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School Of Sciences
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The Curvaton Mechanism in String Inflation

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Abstract

In this Master Degree Thesis we study the Curvaton Mechanism to generate curvature perturbations in String Inflation. First of all, we introduce basic bosonic string theory, which works in 26 dimensions, and then extend it to its 10D supersymmetric version in the RNS formalism. In doing so, we study the spectrum of this theory and we focus in particular on Type IIB superstring theory and the compactification of its 6 extra dimensions. After a brief Mathematical introduction, we focus on the moduli fields which arise from the extra dimensions. We investigate how to stabilise them in order to get a correct and phenomenologically viable inflationary scenario. In particular, we consider 2 models where the inflaton is a Kähler modulus: Non-perturbative and Loop Blow-Up inflation. Finally, we focus on the second model and check if the saxion associated to the inflaton 4-cycle volume can behave as a curvaton field. Its isocurvature perturbations get converted into standard curvature fluctuations when the axion decays. We find the conditions under which this contribution is subdominant with respect to the one arising from inflaton fluctuations, hence guaranteeing compatibility with observational constraints from the Planck satellite.

Abstract (Italian Version)

In questa tesi magistrale implementeremo il meccanismo di curvatone per generare perturbazioni scalari in uno scenario di inflazione in teoria delle stringhe. All'inizio introdurremo la basica teoria delle stringhe bosoniche che necessita di 26D, poi la estenderemo alla sua controparte supersimmetrica in 10D nel formalismo RNS. Nel frattempo studieremo lo spettro di questa teoria e ci concentreremo in particolare nella teoria delle superstringhe Type IIB e nella compattificazione delle sue 6 dimensioni extra. Dopo una piccola introduzione Matematica, ci concentreremo sui moduli, campi che provengono dalle dimensioni extra. Studieremo poi come stabilizzarli in modo da ottenere un corretto scenario inflazionario fenomenologicamente valido. In particolare studieremo nel dettaglio 2 modelli nei quali l'inflatone è un modulo di Kähler: Non-Perturbative e Loop Blow-Up inflation. Infine ci concentreremo sul secondo modello e verificheremo che l'assione partner del modulo che misura il volume del 4-ciclo inflazionario può comportarsi da curvatone. Le sue perturbazioni di entropia vengono convertite nelle perturbazioni standard di curvatura quando l'assione decade. Troveremo le condizioni per le quali il contributo dell'assione alle perturbazioni scalari è non dominante rispetto a quello dell'inflatone, garantendo quindi la compatibilità con i valori osservati dal satellite Planck.

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Contents

Introduction

CMB Power Spectrum and Anisotropies are two of the most evident smoking guns of the inflationary theory consistency. In the years a lot of models have been proposed and succeeded to generate the right amplitude of the CMB power spectra but a lot failed. Among the models that failed the phenomenological check some have very interesting features so David H. Lyth and David Wands introduced in [33] the curvaton mechanism, an alternative mechanism where the curvature perturbations can be generated converting isocurvature perturbations of a field called **curvaton** at the decay of the field itself. This mechanism can reinstate ruled out scenarios, release other bounds on it (for example, making the curvaton generate the whole curvature perturbations we are more free to fine tune parameters on inflaton dynamics) or even compromise the ones which satisfy observational bounds. In fact, if the model features such a field and it already satisfied the Planck satellite value for CMB amplitude, then the risk to generate extra curvature perturbation can break the phenomenological viability of the scenario. In this thesis we are going to implement such a mechanism in String Inflationary scenario. The need of string theory in such scenarios arises from the fact that the ultraviolet (UV) behaviour of gravity is not known and so even of inflation is not well known from classical Quantum Field Theory. In fact, a Quantum Field Theory including gravity will lead to a non-renormalisable UV divergencies, while string theory is a frameworkfeaturing renormalisable finite theory which naturally includes General Relativity and, in particular, graviton in its spectrum. Through the almost 60 years in which string theory has been developed, many versions of it grew up. The first one, which is the **bosonic string theory** features the need of 26 dimensions, a tachyon and even doesn't include fermions in the spectrum which leads to a problematic nature of it. Including supersymmetry (in particular we are going to study the **Type IIB Superstring Theory**) the problems of the fermions absence and of the tachyon have been solved and the necessary and sufficient dimensions fall down to 10. Since our spacetime features only 4D, a natural question can be where the other 6D are. The main idea is that these additional dimensions are compactified, through the Kaluza-Klein Compactification method, to a special complex kind of manifold called **Calabi-Yau threefold** such that the 10D theory becomes a 4D Effective Field Theory. From the additional degrees of freedom of the 10D theory, in particular from the perturbations of the 6D Calabi-Yau metric, a lot of fields, called **moduli** arise naturally with even a large amount of **axions** from the string spectrum. Moduli are divided into 3 classes: **Kähler moduli** which control the deformations of the metric changing Kähler form, **Complex Structure moduli** which control the deformations of the metric changing Complex Structure and **axio-dilaton** which includes the dilaton and a form arising from the string spectrum. These moduli, since their mathematic nature, have a strong geometric meaning, in fact their vacuum expectation value controls the dimension of the Calabi-Yau and, in particular the Kähler ones, of the cycles which are contained in it. However there is a problem, since in the 4D EFT (at tree level and considering no fluxes in the Calabi-Yau) all these moduli are flat directions in the scalar potential and so their vacuum expectation value is undefined leading to a Calabi-Yau that can leave the compactification limit. In addition to this, being these moduli massless and so not decaying, we should have noticed even fifth forces arising from them. What stabilise these moduli are UV effects like 3-form fluxes, loop effects from strings or non-perturbative effects from instantons or branes. Fluxes will stabilise the axio-dilaton and the complex structure moduli, perturnative and non-perturbative effects can stabilise the Kähler moduli and the axions. Given the string theoretical framework, there are different methods to stabilise all these Kähler moduli which lead to different inflationary models. We are going to consider the LVS scenario for Type IIB String Theory which features an Anti-de-Sitter non-supersymmetric minimum, brought to Slightly de Sitter by adding an uplift term. In particular, working in this scenario we are going to study the Loop Blow-Up model which includes a big cycle modulus regulating the overall volume and small cycles which solve singularities and which work as holes in a Swiss-cheese. Here the Kähler moduli are stabilised by the work of both non-perturbative and perturbative effects, both from $(\alpha')^3$ corrections and from string loop ones and, in particular, the real part of these moduli will be our inflaton which has the correct inflationary potential because of the string loop correction. We finally implement the curvaton in this model by using as candidate the axion which is the imaginary counterpart of the inflaton. At this point, given a choice of parameters which is necessary for Loop Blow-Up consistency, the model already satisfies the observational values for the CMB power spectrum amplitude, so we need to check that the curvaton gives subleading values of curvature perturbation in order to keep the model consistent.

The structure of the thesis is almost self consistent: in chapter 1 we are going to study Bosonic String theory, highlighting its problems and how to solve them by the introduction of supersymmetry in it and, in particular, analyzing the spectrum and the different theories that arise from the possible choices in it. In 2 we are going to introduce a little bit of Mathematical background needed for understanding complex geometry and then we are going to study how to apply it on string compactifications and, in the end, we will inspect the moduli space and how to stabilise them in the 2 main different approaches (LVS and KKLT) using various possible corrections. In 3 we are going to study the model in which we include the curvaton both from the point of view of potential and from the one of observables trying to understand how it fits consistently the phenomenological bounds. Finally, in 4 we are going to study the behaviour and the dynamics of the axion which is the curvaton candidate during the inflation and, finally, we are going to use it as a proper curvaton verifying for which values of the free parameters the model keeps its consistency.

Chapter 1

Bosonic String Theory and RNS Superstrings

In this first chapter, following [6], [28] and, for the Supersymmetry part [36], we are going to give some fundamental informations on Bosonic String Theory and RNS formalism for Superstring Theory ending up with GSO Projection and the closed string Bosonic Spectrum.

1.1 The string action

The string action is nothing more then a generalization of a point like particle action, in this section we will start with the relativistic particle and reach finally the Polyakov action.

1.1.1 Relativistic point particle and generalization to p-branes

If we imagine a relativistic particle of mass m moving in a D-dimensional space-time we can imagine the problem as a variational problem where we minimize the action, clearly proportional to invariant length of particle's trajectory since motion

-and so the worldline- is along geodesics.

$$S_0 = -\alpha \int ds \tag{1.1}$$

with $\alpha = const$ and $\hbar = c = 1$.
We recall that S_0 must be $[S_0] = 0 \Rightarrow [\alpha] = 1 \Rightarrow \alpha = m$ the line element can instead be written as:

$$ds^2 = -g_{\mu\nu}(X)dX^\mu dX^\nu; \tag{1.2}$$

$$\mu, \nu = 0, ..., D-1 \tag{1.3}$$

$X^\mu(\tau)$ is usually called the world-line of the particle and it's the particle's trajectory itself. Since the action is independent on the choice of parametrization we can write it as:

$$S_0 = -m \int \sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu} d\tau \tag{1.4}$$

where $\dot{X}^\mu = \frac{dX^\mu}{d\tau}$. Since S_0 in this form contains square root it's difficult to quantize and for m=0 particles is equal to 0 too. So we can introduce an auxiliary part called einbein $e(\tau)$:

$$\tilde{S}_0 = \frac{1}{2} \int d\tau (e^{-1}\dot{X}^2 - m^2 e) \tag{1.5}$$

where we have $\dot{X}^2 = g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu$. Since $\tilde{S}_0$ is parametrization invariant, einbein must transform as $e \to e' = e + \frac{d(\xi e)}{dt}$.
In addition to this by searching for the equation of motion for the einbein:

$$\frac{\delta \tilde{S}_0}{\delta e(\tau)} = 0 \Rightarrow m^2 e^2 + \dot{X}^2 = 0 \tag{1.6}$$

so we get that on-shell (substituting the value of $e(\tau)$ from equation of motion) $S_0 = \tilde{S}_0$.
We can generalize now the action $\tilde{S}_0$ to the case of a string sweeping a two-dimensional world sheet -in analogy of a particle on the world-line- and, more generally, of a p-brane spanning a (p+1)-dimensional world volume where p must be less than D dimension of spacetime.

The action in this case takes the form of:

$$S_p = -T_p \int d\mu_p \tag{1.7}$$

where $d\mu_p = \sqrt{-detG_{\alpha\beta}} d^{p+1}\sigma$ where $G_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \;\; \alpha, \beta = 0, ..., p$ induced metric, so, since the action is mass-dimensionless, $[d\mu_p] = -P - 1 \Rightarrow [T_p] = p + 1$ in natural units. T_p is called p-brane tension.

1.1.2 Nambu-Goto and Polyakov actions

We now deal with a string so a 1-brane propagating in D-dimensional Minkowski spacetime. String will sweep the 2-dimensional surface previously called world-sheet which is parametrized, in analogy to world-line with 1 more dimension, by 2 coordinates: $\sigma^0 = \tau$ timelike and $\sigma^1 = \sigma$ spacelike.

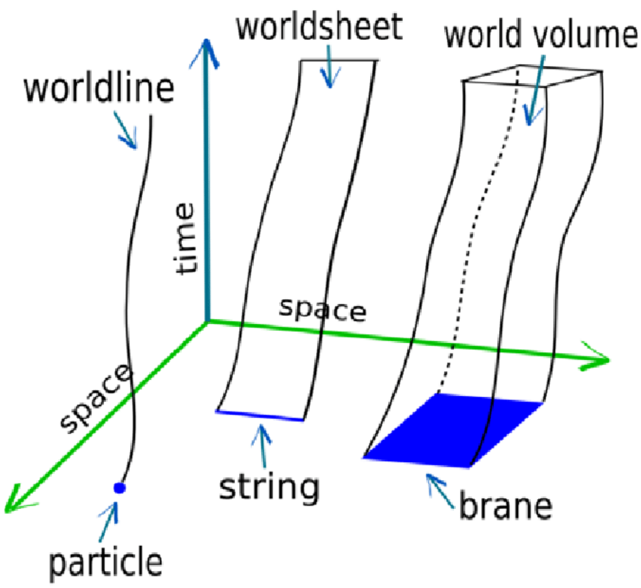

Figure 1.1: Worldsheet spanning of strings and Worldvolume spanning of branes.

If σ is periodic, string is said to be **closed**, if it covers just a finite interval $\sigma \in [-t, t] \;\; t \in \mathbb{R}$ string is said to be **open**. The string world-sheet is clearly embedded inside spacetime, called **target space**, and this embetting is described by functions $X^\mu(\sigma, \tau)$. Using these latter, the action describing a string propagating in a Minkowski spacetime is a generalization of p-brane action S_p obtained previously, which is called **Nambu-Goto action**:

$$S_{NG} = -T \int d\sigma d\tau \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} \tag{1.8}$$

where $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$ *and* $X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$ *and* $\dot{X} \cdot X' = \eta_{\mu\nu} \dot{X}^\mu X'^\nu$.

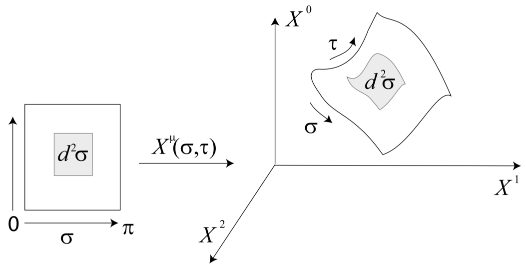

Figure 1.2: X^μ embedding map.

So the classical string motion minimizes (extremizes in a more general sense) the world-sheet area in the same way the particle's one minimizes length of world-line while moving on a geodesic. Even though Nambu-Goto action can be interpreted as the area of string world-sheet, due to square root, quantizing it is very difficult so what is used to do as in [6] is to obtain another equivalent action called string sigma model action or **Polyakov action** by introducing an auxiliary world-sheet metric $h_{\alpha\beta}(\sigma, \tau)$ analogously to the einbein on the particle case:

$$h = det(h_{\alpha\beta}) \ \ and \ \ h^{\alpha\beta} = (h^{-1})_{\alpha\beta} \tag{1.9}$$

with this, we can rewrite the action, classically equivalent to Nambu-Goto as:

$$S_P = -\frac{1}{2} T \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \tag{1.10}$$

1.1.3 Symmetries of Polyakov action

Polyakov action for bosonic string in Minkowski spacetime

$$S_P = -\frac{1}{2}T \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\nu \tag{1.11}$$

has 3 main symmetries+1 gauge redundancy:

- *Poincarè symmetry*:

  This symmetry is global, the action remains invariant while the world-sheet embedding transforms as:

  $$\delta X^\mu = a^\mu_\nu X^\nu + b^\mu \ \ and \ \ \delta h^{\alpha\beta} = 0 \tag{1.12}$$

  where the parameters a^μ_ν describe infinitesimal Lorentz transformations and b^μ infinitesimal space-time translation

- *Reparametrization on the world-sheet*:

  Changing τ and σ the action remains unaffected, in particular:

  $$\sigma^\alpha \to f^\alpha(\sigma) = \sigma'^\alpha \ \ and \ \ h_{\alpha\beta}(\sigma) = \frac{\partial f^\gamma}{\partial \sigma^\alpha} \frac{\partial f^\delta}{\sigma^\beta} h_{\gamma\delta}(\sigma') \tag{1.13}$$

  is a local symmetry called **diffeomorphism** (infinitely differentiable transformations with infinitely differentiable inverse ones) which leaves S_P invariant.

- *Weyl Transformations*:

  These are local transformations of the form:

  $$h_{\alpha\beta} \to e^{\phi(\sigma,\tau)} h_{\alpha\beta} \ \ and \ \ \delta X^\mu = 0 \tag{1.14}$$

  since $\sqrt{-h} \to e^{\phi(\sigma,\tau)}\sqrt{-h}$ transformation cancels with $h^{\alpha\beta} \to e^{\phi(\sigma,\tau)} h^{\alpha\beta}$.

Last two local symmetries can be used to choose a gauge in order to fix the 4 components of $h_{\alpha\beta}$ metric of the world-sheet (3 independent since this metric is a symmetric matrix). Using the reparametrization invariance of the action on we can fix 2 components out of 3 of

$$h_{\alpha\beta} = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix} \tag{1.15}$$

This reparametrization invariance so leaves only one degree of freedom which is analogous to have the metric as conformally flat:

$$h_{\alpha\beta} = e^{\phi(\sigma,\tau)}\eta_{\alpha\beta} \tag{1.16}$$

but using Weyl rescaling invariance we can eliminate that exponential resulting in a metric of the world-sheet looking like if we set $\phi(\sigma, \tau) = 0$:

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.17}$$

the action, since now on $\sqrt{-h} = 1$ *and* $h_{\alpha\beta} = \eta_{\alpha\beta}$, takes the form:

$$S_P = \frac{T}{2}\int d^2\sigma(\dot{X}^2 - X'^2) \tag{1.18}$$

This procedure to obtain a flat world-sheet metric is possible only when $\chi(\Sigma) = 0$ (Euler Characteristic of the world-sheet Σ vanishes) so when topological obstructions like holes are not present. This can happen both in freely propagating closed string case for which worldsheet is a cylinder and in freely propagating open string when it is an infinite strip. However, this is not the end of the gauge freedom story, there is a redundancy yet due to the fact that there is still a gauge transformations class that preserves the choice of the metric, which is:

$$\sigma \to \xi(\sigma) \text{ Coordinate Transformation} \tag{1.19}$$

$$h_{\alpha\beta} \to e^{\phi}h_{\alpha\beta} \text{ Compensating Weyl Rescaling} \tag{1.20}$$

as stated in [37], these coordinate reparametrizations change the world-sheet metric in such a way:

$$\eta_{\alpha\beta} \rightarrow \eta'_{\alpha\beta} = \Omega^2(\sigma)\eta_{\alpha\beta} \tag{1.21}$$

compensated by the Weyl rescaling. How to fix all the gauge freedom will be seen in light cone gauge chapter 1.4.

1.1.4 Equations of motion

Let us suppose that we have no topological obstruction so we can set our $h_{\alpha\beta} = \eta_{\alpha\beta}$ the action is then the already seen:

$$S_P = \frac{T}{2}\int d^2\sigma(\dot{X}^2 - X'^2) = \frac{T}{2}\int d^2\sigma((\partial_\tau - \partial_\sigma)X)^2 \tag{1.22}$$

by varying this action with the respect to X^μ we obtain:

$$\delta_{X^\mu} S_P = \frac{T}{2}\int d^2\sigma(2\dot{X}^\mu\partial_\tau\delta X_\mu - 2X'^\mu\partial_\sigma\delta X_\mu) = T\int d^2\sigma(\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu)\delta X_\mu + \tag{1.23}$$

$$+\text{boundary terms} = T\int d^2\sigma(\partial_\tau^2 - \partial_\sigma^2)X^\mu\delta X_\mu \tag{1.24}$$

since $\delta X_\mu \neq 0$ for arbitrarity of the variation $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$ so

$$\partial_\alpha\partial^\alpha X^\mu = 0 \;\; with \;\; \alpha = 0, 1 \tag{1.25}$$

Which is nothing more than the 2D wave equation.
There are constraints for this equation of motion, they come from the non gauge fixed Polyakov action. From that action in fact we could have derived the equation of motion for our $h_{\alpha\beta}$ and seen that this field can be eliminated, like the einbein in the particle case. Since the world-sheet metric is just an auxiliary field that can be eliminated, it is not physical and thus has no kinetic term so the energy momentum tensor, which is then conserved under spacetime traslation of the action, is:

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_P}{\delta h^{\alpha\beta}} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X = 0 \tag{1.26}$$

so the energy momentum tensor vanishes using the equation of motion for $h_{\alpha\beta}$.

In the gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$ called conformal gauge, this condition becomes a constraint we have to impose since we gauge fixed the world-sheet metric:

$$T_{\alpha\beta} = 0 \Rightarrow T_{01} = T_{10} = \dot{X} \cdot X' = 0 \;\; and \;\; T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \tag{1.27}$$

summing the components we get to the unique constraint:

$$0 = 2\dot{X} \cdot X' + (\dot{X}^2 + X'^2) \Rightarrow (\dot{X} + X')^2 = 0 \tag{1.28}$$

from here we can even see that $Tr(T_{\alpha\beta}) = \eta^{\alpha\beta}T_{\alpha\beta} = T_{11} - T_{00} = 0$ automatically, so the traceless property of Energy Momentum Tensor is automatically guaranteed in this gauge.

In order to simplify more the equation and so the solution it's really convenient to introduce light-cone coordinates on the world-sheet defined as:

$$\sigma^{\pm} = \tau \pm \sigma \tag{1.29}$$

In these coordinates clearly we get:

$$\partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \;\; and \;\; \eta_{\alpha\beta} = \begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.30}$$

and clearly the equation of motion can be rewritten as:

$$\partial_\alpha \partial^\alpha X^\mu = \partial_+ \partial_- X^\mu = 0 \tag{1.31}$$

The constraint which is the vanising of energy momentum tensor is now:

$$\begin{cases} T_{++} = \partial_+ X \cdot \partial_+ X - \frac{1}{2}\eta_{++}\eta^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X = \partial_+ X_\mu \partial_+ X^\mu = 0 \\ T_{+-} = \partial_+ X \cdot \partial_- X - \frac{1}{2}\eta_{+-}\eta^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X = \partial_+ X_\mu \partial_- X^\mu - \partial_+ X_\mu \partial_- X^\mu \equiv 0 \\ T_{-+} = \partial_- X \cdot \partial_+ X - \frac{1}{2}\eta_{-+}\eta^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X = \partial_- X_\mu \partial_+ X^\mu - \partial_- X_\mu \partial_+ X^\mu \equiv 0 \\ T_{--} = \partial_- X \cdot \partial_- X - \frac{1}{2}\eta_{--}\eta^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X = \partial_- X_\mu \partial_- X^\mu = 0 \end{cases} \tag{1.32}$$

While $T_{+-} = T_{-+} = 0$ represent the vanishing of the trace automatically satisfied due to the Weyl Invariance, the other two represent the constraints:

$$T_{++} = \partial_+ X_\mu \partial_+ X^\mu = 0 \tag{1.33}$$

$$T_{--} = \partial_- X_\mu \partial_- X^\mu = 0 \tag{1.34}$$

Now that we found the contraints (1.33), (1.34) and the equation of motion (1.31) the problem is setted almost entirely and so we can derive the general solution:

$$X^\mu(\sigma, \tau) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \tag{1.35}$$

which is sum of right moving part and left moving part. In order to find explicitly both right and left moving parts we need to require the embedding function to be real, impose suitable boundary conditions and the constraints on the energy momentum tensor which now have become:

$$(\partial_- X_R)^2 = (\partial_+ X_L)^2 = 0 \tag{1.36}$$

1.2 Boundary conditions and mode expansions

In order to define properly a full variational problem, boundary conditions of course are needed so in this section we explore different possibilities for boundary conditions and the mode expansion they lead to.

1.2.1 Boundary conditions

As we previously stated string can be both close or open. For convenience we choose $\sigma \in [0, \pi]$ in order to follow [6].

Stationary points of the action as always are determined by demanding invariance of S_P under embedding map shift:

$$X^\mu \to X^\mu + \delta X^\mu \tag{1.37}$$

obtaining, as seen before:

$$\delta_{X^\mu} S_P = \frac{T}{2} \int d^2\sigma (2\dot{X}^\mu \partial_\tau \delta X_\mu - 2X'^\mu \partial_\sigma \delta X_\mu) = \tag{1.38}$$

$$= T \int d^2\sigma \left[(\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu)\delta X_\mu + \partial_\tau(\dot{X}^\mu \delta X_\mu) \right] - T \int d\tau (X'_\mu \delta X^\mu|_{\sigma=\pi} - X'_\mu \delta X^\mu|_{\sigma=0}) \tag{1.39}$$

the total derivative in τ vanishes automatically at $\pm\infty$ but we need even to achieve the non trivial vanishing of

$$-T \int d\tau (X'_\mu \delta X^\mu|_{\sigma=\pi} - X'_\mu \delta X^\mu|_{\sigma=0}) \tag{1.40}$$

This vanishing can be achieved in 3 ways:

- *Closed string boundary condition*:
  In this case embedding functions are periodic in σ:

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + \pi, \tau) \tag{1.41}$$

- *Open string with Neumann boundary conditions*: In this case, component of momentum normal to the world-sheet evaluated at boundary vanishes:

$$X'_\mu|_{\sigma=0,\pi} = 0 \tag{1.42}$$

  Making this choice physically means that no momentum exits the ending of the string and $\forall\mu$ boundary conditions respect Poincarè invariance in D dimensions.

- *Open string with Dirichlet boundary conditions*: In this case, extrema of strings are fixed so $\delta X^\mu = 0$ and:

$$X^\mu|_{\sigma=0} = X^\mu_0 \;\; and \;\; X^\mu|_{\sigma=\pi} = X^\mu_\pi \tag{1.43}$$

  both constant and $\mu = 1, ..., D-p-1$ for the other p+1 coordinates Neumann boundary conditions are imposed. Clearly, since some coordinates are treated differently from others, Poincarè invariance is broken, so in the past these boundary conditions have been abandoned but now, in modern times, X^μ_0 and X^μ_π represent position of Dp-branes which is a special kind of p-brane where string endcaps are attached, this Dp-brane presence can be proved to break Poincarè invariance as previously stated except for p=D-1 so the Dp-brane is spacetime filling (which is exactly our case).

1.2.2 Mode expansions

We start from the closed string mode expansion and then we derive by physical motivations the mode expansion for open string with Neumann boundary conditions. We recall that the Polyakov action can be rewritten, recalling just to add the jacobian multiplication for a curved world-sheet metric ($\sqrt{-h}$), as:

$$S_P = -\frac{1}{2}T\int d^2\sigma\partial_\alpha X^\mu\partial^\alpha X^\mu \tag{1.44}$$

defining T in terms of α' Regge slope parameter and in terms of string length scale l_s as:

$$T = \frac{1}{2\pi\alpha'} \;\; and \;\; \frac{1}{2}l_s^2 = \alpha' \tag{1.45}$$

we can rewrite the action as:

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X^\mu \tag{1.46}$$

Using the closed string boundary conditions (1.41) we recall that any function satisfying such a boundary condition can be Fourier expanded in modes:

$$X^\mu(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} e^{in\sigma} f_n^\mu(\tau) \tag{1.47}$$

Plugging inside the equation of motion $(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})X^\mu(\sigma,\tau) = 0$ this Fourier mode expansion we get the equation of motion for the Fourier modes which is nothing more than the equation for a 1D harmonic oscillator:

$$\partial_\tau^2 f_n^\mu(\tau) + n^2 f_n^\mu(\tau) = 0, \tag{1.48}$$

$$\partial_\tau^2 f_0 = 0 \tag{1.49}$$

The solution for $n \neq 0$ is, as always, a linear combination of imaginary exponential:

$$f_n^\mu(\tau) = \alpha_n^\mu e^{in\tau} + \tilde{\alpha}_n^\mu e^{-in\tau} \tag{1.50}$$

While for $n = 0$ is clearly a linear term with the respect to τ:

$$f_0^\mu(\tau) = x^\mu + p^\mu\tau \tag{1.51}$$

where x^μ will be the **center of mass position** and p^μ which can be proven to be the **total string momentum** via computing the conserved charges with the respect to Poincarè symmetry:

$$p_\mu = T\int_0^\pi d\sigma \dot{X}^\mu(\sigma) \tag{1.52}$$

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + \sum_{n\neq 0} = -\frac{i}{n}(\alpha^\mu_{-n}\alpha_n\nu - \alpha^\nu_{-n}\alpha_n\mu) + \text{ tilded for closed strings} \tag{1.53}$$

where $M^{\mu\nu}$ angular momentum. The exponential parts of (1.47) are the string excitation modes f^μ_n. Putting all this together and introducing convenient factors:

$$X^\mu(\sigma,\tau) = x^\mu + l_s^2 p^\mu \tau + il_s \sum_{n\neq 0} \frac{1}{n}(\alpha^\mu_n e^{-in(\tau-\sigma)} + \tilde{\alpha}^\mu_n e^{-in(\tau+\sigma)}) \tag{1.54}$$

so that we get:

$$X^\mu_R(\sigma,\tau) = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau-\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}(\alpha^\mu_n e^{-in(\tau-\sigma)}) \tag{1.55}$$

$$X^\mu_L(\sigma,\tau) = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau+\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}(\tilde{\alpha}^\mu_n e^{-in(\tau+\sigma)}) \tag{1.56}$$

Requiring X^μ to be real, we derive in the end that x^μ, p^μ are real too and positive and negative modes are complex conjugate of each other because of the change of sign of exponential's argument.

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^* \ \ and\ \ \tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^* \ \ and\ \ \alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \frac{1}{2}l_s p^\mu \tag{1.57}$$

In order to obrain the open string mode expansion we note that the two modes $\alpha^\mu_n, \tilde{\alpha}^\mu_n$ represent in closed modes kind of right and left moving waves propagating.For the open string left and right-moving modes must combine into standing waves since the endcaps of the strings are not connected.

This is analogue to say that $\alpha^\mu_n = \tilde{\alpha}^\mu_n = \alpha^\mu_m$ for open string and so we obtain, imposing this condition in the closed string expansion:

$$X^\mu(\sigma,\tau) = x^\mu + l_s^2 p^\mu \tau + \frac{i}{2}l_s \sum_{m\neq 0} \frac{1}{m}(\alpha^\mu_m e^{-im\tau})(e^{im\sigma} + e^{(-im\sigma)}) \tag{1.58}$$

Using trigonometric identity on cos:

$$X^\mu(\sigma, \tau) = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m \neq 0} \frac{1}{m} (\alpha_m^\mu e^{-im\tau}) cos(m\sigma) \tag{1.59}$$

1.3 Canonical quantization

In this section we try to canonically quantize the theory starting from Poisson Brackets and jumping to commutators, finding out a big problem about negative norm states and solving it by imposing the such called Virasoro Constraints.

1.3.1 Classical Poisson brackets and commutation relations

In order to quantize the theory and so using the Poisson Brackets before, we need to define the canonical conjugate momentum:

$$P^\mu = \frac{\delta S}{\delta \dot{X}^\mu} = T\dot{X}^\mu \tag{1.60}$$

so we get the classical Poisson brackets equal to:

$$[P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)]_{P.B.} = [X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)]_{P.B.} = 0 \tag{1.61}$$

$$[P^\mu(\sigma,\tau), X^\nu(\sigma',\tau)]_{P.B.} = \eta^{\mu\nu}\delta(\sigma-\sigma') \tag{1.62}$$

or, analogously, writing the momentum in terms of $\dot{X}^\mu$:

$$\{\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\}_{P.B.} = \frac{1}{T}\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{1.63}$$

Plugging now inside the poisson brackets, using $\delta(\sigma-\sigma') = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} e^{2in(\sigma-\sigma')}$ we get the ones for the modes:

$$\{\alpha^\mu_m, \alpha^\nu_n\}_{P.B.} = \{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\}_{P.B.} = im\eta^{\mu\nu}\delta_{m+n,0} \tag{1.64}$$

$$\{\alpha^\mu_m, \tilde{\alpha}^\nu_n\}_{P.B.} = 0 \tag{1.65}$$

Now, when we replace Poisson brackets with commutators with the canonical prescription:

$$\{...\}_{P.B.} \rightarrow i\,[...] \tag{1.66}$$

That gives :

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{1.67}$$

$$[\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0 \tag{1.68}$$

We can define analogue of raising and lowering operators by just calling:

$$a^\mu_m = \frac{1}{\sqrt{m}}\alpha^\mu_m \;\; and \;\; a^{\mu\dagger}_m = \frac{1}{\sqrt{m}}\alpha^\mu_{-m} \;\; for \;\; n, m > 0 \tag{1.69}$$

from the commutation relationships for the modes we can derive the one for the a^μ_n and for the $\tilde{a}^\mu_n$ which respect the raising and lowering operator algebra:

$$\left[a^\mu_m, a^{\nu\dagger}_n\right] = \left[\tilde{a}^\mu_m, \tilde{a}^{\nu\dagger}_n\right] = \eta^{\mu\nu}\delta_{m,n} \;\; for \;\; m, n > 0 \tag{1.70}$$

which lead to a big problem due to Minkowski -1 component:

$$\left[a^0_m, a^{0\dagger}_m\right] = -1 \tag{1.71}$$

In fact, by defining the number operators $N = \sum_{k>0} \alpha_{-k} \cdot \alpha_k$ and $\tilde{N} = \sum_{k>0} \tilde{\alpha}_{-k} \cdot \tilde{\alpha}_k$, by a construction à la Fock of the closed string states space

$$\mathcal{H}^{\text{Fock}}_{\text{Closed}} = \text{span}_{\mathbb{C}}\left\{ \prod_{\mu=0}^{D-1} \prod_{n=1}^{\infty} \prod_{m=1}^{\infty} (\alpha^\mu_{-n})^{N^\mu_n} (\tilde{\alpha}^\mu_{-n})^{\tilde{N}^\mu_m} |0\rangle \, | N^\mu_n, \tilde{N}^\mu_m \geq 0 \text{ but finite} \right\} \tag{1.72}$$

(analogously for open string one), we get that a state $a^{0\dagger}_m |0\rangle$ has:

$$\langle 0| a^0_m a^{0\dagger}_m |0\rangle = -1 \tag{1.73}$$

negative norm. Negative-norm states, if not decoupled from the dynamics, can interact in processes with other physical ones generating violation of causality and unitarity. We can remove them in 2 different ways:

- Imposing energy momentum constraints 1.3.2;
- Automatically removing them via light-cone gauge quantization 1.4.

1.3.2 Virasoro constraints

In order to impose the Energy Momentum constraints it is useful to plug the closed (analogously open) string mode expansion (1.54) separated in (1.56) and (1.55) inside (1.33) and (1.34) giving us the Laurent Expansion of the Energy Momentum Tensor:

$$T_{--} = 2l_s^2 \sum_{m=-\infty}^{+\infty} L_m e^{-2im(\tau-\sigma)} \text{ and } T_{++} = 2l_s^2 \sum_{m=-\infty}^{+\infty} \tilde{L}_m e^{-2im(\tau+\sigma)} \tag{1.74}$$

with the coefficients given by:

$$L_m = \frac{T}{2}\int_0^\pi e^{-2im\sigma} T_{--} d\sigma = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \alpha_{m-n}\cdot\alpha_n \tag{1.75}$$

$$\tilde{L}_m = \frac{T}{2}\int_0^\pi e^{-2im\sigma} T_{++} d\sigma = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n \tag{1.76}$$

while, for the open strings, we have just L_m. These L_m and $\tilde{L}_m$ are called Virasoro Generators since they satisfy the Classical Virasoro (or De Witt) Algebra:

$$\{L_m, L_n\}_{\text{P.B.}} = i(m-n)L_{m+n} \tag{1.77}$$

which can be computed by using the Poisson brackets (1.64) and (1.65) and which is the algebra of the transformations corresponding to the residual gauge freedom (1.19). The constraints (1.33) and (1.34) become:

$$L_m = 0 \tag{1.78}$$

$$\tilde{L}_m = 0 \tag{1.79}$$

Up to now everything is classical, however, via passing from the Poisson Brackets to the Commutation Relations (1.66) we get slight complications:

1) The commutation relations for the modes (1.67) and (1.68) imply that we need to define the ill posed Virasoro generator L_0 as normal ordered to solve

the eventual ambiguities:

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\cdot\alpha_n: = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n, \;\; m\neq 0 \tag{1.80}$$

$$\tilde{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n: = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n, \;\; m\neq 0 \tag{1.81}$$

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{+\infty}\alpha_{-n}\cdot\alpha_n = \frac{1}{2}\alpha_0^2 + N \tag{1.82}$$

$$\tilde{L}_0 = \frac{1}{2}\tilde{\alpha}_0^2 + \sum_{n=1}^{+\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n = \frac{1}{2}\tilde{\alpha}_0^2 + \tilde{N} \tag{1.83}$$

2) These expression for the Virasoro generators satisfy the Virasoro Algebra commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n,0} \tag{1.84}$$

which is a central extension of the classical Virasoro Algebra with central charge D spacetime dimension.

3) The classical constraints must have been replaced with Gupta-Bleuler like conditions since the vanishing of all the Virasoro generators does not satisfy the Virasoro Algebra:

$$\langle\phi| L_m |\phi\rangle = 0 \;\; \forall\, |\phi\rangle \in \mathcal{H}_{\text{Phys}} \; \langle\phi| \tilde{L}_m |\phi\rangle = 0 \;\; \forall\, |\phi\rangle \in \mathcal{H}_{\text{Phys}} \tag{1.85}$$

where $\mathcal{H}_{\text{Phys}}$ Fock space of Physical states. This condition by simple manipulation and taking into account the normal ordering problem on L_0 can be rewritten as:

$$L_m |\phi\rangle = 0 \;\; \forall m > 0 \text{ and } \forall\, |\phi\rangle \in \mathcal{H}_{\text{Phys}} \tag{1.86}$$

$$(L_0 - a)|\phi\rangle = 0 \;\; \forall\, |\phi\rangle \in \mathcal{H}_{\text{Phys}} \tag{1.87}$$

with the additional conditions for closed strings:

$$\tilde{L}_m |\phi\rangle = 0 \ \forall m > 0 \text{ and } \forall |\phi\rangle \in \mathcal{H}_{\text{Phys}} \tag{1.88}$$

$$(\tilde{L}_0 - a) |\phi\rangle = 0 \ \forall |\phi\rangle \in \mathcal{H}_{\text{Phys}} \tag{1.89}$$

where a is called normal ordering constant and remains the same both in tilded and non-tilded case in order to avoid gravitational anomalies.

In addition to complications, the introduction of Virasoro generators leads to interesting features:

- Since the 4-momentum p_μ can be rewritten by using (1.52) so as $p^\mu = \frac{\alpha_0^\mu}{\sqrt{2\alpha'}}$ then, using (1.83) the Mass Shell condition can be rewritten as:

$$M^2 = -p_\mu p^\mu = \frac{1}{\alpha'} \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n - a = \frac{1}{\alpha'}(N - a) \tag{1.90}$$

for open strings and

$$M^2 = \frac{4}{\alpha'} \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n - a = \frac{4}{\alpha'} \sum_{n=1}^{+\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n - a = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\tilde{N} - a) \tag{1.91}$$

for closed strings, where $N = \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n$ and $\tilde{N} = \sum_{n=1}^{+\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n$ are the number operators.

This implies that in both open and closed string case, the ground state is tachyonic if $a > 0$:

$$M^2 = -\frac{4}{\alpha'} a \tag{1.92}$$

- Quantising the expression (1.53) we get that:

$$[L_m, M^{\mu\nu}] = 0 \tag{1.93}$$

meaning that this quantisation method is manifestly covariant and that a physical state, a state satisfying Virasoro constraints, remains physical after

Lorentz transformations.

- Taking the constraints (1.87) and (1.89) and subtracting them we get:

$$(L_0 - \tilde{L_0})\,|\phi\rangle = 0 \;\; \forall\,|\phi\rangle \in \mathcal{H}_{\text{Phys}} \tag{1.94}$$

which, from the definition (1.83) implies:

$$N = \tilde{N} \tag{1.95}$$

this is the such called **Level Matching condition** and it has the strong Physical meaning of having the same number of left and right moving modes.

- The normal ordering constant a and the spacetime dimension D can be fixed by demanding the absence of negative norm state. This can be achieved via imposing Virasoro constraints and obtaining the absence of such a unitarity violation states for $a \leq 1$ $1 \leq D \leq 26$, however only the such called critical string theory gives no problems on string interactions so we are going to study this case, which is the one where $\mathbf{a} = \mathbf{1}$ and $\mathbf{D} = \mathbf{26}$ and which then contains a tachyonic ground state:

$$M^2 = -\frac{4}{\alpha'} \tag{1.96}$$

1.4 Light cone gauge quantization

As stated in chapter 1.1.3, bosonic string has residual diffeomorphism symmetries, so residual gauge freedom, after choosing $h_{\alpha\beta} = \eta_{\alpha\beta}$ conformal gauge, this residual symmetry is made, as we said, a reparametrization of the world-sheet parameter σ compensated by a Weyl rescaling. In this chapter we are going to exploit this additional gauge freedom to quantise the theory in an alternative way manifestly free of negative norm states but not manifestly covariant.

1.4.1 Removal of negative norm states

In order to remove this additional gauge freedom we introduce now light-cone coordinates for space time:

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}) \tag{1.97}$$

Where X^μ has now $\mu = +, -, 1, ..., D-2$ and the overall $\frac{1}{\sqrt{2}}$ is due to the fact that we removed the $\frac{1}{2}$ from the definition of $\sigma^\pm$.
Now the spacetime metric becomes in these new coordinates:

$$ds^2 = -2dX^+dX^- + dX^i dX_i \tag{1.98}$$

where $i = 1, ...D-2$. This choice of light-cone coordinates is clearly non manifestly covariant since some coordinates are treated differently (higher $+$ index becomes a lower $-$ under use of metric). In order to proceed to gauge fixing we need before to study better this residual gauge freedom.
The reparametrization infinitesimally can be written as:

$$\sigma^\alpha \to \xi^\alpha = \sigma^\alpha + \gamma^\alpha \Rightarrow h_{\alpha\beta}(\xi) = \frac{\partial(\sigma^\mu + \gamma^\mu)}{\partial(\sigma^\alpha)}\frac{\partial(\sigma^\nu + \gamma^\nu)}{\partial\sigma^\beta}h_{\mu\nu}(\sigma) = \tag{1.99}$$

$$= (\delta_{\alpha\mu} + \partial_\alpha\gamma^\mu)(\delta_{\beta\nu} + \partial_\beta\gamma^\nu)h_{\mu\nu}(\sigma) = h_{\alpha\beta}(\sigma) + \partial_\alpha\gamma_\beta + \partial_\beta\gamma_\alpha \Rightarrow \tag{1.100}$$

$$\Rightarrow \delta h_{\alpha\beta} = \partial_\alpha\gamma_\beta + \partial_\beta\gamma_\alpha \tag{1.101}$$

But in addition to this, considering the infinitesimal Weyl rescaling (with Λ parameter of this transformation) we have:

$$h_{\alpha\beta}(\xi) = (1+\Lambda)h_{\alpha\beta} = h_{\alpha\beta}(\sigma) + \Lambda h_{\alpha\beta}(\sigma) \Rightarrow \delta h_{\alpha\beta} = \Lambda h_{\alpha\beta}(\sigma) = \Lambda\eta_{\alpha\beta} \tag{1.102}$$

after gauge fixing. So in the end, after comparing the two variation of world-sheet metric, the parameters must satisfy:

$$\partial_\alpha\gamma_\beta + \partial_\beta\gamma_\alpha = \Lambda\eta_{\alpha\beta} \tag{1.103}$$

Defining then the world-sheet light cone coordinates again as $\sigma^\pm = \sigma^0 \pm \sigma^1$ the metric becomes $ds^2 = -d\sigma^+ d\sigma^-$ (so the equation works even with high indices $\partial^\alpha\gamma^\beta + \partial^\beta\gamma^\alpha = \Lambda\eta^{\alpha\beta}$) and, analogally, the infinitesimal parameter $\gamma^\pm = \gamma^0 \pm \gamma^1$ we get that the equation for the parameter becomes:

$$\partial^+\gamma^- + \partial^-\gamma^+ = 0 \tag{1.104}$$

$$\partial^+\gamma^+ = \partial^-\gamma^- = 0 \tag{1.105}$$

Specially focusing on:

$$\partial^+\gamma^- = \partial^-\gamma^+ = 0 \tag{1.106}$$

we get that, in the end $\xi^+ = \sigma^+ + \gamma^+ = \xi^+(\sigma^+)$ and $\xi^- = \sigma^- + \gamma^- = \xi^-(\sigma^-)$. This could have be seen even from the fact that as stated in 1.1.3 the reparametrizations connected to residual gauge symmetries are the one that modify the metric in such a way:

$$\eta_{\alpha\beta} \to \Omega(\sigma)\eta_{\alpha\beta} \tag{1.107}$$

so the ones of the form $\sigma^+ \to \xi^+(\sigma^+)\ \ and\ \ \sigma^- \to \xi^-(\sigma^-)$. So the correct parametrization is, in the end:

$$\sigma^+ \to \xi^+(\sigma^+)\ \ and\ \ \sigma^- \to \xi^-(\sigma^-) \tag{1.108}$$

In order to come back to a time/space coordinate couple instead of double null ones we can define:

$$\tilde{\tau} = \frac{1}{2}(\xi^+(\sigma^+) + \xi^-(\sigma^-)); \tag{1.109}$$

$$\tilde{\sigma} = \frac{1}{2}(\xi^+(\sigma^+) - \xi^-(\sigma^-)) \tag{1.110}$$

which means that $\tilde{\tau}$ is a solution of free massless wave equation:

$$\partial_+\partial_-\tilde{\tau} = (\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})\tilde{\tau} = 0 \tag{1.111}$$

However, in the conformal gauge even $X^\mu(\sigma,\tau)$ satisfy 2D wave equation so we can write $X^+(\tilde{\sigma},\tilde{\tau})$ as linearly dependent on $\tilde{\tau}$:

$$X^+(\tilde{\sigma},\tilde{\tau}) = x^+ + l_s^2 p^+\tilde{\tau} \tag{1.112}$$

So all the excited modes are now set to 0 which means:

$$\alpha_n^+ = 0 \;\; for \;\; n \neq 0 \tag{1.113}$$

Since now on we will recall $\tau = \tilde{\tau}, \sigma = \tilde{\sigma}$ for the sake of simplicity. We made now a non covariant gauge choice since the coordinates make us rewrite the metric in a non manifestly Lorentz invariant way. This non covariancy can lead to anomalies breaking the Lorentz invariance, since a Lorentz anomaly in non covariant gauge (light-cone in this case) is analogue to a conformal anomaly (we used conformal gauge for world-sheet metric) in a covariant gauge preserving Lorentz invariance. With this gauge fixing we removed the oscillator modes for X^+ so, if we manage to remove even the one for X^- we can write all the states acting with just **transverse** creation operators on vacuum and so naturally removing the negative norm states. We can do this by simply recalling that the components of X^μ must satisfy wave equations+ energy momentum tensor constraints, so even X^- must:

$$\partial_+\partial_- X^- = 0 \tag{1.114}$$

Leading to the usual solution $X^-(\sigma) = X_L^-(\sigma^+) + X_R^-(\sigma^-)$ constrained by:

$$\partial_+ X^\mu \partial_+ X_\mu = -2\partial_+ X^+ \partial_+ X^- + \partial_+ X^i \partial_+ X_i = 0 \Rightarrow \tag{1.115}$$

$$\Rightarrow 2\partial_+ X^+ \partial_+ X^- = \partial_+ X^i \partial_+ X_i \tag{1.116}$$

Plugging inside $X^+(\sigma,\tau) = x^+ + l_s^2 p^+ \tau$, in particular the shape of left and right moving parts $X_L^+(\sigma^+)) = \frac{1}{2}x^+ + \frac{1}{2}l_s^2 p^+ \sigma^+$, $X_R^+(\sigma^-) = \frac{1}{2}x^+ + \frac{1}{2}l_s^2 p^+ \sigma^-$ this leads to:

$$\partial_+ X_L^- = \frac{1}{l_s^2 p^+} \partial_+ X^i \partial_+ X_i \tag{1.117}$$

Analogously, for the other constraint in double partial derivative in σ_- we get:

$$\partial_- X_R^- = \frac{1}{l_s^2 p^+} \partial_- X^i \partial_- X_i \tag{1.118}$$

Considering now for simplicity an open string and applying now these constraints to the open string expansion for X^- (note that the arbitrary index of the mode is now called n, but before m) which can be written as

$$X^- = x^- + l_s^2 p^- \tau + i l_s \sum_{n=0} \frac{1}{n} \alpha_n^- e^{-in\tau} cos(n\sigma) \tag{1.119}$$

we can get the expression for the excited modes coefficient related to the creation/annihilation operator (classically):

$$\alpha_n^- = \frac{1}{p^+ l_s} \left(\frac{1}{2} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{+\infty} \alpha_{n-m}^i \alpha_m^i \right) \tag{1.120}$$

In the quantum theory however we have normal ordering problems and so inserting a constant appearing due to commutation and normal ordering (and =1 in critical string theory as stated before):

$$\alpha_n^- = \frac{1}{p^+ l_s} \left(\frac{1}{2} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{+\infty} : \alpha_{n-m}^i \alpha_m^i : - a\delta_{n,0} \right) \tag{1.121}$$

So, in the light-cone gauge it's possible even to remove X^+ and X^- in the sense that their modes vanish or can be expressed in terms of transverse modes, so in the end, the time component of the embedding map and so of the creation/annihilation operator a_n^0 is never present implying that unphysical negative norm states are naturally removed in this light-cone gauge even if we lost Lorentz invariance.

1.4.2 Computation of a and D

In order to check the consistency of this approach we are now going to briefly compute the normal ordering constant and the spacetime dimension in this picture.

The starting point is, as always, the Mass shell condition which, in the light cone gauge can be written for the open string as:

$$M^2 = -p_\mu p^\mu = 2p^+p^- - \sum_{i=1}^{D-2} p_i p^i = \frac{1}{\alpha'}(N-a) \tag{1.122}$$

where

$$N = \sum_{i=1}^{D-2} \sum_{n=1}^{+\infty} \alpha_{-n}^i \alpha_n^i \tag{1.123}$$

Since the only independent modes are the transverse one, the first excited state is given by:

$$\alpha_{-1}^i \left|0;p\right\rangle \tag{1.124}$$

which belongs to a D-2 component vector representation of SO(D-2) so it is massless for Lorentz covariance, giving us the value of normal ordering constant:

$$M^2 = \alpha'(1-a) = 0 \Leftrightarrow a = 1 \tag{1.125}$$

Given this the computation of the spacetime dimension comes from heuristic argument from the manual normal ordering of L_0 as showed in [6] and, using Riemann Zeta function we can see how consistently with the previous approach $D = 26$.

1.4.3 Open and closed string spectra

One of the advantages of Light Cone Gauge is that it is pretty straightforward to compute and check for Open and Closed String Spectra.

Open string spectrum

The first 3 mass levels for the open strings are given by:

- **N = 0**:
  Tachyon $|0;p\rangle$ with mass $M^2 = -\frac{1}{\alpha'}$
- **N = 1**:
  Massless Vector Boson $\alpha^i_{-1}|0;p\rangle$.
- **N = 2**:
  Two different possibilities$\Rightarrow$ α^i_{-2} and $\alpha^i_{-1}\alpha^j_{-1}|0;p\rangle$ with $M^2 = \frac{1}{\alpha'}$. These two possibilities represent respectively 24 and 300 states so in total 324 states which is the dimension of the symmetric traceless rank-2 representation of $SO(25)$ $\Rightarrow$ massive spin 2 state.

Closed string spectrum

The Closed String Spectrum construction is totally analogue to the Open String one but with a big difference: the Level Matching condition (1.95) must hold. Taking again the mass shell condition then in critical case:

$$M^2 = \frac{4}{\alpha'}(N-1) = \frac{4}{\alpha'}(\tilde{N}-1) \tag{1.126}$$

the physical states are:

- **N = 0**:
  Tachyon $|0;p\rangle$ with mass $M^2 = -\frac{4}{\alpha'}$
- **N = 1**:
  A tensor $|\Omega^{ij}\rangle = \alpha^i_{-1}\tilde{\alpha}^i_{-1}$ which represents 576 states. This $|\Omega^{ij}\rangle$ can be decomposed as follows:

  $$|\Omega^{ij}\rangle = |\Omega^{(ij)}\rangle + |\Omega^{[ij]}\rangle + \delta_{ij}|\Omega^{ij}\rangle \tag{1.127}$$

  where $|\Omega^{(ij)}\rangle$ is the symmetric traceless part transforming as a massless spin 2 particle under $SO(24)$ $\Rightarrow$ the **Graviton** (which gives an hint on how String

Theory naturally contains General Relativity), $|\Omega^{[ij]}\rangle$ transforms as antisymmetric rank-2 tensor under $SO(24)$ and it is called the **Kalb-Ramond field** $B_{\mu\nu} = B_2$ and $\delta_{ij}\,|\Omega^{ij}\rangle$ is the trace of $|\Omega^{ij}\rangle$ and transforms as a scalar under $SO(24)$; it is called the **Dilaton**.

