda^{\text{dS}}_{\Delta^+\Delta\Delta}}{\lambda^{\text{AdS}}_{\Delta\Delta\Delta^+}\lambda^{\text{AdS}}_{\Delta^+\Delta\Delta}}\, c^{\text{dS-AdS}}_{\Delta^+}}_{\Lambda^+}\ {}^{(\mathbf{t})}A^{\text{AdS}}_{\Delta^+,0}(s,t) + \underbrace{\frac{\lambda^{\text{dS}}_{\Delta\Delta\Delta^-}\lambda^{\text{dS}}_{\Delta^-\Delta\Delta}}{\lambda^{\text{AdS}}_{\Delta\Delta\Delta^-}\lambda^{\text{AdS}}_{\Delta^-\Delta\Delta}}\, c^{\text{dS-AdS}}_{\Delta^-}}_{\Lambda^-}\ {}^{(\mathbf{t})}A^{\text{AdS}}_{\Delta^-,0}(s,t)\,, \tag{3.2.21}$$

where ${}^{(\mathbf{t})}A^{\text{dS}}_{\Delta',0}(s,t) \equiv M(s,t)$, and we renamed the two coefficients as $\Lambda^{\pm}$ for ease of presentation.
Finally, we have also seen that such amplitudes enjoy a conformal partial wave expansion. Embedding it under Mellin-Barnes representation would lead to the searched relation, giving $M(s, 2\Delta)$ in terms of functions fixed by conformal bootstrap. Before to apply the CPW expansions for AdS diagrams we need to Mellin transform the general CPW using the same measure $\rho(s,t)$ used for Mellin amplitudes ${}^{(\mathbf{t})}A^{\text{(A)dS}}_{\Delta',0}(s,t)$ [36]:

$$ {}^{(\mathbf{t})}\mathcal{F}^{\text{AdS}}_{\nu,0}(u,v) = \int_{-i\infty}^{+i\infty} \frac{ds}{4\pi i}\frac{dt}{4\pi i}\,\rho(s,t)\,{}^{(\mathbf{t})}\mathcal{F}^{\text{AdS}}_{\nu,0}(s,t)\,u^{\frac{t}{2}}v^{-\frac{s+t}{2}}\,. \tag{3.2.22}$$

Now we can assemble 3.2.21 with the CPW expansions seen in previous chapter, 2.2.35 and 2.2.36, properly converted in Mellin space. Therefore we have:

$$ \begin{aligned} {}^{(\mathbf{t})}A^{\text{dS}}_{\Delta',0}(s,t) &= \left(\Lambda^+ + \Lambda^-\right)\,{}^{(\mathbf{t})}A^{\text{AdS}}_{\Delta^+,0}(s,t) + \left(2\mu i\,\Lambda^-\right)\,{}^{(\mathbf{t})}\mathcal{F}^{\text{AdS}}_{\mu,0}(s,t) \\ &= \left(\Lambda^+ + \Lambda^-\right)\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\left(\Delta^+-\frac{d}{2}\right)^2}\frac{\nu^2}{\pi}\,{}^{(\mathbf{t})}\mathcal{F}^{\text{AdS}}_{\nu,0}(s,t) + \left(2\mu i\,\Lambda^-\right)\,{}^{(\mathbf{t})}\mathcal{F}^{\text{AdS}}_{\mu,0}(s,t)\,, \end{aligned} \tag{3.2.23}$$

where contact terms are absent due to $l' = 0$.

Inserting this in 3.2.20, we get the anomalous dimension in terms of AdS CPWs:

$$ \begin{aligned} {}^{(\mathbf{s})}\gamma^{\text{dS}}_{0,l} &= \left(\Lambda^+ + \Lambda^-\right)\frac{(-1)^l(2\Delta+l-1)_l}{2^{l-1}(\Delta)_l^2}\int_{-i\infty}^{+i\infty}\frac{ds}{4\pi i}\,\tilde{\rho}\,(s,2\Delta)\ {}^{(\mathbf{t})}A^{\text{AdS}}_{\Delta^+,0}(s,2\Delta)\,Q_l^{(2\Delta,2\Delta,0,0)} \\ &\quad + \left(2\mu i\,\Lambda^-\right)\frac{(-1)^l(2\Delta+l-1)_l}{2^{l-1}(\Delta)_l^2}\int_{-i\infty}^{+i\infty}\frac{ds}{4\pi i}\,\tilde{\rho}\,(s,2\Delta)\ {}^{(\mathbf{t})}\mathcal{F}^{\text{AdS}}_{\mu,0}(s,2\Delta)\,Q_l^{(2\Delta,2\Delta,0,0)}\,. \end{aligned} \tag{3.2.24}$$

[6] Recall that here we assume $\Delta_i \equiv \Delta\,,\ \forall i = 1,\cdots,4$.

The integral in the second term is called the **crossing kernel** of the CPW ${}^{(\mathbf{t})}\mathcal{F}^{\mathrm{AdS}}_{\mu,0}(s,t)$, defined as [30]:

$$ {}^{(\mathbf{t})}\mathcal{J}_{\mu,0|l}(t) = \frac{(-1)^l}{l!}\int_{-i\infty}^{+i\infty}\frac{ds}{4\pi i}\ \tilde{\rho}(s,t)\,{}^{(\mathbf{t})}\mathcal{F}^{\mathrm{AdS}}_{\mu,0}(s,t)Q^{(t,t,0,0)}_{l}(s)\,. \tag{3.2.25}$$

Then, the anomalous dimension is directly expressed in terms of crossing kernels of CPWs:

$$\begin{aligned} {}^{(\mathbf{s})}\gamma^{\mathrm{dS}}_{0,l} &= \left(\Lambda^+ + \Lambda^-\right)\frac{l!(2\Delta+l-1)_l}{2^{l-1}(\Delta)_l^2}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2-\left(\Delta^+-\frac{d}{2}\right)^2}\frac{\nu^2}{\pi}\,{}^{(\mathbf{t})}\mathcal{J}_{\nu,0|l}(2\Delta) \\ &\quad + \left(2\mu i\,\Lambda^-\right)\frac{l!(2\Delta+l-1)_l}{2^{l-1}(\Delta)_l^2}\,{}^{(\mathbf{t})}\mathcal{J}_{\mu,0|l}(2\Delta)\,, \end{aligned} \tag{3.2.26}$$

where the *spectral integral* in the first term encodes the contribution from the normalisable AdS Dirichlet b.c., while the second is a contribution from the single CPW ${}^{(\mathbf{t})}\mathcal{F}^{\mathrm{AdS}}_{\mu,0}(s,t)$, which accounts for the non-normalisable part of the Neumann b.c.

Both ${}^{(\mathbf{t})}\mathcal{J}_{\mu,l'|l}(t)$ and ${}^{(\mathbf{t})}I_{\Delta',l'|l}(t)$ have been computed in literature for a wide class of cases [30, 36]. Following these computation methods, we can now proceed computing their analytic expressions in our specific case.

3.3 Computation of the crossing kernel

To derive the crossing kernel 3.2.25 we have to start from the conformal partial waves. They are defined by 2.2.16 up to normalisation. However we have already used some relations involving ${}^{(\mathbf{t})}\mathcal{F}^{\mathrm{AdS}}_{\mu,0}(s,t)$ while defining the objects which led to the crossing kernel definition, so we have to impose the normalization coherently with them.
Indeed, we have indirectly defined these CPWs via relation 2.2.36. From it we get:

$$ {}^{(\mathbf{t})}\mathcal{F}^{\mathrm{AdS}}_{\mu,0}(s,t) = \frac{2\mu}{i}\left({}^{(\mathbf{t})}A^{\mathrm{AdS}}_{\Delta^+,0}(s,t) - {}^{(\mathbf{t})}A^{\mathrm{AdS}}_{\Delta^-,0}(s,t)\right)\,. \tag{3.3.1}$$