1.5 $D = 26$ target space action

We now want to write an action not from the worldsheet prospective as we did before, but from the 26 dimensional target space. In order to do so, we start focusing on the closed string part. We straightforwardly write a quadratic action starting from the closed bosonic spectrum we have seen before in 1.4.3. Inserting vertices and using Path Integral formalism or checking heuristically that this action gives the correct equation of motions for the closed string spectrum fields as in [28], we can see how the non-linear 2-derivative action, excluding the tachyon, can be rewritten as:

$$S_{26\mathrm{D}} = \frac{1}{k^2}\int d^{26}x\sqrt{-G}e^{-2\phi}\left(R\left[G\right] - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4(\partial\phi)^2\right) \tag{1.128}$$

where k gravitational coupling, $R\left[G\right]$ Ricci scalar, $H_{\mu\nu\rho} = H_3 = dB_2$ a 3-form which is kind of the Field Strength tensor for Kalb-Ramond field, in total analogy with $F_{\mu\nu} = F_2 = dA = dA_\mu$ where d external derivative. The expression (1.128) contains several interesting features:

1) The Kinetic term for the dilaton ϕ is apparently sign mistaken, however this is not a problem since it is due to the fact that this action is written in the such called "String Frame" which is the analogue of the Brans-Dicke frame and just by reparametrizing the action via:

$$G_{\mu\nu} = \tilde{G}_{\mu\nu}e^{\frac{-\phi}{6}} \tag{1.129}$$

we can rewrite (1.128) as:

$$S_{26\mathrm{D}} = \frac{1}{k^2}\int d^{26}x\sqrt{-\tilde{G}}\left(R\left[\tilde{G}\right] - \frac{e^{\frac{-\phi}{3}}}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{6}(\partial\phi)^2\right) \tag{1.130}$$

which has the correct, well known sign and where the Planck Mass is manifestly fixed since the absence of the overall $e^{-2\phi}$ while the mass of excited states changes with the variation of background value of the dilaton.

2) Taking the action (1.128) and setting $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$ and $\phi = 0$ we get:

$$S_{26\mathrm{D}} = \frac{1}{k^2} \int d^{26}x R\left[\eta_{\mu\nu}\right] = 0 \tag{1.131}$$

which gives us the background of our original 2D theory (1.10). However, we can generalize (1.10) in 3 ways:

- Changing $\eta_{\mu\nu} \to G_{\mu\nu}$ in the Polyakov action leading to:

$$S_P = -\frac{1}{2}T \int d^2\sigma \sqrt{-h} h^{\alpha\beta} G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \tag{1.132}$$

  expanding the metric close to $X = 0$ we get:

$$G_{\mu\nu} = \eta_{\mu\nu} + \mathrm{const} \cdot (X^1)^2 \eta_{\mu\nu} + \dots \tag{1.133}$$

  giving an additional quartic interaction term in the worldsheet action.

- By Setting $B_{\mu\nu} = B_{\mu\nu} \neq 0$ as before we can expand it and obtain terms that are no longer quadratic.

- By Setting $\phi \neq 0$ something very interesting happens from the worldsheet perspective and we will discuss it in the next point since it's strongly related with the importance of the dilaton.

3) Since the overall exponential $e^{-2\phi}$ The value of k^2 can be changed by the value of the dilaton itself, which regulates then the value of the 26D Planck Mass with the respect to the mass of first excited modes $\frac{2}{\sqrt{\alpha'}}$ via defining $k^2 = c\alpha'^{12}$. Instead, from a worldsheet perspective, the dilaton enters the game in a very peculiar way. In principle, in the worldsheet action the Einstein Hilbert term

$$S_{\mathrm{E.H.}} = \frac{1}{4\pi} \int d^2\sigma \sqrt{h} R_{2\mathrm{D}} \tag{1.134}$$

appears naturally, however this, since the 2D nature of the worldsheet, it can be rewritten as a total derivative by the use of Einstein equations. In general its value then is a constant different from zero called Euler Characteristic χ which is depending on the genus g of a Riemann surface (number of Handles of it) $\chi = S_{\text{E.H.}} = 2(1-g)$. This term then due to the fact that is a topological invariant does not contribute to the dynamics of the Sigma model, so it is possible to generalize this term straightforwardly by adding a mass dimension 0 element to the action, i.e. a scalar, the dilaton as done in [28]:

$$S_{\text{TOT}} \supset \frac{1}{4\pi} \int d^2\sigma \sqrt{h} \phi(X^\mu) R_{2\text{D}} \tag{1.135}$$

In order to understand the final motivation behind the importance of the dilaton we now follow [35] and we start by taking the Euclidean Polyakov Path Integral:

$$Z = \int dX dg e^{-S} \tag{1.136}$$

where now g_{ab} is the Euclidean correspondent of h_{ab}. Upon switching on only $\phi \neq 0$ the euclidean action can be rewritten as:

$$S = S_P + \lambda\chi \tag{1.137}$$

where λ, using (1.135) is $\lambda = \phi$. The importance of passing into Euclidean description stays in the fact that a nontrivial worldsheet can have a nonsingular euclidean metric but has a singular Minkowskian one so this description is better given. Now adding an handle from a topological point of view corresponds to increasing the genus $g \to g+1 \Rightarrow \chi \to \chi - 2$ and so, using (1.136), a factor $e^{2\phi}$ appears in Z, however, due to the fact that physically adding it corresponds to emission and absorption of a closed string, the amplitude for closed string emission gets a correction of e^{ϕ} coming from the coupling, so this term and in particular the dilaton value controls the string coupling:

$$g_s = e^{\phi} \tag{1.138}$$

1.6 RNS superstring action

In the previous sections we have discussed bosonic string theory which is the most immediate way to discuss a generalization of the classical particle but still has two big problems:

1) Bosonic string spectrum both for closed and for open string contains tachyon and tachyons are symbol of vacuum instability which leads our theory to live in a Universe of false vacuum decaying into real one. Open string tachyon elimination can be traced back to the decay of D-branes into closed-string radiation, but for closed string tachyon the problem remains.

2) Bosonic string theory doesn't take into account fermions which are fundamental constituents of matter in nature.

One can imagine that we can insert fermions by hand but this can be achieved in a more elegant way requiring **Supersymmetry** in the action, a symmetry that relates bosons and fermions. String theories with supersymmetry are called superstring theories. In order to implement Supersymmetry inside string theory we have 2 approaches:

-) Ramond-Neveu-Schwarz (RNS) formalism where we add supersymmetry on the string world-sheet in sense that we include additional fermionic coordinates of the world-sheet related to the generators of supersymmetry.

-) Green-Schwarz (GS) formalism is manifestly supersymmetric instead in 10D Minkowski spacetime and here the "fermionic coordinates" are just fermionic additional embedding maps.

We will focus on RNS formalism.

1.6.1 Ramond-Neveu-Schwarz action

In RNS formalism embedding maps become bosonic fields $X^\mu(\sigma, \tau)$ of the two-dimensional world-sheet theory and they are paired with fermionic partner fields $\psi^\mu(\sigma, \tau)$ which are 2 component spinors on world-sheet (since we are in 2D). As we see from index structure, they are vectors under Lorentz Transformations of D-dimensional spacetime. These fields anticommute -being fermionic- and this is consistent with spin statistics in D=10. Setting now $l_s^2 = 1 \Rightarrow \alpha' = \frac{1}{2} \Rightarrow T = \frac{1}{\pi}$ we can rewrite the bosonic string action in the conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$ as:

$$S_P = -\frac{1}{2\pi}\int d^2\sigma \partial_\alpha X_\mu \partial^\alpha X^\mu \tag{1.139}$$

recalling that, after fixing the gauge, we need to impose the vanishing of energy momentum tensor as constraint in addition to the equation of motion. This is clearly a free field theory in 2D but it's still bosonic. In order to generalize it then we add other degrees of freedom adding fermions on the world-sheet which are D Majorana fermions $\psi^\mu(\sigma,\tau)$ belonging to the $SO(1, D-1)$ vector representation. For the sake of clarity we explicit that in the representation of 2D Dirac algebra a Majorana spinor is equivalent to a real spinor in the sense that depends just on 2 real parameters.

The total action now is obtained by adding the standard Dirac action for D Majorana massless Spinors to the free theory of D massless bosons:

$$S_{\text{TOT}} = -\frac{1}{2\pi}\int d^2\sigma(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu) = S_B + S_F \tag{1.140}$$

where α is a world-sheet component index, μ is the spacetime component index. Here we have that ρ^α *with* $\alpha = 0, 1$ represents the two-dimensional version of γ^μ Dirac matrices, which obey the Clifford algebra:

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta} \tag{1.141}$$

In order to be totally explicit, choosing a convenient basis, we can write the matrices as:

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; and \; \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.142}$$

Which clearly satisfy the algebra.

Let us now talk about the fermionic field in order to rewrite the action in a more convenient way. Classically the world-sheet fermionic field ψ^μ is composed by Grassmann variables which must anticommute:

$$\{\psi^\mu, \psi^\nu\} = 0 \tag{1.143}$$

Of course after quantizing, this must change. This spinor ψ^μ has two components ψ^μ_A where $A = \pm$ spinorial index, which now takes just 2 values since we are in 2D world-sheet:

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} \tag{1.144}$$

by following the procedure explicitly done in A we can rewrite the fermionic part of the action as: and Suppressing Lorentz indices that are just labels from the point of view of world-sheet:

$$S_F = \frac{i}{\pi} \int d^2\sigma (\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+) \tag{1.145}$$

From this action we can easily derive the equations of motion for ψ_+ and ψ_- as done in A:

$$\frac{\delta S_F}{\delta \psi_-} = 0 \Rightarrow \partial_+ \psi_- = 0 \tag{1.146}$$

$$\frac{\delta S_F}{\delta \psi_+} = 0 \Rightarrow \partial_- \psi_+ = 0 \tag{1.147}$$

These equations clearly represent left and right moving waves, for spinors in 2D these are Weyl conditions, so such fields $\psi^\mu_\pm$ are called **Majorana-Weyl spinors** which at a Group theoretical level, are inequivalent real 1D representations of 2D Lorentz group $SPIN(1,1) = GL(1,\mathbb{R}) \Rightarrow SPIN(1,1)/\mathbb{Z}_2 \simeq SO(1,1)$ as stated in [6].

1.6.2 Global world-sheet supersymmetry

The action

$$S_{\text{TOT}} = -\frac{1}{2\pi} \int d^2\sigma (\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu) \tag{1.148}$$

enjoys another symmetry in addition to the ones discussed previously on bosonic string section and preserved here, in fact action remains invariant under transformation:

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \tag{1.149}$$

$$\delta\psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon \tag{1.150}$$

Where ϵ is a constant infinitesimal 2D Majorana spinor which is made by anticommuting Grassmann numbers. This spinor in components can be written as:

$$\epsilon = \begin{pmatrix} \epsilon_- \\ \epsilon_+ \end{pmatrix} \tag{1.151}$$

So now, rewriting the action in terms of light-cone coordinates on the world-sheet and in component $\psi_+^\mu, \psi_-^\mu, \epsilon_+, \epsilon_-$, using the fact that, after gauge fixing, from the definition of $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$,:

$$\partial_\alpha X_\mu \partial^\alpha X^\mu = \partial_\alpha X_\mu \partial_\beta X^\mu \eta^{\alpha\beta} = -\partial_0 X_\mu \partial_0 X^\mu + \partial_1 X_\mu \partial_1 X^\mu = -4\partial_+ X_\mu \partial_- X^\mu \tag{1.152}$$

we get:

$$S_{\text{TOT}} = \frac{1}{\pi}\int d^2\sigma(2\partial_+ X_\mu \partial_- X^\mu + i\psi_- \partial_+ \psi_- + i\psi_+ \partial_- \psi_+) \tag{1.153}$$

And we can easily verify, as done in A, that this action is clearly invariant under the transformation written before.

This symmetry is very peculiar since mixes bosonic X^μ and fermionic ψ^μ degrees of freedom, in fact a variation of the bosonic field depends on fermionic field and viceversa, this is a symmetry that relates particles with different spin and it's called **Supersymmetry**. It has been discovered in 1971 by Gervais and Sakita and by Golfand and Likhtman in Soviet Union from the point of view of Super-Poincarè algebra in the same year. A very important thing about this symmetry that can be noted now is that the algebra of this transformation only closes on-shell as showed explictly in A, in the sense that commutation between two supersymmetry transformations, which are world-sheet translation+anticommuting coordinate translation as we will see later on, gives another world-sheet translation. We conclude the section with a note: ϵ in this case does not depend on σ nor τ so the supersymmetry here is called **global**, in principle we can make it dependent from them and we could have originated **local** supersymmetry called **Supergravity**.

1.7 Superfield formalism for RNS action

What we have said until now can be expressed in a more compact and manifestly supersymmetric way by using the such called superfield formulation and integration over Grassmann fermionic coordinates.

1.7.1 Superspace formalism

Starting from standard Poincarè group with generators $M^{\alpha\beta}, P^{\alpha}$ where $\alpha = 0, 1$ we can add other generators to enlarge the group of symmetry to the such called N=(1,1) SuperPoincarè algebra. This generators we add are fermionic, so are spinors, in particular, in 2D, Majorana-Weyl spinors Q_{-}, Q_{+}. Adding these 2 generators leads to enlarge the spacetime itself by including anti-commuting Grassmann coordinates

$$\theta_A = \begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix} \tag{1.154}$$

$\{\theta_A, \theta_B\} = 0$ forming a Majorana spinor where upper or lower index doesn't give any difference. These new coordinates, in addition to $\sigma^0 = \tau, \sigma_1 = \sigma$ map the such called **superspace**.

The superspace is defined as a coset which is, given two sets G and H, the set of elements in G but not in H, G/H.

For us the superspace is the coset given by

$$SuperPoinc./Lorentz = \{\omega^{\alpha\beta}, a^{\alpha}, Q_+, Q_-\}/\{\omega^{\alpha\beta}\} = \{a^{\alpha} = \sigma^{\alpha}, \theta_+, \theta_-\} \tag{1.155}$$

In this superspace we define **superfields** which are fields acting on it. The great advantages of superspace are 3:

1) The algebra of supersymmetry transformations closes off-shell, in the sense that the commutator between two supersymmetric transformations acting on a superfield gives another supersymmetry transformation without using equations of motion for the fields.

2) Since we will see that supersymmetry transformations correspond to transla-

tions on the world-sheet+translation on Grassmann coordinates this means that the commutator between two translation on world-sheet gives a traslation on world-sheet and all this happens off-shell so without the use of equations of motion, just by inserting another auxiliary field necessary for the consistence of the superfield itself.

3) In addition to this, using this superspace formulation the supersymmetry will be manifest.

1.7.2 Superfields, supercharges and supersymmetry transformations

We want now to start by building models and, as always, the most important brick is the Lagrangian. The first ingredient to build up Lagrangians for our models is the superfield whose most general form is:

$$\Phi^{\mu}(\sigma^{\alpha},\theta) = X^{\mu}(\sigma^{\alpha}) + \bar{\theta}\psi^{\mu}(\sigma^{\alpha}) + \frac{1}{2}\bar{\theta}\theta F^{\mu}(\sigma^{\alpha}) \tag{1.156}$$

(since now on we will suppress Lorentz index μ on superfield since it's just a label for the world-sheet) where we need $\bar{\theta}$ in order to have a coloumn vector multiplying a row one like ψ^{μ} and where we didn't add θ since, for Majorana spinors product, $\bar{\theta}\psi^{\mu} = \bar{\psi}^{\mu}\theta$. No other terms are allowed since, for the anticommuting nature of θ we have that $\theta\theta = \bar{\theta}\bar{\theta} = 0$. Here we see that we have added an auxiliary field $F^{\mu}(\sigma^{\alpha})$ that is very important for off-shell closure of the algebra.

We can derive then the expression of the supercharges following A getting in the end:

$$\mathbb{Q}_{\mathbb{A}} = \partial_{\bar{\theta}} - (\rho^{\alpha}\theta)_A\partial_{\alpha} \tag{1.157}$$

which is:

$$\mathbb{Q}_{\mathbb{A}} = \frac{\partial}{\partial\bar{\theta}} - (\rho^{\alpha}\theta)_A\partial_{\alpha} \tag{1.158}$$

We would like now to study how the transformation acts on the superfield. In order to do that we repeat similar intuition of the one used for a general field φ as seen in A. We let the supersymmetry transformation act on superfield as operator:

$$e^{-\bar{\epsilon}Q}\Phi(\sigma,\theta)e^{\bar{\epsilon}Q} \simeq (1-\bar{\epsilon}Q+o(\bar{\epsilon}^2))\Phi(\sigma,\theta)(1+\bar{\epsilon}Q+o(\bar{\epsilon}^2) = \Phi(\sigma,\theta)-\bar{\epsilon}Q\Phi(\sigma,\theta)+ \tag{1.159}$$

$$+\Phi(\sigma,\theta)\bar{\epsilon}Q+o(\bar{\epsilon}^2) = \Phi(\sigma,\theta)+[\Phi,\bar{\epsilon}Q]+o(\bar{\epsilon}^2) \tag{1.160}$$

Instead considering supersymmetry transformation acting on superfield as a field:

$$e^{\bar{\epsilon}\mathbb{Q}}\Phi(\sigma,\theta) = \Phi'(\sigma,\theta) = \Phi(\sigma,\theta)+\bar{\epsilon}\mathbb{Q}\Phi(\sigma,\theta) \Rightarrow \delta\Phi = \bar{\epsilon}\mathbb{Q} \tag{1.161}$$

comparing the two parts:

$$\delta\Phi = [\Phi,\bar{\epsilon}Q] = \bar{\epsilon}\mathbb{Q}\Phi \tag{1.162}$$

From this transformation:

$$\delta\Phi = \bar{\epsilon}\mathbb{Q}\Phi \tag{1.163}$$

we can derive, by the computations in A, the expression for the transformation of the fields contained in the superfield X^μ, ψ^μ, F^μ:

$$\delta X^\mu(\sigma) = \bar{\epsilon}\psi^\mu(\sigma) \tag{1.164}$$

$$\delta\psi^\mu(\sigma) = \rho^\alpha\partial_\alpha X^\mu(\sigma)\epsilon + F^\mu(\sigma)\epsilon \tag{1.165}$$

$$\delta F^\mu = \bar{\epsilon}\rho^\alpha\partial_\alpha\psi^\mu(\sigma) \tag{1.166}$$

First two formulas for the variation reduce to the one seen in non-superfield formalism if we use the equation of motion of F^μ that, since it is an auxiliary non physical field, is $F^\mu = 0$. From here we can immediately derive the first powerful consequence of adding F^μ field: the algebra of supersymmetry transformations now closes off-shell, since equations of motion are $F^\mu = 0$ and $\rho^\alpha\partial_\alpha\psi^\mu = 0$ so defining $F^\mu = \rho^\alpha\partial_\alpha\psi^\mu$ we get the closure even not using the equations of motion.

1.7.3 RNS action in superfield formalism

We have now almost all the ingredients to write the action we have seen before in superfield formalism, the only problem now is that derivative of a superfield is not a superfield since:

$$\delta(\partial_\alpha S) = [\delta_\alpha S, \bar{\epsilon}\mathbb{Q}] \neq (\bar{\epsilon}\mathbb{Q})\partial_\alpha S \tag{1.167}$$

We need then to define a covariant derivative which compensate the extra terms on commutator so which acting on a superfield gives raise to another superfield:

$$\mathcal{D}_A = \frac{\partial}{\partial\bar{\theta}^A} + (\rho^\alpha\theta)_A\partial_\alpha \tag{1.168}$$

Since the parts in the sum of $\mathcal{D}_A$ are the same of $\mathbb{Q}_A$ generators except for the plus sign in the middle and since they are made by anticommuting variables, of course:

$$\{\mathcal{D}_A, \mathbb{Q}_B\} = 0 \tag{1.169}$$

which tells us, since $\{\mathcal{D}_A, \bar{\epsilon}\} = 0$ that $[\mathcal{D}_A, \bar{\epsilon}\mathbb{Q}] = \bar{\epsilon}^B \{\mathcal{D}_A, \mathbb{Q}_B\} = 0$ and so covariant derivative of a field transforms as the superfield itself:

$$\delta\mathcal{D}_A\Phi = [\mathcal{D}_A\Phi, \bar{\epsilon}\mathbb{Q}] = \mathcal{D}_A [\Phi, \bar{\epsilon}\mathbb{Q}] = \mathcal{D}_A\bar{\epsilon}\mathbb{Q}\Phi = \bar{\epsilon}\mathbb{Q}\mathcal{D}_A\Phi \tag{1.170}$$

In addition to this, covariant derivative has this anticommutators:

$$\{\mathcal{D}_A, \mathcal{D}_B\} = 2i(\rho^\alpha\rho^0)_{AB}\partial_\alpha \tag{1.171}$$

$$\left\{\mathcal{D}_A, \bar{\mathcal{D}}_B\right\} = 2i(\rho^\alpha)_{AB}\partial_\alpha \tag{1.172}$$

Finally, the product of 2 superfields is again a superfield as always, thanks to the Leibnitz rule of the $\bar{\epsilon}\mathbb{Q}$. So the action now, written in terms of superfields is given by:

$$S = \frac{i}{4\pi}\int d^2\sigma d^2\theta \bar{\mathcal{D}}_A\Phi^\mu\mathcal{D}_A\Phi_\mu \tag{1.173}$$

In this formulation action is manifestly supersymmetric since:

$$\delta S = \frac{i}{4\pi}\int d^2\sigma d^2\theta(\bar{\mathcal{D}}_A\delta\Phi^\mu\mathcal{D}_A\Phi_\mu + \bar{\mathcal{D}}_A\Phi^\mu\mathcal{D}_A\delta\Phi_\mu) = \frac{i}{2\pi}\int d^2\sigma d^2\theta\bar{\epsilon}\mathbb{Q}\bar{\mathcal{D}}_A\Phi^\mu\mathcal{D}_A\Phi_\mu \tag{1.174}$$

if suitable boundary conditions are chosen in σ world-sheet coordinate then:

$$\delta S = 0 \tag{1.175}$$

since the integrand is a total derivative owing to the fact that $\mathbb{Q}$ is made by 2 terms which are derivatives, one in θ and one in σ. If not, supersymmetry is broken. The integration in θ follows Grassmann rules:

$$\int d\theta(a + \theta b) = b \tag{1.176}$$

and in our case, the only non zero integral is the one containing one $\bar{\theta}$ and one θ:

$$\int d^2\theta\bar{\theta}\theta = -2i \tag{1.177}$$

since all the other give raise to vanishing term due to excessive number of $\theta, \bar{\theta}$ or to non sufficient number of them under this integral which works like derivation. Applying the covariant derivative to superfield we can write the action 1.173 in terms of the component fields following A, leading us to the action:

$$S_{\text{TOT}} = -\frac{1}{2\pi} \int d^2\sigma(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}_\mu \rho^\alpha \partial_\alpha \psi^\mu - F_\mu F^\mu) \tag{1.178}$$

Varying this action with the respect to F_μ we get that the equation of motion for F^μ is $F^\mu = 0$. We can eliminate then the auxiliary field by this and obtain again the action we found at the chapter of non-superfield formalism for RNS action. However, in doing so, we understand the second motivation behind the importance of this auxiliary field, in fact without it we lose the superfield formulation of the action and the manifest supersymmetry of it.

1.7.4 Worldsheet supergravity

The Supersymmetry transformation that leads the action invariant, up to now is parametrized by the spinor (1.151) which is constant. However, if this spinor depends on local coordinates of the worldsheet things change substantially. The aim that pushes us to do so is to include Gravity on the theory precisely promoting the metric to a field. Even if this seems very straightforward, since we are now working with spinors in addition to bosons, we need to include a vielbein. The need of a vielbein is due to the fact that spinors transform in a non-immediate way under general coordinate transformations since there is no finite-dimensional spinor representation of diffeomorphism group $GL(D,\mathbb{R})$. This vielbein is defined as:

$$h_{\alpha\beta} = (e^\mu)_\alpha (e^\nu)_\beta \eta_{\mu\nu} \tag{1.179}$$

where α, β are curved indices and μ, ν Lorentz frame indices. Since locally, even if in curved background, due to equivalence principle we have Lorentz symmetry, we need a spin connection $\omega_\alpha \in Lie(SO(1,1))$ so we can define covariant derivatives:

$$\nabla_\alpha v^\mu = (\partial_\alpha + \omega_\alpha) v^\mu \tag{1.180}$$

in such a way that the vielbain is covariantly constant:

$$0 \overset{!}{=} \nabla_\alpha e^\mu_\beta = \partial_\alpha e^\mu_\beta + (\omega_\alpha)^\mu_\nu e^\nu_\beta - \Gamma^\delta_{\alpha\beta} e^\mu_\delta \tag{1.181}$$

and from this last equation we can define the spin connection who lives in the same SO(1,1) representation of the object which ∇_α is acting on. Taking our previously seen action (1.140) and adding vielbein and gravity through minimal coupling $\partial_\alpha \to \nabla\alpha$ we get a quadratic action:

$$S_2 = -\frac{1}{2\pi} \int d^2\sigma e (h^{\alpha\beta} (\partial_\alpha X_\mu) \left(\partial_\beta X^\mu\right) + \bar{\psi}^\mu \rho^\alpha \nabla_\alpha \psi_\mu) \tag{1.182}$$

Demanding the invariance under local Supersymmetry (Supergravity) is demanding the invariance under $\xi(\sigma) \to \xi'(\sigma)$. In order to do so we need transformations rule of our gravitational field or analogously the vielbein at least at leading order on perturbation theory around flat space, so we can postulate:

$$\delta_\xi e^\mu_\alpha = 2\bar{\xi}\rho^\mu \chi_\alpha \tag{1.183}$$

which is justified by the fact that χ_α is the **gravitino** and so the supersymmetric partner of the metric. This action is clearly not invariant under Supergravity transformation but, since before minimal coupling and vielbein inclusion was invariant under global supersymmetry, the variation of it must be controlled by the derivative of ξ, so using the well known Noether trick we can compute the variation and the associated Noether current:

$$\delta_\xi S_2 = \frac{2}{\pi}\int d^2\sigma\sqrt{-h}(\nabla_\alpha\bar{\xi})J^\alpha \tag{1.184}$$

with:

$$J^\alpha = -\frac{1}{2}\rho^\beta\rho^\alpha\psi^\mu\partial_\beta X_\mu \tag{1.185}$$

So now we can make the action invariant by adding a piece of third order (in the embedding fields) to the action itself:

$$S_3 = -\frac{2}{\pi}\int d^2\sigma\sqrt{-h}\bar{\chi}_\alpha J^\alpha = \frac{1}{\pi}\int d^2\sigma\sqrt{-h}\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi_\mu\partial_\beta X^\mu \tag{1.186}$$

introducing the transformation law $\delta_\xi\chi_\alpha = \nabla_\alpha\xi$, slightly modifying the variation of ψ^μ including in it the gravitino and adding a quartic term of the action:

$$S_4 = \frac{1}{4\pi}\int d^2\sigma\sqrt{-h}(\bar{\psi}\psi)(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\chi_\beta) \tag{1.187}$$

the theory becomes a Supergravity one so invariant under local Supersymmetry. This method to build Supergravity action is called Noether method.

1.7.5 Superstring boundary conditions and mode expansions

In this section we are going to exactly repeat the same ideas of the previous bosonic string chapter but now considering the superstring worldsheet action. Since the total action (1.140) can be split into bosonic and fermionic part, the boundary conditions and the mode expansions for X^μ are exactly the same seen in section 1.2. Taking now:

$$S_F \simeq \int d^2\sigma(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+) \tag{1.188}$$

and taking the variation with the respect to the fields ψ_-, ψ_+ we get their equations of motion (1.146) and (1.147) in addition to a boundary term:

$$\delta S_F \simeq \int d\tau(\psi_+\delta\psi_+ - \psi_-\delta\psi_-)|_{\sigma=\pi} - (\psi_+\delta\psi_+ - \psi_-\delta\psi_-)|_{\sigma=0} \stackrel{!}{=} 0 \tag{1.189}$$

The ways on which we can achieve this equality depends on the nature of the string.

Open string case

If the string is open, the two terms of (1.189) can't cancel each other so they must vanish separately leading us to the necessity of having $\psi_-^\mu = \pm\psi_+^\mu$. The overall sign is conventional, so we can fix the sign in one endcap $\psi_-^\mu|_{\sigma=0} = \psi_+^\mu|_{\sigma=0}$ and the other relative sign becomes meaningful, giving us 2 possibilities:

- **Ramond (R) boundary conditions**: $\psi_-^\mu|_{\sigma=\pi} = \psi_+^\mu|_{\sigma=\pi}$
  This boundary condition give raise to spacetime fermions and leads to mode expansion for the fermionic field in R sector:

$$\psi_-^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)} \tag{1.190}$$

$$\psi_+^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)} \tag{1.191}$$

  and since the fermions are Majorana, these expansions must be $\in \mathbb{R}$ so we need to have $d_n^\mu = {d_{-n}^\mu}^\dagger$.

- **Neveu-Schwarz (NS) boundary conditions**: $\psi_-^\mu|_{\sigma=\pi} = -\psi_+^\mu|_{\sigma=\pi}$
  This boundary condition give raise to spacetime bosons and leads to mode expansion for the fermionic field in NS sector:

$$\psi_-^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau-\sigma)} \tag{1.192}$$

$$\psi_+^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau+\sigma)} \tag{1.193}$$

  and, again, since the fermions are Majorana, these expansions must be $\in \mathbb{R}$ so we need to have $b_r^\mu = {b_{-r}^\mu}^\dagger$.

Closed string case

Closed string boundary conditions, as we saw before, give raise to two independent set of modes, left and right moving and allow 2 possible boundary conditions $\psi_\pm(\sigma) = \pm\psi_\pm(\sigma+\pi)$ making the term (1.189) vanish. The plus/minus sign define periodic/anti-periodic boundary conditions and we can impose periodicity (R) or anti-periodicity (NS) independently to right and left moving parts. In the end we can take as right movers:

$$\psi^\mu_-(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d^\mu_n e^{-2in(\tau-\sigma)} \text{ or } \psi^\mu_-(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-2ir(\tau-\sigma)} \tag{1.194}$$

and as left

$$\psi^\mu_+(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} \tilde{d}^\mu_n e^{-2in(\tau+\sigma)} \text{ or } \psi^\mu_+(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}^\mu_r e^{-2ir(\tau+\sigma)} \tag{1.195}$$

and all the four different combinations of fermions sign due to boundary conditions are allowed:

$$\text{R-R } \psi_+(\sigma+\pi) = +\psi_+(\sigma) \;\; ; \;\; \psi_-(\sigma+\pi) = +\psi_-(\sigma) \tag{1.196}$$

$$\text{R-NS } \psi_+(\sigma+\pi) = +\psi_+(\sigma) \;\; ; \;\; \psi_-(\sigma+\pi) = -\psi_-(\sigma) \tag{1.197}$$

$$\text{NS-R } \psi_+(\sigma+\pi) = -\psi_+(\sigma) \;\; ; \;\; \psi_-(\sigma+\pi) = +\psi_-(\sigma) \tag{1.198}$$

$$\text{NS-NS } \psi_+(\sigma+\pi) = -\psi_+(\sigma) \;\; ; \;\; \psi_-(\sigma+\pi) = -\psi_-(\sigma) \tag{1.199}$$

In the end, we want to point out that it can be easily seen how the open string case is just a restriction to R-R and NS-NS closed string one as stated in [30].

1.8 Canonical quantisation of superstrings

We already studied the canonical quantisation in the bosonic string case, what we are going to do now is to repeat the same idea for the Superstrings.

1.8.1 Commutation and anti-commutation relations

Repeating now exactly the same steps as in Chapter 1.3.1 we can obtain commutation and anti-commutation rules for the modes promoted to operators for the open strings:

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu} \tag{1.200}$$

$$\{\psi_m^\mu, \psi_n^\nu\} = \delta_{r+s}\eta^{\mu\nu} \Rightarrow \{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \quad \text{and} \quad \{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0} \tag{1.201}$$

$$\text{with} \begin{cases} r,s \in \mathbb{Z} & \text{R} \\ r,s \in \mathbb{Z} + \frac{1}{2} & \text{NS} \end{cases} \tag{1.202}$$

where for the closed ones the tilded modes relations are exactly the same.

1.8.2 Super-Virasoro constraints

Due to the presence of Minkowski metric in (1.201), we have again the problem of negative norm states arising in time components of fermionic modes. The solution is again by applying the constraints derived from the non gauge fixed action.

Starting now from (1.178) and removing F_μ using its equation of motion, we can gauge fix the metric to flat one and set the Gravitino χ_α to 0. Now, since this gauge fixing procedure, the equation of motion for the metric and the Gravitino itself become constraints on Energy Momentum Tensor and on the Supercurrent:

$$T_{\alpha\beta} = (\partial_\alpha X_\mu)(\partial_\beta X^\mu) + \frac{1}{4}\bar{\psi}_\mu \rho_\alpha \partial_\beta \psi^\mu + \frac{1}{4}\bar{\psi}_\mu \rho_\beta \partial_\alpha \psi^\mu - (\text{trace}) \stackrel{!}{=} 0 \tag{1.203}$$

$$(J^\alpha)_A = -\frac{1}{2}(\rho^\beta \rho^\alpha \psi_\mu)_A \partial_\beta X^\mu \stackrel{!}{=} 0 \tag{1.204}$$

As in the Bosonic string case, the constraints can be rewritten in terms of operators:

$$L_m = \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma e^{im\sigma} T_{++} \quad , \quad G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma e^{ir\sigma} J_+ \tag{1.205}$$

plugging inside the expression of energy-momentum tensor and supercurrent in terms of modes:

$$L_m = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} : \alpha_{-n}\cdot\alpha_{m+n} : + \sum_{r\in\mathbb{Z}+\aleph} : \left(r + \frac{m}{2}\right) b_{-r}\cdot b_{m+r} :\right) \tag{1.206}$$

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot b_{r+n} \quad \text{where} \quad \aleph \equiv \begin{cases} 0 & R \\ \frac{1}{2} & NS \end{cases} \tag{1.207}$$

We can immediately note that first of all there is no normal ordering ambiguity in the definition of G. Furthermore, these are exactly the generators of two copies (one for r,s even in Ramond case and one for r,s odd in Neveu-Schwarz case) of the such called **Super-Virasoro Algebra** defined by the following commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + A(m) \tag{1.208}$$

$$\{G_r, G_s\} = 2L_{r+s} + B(r)\delta_{r+s} \tag{1.209}$$

$$[L_m, G_r] = (\tfrac{m}{2} - r)G_{m+r} \tag{1.210}$$

where A and B are the such called anomaly terms that give the Quantum Mechanical extension of the classical algebra

$$A(m) = \frac{D}{8}(m^3 - m) \quad \textit{and} \quad B(r) = \frac{D}{8}(4r^2 - 1) \tag{1.211}$$

Now that we have the commutation relations we can write the constraints à la Gupta-Bleuler:

$$(L_m - a\delta_m)\left|\phi\right\rangle = 0 \;\; m \geq 0, \qquad G_r\left|\phi\right\rangle = 0 \;\; r \geq 0 \;\; \forall\phi\in\mathcal{H}_{\text{Phys}} \tag{1.212}$$

We are not going to repeat all steps, however from these constraints is easy to derive the anomaly factor needed for removing negative norm states:

$$a_{\mathrm{R}} = 0 \tag{1.213}$$

$$a_{\mathrm{NS}} = \tfrac{D-2}{16} \tag{1.214}$$

In order to derive its proper value we need the critical spacetime dimension that can be obtained again by the string spectrum.

NS sector of the open string spectrum

In the Neveu-Schwarz sector the ground state is defined by:

$$|0, k\rangle \text{ such that } \alpha^{\mu}_{m} |0, k\rangle = b^{\mu}_{r} |0, k\rangle = 0 \text{ for } m, r > 0 \tag{1.215}$$

By defining the number operator as:

$$N_{\mathrm{NS}} = \sum_{m=1,2,\ldots} \alpha_{-m} \cdot \alpha_m + \sum_{r=\frac{1}{2},\frac{3}{2}\ldots} r b_{-r} \cdot b_r \tag{1.216}$$

we can rewrite the mass shell condition coming from the 0-th constraint as:

$$0 = (L_0 - a) |0, k\rangle = (\alpha' p^2 + N_{\mathrm{NS}} - a_{\mathrm{NS}}) |0, k\rangle \tag{1.217}$$

which shows us that the ground state is a scalar with mass squared:

$$M^2 = -\frac{a_{\mathrm{NS}}}{\alpha'} \tag{1.218}$$

while the first excited level is a target space vector $\epsilon_\mu b^{\mu}_{-\frac{1}{2}}$ (where ϵ_μ polarization vector) with mass squared:

$$M^2 = \frac{1}{\alpha'} \left(\frac{1}{2} - a_{\mathrm{NS}} \right) \tag{1.219}$$

and, since we want this vector to being massless like in bosonic case we expect, from (1.213) $a = \frac{1}{2} \Rightarrow D = 10$ so a much lower spacetime dimension than in the previous bosonic case. In addition to this we get that the scalar ground state is again a tachyon. In the end, let us note that since all the creation operators α^{μ}_{-m} and b^{ν}_{-r} with $n, m > 0$ transform as spacetime vectors and act on the scalar ground state, the Ramond Sector is made by spacetime bosons.

R sector of the open string spectrum

The R sector case looks similar but hides a non-trivial subtlety. In this case the fermionic modes have integer indices d^{μ}_{n} so the number operator must be rewritten as:

$$N_{\mathrm{R}} = \sum_{m=1,2,...} \alpha_{-m}\cdot\alpha_{m} + \sum_{n=0,1...} n d_{-n}\cdot d_{n} = \sum_{m=1,2,...} \alpha_{-m}\cdot\alpha_{m} + \sum_{n=1,2...} n d_{-n}\cdot d_{n} \quad (1.220)$$

since so, acting with d^{μ}_{0} does not modify the mass of the state since it will commute with the number operator, giving us the intuitive idea of a degenerate ground state. In fact since the modes satisfy the D-dimensional Clifford algebra (apart a factor of 2) $\{d_{\mu}, d^{\nu}_{0}\} = \eta^{\mu\nu}$ then the ground state must be a representation of this algebra so a target space spinor:

$$|a, k\rangle_{R} \qquad \text{with} \qquad a = 1, 2, 3, 4, \ldots 2^{\frac{D}{2}} = 32 \quad (1.221)$$

which in realty is less degenerate due to the Majorana-Weyl condition and due to the such called Dirac-Ramond equation coming from the constraints. This spinorial ground state is massless since $a_{\mathrm{R}} = 0 \Rightarrow M^2 = -\frac{1}{\alpha'} a_R = 0$, by deriving higher order constraints and excited states we can derive again that the critical dimension is $D = 10$. Finally, since again the excited states are obtained by acting with creation operators $\alpha^{\mu}_{-m}, d^{\nu}_{-n}$ with $n, m > 0$ which transform as Spacetime vectors and the ground state is a spinor, than the excited states will be target space spinors too.

1.9 The GSO projection

In the previous section we expressed ideas on the spectrum of RNS open string states which survive the Super-Virasoro constraints. Even if it we got rid of the negative norm states, this spectrum has 2 problems:

- NS sector ground state has a tachyon, a scalar with imaginary mass.
- Spectrum is not manifestly spacetime Supersymmetric but the closed ones contains a massless gravitino which is the Supersymmetric partner of the graviton and so the quantum of gauge field for Supergravity.

1.9.1 Tachyon removal and manifest supersymmetry

In order to solve these issues the main way is to apply the such called GSO Projection introduced by Gliozzi, Scherk and Olive which projects the spectrum in a very specific way based on criteria on the such called G-Parity. The definition of this operator and its effect depends on the sector of the states it acts on and now we will inspect its action on open string spectrum in order to generalise it to the closed one later on.

NS sector

In the NS sector the definition of G-parity is:

$$G = (-1)^{F+1} = (-1)^{\sum_{r=\frac{1}{2}}^{+\infty} b^i_{-r} b^i_r} + 1 \tag{1.222}$$

where F is the worldsheet fermion number since it's the number operator restricted to b-excitations and so it counts whether a state has odd or even worldsheet fermionic quanta. The criterium in the NS sector is to keep only states which have a positive G-parity:

$$G\left|\phi\right\rangle_{\mathrm{NS}} = \left|\phi\right\rangle_{\mathrm{NS}} \Rightarrow (-1)^{F_{\mathrm{NS}}} = -1 \tag{1.223}$$

so only the states with odd number of b-oscillator excitations are projected. This implies immediately that the open-string tachyon is canceled out from the spectrum since:

$$G\,|0,k\rangle_{\mathrm{NS}} = (-1)^{0+1}\,|0,k\rangle = -\,|0,k\rangle \tag{1.224}$$

while the first excited state which was the massless vector $b^i_{-\frac{1}{2}}\,|0,k\rangle_{\mathrm{NS}}$ survives the projection and becomes the ground state of the NS sector. This thing is an hint that we can have a Supersymmetric spectrum since the ground states of NS and R sector are both massless.

R sector

The definition of G-parity in the R sector instead is a little bit more complicated:

$$G = \gamma_{11}(-1)^{\sum_{n=1}^{+\infty} d^i_{-n} d^i_n} \tag{1.225}$$

where $\gamma_{11} = \gamma_0\gamma_1\dots\gamma_9$ is the 10D analogue of the Dirac matrix which in 4D defines the chirality projector. In fact γ_{11}:

- Satisfies idempotency: $\gamma_{11}^2 = \mathbb{I}$
- Has anticommutation relations: $\{\gamma_{11}, \gamma^\mu\} = 0$
- Can define the chirality of a spinor (positive or negative respectively): $\gamma_{11}\psi = \pm\psi$
- Can define a chirality projection operator: $P_\pm = \frac{1}{2}(1 \pm \gamma_{11})$

Let us recall that a spinor with a definite chirality is called a Weyl spinor. The criteria for the R sector, since the different definition of G-parity, are different too, in fact we can project on states with positive or negative G-parity depending on the chirality of the ground state (which is a spinor in R sector as we saw before), so the choice is a pure convention.

As we said before closed string spectrum contains one or two massless gravitinos and so the interacting theory will be inconsistent if we have no supersymmetry and so not the same number of bosonic and fermionic degrees of freedom. However, as we saw before, in the NS sector ground state $b^i_{\frac{1}{2}}\,|0,k\rangle_{\mathrm{NS}}$, since $i = 2, \dots 9$ as we can see in light cone gauge quantisation, we have just 8 propagating degrees of freedom.

The R sector ground state $|\alpha, k\rangle_{\text{R}}$ instead seems to have $2^5 = 32$ complex components, however the spinor must be Majorana, so it could have 32 real components. In dimensions which are $D = 2 \mod 8$ as $D = 10$ a Majorana spinor can be even Weyl, so a chiral ground state has 16 real components and, finally, by imposing the Dirac equation we get an additional halving, leaving us with 8 real degrees of freedom, and so a perfect matching with NS ground state. So the ground state which is massless has the same number of bosonic and fermionic components which are two inequivalent real representations in 8D of the group $Spin(8)$. Despite this is more an heuristic argument that a real proof of supersymmetry, which instead is manifest only in a different formalism called Green-Schwarz (GS) formalism, it can be proven, to be more certain of the result, that this correspondence still holds excited level by excited level and not only at the ground state.

1.9.2 Closed string spectrum and allowed superstring theories

In the previous subsection we analyzed the GSO projection acting on open string spectrum giving us a 10D supersymmetric gauge theory. However this construction must be coupled to a closed string sector and we are now going to study this Hilbert space with the allowed theories on it focusing specially on the such called Type II string theories, the theories where we have 2 supersymmetries so 2 massless gravitinos in 10D as we will see later on. Splitting the NS and R sector in sets with different G-parity, we now have NS_-,NS_+,R_- and R_+. In closed string case we have left and right moving parts so one can think that we can have all the possible combinations between this sectors $(NS_-, NS_-), (NS_-, NS_+), \ldots$ however, since the level matching condition $(L_0 + \tilde{L_0}) |\phi\rangle = 0, \ \forall |\phi\rangle \in \mathcal{H}_{\text{Phys}}$ and since NS_- is the only one containing a tachyon, NS_- can only be coupled to itself and the possibilities for the pairing between left and right moving sectors are just 10. In order to build a theory we can take in principle any subset from these 10 leading every time to a different outcome. However we don't want the tachyons in our theory, in addition to this we want that the interacting theory is consistent too leaving us with only 2 possibilities. Via GSO projection then we can exclude the presence of NS_- in our choices, while for the R sector we can project into states with positive or negative G-parity depending on the chirality of the ground state of the theory itself. We can then build up 2 different theories depending on if the G-parity of left and right moving sector is the same or the opposite; these two theories are called **Type IIB String Theory** and **Type IIA String Theory**.

Type IIA string theory

Type IIA String Theory has left and right moving ground states for R sector which are chosen to have opposite chirality. The massless states in each sector are:

- NS-NS sector: $\tilde{b}^j_{-\frac{1}{2}} \left|0,k\right\rangle_{\text{NS}} \bigotimes b^j_{-\frac{1}{2}} \left|0,k\right\rangle_{\text{NS}}$ these states can be rearranged into a scalar called **dilaton** ϕ, an antisymmetric 2-form gauge field (28 states) called **Kalb-Ramond field** $B_{\mu\nu}$ and a symmetric traceless rank-2 tensor field (35 states) called **graviton** $g_{\mu\nu}$.
- NS-R and R-NS sectors: $\tilde{b}^i_{-\frac{1}{2}} \left|0,k\right\rangle_{\text{NS}} \bigotimes \left|+\right\rangle_{\text{R}}$ and $\left|-\right\rangle_{\text{R}} \bigotimes b^j_{-\frac{1}{2}} \left|0,k\right\rangle_{\text{NS}}$ where $\left|-\right\rangle_{\text{R}}$ and $\left|+\right\rangle_{\text{R}}$ represent the opposite chirality ground states for left and right moving parts. Each of these 2 set of states can be rearranged into a spin $\frac{3}{2}$ field (56 states) called the **gravitino** χ_α and a spin $\frac{1}{2}$ fermion field (8 states) called the **dilatino** Φ_α. The gravitinos in the NS-R sector has opposite chirality with the respect to the one in R-NS sector.
- R-R sector: $\left|-\right\rangle_{\text{R}} \bigotimes \left|+\right\rangle_{\text{R}}$ these states are obtained tensoring a pair of Majorana-Weyl spinors with opposite chirality (the left and right moving ground states) and from this tensor product we can obtain a 1-form (vector) gauge field C_1 (8 states) and a 3-form gauge field C_3 (56 states). So Type IIA String Theory contains **odd** p-form gauge potentials.

Type IIB string theory

Even if Type IIA String Theory is very interesting we are going to work within the setting of Type IIB String Theory which has left and right moving ground states for R sector with the same chirality, choose positive for convention. In this case massless states in each sector are:

- NS-NS sector: $\tilde{b}^j_{-\frac{1}{2}} \left|0,k\right\rangle_{\text{NS}} \bigotimes b^j_{-\frac{1}{2}} \left|0,k\right\rangle_{\text{NS}}$ these states are the same of Type

IIA string case and can be rearranged into a scalar called **dilaton** ϕ, an antisymmetric 2-form gauge field (28 states) called **Kalb-Ramond field** $B_{\mu\nu}$ and a symmetric traceless rank-2 tensor field (35 states) called **graviton** $g_{\mu\nu}$.

- NS-R and R-NS sectors: $\tilde{b}^i_{-\frac{1}{2}} |0,k\rangle_{\mathrm{NS}} \bigotimes |+\rangle_{\mathrm{R}}$ and $|+\rangle_{\mathrm{R}} \bigotimes b^j_{-\frac{1}{2}} |0,k\rangle_{\mathrm{NS}}$ where $|+\rangle_{\mathrm{R}}$ is the 8-component spinorial ground state for Ramond sector. Each of these 2 set of states can be rearranged into a spin $\frac{3}{2}$ field (56 states) called the **gravitino** χ_α and a spin $\frac{1}{2}$ fermion field (8 states) called the **dilatino** Σ_α. The gravitinos in the NS-R sector has, differently from Type IIA String theory, the same chirality with the respect to the one in R-NS sector.

- R-R sector: $|+\rangle_{\mathrm{R}} \bigotimes |+\rangle_{\mathrm{R}}$ these states are obtained tensoring a pair of Majorana-Weyl spinors with the same chirality (the left and right moving ground states) and from this tensor product we can obtain a 0-form (1 state), a scalar, C_0 2-form gauge field C_2 (28 states) and a 4-form gauge field C_4 (35 states) with self-dual field strength $F_5 \coloneqq dC_4 = F_5^\star$. So, differently from the previous case, Type IIB String Theory contains **even** p-form gauge potentials.

In order to conclude this section, let us point out that in all the 2 cases and in all the sectors the total number of physical states is always $8 \times 8 = 64$.

Chapter 2

Type IIB String Compactifications

All the Superstring Theories we have seen up to know need 10 dimensions in order to get rid of negative norm states. One can ask then why are these theories physical and phenomenologically viable given that the observational results we have are obtained from our 4D point of view of the spacetime. The solution is through the use of a technique called **compactification**. We now focus to apply this technique in Type IIB String Theory in order to reduce a 10D non compact manifold $\mathcal{M}_{10}$ into our familiar 4D non-compact spacetime $\mathcal{M}_4$, Cartesian product a tiny 6D compact complex manifold Y_6 called Calabi-Yau $\mathcal{M}_{10} = \mathcal{M}_4 \times Y_6$. We will follow mainly [30], [5], [34] and specific articles cited section by section.

2.1 10D action and Kaluza-Klein compactification

2.1.1 10D action for Type IIB string theory

We already saw in section 1.5 what is the shape and the principles to write a target space action in the bosonic string theory context. In the Supersymmetric case there are few variations but still some things are unchanged:

- There is again the 10D graviton $g_{\mu\nu}$
- There is again the Kalb-Ramond field $B_{\mu\nu} = B_2$ coupling to the worldsheet
- There is again the dilaton which governs the perturbation theory convergence through its vacuum expectation value.

These three elements together form the NS-NS sector as we saw before, however, in addition to these, we have to include inside the action even the C_{p+1} form fields (with $p = 0, 2, 4$ for Type IIB string theory) and the corresponding Dp-branes with their action (called **Dirac-Born-Infeld, or DBI, action**) which are dynamical objects like strings but with different dimensions and a larger tension at weak coupling value.

However the real big difference between bosonic and superstring case is that in superstring case, the 10D theories are unique at second order in derivative due to Supergravity. This happens because realize Supergravity is very hard at high dimension since, for example, the number of spinor components grow exponentially as $2^{\frac{D}{2}}$ and so finding a bosonic structure with the same degrees of freedom is very hard. By going into details it turns out that there exist only 4 Supergravity theories in 10D and all of them come from Type IIA, Type IIB, Type I and Heterotic SO(32), Heterotic E8 (the last 3 are 3 String Theories with only 1 Supersymmetry so we didn't go in detail of their construction). As we said before in 10D a 16 real component spinor exist, which is a spinor with 4 times the number of components of a 4D one, so an $N = 2$ Supergravity theory in 10D can be seen from a 4D point of view as an $N = 8$ one and this is the Supersymmetry case of Type II String theories. The starting point for such theories is a stringy description of real world through one of the models in the **landscape** which is the large set, maybe infinite, of phenomenologically viable models. The most promising landscape has been established in Type IIB String Theory so, as we said before, we are going to focus on it.

We write now the bosonic part of the string-frame Type IIB Action as:

$$S = \frac{1}{2k_{10}^2}\int d^{10}x\sqrt{-g}\left(e^{-2\phi}\left(R+4(\partial\phi)^2-\frac{1}{2\cdot 3!}H_3^2\right)-\frac{1}{2}F_1^2-\frac{1}{2\cdot 3!}\tilde{F_3}^2-\frac{1}{4\cdot 5!}\tilde{F}_5^2\right)+S_{\rm CS}+S_{\rm loc} \tag{2.1}$$

where we have $2k_{10}^2 = (2\pi)^7\alpha'^4$ and, recalling $H_3 = dB_2$, $F_3 = dC_2$, $F_5 = dC_4$:

$$\tilde{F_3} = F_3 - C_0\wedge H_3 \quad , \quad \tilde{F_5} = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3 \tag{2.2}$$

Where $F_i,\ i = 3,5$ RR-forms field strength, and H_3 are gauge invariant which implies that we have a gauge symmetry of the action upon transformation of the potential of the shape:

$$C_2 \to C_2' = C_2 + d\lambda_1 \quad \text{with } C_4 \to C_4' = C_4 + \frac{1}{2}\lambda_1\wedge H_3 \tag{2.3}$$

The S_{CS} term we have in our action is a term not involving the metric called **Chern-Simons** term which is needed in order to have the right amount of propagating fermionic degrees of freedom as stated in [6] and can be written in this case as:

$$S_{\rm CS} = \frac{1}{4k_{10}^2}\int e^{\phi}C_4\wedge H_3\wedge F_3 \tag{2.4}$$

The last part $S_{\rm loc}$ is called the localised part and contains actions of the various branes, for a D3-brane for example the contribution will be:

$$S_{\rm loc} \supset D_{\rm D3} = \frac{1}{2\pi^3\alpha'^2}\int_{D3} C_4 - \int_{D3} d^4\sigma\sqrt{-g}T_3 \tag{2.5}$$

where the D3-brane tension is $T_3 = \frac{1}{(2\pi)^3\alpha'^2}$, g is the determinant of the 10D metric pullback and the integral is interpreted on the D3-brane worldsheet parametrised by σ^i with $i = 0,1,2,3$. Inside $S_{\rm loc}$ obviously other odd-dimensional Dp-brane action have to be added which are analogous to the one seen before but with different string tension $T_p = \frac{e^{(p-3)\frac{\phi}{4}}}{(2\pi)^p\alpha'^{\frac{(p+1)}{2}}}$. After adding all these parts, the pullback of B_2 to the brane and the brane-localised gauge fields+their fermionic partner we get the localised action $S_{\rm loc}$ called Dirac-Born-Infeld action or DBI action which has the form:

$$S_{\mathrm{DBI}} = -T_p \int d^{p+1}\sigma \sqrt{-det(G_{\alpha\beta} + 2\pi\alpha' \mathcal{F}_{\alpha\beta})} \tag{2.6}$$

where $G_{\alpha\beta}$ pullback of the 10D metric and $\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + b_{\alpha\beta}$ where $F_{\alpha\beta}$ usual Field-Strength tensor and $b_{\alpha\beta}$ 2-form needed to make $\mathcal{F}$ Supersymmetric.

2.1.2 Kaluza-Klein compactification

Up to now we have described then the fundamental Type IIB String Theory living in 10D. In order to describe the 4D spacetime we live in, the main idea is to use the compactification method, in particular the Kaluza-Klein Compatification method where we can obtain lower-dimensional Effective Field theories (EFT) from higher-dimensional theories by making compact the extra dimensions featured in the latter.

In order to understand how the mechanism works we start from three simple toy models, a 5D scalar field on a 5D manifold where one dimension is compactified on a circumference and the historical Kaluza-Klein Theory from 2 different points of view:

Scalar field in 5D

We take the scalar field ϕ in $\mathcal{M} = \mathbb{R} \times S^1$, where S^1 has radius R with $x^5 \in (0, 2\pi R)$:

$$S = \int_{\mathcal{M}} d^5x \frac{1}{2}(\partial_M \phi)(\partial^M \phi) \tag{2.7}$$

with $M \in \{0, 1, 2, 3, 5\}$ The equation of motion is clearly the 5D Klein Gordon equation and, Assuming the vacuum expectation value $\langle\phi\rangle = 0$ the fluctuations around the vacuum can be parametrized, via renaming $x^5 \to y$ and $x = \{x^0, x^1, x^2, x^3\}$ as Fourier expanded:

$$\phi(x, y) = \sum_{n=0}^{+\infty} \phi_n^{\cos} \cos\left(\frac{ny}{R}\right) + \sum_{n=1}^{+\infty} \phi_n^{\sin} \sin\left(\frac{ny}{R}\right) \tag{2.8}$$

Plugging inside the action this expansion we get:

$$S = 2\pi R \int_{\mathcal{M}} d^4x \left[\frac{1}{2}(\partial\phi_0^{\cos})^2 + \frac{1}{4}\sum_{n=1}^{+\infty}((\partial\phi_n^{\cos})^2 + m_n^2(\partial\phi_n^{\cos})^2 + (\partial\phi_n^{\sin})^2 + m_n^2(\partial\phi_n^{\sin})^2) \right] \tag{2.9}$$

so our 5D model is equivalent to a 4D model with one massless field $\phi_0^{\cos}$ and a tower of Kaluza-Klein modes which is degenerate and whose modes have mass $m_n = \frac{n}{R}$. The massless mode $\phi_0^{\cos}$ is a flat direction in field space since has no potential and it's called **modulus**. Such moduli usually govern mass and couplings of the rest of 4D theory through their vacuum expectation value which can be an arbitrary constant not having a potential for them.

Kaluza-Klein theory

We now consider a different example starting from 5D Einstein-Hilbert action:

$$S = \frac{M_{\mathrm{P},5}^3}{2} \int_{\mathcal{M}} d^4x dy \sqrt{-g_5} R_5 \tag{2.10}$$

where the subscript 5 means the correspondent 5D quantity to the well known 4D ones. Given this we can now parametrise the 5D metric as:

$$(g_5)_{MN} = \begin{pmatrix} g_{\mu\nu} + \left(\frac{2}{M_P^2}\right)\phi^2 A_\mu A_\nu & \left(\frac{\sqrt{2}}{M_P}\right)\phi^2 A_\mu \\ \left(\frac{\sqrt{2}}{M_P}\right)\phi^2 A_\nu & \phi^2 \end{pmatrix} \tag{2.11}$$

with $M, N \in 0, 1, 2, 3, 5$ and $\mu, \nu \in 0, 1, 2, 3$ and where M_p, A_μ and ϕ are actually just parameters in the 5D metric components. In order to simplify things again we take $y \in (0, 2\pi R)$ parametrising S^1 and we use $g_{\mu\nu} = \eta_{\mu\nu}$, $A_\mu = 0$ and $\phi^2 = g_{55} = 1$. As in the scalar field example we can Fourier expand our field ϕ as function of y and we get a tower of massive mode, plugging inside (2.11) and (2.8) we get that the action (2.10) can be rewritten as:

$$S = \int_{\mathcal{M}} d^4x \sqrt{-g}\phi \left(\frac{M_P^2}{2} R - \frac{1}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} + \frac{M_P^2}{3}\frac{(\partial\phi)^2}{\phi^2} \right) \tag{2.12}$$

in string frame (for going into Einstein frame just a substitution of $g_{\mu\nu} \to \frac{g_{\mu\nu}}{\phi}$ is needed). The 5D metric degrees of freedom then can be seen turning into 4D metric $g_{\mu\nu}$+abelian gauge field A_μ and a scalar ϕ, leading us to the appearance of a $U(1)$ gauge theory. This presence is quite natural since $\mathcal{M} = \mathbb{R} \times S^1$ enjoys clearly a global $U(1)$ symmetry and a diffeomorphism one (we are working in General Relativity picture) so we can rotate $S^1 \Leftrightarrow$ shift y at every point x of our 4D submanifold so our theory must be a local $U(1)$. In addition to this, $\forall R$ fixed radius we have a solution of 5D Einstein equation so we expect that $\phi \leftrightarrow R$ is a scalar degree of freedom with flat potential$\Rightarrow$ a modulus. In fact, parametrising S^1 with dimensionless parameter $y \in (0,1)$ we can set $\langle\phi\rangle = \sqrt{g_{55}} = 2\pi R$ and so the scalar ϕ governs the size of extra dimension, which is a **general** feature of the moduli. Finally we can see how, in the previous action (2.12) we identified $M_P^2 = 2\pi R M_{\mathrm{P},5}^3 \Rightarrow M_{\mathrm{P},5}^3 = \frac{M_P^{\frac{2}{3}}}{\mathcal{V}^{\frac{1}{3}}}$ leading us to the fact that if the extra dimensions are compactified in large volume than the 5D Planck Mass is much lower than the 4D one and so the effects of gravity will be larger.

10D Kaluza-Klein theory

In the previous example we have seen how moduli arise intuitively from a 5D geometry, however now we would like to make contact with what we are going to see soon after in the complex compactifications. We are going to repeat the previous example with a slightly different point of view.

We consider now the ten-dimensional geometry (instead of 5D):

$$G_{MN}dX^M dX^N = e^{-6u(x)} g_{\mu\nu} dx^\mu dx^\nu + e^{2u(x)} \hat{g}_{mn} dy^m dy^n \tag{2.13}$$

where $\hat{g}_{mn}$ is the metric of the 6 extra dimensions compactified at fixed volume:

$$\int_{Y_6} d^6y \sqrt{\hat{g}} \equiv \mathcal{V} \tag{2.14}$$

and $e^{u(x)}$ is a breathing mode which parametrises variation of the compact space as depending on the 4D non-compact spacetime coordinate. We immediately point out instead that the $e^{-6u(x)}$ in the first term is just for having the Einstein-Hilbert action in Einstein frame automatically.

Starting from the 10D Einstein-Hilbert action in the manifold $\mathcal{M} = \mathcal{M}_4 \times Y_6$:

$$S^{10D}_{\text{EH}} = \frac{1}{2k^2_{10}} \int_{\mathcal{M}} d^{10}X \sqrt{-G} e^{-2\phi} R_{10} \tag{2.15}$$

we would like to express the 10D Ricci scalar R_{10} in terms of the 4D and compact 6D ones $R_4, \tilde{R}_6$. By using some Differential Geometry, following [5], we can derive the expression of the 10D action as:

$$S^{10D}_{\text{EH}} = \frac{1}{2k^2_{10}} \int_{\mathcal{M}_4} d^4x \sqrt{-g} \int_{Y_6} d^6y \sqrt{\hat{g}} e^{-2\phi} (R_4 + e^{-8u} \hat{R}_6 + 12 \partial_\mu u \partial^\mu u) \tag{2.16}$$

Considering now the string coupling $g_s = e^\phi$ constant over Y_6 then we can rewrite the 4D Einstein Hilbert action:

$$S^{4D}_{\text{EH}} = \frac{M^2_P}{2} \int_{\mathcal{M}_4} d^4x \sqrt{-g} R_4 \tag{2.17}$$

with $M^2_P \coloneqq \frac{\mathcal{V}}{g^2_s k^2_{10}}$. Recognizing the kinetic term for the scalar field $u(x)$ we can see how, if $\hat{R} = 0$, for example when the compact 6D manifold is Ricci Flat, $u(x)$ is a modulus field and how it corresponds to a deformation in 10D metric as it will be in the moduli we are going to use in the next chapters. In addition to this the kinetic term for u can be originated via Kähler potential $K = -3 \ln(T + \bar{T})$ calling $Re(T) = e^{4u}$ and setting $M_P = 1$. Again this will be very close to what we will do in few sections.

2.2 Complex geometry

Up to now we have analyzed the compactifications of an extra dimension into S^1, however, what is usually done in String Theoretical models which are Phenomenologically viable is to compactify the 6 extra dimension in a complex manifold with peculiar properties. In order to understand better then these kind of compactifications we need before to give some highlights of Complex Geometry and of Homology and Cohomology connected to it.

2.2.1 Complex manifolds

The starting point of our journey is to find a solution of 10D equations of motion which physically correspond to 4D spacetime. If we set all the fields to 0, so we consider just the Einstein-Hilbert action, we must have a metric satisfying $R_{MN} = 0$ to have Einstein's equation solved. This condition is called **Ricci flatness** and its satisfied not only by real trivial geometries but even by a large class of compact 6D complex manifolds called **Calabi Yau Manifolds** Y_6.

In order to understand them completely it is necessary to start from the very basic idea of **complex manifold**, which is simply the generalization of a real differentiable manifold where the charts that give the local Euclidianity are now defined as:

$$(U_i, \phi_i), \quad \phi_i : U_i \to \phi_i(U_i) \subset \mathbb{C}^n \tag{2.18}$$

with U_i in the topology of Y_6 and with holomorphic transition functions $\phi_j \circ \phi_i^{-1}$, so the manifold locally is $\mathbb{C}^n$ with the possibility of having such a holomorphic change of coordinates:

$$z'^i = z'^i \left(z^1, \ldots, z^n\right) \tag{2.19}$$

As we do in real manifolds, we can define complexified tangent $(T_p(Y_6)^{\mathbb{C}} = \{X + iY | X, Y \in T_pY_6\})$ and cotangent spaces and their tensor product with bases:

$$\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^{\bar{\imath}}} \quad \text{and} \quad dz^i, d\bar{z}^{\bar{\imath}} \tag{2.20}$$

and, using this bases, we can define the **almost complex structure** $\mathcal{J}$ as:

$$\mathcal{J} : T_p^* \to T_p^* \quad \forall p \in Y_6 \tag{2.21}$$

$$\mathcal{J} = idz^i \otimes \frac{\partial}{\partial z^i} - id\bar{z}^{\bar{\imath}} \otimes \frac{\partial}{\partial \bar{z}^{\bar{\imath}}} \tag{2.22}$$

which can be written in an imaginary and in a real basis respectively as:

$$\mathcal{J} = \begin{pmatrix} 0 & i\mathbb{I} \\ -i\mathbb{I} & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \tag{2.23}$$

and it satisfies $\mathcal{J}^2 = -\mathbb{I}$. By defining the action of the almost complex structure on the elements of the basis of complexified tangent space we can divide $T_pY_6^{\mathbb{C}}$ into the positive eigenspace $\mathcal{J}\frac{\partial}{\partial z^i} = i\frac{\partial}{\partial z^i}$ spanned by $\{\frac{\partial}{\partial z^i}\}$ and the negative one $\mathcal{J}\frac{\partial}{\partial \bar{z}^i} = -i\frac{\partial}{\partial \bar{z}^i}$ spanned by $\{\frac{\partial}{\partial \bar{z}^i}\}$, the elements of these two eigenspaces are called **vectors**, in particular **holomorphic** and **antiholomorphic** vectors. If we have a real manifold and it features the existence of this almost complex structure $\mathcal{J}$ then the manifold is called **almost complex**. If $\mathcal{J}$ satisfies the vanishing of Nijenhuis tensor ($\Leftrightarrow d = \partial + \bar{\partial}$) then the manifold is called complex and $\mathcal{J}$ the **complex structure** of such a manifold.

2.2.2 Complex differential forms

In analogy on what is well known in real manifolds, it is very useful to extend the concept of differential forms to complex cases. We can in fact define, given two real n-forms α_n, β_n a **complex n-form** $\delta_n \equiv \alpha_n + i\beta_n$ with a complex conjugate $\bar{\delta}_n \equiv \alpha_n - i\beta_n$. We call the vector space of the complexified n-forms as $\Lambda^n_{\mathbb{C}}(Y_6)$. We can even generalise more this concept by defining an **(r,s)-form** which is a complex valued differential form with r holomorphic indices and s antiholomorphic indices whose basis in local coordinates is:

$$dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{\bar{\jmath}_1} \wedge \cdots \wedge d\bar{z}_{\bar{\jmath}_s} \equiv dz_M \wedge d\bar{z}_{\bar{N}} \tag{2.24}$$

using the multi-indices $M = (i_1, \ldots, i_r)$ and $N = (j_1, \ldots, j_s)$. We denote the vector space of this special forms on Y_6 as $\Lambda^{r,s}(Y_6)$. Whatever element of this vector space

$\xi_{r,s} \in \Lambda^{r,s}(Y_6)$ can be rewritten as linear combination in the basis (2.24):

$$\xi_{r,s} = \frac{1}{r!s!}\xi_{i_1,\dots,i_r,\bar{\jmath}_1,\dots,\bar{\jmath}_s} dz^{i_1}\wedge\cdots\wedge dz^{i_r}\wedge d\bar{z}^{\bar{\jmath}_1}\wedge\cdots\wedge d\bar{z}^{\bar{\jmath}_s} = \frac{1}{r!s!}\xi_{M\bar{N}} dz^M \wedge d\bar{z}^{\bar{N}} \quad (2.25)$$

Taking a complex k-form ξ_k, since $\Lambda^k = \bigoplus_{r+s=k} \Lambda^{r,s}$ we can rewrite it as sum of (r,s)-forms as:

$$\xi_k = \sum_{r+s=k} \xi_{r,s} \quad (2.26)$$

It is well known, in the case of real forms that a fundamental operation of them is the exterior derivative $d : \Lambda^{r,s}_{\mathbb{R}} \to \Lambda^{r+1,s}_{\mathbb{R}}$. We can define an analogous operation with the respect to the complex structure via using the **Dolbeault operators** which are maps $\partial : \Lambda^{r,s} \to \Lambda^{r+1,s}$ and $\bar{\partial} : \Lambda^{r,s} \to \Lambda^{r,s+1}$ such that:

$$\partial\xi_{r,s} = \left(\frac{\partial}{\partial z^i}\xi_{M\bar{N}}\right) dz^i \wedge dz^M \wedge d\bar{z}^{\bar{N}} \quad (2.27)$$

$$\bar{\partial}\xi_{r,s} = \left(\frac{\partial}{\partial \bar{z}^{\bar{\imath}}}\xi_{M\bar{N}}\right) d\bar{z}^{\bar{\imath}} \wedge dz^M \wedge d\bar{z}^{\bar{N}} \quad (2.28)$$

In a complex Manifold we can define then the exterior derivative as $d : \Lambda^{r,s} \to \Lambda^{r+1,s}$ such that $d = \partial + \bar{\partial}$ with $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$. Via these definitions we can call a form **holomorphic** (**antiholomorphic**) $\Leftrightarrow$ it is an (r,0)-form ((0,s)-form) and if and only if $\bar{\partial}\xi_{r,0} = 0$ $(\partial\xi_{0,s} = 0)$ with $\xi_{0,0}$ holomorphic 0-form which is a function.

Now that we introduced Dolbeault operators it appears natural to define Dolbeault cohomology classes as the de Rham cohomology ones with the respect to the exterior derivative d. Defining the set of $\bar{\partial}$-closed (r,s)-forms as $Z^{r,s}_{\bar{\partial}}(Y_6)$ and as $E^{r,s}_{\bar{\partial}}(Y_6)$ the set of (r,s)-forms which are exact under $\bar{\partial}$. The **Dolbeault cohomology group** is then the quotient:

$$H^{r,s}_{\bar{\partial}}(Y_6, \mathbb{C}) \equiv Z^{r,s}_{\bar{\partial}}(Y_6)/E^{r,s}_{\bar{\partial}}(Y_6) \quad (2.29)$$

Where the elements are equivalence classes such that:

$$[\omega] = \{\rho \in \Lambda^{r,s}(Y_6) | \bar{\partial}\rho = 0, \omega - \rho = \bar{\partial}\chi, \chi \in \Lambda^{r,s-1}(Y_6)\} \quad (2.30)$$

A crucial point is now to extend Hodge Theory to Complex forms by defining an extension of Hodge star operator to them, we can define it as an isomorphism $\star : \Lambda^{r,s}(Y_6) \to \Lambda^{t-r,t-s}(Y_6)$ (where $t = \dim(Y_6) = 6$ such that it acts on basis element following the rule:

$$\star(dz^{i_1} \wedge \cdots \wedge dz^{i_r} \wedge d\bar{z}^{\bar{\jmath}_{r+1}} \wedge \cdots \wedge d\bar{z}^{\bar{\jmath}_s}) \sim \tag{2.31}$$

$$\epsilon^{i_1 \ldots i_r, \bar{\jmath}_1 \ldots \bar{\jmath}_s}_{k_{r+1} \ldots k_t, \bar{m}_{s+1} \ldots \bar{m}_t} dz^{k_{r+1}} \wedge \cdots \wedge dz^{k_t} \wedge d\bar{z}^{\bar{m}_{s+1}} \wedge \cdots \wedge d\bar{z}^{\bar{m}_t} \tag{2.32}$$

up to a factor proportional to $\frac{\sqrt{|g|}}{(t-r)!(t-s)!}$ where g is the determinant of the metric on the manifold and ϵ is the antisymmetric Levi-Civita symbol. This operator gives a generalisation of the well known isomorphism present in the real cohomology classes giving:

$$H^{r,s}(Y_6, \mathbb{C}) \simeq H^{t-r,t-s}(Y_6, \mathbb{C}) \tag{2.33}$$

At this point we can define through these 2 star operators 2 very important things:

- An inner product of 2 (r,s) forms ω and ξ via $\langle \omega, \xi \rangle \coloneqq \int \omega \wedge \star\xi$
- Two adjoint operators starting from the operators ∂ and $\bar{\partial}$:

$$\partial^\dagger : \Lambda^{r,s} \to \Lambda^{r-1,s} \tag{2.34}$$

$$\bar{\partial}^\dagger : \Lambda^{r,s} \to \Lambda^{r,s-1} \tag{2.35}$$

  whose definitions are in an even dimensional manifold (as in our case where t=6) [34]:

$$\partial^\dagger = - \star \bar{\partial} \star \tag{2.36}$$

$$\bar{\partial}^\dagger = - \star \partial \star \tag{2.37}$$

From these last 2 definition it is possible to define then 2 different kind of Laplacians:

$$\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial \quad , \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial} \tag{2.38}$$

and through them we can define a **∂-harmonic** form as an (r,s)-form ω such that

$\Delta_{\partial}\omega = 0$ (analogously a $\bar{\partial}$**-harmonic** form), it can be written that $\omega \in \mathcal{H}^{r,s}_{\partial}(Y_6)$ ($\omega \in \mathcal{H}^{r,s}_{\bar{\partial}}(Y_6)$). In addition to this we can state a complexified version of the Hodge Theorem stating that

$$H^{r,s}_{\bar{\partial}}(Y_6, \mathbb{C}) \simeq \mathcal{H}^{r,s}_{\bar{\partial}}(Y_6, \mathbb{C}) \tag{2.39}$$

so closed but not exact forms are in 1:1 correspondance with harmonic ones. In particular Hodge Theorem states that a general (r,s)-form ω can be epressed in an unique decomposition as:

$$\omega = \bar{\partial}\alpha + \bar{\partial}^{\dagger}\beta + \gamma \tag{2.40}$$

where $\alpha \in \Lambda^{r,s-1}(Y_6)$, $\beta \in \Lambda^{r,s+1}(Y_6)$ and $\gamma \in \mathcal{H}^{r,s}_{\bar{\partial}}(Y_6)$, so an arbitrary form can be written as sum of exact ($\bar{\partial}\alpha$), coexact ($\bar{\partial}^{\dagger}\beta$) and harmonic ($\gamma$) forms.

Finally it is very important to highlight that the dimensions of Dolbeault cohomology groups are known as **Hodge numbers**:

$$h^{p,q}(Y_6) = \dim(H^{p,q}(Y_6)) \tag{2.41}$$

and they give very important information about the topological characteristics of a manifold. These that can be rearranged in a efficient way in the such called **Hodge Diamond**:

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} \tag{2.42}$$

and due to the Hodge duality (2.33) we can see how $h^{r,s} = h^{t-r,t-s}$ on a manifold of dimension t (for us again t=6).

2.2.3 Holonomy

The last step towards defining Kähler and Calabi-Yau manifolds is the **holonomy**. The concept of holonomy comes from the evolution of tangent vectors under parallel transport, in fact if we want to parallel transport a vector around a triangle on a sphere we will see how its direction changes by a rotation of a certain angle. The same idea happens on a manifold M of dimension $t > 2$ where the tangent vectors can rotate in more than one plane and can both remain in a subspace of tangent space TY_6 or not. This gives us an intuitive idea of the symmetry of the manifold.

We can define the **Holonomy Group** $Hol_p(\nabla)$ at a point $p \in Y_6$ of a certain connection ∇ the set of transformations induced by parallel transport of tangent vectors around closed loops $c \subset Y_6$ such that at the ends $c(0) = c(1) = p$:

$$Hol_p(\nabla) = \{G_c : T_pY_6 \to T_pY_6\} \subset GL(t, \mathbb{R}) \tag{2.43}$$

Roughly speaking, the holonomy group is the group of transformation that sends a parallel transported tangent vector to its pre-parallel transport configuration.

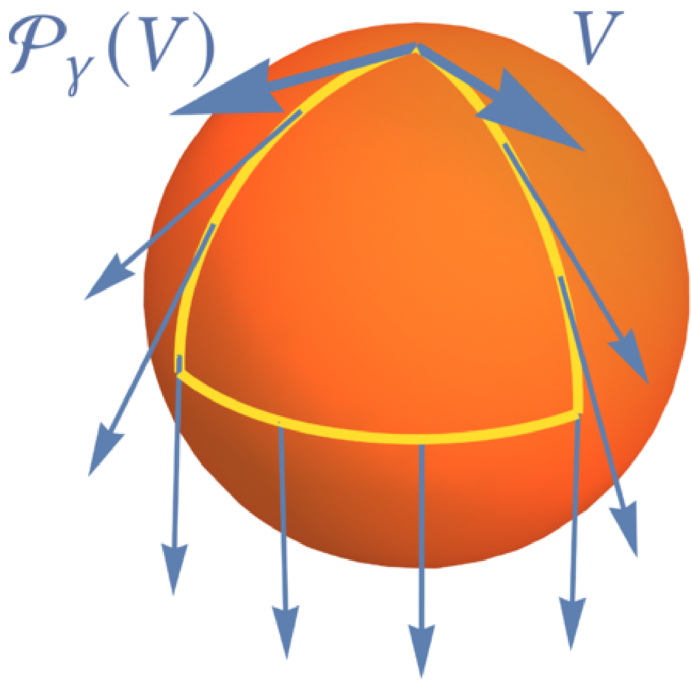

Figure 2.1: Intuitive idea of Holonomy Group on S^2.

It can be shown that holonomy groups at different points of a connected manifold are equal up to a general linear conjugation $Hol_p(\nabla) = gHol_q(\nabla)g^{-1}$, with $g \in GL(t, \mathcal{R})$ so, the holonomy of a connected manifold **does not** depend on the base point.

In the cases we are going to study the connection is the Levi-Civita one compatible with a Riemmaninan metric so the parallel transport leaves the lengths invariant and, given Y_6 an orientable manifold of dimension $t = 6$, then the Holonomy group, called Riemannian Holonomy Group $Hol(Y_6)$ will be $SO(n)$ or a subgroup of it. If Y_6 is a simply connected Riemmanian manifold of dimension t then or Y_6 is a coset space G/H of Lie group G on Lie subgroup $H \subset G$, or the **Berger Classification** of Holonomy groups holds [10]. We are not interested in the whole classification, in fact what we care about are the manifolds with $Hol(Y_6) = SU(3) \subset SO(6)$.

2.2.4 Kähler manifolds

In the previous subsections we have given some general features of complex manifolds and forms on them, however their properties are too general and poor to match the one necessary for a model in the String Landscape so we need to restric the set of Complex manifolds to Kähler one before and Calabi-Yau next.

First of all it is necessary to embed a metric in our construction, in particular we want the metric to be **Hermitian** so a Rimannian metric $g : TY_6 \to TY_6$ which satisfies $g(\mathcal{J}X, \mathcal{J}Y) = g(X, Y)$ and so that can be written in local coordinates as

$$g_{\mu\bar{\nu}} dz^{\mu} \otimes d\bar{z}^{\bar{\nu}} + g_{\bar{\mu}\nu} d\bar{z}^{\bar{\mu}} \otimes dz^{\nu} \tag{2.44}$$

which is real and Hermitian, then, in local coordinates, $g_{\mu\bar{\nu}} = \overline{g_{\bar{\nu}\mu}}$ and $g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0$. Every complex manifold admits an Hermitian metric g which can be constructed from a Riemannian Metric g_0 as $g(X, Y) = g_0(X, Y) + g_0(\mathcal{J}X, \mathcal{J}Y)$, so via equipping the manifold with such an Hermitian metric we can call it an **Hermitian Manifold**. The metric is a fundamental characteristic in this case because, owing to it, we can define the such called **Kähler form** of an Hermitian manifold as a 2-form:

$$J = i g_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\bar{\nu}} \tag{2.45}$$

and if this Kähler form is closed $\Leftrightarrow$ $dJ = 0$ the manifold is called a **Kähler manifold**. We can even say, in holonomy language, that a Kähler manifold is a manifold Y_6 with $Hol(Y_6) = U(t_{\mathbb{C}})$ with $t_{\mathbb{C}}$ now complex dimension of the manifold itself. The metric of such a manifold, called Kähler metric, can be written locally in terms of the such called **Kähler Potential** K:

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K \quad \text{and} \quad J = i\partial\bar{\partial}K \tag{2.46}$$

since these last 2 definitions are invariant under the such called **Kähler transformations** $K \to K' = K + f(z^i) + \bar{f}(\bar{z}^{\bar{\imath}})$, in two intersecting open sets on which we can attach local coordinates the metric is then the same and even and on their intersection, while the Kähler potential, in general defined only locally, is the same up to a Kähler transformation. From the metric the first idea is to define

Christoffel symbols on it, which, on a Kähler manifold, due to Hermiticity, are non-vanishing if and only if:

$$\Gamma^{\rho}_{\mu\nu} = g^{\rho\bar{\sigma}}\partial_{\mu}g_{\nu\bar{\sigma}} \tag{2.47}$$

or

$$\Gamma^{\bar{\rho}}_{\bar{\mu}\bar{\nu}} = g^{\bar{\rho}\sigma}\partial_{\bar{\mu}}g_{\bar{\nu}\sigma} \tag{2.48}$$

Leading to the only non vanishing, independent, components of Riemann tensor:

$$R^{\sigma}_{\mu\nu\bar{\rho}} = -\partial_{\bar{\rho}}\Gamma^{\rho}_{\mu\nu} \tag{2.49}$$

From which we can define the Ricci tensor via contraction of first and second index and then the **Ricci Form**:

$$\mathcal{R} = iR_{\mu\bar{\nu}}dz^{\mu} \wedge d\bar{z}^{\bar{\nu}} \tag{2.50}$$

which can be rewritten, by using $\Gamma^{\nu}_{\mu\nu} = \partial_{\mu}\ln(g) \Rightarrow R_{\mu\bar{\nu}} = -\partial_{\bar{\nu}}\Gamma^{\sigma}_{\mu\sigma} = -\partial_{\bar{\nu}}\partial_{\mu}ln(g)$ with $g = \det(g_{\rho\bar{\sigma}})$ as:

$$\mathcal{R} = i\partial\bar{\partial}\ln(g) \tag{2.51}$$

which leads us to show that $d\mathcal{R} = d^2(\bar{\partial} - \partial)\frac{i}{2}\ln(g) = 0$ and so the Ricci form is closed but not exact since the determinant of the metric g is not a scalar.
The importance of Ricci form $\mathcal{R}$ comes from the fact that we can use it to define the **first Chern Class**:

$$c_1 = \frac{1}{2\pi}[R] \in H^2(Y_6, \mathbb{R}) \tag{2.52}$$

which is a topological invariant and will give a necessary condition for defining a Calabi-Yau manifold in the next chapter.

We end up this chapter with a very important insight on Hodge theory in Kähler manifolds: since we can check that the exterior derivative Laplacian satisfies $\Delta = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}$, we can see how given an (r,s)-form ω which is Harmonic, its complex conjugate is Harmonic too, so via isomorphism of Cohomology classes and Harmonic forms we can see how:

$$h^{r,s} = h^{s,r} \tag{2.53}$$

leading to an Hodge diamond which is vertically symmetric (this is for the case of a 6 dimensional manifold):

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{1,0} & & \\ & h^{2,0} & & h^{1,1} & & h^{2,0} & \\ h^{3,0} & & h^{2,1} & & h^{2,1} & & h^{3,0} \\ & h^{3,1} & & h^{2,2} & & h^{3,1} & \\ & & h^{3,2} & & h^{3,2} & & \\ & & & h^{3,3} & & & \end{array} \tag{2.54}$$

and, in the end, just through these Hodge numbers we can define the Euler Characteristic as:

$$\chi(Y_6) = \sum_{r,s} (-1)^{r+s} h^{r,s} \tag{2.55}$$

2.2.5 Calabi-Yau manifolds

The last Mathematical Step to understand the compactified manifolds in our models is now to define a special class of Kähler manifolds: the **Calabi-Yau k-folds**. We define a Calabi-Yau k-fold as a compact Kähler manifold Y of complex dimension k which satisfies the conditions:

- Admits a Kähler metric with $SU(k)$ holonomy;
- It's Ricci flat, so Admits a Kähler metric with vanishing Ricci curvature;
- Has vanishing first Chern class $c_1(Y)$;
- Admits on it the existence a nowhere vanishing (k,0)-form Ω_3 holomorphic and harmonic.

In particular, we are interested in the case of **Calabi-Yau 3-folds**, so complex Kähler manifolds with **SU(3) holonomy**. The first two conditions are very interesting since both of them give strong consequences when the model we build features such a Calabi-Yau 3-folds, so it is worth to briefly review the consequences of such 2 points.

SU(3) holonomy

Following the idea of section 1.7.4 where the gravitino's transformation under SUGRA is proportional to the covariant derivative of SUSY spinorial parameter $\delta_\xi \chi_\alpha = \nabla_\alpha \xi$ then, to identify such a transformation which preserves the vacuum invariance we need a covariantly constant 4D SUSY spinorial parameter which is a non trivial requirement on a curved manifold but can be show to hold if and only if $SU(3)$ holonomy is present. As a consequence We expect that starting from a 4D N=1 Supersymmetric theory, generalising it to 10D and compactifying to a 4D Supersymmetric Effective Field Theory (E.F.T.) which has a compactification manifold with $SU(3)$ holonomy, we still have N=1 SUSY

$$\text{N=1 SUSY IN 4D} \overset{10D}{\rightarrow} \text{N=4 SUSY IN 4D} \overset{C.Y.Comp.}{\rightarrow} \text{N=1 SUSY in 4D} \tag{2.56}$$

Ricci flatness

Ricci Flatness appear to be fundamental since first of all is equivalent to $SU(k)$ holonomy, but, most importantly implies that einstein equations are solved without sources $R_{MN} = 0 \quad \forall M, N = 0, \ldots, 9$. It has been shown by **Yau's Theorem** [38] that if the first Chern class of a Kähler manifold with J Kähler form vanishes (in the sense that is a trivial class, $c_1 = [0]$) than exist a Ricci flat metric on the manifold with different Kähler form J' but still in the same cohomology class of J: $J - J' = d\alpha$ with α arbitrary form. This is called Calabi-Yau metric and its unique but hard to find explicitly.

Given all these conditions it's now useful to check how they simplify the Hodge diamond shape. In fact, taking as an example a C.Y. 3-fold, from the last condition we have a unique (3,0) form Ω_3 so $h^{3,0} = h^{0,3} = 1$. It can be shown that for $n < 3$ $h^{0,m} = h^{m,0} = 0$, that for a C.Y. 3-fold horizontal symmetry $h^{1,1} = h^{2,2}$ and vertical symmetry (Kähler manifold condition) $h^{r,s} = h^{s,r}$ hold. Finally, recalling the Hodge duality condition $h^{r,s} = h^{t-r,t-s}$we get $h^{0,0} = h^{3,3} = 1$ and $h^{1,2} = h^{2,1}$

giving us such a diamond depending only on $h^{1,1}$ and on $h^{1,2}$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.57}$$

and giving us the expression for the Euler characteristics using (2.55):

$$\chi(\mathrm{CY}_3) = 2(h^{1,1} - h^{2,1}) \tag{2.58}$$

2.2.6 Moduli space of Calabi-Yau manifolds

Given a Calabi-Yau 3-fold, which is the manifold we will compactify the 6 extra dimensions of our 10D Superstring theory, Yau's theorem states that exist a unique Ricci flat metric $g_{i\bar{j}}$ such that $R_{k\bar{s}}(g_{i\bar{j}}) = 0$ given a certain Kähler form J and Complex Structure $\mathcal{J}$. One can ask now if the metric can be deformed mantaining its Ricci-Flatness (since this will lead to the presence of moduli) in the following way:

$$g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} \to g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} + \delta g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} + \delta g_{ij} dz^i dz^j \tag{2.59}$$

we can show, through the Lichnerowicz equation, which is the Ricci Flatness equation $R_{i\bar{j}} = 0$ in a particular gauge, and using the Calabi-Yau conditions, that the equations for the deformations $\delta g_{i\bar{j}}$ and δg_{ij} are decoupled and so independent.

One can think that, due to the possibility of this deformation we have a contradiction to the uniqueness of the metric proved in Yau's theorem on our Calabi-Yau, however this means in realty that this kind of deformation (2.59) must be accompanied by a modification of the harmonic representative of the Kähler form J and of complex structure $\mathcal{J}$. They lead to two different kind of deformations and of moduli:

- A deformation of the metric of the kind $\delta g_{i\bar{\jmath}}$ can be seen as a change of the Kähler form representative:

$$\delta g_{i\bar{\jmath}} = -i\delta J_{i\bar{\jmath}} \tag{2.60}$$

where $\delta J = i\delta g_{i\bar{\jmath}} dz^i \wedge d\bar{z}^{\bar{\jmath}}$ is an harmonic (1,1)-form and so a representative of cohomology class $\delta J \in H^{1,1}_{\bar{\partial}}(Y_6, \mathbb{C})$.
Since the possibilities to choose this representative are one for each cohomology class then we have $h^{1,1}$ independent deformations called **Kähler deformations** which at least are $h^{1,1} > 1$ since we can always rescale the metric making the Calabi-Yau smaller or larger without changing the shape of it.

- A deformation of the metric of the kind $\delta g_{ij} \neq 0$ seems violate the Hermiticity assumption but the concept of Hermitian metric depends strongly on the initial complex structure $\mathcal{J}$ and so, pairing the deformation of the metric with a change of the complex structure $\mathcal{J} \to \mathcal{J} + \delta\mathcal{J}$ we get the Hermiticity again satisfied.
Using the (3,0)-form Ω_3 we can define a (2,1)-form:

$$\delta\chi = \Omega_{ijo}\delta\mathcal{J}^{o}_{\bar{m}} dz^i \wedge dz^j \wedge d\bar{z}^{\bar{m}} = \Omega_{ijo}\delta g_{\bar{k}\bar{m}} g^{o\bar{k}} dz^i \wedge dz^j \wedge d\bar{z}^{\bar{m}} = \tag{2.61}$$

$$= \Omega^{\bar{k}}_{ij}\delta g_{\bar{k}\bar{m}} dz^i \wedge dz^j \wedge d\bar{z}^{\bar{m}} \in H^{2,1}(Y_6, \mathbb{C}) \tag{2.62}$$

which is associated with a deformation of the metric of kind $\delta g_{\bar{l}\bar{m}}$ and by inverting we can write the deformation of the metric in terms of the (2,1)-form's one: $\delta g_{\bar{\imath}\bar{\jmath}} = -\frac{1}{||\Omega_3||^2}\bar{\Omega}^{kl}_{\bar{\imath}}\delta\chi_{kl\bar{\jmath}}$ with $\delta\Omega_3 = \delta\chi$ and with $||\Omega_3||^2 = \frac{1}{3!}\Omega_{ijk}\bar{\Omega}^{ijk}$. We can show that this kind of deformation is a bijection between linearly independent Dolbeault (2,1)-form cohomology classes and independent (not related by reparametrizations) **complex structure deformations**. The number of these complex structure deformations can be counted by the complexified vector space of 3-cycles dimension which is $\dim H^3 = h^{3,0} + h^{1,2} + h^{2,1} + h^{0,3} = 2h^{1,2} + 2$. Two directions are the one of Ω_3 and $\bar{\Omega}_3$ and since the change of one of these 2 is coupled with a change of complex structure, we have $h^{1,2} + 1 - 1 = h^{1,2}$ possible changes (rotations) of Ω_3.

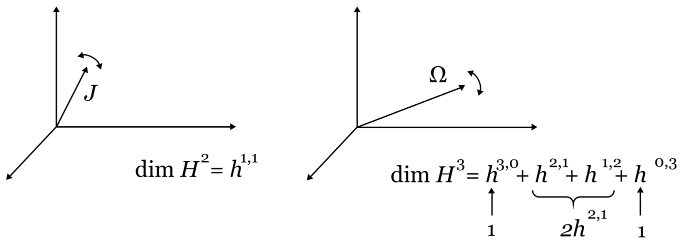

Figure 2.2: A 3D Picture (in general much more dimension are present) from [30] of how the moduli deformation is connected with a motion of J and Ω_3 in the spaces $H^2(Y, \mathbb{C})$ and $H^3(Y, \mathbb{C})$ determining the metric on a Calabi-Yau.

In total then the geometric transformations of a Calabi-Yau 3-fold that preserve the Calabi-Yau condition of Ricci flatness can be parametrised by $h^{1,1} + 2h^{1,2}$ degrees of freedom which are our moduli. In addition to these geometric moduli we can add moduli parametrised by p-forms which are, exactly as the geometric ones, scalar fields in 4D. We will see that at tree level and at classical level, without sources, background fields or fluxes all these fields are massless and this can bring to some problems we can solve through their **stabilisation**.

2.3 Orientifold projection

Before studying the properties of the moduli in full detail and how they enter our String Landscape models, we need a brief study of how to include gauge group and so Standard Model in String Compactifications and the strongly connected concepts of **Orientifold Projections** and of **Orientifold Planes**.

We recall that Type IIB 10D Supergravity contains naturally k-forms with k even and Dp-branes with p odd. Their presence gives a new way of constructing Standard Model as an Effective Field Theory and so even Spontaneous Symmetry Breaking, since a Dp-brane stack represents a dynamical object with a certain tension on which a Super Yang Mills (SYM) theory of dimension p+1 can live. Since the total number of dimension is 10, we can have at maximum D9-branes with 10D gauge fields and gauginos and we have the same number of supercharges for all p. If instead $p < 9$ we will have one (p+1)D gauge field and 9-p scalars compensating the missing bosonic degrees of freedom while the fermionic ones are filled by lower dimensional spinors. One in principle can think that we can compactify a Type IIB theory on a Calabi-Yau to obtain a 4D EFT and, after that, wrap any desired number of D-brane stacks necessary to build the correct Standard Model content. This is too easy to be true and in fact by implementing simply branes wrapping cycles of our manifold, since the RR-charge and the tension the branes carry, we get a non zero charge in compact space which is inconsistant and leads to charge tadpoles (while the non-zero tension to gravitational ones).
The first idea to solve this problem is clearly through including an object with opposite charge like an **antibrane** however this will attract the D-brane and annhilate it, bringing an unstable and temporarily existance of the matter content.

The solution comes by implementing the such called **orientifold planes** or **O-planes** which are objects with opposite RR-charge and tension with the respect to D-branes but they don't annihilate them and in addition to this they don't break additional Supersymmetry with the respect to the half broke by D-branes

$$\text{N=2 SUSY in 10D} \equiv \text{N=8 SUSY in 4D} \overset{\text{C.Y.Comp}}{\rightarrow} \text{N=2 SUSY in 4D} \overset{\text{D-branes and O-Planes}}{\rightarrow} \text{N=1 SUSY in 4D} \tag{2.63}$$

In order to understand better how these O-planes appear it is necessary to study the **orientifold projection** $\mathcal{O}$ which includes the worldsheet orientation reversal Ω_{ws}. The orientifold projections we are interested in are of the shape:

$$\mathcal{O} = (-1)^{F_L} \Omega_{ws} \sigma \tag{2.64}$$

where F_L left moving sector fermion number and σ involution changing the sign of the Ω_3 harmonic, holomorphic (3,0)-form as $\sigma\Omega_3 = \pm\Omega_3$ since this will let us preserve the correct amount of Supersymmetry and the involution in this case is called **holomorphic involution**.

Then the o-planes sit at the singularities on fixed points, or loci, of the orientifold projections which are points of 4-cycles in our 6D Calabi-Yau Y_6. These configurations of our Calabi-Yau, called orbifolds, in principle can have all the O-planes with odd space dimension $l > 4$ since on the other 4 dimensions $\mathcal{O}$ acts trivially; however, depending on the eigenvalue of holomorphic involution we have different o-planes:

- If $\sigma\Omega_3 = +\Omega_3$ we have O5-planes and O9-planes;
- If $\sigma\Omega_3 = -\Omega_3$ we have O3-planes and O7-planes.

If we wish to preserve N=1 SUSY only one of the two must be chosen and we are going to use O3/O7-planes since more interesting Phenomenologically as stated in [30]. Finally, let us cite, without giving further details, that in principle orientifold projection decomposes cohomology groups $H^{1,1} = H^{1,1}_+ \oplus H^{1,1}_-$ with the signs "+" and "-" determining the "parity" of the two-forms under orientifold projection. Since the Hodge duality, we can identify a basis for the cohomology groups as a basis of harmonic form which under orientifold projection separates into even and odd eigenspace with dimension $h^{1,1}_+$ and $h^{1,1}_-$ respectively. However we will take, without loss of generality $h^{1,1}_- = 0 \Rightarrow h^{1,1}_+ = h^{1,1}$ and I will refer $h^{1,2}_-$ as $h^{1,2}$ since now on.

2.4 Calabi-Yau moduli

In the following section we start to study the moduli of a Calabi-Yau Manifold introducing their different kinds, which problem can lead their tree level massless nature and, finally, discussing mechanisms to stabilise them.

2.4.1 Kähler and complex structure moduli

In order to start defining the fields arising now from the compactification we would like to start a treatment of the moduli. We are going to rewrite everything in term of the basis for the cohomology groups (harmonic forms).
For the sake of clarity we can divide the moduli in 2 classes:

- Rewriting (2.60) in terms of the harmonic basis of the cohomology group we obtain $\delta g_{i\bar{\jmath}} = -it^i\omega_i$, by rewriting the Kähler form in terms of the basis $J = t^i\omega_i, \quad \omega_i$ such that $i = 1, \ldots, h^{1,1}$. The coefficients $t^i(x)$ are scalar fields called the **Kähler moduli** which are orientifold invariant and measure the volume of the C.Y. 2-cycles;

- Rewriting instead (2.61) in terms of $H^{1,2}(Y_6, \mathbb{C})$ basis element we get $\delta g^a_{\bar{\imath}\bar{\jmath}} = -\frac{1}{||\Omega_3||^2} z^a(x)(\chi_a)_{kl\bar{\jmath}}\bar{\Omega}^{kl}_{\bar{\imath}}$. In this case the $h^{1,2}$ scalar fields are called the **Complex Structure Moduli**.

So the total geometric moduli space, as stated in previous subsection, can be separated into the two independent parts correspondant to Kähler moduli (Kähler deformations) and Complex Structure Moduli (Complex Structure Deformations) $\mathcal{M}_{\text{Moduli}} = \mathcal{M}_{\mathcal{K}} \times \mathcal{M}_{\text{C.S.}}$. In addition to this geometric moduli we have to include another modulus built up by the dilaton and the 0-form C_0: the **Axio-dilaton** $S = C_0 + ie^{-\varphi} = C_0 + \frac{i}{g_s}$

Moduli however are not the end of the story in String Compactifications, in fact, the integration of p-forms which come from the closed string spectra over $i = \dim(H_2(Y_6))$ p-cycles Σ^i_p of the compact space naturally gives **axions**, pseudoscalar fields enjoying shift symmetry. In our Type IIB case there are 3 forms suitable to build such a shift symmetric fields:

- NS-NS Sector Kalb-Ramond Field B_2: it will give raise to the axion

$$b_i = \frac{1}{\alpha'} \int_{\Sigma_2^i} B_2; \tag{2.65}$$

- R-R Sector 2-form field C_2: it will give raise to the axion

$$c_i = \frac{1}{\alpha'} \int_{\Sigma_2^i} C_2; \tag{2.66}$$

- R-R Sector 4-form Field C_4: it will give raise to the axion

$$\theta_i = \frac{1}{(\alpha')^2} \int_{\Sigma_4^i} C_4 \tag{2.67}$$

  which will be fundamental later on.

2.4.2 General 4D supergravity Kähler potential

We have now all the bricks to start building up models. In order to describe a general model in 4D Supergravity language we need of course to write a Lagrangian:

$$\mathcal{L} = K_{i\bar{j}}(\partial_\mu X^i)(\partial^\mu \bar{X}^{\bar{j}}) + \text{other fields} \tag{2.68}$$

where $K_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ is the Kähler metric coming from the Kähler potential and X^i are the moduli, both Kähler, complex structure and axiodilaton ones. In order to have the explicit Lagrangian then it is necessary to write the Kähler potential and so we need to build it up from scratch with a term for each kind of modulus, working as always in Planck units $M_p = 1$.

Kähler potential for Kähler moduli

Recalling the $J = t^i\omega_i$ with $i = 1, \ldots, h^{1,1}$ we can rewrite the volume of the Calabi-Yau manifold as:

$$\mathcal{V} = \frac{1}{6}\int_{Y_6} J \wedge J \wedge J = \frac{1}{6}k_{ijs}t^i t^j t^s \tag{2.69}$$

where $k_{ijs} = \int_{Y_6} \omega_i \wedge \omega_j \wedge \omega_s$ is called **triple intersection number** since naively counts the points of intersection of the 4-Cycles which are Poincarè dual to the harmonic forms ω_i in the 6D manifold. Instead of using t^i it is more useful to change the Kähler moduli basis and use

$$\tau_i = \frac{1}{2}\int_{\Sigma_4^i} J \wedge J = \frac{1}{2}k_{ijs}t^j t^s \tag{2.70}$$

From a mathematical point of view choosing t^i or τ_i has the deep meaning of choosing an N=1 sub-algebra of the N=2 SUSY on the C.Y. Type IIB compactification and we can in principle find a relation between the two: $t^i = t^i(\tau_1, \ldots, \tau_{h^{1,1}})$. We are going to use the τ_i moduli and we complexify them by adding (2.67):

$$T_j = \tau_j + ic_j \Rightarrow \tau_j = \frac{1}{2}(T_j + \bar{T}_{\bar{j}}) \tag{2.71}$$

so that the volume can be expressed in terms of a real function depending on this new complexified moduli T_j and $\bar{T}_{\bar{j}}$ as $\mathcal{V} = \mathcal{V}(T_j, \bar{T}_{\bar{j}})$ so that we can finally write the Type IIB Kähler potential for the Kähler moduli as:

$$K_K = -2\ln(\mathcal{V}) \tag{2.72}$$

Kähler potential for complex structure moduli

In order to describe the complex structure moduli space $\mathcal{M}_{\text{C.S.}}$ we start from an easier example, the basis of $H_1(R_2)$ the homology group of Riemann Surface with cycles A^1, B_1, A^2, B_2.