If now we perform an inverse Mellin transform, returning to the $\mathbf{x}_i$ coordinates, the CPW will be proportional to the harmonic function $\Omega^{\mathrm{AdS}}_{\mu,0}$ via relation 2.1.27. Then, using the split representation 2.1.42, we find the expected proportionality with the 3-point diagrams $F^{\mathrm{AdS}}_{\Delta_i,\Delta_j,\Delta_k}$ in momentum space:

$$\begin{aligned} {}^{(\mathbf{t})}\mathcal{F}^{\mathrm{AdS}}_{\mu,0}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,u,\mathbf{p},\bar{u},-\mathbf{p},s_3,\mathbf{k}_3,s_4,\mathbf{k}_4) &= F^{\mathrm{AdS}}_{\Delta,\Delta,\frac{d}{2}+i\mu}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,u,\mathbf{p}) \\ &\quad\cdot F^{\mathrm{AdS}}_{\frac{d}{2}-i\mu,\Delta,\Delta}(\bar{u},-\mathbf{p},s_3,\mathbf{k}_3,s_4,\mathbf{k}_4)\,, \end{aligned} \tag{3.3.2}$$

where, in the case of all scalar fields, the 3-point diagrams have the general form[5]:

$$F^{\mathrm{AdS}}_{\Delta_1\Delta_2\Delta_3}(s_1,\mathbf{k}_1,s_2,\mathbf{k}_2,s_3,\mathbf{k}_3) = -\,g\,i\pi\,\delta\left(\frac{d}{4}-s_1-s_2-s_3\right)(2\pi)^d\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) \cdot \prod_{j=1}^{3} K^{\mathrm{AdS}}_{\Delta_j,0}(s_j,\mathbf{k}_j)\,, \tag{3.3.3}$$

and the scalar bulk-to-boundary propagators in Mellin space are:

$$K^{\mathrm{AdS}}_{\Delta_i,0}(s_i,\mathbf{k_i}) = \frac{\Gamma\left(s_i+\frac{1}{2}\left(\frac{d}{2}-\Delta_i\right)\right)\Gamma\left(s_i-\frac{1}{2}\left(\frac{d}{2}-\Delta_i\right)\right)}{2\Gamma\left(\Delta_i-\frac{d}{2}+1\right)}\left(\frac{k_i}{2}\right)^{-2s_i+\Delta_i-\frac{d}{2}}\,. \tag{3.3.4}$$

These expressions have to be recast in terms of the Mellin variables (s,t) related to the harmonic ratios. The resulting CPW[7]is[36]:

$$^{(\mathbf{t})}\mathcal{F}^{\mathrm{AdS}}_{\mu,0}(s,t) = \frac{\pi^{\frac{d}{2}}g^2}{64\mu^2\,\Gamma\left(\Delta-\frac{d}{2}+1\right)^4\,\Gamma(i\mu)\,\Gamma(-i\mu)\,\Gamma\left(\frac{d}{4}+i\frac{\mu}{2}\right)^2\,\Gamma\left(\frac{d}{4}-i\frac{\mu}{2}\right)^2} \cdot \frac{\Gamma\left(\frac{s+t+d-\tau-2\Delta}{2}\right)\Gamma\left(\frac{s+t+\tau-2\Delta}{2}\right)}{\Gamma\left(\frac{s+t}{2}\right)^2}\,. \tag{3.3.6}$$

Inserting this expression in the 3.2.25, the crossing kernel can be derived up to integration. This integral can be solved analytically, but its development is long and complicated, involving Mellin-Barnes integrals of hypergeometric functions. We present here the final result for $t=2\Delta$:

$$^{(\mathbf{t})}\mathcal{J}_{\mu,0|l}(2\Delta) = \frac{(-2)^l\pi^{\frac{d}{2}}g^2(\Delta)_l^2}{64\,l!\,\mu^2\,(2\Delta+l-1)_l\,\Gamma(i\mu)\,\Gamma(-i\mu)\,\Gamma\left(\frac{d}{2}\right)\Gamma\left(\Delta-\frac{d}{2}+1\right)^4} \cdot {}_4F_3\left(\begin{matrix}-l,2\Delta+l-1,\frac{d}{4}-i\frac{\mu}{2},\frac{d}{4}+i\frac{\mu}{2}\\ \Delta,\Delta,\frac{d}{2}\end{matrix};1\right)\,, \tag{3.3.7}$$

where ${}_4F_3$ is a generalised hypergeometric function, defined as[24]:

$${}_4F_3\left(\begin{matrix}a_1,a_2,a_3,a_4\\ b_1,b_2,b_3\end{matrix};z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k(a_3)_k(a_4)_k}{(b_1)_k(b_2)_k(b_3)_k}\frac{z^k}{k!}\,, \tag{3.3.8}$$