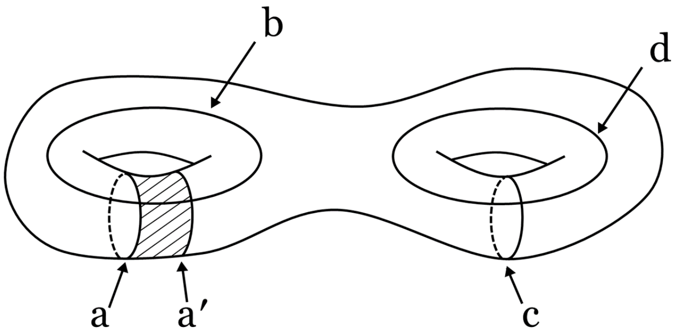

Figure 2.3: Picture of the representative of the 4 linearly independent homology classes in $H_1(R_2)$ with $a \simeq a' = A^1, b = B_1, c = A^2, d = B_2$.

with intersection structure:

$$A^a \cdot A^b = 0 \tag{2.73}$$

$$B_a \cdot B_b = 0 \tag{2.74}$$

$$A^a \cdot B_b = \delta^a_b \tag{2.75}$$

An analogous basis, called **symplectic basis** can be chosen for H_1 with such intersection structure in the form wedge product formalism:

$$\int \omega^a_a A \wedge \omega^b_B = \delta^b_a = - \int \omega^b_B \wedge \omega^A_a \tag{2.76}$$

In particular, in our case of complex Calabi-Yau 3-folds we choose a symplectic basis and we define the **periods**, integrals of the 3-form Ω over 1-cycles, as:

$$z^a = \int_{A^a} \Omega \tag{2.77}$$

$$\mathcal{G}_b = \int_{B_b} \Omega \tag{2.78}$$

The complex periods z^a, $a = 0, \ldots, h^{1,2}$ parametrize the position of the 3-form Ω in the space $H^3(Y_6)$ since one parameter can be set to $z^0 = 1$ by rescaling via complex coefficients Ω, which is then not a geometrical, and so physical, change. Then $h^{1,2}$ parameters only are left over.

The remaining periods $\mathcal{G}_b$ are z^a dependent:

$$\mathcal{G}_b = \mathcal{G}_b(z^0, \ldots, z^{h^{1,2}}) \tag{2.79}$$

and their explicit form can be obtained by solving the **Pichard-Fuchs equations** formulated from topological features of the Calabi-Yau.

We can combine these periods on a total period vector:

$$\Pi = (z^0, \ldots, z^{h^{1,2}}, \mathcal{G}_0(z), \ldots, \mathcal{G}_{h^{1,2}}(z)) \tag{2.80}$$

and, by recalling the symplectic metric form:

$$\Sigma = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \tag{2.81}$$

we get the Kähler potential for the complex structure moduli [30]:

$$K_{\text{C.S.}} = -\ln(i \int_{Y_6} \Omega \wedge \bar{\Omega}) = -\ln(-i\Pi^\dagger \Sigma \Pi) = -\ln(-i\bar{z}^a \mathcal{G}_a(z) + iz^a \overline{\mathcal{G}_a(z)}) \tag{2.82}$$

Kähler potential for the axio-dilaton

The non-geometric axio-dilaton modulus $S = C_0 + ie^{-\varphi} = C_0 + \frac{i}{g_s}$ have a straightforward Kähler potential instead given by:

$$K_S = -\ln(-i(S - \bar{S})) \tag{2.83}$$

WIth all these 3 parts we have finally a full Type IIB Kähler Potential which is quite general and so, for such a 4D Supergravity model:

$$K = K_K(T^i, \bar{T}^{\bar{\jmath}}) + K_{\text{C.S.}}(z^a, \bar{z}^{\bar{a}}) - \ln(-i(S - \bar{S})) \tag{2.84}$$

$$K = -2\ln(\mathcal{V}) - \ln(i \int_{Y_6} \Omega_3 \wedge \bar{\Omega}_3) - \ln(-i(S - \bar{S})) \tag{2.85}$$

2.5 Moduli stabilisation

Moduli are crucial elements of Calabi-Yaus both because they appear naturally as we saw before and because the dynamics of the moduli is crucial from a Cosmological point of view, in fact we can use them as inflaton to drive inflation. Before doing so however, it is clear that a problem arises immediately: up to now, just with a Kähler potential, moduli are flat direction of field space, they have no potential at all. This can lead us to fifth forces which we do not experience today, so to phenomenological inconsistencies. In order to have a proper model which is consistent and viable for phenomenology we then need to find a "source" for the potential of them. The challenge of finding a potential for them such that we can find vacua with all moduli having positive mass squared is called **moduli stabilisation**.

2.5.1 Complex structure and axio-dilaton stabilisation

The idea is to consider now compactifications with non zero components of RR and NS gauge field strength $F_3 = dC_2$ and $H_3 = dB_2$, with metric which is a warped solution. These kind of compactifications feature a 3-form defined as $G_3 = F_3 - SH_3$ which is imaginary self-dual (ISD) $\star_6 G_3 = iG_3$ and, since this, they are called **ISD Compactifications**. The presence of such field strengths different from zero induces us to **turn on fluxes** which means to choose 2 integers $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ such that a flux quantisation like the one in Electromagnetism takes place:

$$\frac{1}{2\pi\alpha'}\int F_3 = 2\pi n \quad , \quad \frac{1}{2\pi\alpha'}\int H_3 = 2\pi m \tag{2.86}$$

The importance of these fluxes is fundamental, in fact we can easily see even from a basic example as in [30] how the fluxes prevent cycles from shrinking and in particular, with various fluxes on various cycles, their presence tends to stabilise the shape of the manifold. It can be now intuitively appear in our mind that, since the complex structure moduli govern the ratios of 3-cycle volumes and the 3-form fluxes stabilise these volumes, the fluxes will stabilise our complex structure moduli giving them mass. Since there is no possibility to generate a scalar potential through Kähler one, this means that fluxes induce a non-zero superpotential W_0

depending on SUGRA models moduli which lead the number of possible models suitable for the landscape to grow exponentially.

This Superpotential is called **Gukov-Vafa-Witten Superpotential** [29] and it has been postulated and Mathematically justified for M-Theory on CY 4-folds, but in our Type IIB case it has been derived from 4D N=1 SUGRA and from 10D theory too, with the shape [27]:

$$W_{GVW} = W_0 = \int_{Y_6} G_3 \wedge \Omega_3 \tag{2.87}$$

Through the usual formula for the scalar potential:

$$V = e^K (K^{i\bar{j}}(D_i W)(D_{\bar{j}}\bar{W}) + K^{a\bar{b}}(D_a W)(D_{\bar{b}}\bar{W}) - 3|W|^2) \tag{2.88}$$

where $D_i = \partial_i + K_i$ covariant derivative and the index i goes through all the Kähler moduli $i = 1, \dots, h^{1,1}$ while the index $a = 0, \dots, h^{1,2}$ goes through complex structure moduli and axio-dilaton, in fact we call since now on $z^a = \{S, z^1, \dots, z^{h^{2,1}}\}$. In addition to this we reabsorb the axio-dilaton term of Kähler potential into the complex structure part of it $K_{\text{C.S.}} = K_{\text{C.S.}} + K_S$. Since W_0 is dependent on z^a (with S included) $W_0 = W_0(S, z^b)$, by using the F-term conditions for SUSY

$$D_a W = 0 \quad \text{for} \quad a = 1, \dots, \frac{b_3}{2} \tag{2.89}$$

with $b_3 = \dim(H_3(Y_6)) = \dim(H^3(Y_6)) = \sum_{r+s=3} h^{r,s} = 2 + 2h^{1,2}$ Betti Number, we get that both the $h^{1,2}$ complex structure moduli and the Axio-dilaton are **stabilised** and so we can integrate them out the Potential at the price of an overall factor $\mathcal{S} = \frac{e^K_{\text{C.S.}} g_s}{8\pi}$ which for now we are going to set $\mathcal{S} = 1$.

2.5.2 Kähler moduli stabilisation

One now can ask if Gukov-Vafa-Witten Potential is enough to stabilise even Kähler moduli, the answer unluckily appears to be no. One in principle can think that, once integrating out z^a, the remaining potential, depending only on $T, \bar{T}$ can be different from 0 since

$$V = e^K (K^{i\bar{j}} D_i W \bar{D}_{\bar{j}} \bar{W} - |W|^2) \tag{2.90}$$

however a strange but fundamental phenomenon appears due to the shape of the Kähler potential for the Kähler moduli $K_K = -\ln(\mathcal{V}^2)$. In fact, since $\mathcal{V}^2$ is usually an homogeneous function of degree 3 of T_i, it appears the such called **no-scale structure**. As shown by computation in B the no-scale structure means that the Kähler Moduli have no potential at all $V = V(T, \bar{T}) \equiv 0$ if the $\mathcal{V}^2$ is an homogeneous function of degree 3. This name comes from the fact that SUSY is broken at an unknown scale, in fact:

$$D^{\bar{T}} W = K^{\bar{T}T} K_T W = \left(\frac{(T + \bar{T})^2}{3} \right) \left(\frac{-3}{(T + \bar{T})} \right) W_0 = -(T + \bar{T}) W \neq 0 \tag{2.91}$$

but $\Lambda_{\not{SUSY}} = m_{\frac{3}{2}} = e^K W_0$ is not fixed since Kähler potential is depending on T which is not stabilised. In order to resume, giving the 10D geometry of a C.Y. we can see how Complex Structure Moduli and Axio-Dilaton are stabilised, while Kähler Moduli, axions (and brane position moduli we have not discussed here) are flat directions in field space.

Given these unstabilised Kähler moduli, a legit doubt is how to avoid the problem of having unobserved fifth forces due to them. Luckily, the presence of quantum corrections is well known and breaks the no-scale structure. These effects are of 2 kinds: perturbative and non-perturbative, the perturbative ones can't affect Superpotential for the Renormalisation theorem so, calling the previously written Kähler potential as $K \to K_0$:

$$\begin{cases} K = K_0 + K_p + K_{np} \\ W = W_0 + W_{np} \end{cases} \tag{2.92}$$

Perturbative corrections

Perturbative corrections arise in 2 different way both from α'^3 expansion of the action and from the g_s expansion, called string loop expansion: $K_p = \delta K_{(\alpha')^3} +$

δK_{gs}. We are going to analyze both of them since these are included in our model in the next chapter.

- $(\alpha')^{\mathbf{3}}$ **correction**: These corrections descend from the 10D action as the quartic invariant part, which is part of the classical 10D SUGRA theory and appear as four-loop correction to the worldsheet σ-model β-function, not as loop correction in the 10D spacetime:

$$S_{\text{Grav}} = \int d^{10}X\sqrt{-G}\left[\frac{M_{10}^2}{2}R + \frac{\zeta(3)}{3\cdot 32}\frac{1}{M^6}\mathcal{R}^4 + \dots\right] \tag{2.93}$$

with $\mathcal{R}^4$ quartic invariant computed from the Riemann tensor, $\zeta(3) = 1.202$ Apéry's constant and M is the mass of Type IIB String First Excited Level. In the 4D theory this correction appears in the Kähler potential as:

$$K = -2\ln\left[\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right] \quad , \quad \xi \equiv -\frac{\chi(Y_6)\zeta(3)}{2(2\pi)^3} \tag{2.94}$$

with $\chi(Y_6)$ Euler Characteristic of the C.Y. 3-fold Y_6. By the computations present in B we can easily see how this kind of perturbative correction spoils the no scale structure of the potential giving a contribute (by recalling $\hat{\xi} = \frac{\xi}{g_s^{3/2}}$) of:

$$\delta V_{(\alpha')^3} \simeq \frac{3\hat{\xi}}{4}\frac{W_0^2}{\mathcal{V}^3} \neq 0 \Leftrightarrow \hat{\xi} \neq 0 \tag{2.95}$$

A very important features of such corrections is that they break Supersymmetry of vacuum when implemented.

- $\mathbf{g_s}$ **correction**: these are perturbative corrections from loop effects in spacetime so from higher genus worldsheets and again, as the previous ones, they will spoil the no-scale structure. These corrections have been explicitly computed in toroidal orientifolds only [7]/[8], like $T^6(\mathbb{Z}_2 \times \mathbb{Z}_2)$ while conjectured in Calabi-Yau ones [9]. In both the cases the Kähler potential correction takes the composite form:

$$\delta K_{\text{gs}} = \delta K_{\text{gs}}^{\text{KK}} + \delta K_{\text{gs}}^{\text{W}} \tag{2.96}$$

where the two kinds of correction come from different sources, in fact $\delta K_{\rm gs}^{\rm KK}$ comes from the exchange of Kaluza-Klein modes (closed strings with K.K. momentum) between D7-branes and D3-branes or their correspondent O3/O7-planes needed for tadpole cancellation. On the other hand $\delta K_{\rm gs}^{\rm W}$ originates by the exchange of winding strings (closed strings with winding number) between intersecting stacks of D7-branes (and even O7-planes). These two terms assume the form:

$$\delta K_{\rm gs}^{\rm KK} = -\frac{1}{128\pi^2}\sum_{i=1}^{3}\frac{\mathcal{E}_i^{\rm KK}(z,\bar{z})}{{\rm Re}(S)\tau_i} \tag{2.97}$$

$$\delta K_{\rm gs}^{\rm W} = -\frac{1}{128\pi^2}\sum_{i=1}^{3}\frac{\mathcal{E}_i^{\rm W}(z,\bar{z})}{\tau_j\tau_k}\bigg|_{j\neq k\neq i} \tag{2.98}$$

where τ_i are the Kähler moduli wrapped by i-th D7-brane and $\mathcal{E}_i^{\rm KK}(z,\bar{z})$ and $\mathcal{E}_i^{\rm W}(z,\bar{z})$ are general functions of the complex structure moduli with complicated form. However, as we can easily notice, the dependence on Kähler moduli is almost trivial.

If we would like to generalise the results to a general Calabi-Yau we need to take the conjectured results:

$$\delta K_{\rm gs}^{\rm KK} \simeq \sum_i C_i^{\rm KK} g_s \frac{\mathcal{P}^i(t^j)}{\mathcal{V}} \tag{2.99}$$

$$\delta K_{\rm gs}^{\rm W} \simeq \sum_i C_i^{\rm W} \frac{1}{\mathcal{G}^i(t^j)\mathcal{V}} \tag{2.100}$$

where $C_i^{\rm KK}$, $C_i^{\rm W}$ functions of complex structure moduli and axio-dilaton while functions $\mathcal{P}^i(t^j)$ and $\mathcal{G}^i(t^j)$ are linear in 2-cycle volume moduli t^j (in general that could be even homogeneous function of degree 1 in t^j).
It can be proven that, since $\delta K_{\rm gs}^{\rm KK}$ homogeneous function of degree -2 in t^j then we can see an **extended no-scale structure** on the correction of the scalar potential $\delta V_{\rm gs}^{\rm KK}$. This means that, even if in SUGRA approximation $t_i >> 1$ where $\delta K_{\rm gs}^{\rm KK} \sim \sum_i \frac{t_i}{\mathcal{V}} > \delta K_{(\alpha')^3} \sim \frac{1}{\mathcal{V}}$ so it seems that string loop correction can change strongly the vacuum structure, these terms remain

subdominant in comparison with the $(\alpha')^3$ corrections due to the cancellation of some terms as showed in B. By using the linearity of the two functions $\mathcal{P}^i(t^j)$ and $\mathcal{G}^i(t^j)$ we can find that these kind of corrections to scalar potential $\delta V_{\rm gs}$ can be written as:

$$\delta V_{\rm gs} = \frac{|W_0|^2 c_{\rm loop}}{\mathcal{V}^3}\frac{1}{\mathcal{V}^{1/3}}\left(\frac{\mathcal{V}^{1/3}}{\sqrt{\tau_i}} + \mathcal{O}\left(\frac{\sqrt{\tau_i}}{\mathcal{V}^{1/s}}\right) + \mathcal{O}(1)\right) \simeq \frac{|W_0|^2 c_{\rm loop}}{\mathcal{V}^3}\frac{1}{\sqrt{\tau_i}} \tag{2.101}$$

where $c_{\rm loop} = C_i^{\rm W}$ or $(g_s C_i^{\rm KK})^2$ depending on where does these loop correction come from. This conjecture seems a little bit like a random guess, instead [17] showed how this result for the potential can match the field theoretical one-loop Coleman-Weinberg potential

$$V^{\rm CW}_{1-{\rm loop}} \simeq \frac{1}{16\pi^2}\Lambda^2\,{\rm STr}(M^2) \tag{2.102}$$

with ${\rm STr}(M^2) \equiv \sum_i (-1)^{2j_i}(2j_i+1)m_i^n$ supertrace in terms of particle with spin j_i and mass eigenvalues m_i and Λ EFT Cutoff which can be seen as the mass of K.K. replicas of open string modes on D7-branes wrapped around different 4-cycles so it depends on the C.Y. structure. For the Calabi Yau we are going to use into Loop Blow-Up Inflation ${\rm STr}(M^2) \simeq m_{3/2}^2 \simeq \frac{W_0^2}{\mathcal{V}^2}$ and $\Lambda_i \simeq \frac{1}{\tau_i^{1/4}\sqrt{\mathcal{V}}}$ giving us $c_{\rm loop} \simeq \frac{1}{16\pi^2}$.

Non-perturbative corrections

As previously stated, due to the non renormalisation theorem, the superpotential receives no $(\alpha')^3$ (due to axionic shift symmetry on imaginary part of Kähler moduli) nor g_s corrections [13] but it can receive Non-perturbative ones and we are going to consider only the one on Superpotential: $W = W_0 + W_{\rm np}$. These non-perturbative contributions usually arise because of 2 phenomena: Gaugino Condensation and E3-Brane Instantons and we are going to give a phenomenological description and the explicit expression for both of the corrections.

- **Gaugino condensation**: We now consider a compactification with a stack of N spacetime filling D7-Branes wrapping a 4-cycle Σ_4. Writing down the action of the D7-branes we get a theory including a Yang-Mills piece with a

4D gauge field A_μ. Given some topological conditions on the 4-cycles, in particular Σ_4 has to be rigid such that no deformations can exists so no charged matter fields, we get that the 4D EFT obtained upon dimensional reduction is a pure $N = 1$ Super Yang-Mills with a non-perturbative potential at low energy of:

$$W_{\rm np} = Ae^{-aT} \tag{2.103}$$

where $a = \frac{2\pi}{N}$, $A = A(z^a, \rho^\alpha) \sim M_P^3$ with ρ^α = brane position and T Kähler modulus whose real part measures the volume of Σ_4. In general, if more than one brane stack is present or some of these wraps more than one cycle then:

$$W_{\rm np} = W_{\rm inst} = \sum_i A_i e^{-a_i T_i} \tag{2.104}$$

with i labelling the wrapped cycles $\Rightarrow i \leq h^{1,1}$.

- **ED3-brane instantons**: If Σ_4 instead is wrapped by **Euclidean D3-branes**, called **ED3-brane instantons**, which are instantonic contributions to the path integral with an action which is Euclidean and with $\text{Re}(S_{\rm inst}) \propto \mathcal{V}_{\Sigma_{p+1}}$ where Σ_{p+1} (p+1)-cycle wrapped by the E3-brane and $\text{Im}(S_{\rm inst}) \propto S_{\rm C.S.}$ Chern-Simons action. In this case the Superpotential can be written as (in the general case with more than one cycle wrapped):

$$W_{\rm np} = W_{\rm E3} = \sum_i A_i e^{-a_i T_i} \tag{2.105}$$

  Where $a_i = 2\pi$ and, again, $A = A(z^a, \rho^\alpha) \sim M_P^3$ with ρ^α = brane position. Again in this case, a rigid cycle guarantees a non-vanishing superpotential contribution and, when fluxes are added, such a sufficient condition can be even relaxed a little bit.

In Principle there can exist even non-perturbative corrections to the Kähler potential but these corrections are negligible with the respect to perturbative ones, so we will not study them.

2.5.3 The KKLT proposal

The KKLT proposal (from Shamit Kachru, Renata Kallosh, Andrei Linde, Sandip P. Trivedi) is a method we can use to stabilise Kähler moduli through non perturbative corrections.
We start focusing on the simplest case of $h^{1,1} = 1$ so of a single Kähler modulus. Considering the complex structure moduli and the Axio-dilaton as integrated out using Gukov-Vafa-Witten Superpotential if the model features

$$K = -3\ln(T + \bar{T}) \quad \text{and} \quad W = W_0 = \text{const} \tag{2.106}$$

then we have the no-scale cancellation $V \equiv 0$ and $\not{S}\not{U}\not{S}\not{Y}$ at energy scale $m_{3/2} = e^{\frac{K}{2}} W_0$.

As previously stated, there are different Quantum corrections that can lift the flat direction of the potential V and so breaking the no-scale structure, in our case we will use the non-perturbative corrections to the Superpotential, leaving us with the following quantities:

$$K = -3\ln(T + \bar{T}) \quad \text{and} \quad W = W_0 + Ae^{-aT} \tag{2.107}$$

where we will not specify a since we want to mantain a general case instead of chosing just or gaugino condensation or instanton corrections. The scalar potential, after stabilising and so integrating out θ_i as done in C, reads as:

$$V = V(\tau) = \frac{g_s e^{K_{\text{C.S.}}}}{8\pi} |W_0|^2 \left(\frac{8a^2 |A|^2 \tau^2 e^{-2a\tau}}{3|W_0|^2 \mathcal{V}} - \frac{4a|A|}{|W_0|} \frac{\tau e^{-a\tau}}{\mathcal{V}^2} \right) \tag{2.108}$$

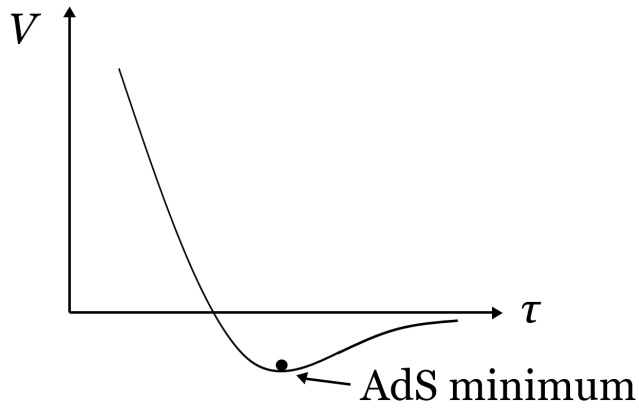

Figure 2.4: Qualitative Picture of The Scalar Potential with an Anti-de Sitter Minimum.

As shown in 2.4 the minimum is at negative values of the potential and it can be proven that this minimum is at a SUSY point where F-term vanishes and so $DW = 0$:

$$DW = \partial_T(Ae^{-aT}) + K_T(W_0 + Ae^{-aT}) = -aAe^{-aT} - \frac{3}{2\tau}(W_0 + Ae^{-aT}) = 0 \Leftrightarrow \tag{2.109}$$

$$\Leftrightarrow W_0 = -(1 + \frac{2}{3}a\tau)Ae^{-aT} \tag{2.110}$$

leading us to the fact that $W_0 \in \mathbb{R}^{<0}$ which is a simple consequence that, deriving th potential we assumed the axion θ_i to be stabilised at $\theta_i = 0$. If instead we considered during the stabilisation process of the axion $W_0 = |W_0|e^{i\varphi} \in \mathbb{C}$ as well as $A = |A|e^{i\sigma} \in \mathbb{C}$ with $\varphi = \arg(W_0), \sigma = \arg(A)$ phases, than the minimum of c would have been in $\theta|_{\min} = \sigma - \varphi + (2k+1)\pi, \quad k \in \mathbb{Z}$. However this is not crucial for us, instead, what is really crucial is the fact that W_0 is exponentially small such that $\tau \sim R^4_{Y_6} >> 1$ and this can be done easily just by fine tuning the fluxes in the landscape.

2.5.4 The Large Volume Scenario

An alternative mechanism to the previously studied KKLT proposal is the Large Volume Scenario or LVS, in which we stabilise the Kähler Moduli not only via Superpotential non-perturbative corrections but also balancing them through the

$(\alpha')^3$ Kähler Potential ones (leading then to a non-supersymmetric vacuum). Through them we can stabilise the overall volume $\mathcal{V}$ at large values such that all the perturbative corrections are subleading in $\mathcal{V}$ to the non-perturbative ones.

The combination of perturbative and non-perturbative effects give a contribution for the scalar potential of:

$$V = V_{\text{np}} + V_{(\alpha')^3} = e^K (K^{j\bar{\imath}}(a_j A_j a_{\bar{\imath}} \bar{A}_{\bar{\imath}} e^{-a_j T_j + a_i \bar{T}_i} - a_j A_j e^{-a_j T_j} \bar{W} \partial_{\bar{\imath}} J + \tag{2.111}$$

$$+ a_{\bar{\imath}} \bar{A}_{\bar{\imath}} e^{-a_i \bar{T}_i} W \partial_j K) + \frac{3}{4} \xi \frac{W_0^2}{\mathcal{V}}) \tag{2.112}$$

where $i \neq 1$, since we changed the moduli basis substituting one modulus with the volume $\{\tau_1, \ldots, \tau_{h^{1,1}}\} \to \{\mathcal{V}, \ldots, \tau_{h^{1,1}}\}$. When volume is large, perturbative $(\alpha')^3$ term (2.95) dominates over all the other terms and this can happen if one or more cycles are smaller than the largest one. The main idea is to use one small cycle, stabilised by non-perturbative corrections and with that stabilise the volume. In order to do so we take the limit, denoting the small 4-cycle volume as $\tau_s = \frac{1}{2}(T_s + \bar{T}_s)$:

$$\mathcal{V} \to \infty, \quad \text{with} \quad a_s \tau_s = \ln(\mathcal{V}) \tag{2.113}$$

and along this ray then $e^{-a_s T_s} \sim \frac{1}{\mathcal{V}}$ so we have all the terms of the scalar potential at the same order. To be complete, in order to do so and contemporarily drive inflation, it is clearly necessary, for an inflationary model, to take a class of Calabi-Yaus with $h^{1,1} \geq 3$, one which parametrises the volume, one small cycle needed to stabilise the first modulus and one inflationary cycle volume as we will see soon. The effect of the $(\alpha')^3$ correction on the scalar potential, as seen from (2.94) is then strongly dependent on the sign of $\hat{\xi}$ and so on the Euler Characteristic of the Calabi-Yau. In our tractation we are going to take $\chi(Y_6) < 0 \Rightarrow \hat{\xi} > 0$ giving us a potential which approaches zero from below in large volume limit and along the direction (2.113). Clearly before assuming that this configuration is suitable for inflationary scenarios, we need to argue the existance of a minimum. We need then 2 things:

- The potential at small $\mathcal{V}$ is positive such that V on the ray of field space (2.113) is minimised to a certain point $\mathcal{V}_{\text{min}}$;

- The potential at $\mathcal{V}_{\min}$ must not decrease on the $h_{+}^{1,1}-1$ directions normal to (2.113).

Intuitively we can give a non-rigorous set of conditions for having a AdS $\not{SUSY}$ minimum at exponentially large values for $\mathcal{V}$:

- At small volume the $(\alpha')^3$ perturbative term (B.16) in V is dominant at small volume so makes the potential positive for small $\mathcal{V}$;
- If the non-perturbative $\mathcal{V}$-leading terms in V are positive and $W = W(T^{\alpha})$, $\alpha \in H^{1,1}(Y_6)/\mathcal{V}$ then, in all the other $h^{1,1}-1$ directions orthogonal to (2.113), potential increases.

These conditions can be made more rigorous by the use of Topology arguments. We can state then that a general Calabi-Yau Y_6 manifold respecting such necessary and sufficient conditions:

1) $\chi(Y_6) < 0 \Rightarrow \hat{\xi} > 0$ or $h^{1,2} > h^{1,1} > 1$ (for driving inflation > 2) so potential goes to zero from below at infinity;
2) It has $h_{1,1} = N_b + N_s$ 4-Cycles where N_b, N_s represent the number of "big" ($\tau_j^b \overset{\mathcal{V}\to\infty}{\to} \infty$) and "small" cycles, at least one of the N_s cycles Σ_i must be a rigid exceptional divisor[1] which arises from the Blow-Up of a Point-Like Singularity in the sense that it arises by replacing (and so smoothing) such a singularity in the Calabi-Yau with the previously cited divisor. Such a cycle must be necessarily wrapped by a sector undergoing gaugino condensation or wrapped by instanton in 0-flux case;
3) The 4-cycle volume corresponding to such a del Pezzo divisor and the other $N_s - 1$ blow-up modes are stabilised small ($M_s < \tau_j^s < \mathcal{V}$ with $j = 1, \ldots, N_s$ both by non-perturbative and $(\alpha')^3$ corrections stabilising even $\mathcal{V} \sim e^{-a\tau_k^s}$, $\forall k \in 1, \ldots, N_s$;

[1]Rigid means $h^{0,1}(\Sigma_i) = h^{0,2}(\Sigma_i) = h^{0,3}(\Sigma_i) = 0$, Exceptional Divisor means $\Sigma_i = \mathbb{P}^2 = dP_0$ or in general $\Sigma_i = dP_n$ del Pezzo Divisor with degree $d = 9 - n$ and $h^{1,1}(dP_n) = 1 + n$ which is a divisor defined from blowing up (making smooth) $0 \leq n \leq 8$ singularities in $dP_0 = \mathbb{P}^2$ and satisfying $\int_{Y_6} \Sigma_i^3 = K_{iii} > 0, \quad \int_{Y_6} D_i^2 D_k 0 \quad \forall k \neq i$.

4) All the other 4-cycles $N_b - 1$ which are fibrations can't be stabilised small even if they have non-perturbative corrections since they are large. They can be stabilised neglecting these kind of non-perturbative effects and by taking into account string loop corrections.

can feature a scalar potential V admitting a set **L of AdS Non-Supersymmetric minima** at exponentially large volume, in particular:

- If $h_{1,1} = h^{1,1} = N_s + 1$ then $L = \{p \in \mathbf{F} \mid p \text{ unique point in field space } \mathbf{F}\}$
- If $h_{1,1} = h^{1,1} > N_s + 1$ then
$L = \{\sigma_i \;\; i = 1, \ldots, h_{1,1}(Y_6) - N_s - 1 \mid \sigma_i \text{ flat directions of field space } \mathbf{F}\}$

The proof of this statement can be found at [16]. We conclude this subsection by explicitly express another difference with KKLT approach (in addition to the non-supersymmetric vacuum owing to the $(\alpha')^3$ corrections): we have no need to set W_0 exponentially small for the model consistency.

2.5.5 Anti-D3 brane uplift

In the previous subsections we have discovered a landscape of SUSY and Non-SUSY vacua which have negative vacuum energy, so negative cosmological constant, this landscape is called **SUSY/$\cancel{SUSY}$ AdS vacua landscape**. However, in order to obtain phenomenological matching with the known universe we need models that have a Minkowski or slightly De Sitter vacuum and this can be obtain via various uplifting techniques. In this subsection we are going to inspect one of the possible approaches which is the **Anti-D3 brane uplift**.

Let us now inspect the case of SUSY vacua (KKLT like) for simplicity. In the previous subchapter 2.3 we saw how the need of a Orientifold projection like the O3-plane one is fundamental in order to cancel charge and gravitational tadpoles and breaks SUSY to $N = 1$ one. Given now this configuration on a Calabi-Yau we can substitute some of the D3-branes with 3-form fluxes since the latter has a Chern-Simons term which reproduce the same tadpole of D3-brane. In doing so we jumped to the world of flux compactifications in $N = 1$ SUSY setting with O3-Planes, D3-branes and fluxes which coherently break the previous $N = 2$

SUSY of the Compactified Type IIB String Theory. However, our actual Standard Model features no SUSY so we need to break the remnant SUSY in order to get a phenomenologically viable model. One can think that a solution is by adding an anti-D3-brane to the previous configuration, however this element will attract a D3-brane and annihilate it releasing energy.
An alternative could be to completely replace D3-branes by fluxes and after this add anti-D3-branes in order to have an uplifting of the potential which lasts long enough; this seems a nice solution but the uplift will be too strong and it will destroy completely the shape of the potential, in such a way that even if the minimum is dS, it will be unstable.
The solution in the end has been given by [27] and it consist of using a CY orientifold which is equipped with 3-form fluxes modeling it with a throat, making the metric **warped**:

$$ds^2 = \Omega^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n \tag{2.114}$$

where x^μ, $\mu = 0, \ldots, 3$ 4D spacetime coordinates, $\eta_{\mu\nu}$, flat metric for the 4D spacetime, y^n, $m = 1, \ldots, 6$ coordinates of the 6D Calabi-Yau manifold and, finally $g_{mn}(y)$ metric for the 6D Calabi-Yau manifold. Even if the topological characteristics of this space are product type $\mathbb{R}^4 \times Y_6$ we have that the metric has not, as we saw, a product structure, even if this holds however, the prefactor of 6D compact part of the manifold we have contains no dependence on the 4D coordinates, so we can easily understand how Poincarè invariance is not broken. We call the prefactor $\Omega(y)$ as the **warp factor**.

As we previously anticipated the 6D compact manifold features a strongly warped region, very usual in the Calabi-Yaus, which is called **Klebanov-Strassler throat** and we can find that these compact throated 6D manifolds are not properly a Calabi-Yau, instead are conformally Calabi-Yaus, in the sense that a change by the conformal factor (which is the warp factor) of the metric does not change the Physical outcome of the theory:

$$g_{mn}(y) = \Omega^{-2}\hat{g}_{mn}(y) \tag{2.115}$$

The warping, which seems just a Geometrial feature, in realty hides crucial physical consequencies like the energy effect of the anti-D3-brane ($\overline{\mathbf{D3}}-\mathbf{brane}$) in the C.Y. In fact, since this geometrical shape, the brane is pulled to the throat where $\Omega << 1$ and so the warping is strong and from the point of view of the unwarped part of the C.Y. its energy content is strongly "redshifted", leading to an uplift (in string units) of:

$$\Omega_{\mathrm{M}}^4 \times \mathcal{O}(1) \tag{2.116}$$

where, from [27]:

$$\Omega_M \sim e^{-\frac{2\pi n}{3mg_s}} \tag{2.117}$$

with $n, m \in \mathbb{Z}$ coming from (2.86) for the 3-fluxes in the throat.

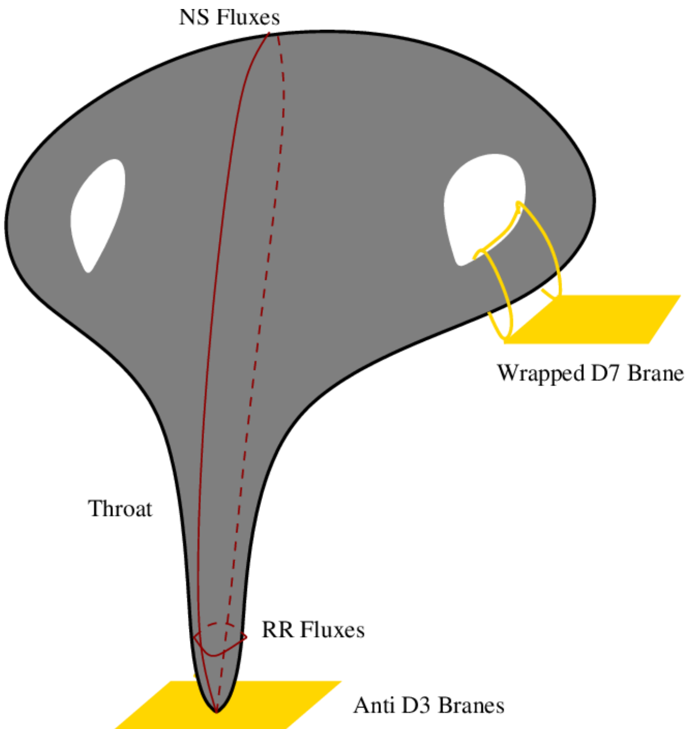

Figure 2.5: Klebanov-Strassler Throat with anti-D3-brane sitting on its tip.

We can prove that the metastability of the uplifted configuration is plausible in the case of no flux backreaction, so we will assume it as a starting point for the SUSY breaking and we want to estimate the magnitude of this latter. Using arguments of dimensional analysis and of physical quantities ratios in different frames [30] it can be proven that naively, the warping suppression is of the order of $\Omega_{\mathrm{M}}\tau$ which leads to a potential term (where we need to multiply by e^K) of:

$$V_{\mathrm{up}}(\tau) = c_{\mathrm{up}}\frac{\Omega_{\mathrm{M}}^4}{\tau^2} \simeq c_{\mathrm{up}}\frac{\Omega_{\mathrm{M}}^4}{\mathcal{V}^{4/3}} \tag{2.118}$$

with $c_{\mathrm{up}} = \mathbf{const} \sim \mathcal{O}(1)$ and $\tau \simeq \mathcal{V}^{2/3}$ in the case of single modulus, with such a scaling the local AdS minimum present in a configuration like KKLT (or LVS) one can become Minkowski or de Sitter (dS) remaining metastable.
In order to conclude in a complete way, every general uplift mechanism, as stated in [23] gives a scalar potential term scaling as:

$$V_{\mathrm{up}} \sim \frac{1}{\mathcal{V}^{\alpha}} \tag{2.119}$$

but in general $\frac{4}{3} \leq \alpha \leq 2$.

Chapter 3

Loop Blow-Up Inflation

In this chapter we are going to use all the background we studied in the previous pages in order to finally build some explicit models of inflation working with a peculiar classes of Calabi-Yaus called **Swiss-Cheese Calabi-Yau Manifolds**. Such manifolds are manifolds where the overall volume can be written in the form:

$$\mathcal{V} = \alpha\tau_b^{3/2} - p_{(3/2)}(\tau_s^r) \tag{3.1}$$

where $\alpha > 0$ and $p_{(3/2)}(\tau_r^s)$ is an homogeneous polynomial of degree $\frac{3}{2}$ in $\tau_r^s, \quad r = 1, \ldots, N_s$. In particular we are going to work with subset of such manifolds in which divisors Σ_s^i are not only del Pezzo but diagonal del Pezzo[1]. These subclass of models feature a volume Mathematically written then as:

$$\mathcal{V} = \alpha\left(\tau_b^{3/2} - \sum_{r=1}^{N_s}\lambda_r(\tau_r^s)^{3/2}\right) \tag{3.2}$$

with α and λ_r topological parameters (in particular they represents ratios of intersection numbers [2]).

[1]Their intersection numbers satisfy $k_{lll}k_{lij} = k_{lli}k_{llj}$ so that it is possible to find a basis of coordinate divisors such that the volume of each 4-cycle is a complete square: $\tau_i = \frac{1}{2}k_{ijl}\tau^j\tau^l = \frac{1}{2k_{iii}}k_{iij}k_{iil}t^jt^l = \frac{1}{k_{iii}}(k_{iij}t^j)^2$ and so the 4-cycle which is a diagonal del Pezzo ddP_n has a volume $\tau_{\rm ddP}$ entering the overall one $\mathcal{V}$ just as a pure power, without any mixing with other 4-cycle volumes.

Now we can easily understand why these manifolds have such a name, in fact their volume shape is made in such a way that τ_b is like the overall "full" cheese volume and τ_r^s like small holes on this Swiss-cheese.

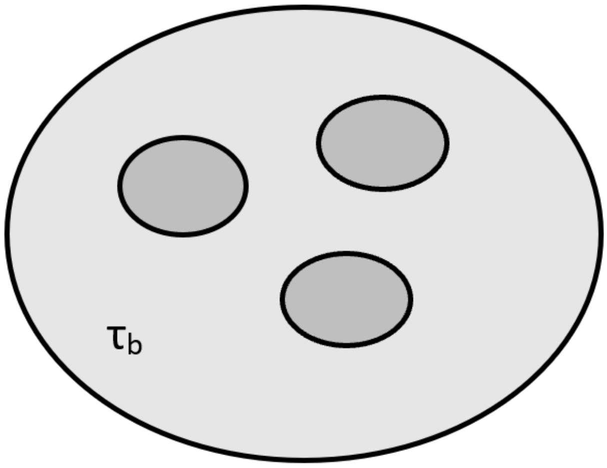

Figure 3.1: Pictorial Representation of a Swiss-Cheese Model Calabi-Yau with 1 big cycle τ_b and 3 small cycles.

First, we give some insights of the baseline model developed by J. Conlon and F. Quevedo which we shall call **Non-perturbative Blow-Up Inflation** [23]. Then we enrich it with string loop (g_s) corrections to develop the model in which we are going to work during the last chapter: **Loop Blow-Up inflation** [2].

3.1 Non-perturbative blow-up inflation

The first Kahler moduli inflationary model is called Blow-Up Inflation and it needs a Calabi-Yau with $h^{2,1} > h^{1,1} > 2$ and features a structure of the scalar potential with exponentially large volume compactifications. At least one Kähler Modulus will be stabilised non-perturbatively and it is a diagonal del Pezzo. Such a model evades even the η problem and matches, upon right fine tuning of the volume, PLANCK measurements on density perturbations. We are going to analize it before in a simple case with the minimum of $h^{1,1} = 3$, then in the general case of

$n > 3$ Kähler moduli.

3.1.1 $h^{1,1} = 3$ case

We start by considering 3 Kähler moduli case, with one big cycle whose volume is fixed by the vacuum expectation value of τ_b and 2 small ones whose volume will work as inflationary τ_ϕ and small overall volume-fixing cycle τ_s.

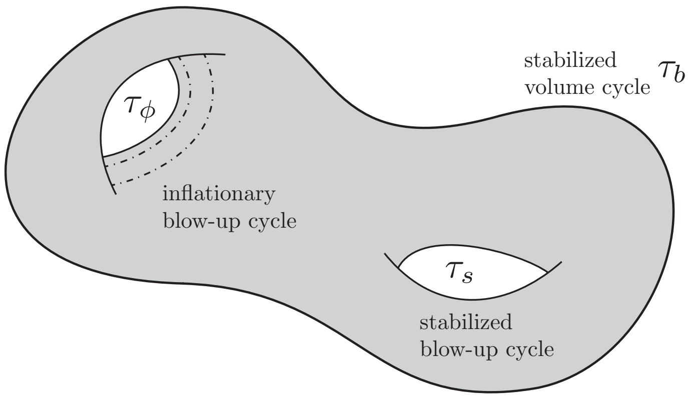

Figure 3.2: Pictorial Representation of a Swiss-Cheese Model Calabi-Yau with 1 big cycle τ_b and 2 small blow ups, with one inflationary cycle τ_ϕ.

We will wrap the two small cycles with brane stacks going under gaugino condensation or by instantons so the model features, after fixing the complex structure moduli and the axio-dilaton at their vacuum expectation value, a Kähler potential and a Superpotential of the shape:

$$K = -2\ln\left(\mathcal{V} + \frac{\hat{\xi}}{2}\right) \tag{3.3}$$

$$W = W_0 + A_\phi e^{-a_\phi T_\phi} + A_s e^{-a_s T_s} \tag{3.4}$$

The overall volume is of the Swiss-Cheese form as:

$$\mathcal{V} = \tau_b^{3/2} - \lambda_\phi \tau_\phi^{3/2} - \lambda_s \tau_s^{3/2} \tag{3.5}$$

where λ_ϕ, λ_s topological constants depending on intersection numbers of the correspondent cycles. Starting from This, the inverse Kähler Metric looks like:

$$(K^{-1})^{i\bar{j}} \approx \begin{pmatrix} \frac{4\tau_b^2}{3} & 4\tau_b\tau_\phi & 4\tau_b\lambda_{\rm s} \\ 4\tau_b\tau_\phi & \frac{8\sqrt{\tau_\phi}\tau_b^{3/2}}{3\lambda_\phi} & 4\tau_\phi\lambda_{\rm s} \\ 4\tau_b\lambda_{\rm s} & 4\tau_\phi\lambda_{\rm s} & \frac{8\sqrt{\lambda_{\rm s}}\tau_b^{3/2}}{3\lambda_{\rm s}} \end{pmatrix}, \tag{3.6}$$

and then we can compute the scalar potential (after neglecting subleading term and after substituting in the moduli base the big cycle volume with the overall volume $\{\tau_b, \tau_\phi, \tau_s\} \rightarrow \{\mathcal{V}, \tau_\phi, \tau_s\}$):

$$\begin{aligned} V(\mathcal{V}, \tau_\phi, \tau_s, \theta_\phi, \theta_s) =& \frac{g_s e^{K_{CS}}}{8\pi} W_0^2 \Bigg(\frac{8a_s^2 A_s^2}{W_0^2 \lambda_s} \frac{\sqrt{\tau_s} e^{-2a_s\tau_s}}{\mathcal{V}} + \frac{4a_s A_s}{W_0} \frac{\tau_s e^{-a_s\tau_s}}{\mathcal{V}^2} \cos(a_s\theta_s) \\ &+ \frac{8a_\phi^2 A_\phi^2}{W_0^2 \lambda_\phi} \frac{\sqrt{\tau_\phi} e^{-2a_\phi\tau_\phi}}{\mathcal{V}} + \frac{4a_\phi A_\phi}{W_0} \frac{\tau_\phi e^{-a_\phi\tau_\phi}}{\mathcal{V}^2} \cos(a_\phi\theta_\phi) + \frac{3\hat{\xi}}{4\mathcal{V}^3} \Bigg) \end{aligned} \tag{3.7}$$

and, after stabilising the axions θ_ϕ and θ_s:

$$\begin{aligned} V(\mathcal{V}, \tau_\phi, \tau_s) =& \frac{g_s e^{K_{CS}}}{8\pi} W_0^2 \Bigg(\frac{8a_s^2 A_s^2}{W_0^2 \lambda_s} \frac{\sqrt{\tau_s} e^{-2a_s\tau_s}}{\mathcal{V}} - \frac{4a_s A_s}{W_0} \frac{\tau_s e^{-a_s\tau_s}}{\mathcal{V}^2} \\ &+ \frac{8a_\phi^2 A_\phi^2}{W_0^2 \lambda_\phi} \frac{\sqrt{\tau_\phi} e^{-2a_\phi\tau_\phi}}{\mathcal{V}} - \frac{4a_\phi A_\phi}{W_0} \frac{\tau_\phi e^{-a_\phi\tau_\phi}}{\mathcal{V}^2} + \frac{3\hat{\xi}}{4\mathcal{V}^3} \Bigg) \end{aligned} \tag{3.8}$$

Now, if we would like to stabilise the volume we need to find the minimum of the potential via partial derivatives. This minimum is clearly non-Supersymmetric since we are working in LVS scenario and in particular with $(\alpha')^3$ corrections:

$$\frac{\partial V}{\partial \mathcal{V}} = 0 \tag{3.9}$$

$$\frac{\partial V}{\partial \tau_s} = 0 \tag{3.10}$$

Leading us to the stabilisation of $\mathcal{V}$:

$$\mathcal{V} = \frac{3\alpha\lambda_s W_0}{a_s A_s} \frac{(1 - a_s\tau_s)}{(1 - 4a_s\tau_s)} \sqrt{\tau_s} e^{a_s\tau_s} \tag{3.11}$$

which, by plugging it in (3.9), gives us:

$$\langle \tau_s \rangle \simeq \left(\frac{\hat{\xi}}{2\lambda_s} \right)^{2/3} \simeq \left(\frac{\xi}{g_s^{3/2}} \right)^{2/3} \tag{3.12}$$

and so:

$$\langle \mathcal{V} \rangle \sim e^{\left(\frac{\xi^{2/3}}{g_s} \right)} \tag{3.13}$$

which means that in the perturbative regime of the theory the overall Calabi-Yau volume is fixed at exponentially large value $g_s \ll 1 \Rightarrow \mathcal{V} \gg 1$. By plugging the vacuum expectation values of the volume and of τ_s in the potential, by minimising it with the respect to τ_ϕ and plugging again this other value, it is possible to show that [23]:

$$\langle V \rangle = \frac{-3W_0^2}{2\langle \mathcal{V} \rangle^3} \left(\left(\frac{\lambda_s}{a_s^{3/2}} + \frac{\lambda_\phi}{a_\phi^{3/2}} \right) (\ln(\mathcal{V}))^{3/2} - \frac{\hat{\xi}}{2} \right) < 0 \tag{3.14}$$

so the minumum is AdS for exponentially large values of the volume.

In order to go to the inflationary regime we need to set the value of $\tau_\phi \gg 1$ so that we are in the plateu of the potential. In such a regime, after stabilising all the other moduli, the double exponential is suppressed and so the only remaining part is:

$$V_{\text{inf}} = V_0 \left(1 - \frac{16 a_\phi A_\phi}{W_0 \hat{\xi}} \tau_\phi \mathcal{V} e^{-a_\phi \tau_\phi} \right) \tag{3.15}$$

with $V_0 = \frac{3W_0^2 \hat{\xi}}{\mathcal{V}^3} = \text{const.}$
Next step is clearly to write the potential in the canonically normalised version and, in order to do this we first need to compute the canonical normalisation. We can do so in 2 different ways, by solving the differential equation:

$$\partial_\mu \phi \partial^\mu \phi = K_{\phi\bar{\phi}} \partial_\mu \tau_\phi \partial^\mu \tau_\phi \tag{3.16}$$

or finding the mass matrix eigenvectors (which is completely equivalent) as done in D, with the difference that what has been computed in D is a pure linear-algebraic result, so in the case of the inflaton its power scale is different due to the need of integration owing to the τ_ϕ dependence of $K_{\phi\phi}$. The canonical normalisation which comes out from this equation (3.16) is:

$$\phi = \sqrt{\frac{4\lambda_\phi}{3\mathcal{V}}}\tau_\phi^{3/4} \tag{3.17}$$

with mass

$$m_\phi \simeq \frac{W_0 \ln(\mathcal{V})}{\mathcal{V}} \tag{3.18}$$

and which leads to a canonically normalised potential of the shape:

$$V_{\text{can.inf.}} = V_0 \left(1 - \frac{16 a_\phi A_\phi}{W_0 \hat{\xi}} \left(\frac{3\mathcal{V}}{4\lambda_\phi}\right)^{2/3} \phi^{4/3} \mathcal{V} e^{-a_\phi \left(\frac{3\mathcal{V}}{4\lambda_\phi}\right)^{2/3} \phi^{4/3}}\right) \tag{3.19}$$

close to textbook potential $V = V_0(1 - e^{-\tau})$.

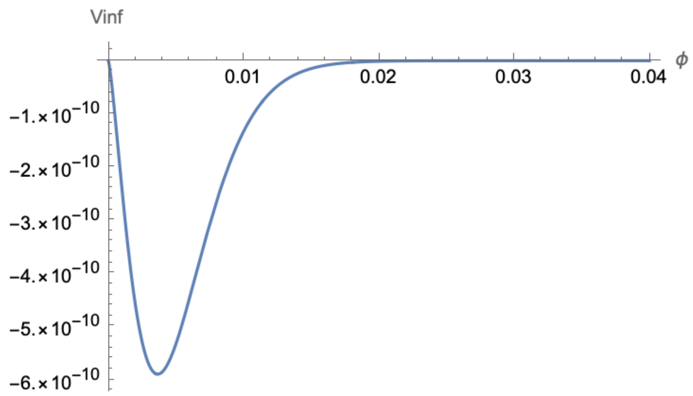

Figure 3.3: Picture of canonically normalised inflaton potential setting $a_\phi = A_\phi = \lambda_\phi = W_0 = \hat{\xi} = 1$ and $\mathcal{V} = 10^5$ with AdS minimum.

3.1.2 General case

We now give some details on the general case of Blow-Up inflation where $h^{1,1} = n \geq 3$ following the original paper [23]. We will notice how physics doesn't change so much from the $h^{1,1} = 3$ case. The Calabi-Yau Swiss-Cheese model volume reads as:

$$\mathcal{V} = \tau_b^{3/2} - \sum_{i=2}^{n} \lambda_i \tau_i^{3/2} = \frac{1}{2\sqrt{2}} \left[(T_b + \bar{T}_b)^{3/2} - \sum_{i=2}^{n} \lambda_i (T_i + \bar{T}_i)^{3/2}\right] \tag{3.20}$$

with τ_b big cycle volume controlling the overall volume, $\tau_2, \ldots, \tau_n$ n-1 small blow-up cycles (diagonal del Pezzo divisors) volumes and $\lambda_i = \text{const} > 0$ topological constant. Again the dilaton and the complex structure moduli are stabilised via Gukov-Vafa-Witten Potential due to fluxes and the blow-up cycles through non-perturbative corrections so the superpotential and the Kähler potential (featuring $(\alpha')^3$ corrections) are written as a generalisation of (3.3), (3.4) as:

$$K = K_{\text{C.S.}} - 2\ln\left(\tau_b^{3/2} - \sum_{i=2}^{n} \lambda_i \tau_i^{3/2} + \frac{\hat{\xi}}{2}\right) \tag{3.21}$$

Working in LVS scenario 2.5.4 we need $\hat{\xi} > 0 \Rightarrow \chi(Y_6) < 0 \Rightarrow h^{2,1} > h^{1,1}$. The scalar potential will read then as:

$$V = e^K (K^{i\bar{j}} \partial_i W \partial_{\bar{j}} \bar{W} + K^{i\bar{j}}((K_i)W)\partial_{\bar{j}}\bar{W} + h.c.) + \frac{3\hat{\xi}W_0^2}{4\mathcal{V}^3} \tag{3.22}$$

where:

$$K^{i\bar{j}} \simeq \frac{8\mathcal{V}\sqrt{\tau_i}}{3\lambda_i}\delta_{ij} + \mathcal{O}(\tau_i\tau_j) \in \mathbb{R}^{(h^{1,1}-1\times h^{1,1}-1)} \tag{3.23}$$

The inverse Kähler metric satisfies, up to subleading terms in $\mathcal{V}$ $K^{i\bar{j}}K_{\bar{j}} = 2\tau_i$, by plugging so in (3.22) we get at leading order:

$$V = \sum_{i=2}^{h^{1,1}} \frac{8(a_iA_i)^2\sqrt{\tau_i}}{3\mathcal{V}\lambda_i} e^{-2a_i\tau_i} - \sum_{i=2}^{h^{1,1}} \frac{4a_iA_iW_0\tau_i}{\mathcal{V}^2} e^{-a_i\tau_i} + \frac{3\hat{\xi}W_0^2}{4\mathcal{V}^3} \tag{3.24}$$

Such that at large τ_i the exponentials get suppressed and so potential for such a modulus features a plateau. By extremising in a generalisation of what done in D we can get, at fixed $\mathcal{V}$:

$$(a_iA_i)e^{-a_i\tau_i} = \frac{3\lambda_i W_0}{\mathcal{V}} \frac{(1 - a_i\tau_i)}{1 - 4a_i\tau_i} \sqrt{\tau_i} \tag{3.25}$$

Taking large volume limit then $a_i\tau_i \sim \ln(\mathcal{V}) \gg 1$ then

$$(a_iA_i)e^{-a_i\tau_i} = \frac{3\lambda_i W_0}{4\mathcal{V}} \sqrt{\tau_i} \tag{3.26}$$

which, plugged inside (3.24), after stabilised all the Fields, it gives:

$$V = -\frac{3W_0^2}{2\mathcal{V}^3}\left(\sum_{i=2}^{n}\left(\frac{\lambda_i}{a_i^{3/2}}\right) - \frac{\hat{\xi}}{2}\right) \tag{3.27}$$

whose vacuum expectation value can be in general, after stabilising even the overall volume, negative, so an uplift term like the anti-D3-brane one is needed again like shown in $h^{1,1} = 3$ case.

Now, to obtain inflation we pick one of the blow-up cycle volumes τ_n and displace far from its minimum. Doing so, considering the potential for the overall volume (3.27) it becomes:

$$V = -\frac{3W_0^2}{2\mathcal{V}^3}\left(\sum_{i=2}^{n-1}\left(\frac{\lambda_i}{a_i^{3/2}}\right) - \frac{\hat{\xi}}{2}\right) + V_{\text{up}} \tag{3.28}$$

Doing so we neglected the inflaton contribute in order to have the volume modulus stable during inflation, in the sense that the evolution of the volume modulus must not depend on the inflaton roll-down, so we clearly need:

$$\rho \coloneqq \frac{\frac{\lambda_n}{a_n^{3/2}}}{\sum_{i=2}^{n}\frac{\lambda_i}{a_i^{3/2}}} < 1 \tag{3.29}$$

So, at least there are needed 3 moduli $\Rightarrow h^{1,1} \geq 3$ so the conditions for a stable volume at exponentially large values and for having a minimum can be resumed in the previously seen $h^{1,2} > h^{1,1} > 2$.

When we drag the inflaton far from its minimum we get that, considering only the terms of the potential depending on it, the second order exponential term is negligible leading to a Scalar Potential which is exactly (3.15) (and (3.19) if inflaton canonically normalised) just with $\tau_n = \tau_\phi$. From such a potential, reinstating $M_p \neq 1$, calling $\delta = \frac{3\hat{\xi}}{4}$ we can derive the slow-roll parameters as in [23] (where

$V_\phi = \frac{dV(\phi)}{d\phi}$):

$$\epsilon = \frac{M_p^2}{2}\left(\frac{V_\phi}{V}\right)^2 = \frac{32\mathcal{V}^3}{3\delta^2 W_0^2} a_n^2 A_n^2 \sqrt{\tau_n}(1-a_n\tau_n)^2 e^{-2a_n\tau_n} \tag{3.30}$$

$$\eta = M_p^2 \frac{V_{\phi\phi}}{V} = -\frac{4a_n A_n \mathcal{V}^2}{2\lambda_n\sqrt{\tau_n}\delta W_0}\left[(1-9a_n\tau_n + 4(a_n\tau_n)^2)e^{-a_n\tau_n}\right] \tag{3.31}$$

which for $\tau_n \gg 1 \Rightarrow \epsilon, \eta \ll 1$ as it correctly should be. Reinstating $M_p = 1$, in the slow-roll approximation the spectral index n_s is given by:

$$n_s - 1 \simeq 2\eta - 6\epsilon \tag{3.32}$$

the tensor-to-scalar ratio is $r \simeq 12.4\epsilon$ and the Numer of E-foldings is given by:

$$N_e = \int_{\phi_{\rm end}}^{\phi} \frac{V}{V_\phi} d\phi = \frac{-3\beta W_0 \lambda_n}{16\mathcal{V}^2 a_n A_n} \int_{\tau_n^{\epsilon\sim 1}}^{\tau_n} \frac{e^{a_n\tau_n}}{\sqrt{\tau_n}(1-a_n\tau_n)} d\tau_n \tag{3.33}$$

Requiring the amplitude of power spectrum for scalar perturbations to match COBE, reinstating again $M_p \neq 1$ we require:

$$\left.\frac{V^{3/2}}{M_p^3 V_\phi}\right|_{N_e=50/60} = 5.2\cdot 10^{-4} \Rightarrow \left(\frac{V}{\epsilon}\right) = 6.6\cdot 10^{16} GeV \tag{3.34}$$

We can solve this last COBE normalisation condition numerically, fixing proper parametrical values, in order to get a range of values for the volume:

$$10^5 l_s^6 \leq \mathcal{V} \leq 10^7 l_s^6 \tag{3.35}$$

with $l_s = (2\pi)\sqrt{\alpha'}$, which can be easily obtained in LVS since exponentially big values of the volume. By using instead all the previous equations, with a range of e-foldings 50-60 we get:

$$0.960 < n < 0.967 \tag{3.36}$$

$$0 < |r| < 10^{-10} \tag{3.37}$$

$$10^{-15} \leq \epsilon \leq 10^{-13} \tag{3.38}$$

$$V_{\rm inf} \sim 10^{13} GeV \tag{3.39}$$

Giving us unobservable gravitational waves in this model. Another important feature of this model is that, since the exponential flatness of the inflaton potential we can get a very large number of e-foldings since $\Delta\phi$ small $\Leftrightarrow \Delta N_e$ big so $N_{\rm e,total} \gg 60$ without a practical upper limit. Moreover, as stated in [25] the lightest (excluding axions) modulus has a mass going as:

$$M \sim \frac{M_p}{\mathcal{V}^{3/2}} \simeq 10^{11} GeV \tag{3.40}$$

which leads us to understand that the cosmological moduli problem[2] is avoided in this case. In addition to this, since we assume that the inflaton is a Kähler modulus rolling down to minimum while the other moduli sit at their vacuum expectation value we have no interference of the latter during inflation, leading to a single field model in this case. In the end the model gives no hints on the cosmological overshoot problem[3].

[2]The Cosmological Moduli problem holds when very light moduli with masses of order $\mathcal{O}(10)$ TeV are present since they will not decay before Big Bang Nucleosynthesis and during it they can decay or in photons which, through photo-dissociation, destroy light nuclei giving the wrong abundance we observe today, or in gravitini which decay in particles destroying light elements. In addition to this, these decays lead to an entropy increase that can destroy baryon-antibaryon asymmetry.

[3]The Cosmological Overshoot Problem is a problem, strongly dependent on initial conditions given by the Universe, which appears if, after the inflaton falls in the well of its potential, it has enough energy to escape it instead of starting oscillating, giving rise to a runaway to the decompactification limit.

3.2 Loop blow-up inflation

Now we study the proper model in which we are going to work in the last chapter. This model starts from the base of previously inspected Blow-Up Inflation, adding however String Loop Correction of the shape (2.101). These corrections in [23] where thought to be negligible, however it appears that they will offer us another inflationary regime even if they are subleading in volume. Discarding the small 4-cycle volumes and treating the overall volume as fixed at exponentially large values a general potential including loop corrections can be written as:

$$V \sim \frac{|W_0|^2}{\mathcal{V}^3}\left(\mathcal{O}(1) - \frac{c_{\rm loop}}{\mathcal{V}^{1/3}} f\left(\frac{\mathcal{V}^{2/3}}{\tau_\phi}\right)\right) \tag{3.41}$$

with f generic function of $\frac{\mathcal{V}^{2/3}}{\tau_\phi}$ and $c_{\rm loop}$ as in (2.101). The constant term $\frac{|W_0|^2}{\mathcal{V}^3}\mathcal{O}(1)$ is the value of the potential of the inflationary plateau. In addition to this, it is possible to show that:

$$\epsilon \equiv \frac{1}{2}\left(\frac{V'(\phi)}{V(\phi)}\right) \simeq \left(\frac{c_{\rm loop}}{\mathcal{V}^{1/3}}\frac{df}{d\phi}\right)^2 \tag{3.42}$$

$$\eta \equiv \frac{V''(\phi)}{V(\phi)} \simeq \frac{c_{\rm loop}}{\mathcal{V}^{1/3}}\frac{d^2 f}{d\phi^2} \tag{3.43}$$

where again $V'(\phi), V''(\phi)$ are derivatives with the respect to ϕ. When LVS inflation condition $\tau_\phi \lesssim \mathcal{V}^{2/3}$ is satisfied then $f'' \sim f' \sim \mathcal{O}(1)$ and so we have slow-roll condition satisfied for $\mathcal{V} \gg 1$. Even adding perturbative corrections depending on the other small moduli we get the inflationary plateau no spoiled by this contributions even if, in order to study the single field dynamics as we will do in this chapter, once again we need to stabilise all the moduli at their minimum except the inflaton.

In order to check that our model is truly single field and so that the volume remains fixed during inflation we need that the corrections that stabilise τ_ϕ stay subdominant with the respect to the leading order potential even if $\tau_\phi \simeq \langle\tau_\phi\rangle$ and in the 2 cases of non-perturbative and string loop stabilisation can happen for 4 reasons:

- In the case of non-perturbative stabilisation of τ_ϕ:
  - -) Leading order instanton contribution vanish due to chiral intersection;
  - -) Gaugino condensation contribution are suppressed because of N_ϕ rank of gauge group on branes wrapping inflaton cycle is much smaller than the ranks of the other sectors that wrap other blow-up cycles $N_\phi \ll N_s$ so that the exponential suppression in the τ_ϕ term is much bigger.
  - -) The Calabi Yau in which we are working on features lots of small cycles $\tau_s \neq \tau_\phi$.
- In the case of string loop stabilisation of τ_ϕ the negligibility of the τ_ϕ terms in the potential is due to the extended no-scale cancellation shown in B.

3.2.1 Inflationary potential

We are now going to introduce a practical easy example of what we have stated in the introduction of this section. This will lead us to the first explicit model of Loop induced Inflation. The general idea is to add loop corrections to [23] and move to large values of τ_ϕ such that the inflationary regime still holds even if the loop corrections could have broken it.

We are going to take a very simple class of Calabi-Yau manifolds where the overall volume can be written as Swiss-Cheese shape with 3 moduli:

$$\mathcal{V} = \tau_b^{3/2} - \lambda_\phi \tau_\phi^{3/2} - \lambda_s \tau_s^{3/2} \tag{3.44}$$

with $\lambda_i,\ i = s, \phi$ topological constants representing, as always, ratios of triple intersection numbers. We assume to include in the Kähler potential of the model both $(\alpha')^3$ corrections and string loop corrections in addition to the non-perturbative corrections to the superpotential, leading us to the quantities:

$$K = K_{\text{C.S.}} - 2\ln\left(\mathcal{V} + \frac{\hat{\xi}}{2}\right) + \delta K_{\text{loop}} \tag{3.45}$$

$$W = W_0 + A_s e^{-a_s T_s} + A_\phi e^{-a_\phi T_\phi} \tag{3.46}$$

with $a_i = 2\pi$ or $a_i = \frac{2\pi}{N_i}$, $\quad i = \phi, s$ depending on if wee have E3-branes or hidden sector undergoing gaugino condensation wrapping the respective cycle.
The scalar potential for this model will be then:

$$V(\mathcal{V}, \tau_\phi, \tau_s, \theta_\phi, \theta_s) = \frac{g_s e^{K_{CS}}}{8\pi} W_0^2 \left(\frac{8a_s^2 A_s^2}{W_0^2 \lambda_s} \frac{\sqrt{\tau_s} e^{-2a_s\tau_s}}{\mathcal{V}} + \frac{4a_s A_s}{W_0} \frac{\tau_s e^{-a_s\tau_s}}{\mathcal{V}^2} \cos(a_s\theta_s) \right. \\ \left. + \frac{8a_\phi^2 A_\phi^2}{W_0^2 \lambda_\phi} \frac{\sqrt{\tau_\phi} e^{-2a_\phi\tau_\phi}}{\mathcal{V}} + \frac{4a_\phi A_\phi}{W_0} \frac{\tau_\phi e^{-a_\phi\tau_\phi}}{\mathcal{V}^2} \cos(a_\phi\theta_\phi) + \frac{3\hat{\xi}}{4\mathcal{V}^3} - \frac{c_{\text{loop}}}{\sqrt{\tau_\phi}} \right) + V_{\text{up}} \tag{3.47}$$

where we have assumed that τ_ϕ is smaller than other nearby cycles in order to have a loop correction depending only on the overall volume and on it. In addition to this, $V_{\text{up}} = \frac{g_s e^{K_{\text{C.S.}}}}{8\pi} \frac{c_{\text{up}} W_0^2}{\mathcal{V}^2}$ is such that at minimum the vacuum is Minkowski and can come from whatever uplift mechanism. In order to be complete, the correct general form of the loop correction should be $\delta V_{\text{loop}} \simeq -\frac{g_s e^{K_{\text{C.S.}}}}{8\pi} \frac{W_0^2}{\mathcal{V}^3} \frac{c_{\text{loop}}}{\mathcal{V}^{1/3}} f\left(\frac{\mathcal{V}^{2/3}}{\tau_\phi}\right)$ with f unknown function which we can took as well approximated, in the limit of not too big τ_ϕ, as $f \simeq \frac{\mathcal{V}^{1/3}}{\sqrt{\tau_\phi}}$.
The factor c_{loop} includes then every $\mathcal{O}(1)$ factor in this function f and it is as seen in the previous chapter of string loop corrections 2.5.2 of the order of $c_{\text{loop}} \simeq \frac{1}{(2\pi)^4}$ as computed in [7] for toroidal orbifolds or $c_{\text{loop}} \simeq \frac{1}{16\pi^2}$ identifying the cutoff as Kaluza-Klein scale $\Lambda_{UV} \simeq \frac{M_p}{\tau_\phi^{1/4}\sqrt{\mathcal{V}}}$.
Another very important thing is that, in [23] it was thought that loop corrections were avoidable however, upon a deeper study, these appear to be necessary. In fact, in order to have a minimum in the potential for stabilising τ_ϕ after the inflation superpotential non-perturbative corrections are needed. These non-perturbative loop corrections come from E3-branes or brane stacks under gaugino condensation as we know well, so, in order to cancel tadpoles, we need an O-plane close to the inflationary cycle which breaks SUSY to $N = 1$. Such a configuration, as stated in [26] necessarily features loop corrections, howerver, if the inflaton cycle is not wrapped by any D7-brane we have no open string correction but closed ones are unavoidable. Choosing as in Blow-Up case, the C.Y. data such that the inflaton potential is so negligible to make the dynamics of the overall volume and of the small cycle decouple ($\lambda_\phi a_\phi^{-3/2} \ll \lambda_s a_s{-}3/2$) we can make τ_ϕ roll while $\mathcal{V}, \tau_s$ are

stabilised at minimum. After stabilising even the axions then we can write the potential, depending only on the inflaton value as:

$$V(\tau_\phi) = V_0 \left(1 + \frac{8a_\phi^2 A_\phi^2}{3W_0^2 \lambda_\phi} \frac{\mathcal{V}^2}{\beta} \sqrt{\tau_\phi} e^{-2a_\phi \tau_\phi} - \frac{4a_\phi A_\phi}{W_0} \frac{\mathcal{V}}{\beta} \tau_\phi e^{-a_\phi \tau_\phi} - \frac{c_{\text{loop}}}{\beta \sqrt{\tau_\phi}} \right) \quad (3.48)$$

$$\begin{aligned} V_0 &:= \left(\frac{g_s e^{K_{CS}}}{8\pi} W_0^2 \left(\frac{8a_s^2 A_s^2}{W_0^2 \lambda_s} \frac{\sqrt{\tau_s} e^{-2a_s \tau_s}}{\mathcal{V}} + \frac{4a_s A_s}{W_0} \frac{\tau_s e^{-a_s \tau_s}}{\mathcal{V}^2} \cos(a_s \theta_s) + \frac{3\hat{\xi}}{4\mathcal{V}^3} \right) + V_{\text{up}} \right) \Bigg|_{\mathcal{V}=\langle \mathcal{V} \rangle, \tau_s = \langle \tau_s \rangle} = \\ &= \frac{g_s e^{K_{CS}}}{8\pi} \frac{W_0^2}{\mathcal{V}^3} \beta = \frac{g_s e^{K_{CS}}}{8\pi} \frac{W_0^2}{\mathcal{V}^3} \frac{3}{2} a_\phi^{-3/2} \lambda_\phi (\ln(\mathcal{V}))^{3/2} \end{aligned} \quad (3.49)$$

$$\beta \simeq \frac{3}{2} a_\phi^{-3/2} \lambda_\phi (\ln(\mathcal{V}))^{3/2} \quad (3.50)$$

Since the Kähler potential is the same, it can be proven that the canonical normalisation is the same, at leading order at least, that the one in Blow-Up inflationary regime as shown in [2]. Using then the canonical normalisation (3.17) we can rewrite the potential in terms of the canonically normalised inflaton and we can easily see that it has 3 inflationary regimes where slow-roll conditions are satisfied. We will see later on that for $c_{\text{loop}} \gtrsim 10^{-6}$ the inflationary regime is at much larger ϕ, in particular in a regime where $\tau_s \ll \tau_\phi < \tau_n$ such that we can neglect the exponentials in the inflaton potential (3.48) and obtain:

$$V(\phi) = V_0 \left(1 - \frac{1}{\beta} \left(\frac{4\lambda_\phi}{3\mathcal{V}} \right)^{1/3} \frac{c_{\text{loop}}}{\phi^{2/3}} \right) = V_0 \left(1 - \frac{b c_{\text{loop}}}{\phi^{2/3}} \right) \quad (3.51)$$

with:

$$b \equiv \frac{1}{\beta} \left(\frac{4\lambda_\phi}{3\mathcal{V}} \right)^{1/3} \quad (3.52)$$

since now on, we assume that this potential can be used for inflation.

3.2.2 Parameters and observational constraints

Inflationary parameters

Given the potential (3.51), we can assume the regime we got if and only if:

- c_{loop} not too small;
- $\tau_\phi \lesssim \mathcal{V}^{2/3} \Leftrightarrow \phi \lesssim 1$

such that the exponentials are negligible and inflation happens inside the Kähler cone. In addition to this the fact that τ_ϕ is far from the walls of Kähler cone during the $N_e \simeq 52$ e-folding of inflation implies even that c_{loop} is constrained from above. We can compute from (3.51) the slow-roll parameters and the spectral index n_s, the tensor-to-scalar ratio r and the number of e-foldings N_e:

$$\epsilon = \frac{1}{2}\left(\frac{V_\phi}{V}\right)^2 \simeq \frac{2}{9}\frac{(bc_{\text{loop}})^2}{\phi^{10/3}} \overset{bc_{\text{loop}}\ll 1}{\ll} 1 \tag{3.53}$$

$$\eta = \frac{V_{\phi\phi}}{V} \simeq -\frac{10}{9}\frac{bc_{\text{loop}}}{\phi^{8/3}} \overset{bc_{\text{loop}}\ll 1}{\ll} \tag{3.54}$$

$$n_s - 1 = 2\eta - 6\epsilon \simeq 1 - \frac{20}{9}\frac{bc_{\text{loop}}}{\phi_*^{8/3}} \tag{3.55}$$

$$r = 16\epsilon \simeq \frac{32}{9}\frac{(bc_{\text{loop}})^2}{\phi_*^{10/3}} \tag{3.56}$$

$$N_e = \int_{\phi_{\text{end}}}^{\phi_*} \frac{V}{V_\phi} d\phi \simeq \frac{9}{16}\frac{\phi_*^{8/3}}{bc_{\text{loop}}} \tag{3.57}$$

with ϕ_* is the value of inflaton field at horizon exit and $\phi_{\text{end}} \ll \phi_*$ at the end of inflation when slow-roll regime is broken $\epsilon \sim \mathcal{O}(1)$.

Observational constraints

What we would like to do now is to try to match all the cosmological bounds we have ensuring to get the right amount of e-foldings and the scalar perturbations amplitude while keeping $\phi_* \lesssim 1$ and while getting a volume big enough to get the LVS regime. We recall that the spectrum of the scalar density perturbation is given by [4]:

$$\Delta_s^2 = P_s \left(\frac{k}{k_*}\right)^{n_s - 1} \tag{3.58}$$

and where the amplitude bound given by Planck is [1]:

$$P_s = 2.105 \pm 0.030 \cdot 10^{-9} \tag{3.59}$$

Recalling ϵ expression in slow roll regime we can rewrite the scalar perturbations power spectrum as:

$$\Delta_s^2 = \frac{1}{24\pi^2}\frac{V}{\epsilon}\bigg|_{\phi=\phi(k)} \overset{\phi=\phi_*}{=} \frac{1}{24\pi^2}\frac{V^3}{V_\phi^2}\bigg|_{\phi=\phi_*} = P_s \tag{3.60}$$

where the last equality holds since the power spectrum at $k = k_*$ is exactly the amplitude. Now using our canonically normalised potential (3.51) and using the approximation $1 - \frac{c_{\text{loop}} b}{\phi_*^{2/3}} \simeq 1$ we get:

$$\frac{9V_0}{48\pi^2}\frac{\phi_*^{10/3}}{(bc_{\text{loop}})^2} = P_s = 2.105 \pm 0.030 \cdot 10^{-9} \simeq 2.1 \cdot 10^{-9} \tag{3.61}$$

Which is a relation between the overall volume value $\mathcal{V}$ and the canonically normalised inflaton value at horizon exit ϕ_*. Another equation to find numerical values for these 2 unknowns is the one for e-foldings, in fact, from post-inflationary history specifically depending on the brane setup of the model we get $N_e = 51.5/53$ and so we get the constraint:

$$N_e = \int_{\phi_{\text{end}}}^{\phi_*} \frac{V}{V_\phi} d\phi \simeq \frac{9}{16}\frac{\phi_*^{8/3}}{bc_{\text{loop}}} \overset{!}{=} 51.5/53 \tag{3.62}$$

By solving this latter constraint in terms of $\mathcal{V}$ we get:

$$\mathcal{V} = \frac{(16N_e)^3}{9^3}\frac{4\lambda_\phi c_{\text{loop}}}{3\beta^3\phi_*^8} \tag{3.63}$$

Plugging this value inside (3.61) and calling $\delta_\phi = \left(\frac{4\lambda_\phi}{3}\right)^{1/3}$ we get:

$$\phi_* = \left(\frac{2^{17}\pi}{3^8}\right)^{1/11}\left(\frac{12\pi^2 P_s N_e^7 (\delta_\phi c_{\text{loop}})^9}{N_Q \beta^{10}}\right)^{1/22} \tag{3.64}$$

where $N_Q := 2\pi g_s e^{K_{\text{C.S.}}} W_0^2$ arises from V_0 and contains all the g_s, W_0 and complex structure moduli dependence of the inflaton at horizon exit. Now, by plugging it inside (3.63) we finally obtain:

$$\mathcal{V} = \left(\frac{1}{144\pi^8} \frac{N_e^5 N_Q^4 \beta^8}{(12\pi^2 P_s) 4 (\delta_\phi c_{\text{loop}})^3} \right)^{1/11} \tag{3.65}$$

One in principle can think we can choose N_Q arbitrarily allowing us to increase the overall volume and decrease the inflaton at horizon exit arbitrarily however this is not true since the bound we have from the orientifold tadpole:

$$N_Q < -Q_3 \sim \mathcal{O}(100) \tag{3.66}$$

since at maximum, in the Kreuzer-Skarke database $Q_3 = -252$.
In our case, since we require perturbative control and so $g_s \lesssim 0.2$ we choose the parameters:

$$\lambda_\phi = 1 \tag{3.67}$$

$$c_{\text{loop}} = \frac{1}{16\pi^2} \tag{3.68}$$

$$\beta = W_0 = g_s e^{K_{\text{C.S.}}} = 2 \Rightarrow N_Q = 16\pi \tag{3.69}$$

$$N_e \simeq 51.5 - 53 \tag{3.70}$$

giving us:

$$\phi_* = 0.06 N_e^{7/22} \sim \mathcal{O}(0.2) \tag{3.71}$$

$$\mathcal{V} = 1743 N_e^{5/11} \sim \mathcal{O}(10^4) \tag{3.72}$$

Leading us to a smaller $\mathcal{V}$ with the respect to Blow-Up inflationary case (3.35).

Kähler cone constraint

We now want to check if we can drive inflation remaining inside the Kähler cone and satisfying the observational constraints. In order to do so it is necessary to take an explicit Calabi-Yau and we are going to take the second one in the table of [21] where the overall volume can be written, after shrinking an exceptional divisor to 0 size, as:

$$\mathcal{V} = \frac{1}{9}\sqrt{\frac{2}{3}}(\tau_b^{3/2} - \sqrt{3}\tau_s^{3/2} - \sqrt{3}\tau_\phi^{3/2}) \tag{3.73}$$

where the divisors are such that we can write the 4-cycles as:

$$\tau_b = \frac{27}{2}t_b^2 \tag{3.74}$$

$$\tau_s = \frac{9}{2}t_s^2 \tag{3.75}$$

$$\tau_\phi = \frac{9}{2}t_\phi^2 \tag{3.76}$$

The canonical normalisation than reads as:

$$\tau_\phi = \left(\frac{\sqrt{3}}{4}\right)^{2/3} \mathcal{V}^{2/3}\phi^{4/3} \simeq \left(\frac{1}{18\sqrt{2}}\right)^{2/3} \tau_b\phi^{4/3} \tag{3.77}$$

since $\mathcal{V} \simeq \frac{1}{9}\sqrt{\frac{2}{3}}\tau_b^{3/2}$. From [21] we can read the Kähler cone conditions as:

$$t_b + t_s > 0 \tag{3.78}$$

$$t_b + t_\phi > 0 \tag{3.79}$$

$$t_s < 0 \tag{3.80}$$

$$t_\phi < 0 \tag{3.81}$$

using the 2 cycle definition (3.74) in (3.77) we get:

$$\frac{|t_\phi|}{t_b} = \left(\frac{1}{18\sqrt{2}}\right)^{1/3} 3^{1/2}\phi^{2/3} = \left(\frac{1}{2\sqrt{6}}\right)^{1/3} \phi^{2/3} \simeq 0.6\phi^{2/3} \tag{3.82}$$

which, evaluated at horizon exit gives:

$$\frac{|t_{\phi*}|}{t_b} \simeq 0.6\phi_*^{2/3} \simeq 0.2 \text{ for } \phi_* \simeq 0.2 \tag{3.83}$$

so, using the previous observational constraints, the whole inflation let the moduli remain well inside the Kähler cone with $|t_\phi| \ll t_b \Rightarrow t_b + t_\phi > 0$. For $\mathcal{V} \sim$

$\mathcal{O}(10^4) \Rightarrow t_b \sim \mathcal{O}(19) > |t_{\phi_*}| \sim \mathcal{O}(3.8)$. In order to be complete, even if will not go into details, it is necessary to cite that, as stated in [2], if we use the anti-D3-brane uplift mechanism another constraint for the volume appears depending on the fluxes contribution in the warped throat.

3.2.3 Standard Model realisation and decay rates

In this section we are going to study the Location of the Standard Model in the extra-dimensions by computing all the moduli couplings and decay rates into S.M. itself and hidden sector particles. This will help us in the future study of the axion decay in the last chapter of this thesis.

As it is showed in [22], the cycle that supports Standard Model is very hard to stabilise through non-perturbative effect since instanton-matter fields chiral intersection will give raise to a null contribution to the superpotential. Since that and since we need non-perturbative effects to generate a minimum when reheating happens at the end of the inflation, it is impossible to realise Standard Model on D7-Branes wrapped around τ_ϕ. We need then another cycle with volume called τ_M which like τ_ϕ is stabilised perturbatively. 2 constructions are possible:

- **Geometric regime** $\Rightarrow$ S.M. lives on D7-branes wrapping the divisor Σ_M;

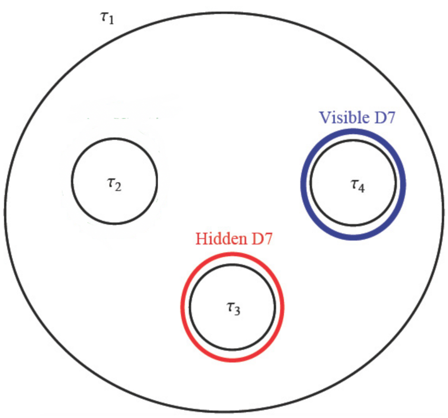

Figure 3.4: Image of our Calabi-Yau with $\tau_1 = \tau_b$, $\tau_2 = \tau_\phi$ non-wrapped, $\tau_3 = \tau_s$ neglected now and, finally, $\tau_4 = \tau_M$.

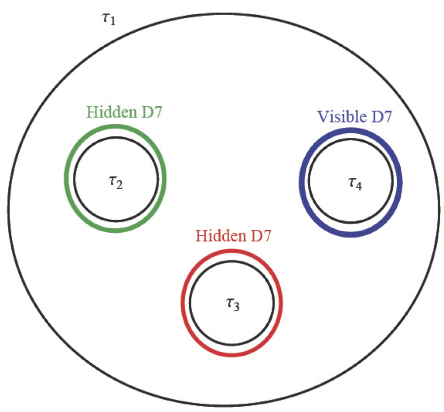

Figure 3.5: Image of our Calabi-Yau with $\tau_1 = \tau_b$, $\tau_2 = \tau_\phi$ wrapped by an hidden sector, $\tau_3 = \tau_s$ neglected now and, finally, $\tau_4 = \tau_M$.

- **Quiver Locus regime** $\Rightarrow$ the S.M. wrapped cycle shrinks and so $\tau_M \to 0$.

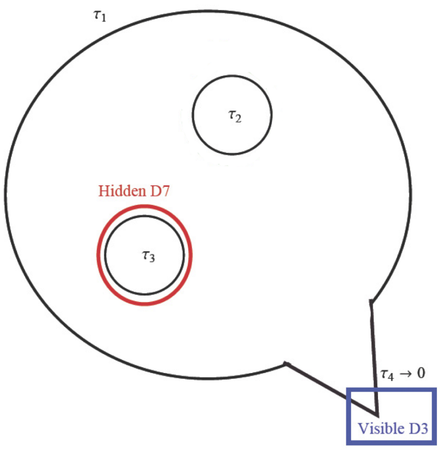

Figure 3.6: Image of our Calabi-Yau with $\tau_1 = \tau_b$, $\tau_2 = \tau_\phi$ non-wrapped, $\tau_3 = \tau_s$ neglected now and, finally, $\tau_4 = \tau_M$ shrunk.

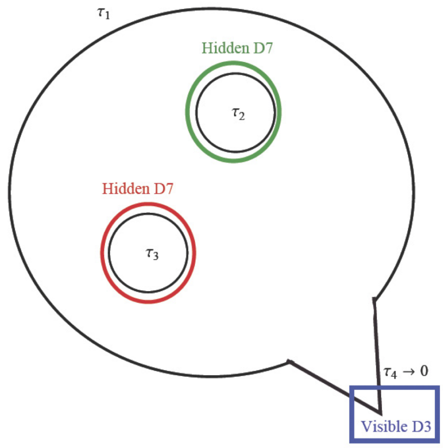

Figure 3.7: Image of our Calabi-Yau with $\tau_1 = \tau_b$, $\tau_2 = \tau_\phi$ wrapped by an hidden sector, $\tau_3 = \tau_s$ neglected now and, finally, $\tau_4 = \tau_M$ shrunk.

The overall Calabi-Yau volume can be written then as:

$$\mathcal{V} = \tau_b^{3/2} - \lambda_s \tau_s^{3/2} - \lambda_\phi \tau_\phi^{3/2} - \lambda_M \tau_M^{3/2} - \lambda_{\text{int}} (\tau_{\text{int}} - \lambda_M \tau_M)^{3/2} \tag{3.84}$$

where we included an additional cycle τ_{int} intersecting with the standard model one. The Standard Model cycle volume is then stabilised by loop contributions of the shape:

$$\delta V_{\text{loop}}^M(\tau_M) = \left(\frac{\mu_1}{\sqrt{\tau_M}} - \frac{\mu_2}{\sqrt{\tau_M} - \mu_3} \right) \frac{|W_0|^2}{\mathcal{V}^3}. \tag{3.85}$$

where $\mu_3 = \sqrt{\langle \tau_s \rangle}$ and where μ_1 and μ_2 loop corrections coefficients. It can be shown that this additional part of the potential admits a minimum at:

$$\langle \tau_s \rangle = \left(1 + \sqrt{\frac{\mu_2}{\mu_1}} \right)^2 \langle \tau_M \rangle \sim \langle \tau_M \rangle \tag{3.86}$$

which means that τ_M is fixed by loops at the non-perturbatively stabilised value $\langle \tau_s \rangle$ which mean that the SM gauge coupling is $g_{\text{SM}}^{-2} \sim \tau_M \sim \tau_s \sim \mathcal{O}(10)$ as it should be.

3.2.4 Kähler moduli decay rates

The only relevant moduli from an energy density point of view are the inflaton and the volume moduli since all the other never dominate the energy density. The mass of canonically normalised inflaton ϕ doesn't change from the one of Blow-Up case and so it is (reinstating $M_P \simeq 2.4 \cdot 10^{18} GeV$):

$$m_\phi \simeq \frac{W_0 \ln \mathcal{V}}{\mathcal{V}} M_p \tag{3.87}$$

while for the volume the mass is:

$$m_\chi \simeq \frac{W_0}{\mathcal{V}^{3/2}\sqrt{\ln(\mathcal{V})}} M_p \tag{3.88}$$

The main Decay rates then are given by:

- **Volume modulus** χ.
  The leading channels are 3:

  1) Volume into its string axions a_b (as in [18], where $\tau_s, \tau_{\text{int}}$ are neglected since small):

  $$\Gamma_{\chi \to a_b a_b} = \frac{1}{48\pi} \frac{m_\chi^3}{M_p^2} \simeq \left(\frac{W_0^3}{48\pi (\ln(\mathcal{V}))^{3/2}} \right) \frac{M_p}{\mathcal{V}^{9/2}} \tag{3.89}$$

  2) Volume into Standard Model Higgs Bosons [20]:

  $$\Gamma_{\chi \to hh} = \frac{c_{\text{loop}}^2}{32\pi} \left(\frac{m_0^4}{m_\chi} \right) \frac{1}{M_p^2} \simeq \left(\frac{\hat{c}_{\text{loop}}^2 W_0^3 \sqrt{\ln(\mathcal{V})}}{32\pi} \right) \frac{M_p}{\mathcal{V}^{5/2}} \tag{3.90}$$

  with $\hat{c}_{\text{loop}} \simeq \frac{1}{16\pi^2}$ a 1-loop factor and m_0 soft ~~SUSY~~ mass.

3) Volume two Higgs bosons H_u and H_d [18]:

$$\Gamma_{\chi\to H_d H_d} = \frac{Z^2}{24\pi}\frac{m_\chi^3}{M_p^2} \simeq \left(\frac{Z^2 W_0^3}{24\pi(\ln\mathcal{V})^{3/2}}\right)\frac{M_p}{\mathcal{V}^{9/2}} \tag{3.91}$$

with Z coefficient.

Which one is the relevant decay rates depend strongly on where the standard model is located:

-) **Geometric Regime**$\Rightarrow$ Standard Model on D7-branes wrapping cycle of volume τ_M:
We can see how $m_0 \simeq m_{3/2} \simeq \frac{W_0 M_p}{\mathcal{V}} \gg m_\chi$ so:

$$\frac{\Gamma_{\chi\to hh}}{\Gamma_{\chi\to a_b a_b}} \simeq (\hat{c}_{\text{loop}}\mathcal{V})^2 \gg 1 \tag{3.92}$$

so the dominant decay rate is (3.90).

-) **Quiver Locus**$\Rightarrow$ Standard Model on D3-branes at the tip of shrinked divisor with volume $\tau_M \to 0$:
We can see how $m_0 \lesssim m_\chi$ so:

$$\frac{\Gamma_{\chi\to hh}}{\Gamma_{\chi\to a_b a_b}} \lesssim \hat{c}_{\text{loop}} \ll 1 \tag{3.93}$$

so the dominant decay rate is (3.91) because of the Z coefficient which enhances it with the respect to (3.89).

- **Inflaton** ϕ:
In this case the dominance of one decay rate with the respect to the other is strongly dependent on how the inflaton 4-cycle is wrapped:

-) When the Inflaton 4-cycle wrapped by an hidden D7-brane stack then the main decay rate of the inflaton is on hidden sector gauge bosons [24]:

$$\Gamma_{\phi\to\gamma_{\text{hid}}\gamma_{\text{hid}}} \simeq \frac{\mathcal{V}}{64\pi}\frac{m_\phi^3}{M_p^2} \simeq \left(\frac{(W_0\ln(\mathcal{V}))^3}{64\pi}\right)\frac{M_p}{\mathcal{V}^2} \tag{3.94}$$

-) When the inflaton is not wrapped by any D7-brane we have that the main decay channels are 2, the one into volume modulus and the one into volume axions [20]:

$$\Gamma_{\phi\to\chi\chi} \simeq \Gamma_{\phi\to a_b a_b} \simeq \frac{(\ln(\mathcal{V}))^{3/2}}{64\pi\mathcal{V}}\frac{m_\phi^3}{M_p^2} \simeq \left(\frac{W_0^3(\ln(\mathcal{V}))^{9/2}}{64\pi}\right)\frac{M_p}{\mathcal{V}^4} \tag{3.95}$$

where, after its production, volume modulus χ will decay as stated in the previous case.
In addition to them, when the Standard Model is realised in D7-branes there are even 3 more decay channels scaling as (3.95) which are [20]:

$$\Gamma_{\phi\to\chi\chi} \simeq \Gamma_{\phi\to a_b a_b} \simeq \Gamma_{\phi\to a_M a_M} \simeq \Gamma_{\phi\to\tau_M\tau_M} \simeq \frac{1}{8N_g}\Gamma_{\phi\to\gamma_{\text{vis}}\gamma_{\text{vis}}} \tag{3.96}$$

where a_M will be the QCD axion and $N_g \geq$ the number of gauge bosons species which is at least 12, the Standard Model one.

3.2.5 Post-inflationary dynamics

The post-inflationary situation then strongly depends on how the Standard Model is built and we have 4 cases:

1) If the Standard Model lives on D7-branes and Inflaton is not wrapped by any D7-branes stack then one can ask which dominates at the end of the inflation. Recalling that the inflaton produces even volume moduli χ with rate $\Gamma_{\phi\to\gamma\gamma} \simeq 8N_g\Gamma_{\phi\to\chi\chi}$ and that, from (3.90), the volume modulus decays into Standard Model Higgses h, we can see how:

$$\frac{\Gamma_{\phi\to\gamma\gamma}}{\Gamma_{\chi\to hh}} \simeq \frac{4N_g(\ln\mathcal{V})^4}{\hat{c}_{\text{loop}}^2\mathcal{V}^{3/2}} \simeq 10^3 \tag{3.97}$$

by using the values $\mathcal{V} \simeq 10^4, \hat{c}_{\text{loop}} \simeq \frac{1}{16\pi^2}, N_g = 12$. This results show us that the volume modulus decays after the inflaton even if, as showed by [2] it will never dominate energy density. What happens then is that, since $\Gamma_{\chi\to hh} \sim \frac{\hat{c}_{\text{loop}}M_p}{\mathcal{V}^{5/2}} \overset{\mathcal{V}\gg 1}{\gg} \Gamma_{\phi\to\chi\chi}$ after ϕ decays into volume moduli they immediately decay

in Higgses never leading to an epoch where volume equates radiation.
In addition to this, studying the Inflaton decay the leading contributes are (3.96) and, since $\Gamma_{\tau_M\to\gamma\gamma} \simeq \Gamma_{\tau_M\to a_M a_M} \sim \frac{M_p}{\mathcal{V}^2} \overset{\mathcal{V}\gg 1}{\gg} \Gamma_{\phi\to\chi\chi}$ and all the other decay rates (3.96), then after the decay of the inflaton into the Standard Model modulus τ_M this latter decays almost instantaneously in photons and QCD axions with the rate computed in [20]:

$$\frac{\Gamma_{\tau_M\to\gamma\gamma}}{\Gamma_{\tau_M\to a_M a_M}} = 8N_g \geq 96 \gg 1 \tag{3.98}$$

The reheating is then set up by the inflaton. This setup leads to a number of e-foldings $N_e \simeq 52$, an overall volume of $\mathcal{V} \simeq 10525$, a spectral index of $n_s \simeq 0.9761$ and a tensor to scalar ratio $r \simeq 1.7 \cdot 10^{-5}$ which leads again to Gravitational waves that are non observable by short-term measurements.

2) If the Standard Model lives on D7-branes and Inflaton is wrapped by a D7-branes stack[4] then the inflaton as we saw decays into hidden sector gauge photons (3.94) and these hidden gauge bosons will be diluited by the decay of the volume into Higgs bosons which leads the reheating. What happens is that, after the inflation, both inflaton and volume modulus start oscillating and since, as stated in [2], energy density of the inflaton is larger then the volume one, the inflaton oscillations drive an era of matter domination until its decay. Then the hidden photon radiation dominates the energy density until the volume oscillations, redshifting as matter and so slower then hidden radiation one, equates it. Then the volume starts dominating the energy density of the universe until its decay which leads to a second, and last, radiation domination epoch and to the reheating. This setup leads to a number of e-foldings $N_e \simeq 53$, an overall volume of $\mathcal{V} \simeq 10616$, a spectral index of $n_s \simeq 0.9765$ and a tensor to scalar ratio $r \simeq 1.7 \cdot 10^{-5}$ which leads to Gravitational waves that are non observable by short-term measurements.

3) If the Standard Model Lives on D3-branes and Inflaton is wrapped by a D7-

[4]We assume now as stated in [2] not to be a pure Super-Yang-Mills theory with mass gap $\Delta\Lambda_{\text{SYM}} > m_\phi$ because if so, from [22] the decay would have been kinematically forbidden and this case would have been the same as if the inflaton was not wrapped by D7-branes stack.

brane stack then the case is the same of 2) but with the fact that, since the dominant decay rate of the volume is (3.91) and that $\Gamma_{\chi \to H_u H_d} \overset{\mathcal{V} \gg 1}{\ll} \Gamma_{\phi \to \chi\chi}$ the volume axion decays much later and diluites the inflaton decay product. This setup leads to a number of e-foldings $N_e \simeq 51.5$, an overall volume of $\mathcal{V} \simeq 10447$, a spectral index of $n_s \simeq 0.9757$ and a tensor to scalar ratio $r \simeq 1.7 \cdot 10^{-5}$ which leads to Gravitational waves that are non observable by short-term measurements.

4) If the Standard Model Lives on D3-branes and Inflaton is not wrapped by any D7-brane stack then the inflaton dominant decay channels are into a pair of volume moduli and into a pair of volume axions (3.95) while for the volume again the dominant decay rate is (3.91). Their ratio is given by:

$$\frac{\Gamma_{\phi \to a_b a_b}}{\Gamma_{\chi \to H_u H_d}} \simeq \frac{3}{8Z^2} (\ln \mathcal{V})^6 \sqrt{\mathcal{V}} \simeq 10^7 \tag{3.99}$$

for

$$Z \simeq 2, \mathcal{V} \simeq 10^4 \tag{3.100}$$

so that the inflaton decays before the volume. Since the inflaton decay products are relativistic they can be considered as radiation and so they redshift quickly and, at a certain equality point they become non-relativistic while, in the meantime, their energy density becomes comparable to the χ particles produced by volume mode. It can be proven than the χ particles produced by the inflaton decay are relativistic, redshifting then as radiation, while the χ particles produced by the volume modulus oscillation are not [2] and this means that after the inflaton domination period we get a radiation domination one, a volume dominated one and, finally, after volume decays, to another radiation dominated period. Again, since $\Gamma_{\chi \to H_u H_d} \overset{\mathcal{V} \gg 1}{\ll} \Gamma_{\phi \to \chi\chi}$ as in 3), the volume decays much later the inflaton leading to a dilution of inflaton decay products. This setup leads to the same inflationary parameter as 3), so to a number of e-foldings $N_e \simeq 51.5$, an overall volume of $\mathcal{V} \simeq 10447$, a spectral index of $n_s \simeq 0.9757$ and a tensor to scalar ratio $r \simeq 1.7 \cdot 10^{-5}$ which leads to Gravitational waves that are non observable by short-term measurements.

As stated in [2] it is possible to note how both the tensor to scalar ratio r and the scalar spectral index n_s are in good agreement with the Planck observations in all cases even if n_s is just slightly more blue than expected.

3.2.6 Dark radiation

We have just studied the decays of the inflaton and the volume modulus and we have seen that, in addition to Standard Model particles, even light axions like a_b and a_M are produced. Such particles are relativistic and they can contribute to Dark Radiation which can push the effective number of neutrino-like species ΔN_{eff} out of Standard Model observational bound. It can happen that the axions arising from the heavies modulus do not contribute to the Dark radiation since they are dilute by lightest modulus decay and in this case ΔN_{eff} is determined by the lightest modulus decay. We will call the lightest modulus Ω and we will call $\Gamma_{\Omega\to\text{SM}}$ its decay width into Standard Model particles and, finally, $\Gamma_{\Omega\to\text{Hid}}$ into hidden sector particles like closed string axions. The axionic contribution to extra dark radiation is then:

$$\Delta N_{\text{eff}} = \frac{43}{7}\frac{\Gamma_{\Omega\to\text{Hid}}}{\Gamma_{\Omega\to\text{SM}}}\left(\frac{10.75}{g_*(T_{\text{rh}})}\right)^{1/3} \tag{3.101}$$

where $g_*(T_{\text{rh}})$ is the number of relativistic degrees of freedom at reheating temperature T_{rh}. This has to match observational bounds $\Delta N_{\text{eff}} \lesssim 0.1 - 0.5$ at 95% of confidence level. The computation of ΔN_{eff} depends again on the Standard model Realisation and so we have 4 cases:

1) Standard Model on D7-branes and inflaton wrapped by D7 branes:
   In this case $\Omega \coloneqq \chi$ and $\Gamma_{\Omega\to\text{Hid}} = \Gamma_{\chi\to a_b a_b}$ (3.89) and $\Gamma_{\Omega\to\text{SM}} = \Gamma_{\chi\to hh}$ (3.90) since their ratio appearing in (3.101) is much small than one:

$$\Delta N_{\text{eff}} \simeq 0 \tag{3.102}$$

   Perfectly matching the bound.

2) Standard Model on D7-branes and inflaton not wrapped by D7 branes:

In this case instead $\Omega \coloneqq \phi$ and the relevant decay rates are:

$$\Gamma_{\Omega\to\mathrm{SM}} = \Gamma_{\phi\to\chi\chi\to hhhh} + \Gamma_{\phi\to\gamma_{\mathrm{vis}}\gamma_{\mathrm{vis}}} + \Gamma_{\phi\to\tau_M\tau_M\to\gamma_{\mathrm{vis}}\gamma_{\mathrm{vis}}\gamma_{\mathrm{vis}}\gamma_{\mathrm{vis}}} \tag{3.103}$$

$$\Gamma_{\Omega\to\mathrm{Hid}} = \Gamma_{\phi\to a_b a_b} + \Gamma_{\phi\to a_M a_M} + \Gamma_{\phi\to\tau_M\tau_M\to a_M a_M a_M a_M} \tag{3.104}$$

in [20] we can find a deep analysis of this case with $\Delta N_{\mathrm{eff}} \simeq 0.14$ with $N_g = 12$ and $g_*(T_{\mathrm{rh}}) = 106.75$ which again matches the observational bound.

3) Standard Model on D3-branes:
When the Standard Model is located at the point where the cycle Σ_M is shrunk, χ is the last modulus to decay so $\Gamma_{\Omega\to\mathrm{Hid}} = \Gamma_{\chi\to a_b a_b}$ (3.89) while $\Gamma_{\Omega\to\mathrm{SM}} = \Gamma_{\chi\to H_u H_d}$ (3.91), doing again the ratios between these two decay rates we get:

$$\Delta N_{\mathrm{eff}} \simeq \frac{1.43}{Z^2} \tag{3.105}$$

for $g_*(T_{\mathrm{rh}}) = 106.75$ since $T_{\mathrm{rh}} \gg \Lambda_{\mathrm{EW}}$ and so here all Standard Model degrees of freedom are relativistic. Imposing the less strict observational constraint $\Delta N_{\mathrm{eff}} \lesssim 0.5$ we get the constraint on Z: $Z \gtrsim 1.7$ which is respected by our previous choice $Z \simeq 2$ (3.100).

Chapter 4

The Curvaton in Loop Blow-Up Inflation

The curvaton mechanism [33] is an alternative mechanism for producing scalar perturbations. In fact, instead of using the inflaton for generating the correct Planck measured scalar power spectrum, we can use a field which is orthogonal to it called curvaton. This curvaton is usually a light field, compared to inflation scale, who is a spectator during inflation since its mass does not exceed H_{inf} and whose quantum fluctuations produce isocurvature perturbations which, upon its decay, are converted into curvature ones. This curvaton field can be used, as stated in [31], both to save models who do not respect Planck bounds on scalar perturbations amplitude (**savior curvaton**) and to check consistency of models which already satisfy this bound thanks to the fluctuations of the inflaton since here the curvaton must generate a subleading amount of scalar perturbations not to exit the bounds (**stealth curvaton**).

In this chapter we are going to implement the curvaton mechanism in the previously reviewed Loop Blow-Up inflationary scenario using as curvaton candidate the inflaton saxion θ_ϕ in its canonically normalised version σ_ϕ. Imposing that for the choice of model parameter made in [2] we get the entire CMB power spectrum from inflaton perturbations, we constraint the remaining free parameters by imposing that the curvaton contribute to the CMB observed amplitude is negligible.