[7]With regard to the function $^{(\mathbf{t})}\mathfrak{M}_{0,0,0,0|0}(s,t)$ and the coefficients $\kappa_{\frac{d}{2}-i\mu}$, $\alpha_{0,0,0;\Delta,\Delta,\frac{d}{2}+i\mu}$ defined in [36], this CPW is equal to:

$$^{(\mathbf{t})}\mathcal{F}^{\mathrm{AdS}}_{\mu,0}(s,t) = \lambda^{\mathrm{AdS}}_{\Delta,\Delta,\frac{d}{2}+i\nu}\lambda^{\mathrm{AdS}}_{\frac{d}{2}-i\nu,\Delta,\Delta}\frac{\pi^{\frac{d}{2}}}{\kappa_{\frac{d}{2}-i\mu}\alpha_{0,0,0;\Delta,\Delta,\frac{d}{2}+i\mu}}\,^{(\mathbf{t})}\mathfrak{M}_{0,0,0,0|0}(s,t)\,, \tag{3.3.5}$$

where the $\lambda^{\mathrm{AdS}}_{\Delta_i,\Delta_j,\Delta_k}$ are the 3-point Witten diagram normalizations 2.2.28.

and it is convergent for $|z| < 1\,,\ z \in \mathbb{C}$.

3.4 Computation of the spectral integral

The first term of 3.2.24 contain a spectral integral of crossing kernels. To evaluate it, it is convenient to set a preparatory normalisation, defining [30]:

$$ {}^{(\mathbf{t})}I_{\mu,0|l}(t) = \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi}\,\chi(\nu)\,{}^{(\mathbf{t})}\hat{\mathcal{J}}_{\frac{d}{2}+i\nu,0|l}(t)\,, \tag{3.4.1} $$

where the spectral function is:

$$ \chi(\nu) = \frac{1}{\nu^2-\mu^2}\frac{\nu^2}{\pi}\lambda^{\text{AdS}}_{\Delta,\Delta,\frac{d}{2}+i\nu}\lambda^{\text{AdS}}_{\frac{d}{2}-i\nu,\Delta,\Delta}\,, \tag{3.4.2} $$

with $\lambda_{\Delta_i,\Delta_j,\Delta_k}$ defined as in 2.2.28, and $i\mu = \Delta^+ - \frac{d}{2} = \Delta' - \frac{d}{2}$. In other words, we are redefining the crossing kernel, extracting the 3-point Witten diagram normalizations. The reason to do this is that the new ${}^{(\mathbf{t})}\hat{\mathcal{J}}_{\frac{d}{2}+i\nu,l'|l}(t)$ can be expressed as a sum of **Wilson polynomials**.
These are defined as:

$$ \begin{aligned} &W_n(x^2;a,b,c,d) = \\ &\quad = (a+b)_n(a+c)_n(a+d)_n\,{}_4F_3\left(\begin{matrix}-n,\ a+b+c+d+n-1,\ a+ix,\ a-ix \\ a+b,\ a+c,\ a+d\end{matrix};1\right)\,, \end{aligned} \tag{3.4.3} $$

In general we have:

$$ {}^{(t)}\hat{\mathcal{J}}_{\frac{d}{2}+i\nu,l'|l}(t) = \sum_{j=1}^{N}\beta_j(t)\mathcal{W}_j\left(\nu^2;a_1,a_2,a_3,a_4\right)\,, \tag{3.4.4} $$

with the $a_i\,,\ i=1,\cdots,4$ suited parameters. The number N of terms depends on l' but not on l. For $l'=0$ we have just one Wilson polynomial in the decomposition. These functions have the very nice property to be orthogonal respect to the Wilson measure $w_{a_i}(\nu)$, which is encoded in $\chi(\nu)$:

$$ w_{a_i}(\nu) = \frac{\Gamma\left(a_1 \pm \frac{i\nu}{2}\right)\Gamma\left(a_2 \pm \frac{i\nu}{2}\right)\Gamma\left(a_3 \pm \frac{i\nu}{2}\right)\Gamma\left(a_4 \pm \frac{i\nu}{2}\right)}{\Gamma(\pm i\nu)}\,, \tag{3.4.5} $$

i.e., for two functions of ν^2:

$$ \langle p(\nu^2)|q(\nu^2)\rangle = \int_{-\infty}^{+\infty}\frac{d\nu}{2\pi}\,w_{a_i}(\nu)\,p(\nu^2)\,q(\nu^2)\,. \tag{3.4.6} $$

Therefore, under these definitions, the spectral integral takes the form of a scalar product:

$$ {}^{(\mathbf{t})}I_{\tau',l'|l}(t) = \left\langle \frac{1}{\nu^2-\mu^2} \middle| {}^{(\mathbf{t})}\hat{\mathcal{J}}_{\frac{d}{2}+i\nu,l'|l}(t) \right\rangle . \tag{3.4.7} $$

The next step consists in defining a *seed integral* to which we can refer more complicated ones, making possible the analytical solution of the general 3.4.7.

$$ \begin{aligned} \phi_l(a_i) &= \left\langle \frac{1}{4}\frac{\Gamma\left(a_5 \pm \frac{i\nu}{2}\right)}{\Gamma\left(1+a_6 \pm \frac{i\nu}{2}\right)} \middle| \mathcal{W}_l(\nu^2;a_i) \right\rangle \\ &= \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi}\, w_{a_i}(\nu)\, \frac{1}{4}\frac{\Gamma\left(a_5 \pm \frac{i\nu}{2}\right)}{\Gamma\left(1+a_6 \pm \frac{i\nu}{2}\right)} \mathcal{W}_l(\nu^2;a_i)\,, \end{aligned} \tag{3.4.8} $$

which has the general result

$$ \begin{aligned} \phi_l(a_i) = {}& \Gamma(a_1+a_2)\Gamma(a_1+a_3)\Gamma(a_1+a_4)\Gamma(a_1+a_5)\Gamma(a_2+a_5)\Gamma(a_3+a_5)\Gamma(a_4+a_5) \\ & \cdot \frac{\Gamma(a_2+a_3+l)\Gamma(a_2+a_4+l)\Gamma(a_3+a_4+l)\Gamma(a_6-a_5+l+1)}{\Gamma(1-a_5+a_6)}\, \psi(a;b,c,d,e,f)\,, \end{aligned} \tag{3.4.9} $$

where the function $\psi(a;b,c,d,e,f)$ is the *Wilson function*, defined by

$$ \begin{aligned} \psi(a;b,c,d,e,f) = {}& \Gamma(a+1)\left[\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\right. \\ & \left. \cdot\, \Gamma(1+a-e)\Gamma(1+a-f)\Gamma(2+2a-b-c-d-e-f)\right]^{-1} \\ & \cdot\, {}_7F_6\left(\begin{matrix} a,\ 1+\frac{a}{2},\ b,\ c,\ d,\ e,\ f \\ \frac{a}{2},\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-f \end{matrix} ; 1 \right) . \end{aligned} \tag{3.4.10} $$

The generalised hypergeometric function ${}_7F_6$ is defined as 3.3.8, but with 7 Pochhammer symbols on the numerator and 6 on the denominator. Finally, the parameters $a, b, \cdots, f$ are determined by:

$$ \begin{aligned} a &= a_1+a_2+a_3+a_4+2a_5+l-1, \\ b &= a_1+a_5, \\ c &= a_2+a_5, \\ d &= a_3+a_5, \\ e &= a_4+a_5, \\ f &= a_1+a_2+a_3+a_4+a_5-a_6+l-1\,. \end{aligned} \tag{3.4.11} $$

For our purposes we will always assume $a_5 = a_6 = i\frac{\mu}{2}$, as in this case the bra in 3.4.8 becomes:

$$\frac{1}{4}\frac{\Gamma\left(a_5 \pm \frac{i\nu}{2}\right)}{\Gamma\left(1 + a_6 \pm \frac{i\nu}{2}\right)} = \frac{1}{\nu^2 - \mu^2}\,, \tag{3.4.12}$$

then, exploiting the Wilson polynomial decomposition, we can always reduce the integration in a linear combination of seed integrals.