4.1 Axion dynamics

Before Starting with the concrete implementation of the curvaton mechanism, it is necessary to study axion dynamics in Loop Blow-Up inflation which comes out to be the similar of the axion dynamics in Blow-Up inflation case since in the potential the curvaton dependence is on the non-perturbative terms. This study is necessary since we always implicitly stabilised the axion in the previous sections obtaining Potential and all the quantities depending only on the Kähler moduli. We are going now to start by computing the canonical normalisation of the axion both from the classical differential equation and then, in the appendix, via linear-algebraic method to check consistency of the obtained result, then we compute the curvaton potential and its value at minimum, its mass and, finally, the decay details on different scenarios.

4.1.1 Axion canonical normalisation

In order to rewrite the potential and all the other quantities in terms of canonically normalised fields we need to compute this canonical normalisation and we can do it analogously of what we have done for obtaining (D.32). The kinetic Lagrangian is analogous to the inflaton case:

$$\mathcal{L}_{kin} = \frac{1}{2}(\partial_\mu \sigma_\phi)^2 + \dots = K_{\phi\phi}((\partial_\mu \theta_\phi)(\partial_\mu \theta_\phi)) + \dots \tag{4.1}$$

by using the Kähler metric, which for Loop Blow-Up inflation is at leading order the same of Blow-Up inflation case (D.12), we can derive (just using the term which is dominant in the overall volume):

$$\mathcal{L}_{kin} = \frac{1}{2}(\partial_\mu \sigma_\phi)^2 + \dots = \frac{3\lambda_\phi}{8\mathcal{V}\sqrt{\tau_\phi}}((\partial_\mu \theta_\phi)(\partial_\mu \theta_\phi)) + \dots \tag{4.2}$$

however now τ_ϕ and θ_ϕ are independent so the integration on differential equation gives:

$$\partial_\mu \sigma_\phi = \sqrt{\frac{3\lambda_\phi}{4\mathcal{V}}} \frac{\partial_\mu \theta_\phi}{\tau_\phi^{\frac{1}{4}}} \tag{4.3}$$

which is much simpler and we can consider all the factors in front of θ_ϕ as independent from the axion itself, giving us the result:

$$\sigma_\phi = \sqrt{\frac{3\lambda_\phi}{4\mathcal{V}}} \frac{\theta_\phi}{\tau_\phi^{\frac{1}{4}}} = f(\tau_\phi)\theta_\phi \tag{4.4}$$

The canonically normalised fields are then:

$$\phi = \sqrt{\frac{4\lambda_\phi}{3\mathcal{V}}} \tau_\phi^{3/4} \Rightarrow \tau_\phi = \left(\frac{3\mathcal{V}}{4\lambda_\phi}\right)^{\frac{2}{3}} \phi^{\frac{4}{3}} \tag{4.5}$$

$$\sigma_\phi = \sqrt{\frac{3\lambda_\phi}{4\mathcal{V}}} \frac{\theta_\phi}{\tau_\phi^{\frac{1}{4}}} \Rightarrow \theta_\phi = \sqrt{\frac{4\mathcal{V}}{3\lambda_\phi}} \tau_\phi^{\frac{1}{4}} \sigma_\phi = \frac{1}{f(\tau_\phi)} \sigma_\phi \tag{4.6}$$

which are coherent with the Appendix B of reference [11] and with the results we found through linear algebra in E.

4.1.2 Axion potential

We would like now to compute the potential for the Axion which will be our Curvaton candidate keeping only the leading terms of its potential.
In the Large Volume Scenario stabilisation scheme, as we saw, we usually include the $(\alpha')^3$ corrections in Kähler potential which can be rewritten as:

$$K = K_{\text{TREE}} + K_{\alpha'^3} = -2\ln\left(\mathcal{V} + \frac{\xi}{2}\right) = -2\ln(\mathcal{V}) - 2\ln\left(1 + \frac{\xi}{2\mathcal{V}}\right) + \delta K_{\text{loop}} \tag{4.7}$$

where $\xi = \frac{\chi(Y_6)\zeta(3)}{2(2\pi)^3 g_s^{\frac{3}{2}}}$. Recalling that we are in Large Volume Scenario $\mathcal{V} \gg 1$ this leads us to:

$$K \simeq K_{\text{TREE}} + K_{\alpha'^3} + \delta K_{\text{loop}} = -2\ln\mathcal{V} - \frac{\xi}{\mathcal{V}} + \delta K_{\text{loop}} \tag{4.8}$$

The superpotential in our case of Loop Blow-Up inflation receives non-perturbative corrections due to gaugino condensation on D7 branes or instantonic Euclidean D3 branes:

$$W = W_0 + W_{\text{np}} = W_0 + \sum_{i \in \{\phi, s\}} e^{-a_i T_i} = W_0 + \sum_{i \in \{\phi, s\}} e^{-a_i \tau_i} e^{-i a_i \theta_i} \tag{4.9}$$

where W_0 is the Gukov-Vafa-Witten superpotential upon fixing all complex structure moduli to their minima, and $a_i = \frac{2\pi}{N_i}$ if we are in the case of gaugino condensation or $a_i = 2\pi$ in the case of ED3 instantons.
The F-term scalar potential is written as always as:

$$V_F = e^K (K^{i\bar{j}} D_i W D_{\bar{j}} \overline{W} - 3|W|^2) \tag{4.10}$$

where as always $D_i W = \partial_i W + K_i W$ and the same for its Hermitian conjugate $D_{\bar{i}} \overline{W} = \partial_{\bar{i}} \overline{W} + K_{\bar{i}} \overline{W}$.
Because of this split structure of Kähler potential, we can rewrite, following [30], all the terms with derivatives of K as:

$$K^{i\bar{j}} K_i W K_{\bar{j}} \overline{W} = K^{i\bar{j}}_{\text{TREE}} K_i^{\text{TREE}} W K_{\bar{j}}^{\text{TREE}} \overline{W} + K^{i\bar{j}}_{\alpha'^3} K_i^{\alpha'^3} W K_{\bar{j}}^{\alpha'^3} \overline{W} \tag{4.11}$$

which gives the result:

$$K^{i\bar{j}} K_i W K_{\bar{j}} \overline{W} = 3|W|^2 + \frac{3\xi W_0^2}{4\mathcal{V}} \tag{4.12}$$

leading us to:

$$V_F = e^K \left(K^{i\bar{j}} \partial_i W \partial_{\bar{j}} \overline{W} + \left(K^{i\bar{j}} \partial_i W K_{\bar{j}} \bar{W} + \text{h.c.} \right) + \frac{3}{4} \frac{W_0^2 \xi}{\mathcal{V}} \right) \tag{4.13}$$

We can imagine then to have this potential, without including loop corrections yet, after changing moduli basis from $\{\tau_b, \tau_\phi, \tau_s\}$ to $\{\mathcal{V}, \tau_\phi, \tau_s\}$ and after stabilizing complex structure moduli, axio-dilaton and the axion θ_s (assuming $W_0, A_i \in$

$\mathbb{R}$, $i = \{s, \phi\}$:

$$V_{\rm TOT}(\mathcal{V},\tau_\phi,\tau_s) = \frac{g_s e^{K_{CS}}}{8\pi} W_0^2 \left(\frac{8a_s^2 A_s^2}{W_0^2 \lambda_s} \frac{\sqrt{\tau_s} e^{-2a_s\tau_s}}{\mathcal{V}} - \frac{4a_s A_s}{W_0} \frac{\tau_s e^{-a_s\tau_s}}{\mathcal{V}^2} \right.$$
$$\left. + \frac{8a_\phi^2 A_\phi^2}{W_0^2 \lambda_\phi} \frac{\sqrt{\tau_\phi} e^{-2a_\phi\tau_\phi}}{\mathcal{V}} - \frac{4a_\phi A_\phi}{W_0} \frac{\tau_\phi e^{-a_\phi\tau_\phi}}{\mathcal{V}^2} h(\theta_\phi) + \frac{3\xi}{4\mathcal{V}^3} \right) + \delta V_{\rm loop} \tag{4.14}$$

where $h(\theta_\phi)$ is a function only of the axion. Notice we have no function of θ_ϕ in the doubly exponentially suppressed term, because in $\partial_i W_{(\rm np)} \partial_{\bar{j}} \overline{W}_{(\rm np)}$ the imaginary exponentials cancel each other.

In principle, one now can should care about 2 things:

1) Mixed terms of θ_ϕ and other axions, arising in (4.10) from terms of the form $K^{\phi\bar{s}} \partial_\phi W \partial_{\bar{s}} \overline{W}$. These terms do appear in the scalar potential in the form:

$$\begin{aligned} V_{\rm TOT} \supset & \frac{a_s A_s a_\phi A_\phi e^{-(a_s\tau_s + a_\phi\tau_\phi)}}{\mathcal{V}^2} (K^{\phi\bar{s}} e^{i(a_\phi\theta_\phi - a_s\theta_s)} + \text{h.c.}) \\ = & 2 \frac{a_s A_s a_\phi A_\phi e^{-(a_s\tau_s + a_\phi\tau_\phi)}}{\mathcal{V}^2} (K^{\phi\bar{s}} \cos(a_\phi\theta_\phi - a_s\theta_s)) \\ \sim & \frac{1}{\mathcal{V}^2} e^{-(a_s\tau_s + a_\phi\tau_\phi)} \cos(a_\phi\theta_\phi - a_s\theta_s) \end{aligned} \tag{4.15}$$

where we used $K^{\phi\bar{s}}$ as stated in [22]. This term is doubly exponentially suppressed in the two τ's, so it is subleading.

2) String Loop Corrections to the Kähler potential. However, as stated in [2] and as saw in the previous chapter about loop corrections, the Kähler potential features a scaling in term of the volume of:

$$\delta K_{\rm loop} \simeq \frac{1}{\mathcal{V}} \tag{4.16}$$

so in the potential the terms containing at least one derivative of loop corrections are (terms in the potential containing $(\delta K_{\rm loop})^{\phi\bar{s}} (\delta K_{\rm loop})_{\bar{s}}$ are doubly suppressed):

$$V_{\rm TOT} \supset \text{terms} \sim \frac{1}{\mathcal{V}}\frac{1}{\mathcal{V}^2} e^{-(a_s\tau_s + a_\phi\theta_\phi)} cos(a_\phi\theta_\phi - a_s\theta_s) \tag{4.17}$$

giving us an highly suppressed term:

$$V_{\rm TOT} \supset \text{terms} \sim \frac{1}{\mathcal{V}^3} e^{-(a_s\tau_s + a_\phi\theta_\phi)} cos(a_\phi\theta_\phi - a_s\theta_s) \tag{4.18}$$

where $\frac{1}{\mathcal{V}^2}$ comes from e^K. Giving us again an highly subleading negligible term in the overall volume.

We can now finally derive the explicit shape of the potential for the axion θ_ϕ at order $\mathcal{O}\left(\frac{1}{\mathcal{V}^2}\right)$:

$$\begin{aligned} V_{\rm ax} =& \frac{g_s e^{K_{CS}}}{8\pi} e^K (K^{\phi\bar{s}}\partial_\phi W_{\rm np} K_{\bar{s}} W_0 + \\ &+ K^{\phi\phi}\partial_\phi W_{\rm np} K_{\bar{\phi}} W_0 + K^{\phi\bar{\phi}}\partial_{\bar{\phi}} W_{\rm np} K_\phi W_0 + K^{s\bar{\phi}}\partial_{\bar{\phi}} W_{\rm np} K_s W_0) + V_{\rm OT}(\tau_\phi) \end{aligned} \tag{4.19}$$

where all the other terms cancel out for no-scale or are included in $V_{\rm OT}(\tau_\phi)$ containing all the potential parts not depending on θ_ϕ. From now on, we will set $\mathcal{S} = \frac{g_s e^{K_{CS}}}{8\pi}$. We can then rewrite V_{θ_ϕ} as:

$$\begin{aligned} V_{\rm ax} &= \frac{\mathcal{S}}{\mathcal{V}^2}\left(2a_\phi A_\phi \tau_\phi W_0 e^{-a_\phi(\tau_\phi - i\theta_\phi)} + 2a_\phi A_\phi \tau_\phi W_0 e^{-a_\phi(\tau_\phi + i\theta_\phi)}\right) + V_{\rm OT} \\ &= \frac{\mathcal{S}}{\mathcal{V}^2}\left(2a_\phi A_\phi \tau_\phi W_0 e^{-a_\phi\tau_\phi}\left(e^{a_\phi\theta_\phi} + e^{-a_\phi i\theta_\phi}\right)\right) + V_{\rm OT} \end{aligned} \tag{4.20}$$

leading us to the final result:

$$V_{\rm ax} = \frac{g_s e^{K_{CS}}}{8\pi}\frac{4W_0 a_\phi A_\phi \tau_\phi}{\mathcal{V}^2} e^{-a_\phi\tau_\phi}\cos(a_\phi\theta_\phi) + V_{\rm OT}(\tau_\phi) \tag{4.21}$$

we will rewrite it, for the sake of shortening, as:

$$V_{\rm ax} = \Lambda(\tau_\phi)\cos(a_\phi\theta_\phi) + V_{\rm OT}(\tau_\phi) \tag{4.22}$$

where the function $\Lambda(\tau_\phi)$ depends only on τ_ϕ and on volume $\mathcal{V}$:

$$\Lambda(\tau_\phi) := \mathcal{S}\frac{4W_0 a_\phi A_\phi \tau_\phi}{\mathcal{V}^2} e^{-a_\phi \tau_\phi} \tag{4.23}$$

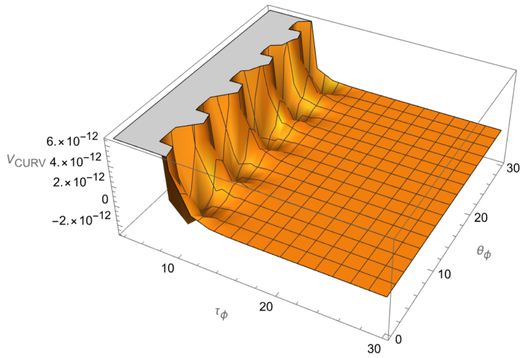

Figure 4.1: Curvaton Potential including V_{OT} for $a_\phi = A_\phi = 1, \mathcal{V} = 10^4$

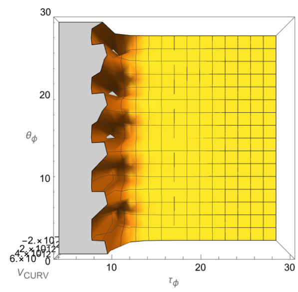

Figure 4.2: Top view of Curvaton Potential.

4.1.3 Axion mass

It is of crucial importance to compute the mass of our fields in order to understand which of them are active during inflation. In fact if a certain field φ has a mass greater than the Hubble scale during inflation $m_\varphi > H_{\mathrm{inf}}$, it will be classically moving during inflation.
In Loop Blow-Up inflation the inflationary potential is:

$$V(\tau_\phi) = V_0\left[1 + \mathcal{A}_\phi \frac{\mathcal{V}^2}{\beta}\sqrt{\tau_\phi}e^{-2a_\phi\tau_\phi} - \mathcal{B}_\phi\frac{\mathcal{V}}{\beta}\tau_\phi e^{-a_\phi\tau_\phi} - \frac{c_{\mathrm{loop}}}{\beta\sqrt{\tau_\phi}}\right]. \tag{4.24}$$

From here, [2] derived the mass, as stated in (3.87), of the inflaton at the end of inflation, when it sits at its non-perturbative minimum.

Now we want to the same for the axion of the inflaton θ_ϕ. We recall that the mass term of the axion in the Lagrangian is:

$$\mathcal{L} \supset \frac{m_{\theta_\phi}^2}{2}\theta_\phi^2 \tag{4.25}$$

If we expand the potential for curvaton around the minimum we get:

$$V_{\rm ax} = V_0 + \partial_{\theta_\phi} V_{\rm ax}\Big|_{min} \theta_\phi + \frac{\partial^2_{\theta_\phi} V_{\rm ax}}{2}\Big|_{min} \theta_\phi^2 + \mathcal{O}(\theta_\phi^3) \simeq V_0 + \frac{\partial^2_{\theta_\phi} V_{\rm ax}}{2}\Big|_{min} \theta_\phi^2 \tag{4.26}$$

hence, we deduce that $m^2_{\theta_\phi} = \partial^2_{\theta_\phi} V_{\rm ax}\big|_{min}$.
We start then by computing the minimum of the potential:

$$V_{\rm ax}(\theta_\phi) = \Lambda(\tau_\phi)\cos(a_\phi \theta_\phi) + V_{\rm OT} \tag{4.27}$$

just by deriving it twice:

$$\partial_{\theta_\phi} V_{\rm ax} = -\Lambda(\tau_\phi) a_\phi \sin(a_\phi \theta_\phi) \tag{4.28}$$

$$\partial^2_{\theta_\phi} V_{\rm ax} = -\Lambda(\tau_\phi) a_\phi^2 \cos(a_\phi \theta_\phi) \tag{4.29}$$

the conditions to have a minimum are:

$$\begin{cases} \partial_{\theta_\phi} V_{\rm ax}|_{\theta_\phi = \langle\theta_\phi\rangle} = 0 \\ \partial^2_{\theta_\phi} V_{\rm ax}|_{\theta_\phi = \langle\theta_\phi\rangle} > 0 \end{cases} \tag{4.30}$$

These are realized when:

$$a_\phi \langle\theta_\phi\rangle = (2k+1)\pi \quad \text{with } k \in \mathbb{Z} \tag{4.31}$$

obtaining the vacuum expectation value of θ_ϕ:

$$\langle\theta_\phi\rangle = \frac{2k+1}{a_\phi}\pi \quad \text{where } k \in \mathbb{Z} \tag{4.32}$$

Now we have to compute the derivatives of the potential with respect to the canonically normalised axion field σ_ϕ in order to retrieve its mass.

In order to do so, let us express the potential in terms of the canonically normalised inflaton and curvaton fields. Renaming now $V_{\rm ax}(\sigma_\phi) \equiv V_{\rm ax}(\sigma_\phi) - V_{\rm OT}$:

$$V_{\rm ax}(\sigma_\phi) = \Lambda(\phi)\cos\left(\frac{a_\phi}{f(\phi)}\sigma_\phi\right) \tag{4.33}$$

where now $\Lambda(\phi), f(\phi)$ are written in function of canonical inflaton:

$$\Lambda(\phi) = \frac{4a_\phi A_\phi \mathcal{S} W_0}{\mathcal{V}^{\frac{4}{3}}}\left(\frac{3}{4\lambda_\phi}\right)^{\frac{2}{3}}\phi^{\frac{4}{3}}e^{-a\phi\left(\frac{3\mathcal{V}}{4\lambda_\phi}\right)^{\frac{2}{3}}\phi^{\frac{4}{3}}} \tag{4.34}$$

$$f(\phi) = \left(\frac{3}{4}\right)^{\frac{1}{3}}\left(\frac{\lambda_\phi}{\mathcal{V}}\right)^{\frac{2}{3}}\frac{1}{\phi^{\frac{1}{3}}} \tag{4.35}$$

In an explicit form then:

$$V_{\rm ax}(\sigma_\phi) = \frac{4a_\phi A_\phi \mathcal{S} W_0}{\mathcal{V}^{\frac{4}{3}}}\left(\frac{3}{4\lambda_\phi}\right)^{\frac{2}{3}}\phi^{\frac{4}{3}}e^{-a\phi\left(\frac{3\mathcal{V}}{4\lambda_\phi}\right)^{\frac{2}{3}}\phi^{\frac{4}{3}}}\cos\left(a_\phi\left(\frac{4}{3}\right)^{\frac{1}{3}}\left(\frac{\mathcal{V}}{\lambda_\phi}\right)^{\frac{2}{3}}\phi^{\frac{1}{3}}\sigma_\phi\right) \tag{4.36}$$

Its derivatives are then, denoting $(V_{\rm ax})_{\sigma_\phi} \equiv \partial_{\sigma_\phi}(V_{\rm ax})$:

$$(V_{\rm ax})_{\sigma_\phi} = -\Lambda(\phi)\frac{a_\phi}{f(\phi)}\sin\left(\frac{a_\phi}{f(\phi)}\sigma_\phi\right) \tag{4.37}$$

$$(V_{\rm ax})_{\sigma_\phi\sigma_\phi} = -\Lambda(\phi)\left(\frac{a_\phi}{f(\phi)}\right)^2\cos\left(\frac{a_\phi}{f(\phi)}\sigma_\phi\right) \tag{4.38}$$

The minimum again is located where:

$$(V_{\rm ax})_{\sigma_\phi}\Big|_{min} = 0 \text{ and } \cos\left(\frac{a_\phi}{f(\phi)}\sigma_\phi\right) = -1 \tag{4.39}$$

so where:

$$\langle\sigma_\phi\rangle = \frac{(2k+1)\pi}{a_\phi}f(\phi) \quad \text{with } k \in \mathbb{Z} \tag{4.40}$$

Notice that, unsurprisingly, the two minima (4.32) and (4.40) coincide, since $f(\phi)$ is precisely the function appearing in the canonical normalisation of the axion.

In order to compute the mass[1] it is necessary to evaluate the second derivative of the potential at $\langle\sigma_\phi\rangle$:

$$m^2_{\sigma_\phi} = (V_{\rm ax})_{\sigma_\phi\sigma_\phi}\Big|_{\sigma_\phi=\langle\sigma_\phi\rangle} = -\Lambda(\phi)\left(\frac{a_\phi}{f(\phi)}\right)^2\cos((2k+1)\pi) = \Lambda(\phi)\left(\frac{a_\phi}{f(\phi)}\right)^2 \tag{4.41}$$

Using (4.34),(4.35) we get the final value:

$$m^2_{\sigma_\phi} = \frac{4\mathcal{S}a^3_\phi A_\phi W_0}{\lambda^2_\phi}\phi^2 e^{-a_\phi\left(\frac{3\mathcal{V}}{4\lambda_\phi}\right)^{\frac{2}{3}}\phi^{\frac{4}{3}}} \tag{4.42}$$

We now plug back τ_ϕ inverting (4.6) and get:

$$m^2_{\sigma_\phi} = \frac{16\mathcal{S}W_0 A_\phi a^3_\phi}{3\lambda_\phi\mathcal{V}}\,\tau^{3/2}_\phi\, e^{-a_\phi\tau_\phi} \tag{4.43}$$

Clearly, since the leading potential for both the inflaton and the curvaton is generated by the same non-perturbative effect, we expect the masses to be exactly equal when all the fields are set to their minima such that:

$$m_{\sigma_\phi} \simeq \frac{W_0\ln(\mathcal{V})}{\mathcal{V}} \simeq m_\phi \tag{4.44}$$

This appears to be true as we can easily show by using (D.27) and (D.25) inside (4.43).

4.1.4 Axion decay rates

In computing the decay rates, through the couplings in F, we are going to follow [20] where the decay rates are splitted in two families:

1) Products of decay are identical and so the Lagrangian reads, in Planck units, as:

$$\mathcal{L}_A \supset g_A\phi_A\psi^2_A \tag{4.45}$$

[1]This is an effective mass during inflation, the physical mass of particles coming out from the axion field is the one at the global minimum of the potential.

and the decay rate as:

$$\Gamma_{\phi_A \to \psi_A \psi_A} = \frac{g_A^2}{8\pi m_{\phi_A}} \tag{4.46}$$

2) Products of decay are different and so the Lagrangian reads, in Planck units, as:

$$\mathcal{L}_B \supset g_B \phi_A \psi_B \chi_B \tag{4.47}$$

and the decay rate as:

$$\Gamma_{\chi \to \psi_B \chi_B} = \frac{g_B^2}{16\pi m_\chi} \tag{4.48}$$

Case 1: no D7s wrapped on the inflaton cycle

We have three non negligible decay channels for the axions:

1) The dominant one for σ_ϕ in this scenario is the one towards SM gauge fields, which is the same as its saxion's and can be computed following the formula in [20]

$$\Gamma_{\sigma_\phi \to \gamma_{\rm vis}\gamma_{\rm vis}} = \frac{g_{\rm vis}^2}{16\pi m_{\sigma_\phi}} \tag{4.49}$$

where $g_{\rm vis} = -\frac{\sqrt{6\lambda_\phi}\tau_\phi^{3/4}\tau_M}{\mathcal{V}^{1/2}} m_{\sigma_\phi}^2$ as in (F.24) and $m_{\sigma_\phi} = \frac{2W_0 a_\phi \tau_\phi}{\tau_b}$, giving us the decay rate:

$$\Gamma_{\sigma_\phi \to \gamma_{\rm vis}\gamma_{\rm vis}} \simeq \frac{3\lambda_\phi W_0^3 a_\phi^3 N_g}{8\pi} \frac{\tau_\phi^{9/2}}{\mathcal{V}^4} M_p \tag{4.50}$$

This decay rate is clearly the dominant one since an higher order of magnitude given by $N_g = 12$ since we have to consider all species of Gauge Bosons.

2) A subdominant by one order of magnitude decay channel is the one of curvaton candidate into volume modulus and volume axion with decay rate:

$$\Gamma_{\sigma_\phi \to \chi \sigma_b} = \frac{g_1}{16\pi m_{\sigma_\phi}} \tag{4.51}$$

where $g_1 \simeq -\frac{\sqrt{3\lambda_\phi}|W_0|^2 a_\phi^2 \tau_\phi^{11/4}}{\tau_b^{15/4}}$ as in (F.8), giving us the final result

$$\Gamma_{\sigma_\phi \to \chi\sigma_b} \simeq \frac{3\lambda_\phi |W_0|^3 a_\phi^3}{32\pi} \frac{\tau_\phi^{9/2}}{\mathcal{V}^4} = \frac{1}{4N_g} \Gamma_{\sigma_\phi \to \gamma_{\text{vis}}\gamma_{\text{vis}}} \tag{4.52}$$

coherently with [20].

3) Another subdominant by one order of magnitude decay channel is the one of curvaton candidate into standard model modulus and standard model axion with decay rate:

$$\Gamma_{\sigma_\phi \to \phi_M \sigma_M} = \frac{g_2}{16\pi m_{\sigma_\phi}} \tag{4.53}$$

where $g_2 \simeq \frac{2\sqrt{3\lambda_\phi}|W_0|^2 a_\phi^2 \tau_\phi^{11/4}}{\tau_b^{15/4}}$ as in (F.19), giving us the final result, reinstating M_P:

$$\Gamma_{\sigma_\phi \to \phi_M \sigma_M} \simeq \frac{3\lambda_\phi |W_0|^3 a_\phi^3}{8\pi} \frac{\tau_\phi^{9/2}}{\mathcal{V}^4} M_P = 4\Gamma_{\sigma_\phi \to \chi\sigma_b} = \frac{1}{N_g} \Gamma_{\sigma_\phi \to \gamma_{\text{vis}}\gamma_{\text{vis}}} \tag{4.54}$$

coherently again with [20].

Which finally let us compute the total decay width of the axion summing (4.50), (4.52), (4.54):

$$\Gamma_{\sigma_\phi} = \Gamma_{\sigma_\phi \to \gamma_{\text{vis}}\gamma_{\text{vis}}} + \Gamma_{\sigma_\phi \to \chi\sigma_b} + \Gamma_{\sigma_\phi \to \phi_M \sigma_M} = (4N_g + 5)\Gamma_{\sigma_\phi \to \chi\sigma_b} \tag{4.55}$$

and for $N_g = 12$ number of Gauge Bosons species we get:

$$\Gamma_{\sigma_\phi} = 53\Gamma^{\sigma_\phi \to \chi\sigma_b} = \Gamma_\phi \tag{4.56}$$

coherently with table 2 page 30 of [20]. We can then notice how the axion and the inflaton decay exactly at the same moment.

Case 2: D7s wrapped on the inflaton cycle

While the decay rates (4.52) and (4.54), (4.50) of course still hold, another dominant decay rate appear in this case and it will be the one for $\sigma_\phi \to \gamma_{\text{hid}}\gamma_{\text{hid}}$ which is:

$$\Gamma_{\sigma_\phi \to \gamma_{\rm hid}\gamma_{\rm hid}} = \frac{g_{\rm hid}}{16\pi m_{\sigma_\phi}} \tag{4.57}$$

where $g_{\rm hid} = \frac{4|W_0|^2 a_\phi^2}{\sqrt{3\lambda_\phi}} \frac{\tau_\phi^{5/4}}{\tau_b^{9/4}}$ as in (F.27), giving us the decay rate:

$$\Gamma_{\sigma_\phi \to \gamma_{\rm hid}\gamma_{\rm hid}} \simeq \frac{W_0^3 a_\phi^3 N_g^{\rm hid}}{6\pi\lambda_\phi} \frac{\tau_\phi^{5/2}}{\mathcal{V}^2} M_p = \frac{4}{9} \frac{N_g^{\rm hid}}{\lambda_\phi^2} \frac{\mathcal{V}^2}{\tau_\phi^2} \Gamma_{\sigma_\phi \to \phi_M \sigma_M} \tag{4.58}$$

coherently, again, with the one of the inflaton in [22] (so even now inflaton and axion decay together) and where $N_g^{\rm hid}$ is the number of hidden gauge bosons species. We now compute the total decay width as the one in scenario 1) (4.55) plus this last decay:

$$\Gamma_{\sigma_\phi} = \left(4N_g + 5 + \frac{4}{9} \frac{N_g^{\rm hid}}{\lambda_\phi^2} \frac{\mathcal{V}^2}{\tau_\phi^2} \right) \Gamma_{\sigma_\phi \to \chi\sigma_b} \tag{4.59}$$

4.2 The axion σ_ϕ as a curvaton candidate

In our model, given the choice of parameters:

$$\beta = W_0 = g_s e^{K_{C.S.}} = 2 \tag{4.60}$$

$$\lambda_\phi = 1 \tag{4.61}$$

$$\phi_* \sim O(0.2) \Rightarrow \tau_\phi = 44.81 \tag{4.62}$$

$$\mathcal{V} \sim 10^4 \tag{4.63}$$

we get, as seen in the previous chapter, both the Kähler cone constraints satisfied and the correct amplitude of curvature perturbations given by Planck. It is strictly fundamental then to check that the slice of parameter space we choose for our model is geometrically and phenomenologically consistent, in the sense that we don't exit the Kähler cone and that the curvature perturbations generated by our candidate curvaton σ_ϕ are negligible and so our axion will be a stealth curvaton. If not, the parameter subspace we took or the model itself has to be corrected.

4.2.1 Axion isocurvature perturbations

As we said in the introduction of this chapter the curvaton mechanism is divided into 2 parts:

- Generation of isocurvature perturbations;
- Conversion to curvature perturbations because of curvaton decay.

We are going now to study the first point by deriving the isocurvature power spectrum amplitude.

The candidate curvaton field σ_ϕ (non-canonically normalised θ_ϕ) during inflation has practically 0 effective mass since the exponential suppression on it in (4.43). The Hubble scale during inflation is:

$$H_{\text{inf}} = \frac{1}{\sqrt{3}}\sqrt{V_{\text{inf}}} \simeq \frac{W_0\sqrt{\beta\mathcal{S}}}{\sqrt{3}\mathcal{V}^{\frac{3}{2}}} \tag{4.64}$$

since:

$$V_{\rm inf} = V_0 \left(1 - \frac{c_{\rm loop}}{\beta\sqrt{\tau_\phi}}\right) \overset{\tau_\phi \gg 1}{\simeq} V_0 = \frac{\mathcal{S} W_0^2 \beta}{\mathcal{V}^3} \tag{4.65}$$

This show us clearly how:

$$m_{\sigma_\phi}^2 \sim |(V_{ax})_{\sigma_\phi \sigma_\phi}| \ll H_{\rm inf}^2 \tag{4.66}$$

and so, as stated in [32] and as computed in [14], at super-horizon scales the fluctuations of the axion induce a Gaussian Perturbation with scale independent power spectrum amplitude:

$$P_{\delta\sigma_\phi}^{1/2}(k) \simeq \frac{H_{\rm inf}^*}{2\pi} \tag{4.67}$$

where $H_{\rm inf}^* = H_{\rm inf}$ is the Hubble scale at horizon exit but for us, during whole inflation the Hubble parameter will be $H_{\rm inf}$. This spectrum can be shown to be practically scale dependent by computing its spectral index:

$$n_{\delta\sigma_\phi} - 1 \equiv \frac{d\ln(P_{\delta\sigma_\phi})}{d\ln k} = 2\frac{(V_{ax})_{\sigma_\phi\sigma_\phi}}{3H_{\rm inf}^2} - 2\epsilon \tag{4.68}$$

but $\epsilon \ll 1$ and even $|(V_{\rm ax})_{\sigma_\phi\sigma_\phi}| \ll 1$ so $n_{\delta\sigma_\phi} \simeq 1$. When all the cosmological scales have left the horizon, and so they are stretched out, the curvaton candidate starts oscillating leading to an evolution for our axion which is on an unperturbed regime. This means that our inflaton-orthogonal axion evolves under the Klein-Gordon equation on an expanding FLRW metric:

$$\ddot{\sigma}_\phi + 3H\dot{\sigma}_\phi + V_{\sigma_\phi} = 0 \tag{4.69}$$

and using the first order approximation $(\delta V_{\rm ax})_{\sigma_\phi} \simeq (V_{\rm ax})_{\sigma_\phi\sigma_\phi}\delta\sigma_\phi$ we can find that the perturbations of our axion follow the equation:

$$\ddot{\delta\sigma}_\phi + 3H\dot{\delta\sigma}_\phi + (V_{\rm ax})_{\sigma_\phi\sigma_\phi}\delta\sigma_\phi = 0 \tag{4.70}$$

which is the same as (4.69) if the potential is quadratic or is sufficiently flat. As we saw, the exponential suppression during inflation makes the potential almost exactly flat so, in our case the field and the inhomogeneous perturbation satisfy the same equation:

$$\ddot{\aleph} + 3H\dot{\aleph} \simeq 0 \tag{4.71}$$

where $\aleph = \sigma_\phi$ or $\aleph = \delta\sigma_\phi$. Since they both satisfy the same equation, then their ratio will be constant:

$$\left(\frac{\delta\sigma_\phi}{\sigma_\phi}\right) = \left(\frac{\delta\sigma_\phi}{\sigma_\phi}\right)_{\text{in}} \tag{4.72}$$

Then, as stated in [32] the isocurvature power spectrum is given by:

$$P_{\delta\sigma_\phi/\sigma_\phi} := P_{\text{iso}} = \frac{H_{\text{inf}}^2}{(2\pi\sigma_\phi^{\text{in}})^2} = \frac{W_0^2\beta\mathcal{S}}{12\pi^2\mathcal{V}^3{\sigma_\phi^{\text{in}}}^2} = \frac{W_0^2\beta\sqrt{\tau_\phi^*}}{9\pi^2\lambda_\phi\mathcal{V}^2{\theta_\phi^{\text{in}}}^2} \simeq \frac{6.03\cdot 10^{-9}}{{\theta_\phi^{\text{in}}}^2} \tag{4.73}$$

4.2.2 Initial conditions and stochastic behaviour

In the classical curvaton mechanism applications like [12] the value σ_ϕ^{in} is computed by solving the classical motion value neglecting only the second derivative term of the inhomogeneous perturbations Klein-Gordon equation (4.69):

$$\dot{\sigma_\phi} = \frac{\Delta\sigma_\phi}{\Delta t} \simeq -\frac{(V_{\text{ax}})_{\sigma_\phi}}{3H_{\text{inf}}} \tag{4.74}$$

which for one Hubble time $\Delta t = 1$ gives:

$$\dot{\sigma_\phi} = \Delta\sigma_\phi \simeq -\frac{(V_{\text{ax}})_{\sigma_\phi}}{3H_{\text{inf}}} \tag{4.75}$$

In the same time, quantum fluctuations make the field move of an amount:

$$\delta\sigma_\phi \simeq \frac{H_{\text{inf}}}{2\pi} \tag{4.76}$$

so, equating these two movements, $\Delta\sigma_\phi \simeq \delta\sigma_\phi$ we can find kind of a point of equilibrium between classical and quantum motion. However, in our case, this can't be done at CMB scale since the fact that at this point the potential is too flat for the axion to make the classical value equate the quantum one, in fact:

$$\dot{\sigma_\phi} = \Delta\sigma_\phi \simeq -\frac{(V_{\text{ax}})_{\sigma_\phi}}{3H_{\text{inf}}} \overset{\tau_\phi \gg 1}{\simeq} 0 \tag{4.77}$$

due to the usual exponential suppression in the potential. This means that $\Delta\sigma_\phi \ll \delta\sigma_\phi$ and so we can't solve this inequality as an equation finding σ_ϕ^{in}, which instead should be fixed a priori by hand since "chosen by Nature", owing to the fact that the motion is totally dominated by the quantum fluctuations. This happens until the curvaton starts to move classically, so when:

$$m_{\sigma_\phi} = H \tag{4.78}$$

which it is possible to check it happens close to the end of inflation. It is now interesting to study how these quantum fluctuations make the axion move and in order to do so we need to study the stochastic behaviour of the candidate curvaton Brownian motion in field space.

As just said since $H_{\text{inf}} = \text{const}$ and since, reinstating M_p momentarily, $V_{\text{ax}} \ll H_{\text{inf}}^2 M_p^2$ at large τ_ϕ then the axion is frozen during inflation and its behaviour can be described, like in [19], by the Langevin equation:

$$\frac{\partial\sigma_\phi}{\partial N_e} = -\frac{(V_{\text{ax}})_{\sigma_\phi}}{3H_{\text{inf}}^2} + \frac{H_{\text{inf}}}{2\pi}\xi \tag{4.79}$$

with ξ is a stochastic variable with variance $\langle\xi(N_e)_1\xi(N_e)_2\rangle = \delta((N_e)_1 - (N_e)_2)$ and zero mean $\langle\xi(N_e)\rangle = 0$. The last term of the Langevin equation describes the stochastic quantum "kick" the field is sbuject to during its dynamics. Since this stochastic nature of the process the system can be described by the Fokker-Planck equation for the probability density function $P(\sigma_\phi, \sigma_\phi^{in}, N_e)$:

$$\frac{\partial P}{\partial N_e} = -\frac{\partial((V_{\text{ax}})_{\sigma_\phi}P)}{\partial\sigma_\phi}\frac{1}{3H_{\text{inf}}^2} + \frac{H_{\text{inf}}^2}{8\pi^2}P_{\sigma_\phi\sigma_\phi} \tag{4.80}$$

where σ_ϕ^{in} is fixed during the inflation. Once we know the solution of the Fokker-Planck equation we can compute all the statistical quantities of the distribution like the mean and the variance via the general formula:

$$\langle\sigma_\phi^n\rangle(N_e) = \int d\sigma_\phi \sigma_\phi^n P(\sigma_\phi, \sigma_\phi^{in}, N_e) \tag{4.81}$$

In particular for $n = 1$ we will obtain the mean $\mathrm{E}(\sigma_\phi) = \langle\sigma_\phi\rangle$ and by the value $\mathrm{Var}(\sigma_\phi) = \langle\sigma_\phi\rangle^2 - \langle\sigma_\phi^2\rangle$ the variance. Since the axion is a spectator during inflation it undergoes Brownian motion described by the Langevin equation and so the equation (4.79) is now reduced to:

$$\frac{\partial\sigma_\phi}{\partial N_e} = \frac{H_{\mathrm{inf}}}{2\pi}\xi \tag{4.82}$$

or, from a distribution point of view, the Fokker-Planck equation has no second term becoming the 1D heat equation:

$$\frac{\partial P}{\partial N_e} = \frac{H_{\mathrm{inf}}^2}{8\pi^2} P_{\sigma_\phi\sigma_\phi} \tag{4.83}$$

whose solution is a gaussian:

$$P(\sigma_\phi, \sigma_\phi^{\mathrm{in}}, N_e) = \sqrt{\frac{2\pi}{N_e H_{\mathrm{inf}}^2}}\, e^{-\frac{2\pi^2}{N_e}\frac{(\sigma_\phi - \sigma_\phi^{\mathrm{in}})^2}{H_{\mathrm{inf}}^2}} \tag{4.84}$$

so we can easily see that $\mathrm{E}(\sigma_\phi) = \langle\sigma_\phi\rangle = \sigma_\phi^{\mathrm{in}}$ initial condition since the field is classically frozen in the initial value given by Nature. It is possible to show even that since the distribution is a Gaussian:

$$\langle\sigma_\phi^2\rangle = \left(\frac{H_{\mathrm{inf}}}{2\pi}\right)^2 N_e + \sigma_\phi^{\mathrm{in}\,2} \tag{4.85}$$

implying that:

$$\mathrm{Std}(\sigma_\phi) = \sqrt{\mathrm{Var}(\sigma_\phi)} = \sqrt{\langle\sigma_\phi^2\rangle - \langle\sigma_\phi\rangle^2} = \frac{H_{\mathrm{inf}}}{2\pi}\sqrt{N_e} \tag{4.86}$$

so that in N_e e-foldings the axion is kicked by quantum perturbations in average of $\frac{H_{\mathrm{inf}}}{2\pi}\sqrt{N_e}$ from its initial value.
At this point it's fundamental to check the order of this standard deviation in order to see if the quantum kick makes the axion go far away from its initial value

or if it stays in a neighborhood of the field space point σ_ϕ^{in}. Recalling:

$$H_{\text{inf}} = \frac{\sqrt{V_0}}{\sqrt{3}} = \frac{W_0\sqrt{\beta S}}{\sqrt{3}\mathcal{V}^{3/2}} \tag{4.87}$$

we got that for the parameter choice (4.60):

$$H_{\text{inf}} \simeq 1.63 \cdot 10^{-6} \tag{4.88}$$

So now we just need to compute the maximum number of possible e-foldings for the inflationary regime and see how much is the quantum kick. This can be done by computing the possible values of ϕ where at least one of these two conditions hold:

1) The slow roll approximation breaks ($\epsilon = 1$ for large ϕ);

2) We exit the Kähler cone.

The maximum value of ϕ from where one of these two conditions are not satisfied anymore is the last value where the physics of our model still holds and so the value where the axion stochastic motion starts. Beyond this value the physics is not understandable from our model.

Let us analyze in detail the two conditions:

1) **Slow roll breaking**:
Since the potential features an infinite plateau going at bigger and bigger values in ϕ, then the slow-roll condition breaks only for (after setting $c_{\text{loop}} = \frac{1}{16\pi^2}$ as in the field theoretic interpretation):

$$\epsilon = \frac{1}{2}\frac{V_\phi}{V} \simeq \frac{2}{9}\frac{(bc_{\text{loop}})^2}{\phi^{10/3}} \stackrel{!}{=} 1 \Leftrightarrow \phi_{\text{end}} = 3.4 \cdot 10^{-3} \tag{4.89}$$

differently from Fibre Inflation where the inflationary plateau is followed by a steep growth for large ϕ. Note how the value of ϕ_{end} does not depend on the case of wrapped or non-wrapped inflaton cycle and that we are not anymore in slow-roll inflationary regime $\forall \phi < 3.4 \cdot 10^{-3}$.

2) **Kähler cone exit**:
The second condition instead is not general, in fact, it is needed to take an explicit particular Calabi-Yau to check it is satisfied. Again, we are going to take, the second one in the table in [21], where the Kähler cone constraint $t_b + t_\phi > 0$ is given by:

$$\frac{|t_{\phi *}|}{t_b} = 0.6\phi_*^{2/3} < 1 \tag{4.90}$$

So we are at the boundary of the Kähler cone if $0.6\phi_*^{2/3} = 1 \Leftrightarrow \phi_* = \phi_{\max} = 2.15$. It is possible then to compute the maximum number of e-foldings available for inflation and this can be done starting from the standard e-folding formula and solving it numerically[2]:

$$N_e\big|_{\max} = \int_{\phi_{\text{end}}}^{\phi_{\max}} \frac{V_\phi}{V} d\phi \simeq 26798.6 \tag{4.91}$$

Plugging this latter result inside (4.86) we get:

$$\text{Std}(\sigma_\phi)\big|_{\max} = \sqrt{\text{Var}(\sigma_\phi)}\big|_{\max} = \sqrt{\langle\sigma_\phi^2\rangle - \langle\sigma_\phi\rangle^2} = \frac{H_{\text{inf}}}{2\pi}\sqrt{N_e\big|_{\max}} \simeq 4.25 \cdot 10^{-4} \tag{4.92}$$

which tells us that during the 26798.6 possible e-foldings we have very small displacement from the initial condition via quantum kick, so we will consider the curvaton as still in the mean value σ_ϕ^{in} of the distribution even at the end of the inflation since the point where (4.78) holds, and so the axion starts moving, is very close to the slow-roll breaking point as stated before.

4.2.3 From axion isocurvature to curvature perturbations

In this subsection we will explore the last part of curvaton mechanism, the conversion of isocurvature entropy perturbations generated by field σ_ϕ into curvature ones at the time of its decay. While the curvaton is still frozen by Hubble friction $m_{\sigma_\phi} = H_{\text{inf}}$, since the inflaton dominates the energy density of the universe

[2]This is in realty an overkill, in fact the Number of e-foldings should be computed until the curvaton starts moving which is a little bit before the end of inflation, however we exceeded in order to have a larger estimate.

then:

$$\zeta = \zeta_\phi = -H_* \frac{\delta\phi}{\dot{\phi}_*} \tag{4.93}$$

where ζ is the overall curvature perturbation and ζ_i for $i \in \{\phi, \sigma_\phi\}$ the curvature perturbation generated from inflaton or curvaton. When the curvaton mass falls under the Hubble scale $m_{\sigma_\phi} \ll H_{\text{inf}}$ then the curvaton starts to oscillate until it decays. At this point $H_{\text{inf}} = \Gamma_{\sigma_\phi}$, then the isocurvature perturbation of the candidate curvaton fluid is converted into curvature perturbation. The global curvature perturbation is then sourced both by the inflaton and by the curvaton and reads as [32]:

$$\zeta = \zeta_\phi + \frac{2\Omega_{\text{cv}}}{3} S_{iso} + \mathcal{O}(S_{\text{iso}}^2) \tag{4.94}$$

where S_{iso} isocurvature perturbations of the curvaton and where Ω_{cv} is the linear convertion factor between isocurvature and curvature perturbations of the curvaton field and reads as [32]:

$$\Omega_{\text{cv}} = \frac{3r_{\text{ed}}^{\text{dec}}}{4 - r_{\text{ed}}^{\text{dec}}} \tag{4.95}$$

where $r_{\text{ed}}^{\text{dec}} = \left.\frac{\rho_{\sigma_\phi}}{\rho_{\text{tot}}}\right|_{H_{\text{inf}}=\Gamma_{\sigma_\phi}}$. By plugging inside (4.95) the definition of $r_{\text{ed}}^{\text{dec}}$ we get, considering only first order [32]:

$$\Omega_{\text{cv}} \simeq \left.\left(\frac{\rho_{\sigma_\phi}}{\rho_{\text{tot}}}\right)\right|_{H_{\text{inf}}=\Gamma_{\sigma_\phi}} \simeq \left.\left(\frac{\rho_{\sigma_\phi}}{\rho_\phi}\right)\right|_{H_{\text{inf}}=\Gamma_{\sigma_\phi}} \tag{4.96}$$

where the last equality holds if we have the inflaton dominating while the curvaton decays.

Computing the correlation function $\langle \zeta_k \zeta_{k'} \rangle$ we get the power spectrum:

$$P_\zeta = P_{\zeta_\phi} + P_{\zeta_\sigma} = P_{\zeta_\phi} + \frac{4}{9}\Omega_{\text{cv}}^2 P_{\text{iso}} = P_{\zeta_\phi} + \frac{4}{9}\left.\left(\frac{\rho_{\sigma_\phi}}{\rho_{\text{tot}}}\right)^2\right|_{H_{\text{inf}}=\Gamma_{\sigma_\phi}} P_{\text{iso}} \tag{4.97}$$

Since the field content, at the end of inflation, is given by just the inflaton and the axion:

$$\left.\left(\frac{\rho_{\sigma_\phi}}{\rho_{\text{tot}}}\right)^2\right|_{H_{\text{inf}}=\Gamma_{\sigma_\phi}} < 1 \tag{4.98}$$

Leading us to a power spectrum:

$$P_{\zeta_\sigma} < \frac{4}{9} P_{\text{iso}} \tag{4.99}$$

The final step to perform is now to compute the conversion ratio and plugging inside our numerical values to check if the curvaton contribute to scalar perturbations (4.97) is negligible with the respect to the one of inflaton for our choice of parameters (4.60) or if we need additional fine tuning on the free ones.
We start recalling that, in general, since both the curvaton and the inflaton behave as matter they redshift as:

$$\rho_i|_{\text{dec}} = \rho_i|_{\text{end}} e^{-3N_\phi} \tag{4.100}$$

where $i \in \{\phi, \sigma_\phi\}$, $\rho_i|_{\text{dec}} = \rho_i|_{H_{\text{inf}}=\Gamma_{\sigma_\phi}}$, $\rho_i|_{\text{end}}$ is the value of the energy density at the end of the inflation and N_ϕ is the number of e-foldings of the inflaton domination which coincides in all scenarios with the number of e-foldings between the end of inflation and the decay time since inflaton and axion always decay together in geometric regime. Given this, it is evident that the ratio between the two energy densities is not dependent on the number of e-foldings, instead the one at decay time is exactly equal to the ratio of the energy densities at the end of inflation:

$$\frac{\rho_{\sigma_\phi}|_{\text{dec}}}{\rho_\phi|_{\text{dec}}} = \frac{\rho_{\sigma_\phi}|_{\text{end}}}{\rho_\phi|_{\text{end}}} \frac{\cancel{e^{-3N_\phi}}}{\cancel{e^{-3N_\phi}}} = \frac{\rho_{\sigma_\phi}|_{\text{end}}}{\rho_\phi|_{\text{end}}} \tag{4.101}$$

Now, at the end of inflation, two cases appear depending on the initial conditions of the axion:

1) The axion is set initially in a value $\frac{\pi}{2} + 2k\pi < a_\phi\theta_\phi < \frac{3}{2}\pi + 2k\pi, \;\; k \in \mathbb{Z}$ where the boundary terms of this interval $\frac{\pi}{2}, \frac{3}{2}\pi$ are terms where the second derivative of the axion potential changes sign. Then the curvaton is close to its minimum in the potential and, since we are at the end of the inflaton, the curvaton and the inflaton potentials are strongly dominated by non-perturbative corrections, so we expect the case to be very similar to [3] where the axion energy density is much less than the one for the inflaton due to preheating effects, justifying a posteriori (4.96) and so giving us:

$$\Omega_{\text{cv}} \simeq 0 \tag{4.102}$$

and so:

$$P_\zeta = P_{\zeta_\phi} \tag{4.103}$$

saving the consistency of the model in our slice of parameter space.