In our case, ${}^{(\mathbf{t})}\hat{\mathcal{J}}_{\frac{d}{2}+i\nu,0|l}(t)$ is equal to:

$$^{(t)}\hat{\mathcal{J}}_{\frac{d}{2}+i\nu,0|l}(t)\,|l(t)\rangle = a^{(0)}_{0,l}\beta(t)\,\mathcal{W}_l(\nu^2; a_1, a_2, a_3, a_4)\,, \tag{3.4.13}$$

with

$$\beta(t) = \frac{(-1)^l\, 2^{2\Delta-t}\, \pi^{\frac{d}{2}}\, \Gamma(\Delta)^2 \Gamma\left(l + \Delta - \frac{1}{2}\right) \Gamma\left(l + \frac{t}{2}\right) \Gamma(l + t - 1)}{\Gamma\left(\frac{t}{2}\right)^2 \Gamma(l + \Delta)\Gamma(l + 2\Delta - 1)\Gamma\left(l + \frac{t}{2} - \frac{1}{2}\right)\Gamma\left(\frac{d}{2} + t - 2\Delta\right)}\,, \tag{3.4.14}$$

$$a_1 = \frac{d}{4} - \Delta + \frac{t}{2}, \quad a_2 = \frac{d}{4} - \Delta + \frac{t}{2}, a_3 = \Delta - \frac{d}{4}, \quad a_4 = \Delta - \frac{d}{4}\,. \tag{3.4.15}$$

Then, the analytical result for the crossing kernel is:

$$\begin{aligned} ^{(\mathbf{t})}I_{\mu,0|l}(t) = -a^{(0)}_{0,l}\, &\frac{(-1)^l\, 2^{2\Delta-t}\, \pi^{\frac{d}{2}}\, \Gamma(\Delta)^2\, g^2\, \Gamma\left(l + \Delta - \frac{1}{2}\right)\, \Gamma\left(l + \frac{t}{2}\right)\, \Gamma(l + t - 1)}{\Gamma\left(\frac{t}{2}\right)^2\, \Gamma(l + \Delta)\, \Gamma(l + 2\Delta - 1)\, \Gamma\left(l + \frac{t}{2} - \frac{1}{2}\right)\, \Gamma\left(\frac{d}{2} + t - 2\Delta\right)} \\ &\cdot\, \phi_l\left(\frac{d}{4} - \Delta + \frac{t}{2}, \frac{d}{4} - \Delta + \frac{t}{2}, \Delta - \frac{d}{4}, \Delta - \frac{d}{4}, i\frac{\mu}{2}, i\frac{\mu}{2}\right)\,. \end{aligned} \tag{3.4.16}$$

Evaluating this for $t = 2\Delta$ we get:

$$^{(\mathbf{t})}I_{\Delta',0|l}(2\Delta) = -a^{(0)}_{0,l}\, \frac{(-1)^l\, \pi^{\frac{d}{2}}\, g^2}{\Gamma\left(\frac{d}{2}\right)}\, \phi_l\left(\frac{d}{4} - \Delta + \frac{t}{2}, \frac{d}{4} - \Delta + \frac{t}{2}, \Delta - \frac{d}{4}, i\frac{\mu}{2}, i\frac{\mu}{2}\right)\,. \tag{3.4.17}$$

Substituting both 3.4.17 and 3.3.7 in 3.2.24, we get the final result for the anomalous dimension of a double-trace operator $[OO]_{0,l}$ belonging to the **s** channel expansion of a **t** channel exchange of a massive scalar in dS_{d+1}:

$$\begin{aligned} ^{(\mathbf{s})}\gamma^{\mathrm{dS}}_{0,l} = &\frac{(-1)^l\, \pi^{\frac{d}{2}} g^2}{\Gamma\left(\frac{d}{2}\right)} \left[-2\left(\Lambda^+ + \Lambda^-\right)\, \phi_l\left(\frac{d}{4} - \Delta + \frac{t}{2}, \frac{d}{4} - \Delta + \frac{t}{2}, \Delta - \frac{d}{4}, i\frac{\mu}{2}, i\frac{\mu}{2}\right)\right. \\ &\left. + \frac{i\Lambda^-}{16\,\mu\Gamma\left(i\mu\right)\Gamma\left(-i\mu\right)\Gamma\left(\Delta - \frac{d}{2} + 1\right)^4}\; {}_4F_3\left(\begin{matrix} -l, 2\Delta + l - 1, \frac{d}{4} - i\frac{\mu}{2}, \frac{d}{4} + i\frac{\mu}{2} \\ \Delta, \Delta, \frac{d}{2} \end{matrix}; 1\right)\right]\,, \end{aligned} \tag{3.4.18}$$

with the coefficients $\Lambda^{\pm}$ computed using 2.2.30 and 2.1.18.

To summarise: the just computed anomalous dimensions give the complete scaling dimension of the double-trace operators in the OPE dual to a desitterian exchange. The analysis of their properties and of the consequences of the result 3.4.18 could proceed in various directions, with both formal and phenomenological applications. For sure, the most immediate and important would regard the extraction of information about the stability of bound states.
Indeed, analysing the complete scaling dimension of $[OO]_{0,l}$, in function of l and or of the external dimension Δ, we could explore its behaviour respect to the dS unitarity constraints presented in section 1.2: stable states would necessarily fit into one of the possible unitary series, while the unstable ones would violate them.

Conclusions

Despite its potential capability of probing the highest energy scales ever existed in the history of Universe, leading to a better understanding on various branches of open research (primordial cosmology, beyond Standard Model physics, string theory, quantum gravity), the quantum field theory on de Sitter spacetime has been decades too much underdeveloped. Without the possibility of derive enough efficient predictions on cosmological observables from the dynamical hypotheses of inflationary models, primordial cosmology has been severed, limiting its main business Gaussianities and power spectra. As we have seen in this thesis, to directly address the study of dynamical fields embedded in the inflationary presents multiple limits and difficulties. For first one have, in fact, to circumvent the impossibility of preparing a scattering experiment, changing the whole formalism describing interactions in the theory. But the worst problem regards the mathematical complexity of dS QFT, making unfeasible the analytical computation of general 1-loop, or even tree-level, Feynman diagrams.

It is for such reasons that desitterian QFT needs an innovative approach. This work has presented a promising one which has an immediate application to cosmology. Indeed, besides its intrinsic interest for some of the most deep questions of nowadays physics, the holographic principle fits extremely well the exotic observational conditions of inflationary era. Even if perturbative, this holographic approach seems to bring a revolution on the amount of information we are today able to extract from the newborn Universe. Exploiting an already beaten path, AdS/CFT correspondence, the dS/CFT is a young and promising formal theory with a huge potential of application.
On the other hand, it is astonishing how this very general approach comes from a simple geometrical relation of analytic continuation. Further, we have seen how the key tool to deal with dS quantum physics, perturbatively or not, is the Mellin-Barnes representation, which suits well the symmetries of this spacetime.

This representation, together with these two fundamental results:

$$K^{\pm}_{\Delta,J}\left(s,\mathbf{k}\right)=c^{\text{dS-AdS}}_{\Delta}\,e^{\mp\left(s+\frac{1}{2}\left(\Delta-\frac{d}{2}\right)\right)\pi i}\,K^{\text{AdS}}_{\Delta,J}\left(s,\mathbf{k}\right)\,, \tag{3.4.19}$$

$$
\begin{aligned}
\Pi^{\pm\hat{\pm}}_{\mu,J}(u,\mathbf{p};\bar{u},-\mathbf{p}) = & \, c^{\text{dS-AdS}}_{\frac{d}{2}+i\mu} \; \mathrm{e}^{\mp\left(u+\frac{i\mu}{2}\right)\pi i} \; \mathrm{e}^{\hat{\mp}\left(u+\frac{i\mu}{2}\right)\pi i} \Pi^{\text{AdS}}_{\frac{d}{2}+i\mu,J}\left(u,\mathbf{p};\bar{u},-\mathbf{p}\right) \\
& + \left(\mu \to -\mu\right),
\end{aligned}
\tag{3.4.20}
$$

are what needed to translate whatever dS diagram in an EAdS one, and vice-versa. The built machinery allows a perturbative computation of desitterian Schwinger-Keldysh correlators, with the help of tools provided by AdS/CFT correspondence.