2) If instead the axion is such that: $2k\pi \leq a_\phi\theta_\phi^{\text{in}} \leq \frac{\pi}{2}+2$ or $\frac{3}{2}\pi+2k\pi \leq a_\phi\theta_\phi^{\text{in}} \leq 2k\pi,\ \ k \in \mathbb{Z}$ we need a full estimation of preheating effects which is beyond the scope of this thesis. However we can estimate the conversion factor by considering then (4.95) with:

$$\Omega_{\text{cv}} = \left.\frac{3\rho_{\sigma_\phi}}{4\rho_\phi + 3\rho_{\sigma_\phi}}\right|_{end} \tag{4.104}$$

However, at the end of inflation we have:

$$\rho_\phi|_{\text{end}} = V(\phi_{\text{end}}) \simeq V_0 \tag{4.105}$$

and, for the curvaton:

$$\rho_{\sigma_\phi}|_{\text{end}} = V_{\text{ax}}|_{\text{end}} = V_{\text{ax}}(\sigma_\phi^{\text{end}}) \tag{4.106}$$

The problem then reduces to find σ_ϕ^{end}. Again, since the axion is frozen almost until the end of the inflation, without loss of generality, we can set this value to:

$$\sigma_\phi^{\text{end}} = \sigma_\phi^{\text{in}} \tag{4.107}$$

An interesting choice of σ_ϕ^{in} could be exactly the one which is given at the boundary between the two cases [3] like $a_\phi\theta_\phi^{\text{in}} = \frac{\pi}{2}$ but in this case $V_{\text{ax}}(\sigma_\phi^{\text{in}}) = 0$ so again:

$$\Omega_{\text{cv}} = 0 \tag{4.108}$$

We will take instead:

$$\sigma_\phi^{\text{in}} : a_\phi\theta_\phi^{\text{in}} = \frac{\pi}{4} \Rightarrow \theta_\phi^{\text{in}} = \frac{\pi}{4a_\phi} = \begin{cases} \frac{1}{8} \text{ if inflaton cycle not-wrapped} \\ \frac{N}{8} \text{ if inflaton cycle wrapped} \end{cases} \tag{4.109}$$

We get then 2 subcases:

– If inflaton cycle is not wrapped by a D7-brane stack then we have instanton non-perturbative corrections and so $a_\phi = 2\pi$ leading us to an isocurvature perturbation (4.73) of:

$$P_{\rm iso} = \frac{6.03 \cdot 10^{-9}}{\theta_\phi^{{\rm in}\,2}} = 6.03 \cdot 10^{-9} \cdot 8 = 3.6 \cdot 10^{-7} \tag{4.110}$$

instead obtaining[3]:

$$V_{\rm ax}\left(\frac{\pi}{4}\right)|_{\rm end} = \frac{\sqrt{2}}{2}\Lambda(\tau_\phi^{\rm end}) = \frac{\sqrt{2}}{2}\mathcal{S}\frac{8W_0\pi A_\phi \tau_\phi^{\rm end}}{\mathcal{V}^2}e^{-2\pi\tau_\phi^{\rm end}} \tag{4.111}$$

$$\Omega_{\rm cv} = \left(1 + \frac{4\beta W_0}{3\frac{\sqrt{2}}{2}8\pi A_\phi \mathcal{V}\tau_\phi^{\rm end}e^{-2\pi\tau_\phi^{\rm end}}}\right)^{-1} \tag{4.112}$$

With $\tau_\phi^{\rm end} = 0.19$ such that $\phi = 0.0034$. At this point, in order to keep consistency with Planck measurement, so to have the curvaton with subleading contribution of the CMB spectrum, we need the condition:

$$P_{\zeta_\sigma} = \frac{4}{9}\Omega_{\rm cv}^2 P_{\rm iso} \le 10^{-10} \tag{4.113}$$

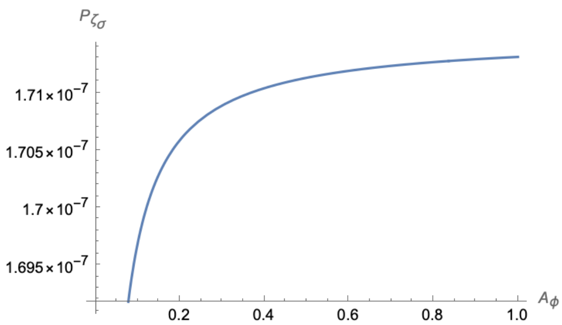

Figure 4.3: Plot of the curvature perturbation power spectrum amplitude P_{ζ_σ} generated by the curvaton-axion σ_ϕ with the respect to A_ϕ.

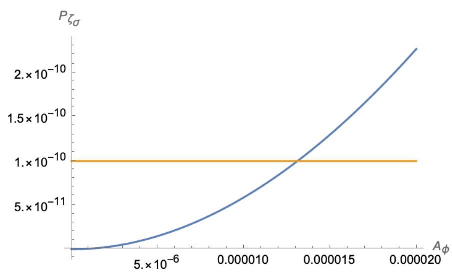

Figure 4.4: Plot of the region where curvature perturbation power spectrum amplitude P_{ζ_σ} generated by the axion becomes of an order lower than $\mathcal{O}(10^{-10})$.

[3] We write everything depending on $\tau_\phi^{\rm end}$ in order to have a more compact notation, if $\phi_{\rm end} \simeq 0.0034 \Rightarrow \tau_\phi^{\rm end} = 0.19$.

which, plugging inside all the values (4.60) translates into a constraint for the free parameter A_ϕ:

$$0 < A_\phi \leq 1.31 \cdot 10^{-5} \tag{4.114}$$

– The inflaton cycle is wrapped instead by a D7-brane stack: $a_\phi = \frac{2\pi}{N_\phi}$ where N_ϕ is the number of D7-branes wrapped around the cycle. We want now to have a theoretical bound for the value of N_ϕ and so for the number of D7-branes wrapped around the inflaton cycle. The isocurvature perturbations are given in this case by:

$$P_{\text{iso}} = \frac{3.86 \cdot 10^{-7}}{N_\phi^2} \tag{4.115}$$

While the conversion factor, with the choice $A_\phi = 1$[4], since the axion potential is the same as before reads as:

$$\Omega_{\text{cv}} = \frac{6784.6}{e^{\frac{1.27932}{N_\phi}} N_\phi + 6784.6} \tag{4.116}$$

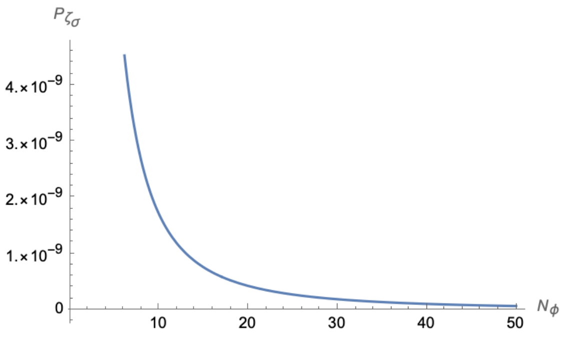

Figure 4.5: Plot of the curvature perturbation power spectrum amplitude P_{ζ_σ} generated by the curvaton-axion σ_ϕ with the respect to N_ϕ.

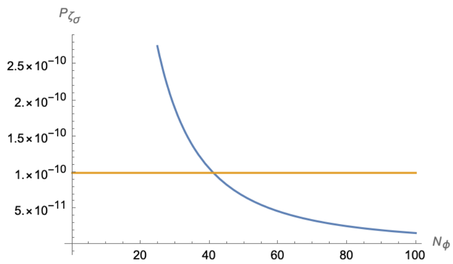

Figure 4.6: Plot of the region where curvature perturbation power spectrum amplitude P_{ζ_σ} generated by the axion becomes of an order lower than $\mathcal{O}(10^{-10})$.

[4]The previous constraint on it does not apply, the brane setting is different.

which leads to a constraint:

$$P_{\zeta_\sigma} = \frac{7.89407}{N^2 \left(e^{\frac{1.27932}{N_\phi}} N_\phi + 6784.6\right)^2} \leq 10^{-10} \tag{4.117}$$

holding for:

$$N_\phi \geq 41 \Leftrightarrow a_\phi \leq 0.15 \tag{4.118}$$

Chapter 5

Conclusions

In this Master Degree Thesis we studied the dynamics of the axion associated to the inflaton modulus in Loop Blow-Up inflation and how it can be seen a curvaton in such an inflationary scenario. We showed how the entire behaviour of the axion, which is totally non-trivial, can be splitted into two parts:

- At the starting point of the inflation the axion is ultralight, $m_{\sigma_\phi} \ll H_{\rm inf}$ and so it's classically frozen, however, since its potential depends on the value of the inflaton with a negative exponential, it's an almost flat direction. The inflaton then moves due to the stochastic quantum fluctuations even though the deviation in field space is of the order of $\mathcal{O}(10^{-4})$ from the initial value. During this period then the isocurvature perturbations are produced.
- Slightly before the end of the inflation, the axion unfreezes, $m_{\sigma_\phi} \gg H_{\rm inf}$, and it starts oscillating in its potential which now is not flat anymore. After some e-folds it decays with the same couplings and so exactly at the same time of the inflaton. This decay converts the whole isocurvature perturbation into curvature one with a conversion factor $\Omega_{\rm cv}$.

We then computed the amount of curvature perturbations in the 2 possible cases depending on the initial conditions and the results can be resumed as:

- If the axion initial condition is close enough to the minimum of its potential

($\frac{\pi}{2} < a_\phi \theta_\phi^{\text{in}} < \frac{3}{2}\pi$) the case is the same of [3] where the curvaton energy density is so small because of preheating effects that the conversion ratio is $\Omega_{\text{cv}} = 0$ and so the axion does not contribute to the curvature perturbations leading to a safe model for each choice of the remaining free parameters A_ϕ and a_ϕ.

- If the axion initial condition is far from the minimum ($0 < a_\phi \theta_\phi^{\text{in}} < \frac{\pi}{2}$ or $\frac{3}{2}\pi < a_\phi \theta_\phi^{\text{in}} < 2\pi$), a detailed, more general analysis of preheating effects is needed and goes beyond the scope of this thesis, however, we made a rough estimate of the results for $a_\phi \theta_\phi^{\text{in}} = \frac{\pi}{4}$. These can be divided in 2 subcases depending on the brane:
  - If the inflaton cycle is wrapped by an E3-Brane then we know that $a_\phi = 2\pi$ and, imposing that the curvature perturbations generated by the axion are subleading, we obtained the constraint for the only free parameter $0 < A_\phi \leq 1.31 \cdot 10^{-5}$. However we must take it even not too small in order to have non-perturbative corrections big enough to mantain the existence of the minimum.
  - If the inflaton cycle is wrapped by a D7-Brane stack then we don't know a_ϕ, which depends on the number of branes on the stack N_ϕ, however, picking $A_\phi \sim \mathcal{O}(1)$ as done in some numerical evaluation and plots in [2], imposing that the curvature perturbations generated by the axion are subleading, we obtained the constraint: $N_\phi \geq 41 \Leftrightarrow 0 < a_\phi \leq 0.15$, again, always recalling not to pick even a_ϕ too small to have suppressed non-perturbative corrections to V and so the spoil of its minimum. As a matter of completeness, if we would like to pick $A_\phi \leq \mathcal{O}(10^{-5})$ even in this case, it is possible to find that there is no constraint on N_ϕ and so on a_ϕ: the curvature perturbation of the axion is always subleading.

There are very interesting further developments of this work which can be analyzed. First of all a detailed study of the preheating case similarly to [3] will be very intriguing in order to reckon precisely the computation of the conversion factor in the second case. In addition to this, a possible and interesting further development can be to implement this mechanism in other Kähler moduli String Inflationary scenarios like Fibre Inflation [15], since they feature naturally the presence of axions with similar characteristics to the one we used here as curvaton candidate.

Appendix A

RNS formalism computations

Fermionic part of the RNS action

We can start by defining the Dirac conjugate of a spinor as:

$$\bar{\psi}^{\mu} = \psi^{\mu\dagger}\beta = \psi^{\mu\dagger}i\rho^{0} = \psi^{\mu T}\beta \tag{A.1}$$

where the last equality is true since ψ^{μ} Majorana spinor so $\psi_{+}^{\mu *} = \psi_{+}^{\mu}$ *and* $\psi_{-}^{\mu *} = \psi_{-}^{\mu}$ so:

$$\bar{\psi}^{\mu} = i\begin{pmatrix}\psi_{-}^{\mu} & \psi_{+}^{\mu}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}\psi_{+}^{\mu} & -i\psi_{-}^{\mu}\end{pmatrix} \tag{A.2}$$

in this notation, pointing out that that:

$$\rho^{\alpha}\partial_{\alpha} = \partial_{0}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + \partial_{1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & \partial_{1}-\partial_{0}\\ \partial_{1}+\partial_{0} & 0\end{pmatrix} = \begin{pmatrix}0 & -\partial_{-}\\ \partial_{+} & 0\end{pmatrix} \tag{A.3}$$

we can rewrite the fermionic action as:

$$S_{F} = \int d^{2}(\sigma\bar{\psi}^{\mu}\rho^{\alpha}\partial_{\alpha}\psi_{\mu}) = \frac{i}{\pi}\int d^{2}\sigma(\psi_{-}^{\mu}\partial_{+}\psi_{-\mu} + \psi_{+}^{\mu}\partial_{-}\psi_{+\mu}) \tag{A.4}$$

Equations of motion for fermionic field

The computation of equations of motion is as always obtained by variation of the action:

$$\delta S_F = \frac{i}{\pi}\int d^2\sigma(\delta\psi_-\partial_+\psi_- + \psi_-\partial_+\delta\psi_- + \delta\psi_+\partial_-\psi_+ + \psi_+\partial_-\delta\psi_+) = \tag{A.5}$$

integrating by parts and removing boundary terms that vanish:

$$= \frac{i}{\pi}\int d^2\sigma(\delta\psi_-\partial_+\psi_- - \partial_+\psi_-\delta\psi_- + \delta\psi_+\partial_-\psi_+ - \partial_-\psi_+\delta\psi_+) = \tag{A.6}$$

due to the fact that ψ_A^μ $A = 1, 2$ are fermionic variables, they anticommute with their variation:

$$= \frac{-2i}{\pi}\int d^2\sigma(\partial_+\psi_-)\delta\psi_- + (\partial_-\psi_+)\delta\psi_+ \stackrel{!}{=} 0 \tag{A.7}$$

giving us the result we saw in 1.

Global Supersymmetry of action

The transformation on which the action is invariant can be rewritten in component as:

$$\delta X^\mu = i(\epsilon_+\psi_-^\mu - \epsilon_-\psi_+^\mu) \tag{A.8}$$

$$\delta\psi_-^\mu = -2\partial_- X^\mu\epsilon_+ \tag{A.9}$$

$$\delta\psi_+^\mu = 2\partial_+ X^\mu\epsilon_- \tag{A.10}$$

We can see that this is a symmetry, in fact, suppressing again Lorentz indices:

$$\delta S = \frac{1}{\pi}\int d^2\sigma(2\partial_+\delta X\partial_- X + 2\partial_+ X\partial_-\delta X + i\delta\psi_-\partial_+\psi_- + i\psi_-\partial_+\delta\psi_- + i\delta\psi_+\partial_-\psi_+ + \tag{A.11}$$

$$+ i\psi_+\partial_-\delta\psi_+) = \frac{2i}{\pi}\int d^2\sigma(\epsilon_+\partial_+\psi_-\partial_- X - \epsilon_-\partial_+\psi_+\partial_- X + \epsilon_+\partial_+ X\partial_-\psi_+ \tag{A.12}$$

$$-\epsilon_-\partial_+ X\partial_-\psi_+ - \partial_- X\epsilon_+\partial_+\psi_- - \psi_-\partial_+\partial_- X\epsilon_+ + \partial_+ X\epsilon_-\partial_-\psi_+ + \psi_+\partial_-\partial_+ X\epsilon_-) = \tag{A.13}$$

paying attention to the sixth and last term, moving $\epsilon_\pm$ at first place of the term getting there 2 minuses:

$$= \frac{2i}{\pi} \int d^2\sigma \epsilon_+ (\cancel{\partial_+ \psi_- \partial_- X} + \partial_+ X \partial_- \psi_- - \cancel{\partial_- X \partial_+ \psi_-} + \psi_- \partial_+ \partial_- X) + \tag{A.14}$$

$$+\epsilon_- (-\partial_+ \psi_+ \partial_- X - \cancel{\partial_+ X \partial_- \psi_+} + \cancel{\partial_+ X \partial_- \psi_+} - \psi_+ \partial_- \partial_+ X) = \tag{A.15}$$

$$= \frac{2i}{\pi} \int d^2\sigma \epsilon_+ (\partial_- (\psi_- \partial_+ X)) - \epsilon_- (\partial_+ (\psi_+ \partial_- X)) = 0 \tag{A.16}$$

since it's a sum of 2 total derivatives, so it's boundary term that vanishes.

On-Shell Closure of Global Supersymmetry

We start by recalling the transformation:

$$\delta X^\mu = \bar{\epsilon} \psi^\mu \tag{A.17}$$

$$\delta \psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon \tag{A.18}$$

and by computing the commutator of two supersymmetry transformations acting on X^μ and ψ^μ:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \psi^\mu = \delta_{\epsilon_1}(\delta_{\epsilon_2} \psi^\mu) - \delta_{\epsilon_2}(\delta_{\epsilon_1} \psi^\mu) = \tag{A.19}$$

$$\delta_{\epsilon_1}(\rho^\alpha \partial_\alpha X^\mu \epsilon_2) - \delta_{\epsilon_2}(\rho^\alpha \partial_\alpha X^\mu \epsilon_1) = \rho^\alpha \epsilon_2 \partial_\alpha \delta_{\epsilon_1} X^\mu - \rho^\alpha \epsilon_1 \partial_\alpha \delta_{\epsilon_2} X^\mu = \rho^\alpha (\epsilon_2 \bar{\epsilon}_1 - \epsilon_1 \bar{\epsilon}_2) \partial_\alpha \psi^\mu = \tag{A.20}$$

Now using the spinor identity $\epsilon_2 \bar{\epsilon_1} - \epsilon_1 \bar{\epsilon_2} = -\bar{\epsilon_1} \rho_\beta \epsilon_2 \rho^\beta$and $\left\{\rho^\alpha, \rho^\beta\right\} = 2\eta^{\alpha\beta}$:

$$= -\bar{\epsilon}_1 \rho_\beta \epsilon_2 \rho^\alpha \rho^\beta \partial_\alpha \psi^\mu = -\bar{\epsilon}_1 \rho_\beta \epsilon_2 \left\{\rho^\alpha \rho^\beta\right\} \partial_\alpha \psi^\mu + \bar{\epsilon}_1 \rho_\beta \epsilon_2 \rho^\beta \rho^\alpha \partial_\alpha \psi^\mu = \tag{A.21}$$

$$= -\bar{\epsilon}_1 \rho_\beta \epsilon_2 2\eta_{\alpha\beta} \partial_\alpha \psi^\mu + \bar{\epsilon}_1 \rho_\beta \epsilon_2 \rho^\beta \rho^\alpha \partial_\alpha \psi^\mu = -2\bar{\epsilon}_1 \rho^\alpha \epsilon_2 \partial_\alpha \psi^\mu + \bar{\epsilon}_1 \rho_\beta \epsilon_2 \rho^\beta \rho^\alpha \partial_\alpha \psi^\mu = \tag{A.22}$$

$$= a^\alpha \partial_\alpha \psi^\mu + \bar{\epsilon}_1 \rho_\beta \epsilon_2 \rho^\beta \rho^\alpha \partial_\alpha \psi^\mu \tag{A.23}$$

Where $a^\alpha = -2\bar{\epsilon}_1 \rho^\alpha \epsilon_2$ can be interpreted as a parameter of the translation $-2\bar{\epsilon}_1 \rho^\alpha \epsilon_2 \partial_\alpha = a^\alpha \partial_\alpha$ and the second term vanishes on-shell which means using the equation of motion of ψ^μ which is $\rho^\alpha \partial_\alpha \psi^\mu = 0$.

So on-shell the commutator of 2 supersymmetry transformations lead to a translation on the world-sheet which is another supersymmetry transformation.

For the other field X^μ:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] X^\mu = \delta_{\epsilon_1}(\delta_{\epsilon_2} X^\mu) - \delta_{\epsilon_2}(\delta_{\epsilon_1} X^\mu) = \bar{\epsilon_2}\delta_{\epsilon_1}\psi^\mu - \bar{\epsilon_1}\delta_{\epsilon_2}\psi^\mu = \bar{\epsilon}_2\rho^\alpha\partial_\alpha X^\mu \epsilon_1 - 1 \leftrightarrow 2 = \tag{A.24}$$

using the spinor identity $\bar{\epsilon}_2\rho^\alpha\epsilon_1 = -\bar{\epsilon_1}\rho^\alpha\epsilon_2$:

$$= -2\bar{\epsilon}_1\rho^\alpha\epsilon_2\partial_\alpha X^\mu = a^\alpha\partial_\alpha X^\mu \tag{A.25}$$

which is again a translation of the amount $a^\alpha = -2\bar{\epsilon}_1\rho^\alpha\epsilon_2$.
So, in the end, the algebra of this transformation is closed on-shell.

Supercharges Explicit Expression

In order to have an explicit expression for the representation of the supercharges we start from the trivial case of transformation of a field under translations. We have that a field, scalar in order to simplify notation, ϕ transforms under a spacetime translation $\sigma^\alpha \to \sigma'^\alpha = \sigma^\alpha + a^\alpha$ as an operator as:

$$\varphi \to \varphi' = e^{-ia_\alpha P^\alpha}\varphi e^{ia_\alpha P^\alpha} \sim 1 - ia_\alpha P^\alpha\varphi 1 + ia_\alpha P^\alpha = \varphi + i\left[\varphi, a_\alpha P^\alpha\right] \tag{A.26}$$

As a field of Hilbert space instead as:

$$\varphi(\sigma^\alpha) \to \varphi' = e^{ia_\mu \mathcal{P}^\mu}\varphi = \varphi(\sigma^\alpha + a^\alpha) \tag{A.27}$$

where $\mathcal{P}$ is the infinite dimensional representation of momentum operator $\mathcal{P} = -i\partial_\alpha$. Equating the two expressions we get:

$$i\left[\varphi, a_\alpha P^\alpha\right] = ia_\alpha \mathcal{P}^\alpha\varphi = a^\alpha\partial_\alpha\varphi = \delta\phi \tag{A.28}$$

In a similar manner we can derive the expression of supersymmetry transformation for field, of supercharges and even the meaning of supersymmetry as world-sheet transformation.
Let us start with computing the effect of a supersymmetry transformation generated by $\bar{\epsilon}Q$ (which is a sum of a transformation generated by $-\epsilon_+ Q_-$ and one by

$-\epsilon_- Q_+$ so written in the form $e^{i\epsilon_A \rho^0 Q_A} = e^{\bar{\epsilon} Q}$):

$$\Phi(\sigma+\delta\sigma, \theta_+ + \delta\theta_+, \theta_- + \delta\theta_-) = e^{-i(\epsilon_+ Q_- - \epsilon_- Q_+)} \Phi(\sigma, \theta_+, \theta_-) e^{i(\epsilon_+ Q_- - \epsilon_- Q_+)} = \tag{A.29}$$

$$= e^{-i(\epsilon_+ Q_- - \epsilon_- Q_+)} e^{-i(\sigma^\alpha P_\alpha + \theta_+ Q_- + \theta_- Q_+)} \Phi(0,0,0) e^{i(\sigma^\alpha P_\alpha + \theta_+ Q_- + \theta_- Q_+)} e^{i(\epsilon_+ Q_- - \epsilon_- Q_+)} = \tag{A.30}$$

where we get that transformations on fermionic coordinate are made by a barred parameter $\bar{\epsilon}, \bar{\theta}$ and a generator because of the multiplication that needs a row and a coloumn vector. We now consider just the last two exponentials and use Becker-Campbell-Hausdorff formula which is:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{3!}(\frac{1}{2}[A,[A,B]])-\frac{1}{3!}(\frac{1}{2}[B,[B,A]])} \tag{A.31}$$

at just first order in commutator:

$$e^{i(\sigma^\alpha P_\alpha + \theta_+ Q_- + \theta_- Q_+)} e^{i(\epsilon_+ Q_- - \epsilon_- Q_+)} = e^{i(\sigma^\alpha P_\alpha + (\epsilon_+ + \theta_+) Q_- - (\epsilon_- + \theta_-) Q_+) + \frac{1}{2}[\epsilon_+ Q_-, \theta_- Q_+] + \frac{1}{2}[\epsilon_- Q_+, \theta_+ Q_-]} = \tag{A.32}$$

remembering that since $\epsilon_-, \epsilon_+, \theta_-, \theta_+$ are anticommuting variables: $[\epsilon_A Q_B, \epsilon_B Q_A] = \{\epsilon_A Q_B, \theta_B Q_A\} = 2\epsilon_A \theta_B \rho^\alpha P_\alpha (1-\delta_{A,B})$ where the expression of the anticommutator is obtained just because is the only one coherent with the index structure and that makes $\{Q_A, Q_A\} = 0$ as in 4D case we get:

$$e^{i\left[\sigma^\alpha P_\alpha + (\epsilon_+ + \theta_+) Q_- - (\epsilon_- + \theta_-) Q_+ - \frac{i}{\not{2}} \epsilon_+ \theta_- \not{2} \rho^\alpha P_\alpha + \frac{i}{\not{2}} \epsilon_- \theta_+ \not{2} \rho^\alpha P_\alpha\right]} = \tag{A.33}$$

$$e^{i[\sigma^\alpha P_\alpha + (\epsilon_+ + \theta_+) Q_- - (\epsilon_- + \theta_-) Q_+ + i(-\epsilon_+ \theta_- + \epsilon_- \theta_+) \rho^\alpha P_\alpha]} = e^{\left(\bar{\epsilon} + \bar{\theta}\right) Q + \left(\sigma^\alpha + \bar{\theta} \epsilon \rho^\alpha\right) P_\alpha} \tag{A.34}$$

Which leads us to the fact that a supersymmetry transformation is nothing more than a world-sheet translation, so a **geometrical superspace transformation**, of the amount:

$$\begin{cases} \sigma^\alpha \to \sigma'^\alpha = \sigma^\alpha + \bar{\theta} \rho^\alpha \epsilon = \sigma^\alpha + i\left(-\epsilon_+ \rho^\alpha \theta_- + \epsilon_- \rho^\alpha \theta_+\right) \Rightarrow \delta\sigma^\alpha = \bar{\theta} \rho^\alpha \epsilon = -\bar{\epsilon} \rho^\alpha \theta \\ \theta^A \to \theta'^A = \theta^A + \epsilon^A \Rightarrow \delta\theta^A = \epsilon^A \end{cases} \tag{A.35}$$

Now we want to derive the expression for the supercharges. In doing so we will compare the expression given by both transformations and the infinitesimal effect they have starting with $\mathbb{Q}_-$ (which is the representation of the generator Q_-):

$$e^{i\epsilon_+\mathbb{Q}_-}\Phi(\sigma,\theta) = (1+i\epsilon_+\mathbb{Q}_-)\Phi(\sigma,\theta) = \Phi(\sigma,\theta) + i\epsilon_+\mathbb{Q}_-\Phi(\sigma,\theta) = \Phi(\sigma - i\epsilon_+\rho^\alpha\theta_-, \theta_+ + \epsilon_+) \tag{A.36}$$

by expanding infinitesimally according to the law $\phi(t+\delta t, x+\delta x) = \phi(t,x) + \delta t\partial_t\phi(t,x) + \delta x\partial_x\phi(t,x)$:

$$\Phi(\sigma - i\epsilon_+\rho^\alpha\theta_-, \theta_+ + \epsilon_+) = \Phi(\sigma,\theta) - i\epsilon_+\rho^\alpha\theta_-\partial_\alpha\Phi(\sigma,\theta) + \epsilon_+\partial_{\theta_+}\Phi(\sigma,\theta) \tag{A.37}$$

Comparing this result with the previous one at all orders in ϵ_+:

$$\mathbb{Q}_- = -\rho^\alpha\theta_-\partial_\alpha + i\partial_{\theta_+} = \partial_{\bar{\theta}_-} - \rho^\alpha\theta_-\partial_\alpha \tag{A.38}$$

Repeating the same idea for $\mathbb{Q}_+$ (which is the representation of the generator Q_+):

$$e^{-i\epsilon_+\mathbb{Q}_+}\Phi(\sigma,\theta) = (1-i\epsilon_+\mathbb{Q}_+)\Phi(\sigma,\theta) = \Phi(\sigma,\theta) - i\epsilon_+\mathbb{Q}_+\Phi(\sigma,\theta) = \Phi(\sigma + i\epsilon_-\rho^\alpha\theta_+, \theta_- + \epsilon_-) \tag{A.39}$$

by expanding infinitesimally according to the law $\phi(t+\delta t, x+\delta x) = \phi(t,x) + \delta t\partial_t\phi(t,x) + \delta x\partial_x\phi(t,x)$:

$$\Phi(\sigma + i\epsilon_-\rho^\alpha\theta_+, \theta_- + \epsilon_-) = \Phi(\sigma,\theta) + i\epsilon_-\rho^\alpha\theta_+\partial_\alpha\Phi(\sigma,\theta) + \epsilon_-\partial_{\theta_-}\Phi(\sigma,\theta) \tag{A.40}$$

Comparing this result with the previous one at all orders in ϵ_+:

$$\mathbb{Q}_+ = -\rho^\alpha\theta_+\partial_\alpha + i\partial_{\theta_-} = \partial_{\bar{\theta}_+} - \rho^\alpha\theta_+\partial_\alpha \tag{A.41}$$

Supersymmetric transformations on Superfield components

We start from the transformation on superfield:

$$\delta\Phi(\sigma,\theta) = \delta X^\mu(\sigma) + \bar{\theta}\delta\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta\delta F^\mu(\sigma) = \bar{\epsilon}\mathbb{Q}\Phi(\sigma,\theta) = \bar{\epsilon}\mathbb{Q}(X^\mu(\sigma)) + \bar{\epsilon}\mathbb{Q}(\bar{\theta}\psi^\mu(\sigma)) + \tag{A.42}$$

$$+\bar{\epsilon}\mathbb{Q}(\frac{1}{2}\bar{\theta}\theta F^\mu(\sigma)) = \frac{\partial X^\mu}{\partial\bar{\theta}} - (\rho^\alpha\theta)_A\partial_\alpha X^\mu + \frac{\partial(\bar{\theta}\psi^\mu(\sigma))}{\partial\bar{\theta}} - (\rho^\alpha\theta)_A\partial_\alpha(\bar{\theta}\psi^\mu(\sigma)) + \frac{\partial(\frac{1}{2}\bar{\theta}\theta F^\mu(\sigma))}{\partial\bar{\theta}} + \tag{A.43}$$

$$-(\rho^\alpha\theta)_A\partial_\alpha(\frac{1}{2}\bar{\theta}\theta F^\mu(\sigma)) = \bar{\epsilon}^A\psi^\mu_A(\sigma) - \bar{\epsilon}^A(\rho^\alpha\theta)_A\partial_\alpha X^\mu(\sigma) + \bar{\epsilon}^A\theta_A F^\mu(\sigma) - \bar{\epsilon}^A(\rho^\alpha\theta)_A\bar{\theta}^B\partial_\alpha\psi^\mu_B(\sigma) \tag{A.44}$$

since $\partial_{\bar{\theta}}\theta = -i(\rho^\alpha)^{-1}$ and all the other terms get higher power of θ or $\bar{\theta}$ and so vanish.
We can continue the computation rewriting in component the term (or simply using the Fierz relation in 2D $\theta_A\bar{\theta}_B = -\frac{1}{2}\delta_{AB}\bar{\theta}_C\theta_C$):

$$\bar{\epsilon}^A(\rho^\alpha\theta)_A\bar{\theta}^B\partial_\alpha\psi^\mu_F(\sigma) = -(\epsilon_+\theta_- - \epsilon_-\theta_+)\rho^\alpha(\theta_+\partial_\alpha\psi^\mu_- - \theta_-\partial_\alpha\psi^\mu_+) = \tag{A.45}$$

owing to the anticommutating nature of the $\epsilon_-, \theta_-, \epsilon_+, \theta_+$ and so no more than 1 copy can be in each term:

$$= -(\epsilon_+\theta_-\theta_+\rho^\alpha\partial_\alpha\psi_- + \epsilon_-\theta_+\theta_-\rho^\alpha\partial_\alpha\psi^\mu_+) = \frac{1}{2}\bar{\theta}\theta\bar{\epsilon}\rho^\alpha\partial_\alpha\psi^\mu(\sigma) \tag{A.46}$$

Using this and the identity $\bar{\epsilon}\rho^\alpha\theta = -\bar{\theta}\rho^\alpha\epsilon$ we can rewrite finally the variation of the superfield as:

$$\delta\Phi(\sigma,\theta) = \bar{\epsilon}\psi^\mu(\sigma) + \bar{\theta}\rho^\alpha\epsilon\partial_\alpha X^\mu(\sigma) + \bar{\theta}\epsilon F^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta\bar{\epsilon}\rho^\alpha\partial_\alpha\psi^\mu(\sigma) \tag{A.47}$$

comparing with the previously obtained:

$$\delta\Phi(\sigma,\theta) = \delta X^\mu(\sigma) + \bar{\theta}\delta\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta\delta F^\mu(\sigma) \tag{A.48}$$

at each order of $\bar{\theta}$ we get the searched result.

Action in component fields

so we need to compute the effect of covariant derivatives $\mathcal{D}_A = \frac{\partial}{\partial\bar{\theta}^A} + (\rho^\alpha\theta)_A\partial_\alpha$ on superfield $\Phi^\mu(\sigma^\alpha,\theta) = X^\mu(\sigma^\alpha) + \bar{\theta}\psi^\mu(\sigma^\alpha) + \frac{1}{2}\bar{\theta}\theta F^\mu(\sigma^\alpha)$:

$$\mathcal{D}\Phi^\mu = \left(\frac{\partial}{\partial\bar{\theta}} + (\rho^\alpha\theta)\partial_\alpha\right)(X^\mu(\sigma^\alpha) + \bar{\theta}\psi^\mu(\sigma^\alpha) + \frac{1}{2}\bar{\theta}\theta F^\mu(\sigma^\alpha)) = \tag{A.49}$$

$$= \frac{\partial X^\mu}{\partial \bar{\theta}} + \psi^\mu + \frac{1}{2}\theta F^\mu + \frac{1}{2}\frac{\bar{\theta}}{i\rho_\alpha}F^\mu + (\rho^\alpha\theta)\partial_\alpha X^\mu + (\rho^\alpha\theta)_A\bar{\theta}^B\partial_\alpha\psi^\mu_B + (\rho^\alpha\theta)\frac{1}{2}\partial_\alpha\bar{\theta}\theta F^\mu = \tag{A.50}$$

where we cancelled first term because X^μ does not depend on $\theta, \bar{\theta}$ and last term since contains 2 $\bar{\theta}$. Using 2D Fierz transformation $\theta_A\bar{\theta_B} = -\frac{1}{2}\delta_{AB}\bar{\theta}_C\theta_C$:

$$\mathcal{D}\Phi^\mu = \psi^\mu + \theta F^\mu + \rho^\alpha\theta\partial_\alpha X^\mu - \frac{1}{2}\bar{\theta}\theta\rho^\alpha\partial_\alpha\psi^\mu \tag{A.51}$$

and analogally, just by multiplying by $i\rho^0$ correctly and switching according to anticommutation rules:

$$\bar{\mathcal{D}}\Phi^\mu = \bar{\psi}^\mu + B^\mu\bar{\theta} - \bar{\theta}\partial_\alpha X^\mu\rho^\alpha + \frac{1}{2}\bar{\theta}\theta\partial_\alpha\bar{\psi}^\mu\rho^\alpha \tag{A.52}$$

Plugging this inside the action and keeping just terms containing $\bar{\theta}\theta$ since the other ones vanish upon integration on $d^2\theta$:

$$S = \frac{i}{4\pi}\int d^2\sigma d^2\theta\bar{\mathcal{D}}_A\Phi^\mu\mathcal{D}_A\Phi_\mu = \frac{i}{4\pi}\int d^2\sigma d^2(\psi^\mu + \theta F^\mu + \rho^\alpha\theta\partial_\alpha X^\mu - \frac{1}{2}\bar{\theta}\theta\rho^\alpha\partial_\alpha\psi^\mu)(\bar{\psi}^\mu + \tag{A.53}$$

$$+B^\mu\bar{\theta} - \bar{\theta}\partial_\alpha X^\mu\rho^\alpha + \frac{1}{2}\bar{\theta}\theta\partial_\alpha\bar{\psi}^\mu\rho^\alpha) = \tag{A.54}$$

integrating by parts $\frac{i}{4\pi}\int d^2\sigma d^2\theta\frac{1}{2}\bar{\theta}\theta\partial_\alpha\bar{\psi}^\mu\rho^\alpha\psi_\mu = -\frac{i}{4\pi}\int d^2\sigma d^2\theta\frac{1}{2}\bar{\theta}\theta\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu$ we get:

$$= \frac{i}{4\pi}\int d^2\sigma d^2\theta(-\bar{\theta}\theta\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi^\mu - \bar{\theta}\theta F_\mu F^\mu + F^\mu\rho^\alpha\bar{\theta}\theta\partial_\alpha X_\mu - \bar{\theta}\theta F_\mu\rho^\alpha\partial_\alpha X^\mu + \tag{A.55}$$

$$-\bar{\theta}\theta\rho^\alpha\rho^\beta\partial_\alpha X_\mu\partial_\beta X^\mu) \tag{A.56}$$

recalling that for $\alpha \neq \beta \Rightarrow \rho^\alpha\rho^\beta\partial_\alpha X_\mu\partial_\beta X^\mu = 0$ since derivatives commute while, for $\alpha \neq \beta$ Dirac matrices not we get that $\rho^\alpha\rho^\beta\partial_\alpha X_\mu\partial_\beta X^\mu = (\rho^0)^2\partial_0 X_\mu\partial_0 X^\mu + (\rho^1)^2\partial_1 X_\mu\partial_1 X^\mu = -\partial_0 X_\mu\partial_0 X^\mu + \partial_1 X_\mu\partial^1 X^\mu = \partial_\alpha X_\mu\partial^\alpha X^\mu$ so, applying this to the calculations and using $\int d^2\theta\bar{\theta}\theta = -2i$ we get:

$$S = \frac{i}{4\pi}\int d^2\sigma(2i\bar{\psi}_\mu\rho^\alpha\partial_\alpha\psi^\mu - 2iF_\mu F^\mu + 2i\partial_\alpha X_\mu\partial^\alpha X^\mu) \tag{A.57}$$

Appendix B

No-scale structure

Single Kähler modulus

We start from the simplest case of a single Kähler modulus in a model with Kähler potential $K = -\ln(T+\bar{T})^n$ and $W = W_0$ Gukov-Vafa-Witten Superpotential (2.87). Having $K = -n\ln(T+\bar{T}) \Rightarrow K_T = K_{\bar{T}} = \frac{-n}{T+\bar{T}} \Rightarrow K_{T\bar{T}} = \frac{n}{(T+\bar{T})^2} \Rightarrow K^{T\bar{T}} = \frac{(T+\bar{T})^2}{n}$ we can compute the potential, after stabilising complex structure moduli and axio-dilaton and after noticing that since $W_0 = W_0(z^a) \Rightarrow \partial_T W_0 = \partial_{\bar{T}} W_0 = 0$:

$$V(T,\bar{T}) = e^K(K^{T\bar{T}}|K_T W_0|^2 - 3|W_0|^2) = e^K\left(\frac{(T+\bar{T})^2}{n}|\frac{-n}{T+\bar{T}}|^2|W_0|^2 - 3|W_0|^2\right) \tag{B.1}$$

$$V(T,\bar{T}) = e^K|W_0|^2(n-3) \tag{B.2}$$

In our case, the volume squared $\mathcal{V}^2 = (T+\bar{T})^3$ so $n = 3$ which leads to $V(T,\bar{T}) = 0$.

Multiple Kähler moduli

Even if the case of multiple Kähler Moduli seems so much difficult, Mathematics comes in our help. In fact given:

$$K = -\ln(f(T^1+\bar{T}^1,\ldots,T^j+\bar{T}^j) \tag{B.3}$$

we can use the Homogeinity of the function f to get:

$$f(\alpha(T^1+\bar{T}^1),\dots,\alpha(T^j+\bar{T}^j)) = \alpha^n f(T^1+\bar{T}^1,\dots,T^j+\bar{T}^j) \tag{B.4}$$

and by using Euler's theorem for the homogeneous functions:

$$(T^i+\bar{T}^{\bar{i}})\partial_i(e^{-K}) = (T^i+\bar{T}^{\bar{i}}))K_i e^{-K} = ne^{-K} \Rightarrow (T^i+\bar{T}^{\bar{i}})K_i = -n \tag{B.5}$$

which gives, upon differentiating in $\bar{T}^{\bar{j}}$ and recalling $K_i = K_{\bar{i}}$: $K_{\bar{j}} + (T+\bar{T})^i K_{i\bar{j}}$. Finally, multiplying by the inverse Kähler metric we get the fundamental relation:

$$K^{i\bar{j}}K_{\bar{j}} + (T+\bar{T})^i = 0 \tag{B.6}$$

and, using (B.5) after multiplying (B.6):

$$K_i K^{i\bar{j}} K_{\bar{j}} = n \tag{B.7}$$

From this last equation we can finally obtain the generalisation of the single Kähler modulus result:

$$V = e^K(K^{i\bar{j}}(K_i W_0)(K_{\bar{j}}\bar{W}_0) - 3|W_0|^2) = e^K|W_0|^2(n-3) \overset{n=3}{=} 0 \tag{B.8}$$

$(\alpha')^3$ No-Scale Breakdown

Starting from the $(\alpha')^3$ corrected Kähler potential $K = -2\ln(\mathcal{V}+y)$ where $y = \frac{\hat{\xi}}{2}$ we can repeat the idea of the previous section:

$$K_i = (K_0)_i \frac{\mathcal{V}}{\mathcal{V}+y} = (K_0)_i \frac{1}{1+\frac{y}{\mathcal{V}}} \simeq (K_0)_i \left(1 - \frac{y}{\mathcal{V}}\right) \tag{B.9}$$

by taking (B.5) (note that since now on what we find in (B.5) named K_i is now $(K_0)_i$) and by substituting $(K_0)_i = \frac{K_i}{(1-\frac{y}{\mathcal{V}})}$ in it we get:

$$(T^i+\bar{T}^{\bar{i}})K_i i = -3\left(1 - \frac{y}{\mathcal{V}}\right) \tag{B.10}$$

by repeating exactly the same steps as before, differentiating with the respect to $\bar{T}_{\bar{\jmath}}$, multiplying by the inverse metric and neglecting the subleading terms in $\mathcal{V}$ we get:

$$K_i K^{i\bar{\jmath}} K_{\bar{\jmath}} - 3\left(1 - \frac{y}{\mathcal{V}}\right) = K_i K^{i\bar{\jmath}} \left(\frac{3y}{\mathcal{V}}\right)_{\bar{\jmath}} = K_i K^{i\bar{\jmath}} K_{\bar{\jmath}} \frac{3y}{2\mathcal{V}} \tag{B.11}$$

reshuffling:

$$K_i K^{i\bar{\jmath}} K_{\bar{\jmath}} = 3\left(1 + \frac{\hat{\xi}}{4\mathcal{V}}\right) \tag{B.12}$$

Giving us the final expression for $\delta V_{(\alpha')^3}$:

$$\delta V_{(\alpha')^3} \simeq e^K (K_i K^{i\bar{\jmath}} K_{\bar{\jmath}} - 3)|W_0|^2 = \frac{3\hat{\xi}}{4}\frac{|W_0|^2}{\mathcal{V}^3} \neq 0 \Leftrightarrow \hat{\xi} \neq 0 \tag{B.13}$$

Extended no-scale structure

In the following part we are going to study the extended no-scale structure following the line of [17]. First of all it is preferable to give a rigorous definition of the extended no-scale structure following a Mathematical pattern.
Let Y_6 to be a Calabi-Yau 3-fold and considering our usual Type IIB String compaactification leading to an N=1 4D SUGRA where we have such a model:

$$K = K_0 + \delta K \tag{B.14}$$

$$W = W_0 \tag{B.15}$$

then we have that $\delta V_{\text{gs}} = 0$ if and only if the general correction δK is an homogeneous function of degree -2 in t^i 2-cycle volumes.

It's not easy to prove this statement, but we caqn do this rigorously using perturbative tools. Since we would like to focus only on the corrections of scalar potential we have to keep the focus on the part:

$$\delta V_{\text{gs}} = \frac{|W_0|^2}{\mathcal{V}^2}(K^{ij}\partial_i K \partial_j K - 3) \tag{B.16}$$

We clearly consider $K = -2\ln(\mathcal{V}) + \delta K_{\text{gs}}$ and, in order to get the Kähler metric with quantum corrections we introducing an expansion parameter ϵ. We have than:

$$K^{ij} = \left(\frac{\partial^2 K_0}{\partial\tau_i\partial\tau_j} + \epsilon\frac{\partial^2\delta K_{\rm gs}}{\partial\tau_i\partial\tau_j}\right)^{ij} = \left(\frac{\partial^2 K_0}{\partial\tau_i\partial\tau_j}\left(1+\epsilon\left(\left(\frac{\partial^2 K_0}{\partial\tau_i\partial\tau_j}\right)^{-1}\frac{\partial^2\delta K_{\rm gs}}{\partial\tau_i\partial\tau_j}\right)\right)\right)^{ij} \tag{B.17}$$

$$= \left(1+\epsilon\left(\left(\frac{\partial^2 K_0}{\partial\tau_i\partial\tau_j}\right)^{-1}\frac{\partial^2\delta K_{\rm gs}}{\partial\tau_i\partial\tau_j}\right)\right)^{il}\left(\frac{\partial^2 K_0}{\partial\tau_i\partial\tau_j}\right)^{lj} \tag{B.18}$$

And by using the Neumann series to compute the inverse matrix we get:

$$\left(1+\epsilon\left(\left(\frac{\partial^2 K_0}{\partial\tau_i\partial\tau_j}\right)^{-1}\frac{\partial^2\delta K_{\rm gs}}{\partial\tau_i\partial\tau_j}\right)\right)^{il} = \delta^i_l - \epsilon K_0^{in}\delta K_{nl} + \epsilon^2 K_0^{in}\delta K_{np}K_0^{pq}\delta K_{ql} + \mathcal{O}(\epsilon^3) \tag{B.19}$$

plugging this expansion, recalling $\delta K_{gs} = \delta K$ in (B.17) we are going to find:

$$K^{ij} = K_0^{ij} - \epsilon K_0^{in}\delta K_{nl}K_0^{lj} + \epsilon^2 K_0^{in}\delta K_{np}K_0^{pq}\delta K_{ql}K_0^{lj} + \mathcal{O}(\epsilon^3) \tag{B.20}$$

and by using this last expression in (B.16) we finally get:

$$\delta V_{\rm gs} = V_0 + \epsilon\delta V_1 + \epsilon^2\delta V_2 + \mathcal{O}(\epsilon^3) = \epsilon\delta V_1 + \epsilon^2\delta V_2 + \mathcal{O}(\epsilon^3) \tag{B.21}$$

since $V_0 = \frac{|W_0|^2}{\mathcal{V}^2}(K_0^{ij}(K_0)_i(K_0)_j) = 0$ due to no-scale structure we prove in the previous part (B.8) and where we have:

$$\delta V_1 = (2K_0^{ij}(K_0)_i\delta K_j - K_0^{in}\delta K_{nl}K_0^{lj}(K_0)_i(K_0)_j) \tag{B.22}$$

$$\delta V_2 = (K_0^{lj}\delta K_i\delta K_j - 2K_0^{in}\delta K_{nl}K_0^{lj}(K_0)_i\delta(K_0)_j + K_0^{in}\delta K_{np}K_0^{pq}\delta K_{ql}K_0^{lj}(K_0)_i(K_0)_j) \tag{B.23}$$

By using (B.6) we get that δV_1 can be rewritten in simpler way as:

$$\delta V_1 = -\frac{|W_0|^2}{\mathcal{V}^2}\left(-2\tau_j\frac{\partial(\delta K)}{\partial\tau_j} + \tau_n\tau_l\frac{\partial^2(\delta K)}{\partial\tau_n\tau_l}\right) \tag{B.24}$$

and by changing basis of the field space, defining $A_i j = \frac{\partial\tau_i}{\partial t^j} = \int_{Y_6} D_i\wedge D_j\wedge J = k_{ijl}t^l$ homogeneous function of degree 1 $\forall i,j$ (so the elements of the inverse A^{ij} of degree

-1), using the relations $t^i\tau_i = 3\mathcal{V},\ \ A_{ij}t^j = 2\tau_i$ and $A_{ij}t^it^j = 6\mathcal{V}$ we can go into 2-cycle volumes' one:

$$2\tau_j \frac{\partial}{\partial \tau_j} = t_l \frac{\partial}{\partial t_l} \tag{B.25}$$

$$\tau_n \tau_l \frac{\partial^2}{\partial \tau_n \partial \tau_l} = \frac{1}{4} t_i t_k \frac{\partial^2}{\partial t_i \partial t_k} + \frac{1}{4} A_{ij} t_i t_k \frac{\partial (A^{lp})}{\partial t_k} \frac{\partial}{\partial t_p} \tag{B.26}$$

Using Euler's theorem for homogeneous functions $t_k \frac{\partial (A^{lp})}{\partial t_k} = (-1)A^{lp}$ we get $\tau_n\tau_l \frac{\partial^2}{\partial \tau_n \tau_l} = \frac{1}{4} t_i t_k \frac{\partial^2}{\partial t_i \partial t_k} - \frac{1}{4} t_p \frac{\partial}{\partial t_p}$ and so:

$$\delta V_1 = -\frac{1}{4} \frac{|W_0|^2}{\mathcal{V}^2} \left(3 t_l \frac{\partial (\delta K)}{\partial t_l} + t_i t_k \frac{\partial^2 (\delta K)}{\partial t_i \partial t_k} \right) \tag{B.27}$$

and by reusing Euler's theorem, with m degree of homogeneity in 2-cycle volumes:

$$\delta V_1 = -\frac{|W_0|^2}{\mathcal{V}^2} \frac{1}{4} (3m + m(m-1)) \delta K = -\frac{1}{4} \frac{|W_0|^2}{\mathcal{V}^2} (m(m+2)) \delta K \tag{B.28}$$

Since we know that conjectured String Loop corrections to Kähler form are homogeneous of degree

$$\begin{cases} m = -2 \text{ for } \delta K^{\text{KK}}_{\text{gs}} \\ m = -4 \text{ for } \delta K^{\text{W}}_{\text{gs}} \end{cases} \tag{B.29}$$

then:

$$\begin{cases} \delta (V^{\text{KK}}_{\text{gs}})_1 = 0 \\ \delta (V^{\text{KK}}_{\text{gs}})_1 = -2 \frac{|W_0|^2}{\mathcal{V}^2} \delta K^{\text{W}}_{\text{gs}} \end{cases} \tag{B.30}$$

by computing the second order term in ϵ in (B.16) we can get the first non-vanishing contribute given by the exchange of strings with Kaluza Klein momentum giving the final potential of:

$$\delta V^{1Loop}_{\text{gs}} = \frac{|W_0|^2}{\mathcal{V}^2} \sum_i \left((g_s C_i)^{\text{KK}} (K_0)_{ii} - 2 \delta K^{\text{W}}_{\text{gs}} \right) \tag{B.31}$$

Appendix C

KKLT computations

We start from our Kähler potential and Superpotential with a single Kähler modulus T:

$$K = -2\ln(\mathcal{V}) = -3\ln(T+\bar{T}) = -3\ln(\tau) \quad \text{and} \quad W = W_0 + Ae^{-aT} \tag{C.1}$$

The consequent Scalar Potential reads as:

$$V = e^K (K^{i\bar{j}}\partial_i W \partial_{\bar{j}}\bar{W} + K^{i\bar{j}}(\partial_i W K_{\bar{j}}\bar{W} + \partial_i \bar{W} K_{\bar{j}} W)) = \tag{C.2}$$

$$= e^K \left(K^{T\bar{T}}\partial_T W \partial_{\bar{T}}\bar{W} + K^{T\bar{T}}K_{\bar{T}}((\partial_T W)\bar{W} + (\partial_T \bar{W})W) \right) \tag{C.3}$$

by neglecting the suppressed terms in the second part of the sum containing 2 non-perturbative corrections since subleading, recalling $K^{T\bar{T}}K_{\bar{T}} = -2\tau$, $K^{T\bar{T}} = \frac{4\tau^2}{3}$ and using (C.1) we get:

$$V = e^K \left(\frac{4\tau^2}{3} a^2 |A|^2 e^{-2aT} - 2\tau |W_0| (-aAe^{-aT}\bar{W}_0 - a\bar{A}e^{-a\bar{T}}W_0) \right) \tag{C.4}$$

by rewriting $W_0 = |W_0|e^{i\,\arg(W_0)}$, $A = |A|e^{i\,\arg(A)}$

$$V = e^K \left(\frac{4\tau^2}{3} a^2 |A|^2 e^{-2aT} + 2\tau a |A||W_0| e^{-a\tau} (e^{-i(\theta - \arg(A) + \arg(W_0))} + e^{+i(\theta - \arg(A) + \arg(W_0))} \right) = \tag{C.5}$$

$$= \frac{g_s e^{K_{\text{C.S.}}}}{8\pi} e^{K_K} |W_0|^2 \left(\frac{4\tau^2}{3|W_0|^2} a^2 |A|^2 e^{-2aT} + 2\tau \frac{a|A|}{|W_0|} e^{-a\tau} cos(\theta - \arg(A) + \arg(W_0)) \right) \tag{C.6}$$