We have then focused on exchange diagrams, as they encode much physical information of the bulk theory. We so derived the other very general relation:

$$
A^{\text{dS}}_{\mu,J} = \frac{\lambda^{\text{dS}}_{\Delta_1\Delta_2\Delta^+}\lambda^{\text{dS}}_{\Delta^+\Delta_3\Delta_4}}{\lambda^{\text{AdS}}_{\Delta_1\Delta_2\Delta^+}\lambda^{\text{AdS}}_{\Delta^+\Delta_3\Delta_4}} c^{\text{dS-AdS}}_{\Delta^+} \, A^{\text{AdS}}_{\Delta^+,J} + \frac{\lambda^{\text{dS}}_{\Delta_1\Delta_2\Delta^-}\lambda^{\text{dS}}_{\Delta^-\Delta_3\Delta_4}}{\lambda^{\text{AdS}}_{\Delta_1\Delta_2\Delta^-}\lambda^{\text{AdS}}_{\Delta^-\Delta_3\Delta_4}} c^{\text{dS-AdS}}_{\Delta^-} \, A^{\text{AdS}}_{\Delta^-,J} \,. \tag{3.4.21}
$$

Finally, to witness the application capability of the results derived in chapter 2, we have pushed this method to the computation of an intrinsic and original quantity of the boundary CFT, the anomalous dimension induced by a desitterian exchange of a massive scalar field. These are extra terms in the scaling dimension of composite operators appearing in the OPE of the boundary CFT, related to the exchange of bound states in dual bulk theory. Besides its importance as a formal exemplification, the anomalous dimension could be further investigated to extract physical information on the mass spectrum of bound states, as well to their stability in the dS spacetime. We recall here the found expression:

$$
\begin{aligned}
{}^{(\mathbf{s})}\gamma^{\text{dS}}_{0,l} = & \frac{(-1)^l \, \pi^{\frac{d}{2}} g^2}{\Gamma\left(\frac{d}{2}\right)} \Bigg[-2\left(\Lambda^+ + \Lambda^-\right) \, \phi_l\left(\frac{d}{4}-\Delta+\frac{t}{2}, \frac{d}{4}-\Delta+\frac{t}{2}, \Delta-\frac{d}{4}, i\frac{\mu}{2}, i\frac{\mu}{2}\right) \\
& + \frac{i\Lambda^-}{16\,\mu\Gamma\left(i\mu\right)\Gamma\left(-i\mu\right)\Gamma\left(\Delta-\frac{d}{2}+1\right)^4} \; {}_4F_3\left(\begin{matrix} -l, 2\Delta+l-1, \frac{d}{4}-i\frac{\mu}{2}, \frac{d}{4}+i\frac{\mu}{2} \\ \Delta, \Delta, \frac{d}{2} \end{matrix}; 1\right) \Bigg] \,.
\end{aligned}
\tag{3.4.22}
$$

Further developments in dS/CFT physics could range from the phenomenological applications in the inflationary models, but also in dark energy models describing the current expanding era, to a largeness of formal extensions and secondary results. To cite the most important: the still obscure unitarity and causality structure of the boundary CFT, which are strongly related to the topic of time emergence, as well of quantum gravity; the possibility to extend the holographic approach to all values of the cosmological constant, i.e. to even to flat spacetime [37, 38, 39]; the search, via the dS – EAdS map, for a non-perturbative and exact dS/CFT.

Acknowledgements

I offer my sincere thanks to my supervisors. I would like to thank Prof. Sleight very much for her great helpfulness and kindness in welcoming me and following me throughout my thesis. Her advice and explanations were invaluable and fundamental in achieving this goal. I would also like to thank Prof. Cicoli for being a safe reference, both during these two years of my Master's degree and during the writing of my thesis.

A special thanks goes to my girlfriend Erika for making me believe in myself, as well for all her patience. I thank my family for having dreamed this goal with me and for all the sacrifices spent on my behalf. I thank my colleagues and friends Mohammad Kassir, Alessandro Borys and Thomas Tawfik for all the lessons, exams and laughs spent with you.

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