Recalling $e^{K_K} = \frac{1}{\mathcal{V}^2}$:

$$V = \frac{g_s e^{K_{\text{C.S.}}}}{8\pi} |W_0|^2 \left(\frac{4\tau^2}{3\mathcal{V}^2 |W_0|^2} a^2 |A|^2 e^{-2aT} + 2\tau \frac{a|A|}{\mathcal{V}^2 |W_0|} e^{-a\tau} cos(\theta - \arg(A) + \arg(W_0)) \right) \tag{C.7}$$

minimising it in $\theta \Rightarrow \frac{\partial V}{\partial \theta} = 0 \;\; \& \;\; \cos(\theta - \arg(A) + arg(W_0)) = -1$
$\theta = \arg(A) - \arg(W_0) + (2k+1)\pi, \quad k \in \mathbb{Z}$ we get:

$$V = \frac{g_s e^{K_{\text{C.S.}}}}{8\pi} |W_0|^2 \left(\frac{4\tau^2}{3\mathcal{V}^2 |W_0|^2} a^2 |A|^2 e^{-2aT} - 2\tau \frac{a|A|}{\mathcal{V}^2 |W_0|} e^{-a\tau} \right) \tag{C.8}$$

Appendix D

Non-perturbative blow-up inflation: computations for $h^{1,1} = 3$ case

Kähler metric

We start by assuming the Kähler potential of the form:

$$K = -2 \ln \mathcal{V} \tag{D.1}$$

In order to compute the Kähler metric need the second derivatives of K with respect to the T_i. We start from first derivatives:

$$K_b = -2\,\partial_b \ln \mathcal{V} = -\frac{3\sqrt{\tau_b}}{2\mathcal{V}} \tag{D.2}$$

$$K_\phi = -2\,\partial_\phi \ln \mathcal{V} = \frac{3\lambda_\phi\sqrt{\tau_\phi}}{2\mathcal{V}} \tag{D.3}$$

$$K_s = -2\,\partial_s \ln \mathcal{V} = \frac{3\lambda_s\sqrt{\tau_s}}{2\mathcal{V}} \tag{D.4}$$

where $\partial_i \equiv \partial_{T_i} = \frac{1}{2}\partial_{\tau_i}$. The second order derivatives instead, read:

$$K_{b\bar{b}} = -\frac{3}{8\mathcal{V}\sqrt{\tau_b}} + \frac{9\tau_b}{8\mathcal{V}^2} \simeq -\frac{3}{8\mathcal{V}\sqrt{\tau_b}} \tag{D.5}$$

$$K_{\phi\bar{\phi}} = +\frac{3\lambda_\phi}{8\mathcal{V}\sqrt{\tau_\phi}} - \frac{9\lambda_\phi^2\tau_\phi}{8\mathcal{V}^2} \simeq \frac{3\lambda_\phi}{8\mathcal{V}\sqrt{\tau_\phi}} \tag{D.6}$$

$$K_{s\bar{s}} = +\frac{3\lambda_s}{8\mathcal{V}\sqrt{\tau_s}} - \frac{9\lambda_s^2\tau_s}{8\mathcal{V}^2} \simeq \frac{3\lambda_s}{8\mathcal{V}\sqrt{\tau_s}} \tag{D.7}$$

$$K_{b\bar{\phi}} = K_{\phi b} = -\frac{9\lambda_\phi\sqrt{\tau_b\tau_\phi}}{8\mathcal{V}^2} \tag{D.8}$$

$$K_{b\bar{s}} = K_{\bar{s}b} = -\frac{9\lambda_s\sqrt{\tau_b\tau_s}}{8\mathcal{V}^2} \tag{D.9}$$

$$K_{\phi\bar{s}} = K_{s\bar{\phi}} = \frac{9\lambda_\phi\lambda_s\sqrt{\tau_\phi\tau_s}}{8\mathcal{V}^2} \tag{D.10}$$

where we neglected terms suppressed by higher powers of $\mathcal{V}$. These computations lead then, with the crossing terms, to the Kähler metric:

$$K_{i\bar{j}} = \begin{pmatrix} K_{b\bar{b}} & K_{b\bar{\phi}} & K_{b\bar{s}} \\ K_{\phi\bar{b}} & K_{\phi\bar{\phi}} & K_{\phi\bar{s}} \\ K_{s\bar{b}} & K_{s\bar{\phi}} & K_{s\bar{s}} \end{pmatrix} \tag{D.11}$$

$$K_{i\bar{j}} = \begin{pmatrix} -\frac{3}{8\mathcal{V}\sqrt{\tau_b}} + \frac{9\tau_b}{2\mathcal{V}^2} & -\frac{9\lambda_\phi\sqrt{\tau_b\tau_\phi}}{8\mathcal{V}^2} & -\frac{9\lambda_s\sqrt{\tau_b\tau_s}}{8\mathcal{V}^2} \\ -\frac{9\lambda_\phi\sqrt{\tau_\phi\tau_b}}{8\mathcal{V}^2} & +\frac{3\lambda_\phi}{8\mathcal{V}\sqrt{\tau_\phi}} - \frac{9\lambda_\phi^2\tau_\phi}{2\mathcal{V}^2} & \frac{9\lambda_\phi\lambda_s\sqrt{\tau_\phi\tau_s}}{8\mathcal{V}^2} \\ -\frac{9\lambda_s\sqrt{\tau_s\tau_b}}{8\mathcal{V}^2} & \frac{9\lambda_\phi\lambda_s\sqrt{\tau_\phi\tau_s}}{8\mathcal{V}^2} & \frac{3\lambda_s}{8\mathcal{V}\sqrt{\tau_s}} - \frac{9\lambda_s^2\tau_s}{2\mathcal{V}^2} \end{pmatrix} \tag{D.12}$$

Whose leading inverse terms are:

$$(K^{-1})^{ij} \approx \begin{pmatrix} \frac{4\tau_b^2}{3} & 4\tau_b\tau_\phi & 4\tau_b\lambda_s \\ 4\tau_b\tau_\phi & \frac{8\sqrt{\tau_\phi}\tau_b^{3/2}}{3\lambda_\phi} & 4\tau_\phi\lambda_s \\ 4\tau_b\lambda_s & 4\tau_\phi\lambda_s & \frac{8\sqrt{\lambda_s}\tau_b^{3/2}}{3\lambda_s} \end{pmatrix}, \tag{D.13}$$

Minimum of the Scalar Potential

Now what we can easily do, before computing the canonical normalisation, is to find the minimum conditions via setting all the partial derivatives to zero:

$$\frac{\partial V}{\partial \mathcal{V}} = 0 \tag{D.14}$$

$$\frac{\partial V}{\partial \tau_\phi} = 0 \tag{D.15}$$

$$\frac{\partial V}{\partial \tau_s} = 0 \tag{D.16}$$

First of all it's important to rewrite the scalar potential after fixing the axions:

$$\begin{aligned} V(\mathcal{V}, \tau_\phi, \tau_s) =& \frac{g_s e^{K_{CS}}}{8\pi} W_0^2 \left(\frac{8a_s^2 A_s^2}{W_0^2 \lambda_s} \frac{\sqrt{\tau_s} e^{-2a_s\tau_s}}{\mathcal{V}} - \frac{4a_s A_s}{W_0} \frac{\tau_s e^{-a_s\tau_s}}{\mathcal{V}^2} \right. \\ & \left. + \frac{8a_\phi^2 A_\phi^2}{W_0^2 \lambda_\phi} \frac{\sqrt{\tau_\phi} e^{-2a_\phi\tau_\phi}}{\mathcal{V}} - \frac{4a_\phi A_\phi}{W_0} \frac{\tau_\phi e^{-a_\phi\tau_\phi}}{\mathcal{V}^2} + \frac{3\hat{\xi}}{4\mathcal{V}^3} \right) \end{aligned} \tag{D.17}$$

At this point, setting $\mathcal{S} = 1$ the second and third equations (D.15),(D.16) give us:

$$-\frac{16a_s^3 A_s^2}{W_0^2 \lambda_s} \frac{\sqrt{\tau_s} e^{-2a_s\tau_s}}{\mathcal{V}} + \frac{4a_s^2 A_s^2}{W_0^2 \lambda_s} \frac{e^{-2a_s\tau_s}}{\sqrt{\tau_s}\mathcal{V}} = -\frac{4a_s^2 A_s}{W_0} \frac{\tau_s e^{-a_s\tau_s}}{\mathcal{V}^2} + \frac{4a_s A_s}{W_0} \frac{e^{-a_s\tau_s}}{\mathcal{V}^2} \tag{D.18}$$

and (here we show the right simplification for proceeding):

$$-\frac{\cancel{16}^{4} a_\phi^{\cancel{3}^{2}} A_\phi^{\cancel{2}}}{W_0^{\cancel{2}} \lambda_\phi} \frac{\sqrt{\tau_\phi} e^{-\cancel{2}a_\phi\tau_\phi}}{\cancel{\mathcal{V}}} + \frac{\cancel{4} a_\phi^{\cancel{2}} A_\phi^{\cancel{2}}}{W_0^{\cancel{2}} \lambda_\phi} \frac{e^{-\cancel{2}a_\phi\tau_\phi}}{\sqrt{\tau_\phi}\cancel{\mathcal{V}}} = -\frac{\cancel{4} a_\phi^{\cancel{2}} \cancel{A_\phi}}{\cancel{W_0}} \frac{\tau_\phi \cancel{e^{-a_\phi\tau_\phi}}}{\mathcal{V}^{\cancel{2}}} + \frac{\cancel{4} \cancel{a_\phi} \cancel{A_\phi}}{\cancel{W_0}} \frac{\cancel{e^{-a_\phi\tau_\phi}}}{\mathcal{V}^{\cancel{2}}} \tag{D.19}$$

Leading us to:

$$\frac{a_s A_s e^{-a_s\tau_s}}{W_0 \lambda_s \sqrt{\tau_s}} (1 - 4a_s\tau_s) = \frac{1}{\mathcal{V}} (1 - a_s\tau_s) \tag{D.20}$$

$$\frac{a_\phi A_\phi e^{-a_\phi\tau_\phi}}{W_0 \lambda_\phi \sqrt{\tau_\phi}} (1 - 4a_\phi\tau_\phi) = \frac{1}{\mathcal{V}} (1 - a_\phi\tau_\phi) \tag{D.21}$$

Which can be rewritten as:

$$a_s A_s e^{-a_s\tau_s} = \frac{W_0\lambda_s\sqrt{\tau_s}}{\mathcal{V}}\frac{(1-a_s\tau_s)}{(1-4a_s\tau_s)} \tag{D.22}$$

$$a_\phi A_\phi e^{-a_\phi\tau_\phi} = \frac{W_0\lambda_\phi\sqrt{\tau_\phi}}{\mathcal{V}}\frac{(1-a_\phi\tau_\phi)}{(1-4a_\phi\tau_\phi)} \tag{D.23}$$

And, finally, recalling to evaluate them in the minimum, to:

$$\langle\mathcal{V}\rangle = \frac{W_0\lambda_s\sqrt{\langle\tau_i\rangle}}{a_i A_i}\frac{(1-a_i\langle\tau_i\rangle)}{(1-4a_i\langle\tau_i\rangle)}e^{a_i\langle\tau_i\rangle} \quad \forall i = \phi, s \tag{D.24}$$

Since during inflation the inflaton value τ_ϕ is big this can be even approximated by [30]:

$$\langle\mathcal{V}\rangle = \frac{W_0\lambda_s\sqrt{\langle\tau_i\rangle}}{4a_i A_i}e^{a_i\langle\tau_i\rangle} \tag{D.25}$$

Instead, the first equation (D.14), leads us to:

$$-\frac{8a_s^2A_s^2}{W_0^2\lambda_s}\frac{\sqrt{\tau_s}e^{-2a_s\tau_s}}{\mathcal{V}^2}+\frac{8a_sA_s}{W_0}\frac{\tau_s e^{-a_s\tau_s}}{\mathcal{V}^3} = +\frac{8a_\phi^2A_\phi^2}{W_0^2\lambda_\phi}\frac{\sqrt{\tau_\phi}e^{-2a_\phi\tau_\phi}}{\mathcal{V}^2}-\frac{8a_\phi A_\phi}{W_0}\frac{\tau_\phi e^{-a_\phi\tau_\phi}}{\mathcal{V}^3}+\frac{9\hat{\xi}}{4\mathcal{V}^4} \tag{D.26}$$

Solving in $\mathcal{V}$ and plugging its vacuum expectation value depending on $\langle\tau_s\rangle, \langle\tau_\phi\rangle$ into (D.24) we finally get:

$$\langle\tau_s\rangle \simeq \langle\tau_\phi\rangle \simeq \left(\frac{\hat{\xi}}{2\lambda_s}\right)^{2/3} \simeq \left(\frac{\xi}{g_s^{3/2}}\right)^{2/3} \tag{D.27}$$

By substituting all these values inside the potential (D.17) we get, following [23] in $h^{1,1} = 3$ case:

$$\langle V\rangle = \frac{-3W_0^2}{2\mathcal{V}^3}\left(\left(\frac{\lambda_\phi}{a_\phi^{3/2}}+\frac{\lambda_s}{a_s^{3/2}}\right)\ln(\langle\mathcal{V}\rangle)^{3/2}-\frac{\xi}{2}\right) \overset{\mathcal{V}\gg 1}{<} 0 \tag{D.28}$$

However, adding the uplift term $V_{\text{up}} \simeq \frac{cW_0}{\mathcal{V}^2}$ and fine tuning the coefficient c_{up} we can get a Minkowski or slightly De Sitter vacuum.

Canonical Normalisation: Differential Equation Approach

As we previously stated, the inflaton for us will be the modulus τ_ϕ and we want now to derive their canonical normalisation. In order to rewrite the potential and all the other quantities we are going to compute in terms of canonically normalised fields.

The Kinetic Lagrangian for the canonically normalised inflaton ϕ can be written as:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}(\partial_\mu \phi)^2 + \dots = K_{\phi\phi}((\partial_\mu \tau_\phi)(\partial^\mu \tau_\phi)) + \dots \tag{D.29}$$

where dots indicate kinetic terms of other moduli. Substituting (D.6) in:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}(\partial_\mu \phi)^2 + \dots = \frac{3\lambda_\phi}{8\mathcal{V}\sqrt{\tau_\phi}}((\partial_\mu \tau_\phi)(\partial^\mu \tau_\phi)) + \dots \tag{D.30}$$

so, in order to find the expression for ϕ in terms of τ_ϕ we need to solve this differential equation:

$$\partial_\mu \phi = \sqrt{\frac{3\lambda_\phi}{4\mathcal{V}}} \frac{\partial_\mu \tau_\phi}{\tau_\phi^{\frac{1}{4}}} \tag{D.31}$$

which gives:

$$\phi = \sqrt{\frac{4\lambda_\phi}{3\mathcal{V}}} \tau_\phi^{3/4} \tag{D.32}$$

With such a canonical normalisation the inflaton scalar Potential

$$V_{\text{inf}} = V_0 \left(1 - \frac{16 a_\phi A_\phi}{W_0 \hat{\xi}} \tau_\phi \mathcal{V} e^{-a_\phi \tau_\phi}\right) \tag{D.33}$$

can be written, just by easily substituting τ_ϕ using (D.32) as:

$$V_{\text{can.inf.}} = V_0 \left(1 - \frac{16 a_\phi A_\phi}{W_0 \hat{\xi}} \left(\frac{3\mathcal{V}}{4\lambda_\phi}\right)^{2/3} \phi^{4/3} \mathcal{V} e^{-a_\phi \left(\frac{3\mathcal{V}}{4\lambda_\phi}\right)^{2/3} \phi^{4/3}}\right) \tag{D.34}$$

Canonical Normalisation: Linear-Algebraic Approach close to Minimum

As we said before, the canonical normalisation can be computed, around the minimum, even through the mass matrix eigenvalues, let us do so, following [22], starting expanding each modulus around its vacuum expectation value:

$$\tau_i = \langle \tau_i \rangle + \delta\tau_i, \quad \forall i = b, \phi, s. \tag{D.35}$$

Plugging it inside the Lagrangian we get[1]:

$$\mathcal{L} = K_{ij}\partial_\mu (\delta\tau_i)\, \partial^\mu (\delta\tau_j) - \langle V \rangle - \frac{1}{2} V_{ij}\delta\tau_i\delta\tau_j + \mathcal{O}(\delta\tau^3), \tag{D.36}$$

using the Kähler metric (D.12) we can rewrite the original moduli $\delta\tau_i$ in terms of the canonically normalised fields around the minimum $\delta\phi_i$ as:

$$\delta\tau_i = \frac{1}{\sqrt{2}} P_{ij}\delta\phi_j \tag{D.37}$$

or, explicitly, as:

$$\begin{pmatrix} \delta\tau_b \\ \delta\tau_\phi \\ \delta\tau_s \end{pmatrix} = \begin{pmatrix} \\ \vec{v}_b \\ \\ \end{pmatrix} \frac{\delta\chi}{\sqrt{2}} + \begin{pmatrix} \\ \vec{v}_\phi \\ \\ \end{pmatrix} \frac{\delta\phi}{\sqrt{2}} + \begin{pmatrix} \\ \vec{v}_s \\ \\ \end{pmatrix} \frac{\delta\phi_s}{\sqrt{2}}, \tag{D.38}$$

the Lagrangian (D.36) takes the canonical form:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu (\delta\chi)\, \partial^\mu (\delta\chi) + \partial_\mu (\delta\phi)\, \partial^\mu (\delta\phi) + \partial_\mu (\delta\phi_s)\, \partial^\mu (\delta\phi_s)\right) - \langle V \rangle - \left(\frac{m_\chi^2}{2}\delta\chi^2 + \right.$$
$$\left. + \frac{m_\phi^2}{2}\delta\phi^2 + \frac{m_s^2}{2}\delta\phi_s^2\right), \tag{D.39}$$

with $\vec{v}_i$ eigenvectors and m_i^2 eigenvalues of the mass-squared matrix $(M^2)_{ij} \equiv$

[1] Since now on, in this appendix, in order to make the notation easier, we are recalling $K_{i\bar{j}} = K_{ij}$ and the same for V_{ij}, M_{ij} since, even if derivatives are on $T_i, \bar{T}_j$ we are working with quantities dependent on τ_i which is a real field.

$\frac{1}{2}\left(K^{-1}\right)_{ik} V_{kj}$. These eigenvectors $\vec{v_i}$ have to be normalised as $(\vec{v}_i^T)^m \frac{\partial K}{\partial T_m \partial T_l}(\vec{v}_j)^l = \delta_{ij}$.

Using instead the inverse Kähler metric at leading order (D.13) we can compute the Hessian of the scalar potential evaluated at the global minimum (D.24), (D.27), which, at leading order, looks like:

$$V_{ij} = \frac{1}{\langle\tau_b\rangle^{13/2}} \begin{pmatrix} c_b - c_\phi \langle\tau_\phi\rangle^{3/2} - c_s \langle\tau_s\rangle^{3/2} & -\frac{4a_\phi c_\phi \langle\tau_b\rangle\langle\tau_\phi\rangle^{3/2}}{27} & -\frac{4a_s c_s \langle\tau_b\rangle\langle\tau_s\rangle^{3/2}}{27} \\ -\frac{4a_\phi c_\phi \langle\tau_b\rangle\langle\tau_\phi\rangle^{3/2}}{27} & \frac{8a_\phi^2 c_\phi \langle\tau_b\rangle^2\langle\tau_\phi\rangle^{3/2}}{81} & 0 \\ -\frac{4a_s c_s \langle\tau_b\rangle\langle\tau_s\rangle^{3/2}}{27} & 0 & \frac{8a_s^2 c_s \langle\tau_b\rangle^2\langle\tau_s\rangle^{3/2}}{81} \end{pmatrix}, \tag{D.40}$$

where:

$$c_b \equiv \frac{99\nu}{4}, \qquad c_\phi \equiv \frac{81(4a_\phi A_\phi W_0)^2}{16\lambda_\phi}, \qquad c_s \equiv \frac{81(4a_s A_s W_0)^2}{16\lambda_s}. \tag{D.41}$$

As we said before multiplying (D.13) by (D.40) we can get the leading order mass (setting without loss of generality $\gamma_\phi = \gamma_s = A_\phi = A_s = W_0 = 1$):

$$\mathcal{M}^2 = \frac{1}{\langle\tau_b\rangle^{9/2}} \begin{pmatrix} -9\left(a_\phi\langle\tau_\phi\rangle^{5/2} + a_s\langle\tau_s\rangle^{5/2}\right)(1-7\delta) & 6a_\phi^2\langle\tau_b\rangle\langle\tau_\phi\rangle^{5/2}(1-5\delta) & 6a_s^2\langle\tau_b\rangle\langle\tau_s\rangle^{5/2}(1-5\delta) \\ -6a_\phi\sqrt{\langle\tau_b\rangle}\langle\tau_\phi\rangle^2(1-5\delta) & 4a_\phi^2\langle\tau_b\rangle^{3/2}\langle\tau_\phi\rangle^2(1-3\delta) & 6a_s^2\langle\tau_\phi\rangle\langle\tau_s\rangle^{5/2} \\ -6a_s\sqrt{\langle\tau_b\rangle}\langle\tau_s\rangle^2(1-5\delta) & 6a_\phi^2\langle\tau_s\rangle\langle\tau_\phi\rangle^{5/2} & 4a_s^2\langle\tau_b\rangle^{3/2}\langle\tau_s\rangle^2(1-3\delta) \end{pmatrix} \tag{D.42}$$

where $\delta \equiv \frac{1}{4a_\phi\langle\tau_\phi\rangle} = \frac{1}{4a_s\langle\tau_s\rangle} \simeq \frac{1}{4\ln(\mathcal{V})} \ll 1$. The two small blow-up modes τ_ϕ and τ_s are both stabilised non perturbatively so their dynamic behaviour is the same, then they will have the same mass which will be heavier that the large overall volume mode τ_b: $m_\phi \sim m_s \gg m_\chi$. Therefore we can work out the leading order volume scaling of the moduli mass spectrum, reinstating $M_p \neq 1$ for a better point of view:

$$m_\phi^2 \sim \ m_s^2 \sim \mathrm{Tr}[\mathcal{M}^2] = m_\chi^2 + m_\phi^2 + m_s^2 \sim \frac{a_\phi^2\langle\tau_\phi\rangle^2}{\langle\tau_b\rangle^3} \sim \frac{a_s^2\langle\tau_s\rangle^2}{\langle\tau_b\rangle^3} \sim \left(\frac{\ln\mathcal{V}}{\mathcal{V}}\right)^2 M_P^2, \tag{D.43}$$

$$m_\chi^2 \sim \ \frac{\mathrm{Det}[\mathcal{M}^2]}{\mathrm{Tr}[\mathcal{M}^2]^2} \sim \frac{m_\chi^2 m_\phi^2 m_s^2}{m_\phi^2 m_s^2} \sim \frac{(\langle\tau_\phi\rangle^{3/2}+\langle\tau_s\rangle^{3/2})}{a_\phi\langle\tau_\phi\rangle\langle\tau_b\rangle^{9/2}} \sim \frac{(\langle\tau_\phi\rangle^{3/2}+\langle\tau_s\rangle^{3/2})}{a_s\langle\tau_s\rangle\langle\tau_b\rangle^{9/2}} \sim \frac{M_P^2}{\mathcal{V}^3\ln\mathcal{V}}. \tag{D.44}$$

Let us now derive the corresponding eigenvectors from the classical eigenvector equation $\mathcal{M}^2\vec{v}_i = m_i^2\vec{v}_i \quad \forall i = b, \phi, s$. For the eigenvalue m_χ^2 we have (for $\vec{v}_b =$

(x_b, y_b, z_b)):

$$\mathcal{M}^2 \vec{v}_b = m_\chi^2 \vec{v}_b \Leftrightarrow \begin{cases} x_b \simeq a_s \langle \tau_b \rangle (y_b + z_b) \\ y_b = z_b \\ z_b \end{cases}, \tag{D.45}$$

where without loss of generality we have set $a_\phi = a_s$ and $\langle \tau_\phi \rangle = \langle \tau_s \rangle$.
Next, for the eigenvalue m_ϕ^2 we get the eigenvector (for $\vec{v}_\phi = (x_\phi, y_\phi, z_\phi)$):

$$\mathcal{M}^2 \vec{v}_\phi = m_\phi^2 \vec{v}_\phi \Leftrightarrow \begin{cases} x_\phi \simeq \frac{\langle \tau_s \rangle}{\sqrt{\langle \tau_b \rangle}} y_\phi \\ y_\phi \\ z_2 \simeq \frac{\langle \tau_s \rangle}{\langle \tau_b \rangle}^{3/2} y_\phi \ll y_\phi \end{cases}. \tag{D.46}$$

In the end the eigenvector correspondent to the eigenvalue m_s^2 is (for $\vec{v}_3 = (x_s, y_s, z_s)$):

$$\mathcal{M}^2 \vec{v}_s = m_s^2 \vec{v}_s \Leftrightarrow \begin{cases} x_s \simeq \frac{\langle \tau_s \rangle}{\sqrt{\langle \tau_b \rangle}} z_s \\ y_s \simeq \frac{\langle \tau_s \rangle}{\langle \tau_b \rangle}^{3/2} z_s \ll z_s \\ z_s \end{cases}. \tag{D.47}$$

The non fixed remaning components z_b, y_ϕ, z_s can be worked out via eigenvectors normalisation as (recalling $\mathcal{K} = \frac{\partial K}{\partial T_m \partial T_l}$):

$$\begin{cases} \vec{v}_b \cdot \mathcal{K} \cdot \vec{v}_b = 1 \Leftrightarrow z_b \simeq \frac{1}{a_s}, \\ \vec{v}_\phi \cdot \mathcal{K} \cdot \vec{v}_\phi = 1 \Leftrightarrow y_2 \simeq \langle \tau_s \rangle \langle \tau_b \rangle^{3/4}, \\ \vec{v}_s \cdot \mathcal{K} \cdot \vec{v}_s = 1 \Leftrightarrow z_s \simeq \langle \tau_s \rangle^{1/4} \langle \tau_b \rangle^{3/4}. \end{cases} \tag{D.48}$$

Therefore the general form (D.38) for the canonical normalisation takes the form:

$$\begin{pmatrix} \delta\tau_b \\ \delta\tau_\phi \\ \delta\tau_s \end{pmatrix} = \begin{pmatrix} \langle \tau_b \rangle \\ \frac{1}{a_s} \\ \frac{1}{a_s} \end{pmatrix} \frac{\delta\phi_1}{\sqrt{2}} + \begin{pmatrix} \langle \tau_b \rangle^{1/4} \langle \tau_s \rangle^{3/4} \\ \langle \tau_b \rangle^{3/4} \langle \tau_s \rangle^{1/4} \\ \frac{\langle \tau_s \rangle^{7/4}}{\langle \tau_b \rangle^{3/4}} \end{pmatrix} \frac{\delta\phi_2}{\sqrt{2}} + \begin{pmatrix} \langle \tau_b \rangle^{1/4} \langle \tau_s \rangle^{3/4} \\ \frac{\langle \tau_s \rangle^{7/4}}{\langle \tau_b \rangle^{3/4}} \\ \langle \tau_b \rangle^{3/4} \langle \tau_s \rangle^{1/4} \end{pmatrix} \frac{\delta\phi_3}{\sqrt{2}}, \tag{D.49}$$

which, since we have $\left(\frac{\langle \tau_s \rangle}{\langle \tau_b \rangle}\right)^{3/4} = \left(\frac{1}{g_s^{3/4}\sqrt{\mathcal{V}}}\right) \ll 1$ in terms of factors of the overall volume[2] scales as:

$$\frac{\delta\tau_b}{\langle\tau_b\rangle} \simeq \mathcal{O}(1)\,\delta\chi + \mathcal{O}\left(\frac{1}{g_s^{3/4}\sqrt{\mathcal{V}}}\right)\delta\phi + \mathcal{O}\left(\frac{1}{g_s^{3/4}\sqrt{\mathcal{V}}}\right)\delta\phi_s \simeq \mathcal{O}(1)\,\delta\chi, \tag{D.50}$$

$$\frac{\delta\tau_\phi}{\langle\tau_\phi\rangle} \simeq \mathcal{O}\left(\frac{1}{\ln(\mathcal{V})}\right)\delta\chi + \mathcal{O}\left(g_s^{-1/4}\sqrt{\mathcal{V}}\right)\delta\phi + \mathcal{O}\left(\frac{1}{g_s^{7/4}\sqrt{\mathcal{V}}}\right)\delta\phi_s \simeq \mathcal{O}\left(\frac{\sqrt{\mathcal{V}}}{g_s^{1/4}}\right)\delta\phi, \tag{D.51}$$

$$\frac{\delta\tau_s}{\langle\tau_s\rangle} \simeq \mathcal{O}\left(\frac{1}{\ln(\mathcal{V})}\right)\delta\chi + \mathcal{O}\left(\frac{1}{g_s^{7/4}\sqrt{\mathcal{V}}}\right)\delta\phi + \mathcal{O}\left(g_s^{-1/4}\sqrt{\mathcal{V}}\right)\delta\phi_s \simeq \mathcal{O}\left(\frac{1}{g_s^{7/4}\sqrt{\mathcal{V}}}\right)\delta\phi_s, \tag{D.52}$$

Giving the right volume scaling found in (D.32).

These expressions are not only useful in the context of canonical normalisation per se, in fact they have a very interesting geometric and physical meaning. From (D.50), we see that the overall volume mode is mostly given by $\delta\chi$ and then it mixes at subleading order with $\delta\phi$ and $\delta\phi_s$ in the same way; this have the meaning that the volume is bigger than both the blow-up modes in the same way (excluding the evolution during the inflation). From the other point of view, from (D.51) and (D.52), we realise that each blow-up mode is mostly given by $\delta\phi$, or $\delta\phi_s$ respectively, then it mixes with the overall volume, with an even more suppressed mixing with the other blow-up mode, which let us understand better the geometric separation between the two blow-up modes smoothing two singularities in different points of the Calabi-Yau three-fold. In addition to this, as stated in [22], since $\delta\tau_\phi$ is our non-canonically normalised inflaton, when inflation ends, reached its minimum, after some oscillations, the field τ_ϕ stops oscillating producing $\delta\tau_\phi$ particles. However the canonical normalisation (D.51) let us show how, at this point, since the enhanced coupling, the universe is filled by $\delta\phi$ particles and much fewer $\delta\chi$ and $\delta\phi_s$ such that the field $\delta\phi$ (and not the volume or small modulus) starts dominating the energy density to the universe.

[2]For $\tau_s \sim g_s^{-1} \sim \mathcal{O}(10)$ and $\tau_b \sim (\mathcal{V})^{2/3}$.

Appendix E

Axion canonical normalisation

We would like now to check the consistency of the result (4.6) with the linear algebra method. Differently from the inflaton case, in this case the two methods will perfectly give the same result since $K_{\phi\phi}$ does not depend on inflaton axion and so the differential equation solution is trivial as we saw before and matches the eigenvector solution.
We start from the potential which is the curvaton potential (4.21) and we compute his Hessian matrix with the respect to the axions at leading order given by[1]:

$$\langle V_{ij}^{\mathrm{c}} \rangle \equiv \left\langle \frac{\partial^2 V}{\partial\theta_i \partial\theta_j} \right\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3\lambda_\phi |W_0|^2 a_\phi^2 \tau_\phi^{3/2}}{\tau_b^{9/2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{E.1}$$

where we have, again, used the relations (D.25) and (D.27) after applying the second derivatives. The transformation to canonical fields is given by

$$\begin{pmatrix} \delta\theta_b \\ \delta\theta_\phi \\ \delta\theta_M \end{pmatrix} = \begin{pmatrix} \\ \vec{w}_b \\ \\ \end{pmatrix} \frac{\delta\sigma_b}{\sqrt{2}} + \begin{pmatrix} \\ \vec{w}_\phi \\ \\ \end{pmatrix} \frac{\delta\sigma_\phi}{\sqrt{2}} + \begin{pmatrix} \\ \vec{w}_M \\ \\ \end{pmatrix} \frac{\delta\sigma_M}{\sqrt{2}} \tag{E.2}$$

[1]We are going to neglect the small cycle axion θ_s, since we are interested, in the end, in the canonical normalisation of θ_ϕ.

This can be rewritten shortly as

$$\delta\theta_i = \frac{1}{\sqrt{2}} Q_{ij} \delta\sigma_j \tag{E.3}$$

where Q is the matrix that contains the vectors $\vec{w}_j$ as columns. They are the eigenvectors of the mass matrix $(M_\mathrm{c}^2)_{ij} \equiv \frac{1}{2} \langle (K^{-1})_{ik} V_{kj}^{\mathrm{ax}} \rangle$ whose eigenvalues are the axion masses. The eigenvectors fulfill the normalization condition

$$\vec{w}_i^{\mathsf{T}} \cdot \langle K \rangle \cdot \vec{w}_j \equiv Q_{ki} \langle K_{kl} \rangle Q_{lj} = \delta_{ij}. \tag{E.4}$$

The mass matrix at leading order is given by

$$(M_\mathrm{c}^2)_{ij} \approx \begin{pmatrix} 0 & \frac{6\lambda_\phi |W_0|^2 a_\phi^2 \tau_\phi^{5/2}}{\tau_b^{7/2}} & 0 \\ 0 & \frac{4|W_0|^2 a_\phi^2 \tau_\phi^2}{\tau_b^3} & 0 \\ 0 & \frac{6\lambda_\phi |W_0|^2 a_\phi^2 \tau_\phi^{5/2} \tau_M}{\tau_b^{9/2}} & 0 \end{pmatrix}. \tag{E.5}$$

The corresponding eigenvalues and eigenvectors are

$$m_{\theta_b}^2 = 0, \qquad \vec{w}_b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{E.6}$$

$$m_{\theta_\phi}^2 = \frac{4|W_0|^2 a_\phi^2 \tau_\phi^2}{\tau_b^3}, \qquad \vec{w}_\phi = \begin{pmatrix} \tau_b / \tau_M \\ \frac{2\tau_b^{3/2}}{3\lambda_\phi \sqrt{\tau_\phi} \tau_M} \\ 1 \end{pmatrix}, \tag{E.7}$$

$$m_{\theta_M}^2 = 0, \qquad \vec{w}_M = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{E.8}$$

After imposing the normalisation condition (E.4), the normalized eigenvectors read as:

$$\vec{w}_b \equiv \frac{\vec{w}_b}{\sqrt{\vec{w}_b^{\mathsf{T}} \cdot \langle K \rangle \cdot \vec{w}_b}} \approx \frac{2\tau_b}{\sqrt{3}} \vec{w}_b = \begin{pmatrix} \frac{2\tau_b}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}, \tag{E.9}$$

$$\vec{w}_\phi \equiv \frac{\vec{w}_\phi}{\sqrt{\vec{w}_\phi^{\mathsf{T}} \cdot \langle K \rangle \cdot \vec{w}_\phi}} \approx \frac{\sqrt{6\lambda_\phi}\tau_\phi^{3/4}\tau_M}{\tau_b^{3/4}} \vec{w}_\phi = \begin{pmatrix} \sqrt{6\lambda_\phi}\tau_\phi^{3/4}\tau_b^{1/4} \\ \frac{2\sqrt{2}\tau_\phi^{1/4}\tau_b^{3/4}}{\sqrt{3\lambda_\phi}} \\ \frac{\sqrt{6\lambda_\phi}\tau_\phi^{3/4}\tau_M}{\tau_b^{3/4}} \end{pmatrix}, \tag{E.10}$$

$$\vec{w}_M \equiv \frac{\vec{w}_M}{\sqrt{\vec{w}_M^{\mathsf{T}} \cdot \langle K \rangle \cdot \vec{w}_M}} \approx \frac{2\sqrt{2}\tau_M^{1/4}\tau_b^{3/4}}{\sqrt{3\lambda_M}} \vec{w}_M = \begin{pmatrix} 0 \\ 0 \\ \frac{2\sqrt{2}\tau_M^{1/4}\tau_b^{3/4}}{\sqrt{3\lambda_M}} \end{pmatrix}. \tag{E.11}$$

Giving us in particular:

$$\delta\theta_b \sim \mathcal{O}(\mathcal{V}^{2/3})\delta\sigma_b + \mathcal{O}(\mathcal{V}^{1/6})\delta\sigma_\phi \sim \mathcal{O}(\mathcal{V}^{2/3})\delta\sigma_b, \tag{E.12}$$

$$\delta\theta_\phi = \left(\frac{4\mathcal{V}}{3\lambda_\phi}\right)^{1/2} \tau_\phi^{1/4}\delta\sigma_\phi, \tag{E.13}$$

$$\delta\theta_M \sim \mathcal{O}(\mathcal{V}^{1/2})\delta\sigma_M + \mathcal{O}(\mathcal{V}^{-1/2})\delta\sigma_\phi \sim \mathcal{O}(\mathcal{V}^{1/2})\delta\sigma_M \tag{E.14}$$

which is coherent with the normalisation we computed (4.6).

Appendix F

Axion couplings

The decay possibilities for the axion and so the couplings, depend on the brane setting so, again, since we consider only the geometric regime, we have 2 possibilities which we have to inspect[1].

No D7-Branes Wrapped Around The Inflaton

Using the previosuly defined Kähler metric (D.12), the matrices P_{ij} in appendix D.2 of [20], Q_{kl}(E.3) and following the notation in [20] where indices on these 3 elements have the meaning of derivatives, the kinetic and potential trilinear coupling terms we need to explicitly compute are given by:

$$\begin{aligned}
\mathcal{L}_{\text{int,kin}} &= \langle \partial_{\tau_m} K_{np} \rangle \delta\tau_m \partial_\mu \delta\theta_n \partial^\mu \delta\theta_p \\
&= \frac{1}{2^{3/2}} K_{mnp} P_{mi} Q_{nj} Q_{pk} \delta\phi_i \partial_\mu \delta\sigma_j \partial^\mu \delta\sigma_k , \qquad \text{(F.1)}\\
\mathcal{L}_{\text{int,pot}} &= -\frac{1}{2} \left\langle \frac{\partial^3 V}{\partial\tau_m \partial\theta_n \partial\theta_p} \right\rangle \delta\tau_m \delta\theta_n \delta\theta_p \\
&= -\frac{1}{2^{5/2}} \left\langle \frac{\partial^3 V}{\partial\tau_m \partial\theta_n \partial\theta_p} \right\rangle P_{mi} Q_{nj} Q_{pk} \delta\phi_i \delta\sigma_j \delta\sigma_k \,. \qquad \text{(F.2)}
\end{aligned}$$

[1]Note that we are going to neglect the small cycle axion since, as stated in [20], this choice will not impact the couplings values.

Let us first argue that the potential couplings to the volume and SM axions vanish: since V does not depend on θ_b or θ_M but only on θ_ϕ, the indices n and p in (F.2) must both take on the value "ϕ". However, the components $Q_{\phi b}$ and $Q_{\phi M}$ vanish, so that there are no potential couplings $\sim\delta\phi\delta\sigma_b\delta\sigma_b$ or $\sim\delta\phi\delta\sigma_M\delta\sigma_M$. However we don't care too much about it since we want terms with $\delta\sigma_\phi$ inside so the only important note is that the curvaton $\delta\sigma_\phi$ can't decay, since it has equal mass, in the inflaton ϕ+something or in itself+something, so no terms can contain σ_ϕ or ϕ. We now compute the third derivatives of the Kähler potential now:

$$K_{bbb}=-\frac{3}{2\tau_b^3},\quad K_{bb\phi}=\frac{45\lambda_\phi\sqrt{\tau_\phi}}{16\tau_b^{7/2}},\quad K_{bbM}=\frac{45\lambda_M\sqrt{\tau_M}}{16\tau_b^{7/2}},\quad K_{b\phi\phi}=-\frac{9\lambda_\phi}{16\sqrt{\tau_\phi}\tau_b^{5/2}},$$
$$K_{b\phi M}=-\frac{27\lambda_\phi\lambda_M\sqrt{\tau_\phi\tau_M}}{8\tau_b^4},\quad K_{bMM}=-\frac{9\lambda_M}{16\sqrt{\tau_M}\tau_b^{5/2}},\quad K_{\phi\phi\phi}=-\frac{3\lambda_\phi}{16\tau_\phi^{3/2}\tau_b^{3/2}},\qquad \text{(F.3)}$$
$$K_{\phi\phi M}=\frac{9\lambda_\phi\lambda_M\sqrt{\tau_M}}{16\sqrt{\tau_\phi}\tau_b^3},\quad K_{\phi MM}=\frac{9\lambda_\phi\lambda_M\sqrt{\tau_\phi}}{16\sqrt{\tau_M}\tau_b^3},\quad K_{MMM}=-\frac{3\lambda_M}{16\tau_M^{3/2}\tau_b^{3/2}}$$

The trilinear couplings of the inflaton axion always involve exactly one other axion and one modulus field. The relevant coupling terms are given in (F.1) and (F.2). In analogy to the argument above, the potential coupling terms (F.2) vanish because the indices n and p must be "ϕ" for having a non vanishing potential derivative, while on of the indices j and k must either take on the value "b" or "M" for not having a decay of the inflaton axion in itself. This gives rise to either a factor "$Q_{\phi b}$" or "$Q_{\phi M}$", both of which are zero.

From the kinetic coupling terms of the inflaton axion are induced from (F.1). There are always two possibilities how $\delta\phi_i\partial_\mu\delta\sigma_j\partial^\mu\delta\sigma_k$ can contribute to a decay of σ_ϕ corresponding to $j=\phi$ or $k=\phi$. Before doing so we must eliminate the derivatives through a very easy procedure, using Klein Gordon equation and integration by parts in fact we can rewrite:

$$\delta\phi_i(\partial_\mu\delta\sigma_j)(\partial^\mu\delta\sigma_k)=\frac{1}{2}\left(m_i^2-m_j^2-m_k^2\right)\delta\phi_i\delta\sigma_j\delta\sigma_k, \qquad \text{(F.4)}$$

The individual coupling terms are given as follows:

- Decay $\delta\sigma_\phi \to \delta\chi\delta\sigma_b$:
  Here we have, since the masses $m_\chi, m_{\sigma_b} \ll m_{\sigma_\phi}$:

$$\mathcal{L}^{(\sigma_\phi\to\chi\sigma_b)}_{\text{int,kin},(c)} = -\frac{1}{2^{3/2}} K_{mnp} P_{mb} Q_{n\phi} Q_{pb} m^2_{\theta_\phi} \delta\chi\delta\sigma_\phi\delta\sigma_b. \tag{F.5}$$

  Since $Q_{Mb} = Q_{\phi b} = 0$, the index p is forced to take on the value "b" so that we obtain

$$\mathcal{L}^{(\sigma_\phi\to\chi\sigma_b)}_{\text{int,kin},(c)} = -\frac{1}{2^{3/2}} K_{mnb} P_{mb} Q_{n\phi} Q_{bb} m^2_{\theta_\phi} \delta\chi\delta\sigma_\phi\delta\sigma_b \tag{F.6}$$

$$\approx -\frac{1}{2^{3/2}} \left(K_{bbb} P_{bb} Q_{b\phi} + K_{bIb} P_{bb} Q_{\phi\phi}\right) Q_{bb} m^2_{\theta_\phi} \delta\chi\delta\sigma_\phi\delta\sigma_b \tag{F.7}$$

$$\approx -\frac{\sqrt{3\lambda_\phi}|W_0|^2 a^2_\phi \tau^{11/4}_\phi}{\tau^{15/4}_b} \delta\chi\delta\sigma_\phi\delta\sigma_b\,. \tag{F.8}$$

- Decay $\delta\sigma_\phi \to \delta\chi\delta\sigma_M$:
  This decay is given by (neglecting all the subleading masses):

$$\mathcal{L}^{(\sigma_\phi\to\chi\sigma_M)}_{\text{int,kin},(c)} = -\frac{1}{2^{3/2}} K_{mnp} P_{mb} Q_{n\phi} Q_{pM} m^2_{\theta_\phi} \delta\chi\delta\sigma_\phi\delta\sigma_M. \tag{F.9}$$

  Since $Q_{\phi M} = Q_{bM} = 0$, the index p is forced to take on the value "M" so that we have

$$\mathcal{L}^{(\sigma_\phi\to\chi\sigma_M)}_{\text{int,kin},(c)} = -\frac{1}{2^{3/2}} K_{mnp} P_{mb} Q_{n\phi} Q_{MM} m^2_{\theta_\phi} \delta\chi\delta\sigma_\phi\delta\sigma_M \tag{F.10}$$

$$\sim \tau_b^{-9/2} \delta\chi\delta\sigma_\phi\delta\sigma_M\,. \tag{F.11}$$

- Decay $\delta\sigma_\phi \to \delta\phi_M\delta\sigma_b$:
  The coupling terms, neglecting all the subleading masses again, read as:

$$\mathcal{L}^{(\sigma_\phi\to\phi_M\sigma_b)}_{\text{int,kin},(c)} = -\frac{1}{2^{3/2}} K_{mnp} P_{mM} Q_{n\phi} Q_{pb} m^2_{\theta_\phi} \delta\phi_M\delta\sigma_\phi\delta\sigma_b. \tag{F.12}$$

Here the index p is again forced to take on the value "b" and we obtain

$$\mathcal{L}^{(\sigma_\phi \to \phi_M \sigma_b)}_{\text{int,kin,}(c)} = -\frac{1}{2^{3/2}} K_{mnp} P_{mM} Q_{n\phi} Q_{pb} m^2_{\theta_\phi} \delta\phi_M \delta\sigma_\phi \delta\sigma_b \tag{F.13}$$

$$\sim \tau_b^{-9/2} \delta\phi_M \delta\sigma_\phi \delta\sigma_b \, . \tag{F.14}$$

- Decay $\delta\sigma_\phi \to \delta\phi_M \delta\sigma_M$:
  For this decay we have (neglecting all the subleading masses):

$$\mathcal{L}^{(\sigma_\phi \to \phi_M \sigma_M)}_{\text{int,kin,}(c)} = -\frac{1}{2^{3/2}} K_{mnp} P_{mM} Q_{n\phi} Q_{pM} m^2_{\theta_\phi} \delta\phi_M \delta\sigma_\phi \delta\sigma_M . \tag{F.15}$$

The index p must take on the value "M" and the coupling terms are given by:

$$\mathcal{L}^{(\sigma_\phi \to \phi_M \sigma_M)}_{\text{int,kin,}(c)} = -\frac{1}{2^{3/2}} K_{mnp} P_{mM} Q_{n\phi} Q_{MM} m^2_{\theta_\phi} \delta\phi_M \delta\sigma_\phi \delta\sigma_M \tag{F.16}$$

$$\approx -\frac{1}{2^{3/2}} \Big(K_{MbM} P_{MM} Q_{b\phi} + K_{M\phi M} P_{MM} Q_{\phi\phi} + \tag{F.17}$$

$$+ K_{MMM} P_{MM} Q_{M\phi} \Big) Q_{MM} m^2_{\theta_\phi} \delta\phi_M \delta\sigma_\phi \delta\sigma_M \approx \tag{F.18}$$

$$\approx \frac{2\sqrt{3\lambda_\phi} |W_0|^2 a_\phi^2 \tau_\phi^{11/4}}{\tau_b^{15/4}} \delta\phi_M \delta\sigma_\phi \delta\sigma_M \, . \tag{F.19}$$

So the dominant decays are the first and the last ones.
However an allowed decay is still missing from the list, which will be the one seen in (4.50). This decay is the one arising from the Gauge Kinetic function term:

$$(f W_\alpha W^\alpha \big|_F) + \text{h.c.} \tag{F.20}$$

Where W_α Supersymmetric generalisation of Field Strength tensor (now in the abelian case for simplicity, in the non-abelian is needed just to add a trace on the contraction of the tensors) and $f = T_{\text{M}}$ gauge kinetic function.

Expanding the expression we get:

$$(fW_\alpha W^\alpha\big|_F) + \text{h.c.} = -\tau_M F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}(f - f^*)F_{\mu\nu}\tilde{F}^{\mu\nu} = -\tau_M F_{\mu\nu}F^{\mu\nu} + \theta_M F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{F.21}$$

From here we can rewrite the Canonically Normalised Field Strength Tensor as $G_{\mu\nu} = \sqrt{2\langle\tau_{\text{M}}\rangle}F_{\mu\nu}$ giving us:

$$(fW_\alpha W^\alpha\big|_F) + \text{h.c.} = -\frac{\tau_M}{2\langle\tau_M\rangle}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_M}{2\langle\tau_M\rangle}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{F.22}$$

Where we kept the name $F_{\mu\nu}$ even for the canonically normalised field strength tensor $G_{\mu\nu} \to F_{\mu\nu}$. Expanding now around the minimum $\tau_M = \langle\tau_M\rangle + \delta\tau_M$ and $\theta_M = \langle\theta_M\rangle + \delta\theta_M$ and using the non-dominant canonical normalisation component of τ_M in terms of σ_ϕ in (E.10) we get:

$$\mathcal{L} \supset -\frac{\sqrt{6\lambda_\phi}\tau_\phi^{3/4}\tau_M}{\tau_b^{3/4}}\delta\sigma_\phi F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{F.23}$$

Eliminating derivatives of the gauge field A_μ in $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and in $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ using (F.4) and so adding the only mass different from zero m_{σ_ϕ} we get the coupling:

$$g_{\text{vis}} = -\frac{\sqrt{6\lambda_\phi}\tau_\phi^{3/4}\tau_M}{\mathcal{V}^{1/2}}m_{\sigma_\phi}^2 \tag{F.24}$$

D7-Branes Wrapped Around Inflaton Cycle

In the following scenario we instead have the inflaton wrapped by an hidden sector. Even though this change, we will obtain the same results for every coupling but we have even an additional one coming from the gauge kinetic function term where the inflaton τ_ϕ couples with hidden sector gauge bosons γ_{vis}. In this case then we get a term:

$$\mathcal{L} \supset -\tau_\phi F_{\mu\nu}F^{\mu\nu} - \theta_\phi F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{F.25}$$

Proceeding in analogy with the steps done to obtain (F.24) and using the canonical normalisation of the curvaton in (E.10) we get (again after removing the derivatives and inserting the mass):

$$\mathcal{L} \supset \frac{\tau_\phi^{-3/4}\tau_b^{3/4}}{\sqrt{3\lambda_\phi}} m_{\sigma_\phi}^2 \delta\sigma_\phi \delta A_\mu \delta A^\mu \tag{F.26}$$

where the coupling is then:

$$g_{\text{hid}} = \frac{\tau_\phi^{-3/4}\tau_b^{3/4}}{\sqrt{3\lambda_\phi}} m_{\sigma_\phi}^2 = \frac{4|W_0|^2 a_\phi^2}{\sqrt{3\lambda_\phi}} \frac{\tau_\phi^{5/4}}{\tau_b^{9/4}} = \frac{4|W_0|^2 a_\phi^2}{\sqrt{3\lambda_\phi}} \frac{\tau_\phi^{5/4}}{\mathcal{V}^{3/2}} \tag{F.27}$$

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